RECOVERY OF INTERGRADIENT AND INTERBLOCK INFORMATION IN INCOMPLETE BLOCK AND LATTICE RECTANGLE DESIGNED EXPERIMENTS

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BU-1265-MA

March 1995

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SUMMARY

Spatial analysis and blocking analysis of experimental results are treated separately in the literature. Here we combine these analyses into a single analysis. In addition, the information arising from the distributional properties of differential gradients within incomplete blocks is used to adjust treatment means. We extend Cox's (1958) idea of differential gradients within columns from a Latin square to within blocks for incomplete block and row-columns designed experiments. With this analysis, the restrictions on randomization due to blocking are taken into consideration whereas they are ignored in spatial analysis literature. Some comments on designing experiments and analyzing experimental results to control heterogeneity are presented. A numerical example is used to illustrate the computational procedure.

Key words and phrases: Statistical analyses; Post-blocking; Covariates; Differential trends; Modeling; Design of experiment.

1. Introduction

Trends or gradients in experimental material are sometimes encountered during the conduct of an experiment. Much has been published on spatial analysis but little on connecting the design of experiments with spatial methods of analysis as an effective method of dealing with gradients within an experiment.

Fisher (1935) stated that a Latin square design would be the chief if not the universal experiment design used if the number of treatments in an experiment was in the range of four to eight. Yates (1940) showed how to extend the benefits of the Latin square principle of row and column blocking in an experiment by introducing a new class of designs denoted as lattice square designs. Here the number of treatments $v = k^2$, k a prime or prime power, can be accommodated in k rows and k columns within each of r complete blocks. The balanced lattice square designs require $r = k + 1$ complete block arrangements in which each treatment occurs once with every other treatment in a row and in a column. The semi-balanced lattice square designs require $r = (k+1)/2$ complete block arrangements in which each treatment occurs once with every other treatment either in a row or in a column. Cochran (1943), Kempthorne and Federer (1948), Federer (1950, 1955), Kempthorne (1952), and Federer and Raktoe (1966) presented lattice square designs and analyses for any r and k. To alleviate the restriction that $v = k^2$, Na Nagara (1957) constructed a class of lattice rectangle designs of k rows and s columns and developed the statistical analysis for $v = ks$ treatments, $s < k$. Federer and Raktoe (1965) presented a class of lattice rectangle designs and corresponding analyses for $v = s^m$ treatments in s^r rows and s^c columns where $m = r + c$. Recently John and Whitaker (1993) and Nguyen and Williams (1993) have shown how to construct lattice rectangle designs for $v = ks$ treatments in k rows and s columns with r replicates. A computer program, GENDEX, for constructing lattice rectangle designs (resolvable row-column designs) may be obtained from N-K. Nguyen. Federer and Wright (1988) show how to construct augmented lattice rectangle designs from lattice squares. A general method for recovering interrow and intercolumn information is straightforward by following the generalized procedure for recovering interblock information as given, e.g., by Khare and Federer (1981).

None of the above authors discuss statistical analyses for situations where trends occur within incomplete blocks or within rows or columns. Cox (1958), in an interesting paper, presented a statistical analysis for Latin square designs with differential curvatures (trends) in each column (row) of the Latin square. Since two gradients for blocking do not always appear at right angles to each other, the gradients of one category may vary for each level of the second category. For example, if cows are in different parts of their lactation curve, there will be different curvatures for each cow; if insects or disease penetrate from one corner of the experiment, there will be differential gradients of damage in the various blocks or rows and columns. In the following, we show how to design for possible gradients and then how to remove the effect of remaining gradients through statistical analyses. Suggestions have been made in published papers that blocking be ignored and a spatial analysis be performed as if there were no blocking. This violates the randomization theory in that certain restrictions were used in blocking a set of experimental material and degrees of freedom must be allocated to account for these restrictions. This fact appears to have escaped the notice of persons writing on spatial analyses.

In many situations, incomplete blocks of size two will suffice to control gradients, spotty patches, and other forms of blocking variables causing extraneous variation in an experiment. Federer (1994) presents a simple construction method for incomplete blocks of size two for all even v and for incomplete blocks of size k = 3 for v = 3t. If interblock information is recovered, and it should be, these designs will have high efficiencies (see Federer and Speed, 1987). It is desirable to utilize the blocking qualities of the Latin square, other row-column arrangements, lattice square, and lattice rectangle experiment designs whenever possible to control heterogeneity. In the cases where differential gradients occur within the blocking categories or where an area of the experiment needs to be put into another block, proper statistical analyses can be provided to take care of this extraneous variation. This will be illustrated in the following.

