THE EFFECT OF MEMORY ON LARGE DEVIATIONS OF MOVING AVERAGE PROCESSES AND INFINITELY DIVISIBLE PROCESSES

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Long range dependence is a very important phenomenon that has been observed in many real life situations. The large deviation principle is a very important probabilistic tool for dealing with rare events. The interaction between the two topics is investigated. We study the effect of the dependence structure of the process on large deviations in the perspective of the moving average process and the infinitely divisible process with exponentially light tails.

The large deviations of an infinite moving average process with exponentially light tails are very similar to those of an i.i.d. sequence as long as the coefficients decay fast enough. If they do not, the large deviations change dramatically. We study this phenomenon in the context of functional large, moderate and huge deviation principles. We apply the results to study the rate of growth of long strange segments and the rate of decay of ruin probabilities and the effect of memory on those.

We study the functional large deviation principle for a general class of long range dependent infinitely divisible processes driven by a null recurrent Markov chain. We also apply the principle to obtain the rate of decay of the probability of ruin for these models.

BIOGRAPHICAL SKETCH

Souvik Ghosh was born on October 14, 1980 to Jharna Ghosh and Sujit K. Ghosh in Kolkata, India. He studied in Lycée until the 10th standard and then completed his schooling in The Future Foundation School in 1998.

Although he always had a strong liking of mathematics, but never thought of taking it up as his subject of choice before getting admission for the Bachelor of Statistics program in the Indian Statistical Institute, Kolkata. After completing B.Stat in 2001 he continued to complete the Master of Statistics degree in the same institute in 2003. He also participated in several summer courses in mathematics offered by the Tata Institute for Fundamental Research, Mumbai. The five years at ISI, Kolkata and occasional trips to TIFR convinced him to pursue higher studies.

Souvik came to Cornell University, Ithaca, NY in 2003 to pursue his Ph.D. in Statistics and Probability in the School of OR&IE. ISI, Kolkata was also the place where he met Romita Mukherjee whom he later married in 2004.

Souvik has accepted a position of assistant professor in the Department of Statistics at Columbia University, New York and he plans to join there soon after completing his Ph.D.

I dedicate this dissertation to Romita.

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TABLE OF CONTENTS

LIST OF TABLES

CHAPTER 1

INTRODUCTION

1.1 Long Range Dependence

The notion of *long range dependence* or *long memory* was introduced by B. B. Mandelbrot and his co-authors (e.g. Mandelbrot and Wallis (1968) and Mandelbrot and Ness (1968)) during the 1960s. As the name suggests, this refers to certain stationary stochastic processes $\{X_n, n \in \mathbb{Z}\}$ for which the memory lingers for an unusually long amount of time. Their goal was to explain a strange phenomenon observed by Harold Hurst while studying a data set on the level of water in the Nile river. Harold Hurst, a hydrologist by profession, considered the "range of the data" for the annual minima of the water level in the river from 622 A.D. to 1281 A.D. and noticed that the rate of growth of this statistic is much faster than what it should be had the observations been independent and identically distributed. This phenomenon, which later became known as the Hurst phenomenon, is believed to be the beginning of long range dependence.

Over the last decade people have observed "similar" phenomenon in many real life processes and that has triggered a lot of interest among both statisticians and probabilists towards the study of long range dependence and its effects. Long range dependence has been observed in fields as diverse as finance, internet modeling, DNA sequencing, linguistics, geology and climate studies. See Samorodnitsky (2007) for references and a detailed study on this topic.

Even though this phenomenon is very intuitive and has been observed in many different areas, there is no universally accepted definition of long range dependence.

A recent survey in Guégan (2005) gives 11 different definitions of long memory. Many of these definitions are "similar", but they are certainly not equivalent. In most cases, people try to define this phenomenon by using some second-order properties (e.g. covariances, spectral density etc.) of the process. One of the severe criticisms of these methods is that the amount of information captured by these second order properties is arguably limited if the process is not Gaussian. In spite of their short comings, these definitions are popular because they are statistically verifiable. On the other hand, efforts to define long memory using ergodic theoretic notions have not become popular. Even though they are theoretically sound, it is difficult to deal with them in statistical terms.

In this dissertation we want to look at the problem from a different point of view, as suggested in Samorodnitsky (2004). Suppose $(\mathcal{P}_\theta, \theta \in \Theta)$ is a family of laws of stationary stochastic processes indexed by some parameter space Θ. Assume also that the parameter space can be partitioned into Θ_0 and Θ_1 in a way such that certain properties of the process differ vastly in these parameter regimes. If in one of these regimes the behavior is similar to those of i.i.d random variables then it would be reasonable to term that as short memory and the other as the long memory regime. We would then use the boundary between Θ_0 and Θ_1 to define the boundary between short and long memory. Keeping this goal in mind we study the *large deviation principle* and some of its applications in the context of *moving average processes* and *infinitely divisible processes*.

1.2 Large Deviation Principle

A sequence of probability measures $\{\mu_n\}$ on the Borel subsets of a topological space is said to satisfy the *large deviation principle*, or LDP, with *speed* b_n, and upper and lower *rate function* $I_u(\cdot)$ and $I_l(\cdot)$, respectively, if for any Borel set A,

$$-\inf_{x\in A^\circ} I_l(x) \leq \liminf_{n\to\infty} \frac{1}{b_n} \log \mu_n(A) \leq \limsup_{n\to\infty} \frac{1}{b_n} \log \mu_n(A) \leq -\inf_{x\in \bar{A}} I_u(x), \quad (1.1)$$

where A° and $\bar{A}$ are, respectively, the interior and closure of A. A rate function is a non-negative lower semi-continuous function. Recall that a function is said to be lower semicontinuous if its level sets are closed sets. A rate function is said to be *good* if it has compact level sets. We refer the reader to Varadhan (1984), Deuschel and Stroock (1989) or Dembo and Zeitouni (1998) for a detailed treatment of large deviations.

The theory of large deviations is related to rare events. It owes much of its foundation to M. D. Donsker, S. R. S. Varadhan, M. I. Freidlin and A. D. Wentzell. Because of its ability to deal with rare events, large deviations are extremely important tools with limitless applications. We will discuss two such important applications of large deviations, namely *long strange segments* and *ruin probabilities*.

In many cases, the sequence of measures $\{\mu_n\}$ is the sequence of the laws of the normalized partial sums $a_n^{-1}(X_1 + \ldots + X_n)$, for some appropriate normalizing sequence (a_n). Large deviations can also be formulated in function spaces, or in measure spaces. The normalizing sequence has to grow faster than the rate of growth required to obtain a non-degenerate weak limit theorem for the normalized partial sums. There is, usually, a boundary for the rate of growth of the normalizing sequence, that separates the "proper large deviations" from the so-called "moderate deviations". In the moderate deviations regime the normalizing sequence (a_n) grows slowly enough so as to make the underlying weak limit felt, and Gaussian-like rate functions appear. This effect disappears at the boundary, which corresponds to the proper large deviations. Normalizing sequences that grow even faster lead to the so-called "huge deviations". For the i.i.d. sequencies $X_1, X_2, \ldots$ the proper large deviations regime corresponds to the linear growth of the normalizing sequence; see Theorem 2.2.3 (Cramér's Theorem) in Dembo and Zeitouni (1998). The same remains true for certain short memory processes. Later we will see that for certain long memory processes the natural boundary is not the linear normalizing sequence.

1.3 Moving Average Processes

An infinite moving average process (X_n) is defined by

$$X_n := \sum_{i=-\infty}^{\infty} \phi_i \, Z_{n-i}, n \in \mathbb{Z}. \tag{1.2}$$

The innovations $\{Z_i, i \in \mathbb{Z}\}$ are assumed to be i.i.d. $\mathbb{R}^d$-valued light-tailed random variables with 0 mean and covariance matrix Σ. In this setup square summability of the coefficients (ϕ_i)

$$\sum_{i=-\infty}^{\infty} \phi_i^2 < \infty \tag{1.3}$$

is well known to be necessary and sufficient for convergence of the series in (1.2). Under these assumptions (X_n) is a well defined stationary process, also known as a linear process; see Brockwell and Davis (1991). It is common to think of a linear process as a short memory process when it satisfies the stronger condition of absolute summability of coefficients,

$$\sum_{n\in\mathbb{Z}} |\phi_i| < \infty. \tag{1.4}$$

One can easily check that absolute summability of coefficients implies that the covariances are absolutely summable:

$$\sum_{n=0}^{\infty} |Cov(X_0, X_n)| < \infty. \tag{1.5}$$

It is also easy to exhibit a broad class of examples where (1.4) fails and the covariances are not summable.

Instead of covariances, we are interested in understanding how the large deviations of a moving average process change as the coefficients decay slower and slower. Information obtained in this way is arguably more substantial than that obtained via covariances alone.

We assume that the moment generating function of a generic noise variable Z_0, is finite in a neighborhood of the origin. We denote its log-moment generating function by $\Lambda(\lambda) := \log E\big(\exp(\lambda \cdot Z_0)\big)$, where $x \cdot y$ is the scalar product of two vectors, x and y. For a function $f : \mathbb{R}^d \to (-\infty, \infty]$, define the Fenchel-Legendre transform of f by $f^*(x) = \sup_{\lambda \in \mathbb{R}^d} \big\{\lambda \cdot x - f(\lambda)\big\}$, and the set $\mathcal{F}_f := \{x \in \mathbb{R}^d : f(x) < \infty\} \subset \mathbb{R}^d$. The imposed assumption $0 \in \mathcal{F}_\Lambda^\circ$, the interior of $\mathcal{F}_\Lambda$, is then the formal statement of our comment that the innovations (Z_i) are light-tailed. Section 2.2 in Dembo and Zeitouni (1998) summarizes the properties of Λ and Λ^*.

There exists rich literature on large deviation for moving average processes, going back to Donsker and Varadhan (1985). They considered Gaussian moving averages and proved LDP for the random measures $n^{-1} \sum_{i \le n} \delta_{X_i}$, under the assumption that the spectral density of the process is continuous. Burton and Dehling (1990) considered a general one-dimensional moving average process with $\mathcal{F}_\Lambda = \mathbb{R}$, assuming that (1.4) holds. They also assumed that

$$\sum_{n \in \mathbb{Z}} \phi_i = 1; \tag{1.6}$$

the only substantial part of the assumption being that the sum of the coefficients in non-zero. In that case $\{\mu_n\}$, the laws of $n^{-1}S_n = n^{-1}(X_1 + \ldots + X_n)$, satisfy LDP with a good rate function $\Lambda^*(\cdot)$. The work of Jiang et al. (1995) handled the case of $\{Z_i, i \in \mathbb{Z}\}$, taking values in a separable Banach space. Still assuming (1.4) and (1.6), they proved that the sequence $\{\mu_n\}$ satisfies a large deviation lower bound with the good rate function $\Lambda^*(\cdot)$, and, under an integrability assumption, a large deviation upper bound also holds with a certain good rate function $\Lambda^{\#}(\cdot)$. In a finite dimensional Euclidian space, the integrability assumption is equivalent to $0 \in \mathcal{F}_\Lambda^\circ$, and the upper rate function is given by

$$\Lambda^{\#}(x) := \sup_{\lambda \in \Pi}\{\lambda \cdot x - \Lambda(\lambda)\}, \tag{1.7}$$

where $\Pi = \{\lambda \in \mathbb{R}^d$: there exists N_λ such that $\sup_{n \geq N_\lambda, i \in \mathbb{Z}} \Lambda(\lambda\phi_{i,n}) < \infty\}$ with $\phi_{i,n} := \phi_{i+1} + \cdots + \phi_{i+n}$. Observe that, if $\mathcal{F}_\Lambda = \mathbb{R}^d$, then $\Lambda^{\#} \equiv \Lambda^*$.

In their paper, Djellout and Guillin (2001) went back to the one-dimensional case. They worked under the assumption that the spectral density is continuous and non-vanishing at the origin. Assuming also that the noise variables have a bounded support, they showed that the LDP of Burton and Dehling (1990) still holds, and also established a moderate deviation principle.

Wu (2004) extended the results of Djellout and Guillin (2001) and proved a large deviation principle for the occupation measures of the moving average processes. He worked in an arbitrary dimension $d \geq 1$, with the same assumption on the spectral density but replaced the assumption of the boundedness of the support of the noise variables with the strong integrability condition, $E[\exp(\delta|Z_0|^2)] < \infty$, for some $\delta > 0$. It is worth noting that an explicit rate function could be obtained only under the absolute summability assumption (1.4).

Further, Jiang et al. (1992) considered moderate deviations in one dimension under the absolute summability of the coefficients, and assuming that $0 \in \mathcal{F}_\Lambda^\circ$. Finally, Dong et al. (2005) showed that, under the same summability and integrability assumptions, the moving average "inherits" its moderate deviations from the noise variables even if the latter are not necessarily i.i.d.

As is very evident, there has been very little work in the case when the assumption (1.4) fails. A functional LDP under the assumption $\mathcal{F}_\Lambda = \mathbb{R}^d$ was obtained by Barbe and Broniatowski (1998) for a non-stationary fractional ARIMA model. In a forthcoming paper Merlevède and Peligrad (2008) proves a moderate deviation principle for certain linear processes in which the innovations are bounded martingale differences. We give a complete picture of large, moderate and huge deviations for moving average processes and outline their differences in long and short memory regimes.

1.4 Infinitely Divisible Processes

A $\mathbb{R}^d$-valued random variable X defined on some probability space $(\Omega, \mathcal{F}, P)$ is said to have an infinitely divisible (ID) distribution, if for any integer $k > 0$ it has the same distribution as the sum of k independent random variables, that is, there exists $W_1^{(k)}, \dots, W_k^{(k)}$ such that

$$X \stackrel{d}{=} W_1^{(k)} + \cdots + W_k^{(k)}.$$

The class of infinitely divisible distributions is an extremely broad class. For example, the Gaussian, Cauchy, Stable, Poisson, binomial, negative binomial, gamma, Student's t, F, Gumbel, Weibull log-normal, Pareto, logistic distributions are all infinitely divisible. Sato (1999) gives an excellent exposure to this topic.

A process $\{X_n, n \in \mathbb{Z}\}$ is said to be an infinitely divisible process if every finite dimensional marginals of this process is infinitely divisible. Obviously this is again an extremely broad class of processes, with the most popular being the Lévy process. Although Lévy processes are important in their own right, they are not so interesting for the purpose of studying long range dependence because of the independent increments structure. For this reason we concentrate on a class of infinitely divisible processes which are not Lévy processes.

Maruyama (1970) started the study of infinitely divisible processes. Since then many authors have been looking at the structure of these processes and criteria for ergodicity and mixing; see for e.g. Rosiński (1990), Rosiński and Żak (1996) and Rosiński and Żak (1997). The area of large deviations for infinitely divisible processes is largely open.

The *Lévy-Khintchine* representation is pivotal for the study of an infinitely divisible distribution; see Theorem 8.1 in Sato (1999). Maruyama (1970) extended this representation and proved the existence of a Lévy measure for an infinitely divisible process. It turns out that an infinitely divisible process is the independent sum of a Gaussian process and a Poissonian process, where the latter is uniquely determined by its Lévy measure. We consider a stationary infinitely divisible process $\{X_n, n \in \mathbb{Z}\}$ without a Gaussian component. Chapter 2 gives an almost complete picture of the large deviations for stationary Gaussian processes. We assume that the marginals are light-tailed, that is,

$$E\left[\exp(\lambda \cdot X_0)\right] < \infty, \text{ for all } \lambda \in \mathbb{R}^d. \tag{1.8}$$

de Acosta (1994) proved functional large deviations for the Lévy processes taking values in some Banach space under certain integrability assumptions, which in the finite dimensional Euclidean space, is identical to (1.8). A general class of infinitely divisible processes is a truly huge class and this is one of the first attempts in studying large deviations for such a big class of processes.

1.5 Long Strange Segments

One of the applications of large deviations that we discuss is the *long strange segments*. Suppose that $\{X_n, n \in \mathbb{Z}\}$ is a $\mathbb{R}^d-$valued, stationary and ergodic stochastic process defined on some probability space $(\Omega, \mathcal{F}, P)$. Given any measurable set $A \subset \mathbb{R}^d$, the long strange segments are random variables, defined as

$$T_n(A) := \inf\Big\{l : \exists k, 0 \le k \le l-n, \frac{S_l - S_k}{l-k} \in A\Big\}, \tag{1.9}$$

where $S_k = X_1 + \cdots + X_k$, denotes the partial sums. $T_n(A)$ is the minimum number of observations required to have a segment of length at least n, whose average is in the set A. With out loss of any generality we can assume that X_0 is centered. To understand the justification for the name long strange segments, consider any set A bounded away from the origin, that is $0 \notin \bar{A}$, where $\bar{A}$ is the closure of A. Since the process is ergodic, we would not expect that the average of a long segment to be in A, and it is strange if that happens. If we use the process to model a system, then the long strange segments detect the time intervals where the system runs at a different work load than is anticipated. This results in a variety of applications for the study of these functionals, for example in manufacturing, insurance and finance. As mentioned in Mansfield et al. (2001) the long strange segments also find significant use in other areas such as DNA sequence matching and comparing computer search algorithms.

It is easy to show that (see Theorem 3.2.1 in Dembo and Zeitouni (1998)) if

$\{X_n\}$ are i.i.d. random variables for which,

$$\text{there exists } \epsilon > 0, \text{ such that } E\left[e^{tX_0}\right] < \infty, \text{ for all } t \in (-\epsilon, \epsilon),$$

then

$$-\limsup_{n\to\infty} \frac{1}{n} \log P\left[S_n/n \in A\right] \leq \liminf_{n\to\infty} \frac{1}{n} \log T_n(A), P - a.s. \tag{1.10}$$

and

$$-\liminf_{n\to\infty} \frac{1}{n} \log P\left[S_n/n \in A\right] \geq \limsup_{n\to\infty} \frac{1}{n} \log T_n(A), P - a.s. \tag{1.11}$$

This result gives a clear relation between the long strange segments and large deviations of the process. With the aim of studying the long range dependence of moving average processes, Mansfield et al. (2001) and Rachev and Samorodnitsky (2001) considered similar functionals:

$$R_n(A) := \sup\left\{j - i : 0 \leq i < j \leq n, \frac{S_j - S_i}{j - i} \in A\right\}. \tag{1.12}$$

$R_n(A)$ is the maximum length of a segment from the first n observations whose average is in A. It is a simple exercise to check that there is a duality relation between the rate of growth of T_n and the rate of growth of R_n. They consider the process

$$X_n = \sum_{i\in\mathbb{Z}} \phi_i Z_{n-i}, n \in \mathbb{Z},$$

where $\{Z_i, i \in \mathbb{Z}\}$ have balanced regular varying tails with exponent $-\beta(\beta > 1)$. Mansfield et al. (2001) showed that if

$$\sum_{i\in\mathbb{Z}} |\phi_i| < \infty, \tag{1.13}$$

then for any $y > 0$ and $x > 0$

$$P\left(a_n^{-1} R_n((y, \infty)) \leq x\right) \to \exp(-C_s y^{-\beta} x^{-\beta}) \tag{1.14}$$

for some $C_s > 0$, where $\{a_n\}$ is regular varying at infinity with index β^{-1}. We say that a sequence $\{a_n\}$ is regular varying with index a, or $\{a_n\} \in RV_a$, if

$$\lim_{n\to\infty} \frac{a_{\lfloor un \rfloor}}{a_n} = u^a, \text{ for all } u > 0.$$

See Resnick (1987) or Bingham et al. (1989) for details on regular variation. This means that the weak limit of $a_n^{-1} R_n$ might depend on the coefficients, but the rate of growth of R_n does not. In particular, if we take $\phi_0 = 1$ and $\phi_i = 0$ for all $i \neq 0$ and then we see that for a sequence of i.i.d. random variables, R_n grows like a_n. In another article, Rachev and Samorodnitsky (2001) considered the case when (1.13) fails to hold, and in particular, $\{\phi_i\}$ is balanced regular varying at infinity with exponent $-\alpha$, satisfying $\max\left\{\frac{1}{\beta}, \frac{1}{2}\right\} < \alpha \leq 1$, that is,

$$\exists \psi(t) \in RV_{-\alpha}, \text{ such that } \frac{\phi_n}{\psi(n)} \to p, \ \frac{\phi_{-n}}{\psi(n)} \to 1-p, \text{ where } 0 \leq p \leq 1. \tag{1.15}$$

We say that a function $f : (0,\infty) \to (0,\infty)$ is in RV_a, if

$$\lim_{t\to\infty} \frac{f(ut)}{f(t)} = u^a, \forall u > 0.$$

They showed under this assumption, that for any $y > 0$ and $x > 0$,

$$P\big(b_n^{-1} R_n((y,\infty)) \leq x\big) \to \exp(C_l y^{-\beta} x^{-\beta\alpha}), \tag{1.16}$$

for some $\{b_n\} \in RV_{(\alpha\beta)^{-1}}$. This exhibits a marked difference in the rate of growth of long strange segments when the coefficients of the moving average process are not absolutely summable, which justifies the term *short memory* in that case.

We, on the other hand, take $\{X_n\}$ to be the moving average process described in Section 1.3 and find the rate of growth of

$$T_n(A) := \inf\left\{ l : \exists k, 0 \leq k \leq l-n, \frac{S_l - S_k}{a_{l-k}} \in A \right\}, \tag{1.17}$$

for suitable choices of the normalizing sequence $\{a_n\}$. As in the case of large deviations the difference in the two regimes (long and short memory) will be obvious.

1.6 Ruin Probabilities

The second application of large deviations that we consider is that of *ruin probabilities* or *level crossings*. For some one dimensional stochastic process $\{Y_n\}$, level crossing of u denotes the event that the process takes some value greater than u at some point of time. We are interested in the rate of decay of the probability of this event as u tends to infinity. We define the probability of ruin as

$$\rho(u) = P\left[Y_n > u, \text{ for some } n \geq 1\right].$$

This derives its name from its prime application: if Y_n denotes the total losses incurred by a company until time n, and u is the initial capital of the firm, then $Y_n > u$ will imply that the company is going bankrupt. Of particular interest to us is the case when

$$Y_n = \sum_{i=1}^{n} X_i - a_n\mu,$$

for some $\mu > 0$, a suitable choice of $\{a_n\}$ and certain stationary sequences $\{X_n\}$. An example would be a simple model for an insurance company. Suppose $a_n = n$ and let X_n stand for the total claims in the nth year. Furthermore, suppose μ is the total premium earned in a year. Then Y_n is the cumulative loss for the company until the nth year and when that is more than the initial capital u, of the company, then the company is in ruin. One may use different sequences a_n to incorporate growth of customers or inflation etc.

There exists an extremely rich literature on ruin probabilities and it will be foolish to try attempt to mention all of them. We mention a very selected few below that are closely related to our topic. The classic results in this area are due to Lundberg and Cramèr (see Cramèr (1955) and the references therein for earlier works). There have been many refinements and improvements since. Embrechts et al. (1997) and Asmussen (2000) gives a good exposition to the subject.

Gerber (1982) showed exponential rate of decay of $\rho(u)$ for an ARMA(p, q) process. Michna (1998) studied the effect of long range dependence on the probability of ruin. Assuming the claim process has long range dependence, he estimated the risk process by a self-similar process and also gave estimates for the probability of ruin in finite time for a fractional Brownian motion. The references mentioned in this article gives a good account of empirical evidence of non-standard dependence structures in certain real-life processes and interactions of long memory with risk theory for them.

Hüsler and Piterbarg (2004, 2007) gives exact asymptotic behavior of the probability of ruin under the assumption that the claims come from a long range dependent Gaussian process. Mikosch and Samorodnitsky (2000) and Alparslan and Samorodnitsky (2007a,b) discusses the effect of memory on the rate of decay of the probability of ruin when the claims come from a stable process.

Nyrhinen (1994, 1995) uses large deviation techniques to prove exponential rate of decay of a general class of claim processes. They also work out the example of a moving average process satisfying (1.4). Collamore (1996) extended those results to the multidimensional setting. But those results are not valid when we are in the long memory regime. We extend those techniques to get the rate of decay of the ruin probability in the long range dependent models that we consider. In the long range dependent regimes the probability of ruin decays at a much slower rate that in the short memory regime. The effect of the dependence structure of the claim process on the ruin probability is therefore made clearly visible.

1.7 Connecting the Dots

As we have already explained the central theme of this dissertation is to study the effect of memory on large deviation principle and few of its applications. To see this we look at moving average processes and infinitely divisible processes.

Chapter 2 discusses this for moving average processes. We prove a large, moderate and huge deviation principle for the partial sums process of an infinite moving average process with i.i.d. innovations having exponentially light tails. Depending on whether 1.4 holds or not we get two different regimes which we call the short and long memory regimes. Theorem 2.2.2 states the results for the short memory case and Theorem 2.2.4 discusses the long memory situation. In the following sections we discuss long strange segments and ruin probabilities as an application of large deviations and we see that the effect of memory trickles down in these applications as well.

In Chapter 3 we prove a functional version of the large deviation principle for a general class of infinitely divisible processes with long memory. Theorem 3.3.1 states the central result of this chapter. We then discuss a few examples and the ruin probabilities for this class of processes.

CHAPTER 2

MOVING AVERAGE PROCESSES

2.1 Introduction

This chapter discusses the large, moderate and huge deviation principles for the sample paths of the moving average process. We define a moving average process $\{X_n, n \in \mathbb{Z}\}$ as

$$X_n = \sum_{i\in\mathbb{Z}} \phi_i Z_{n-i}, n \in \mathbb{Z} \tag{2.1}$$

where $\{Z_n, n \in \mathbb{Z}\}$ are i.i.d., known as the innovations or white noises, and the coefficients (ϕ_i) satisfying

$$\sum_{i\in\mathbb{Z}} \phi_i^2 < \infty. \tag{2.2}$$

We assume that the innovations have mean 0, variance Σ and that the moment generating function exists in a neighborhood of 0, that is, there exists $\epsilon > 0$ such that

$$\Lambda(\lambda) := \log E\left[\exp\left(\lambda \cdot Z_0\right)\right] < \infty \text{ for all } |\lambda| < \epsilon. \tag{2.3}$$

Section 2.2 gives the main theorems for both the short and long memory regime and Section 2.3 contains some lemmas which are essential for the proofs of the theorems in Section 2.2. We then look at two important applications of the the large deviations. Section 2.4 discusses the rate of growth of long strange segments and Section 2.5 discusses the rate of decay of ruin probabilities for moving average processes.

2.2 Functional Large Deviation Principle

We study the step process $\{Y_n\}$

$$Y_n(t) = \frac{1}{a_n} \sum_{i=1}^{[nt]} X_i, t \in [0,1], \tag{2.4}$$

and its polygonal path counterpart

$$\tilde{Y}_n(t) = \frac{1}{a_n} \sum_{i=1}^{[nt]} X_i + \frac{1}{a_n}(nt - [nt])X_{[nt]+1}, t \in [0,1]. \tag{2.5}$$

Here (a_n) is an appropriate normalizing sequence. We will use the notation μ_n and $\tilde{\mu}_n$ to denote the laws of Y_n and $\tilde{Y}_n$, respectively, in the function space appropriate to the situation at hand, equipped with the cylindrical σ-field.

Various parts of the theorems in this section will work with several topologies on the space $\mathcal{BV}$ of all $\mathbb{R}^d$-valued functions of bounded variation defined on the unit interval $[0,1]$. To ensure that the space $\mathcal{BV}$ is a measurable set in the cyindrical σ-field of all $\mathbb{R}^d$-valued functions on $[0,1]$, we use only rational partitions of $[0,1]$ when defining variation. We will use subscripts to denote the topology on the space. Specifically, the subscripts S, P and L will denote the sup-norm topology, the topology of pointwise convergence and, finally, the topology in which f_n converges to f if and only if f_n converges to f both pointwise and in L_p for all $p \in [1,\infty)$.

We call a function $f : \mathbb{R}^d \to \mathbb{R}$ *balanced regular varying* with exponent $\beta > 0$, if there exists a non-negative bounded function ζ_f defined on the unit sphere on $\mathbb{R}^d$ and a function $\tau_f : [0,\infty) \to [0,\infty)$ satisfying

$$\lim_{t\to\infty} \frac{\tau_f(tx)}{\tau_f(t)} = x^\beta \tag{2.6}$$

for all $x > 0$ (i.e. τ_f is regularly varying with exponent β) such that for any $(\lambda_t) \subset \mathbb{R}^d$ converging to λ, with $|\lambda_t| = 1$ for all t, we have

$$\lim_{t\to\infty} \frac{f(t\lambda_t)}{\tau_f(t)} = \zeta_f(\lambda). \tag{2.7}$$

We will typically omit the subscript f if doing so is not likely to cause confusion.

The following assumption describes the short memory scenarios we consider. In addition to the summability of the coefficients, the different cases arise from the "size" of the normalizing constants (a_n) in (2.4), the resulting speed sequence (b_n) and the integrability assumptions on the noise variables.

Assumption 2.2.1. *All the scenarios below assume that*

$$\sum_{i\in\mathbb{Z}} |\phi_i| < \infty \text{ and } \sum_{i\in\mathbb{Z}} \phi_i = 1. \tag{2.8}$$

*S*1. $a_n = n, 0 \in \mathcal{F}_\Lambda^\circ$ *and* $b_n = n$.

*S*2. $a_n = n, \mathcal{F}_\Lambda = \mathbb{R}^d$ *and* $b_n = n$.

*S*3. $a_n/\sqrt{n} \to \infty,\ a_n/n \to 0,\ 0 \in \mathcal{F}_\Lambda^\circ$ *and* $b_n = a_n^2/n$.

*S*4. $a_n/n \to \infty$, $\Lambda(\cdot)$ *is balanced regular varying with exponent* $\beta > 1$ *and* $b_n = n\tau(\gamma_n)$, *where*

$$\gamma_n = \sup\{x : \tau(x)/x \le a_n/n\}. \tag{2.9}$$

Next, we introduce a new notation required to state our first result. For $i \in \mathbb{Z}$ and $n \ge 1$ we set $\phi_{i,n} := \phi_{i+1} + \cdots + \phi_{i+n}$. Also for $k \ge 1$ and $0 < t_1 < \cdots < t_k \le 1$, a subset $\Pi_{t_1,\ldots,t_k} \subset (\mathbb{R}^d)^k$ is defined by

$$\Pi_{t_1,\ldots,t_k} := \Big\{ \underline{\lambda} = (\lambda_1, \ldots, \lambda_k) \in (\mathcal{F}_\Lambda)^k : \Lambda \text{ is continuous on } \mathcal{F}_\Lambda \text{ at each } \lambda_j,$$

$$\text{and for some } N \ge 1, \sup_{n\ge N,\, j\in\mathbb{Z}} \Lambda\Big(\sum_{i=1}^{k} \lambda_i \phi_{j+[nt_i],[nt_i]-[nt_{i-1}]} \Big) < \infty \Big\}. \tag{2.10}$$

We view the next theorem as describing the sample path large deviations of (the partial sums of) a moving average process in the short memory case. The long memory counterpart is Theorem 2.2.4 below.

Theorem 2.2.2. *(i) If* $S1$ *holds, then* $\{\mu_n\}$ *satisfy in* $\mathcal{BV}_L$*, LDP with speed* $b_n \equiv n$*, good upper rate function*

$$G^{sl}(f) = \sup_{k\geq 1, t_1,\ldots,t_k} \left\{ \sup_{\underline{\lambda}\in\Pi_{t_1,\ldots,t_k}} \sum_{i=1}^{k} \Big\{ \lambda_i \cdot \big(f(t_i) - f(t_{i-1})\big) - (t_i - t_{i-1})\Lambda(\lambda_i) \Big\} \right\} \tag{2.11}$$

if $f(0) = 0$ *and* $G^{sl}(f) = \infty$ *otherwise, and with good lower rate function*

$$H^{sl}(f) = \begin{cases} \int_0^1 \Lambda^*(f'(t))dt & if \quad f \in \mathcal{AC}, f(0) = 0 \\ \infty & otherwise, \end{cases}$$

where $\mathcal{AC}$ *is the set of all absolutely continuous functions, and* f' *is the coordinate-wise derivative of* f*.*

(ii) If $S2$ *holds, then* $H^{sl} \equiv G^{sl}$ *and* $\{\mu_n\}$ *satisfy LDP in* $\mathcal{BV}_S$*, with speed* $b_n \equiv n$ *and good rate function* $H^{sl}(\cdot)$*.*

(iii) Under Assumption $S3$*,* $\{\mu_n\}$ *satisfy in* $\mathcal{BV}_S$*, LDP with speed* b_n *and good rate function*

$$H^{sm}(f) = \begin{cases} \int_0^1 \frac{1}{2} f'(t) \cdot \Sigma^{-1} f'(t)dt & if \quad f \in \mathcal{AC}, f(0) = 0 \\ \infty & otherwise. \end{cases}$$

Here Σ *is the covariance matrix of* Z_0*, and we understand* $a \cdot \Sigma^{-1} a$ *to mean* ∞ *if* $a \in K_\Sigma := \{x \in \mathbb{R}^d - \{0\} : \Sigma x = 0\}$*.*

(iv) Under Assumption $S4$*,* $\{\mu_n\}$ *satisfy in* $\mathcal{BV}_S$*, LDP with speed* b_n *and good rate function*

$$H^{sh}(f) = \begin{cases} \int_0^1 (\Lambda^h)^*(f'(t))dt & if \quad f \in \mathcal{AC}, f(0) = 0 \\ \infty & otherwise, \end{cases} .$$

where $\Lambda^h(\lambda) = \zeta_\Lambda\left(\frac{\lambda}{|\lambda|}\right)|\lambda|^\beta$ *for* $\lambda \in \mathbb{R}^d$ *(defined as zero for* $\lambda = 0$*).*

A comparison with the LDP for i.i.d. sequences (see Mogulskii (1976) or Theorem 5.1.2 in Dembo and Zeitouni (1998)) reveals that the rate function stays the same as long as the coefficients in the moving average process stay summable.

We also note that an application of the contraction principle gives, under scenario S1, a marginal LDP for the law of $n^{-1}S_n$ in $\mathbb{R}^d$ with speed n, upper rate function $G_1^{sl}(x) = \sup_{\lambda \in \Pi_1} \Big\{ \lambda \cdot x - \Lambda(\lambda) \Big\}$, and lower rate function $\Lambda^*(\cdot)$, recovering the statement of Theorem 1 in Jiang et al. (1995) in the finite-dimensional case.

Next, we consider what happens when the absolute summability fails, in a major way. We will assume that the coefficients are balanced regular varying with an appropriate exponent. The following assumption is parallel to Assumption 2.2.1 in the present case, dealing, once again, with the various cases that may arise.

Assumption 2.2.3. *All the scenarios assume that the coefficients* $\{\phi_i\}$ *are balanced regular varying with exponent* $-\alpha, 1/2 < \alpha \le 1$ *and* $\sum_{i=-\infty}^{\infty} |\phi_i| = \infty$. *Specifically, there is* $\psi : [0,\infty) \to [0,\infty)$ *and* $0 \le p \le 1$, *such that*

$$\left.\begin{array}{c} \lim_{t\to\infty} \frac{\psi(tx)}{\psi(t)} = x^{-\alpha}, \text{for all } x > 0 \\ \lim_{n\to\infty} \frac{\phi_n}{\psi(n)} = p \text{ and } \lim_{n\to\infty} \frac{\phi_{-n}}{\psi(n)} = q := 1 - p. \end{array}\right\} \tag{2.12}$$

Let $\Psi_n := \sum_{1 \le i \le n} \psi(i)$.

*R*1. $a_n = n\Psi_n, 0 \in \mathcal{F}_\Lambda^\circ$ *and* $b_n = n$.

*R*2. $a_n = n\Psi_n, \mathcal{F}_\Lambda = \mathbb{R}^d$ *and* $b_n = n$.

*R*3. $a_n/\sqrt{n}\Psi_n \to \infty, a_n/(n\Psi_n) \to 0, 0 \in \mathcal{F}_\Lambda^\circ$ *and* $b_n = a_n^2/(n\Psi_n^2)$.

