

**Gravitational Waves
From Binary Systems:
The Conservative Self-Force
via Worldline Actions
and Hamiltonians
and
Post-Newtonian
Tidal Effects**

A Dissertation

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Gravitational Waves from Binary Systems: The Conservative Self-Force via Worldline Actions and Hamiltonians, and Post-Newtonian Tidal Effects.

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Binary systems of compact massive objects like black holes and neutron stars produce strong gravitational waves, propagating disturbances in the fabric of spacetime, which are likely to be directly detected for the first time in this decade. While the expected gravitational wave signals should contain information that can help to answer many outstanding questions in physics and astrophysics, it is crucial, to that end, to be able to predict in great detail the general forms of the expected signals. Chapter 1 of this thesis broadly reviews some of the analytic approximation techniques used to solve the relativistic two-body problem. This sets the stage for the presentation of the original work in the following chapters, which concerns two separate aspects gravitational wave generation by binary systems.

Chapters 3 and 4 concern tidal effects in relativistic binaries. Tidal forces significantly influence the orbital dynamics and gravitational wave emissions of close binaries containing neutron stars, and this imprints on the gravitational wave signal information about the mysterious interior structure of a neutron star. We derive the tidal perturbations to binary orbital dynamics, working in the first-post-Newtonian approximation to general relativity, and working to linear order in the spins and mass-quadrupoles of the constituent bodies. While many of our important results are valid for arbitrarily structured bodies, we also specialize to the case in which the quadrupolar deformations respond linearly and instantaneously to the applied tidal fields. In that case, we derive the quadrupolar tidal corrections to the gravitational wave signal from a neutron star binary in a circular orbit.

Chapter 2 considers the following question: For a small body orbiting a large black hole, can the conservative part of the self-force experienced by the small body be encoded in an action principle or a Hamiltonian system involving only the body's worldline? Such formulations of the

conservative self-force dynamics could lead to more efficient computational methods for solving the extreme-mass-ratio two-body problem. We present our interpretations and investigations of this question, exploring various further questions that arise. We present a proof that, for a small body orbiting a Schwarzschild black hole, there does exist a Hamiltonian system which encodes the linear-order conservative self-force sourced by the osculating geodesic.

BIOGRAPHICAL SKETCH

Justin was born to Rose and Lanny Vines in Cabot, Arkansas on June 26, 1984. He attended Mills High School in Little Rock, Arkansas, where he intensely focused on playing the violin, learning everything he knows about music from Mr. Hatch and his good friend Klansee Reynolds. While planning to study music in college, in his senior year of high school, his AP physics class with Mr. Gaston changed his mind.

He attended the University of Arkansas, in Fayetteville, obtaining a Bachelor of Science degree in Physics, with minors in Mathematics and French. He did research in theoretical quantum optics with Reeta Vyas, and was awarded the Barry Goldwater Scholarship for that work. He was a physics TA in his senior year, working closely with and learning much from John and Gay Stewart.

He came to Cornell University in Ithaca, New York as a physics Ph.D. candidate in 2006. In addition to learning much about all kinds of physics, TAing for many different physics classes (from freshman mechanics to graduate general relativity, multiple times for most) and loving it, and conducting research in general relativity, he has enjoyed playing music with his friends and in some school concerts and exploring the numerous incredible gorges of the Ithaca area.

This thesis is dedicated to my parents Lanny and Rose Vines and my grandparents Vernon and Alice Driskill.

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PREFACE

Citations to Previously Published Work

Chapters 3 and 4 of this thesis have been adapted from the following two previously published or submitted articles, respectively.

Justin Vines, Tanja Hinderer, and Éanna Flanagan. Post-1-Newtonian tidal effects in the gravitational waveforms from binary inspirals. *Physical Review D*, 83:084051, 2011.

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Justin Vines and Éanna Flanagan. Post-1-Newtonian quadrupole tidal interactions in binary systems. Submitted to *Physical Review D*.

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Permission from coauthors has been granted for these works to be included in this dissertation.

List of Abbreviations

The following abbreviations are commonly used in this document.

- BDF – *Bini-Damour-Faye*, referring to their work on post-Newtonian tidal effects [52].
- BH – *Black Hole*.
- CoM – *Center of Mass* (-Energy).
- DN – *Damour-Nagar*, referring to their work on post-Newtonian tidal effects [37, 94].
- DSX – *Damour-Soffel-Xu*, referring to their framework for post-Newtonian celestial mechanics [29, 33, 47, 48].
- DW – *Detweiler-Whiting*, referring to their self-force regularization method [67].
- EOB – *Effective-One-Body*, referring to the formalism for treating relativistic binaries [22].
- EoM – *Equation of Motion*.
- EMRI – *Extreme-Mass-Ratio Inspiral*.
- GR – *General Relativity*.
- GW – *Gravitational Wave*.
- NS – *Neutron Star*.
- PM – *Post-Minkowskian*, with n PM referring to the n th post-Minkowskian approximation.
- PN – *Post-Newtonian*, with n PN referring to the n th post-Newtonian approximation.
- RF – *Racine-Flanagan*, referring to their work on post-Newtonian celestial mechanics [31].
- VF – *Vines-Flanagan*, referring to their work on post-Newtonian tidal effects [114].
- w.r.t. – *with respect to*.

CHAPTER 1

THE TWO-BODY PROBLEM IN GENERAL RELATIVITY AND GRAVITATIONAL WAVES

1.1 Motivation

The existence of gravitational waves (GWs), ripples in the geometry of spacetime, was predicted by Einstein in 1915 as a central element of his theory of general relativity (GR) [1]. Though there is strong evidence confirming his predictions (from observations of pulsars in binary systems [2]), GWs have yet to be directly detected by scientists on Earth. This is the goal of contemporary ground-based experiments like LIGO [3], VIRGO [4], and GEO [5], and future space-based experiments like (e)LISA [6], which use large-scale laser interferometry to ‘listen’ for GWs reaching the Earth from sources in distant parts of the Universe.

The advent of GW astronomy will open an entirely new window on the Universe, and we may see in it things we have yet to imagine. There are, however, several sources of GWs we have imagined and expect or hope to detect with these experiments.¹ Along with the excitement of the first direct detections, the expected GW signals should bring a wealth of insights into astrophysics and cosmology which are inaccessible by electromagnetic observations.

Probably the most certain sources for these detectors, those for which the detectors are designed (more or less), are binaries containing black holes (BHs) of various sizes and neutron stars (NSs).² Being the densest (known) astronomical bodies makes close binaries of these objects the sources of the strongest GWs. Such binaries undergo inspiral, an initially slow but accelerating decay of the

¹In addition to sources falling under the broad category of inspiraling binaries, we expect to detect GWs from, for example, supernovae, spinning deformed neutron stars, various accretion processes, and perhaps more exotic sources, like cosmic strings or a cosmic GW background from the early Universe [3, 6].

²Space-based GW detectors should also detect white dwarf binaries within the Milky Way [6].

orbit due to GW emission, emitting ever stronger GWs as the bodies approach one another and eventually merge. A primary LIGO-type source is a binary containing NSs and/or few-solar-mass BHs, in any of the pairings NS-NS, BH-NS, or BH-BH, whose signal will be in the 10^1 - 10^3 Hz band in the last few minutes before merger. A primary source for space-based detectors is a few-solar-mass compact object spiraling into a supermassive BH (10^5 - $10^8 M_\odot$), known as an extreme-mass-ratio inspiral (EMRI), which can be in the 10^{-4} - 10^{-1} Hz band for years.

To extract a signal from the detector noise, especially in these early days of GW astronomy, GW detectors must already know what they are looking for; they must use matched filtering using theoretical templates with as few adjustable parameters as possible (for some classes of sources). Thus, to predict in great detail the GW signals from binary inspirals has been an occupation for many a physicist in recent decades. This opening chapter will selectively review some aspects of this historied and ongoing effort to solve the relativistic two-body problem.

The basic assumption in the field is that inspiraling binaries will be governed by the equations of GR as written down by Einstein in 1915. This is motivated by the success of GR in every experimental test to which it has yet been subjected [7]; probably most relevant here are the decades of observations of the decaying orbits of binary pulsars [2, 8]. It is possible that deviations from GR might become important only in the strong-field regime which GW detectors may be the first to probe, and this exciting prospect has been studied at length [9, 10]. But still we expect that GR will survive as the leading-order theory of gravity with only small potential corrections in most of the regimes to be probed.

For some types of sources, gravity is the only physics needed to predict the GW signal. This is true of all binary systems sufficiently early in their lifetimes, when the bodies can be treated effectively as spinning point masses, and the inspiral evolution and GW emissions are determined only by the bodies' initial masses, positions, and spins and by Einstein's equations. Remarkably, the dynamics of a BH-BH binary, which is vacuum everywhere, is *completely* determined by the BHs' initial masses, positions, and spins and by Einstein's equations, for all times, including the eventual merger to form a single BH (assuming unmodified GR).³ Even if measured BH-BH signals completely conform to GR expectations, they will still provide valuable astrophysical information. In

³This is because, in GR, the structure of an isolated BH is *entirely* determined by its mass and spin (and the external gravitational fields it experiences). A BH can have no other internal degrees of freedom (save an [astrophysically irrelevant] electric charge), as stated by the 'no-hair theorem' [11].

addition to yielding statistics about binary populations, inspiral GWs can act as a kind of ‘standard siren’, allowing distance measurements which probe cosmological parameters [12, 13].

The above picture, with all properties of the orbiting bodies ‘effaced’ save their masses and spins, remains mostly true even when the BHs are replaced by material bodies like NSs, but not entirely. The internal physics of a NS will begin to influence the GW signal late in an inspiral when it develops a tidal deformation, and during merger. It is hoped that such signals will provide valuable insights into NS internal structure, which is still relatively unconstrained by electromagnetic observations [14, 15].

1.2 Regimes and solution techniques

The endeavor to predict GW signals from binary inspirals boils down to the task of solving Einstein’s equations. Since there are no exact analytic solutions describing any realistic version of a binary inspiral, one has two options: direct numerical integration of Einstein’s equations, or use of analytic approximations valid in certain regimes.

Regardless of the nature of the bodies composing the binary, the choice of solution technique depends most importantly on the bodies’ masses M_1 and M_2 and their typical orbital separation r . These scales define two useful dimensionless parameters. First is the symmetric mass ratio,

$$\nu \equiv \frac{M_1 M_2}{M^2} = \frac{\mu}{M},$$

where $M = M_1 + M_2$ is the total mass and μ is the reduced mass. The second, the ‘post-Newtonian’ parameter,

$$\epsilon \equiv \frac{GM}{rc^2} \sim \frac{v^2}{c^2},$$

characterizes the strength of *orbital* gravity (as opposed to the bodies’ self gravity), which relates to typical orbital speeds v . A diagram of the parameter space for the two-body problem is shown in Figure 1.

When neither of these parameters is small, for comparable masses ($\nu \sim 1$) in the strong-field, relativistic regime ($\epsilon \sim 1$), one has no choice but to solve the full Einstein equations numerically. This effort has seen much activity and great successes in recent years, treating the last few orbits and merger of comparable-mass BH-BH, BH-NS, and NS-NS binaries. The BH-BH merger problem

		strong-field, relativistic $\epsilon \sim 1$	weak-field, slow-motion $\epsilon \ll 1$	$\epsilon \rightarrow 0$
comparable masses	$\nu \sim 1$	Numerical Relativity	Post- Newtonian Gravity	Newtonian Gravity
extreme mass ratio	$\nu \ll 1$	BH Pert. Th. & 2-Timescale Exp.		
test mass limit	$\nu \rightarrow 0$	Geodesic Motion in BH Sptm.		

Figure 1: The regimes and solution techniques for relativistic binary systems, parametrized by the symmetric mass ratio ν and the post-Newtonian expansion parameter ϵ .

has finally been solved by the landmark work of Refs. [16, 17, 18], and reviews of the ongoing investigations of BH-NS and NS-NS mergers can be found e.g. in Refs. [19, 20].

In the weak-field, slow motion regime ($\epsilon \ll 1$), for any mass ratio, one can employ the post-Newtonian (PN) approximation. Formally, GR reduces to Newtonian gravity in the limit where the speed of light c goes to infinity ($\epsilon \rightarrow 0$), and n th-post-Newtonian (n PN) gravity is the theory obtained by keeping the $O(c^{-2n}) = O(\epsilon^n)$ corrections.⁴ Post-Newtonian theory provides a good description of the earlier stages of an inspiral, before the binary evolves into the strong-field regime at the end of its lifetime. This approach has also been vigorously and fruitfully pursued in recent years, attempting to push it ever closer to the strong-field regime by going to ever higher PN orders [21].⁵

Finally, for the case of an EMRI, having very different masses ($\nu \ll 1$), with a small body in the strong-field ($\epsilon \sim 1$) region near a large BH, the PN expansion fails, and it is prohibitively difficult for numerical simulations to resolve the disparate scales involved. But another approximation technique becomes useful in this regime, one that uses the mass ratio ν as a small expansion parameter. In

⁴The PN approximation is appropriate for describing the orbital dynamics of a binary with $\epsilon \ll 1$, but not for describing the propagation of GWs away from such a binary. A post-Newtonian description of the ‘near zone’ must be matched onto a post-Minkowskian (PM) description of the ‘far zone’ gravitational field. While PM gravity is obtained by perturbation theory on a background Minkowski spacetime with G as the expansion parameter (at fixed c) [see Secs. 1.3 and 1.5], PN gravity is obtained by expanding the theory about a set of Newtonian structures with $1/c^2$ as the expansion parameter (at fixed G) [see Secs. 1.3 and 1.5].

⁵Attempts to somehow extrapolate from PN gravity into the strong-field regime, aided by data from NR simulations, have led to the effective-one-body (EOB) formalism [22].

the limit $\nu \rightarrow 0$, the problem reduces to a test particle moving geodesically in the strongly curved but stationary spacetime of the large BH, which in most astrophysical circumstances will be the Kerr spacetime of a spinning BH. When $\nu \ll 1$, one can use BH perturbation theory, considering the $O(\nu)$ (or higher-order) perturbations to a Kerr background created by an orbiting object of small but finite mass. These perturbations carry GWs out to infinity and exert a ‘self-force’ which causes the object to deviate from geodesic motion and spiral into the BH.⁶ Computing inspirals of small bodies into a Kerr BH is an active area of current research, reviewed e.g. in Refs. [24, 25].

In either PN theory or BH perturbation theory, the simplest formulation of an inspiral treats the orbiting bodies (both bodies for a PN binary, or the small body in an EMRI) as structureless point particles with conserved rest masses.⁷ Even for realistic extended bodies, the point-particle model still gives the correct orbits for the bodies’ center-of-mass (CoM) worldlines, to leading order in R/r , the bodies’ sizes R over their separations r . This is in part because the external gravitational field produced by any body, to leading order in R/r , is a spherically symmetric monopole field, identical to that of a point particle.