2. Standard Statistical Analyses with Recovery of Interblock Information

The usual linear response models for a resolvable incomplete block experiment design (RIBED) is

$$Y_{ghi} = \mu + \beta_g + \rho_{gh} + \tau_i + \epsilon_{ghi} , \qquad (1)$$

where μ is a general mean effect, β_g is the gth replicate (complete block) effect, ρ_{gh} is the hth incomplete random block within the gth replicate effect and is distributed with mean zero and variance σ_ρ^2, τ_i is the effect of the ith treatment, and ϵ_{ghi} is a random error effect distributed with mean zero and variance σ_ϵ^2. When the design is not resolvable, simply drop the ρ_{gh} term and consider β_g as the incomplete block effect.

The usual response model for a resolvable lattice square or lattice rectangle (row-column) experiment design is

$$Y_{ghij} = \mu + \beta_g + \rho_{gh} + \gamma_{gi} + \tau_j + \epsilon_{ghij}\,; \tag{2}$$

where ρ_{gh} is the hth row effect within replicate g, γ_{gi} is the ith random column effect within replicate g, and the other effects are defined as for (1). Analyses of variance, ANOVAs, for (1) and (2) are given in Table 1.

Insert Table 1 here

Using the values $Y_{ghi} - \bar{y}_{g..}$ to remove the μ and β_g effects from (1), the resulting normal equations are

$$\begin{bmatrix} k\mathbf{I}_{rb} & \mathbf{N}_{rb\times v} \\ \mathbf{N}' & r\mathbf{I}_v \end{bmatrix} \begin{bmatrix} \rho_{rb\times 1} \\ \tau_{v\times 1} \end{bmatrix} = \begin{bmatrix} \mathbf{YB}_{rb\times 1} \\ \mathbf{YT}_{v\times 1} \end{bmatrix}. \tag{3}$$

The block (eliminating treatment effects) sum of squares is

$$\hat{\rho}'\left[\mathbf{YB} - \mathbf{N}\times\mathbf{YT}\big/ r\right] \tag{4}$$

where

$$\hat{\rho} = \left[k\mathbf{I}_{rb} - \mathbf{N}\,\mathbf{N}'\big/ r + \mathbf{J}\big/ r\right]^{-1}\left[\mathbf{YB} - \mathbf{N}\times\mathbf{YT}\big/ r\right], \tag{5}$$

where J is a matrix of ones. Treatment effects with recovery of interblock information are obtained from (3) by first substituting $\left(k + \hat{\sigma}_\epsilon^2 \big/ \hat{\sigma}_\rho^2\right)$ for k and then solving the equations as before. The expected value of B in Table 1 is $\sigma_\epsilon^2 + k(r-1)\sigma_\rho^2 \big/ r$ and of E is σ_ϵ^2.

Again, using values of $Y_{ghij} - \bar{y}_{g...} = Y_{ghij} - (\hat{\mu} + \hat{\beta}_g)$ to remove the μ and β_g effects from (2), the resulting normal equations are:

$$\begin{bmatrix} b\mathbf{I}_{rk} & \mathbf{RC}_{rk\times rb} & \mathbf{NR}_{rk\times v} \\ \mathbf{RC}' & k\mathbf{I}_{rb} & \mathbf{NC}_{rb\times v} \\ \mathbf{NR}' & \mathbf{NC}' & r\mathbf{I}_v \end{bmatrix} \begin{bmatrix} \rho_{rk\times v} \\ \gamma_{rb\times 1} \\ \tau_{v\times 1} \end{bmatrix} = \begin{bmatrix} \mathbf{YR}_{rk\times 1} \\ \mathbf{YC}_{rb\times 1} \\ \mathbf{YT}_{v\times 1} \end{bmatrix}. \tag{6}$$