*R*4. $a_n/(n\Psi_n) \to \infty$, $\Lambda(\cdot)$ *is balanced regular varying with exponent* $\beta > 1$ *and* $b_n = n\tau(\Psi_n\gamma_n)$, *where*

$$\gamma_n = \sup\{x : \tau(\Psi_n x)/x \le a_n/n\}. \tag{2.13}$$

Similar to (2.10) we define

$$\Pi^{\alpha}_{t_1,\dots,t_k} := \Big\{ \underline{\lambda} = (\lambda_1,\dots,\lambda_k) : (p\wedge q)\lambda_i \in \mathcal{F}^{\circ}_{\Lambda},\ i=1,\dots,k,\ \text{and}$$

$$\text{for some } N=1,2,\dots \sup_{n\geq N,\, j\in\mathbb{Z}} \Lambda\Big(\frac{1}{\Psi_n}\sum_{i=1}^{k}\lambda_i\phi_{j+[nt_i],[nt_i]-[nt_{i-1}]}\Big) < \infty \Big\} \tag{2.14}$$

for $1/2 < \alpha < 1$, while for $\alpha = 1$, we define

$$\Pi^{1}_{t_1,\dots,t_k} := \Big\{ \underline{\lambda} = (\lambda_1,\dots,\lambda_k) \in (\mathcal{F}_{\Lambda})^k : \Lambda \text{ is continuous on } \mathcal{F}_{\Lambda} \text{ at each } \lambda_j$$

$$\text{and for some } N=1,2,\dots \sup_{n\geq N,\, j\in\mathbb{Z}} \Lambda\Big(\frac{1}{\Psi_n}\sum_{i=1}^{k}\lambda_i\phi_{j+[nt_i],[nt_i]-[nt_{i-1}]}\Big) < \infty \Big\} \tag{2.15}$$

Also for $1/2 < \alpha < 1$, any $k \geq 1$, $0 < t_1 \leq \cdots \leq t_k \leq 1$, and $\underline{\lambda} = (\lambda_1,\dots,\lambda_k) \in (\mathbb{R}^d)^k$ let

$$h_{t_1,\dots,t_k}(x;\underline{\lambda}) := (1-\alpha)\sum_{i=1}^{k}\lambda_i \int_{x+t_{i-1}}^{x+t_i} |y|^{-\alpha}(pI_{[y\geq 0]} + qI_{[y<0]})dy. \tag{2.16}$$

For any $\mathbb{R}^d$-valued convex function Γ, any function $\varphi \in L_1[0,1]$ and $1/2 < \alpha < 1$ we define ,

$$\Gamma^*_{\alpha}(\varphi) = \sup_{\psi\in L_\infty[0,1]} \Big\{ \int_0^1 \psi(t)\cdot\varphi(t)\,dt \tag{2.17}$$

$$- \int_{-\infty}^{\infty} \Gamma\left(\int_0^1 \psi(t)(1-\alpha)|x+t|^{-\alpha}\Big[pI_{[x+t\geq 0]} + qI_{[x+t<0]}\Big]\,dt\right)dx \Big\},$$

whereas for $\alpha = 1$ we put

$$\Gamma^*_1(\varphi) = \int_0^1 \Gamma^*(\varphi(t))\,dt\,. \tag{2.18}$$

We view the following result as describing the large deviations of moving averages in the long memory case.

Theorem 2.2.4. *(i) If R1 holds, then $\{\mu_n\}$ satisfy in $\mathcal{BV}_L$, LDP with speed $b_n = n$, good upper rate function*

$$G^{rl}(f) = \sup_{k\geq 1, t_1,\dots,t_k} \Big\{ \sup_{\underline{\lambda}\in\Pi^{\alpha}_{t_1,\dots,t_k}} \sum_{i=1}^{k} \lambda_i \cdot \big(f(t_i) - f(t_{i-1})\big) - \Lambda^{rl}_{t_1,\dots,t_k}(\lambda_1,\dots,\lambda_k)\Big\} \tag{2.19}$$

if $f(0) = 0$ and $G^{rl}(f) = \infty$ otherwise, where

$$\Lambda^{rl}_{t_1,\cdots,t_k}(\lambda_1,\cdots,\lambda_k) \;:=\; \begin{cases} \int_{-\infty}^{\infty} \Lambda\Big(h_{t_1,\dots,t_k}(x;\underline{\lambda})\Big)dx & if \quad \alpha < 1 \\ \sum_{i=1}^{k}(t_i - t_{i-1})\Lambda(\lambda_i) & if \quad \alpha = 1, \end{cases} \tag{2.20}$$

and good lower rate function

$$H^{rl}(f) = \begin{cases} \Lambda^*_\alpha(f') & if \quad f \in \mathcal{AC}, f(0) = 0 \\ \infty & otherwise. \end{cases}$$

(ii) If R2 holds, then $H^{rl} \equiv G^{rl}$ and $\{\mu_n\}$ satisfy LDP in $\mathcal{BV}_S$, with speed $b_n = n$ and good rate function $H^{rl}(\cdot)$.

(iii) Under Assumption R3, $\{\mu_n\}$ satisfy in $\mathcal{BV}_S$, LDP with speed b_n and good rate function

$$H^{rm}(f) = \begin{cases} (G_\Sigma)^*_\alpha(f') & if \quad f \in \mathcal{AC}, f(0) = 0 \\ \infty & otherwise, \end{cases}$$

where $G_\Sigma(\lambda) = \frac{1}{2}\lambda \cdot \Sigma\lambda$, $\lambda \in \mathbb{R}^d$.

(iv) Under Assumption R4, $\{\mu_n\}$ satisfy in $\mathcal{BV}_S$, LDP with speed b_n and good rate function

$$H^{rh}(f) = \begin{cases} (\Lambda^h)^*_\alpha(f') & if \quad f \in \mathcal{AC}, f(0) = 0 \\ \infty & otherwise, \end{cases}$$

with Λ^h as in Theorem 2.2.2.

We note that a functional LDP under the Assumption $R2$, but for a non-stationary fractional ARIMA model was obtained by Barbe and Broniatowski (1998).

Remark 2.2.5. The proof of Theorem 2.2.4 below shows that, under the Assumption $R1$, the laws of $(n\Psi_n)^{-1}S_n$ satisfy LDP with speed n, good lower rate function $\Lambda_1^{rl*}(\cdot)$ and good upper rate function $G_1^{rl}(x) := \sup_{\lambda\in\Pi_1^\alpha}\left\{\lambda\cdot x\Lambda_1^{rl}(\lambda)\right\}$. If $R2$ holds, then $\Pi_1^\alpha = \mathbb{R}^d$ and $G_1^{rl} \equiv (\Lambda_1^{rl})^*$.

Remark 2.2.6. It is interesting to note that under the Assumption $R3$ it is possible to choose $a_n = n$, and, hence, compare the large deviations of the sample means of moving average processes with summable and non-summable coefficients. We see that the sample means of moving average processes with summable coefficients satisfy LDP with speed $b_n = n$, while the sample means of moving average processes with non-summable coefficients (under Assumption $R3$) satisfy LDP with speed $b_n = n/\Psi_n^2$, which is regular varying with exponent $2\alpha - 1$. The markedly slower speed function in the latter case (even for $\alpha = 1$ one has $b_n = nL(n)$, with a slowly varying function $L(\cdot)$ converging to zero) demonstrates a phase transition occurring here.

Remark 2.2.7. Lemma 2.2.8 at the end of this section describes certain properties of the rate function $(G_\Sigma)_\alpha^*$, which is, clearly, also the rate function in *all* scenarios in the Gaussian case.

The proofs of Theorems 2.2.2 and 2.2.4 rely on lemmas appearing in section 3.

Proof of Theorem 2.2.2. (ii), (iii) and (iv): Let $\mathcal{X}$ be the set of all $\mathbb{R}^d$-valued functions defined on the unit interval $[0, 1]$ and let $\mathcal{X}^o$ be the subset of $\mathcal{X}$, of functions which start at the origin. Define $\mathcal{J}$ as the collection of all ordered finite subsets of

$(0,1]$ with a partial order defined by inclusion. For any $j = \{0 < t_1 < \ldots < t_{|j|} \leq 1\}$ define the projection $p_j : \mathcal{X}^o \to \mathcal{Y}_j$ as $p_j(f) = (f(t_1), \ldots, f(t_{|j|}))$, $f \in \mathcal{X}^o$. So $\mathcal{Y}_j$ can be identified with the space $(\mathbb{R}^d)^{|j|}$ and the projective limit of $\mathcal{Y}_j$ over $j \in \mathcal{J}$ can be identified with $\mathcal{X}^o$ equipped with the topology of pointwise convergence. Note that $\mu_n \circ p_j^{-1}$ is the law of

$$Y_n^j = (Y_n(t_1), \ldots, Y_n(t_{|j|}))$$

and let

$$V_n = \big(Y_n(t_1), Y_n(t_2) - Y_n(t_1), \cdots, Y_n(t_{|j|}) - Y_n(t_{|j|-1})\big). \tag{2.21}$$

By Lemma 2.3.5 we see that for any $\underline{\lambda} = (\lambda_1, \ldots, \lambda_{|j|}) \in (\mathbb{R}^d)^{|j|}$

$$\begin{aligned}
\lim_{n\to\infty} \frac{1}{b_n} \log E\big(\exp\big[b_n \underline{\lambda} \cdot V_n\big]\big) &= \lim_{n\to\infty} \frac{1}{b_n} \log E \exp\Big[\frac{b_n}{a_n} \sum_{i=1}^{|j|} \lambda_i \cdot \Big(\sum_{k=[nt_{i-1}]+1}^{[nt_i]} X_k\Big)\Big] \\
&= \lim_{n\to\infty} \frac{1}{b_n} \sum_{l=-\infty}^{\infty} \Lambda\Big(\frac{b_n}{a_n} \sum_{i=1}^{|j|} \lambda_i \phi_{l+[nt_{i-1}],[nt_i]-[nt_{i-1}]}\Big) \\
&= \sum_{i=1}^{|j|} (t_i - t_{i-1}) \Lambda^v(\lambda_i) := \Lambda^v_{t_1,\ldots,t_{|j|}}(\underline{\lambda}),
\end{aligned}$$

where $t_0 = 0$ and for any $\lambda \in \mathbb{R}^d$,

$$\Lambda^v(\lambda) = \begin{cases} \Lambda(\lambda) & \text{in part (ii)}, \\ \frac{1}{2}\lambda \cdot \Sigma\lambda & \text{in part (iii)}, \\ \zeta\Big(\frac{\lambda}{|\lambda|}\Big)|\lambda|^\beta & \text{in part (iv)}. \end{cases}$$

By the Gartner-Ellis Theorem (see Theorem II.2 in Ellis (1984)), the laws of (V_n) satisfy LDP with speed b_n and good rate function

$$\Lambda^{v*}_{t_1,\cdots,t_{|j|}}(w_1, \ldots, w_{|j|}) = \sum_{i=1}^{|j|} (t_i - t_{i-1}) \Lambda^{v*}\Big(\frac{w_i}{t_i - t_{i-1}}\Big),$$

where $(w_1, \ldots, w_{|j|}) \in (\mathbb{R}^d)^{|j|}$. The map $V_n \mapsto Y_n^j$ from $(\mathbb{R}^d)^{|j|}$ onto itself is one to one and continuous. Hence the contraction principle tells us that $\{\mu_n \circ p_j^{-1}\}$ satisfy

LDP in $(\mathbb{R}^d)^{|j|}$ with good rate function

$$H^v_{t_1,\ldots,t_{|j|}}(y_1,\ldots,y_{|j|}) := \sum_{i=1}^{|j|}(t_i - t_{i-1})\Lambda^{v*}\Big(\frac{y_i - y_{i-1}}{t_i - t_{i-1}}\Big), \tag{2.22}$$

where we take $y_0 = 0$. By Lemma 2.3.1, the same holds for the measures $\{\tilde{\mu}_n \circ p_j^{-1}\}$. Proceeding as in Lemma 5.1.6 in Dembo and Zeitouni (1998) this implies that the measures $\{\tilde{\mu}_n\}$ satisfy LDP in the space $\mathcal{X}^o$ equipped with the topology of pointwise convergence, with speed b_n and the rate function described in the appropriate part of the theorem. As $\mathcal{X}^o$ is a closed subset of $\mathcal{X}$, the same holds for $\{\tilde{\mu}_n\}$ in $\mathcal{X}$ and the rate function is infinite outside $\mathcal{X}^o$. Since $\tilde{\mu}_n(\mathcal{BV}) = 1$ for all $n \geq 1$ and the 3 rate functions in parts (ii), (iii) and (iv) of the theorem are infinite outside of $\mathcal{BV}$, we conclude that $\{\tilde{\mu}_n\}$ satisfy LDP in $\mathcal{BV}_P$ with the same rate function. The sup-norm topology on $\mathcal{BV}$ is stronger than that of pointwise convergence and by Lemma 2.3.2, $\{\tilde{\mu}_n\}$ is exponentially tight in $\mathcal{BV}_S$. So by corollary 4.2.6 in Dembo and Zeitouni (1998), $\{\tilde{\mu}_n\}$ satisfy LDP in $\mathcal{BV}_S$ with speed b_n and good rate function $H^v(\cdot)$. Finally, applying Lemma 2.3.1 once again, we conclude that the same is true for the sequence $\{\mu_n\}$.

(i): We use the above notation. It follows from Lemma 2.3.5 that for any partition j of $(0,1]$ and $\underline{\lambda} = (\lambda_1,\ldots,\lambda_{|j|}) \in (\mathbb{R}^d)^{|j|}$,

$$\limsup_{n\to\infty} \frac{1}{n} \log E\big[\exp\big(n\underline{\lambda}\cdot V_n\big)\big] \leq \chi(\underline{\lambda}),$$

where

$$\chi(\underline{\lambda}) = \begin{cases} \sum_{i=1}^{|j|}(t_i - t_{i-1})\Lambda(\lambda_i) & \text{if } \underline{\lambda} \in \Pi_{t_1,\ldots,t_{|j|}} \\ \infty & \text{otherwise.} \end{cases}$$

The law of V_n is exponentially tight since by Jiang et al. (1995) the law of $Y_n(t_i) - Y_n(t_{i-1})$ is exponentially tight in $\mathbb{R}^d$ for every $1 \leq i \leq |j|$. Thus by Theorem 2.1 of de Acosta (1985) the laws of (V_n) satisfy a LD upper bound with speed n and rate function

$$\sup_{\underline{\lambda}\in\Pi_{t_1,\dots,t_{|j|}}} \left\{ \underline{\lambda}\cdot\underline{w} - \sum_{i=1}^{|j|}(t_i - t_{i-1})\Lambda(\lambda_i) \right\},$$

which is, clearly, good. Therefore, the laws of $(Y_n(t_1),\dots,Y_n(t_{|j|}))$ satisfy a LD upper bound with speed n and good rate function

$$G^{sl}_{t_1,\dots,t_{|j|}}(\underline{y}) := \sup_{\underline{\lambda}\in\Pi_{t_1,\dots,t_{|j|}}} \Big\{ \sum_{i=1}^{|j|}\lambda_i\cdot(y_i - y_{i-1}) - \sum_{i=1}^{|j|}(t_i - t_{i-1})\Lambda(\lambda_i)\Big\}. \tag{2.23}$$

Using the upper bound part of the Dawson-Gartner Theorem, we see that $\{\mu_n\}$ satisfy LD upper bound in $\mathcal{X}_P^o$ with speed n and good rate rate function

$$G^{sl}(f) = \sup_{j\in J} G^{sl}_{t_1,\dots,t_{|j|}}\big(f(t_1),\cdots,f(t_{|j|})\big)$$

and, as before, the same holds in $\mathcal{X}_P$ as well.

Next we prove that $(Y_n(t_1),\dots,Y_n(t_{|j|}))$ satisfy a LD lower bound with speed n and rate function $H^v_{t_1,\dots,t_{|j|}}(\cdot)$ defined in (2.22) for part (ii). Let

$$V_n' = \frac{1}{n}\Big(\sum_{|i|\le 2n}\phi_{i,[nt_1]}Z_{-i}, \sum_{|i|\le 2n}\phi_{i+[nt_1],[nt_2]-[nt_1]}Z_{-i},\cdots,$$

$$\sum_{|i|\le 2n}\phi_{i+[nt_{|j|-1}],[nt_{|j|}]-[nt_{|j|-1}]}Z_{-i}\Big)$$

and observe that the laws of (V_n) and of (V_n') are exponentially equivalent.

For $k>0$ large enough so that $p_k := P(|Z_0|\le k)>0$ we let $\upsilon_k = E\big(Z_0\big|\,|Z_0|\le k\big)$, and note that $|\upsilon_k|\to 0$ as $k\to\infty$.

Let

$$V_n'^{,k} = \frac{1}{n}\Big(\sum_{|i|\le 2n}\phi_{i,[nt_1]}(Z_{-i}-\upsilon_k), \sum_{|i|\le 2n}\phi_{i+[nt_1],[nt_2]-[nt_1]}(Z_{-i}-\upsilon_k),\cdots,$$

$$\sum_{|i|\le 2n}\phi_{i+[nt_{|j|-1}],[nt_{|j|}]-[nt_{|j|-1}]}(Z_{-i}-\upsilon_k))\Big) := V_n' - a_{n,k},$$

where $a_{n,k} = (b_1^{(n)}\mu_k, b_2^{(n)}\mu_k, \ldots, b_{|j|}^{(n)}\mu_k) \in (R^d)^{|j|}$ with some $|b_i^{(n)}| \le c$, a constant independent of i and n. We define a new probability measure

$$\nu_n^k(\cdot) = P\Big(V_n^{\prime,k} \in \cdot, |Z_i| \le k, \ \text{ for all } |i| \le 2n\Big) p_k^{-(4n+1)} .$$

Note that for all $\underline{\lambda} \in (\mathbb{R}^d)^{|j|}$ by (the proof of part (i) of) Lemma 2.3.5,

$$\begin{aligned}
&\lim_{n\to\infty} \tfrac{1}{n} \log \Big\{ p_k^{-(4n+1)} E\Big[\exp\big(n\underline{\lambda} \cdot V_n'\big) I_{[|Z_i| \le k, \ |i| \le 2n]} \Big] \Big\} \\
&\quad = \sum_{l=1}^{|j|} (t_l - t_{l-1}) \Big(L^k(\lambda_l) - \lambda_l \upsilon_k \Big) - t_{|j|} \log p_k ,
\end{aligned}$$

where $L^k(\lambda) := \log E\big[\exp(\lambda \cdot Z_0) I_{[|Z_0| \le k]}\big]$, and so for every $k \ge 1$, $\{\nu_n^k, n \ge 1\}$ satisfy LDP with speed n and good rate function

$$\begin{aligned}
&\sup_{\underline{\lambda}} \Big\{ \underline{\lambda} \cdot \underline{x} - \sum_{l=1}^{|j|} (t_l - t_{l-1}) \Big(L^k(\lambda_l) - \lambda_l \upsilon_k \Big) \Big\} + t_{|j|} \log p_k \\
&\quad = \sum_{l=1}^{|j|} (t_l - t_{l-1}) L^{k*}\Big(\tfrac{x_l + t_{|j|} \upsilon_k}{t_l - t_{l-1}} \Big) + t_{|j|} \log p_k . \qquad (2.24)
\end{aligned}$$

Since for any open set G

$$\liminf_{n\to\infty} \frac{1}{n} \log P(V_n^{\prime,k} \in G) \ge \liminf_{n\to\infty} \frac{1}{n} \log \nu_n^k(G) + 4 \log p_k ,$$

we conclude that for any x and $\epsilon > 0$, for all k large enough,

$$\liminf_{n\to\infty} \frac{1}{n} \log P(V_n' \in B(\underline{x}, 2\epsilon)) \ge \liminf_{n\to\infty} \frac{1}{n} \log \nu_n^k(B(\underline{x}, \epsilon)) + 4 \log p_k ,$$

where $B(\underline{x}, \epsilon)$ is an open ball centered at x with radius ϵ.

Now note that for every $\lambda \in \mathbb{R}^d$, $L^k(\lambda)$ is increasing to $\Lambda(\lambda)$ with k. So by Theorem B3 in de Acosta (1988), there exists $\{\underline{x}^k\} \subset (\mathbb{R}^d)^{|j|}$, such that $\underline{x}^k \to \underline{x}$, and

$$\limsup_{k\to\infty} \sum_{l=1}^{|j|} (t_l - t_{l-1}) L^{k*}\Big(\frac{x_l^k}{t_l - t_{l-1}} \Big) \le \sum_{l=1}^{|j|} (t_l - t_{l-1}) L^{*}\Big(\frac{x_l}{t_l - t_{l-1}} \Big) .$$

Since $\underline{x}^k - t_{|j|}\underline{\upsilon}_k \in B(\underline{x}, 2\epsilon)$ for k large, where $\underline{\upsilon}_k = (\upsilon_k, \ldots, \upsilon_k) \in (R^d)^{|j|}$, we conclude that

$$\liminf_{n\to\infty} \frac{1}{n} \log P(V_n' \in B(\underline{x}, \epsilon)) \geq -\sum_{l=1}^{|j|} (t_l - t_{l-1})\Lambda^*\Big(\frac{x_l}{t_l - t_{l-1}}\Big) .$$

Furthermore, because the laws of (V_n) and of (V_n') are exponentially equivalent, the same statement holds with V_n replacing V_n'. We have, therefore, established that the laws of $(Y_n(t_1), \ldots, Y_n(t_{|j|}))$ satisfy a LD lower bound with speed n and good rate function $H^v_{t_1,\ldots,t_{|j|}}(\cdot)$ defined in (2.22) for part (ii). By the lower bound part of the Dawson-Gärtner theorem, $\{\mu_n\}$ satisfy a LD lower bound in $\mathcal{X}_P$ with speed n and rate function $\sup_{j\in J} H^v_{t_1,\ldots,t_{|j|}}(f(t_1), \ldots, f(t_{|j|}))$. This rate function is identical to H^{sl}.

It is to observe that the lower rate function H^{sl} is infinite outside of the space $\cap_{p\in[1,\infty)} L_p[0,1]$, and by Lemma 2.3.4, the same is true for the upper rate function G^{sl} (we view $\cap_{p\in[1,\infty)} L_p[0,1]$ as a measurable subset of $\mathcal{X}$ with respect to the universal completion of the cylindrical σ-field). We conclude that the measures $\{\mu_n\}$ satisfy a LD lower bound in $\cap_{p\in[1,\infty)} L_p[0,1]$ with the topology of pointwise convergence. Since this topology is coarser than the L topology, we can use Lemma 2.3.3 to conclude that the LD upper bound and the LD lower bound also hold in $\cap_{p\in[1,\infty)} L_p[0,1]$ equipped with L topology. Finally, the rate functions are also infnite outside of the space $\mathcal{BV}$, and so the measures $\{\mu_n\}$ satisfy the LD bounds in $\mathcal{BV}$ equipped with L topology. □

Proof of Theorem 2.2.4. The proof of parts (ii), (iii) and (iv) is identical to the proof of the corresponding parts in Theorem 2.2.2, except that now Lemma 2.3.6 is used instead of Lemma 2.3.5, and we use Lemma 2.3.8 to identify the rate function.

We now prove part (i) of the theorem. We start by proving the finite dimensional LDP for the laws of V_n in (2.21). An inspection of the proof of the corresponding statement on Theorem 2.2.2 shows that the only missing ingredient needed to obtain the upper bound part of this LDP is the exponential tightness of $Y_n(1)$ in $\mathbb{R}^d$. Notice that for $s > 0$ and small $\lambda > 0$

$$P\Big(Y_n(1) \notin [-s,s]^d\Big) \le e^{-\lambda ns} \sum_{l=1}^{d} E\Big(e^{\lambda Y_n^{(l)}(1)} + e^{-\lambda Y_n^{(l)}(1)}\Big),$$

where $Y_n^{(l)}(1)$ is the lth coordinate of $Y_n(1)$. Since $0 \in \mathcal{F}_\Lambda^o$, by part (i) of Lemma 2.3.6 we see that

$$\lim_{s\to\infty} \limsup_{n\to\infty} \frac{1}{n} \log P\Big(Y_n(1) \notin [-s,s]^d\Big) = -\infty\,,$$

which is the required exponential tightness. It follows that the laws of (V_n) satisfy a LD upper bound with speed n and rate function

$$\sup_{\underline{\lambda}\in\Pi^{rl}_{t_1,\ldots,t_{|j|}}} \Big\{\underline{\lambda}\cdot\underline{w} - \Lambda^{rl}_{t_1,\ldots,t_{|j|}}(\lambda_1,\ldots,\lambda_{|j|})\Big\}\,.$$

Next we prove a LD lower bound for the laws of (V_n). The proof in the case $\alpha = 1$ follows the same steps as the corresponding argument in Theorem 2.2.2, so we will concentrate on the case $1/2 < \alpha < 1$. For $m \ge 1$ let

$$V'_{n,m} = \frac{1}{n\Psi_n}\Big(\sum_{|i|\le mn} \phi_{i,[nt_1]} Z_{-i}, \sum_{|i|\le mn} \phi_{i+[nt_1],[nt_2]-[nt_1]} Z_{-i}, \cdots,$$
$$\sum_{|i|\le mn} \phi_{i+[nt_{|j|-1}],[nt_{|j|}]-[nt_{|j|-1}]} Z_{-i}\Big)\,.$$

Observe that $V_n = V'_{n,m} + R'_{n,m}$ for some $R'_{n,m}$ independent of $V'_{n,m}$ and such that for every m, $R'_{n,m} \to 0$ in probability as $n \to \infty$. We conclude that for any $\underline{x} = (x_1,\cdots,x_{|j|}) \in (\mathbb{R}^d)^{|j|}$, $\epsilon > 0$, and n sufficiently large, one has

$$P(V_n \in B(\underline{x},2\epsilon)) \ge \frac{1}{2} P(V'_{n,m} \in B(\underline{x},\epsilon))\,. \tag{2.25}$$

For $k \geq 1$ we define p_k and υ_k as in the proof of Theorem 2.2.2, and once again we choose k large enough so that $p_k > 0$. We also define

$$V'^{,k}_{n,m} = \frac{1}{n\Psi_n}\Big(\sum_{|i|\leq mn} \phi_{i,[nt_1]}\big(Z_{-i} - \upsilon_k\big), \sum_{|i|\leq mn} \phi_{i+[nt_1],[nt_2]-[nt_1]}\big(Z_{-i} - \upsilon_k\big), \cdots ,$$

$$\sum_{|i|\leq mn} \phi_{i+[nt_{|j|-1}],[nt_{|j|}]-[nt_{|j|-1}]}\big(Z_{-i} - \upsilon_k\big)\Big) := V'_{n,m} - a^{(m)}_{n,k} ,$$

where $a^{(m)}_{n,k} = (b^{(n,m)}_1 \upsilon_k, b^{(n,m)}_2 \upsilon_k, \ldots, b^{(n,m)}_{|j|} \upsilon_k) \in (R^d)^{|j|}$ with some $|b^{(n,m)}_i| \leq c_m$, a constant independent of i and n.

Once again we define a new probability measure by

$$\nu^{k,m}_n(\cdot) = P\Big(V'^{,k}_{n,m} \in \cdot, |Z_i| \leq k, \ \text{ for all } |i| \leq mn\Big) p_k^{-(2mn+1)} .$$

Note that for all $\underline{\lambda} \in (\mathbb{R}^d)^{|j|}$, by (the proof of) Lemma 2.3.6,

$$\begin{aligned}
& \lim_{n\to\infty} \frac{1}{n} \log \Big\{ p_k^{-(2mn+1)} E\Big[\exp\big(n\underline{\lambda} \cdot V'^{,k}_{n,m}\big) I_{[|Z_i|\leq k,\, |i|\leq mn]}\Big]\Big\} \\
= & \int_{-m}^{m} L^k \left((1-\alpha) \sum_{i=1}^{|j|} \lambda_i \int_{x+t_{i-1}}^{x+t_i} |y|^{-\alpha}(pI_{[y\geq 0]} + qI_{[y<0]})dy \right) dx \\
& -(1-\alpha) \sum_{l=1}^{|j|} \lambda_i \cdot \upsilon_k \int_{-m}^{m} \left(\int_{x+t_{i-1}}^{x+t_i} |y|^{-\alpha}(pI_{[y\geq 0]} + qI_{[y<0]})dy \right) dx - 2m \log p_k \\
= & \ Q^{k,m}(\underline{\lambda}) - \upsilon_k \cdot R^m(\underline{\lambda}) - 2m \log p_k \quad \text{(say)}
\end{aligned}$$

where $L^k(\lambda) = \log E\big[\exp(\lambda \cdot Z_0) I_{[|Z_0|\leq k]}\big]$, as defined before. Therefore, for every $k \geq 1$, $\{\nu^{k,m}_n, n \geq 1\}$ satisfy LDP with speed n and good rate function $(Q^{k,m})^*(\underline{x} - \underline{c}_{k,m}) + 2m \log p_k$, where $\underline{c}_{k,m} = (c^m_1 \upsilon_k, c^m_2 \upsilon_k, \ldots, c^m_{|j|} \upsilon_k) \in (\mathbb{R}^d)^{|j|}$ with

$$c^m_i = (1-\alpha) \int_{-m}^{m} \left(\int_{x+t_{i-1}}^{x+t_i} |y|^{-\alpha}(pI_{[y\geq 0]} + qI_{[y<0]})dy \right) dx .$$

Note that as k increases, for every $\lambda \in \mathbb{R}^d$, $L^k(\lambda)$ is increasing to $\Lambda(\lambda)$ and $Q^{k,m}(\underline{\lambda})$ is increasing to

$$\Lambda^{rl,m}_{t_1,\ldots,t_{|j|}}(\underline{\lambda}) = \int_{-m}^{m} \Lambda\Big(h_{t_1,\ldots,t_k}(x; \underline{\lambda})\Big) dx.$$

An application of Theorem B3 in de Acosta (1988) shows, as in the proof of Theorem 2.2.2, that for any ball centered at x with radius ϵ

$$\liminf_{n\to\infty} \frac{1}{n} \log P(V'_{n,m} \in B(\underline{x}, \epsilon)) \geq -(\Lambda^{rl,m}_{t_1,\dots,t_{|j|}})^*(\underline{x}).$$

Appealing to (2.25) gives us

$$\liminf_{n\to\infty} \frac{1}{n} \log P(V_n \in B(\underline{x}, 2\epsilon)) \geq -(\Lambda^{rl,m}_{t_1,\dots,t_{|j|}})^*(\underline{x})$$

for all $m \geq 1$. We now apply the above argument once again: for every $\lambda \in \mathbb{R}^d$, $\Lambda^{rl,m}_{t_1,\dots,t_{|j|}}(\underline{\lambda})$ increases to $\Lambda^{rl}_{t_1,\dots,t_{|j|}}(\underline{\lambda})$, and yet another appeal to Theorem B3 in de Acosta (1988) gives us the desired LD lower bound for the laws of (V_n) in the case $1/2 < \alpha < 1$.

Continuing as in the proof of Theorem 2.2.2 we conclude that $\{\mu_n\}$ satisfy a LD lower bound in $\mathcal{X}_P$ with speed n and rate function $\sup_{j\in J}(\Lambda^{rl}_{t_1,\dots,t_{|j|}})^*(f(t_1), f(t_2)-f(t_1),\dots,f(t_{|j|})-f(t_{|j|-1}))$. By Lemma 2.3.8 this is equal to $H^{rl}(f)$ in the case $1/2 < \alpha < 1$, and in the case $\alpha = 1$ the corresponding statement is the same as in Theorem 2.2.2. The fact that the LD lower bound holds also in $\mathcal{BV}_L$ follows in the same way as in Theorem 2.2.2. This completes the proof. □

The next lemma discusses some properties of the rate function $(G_\Sigma)^*_\alpha$ in Theorem 2.2.4. For $0 < \theta < 1$, let

$$H_\theta = \left\{ \psi : [0,1] \to \mathbb{R}^d, \text{ measurable, and } \int_0^1 \int_0^1 \frac{|\psi(t)||\psi(s)|}{|t-s|^\theta} dt\, ds < \infty \right\}.$$

If Σ is a nonnegative definite matrix, we define an inner product on H_θ by

$$(\psi_1, \psi_2)_\Sigma = \int_0^1 \int_0^1 \frac{\psi_1(t) \cdot \Sigma \psi_2(s)}{|t-s|^\theta} dt\, ds\, .$$

This results in an incomplete inner product space; see Landkof (1972). Observe also that $L_\infty[0,1] \subset H_\theta \subset L_2[0,1]$, and that

$$(\psi_1, \psi_2)_\Sigma = (\psi_1, T_\theta \psi_2)\, ,$$

where

$$(\psi_1, \psi_2) = \int_0^1 \psi_1(t) \cdot \psi_2(t)\, dt$$

is the inner product in $L_2[0,1]$, and $T_\theta : H_\theta \to H_\theta$ is defined by

$$T_\theta \psi(t) = \int_0^1 \frac{\Sigma\psi(s)}{|t-s|^\theta} ds\,. \tag{2.26}$$

Lemma 2.2.8. *For $\varphi \in L_1[0,1]$ and $1/2 < \alpha < 1$,*

$$(G_\Sigma)^*_\alpha(\varphi) = \sup_{\psi \in L_\infty[0,1]} (\psi, \varphi) - \frac{\sigma^2}{2}\Big(\psi, T_{2\alpha-1}\psi\Big)\,, \tag{2.27}$$

where

$$\sigma^2 = (1-\alpha)^2 \int_{-\infty}^{\infty} |x+1|^{-\alpha}|x|^{-\alpha}\Big[pI_{[x+1\geq 0]} + qI_{[x+1<0]}\Big]\Big[pI_{[x\geq 0]} + qI_{[x<0]}\Big]\, dx\,,$$

ψ is regarded as an element of the dual space $L_1[0,1]'$, and $T_{2\alpha-1}$ in (2.26) *is regarded as a map $L_\infty[0,1] \to L_1[0,1]$.*

(i) Suppose that $\varphi \in T_{2\alpha-1}H_{2\alpha-1}$. Then

$$(G_\Sigma)^*_\alpha(\varphi) = \frac{1}{2\sigma^2}\|h\|^2_\Sigma\,,$$

where $\varphi = T_{2\alpha-1}h$.

(ii) Suppose that $Leb\{t \in [0,1] : \varphi(t) \in K_\Sigma\} > 0$, where $K_\Sigma = Ker(\Sigma) - \{0\}$ is as defined in (2.2.2). Then $(G_\Sigma)^*_\alpha(\varphi) = \infty$.

Proof. Note that for $\varphi \in L_1[0,1]$

$$\int_{-\infty}^{\infty} G_\Sigma\left(\int_0^1 \psi(t)(1-\alpha)|x+t|^{-\alpha}\Big[pI_{[x+t\geq 0]} + qI_{[x+t<0]}\Big]\, dt\right)$$

$$= \frac{1}{2}(1-\alpha)^2 \int_0^1\int_0^1 \psi(s)\cdot\Sigma\psi(t)\left(\int_{-\infty}^{\infty} |x+s|^{-\alpha}|x+t|^{-\alpha}\Big[pI_{[x+s\geq 0]} + qI_{[x+s<0]}\Big]\right.$$

$$\left.\Big[pI_{[x+t\geq 0]} + qI_{[x+t<0]}\Big]\, dx\right) ds\, dt = \frac{\sigma^2}{2}\int_0^1\int_0^1 \frac{\psi(s)\cdot\Sigma\psi(t)}{|t-s|^\theta} ds\, dt\,,$$

and so (2.27) follows.

For part (i), suppose that $\varphi = T_{2\alpha-1}h$ for $h \in H_{2\alpha-1}$. For $\psi \in H_{2\alpha-1}$ we have

$$(\psi, \varphi) - \frac{\sigma^2}{2}(\psi, T_{2\alpha-1}\psi) = \frac{1}{2\sigma^2}\big(h, T_{2\alpha-1}h\big) - \frac{\sigma^2}{2}\Big((\psi - \frac{1}{\sigma^2}h), T_{2\alpha-1}(\psi - \frac{1}{\sigma^2}h)\Big)$$

because the operator $T_{2\alpha-1}$ is self-adjoint. Therefore,

$$\sup_{\psi \in H_{2\alpha-1}} (\psi, \varphi) - \frac{\sigma^2}{2}(\psi, T_{2\alpha-1}\psi) = \frac{1}{2\sigma^2}\big(h, T_{2\alpha-1}h\big)\,,$$

achieved at $\psi_0 = h/\sigma^2$, and so by (2.27),

$$(G_\Sigma)^*_\alpha(\varphi) \le \frac{1}{2\sigma^2}\big(h, T_{2\alpha-1}h\big)\,.$$

On the other hand, for $M > 0$ let $\psi_0^{(M)} = \psi_0 I\big(|\psi_0| \le M\big) \in L_\infty[0,1]$. Then

$$\begin{aligned}(G_\Sigma)^*_\alpha(\varphi) &\ge \limsup_{M\to\infty} \psi_0^{(M)}(\varphi) - \frac{\sigma^2}{2}\psi_0^{(M)}\Big(T_{2\alpha-1}\psi_0^{(M)}\Big)\\ &= (\psi_0, \varphi) - \frac{\sigma^2}{2}(\psi_0, T_{2\alpha-1}\psi_0) = \frac{1}{2\sigma^2}\big(h, T_{2\alpha-1}h\big)\,,\end{aligned}$$

completing the proof of part (i).