Finite-size corrections, which couple the orbits to the bodies’ internal structure and dynamics, can be usefully organized into multipole expansions. Following the mass monopole (the total mass-energy) and the mass dipole (which encodes the CoM worldline), a body’s higher-order ‘mass multipole moments’ (the quadrupole, octupole, etc.) characterize the non-sphericity of its mass-energy distribution. The non-sphericity (which can be either intrinsic or induced by external gravity) gives rise to tidal perturbations to the body’s orbit, with the effects of higher-order moments suppressed by increasing powers of R/r . The internal distribution of momentum (or mass current) also affects a body’s motion in GR. A body’s spin is its first ‘current multipole moment’, followed by the current quadrupole, etc., and these moments give rise to orbital perturbations which again decrease with higher powers of R/r for higher-order moments.⁸

⁶Black hole perturbation theory can be used to compute inspirals over short timescales, but must be supplemented by two timescale techniques to enable computations of complete inspirals [23].

⁷Though there are fundamental problems with the idea of a finite-mass point particle in full nonlinear GR (since bodies will always form BHs of finite size if sufficiently compressed), one can still make sense of such an idea in both PN gravity and in linear perturbation theory (requiring much more effort in the latter case).

⁸The mass and current multipoles of a mass-energy distribution in GR are analogous, respectively, to the electric and magnetic multipoles of a charge distribution in electromagnetism (to a certain extent). Such moments can be defined within the context of post-Newtonian gravity, post-Minkowskian gravity (see e.g. Ref. [21]), or perturbation theory on a stationary curved background (see e.g. Refs. [26, 27]).

Black holes and neutron stars have strong internal gravity, and thus, their internal structure cannot be accurately described either by PN theory or by perturbation theory over the external gravitational field in which the BHs or NSs are located. Nevertheless, these approximations can be used to accurately describe the *orbits* of BHs and NSs, in certain regimes. This is because, at sufficient distances from a BH or NS, its gravity does become sufficiently weak to be described by PN gravity or perturbation theory. Within either scheme, one can define an effective CoM worldline and effective multipole moments which characterize a body—these quantities are encoded in and defined by the body’s weak gravitational field at sufficient distances, without reference to its interior. One can then show that the body’s CoM worldline obeys the same equation of motion (involving its multipoles and the external spacetime) that would be derived for a body whose interior can be described by the given weak-field approximation [28].

The remainder of this opening chapter reviews some aspects of each of the aforementioned approximation techniques, gathering foundations for the original work presented in the subsequent chapters:

- Chapters 3 and 4 are concerned with tidal effects in binaries in the PN regime, targeted toward the tidal interactions of NSs in LIGO-band binaries. Working to 1PN order in the orbital gravity, and keeping the mass, spin, and mass-quadrupole terms in the bodies’ multipole expansions, Chapter 3 presents a derivation of the orbital dynamics of such a binary, in various levels of specialization, from complete generality to the case of bodies with adiabatically induced tidal deformations on circular orbits. Chapter 4 derives the gravitational wave signal from such a binary.
- Chapter 2 presents an exploration of some issues concerning EMRIs. Focusing on a small body coupled to a scalar field in a stationary vacuum spacetime, a toy model with many of the features of gravitational perturbations on such a spacetime, we discuss the computation of the self-force experienced by the body. We explore the possibility of encoding the self-force equation of motion, or rather a version of it valid in the adiabatic limit, into an action principle or Hamiltonian system involving only the body’s worldline. The goal of these explorations is to formulate computational methods that are more efficient than methods currently being explored [24].

With those ends in mind, Chapter 1 continues as follows, outlining foundational results and hoping to draw useful connections:

- Section 1.3 briefly discusses GR and its relation to Newtonian gravity.
- Section 1.4 discusses the multipole decomposition of the Newtonian gravitational field surrounding an isolated system, and how it can be used to formulate the orbital equations of motion for arbitrary bodies in Newtonian orbits, even bodies with strong internal gravity.
- Section 1.5 describes the post-Newtonian expansion and its corrections to binary orbital dynamics, and outlines the work of Chapter 3.
- Section 1.6 reviews the description of freely propagating GWs on a flat background via the linearized or first-post-Minkowskian approximation, and the extension to the n th-post-Minkowskian approximation.
- Section 1.7 discusses the calculation of GW signals from post-Newtonian sources, achieved by matching a near-zone post-Newtonian spacetime to a far-zone post-Minkowskian spacetime, yielding the famous Einstein quadrupole formulae for GW emission and their post-Newtonian corrections. This section fills in details of the justification of the methods used in Chapter 4.
- Finally, Section 1.8 reviews the description of extreme-mass-ratio inspirals using BH perturbation theory to calculate the self-force on and GWs emitted by an orbiting point-particle, providing context for the investigations of Chapter 2.

1.3 General relativity and Newtonian gravity

General relativity describes spacetime as a pseudo-Riemannian geometry which is locally Minkowskian, with the effects of gravity arising from its curvature. With coordinates x^μ with $\mu = 0, 1, 2, 3$, the geometry is specified by the metric components $g_{\mu\nu}(x)$,⁹ which give the ‘interval’ ds between infinitesimally separated events according to

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu. \quad (1.3.1)$$

The dynamics of the geometry is governed by Einstein’s field equation

$$G^{\mu\nu} = \frac{8\pi G}{c^4} T^{\mu\nu}, \quad (1.3.2)$$

where $G^{\mu\nu}$ is the Einstein curvature tensor constructed from the metric and $T^{\mu\nu}$ is the stress-energy tensor of matter. This equation implies (among other things) the local conservation of stress-energy, $\nabla_\mu T^{\mu\nu} = 0$, which implies (among other things) that test masses move along geodesics.

A simple and invaluable approximate solution to Eq. (1.3.2) is a ‘Newtonian spacetime’, for which the equations of GR reduce to those of Newtonian gravity, giving the zeroth-order approximation in the post-Newtonian expansion. With a time coordinate $x^0 = t$ and Cartesian spatial coordinates x^i with $i = 1, 2, 3$, the metric (1.3.1) of a Newtonian spacetime reads

$$ds^2 = - \left(1 + \frac{2\phi}{c^2} \right) c^2 dt^2 + dx^i dx^i + O(c^{-2}), \quad (1.3.3)$$

where $\phi(t, \mathbf{x})$ is the Newtonian gravitational potential, with $1/c^2$ being used as a small expansion parameter. Einstein’s field equation (1.3.2) for this metric reduces to the Newtonian field equation

$$\nabla^2 \phi = 4\pi G \rho, \quad (1.3.4)$$

a Poisson equation sourced by the (rest) mass density $\rho(t, \mathbf{x}) = T^{00} + O(c^{-2})$.

⁹Concerning notation and conventions: 4D spacetime indices are denoted by Greek letters, $\mu, \nu, \rho, \sigma, \dots$, and 3D spatial indices by Latin letters, i, j, k, m, \dots , with the Einstein summation convention in force for both types. Spatial indices are always contracted with the Kronecker delta δ_{ij} , and up or down placement of spatial indices has no meaning whatsoever (usually being decided by aesthetics). Sometimes \vec{v} is used for a 4-vector v^μ , and \mathbf{v} for a 3-vector v^i . For a 3-vector \mathbf{v} , $v^2 = v_i v_i = \delta_{ij} v_i v_j$.

Spacetime covariant derivatives are denoted by ∇_μ . Partial derivatives are denoted by ∂_μ for spacetime, by ∂_i for space, and by overdots (or ∂_t) for the time coordinate t . Until indicated otherwise (in Sec. 1.8), ∇^2 denotes the flat 3D Laplacian, $\nabla^2 = \delta_{ij} \partial_i \partial_j$.

The proper time τ along a timelike curve $x^\mu = z^\mu(t) = (t, \mathbf{z}(t))$ in the geometry (1.3.3) is

$$\begin{aligned}\tau &= \frac{1}{c} \int dt \sqrt{-g_{\mu\nu}(z) \frac{dz^\mu}{dt} \frac{dz^\nu}{dt}} = \int dt \left(1 + \frac{\phi(z)}{c^2} - \frac{\mathbf{v}^2}{2c^2} + O(c^{-4}) \right) \\ &= t - \frac{1}{mc^2} \int dt \left(\frac{m\mathbf{v}^2}{2} - m\phi(z) \right) + O(c^{-4}),\end{aligned}\tag{1.3.5}$$

where $\mathbf{v} = d\mathbf{z}/dt$, and we have inserted factors of a mass to be suggestive. At leading order in $1/c^2$, we have $\tau = t$, which gives Newton's absolute time, the same for all observers on any trajectory. The $O(c^{-2})$ terms encode the leading-order relativistic time dilation due to motion and to gravity. While one does not consider these terms in defining time in a strictly Newtonian interpretation, these relativistic corrections to τ give rise to the Newtonian equations of motion, in the following sense. Test masses in GR move along timelike geodesics, which are curves which extremize τ , the functional (1.3.5) of the trajectory $\mathbf{z}(t)$. But we recognize this as $\tau = t - S/mc^2 + O(c^{-4})$, where S is the action of a test mass in Newtonian gravity, whose extremization leads to the equation of motion

$$m \frac{d^2 z_i}{dt^2} = -m \partial_i \phi,\tag{1.3.6}$$

giving the Newtonian gravitational force on a free point mass.¹⁰

Roughly speaking, Newtonian gravity (and by extension, PN gravity) will provide a good description of a binary system whenever speeds and potentials in the system obey $v^2 \ll c^2$ and $\phi \ll c^2$, but this statement has important caveats.

Firstly, the instantaneous action at a distance inherent in the Newtonian field equation (1.3.4) is replaced in GR by field information traveling at the speed of light. Sufficiently far from the system, no matter how weak its gravity, retardation effects must be taken into account (along with other GR effects). This restricts the validity of a Newtonian (or PN) description of the gravitational field to the near zone, out to distances D from the system for which the light travel time is much less than the timescale on which the gravitational field is changing, $D/c \ll \phi/\dot{\phi} \sim r/v \sim (\text{GW wavelength})/c$.¹¹ Beyond this region, in the far zone, one must switch to a post-Minkowskian description of the gravitational field to properly describe GWs propagating to infinity (see Secs. 1.6 and 1.7).

¹⁰More generally, for a continuous distribution of matter, also interacting via forces other than gravity, Newtonian gravity is completely described by [in addition to the field equation (1.3.4)] the Euler equation (3.2.2) and the continuity equation (3.2.3), which express stress-energy conservation at Newtonian order, discussed further in Sec. 3.2.I below.

¹¹Note that the condition that a binary be within its own near zone, $r/c \ll \phi/\dot{\phi} \sim r/v \sim \lambda_{\text{GW}}/c$ is equivalent to (the square root of) the slow-motion, weak-field condition $v^2/c^2 \sim \phi/c^2 \ll 1$.

Secondly, it need not be the case that gravity is weak throughout the system. In particular, even if $\phi/c^2 \sim 1$ near or within the two bodies (as for BHs and NSs), the orbital dynamics will still be nearly Newtonian as long as there exists a region surrounding and between the bodies where $\phi/c^2 \ll 1$. The next section explains how this is so, employing the multipole decomposition of the gravitational field.

1.4 Multipole expansions and black holes in Newtonian orbits

Consider an isolated system \mathcal{S} (which could be a body in a binary, or the whole binary), and say, for the moment, that Newtonian gravity is valid throughout the system. In a finite vacuum region that surrounds the system, a buffer region, the solution to the field equation (1.3.4) for the Newtonian potential ϕ can be cleanly split [thanks to the linearity of Eq. (1.3.4)] into two parts, $\phi = \phi_{\text{int}} + \phi_{\text{ext}}$, each satisfying the vacuum field equation $\nabla^2\phi = 0$. The internal part ϕ_{int} is generated by the mass density ρ within the system, given by the standard inversion of Eq. (1.3.4),¹²

$$\phi_{\text{int}}(t, \mathbf{x}) = - \int_{\mathcal{S}} d^3x' \frac{\rho(t, \mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|}, \quad (1.4.1)$$

and the external part ϕ_{ext} is generated by the rest of the Universe, beyond the buffer region.

The solution (1.4.1) for ϕ_{int} can be expressed as a multipole expansion at large \mathbf{x} by using the Taylor series

$$\frac{1}{|\mathbf{x} - \mathbf{x}'|} = \frac{1}{r} - x'_i \partial_i \frac{1}{r} + \frac{1}{2} x'_{\langle i} x'_{j \rangle} \partial_i \partial_j \frac{1}{r} - \dots + \frac{(-)^\ell}{\ell!} x'_{\langle i_1} \dots x'_{i_\ell \rangle} \partial_{i_1} \dots \partial_{i_\ell} \frac{1}{r} + \dots,$$

where $r = |\mathbf{x}|$, $\partial_i = \partial/\partial x_i$, angular brackets denote symmetric-tracefree (STF) projection of enclosed indices¹³, and i_1, \dots, i_ℓ are ℓ different spatial indices. With this, Eq. (1.4.1) becomes

$$\phi_{\text{int}} = -\frac{M}{r} + M_i \partial_i \frac{1}{r} - \frac{1}{2} M_{ij} \partial_i \partial_j \frac{1}{r} + \dots - \frac{(-)^\ell}{\ell!} M_{i_1 \dots i_\ell} \partial_{i_1} \dots \partial_{i_\ell} \frac{1}{r} + \dots \quad (1.4.2)$$

¹²Here and from now on (except when otherwise noted), we set $G = 1$.

¹³The presence of the STF projections arises as follows: $\partial_{i_1} \dots \partial_{i_\ell}(1/r)$ is a STF tensor because partial derivative commute and because $\nabla^2(1/r) = 0$ (away from $\mathbf{x} = 0$). Contraction of an arbitrary tensor with a STF tensor projects out the STF part of the former. Further discussion of STF projections is given below in Secs. 3.1.V and 3.2.II, or see Ref. [28] or [29].

where

$$\begin{aligned}
M(t) &= \int_{\mathcal{S}} d^3x \rho(t, \mathbf{x}), & M_i(t) &= \int_{\mathcal{S}} d^3x x_i \rho(t, \mathbf{x}), & M_{ij}(t) &= \int_{\mathcal{S}} d^3x x_{\langle i} x_{j \rangle} \rho(t, \mathbf{x}), \\
&\dots & M_{i_1 \dots i_\ell}(t) &= \int_{\mathcal{S}} d^3x x_{\langle i_1} \dots x_{i_\ell \rangle} \rho(t, \mathbf{x}), & \dots &
\end{aligned} \tag{1.4.3}$$

are the system's STF mass multipole moments: the monopole M ($\ell = 0$), dipole M_i ($\ell = 1$), quadrupole M_{ij} ($\ell = 2$), etc. The internal potential (1.4.2) depends on time only through these moments (which depend only on time), with the spatial dependence given by the derivatives of $1/r$. The monopole M is the system's total mass, the dipole M_i its center of mass position (times M), and the higher-order moments characterize the non-sphericity of the system's mass distribution. We see that the ℓ th moment scales at most like MR^ℓ where R is the size of the system, and its contribution to the potential then scales like $(M/r)(R/r)^\ell$ at most.