The row (eliminating treatment and column effects) sums of squares is

$$\hat{\rho}'\left[\mathbf{YR}-(\mathbf{RC}\ \ \mathbf{NR})\begin{bmatrix}\mathrm{k}\mathbf{I}_{rb} & \mathbf{NC}-\mathbf{J}\\ \mathbf{NC}' & \mathrm{r}\mathbf{I}_v\end{bmatrix}^{-1}\begin{bmatrix}\mathbf{YC}\\ \mathbf{YT}\end{bmatrix}\right], \tag{7}$$

where

$$\hat{\rho} = \left[\mathrm{b}\mathbf{I}_{rk}-(\mathbf{RC}\ \ \mathbf{NR})\begin{bmatrix}\mathrm{k}\mathbf{I}_{rb} & \mathbf{NC}-\mathbf{J}\\ \mathbf{NC}' & \mathrm{r}\mathbf{I}_v\end{bmatrix}^{-1}\begin{bmatrix}\mathbf{RC}'\\ \mathbf{NR}'\end{bmatrix}+\mathbf{J}/\mathrm{r}\right]^{-1}\left[\mathbf{YR}-(\mathbf{RC}\ \ \mathbf{NR})\begin{bmatrix}\mathrm{k}\mathbf{I}_{rb} & \mathbf{NC}-\mathbf{J}\\ \mathbf{NC}' & \mathrm{r}\mathbf{I}_v\end{bmatrix}^{-1}\begin{bmatrix}\mathbf{YC}\\ \mathbf{YT}\end{bmatrix}\right]. \tag{8}$$

The sum of squares for column (eliminating row and treatment effects) is

$$\hat{\gamma}'\times\left[\mathbf{YC}-(\mathbf{RC}'\ \ \mathbf{NC})\begin{bmatrix}\mathrm{k}\mathbf{I}_{rb} & \mathbf{NR}-\mathbf{J}\\ \mathbf{NR}' & \mathrm{r}\mathbf{I}_v\end{bmatrix}^{-1}\begin{bmatrix}\mathbf{YR}\\ \mathbf{YT}\end{bmatrix}\right], \tag{9}$$

where

$$\hat{\gamma} = \left[\mathrm{k}\mathbf{I}_{rb}-(\mathbf{RC}'\ \ \mathbf{NC})\begin{bmatrix}\mathrm{k}\mathbf{I}_{rb} & \mathbf{NR}-\mathbf{J}\\ \mathbf{NR}' & \mathrm{r}\mathbf{I}_v\end{bmatrix}^{-1}\begin{bmatrix}\mathbf{RC}\\ \mathbf{NC}'\end{bmatrix}+\mathbf{J}/\mathrm{r}\right]^{-1}\left[\mathbf{YC}-(\mathbf{RC}'\ \ \mathbf{NC})\begin{bmatrix}\mathrm{k}\mathbf{I}_{rb} & \mathbf{NR}-\mathbf{J}\\ \mathbf{NR}' & \mathrm{r}\mathbf{I}_v\end{bmatrix}^{-1}\begin{bmatrix}\mathbf{YR}\\ \mathbf{YT}\end{bmatrix}\right]. \tag{10}$$

The adjusted treatment effects recovering row and column information is

$$\hat{\tau}^* = \left[\mathrm{r}\mathbf{I}_v-(\mathbf{NR}'\ \ \mathbf{NC}')\begin{bmatrix}\mathrm{b}^*\mathbf{I}_{rk} & \mathbf{RC}-\mathbf{J}\\ \mathbf{RC}' & \mathrm{k}^*\mathbf{I}_{rb}\end{bmatrix}^{-1}\begin{bmatrix}\mathbf{NR}\\ \mathbf{NC}\end{bmatrix}+\mathbf{J}\big/\mathrm{k}\right]^{-1}$$

$$\times\left[\mathbf{YT}-(\mathbf{NR}'\ \ \mathbf{NC}')\begin{bmatrix}\mathrm{b}^*\mathbf{I}_{rk} & \mathbf{RC}-\mathbf{J}\\ \mathbf{RC}' & \mathrm{k}^*\mathbf{I}_{rb}\end{bmatrix}^{-1}\begin{bmatrix}\mathbf{YR}\\ \mathbf{YC}\end{bmatrix}\right], \tag{11}$$

where

$$\mathrm{b}^* = \mathrm{b} + \hat{\sigma}_\epsilon^2\big/\hat{\sigma}_\rho^2$$

and

$$\mathrm{k}^* = \mathrm{k} + \hat{\sigma}_\epsilon^2\big/\hat{\sigma}_\gamma^2\,.$$

Variances of differences of treatment effects are obtained as $\hat{\sigma}_\epsilon^2$ times the first factor in (11). An approximate average variance of a difference, which is less than or equal to the correct one, may be obtained as $2\hat{\sigma}_\epsilon^2$ times the 1/v th root of the determinant of the first factor in (11).