For part (ii), note that using (2.27) and choosing for $c > 0$, $\psi(t) = c\varphi(t)/|\varphi(t)|$ if $\varphi(t) \in K_\Sigma$, and $\psi(t) = 0$ otherwise, we obtain

$$(G_\Sigma)^*_\alpha(\varphi) \ge c\int_A |\varphi(t)|\,dt\,,$$

where $A = \{t \in [0,1] : \varphi(t) \in K_\Sigma\}$. The proof is completed by letting $c \to \infty$. □

2.3 Lemmas and their Proofs

In this section we prove the lemmas used in Section 2.2. We retain the notation of Section 2.2.

Lemma 2.3.1. *Under any of the Assumptions $S2, S3, S4, R2, R3$ or $R4$, the families $\{\mu_n\}$ and $\{\tilde{\mu}_n\}$ are exponentially equivalent in $\mathcal{D}_S$, where $\mathcal{D}$ is the space of all right-continuous functions with left limits and, as before, the subscript denotes the sup-norm topology on that space.*

Proof. It is clearly enough to consider the case $d = 1$. For any $\delta > 0$ and $\lambda \in \mathcal{F}_\Lambda \cap -\mathcal{F}_\Lambda$, $\lambda \neq 0$,

$$\begin{aligned}
& \limsup_{n\to\infty} \frac{1}{b_n} \log P\Big(||Y_n - \tilde{Y}_n|| > \delta\Big) \\
\leq \;& \limsup_{n\to\infty} \frac{1}{b_n} \log P\Big(\frac{1}{a_n} \max_{1\leq i\leq n} |X_i| > \delta\Big) \\
\leq \;& \limsup_{n\to\infty} \frac{1}{b_n} \log \Big(n\, P(|X_1| > a_n\delta)\Big) \\
\leq \;& \limsup_{n\to\infty} \frac{1}{b_n} \Big(\log n - a_n\lambda\delta + \Lambda(\lambda) + \Lambda(-\lambda))\Big) \\
= \;& \limsup_{n\to\infty} \frac{1}{b_n} \Big(- a_n\lambda\delta\Big).
\end{aligned}$$

Under the Assumptions $S3, S4, R3$ or $R4$ we have $a_n/b_n \to \infty$, so the above limit is equal to $-\infty$. Under the Assumptions $S2$ and $R2$, $a_n = b_n$, but we can let $\lambda \to \infty$ after taking the limit in n. □

Lemma 2.3.2. *Under any of the Assumptions $S2, S3, S4, R2, R3$ or $R4$, the family $\{\tilde{\mu}_n\}$ is exponentially tight in $\mathcal{D}_S$, i.e, for every $\pi > 0$ there exists a compact $K_\pi \subset \mathcal{D}_S$, such that*

$$\lim_{\pi\to\infty} \limsup_{n\to\infty} \frac{1}{b_n} \log \tilde{\mu}_n(K_\pi^c) = -\infty.$$

Proof. We first prove the lemma assuming that $d = 1$. We use the notation $w(f, \delta) := \sup_{s,t\in[0,1],|s-t|<\delta} |f(s) - f(t)|$ for the modulus of continuity of a function $f : [0, 1] \to \mathbb{R}^d$. First we claim that for any $\epsilon > 0$,

$$\lim_{\delta\to 0} \limsup_{n\to\infty} \frac{1}{b_n} \log P\Big(w(\tilde{Y}_n, \delta) > \epsilon\Big) = -\infty, \tag{2.28}$$

where $\tilde{Y}_n$ is the polygonal process in (2.5). Let us prove the lemma assuming that the claim is true. By (2.28) and the continuity of the paths of $\tilde{Y}_n$, there is $\delta_k > 0$ such that for all $n \geq 1$

$$P\big(w(\tilde{Y}_n, \delta_k) \geq k^{-1}\big) \leq e^{-\pi b_n k},$$

and set $A_k = \{f \in \mathcal{D} : w(f, \delta_k) < k^{-1}, f(0) = 0\}$. Now the set $K_\pi := \overline{\cap_{k\geq 1} A_k}$ is compact in $\mathcal{D}_S$ and by the union of events bound it follows that

$$\limsup_{n\to\infty} \frac{1}{b_n} \log P(\tilde{Y}_n \notin K_\pi) \leq -\pi,$$

establishing the exponential tightness. Next we prove the claim (2.28). Observe that for any $\epsilon > 0$, $\delta > 0$ small and $n > 2/\delta$

$$\begin{aligned}
P\quad & \big(w(\tilde{Y}_n, \delta) > \epsilon\big) \leq P\Big(\max_{0\leq i<j\leq n, j-i\leq [n\delta]+2} \frac{1}{a_n}\Big|\sum_{k=i}^{j} X_k\Big| > \epsilon\Big) \\
\leq\quad & n \sum_{i=1}^{[2n\delta]} P\Big(\frac{b_n}{a_n}\Big|\sum_{k=1}^{i} X_k\Big| > b_n\epsilon\Big) \\
\leq\quad & ne^{-b_n\lambda\epsilon} \sum_{i=1}^{[2n\delta]} E\Big[\exp\Big(\frac{\lambda b_n}{a_n}\sum_{k=1}^{i} X_k\Big) + \exp\Big(-\frac{\lambda b_n}{a_n}\sum_{k=1}^{i} X_k\Big)\Big] \\
=\quad & ne^{-b_n\lambda\epsilon} \sum_{i=1}^{[2n\delta]} \Big(\exp\Big[\sum_{j\in\mathbb{Z}} \Lambda\Big(\frac{\lambda b_n}{a_n}\phi_{j,i}\Big)\Big] + \exp\Big[\sum_{j\in\mathbb{Z}} \Lambda\Big(-\frac{\lambda b_n}{a_n}\phi_{j,i}\Big)\Big]\Big) \\
\leq\quad & \frac{2n^2\delta}{e^{b_n\lambda\epsilon}}\Big(\exp\Big[\sum_{j\in\mathbb{Z}} \Lambda\Big(\frac{|\lambda| b_n}{a_n}|\phi|_{j,[2n\delta]}\Big)\Big] + \exp\Big[\sum_{j\in\mathbb{Z}} \Lambda\Big(-\frac{|\lambda| b_n}{a_n}|\phi|_{j,[2n\delta]}\Big)\Big]\Big)
\end{aligned}$$

by convexity of Λ (we use the notation $|\phi|_{i,n} = |\phi_{i+1}| + \cdots + |\phi_{i+n}|$ for $i \in \mathbb{Z}$ and $n \geq 1$). Therefore by Lemmas 2.3.5 and 2.3.6 we have

$$\lim_{\delta\to 0}\limsup_{n\to\infty} \frac{1}{b_n} \log P\big(w(\tilde{Y}_n, \delta) > \epsilon\big) \leq -\lambda\epsilon.$$

Now, letting $\lambda \to \infty$ we obtain (2.28).

If $d \geq 1$ then $\{\tilde{\mu}_n\}$ is exponentially tight since $\{\tilde{\mu}_n^k\}$, the law of the kth coordinate of $\tilde{Y}_n$, is exponentially tight for every $1 \leq k \leq d$. □

Lemma 2.3.3. *Under the Assumptions* $S1$ *or* $R1$ *the family* $\{\mu_n\}$ *is, for any* $p \in [1, \infty)$*, exponentially tight in the space of functions in* $\cap_{p\in[1,\infty)} L_p[0,1]$*, equipped with the topology* L*, where* f_n *converges to* f *if and only if* f_n *converges to* f *both pointwise and in* $L_p[0,1]$ *for all* $p \in [1, \infty)$*.*

Proof. Here $a_n = n$ under the Assumption $S1$, $a_n = n\Psi_n$ under the Assumption $R1$, and $b_n = n$ in both cases. As before, it is enough to consider the case $d = 1$. We claim that for any $p \in [1, \infty)$,

$$\lim_{x\downarrow 0} \limsup_{n\to\infty} \frac{1}{n} \log P\Bigg[\int_0^{1-x} \Big|Y_n(t+x) - Y_n(t)\Big|^p dt \tag{2.29}$$
$$+ \int_0^x \Big|Y_n(t)\Big|^p dt + \int_{1-x}^1 \Big|Y_n(t)\Big|^p dt > \epsilon \Bigg] = -\infty,$$

for any $\epsilon > 0$, while

$$\lim_{M\uparrow\infty} \limsup_{n\to\infty} \frac{1}{n} \log P\Big(\sup_{0\le t\le 1} |Y_n(t)| > M \Big) = -\infty\,. \tag{2.30}$$

Assuming that both claims are true, for any $\pi > 0$, $m \geq 1$ and $k \geq 1$, we can choose (using the fact that $Y_n \in L^\infty[0,1]$ a.s. for all $n \geq 1$) $0 < x_k^{(m)} < 1$ such that for all $n \geq 1$,

$$P\Bigg[\int_0^{1-x_k^{(m)}} \Big|Y_n(t + x_k^{(m)}) - Y_n(t)\Big|^m dt$$
$$+ \int_0^{x_k^{(m)}} \Big|Y_n(t)\Big|^m dt + \int_{1-x_k^{(m)}}^1 \Big|Y_n(t)\Big|^m dt > k^{-1} \Bigg] \leq e^{-\pi knm},$$

and $M_\pi > 0$ such that for all $n \geq 1$

$$P\Big(\sup_{0\le t\le 1} |Y_n(t)| > M_\pi \Big) \leq e^{-\pi n}.$$

Now define sets

$$K_\pi = \overline{\cap_{k,m\geq 1} A_{k,m}}, \qquad \text{for all } k, m \geq 1,$$

where

$$A_{k,m} = \Big\{ f \in \cap_{p\geq 1} L_p[0,1] : \int_0^{1-x_k^{(m)}} \Big| f(t + x_k^{(m)}) - f(t) \Big|^m dt$$

$$+ \int_0^{x_k^{(m)}} \Big| f(t) \Big|^m dt + \int_{1-x_k^{(m)}}^1 \Big| f(t) \Big|^m dt \leq k^{-1}, \ \sup_{0\leq t\leq 1} |f(t)| \leq M_\pi \Big\}.$$

Then K_π is compact for every $\pi > 0$ by Tychonov's Theorem (see Theorem 19, p. 166 in Royden (1968) and Theorem 20, p. 298 in Dunford and Schwartz (1988)). Furthermore,

$$\limsup_{n\to\infty} \frac{1}{n} \log P[Y_n \notin K_\pi] \leq -\pi.$$

This will complete the proof once we prove (2.29) and (2.30). We first prove (2.29) for $p = 1$. Observe that

$$\begin{aligned}
& P\Big[\int_0^{1-x} |Y_n(t+x) - Y_n(t)| dt > \epsilon \Big] \leq P\Big[\frac{[nx]}{n} \frac{1}{a_n} \sum_{i=1}^n |X_i| > \epsilon \Big] \\
\leq \ & e^{-\lambda n \epsilon / x} E\Big[\exp\Big(\lambda \frac{b_n}{a_n} \sum_{i=1}^n |X_i| \Big) \Big] \leq e^{-\lambda n\epsilon/x} E\Big[\prod_{i=1}^n \exp\Big(\frac{\lambda b_n}{a_n} |X_i| \Big) \Big] \\
\leq \ & e^{-\lambda n\epsilon/x} E\Big[\prod_{i=1}^n \Big(\exp\Big(\frac{\lambda b_n}{a_n} X_i \Big) + \exp\Big(- \frac{\lambda b_n}{a_n} X_i \Big) \Big) \Big] \\
= \ & e^{-\lambda n\epsilon/x} \sum_{l_i = \pm 1} E\Big[\exp\Big(\frac{\lambda b_n}{a_n} \sum_{i=1}^n l_i X_i \Big) \Big] \\
= \ & e^{-\lambda n\epsilon/x} \sum_{l_i = \pm 1} \exp\Big(\sum_{j\in\mathbb{Z}} \Lambda\Big(\frac{\lambda b_n}{a_n} (\phi_{j+1} l_1 + \cdots + \phi_{j+n} l_n) \Big) \Big) \\
\leq \ & 2^n e^{-\lambda n\epsilon/x} \exp\Big(\sum_{j\in\mathbb{Z}} \Lambda\Big(\frac{\lambda b_n}{a_n} |\phi|_{j,n} \Big) + \sum_{j\in\mathbb{Z}} \Lambda\Big(- \frac{\lambda b_n}{a_n} |\phi|_{j,n} \Big) \Big).
\end{aligned}$$

Therefore,

$$\begin{aligned}
& \limsup_{n\to\infty} \frac{1}{n} \log P\Big[\int_0^{1-x} |Y_n(t+x) - Y_n(t)| dt > \epsilon \Big] \\
\leq \ & \log 2 - \frac{\lambda\epsilon}{x} + \limsup_{n\to\infty} \frac{1}{n} \sum_{j\in\mathbb{Z}} \Lambda\Big(\frac{\lambda b_n}{a_n} |\phi|_{j,n} \Big) + \limsup_{n\to\infty} \frac{1}{n} \sum_{j\in\mathbb{Z}} \Lambda\Big(- \frac{\lambda b_n}{a_n} |\phi|_{j,n} \Big).
\end{aligned}$$

Keeping $\lambda > 0$ small, using Lemma 2.3.5 and Lemma 2.3.6 and then letting $x \to 0$

one establishes the limit

$$\limsup_{n\to\infty} \frac{1}{n} \log P\Big[\int_0^{1-x} |Y_n(t+x) - Y_n(t)| dt > \epsilon \Big] = -\infty.$$

It is simpler to show a similar inequality for the second and the third integrals under the probability of the equation (2.29). The proof of (2.30) is similar, starting with

$$P\Big(\sup_{0\le t\le 1} |Y_n(t)| > M \Big) \le P\Big(\frac{1}{a_n} \sum_{i=1}^n |X_i| > M \Big).$$

Now one establishes (2.29) for $p \ge 1$ by writing, for $M > 0$,

$$P\Big[\int_0^{1-x} \Big|Y_n(t+x) - Y_n(t)\Big|^p dt + \int_0^x \Big|Y_n(t)\Big|^p dt + \int_{1-x}^1 \Big|Y_n(t)\Big|^p dt > \epsilon \Big]$$

$$\le P\Big[\int_0^{1-x} \Big|Y_n(t+x) - Y_n(t)\Big| dt + \int_0^x \Big|Y_n(t)\Big| dt + \int_{1-x}^1 \Big|Y_n(t)\Big| dt > \frac{\epsilon}{2M^{p-1}} \Big]$$
$$+ P\Big[\sup_{0\le t\le 1} |Y_n(t)| > M \Big],$$

and letting first $n \to \infty$, $x \downarrow 0$, and then $M \uparrow \infty$. □

Lemma 2.3.4. *Under the Assumptions $S1$ or $R1$, the corresponding upper rate functions, G^{sl} in (2.11) and G^{rl} in (2.19), are infinite outisde of the space $\mathcal{BV}$.*

Proof. Let $f \notin \mathcal{BV}$. Choose $\delta > 0$ small enough such that any λ with $|\lambda| \le \delta$ is in $\mathcal{F}_\Lambda^\circ$ and a vector with k identical components $(\lambda, \ldots, \lambda)$ is in the interiors of both $\Pi_{t_1,\ldots,t_k}$ in (2.10) and $\Pi_{t_1,\ldots,t_k}^{r,\alpha}$ in (2.14) and (2.15). For $M > 0$ choose a partition $0 < t_1 < \cdots < t_k = 1$ of $[0,1]$ such that $\sum_{i=1}^k \big| f(t_i) - f(t_{i-1}) \big| > M$. For $i = 1, \ldots, k$ such that $f(t_i) - f(t_{i-1}) \ne 0$ choose λ_i of length δ in the direction of $f(t_i) - f(t_{i-1})$. Then under, say, Assumption $S1$,

$$G^{sl}(f) \ge \sup_{\underline{\lambda}\in\Pi_{t_1,\ldots,t_k}} \sum_{i=1}^k \Big\{ \lambda_i \cdot \big(f(t_i) - f(t_{i-1})\big) - (t_i - t_{i-1})\Lambda(\lambda_i) \Big\}$$
$$\ge \delta M - \sup_{|\lambda|\le\delta} \Lambda(\lambda).$$

Letting $M \to \infty$ proves the statement under the Assumption $S1$, and the argument under the Assumption $R1$ is similar. □

Lemma 2.3.5. *Suppose $\Lambda : \mathbb{R}^d \to \mathbb{R}$ is the log-moment generating function of a mean zero random variable Z, with $0 \in \mathcal{F}_\Lambda^\circ$, $\sum_{i=-\infty}^{\infty} |\phi_i| < \infty$ with $\sum_{i=-\infty}^{\infty} \phi_i = 1$ and $0 < t_1 < \cdots < t_k \leq 1$.*

(i) For all $\underline{\lambda} = (\lambda_1, \ldots, \lambda_k) \in \Pi_{t_1,\ldots,t_k} \subset (\mathbb{R}^d)^k$,

$$\lim_{n\to\infty} \frac{1}{n} \sum_{l=-\infty}^{\infty} \Lambda\Big(\sum_{i=1}^{k} \lambda_i \phi_{l+[nt_{i-1}],[nt_i]-[nt_{i-1}]}\Big) = \sum_{i=1}^{k} (t_i - t_{i-1})\Lambda(\lambda_i).$$

(ii) If $a_n/\sqrt{n} \to \infty$ and $a_n/n \to 0$ then for all $\underline{\lambda} \in (\mathbb{R}^d)^k$,

$$\lim_{n\to\infty} \frac{n}{a_n^2} \sum_{l=-\infty}^{\infty} \Lambda\Big(\frac{a_n}{n} \sum_{i=1}^{k} \lambda_i \phi_{l+[nt_{i-1}],[nt_i]-[nt_{i-1}]}\Big) = \sum_{i=1}^{k} (t_i - t_{i-1})\lambda_i \cdot \Sigma\lambda_i,$$

where Σ is the covaraince matrix of Z.

(iii) If $\Lambda(\cdot)$ is balanced regular varying at ∞ with exponent $\beta > 1$, $a_n/n \to \infty$ and b_n is as defined as defined in Assumption S4, then for all $\underline{\lambda} \in (\mathbb{R}^d)^k$,

$$\lim_{n\to\infty} \frac{1}{b_n} \sum_{l=-\infty}^{\infty} \Lambda\Big(\frac{b_n}{a_n} \sum_{i=1}^{k} \lambda_i \phi_{l+[nt_{i-1}],[nt_i]-[nt_{i-1}]}\Big)$$
$$= \sum_{i=1}^{k} (t_i - t_{i-1})\zeta\Big(\frac{\lambda_i}{|\lambda_i|}\Big)|\lambda_i|^\beta.$$

Proof. (i) We begin by making a few observations:

(a) For every $\delta > 0$ there exists N_δ such that for all $n > N_\delta$

$$\sum_{|i|>(n \min_j (t_j - t_{j-1}))^{1/2}} |\phi_i| < \delta. \tag{2.31}$$

(b) For fixed $\underline{\lambda} = (\lambda_1, \ldots, \lambda_k) \in \Pi_{t_1,\ldots,t_k}$, there exists $M > 0$ such that for all $l \in \mathbb{Z}$ and all n large enough

$$\Big|\Lambda\Big(\sum_{i=1}^{k} \lambda_i \phi_{l+[nt_{i-1}],s_i}\Big)\Big| \leq M, \tag{2.32}$$

where $s_i = s_i(n) = [nt_i] - [nt_{i-1}]$. Since the zero mean of Z means that $\Lambda(x) = o(|x|)$ as $|x| \to 0$, it follows from (2.32) that there exists $C > 0$ such that in the same range of n and for all $l \in \mathbb{Z}$

$$\Big|\Lambda\Big(\sum_{i=1}^{k} \lambda_i \phi_{l+[nt_{i-1}],s_i}\Big)\Big| \leq C\Big|\sum_{i=1}^{k} \lambda_i \phi_{l+[nt_{i-1}],s_i}\Big|. \tag{2.33}$$

Let $L = \Big(|\lambda_1| + \cdots + |\lambda_k|\Big)$. Since Λ is continuous at λ_j, given $\epsilon > 0$ we can choose $\delta > 0$ so that for n large enough,

$$\Big|\sum_{i=1}^{k} \lambda_i \phi_{l+[nt_{i-1}],s_i} - \lambda_j\Big| < \delta$$

for all $-[nt_j] + \sqrt{s_j} < l < -[nt_{j-1}] - \sqrt{s_j}$, and then

$$\Big|\frac{1}{n}\sum_{l=-[nt_j]+\sqrt{s_j}}^{-[nt_{j-1}]-\sqrt{s_j}} \Lambda\Big(\sum_{i=1}^{k} \lambda_i \phi_{l+[nt_{i-1}],s_i}\Big) - \frac{s_j - 2\sqrt{s_j}}{n}\Lambda(\lambda_j)\Big| < \epsilon.$$

Therefore for $j = 1, \ldots, k$

$$\lim_{n\to\infty} \frac{1}{n}\sum_{l=-[nt_j]+\sqrt{s_j}}^{-[nt_{j-1}]-\sqrt{s_j}} \Lambda\Big(\sum_{i=1}^{k} \lambda_i \phi_{l+[nt_{i-1}],s_i}\Big) = (t_j - t_{j-1})\Lambda(\lambda_j). \tag{2.34}$$

Note that

$$\Big|\frac{1}{n}\sum_{l=-[nt_j]-\sqrt{s_j}}^{-[nt_j]+\sqrt{s_{j+1}}} \Lambda\Big(\sum_{i=1}^{k} \lambda_i \phi_{l+[nt_{i-1}],s_i}\Big)\Big| \overset{(2.32)}{\leq} \frac{\sqrt{s_j} + \sqrt{s_{j+1}}}{n} M \overset{n\to\infty}{\longrightarrow} 0\,. \tag{2.35}$$

Finally, observe that for large n,

$$\begin{aligned}
&\Big|\frac{1}{n}\sum_{l=-\infty}^{-[nt_k]-\sqrt{s_k}} \Lambda\Big(\sum_{i=1}^{k} \lambda_i \phi_{l+[nt_{i-1}],s_i}\Big)\Big| \\
\overset{(2.33)}{\leq}\quad & C\frac{1}{n}\sum_{l=-\infty}^{-[nt_k]-\sqrt{s_k}} \Big|\sum_{i=1}^{k} \lambda_i \phi_{l+[nt_{i-1}],s_i}\Big| \\
\leq\quad & CL \sum_{l=-\infty}^{-\sqrt{s_k}} |\phi_l| \overset{(i)}{\to} 0.
\end{aligned} \tag{2.36}$$

and

$$
\begin{aligned}
&\Big|\frac{1}{n}\sum_{l=\sqrt{s_1}}^{\infty}\Lambda\Big(\sum_{i=1}^{k}\lambda_i\phi_{l+[nt_{i-1}],s_i}\Big)\Big| \\
\overset{(2.33)}{\leq}\;& C\frac{1}{n}\sum_{l=\sqrt{s_1}}^{\infty}\Big|\sum_{i=1}^{k}\lambda_i\phi_{l+[nt_{i-1}],s_i}\Big| \\
\leq\;& CL\sum_{l=\sqrt{s_1}}^{\infty}|\phi_l| \to 0.
\end{aligned} \tag{2.37}
$$

Thus, combining (2.34), (2.35), (2.36) and (2.37) we have

$$
\lim_{n\to\infty}\frac{1}{n}\sum_{l=-\infty}^{\infty}\Lambda\Big(\sum_{i=1}^{k}\lambda_i\phi_{l+[nt_{i-1}],[nt_i]-[nt_{i-1}]}\Big) = \sum_{i=1}^{k}(t_i - t_{i-1})\Lambda(\lambda_i).
$$

(ii) Since $\Lambda(x) \sim x\cdot\Sigma x/2$ as $|x|\to 0$, we see that for every $1\leq j\leq k$,

$$
\lim_{n\to\infty}\frac{n}{a_n^2}\sum_{l=-[nt_j]+\sqrt{s_j}}^{-[nt_{j-1}]-\sqrt{s_j}}\Lambda\Big(\frac{a_n}{n}\sum_{i=1}^{k}\lambda_i\phi_{l+[nt_{i-1}],[nt_i]-[nt_{i-1}]}\Big) = (t_j - t_{j-1})\frac{1}{2}\lambda_j\cdot\Sigma\lambda_j.
$$

The rest of the proof is similar to the proof of part (i).

(iii) Since $\Lambda(\lambda)$ is regular varying at infinity with exponent $\beta > 1$, for every $1\leq j\leq k$,

$$
\begin{aligned}
&\lim_{n\to\infty}\frac{1}{b_n}\sum_{l=-[nt_j]+\sqrt{s_j}}^{-[nt_{j-1}]-\sqrt{s_j}}\Lambda\Big(\frac{b_n}{a_n}\sum_{i=1}^{k}\lambda_i\phi_{l+[nt_{i-1}],[nt_i]-[nt_{i-1}]}\Big) \\
&\qquad = (t_j - t_{j-1})\zeta\Big(\frac{\lambda_j}{|\lambda_j|}\Big)|\lambda_j|^{\beta}.
\end{aligned}
$$

The rest of the proof is, once again, similar to the proof of part (i). □

Lemma 2.3.6. *Suppose $\Lambda:\mathbb{R}^d\to\mathbb{R}$ is the log-moment generating function of a mean zero random variable, with $0\in\mathcal{F}_{\Lambda}^{\circ}$, the coefficients of the moving average are balanced regularly varying with exponent α as in Assumption 2.2.3, and $0 < t_1 < \cdots < t_k \leq 1$.*

(i) For all $\underline{\lambda} = (\lambda_1, \dots, \lambda_k) \in \Pi^{r,\alpha}_{t_1,\dots,t_k} \subset (\mathbb{R}^d)^k$,

$$\lim_{n\to\infty} \frac{1}{n} \sum_{l=-\infty}^{\infty} \Lambda\Big(\frac{1}{\Psi_n} \sum_{i=1}^{k} \lambda_i \phi_{l+[nt_{i-1}],[nt_i]-[nt_{i-1}]}\Big) = \Lambda^{rl}_{t_1,\cdots,t_k}(\underline{\lambda}).$$

(ii) If $a_n/\sqrt{n} \to \infty$ *and* $a_n/n \to 0$ *then for all* $\underline{\lambda} \in (\mathbb{R}^d)^k$,

$$\lim_{n\to\infty} \frac{n\Psi_n^2}{a_n^2} \sum_{l=-\infty}^{\infty} \Lambda\Big(\frac{a_n}{n\Psi_n^2} \sum_{i=1}^{k} \lambda_i \phi_{l+[nt_{i-1}],[nt_i]-[nt_{i-1}]}\Big)$$

$$= \begin{cases} \int\limits_{-\infty}^{\infty} G_\Sigma\Big(h_{t_1,\dots,t_k}(x;\underline{\lambda})\Big)dx & if \quad \alpha < 1 \\ \sum\limits_{i=1}^{k} (t_i - t_{i-1}) G_\Sigma(\lambda_i) & if \quad \alpha = 1, \end{cases}.$$

(iii) If $a_n/n \to \infty$, b_n *is as defined in Assumption R4, and* $\Lambda(\cdot)$ *is balanced regular varying at* ∞ *with exponent* $\beta > 1$*, then for all* $\underline{\lambda} \in (\mathbb{R}^d)^k$,

$$\lim_{n\to\infty} \frac{1}{b_n} \sum_{l=-\infty}^{\infty} \Lambda\Big(\frac{b_n}{a_n} \sum_{i=1}^{k} \lambda_i \phi_{l+[nt_{i-1}],[nt_i]-[nt_{i-1}]}\Big)$$

$$= \begin{cases} \int\limits_{-\infty}^{\infty} \Lambda^h\Big(h_{t_1,\dots,t_k}(x;\underline{\lambda})\Big)dx & if \quad \alpha < 1 \\ \sum\limits_{i=1}^{k} (t_i - t_{i-1}) \Lambda^h(\lambda_i) & if \quad \alpha = 1, \end{cases}.$$

Proof. (i) We may (and will) assume that $t_k = 1$, since we can always add an extra point with the zero vector λ corresponding to it. Let us first assume that $\alpha < 1$. Note that for any $m \geq 1$ and large n,

$$\begin{aligned} & \frac{1}{n} \sum_{j=nm+1}^{n(m+1)} \Lambda\Big(\frac{1}{\Psi_n} \sum_{i=1}^{k} \lambda_i \phi_{j+[nt_{i-1}],[nt_i]-[nt_{i-1}]}\Big) \\ & \frac{1}{n} \sum_{j=nm+1}^{n(m+1)} \Lambda\Big(\frac{n\psi(n)}{\Psi_n \psi(n) n} \sum_{i=1}^{k} \lambda_i \phi_{j+[nt_{i-1}],[nt_i]-[nt_{i-1}]}\Big) \\ = \; & \frac{1}{n} \sum_{j=nm+1}^{n(m+1)} \Lambda\Big(\sum_{i=1}^{k} \lambda_i \frac{n\psi(n)}{\Psi_n} \frac{1}{n} \big(\frac{\phi_{j+[nt_{i-1}]+1}}{\psi(n)} + \cdots + \frac{\phi_{j+[nt_i]}}{\psi(n)}\big)\Big) \\ = \; & \int_m^{m+1} f_n(x)\, dx\,, \end{aligned}$$

where

$$f_n(x) = \Lambda\Big(\frac{1}{\Psi_n}\sum_{i=1}^{k}\lambda_i\phi_{j+[nt_{i-1}],[nt_i]-[nt_{i-1}]}\Big)$$

if $(j-1)/n < x \le j/n$ for $j = nm+1, \dots, n(m+1)$.

Notice that by Karamata's Theorem (see Resnick (1987)), $n\psi(n)/\Psi_n \to 1-\alpha$ as $n \to \infty$. Furthermore, given $0 < \epsilon < \alpha$, we can use Potter's bounds (see Proposition 0.8 *ibid*) to check that there is n_ϵ such that for all $n \ge n_\epsilon$, for all $k = [nt_{i-1}]+1, \dots, [nt_i]$, $m-1 < x \le m$ and $(j-1)/n < x \le j/n$

$$\frac{\phi_{j+k}}{\psi(n)} = \frac{\phi_{j+k}}{\psi(j+k)}\frac{\psi(j+k)}{\psi(j)}\frac{\psi(j)}{\psi(n)}$$

$$\in \left((1-\epsilon)\,p\left(\frac{j+k}{j}\right)^{-(\alpha+\epsilon)}x^{-\alpha},\ (1+\epsilon)\,p\left(\frac{j+k}{j}\right)^{-(\alpha-\epsilon)}x^{-\alpha}\right),$$

and so for n large enough,

$$\frac{1}{n}\left(\frac{\phi_{j+[nt_{i-1}]+1}}{\psi(n)} + \cdots + \frac{\phi_{j+[nt_i]}}{\psi(n)}\right) \tag{2.38}$$

$$\in \left((1-\epsilon)\,p\int_{t_{i-1}}^{t_i}\left(\frac{y+x}{x}\right)^{-(\alpha+\epsilon)}x^{-\alpha}\,dy,\ (1+\epsilon)\,p\int_{t_{i-1}}^{t_i}\left(\frac{y+x}{x}\right)^{-(\alpha-\epsilon)}x^{-\alpha}\,dy\right).$$

Therefore,

$$\frac{1}{\Psi_n}\sum_{i=1}^{k}\lambda_i\phi_{j+[nt_{i-1}],[nt_i]-[nt_{i-1}]} \to (1-\alpha)\,p\sum_{i=1}^{k}\lambda_i\int_{t_{i-1}}^{t_i}(y+x)^{-\alpha}\,dy$$

$$= p\sum_{i=1}^{k}\lambda_i\Big((t_i+x)^{1-\alpha} - (t_{i-1}+x)^{1-\alpha}\Big).$$

This last vector is a convex linear combination of the vectors $p\big((1+x)^{1-\alpha} - x^{1-\alpha}\big)\lambda_i$, $i = 1\dots,k$. By the definition of the set $\Pi^{r,\alpha}_{t_1,\dots,t_k}$, each one of these vectors belongs to $\mathcal{F}^\circ_\Lambda$ and, by convexity of Λ, so does the convex linear combination. Therefore,

$$\Lambda\Big(\frac{1}{\Psi_n}\sum_{i=1}^{k}\lambda_i\phi_{j+[nt_{i-1}],[nt_i]-[nt_{i-1}]}\Big) \to \Lambda\Big(p\sum_{i=1}^{k}\lambda_i\Big((t_i+x)^{1-\alpha} - (t_{i-1}+x)^{1-\alpha}\Big)\Big).$$

This convexity argument also shows that the function f_n is uniformly bounded on $(m, m+1]$ for large enough n, and so we conclude that for any $m \geq 1$

$$\frac{1}{n}\sum_{j=nm+1}^{n(m+1)} \Lambda\Big(\frac{1}{\Psi_n}\sum_{i=1}^{k}\lambda_i\phi_{j+[nt_{i-1}],[nt_i]-[nt_{i-1}]}\Big)$$

$$\to \int_m^{m+1} \Lambda\Big((1-\alpha)\sum_{i=1}^{k}\lambda_i \int\limits_{x+t_{i-1}}^{x+t_i} py^{-\alpha}dy\Big)dx.$$

Similar arguments show that for $m \leq -3$

$$\frac{1}{n}\sum_{j=nm+1}^{n(m+1)} \Lambda\Big(\frac{1}{\Psi_n}\sum_{i=1}^{k}\lambda_i\phi_{j+[nt_{i-1}],[nt_i]-[nt_{i-1}]}\Big)$$

$$\to \int_m^{m+1} \Lambda\Big((1-\alpha)\sum_{i=1}^{k}\lambda_i \int\limits_{x+t_{i-1}}^{x+t_i} q|y|^{-\alpha}dy\Big)dx,$$

and that for any $\delta > 0$,

$$\frac{1}{n}\sum_{j=-2n+1}^{-n-n\delta} \Lambda\Big(\frac{1}{\Psi_n}\sum_{i=1}^{k}\lambda_i\phi_{j+[nt_{i-1}],[nt_i]-[nt_{i-1}]}\Big)$$

$$\to \int_{-2}^{-1-\delta} \Lambda\Big((1-\alpha)\sum_{i=1}^{k}\lambda_i \int\limits_{x+t_{i-1}}^{x+t_i} q|y|^{-\alpha}dy\Big)dx$$

and

$$\frac{1}{n}\sum_{j=n\delta}^{n} \Lambda\Big(\frac{1}{\Psi_n}\sum_{i=1}^{k}\lambda_i\phi_{j+[nt_{i-1}],[nt_i]-[nt_{i-1}]}\Big) \to \int_\delta^1 \Lambda\Big((1-\alpha)\sum_{i=1}^{k}\lambda_i \int\limits_{x+t_{i-1}}^{x+t_i} py^{-\alpha}dy\Big)dx.$$

Using once again the same argument we see that for small δ

$$\frac{1}{n}\sum_{j=-n}^{0} I\Big(\Big|\frac{j}{n}+t_i\Big| > \delta \ \text{ all } i=1,\ldots,k\Big)\Lambda\Big(\frac{1}{\Psi_n}\sum_{i=1}^{k}\lambda_i\phi_{j+[nt_{i-1}],[nt_i]-[nt_{i-1}]}\Big)$$

$$\to \int_{-1}^{0} I\Big(|x+t_i| > \delta \ \text{ all } i=1,\ldots,k\Big)$$

$$\Lambda\Big((1-\alpha)\sum_{i=1}^{k}\lambda_i \int\limits_{x+t_{i-1}}^{x+t_i} |y|^{-\alpha}\big(pI_{[y\geq 0]}+qI_{[y<0]}\big)dy\Big)dx.$$

We have covered above all choices of the subscript j apart from a finite number of stretches of j of length at most $n\delta$ each. By the definition of the set $\Pi^{r,\alpha}_{t_1,\dots,t_k}$ we see that there is a finite K such that for all n large enough,

$$\frac{1}{n}\sum_{j \text{ not yet considered}} \Lambda\Big(\frac{1}{\Psi_n}\sum_{i=1}^{k}\lambda_i\phi_{j+[nt_{i-1}],[nt_i]-[nt_{i-1}]}\Big) \le K\delta.$$

It follows from (2.38) and the fact that $\Lambda(\lambda) = O(|\lambda|^2)$ as $\lambda \to 0$ that for all $|m|$ large enough there is $C \in (0,\infty)$ such that

$$\frac{1}{n}\sum_{nm+1}^{n(m+1)} \Lambda\Big(\frac{1}{\Psi_n}\sum_{i=1}^{k}\lambda_i\phi_{j+[nt_{i-1}],[nt_i]-[nt_{i-1}]}\Big) \le C|m|^{-2\alpha}$$

for all n large enough. This is summable by the assumption on α, and so the dominated convergence theorem gives us the result.