Without regard for how it is produced by the external Universe, we can similarly expand the external potential ϕ_{ext} in a Taylor series about $\mathbf{x} = 0$:

$$\phi_{\text{ext}} = -G_{(0)} - G_i x_i - \frac{1}{2} G_{ij} x_i x_j - \dots - \frac{1}{\ell!} G_{i_1 \dots i_\ell} x_{i_1} \dots x_{i_\ell} - \dots \tag{1.4.4}$$

where

$$G_{i_1 \dots i_\ell}(t) = - [\partial_{i_1} \dots \partial_{i_\ell} \phi_{\text{ext}}(t, \mathbf{x})]_{\mathbf{x}=0} \tag{1.4.5}$$

is the ℓ th tidal moment. The tidal moments are also STF tensors, as follows from $\nabla^2 \phi_{\text{ext}} = 0$. The moment $G_{(0)}$ ($\ell = 0$) is a spatially constant piece of the potential, G_i ($\ell = 1$) gives a uniform external gravitational field, and the higher-order moments characterize tidal fields from the external Universe. The expansions (1.4.2) and (1.4.4) in terms of STF tensors are entirely equivalent to expansions in terms of spherical harmonics [30].

Through Eqs. (1.4.2) and (1.4.4), the multipole moments, characterizing the system, and the tidal moments, characterizing the external Universe, are encoded in the vacuum gravitational field surrounding the system.

In Eq. (1.4.3), the system's multipole moments are defined as integrals over the interior of the system of its mass density $\rho(t, \mathbf{x})$, assuming that the system can be described by such a Newtonian matter distribution. Now suppose that this is not the case: the system may have strong internal gravity, and could even contain a BH, having no matter but still producing a gravitational field.

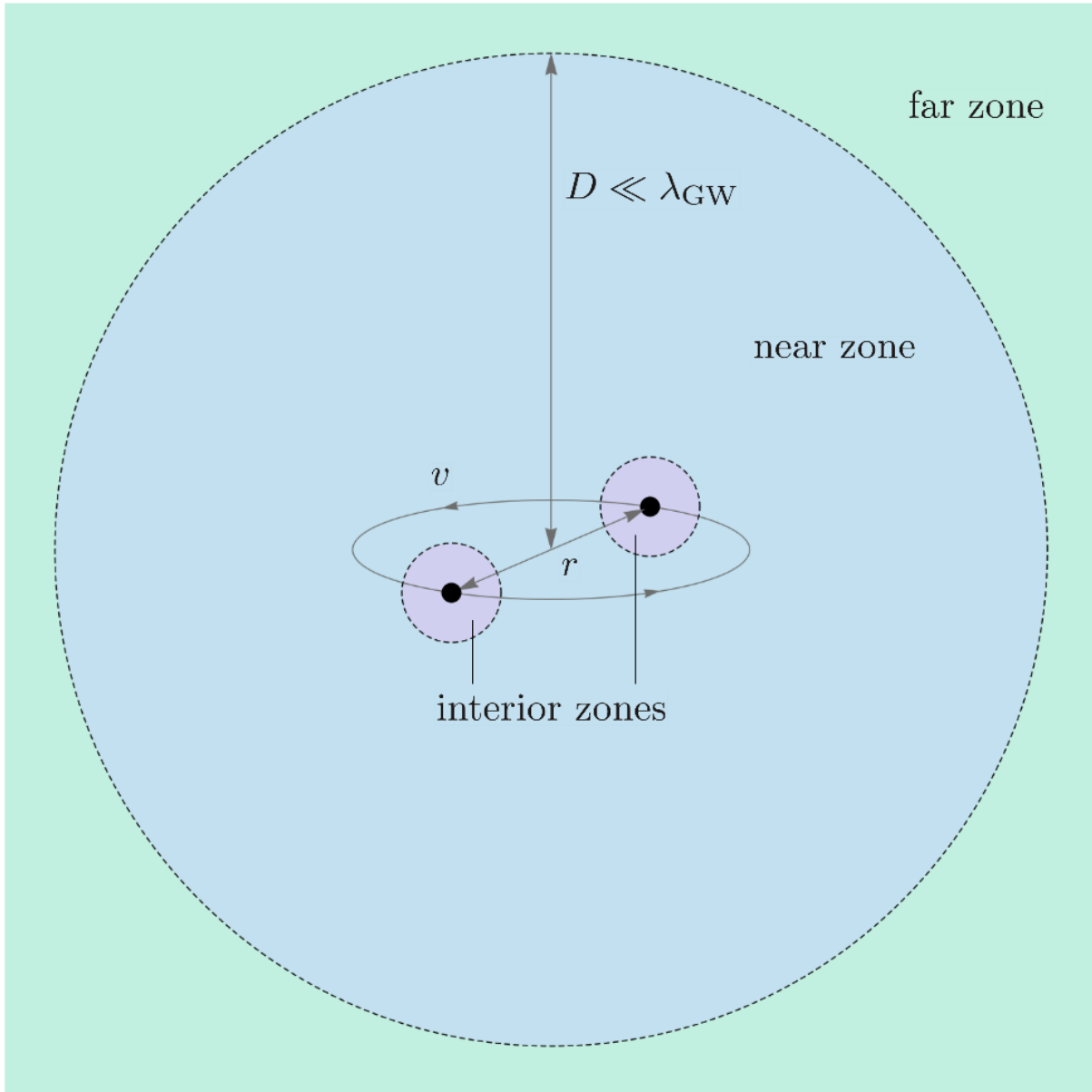


Figure 2: A (post-)Newtonian binary of bodies with strong internal gravity (like neutron stars or black holes). The regions of strong gravity are confined to the (violet) interior zones, which include vacuum regions surrounding the bodies. In the (blue) near zone, gravity is sufficiently weak to be described by (post-)Newtonian gravity. In this ‘buffer’ region, effective multipole moments and center-of-mass worldlines for the bodies are defined in terms of the vacuum gravitational field. Beyond distances $D \sim \lambda_{\text{GW}} \sim rc/v$, in the far zone, the (post-)Newtonian description of gravity (described here and in Sec. 1.5) must be replaced by a post-Minkowskian description (described in Secs. 1.6 and 1.7).

Even then, there may exist a vacuum buffer region, surrounding but not too close to the system, in which the spacetime is approximately described by a Newtonian gravitational field with $\phi \ll c^2$. In that case, the multipole moments can be operationally defined by the expansion (1.4.2) of the internal potential in the vacuum buffer region.¹⁴ The sum of Eqs. (1.4.2) and (1.4.4) represents the most general solution to $\nabla^2\phi = 0$ in the buffer region (with all the STF moments being arbitrary functions of time [for now]), so that any system with a Newtonian buffer region can be characterized by such moments.

The multipole and tidal moments are useful because the system’s translational equation of motion (EoM) can be written solely in terms of these moments. By integrating the equations of Newtonian stress-energy conservation, Eqs. (3.2.2) and (3.2.3), over the interior of the system, one can derive the relations $\dot{M} = 0$, expressing the conservation of the system’s mass, and

$$\ddot{M}_i = MG_i + M_j G_{ij} + \frac{1}{2}M_{jk}G_{ijk} + \dots + \frac{1}{\ell!}M_{j_1\dots j_\ell}G_{ij_1\dots j_\ell} + \dots \quad (1.4.6)$$

which relates to conservation of momentum and serves as an EoM for the system’s center of mass.

The EoMs $\dot{M} = 0$ and (1.4.6) determine the time evolution of the system’s mass monopole and dipole, with the effects of the system’s internal structure entering only through the presence of the higher-order mass multipole moments in Eq. (1.4.6). The evolution of those higher moments depends on the details of the system’s internal dynamics.

Before translating the EoM (1.4.6) into a more familiar form, let us consider systems with strong internal gravity. As discussed above, one can define Newtonian multipole moments for such a system via its vacuum gravitational field in a Newtonian buffer region (provided such a region exists), without reference to its interior. It turns out that one can also derive the EoM (1.4.6) (as well as $\dot{M} = 0$) without reference to the interior: instead of using Newtonian stress-energy conservation in the interior, one can derive the same Newtonian EoM by using the first-post-Newtonian (1PN) vacuum field equations in the buffer region.¹⁵ This means that the EoM (1.4.6) is equally valid for bodies with strong internal gravity, provided they admit a Newtonian buffer region¹⁶—*even binary BHs will follow Newtonian orbits if the BHs are separated by a sufficient distance.*

¹⁴To make this precise, one can define the multipole moments as surface integrals of the vacuum gravitational field in the buffer region, as shown for the Newtonian and 1PN cases in Appendix E of Ref. [31].

¹⁵This fact seems to have been first demonstrated by Thorne and Hartle [28]; see also Futamase [32] and Racine and Flanagan [31].

¹⁶This condition fails, e.g., when the body is emitting a strong burst of GWs.

Now, consider a binary in which body ‘1’ is spherically symmetric (for simplicity), while body ‘2’ is arbitrarily structured, having a quadrupole $M_2^{ij}(t)$, octupole $M_2^{ijk}(t)$, etc. By applying the EoM (1.4.6) separately to each body (each body being the system \mathcal{S}), one can derive the EoM for the separation $z^i(t) = z_2^i(t) - z_1^i(t)$ between the bodies’ CoM worldlines $z_1^i(t)$ and $z_2^i(t)$:¹⁷

$$\begin{aligned} \mu \ddot{z}^i = & -\frac{M_1 M_2}{r^2} n^i - \frac{15 M_1}{2 r^4} M_2^{jk} n^{<i} n^j n^{k>} - \dots \\ & \dots - \frac{(-)^\ell (2\ell + 1)!!}{\ell!} \frac{M_1}{r^{\ell+2}} M_2^{j_1 \dots j_\ell} n^{<i} n^{j_1} \dots n^{j_\ell>} - \dots \end{aligned} \quad (1.4.7)$$

The first term here gives the point-particle EoM (or the EoM for perfectly spherical bodies) leading to the well-known Keplerian orbits. The remaining terms, involving the higher-order mass multipoles, give the tidal perturbations.

The orbital EoM (1.4.7) by itself does not provide a complete description of the dynamics, as one needs also to determine the evolution of the quadrupole $M_2^{ij}(t)$ and higher multipoles. In general, this will require a detailed model of the body’s internal dynamics. However, for a body which develops non-sphericity in response to a slowly varying external tidal field, one can use a model with adiabatic (instantaneous) linear response of the multipoles. In that case, one can show that the ℓ th mass multipole moment ($\ell \geq 2$) will respond according to

$$M^{i_1 \dots i_\ell}(t) = \lambda_\ell G^{i_1 \dots i_\ell}(t), \quad (1.4.8)$$

where $G^{i_1 \dots i_\ell}$ is the ℓ th tidal moment, and λ_ℓ is a constant ‘tidal deformability’ response coefficient.¹⁸ With adiabatically induced mass multipoles as in Eq. (1.4.8), the orbital EoM (1.4.7) simplifies to

$$\mu \ddot{z}^i = -\frac{M_1 M_2}{r^2} n^i - \frac{9 \lambda_2 M_1^2}{r^7} n^i - \dots - (\ell + 1)(2\ell - 1)!! \frac{\lambda_\ell M_1^2}{r^{2\ell+3}} n^i - \dots \quad (1.4.9)$$

which now does provide a complete description of the dynamics and can be directly integrated to solve for the orbit.

¹⁷Note in Eq. (1.4.7) that $M = M_1 + M_2$, $\mu = M_1 M_2 / M$, $r = |\mathbf{z}|$, and $n^i = z^i / r$, and we are in the binary’s CoM frame. Note also that the higher-order multipole moments appearing in Eq. (1.4.7) are those measured about the body’s center of mass, as in Eqs. (3.2.6, 3.2.7) below, rather than about a fixed origin, as in Eqs. (1.4.2, 1.4.3). A derivation of the EoM (1.4.7) which works in a global inertial frame is given below in Sec. 3.2, and derivation which works in accelerated frames attached to the bodies’ CoMs is given in Sec. III of Ref. [33].

¹⁸Calculation of the tidal deformability coefficients for neutron stars (in a fully relativistic context) is discussed briefly below in Secs. 3.1.I and 3.1.II, and in detail in Refs. [34, 35, 36, 37]. The responses of black holes to tidal fields is discussed e.g. in Refs. [38, 39, 40].

The extension of the orbital EoMs (1.4.7) and (1.4.9) to include the effects of first-post-Newtonian gravity, at quadrupolar order in the multipole expansion, is the subject of Chapter 3. The coming section reviews some of the basics of the post-Newtonian description of binary orbits, closely paralleling our discussion of the Newtonian case, and outlines the content of Chapter 3.

Before leaving the subject of Newtonian multipole expansions, we should note another use of the multipole and tidal moments. Just as for a single body in a binary, an entire binary system can be characterized by its mass multipole moments, and the external gravitational influences on it by its tidal moments. These are defined by the vacuum gravitational field surrounding the binary (which is now the system \mathcal{S}), expanded in the form of Eqs. (1.4.2) and (1.4.4).

We will see in Sec. 1.7 that the GWs emitted by a Newtonian binary are determined by the evolution of its Newtonian mass quadrupole $M^{ij}(t)$. Also, the backreaction on the orbits due to GW emission can be encoded in the system's quadrupolar tidal moment $G^{ij}(t)$.

1.5 The post-Newtonian approximation

The post-Newtonian approximation has the same realm of applicability (qualitatively) as the Newtonian approximation: it assumes $\epsilon \sim v^2/c^2 \sim GM/rc^2 \ll 1$, its validity is restricted to the near zone, and it can (sometimes) describe the orbits of bodies with strong internal gravity. The Newtonian spacetime metric (1.3.3) was itself an expansion in $1/c^2$ (or in ϵ), accurate to $O(c^0)$, and the n th-post-Newtonian metric extends this accuracy to $O(c^{-2n})$.