If the treatment (eliminating row and column effects) sum of squares is desired, obtain intrablock solutions $\hat{\tau}$ from (11) by using b for b* and k for k*. Then the sum of squares is

$$\hat{\tau}'\left[\mathbf{YT} - (\mathbf{NR}' \ \ \mathbf{NC}')\begin{bmatrix} b\mathbf{I}_{rk} & \mathbf{RC}-\mathbf{J} \\ \mathbf{RC}' & k\mathbf{I}_{rb} \end{bmatrix}^{-1}\begin{bmatrix} \mathbf{YR} \\ \mathbf{YC} \end{bmatrix}\right]. \tag{12}$$

3. Statistical Analyses with Recovery of Intergradient and Interblock Information

The following statistical analysis applies equally well to block designs as to row-column designs. The linear model used in place of model (1) or (2) is either

$$Y_{ghi} = \mu + \beta_g + \rho_{gh} + \pi_{gh} a_{ghi} + \tau_i + \epsilon_{ghi} , \tag{13}$$

$$Y_{ghi} = \mu + \beta_g + \rho_{gh} + \pi_{gh} a_{ghi} + \pi_{g\cdot} a_{ghi} + \tau_i + \epsilon_{ghi} , \tag{14}$$

or

$$Y_{ghi} = \mu + \beta_g + \rho_{gh} + \pi_{gh} a_{ghi} + \pi_{\cdot\cdot} a_{ghi} + \tau_i + \epsilon_{ghi} , \tag{15}$$

where the a_{ghi} are the centered linear regression values of position within block (or row) gh (e.g., for k = 3, the values are −1, 0, and 1 and for k = 4, the a_{ghi} values are −3, −1, 1, and 3 for any block or row), π_{gh} is the linear regression coefficient for block gh and is a random effect distributed with mean zero (13), mean $\pi_{g\cdot}$ (14), or mean $\pi_{\cdot\cdot}$ (15), and variance σ_π^2, and the other effects are defined as for (1) and (2). Note that the orthogonal polynomial values b_{ghi} for quadratic (or higher) regressions δ_{gh} could be added to equations (13) – (15) as well if the situation warranted differential curvilinear regressions within blocks.

The resulting normal equations for equation (13) with $Y_{ghi} - \bar{y}_{g\cdot\cdot} = Y_{ghi} - \hat{\mu} - \hat{\beta}_g$ values are

$$\begin{bmatrix} k\mathbf{I}_{rb} & \mathbf{0}_{rb \times rb} & \mathbf{NB}_{rb \times v} \\ \mathbf{0} & \Sigma a_{ghi}^2 \mathbf{I}_{rb} & \mathbf{NG}_{rb \times v} \\ \mathbf{NB}' & \mathbf{NG}' & r\mathbf{I}_v \end{bmatrix}\begin{bmatrix} \rho_{rb \times 1} \\ \pi_{rb \times 1} \\ \tau_{v \times 1} \end{bmatrix} = \begin{bmatrix} \mathbf{YB}_{rb \times 1} \\ \mathbf{YG}_{rb \times 1} \\ \mathbf{YT}_{v \times 1} \end{bmatrix}, \tag{16}$$

where **0** is a matrix of zeros since the sum of the a_{ghi} in each block gh is zero, **NB** is the block-by-treatment design matrix, **NG** is a matrix of a_{ghi} values for treatment i in block gh, **YB** is a vector of block totals for $Y_{ghi}-\bar{y}_{g\cdot\cdot}$ values, **YG** is a vector of sums of products of a_{ghi} and Y_{ghi} values for each block gh, and the other terms are as defined previously. The restrictions

$$\sum_{h=1}^{b} \hat{\rho}_{gh} = \sum_{i=1}^{v} \hat{\tau}_i = 0$$

are used to obtain the intrablock analysis of variance in Table 2. Intrablock solutions for the various effects are:

$$\hat{\boldsymbol{\rho}} = \left[k\mathbf{I}_{rb} - \mathbf{NB}\left[r\mathbf{I}_v - \mathbf{NG}'\ \mathbf{NG}/C\right]^{-1}\mathbf{NB}' + \mathbf{J}/r\right]^{-1}$$

$$\times\left[\mathbf{YR} - \mathbf{NB}\left[r\mathbf{I}_v - \mathbf{NG}'\ \mathbf{NG}/C\right]^{-1}\left[\mathbf{YT} - \mathbf{NG}'\ \mathbf{YG}/C\right]\right], \tag{17}$$

where $C = \Sigma a_{ghi}^2$ = sum of squares of coefficients in block gh, e.g., $(1)^2 + 0^2 + 1^2 = 2$ for $k = 3$ and $(-3)^2 + (-1)^2 + 1^2 + 3^2 = 20$ for $k = 4$.