Next we move our attention to the case $\alpha = 1$. Choose any $\delta > 0$. By the slow variation of Ψ_n we see that

$$\sup_{j>\delta n \text{ or } j<-(1+\delta)n} \frac{|\phi_{j,n}|}{\Psi_n} \to 0\,,$$

while for any $0 < x < 1$ we have

$$\frac{\phi_{0,[nx]}}{\Psi_n} \to p \text{ and } \frac{\phi_{-[nx],[nx]}}{\Psi_n} \to q.$$

Write

$$\frac{1}{n}\sum_{j=-n+1}^{0} \Lambda\Big(\frac{1}{\Psi_n}\sum_{i=1}^{k}\lambda_i\phi_{j+[nt_{i-1}],[nt_i]-[nt_{i-1}]}\Big)$$
$$= \sum_{m=1}^{k}\frac{1}{n}\sum_{j=-[nt_m]+1}^{j=-[nt_{m-1}]} \Lambda\Big(\sum_{i=1}^{k}\lambda_i\frac{\phi_{j+[nt_{i-1}],[nt_i]-[nt_{i-1}]}}{\Psi_n}\Big)\,.$$

Fix $m = 1,\dots,k$, and observe that for any $\epsilon > 0$ and n large enough,

$$\frac{1}{n}\sum_{j=-[nt_m]+1}^{-[nt_{m-1}]} \Lambda\Big(\sum_{i=1}^{k}\lambda_i\frac{\phi_{j+[nt_{i-1}],[nt_i]-[nt_{i-1}]}}{\Psi_n}\Big) = \int_{-t_m-\epsilon}^{-t_{m-1}} f_n(x)\,dx\,,$$

where this time

$$f_n(x) = I\Big(-\frac{[nt_m]}{n} < x \le -\frac{[nt_{m-1}]}{n}\Big)\Lambda\Big(\sum_{i=1}^{k} \lambda_i \frac{\phi_{j+[nt_{i-1}],[nt_i]-[nt_{i-1}]}}{\Psi_n}\Big)$$

if $(j-1)/n < x \le j/n$ for $j = -[nt_m]+1, \ldots, -[nt_{m-1}]$, otherwise $f_n(x) = 0$. Clearly, $f_n(x) \to 0$ as $n \to \infty$ for all $-t_m - \epsilon < x < -t_m$. Furthermore,

$$\frac{\phi_{j+[nt_{i-1}],[nt_i]-[nt_{i-1}]}}{\Psi_n} \to 0$$

uniformly in $i \ne m$ and $j = -[nt_m]+1, \ldots, -[nt_{m-1}]$, while for every $-t_m < x < -t_{m-1}$,

$$\frac{\phi_{j+[nt_{m-1}],[nt_i]-[nt_{m-1}]}}{\Psi_n} \to p + q = 1\,.$$

By the definition of the set $\Pi^{r,1}_{t_1,\ldots,t_k}$ we see that $f_n \to I_{(-t_m,-t_{m-1})}\Lambda(\lambda_m)$ a.e., and that the functions f_n are uniformly bounded for large n. Therefore,

$$\frac{1}{n}\sum_{j=-n+1}^{0} \Lambda\Big(\frac{1}{\Psi_n}\sum_{i=1}^{k} \lambda_i \phi_{j+[nt_{i-1}],[nt_i]-[nt_{i-1}]}\Big) \to \sum_{m=1}^{k}(t_m - t_{m-1})\Lambda(\lambda_m)\,.$$

Finally, the argument above, using Potter's bounds and the fact that $\Lambda(\lambda) = O(|\lambda|^2)$ as $\lambda \to 0$, shows that

$$\frac{1}{n}\sum_{j \notin [-n,0]} \Lambda\Big(\frac{1}{\Psi_n}\sum_{i=1}^{k} \lambda_i \phi_{j+[nt_{i-1}],[nt_i]-[nt_{i-1}]}\Big) \to 0.$$

This completes the proof of part (i).

For part (ii) consider, once again, the cases $1/2 < \alpha < 1$ and $\alpha = 1$ separately. If $1/2 < \alpha < 1$, then for every $m \ge 1$ we use the regular variation and the fact that $\Lambda(x) \sim x \cdot \Sigma x/2$ as $|x| \to 0$ to obtain

$$\frac{n\Psi_n^2}{a_n^2}\sum_{j=nm+1}^{n(m+1)} \Lambda\Big(\frac{a_n}{n\Psi_n^2}\sum_{i=1}^{k} \lambda_i \phi_{l+[nt_{i-1}],[nt_i]-[nt_{i-1}]}\Big) \to$$

$$\int_m^{m+1} \Big((1-\alpha)\sum_{i=1}^{k} \lambda_i \int_{x+t_{i-1}}^{x+t_i} py^{-\alpha}dy\Big) \cdot \Sigma\Big((1-\alpha)\sum_{i=1}^{k} \lambda_i \int_{x+t_{i-1}}^{x+t_i} py^{-\alpha}dy\Big)/2\,dx\,,$$

and we proceed as in the proof of part (i), considering the various other ranges of m, obtaining the result. If $\alpha = 1$, then for any $m = 1, \ldots, k$, by the regular variation and the fact that $\Lambda(x) \sim x \cdot \Sigma x/2$ as $|x| \to 0$, one has

$$\frac{n\Psi_n^2}{a_n^2} \sum_{j=-[nt_m]+1}^{[nt_{m-1}]} \Lambda\Big(\frac{a_n}{n\Psi_n} \sum_{i=1}^{k} \lambda_i \frac{\phi_{j+[nt_{i-1}],[nt_i]-[nt_{i-1}]}}{\Psi_n}\Big) \to \int_{-t_m}^{-t_{m-1}} \frac{1}{2} \lambda_m \cdot \Sigma \lambda_m \, dx \,,$$

and so

$$\frac{n\Psi_n^2}{a_n^2} \sum_{j=-n+1}^{0} \Lambda\Big(\frac{a_n}{n\Psi_n^2} \sum_{i=1}^{k} \lambda_i \phi_{j+[nt_{i-1}],[nt_i]-[nt_{i-1}]}\Big) \to \frac{1}{2} \sum_{m=1}^{k} (t_m - t_{m-1}) \lambda_m \cdot \Sigma \lambda_m \,.$$

As in part (i), by using Potter's bounds and the fact that $\Lambda(\lambda) = O(|\lambda|^2)$ as $\lambda \to 0$, we see that

$$\frac{n\Psi_n^2}{a_n^2} \sum_{j \notin [-n,0]} \Lambda\Big(\frac{a_n}{n\Psi_n^2} \sum_{i=1}^{k} \lambda_i \phi_{j+[nt_{i-1}],[nt_i]-[nt_{i-1}]}\Big) \to 0,$$

giving us the desired result.

We proceed in a similar fashion in part (iii). If $1/2 < \alpha < 1$, then, for example, for $m \geq 1$, by the regular variation at infinity,

$$\frac{1}{b_n} \sum_{j=nm+1}^{n(m+1)} \Lambda\Big(\frac{b_n}{a_n} \sum_{i=1}^{k} \lambda_i \phi_{l+[nt_{i-1}],[nt_i]-[nt_{i-1}]}\Big) \to$$

$$\int_m^{m+1} \zeta \left(\frac{(1-\alpha) \sum_{i=1}^k \lambda_i \int_{x+t_{i-1}}^{x+t_i} p y^{-\alpha} dy}{\left|(1-\alpha) \sum_{i=1}^k \lambda_i \int_{x+t_{i-1}}^{x+t_i} p y^{-\alpha} dy\right|} \right) \left|(1-\alpha) \sum_{i=1}^{k} \lambda_i \int_{x+t_{i-1}}^{x+t_i} p y^{-\alpha} dy\right|^\beta$$

(if the argument of the function ζ is $0/0$, then the integrand is set to be equal to zero), and we treat the other ranges of m in a manner similar to what has been done in part (ii). This gives us the stated limit. For $\alpha = 1$ we have for any $m = 1, \ldots, k$, by the regular variation at infinity,

$$\frac{1}{b_n} \sum_{j=-[nt_m]+1}^{[nt_{m-1}]} \Lambda\Big(\frac{b_n}{a_n} \sum_{i=1}^{k} \lambda_i \phi_{j+[nt_{i-1}],[nt_i]-[nt_{i-1}]}\Big) \to \int_{-t_m}^{-t_{m-1}} \zeta\left(\frac{\lambda_m}{|\lambda_m|}\right) |\lambda_m|^\beta \, dx \,,$$

and so

$$\frac{1}{b_n}\sum_{j=-n+1}^{0}\Lambda\Big(\frac{b_n}{a_n}\sum_{i=1}^{k}\lambda_i\phi_{j+[nt_{i-1}],[nt_i]-[nt_{i-1}]}\Big)\to\sum_{m=1}^{k}(t_m-t_{m-1})\zeta\left(\frac{\lambda_m}{|\lambda_m|}\right)|\lambda_m|^{\beta},$$

while the sum over the rest of the range of j contributes only terms of a smaller order. Hence the result. $\square$

Remark 2.3.7. The argument in the proof shows also that the statements of all three parts of the lemma remain true if the sums $\sum_{l=-\infty}^{\infty}$ are replaced by sums $\sum_{l=-A_n}^{A_n}$ with $n/A_n\to 0$ as $n\to\infty$.

Lemma 2.3.8. *For $1/2<\alpha<1$, let $h_{t_1,\ldots,t_k}$ be defined by* (2.16)*, and $\Lambda^{rl}_{t_1,\cdots,t_k}$ defined by* (2.20)*. Then for any function f of bounded variation on $[0,1]$ satisfying $f(0)=0$,*

$$\sup_{j\in\mathcal{J}}(\Lambda^{rl}_{t_1,\ldots,t_{|j|}})^*\big(f(t_1),f(t_2)-f(t_1),\ldots,f(t_{|j|})-f(t_{|j|-1})\big)$$

$$=\begin{cases}\Lambda^*_\alpha(f') & if \quad f\in\mathcal{AC},\\ \infty & \qquad otherwise,\end{cases}$$

where Λ^*_α is defined by (2.17)*.*

Proof. First assume that $f\in\mathcal{AC}$. It is easy to see that the inequality $\Lambda^*_\alpha(f')\geq\sup_{j\in\mathcal{J}}(\Lambda^{rl}_{t_1,\cdots,t_{|j|}})^*(f(t_1),f(t_2)-f(t_1),\ldots,f(t_{|j|})-f(t_{|j|-1}))$ holds by considering a function $\psi\in L_\infty[0,1]$, which takes the value λ_i in the interval $(t_{i-1},t_i]$. For the other inequality, we start by observing that the supremum in the definition of Λ^*_α in (2.17) is achieved over those $\psi\in L_\infty[0,1]$ for which the integral

$$I_x=\int_0^1\psi(t)(1-\alpha)|x+t|^{-\alpha}\Big[pI_{[x+t\geq 0]}+qI_{[x+t<0]}\Big]\,dt\in\mathcal{F}_\Lambda$$

for almost all real x, and, hence, also over those $\psi\in L_\infty[0,1]$ for whch $I_x\in\mathcal{F}^\circ_\Lambda$ for almost every x.

For any ψ as above choose a sequence of uniformly bounded functions ψ^n converging to ψ almost everywhere on $[0,1]$, such that for every n, ψ^n is of the form $\sum_i \lambda_i^n I_{A_i^n}$, where $A_i^n = (t_{i-1}^n, t_i^n]$, for some $0 < t_1^n < t_2^n < \cdots < t_{k_n}^n = 1$. Then by the continuity of Λ over $\mathcal{F}_\Lambda^\circ$ and Fatou's Lemma,

$$\begin{aligned}
&\int_0^1 \psi(t) f'(t)dt - \int_{-\infty}^{\infty} \Lambda\left(\int_0^1 \psi(t)(1-\alpha)|x+t|^{-\alpha}\left[pI_{[x+t\geq 0]} + qI_{[x+t<0]}\right]dt\right)dx \\
&= \int_0^1 \lim_n \psi^n(t) f'(t)dt \\
&\quad - \int_{-\infty}^{\infty} \Lambda\left(\int_0^1 \lim_n \psi^n(t)(1-\alpha)|x+t|^{-\alpha}\left[pI_{[x+t\geq 0]} + qI_{[x+t<0]}\right]dt\right)dx \\
&= \lim_n \int_0^1 \psi^n(t) f'(t)dt \\
&\quad - \int_{-\infty}^{\infty} \lim_n \Lambda\left(\int_0^1 \psi^n(t)(1-\alpha)|x+t|^{-\alpha}\left[pI_{[x+t\geq 0]} + qI_{[x+t<0]}\right]dt\right)dx \\
&\leq \lim_n \int_0^1 \psi^n(t) f'(t)dt \\
&\quad - \limsup_n \int_{-\infty}^{\infty} \Lambda\left(\int_0^1 \psi^n(t)(1-\alpha)|x+t|^{-\alpha}\left[pI_{[x+t\geq 0]} + qI_{[x+t<0]}\right]dt\right)dx \\
&= \liminf_n \left\{\sum_{i=1}^{k_n} \lambda_i^n \cdot \big(f(t_i^n) - f(t_{i-1}^n)\big) - \Lambda^{rl}_{t_1^n,\cdots,t_n^n}(\lambda_1^n,\cdots,\lambda_n^n)\right\} \\
&\leq \sup_{j\in\mathcal{J}} (\Lambda^{rl}_{t_1,\cdots,t_{|j|}})^*\big(f(t_1), f(t_2)-f(t_1),\ldots,f(t_{|j|})-f(t_{|j|-1})\big).
\end{aligned}$$

Now suppose that f is not absolutely continuous. That is, there exists $\epsilon > 0$ and $0 \leq r_1^n < s_1^n \leq r_2^n < \cdots \leq r_{k_n}^n < s_{k_n}^n \leq 1$, such that $\sum_{i=1}^{k_n}(s_i^n - r_i^n) \to 0$ but $\sum_{i=1}^{k_n}|f(s_i^n) - f(r_i^n)| \geq \epsilon$. Let j^n be such that $t_{2p}^n = s_p^n$ and $t_{2p-1}^n = r_p^n$ (so that $|j^n| = 2k_n$). Now

$$\begin{aligned}
&\sup_{j\in\mathcal{J}} (\Lambda^{rl}_{t_1,\cdots,t_{|j|}})^*\big(f(t_1), f(t_2)-f(t_1),\ldots,f(t_{|j|})-f(t_{|j|-1})\big) \\
&\geq \limsup_n \left\{\sup_{\underline{\lambda}^n\in\mathbb{R}^{2k_n}} \sum_{i=1}^{2k_n} \lambda_i^n \cdot \big(f(t_i^n) - f(t_{i-1}^n)\big) - \Lambda^{rl}_{t_1,\cdots,t_{2k_n}}(\underline{\lambda}^n)\right\} \\
&\geq \limsup_n \left\{A\sum_{i=1}^{k_n} \big|f(s_i^n) - f(r_i^n)\big| - \Lambda^{rl}_{t_1,\cdots,t_{2k_n}}(\underline{\lambda}^{n*})\right\} \geq A\epsilon,
\end{aligned}$$

where $\lambda^{n*}_{2p-1} = 0$ and $\lambda^{n*}_{2p} = A\big(f(s^n_i) - f(r^n_i)\big)/|f(s^n_i) - f(r^n_i)|$ ($= 0$ if $f(s^n_i) - f(r^n_i) = 0$) for any $A > 0$. The last inequality follows from an application of dominated convergence theorem, quadratic behaviour of Λ at 0 and the fact that $h_{t_1,\cdots,t_{2k_n}}(x; \underline{\lambda}^{n*}) \to 0$ as $n \to \infty$ for every $x \in \mathbb{R}$. This completes the proof since A is arbitrary. □

2.4 Long Strange Segments

This section discusses the rate of growth of long strange segments for a moving average process $\{X_n, n \in \mathbb{Z}\}$ as defined in (2.1). We retain the notation of Section 2.2 with the only exception being that μ_n denote the law of $a_n^{-1} S_n$. For any measurable set $A \subset \mathbb{R}^d$, define the length of the longest "strange" segment by

$$R_m(A; \underline{a}) := \sup\left\{n : \frac{S_l - S_{l-n}}{a_n} \in A \text{ for some } l = n, \dots, m\right\} \tag{2.39}$$

and the equivalent characterization

$$T_r(A; \underline{a}) := \inf\left\{l : \text{ there exists } k, 0 \le k \le l - r, \frac{S_l - S_k}{a_{l-k}} \in A\right\}. \tag{2.40}$$

Notice that $\{R_m(A; \underline{a}) \ge r\}$ if and only if $\{T_r(A; \underline{a}) \le m\}$. We will often refer to $R_m(A; \underline{a})$ as R_m and to $T_r(A; \underline{a})$ as T_r, as long as the set A and the sequence $\{a_n\}$ under consideration are obvious.

Following the trend set in Section 2.2 here also we consider two different situations, corresponding to what we view as a short and a long range dependent moving average process. The assumptions we impose correspond, roughly, to those in Assumptions 2.2.1 and 2.2.3. The only difference is that the Assumptions $\widehat{S3}$ and $\widehat{R3}$ are slightly stringer than the Assumptions $S3$ and $R3$, respectively. The Assumptions $S1, S2, S4, R1, R2$ and $R4$ remain unchanged.

We start with the assumption describing the short memory case.

Assumption 2.4.1. *Assume that*

$$\sum_{i\in\mathbb{Z}} |\phi_i| < \infty \text{ and } \sum_{i\in\mathbb{Z}} \phi_i = 1 \tag{2.41}$$

and

$\widehat{S3}$. $a_n/\sqrt{n\log n} \to \infty,\ a_n/n \to 0,\ 0 \in \mathcal{F}_\Lambda^\circ$ *and* (b_n) *an increasing positive sequence such that* $b_n \sim a_n^2/n$ *as* $n \to \infty$.

The next assumption describes the long memory case.

Assumption 2.4.2. *Assume that the coefficients* $\{\phi_i\}$ *are balanced regular varying with exponent* $-\alpha, 1/2 < \alpha \le 1$ *and* $\sum_{i=-\infty}^{\infty} |\phi_i| = \infty$. *Specifically, there is* $\psi : [0,\infty) \to [0,\infty)$ *and* $0 \le p \le 1$, *such that*

$$\left.\begin{aligned} \lim_{t\to\infty} \tfrac{\psi(tx)}{\psi(t)} = x^{-\alpha},\ \text{for all } x > 0 \\ \lim_{n\to\infty} \tfrac{\phi_n}{\psi(n)} = p \text{ and } \lim_{n\to\infty} \tfrac{\phi_{-n}}{\psi(n)} = q := 1-p. \end{aligned}\right\} \tag{2.42}$$

Let $\Psi_n := \sum_{1\le i\le n} \psi(i)$.

$\widehat{R3}$. $a_n/\big(\sqrt{n\log n}\Psi_n\big) \to \infty, a_n/(n\Psi_n) \to 0, 0 \in \mathcal{F}_\Lambda^\circ$ *and* (b_n) *is an increasing positive sequence such that* $b_n \sim a_n^2/(n\Psi_n^2)$ *as* $n \to \infty$.

Let $\mu_n(\cdot) \equiv \mu_n(\cdot;\underline{a})$ denote the law of $a_n^{-1}S_n$. We quote the "marginal version" of the functional results in chapter 2. The sequence (μ_n) satisfies the large deviation principle on $\mathbb{R}^d$:

$$-\inf_{x\in A^\circ} I_l(x) \le \liminf_{n\to\infty} \frac{1}{b_n}\log\mu_n(A;\underline{a}) \le \limsup_{n\to\infty} \frac{1}{b_n}\log\mu_n(A;\underline{a}) \le -\inf_{x\in\bar{A}} I_u(x) \tag{2.43}$$

with a good lower rate function I_l and a good upper rate function I_u given by

$$
\begin{aligned}
I_l = \Lambda^*,\ I_u = \Lambda^\sharp &\qquad \text{under the Assumption } S1 \\
I_l = I_u = \Lambda^* &\qquad \text{under the Assumption } S2 \\
I_l = I_u = \big(G_\Sigma\big)^* &\qquad \text{under the Assumption } S3 \text{ or } \widehat{S3} \\
I_l = I_u = \big(\Lambda^h\big)^* &\qquad \text{under the Assumption } S4 \\
I_l = \big(\Lambda_\alpha\big)^*,\ I_u = \Lambda_\alpha^\sharp &\qquad \text{under the Assumption } R1 \\
I_l = I_u = \big(\Lambda_\alpha\big)^* &\qquad \text{under the Assumption } R2 \\
I_l = I_u = \big((G_\Sigma)_\alpha\big)^* &\qquad \text{under the Assumption } R3 \text{ or } \widehat{R3} \\
I_l = I_u = \big((\Lambda^h)_\alpha\big)^* &\qquad \text{under the Assumption } R4
\end{aligned}
\tag{2.44}
$$

As before, for a function $f : \mathbb{R}^d \to (-\infty, \infty]$ we denote by f^* its Legendre transform $f^*(x) = \sup_{\lambda \in \mathbb{R}^d}\big\{\lambda \cdot x - f(\lambda)\big\}$, $x \in \mathbb{R}^d$. Further, under the Assumption $S1$, $f^\sharp(x) = \sup_{\lambda \in \Pi}\big\{\lambda \cdot x - f(\lambda)\big\}$, with

$$\Pi = \Big\{\lambda \in \mathbb{R}^d :\ \text{for some } N_\lambda,\ \sup_{n \ge N_\lambda,\, i \in \mathbb{Z}} \Lambda(\lambda\phi_{i,n}) < \infty\Big\},$$

with $\phi_{i,n} = \phi_{i+1} + \cdots + \phi_{i+n}$, a partial sum of the moving average coefficients. Under the Assumptions $\widehat{S3}$ and $\widehat{R3}$, G_Σ is the log-moment generating function of a zero mean Gaussian random vector in $\mathbb{R}^d$ with the same variance-covariance matrix as that of Z_0. Next, under the Assumptions $S4$ and $R4$, $\Lambda^h(\lambda) = \zeta_\Lambda(\lambda/\|\lambda\|)\|\lambda\|^\beta$. Under the Assumptions $R1, R2, \widehat{R3}$ and $R4$, for a nonnegative measurable function f on $\mathbb{R}^d$ we define

$$f_\alpha(\lambda) = \int_{-\infty}^{\infty} f\left(\lambda(1-\alpha)\int_x^{x+1} |y|^{-\alpha}\big(pI(y \ge 0) + qI(y < 0)\big)\, dy\right) dx$$

if $1/2 < \alpha < 1$ and $f_1 = f$. Finally, under the Assumption $R1$,

$$\Lambda_\alpha^\sharp(x) = \sup_{\lambda \in \Pi_\alpha}\big\{\lambda \cdot x - \Lambda_\alpha(\lambda)\big\},$$

with Π_α given by

$$\Pi_\alpha := \Big\{\lambda : (p \wedge q)\lambda \in \mathcal{F}_\Lambda^\circ, \text{ and for some } N \geq 1, \sup_{n \geq N, j \in \mathbb{Z}} \Lambda\Big(\frac{1}{\Psi_n}\lambda\phi_{j,n}\Big) < \infty\Big\}.$$

We are now ready to state the main result of this section. The following theorem considers the various cases in Assumptions 2.2.1, 2.2.3, 2.4.1 and 2.4.2 and gives us the rate of growth of long strange segments in each of the cases. For a Borel set A in $\mathbb{R}^d$ and $\eta > 0$ we denote

$$A(\eta) := \big\{x : d(x, A^c) > \eta\big\}, \tag{2.45}$$

where $d(x, A^c)$ is the distance from the point x to the complement A^c.

Theorem 2.4.3. *If $S1$, $S2$, $\widehat{S3}$, $S4$, $R1$, $R2$, $\widehat{R3}$ or $R4$ hold, then for any Borel set $A \subset \mathbb{R}^d$,*

$$I_* \leq \liminf_{r \to \infty} \frac{\log T_r(A; \underline{a})}{b_r} \leq \limsup_{r \to \infty} \frac{\log T_r(A; \underline{a})}{b_r} \leq I^* \tag{2.46}$$

and

$$\frac{1}{I^*} \leq \liminf_{m \to \infty} \frac{b_{R_m}}{\log m} \leq \limsup_{m \to \infty} \frac{b_{R_m}}{\log m} \leq \frac{1}{I_*} \tag{2.47}$$

with probability 1, where, under the Assumptions $S2, \widehat{S3}, S4, R2, \widehat{R3}$ and $R4$,

$$I_* = -\inf_{x \in \bar{A}} I_u(x) \quad \textit{and} \quad I^* = -\lim_{\eta \downarrow 0} \inf_{x \in A(\eta)} I_l(x),$$

with I_l and I_u as in (2.44). Under the Assumption $S1$, I_* is defined in the same way, while I^* is defined now as follows. Let $\lambda^* = \sup\{\lambda : \lambda \in \Pi\} > 0$. Then

$$I^* = -\inf_{\eta \in \Theta} \inf_{x \in A(\eta)} I_l(x),$$

where $\Theta = \{\eta > 0 : \eta > (\lambda^*)^{-1} \inf_{x \in A(\eta)} I_l(x)\}$. Finally, under the Assumption $R1$, I_* is defined in the same way, and with $\lambda_\alpha^* = \sup\{\lambda : \lambda \in \Pi_\alpha\} > 0$, and $\Theta_\alpha = \{\eta > 0 : \eta > (\lambda_\alpha^*)^{-1} \inf_{x \in A(\eta)} I_l(x)\}$, one sets

$$I^* = -\inf_{\eta \in \Theta_\alpha} \inf_{x \in A(\eta)} I_l(x).$$

Remark 2.4.4. In certain cases it turns out that $I_* = I^*$ in Theorem 2.4.3, and then its conclusions may be strengthened. For example, under the Assumptions $S2, S3, S4, R2, R3$ or $R4$, Suppose that for some Borel set A,

$$\inf_{x\in A^\circ} I_l(x) = \inf_{x\in \bar{A}} I_u(x) = I \text{ (say)}.$$

We claim that then, with probability 1,

$$\lim_{r\to\infty} \frac{\log T_r}{b_r} = I \tag{2.48}$$

and

$$\lim_{m\to\infty} \frac{b_{R_m}}{\log m} = \frac{1}{I}\,. \tag{2.49}$$

We will check (2.48); the statement (2.49) is an immediate consequence. Start with observing that by Theorem 2.4.3

$$I = \inf_{x\in \bar{A}} I_u(x) \le -\limsup_{n\to\infty} \frac{1}{b_n} \log \mu_n(A) \le \liminf_{r\to\infty} \frac{\log T_r}{b_r}.$$

If $I = \infty$, this is all one needs. If $I < \infty$, for $\epsilon > 0$ we can choose $x \in A^\circ$ with $I_l(x) \le I + \epsilon$. For all $\eta > 0$ small enough we will have $x \in A(\eta)^\circ$, which implies that

$$\liminf_{n\to\infty} \frac{1}{b_n} \log \mu_n\big(A(\eta)\big) \ge -(I+\epsilon)\,,$$

and so

$$\limsup_{r\to\infty} \frac{\log T_r}{b_r} \le -\lim_{\eta\downarrow 0} \liminf_{n\to\infty} \frac{1}{b_n} \log \mu_n\big(A(\eta)\big) \le I\,,$$

establishing (2.48).

Because of the large deviation principle for the sequence (μ_n), the sequence (b_n) is the "right" normalization to use in the Theorem 2.4.3. In particular, if, for instance, the set A is bounded away from the origin (which we recall to be the mean of the moving average process), then the quantity I_* is strictly positive.

Under further additional assumptions on the set A the quantity I^* will be finite, and then (2.46) and (2.47) give us precise information on the order of magnitude of long strange segments.

Notice that under the "usual" normalization $a_n = n$, Theorem 2.4.3 says that R_m grows like $\log m$ in the short memory case (i.e. under the Assumption $S1$; see also Theorem 3.2.1 in Dembo and Zeitouni (1998). On the other hand, in the long memory case (i.e. under the Assumption $R2$) the length R_m of long strange segments grows at the rate $\phi(\log m)$, where ϕ is regularly varying at infinity with exponent $1/(2\alpha - 1)$. Therefore, long strange segments are much longer in the long memory case than in the short memory case. In fact, to get long strange segments with length of order $\log m$ in the long memory case one needs to use a stronger normalization $a_n = n\Psi_n$ (the Assumption $R1$). This phase transition property is directly inherited from the similar phenomenon for large deviations; see chapter 2.

Table 2.1: The effect of memory on the rate of growth of Long Strange Segments of a Moving Average Process

Range of b	Short range dependent	Long range dependent
$\frac{1}{2} \leq b \leq \frac{3}{2} - \alpha$	$\theta = \frac{1}{2b-1}$	$\theta = \infty$
$\frac{3}{2} - \alpha \leq b \leq 1$	$\theta = \frac{1}{2b-1}$	$\theta = \frac{1}{2b+2\alpha-3}$
$1 \leq b \leq 2 - \alpha$	$\theta = \frac{\beta-1}{\beta b-1}$	$\theta = \frac{1}{2b+2\alpha-3}$
$b \geq 2 - \alpha$	$\theta = \frac{\beta-1}{\beta b-1}$	$\theta = \frac{\beta-1}{\beta(b+\alpha-1)-1}$

To emphasize more generally the difference between the length of the long strange segments in the two cases we summarize in Table 2.1 the corresponding statements of Theorem 2.4.3 for (a_n) being a regularly varying sequence with exponent $b \geq 1/2$ of regular variation. We will implicitely assume that the appropriate assumptions of the theorem hold in each case, and that the limits I_* and I^* are positive and finite. The general statement is that, with probability 1, R_m is of the order $\chi(\log m)$, where χ is regularly varying at infinity with some exponent θ. We describe θ as a function of b in all cases. In all cases the long strange segments are much longer in the long memory case than in the short memory case. The value $\theta = \infty$ corresponds to R_m growing faster than any power of $\log m$.

Proof of Theorem 2.4.3. The duality relation $\{R_m(A;\underline{a}) \geq r\} = \{T_r(A;\underline{a}) \leq m\}$ and monotonicty of the sequence (b_n) imply that the statements (2.46) and (2.47) are equivalent. We will, therefore, concentrate on proving (2.46). The proof of the lower bound is standard, and does not rely on the fact that the underlying process is a moving average; see Theorem 3.2.1 in Dembo and Zeitouni (1998). We include the argument for completeness. Note that for every $r, m \geq 1$

$$P\big(T_r(A;\underline{a}) \leq m\big) \leq m \sum_{n=r}^{\infty} \mu_n(A;\underline{a})\,.$$

If $I_* = 0$, there is nothing to prove. Suppose that $0 < I_* < \infty$. Choose $0 < \varepsilon < I_*$. By the definition of I_* and the large deviation principle (2.43), we know that there is $c = c_\varepsilon \in (0,\infty)$ such that $\mu_n(A;\underline{a}) \leq ce^{-b_n(I_*-\varepsilon/2)}$ for all $n \geq 1$. Choosing $m = \lfloor e^{b_r(I_*-\varepsilon)} \rfloor$ gives us

$$\begin{aligned} \sum_{r=1}^{\infty} P(T_r \leq e^{b_r(I_*-\varepsilon)}) &\leq \sum_{r=1}^{\infty} e^{b_r(I_*-\varepsilon)} \sum_{n=r}^{\infty} ce^{-b_n(I_*-\varepsilon/2)} \\ &\leq c' \sum_{r=1}^{\infty} e^{-b_r\varepsilon/2} < \infty \end{aligned}$$

for some positive constant c' (depending on ε). Using the first Borel-Cantelli Lemma and letting $\varepsilon \downarrow 0$ established the lower bound in (2.46). When $I_* = \infty$, we take any $\varepsilon > 0$ and observe that by the definition of I_* there is $c = c_\varepsilon \in (0,\infty)$

such that $\mu_n(A;\underline{a}) \le ce^{-2b_n/\varepsilon)}$ for all $n \ge 1$. Choose now $m = \lfloor e^{b_r/\varepsilon} \rfloor$ and proceed as above to conclude that

$$\sum_{r=1}^{\infty} P(T_r \le e^{b_r/\varepsilon}) < \infty,$$

after which one uses, once again, the first Borel-Cantelli Lemma and lets $\varepsilon \downarrow 0$ to obtain the lower bound in (2.46).

For the upper bound in (2.46), we only need to consider the case $I^* < \infty$. In that case the set A has nonempty interior.

Define two new probability measures by

$$\mu_n'(\cdot) := P\Big(\frac{1}{a_n}\sum_{|i|\le n^2} \phi_{i,n} Z_i \in \cdot\Big) \text{ and } \mu_n''(\cdot) := P\Big(\frac{1}{a_n}\sum_{|i|> n^2} \phi_{i,n} Z_i \in \cdot\Big),$$

where $\phi_{i,n} = \phi_{i+1} + \cdots + \phi_{i+n}$.

For any sequence $\{k_n\}$ of integers, with $k_n/n \to \infty$, and any $\lambda > 0$ under the Assumptions $S2, \widehat{S3}, S4, R2, \widehat{R3}$ and $R4$, any $\lambda \in \Pi$ under the Assumption $S1$, or any $\lambda \in \Pi_\alpha$ under the Assumption $R1$,

$$\lim_{n\to\infty} \frac{1}{b_n} \sum_{i=-k_n}^{k_n} \Lambda\Big(\frac{b_n}{a_n}\lambda\phi_{i,n}\Big) = \lim_{n\to\infty} \frac{1}{b_n} \sum_{i=-\infty}^{\infty} \Lambda\Big(\frac{b_n}{a_n}\lambda\phi_{i,n}\Big); \tag{2.50}$$

see Remark 2.3.7. This means that the sequence $\{\mu_n'\}$ satisfies the LDP with speed b_n and same upper rate functions I_u given in (2.44) as the sequence $\{\mu_n\}$. The fact that the same is true for the lower rate functions in (2.44) follows from the argument in Theorems 2.2.2 and 2.2.4.