To the Newtonian metric $ds^2 = -(c^2 + 2\phi)dt^2 + \delta_{ij}dx^i dx^j + O(c^{-2})$, the 1PN metric adds $O(c^{-2})$ corrections:

$$ds^2 = - \left(c^2 + 2\Phi + \frac{2\Phi^2}{c^2} \right) dt^2 + \frac{2\zeta_i}{c^2} dt dx^i + \left(1 + \frac{2\Phi}{c^2} \right) \delta_{ij} dx^i dx^j + O(c^{-4}). \quad (1.5.1)$$

The potential Φ is related to the Newtonian potential ϕ by $\Phi = \phi + O(c^{-2})$, being the same at Newtonian order but differing at 1PN order. The nonlinear parametrization $g_{00} = -c^2 - 2\Phi - 2\Phi^2/c^2$ is chosen to linearize field equation for Φ . Through the Φ^2 term, 1PN gravity encodes nonlinear GR effects even though its field equations are linear. The new degrees of freedom are the 3-vector potential ζ_i , known as the gravito-magnetic potential, appearing in g_{0i} . While g_{ij} does get $O(c^{-2})$ corrections, the Einstein equations dictate that these are determined by the same potential Φ

appearing in g_{00} , introducing no more new degrees of freedom at 1PN order.¹⁹

The 1PN metric (1.5.1) solves Einstein's equations to $O(c^{-2})$ when the potentials solve the field equations²⁰

$$\nabla^2\Phi = 4\pi G \left(T^{00} + \frac{1}{c^2} T^{ii} \right) + \frac{1}{c^2} \frac{\partial^2\Phi}{\partial t^2} + O(c^{-4}), \quad \nabla^2\zeta^i = 16\pi G T^{0i} + O(c^{-2}), \quad (1.5.2)$$

and the harmonic gauge condition²¹

$$\partial_\mu(\sqrt{-g}g^{\mu\nu}) = 0 \quad \Rightarrow \quad 4\dot{\Phi} + \partial_i\zeta_i = O(c^{-2}). \quad (1.5.3)$$

Just as in Newtonian gravity [cf. Eq. (1.3.5)], the geodesic worldline $x^i = z^i(t)$ of a free test-mass can be found by extremizing its proper time $\tau = t - S/mc^2$, where the action $S[\mathbf{z}(t)]$ at 1PN order is

$$S = m \int dt \left[\left(\frac{v^2}{2} - \Phi \right) + \frac{1}{c^2} \left(\frac{v^4}{8} - \frac{\Phi^2}{2} - \frac{3\Phi v^2}{2} + v_i\zeta_i \right) + O(c^{-4}) \right]. \quad (1.5.4)$$

When this is combined with a solution to Eqs. (1.5.2) with point-particle sources,²² the resultant EoMs for a point-particle binary are typically referred to as the (2-body) Einstein-Infeld-Hoffman (EIH) equations, from their 1938 derivation [42], though they were first derived by Lorentz and Droste in 1917 [43]. The harmonic-gauge forms of these EoMs are given in Eqs. (3.3.33c, 3.3.33d) or Eq. (3.5.9b) below. Point-particle EoMs are now known through 3PN order, as detailed in the review by Blanchet [21].

¹⁹Note that Eq. (1.5.1) incorporates a gauge specialization, the conformally Cartesian gauge condition [29].

²⁰The first of the field equations (1.5.2) resembles a wave equation, but when treated perturbatively in $1/c^2$ gives only a leading order retardation effect and not actual wave-like behavior. [In a perturbative treatment, one first solves the zeroth-order equation $\nabla^2\Phi = T^{00} + O(c^{-2})$. That solution is substituted into the $c^{-2}(\partial^2\Phi/\partial t^2)$ term in Eq. (1.5.2a), which then serves as a fixed source for the Poisson equation for the 1PN-accurate Φ .]

The material source for $\nabla^2\Phi$, being simply rest-mass density ρ at Newtonian order, receives relativistic corrections from all forms of energy and pressure (or stress). For a perfect fluid, for example,

$$T^{00} + c^{-2}T^{ii} = \rho + c^{-2}(2\rho v^2 - 2\rho\Phi + \varepsilon + 3p) + O(c^{-4}),$$

where ρ is proper rest-mass density, $v^i = dx^i/dt$, ε is internal energy density, and p is pressure.

The Poisson equation (1.5.2b) for the gravito-magnetic potential ζ_i is sourced by the momentum density (or mass current) $T^{0i} = \rho v^i + O(c^{-2})$. The name comes from the close analogy with the 3-vector potential A^i of magnetostatics, with $\nabla^2 A^i = -4\pi J^i$, J^i being electric current density.

Textbook treatments of PN gravity, drawn upon here, can be found in Chapter 39 of MTW [11] and Chapter 9 of Weinberg [41].

²¹Other gauge choices are possible, such as the 'standard PN gauge' $3\dot{\Phi} + \partial_i\zeta_i = O(c^{-2})$, but will lead to different field equations.

²²The solutions to the field equations (1.5.2) with point-particle sources diverge at the locations of the particles. This problem is easily remedied by simply dropping all the divergent terms in the geodesic action (1.5.4).

The 1PN orbital dynamics for a system of non-spherical bodies presents a somewhat greater challenge than the point-particle dynamics, primarily due to issues in disentangling physical effects from coordinate or gauge effects in describing the coupling between internal and orbital degrees of freedom. These issues were resolved in the early 1990s by Brumberg and Kopeikin [44, 45, 46] and Damour, Soffel, and Xu (DSX) [29, 33, 47, 48], who developed comprehensive theories of 1PN reference frames, relating a global inertial frame to frames attached and specially adapted to moving bodies. The work of DSX provided a framework and prescription to compute orbital EoMs for arbitrarily structured bodies, based on multipole decompositions of the 1PN gravitational field.

The general solutions to the 1PN field equations (1.5.2) in a vacuum region surrounding any system can be written in the forms $\Phi = \Phi_{\text{int}} + \Phi_{\text{ext}}$ and $\zeta^i = \zeta_{\text{int}}^i + \zeta_{\text{ext}}^i$, expanded as follows. The multipole expansion of Φ_{int} defines the system's 1PN-accurate mass multipole moments,²³

$$\begin{aligned} \Phi_{\text{int}} = & -\frac{M}{r} + M_i \partial_i \frac{1}{r} - \frac{1}{2} M_{ij} \partial_i \partial_j \frac{1}{r} + \dots - \frac{(-)^{\ell}}{\ell!} M_L \partial_L \frac{1}{r} + \dots \\ & + \frac{1}{2c^2} \left[\ddot{M}_i \partial_i r - \frac{1}{2} \ddot{M}_{ij} \partial_i \partial_j r + \dots - \frac{(-)^{\ell}}{\ell!} \ddot{M}_L \partial_L r + \dots \right] + O(c^{-4}). \end{aligned} \quad (1.5.5a)$$

The first line gives the general solution to the Laplace equation that goes to zero at infinity, and adding the second line yields a particular solution (perturbatively) to the vacuum field equation $\nabla^2 \Phi = c^{-2}(\partial \Phi / \partial t) + O(c^{-4})$. The expansion (1.5.5a) defines the system's mass monopole $M(t)$, dipole $M_i(t)$, quadrupole $M_{ij}(t)$, etc., to 1PN order, from the vacuum gravitational field outside the system. The 1PN-accurate 'electric-type' tidal moments G_L experienced by the system are defined by the expansion

$$\begin{aligned} \Phi_{\text{ext}} = & -G_{(0)} - G_i x_i - \frac{1}{2} G_{ij} x_i x_j - \dots - \frac{1}{\ell!} G_L x_L + \dots \\ & - \frac{r^2}{2c^2} \left[\frac{1}{3} \ddot{G}_{(0)} + \frac{1}{5} \ddot{G}_i x_i + \frac{1}{7} \ddot{G}_{ij} x_i x_j + \dots + \frac{1}{2\ell + 3} \ddot{G}_L x_L + \dots \right] + O(c^{-4}), \end{aligned} \quad (1.5.5b)$$

the first line being the general solution to the Laplace equation which is regular at the origin and the second giving a particular solution for the retardation term.

²³The 1PN mass multipole moments coincide with the Blanchet-Damour [49] moments for bodies with weak internal gravity, and their definitions were extended to bodies with strong internal gravity by RF [31]. Note that the gauge of Eqs. (1.5.5) has been somewhat specialized (within harmonic gauge) to make the physical information clearer. Gauge issues that we gloss over here are discussed in detail in Section 3.3.

Here we have introduced the 'multi-index' notation $L = a_1 \dots a_{\ell}$ for ℓ different spatial indices, discussed further in Chapter 3, or in Ref. [29].

One can similarly perform a multipole decomposition of the gravito-magnetic potential ζ^i :

$$\begin{aligned} \zeta_{\text{int}}^i &= 2\epsilon_{ijk}S_j\partial_k\frac{1}{r} - \frac{4}{3}\epsilon_{ijk}S_{mj}\partial_k\partial_m\frac{1}{r} + \dots - \frac{4\ell(-)^\ell}{(\ell+1)!}\epsilon_{ij<a_\ell}S_{L-1>j}\partial_L\frac{1}{r} + \dots \\ &\quad - \frac{4\dot{M}_i}{r} + 2\dot{M}_{ij}\partial_j\frac{1}{r} + \dots - \frac{4(-)^\ell}{(\ell+1)!}\dot{M}_{iL}\partial_L\frac{1}{r} + \dots + O(c^{-2}). \end{aligned} \quad (1.5.5c)$$

This expansion²⁴ defines the system's STF current multipole moments: S_i is the system's spin (or angular momentum, or current dipole), S_{ij} its current quadrupole, etc. (with no current monopole). The second line shows that the system's linear momentum \dot{M}_i also contributes to ζ^i , as well as time-derivatives of its other mass multipoles. Finally, the 'magnetic-type' tidal moments H_L ($\ell \geq 1$) experienced by the system are defined by the expansion

$$\begin{aligned} \zeta_{\text{ext}}^i &= \frac{1}{2}\epsilon_{ijk}H_jx_k + \frac{1}{3}\epsilon_{ijk}H_{mj}x_kx_m + \dots + \frac{\ell}{(\ell+1)!}\epsilon_{ij<a_\ell}H_{L-1>x_L} + \dots \\ &\quad - \frac{4}{3}\dot{G}_{(0)}x_i - \frac{6}{5}\dot{G}_jx_ix_j - \frac{10}{21}\dot{G}_{jk}x_ix_jx_k - \dots - \frac{4(2\ell-1)}{\ell!(2\ell+1)}\delta_{i<a_\ell}\dot{G}_{L-1>x_L} + \dots + O(c^{-2}) \end{aligned} \quad (1.5.5d)$$

The moment H_i gives a uniform gravito-magnetic field (equivalent to placing the system in a rotating reference frame), and the higher-order moments encode gravito-magnetic tidal forces acting on the system.

A system's 1PN mass and current multipole moments can be defined as integrals over its interior stress-energy distribution [see Eqs. (3.3.12)], provided 1PN gravity is valid in its interior. If the system has strong internal gravity, but is surrounded by a vacuum buffer region where 1PN gravity is valid, one can instead use the multipole expansions (1.5.5) of the vacuum 1PN potentials to define these moments.¹⁴

Just as in Newtonian gravity, a system's translational equation of motion can be written solely in terms of its multipole and tidal moments (now of electric- and magnetic-type). By integrating the equations of 1PN stress-energy conservation over the system's interior, DSX [33] derived equations

²⁴The expansion (1.5.5c) of ζ_{int}^i also is simply a general solution to the Laplace equation, as is more evident when written in the form

$$\zeta_{\text{int}}^i = -Z_i\frac{1}{r} + Z_{ij}\partial_j\frac{1}{r} - \frac{1}{2}Z_{ijk}\partial_j\partial_k\frac{1}{r} + \dots - \frac{(-)^\ell}{\ell!}Z_{ij_1\dots j_\ell}\partial_{i_1}\dots\partial_{i_\ell}\frac{1}{r} + \dots$$

The tensors $Z_{ij_1\dots j_\ell}$ are split into parts according to their properties under interchange or contraction of the i index with one of $j_1 \dots j_\ell$. The antisymmetric part defines the current multipoles S_L , the STF part is related to $\dot{M}_{(\ell+1)}$ (required by the harmonic gauge condition), and the trace part is gauge (omitted here). A similar decomposition applies to ζ_{ext}^i in Eq. (1.5.5d). See Sec. 3.3 for details.

of the form

$$\begin{aligned}\dot{M} &= F[\{M_L\}, \{G_L\}] + O(c^{-4}), \\ \ddot{M}_i &= \mathcal{F}_i[\{M_L\}, \{S_L\}, \{G_L\}, \{H_L\}] + O(c^{-4}).\end{aligned}\tag{1.5.6}$$

The first of these relates to Newtonian energy conservation, and the second gives the 1PN corrections to Eq. (1.4.6), given explicitly by Eq. (3.3.22b) below. To extend the validity of these equations to systems with strong internal gravity, Racine and Flanagan (RF) [31] presented an alternate derivation which uses the 2PN vacuum Einstein equations in a buffer region surrounding the system.

The relation (1.5.6) can be used to derive the 1PN orbital equations of motion of a binary with arbitrarily structured bodies [to derive the 1PN corrections to the Newtonian Eq. (1.4.7)]. It was first used by DSX [47] to derive the 1PN spin-orbit coupling [reproduced in Eqs. (3.3.33e,3.3.33f) or Eq. (3.5.9c) below]. Subsequently, Refs. [50, 51] presented the terms in the orbital EoMs arising from the bodies’ mass quadrupole moments, and later Ref. [31] gave expressions for the terms from arbitrarily high-order mass and current multipole moments. However, the results of Refs. [50, 51, 31] contained errors, revealed by the work of Chapter 3 of this thesis. Chapter 3 also goes beyond the scope of Refs. [50, 51, 31] in several ways, as discussed in the last three bullets of this brief summary:

- We begin by reviewing the literature on tidal effects in LIGO-band binaries and how they can be exploited by GW detectors to probe the equation of state of neutron star matter. (Sec. 3.1)
- We then review some further aspects of Newtonian tidal effects: from orbital EoMs to an action principle for the orbital dynamics, the evolution of a body’s spin (due to tidal torques) and internal energy (due to ‘tidal heating’), and adiabatic linear response to tidal fields. (Sec. 3.2)
- Following the DSX formalism [29, 33] for 1PN celestial mechanics and some of its modifications by RF [31], we review the precise definitions of a body’s multipole and tidal moments, defined in body-adapted coordinates, and its CoM worldline, defined in global inertial coordinates, and the relationship between those coordinates. We derive the 1PN orbital EoMs for a binary of arbitrary bodies, keeping the mass monopole, spin, and mass quadrupole terms in the multipole expansions, working to linear order in the spin and quadrupole. We discuss why a consistent treatment of quadrupolar tidal effects at 1PN order necessitates the inclusion of spin-orbit coupling effects. (Sec. 3.3)

- We then derive formulae that relate the bodies’ multipole moments and CoM worldlines to the multipole moments of the entire binary system. The system’s 1PN mass monopole is seen to equal its total rest mass plus total Newtonian energy/ c^2 . The time derivative of its mass dipole is the 1PN-accurate total momentum. The constancy of this momentum should be implied by the orbital EoMs, and this serves as an important check that our EoMs are correct (and one not satisfied by the results of Refs. [50, 51, 31]). (Sec. 3.4)

The system’s 1PN-accurate mass quadrupole (as well as its mass octupole and current quadrupole at Newtonian order) are derived from the results of Sec. 3.4 and used in the calculations of gravitational radiation in Chapter 4.