$$\hat{\boldsymbol{\pi}} = \left[C\mathbf{I}_{rb} - \mathbf{NG}\left[r\mathbf{I}_v - \mathbf{NB}'\ \mathbf{NB}/k + \mathbf{J}/k\right]^{-1}\mathbf{NG}'\right]^{-1}$$

$$\times\left[\mathbf{YG} - \mathbf{NG}\left[r\mathbf{I}_v - \mathbf{NB}'\ \mathbf{NB}/k + \mathbf{J}/k\right]^{-1}\left[\mathbf{YT} - \mathbf{NB}'\ \mathbf{YB}/k\right]\right]. \tag{18}$$

The sum of squares for block (eliminating treatment and gradient effects) in Table 2 is

Insert Table 2 here

$$\hat{\boldsymbol{\rho}}'\left[\mathbf{YB} - \mathbf{NB}\left[r\mathbf{I}_v - \mathbf{NG}'\ \mathbf{NG}/C\right]^{-1}\left[\mathbf{YT} - \mathbf{NG}'\ \mathbf{YG}/C\right]\right]. \tag{19}$$

The sum of squares for gradient (eliminating block and treatment effects) in Table 2 is

$$\hat{\boldsymbol{\pi}}'\left[\mathbf{YG} - \mathbf{NG}\left[r\mathbf{I}_v - \mathbf{NB}'\ \mathbf{NB}/k + \mathbf{J}/k\right]^{-1}\left[\mathbf{YT} - \mathbf{NB}'\ \mathbf{YB}/k\right]\right]. \tag{20}$$

The block (eliminating treatment but ignoring gradient effects) sum of squares is

$$\left[\mathbf{YB} - \mathbf{NB}\ \mathbf{YT}/r\right]'\left[k\mathbf{I}_{rb} - \mathbf{NB}\ \mathbf{NB}'/r + \mathbf{J}/r\right]^{-1}\left[\mathbf{YB} - \mathbf{NB}\ \mathbf{YT}/r\right]. \tag{21}$$

Note that block and gradient effects are orthogonal to each other but both are nonorthogonal to treatment effects, which accounts for the above form of equations (17) – (20) as opposed to the form for equations (6) – (10). The design matrix being a zero matrix as opposed to being of the form **RC** accounts for this difference.

The expected value of the error mean square E is taken to be σ_ϵ^2. The expected value of the gradient (eliminating block and treatment effects) mean square G has the form

$$\sigma_\epsilon^2 + g_0\sigma_\pi^2 = \sigma_\epsilon^2 + \sigma_\pi^2 C(rk-k-1)/rb \; , \tag{22}$$

as found using *Mathematica*. The expected value of the block (eliminating gradient and treatment effects) mean square B has the following form:

$$\sigma_\epsilon^2 + k_0\sigma_\rho^2 \; , \tag{23}$$

where k_0 for a number of examples using *Mathematica* is given in Table 3. For $v = 16$, $k = 4$, and $r = 5$ for a lattice square design the expected value of the columns (or rows) mean square after eliminating rows (or columns) and treatment effects is $\sigma_\epsilon^2 + 3\sigma_\gamma^2$. The coefficient of 3 is slightly larger than $k_0 = 2.8149$. For g_0, the value is 15 for this example as compared to $C = 20$. When r becomes large, k_0 approaches k and g_0 approaches Σa_{ghi}^2. Solutions for the various variance components then allow for recovery of interblock and inter-gradient information as follows:

$$\hat{\tau} = \left[r\mathbf{I}_v - \mathbf{NB}' \; \mathbf{NB} \Big/ \left(k + \hat{\sigma}_\epsilon^2/\hat{\sigma}_\rho^2 \right) - \mathbf{NG}' \; \mathbf{NG} \Big/ \left(C + \hat{\sigma}_\epsilon^2/\hat{\sigma}_\pi^2 \right) + \mathbf{J} \Big/ k \right]^{-1}$$

$$\times \left[\mathbf{YT} - \mathbf{NB}' \; \mathbf{YB} \Big/ \left(k + \hat{\sigma}_\epsilon^2/\hat{\sigma}_\rho^2 \right) - \mathbf{NG}' \; \mathbf{YG} \Big/ \left(C + \hat{\sigma}_\epsilon^2/\hat{\sigma}_\pi^2 \right) \right] . \tag{24}$$