For fixed integers r, q, and $l = 1, \ldots, \lfloor q/(2r^2+1) \rfloor$, define

$$B_l := \frac{1}{a_r} \sum_{i=1+(l-1)(2r^2+1)}^{r+(l-1)(2r^2+1)} X_i,$$

and

$$B'_l := \frac{1}{a_r} \sum_{j=-r^2}^{r^2} \phi_{j,r} Z_{-j+(l-1)(2r^2+1)}.$$

Since the B'_l are independent, for any r and q we have,

$$\begin{aligned}
& P\big[T_r > q\big] \\
\leq\ & P\left[B_l \notin A, l = 1, \ldots, \left\lfloor \frac{q}{2r^2+1} \right\rfloor\right] \\
\leq\ & P\left[B'_l \notin A(\eta), l = 1, \ldots, \left\lfloor \frac{q}{2r^2+1} \right\rfloor\right] + \sum_{l=1}^{\lfloor q/(2r^2+1) \rfloor} P\Big[|B_l - B'_l| > \eta\Big] \\
=\ & \Big(1 - \mu'_r\big(A(\eta)\big)\Big)^{\lfloor q/(2r^2+1) \rfloor} + \sum_{l=1}^{\lfloor q/(2r^2+1) \rfloor} P\Big[|B_l - B'_l| > \eta\Big] \\
\leq\ & \exp\Big(- \frac{q}{2r^2+1} \mu'_r\big(A(\eta)\big)\Big) + \frac{q}{2r^2+1} \mu''_r\big(\{x : |x| > \eta\}\big).
\end{aligned}$$

By the definition of I^* and the large deviation principle (2.43), for any $\varepsilon > 0$ there is $c = c_\varepsilon \in (0, \infty)$ such that for all $\eta > 0$ small enough, $\mu_n(A(\eta)) \geq c e^{-b_n(I^* + \varepsilon/2)}$ for all n large than some n_ε. Therefore, fixing $\varepsilon > 0$ and using the bound above with $q = e^{b_r(I^*+\varepsilon)}$, we see that for some $C = C_\varepsilon \in (0, \infty)$, for all $\eta > 0$ small enough,

$$\begin{aligned}
\sum_{r=1}^{\infty} \exp\Big(- \frac{e^{b_r(I^*+\epsilon)}}{2r^2+1} \mu'_r\big(A(\eta)\big)\Big) \ &\leq\ C \sum_{r=1}^{\infty} \exp\Big(- c \frac{e^{b_r(I^*+\epsilon)}}{2r^2+1} e^{-b_r(I^*+\epsilon/2)}\Big) \\
&=\ C \sum_{r=1}^{\infty} \exp\Big(- c \frac{e^{b_r(\epsilon/2)}}{2r^2+1}\Big) < \infty. \qquad (2.51)
\end{aligned}$$

Suppose first that we are under the Assumptions $S2, \widehat{S3}, S4, R2, \widehat{R3}$ or $R4$. Fixing $\varepsilon > 0$ and choosing $\eta > 0$ small enough for the above to hold, we see that

$$\begin{aligned}
& \limsup_{n\to\infty} \frac{1}{b_n} \log \mu''_n\big(\{x : |x| > \eta\}\big) \\
\leq\ & \limsup_{n\to\infty} \frac{1}{b_n} \log \left(e^{-b_n \lambda \eta} E\Big[\exp\Big\{\lambda \frac{b_n}{a_n} \sum_{|i|>n^2} \phi_{i,n} Z_i\Big\}\Big]\right) \\
=\ & -\lambda\eta + \limsup_{n\to\infty} \frac{1}{b_n} \sum_{|i|>n^2} \Lambda\Big(\frac{b_n}{a_n} \lambda \phi_{i,n}\Big) = -\lambda\eta,
\end{aligned}$$

with the last equality following from (2.50). Choosing now $\lambda > (I^*+\epsilon)/\eta$ (which is possible under the current assumptions no matter how small $\eta > 0$ is), we obtain

$$\sum_{r=1}^{\infty} \frac{e^{b_r(I^*+\epsilon)}}{2r^2+1} \mu_r''\big(\{x : |x| > \eta\}\big) < \infty. \tag{2.52}$$

Combining (2.51) and (2.52) we have $\sum_{r=1}^{\infty} P\Big[T_r > e^{b_r(I^*+\epsilon)}\Big] < \infty$, so that using the first Borel-Cantelli Lemma gives and letting $\varepsilon \downarrow 0$ proves the upper bound in (2.46). The cases of the Assumptions $S1$ and $R1$ are the same, except now λ cannot be taken to be arbitrarily large, which restricts the feasible values of $\eta > 0$. This completes the proof. □

2.5 Ruin Probabilities

This section discusses the rate of decay ruin probability for a moving average process $\{X_n, n \in \mathbb{Z}\}$. Throughout this section we will assume $d = 1$, that is, $\{X_n\}$ is a one dimensional process. We study the probability of ruin in infinite time, defined as

$$\rho(u) = P\left[S_n > n\mu + u \text{ for some } n \geq 1\right] \tag{2.53}$$

where $\mu > 0$ is a constant. Throughout this section T plays the important role of the time of ruin, that is,

$$T(u) = \inf\left\{n : S_n > n\mu + u\right\}.$$

This means $\rho(u) = P[T(u) < \infty]$. Specifically, we are interested in the rate of decay of $\rho(u)$ as u increases to infinity. We retain the assumptions on the white noises $\{Z_n\}$ and the notations used in Section 2.4. Theorems 2.5.1 and 2.5.2 state the results for the two different scenarios which we call the short and long memory regimes. The result in the short memory case has been worked out in Nyrhinen (1994), but we give the details below for completeness.

Theorem 2.5.1. *Suppose that the moving average process $\{X_n\}$ satisfies* (2.8)*.*

(i) Then

$$\limsup_{u\to\infty} \frac{1}{u}\log\rho(u) = -\sup\Big\{t \geq 0 : \sup_{n\geq 1}\Big\{\sum_{i\in\mathbb{Z}} \Lambda\left(t\phi_{i,n}\right) - tn\mu\Big\} < \infty\Big\}$$

(ii) Assume that there exists $w > 0$ such that $\Lambda(w) = w\mu$ and $\delta > 0$ such that $w + \delta \in \Pi_1$, where Π_1 is as defined in (2.10)*. Then*

$$\liminf_{u\to\infty} \frac{1}{u}\log\rho(u) \geq -w.$$

Theorem 2.5.2. *If the moving average process $\{X_n\}$ satisfies* (2.12) *then*

$$\lim_{u\to\infty} \frac{\Psi_u^2}{u}\log\rho(u) = -\frac{2}{\gamma(3-2\alpha)^{3-2\alpha}(2\alpha-1)^{2\alpha-1}}\mu^{3-2\alpha},$$

where,

$$\gamma = \begin{cases} Var(Z_0)(1-\alpha)^2 \int\limits_{-\infty}^{\infty}\left(\int\limits_{x}^{x+1} |y|^{-\alpha}(pI_{[y\geq 0]} + qI_{[y<0]})dy\right)^2 dx & \text{if } \alpha < 1 \\ Var(Z_0) & \text{if } \alpha = 1. \end{cases}$$

Here, for any non-integer u, we take $\Psi_u := \Psi_{[u]}$.

Remark 2.5.3. Theorem 2.5.1 is a direct consequence of Theorems 3.1 and 3.2 in Nyrhinen (1994). Although the theorem gives an exact value of the 'limsup', it only gives a lower bound for the 'liminf'. But along with assumptions of Theorem 2.5.1 if we assume $t\phi_{i,n} \in \mathcal{F}_\Lambda^\circ$ for every $0 < t < w$, $i \in \mathbb{Z}$ and $n \geq 1$, then one easily verify that

$$\lim_{u\to\infty} \frac{1}{u}\log\rho(u) = -w. \tag{2.54}$$

In particular, if $\Lambda(t) < \infty$ for every $t \in \mathbb{R}$ then (2.54) holds. To see why this is true we refer to the discussion following (3.1) in Nyrhinen (1994) and show that

$$w = \sup\Big\{t \geq 0 : \sup_{n\geq 1}\Big\{\sum_{i\in\mathbb{Z}} \Lambda\left(t\phi_{i,n}\right) - tn\mu\Big\} < \infty\Big\}. \qquad (2.55)$$

From Lemma 2.3.5(i) we know that

$$\lim_{n\to\infty}\frac{1}{n}\sum_{i\in\mathbb{Z}}\Lambda(t\phi_{i,n}) = \Lambda(t)$$

for every $t > 0$ satisfying

$$t\phi_{i,n} \in \mathcal{F}_\Lambda^\circ \qquad \text{for all } i \in \mathbb{Z}, n \geq 1. \qquad (2.56)$$

Since (2.56) holds for every $0 < t < w$ fix any t in that range. Following the arguments used to prove Lemma 2.3.5 it is easy to check that for any $n \geq 1$, $\sum_{i\in\mathbb{Z}} \Lambda(t\phi_{i,n}) < \infty$. Furthermore, since

$$\lim_{n\to\infty}\frac{1}{n}\sum_{i\in\mathbb{Z}}\Lambda(t\phi_{i,n}) - t\mu < 0$$

we have $\sup_{n\geq 1}\big\{\sum_{i\in\mathbb{Z}}\Lambda\left(t\phi_{i,n}\right) - tn\mu\big\} < \infty$. That implies

$$w \leq \sup\Big\{t \geq 0 : \sup_{n\geq 1}\Big\{\sum_{i\in\mathbb{Z}} \Lambda\left(t\phi_{i,n}\right) - tn\mu\Big\} < \infty\Big\}.$$

The other inequality is obvious and hence (2.54) is true.

Remark 2.5.4. If the conditions of Theorem 2.5.1(ii) are not satisfied then the following lower bound holds trivially:

$$\liminf_{u\to\infty}\frac{1}{u}\log\rho(u) \geq -\inf_{t>0}\frac{1}{t}\Lambda^*(\mu + t).$$

In order to get this bound observe that for any $t > 0$ and $\epsilon > 0$

$$\begin{aligned}\liminf_{u\to\infty}\frac{1}{u}\log\rho(u) &\geq \liminf_{n\to\infty}\frac{1}{nt}\log P\big[S_n - n\mu > nt\big]\\ &= \frac{1}{t}\liminf_{n\to\infty}\frac{1}{n}\log P\big[S_n/n > \mu + t\big]\\ &\geq -\frac{1}{t}\Lambda^*(\mu + t + \epsilon),\end{aligned}$$

where the last inequality follows form the marginal version of Theorem 2.2.2(i) which is described in (2.44). Since $t > 0$ and $\epsilon > 0$ are arbitrary and Λ^* is convex, we have the result.

Remark 2.5.5. The Theorems 2.5.1 and 2.5.2 again clearly demonstrate the phenomenon of short and long range dependence. When the coefficients are summable then $\log\rho(u)$ decreases linearly in u, whereas under the assumption (2.12) $\log\rho(u)$ decreases like u/Ψ_u^2 which is regular varying of index $2\alpha - 1$.

Proof of Theorem 2.5.1. (i) By Theorem 3.1 in Nyrhinen (1994) we know that

$$\limsup_{u\to\infty}\frac{1}{u}\log\rho(u) = -\sup\Big\{t \geq 0 : \sup_{n\geq 1} E\exp\Big(t\big(S_n - n\mu\big)\Big) < \infty\Big\}.$$

and for the moving average process we obviously have

$$\log E\exp\Big(t\big(S_n - n\mu\big)\Big) = \sum_{i\in\mathbb{Z}}\Lambda\left(t\phi_{i,n}\right) - tn\mu.$$

(ii) Theorem 3.2 in Nyrhinen (1994) tells us that if

$$\bar{w} = \sup\Big\{t > 0 : \limsup_{n\to\infty}\frac{1}{n}\log E\exp\Big(t(S_n - n\mu)\Big) \leq 0\Big\} < \infty$$

and there exists $\delta > 0$ such that

$$\lim_{n\to\infty}\frac{1}{n}\log E\exp\Big(t(S_n - n\mu)\Big) < \infty \qquad \text{for all } t \in (\bar{w}, \bar{w}+\delta) \tag{2.57}$$

then

$$\liminf_{u\to\infty}\frac{1}{u}\log\rho(u) \geq \bar{w}.$$

We claim that if the assumption of part (ii) of the theorem holds then such a $\bar{w}$ exists and it equals w. We know that there exists $\delta > 0$ such that $w + \delta \in \Pi_1$. Hence by Lemma 2.3.5(i) we get that for any $w < t < w + \delta$

$$\lim_{n\to\infty}\frac{1}{n}\log E\exp\Big(t(S_n - n\mu)\Big) = \Lambda(t) - t\mu > 0.$$

By a similar argument for any $0 < t < w$

$$\lim_{n\to\infty} \frac{1}{n} \log E \exp\Big(t(S_n - n\mu)\Big) = \Lambda(t) - t\mu < 0.$$

Therefore $\bar{w} < \infty$ and $\bar{w} = w$. □

Proof of Theorem 2.5.2. The proof consists of two parts. First we consider the easier half and prove that

$$\liminf_{u\to\infty} \frac{\Psi_u^2}{u} \log \rho(u) \geq -\frac{2}{\gamma(3-2\alpha)^{3-2\alpha}(2\alpha-1)^{2\alpha-1}}\mu^{3-2\alpha}. \tag{2.58}$$

For that purpose take any $a > 0$ and observe that

$$\begin{aligned}\liminf_{u\to\infty} \frac{\Psi_u^2}{u} \log \rho(u) &= \liminf_{n\to\infty} \frac{\Psi_{an}^2}{an} \log P\big[T(an) < \infty\big] \\ &\geq a^{1-2\alpha} \liminf_{n\to\infty} \frac{\Psi_n^2}{n} \log P\left[\frac{S_n}{n} > \mu + a\right],\end{aligned}$$

From the marginal version of Theorem 2.2.4(iii) we get that $P\big[S_n/n \in \cdot\big]$ satisfies LDP on $\mathbb{R}$ with speed n/Ψ_n^2 and rate function $G(x) = x^2/2\gamma$. This is easy to check. By putting $a_n = n$ and $b_n = n/\Psi_n^2$ in Lemma 2.3.6(iii) we get

$$\lim_{n\to\infty} \frac{\Psi_n^2}{n} \log E\left[\exp\left(\frac{\lambda}{\Psi_n^2} S_n\right)\right] = \frac{\gamma}{2}\lambda^2 \qquad \text{for all } \lambda \in \mathbb{R} \tag{2.59}$$

and hence by the Gartner-Ellis Theorem $P\big[S_n/n \in \cdot\big]$ satisfies LDP on $\mathbb{R}$ with speed n/Ψ_n^2 and rate function $G(x) = x^2/2\gamma$. This implies

$$\liminf_{n\to\infty} \frac{\Psi_n^2}{n} \log P\left[\frac{S_n}{n} > \mu + a\right] \geq -\frac{1}{2\gamma}(\mu + a)^2$$

which in turn gives us

$$\liminf_{u\to\infty} \frac{\Psi_u^2}{u} \log \rho(u) \geq -a^{1-2\alpha}(\mu + a)^2/2\gamma$$

Since $a > 0$ is arbitrary, the best result is achieved by maximizing the function on the right hand side over $a > 0$. That is attained at $a = \mu(2\alpha - 1)/(3 - 2\alpha)$ which gives (2.58).

We now prove the other inequality. Let us first consider the case when $\alpha < 1$. Note that there exists $t^* > 0$ sufficiently small such that

$$\sup_{k\geq 1} E\Big[e^{t^*(S_k - k\mu)/\Psi_k^2}\Big] < \infty.$$

Now fix any $\delta > 0$. Then

$$\begin{aligned}
\limsup_{n\to\infty} \frac{\Psi_n^2}{n} \log P\big[T(n) \leq n\delta\big] &\leq \limsup_{n\to\infty} \frac{\Psi_n^2}{n} \log \sum_{k=1}^{[n\delta]} P\big[S_k - k\mu > n\big] \\
&\leq \limsup_{n\to\infty} \frac{\Psi_n^2}{n} \log \sum_{k=1}^{[n\delta]} e^{-t^* n/\Psi_k^2} E\Big[e^{t^*(S_k - k\mu)/\Psi_k^2}\Big] \\
&\leq -t^* \limsup_{n\to\infty} \frac{\Psi_n^2}{\Psi_{[n\delta]}^2} = -t^* \delta^{2\alpha - 2}, \qquad (2.60)
\end{aligned}$$

which decreases to $-\infty$ as δ decreases to 0. Now fix any t such that $\frac{\gamma}{2}t^2 - \mu t < 0$ and get ϵ such that $0 < \epsilon < \mu t - \frac{\gamma}{2}t^2$. From (2.59) it is possible to get $N \geq 1$ such that $k \geq N$ implies

$$\frac{\Psi_k^2}{k} \log E\Big[e^{tS_k/\Psi_k^2}\Big] \leq \frac{\gamma}{2}t^2 + \epsilon.$$

Now note that

$$\begin{aligned}
&\limsup_{n\to\infty} \frac{\Psi_n^2}{n} \log P\big[n\delta < T(n) < \infty\big] \\
\leq\; &\limsup_{n\to\infty} \frac{\Psi_n^2}{n} \log \sum_{k=[n\delta]+1}^{\infty} P\big[T(n) = k\big] \\
\leq\; &\limsup_{n\to\infty} \frac{\Psi_n^2}{n} \log \sum_{k=[n\delta]+1}^{\infty} P\big[S_k - k\mu > n\big] \\
\leq\; &\limsup_{n\to\infty} \frac{\Psi_n^2}{n} \log \sum_{k=[n\delta]+1}^{\infty} \exp\Big\{-\frac{n}{\Psi_k^2}t + \frac{k}{\Psi_k^2}\Big(\frac{\gamma}{2}t^2 - \mu t + \epsilon\Big)\Big\},
\end{aligned}$$

where the last inequality follows by an application of exponential Markov inequality. Now if $n > N/\delta$ then for $i \geq 1$

$$\begin{aligned}
&\sum_{k=i[n\delta]+1}^{(i+1)[n\delta]} \exp\Big\{-\frac{n}{\Psi_k^2}t + \frac{k}{\Psi_k^2}\Big(\frac{\gamma}{2}t^2 - \mu t + \epsilon\Big)\Big\} \\
\leq\; &n\delta \exp\Big\{-\frac{n}{\Psi_{(i+1)[n\delta]}^2}t + \frac{i[n\delta]+1}{\Psi_{(i+1)[n\delta]}^2}\Big(\frac{\gamma}{2}t^2 - \mu t + \epsilon\Big)\Big\}
\end{aligned}$$

Since $1/2 < \alpha < 1$ we can get $\eta > 0$ such that $2\alpha - 2 + \eta < 0$ and $2\alpha - 2 - \eta > -1$. Then we can get $N_1 \geq 1$ such that for every $n \geq N_1$

$$\frac{\Psi_n^2}{\Psi_{(i+1)[n\delta]}^2} \geq x_{i,\delta}(\eta) := (1-\eta)\min\Big\{\big((i+1)\delta\big)^{2\alpha-2-\eta}, \big((i+1)\delta\big)^{2\alpha-2+\eta}\Big\}$$

and

$$\frac{i[n\delta]+1}{n} \geq i\delta(1-\eta).$$

This implies that for $n > \max\{N/\delta, N_1\}$

$$\begin{aligned}
&\sum_{k=i[n\delta]+1}^{(i+1)[n\delta]} \exp\Big\{-\frac{n}{\Psi_k^2}t + \frac{k}{\Psi_k^2}\Big(\frac{\gamma}{2}t^2 - \mu t + \epsilon\Big)\Big\} \\
\leq\ & n\delta \exp\Big\{-\frac{n}{\Psi_n^2}x_{i,\delta}(\eta)\Big(t + i\delta(1-\eta)\Big(\frac{\gamma}{2}t^2 - \mu t + \epsilon\Big)\Big\}
\end{aligned}$$

Define $y_i = -x_{i,\delta}(\eta)\Big(t + i\delta(1-\eta)\Big(\frac{\gamma}{2}t^2 - \mu t + \epsilon\Big)$ and let $y^* = \max_{i\geq 1} y_i$. Observe that $y^* < 0$ and there exists $i^* \geq 1$ such that $y^* = y_{i^*}$. Then

$$\begin{aligned}
&\sum_{k=[n\delta]+1}^{\infty} \exp\Big\{-\frac{n}{\Psi_k^2}t + \frac{k}{\Psi_k^2}\Big(\frac{\gamma}{2}t^2 - \mu t + \epsilon\Big)\Big\} \\
\leq\ & n\delta \sum_{i=1}^{\infty} \exp\Big\{\frac{n}{\Psi_n^2}y_i\Big\} \\
=\ & n\delta \exp\Big\{\frac{n}{\Psi_n^2}y^*\Big\} \sum_{i=1}^{\infty} \exp\Big\{\frac{n}{\Psi_n^2}(y_i - y^*)\Big\}.
\end{aligned}$$

Now there exists $c > 0$ such that $\exp\big\{\frac{n}{\Psi_n^2}(y_i - y^*)\big\} \leq \exp(cy_i - cy^*)$ for every $n \geq 1$, and because of the choice of η we have

$$\sum_{i=1}^{\infty} \exp(cy_i - cy^*) < \infty.$$

Therefore by the Dominated Convergence Theorem we get

$$\sum_{i=1}^{\infty} \exp\Big\{\frac{n}{\Psi_n^2}(y_i - y^*)\Big\} = 1 + \sum_{i\neq i^*} \exp\Big\{\frac{n}{\Psi_n^2}(y_i - y^*)\Big\} \to 1.$$

Hence we get

$$\limsup_{n\to\infty} \frac{\Psi_n^2}{n} \log \sum_{k=[n\delta]+1}^{\infty} \exp\Big\{-\frac{n}{\Psi_k^2}t + \frac{k}{\Psi_k^2}\Big(\frac{\gamma}{2}t^2 - \mu t + \epsilon\Big)\Big\} \leq y^* \tag{2.61}$$

and therefore by combining (2.60) and (2.61) we get

$$\limsup_{u\to\infty} \frac{\Psi_u^2}{u} \log \rho(u) = \limsup_{u\to\infty} \frac{\Psi_u^2}{u} \log P\big[T(u) < \infty\big] \le \max\big\{ -t^*\delta^{2\alpha-2}, y^*\big\}.$$

Finally, by letting ϵ, δ and η to 0, we get

$$\begin{aligned} \limsup_{u\to\infty} \frac{\Psi_u^2}{u} \log \rho(u) &\le \sup_{u>0} \Big\{ -tu^{2\alpha-2} + u^{2\alpha-1}\Big(\frac{\gamma}{2}t^2 - \mu t\Big)\Big\} \\ &= -\frac{(2-2\alpha)^{2\alpha-2}}{(2\alpha-1)^{2\alpha-1}} t^{2\alpha-1}\Big(\mu t - \frac{\gamma}{2}t^2\Big)^{2-2\alpha} \end{aligned}$$

which is attained at $u = \frac{2-2\alpha}{2\alpha-1}\frac{t}{\mu t - \frac{\gamma}{2}t^2}$. The best result is obtained by taking infimum over t such that $\frac{\gamma}{2}t^2 - \mu t < 0$. That is attained at $t = \frac{2\mu}{\gamma(3-2\alpha)}$ which gives us

$$\begin{aligned} &\limsup_{n\to\infty} \frac{\Psi_n^2}{n} \log \sum_{k=n\delta+1}^{\infty} \exp\Big\{ -\frac{n}{\Psi_k^2}t + \frac{k}{\Psi_k^2}\Big(\frac{\gamma}{2}t^2 - \mu t + \epsilon\Big)\Big\} \\ = \quad & -\frac{2}{\gamma(3-2\alpha)^{3-2\alpha}(2\alpha-1)^{2\alpha-1}}\mu^{3-2\alpha}, \end{aligned}$$

and that completes the proof when $\alpha < 1$.

In the last step we consider the case when $\alpha = 1$. We claim that

$$\lim_{M\to\infty} \limsup_{n\to\infty} \frac{\Psi_n^2}{n} \log P\big[nM < T(n) < \infty\big] = -\infty. \tag{2.62}$$

To see this choose $t > 0$ and $\epsilon > 0$ such that $\frac{\gamma}{2}t^2 - \mu t + \epsilon < 0$. Then for all n large we have

$$\begin{aligned} P\big[nM < T(n) < \infty\big] &\le \sum_{k=nM+1}^{\infty} \exp\Big\{ -\frac{n}{\Psi_k^2}t + \frac{k}{\Psi_k^2}\Big(\frac{\gamma}{2}t^2 - \mu t + \epsilon\Big)\Big\} \\ &\le \sum_{k=nM+1}^{\infty} \exp\Big\{ \frac{k}{\Psi_k^2}\Big(\frac{\gamma}{2}t^2 - \mu t + \epsilon\Big)\Big\} \end{aligned}$$

By Theorem 4.12.10 in Bingham et al. (1989) we get that

$$\log \sum_{k=nM+1}^{\infty} \exp\Big\{ \frac{k}{\Psi_k^2}\Big(\frac{\gamma}{2}t^2 - \mu t + \epsilon\Big)\Big\} \sim \frac{nM}{\Psi_{nM}^2}\Big(\frac{\gamma}{2}t^2 - \mu t + \epsilon\Big)$$

which implies

$$\begin{aligned}
&\limsup_{n\to\infty} \frac{\Psi_n^2}{n} \log P\big[nM < T(n) < \infty\big] \\
\le\ & \limsup_{n\to\infty} \frac{\Psi_n^2}{n} \log \sum_{k=nM+1}^{\infty} \exp\left\{ \frac{k}{\Psi_k^2}\Big(\frac{\gamma}{2}t^2 - \mu t + \epsilon\Big)\right\} \\
=\ & M\Big(\frac{\gamma}{2}t^2 - \mu t + \epsilon\Big)
\end{aligned}$$

which proves (2.62). Next we claim that

$$\limsup_{M\to\infty} \limsup_{n\to\infty} \frac{\Psi_n^2}{n} \log P\big[T(n) \le nM\big] \le -\frac{2}{\gamma}\mu, \tag{2.63}$$

where $\gamma = Var(Z_0)$. We have, for a fixed $M > 0$ and $N \ge 1$ such that $nM > N$

$$P\big[T(n) \le nM\big] \le P\big[N < T(n) \le nM\big] + P\big[T(n) \le N\big].$$

Clearly for any $N \ge 1$ there exists $c_N > 0$ and $\beta > 0$ such that

$$P\big[T(n) \le N\big] \le c_N e^{-\beta n}.$$

Since $\Psi_n^2 \to \infty$ as $n \to \infty$ we have

$$\lim_{n\to\infty} \frac{\Psi_n^2}{n} \log P\big[T(n) \le N\big] = -\infty \qquad \text{for all } N \ge 1. \tag{2.64}$$

Therefore, in order to prove (2.63) it suffices to show that

$$\limsup_{M\to\infty} \limsup_{N\to\infty} \limsup_{n\to\infty} \frac{\Psi_n^2}{n} \log P\big[N < T(n) \le nM\big] \le -\frac{2}{\gamma}\mu. \tag{2.65}$$

Now notice that

$$\begin{aligned}
&P\big[N < T(n) \le nM\big] \\
=\ & P\big[S_k > k\mu + n \text{ for some } N < k \le nM\big] \\
=\ & P\left[S_{[nMt]} > [nMt]\mu + n \text{ for some } \frac{N}{nM} < t \le 1\right] \\
=\ & P\left[Y_{nM}(t) > \frac{1}{M} + \frac{[nMt]}{nM}\mu \text{ for some } \frac{N}{nM} < t \le 1\right] \\
\le\ & P\left[Y_{nM}(t) > \frac{1}{M} + \frac{nMt-1}{nM}\mu \text{ for some } \frac{N}{nM} < t \le 1\right]
\end{aligned}$$

Now fix $M > 0$ and $0 < \eta < 1$ such that $M\mu(1-\eta) > 1$ and then choose $N \geq \frac{M}{\eta}$. Then for $t \geq \frac{N}{nM}$ we have $nMt - 1 \geq (1-\eta)nMt$. Therefore

$$\begin{aligned} & P\big[N < T(n) \leq nM\big] \\ \leq\ & P\left[Y_{nM}(t) > \frac{1}{M} + (1-\eta)\mu t \text{ for some } 0 \leq t \leq 1\right] \\ =\ & P\big[Y_{nM} \in A\big] \end{aligned}$$

where

$$A = \left\{ f \in \mathcal{BV} : f(t) > \frac{1}{M} + (1-\eta)\mu t \text{ for some } 0 \leq t \leq 1 \right\}.$$

By the functional large deviation principle (Theorem 2.2.4(iii)) we then get

$$\limsup_{n\to\infty} \frac{\Psi_n^2}{n} P\big[N < T(n) \leq nM\big] \leq -M \cdot \inf_{f \in \bar{A}} I(f),$$

where

$$I(f) = \begin{cases} \frac{1}{2\gamma}\int_0^1 f'(t)^2 dt & \text{if } f \in \mathcal{AC}, f(0) = 0 \\ \infty & \text{otherwise.} \end{cases}$$

Now for every $f \in \mathcal{AC}$ such that $f(0) = 0$, $f \in \bar{A}$ and $f(t_0) = \frac{1}{M} + (1-\eta)\mu t_0$ for some $0 < t \leq 1$,

$$\begin{aligned} I(f) &\geq \frac{1}{2\gamma}\int_0^{t_0} f'(t)^2 dt \quad \geq \quad \frac{1}{2\gamma}\frac{1}{t_0}\left(\int_0^{t_0} f'(t)dt\right)^2 \\ &= \frac{1}{2\gamma}\frac{1}{t_0} f(t_0)^2 \quad = \quad \frac{1}{2\gamma}\frac{1}{t_0}\left(\frac{1}{M} + (1-\eta)\mu t_0\right)^2 \end{aligned}$$

The minimum is achieved at $t_0 = \frac{1}{M\mu(1-\eta)} \in (0,1)$. Then

$$I(f) \geq (1-\eta)\frac{2}{\gamma M}\mu,$$

and so for any $0 < \eta < 1$ and M sufficiently large we have

$$\limsup_{N\to\infty} \limsup_{n\to\infty} \frac{\Psi_n^2}{n} \log P\big[N < T(n) \leq nM\big] \leq -(1-\eta)\frac{2}{\gamma}\mu$$

and finally by letting $\eta \to 0$ the proof is complete. □

CHAPTER 3

INFINITELY DIVISIBLE PROCESSES

3.1 Introduction

This chapter is dedicated to studying the functional large deviation principle for certain stationary infinitely divisible processes. A random variable X is infinitely divisible if for every $k \geq 1$, there exists i.i.d. random variables $W_1^{(k)}, \ldots, W_k^{(k)}$ such that

$$X \stackrel{d}{=} W_1^{(k)} + \cdots + W_k^{(k)}.$$

A process $(X_n, n \in \mathbb{Z})$ is infinitely divisible if for every $-\infty < t_1 < \cdots < t_k < \infty$, $(X_{t_1}, \ldots, X_{t_k})$ is infinitely divisible. We assume that the marginals satisfy

$$E[\exp(\lambda X_0)] < \infty \text{ for all } \lambda \in \mathbb{R}. \tag{3.1}$$

Section 3.2 discusses the required background materials. The main results are stated in Section 3.3 and Section 3.4 discusses some interesting examples. In Section 3.5 we discuss ruin probabilities as an application of large deviations.

3.2 Background Materials

3.2.1 The Lévy-Khintchine Representation

The Lévy-Khintchine representation is a vital tool for the study of infinitely divisible distributions and processes. We state the result below but refer to Theorem 8.1 and the following remarks in Sato (1999) for further details.

Theorem 3.2.1. *For any $\mathbb{R}^d$-valued infinitely divisible random variable X there exists a unique triplet (Σ, ν, υ), such that Σ is a $d \times d$ non-negative definite matrix, $\upsilon \in \mathbb{R}^d$ and ν is a measure on $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$ satisfying*

$$\int_{\mathbb{R}^d} |[\![z]\!]|^2 \, \nu(dz) < \infty,$$

where

$$[\![z]\!] := \frac{z}{|z| \vee 1} = \begin{cases} z & \text{if } |z| \leq 1 \\ z/|z| & \text{if } |z| > 1 \end{cases}$$

such that

$$E\left(e^{i\lambda \cdot X}\right) = \exp\left\{-\frac{1}{2}\lambda \cdot \Sigma\lambda + i\lambda \cdot \upsilon + \int_{\mathbb{R}^d} \left(e^{i\lambda \cdot z} - 1 - i\lambda \cdot [\![z]\!]\right) \nu(dz)\right\} \tag{3.2}$$

The triplet (Σ, ν, υ) uniquely determines the distribution of X and is called the generating triplet of X. From the expression of the characteristic function of X in (3.2) it is evident that $X \stackrel{d}{=} X^1 + X^2$, where X^1 has a Gaussian distribution with mean υ and covariance matrix Σ and X^2 is independent of X^1. Furthermore, this decomposition of X is unique. X^1 and X^2 are called the Gaussian and Poissonian component of X, respectively. The measure ν is called the Lévy measure of X, and it determines the Poissonian component. We will use the notation $X \sim (\Sigma, \nu, \upsilon)$ to signify that X is an infinitely divisible random variable with generating triplet (Σ, ν, υ). If X satisfies

$$E[\exp(\lambda \cdot X)] < \infty, \text{ for all } \lambda \in \mathbb{R},$$

then by Theorem 25.17 in Sato (1999) we can get a representation for the moment generating function of X:

$$E[\exp(\lambda \cdot X)] = \exp\left\{\frac{1}{2}\lambda \cdot \Sigma\lambda + \lambda \cdot \upsilon + \int_{\mathbb{R}^d} \left(e^{\lambda \cdot z} - 1 - \lambda \cdot [\![z]\!]\right) \nu(dz)\right\}. \tag{3.3}$$

This result is vital for the study of large deviations of infinitely divisible processes.

3.2.2 Infinitely Divisible Random Measure

We introduce the notion of an *infinitely divisible random measure* or *IDRM*. Rosiński and Samorodnitsky (2007) provides a comprehensive treatment of this topic.

Definition 3.2.2. *Suppose $(\Omega, \mathcal{B}, P)$ is a probability space. Let S be a set and and $\mathcal{S}_0$ be a σ-ring of measurable subsets of S. $\{M(A), A \in \mathcal{S}_0\}$ is an infinitely divisible random measure on $(S, \mathcal{S}_0)$ with control measure m if*

(i) $M(\emptyset) = 0, P - a.s.$

(ii) For every $\{A_i\} \subset \mathcal{S}_0$ pairwise disjoint, $\{M(A_i)\}$ forms a sequence of independent random variables and if $\cup_i A_i \in \mathcal{S}_0$ then

$$M\big(\bigcup_{i=1}^{\infty} A_i\big) = \sum_{i=1}^{\infty} M(A_i), P - a.s.$$

(iii) For every $A \in \mathcal{S}_0$, $M(A)$ has an infinitely divisible distribution.

(iv) For a set $A \in \mathcal{S}_0$, $m(A) = 0$ if and only if $M(A') = 0, P - a.s.$ for every $A' \subset A, A' \in \mathcal{S}_0$.

The following theorem characterizes the generating triplet of an IDRM.

Theorem 3.2.3. *(a) Let M be an IDRM on $(S, \mathcal{S}_0)$ such that for every $A \in \mathcal{S}_0$,*

$$M(A) \sim \big(\Sigma(A), \nu(A), \upsilon(A)\big). \tag{3.4}$$

Then

(i) $\Sigma : \mathcal{S}_0 \to \mathbb{R}_+$ is a measure.

(ii) ν is a bi-measure, i.e., $\forall A \in \mathcal{S}_0$, $\nu(A, \cdot)$ is a measure on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ and $\forall B \in \mathcal{B}(\mathbb{R}^d)$, $\nu(\cdot, B)$ is a measure on $(S, \mathcal{S}_0)$.

(iii) $\upsilon : \mathcal{S}_0 \to \mathbb{R}$ *is a signed measure.*

(b) If (Σ, ν, υ) *satisfy the conditions given in (a), then there exists a unique (in the sense of finite dimensional distributions) IDRM* M *such that* (3.4) *holds.*

(c) Let (Σ, ν, υ) *be as in (a). Define a measure*

$$m(A) = \Sigma(A) + |\upsilon|(A) + \int_{\mathbb{R}} [\![x]\!]^2 \nu(A, dx) \qquad A \in \mathcal{S}_0, \tag{3.5}$$

where $|\upsilon| = \upsilon^+ + \upsilon^-$ *is the Jordan decomposition of measure* υ *into positive and negative parts. Then* $m(\cdot)$ *is a control measure of* M*.*

Since we prefer to work with σ-finite measures we assume that there exists an increasing sequence $\{S_n\} \subset \mathcal{S}_0$ such that

$$S = \bigcup_n S_n. \tag{3.6}$$

We can extend ν to a measure (by an abuse of notation we will call this ν as well) on $(S \times \mathbb{R}, \mathcal{S} \times \mathcal{B}(\mathbb{R}))$ such that

$$\nu(A \times B) = \nu(A, B), \qquad A \in \mathcal{S}, B \in \mathcal{B}(\mathbb{R}), \tag{3.7}$$

where $\mathcal{S} := \sigma(\mathcal{S}_0)$. Similarly, it is also possible to extend the measurs Σ and $|\upsilon|$ on $(S, \mathcal{S})$. Since all the measures are σ-finite, we can define measurable functions

$$\sigma^2(s) := \frac{d\Sigma}{dm}(s) \tag{3.8}$$

$$\eta(s) := \frac{d\upsilon}{dm}(s) \tag{3.9}$$

and a measure kernel $\rho(x, dx)$ on $(S, \mathcal{B}(\mathbb{R}))$ such that

$$\nu(ds, dx) = \rho(s, dx) m(ds). \tag{3.10}$$

(σ^2, ρ, η) is called the *local characteristic* of M with respect to the control measure m. Intuitively, we can think that

$$M(ds) \sim (\sigma^2(s) m(ds), \rho(s, \cdot) m(ds), \eta(s) m(ds)).$$

The following theorem makes this statement more precise.