- The conservation of 1PN momentum allows us to specialize to the system’s CoM frame. We show that the simplified orbital EoMs in the CoM frame can be encoded in an action principle for the CoM worldline. (Sec. 3.5)
- With all previous results allowing completely generic internal dynamics, we then specialize to the case where a body’s quadrupole is adiabatically induced, with instantaneous linear response to the external tidal field. We show how the resultant dynamics can also be incorporated into a rather simple action principle. (Sec. 3.6)

Since the completion of this work, some of our results have been verified by the work of Bini, Damour, and Faye (BDF) [52]; similar results specialized to circular orbits were previously presented by Damour and Nagar (DN) [37]. These authors’ results are obtained through effective action techniques, taking the point-particle action and supplementing it by terms quadratic in the Riemann tensor evaluated along the worldline; this gives the tidal coupling for bodies with adiabatic linear response to tidal fields, but does not allow a determination of the tidal coupling for arbitrary internal dynamics. The work of BDF has also presented results for the adiabatic quadrupolar tidal coupling to 2PN order. Appendix 3.8 below demonstrates the agreement between the 1PN results of DN and BDF with ours.

1.6 Gravitational waves in the post-Minkowskian approximation

While the post-Newtonian approximation can provide a good description of the orbital dynamics of a weak-field, slow motion binary, it cannot describe the GWs emitted by such a binary. For this purpose, one must turn to the *linearized* or *first-post-Minkowskian* approximation (or the higher-order n th-post-Minkowskian approximations). Post-Newtonian gravity fails to describe GWs because its field variables obey instantaneous Poisson equations rather than wave equations (with perturbative retardation effects coming in at higher PN orders, valid only in the near zone). While the field variables of linearized GR do obey wave equations and describe far-zone GWs, we will see shortly how linearized GR fails to describe binary orbital dynamics.

The linearized approximation is based on the metric ansatz

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} + O(h^2), \quad (1.6.1)$$

positing that the spacetime metric $g_{\mu\nu}$ differs from the Minkowski metric $\eta_{\mu\nu} = \text{diag}(-1, 1, 1, 1)$ by a small perturbation $h_{\mu\nu}$, working to linear order in h .²⁵ Extending this to $O(h^n)$ defines the n th-post-Minkowskian approximation. The metric perturbation $h_{\mu\nu}$ can be thought of as a tensor field living on Minkowski space, much like the electromagnetic vector potential A_μ in special relativity, and its dynamics have many parallels with those of A_μ .

From the general covariance of full GR, linearized GR inherits a linear gauge invariance. The theory is invariant under linearized coordinate transformations of the form $x^\mu \rightarrow x^\mu + \xi^\mu(x)$, for any (small) vector ξ , which transform the metric according to $g_{\mu\nu} \rightarrow g_{\mu\nu} - 2\partial_{(\mu}\xi_{\nu)}$. This means that of the ten degrees of freedom in the metric g (a symmetric 4-by-4 matrix), six are physical and four (corresponding to the gauge vector ξ) are pure gauge.²⁶ One useful way to fix (most of) the

²⁵Here we switch to units where $G = c = 1$. Note that the Newtonian metric (1.3.3) is also of the form (1.6.1), the Minkowski metric plus a perturbation, the difference being that the Newtonian metric only allows a perturbation to the time-time component, $h_{00} = -2\phi$. This is because the small expansion parameter in (post-)Newtonian gravity is $1/c^2$, and since $\eta_{\mu\nu} = \text{diag}(-c^2, 1, 1, 1)$, this makes spatial directions ‘higher-order’ than time directions—this in essence is the slow-motion approximation. In linearized GR, space and time are on more equal footing.

One can use Newton’s constant G (the coupling constant between gravity and matter) as a formal expansion coefficient for linearized GR (like $1/c^2$ for PN theory), but here we set $G = 1$ and use the scale of the metric perturbation h as our small parameter. The indices of $h_{\mu\nu}$ (and ∂_μ) are raised and lowered with the Minkowski metric. Textbook treatments of linearized gravity (drawn upon in this section) can be found, e.g., in Chapter 18 of MTW [11], Chapter 10 of Weinberg [41], and Section 4.4 of Wald [53].

²⁶This is analogous to the gauge freedom $A_\mu \rightarrow A_\mu + \partial_\mu\lambda$ in electromagnetism. There, three of the four degrees of freedom in A_μ are physical, and one (corresponding to the gauge function λ) is pure gauge. The harmonic gauge condition (1.6.2) is analogous to the Lorentz gauge condition $\partial_\mu A^\mu = 0$.

gauge freedom is to impose the harmonic gauge condition,

$$0 = \partial_\mu(\sqrt{-g}g^{\mu\nu}) = -\partial_\mu\bar{h}^{\mu\nu} + O(h^2), \quad \bar{h}_{\mu\nu} = h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}h_\rho{}^\rho, \quad (1.6.2)$$

where $\bar{h}_{\mu\nu}$ is the trace-reversed metric perturbation, often a more useful field variable than $h_{\mu\nu}$ (and a fully equivalent one because $h_{\mu\nu} = \bar{h}_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}\eta^{\rho\sigma}\bar{h}_{\rho\sigma}$).

The Einstein equations (1.3.2) for the linearized metric (1.6.1) with the gauge condition (1.6.2) reduce to

$$\square\bar{h}^{\mu\nu} = -16\pi T^{\mu\nu}, \quad (1.6.3)$$

where $\square = \eta^{\mu\nu}\partial_\mu\partial_\nu$ is the flat-space wave operator. Operating with ∂_μ on this equation and using the harmonic gauge condition (1.6.2) yields

$$\partial_\mu T^{\mu\nu} = 0, \quad (1.6.4)$$

which is the *flat-space* conservation equation for the matter stress-energy tensor.²⁷ Flat-space stress-energy conservation implies that test-masses move along flat-space geodesics (straight lines)—they are not deflected by gravity. In linearized GR, though the field equation (1.6.3) tells us that matter does generate a gravitational field, the influence of the gravitational field on matter is a higher-order effect. This is why linearized GR cannot consistently describe the orbital dynamics of a binary.

Of the six physical degrees of freedom in the metric perturbation, it turns out that two are independent dynamical degrees of freedom, while the other four are ‘slave’ degrees of freedom, being determined non-dynamically by the matter degrees of freedom. A careful analysis of the gauge-invariant content of the field equations (1.6.3) reveals that only two gauge-invariant degrees of freedom obey wave equations while the other four obey instantaneous Poisson equations.²⁸

The two dynamical degrees of freedom, those that describe actual gravitational radiation (in vacuum), are gauge-invariantly associated with the transverse-traceless (TT) part of the space-space part of the metric perturbation, i.e. the part h_{ij}^{TT} of h_{ij} which obeys

$$\partial_i h_{ij}^{\text{TT}} = 0, \quad h_{ii}^{\text{TT}} = 0, \quad (1.6.5)$$

²⁷Eqs. (1.6.3) and (1.6.4) are analogous to $\square A^\mu = -4\pi J^\mu$ and $\partial_\mu J^\mu = 0$ in the EM case.

²⁸A succinct derivation of this result is given in the review by Flanagan and Hughes [54]. This also has an analog in electromagnetism, where, of the three physical degrees of freedom in A_μ , two are dynamical (the two independent polarization states of EM waves) and one is ‘slave’.

and is the same as \bar{h}_{ij}^{TT} in vacuum. From this and Eq. (1.6.3), we see that a vacuum gravitational plane wave in linearized theory, moving in the z -direction, takes the form (of the real part of)

$$h_{ij}^{\text{TT}}(t, x, y, z) = \left[h_+ \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} + h_\times \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \right] e^{i\omega(z-t)}. \quad (1.6.6)$$

Here, h_+ and h_\times are the amplitudes of the two independent polarization states, called ‘plus’ and ‘cross’. The polarization tensors show that the z -directed wave shears spacetime in the x - and y -directions, reflecting the transverse nature of GWs. A succinct review of how such plane waves interact with matter, and in particular with GW detectors, can be found in the review by Buonanno [55].

Gravitational waves carry energy, momentum, and angular momentum, as can be described by their effective stress-energy tensor (interpreted on the Minkowski background):

$$T_{\mu\nu}^{\text{GW}} = \frac{1}{32\pi} \langle \partial_\mu h_{ij}^{\text{TT}} \partial_\nu h_{ij}^{\text{TT}} \rangle, \quad (1.6.7)$$

where the angular brackets denote a spatial average over several wavelengths of the GWs.²⁹ This stress-energy tensor can be used to calculate the flux of energy (or momentum, or angular momentum) carried by linearized GWs away from a radiating system to infinity. Equating that flux to the rate of change of the system’s energy (etc.) is often a useful (and simple) way of determining the backreaction on the system due to GW emission (and is the method used in Chapter 3 below).

The energy-momentum in GWs, or more generally, in the gravitational field, also serves as a source for more gravity. Such nonlinear effects enter in higher-order post-Minkowskian theory, which can be formulated as follows.³⁰ Given the exact spacetime metric $g_{\mu\nu}$, we define the potentials $\bar{h}^{\mu\nu} \equiv \eta^{\mu\nu} - \sqrt{-g}g^{\mu\nu}$, which coincide with the trace-reversed metric perturbation at linear order, and impose the harmonic gauge condition $\partial_\mu \bar{h}^{\mu\nu} = 0$. The exact Einstein equations can then be written in the form

$$\square \bar{h}^{\mu\nu} = -16\pi(-g)(T^{\mu\nu} + \tau^{\mu\nu}), \quad (1.6.8)$$

where $\square = \eta^{\mu\nu} \partial_\mu \partial_\nu$, and $\tau^{\mu\nu}$ is an effective stress-energy pseudo-tensor for the gravitational field, which is composed of terms at least quadratic in $\bar{h}^{\mu\nu}$ and its derivatives. (This $\tau^{\mu\nu}$ differs from

²⁹It is a characteristic feature of GR that the energy-momentum content of the gravitational field itself cannot be meaningfully localized. For linearized GWs, the ambiguity in the location of energy-momentum is on the scale of a single GW wavelength, and one can show that the averaging over several wavelengths in Eq. (1.6.7) then gives an unambiguous definition of GW stress-energy.

³⁰See e.g. Refs. [21, 56, 57].

the Landau-Lifshitz pseudo-tensor [58] only by two terms quadratic in $\bar{h}^{\mu\nu}$, having terms up to sixth order in $\bar{h}^{\mu\nu}$.) Thus, at $O(\bar{h}^1)$, this gives the linearized field equations (1.6.3). One can then iteratively generate the higher-order approximations, with the source $\tau^{\mu\nu}$ at each order being determined by the lower-order solutions.³¹

1.7 Generation of gravitational waves by post-Newtonian sources

A rigorous description of GW emission from post-Newtonian binaries can be found by matching a post-Newtonian treatment of the near-zone orbital dynamics onto a post-Minkowskian treatment of the far-zone GWs.³² An important feature of the results is that increasing orders in the post-Newtonian expansion are coupled to increasing orders in the *multipole* expansion of the post-Minkowskian gravitational field (as well as to increasing post-Minkowskian orders).

Similar to the vacuum post-Newtonian potentials in Eqs. (1.5.5), the multipole expansion of the post-Minkowskian potentials $\bar{h}^{\mu\nu}$ in a binary's far zone can be parametrized by a set of mass multipole moments I_L ($\ell \geq 0$) and current multipole moments J_L ($\ell \geq 1$) [renamed $M \rightarrow I$ and $S \rightarrow J$ because they generally differ from their PN counterparts] which characterize the binary. These moments are defined by the following general solution³³ to the vacuum linearized field equations (1.6.3) (which are simply homogeneous wave equations) and the harmonic gauge condition (1.6.2) (with some extra gauge specialization here for simplicity):

$$\begin{aligned}\bar{h}^{00} &= 4 \sum_{\ell=0}^{\infty} \frac{(-)^{\ell}}{\ell!} \partial_L \left(\frac{I_L(u)}{r} \right), \\ \bar{h}^{0i} &= -4 \sum_{\ell=1}^{\infty} \frac{(-)^{\ell}}{\ell!} \left[\partial_{L-1} \left(\frac{\dot{I}_{i(L-1)}(u)}{r} \right) + \frac{\ell}{\ell+1} \epsilon_{ijk} \partial_{j(L-1)} \left(\frac{J_{k(L-1)}(u)}{r} \right) \right], \\ \bar{h}^{ij} &= 4 \sum_{\ell=2}^{\infty} \frac{(-)^{\ell}}{\ell!} \left[\partial_{L-2} \left(\frac{\ddot{I}_{ij(L-2)}(u)}{r} \right) + \frac{2\ell}{\ell+1} \partial_{k(L-2)} \left(\frac{\epsilon_{km(i} \dot{J}_{j)m(L-2)}(u)}{r} \right) \right],\end{aligned}\tag{1.7.1}$$

where all the moments are functions of the retarded time $u = t - r$, and dots denote u -derivatives.

Such moments can also be used to parametrize the general n th-post-Minkowskian solution to the

³¹This general formulation of the Einstein equations can also be used to generate the post-Newtonian expansion, when the post-Minkowskian solution is re-expanded in powers of $1/c$ [21].

³²The material in this section draws from the reviews by Blanchet [21] and Will and Wiseman [56] and the lecture notes by Poisson [57].

³³Note that there are no 'tidal' terms (no terms growing with r) in Eq. (1.7.1), since this solution is valid as $r \rightarrow \infty$, and we assume the spacetime is asymptotically flat, using asymptotically inertial coordinates.

field equation (1.6.8).³⁴ When matching onto a source described by 1PN gravity, the linearized solution (1.7.1) is sufficient to describe the radiation propagating to infinity.³⁵

The matching procedure, to determine the PM moments I_L and J_L from the interior PN dynamics, is accomplished as follows. One performs a *near-zone expansion*, an expansion in small r , of the PM solution; this is closely related to the $1/c$ expansion, since the retarded time appearing in the PM solution (1.7.1) is really $u = t - r/c$ with cs restored. The result can then be directly identified with the far-zone expansion of the interior PN gravitational field.

With details given in Refs. [21, 56, 57], the results relevant to our purposes are

$$I_L(u) = M_L(u) + O(c^{-3}), \quad J_L(u) = S_L(u) + O(c^{-3}), \quad (1.7.2)$$

where the 1PN moments M_L and S_L , defined as functions of t , are evaluated at $t = u$. While these expressions may seem somewhat anticlimactic, this is because we are working only to 1PN order. At 1.5PN order and higher, these relations pick up corrections due to nonlinear memory effects and tail effects (or multipole-multipole couplings).³⁶

With these identifications, the gravitational radiation at infinity is given by the transverse-traceless (TT) part of the $1/r$ part of h_{ij} from Eq. (1.5.5), which becomes (with all G s and cs restored)

$$h_{ij}^{\text{TT}} = \frac{4G}{c^4 r} P_{ijklm} \left(\frac{1}{2} \ddot{M}_{km} + \frac{1}{6c} \ddot{M}_{kmp} n_p - \frac{2}{3c} \epsilon_{pq(k} \ddot{S}_{m)q} n_p + O(c^{-2}) \right), \quad (1.7.3)$$

where n_i is the radial unit vector and $P_{ijklm} = P_{ik}P_{jm} - (1/2)P_{ij}P_{km}$, $P_{ij} = \delta_{ij} - n_i n_j$, is the TT projection operator for radial waves. Using this waveform in the effective GW stress-energy tensor (1.6.7) and integrating over a sphere at infinity yields the total radiated power, or the rate of energy

³⁴This is because the n PM solution to Eq. (1.6.8) will consist of particular solutions for the $\tau^{\mu\nu}$ source terms (determined by lower-order solutions) plus a solution to the homogeneous wave equation of the form (1.5.5). Once fixed choices are made for generating the particular solutions (when a particular inverse wave operator is used) the general n PM solution is completely determined by the mass and current moments parametrizing the homogeneous solution.