The variance-covariance matrix for $\hat{\tau}$ is

$$\hat{\sigma}_\epsilon^2 \left[r\mathbf{I}_v - \mathbf{NB}' \; \mathbf{NB} \Big/ \left(k + \hat{\sigma}_\epsilon^2/\hat{\sigma}_\rho^2 \right) - \mathbf{NG}' \; \mathbf{NG} \Big/ \left(C + \hat{\sigma}_\epsilon^2/\hat{\sigma}_\pi^2 \right) + \mathbf{J} \Big/ k \right]^{-1} . \tag{25}$$

Insert Table 3 here

4. A Numerical Example

The numerical example used to illustrate the statistical analysis with recovery of interblock and intergradient information is the one presented by Wadley (1946) and given in Table 12.5 of Cochran and Cox (1957) for an experiment designed as a lattice square. We consider differential gradients within rows as an alternative analysis for columns. Also, since a rather high coefficient of variation, 44%, was obtained from the analysis on counts for the standard lattice square analysis, some alternative analysis and/or transformation of data appears to be required. In addition, the intrarow-column error mean square was slightly larger than the treatment (eliminating row and column effects) mean square. This hardly appears logical since it is unlikely that the null hypothesis would be true for 16 different chemical spray treatments involving a check treatment. The lattice square analyses for counts, square roots of counts, and the arcsine transformation of counts is given in Table 4. The analysis of Section 3 is given in Table 5.

Insert Table 4 here

The experiment was designed as a balanced lattice square with $v = 16$ treatments in $k = 4$ rows and $k = 4$ columns in each $r = 5$ replicates. The 16 treatments were arsenical insecticides applied to cotton plants with a hand dusting machine. The experimental units (plots) were ten rows wide by 70 feet long (about 1/18 of an acre). To allow for border effects, the responses are for the four center rows only. The data are counts of young buds showing attack from boll weevils. Twenty-five squares from each of four rows were counted at three different times and averaged over time. The counts were made in August.

Since the data are average counts per 100 squares or percentages, it is possible that some transformation of the data would be desirable. Two transformations that come to mind are square root and arcsine. The ANOVAs for counts, square root of counts, and arcsine of percentages are given in Table 4. For counts, the coefficient of variation is high, 44%. The statistic is reduced when square roots or arcsines are used but is still rather high, 24% and 25%. One unsettling result of the ANOVAs in Table 4 is that the treatment (eliminating row and column effects) mean square is smaller than the error term. It is unlikely that an experimenter would select 16 insecticides which did not differ. A more likely explanation is that the model, (2), is inappropriate for the experiment.

One possible alternative model is that there are differential linear trends in each row (or column) of the lattice square designed experiment. The polynomial linear regression coefficients are −3, −1, 1, and 3 with a sum of squares of $C = 20$. An ANOVA on counts using model (13) is given in Table 5. The intrarow-gradient error mean square is reduced from 22.67 to 18.97. The treatment (eliminating row and differential gradient effects) mean square, 22.15, is larger than the error mean square. However, the reduction in the error mean square is not large enough. For example, if the counts have a Poisson distribution, the variance should equal the mean and here the variance is approximately twice the mean, 10.905. Using the square root transformation, the theoretical variance should be 1/4 instead of 0.5759, and using the arcsine transformation the theoretical variance is 821/100 versus 21.19. These results indicate that there is extraneous variation present.

Insert Table 5 here

Model (13) appears to be more appropriate than (2) for these data but a more explanatory one perhaps needs to be obtained. Perhaps differential quadratic (curvilinear) regressions need to be included in (13). Since our purpose of demonstrating how to recover interblock and intergradient information has been achieved, we shall not pursue modeling the results further.

5. Post-Blocking in Experiments

During the course of conducting an experiment, events occur which were not controlled by the original blocking for the experiment. For example, a field experiment on alfalfa exhibited a patch of yellowing in a part of the experiment which was probably caused by excessive rain during the previous year. There are two ways of handling this problem. First and probably best is to obtain a measure of amount of yellowing on each experimental unit and then use this measurement as a covariate. Second, a new block for the yellowed area of the experiment can be designated and this can then be taken care of in the analysis as an additional block. Note that this is equivalent to using covariance with a 0 or 1 independent variate to signify the presence or absence of yellowing. The same procedure can be used to handle other situations such as water standing in part of the experiment and insect, disease, or animal damage to a part of the experiment. In marketing experiments, a part of the experiment may be damaged by fire, water, or wind and the part affected can be handled as described above. The following axiom is useful in determining whether or not to use additional blocking:

> **Axiom:** Any event occurring during the course of an experiment which is not caused by or is a response of the treatments in the experiment is a candidate for removal by blocking or covariance.