Theorem 3.2.4. *Under the above notation and condition* (3.6), $\big(\sigma^2(s), \rho(s,\cdot), \eta(s)\big)$ *is a generating triplet of some infinitely divisible distribution* $\mu(s,\cdot)$ *on* $\mathbb{R}$, *m-a.e. Moreover,*

$$\sigma^2(s) + \int_{\mathbb{R}} [\![x]\!]^2 \rho(s,dx) + |\upsilon|(s) = 1, \qquad m - a.e. \tag{3.11}$$

For every $B \in \mathcal{B}(\mathbb{R})$, $s \mapsto \mu(s,B)$ *is measurable and thus* μ *is a probability kernel on* $S \times \mathcal{B}(\mathbb{R})$. *If*

$$C(s,\lambda) = -\frac{1}{2}\sigma^2(s)\lambda^2 + i\eta(s)\lambda + \int_{\mathbb{R}} (e^{i\lambda x} - 1 - i\lambda [\![x]\!])\rho(s,dx) \tag{3.12}$$

then $\int_{\mathbb{R}} e^{i\lambda x}\mu(s,dx) = \exp C(s,\lambda)$ *and*

$$E\big[\exp(i\lambda M(A))\big] = \exp \int_A C(s,\lambda) m(ds). \tag{3.13}$$

3.2.3 Integration with respect to an IDRM

Suppose M is an infinitely divisible random measure on $(S, \mathcal{S}_0)$ with control measure m and local characteristics (σ^2, ρ, η). We will define integration with respect to M of a deterministic function $f : S \to \mathbb{R}$. As is often the case, we begin by defining the integral for a simple function. By a simple function we understand a finite linear combination of sets from $\mathcal{S}_0$,

$$f(s) = \sum_{j=1}^{n} a_j I_{A_j}(s), \qquad A_j \in \mathcal{S}_0.$$

For such a function the integral is defined in an obvious way:

$$\int_S f(s) M(ds) = \sum_{j=1}^{n} a_j M(A_j).$$

In order to extend to integral beyond simple functions we need to define a distance, say d_M, such that if $d_M(f_n, f) \to 0$, then $\int f_n(s)M(ds)$ converges in probability to some random variable X. Then we define $\int f(s)M(ds) = X$.

For a random variable X, let $\|X\|_0 := E\big[|X| \wedge 1\big]$. Clearly $\|X_n - X\| \to 0$ if and only if $X_n \xrightarrow{P} X$. Define for a simple function $f : S \to \mathbb{R}$,

$$\|f\|_M := \sup_{\phi\in\Delta} \| \int_S \phi f M(ds)\|_0, \tag{3.14}$$

where

$$\Delta := \{\phi : S \to \mathbb{R} \text{ such that } |\phi| \le 1 \text{ and has finite range}\}. \tag{3.15}$$

Notice that $s \mapsto \phi(s)f(s)$ is a simple function and by our definition, $\|f\|_M$ is well-defined. It is easy to verify that for any simple functions f and g,

(i) $\|f\|_M = 0 \iff f = 0 \;\; m - a.e.$

(ii) $\|f + g\|_M \le \|f\|_M + \|g\|_M$.

(iii) $\|\theta f\|_M \le \|f\|_M$, for any $|\theta| \le 1$.

These are properties of an F-norm on a vector space. Naturally, $d_M(f, g) := \|f - g\|_M$ is a metric on the vector space of simple functions.

Definition 3.2.5. *We say that a function $f : S \to \mathbb{R}$ is M-integrable if there exists a sequence $\{f_n\}$ of simple functions such that*

(a) $f_n \to f \;\; m$*-a.e.*

(b) $\lim_{k,n\to\infty} \|f_n - f_k\|_M = 0$.

If (a)-(b) hold, then we define

$$\int_S f(s)M(ds) = \lim_{n\to\infty} \int_S f_n(s)M(ds), \tag{3.16}$$

where the limit is taken in probability.

We now state a necessary and sufficient condition for the existence of $\int f dM$.

Theorem 3.2.6. *A measurable function $f : S \to \mathbb{R}$ is M-integrable if and only if*

$$\int_S \Phi_M(s, f(s)) m(ds) < \infty, \tag{3.17}$$

where

$$\Phi_M(s,x) = \sigma^2(s)x^2 + \int_{\mathbb{R}} [\![xy]\!]^2 \rho(s,dy) + \left|\eta(s)x + \int_{\mathbb{R}} \big([\![xy]\!] - x[\![y]\!]\big)\rho(s,dy)\right|$$

If f is M-integrable, then the integral $\int f dM$ is well-defined by (3.16)*, i.e., it does not depend on a choice of a sequence $\{f_n\}$. The integral has an infinitely divisible distribution with characteristic function*

$$E\Big[\exp\Big(i\lambda \int_S f dM\Big)\Big] = \exp\left\{\int_S C(s, \lambda f(s)) m(ds)\right\}, \tag{3.18}$$

where the function C is as defined in (3.12)*.*

3.3 Functional Large Deviation Principle

We consider a special class of infinitely divisible processes. Let (Z_n) be an irreducible Harris null-recurrent Markov chain on $(E, \mathcal{E})$, with transition probabilities $P(x, \cdot)$, and $\sigma-$finite invariant measure π. Define a set $S := E^{\mathbb{Z}}$ and let $\mathcal{S}$ be the cylindrical $\sigma-$field on S. Let m be a shift invariant measure on $(S, \mathcal{S})$ defined as

$$m(s : (s_n, \dots, s_{n+k}) \in A_0 \times \cdots \times A_k) = \int_{A_0} \cdots \int_{A_k} \pi(ds_0) P(s_0, ds_1) \cdots P(s_{k-1}, ds_k)$$
$$\text{for all } n \in \mathbb{Z}, k \geq 1, \text{ and } A_0, \cdots, A_k \in \mathcal{E}. \tag{3.19}$$

Suppose that $\mathcal{S}_0 = \{A \in \mathcal{S} : m(A) < \infty\}$ and that $\{M(A) : A \in \mathcal{S}_0\}$ is an infinitely divisible random measure on $(S, \mathcal{S}_0)$ with control measure m and local characteristics $(0, \rho, 0)$. That means for any $A \in \mathcal{S}_0$

$$E\big(\exp(i\lambda M(A))\big) = \exp\left\{\int_A \int_{\mathbb{R}} \left(e^{i\lambda z} - 1 - i\lambda[\![z]\!]\right) \rho(s, dz) m(ds)\right\}. \qquad (3.20)$$

We assume that ρ is a Lévy measure on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ such that

$$\rho(s, \cdot) = \rho(\cdot), \qquad m - a.s.$$

and

$$\int_{|z|>1} e^{\lambda z} \rho(dz) < \infty, \qquad \text{for all } \lambda \in \mathbb{R} \qquad (3.21)$$

Then by (3.3), $E[\exp(\lambda M(A))] < \infty$ for all $\lambda \in \mathbb{R}$ and

$$E\left[\exp(\lambda M(A))\right] = \exp\left\{\int_A \int_{\mathbb{R}} \left(e^{\lambda z} - 1 - \lambda[\![z]\!]\right) \rho(dz) m(ds)\right\} \qquad (3.22)$$

We will use $g : \mathbb{R} \to \mathbb{R}$ to denote the function

$$g(\lambda) = \int_{\mathbb{R}} \left(e^{\lambda z} - 1 - \lambda[\![z]\!]\right) \rho(dz) \qquad (3.23)$$

and that implies,

$$E\left[\exp(\lambda M(A))\right] = \exp\left\{g(\lambda) m(A)\right\}.$$

We will now state certain facts about Markov chains and make certain assumptions on (Z_n):

(i) Theorem 2.1 in Nummelin (1984) states that the Markov kernel $P(x, \cdot)$ satisfies a minorization condition, i.e, there exists a set $C \in \mathcal{E}$, satisfying $0 < \pi(C) < \infty$, a probability measure ν on $(E, \mathcal{E})$ with $\nu(C) > 0$, such that for some $0 < b \leq 1$ and $n_0 \geq 1$,

$$P^{n_0}(x, A) \geq b I_C(x) \nu(A), \qquad \text{for all } x \in E, A \in \mathcal{E}. \qquad (3.24)$$

To simplify arguments we will assume that (3.24) holds with $n_0 = 1$. The results in this chapter hold for general n_0 as well. By the split-chain technique it is possible to embed (Z_n) into a larger probability space on which one can define a sequence of Bernoulli random variables $(\tilde{Z}_n)$, such that $(Z'_n) = (Z_n, \tilde{Z}_n)$ forms a Harris recurrent Markov chain on $E' := E \times \{0, 1\}$ with an atom $a = E \times \{1\}$; see Athreya and Ney (1978), Nummelin (1978) and section 4.4 in Nummelin (1984). By saying that a is an atom, we mean that

$$P'(x, \cdot) = P'(y, \cdot), \qquad \text{for all } x, y \in a, \tag{3.25}$$

where $P'(x, \cdot)$ is the transition kernel of this augmented chain (Z'_n). We also have

$$P'^k(a, a) = b\nu P^{k-1}(C), \qquad k \geq 1,$$

where P'^k is the k-step transition of (Z'_n). Moreover, there is an invariant measure π' of (Z'_n) such that the marginal of π' on E is π and

$$\pi'(a) = b\pi(C), \tag{3.26}$$

(ii) Define

$$T_0 = 0 \text{ and } T_k := \inf\{n > T_{k-1} : Z'_n \in a\},$$

which is the time taken by (Z'_n) to hit a for the k-th time. Sometimes we will also use T to denote T_1. We will assume that (Z_n) is α-regular, that is,

$$\gamma(x) := \left(b\pi(C)P'_a[T > x]\right)^{-1} \in RV_\alpha \tag{3.27}$$

for some $0 < \alpha < 1$. As explained in equation (5.17) in Chen (1999) and the proof of Theorem 2.3 in the same article

$$\frac{1}{\pi'(a)} \sum_{k=1}^{n} P'^k(a, a) = \frac{1}{\pi(C)} \sum_{k=1}^{n} \nu P^{k-1}(C) \sim \gamma(n). \tag{3.28}$$

Because of this without loss of any generality we can assume that (Z_n) has an atom a.

(iii) We assume that there exists a partition $\{E_n, n \geq 0\}$ of E such that the following holds:

(a) $\pi(E_n) < \infty$ for every $n \geq 0$.

(b) There exists a monotone function $\psi \in RV_\beta$ with $\beta > 0$ such that

$$Q_n(\cdot) := P_{\pi_n}[T/\psi(n) \in \cdot] \Longrightarrow Q(\cdot), \tag{3.29}$$

where $\pi_n(\cdot) = \pi(\cdot \cap E_n)/\pi(E_n)$. Define the inverse function of ψ as

$$\psi^{\leftarrow}(n) := \inf\{k \geq 1 : \psi(k) \geq n\} \in RV_{1/\beta}. \tag{3.30}$$

Throughout this chapter we will use V to denote a random variable having distribution $Q(\cdot)$.

(c) There exists $\zeta > -1$ such that

$$\frac{\pi(E_{[rn]})}{\pi(E_n)} \to r^\zeta, \text{ for all } r > 0.$$

(d) There exists $\epsilon' > 0$, $c > 0$, $N > 1$, $k > 0$ and $\epsilon > 0$ such that for every $r \geq 1$

$$\sup_{n \geq Nr} P_{\pi_n}\left[\frac{T}{\psi(n)} \leq cr^{-\beta+\epsilon'}\right] < kr^{-\zeta-1-\epsilon}. \tag{3.31}$$

If $S_k = \{s \in S : s_0 \in E_k\}$ it is easy to verify that $m(S_k) = \pi(E_k) < \infty$. One can also check that the probability measures m_k defined on $(S, \mathcal{S})$ by

$$m_k(A) = \frac{m(A \cap S_k)}{m(S_k)} \text{ for every } k \geq 1$$

is the law of the process (Z_n) given that Z_0 has the law π_k. Finally we define a mean-zero stationary infinitely divisible process

$$X_n = \int_S f(s_n) M(ds), n \in \mathbb{Z} \tag{3.32}$$

where $f : E \to \mathbb{R}$ is a measurable function satisfying:

(a) $f \in L_1(E, \mathcal{E}, \pi)$ with

$$c_f := \int_E f(x)\pi(dx) \neq 0. \tag{3.33}$$

(b) f satisfies the conditions of Theorem 3.2.6.

(c) For every $\lambda > 0$ there exists $k > 0$, $N > 1$ and $\epsilon > 0$ such that for every $r \geq 1$

$$\sup_{n \geq Nr} E_{\pi_n}\left[g\Big(\lambda \sum_{i=1}^{T \wedge \psi(n/r)} \big|f(Z_i)\big|\Big)\right] \leq kr^{-\zeta-1-\epsilon}. \tag{3.34}$$

(d) There exists $1 \leq \delta < (1-\alpha)^{-1}$ such that for every $\lambda \in \mathbb{R}$,

$$\sup_{x \in E} E_x\Big[\exp\Big(\lambda \sum_{i=1}^{T} \big|f(Z_i)\big|\Big)^{\delta}\Big] < \infty \tag{3.35}$$

(e) The function g defined in (3.23) satisfies the integrability condition

$$\int_0^\infty \exp\left(-k_0 \bar{g}(t)^\delta\right) dt < \infty, \tag{3.36}$$

for some $k_0 > 0$, where $\bar{g}(t) = \min\{|s| : g(s) = t\}$.

We study the step process $\{Y_n\}$

$$Y_n(t) = \frac{1}{a_n} \sum_{i=1}^{[nt]} X_i, t \in [0,1], \tag{3.37}$$

and its polygonal path counterpart

$$\tilde{Y}_n(t) = \frac{1}{a_n} \sum_{i=1}^{[nt]} X_i + \frac{1}{a_n}(nt - [nt])X_{[nt]+1}, t \in [0,1], \tag{3.38}$$

where

$$a_n := \pi(E_{[\psi^{\leftarrow}(n)]})\gamma(n)\psi^{\leftarrow}(n) \in RV_{(\zeta+1+\alpha\beta)/\beta}. \tag{3.39}$$

Let μ_n be the law of Y_n and $\tilde{\mu}_n$ be the law of $\tilde{Y}_n$, in some appropriate function space equipped with the cylindrical $\sigma-$field. We use $\mathcal{BV}$ to denote the space of all real valued functions of bounded variation defined on the unit interval $[0,1]$. To ensure that the space $\mathcal{BV}$ is a measurable set in the cylindrical σ-field of all real-valued functions on $[0,1]$, we use only rational partitions of $[0,1]$ when defining variation. $\mathcal{D}$ will denote the space of all function on $[0,1]$ which are right continuous with left limits. We will use subscripts to denote the topology on the space. Specifically, the subscripts S, Sk and P will denote the sup-norm topology, the Skorohod topology and the topology of pointwise convergence.

Theorem 3.3.1. *Under the assumptions stated above, $\{\mu_n\}$ satisfies large deviation principle in $\mathcal{BV}_S$ with speed $b_n = \pi(E_{[\psi^{\leftarrow}(n)]})\psi^{\leftarrow}(n)$ and good rate function*

$$H(\xi) = \begin{cases} \Lambda^*_{\alpha,\beta}(\xi') & \text{if } \xi \in \mathcal{AC}, \xi(0) = 0 \\ \infty & \text{otherwise.} \end{cases} \tag{3.40}$$

where for any $\varphi \in L_1[0,1]$

$$\Lambda^*_{\alpha,\beta}(\varphi) = \sup_{\psi \in L_\infty[0,1]} \left\{ \int_0^1 \psi(t)\varphi(t)dt - \int_0^\infty r^\zeta E\Big[I_{[r^\beta V \leq 1]} g\Big(c_f \int_0^{1-r^\beta V} \psi(t)U(dt)\Big)\Big] dr \right\}. \tag{3.41}$$

Here $U(t) := \inf\{x : S_\alpha(x) \geq t\}, 0 \leq t \leq 1,$ is the inverse time $\alpha-$stable subordinator with

$$E\left\{\exp\left(-\lambda S_\alpha(1)\right)\right\} = \exp\left\{-\frac{\lambda^\alpha}{\Gamma(1+\alpha)}\right\}, \forall \lambda \geq 0,$$

and V is independent of $\{U(t) : 0 \leq t \leq 1\}$ having distribution $Q(\cdot)$.

Proof. Let $\mathcal{X}$ be the set of all $\mathbb{R}^d$-valued functions defined on the unit interval $[0,1]$ and let $\mathcal{X}^o$ be the subset of $\mathcal{X}$, of functions which start at the origin. Define $\mathcal{J}$ as the collection of all ordered finite subsets of $(0,1]$ with a partial order defined by inclusion. For any $j = \{0 < t_1 < \ldots < t_{|j|} \leq 1\}$ define the projection $p_j : \mathcal{X}^o \to \mathcal{Y}_j$ as $p_j(\xi) = (\xi(t_1), \ldots, \xi(t_{|j|}))$, $\xi \in \mathcal{X}^o$. So $\mathcal{Y}_j$ can be identified with the space $\mathbb{R}^{|j|}$ and the projective limit of $\mathcal{Y}_j$ over $j \in \mathcal{J}$ can be identified with $\mathcal{X}^o_P$, that is, $\mathcal{X}^o$

equipped with the topology of pointwise convergence. Note that $\mu_n \circ p_j^{-1}$ is the law of

$$Y_n^j = (Y_n(t_1), \ldots, Y_n(t_{|j|})).$$

Define the vector V_n as

$$V_n := \big(Y_n(t_1), Y_n(t_2) - Y_n(t_1), \cdots, Y_n(t_{|j|}) - Y_n(t_{|j|-1})\big) \tag{3.42}$$

and observe that for any $\underline{\lambda} = (\lambda_1, \ldots, \lambda_{|j|}) \in \mathbb{R}^{|j|}$

$$\begin{aligned}
&\log E\big(\exp\big[b_n\underline{\lambda}\cdot V_n\big]\big)\\
=\ &\log E\exp\Big[\frac{b_n}{a_n}\sum_{i=1}^{|j|}\lambda_i\Big(\sum_{k=[nt_{i-1}]+1}^{[nt_i]}X_k\Big)\Big]\\
=\ &\int_S\int_{\mathbb{R}}\left(\exp\Big(\frac{b_n}{a_n}z\sum_{i=1}^{|j|}\lambda_i\sum_{k=[nt_{i-1}]+1}^{[nt_i]}f(s_k)\Big)-1\right.\\
&\qquad\left.-\frac{b_n}{a_n}[\![z]\!]\sum_{i=1}^{|j|}\lambda_i\sum_{k=[nt_{i-1}]+1}^{[nt_i]}f(s_k)\right)\rho(dz)m(ds)\\
=\ &\int_S g\left(\frac{b_n}{a_n}\sum_{i=1}^{|j|}\lambda_i\sum_{k=[nt_{i-1}]+1}^{[nt_i]}f(s_k)\right)m(ds)
\end{aligned}$$

where $t_0 = 0$. By Lemma 3.3.3

$$\lim_{n\to\infty}\frac{1}{b_n}\log E\big(\exp\big[b_n\underline{\lambda}\cdot V_n\big]\big) = \Lambda_j(\underline{\lambda})$$

with

$$\Lambda_j(\underline{\lambda}) := \int_0^\infty r^\zeta E\Big[I_{[r^\beta V\le 1]}g\Big(c_f\sum_{i=1}^{|j|}\lambda_i U(t_i - r^\beta V) - U(t_{i-1} - r^\beta V)\Big)\Big]dr$$

where V and $\{U(t) : 0 \le t \le 1\}$ are as described in the statement of the theorem. We understand $U(t) = 0$ for all $t < 0$. By the Gartner-Ellis Theorem the laws of (V_n) satisfy LDP with speed b_n and good rate function

$$\Lambda_j^*(w_1, \ldots, w_{|j|}) = \sup_{\underline{\lambda}}\Big\{\sum_{i=1}^{|j|}w_i\lambda_i - \Lambda_j(\underline{\lambda})\Big\}, \tag{3.43}$$

where $(w_1, \ldots, w_{|j|}) \in \mathbb{R}^{|j|}$. The map $V_n \mapsto Y_n^j$ from $\mathbb{R}^{|j|}$ onto itself is one to one and continuous. Hence the contraction principle tells us that $\{\mu_n \circ p_j^{-1}\}$ satisfy LDP in $\mathbb{R}^{|j|}$ with good rate function

$$H_j(y_1, \ldots, y_{|j|}) := \Lambda_j^*(y_1, y_2 - y_1, \ldots, y_{|j|} - y_{|j|-1}). \tag{3.44}$$

By Lemma 3.3.2, the same holds for the measures $\{\tilde{\mu}_n \circ p_j^{-1}\}$. By the Dawson-Gartner Theorem (Theorem 4.6.1 in Dembo and Zeitouni (1998)) this implies that the measures $\{\tilde{\mu}_n\}$ satisfy LDP in the space $\mathcal{X}^o$ equipped with the topology of pointwise convergence, with speed b_n and the rate function

$$\sup_{j \in \mathcal{J}} H_j(p_j(\cdot))$$

which by Lemma 3.3.5 is same as the function $H(\cdot)$ described in (3.40). As $\mathcal{X}^o$ is a closed subset of $\mathcal{X}$, the same holds for $\{\tilde{\mu}_n\}$ in $\mathcal{X}$ and the rate function is infinite outside $\mathcal{X}^o$. Since $\tilde{\mu}_n(\mathcal{BV}) = 1$ for all $n \geq 1$ and $H(\cdot)$ is infinite outside of $\mathcal{BV}$, we conclude that $\{\tilde{\mu}_n\}$ satisfy LDP in $\mathcal{BV}_P$ with the same rate function. The sup-norm topology on $\mathcal{BV}$ is stronger than that of pointwise convergence and by Lemma 3.3.4, $\{\tilde{\mu}_n\}$ is exponentially tight in $\mathcal{BV}_S$. So by corollary 4.2.6 in Dembo and Zeitouni (1998), $\{\tilde{\mu}_n\}$ satisfy LDP in $\mathcal{BV}_S$ with speed b_n and good rate function $H(\cdot)$. Finally, applying Lemma 3.3.2 once again, we conclude that the same is true for the sequence $\{\mu_n\}$. □

Lemma 3.3.2. *The families $\{\mu_n\}$ and $\{\tilde{\mu}_n\}$ are exponentially equivalent in $\mathcal{D}_S$, where $\mathcal{D}$ is the space of all right-continuous functions with left limits and, as before, the subscript denotes the sup-norm topology on that space.*

Proof. Observe that for every $n \geq 1$

$$||Y_n - \tilde{Y}_n|| \leq \frac{1}{a_n} \max_{1 \leq i \leq n} |X_i|.$$

Therefore for any $\delta > 0$ and $\lambda > 0$

$$\begin{aligned}
& \limsup_{n\to\infty} \frac{1}{b_n} \log P\Big(||Y_n - \tilde{Y}_n|| > \delta\Big) \\
\leq \ & \limsup_{n\to\infty} \frac{1}{b_n} \log P\Big(\frac{1}{a_n} \max_{1\le i\le n} |X_i| > \delta\Big) \\
\leq \ & \limsup_{n\to\infty} \frac{1}{b_n} \log \Big(n\, P(|X_1| > a_n\delta)\Big) \\
\leq \ & \limsup_{n\to\infty} \frac{1}{b_n} \Big(\log n - a_n\lambda\delta + \log E\big(e^{\lambda X_1}\big) + \log E\big(e^{-\lambda X_1}\big)\Big) \\
= \ & \limsup_{n\to\infty} \frac{1}{b_n} \Big(- a_n\lambda\delta\Big) .
\end{aligned}$$

By the definition of a_n and b_n we have $a_n/b_n \to \infty$, so the above limit is equal to $-\infty$ and that completes the proof. □

Lemma 3.3.3. *For any $j = \{0 = t_0 < t_1 < \cdots < t_{|j|}\} \in \mathcal{J}$ and $(\lambda_1, \ldots, \lambda_{|j|}) \in \mathbb{R}^{|j|}$,*

$$\begin{aligned}
& \lim_{n\to\infty} \frac{1}{b_n} \int_S g\left(\frac{b_n}{a_n} \sum_{i=1}^{|j|} \lambda_i \sum_{k=[nt_{i-1}]+1}^{[nt_i]} f(s_k) \right) m(ds) \\
= \ & \int_0^\infty r^\zeta E\left[I_{[r^\beta V \le 1]} g\left(c_f \sum_{i=1}^{|j|} \lambda_i \Big(U\big(t_i - r^\beta V\big) - U\big(t_{i-1} - r^\beta V\big)\Big)\right)\right] dr < \infty,
\end{aligned}$$

where $U(t) = \inf\{x : S_\alpha(x) \ge t\}, 0 \le t \le 1,$ (we understand $U(t) = 0, \forall t < 0$) is the inverse time $\alpha-$stable subordinator with

$$E\left\{\exp\left(-\lambda S_\alpha(1)\right)\right\} = \exp\left\{ -\frac{\lambda^\alpha}{\Gamma(1+\alpha)} \right\}, \forall \lambda \ge 0,$$

and V is independent of $\{U(t) : 0 \le t \le 1\}$ with distribution $Q(\cdot)$.

Proof. We begin by observing that

$$\begin{aligned}
& \int_S g\Big(\frac{b_n}{a_n} \sum_{i=1}^{|j|} \lambda_i \sum_{k=[nt_{i-1}]+1}^{[nt_i]} f(s_k)\Big) m(ds) \\
= \ & \sum_{l=0}^{\infty} \int_{S_l} g\Big(\frac{b_n}{a_n} \sum_{i=1}^{|j|} \lambda_i \sum_{k=[nt_{i-1}]+1}^{[nt_i]} f(s_k)\Big) m(ds)
\end{aligned}$$

and since $m_l(\cdot) = m(\cdot \cap S_l)/m(S_l)$ we get

$$\begin{aligned}
&\int_S g\Big(\frac{b_n}{a_n}\sum_{i=1}^{|j|}\lambda_i \sum_{k=[nt_{i-1}]+1}^{[nt_i]} f(s_k)\Big) m(ds) \\
= &\sum_{l=0}^{\infty} m(S_l)\int_{S_l} g\Big(\frac{1}{\gamma(n)}\sum_{i=1}^{|j|}\lambda_i \sum_{k=[nt_{i-1}]+1}^{[nt_i]} f(s_k)\Big) m_l(ds).
\end{aligned}$$

Now note that for any $r > 0$

$$\begin{aligned}
&\lim_{n\to\infty}\int_{S_{[r\psi^{\leftarrow}(n)]}} g\Big(\frac{1}{\gamma(n)}\sum_{i=1}^{|j|}\lambda_i \sum_{k=[nt_{i-1}]+1}^{[nt_i]} f(s_k)\Big) m_{[r\psi^{\leftarrow}(n)]}(ds) \\
= &\lim_{n\to\infty} E_{\pi_{[r\psi^{\leftarrow}(n)]}}\Big[g\Big(\frac{1}{\gamma(n)}\sum_{i=1}^{|j|}\lambda_i \sum_{k=[nt_{i-1}]+1}^{[nt_i]} f(Z_k)\Big)\Big] \\
= &\lim_{n\to\infty} E_{\pi_{[r\psi^{\leftarrow}(n)]}}\Big[g\Big(\frac{1}{\gamma(n)}\sum_{i=1}^{|j|}\lambda_i \sum_{k=[nt_{i-1}]\wedge T+1}^{[nt_i]\wedge T} f(Z_k) \\
&\quad + \frac{1}{\gamma(n)}\sum_{i=1}^{|j|}\lambda_i \sum_{k=[nt_{i-1}]\vee T+1}^{[nt_i]\vee T} f(Z_k)\Big)\Big]
\end{aligned}$$

where for any sequence (x_n) we understand $\sum_{k=i}^{j} x_k = 0$ if $j < i$. From assumption (3.34) it is easy to see that

$$\frac{1}{\gamma(n)}\sum_{i=1}^{|j|}\lambda_i \sum_{k=[nt_{i-1}]\wedge T+1}^{[nt_i]\wedge T} f(Z_k) \xrightarrow{P_x} 0 \qquad \text{for all } x \in E. \tag{3.45}$$

Next we concentrate on the second component. Define the function $\Psi : \mathcal{D} \to \mathcal{D}$ as

$$\Psi(h)(t) := \sum_{i=1}^{|j|}\lambda_i\Big(h\big((t_i - t)\vee 0\big) - h\big((t_{i-1} - t)\vee 0\big)\Big), \qquad \text{for all } h \in \mathcal{D}, t \in [0,1].$$

Note that

$$\frac{1}{\gamma(n)}\sum_{i=1}^{|j|}\lambda_i \sum_{k=[nt_{i-1}]+1}^{[nt_i]} f(Z_k) = \Psi(L_n)(0),$$

where

$$L_n(t) := \frac{1}{\gamma(n)}\sum_{k=1}^{[nt]} f(Z_k), \qquad \text{for all } t \in [0,1].$$

Since a is an atom T is independent of $\sigma(Z_n : n \geq T)$ and therefore for any measurable set $A \subset \mathbb{R}$

$$\begin{aligned} & P_{\pi_{[r\psi^{\leftarrow}(n)]}}\left[\frac{1}{\gamma(n)}\sum_{i=1}^{|j|}\lambda_i \sum_{k=[nt_{i-1}]\vee T+1}^{[nt_i]\vee T} f(Z_k) \in A\right] \\ = \; & P\left[\Psi(L_n)\Big((T^n/n)\wedge 1\Big) \in A \Big| Z_0 \in a\right] \end{aligned}$$

where T^n is a random variable independent of $(Z_n, n \geq 0)$ such that

$$P[T^n \in \cdot] = P_{\pi_{[r\psi^{\leftarrow}(n)]}}\big[T \in \cdot\big].$$

Furthermore, if h is continuous and $h_n \to h$ in $\mathcal{D}_{Sk}$ then $\Psi(h_n) \to \Psi(h)$ in $\mathcal{D}_{Sk}$. This is easy to verify. If h is continuous and $h_n \to h$ in $\mathcal{D}_{Sk}$ then $\big\|h_n - h\big\| \to 0$ and hence $\big\|\Psi(h_n) - \Psi(h)\big\| \to 0$. By Lemma 3.3.6 we know that $L_n \Longrightarrow c_f U$ in $\mathcal{D}_{Sk}$, where $\{U(t), 0 \leq t \leq 1\}$ is as in the statement of this lemma. Since the $\{U(t), 0 \leq t \leq 1\}$ is almost surely continuous we can apply the continuous mapping theorem (see Theorem 2.7 in Billingsley (1999)) to get

$$\Psi(L_n) \Longrightarrow \Psi(c_f U) \qquad \text{in } \mathcal{D}_{Sk}. \tag{3.46}$$

Let c_n be defined as $c_n := \psi(r\psi^{\leftarrow}(n))$. Since $\psi \in RV_\beta$ it follows immediately that

$$\frac{c_n}{n} \longrightarrow r^\beta \qquad \text{as } n \to \infty.$$

By assumption (3.29) we get

$$\frac{T^n}{n} = \frac{T^n}{c_n}\frac{c_n}{n} \Longrightarrow r^\beta V. \tag{3.47}$$

Furthermore, since T^n is independent of $\{Z_n, n \geq 0\}$ we get

$$\Big((T^n/n)\wedge 1, \Psi(L_n)\Big) \Longrightarrow \Big(r^\beta V \wedge 1, \Psi(c_f U)\Big) \qquad \text{in } [0,1]\times\mathcal{D}.$$

Also, the map $\tilde{\psi} : [0,1]\times\mathcal{D} \to \mathbb{R}$ defined by $\tilde{\psi}(t,h) = h(t)$ is continuous at (t,h) if h is continuous. Hence another application of the continuous mapping theorem gives us

$$\Psi\big(L_n\big)\Big((T^n/n)\wedge 1\Big) \Longrightarrow \Psi\big(c_f U\big)\Big(r^\beta V \wedge 1\Big)$$

This combined with (3.45) and Lemma 3.3.7 gives us

$$\begin{aligned}
&\lim_{n\to\infty} \int_{S_{[r\psi^{\leftarrow}(n)]}} g\Big(\frac{1}{\gamma(n)} \sum_{i=1}^{|j|} \lambda_i \sum_{k=[nt_{i-1}]+1}^{[nt_i]} f(s_k)\Big) m_{[r\psi^{\leftarrow}(n)]}(ds) \\
= \;& E\left[g\left(\Psi\big(c_f U\big)\Big(r^\beta V \wedge 1\Big)\right)\right] \\
= \;& E\left[I_{[r^\beta V \le 1]} g\left(c_f \sum_{i=1}^{|j|} \lambda_i \Big(U\big(t_i - r^\beta V\big) - U\big(t_{i-1} - r^\beta V\big)\Big)\right)\right].
\end{aligned}$$

Now for the final step.

$$\begin{aligned}
& \frac{1}{b_n} \int_S g\left(\frac{b_n}{a_n} \sum_{i=1}^{|j|} \lambda_i \sum_{k=[nt_{i-1}]+1}^{[nt_i]} f(s_k)\right) m(ds) \\
= \;& \frac{1}{b_n} \sum_{l=0}^{\infty} m(S_l) \int_{S_l} g\Big(\frac{1}{\gamma(n)} \sum_{i=1}^{|j|} \lambda_i \sum_{k=[nt_{i-1}]+1}^{[nt_i]} f(s_k)\Big) m_l(ds) \\
= \;& \frac{1}{\psi^{\leftarrow}(n)} \sum_{l=0}^{\infty} \frac{m(S_l)}{m(S_{[\psi^{\leftarrow}(n)]})} \int_{S_l} g\Big(\frac{1}{\gamma(n)} \sum_{i=1}^{|j|} \lambda_i \sum_{k=[nt_{i-1}]+1}^{[nt_i]} f(s_k)\Big) m_l(ds) \\
= \;& \int_0^{\infty} h_n(r)dr,
\end{aligned}$$

where for every $r > 0$

$$\begin{aligned}
h_n(r) \;&=\; \frac{m\big(S_{[r\psi^{\leftarrow}(n)]}\big)}{m\big(S_{[\psi^{\leftarrow}(n)]}\big)} \int\limits_{S_{[r\psi^{\leftarrow}(n)]}} g\Big(\frac{1}{\gamma(n)} \sum_{i=1}^{|j|} \lambda_i \sum_{k=[nt_{i-1}]+1}^{[nt_i]} f(s_k)\Big) m_l(ds) \\
&=\; \frac{m\big(S_{[r\psi^{\leftarrow}(n)]}\big)}{m\big(S_{[\psi^{\leftarrow}(n)]}\big)} E_{\pi_{[r\psi^{\leftarrow}(n)]}} \left[g\Big(\frac{1}{\gamma(n)} \sum_{i=1}^{|j|} \lambda_i \sum_{k=[nt_{i-1}]+1}^{[nt_i]} f(s_k)\Big)\right].
\end{aligned}$$