³⁵A part of the 1PN solution must actually be matched onto a 2PM solution, but this part only concerns the metric's 'slave' degrees of freedom (specifically the monopole-monopole nonlinearity) and not GWs.

At higher PN orders, one must also make a change of gauge, from harmonic to 'radiative' coordinates, to best describe the radiation at infinity, but this is not necessary at 1PN order.

³⁶Note, in the language of Blanchet [21] to be concrete, we are not differentiating here between the radiative multipole moments, U_L and V_L , and the source moments I_L and J_L , since these differ only at 1.5PN order and higher. Note also that our use of M_L and S_L for the PN moments does not follow Blanchet; his M_L and S_L and I_L and J_L are all PM moments which differ due to gauge issues which are ignored here.

loss from the system:

$$\dot{E} = -\frac{G}{5c^5}\ddot{M}_{ij}\ddot{M}_{ij} - \frac{G}{189c^7}\ddot{M}_{ijk}\ddot{M}_{ijk} + \frac{16G}{45c^7}\ddot{S}_{ij}\ddot{S}_{ij} + O(c^{-8}). \quad (1.7.4)$$

Similar formulae can be derived for the loss of angular and linear momentum [21].

The leading-order terms in $1/c$ in Eqs. (1.7.3) and (1.7.4) give the famous Einstein quadrupole formulae for the gravitational radiation from a Newtonian source, where one needs only to use the Newtonian-order value for the source's mass quadrupole M_{ij} . The extension to 1PN-accuracy requires the 1PN-accurate M_{ij} and the Newtonian values for the mass octupole M_{ijk} and current quadrupole S_{ij} .

The matching procedure also determines the radiation reaction experienced by the source, which can be succinctly encoded in the near-zone tidal moments felt by the source. At leading order, the source feels a quadrupolar tidal moment [contributing to the near-zone metric through Eqs. (1.5.5b) and (1.5.1)] given by

$$G_{ij} = -\frac{2G}{5c^5}\partial_t^5 M_{ij} + O(c^{-7}), \quad (1.7.5)$$

showing that the leading-order radiation reaction is formally a 2.5PN effect. The source's reaction to this radiation 'tidal field' causes it to lose energy at a rate which precisely matches the radiated power formula (1.7.4). In situations with high symmetry, in particular for binaries in circular orbits, this fact can be used to circumvent the direct calculation of the backreaction; one can instead use *energy balance*, equating the radiated power formula (1.7.4) to the rate of change of the source's energy, to determine the gravitational waveform.

A primary observable that can be calculated by the energy balance method, for circular orbits, is the phase ϕ of the Fourier transformed gravitational waveform, $\tilde{h} = \mathcal{A}e^{i\Psi}$, as a function of the orbital frequency ω as measured by observers at infinity. In the stationary phase approximation [59, 60], this is found by solving

$$\frac{d^2\Psi}{d\omega^2} = \frac{2}{\dot{E}} \frac{dE}{d\omega}, \quad (1.7.6)$$

where E is the source's energy, expressed as a function of ω .

Chapter 4 of this thesis uses the energy balance method to determine the 1PN-accurate waveform from a circular binary containing bodies with adiabatically induced quadrupoles, building on the orbital dynamics of such a system developed in Chapter 3.

1.8 Point mass inspirals in black hole spacetimes and the self-force

When a large BH of mass M is orbited by a small body of mass $m \ll M$, one can mostly ignore the gravitational field produced by the small mass. In the limit $m \rightarrow 0$, the small body moves along a geodesic (equivalent to moving locally inertially) in the fixed background geometry that would be the spacetime of the large BH in isolation.

A great advantage in this version of the two-body problem is that we have *exact* solutions to the Einstein equations, the Schwarzschild/Kerr solutions, that describe the spacetimes of isolated static/spinning BHs. The geodesic equations in these geometries are relatively easily solved, yielding the leading-order bound orbits for a small body, which do not spiral into the BH.

Beyond this $m/M \rightarrow 0$ limit, one must consider the perturbations to the background BH spacetime produced by the small body. These perturbations will encode, for example, the motion of the BH in response to the small body and the corresponding corrections to the small body's orbit (conservative effects), and the emission of GWs and the corresponding inspiral of the orbit (dissipative effects).

The small body can be modeled as point particle, and one can solve the perturbed Einstein equations for the linear metric perturbation produced by a point-particle source.³⁷ This field, however, diverges at the location of the particle, which presents a difficulty because it is precisely that self-field that will exert a 'self-force' on the particle causing it to deviate from geodesic motion.

This kind of problem is well known, say, from flat-space electromagnetism, in which a point charge's self-field also diverges. Historically, issues of self-forces and radiation reaction for point particles were first understood in electromagnetism, both in flat spacetime and in a curved background.

³⁷Given a curved background metric $g_{\mu\nu}(x)$, the full metric can be written as $g_{\mu\nu} + h_{\mu\nu}$ where the perturbation $h_{\mu\nu}(x)$ acts as a tensor field living on the curved background. Defining the trace-reversed metric perturbation $\bar{h}_{\mu\nu} = h_{\mu\nu} - g_{\mu\nu}g^{\rho\sigma}h_{\rho\sigma}$, the full Einstein equations sourced by a point mass in Lorentz gauge read

$$\nabla^\rho \nabla_\rho \bar{h}^{\mu\nu} + 2R^{\mu\rho\nu\sigma} \bar{h}_{\rho\sigma} = -16\pi T_{\text{p.m.}}^{\mu\nu} + O(h^2), \quad T_{\text{p.m.}}^{\mu\nu} = m \int d\tau u^\mu u^\nu \delta_4(x, z(\tau)), \quad (1.8.1)$$

where $x = z(\tau)$ and $u^\mu(\tau)$ are the particle's geodesic worldline and normalized 4-velocity, ∇_μ and $R^{\mu\rho\nu\sigma}$ are the covariant derivative and Riemann tensor, and $\delta_4(x, z)$ is the invariant 4D Dirac delta function,⁴ all defined w.r.t. the background metric. This reduces to the first-post-Minkowskian field equation (1.6.3) in flat spacetime.

Note that the consistent solution to the Einstein equations at linear order in the mass ratio consists of the particle of mass m moving strictly along a geodesic, along with the linearized metric perturbation sourced by that geodesic worldline. The deviation of the worldline from a geodesic is a higher-order effect.

In flat spacetime, a point charge q which moves inertially feels no effect from its self-field and emits no EM radiation. If it is accelerated by an external 3-force \mathbf{f}_{ext} , then it experiences a radiation reaction force given (in the charge's instantaneous rest frame) by

$$m\mathbf{a} = \mathbf{f}_{\text{ext}} + \frac{2q^2}{3c^3}\dot{\mathbf{a}} + O(q^2c^{-5}) + O(q^4c^{-3}), \quad (1.8.2)$$

where \mathbf{a} is its 3-acceleration. This is a version of the (Abraham-)Lorentz-Dirac equation.³⁸ The self-force is caused by the emission of EM radiation, with the emitted power being given by the Larmor formula $P = 2q^2\mathbf{a}^2/3c^3$.

These results can be derived from the fact (proven in Ref. [61]) that the charge responds only to the 'radiative' part of its self-field, given by the future/past antisymmetric combination

$$A_{\text{rad}}^\mu = \frac{1}{2}(A_{\text{ret}}^\mu - A_{\text{adv}}^\mu), \quad (1.8.3)$$

where A_{ret}^μ and A_{adv}^μ are the retarded and advanced solutions to Maxwell's equations (in Lorentz gauge, $\square A^\mu = -4\pi J^\mu$, $\partial_\mu A^\mu = 0$). The divergent ($1/r$) parts of A_{ret}^μ and A_{adv}^μ cancel each other in this combination, and the resultant A_{rad}^μ is well-behaved at the location of the particle and yields the correct Lorentz-Dirac radiation reaction force (1.8.2) via the Lorentz force law $ma^\mu = F^{\mu\nu}u_\nu = 2\partial^{[\mu}A^{\nu]}$. The remainder of the physical field, $\frac{1}{2}(A_{\text{ret}}^\mu + A_{\text{adv}}^\mu)$, which is divergent, contains incoming and outgoing radiation in equal amounts, thus leaving the particle's energy-momentum unchanged and having no effect on the motion.

Now consider the case of a point charge moving in a fixed curved spacetime (ignoring gravitational perturbations). With no external (non-gravitational) forces on the charge, one expects that it should move on a geodesic in the curved spacetime, apart from possible EM self-field effects. Geodesic motion appears to be 'accelerated', due to gravitational 'forces', but actually is locally inertial motion, with the acceleration a^μ being zero. Thus, locally applying the flat-space Lorentz-Dirac result (1.8.2) would suggest there is no self-force on or radiation from a point charge in geodesic motion. Put another way, the *equivalence principle* would seem to suggest that a stationary charge in a freely falling elevator will not experience a self-force.

This suggestion turns out to be wrong, as first rigorously demonstrated by DeWitt and Brehme in 1960 [62].³⁹ They showed that a freely falling charge in curved spacetime generally does experience

³⁸Poisson gives a thorough modern review of the Lorentz-Dirac equation in Ref. [61].

³⁹The results of DeWitt and Brehme [62] were corrected by Hobbs [63] in 1968, in a way that affects only non-vacuum (non-Ricci-flat) spacetimes.

a self-force, in a nutshell, because its EM field (or the field information, leaving the charge at the speed of light [never returning in flat spacetime]) ‘scatters’ off of spacetime curvature and returns to influence the charge. This effect is missed by a naive application of the equivalence principle because of its nonlocal nature.

Using essentially the same methods as DeWitt and Brehme’s treatment of the EM case (though 37 years later), first Mino, Sasaki and Tanaka (MiSaTa) [64] and then Quinn and Wald (QuWa) [65] derived the analogous results for linear gravitational perturbations in a curved background, giving the gravitational radiation reaction force on a free point mass in curved spacetime, now known as the MiSaTaQuWa self-force. Analogous results for the third case of a scalar field coupled to a point scalar charge in a curved background were then derived by Quinn [66]. We will focus on the example of the scalar case to highlight important features of the results for all three cases. We rely heavily on the excellent review by Poisson et al. [25], mostly following its notation.

Consider a scalar field $\psi(x)$ obeying the curved-spacetime wave equation [the analog of Maxwell’s equations for the EM case or the perturbed Einstein equations (1.8.1) for the gravitational case], sourced by a point scalar charge q moving along the worldline $x = z(\tau)$:

$$\nabla_\mu \nabla^\mu \psi = -4\pi q \int d\tau \delta_4(x, z), \quad (1.8.4)$$

The scalar force on the particle [analogous to the Lorentz force, or to the gravitational ‘force’ encoded in the geodesic equation] is given formally by

$$[m_0 - q\psi(z)] a^\mu = (g^{\mu\nu} + u^\mu u^\nu) \nabla_\nu \psi(z), \quad (1.8.5)$$

with u^μ being the velocity, a^μ the acceleration, and m_0 the particle’s bare rest mass. These formal EoMs can both be derived from the action

$$S[\psi(x), z(\tau)] = -\frac{1}{8\pi} \int d^4x \sqrt{-g} \nabla_\mu \psi \nabla^\mu \psi - \int d\tau [m_0 - q\psi(z)], \quad (1.8.6)$$

as discussed in Sec. 2.2 below. As it stands, the equation of motion (1.8.5) is useless because the solution ψ to the field equation (1.8.4) diverges at the charge’s location.

Given a worldline $z(\tau)$, the field equation (1.8.4) can be solved with a Green’s function $G(x, y)$. If we define G_{ret} to be the solution to $\nabla^2 G(x, y) = -4\pi \delta_4(x, y)$ with *retarded* boundary conditions (no incoming radiation from past null infinity), then the retarded solution to Eq. (1.8.4) for the

scalar field is

$$\psi_{\text{ret}}(x) = q \int d\tau G_{\text{ret}}(x, z). \quad (1.8.7)$$

In a smooth spacetime, the retarded Green's function can be written (locally) in the Hadamard form

$$G_{\text{ret}}(x, y) = U(x, y) \delta_+(\sigma) + V(x, y) \theta_+(-\sigma), \quad (1.8.8)$$

where U and V are smooth functions, $\delta_+(\sigma)$ is a delta function for x to lie *on* the future light-cone of y , and $\theta_+(-\sigma)$ is a step function which is one if x is *inside* the future light-cone of y and zero otherwise.⁴⁰ The U term, on the light-cone, is known as the *direct* part, and gives rise to the divergent ($\sim 1/r$) part of the field. The V term, inside the light-cone, is known as the *tail* part, which is only non-zero in curved spacetime, arising from field information not traveling just along light-cones but being scattered inside the light-cones by curvature.

For a point particle with no external forces, in a vacuum background, the Quinn/DeWitt-Brehme/MiSaTaQuWa results showed that a scalar/EM/gravitational self-force on the particle arises only from the tail part of its self-field, with the direct part having no effect. Quinn's result (for this free, vacuum case) showed that, instead of the formal EoM (1.8.5) with the divergent field (1.8.7) plugged in for ψ (which gives nonsense), the scalar charge will obey the EoM⁴¹

$$[m - q\psi(z)] a^\mu(\tau) = q^2 (g^{\mu\nu} + u^\mu u^\nu) \int_{-\infty}^{\tau-0^+} d\tau' \nabla_\nu^{(1)} G_{\text{ret}}(z(\tau), z(\tau')), \quad (1.8.10)$$

which effectively cuts off the field's direct part, leaving only the tail part, by bringing the derivative inside the integral and stopping the integration just before $\tau' = \tau$. To calculate this tail-resultant self-force at any given instant, *one must integrate over the entire past history of the charge's worldline.*

A significant advance in the understanding and practical computation of the self-force was provided by the work of Detweiler and Whiting (DW) [67], who showed that there exists a regularized self-field $\psi_{\text{R}}(x)$ which when substituted into the formal scalar charge EoM (1.8.5) (as though it

⁴⁰In flat spacetime, $U = 1$, $V = 0$, and, in Cartesian coordinates, we have the usual retarded Green's function

$$G_{\text{ret}}(x, y) = \delta_+(\sigma) = \theta(x^0 - y^0) \delta(\sigma) = \frac{\delta(x^0 - y^0 - |\mathbf{x} - \mathbf{y}|)}{|\mathbf{x} - \mathbf{y}|}, \quad (1.8.9)$$

where $\sigma(x, y) = \frac{1}{2}(x - y)^\mu (x - y)_\mu$ in flat spacetime. In general, the function $\sigma = \sigma(x, y)$, Synge's 'world function', gives half the squared interval along the geodesic connecting x and y , discussed further in Sec. 2.9.