Likewise, patchy or spotty occurrences can be accounted for in a similar manner. One word of caution here is that some of the events which occur cause a treatment by event interaction and thus should not be treated as blocking variables. One such event is winter heaving (or kill) for a group of perennial varieties such as alfalfa, winter wheat, winter rye, etc. In many of these cases it will not be possible to determine the size of these interactions owing to the complete confounding of some effects.

6. Discussion

Control of within-complete-block heterogeneity is best accomplished with a row-column arrangement within each complete block. Such designs have been denoted as resolvable row-column experiment designs. These designs have the desirable properties of the Latin square design. Hence, as a measure of insurance, resolvable row-column designs should be used whenever the experimenter even suspects that there may be removable variation in two directions with possibly differential trends or gradients in one direction. If an incomplete block design has been used and then trends occur within some or all of the incomplete blocks, the procedure of Section 3 will be useful in removing this type of experimental variation. Differential trends in two directions can also be handled by appropriate statistical analyses. Recovery of interrow and intercolumn information or of interblock and inter-gradient information should always be done when analyzing data. Ignoring this type of information is an inefficient use of resources and information. The information is there, use it!

An alternate analysis [see Fisher (1944), Federer and Schlottfeldt (1954), Outwaithe and Rutherford (1955), Federer *et al.* (1962), and Cox and Meeker (1992) for some related ideas for row-column designs] to that presented in Section 3 is to fit a low degree polynomial, e.g., quadratic, to the rows and to the columns in each complete block of an experiment designed as a lattice rectangle (or an incomplete block in a rectangular layout) design. Then, include differential row-column interaction terms such as linear × linear, linear × quadratic, quadratic × linear, and/or quadratic × quadratic for each complete block to obtain an ANOVA such as the one on the next page:

Source of variation	Degrees of freedom	Mean square
Total	rv	—
Correction for mean	1	—
Complete blocks = R	r−1	—
Treatment (ignoring regressions)	v − 1	—
Row regressions within R+ Column regressions with R+ Row-column interactions Within R (eliminating treatments)	8r	RMS
Intrarow-column regression	(r − 1)(v − 1) − 8r	EMS

In the above, all the blocking (gradient), regression variables have been pooled to obtain RMS. Then for random regression effects within complete blocks, interregression information may be recovered in the same manner as for interblock information. The expected value of RMS may be evaluated using some such program as MATHEMATICA, as was done by Federer (1995).

The computational procedures described in Sections 3 and 4 are easily programmed in GAUSS. Such programs are available upon request from the author. In this connection the computational procedures for incomplete block designs for v treatments in b incomplete blocks of size k with r replicates and of resolvable row-column designs with recovery of interblock and intergradient or of interrow and intercolumn information as given in textbooks is outdated and too specific with the availability of PCs and software packages such as GAUSS.

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Table 1. Standard ANOVAs

Incomplete block designs with v = kb in incomplete blocks of size k.

Source of variation	Degrees of freedom	Sum of squares	Mean square
Total	rv	$\sum_{g=1}^{r} \sum_{i=1}^{v} Y^2_{ghi}$	–
Correction for mean	1	$Y^2_{\cdots} / rv$	–
Replicate = R	r−1	$\sum_{g=1}^{r} Y^2_{g\cdot\cdot} / v - Y^2_{\cdots} / rv$	–
Treatment (ignoring incomplete blocks)	v−1	$\sum_{i=1}^{v} Y^2_{\cdot\cdot i} / r - Y^2_{\cdots} / rv$	–
Incomplete blocks (eliminating treatments)	r(b−1)	See text (4)	B
Intrablock error	(r−1)(v−1)−r(b−1)	By subtraction	E

Resolvable row-column designs with v = kb treatments in k rows and b columns.

Source of variation	Degrees of freedom	Sum of squares	Mean square
Total	rv	$\sum_{g=1}^{r} \sum_{j=1}^{v} Y^2_{ghij}$	–
Correction for mean	1	$Y^2_{\cdots\cdot} / rv$	–
Replicate	r−1	$\sum_{g=1}^{r} Y^2_{g\cdots} / v - Y^2_{\cdots\cdot} / rv$	–
Treatment (ignoring row and column effects)	v−1	$\sum_{j=1}^{v} Y^2_{\cdots j} / r - Y^2_{\cdots\cdot} / rv$	–
Row (eliminating treatment; ignoring column)	r(k−1)	See text (4)	–
Column (eliminating row and treatment effects)	r(b−1)	See text (9)	C
Intrarow-column error	(r−1)(v−1) −r(k−1)−r(b−1)	By subtraction	E
Row (eliminating column and treatment effects)	r(k−1)	See text (7)	R

Table 2.