We already know that for any $r > 0$

$$\lim_{n\to\infty} h_n(r) = r^\zeta E\left[I_{[r^\beta V \le 1]} g\left(c_f \sum_{i=1}^{|j|} \lambda_i \Big(U\big(t_i - r^\beta V\big) - U\big(t_{i-1} - r^\beta V\big)\Big)\right)\right]$$

which means that we will get the required result if we are able to prove that the functions h_n are dominated by an integrable function. For that purpose note that it suffices to consider $j = \{1\}$ since

$$\begin{aligned} & g\Big(\frac{1}{\gamma(n)}\sum_{i=1}^{|j|}\lambda_i \sum_{k=[nt_{i-1}]+1}^{[nt_i]} f(Z_k)\Big) \\ \le \quad & \max\left\{ g\Big(\frac{\bar\lambda}{\gamma(n)}\sum_{k=1}^{n}\big|f(Z_k)\big|\Big), g\Big(\frac{-\bar\lambda}{\gamma(n)}\sum_{k=1}^{n}\big|f(Z_k)\big|\Big)\right\}, \end{aligned}$$

where $\bar\lambda = \max_{1\le k\le |j|}\big|\lambda_k\big|$. By Lemma 3.3.7 we know that there exists $K > 0$ such that

$$\sup_{x\in E, n\ge 1} E_x\left[g\Big(\frac{2\lambda}{\gamma(n)}\sum_{k=1}^{n}\big|f(Z_k)\big|\Big)\right] \le K. \tag{3.48}$$

Using (3.31) and (3.34) we can get constants $\epsilon' > 0$, $c > 0$, $N > 1$, $k_0 > 0$ and $0 < \epsilon < \zeta + 1$ such that for every $r \ge 1$

$$\sup_{n\ge Nr} P_{\pi_n}\left[\frac{T}{\psi(n)} \le c r^{-\beta+\epsilon'}\right] \le k_0 r^{-\zeta-1-\epsilon} \tag{3.49}$$

and

$$\sup_{n\ge Nr} E_{\pi_n}\left[g\Big(\lambda \sum_{i=1}^{T\wedge\psi(n/r)} \big|f(Z_i)\big|\Big)\right] \le k_0 r^{-\zeta-1-\epsilon}. \tag{3.50}$$

Furthermore, from Potter bounds it is possible to get $N_1 > 0$ and $k_1 > 0$ such that

$$\sup_{n\ge N_1} \frac{m(S_{[r\psi^{\leftarrow}(n)]})}{m(S_{[\psi^{\leftarrow}(n)]})} \le \begin{cases} k_1 r^{(\zeta-\epsilon)\wedge 0} & \text{if } r\in(0,1) \\ k_1 r^{\zeta+\epsilon/2} & \text{if } r \ge 1. \end{cases} \tag{3.51}$$

Combining (3.48) and (3.51) we get that $n \ge N_1$ implies

$$h_n(r) \le K k_1 r^{(\zeta-\epsilon)\wedge 0} \qquad \text{for all } r\in(0,1).$$

For $r \ge 1$ we use the convexity of g to get for $\lambda \in \mathbb{R}$

$$\begin{aligned} & E_{\pi_{[r\psi^{\leftarrow}(n)]}}\left[g\Big(\frac{\lambda}{\gamma(n)}\sum_{k=1}^{n}\big|f(Z_k)\big|\Big)\right] \\ \le \quad & \frac{1}{2}E_{\pi_{[r\psi^{\leftarrow}(n)]}}\left[g\Big(\frac{2\lambda}{\gamma(n)}\sum_{k=1}^{T\wedge n}\big|f(Z_k)\big|\Big)\right] + \frac{1}{2}E_{\pi_{[r\psi^{\leftarrow}(n)]}}\left[g\Big(\frac{2\lambda}{\gamma(n)}\sum_{k=T\wedge n+1}^{n}\big|f(Z_k)\big|\Big)\right] \end{aligned}$$

Then using (3.50) get $N_2 > 0$ such that

$$\sup_{n\ge N_2} E_{\pi_{[r\psi^{\leftarrow}(n)]}}\left[g\Big(\frac{2\lambda}{\gamma(n)}\sum_{i=1}^{T\wedge n}\big|f(Z_i)\big|\Big)\right] \le k_0 r^{-\zeta-1-\epsilon}.$$

For the second component observe that

$$\begin{aligned}
& E_{\pi_{[r\psi^{\leftarrow}(n)]}}\left[g\Big(\frac{2\lambda}{\gamma(n)}\sum_{k=T\wedge n+1}^{n}\big|f(Z_k)\big|\Big)\right] \\
= \; & E_{\pi_{[r\psi^{\leftarrow}(n)]}}\left[I_{[T\le n]}g\Big(\frac{2\lambda}{\gamma(n)}\sum_{k=T+1}^{n}\big|f(Z_k)\big|\Big)\right] \\
\le \; & E_{\pi_{[r\psi^{\leftarrow}(n)]}}\left[I_{[T\le n]}g\Big(\frac{2\lambda}{\gamma(n)}\sum_{k=T+1}^{T+n}\big|f(Z_k)\big|\Big)\right] \\
= \; & P_{\pi_{[r\psi^{\leftarrow}(n)]}}\left[T\le n\right] E_a\left[g\Big(\frac{2\lambda}{\gamma(n)}\sum_{k=1}^{n}\big|f(Z_k)\big|\Big)\right]
\end{aligned}$$

By another application of Potter bounds we can get $N_3 > 0$ such that

$$\sup_{n\ge N_3}\frac{n}{\psi\big(r\psi^{\leftarrow}(n)\big)} \le cr^{-\beta+\epsilon'}$$

and this combined with (3.49) gives us that there exists $N_4 > 0$ such that

$$\begin{aligned}
\sup_{n\ge N_4} P_{\pi_{[r\psi^{\leftarrow}(n)]}}\big[T\le n\big] \; &= \; \sup_{n\ge N_4} P_{\pi_{[r\psi^{\leftarrow}(n)]}}\left[\frac{T}{\psi\big(r\psi^{\leftarrow}(n)\big)} \le \frac{n}{\psi\big(r\psi^{\leftarrow}(n)\big)}\right] \\
&\le \; k_0 r^{-\zeta-1-\epsilon}.
\end{aligned}$$

Therefore, we get that $h_n(r) \le h(r)$ for all $n \ge \max_{1\le i\le 4} N_i$ and $r > 0$, where

$$h(r) = \begin{cases} Kk_1 r^{(\zeta-\epsilon)\wedge 0} & \text{if } r\in(0,1) \\ k_1k_0r^{-1-\epsilon/2} + Kk_0k_1r^{-1-\epsilon/2} & \text{if } r\ge 1 \end{cases}$$

Observe that h is integrable because $\zeta - \epsilon > -1$. Finally, we apply the dominated convergence theorem to get,

$$\begin{aligned}
& \lim_{n\to\infty}\frac{1}{b_n}\int_S g\left(\frac{b_n}{a_n}\sum_{i=1}^{|j|}\lambda_i\sum_{k=[nt_{i-1}]+1}^{[nt_i]} f(s_k)\right) m(ds) \\
= \; & \int_0^\infty r^{\zeta} E\left[I_{[r^\beta V\le 1]} g\left(c_f\sum_{i=1}^{|j|}\lambda_i\Big(U\big(t_i - r^\beta V\big) - U\big(t_{i-1}-r^\beta V\big)\Big)\right)\right] dr
\end{aligned}$$

and that completes the proof of the lemma. □

Lemma 3.3.4. *The family $\{\tilde{\mu}_n\}$ is exponentially tight in $\mathcal{D}_S$, i.e, for every $\pi > 0$ there exists a compact $K_\pi \subset \mathcal{D}_S$, such that*

$$\lim_{\pi\to\infty} \limsup_{n\to\infty} \frac{1}{b_n} \log \tilde{\mu}_n(K_\pi^c) = -\infty.$$

Proof. We use the notation $w(h,u) := \sup_{s,t\in[0,1],|s-t|<u} |h(s)-h(t)|$ for the modulus of continuity of a function $h : [0,1] \to \mathbb{R}^d$. First we claim that for any $\epsilon > 0$,

$$\lim_{u\to 0} \limsup_{n\to\infty} \frac{1}{b_n} \log P\big(w(\tilde{Y}_n, u) > \epsilon\big) = -\infty, \tag{3.52}$$

where $\tilde{Y}_n$ is the polygonal process in (3.38). Let us prove the lemma assuming that the claim is true. By (3.52) and the continuity of the paths of $\tilde{Y}_n$, there is $u_k > 0$ such that for all $n \geq 1$

$$P\big(w(\tilde{Y}_n, u_k) \geq k^{-1}\big) \leq e^{-\pi b_n k},$$

and set $A_k = \{\xi \in \mathcal{D} : w(\xi, u_k) < k^{-1}, \xi(0) = 0\}$. Now the set $K_\pi := \overline{\cap_{k\geq 1} A_k}$ is compact in $\mathcal{D}_S$ and by the union of events bound it follows that

$$\limsup_{n\to\infty} \frac{1}{b_n} \log P(\tilde{Y}_n \notin K_\pi) \leq -\pi,$$

establishing the exponential tightness. Next we prove the claim (3.52). Observe that for any $\epsilon > 0$, $u > 0$ small, $\lambda > 0$ and $n > 2/u$

$$\begin{aligned}
P\big(w(\tilde{Y}_n, u) > \epsilon\big) &\leq P\Big(\max_{0\leq i<j\leq n, j-i\leq[nu]+2} \frac{1}{a_n}\Big|\sum_{k=i}^{j} X_k\Big| > \epsilon\Big) \\
&\leq n\sum_{i=1}^{[2nu]} P\Big(\frac{b_n}{a_n}\Big|\sum_{k=1}^{i} X_k\Big| > b_n\epsilon\Big) \\
&\leq ne^{-b_n\lambda\epsilon}\sum_{i=1}^{[2nu]} E\Big[\exp\Big(\frac{\lambda b_n}{a_n}\sum_{k=1}^{i} X_k\Big) + \exp\Big(-\frac{\lambda b_n}{a_n}\sum_{k=1}^{i} X_k\Big)\Big] \\
&= ne^{-b_n\lambda\epsilon}\sum_{i=1}^{[2nu]} \exp\Big\{\int_S g\Big(\frac{\lambda b_n}{a_n}\sum_{k=1}^{i} f(s_i)\Big)m(ds)\Big\} \\
&\quad + ne^{-b_n\lambda\epsilon}\sum_{i=1}^{[2nu]} \exp\Big\{\int_S g\Big(-\frac{\lambda b_n}{a_n}\sum_{k=1}^{i} f(s_i)\Big)m(ds)\Big\}\Big).
\end{aligned}$$

Now using the convexity of g we get

$$\begin{aligned} & P\Big(w(\tilde{Y}_n, u) > \epsilon\Big) \\ \leq \; & \frac{4n^2 u}{e^{b_n \lambda \epsilon}} \exp\Big\{ \int_S g\Big(\frac{\lambda b_n}{a_n} \sum_{k=1}^{[2nu]} |f(s_i)|\Big) m(ds)\Big\} \\ + \; & \frac{4n^2 u}{e^{b_n \lambda \epsilon}} \exp\Big\{ \int_S g\Big(-\frac{\lambda b_n}{a_n} \sum_{k=1}^{[2nu]} |f(s_i)|\Big) m(ds)\Big\}. \end{aligned}$$

Therefore by Lemma 3.3.3 we have

$$\lim_{u\to 0} \limsup_{n\to\infty} \frac{1}{b_n} \log P\Big(w(\tilde{Y}_n, u) > \epsilon\Big) \leq -\lambda\epsilon.$$

Now, letting $\lambda \to \infty$ we obtain (3.52). □

Lemma 3.3.5. *Suppose Λ_j^* is as defined in* (3.43)*. Then for any $j = \{0 = t_0 < t_1 < \cdots < t_1 \leq 1\} \in \mathcal{J}$ and any function ξ of bounded variation on $[0,1]$ satisfying $\xi(0) = 0$,*

$$\sup_{j\in\mathcal{J}} \Lambda_j^*\Big(\xi(t_1), \xi(t_2) - f(t_1), \ldots, \xi(t_{|j|}) - \xi(t_{|j|-1})\Big)$$

$$= \begin{cases} \Lambda_{\alpha,\beta}^*(\xi') & \text{if } \xi \in \mathcal{AC} \\ \infty & \text{otherwise.} \end{cases}$$

where $\Lambda_{\alpha,\beta}^*(\cdot)$ is as defined in (3.41)*.*

Proof. First assume that $\xi \in \mathcal{AC}$. It is easy to see that the inequality $\Lambda_{\alpha,\beta}^*(\xi') \geq \sup_{j\in\mathcal{J}} \Lambda_j^*(\xi(t_1), \xi(t_2) - \xi(t_1), \ldots, \xi(t_{|j|}) - \xi(t_{|j|-1}))$ holds by considering a function $\psi \in L_\infty[0,1]$, which takes the value λ_i in the interval $(t_{i-1}, t_i]$. For the other inequality, take any $\psi \in L_\infty[0,1]$ and choose a sequence of uniformly bounded functions ψ^n converging to ψ almost everywhere on $[0,1]$, such that for every n, ψ^n is of the form $\sum_i \lambda_i^n I_{A_i^n}$, where $A_i^n = (t_{i-1}^n, t_i^n]$, for some

$$j^n = \Big\{0 = t_0 < t_1^n < t_2^n < \cdots < t_{k_n}^n = 1\Big\}.$$

Then by the continuity of Λ over $\mathcal{F}_\Lambda^\circ$ and Fatou's Lemma,

$$\begin{aligned}
&\int_0^1 \psi(t)\xi'(t)dt - \int_0^\infty r^\zeta E\Big[I_{[r^\beta V\le 1]} g\Big(c_f \int_0^{1-r^\beta V} \psi(t)U(dt)\Big)\Big]dr \\
=\ & \int_0^1 \lim_n \psi^n(t)\xi'(t)dt - \int_0^\infty r^\zeta E\Big[I_{[r^\beta V\le 1]} g\Big(c_f \int_0^{1-r^\beta V} \lim_n \psi^n(t)U(dt)\Big)\Big]dr \\
=\ & \lim_n \int_0^1 \psi^n(t)\xi'(t)dt - \int_0^\infty r^\zeta E\Big[I_{[r^\beta V\le 1]} \lim_n g\Big(c_f \int_0^{1-r^\beta V} \psi^n(t)U(dt)\Big)\Big]dr \\
\le\ & \lim_n \int_0^1 \psi^n(t)\xi'(t)dt - \limsup_n \int_0^\infty r^\zeta E\Big[I_{[r^\beta V\le 1]} g\Big(c_f \int_0^{1-r^\beta V} \psi^n(t)U(dt)\Big)\Big]dr \\
=\ & \liminf_n \left\{ \sum_{i=1}^{k_n} \lambda_i^n \cdot \big(\xi(t_i^n) - \xi(t_{i-1}^n)\big) - \Lambda_{j^n}(\lambda_1^n, \cdots, \lambda_n^n) \right\} \\
\le\ & \sup_{j\in\mathcal{J}} \Lambda_j^*\big(\xi(t_1), \xi(t_2) - \xi(t_1), \ldots, \xi(t_{|j|}) - \xi(t_{|j|-1})\big).
\end{aligned}$$

Now suppose that ξ is not absolutely continuous. That is, there exists $\epsilon > 0$ and $0 \le r_1^n < s_1^n \le r_2^n < \cdots \le r_{k_n}^n < s_{k_n}^n \le 1$, such that $\sum_{i=1}^{k_n}(s_i^n - r_i^n) \to 0$ but $\sum_{i=1}^{k_n} |\xi(s_i^n) - \xi(r_i^n)| \ge \epsilon$. Let j^n be such that $t_{2p}^n = s_p^n$ and $t_{2p-1}^n = r_p^n$ (so that $|j^n| = 2k_n$). Now

$$\begin{aligned}
& \sup_{j\in\mathcal{J}} \Lambda_j^*\big(\xi(t_1), \xi(t_2) - \xi(t_1), \ldots, \xi(t_{|j|}) - \xi(t_{|j|-1})\big) \\
\ge\ & \limsup_n \left\{ \sup_{\underline{\lambda}^n \in \mathbb{R}^{2k_n}} \sum_{i=1}^{2k_n} \lambda_i^n \cdot \big(\xi(t_i^n) - \xi(t_{i-1}^n)\big) - \Lambda_{j^n}(\underline{\lambda}^n) \right\} \\
\ge\ & \limsup_n \left\{ A \sum_{i=1}^{k_n} \big|\xi(s_i^n) - \xi(r_i^n)\big| - \Lambda_{j^n}(\underline{\lambda}^{n*}) \right\} \ge A\epsilon,
\end{aligned}$$

where $\lambda_{2p-1}^{n*} = 0$ and $\lambda_{2p}^{n*} = A\big(\xi(s_i^n) - \xi(r_i^n)\big)/|\xi(s_i^n) - \xi(r_i^n)|$ ($= 0$ if $\xi(s_i^n) - \xi(r_i^n) = 0$) for any $A > 0$. The last inequality holds since $\Lambda_j(\underline{\lambda}^{n*}) \to 0$ as $n \to \infty$, which follows from an application of dominated convergence theorem and the fact that g is continuous at 0 with $g(0) = 0$. This completes the proof since A is arbitrary. $\square$

Lemma 3.3.6. *Suppose $f : E \to \mathbb{R}$ is $L_1(E, \mathcal{E}, \pi)$ and $c_f = \int_E f(x)\pi(dx) \neq 0$. Then for any initial distribution ν of Z_0*

$$\left(\frac{1}{\gamma(n)} \sum_{k=1}^{[nt]} f(Z_k), t \in [0,1] \right) \Longrightarrow c_f \Big(U(t), t \in [0,1] \Big)$$

in $\mathcal{D}_{Sk}$, where $U(t) = \inf\{x : S_\alpha(x) \geq t\}, 0 \leq t \leq 1$, is the inverse time $\alpha-$stable subordinator with

$$E\left\{\exp\left(-\lambda S_\alpha(1)\right)\right\} = \exp\left\{ -\frac{\lambda^\alpha}{\Gamma(1+\alpha)} \right\}, \qquad \forall \lambda \geq 0. \tag{3.53}$$

Proof. This lemma is an extension of Theorem 2.3 in Chen (1999) which states that for any initial distribution ν of Z_0

$$\frac{1}{\gamma(n)} \sum_{k=1}^{n} f(Z_k) \Longrightarrow c_f U(1).$$

We proceed in a way similar to the proof of that theorem. By a well known ratio limit theorem (see e.g. Theorem 17.3.2 in Meyn and Tweedie (1993)) we know that if $g_1, g_2 \in L_1(E, \mathcal{E}, \pi)$ with $\int g_2(x)\pi(dx) \neq 0$ then

$$\lim_{n\to\infty} \sum_{k=1}^{n} g_1(Z_k) \Big/ \sum_{k=1}^{n} g_2(Z_k) = \frac{\int g_1(x)\pi(dx)}{\int g_2(x)\pi(dx)}$$

Therefore it suffices to consider the function $f : E' \to \mathbb{R}$ as $f(x) = I_a(x)$.

Now, suppose $I_n = \sum_{k=1}^{n} f(Z_k) = \sum_{k=1}^{n} I_{[Z_k \in a]}$. By Theorem 2.3 in Chen (1999) we get that for any $j = \{0 < t_1 < \cdots < t_{|j|} \leq 1\} \in \mathcal{J}$ and $(x_1, \ldots, x_{|j|}) \in \mathbb{R}^{|j|}$

$$\begin{aligned}
& P_\nu\left[(I_{[nt_1]}, \ldots, I_{[nt_{|j|}]}) \leq \gamma(n)(x_1, \ldots, x_{|j|}) \right] \\
= \;& P_\nu\left[(T_{[\gamma(n)x_1]}, \ldots, T_{[\gamma(n)x_{|j|}]}) \geq ([nt_1], \ldots, [nt_{|j|}]) \right] \\
\sim \;& P_\nu\left[\frac{1}{\gamma^{\leftarrow}(k)} \left(T_{[kx_1]}, \ldots, T_{[kx_{|j|}]}\right) \geq \frac{1}{\gamma^{\leftarrow}(k)} \left([\gamma^{\leftarrow}(k)t_1], \ldots, [\gamma^{\leftarrow}(k)t_{|j|}]\right) \right] \\
\to \;& P\Big[\Big(S_\alpha(x_1), \ldots, S_\alpha(x_{|j|})\Big) \geq (t_1, \ldots, t_{|j|}) \Big] \\
= \;& P\Big[\Big(U(t_1), \ldots, U(t_{|j|})\Big) \leq (x_1, \ldots, x_{|j|}) \Big]
\end{aligned}$$

Therefore,

$$\Big(\frac{1}{\gamma(n)} I_{[nt_i]}, i = 1, \dots |j|\Big) \Longrightarrow \Big(U(t_i), i = 1, \dots, |j|\Big),$$

which in turn implies

$$\left(\frac{1}{\gamma(n)} \sum_{k=1}^{[nt_i]} f(Z_k), i = 1, \dots |j|\right) \Longrightarrow c_f \Big(U(t_i), i = 1, \dots, |j|\Big). \tag{3.54}$$

We now need to prove tightness in the space $\mathcal{D}_{Sk}$. For that purpose consider the polygonal process

$$\tilde{L}_n(t) = \frac{1}{\gamma(n)} \Big(\sum_{k=1}^{[nt]} f(Z_k) + (nt - [nt]) f(Z_{[nt]+1}) \Big) \qquad \text{for all } t \in [0, 1].$$

Let $w(h, u) = \sup_{s,t \in [0,1], |s-t|<u} |h(s) - h(t)|$, be the modulus of continuity of a function $h : [0, 1] \to \mathbb{R}$. Note that if suffices to prove that for any $\epsilon > 0$

$$\lim_{u \to 0} \limsup_{n \to \infty} P_\nu \big[w(\tilde{L}_n, u) > \epsilon\big] = 0. \tag{3.55}$$

For that purpose observe that

$$\begin{aligned}
& P_\nu \big[w(\tilde{L}_n, u) > \epsilon\big] \\
\leq \; & P_\nu \Big(\max_{0 \leq i < j \leq n, j-i \leq [nu]+2} \frac{1}{\gamma(n)} \Big| \sum_{k=i}^{j} f(Z_k) \Big| > \epsilon \Big) \\
\leq \; & P_\nu \Big(\max_{0 \leq i < j \leq n, j-i \leq [nu]+2} \frac{1}{\gamma(n)} \sum_{k=i}^{j} \big| f(Z_k) \big| > \epsilon \Big) \\
\leq \; & P_\nu \Big(\max_{0 \leq i \leq n-[nu]-2} \frac{1}{\gamma(n)} \sum_{k=i}^{i+[nu]+2} \big| f(Z_k) \big| > \epsilon \Big)
\end{aligned}$$

It is easy to check that for any non-decreasing function $h : \mathbb{R}_+ \to \mathbb{R}_+$ and $u \in [0, 1]$

$$\sup_{0 \leq t \leq 1-u} \big\{h(t + u) - h(t)\big\} \leq 2 \max_{1 \leq i \leq [1/u]+1} \big\{h(iu) - h((i - 1)u)\big\}$$

which implies

$$\begin{aligned}
&P_\nu\Big(\max_{0\le i\le n-[nu]-2}\frac{1}{\gamma(n)}\sum_{k=i}^{i+[nu]+2}\big|f(Z_k)\big|>\epsilon\Big)\\
\le\ & P_\nu\Big(\max_{1\le i\le[1/u]+1}\sum_{k=(i-1)[nu]+1}^{i[nu]}\big|f(Z_k)\big|>\epsilon/2\Big)\\
\le\ & \Big(\frac{1}{u}+1\Big)\sup_x P_x\Big(\frac{1}{\gamma(n)}\sum_{k=1}^{[nu]}\big|f(Z_k)\big|>\epsilon/2\Big)
\end{aligned}$$

Now observe that

$$\begin{aligned}
&\sup_x P_x\Big(\frac{1}{\gamma(n)}\sum_{k=1}^{[nu]}\big|f(Z_k)\big|>\epsilon/2\Big)\\
\le\ & \sup_x P_x\Big(\frac{1}{\gamma(n)}\sum_{k=1}^{[nu]\wedge T}\big|f(Z_k)\big|>\epsilon/4\Big)+P_a\Big(\frac{1}{\gamma(n)}\sum_{k=0}^{[nu]\vee T-T}\big|f(Z_k)\big|>\epsilon/4\Big)\\
\le\ & \sup_x P_x\Big(\frac{1}{\gamma(n)}\sum_{k=1}^{T\wedge n}\big|f(Z_k)\big|>\epsilon/4\Big)+P_a\Big(\frac{1}{\gamma(n)}\sum_{k=1}^{[nu]}\big|f(Z_k)\big|>\epsilon/4\Big)
\end{aligned}$$

Again by assumption (3.34) it we get that

$$\limsup_{n\to\infty}\sup_x P_x\Big(\frac{1}{\gamma(n)}\sum_{k=1}^{T\wedge n}\big|f(Z_k)\big|>\epsilon/4\Big)=0,$$

and by (3.54)

$$\limsup_{n\to\infty}P_a\Big(\frac{1}{\gamma(n)}\sum_{k=0}^{[nu]}\big|f(Z_k)\big|>\epsilon/4\Big)=P\big(c_{|f|}U(u)>\epsilon/4\big),$$

where $c_{|f|}=\int_E|f(s|)\pi(ds)$. Therefore,

$$\begin{aligned}
&\limsup_{n\to\infty}P_\nu\big[w(\tilde{L}_n,u)>\epsilon\big]\\
\le\ & \limsup_{n\to\infty}\Big(\frac{1}{u}+1\Big)\sup_x P_x\Big(\frac{1}{\gamma(n)}\sum_{k=1}^{[nu]}\big|f(Z_k)\big|>\epsilon/2\Big)\\
\le\ & \Big(\frac{1}{u}+1\Big)P\big(c_{|f|}U(u)>\epsilon/4\big)\\
=\ & \Big(\frac{1}{u}+1\Big)P\big(S_\alpha(\epsilon/4c_{|f|})\le u\big)
\end{aligned}$$

Finally by Theorem 2.5.3 in Zolotarev (1986) we get

$$\lim_{u\to 0}\limsup_{n\to\infty} P_\nu\big[w(\tilde{L}_n,u)>\epsilon\big]=0$$

and that completes the proof of the lemma. $\square$

Lemma 3.3.7. *For any* $j=\{0=t_0<t_1<\cdots<t_{|j|}\le 1\}\in\mathcal{J}$*, as* $N\to\infty$

$$\sup_{x\in E,n\ge 1} E_x\left[g\Big(\frac{1}{\gamma(n)}\sum_{i=1}^{|j|}\lambda_i\sum_{k=[nt_{i-1}]+1}^{[nt_i]} f(Z_k)\Big)I_{\left[g\big(\frac{1}{\gamma(n)}\sum_{i=1}^{|j|}\lambda_i\sum_{k=[nt_{i-1}]+1}^{[nt_i]} f(Z_k)\big)\ge N\right]}\right]\longrightarrow 0$$

Proof. It suffices to prove that for any $\lambda\in\mathbb{R}$,

$$\sup_{x\in E,n\ge 1} E_x\left[g\Big(\frac{\lambda}{\gamma(n)}\sum_{k=1}^{n} f(Z_k)\Big)I_{\left[g\big(\frac{\lambda}{\gamma(n)}\sum_{k=1}^{n} f(Z_k)\big)\ge N\right]}\right]\longrightarrow 0$$

as $N\to\infty$. For that purpose, we look at

$$\begin{aligned}
& P_x\left[g\Big(\frac{\lambda}{\gamma(n)}\sum_{i=1}^{n} f(Z_i)\Big)>t\right]\\
\le\ & P_x\left[\frac{|\lambda|}{\gamma(n)}\sum_{i=1}^{n}\big|f(Z_i)\big|>\bar{g}(t)\right]\\
=\ & P_x\left[\Big(\frac{|\lambda|}{\gamma(n)}\sum_{i=1}^{n}\big|f(Z_i)\big|\Big)^\delta>(\bar{g}(t))^\delta\right]\\
\le\ & P_x\left[\frac{|\lambda|^\delta}{\gamma(n)^\delta}\Big(\sum_{i=1}^{I_n+1}\sum_{k=T_{i-1}+1}^{T_i}\big|f(Z_k)\big|\Big)^\delta>(\bar{g}(t))^\delta\right]
\end{aligned}$$

where, as before, $I_n=\sum_{k=1}^n I_{[X_k\in a]}$. By applying Holder's inequality we get

$$\begin{aligned}
& P_x\left[\frac{|\lambda|^\delta}{\gamma(n)^\delta}\Big(\sum_{i=1}^{I_n+1}\sum_{k=T_{i-1}+1}^{T_i}\big|f(Z_k)\big|\Big)^\delta>(\bar{g}(t))^\delta\right]\\
\le\ & P_x\left[\frac{|\lambda|^\delta}{\gamma(n)^\delta}2^{\delta-1}\left\{\Big(\sum_{k=1}^{T_1}\big|f(Z_k)\big|\Big)^\delta+\Big(\sum_{i=2}^{I_n+1}\sum_{k=T_{i-1}+1}^{T_i}\big|f(Z_k)\big|\Big)^\delta\right\}>(\bar{g}(t))^\delta\right]\\
\le\ & P_x\left[\frac{|\lambda|^\delta}{\gamma(n)^\delta}2^{\delta-1}\Big(\sum_{k=1}^{T_1\wedge n}\big|f(Z_k)\big|\Big)^\delta>(\bar{g}(t))^\delta/2\right]\\
&\quad +\ P_x\left[\frac{|\lambda|^\delta}{\gamma(n)^\delta}2^{\delta-1}\Big(\sum_{i=2}^{I_n+1}\sum_{k=T_{i-1}+1}^{T_i}\big|f(Z_k)\big|\Big)^\delta>(\bar{g}(t))^\delta/2\right]
\end{aligned}$$

The assumption (3.35) then implies

$$\begin{aligned} & P_x\Big[\frac{|\lambda|^\delta}{\gamma(n)^\delta}2^{\delta-1}\Big(\sum_{k=1}^{T_1\wedge n}\big|f(Z_k)\big|\Big)^\delta > (\bar g(t))^\delta/2\Big] \\ \leq\ & \exp(-k_0\bar g(t)^\delta)E_x\Big[\exp\Big\{\frac{1}{k_0}\Big(\frac{2|\lambda|}{\gamma(n)}\Big)^\delta\Big(\sum_{k=1}^{T_a\wedge n}\big|f(Z_k)\big|\Big)^\delta\Big\}\Big] \end{aligned} \tag{3.56}$$

where k_0 is as in (3.36). By another application of Holder's inequality and using the fact that $I_n \leq n$ for every n we get

$$\begin{aligned} & P_x\Big[\frac{|\lambda|^\delta}{\gamma(n)^\delta}2^{\delta-1}\Big(\sum_{i=2}^{I_n+1}\sum_{k=T_{i-1}+1}^{T_i}\big|f(Z_k)\big|\Big)^\delta > (\bar g(t))^\delta/2\Big] \\ \leq\ & P_a\Big[\frac{|\lambda|^\delta}{\gamma(n)^\delta}2^{\delta-1}\Big(\sum_{i=1}^{I_n+1}\sum_{k=T_{i-1}+1}^{T_i}\big|f(Z_k)\big|\Big)^\delta > (\bar g(t))^\delta/2\Big] \\ \leq\ & P_a\Big[\Big(\frac{2|\lambda|}{\gamma(n)}\Big)^\delta(I_n+1)^{\delta-1}\sum_{i=1}^{I_n+1}\Big(\sum_{k=T_{i-1}+1}^{T_i}\big|f(Z_k)\big|\Big)^\delta > (\bar g(t))^\delta\Big] \\ \leq\ & \sum_{l=1}^{n/\gamma(n)} P_a\Big[\frac{(2|\lambda|)^\delta}{\gamma(n)}l^{\delta-1}\sum_{i=1}^{l\gamma(n)}\Big(\sum_{k=T_{i-1}+1}^{T_i}\big|f(Z_k)\big|\Big)^\delta > (\bar g(t))^\delta, l-1 < \frac{I_n+1}{\gamma(n)} \leq l\Big] \\ \leq\ & \sum_{l=1}^{n/\gamma(n)} P_a\Big[\frac{(2|\lambda|)^\delta}{\gamma(n)}l^{\delta-1}\sum_{i=1}^{l\gamma(n)}\Big(\sum_{k=T_{i-1}+1}^{T_i}\big|f(Z_k)\big|\Big)^\delta > (\bar g(t))^\delta\Big]^{1/p} \\ & \times\ P_a\Big[l-1 < \frac{I_n+1}{\gamma(n)} \leq l\Big]^{1-1/p} \end{aligned}$$

where $p > 1$. We can now apply an exponential Markov inequality to get the bound

$$\begin{aligned} & P_a\Big[\frac{(2|\lambda|)^\delta}{\gamma(n)}l^{\delta-1}\sum_{i=1}^{l\gamma(n)}\Big(\sum_{k=T_{i-1}+1}^{T_i}\big|f(Z_k)\big|\Big)^\delta > (\bar g(t))^\delta\Big] \\ \leq\ & l\gamma(n)\Gamma\Big(\frac{1}{k_0 p}\frac{(2|\lambda|)^\delta}{\gamma(n)}l^{\delta-1}\Big)\exp(-k_0 p\bar g(t)^\delta) \end{aligned}$$

where

$$\Gamma(\lambda) := \log E_a\Big[\exp\Big(\lambda\sum_{k=1}^{T}\big|f(Z_k)\big|\Big)^\delta\Big].$$

Since $\delta < (1-\alpha)^{-1}$ and l is atmost $n/\gamma(n)$ for any $n \geq 1$, there exists $K_1 > 0$ such that

$$\frac{l^{\delta-1}}{\gamma(n)} \leq \frac{n^{\delta-1}}{\gamma(n)^\delta} \leq K_1 \qquad \text{for all } n \geq 1.$$

Now using convexity of Γ and the fact that $\Gamma(0) = 0$ we can get $K_2 > 0$ such that

$$\Gamma(x) \leq K_2 x \qquad \text{for all } 0 \leq x \leq (2|\lambda|)^\delta K_1/k_0 p.$$

Therefore,

$$\begin{aligned} & P_a\left[\frac{(2|\lambda|)^\delta}{\gamma(n)} l^{\delta-1} \sum_{i=1}^{l\gamma(n)} \Big(\sum_{k=T_{i-1}+1}^{T_i} \big|f(Z_k)\big|\Big)^\delta > (\bar{g}(t))^\delta\right] \\ \leq\ & l\gamma(n) K_2 \frac{1}{k_0 p} \frac{(2|\lambda|)^\delta}{\gamma(n)} l^{\delta-1} \exp(-k_0 p \bar{g}(t)^\delta) \\ =\ & K_2 \frac{1}{k_0 p} (2|\lambda|)^\delta l^\delta \exp(-k_0 p \bar{g}(t)^\delta) \end{aligned} \tag{3.57}$$

We also know that

$$P_a\left[I_n \geq l\gamma(n)\right] = P_a\left[T_{[l\gamma(n)]} \leq n\right] \leq P_a\left[T_{l[\gamma(n)]} \leq n\right].$$

Let $W_k := T_k - T_{k-1}$ for $k \geq 1$. Since a is an atom, $\{W_l\}$ is a sequence of i.i.d random variables. For any $x > 1/n$

$$\begin{aligned} P\big[W_1 + \cdots + W_{[\gamma(n)]} \leq nx\big] &\leq P\left[\max_{1 \leq i \leq [\gamma(n)]} W_i \leq nx\right] \\ &= \left(1 - \frac{1}{b\pi(C)\gamma(nx)}\right)^{[\gamma(n)]} \end{aligned}$$

There exists $c_1 > 0$ such that for any $n \geq 1$ and $1/n < x \leq 2$,

$$P\big[W_1 + \cdots + W_{[\gamma(n)]} \leq nx\big] \leq \exp\left(-c_1 \frac{[\gamma(n)]}{\gamma(nx)}\right) \tag{3.58}$$

Fix $\epsilon > 0$ such that $\alpha - \epsilon > 0$. Using Potter bounds (see Theorem 1.5.6 in Bingham et al. (1989)) we get $c_2 > 1$ such that for $x_1 > x_2 > 1$

$$c_2 \left(\frac{x_1}{x_2}\right)^{\alpha-\epsilon} \leq \frac{\gamma(x_1)}{\gamma(x_2)}. \tag{3.59}$$

Hence, it is easy to get $c_3 > 0$ such that for all $1/n < x \leq 2$,

$$P\big[W_1 + \cdots + W_{[\gamma(n)]} \leq nx\big] \leq \exp\left(-c_3 x^{\alpha-\epsilon}\right). \tag{3.60}$$

Now if $V_k^n := (W_{(k-1)[\gamma(n)]+1} + \cdots + W_{k[\gamma(n)]})/n$ then

$$P\big[T_{l[\gamma(n)]} \leq n\big] = P\big[V_1^n + \cdots + V_l^n \leq 1\big].$$

By (3.60) there exists $\sigma > 0$ such that for any $0 < x \leq 2$

$$P\Big[\frac{1}{n} + V_l^n \leq x\Big] \leq P\big[\sigma S_{\alpha-\epsilon} \leq x\big]$$

where $S_{\alpha-\epsilon}$ is a right skewed $(\alpha-\epsilon)$-stable random variable satisfying

$$E\big[\exp(-tS_{\alpha-\epsilon})\big] = \exp(-t^{\alpha-\epsilon}) \qquad \text{for all } t > 0.$$

Using the fact that $l \leq n/\gamma(n)$ for any $n \geq 1$

$$\begin{aligned} P\big[T_{l[\gamma(n)]} \leq n\big] &= P\big[V_1^n + \cdots + V_l^n \leq 1\big] \\ &\leq P\Big[\Big(\frac{1}{n} + V_1^n\Big) + \cdots + \Big(\frac{1}{n} + V_l^n\Big) \leq 2\Big] \\ &\leq P\big[S_{\alpha-\epsilon} l^{1(\alpha-\epsilon)} \leq 2/\sigma\big] \end{aligned}$$

By Theorem 2.5.3 in Zolotarev (1986) there exists $c_4 > 0$ and $c_5 > 0$ such that

$$P\big[T_{l[\gamma(n)]} \leq n\big] \leq c_4 \exp\left(-c_5 l^{\frac{1}{1-\alpha+\epsilon}}\right). \tag{3.61}$$

Therefore, by combining (3.57) and (3.61) we get

$$\begin{aligned} & P_x\left[\frac{|\lambda|^\delta}{\gamma(n)^\delta} 2^{\delta-1} \Big(\sum_{i=2}^{I_n+1} \sum_{k=T_{i-1}+1}^{T_i} \big|f(Z_k)\big|\Big)^\delta > (\bar{g}(t))^\delta/2\right] \qquad (3.62) \\ \leq \ & \exp(-k_0 \bar{g}(t)^\delta) \sum_{l=1}^{\infty} K_2 \frac{1}{k_0 p} (2|\lambda|)^\delta l^\delta c_4 \exp\left(-c_5(1-1/p) l^{\frac{1}{1-\alpha+\epsilon}}\right) \end{aligned}$$

The series in (3.62) surely converges to a finite number. Finally,

$$\begin{aligned} & E_x\left[g\Big(\frac{\lambda}{\gamma(n)} \sum_{i=1}^n f(Z_i)\Big) I_{\left[g\left(\frac{\lambda}{\gamma(n)} \sum_{i=1}^n f(Z_i)\right) > N\right]}\right] \\ = \ & \int_N^\infty P_x\left[g\Big(\frac{\lambda}{\gamma(n)} \sum_{i=1}^n f(Z_i)\Big) > t\right] dt + N P_x\left[g\Big(\frac{\lambda}{\gamma(n)} \sum_{i=1}^n f(Z_i)\Big) > N\right] \end{aligned}$$

and that converges 0 as $N \to \infty$ by combining (3.56), (3.62) and assumption (3.36). □

3.4 Examples

Example 3.4.1 (Simple symmetric random walk on $\mathbb{Z}$)**.** Suppose $E = \mathbb{Z}$ and $\mathcal{E}$ is the power set of E. Let (Z_n) be the simple symmetric random walk on $\mathbb{Z}$, that is, it is a markov chain with transition kernel

$$p(i,j) = \begin{cases} 1/2 & \text{if } j = i+1 \text{ or } j = i-1 \\ 0 & \text{otherwise.} \end{cases}$$

Then the counting measure π on $(E, \mathcal{E})$ is an invariant measure for this kernel $p(\cdot,\cdot)$. Here, we can take $E_n = \{-n, n\}$ and $E_0 = 0$, which means $\zeta = 0$. Furthermore, $a = \{0\}$ is an atom for (Z_n). From the arguments proving Proposition 2.4 in Le Gall and Rosen (1991) we get

$$\gamma(n) \sim \sum_{k=1}^{n} P_0[X_k = 0] \sim \sqrt{n}\sqrt{2/\pi} \in RV_{1/2}.$$

and hence $\alpha = 1/2$. It is also well known that

$$P_{\pi_n}\big[T/n^2 \in \cdot\big] \Rightarrow S_{1/2},$$

where $S_{1/2}$ is a right-skewed 1/2-stable distribution with density

$$h(x) = \frac{1}{\sqrt{2\pi}} x^{-3/2} \exp\big\{ -(2x)^{-1} \big\}, \qquad \forall x > 0.$$

Therefore, $\psi(n) = n^2$, $\beta = 2$ and $Q(\cdot)$ is the law of $S_{1/2}$. By the arguments used to prove the statement (3.61) we get that assumption (3.31) is satisfied. Now suppose that $\rho(\cdot)$ is a Lévy measure on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ such that

$$g(\lambda) = \int_{\mathbb{R}} \big(e^{\lambda z} - 1 - \lambda[\![z]\!]\big)\rho(dz) < \infty, \qquad \forall \lambda \in \mathbb{R},$$

and

$$\int_0^\infty \exp\big(-k_0\bar{g}(t)^\delta\big)dt < \infty,$$

for some $\delta < 2$ and $k_0 > 0$ where $\bar{g}(t) = \min\{|s| : g(s) = t\}$.