⁴¹The (1) in $\nabla_\nu^{(1)}$ indicates differentiation w.r.t. the first of the two arguments. The notation $\tau - 0^+$ for the upper limit of integration denotes the limit as $\epsilon \rightarrow 0$, from the positive side, of $\tau - \epsilon$.

were an external field) gives a self-force identical to Quinn’s EoM (1.8.10) [also working for non-free, non-vacuum cases].

In flat spacetime, the DW regular field ψ_R would be the same as the ‘radiative’ field $\psi_{\text{rad}} = (\psi_{\text{ret}} - \psi_{\text{adv}})/2$ where ψ_{adv} is the advanced solution. As for the flat EM case from Eq. (1.8.3), the divergent parts cancel in the combination $\psi_{\text{rad}} = (\psi_{\text{ret}} - \psi_{\text{adv}})/2$, and this future/past antisymmetric field is responsible for the entire self-force. The remainder of the physical field ψ_{ret} , namely the future/past symmetric part $(\psi_{\text{ret}} + \psi_{\text{adv}})/2$, containing the divergence, has no effect on the motion. In curved spacetime, however, a part of $(\psi_{\text{ret}} + \psi_{\text{adv}})/2$ does affect the motion.

The proper definition of the DW regular field is $\psi_R = \psi_{\text{ret}} - \psi_S$, where ψ_S is the ‘singular’ field, which DW defined and showed to have no effect on the motion. Without giving its precise definition, we note for now only that ψ_S differs from $(\psi_{\text{ret}} + \psi_{\text{adv}})/2$ [due to tails], while sharing the property of future/past symmetry. The regular field ψ_R can be split into (antisymmetric) radiative and (symmetric) conservative parts,

$$\psi_R = \psi_{\text{ret}} - \psi_S = \psi_{\text{rad}} + \psi_{\text{cons}}, \quad \psi_{\text{rad}} = \frac{1}{2}(\psi_{\text{ret}} - \psi_{\text{adv}}), \quad \psi_{\text{cons}} = \frac{1}{2}(\psi_{\text{ret}} + \psi_{\text{adv}}) - \psi_S,$$

with ψ_{rad} associated with secular dissipation due to radiation, and ψ_{cons} with conservative effects.

While the DW method allows a computation of the full self-force, regularizing the full self-field by subtracting off the appropriate singular part,⁴² this (comparatively difficult) regularization procedure is not necessary to compute only the time-averaged radiative part of the self-force. That part can be found by calculating the averaged fluxes of energy and angular momentum to infinity (and down the BH horizon), which requires only knowledge of the unregularized retarded field, and using those fluxes to compute the evolution of the particle’s orbital parameters. For the Kerr spacetime, Mino [70], building on the work of Gal’tsov [71], showed that those averaged fluxes agree with the averaged rates of change of the particle’s orbital parameters caused by the full self-force, and that

⁴²The DW decomposition provides a powerful method for computing self-forces in BH spacetimes when combined with methods to calculate (or approximate) the retarded and singular fields. The most popular method is to use a mode sum, taking advantage of the separability of the wave equation for the retarded field to reduce it to ODEs for radial mode functions, using spherical (or spheroidal) harmonics and a Fourier transform w.r.t. the time coordinate, as discussed for the Schwarzschild case in Sec. 2.7 below. The singular (or direct) field can be computed from a local expansion. Important early mode sum regularization calculations in the Schwarzschild spacetime were accomplished by Barack and Ori [68] (predating the DW decomposition) and Detweiler, Messaritaki, and Whiting [69]. Since then, many works have advanced the method in numerous ways, also extending it to the Kerr spacetime, as reviewed e.g. by Poisson et al. [25] and Barack [24]. These reviews also discuss other useful schemes, such as the ‘effective source’ method and 1+1, 2+1, or 3+1 numerical evolution of the field equations.

the radiative self-force can be constructed from the radiative field, needing no regularization. The necessary fluxes have been computed numerically in Kerr, e.g., in the work of Drasco and Hughes [72].

The radiative self-force is in a sense more important than the conservative self-force: radiative effects accumulate from orbit to orbit and drive the inspiral, while conservative effects will generally average out over one orbital period. A rigorous derivation of this fact is provided e.g. by the two timescale analysis of Hinderer and Flanagan [23], which establishes a consistent formulation of an adiabatic approximation and its ‘post-adiabatic’ corrections. Writing the gravitational self-force as a sum of radiative and conservative pieces, at first and second order in the mass ratio,

$$\vec{f} = \frac{m}{M}(\vec{f}_{\text{rad}}^{(1)} + \vec{f}_{\text{cons}}^{(1)}) + \frac{m^2}{M^2}(\vec{f}_{\text{rad}}^{(2)} + \vec{f}_{\text{cons}}^{(2)}) + \dots$$

Hinderer and Flanagan showed that the GW phase Ψ , the primary observable, takes the form

$$\Psi = \frac{M}{m} \left(\Psi^{(0)} + \frac{m}{M} \Psi^{(1)} + \dots \right),$$

where the ‘adiabatic order’ piece $\Psi^{(0)}$ depends only on the time-averaged part of $\vec{f}_{\text{rad}}^{(1)}$, while the ‘first-post-adiabatic order’ piece $\Psi^{(1)}$ depends on $\vec{f}_{\text{cons}}^{(1)}$, the oscillatory part of $\vec{f}_{\text{rad}}^{(1)}$, and the time-averaged part of $\vec{f}_{\text{rad}}^{(2)}$. It is estimated that a knowledge of the averaged radiative self-force will be sufficient for signal detection, but that parameter extraction will require knowledge of the next-to-leading-order effects.

The adiabatic approximation makes use of the *osculating geodesic* method (see e.g. Ref. [73]), described as follows. In Quinn’s EoM (1.8.10), the self-force on the particle at a given point is a *nonlocal* functional of the entire past history of the worldline. To render this a *local* EoM, instead of using the actual self-forced worldline to calculate the self-force, one can use the geodesic which is tangent to the actual worldline at the point where one is calculating the self-force. This osculating geodesic agrees with the actual worldline at zeroth order, making it valid to replace the latter with the former in computing the adiabatic-order (radiative) self-force, and in computing the conservative self-force contributions at first-post-adiabatic order. The resultant self-force contributions result in a local EoM, in the sense that the force only depends on the particle’s instantaneous position and velocity.

Consider the conservative part of the self-force (with the radiative part ‘turned off’) sourced by the osculating geodesic. Since the resultant EoM for the particle worldline is local and (in some sense) conservative, one could be led to wonder whether or not that EoM can be encoded in either an action principle or a Hamiltonian system for only the worldline degrees of freedom.

If this were true, it might present a new method for computing the conservative self-force, with potentially improved efficiency obtained by moving the regularization procedure from the level of the EoM to the level of the action or Hamiltonian. Also, the simple fact that there exists a Hamiltonian for the conservative dynamics (which is a foundational assumption of the EOB formalism [22]) can be used to elucidate certain physical consequences of the conservative self-force, such as the shifts in perihelion precession and the innermost stable circular orbit, as in the work of Le Tiec [74].

Chapter 2 of this thesis summarizes our efforts to find an action or Hamiltonian for the conservative self-force. We show that there does exist a Hamiltonian system encoding the conservative self-force dynamics in the Schwarzschild spacetime. Extensions of this result to the Kerr spacetime are in progress, as described below.

CHAPTER 2

THE CONSERVATIVE SELF-FORCE, ACTIONS, AND HAMILTONIANS

As motivated in the previous chapter, this chapter investigates the possibility of formulating an action principle or Hamiltonian system for a particle worldline which encodes the conservative self-force dynamics of a point particle orbiting a black hole.

We begin in Sec. 2.1 by discussing the action principle for geodesic motion, the zeroth-order motion in the self-force problem. We also introduce some helpful covariant techniques for working with worldline actions, in particular, the use of the horizontal covariant derivative $\tilde{\nabla}$ acting on functions on the tangent bundle of spacetime.

In Sec. 2.2, we review the formal description of a scalar field ψ coupled to a point scalar charge in a curved background, a simplified model that we use throughout this chapter which encompasses many important features of the gravitational self-force problem. Section 2.3 then reviews the solution of the scalar wave equation in terms of Green's functions, the local structure of Green's functions in curved spacetime, and the Detweiler-Whiting method for regularizing the self-field and self-force.

Section 2.4 discusses how the nonlocal conservative self-force EoM can be encoded in a nonlocal action principle, though this action seems (as yet) to be of little use for practical computations. Section 2.5 then considers the osculating geodesic method and the resultant local EoM. We ask whether or not this local EoM can be encoded in some action principle or Hamiltonian system. We present a naive conjecture for such an action, which results from a reduction of order (using the osculating geodesic) of the nonlocal action in the same way that the local EoM is obtained by reducing order in the nonlocal EoM. We present in Sec. 2.6 simple examples (in flat spacetime with a boundary, and in the linearized Schwarzschild solution) where the conjectured local action does in

fact reproduce the correct osculating-geodesic conservative self-force.

We consider the case of the Schwarzschild spacetime in Sec. 2.7, reviewing the explicit construction of the full scalar self-force using mode sum regularization in the Detweiler-Whiting framework. Based on the (preliminary) results of our own numerical implementation of the Schwarzschild scalar self-force computation (for mildly eccentric orbits), we find that the conjectured local action of Sec. 2.5 does not yield the correct conservative self-force in Schwarzschild. Section 2.7 also reviews the proof, in the Schwarzschild case, that the time-averaged change in the geodesic ‘constants’ of motion (the energy and angular momentum) caused by the conservative self-force is zero.

In Sec. 2.8, we employ the formulation of Hamiltonian mechanics in terms of symplectic geometry to address the question of whether or not there exists a Hamiltonian system encoding the local conservative self-force. We conclude that such a system does exist for the case of the Schwarzschild spacetime, while extending this result to the more astrophysically relevant Kerr spacetime will require further investigation.

Section 2.9 discusses the properties of the ‘geodesic function’ which gives the solution to the geodesic equation given initial data (a point and a velocity), and which is used (often implicitly) in the osculating geodesic method. These properties are used to accomplish some interesting simplifications in the EoM resulting from the conjectured action of Sec. 2.5. The properties of the geodesic function are also used in the discussion of geodesic deviation in Sec. 2.10, where we consider how the dynamics of a worldline can be formulated in terms of a deviation vector field along a fiducial geodesic.

2.1 Geodesic motion and covariant techniques

A timelike worldline $z(\lambda)$, with coordinates $z^\mu(\lambda)$, with λ an arbitrary parameter, will be a geodesic if it extremizes its proper time τ , or equivalently the action for a test mass m ,

$$S[z] = -m\tau[z] = -m \int d\lambda \sqrt{-\dot{z}^2} = -m \int d\lambda \sqrt{-g_{\mu\nu}(z)\dot{z}^\mu\dot{z}^\nu}, \quad (2.1.1)$$

where $\dot{z}^\mu = dz^\mu/d\lambda$ is the tangent vector, and $g_{\mu\nu}(z)$ is the (fixed background) spacetime metric evaluated along the worldline. This action is of the general type

$$S[z] = \int d\lambda L(z, \dot{z}),$$

with L a spacetime scalar, which we will encounter several times below (along with more general actions). It will be worthwhile to consider now techniques for varying such actions that maintain spacetime general covariance at each step of the manipulations.

The usual techniques proceed by using partial derivatives w.r.t. the coordinates z^μ while holding fixed the components \dot{z}^μ of the tangent vector, and vice versa. The variation of the Lagrangian $L(z, \dot{z})$ under $z \rightarrow z + \delta z$ would be written as

$$\delta L = \frac{\partial L}{\partial z^\mu} \delta z^\mu + \frac{\partial L}{\partial \dot{z}^\mu} \delta \dot{z}^\mu. \quad (2.1.2)$$

The worldline variation $\delta z^\mu(\lambda)$ can be considered, at linear order, as a vector at the point $z(\lambda)$. It is then clear that $\delta \dot{z}^\mu = (d/d\lambda)\delta z^\mu$ is not a vector, i.e. it does not transform covariantly. It turns out that $(\partial/\partial z^\mu)L(z, \dot{z})$ [unlike $(\partial/\partial z^\mu)f(z) = \nabla_\mu f(z)$] is not a vector either. This can be remedied by the rearrangement

$$\begin{aligned} \delta L &= \left(\frac{\partial L}{\partial z^\mu} - \dot{z}^\nu \Gamma_{\nu\mu}^\rho \frac{\partial L}{\partial \dot{z}^\rho} \right) \delta z^\mu + \frac{\partial L}{\partial \dot{z}^\mu} \left(\delta \dot{z}^\mu + \dot{z}^\nu \Gamma_{\nu\rho}^\mu \delta z^\rho \right) \\ &\equiv \tilde{\nabla}_\mu L \delta z^\mu + \frac{\partial L}{\partial \dot{z}^\mu} D_\lambda \delta z^\mu, \end{aligned} \quad (2.1.3)$$

where $D_\lambda \equiv D/d\lambda = \dot{z}^\nu \nabla_\nu$ is the usual covariant parameter derivative, and $\tilde{\nabla}_\mu$ is the less often encountered ‘horizontal covariant derivative’ of functions on the tangent bundle of spacetime. The operator $\tilde{\nabla}_\mu$ compares the values of L at different points z while parallel propagating the vector \dot{z} between them.¹

¹Consider a function $L(x, v)$ of a spacetime point x and a vector v at that point. If we wish to compare the values of L at two neighboring points x and $x + \Delta x$ in a covariant way, we cannot simply use the same v component-wise at both points (as the partial derivative $\partial L/\partial x^\mu$ does) because we have two distinct tangent spaces at the two points. Instead, we can parallel transport v from x to $x + \Delta x$. To linear order, we can consider the path $x^\mu(\lambda) = x^\mu + \lambda \Delta x^\mu$ with $0 \leq \lambda \leq 1$, whose tangent is Δx^μ . Parallel transporting v means

$$0 = D_\lambda v^\mu = \Delta x^\nu \nabla_\nu v^\mu = \frac{d}{d\lambda} v^\mu + \Delta x^\nu \Gamma_{\nu\rho}^\mu v^\rho \quad \Rightarrow \quad v^\mu(x + \Delta x) = v^\mu(x) - \Delta x^\nu \Gamma_{\nu\rho}^\mu v^\rho \equiv \Psi v^\mu.$$

To linear order, the parallel transport map Ψ is path-independent. We then define the operator $\tilde{\nabla}_\mu$ by

$$\Delta x^\mu \tilde{\nabla}_\mu L(x, v) \equiv L(x + \Delta x, \Psi v) - L(x, v) = \Delta x^\mu \left[\frac{\partial L}{\partial x^\mu} - v^\nu \Gamma_{\mu\nu}^\rho \frac{\partial L}{\partial v^\rho} \right].$$

With this holding for all Δx , we see that $\tilde{\nabla}_\mu L$ equals the quantity in square brackets, which can be shown to properly transform as a 1-form on spacetime. This construction easily generalizes from scalar functions to arbitrary tensor-valued functions on the tangent bundle, which $\tilde{\nabla}_\mu$ acts on according to

$$\tilde{\nabla}_\mu T^{\nu\dots}_{\rho\dots}(x, v) = \left(\frac{\partial}{\partial x^\mu} T^{\nu\dots}_{\rho\dots} + \Gamma_{\mu\sigma}^\nu T^{\sigma\dots}_{\rho\dots} - \Gamma_{\mu\rho}^\sigma T^{\nu\dots}_{\sigma\dots} + \dots \right) - v^\xi \Gamma_{\mu\xi}^\sigma \frac{\partial}{\partial v^\sigma} T^{\nu\dots}_{\rho\dots},$$

The variation of the action $S[z] = \int d\lambda L(z, \dot{z})$ can then be written covariantly as

$$\delta S = \int d\lambda \left[\tilde{\nabla}_\mu L \delta z^\mu + \frac{\partial L}{\partial \dot{z}^\mu} D_\lambda \delta z^\mu \right] = \int d\lambda \left[\left(\tilde{\nabla}_\mu L - D_\lambda \frac{\partial L}{\partial \dot{z}^\mu} \right) \delta z^\mu + D_\lambda \left(\frac{\partial L}{\partial \dot{z}^\mu} \delta z^\mu \right) \right].$$

Noting that D_λ coincides with $d/d\lambda$ when acting on a scalar, we see that the final term is a boundary term which vanishes when δz^μ is held fixed on the boundary. Requiring that $\delta S = 0$ for all δz^μ then implies the EoM

$$D_\lambda \frac{\partial L}{\partial \dot{z}^\mu} = \tilde{\nabla}_\mu L, \quad (2.1.4)$$

the covariant version of the worldline's Euler-Lagrange equation.