ANOVA for differential gradients for incomplete blocks of size k using (13) for v = bk.

Source of variation	Degrees of freedom	Sum of squares	Mean square
Total	rv	$\sum_{ghi} Y^2_{ghi}$	–
Correction for mean	1	$Y^2_{...} / rv$	–
Replicate	r − 1	$\sum_{g=1}^{r} Y^2_{g..} / v - Y^2_{...} / rv$	–
Treatment	v − 1	$\sum_{i=1}^{v} Y^2_{..i} / r - Y^2_{...} / rv$	–
Block (eliminating treatments, ignoring gradients)	r(b − 1)	Equation (21) or (4)	–
Gradient (eliminating block and treatment effects)	rb	Equation (20)	G
Error	(r − 1)(v − 1) −rb − r(b − 1)	By subtraction	E
Block (eliminating treatment and gradient effects)	r(b − 1)	Equation (19)	R

Table 3.

Results from *Mathematica* for coefficients of σ_ρ^2 and σ_γ^2.

Design v, k, r	σ_γ^2 Σa_{ghi}^2	coef. = g_0^*	σ_ρ^2 r(k−1)coef.*	k_0^* = coef.	k(r−1)/r
9, 3, 3	2	$\frac{10}{9}$	$\frac{2,707}{390}$	1.1568	2
9, 3, 4	2	$\frac{16}{12}$	$\frac{1,277,310}{101,537}$	1.5725	$\frac{9}{4}$
16, 4, 2	20	$\frac{60}{8}$	$\frac{116}{17}$	1.1373	2
16, 4, 3	20	$\frac{140}{12}$	$\frac{117,839,479}{6,425,622}$	2.0377	$\frac{8}{3}$
16, 4, 4	20	$\frac{220}{16}$	$\frac{70,802,011,943,381}{2,318,100,987,513}$	2.5453	3
16, 4, 5	20	$\frac{300}{20}$	$\frac{413,177,907,339,875,760}{9,785,630,262,982,549}$	2.8149	$\frac{16}{5}$

* coef. = coefficient of corresponding component of variance.

Table 4.

ANOVAs for counts Y, square root of counts, and arcsine of counts of data in Table 12.5 of Cochran and Cox (1957).

Source of variation	Degrees of freedom	Mean squares		
		Y	$\sqrt{Y}$	arcsine Y
Replicate	4	7.89	0.0811	3.39
Treatment (ignoring rows and columns)	15	82.95	1.7833	67.07
Row (eliminating treatment, ignoring column)	15	72.87	1.5727	59.03
Column (eliminating row and treatment)	15	37.31	0.9450	34.70
Intrarow-column error	30	22.67	0.5759	21.19
Row (eliminating column and treatment)	15	68.45	1.5061	56.39
Treatment (eliminating row and column)	15	21.30	0.4716	17.49
Coefficient of variation		44%	24%	25%

Table 5.

ANOVA for differential gradients within rows of the lattice square designed experiment of Table 12.5 (Cochran and Cox, 1957).

Source of variation	Degrees of freedom	Mean square	Expected value of Mean Square*
Replicate	4	–	–
Treatment (ignoring row and gradient)	15	82.95	–
Row (eliminating treatment, ignoring gradient)	15	72.87	–
Gradient (eliminating row and treatment)	20	38.28	$\sigma_\epsilon^2 + g_0\sigma_\pi^2$
Intrarow-gradient error	25	18.97	σ_ϵ^2
Row (eliminating gradient and treatment)	15	58.94	$\sigma_\epsilon^2 + k_0\sigma_\rho^2$
Treatment (eliminating and column)	15	23.15	–

Coefficient of variation 40%.

* $g_0 = \Sigma a_{ghi}^2(kr-k-1)/rb = 20(20-4-1)/20 = 15.$

$k_0 = 2.8149$ from Table 3.

$\hat{\sigma}_\epsilon^2 = 18.97,\ \hat{\sigma}_\rho^2 = 14.20,\ \hat{\sigma}_\pi^2 = 1.29.$