Suppose $\{X_n, n \in \mathbb{Z}\}$ is an ID process where

$$X_n = \int_S f(s_n)M(ds), \qquad n \in \mathbb{Z},$$

where $f(x) = cI_{\{0\}}(x)$ for some $c \neq 0$. It is easy to check that only a function of this form satisfies conditions (3.33)-(3.35). μ_n is the law of Y_n in $\mathcal{BV}$, where

$$Y_n(t) = \frac{1}{n}\sum_{i=1}^{[nt]} X_i, \qquad t \in [0,1].$$

Then $\{\mu_n\}$ satisfies LDP in $\mathcal{BV}$ with speed $\sqrt{n}$ and good rate function

$$H(\xi) = \begin{cases} \Lambda^*(\xi') & \text{if } \xi \in \mathcal{AC}, \xi(0) = 0 \\ \infty & \text{otherwise.} \end{cases} \tag{3.63}$$

where for any $\varphi \in L_1[0,1]$

$$\begin{aligned} \Lambda^*(\varphi) &= \sup_{\psi \in L_\infty[0,1]} \Bigg\{ \int_0^1 \psi(t)\varphi(t)dt \\ &\quad - \int_0^\infty 2E\Big[I_{[r^2 S^*_{1/2} \le 1]}\, g\Big(c\sqrt{\frac{\pi}{2}}\int_0^{1-r^2 S^*_{1/2}} \psi(t)U(dt)\Big)\Big]dr \Bigg\}. \end{aligned} \tag{3.64}$$

Here $U(t) := \inf\{x : S_{1/2}(x) \geq t\}, 0 \leq t \leq 1$, is the inverse time 1/2-stable subordinator, where

$$E\left\{\exp\left(-\lambda S_{1/2}(1)\right)\right\} = \exp\left\{-\frac{2}{\sqrt{\pi}}\lambda^{1/2}\right\}, \forall \lambda \geq 0,$$

and $S^*_{1/2}$ is independent of $\{U(t) : 0 \leq t \leq 1\}$ having the same distribution as $S_{1/2}(1)$.

Example 3.4.2. Suppose that (Z_n) is a Markov chain on $E = \mathbb{Z}_+$ with transition probabilities

$$P(i,j) = \begin{cases} p_i q_i & \text{if } i \neq 0, j = i+1 \\ p_i(1-q_i) & \text{if } i \neq 0, j = 0 \\ 1 - p_i & \text{if } j = i \\ 1 & \text{if } i = 0, j = 1 \\ 0 & \text{otherwise} \end{cases} \tag{3.65}$$

where (p_n) and (q_n) are two sequences of real numbers between 0 and 1 ($p_0 = 1, q_0 = 1$). (Z_n) is an irreducible and recurrent chain. To see why it recurrent observe the following. Whenever the chain hits a state i, it stays there for τ_i amount of time where $\tau_i \sim$ geometric(p_i). When it leaves i, it jumps to $i+1$ with probability q_i or goes to 0 with probability $1 - q_i$. We can take $a = \{0\}$ to be an atom. Therefore, given that $X_0 = 0$, we can write

$$T \overset{d}{=} 1 + \sum_{k=1}^{N} \tau_k,$$

where N is a random variable independent of $\{\tau_k, k \geq 1\}$ and having distribution

$$P[N = n] = q_1 \cdots q_{n-1}(1 - q_n), \qquad \text{for every } n \geq 1.$$

Clearly that means $P_0[T < \infty] = P_0[N < \infty] = 1$ if

$$\prod_{k=0}^{\infty} q_n = 0. \tag{3.66}$$

If (3.66) holds, whether the chain is positive or null-recurrent will depend on the choice of both sequences (p_n) and (q_n):

$$E_0[T] = 1 + \sum_{n=1}^{\infty} P[N = n] \sum_{k=1}^{n} \frac{1}{p_k}.$$

It is also easy to check that

$$\pi(0) = 1 \text{ and } \pi(n) = q_1 \cdots q_{n-1}/p_n, \qquad \text{for all } n \geq 1$$

is an invariant measure for this Markov chain.

We now discuss a special case in details. Suppose

$$p_n = \frac{1}{(n+1)^s} \text{ and } q_n = \Big(\frac{n}{n+1}\Big)^t \qquad \text{for every } n \geq 1.$$

where $s > 0$ and $t > 0$ satisfies $1/2 < t - s < 1$. In this setup it is easy to check that

$$P\big[N = n\big] \sim \frac{t}{n^{t+1}} \in RV_{-(t+1)}$$

which implies

$$P\big[N > n\big] \sim \frac{1}{n^t} \in RV_{-t}.$$

In order to find α we need to estimate the tail probability of the random variable T. Note that

$$T \stackrel{d}{=} 1 + \sum_{k=1}^{N} \tau_k = \sum_{k=1}^{N} \big(\tau_k - (k+1)^s\big) + h(N)$$

where h is defined as

$$h(n) := 1 + \sum_{k=1}^{n} (k+1)^s \sim \frac{1}{s+1} n^{s+1} \in RV_{s+1}.$$

It is easy to check that $P[h(N) > n] \sim \big((s+1)n\big)^{-t/(s+1)} \in RV_{-t/(s+1)}$. If we can show that $\sum_{k=1}^{N}(\tau_k - (k+1)^s)$ has a lighter tail then it would follow that $P[T > n] \in RV_{-t/(s+1)}$ which would in turn imply $\alpha = t/(s+1)$. For that purpose observe that $t/(s+1) < 1 < t/(s+1/2)$. Then by Cauchy Schwartz inequality

$$E\left[\Big|\sum_{k=1}^{N}\Big(\tau_k - (k+1)^s\Big)\Big|\right] \;\leq\; E\left[\sum_{k=1}^{N} E\Big(\tau_k - (k+1)^s\Big)^2\right]^{1/2}$$

Since $Var(\tau_k) = (k+1)^{2s}$ we get

$$\begin{aligned} E\left[\Big|\sum_{k=1}^{N}\Big(\tau_k - (k+1)^s\Big)\Big|\right] &\leq E\left[\sum_{k=1}^{N}(k+1)^{2s}\right]^{1/2} \\ &\leq cE\left[N^{2s+1}\right]^{1/2} \\ &= cE\left[N^{s+1/2}\right] < \infty. \end{aligned}$$

where $c > 0$ is a constant such that

$$\sum_{k=1}^{n} (k+1)^{2s} \le cn^{2s+1} \qquad \text{for all } n \ge 1.$$

Therefore we have

$$\gamma(n) = (s+1)n^{t/(s+1)}.$$

We can take $E_n = \{n\}$ for every $n \ge 0$ and

$$\pi(n) = \frac{(n+1)^s}{n^t} \in RV_{s-t}$$

which means $\zeta = s - t > -1$. We now claim that $\beta = s + 1$. Note that given $X_0 = n$

$$T \stackrel{d}{=} \sum_{k=0}^{N_n} \tau_{n+k}$$

where N_n is a random variable such that for $k \ge 1$

$$P\big[N_n = k\big] = q_n \cdots q_{n+k-1}(1 - q_{n+k}) = P\big[N = n + k \big| N \ge n\big]$$

It follows immediately that

$$P\big[N_n/n > x\big] \sim \frac{1}{(1+x)^t} \qquad \text{for all } x > 0. \tag{3.67}$$

Following the same argument as above it is easy to check that

$$E\left|\frac{1}{n^{s+1}} \sum_{k=0}^{N_n} \Big(\tau_{n+k} - (n+k)^s\Big)\right| \longrightarrow 0. \tag{3.68}$$

Therefore for any $x > 0$

$$\begin{aligned}
\lim_{n\to\infty} P_{\pi_n}\left[T/n^{s+1} > x\right] &= \lim_{n\to\infty} P\left[\frac{1}{n^{s+1}} \sum_{k=0}^{N_n} (n+k)^s > x\right] \\
&= \lim_{n\to\infty} P\left[\frac{1}{n^{s+1}} \Big(h(n+N_n) - h(n-1)\Big) > x\right] \\
&= \lim_{n\to\infty} P\left[\left(1 + \frac{N_n}{n}\right)^{s+1} > 1 + (s+1)x\right] \\
&= \big(1 + (s+1)x\big)^{-\frac{t}{s+1}}.
\end{aligned}$$

Hence $\psi(n) = n^{s+1}$ and $Q(\cdot)$ is a measure on $(\mathbb{R}_+, \mathcal{B}(\mathbb{R}_+))$ such that

$$Q((x,\infty)) = \big(1 + (s+1)x\big)^{-\frac{t}{s+1}} \qquad \text{for all } x > 0.$$

We need to check that assumption (3.31) is satisfied. Fix any $0 < \epsilon < t$. Then for any $r \geq 1$

$$\begin{aligned} P_{\pi_n}\left[\frac{T}{n^{s+1}} \leq cr^{-s-1+\epsilon}\right] &= P\left[\frac{1}{n^{s+1}}\sum_{k=0}^{N_n} \tau_{n+k} \leq cr^{-s-1+\epsilon}\right] \\ &\leq P\left[\frac{1}{n^{s+1}}\sum_{k=0}^{N_n'} W_k \leq cr^{-s-1+\epsilon}\right] \end{aligned}$$

where $\{W_i, i \geq 0\}$ are i.i.d geometric(p_n) and N_n' is geometric($1 - q_n$) and is independent of $\{W_i\}$. Now

$$\sum_{k=0}^{N_n'} W_k \sim \text{ geometric}(p_n(1-q_n))$$

and therefore it is possible to get $c' > 0$ and $K > 1$ such that

$$\sup_{n \geq Kr} P\left[\frac{1}{n^{s+1}}\sum_{k=0}^{N_n'} W_k \leq cr^{-s-1+\epsilon}\right] \leq 1 - \exp\left\{c'cr^{-s-1+\epsilon}n^{s+1}p_n(1-q_n)\right\}.$$

Observe that $n^{s+1}p_n(1-q_n) \to t$ as $n \to \infty$. Therefore one can get $c'' > 0$ such that

$$\sup_{n \geq Kr} P\left[\frac{1}{n^{s+1}}\sum_{k=0}^{N_n'} W_k \leq cr^{-s-1+\epsilon}\right] \leq c''r^{-s-1+\epsilon}.$$

Since $\zeta = s - t$ and we assumed that $t > \epsilon$, we see that (3.31) is satisfied. Now suppose that g satisfies (3.36) with $\delta < (s+1)/(s-t+1)$ and $f : \mathbb{Z}_+ \to \mathbb{R}$ is of the form $f(x) = cI_{\{0\}}(x)$ for some $c \neq 0$. It is easy to check that a function satisfying assumptions (3.33)-(3.35) must be of this form. From all these we get

$$a_n = \pi(E_{[\psi^{\leftarrow}(n)]})\gamma(n)\psi^{\leftarrow}(n) \sim (s+1)n.$$

Now let μ_n be the law of $Y_n \in \mathcal{BV}$ where

$$Y_n(t) = \frac{1}{n}\sum_{i=1}^{[nt]} X_i \qquad \text{for all } t \in [0,1].$$

Then $\{\mu_n\}$ satisfies LDP in $\mathcal{BV}$ with speed $n^{\frac{s-t+1}{s+1}}$ and good rate function

$$H(\xi) = \begin{cases} \Lambda^*(\xi') & \text{if } \xi \in \mathcal{AC}, \xi(0) = 0 \\ \infty & \text{otherwise.} \end{cases} \tag{3.69}$$

where for any $\varphi \in L_1[0,1]$

$$\begin{aligned} \Lambda^*(\varphi) &= \sup_{\psi \in L_\infty[0,1]} \left\{ \int_0^1 \psi(t)\varphi(t)dt \right. \\ &\quad \left. - \int_0^\infty r^{s-t} E\Big[I_{[r^{s+1}V \le 1]} g\Big(\frac{c}{s+1} \int_0^{1-r^{s+1}V} \psi(x)U(dx) \Big) \Big] dr \right\}. \end{aligned} \tag{3.70}$$

Here $U(x) := \inf\{y : S_{t/(s+1)}(y) \ge x\}, 0 \le x \le 1$, is the inverse time $t/(s+1)$-stable subordinator and V is independent of $\{U(x) : 0 \le x \le 1\}$ having distribution $Q(\cdot)$.

3.5 Ruin Probabilities

This section discusses the rate of decay ruin probability for an infinitely divisible process $\{X_n, n \in \mathbb{Z}\}$ defined in (3.32). We retain the assumptions and notations of Section 3.3. We study the probability of ruin in infinite time, defined as

$$\rho(u) = P\left[S_n > a_n\mu + u \text{ for some } n \ge 1\right] \tag{3.71}$$

where $\mu > 0$ is a constant and $a_n = \pi(E_{[\psi^{\leftarrow}(n)]})\gamma(n)\psi^{\leftarrow}(n)$ is as defined in (3.39). As before, T denotes the time of ruin, that is,

$$T(u) = \inf\left\{n : S_n > a_n\mu + u\right\}.$$

Theorem 3.5.1 gives the rate of decay of $\rho(u)$ as u increases.

Theorem 3.5.1. *Under the assumptions of Section 3.3*

$$\lim_{u\to\infty} \frac{1}{b_{a^{\leftarrow}(u)}} \log \rho(u) = -\inf_{t>0} t^{-\frac{\zeta+1}{\zeta+1+\alpha\beta}} \Lambda_1^*(\mu + t)$$

where Λ_1^* *is the Fenchel-Legendre transform of*

$$\Lambda_1(\lambda) = \int_0^\infty r^\zeta E\Big[I_{[r^\beta V \le 1]} g\Big(c_f \lambda U(1 - r^\beta V)\Big)\Big] dr,$$

and $a^{\leftarrow}(u) = \inf\{k \ge 1 : a_k \ge u\}$. *As before* $b_n = \pi(E_{[\psi^{\leftarrow}(n)]})\psi^{\leftarrow}(n)$.

Remark 3.5.2. Under the assumptions made in Section 3.3 we have

$$0 < \inf_{t>0} t^{-\frac{\zeta+1}{\zeta+1+\alpha\beta}} \Lambda_1^*(\mu + t) < \infty.$$

This is not difficult see. Let $h(t) = t^{-\frac{\zeta+1}{\zeta+1+\alpha\beta}} \Lambda_1^*(\mu + t)$. Since $\Lambda_1(\lambda) < \infty$ for every $\lambda \in \mathbb{R}$, we get that $0 < \Lambda_1^*(x) < \infty$ for every $x > 0$ (see Section 2.2 in Dembo and Zeitouni (1998)). Therefore $h(t) \to \infty$ as $t \to 0$. Furthermore, as Λ_1^* is convex and $\frac{\zeta+1}{\zeta+1+\alpha\beta} < 1$ we get $h(t) \to \infty$ as $t \to \infty$. This means that the infimum is achieved at some point $0 < t^* < \infty$ and that proves the claim.

Remark 3.5.3. People typically use $a_n = n$ in the definition of $\rho(\cdot)$ in (3.71). Unfortunately, we only have the result when a_n is of the form described in (3.39). In the examples below we indeed see that $a_n = n$ but we do not know for sure the possible ranges of a_n. As will be evident in the proof of the theorem, the central tool used is the large deviation principle proved in Theorem 3.3.1. If the normalizing sequence a_n grows slower than that in (3.39) but faster than what gives weak convergence, then it would be in the regime of moderate deviations. Although we have not proved a moderate deviations result here, we have every reason to believe that a moderate deviation principle holds for this model. And in that case the argument used to prove Theorem 3.5.1 will also work for the normalizing sequences a_n that the moderate deviations will allow.

Example 3.5.4. We can easily apply Theorem 3.5.1 to the examples described in Section 3.4. If the process $\{X_n\}$ is the infinitely divisible process described in Example 3.4.1 then

$$\lim_{n\to\infty} \frac{1}{\sqrt{n}} \log \rho(n) = -\inf_{t>0} \frac{1}{\sqrt{t}} \Lambda_1^*(\mu + t),$$

where

$$\rho(u) = P\Big[\sum_{i=1}^{n} X_i - n\mu > u \text{ for some } n \geq 1\Big]$$

and Λ_1^* is the Fenchel-Legendre transform of

$$\Lambda_1(\lambda) = \int_0^\infty 2E\Big[I_{[r^2 S^*_{1/2} \leq 1]} g\Big(c\sqrt{\frac{\pi}{2}}\lambda U\big(1 - r^2 S^*_{1/2}\big)\Big)\Big] dr.$$

Here $U(t) := \inf\{x : S_{1/2}(x) \geq t\}, 0 \leq t \leq 1$, is the inverse time 1/2-stable subordinator, where

$$E\left\{\exp\left(-\lambda S_{1/2}(1)\right)\right\} = \exp\left\{-\frac{2}{\sqrt{\pi}}\lambda^{1/2}\right\}, \forall \lambda \geq 0,$$

and $S^*_{1/2}$ is independent of $\{U(t) : 0 \leq t \leq 1\}$ having the same distribution as $S_{1/2}(1)$.

Example 3.5.5. If $\{X_n\}$ is the infinitely divisible process described in Example 3.4.2 then retaining the notation used in the example we get that

$$\lim_{n\to\infty} n^{-\frac{s-t+1}{s+1}} \log \rho(n) = -\inf_{u>0} u^{-\frac{s-t+1}{s+1}} \Lambda_1^*(\mu + u),$$

where

$$\rho(u) = P\Big[\sum_{i=1}^{n} X_i - n\mu > u \text{ for some } n \geq 1\Big]$$

and Λ_1^* is the Fenchel-Legendre transform of

$$\Lambda_1(\lambda) = \int_0^\infty r^{s-t} E\Big[I_{[r^{s+1}V \leq 1]} g\Big(\frac{c}{s+1}\lambda U\big(1 - r^{s+1}V\big)\Big)\Big] dr.$$

Proof of Theorem 3.5.1. The proof is very similar to the proof of Theorem 2.5.2. First we consider the easier half and prove that

$$\liminf_{u\to\infty} \frac{1}{b_{a^{\leftarrow}(u)}} \log \rho(u) \geq -\inf_{t>0} t^{-\frac{\zeta+1}{\zeta+1+\alpha\beta}} \Lambda_1^*(\mu + t). \tag{3.72}$$

For that purpose take any $t > 0$ and observe that

$$\begin{aligned}\liminf_{u\to\infty} \frac{1}{b_{a^{\leftarrow}(u)}} \log \rho(u) &= \liminf_{n\to\infty} \frac{1}{b_{a^{\leftarrow}(ta_n)}} \log P\big[T(ta_n) < \infty\big] \\ &\geq \liminf_{n\to\infty} \frac{b_n}{b_{a^{\leftarrow}(ta_n)}} \frac{1}{b_n} \log P\left[\frac{S_n}{a_n} > \mu + t\right],\end{aligned}$$

From the marginal version of Theorem 3.3.1 (see the proof of the theorem) we get that $P\big[S_n/a_n \in \cdot\big]$ satisfies LDP on $\mathbb{R}$ with speed b_n and rate function

$$\Lambda_1^*(w) = \sup_{\lambda\in\mathbb{R}} \big\{\lambda w - \Lambda_1(\lambda)\big\},$$

where

$$\Lambda_1(\lambda) = \int_0^\infty r^\zeta E\Big[I_{[r^\beta V\le 1]} g\Big(c_f \lambda U(1-r^\beta V)\Big)\Big] dr.$$

This implies

$$\liminf_{n\to\infty} \frac{1}{b_n} \log P\left[\frac{S_n}{a_n} > \mu + t\right] \ge -\Lambda_1^*(\mu+t).$$

Now, since $\{a_n\} \in RV_{(\zeta+1+\alpha\beta)/\beta}$ and $\{b_n\} \in RV_{(\zeta+1)/\beta}$ it follows that $\{b_{a^{\leftarrow}(n)}\} \in RV_{(\zeta+1)/(\zeta+1+\alpha\beta)}$, which in turn gives us

$$\lim_{n\to\infty} \frac{b_n}{b_{a^{\leftarrow}(ta_n)}} = t^{-\frac{\zeta+1}{\zeta+1+\alpha\beta}}.$$

Therefore, we get

$$\liminf_{u\to\infty} \frac{1}{b_{a^{\leftarrow}(u)}} \log \rho(u) \ge -t^{-\frac{\zeta+1}{\zeta+1+\alpha\beta}} \Lambda_1^*(\mu+t).$$

Since $t > 0$ is arbitrary, the best result is achieved by maximizing the function on the right hand side over $t > 0$, which gives us (3.72).

We now prove the other inequality. Note that there exists $t^* > 0$ sufficiently small such that

$$\sup_{k\ge 1} E\big[e^{t^* b_k (S_k - a_k\mu)/a_k}\big] < \infty.$$

Now fix any $\delta > 0$. Observe that from the definition of a_n and b_n we have $a_n/b_n = \gamma(n)$ where $\gamma(\cdot)$ is as defined in (3.27). Then

$$\begin{aligned}
&\limsup_{n\to\infty} \frac{1}{b_n} \log P\big[T(a_n) \le n\delta\big] \\
\le\ & \limsup_{n\to\infty} \frac{1}{b_n} \log \sum_{k=1}^{[n\delta]} P\big[S_k - a_k\mu > a_n\big] \\
\le\ & \limsup_{n\to\infty} \frac{1}{b_n} \log \sum_{k=1}^{[n\delta]} \exp\left\{-t^* \frac{a_n b_k}{a_k}\right\} E\left[\exp\left\{t^* \frac{b_k}{a_k}(S_k - a_k\mu)\right\}\right]
\end{aligned} \tag{3.73}$$

$$
\begin{aligned}
&\leq \limsup_{n\to\infty} \frac{1}{b_n} \log \left\{ n\delta \sup_{k\geq 1} E\left[\exp\left\{t^* \frac{b_k}{a_k}(S_k - a_k\mu)\right\}\right] \exp\left\{-t^* \frac{a_n}{\gamma([n\delta])}\right\}\right\} \\
&\leq -t^* \limsup_{n\to\infty} \frac{a_n}{b_n\gamma([n\delta])} = -t^* \limsup_{n\to\infty} \frac{\gamma(n)}{\gamma(n\delta)} = -t^*\delta^{-\alpha}, \qquad (3.74)
\end{aligned}
$$

which decreases to $-\infty$ as δ decreases to 0. Now fix any $\lambda > 0$ such that $\Lambda_1(\lambda) - \mu\lambda < 0$ and get ϵ such that $0 < \epsilon < \mu\lambda - \Lambda_1(\lambda)$. From Lemma 3.3.3 we have

$$
\lim_{k\to\infty} \frac{1}{b_k} \log E\left[\exp\left\{\lambda \frac{b_k}{a_k} S_k\right\}\right] = \Lambda_1(\lambda),
$$

and therefore it is possible to get $N \geq 1$ such that $k \geq N$ implies

$$
\frac{1}{b_k} \log E\left[\exp\left\{\lambda \frac{b_k}{a_k} S_k\right\}\right] \leq \Lambda_1(\lambda) + \epsilon.
$$

Now note that

$$
\begin{aligned}
&\limsup_{n\to\infty} \frac{1}{b_n} \log P\big[n\delta < T(a_n) < \infty\big] \\
\leq\ & \limsup_{n\to\infty} \frac{1}{b_n} \log \sum_{k=[n\delta]+1}^{\infty} P\big[T(a_n) = k\big] \\
\leq\ & \limsup_{n\to\infty} \frac{1}{b_n} \log \sum_{k=[n\delta]+1}^{\infty} P\big[S_k - a_k\mu > a_n\big] \\
\leq\ & \limsup_{n\to\infty} \frac{1}{b_n} \log \sum_{k=[n\delta]+1}^{\infty} \exp\Big\{ - \frac{a_n b_k}{a_k}\lambda + b_k\Big(\Lambda_1(\lambda) - \mu\lambda + \epsilon\Big)\Big\},
\end{aligned}
$$

where the last inequality follows by an application of exponential Markov inequality. Now if $n > N/\delta$ then for $i \geq 1$

$$
\begin{aligned}
&\sum_{k=i[n\delta]+1}^{(i+1)[n\delta]} \exp\Big\{ - \frac{a_n b_k}{a_k}\lambda + b_k\Big(\Lambda_1(\lambda) - \mu\lambda + \epsilon\Big)\Big\} \\
\leq\ & n\delta \exp\Big\{ - \frac{a_n}{\gamma\big((i+1)[n\delta]\big)}\lambda + \inf_{i[n\delta]+1\leq k\leq (i+1)[n\delta]} b_k\Big(\Lambda_1(\lambda) - \mu\lambda + \epsilon\Big)\Big\}
\end{aligned}
$$

Now get $\eta > 0$ such that $(\zeta + 1)/\beta - \eta > 0$. Then we can get $N_1 \geq 1$ such that for every $n \geq N_1$

$$
\frac{\gamma(n)}{\gamma\big((i+1)[n\delta]\big)} \geq x^1_{i,\delta}(\eta) := (1-\eta)\min\Big\{\big((i+1)\delta\big)^{-\alpha-\eta}, \big((i+1)\delta\big)^{-\alpha+\eta}\Big\}
$$

and

$$\frac{1}{b_n} \inf_{i[n\delta]+1 \le k \le (i+1)[n\delta]} b_k \ge x^2_{i,\delta}(\eta) := (1-\eta) \min \Big\{ \big(i\delta\big)^{(\zeta+1)/\beta-\eta}, \big(i\delta\big)^{(\zeta+1)/\beta+\eta} \Big\}.$$

This implies that for $n > \max\{N/\delta, N_1\}$

$$\begin{aligned} & \sum_{k=i[n\delta]+1}^{(i+1)[n\delta]} \exp\Big\{ -\frac{a_n b_k}{a_k}\lambda + b_k\Big(\Lambda_1(\lambda) - \mu\lambda + \epsilon\Big)\Big\} \\ \le \ & n\delta \exp\Big\{ - b_n x^1_{i,\delta}(\eta)\lambda + b_n x^2_{i,\delta}(\eta)\Big(\Lambda_1(\lambda) - \mu\lambda + \epsilon\Big)\Big\} \end{aligned}$$

Define $y_i = -x^1_{i,\delta}(\eta)\lambda + x^2_{i,\delta}(\eta)\Big(\Lambda_1(\lambda) - \mu\lambda + \epsilon\Big)$ and let $y^* = \max_{i\ge 1} y_i$. Observe that $y^* < 0$ and there exists $i^* \ge 1$ such that $y^* = y_{i^*}$. Then

$$\begin{aligned} & \sum_{k=[n\delta]+1}^{\infty} \exp\Big\{ -\frac{a_n b_k}{a_k}\lambda + b_k\Big(\Lambda_1(\lambda) - \mu\lambda + \epsilon\Big)\Big\} \\ \le \ & n\delta \sum_{i=1}^{\infty} \exp\Big\{b_n y_i\Big\} \\ = \ & n\delta \exp\Big\{b_n y^*\Big\} \sum_{i=1}^{\infty} \exp\Big\{b_n(y_i - y^*)\Big\}. \end{aligned}$$

Now there exists $c > 0$ such that $\exp\big\{b_n(y_i - y^*)\big\} \le \exp(cy_i - cy^*)$ for every $n \ge 1$, and because of the choice of η we have

$$\sum_{i=1}^{\infty} \exp(cy_i - cy^*) < \infty.$$

Therefore by the Dominated Convergence Theorem we get

$$\sum_{i=1}^{\infty} \exp\Big\{b_n(y_i - y^*)\Big\} = 1 + \sum_{i \ne i^*} \exp\Big\{b_n(y_i - y^*)\Big\} \to 1.$$

Hence we get

$$\limsup_{n\to\infty} \frac{1}{b_n} \log \sum_{k=[n\delta]+1}^{\infty} \exp\Big\{ -\frac{a_n b_k}{a_k}\lambda + b_k\Big(\Lambda_1(\lambda) - \mu\lambda + \epsilon\Big)\Big\} \le y^* \tag{3.75}$$

and therefore by combining (3.74) and (3.75) we get

$$\limsup_{n\to\infty} \frac{1}{b_n} \log \rho(a_n) = \limsup_{n\to\infty} \frac{1}{b_n} \log P\big[T(a_n) < \infty\big] \le \max\big\{ -t^*\delta^{-\alpha}, y^*\big\}.$$

Hence, by letting ϵ, δ and η to 0, we get

$$\begin{aligned}\limsup_{n\to\infty}\frac{1}{b_n}\log\rho(a_n) &\leq \sup_{s>0}\Big\{-\lambda s^{-\alpha}+s^{(\zeta+1)/\beta}\Big(\Lambda_1(\lambda)-\mu\lambda\Big)\Big\}\\ &= \sup_{t>0} t^{-\frac{\zeta+1}{\zeta+1+\alpha\beta}}\Big(\Lambda_1(\lambda)-\lambda(\mu+t)\Big).\end{aligned}$$

Since the above result is true for every $\lambda\in C$ where $C:=\{\lambda>0:\Lambda_1(\lambda)-\lambda\mu<0\}$, we get that

$$\limsup_{n\to\infty}\frac{1}{b_n}\log\rho(a_n)\leq\inf_{\lambda\in C}\sup_{t>0}t^{-\frac{\zeta+1}{\zeta+1+\alpha\beta}}\Big(\Lambda_1(\lambda)-\lambda(\mu+t)\Big).$$

Let $R(t,\lambda)=t^{-\frac{\zeta+1}{\zeta+1+\alpha\beta}}\Big(\Lambda_1(\lambda)-\lambda(\mu+t)\Big)$. It is easy to check that for every fixed $t>0$, $R(t,\cdot)$ is a convex function and for every fixed $\lambda\in\bar{C}$, $R(\cdot,\lambda)$ is a quasi-concave function. Furthermore, we see that

$$\inf_{\lambda\in C}\sup_{t>0}R(t,\lambda)=\inf_{\lambda\in\bar{C}}\sup_{t>0}R(t,\lambda).$$

Therefore, by Sion's Minimax Theorem (see Sion (1958)) we get that

$$\inf_{\lambda\in\bar{C}}\sup_{t>0}R(t,\lambda)=\sup_{t>0}\inf_{\lambda\in\bar{C}}R(t,\lambda)$$

It is also easy to check that

$$\inf_{\lambda\in\bar{C}}R(t,\lambda)=\inf_{\lambda\in\mathbb{R}}R(t,\lambda)=-t^{-\frac{\zeta+1}{\zeta+1+\alpha\beta}}\Lambda_1^*(\mu+t)$$

which implies

$$\limsup_{n\to\infty}\frac{1}{b_n}\log\rho(a_n)=-\inf_{t>0}t^{-\frac{\zeta+1}{\zeta+1+\alpha\beta}}\Lambda_1^*(\mu+t).$$

This is equivalent to the statement

$$\limsup_{n\to\infty}\frac{1}{b_{a^{\leftarrow}(u)}}\log\rho(u)=-\inf_{t>0}t^{-\frac{\zeta+1}{\zeta+1+\alpha\beta}}\Lambda_1^*(\mu+t)$$

and that completes the proof. □

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