Now consider the geodesic action

$$S[z] = -m \int d\tau, \quad d\tau \equiv d\lambda \sqrt{-\dot{z}^2} = d\lambda \sqrt{-g_{\mu\nu}(z) \dot{z}^\mu \dot{z}^\nu}. \quad (2.1.5)$$

Applying the variation (2.1.3) with $L = \sqrt{-\dot{z}^2}$, noting $\tilde{\nabla}_\mu g_{\nu\rho}(z) = 0$, we have the result

$$\delta \sqrt{-\dot{z}^2} = -\frac{\dot{z}_\mu}{\sqrt{-\dot{z}^2}} D_\lambda \delta z^\mu,$$

or in a tidier form (used frequently below), using the definition (2.1.5) of $d\tau$,

$$\delta d\tau = -d\tau u_\mu D_\tau \delta z^\mu, \quad (2.1.6)$$

where $D_\tau = (1/\sqrt{-\dot{z}^2})D_\lambda$ and $u^\mu \equiv \dot{z}^\mu/\sqrt{-\dot{z}^2} = dz^\mu/d\tau$ is the normalized velocity vector. [*Note:* We will usually work with an arbitrarily parametrized worldline $z(\lambda)$, while defining (and using) quantities usually associated with the proper time parametrization (like $d\tau$, D_τ , u^μ , a^μ , etc.) in terms of $z(\lambda)$.] The variation of the geodesic action (2.1.5) is then given by

$$\delta S = m \int d\tau u_\mu D_\tau \delta z^\mu = m \int d\tau \left[- (D_\tau u_\mu) \delta z^\mu + D_\tau (u_\mu \delta z^\mu) \right],$$

where the parentheses give the usual covariant derivative ∇T with v held fixed component-wise, and the last term gives the v -transport correction. The horizontal derivative $\tilde{\nabla}$ is discussed e.g. by Dixon [75].

Note that the partial derivatives $\partial L/\partial v^\mu$ w.r.t. the vector components, unmodified, transform properly as a 1-form, and similarly with $L \rightarrow T^\nu_{\rho\dots}$. For intuition, if

$$L(x, v) = L_0(x) + L_\mu(x) v^\mu + L_{\mu\nu}(x) v^\mu v^\nu + \dots$$

with all the L s necessarily being covariant tensors, then

$$\tilde{\nabla}_\mu L = \nabla_\mu L_0 + (\nabla_\mu L_\nu) v^\nu + (\nabla_\mu L_{\nu\rho}) v^\nu v^\rho + \dots, \quad \frac{\partial L}{\partial v^\mu} = L_\mu + 2L_{\mu\nu} v^\nu + \dots$$

which, dropping the boundary term,² leads to the geodesic equation

$$0 = a^\mu \equiv D_\tau u^\mu = \frac{1}{\sqrt{-\dot{z}^2}} D_\lambda \left(\frac{\dot{z}^\mu}{\sqrt{-\dot{z}^2}} \right), \quad (2.1.7)$$

where a^μ is the acceleration vector.

Finally, consider the general reparametrization-invariant action

$$S[z] = \int d\tau \mathcal{L}(z, u) = \int d\lambda \sqrt{-\dot{z}^2} \mathcal{L} \left(z, \frac{\dot{z}}{\sqrt{-\dot{z}^2}} \right), \quad (2.1.8)$$

where $\mathcal{L}(z, u)$ is a spacetime scalar and is reparametrization-invariant by virtue of being a function of $u^\mu = \dot{z}^\mu / \sqrt{-\dot{z}^2}$. The variation of such a function can be written as

$$\delta \mathcal{L}(z, u) = \delta z^\mu \tilde{\nabla}_\mu \mathcal{L} + (D_\lambda \delta z^\mu) \frac{\partial \mathcal{L}}{\partial \dot{z}^\mu} = \delta z^\mu \tilde{\nabla}_\mu \mathcal{L} + (D_\tau \delta z^\mu) P_\mu^\nu \frac{\partial \mathcal{L}}{\partial u^\nu}, \quad (2.1.9)$$

where

$$P_\mu^\nu \equiv \delta_\mu^\nu + u_\mu u^\nu = \sqrt{-\dot{z}^2} \frac{\partial u^\nu}{\partial \dot{z}^\mu} \quad (2.1.10)$$

is the projection operator orthogonal to the velocity u . [Note $P_\mu^\nu u^\mu = 0$ because $u^2 = -1$.]

Such covariant and reparametrization-invariant techniques will significantly streamline later calculations, avoiding the appearance of many terms with $\Gamma_{\nu\rho}^\mu$, $\partial_\mu g_{\nu\rho}$, $\sqrt{-\dot{z}^2}$, etc. (as was done in the above derivation of the geodesic equation). The basic results we will use repeatedly below, for any covariant, reparametrization-invariant, tensor-valued function $T^{\nu\dots}_{\rho\dots}(z, u)$ of a worldline point $z(\lambda)$ and the velocity u at that point, are the variation

$$\delta T^{\nu\dots}_{\rho\dots} = \delta z^\mu \tilde{\nabla}_\mu T^{\nu\dots}_{\rho\dots} + (D_\tau \delta z^\mu) P_\mu^\sigma \frac{\partial}{\partial u^\sigma} T^{\nu\dots}_{\rho\dots}, \quad (2.1.11)$$

and the related identity

$$D_\tau T^{\nu\dots}_{\rho\dots} = u^\mu \tilde{\nabla}_\mu T^{\nu\dots}_{\rho\dots} + a^\mu \frac{\partial}{\partial u^\mu} T^{\nu\dots}_{\rho\dots}, \quad (2.1.12)$$

where we can drop the projection operator in the last term because $P_\mu^\nu a^\mu = a^\nu$ since $u_\mu a^\mu = 0$.

Note that the derivative $\partial/\partial u^\mu$ w.r.t. the components of the normalized velocity vector is generally not well-defined as a 4-dimensional derivative, because u^μ is really 3-dimensional, being constrained by $u^2 = -1$. What is well-defined is the projection of $\partial/\partial u^\mu$ orthogonal to u^μ , as appears in Eqs. (2.1.9) and (2.1.11). The u -derivative will always be projected in this way.

²Note that $\int d\tau D_\tau f(z, \dot{z}) = \int d\lambda D_\lambda f(z, \dot{z})$, and is thus a true boundary term.

2.2 A scalar field coupled to a point scalar charge

Now we consider a scalar field $\psi(x)$ on spacetime which is coupled to a point particle with a scalar charge q and bare rest mass m_0 moving on the worldline $z(\lambda)$ in a fixed background spacetime with metric $g_{\mu\nu}(x)$. The total action for this system is³

$$S[\psi, z] = -\frac{q^2}{8\pi} \int d^4x \sqrt{-g} \nabla_\mu \psi \nabla^\mu \psi - \int d\tau [m_0 - q^2 \psi(z)], \quad (2.2.1)$$

which is generally covariant and worldline-reparametrization-invariant [with $d\tau$ defined by Eq. (2.1.5)].

The variation of $S[\psi, z]$ under $\psi \rightarrow \psi + \delta\psi$ is

$$\begin{aligned} \delta_\psi S &= -\frac{q^2}{4\pi} \int d^4x \sqrt{-g} \nabla_\mu \psi \nabla^\mu \delta\psi + q^2 \int d\tau \delta\psi(z) \\ &= q^2 \int d^4x \sqrt{-g} \left\{ \left[\frac{1}{4\pi} \nabla^2 \psi + \int d\tau \delta_4(x, z) \right] \delta\psi - \frac{1}{4\pi} \nabla_\mu (\nabla^\mu \psi \delta\psi) \right\}, \end{aligned} \quad (2.2.2)$$

where $\nabla^2 = \nabla_\mu \nabla^\mu$, and the final term is a boundary term.⁴ This yields the EoM for the scalar field,

$$\nabla^2 \psi = -4\pi \int d\tau \delta_4(x, z), \quad (2.2.3)$$

which determines $\psi(x)$ once the worldline $z(\lambda)$ is specified.

The variation of S under $z \rightarrow z + \delta z$ is

$$\begin{aligned} \delta_z S &= - \int d\tau \{ -u_\mu D_\tau \delta z^\mu [m_0 - q^2 \psi(z)] - q^2 \nabla_\mu \psi(z) \delta z^\mu \} \\ &= \int d\tau \left[\left(-D_\tau \{ [m_0 - q^2 \psi(z)] u_\mu \} + q^2 \nabla_\mu \psi(z) \right) \delta z^\mu + D_\tau \{ [m_0 - q^2 \psi(z)] u_\mu \delta z^\mu \} \right], \end{aligned} \quad (2.2.4)$$

where the final term is a boundary term. This yields the EoM for the worldline,

$$D_\tau (m u_\mu) = q^2 \nabla_\mu \psi(z), \quad m \equiv m_0 - q^2 \psi(z), \quad (2.2.5)$$

where we have defined the dynamical rest mass m . In the $q^2 \rightarrow 0$ limit, this reduces to the geodesic equation (2.1.7). We will ultimately be interested in the case where q^2 is small, so that the scalar force yields small corrections to geodesic motion.

³Note that the normalization of our scalar field here differs from the usual conventions, e.g. of Poisson [25] or Quinn [66] (whose conventions we used in Chapter 1), according to $q\psi = \psi_{\text{usual}}$, so that our ψ is independent of q to leading order. This makes more obvious the order counting in a small q expansion.

⁴In Eq. (2.2.2), $\delta_4(x, y)$ is the covariant 4D Dirac delta function, defined by $\int_{\mathcal{V}} d^4x \sqrt{-g} f(x) \delta_4(x, y) = f(y)$ if \mathcal{V} contains y , and $= 0$ otherwise. We have also used the identity $\sqrt{-g} \nabla_\mu A^\mu = \partial_\mu (\sqrt{-g} A^\mu)$ for spacetime covariant integration by parts.

The component of Eq. (2.2.5) perpendicular to the velocity [found by expanding out $D_\tau(mu_\nu)$ and contracting with P_μ^ν] yields the acceleration,

$$ma_\mu = q^2 P_\mu^\nu \nabla_\nu \psi(z), \quad P_\mu^\nu \equiv \delta_\mu^\nu + u_\mu u^\nu. \quad (2.2.6)$$

The component parallel to the velocity [found by contracting Eq. (2.2.5) with u^μ] yields the evolution equation for the dynamical rest mass,

$$D_\tau m = -q^2 u^\mu \nabla_\mu \psi(z), \quad (2.2.7)$$

which is consistent with the definition (2.2.5). The fact that the effective inertial mass of a scalar charge is not conserved, discovered by Quinn [66], is associated with the emission of monopolar scalar radiation (which does not occur in the electromagnetic and gravitational cases, where the lowest-order radiation is dipolar and quadrupolar, respectively, and particles' inertial masses are conserved).

The worldline EoM (2.2.5) is ill-defined as it stands because the solution to Eq. (2.2.3) for the scalar field ψ will diverge on the worldline. Detweiler and Whiting [67], building on the work of Quinn [66], showed that the correct EoM to leading order in q can be obtained by replacing the full divergent (retarded) solution for ψ with a regularized field ψ_R , defined in terms of a Green's function decomposition.

2.3 Scalar Green's functions in curved spacetime and the Detweiler-Whiting decomposition

The wave equation (2.2.3) for the scalar field, $\nabla^2 \psi(x) = -4\pi \int d\tau \delta_4(x, z)$, can be solved by means of a Green's function. If we can find a function $G(x, y)$ satisfying

$$\nabla^2 G(x, y) = -4\pi \delta_4(x, y), \quad (2.3.1)$$

then a solution for $\psi(x)$, given the worldline $z(\lambda)$, will be

$$\psi[z](x) = \int d\tau G(x, z). \quad (2.3.2)$$

The wave equation (2.3.1) for the Green's function does not specify a unique solution and must be supplemented by boundary conditions. The retarded Green's function $G_{\text{ret}}(x, y)$, which gives the

physical solution for the scalar field, is defined by being nonzero only when the field point x is in the future of the source point y , so that the resultant field ψ_{ret} will contain no incoming radiation from past null infinity. Similarly, the advanced Green's function $G_{\text{adv}}(x, y)$ is nonzero only when x is in the past of y , giving no radiation to future null infinity. Both of these Green's functions are defined globally, for any two spacetime points x and y .

When x is in the normal convex neighborhood of y (when there is a unique geodesic connecting x and y), the retarded and advanced Green's functions can be written in the Hadamard form

$$G_{\text{ret/adv}}(x, y) = U(x, y)\delta_{\pm}(\sigma) + V(x, y)\theta_{\pm}(-\sigma), \quad (2.3.3)$$

with the upper signs for G_{ret} and lower signs for G_{adv} , where $U(x, y)$ and $V(x, y)$ are smooth functions, both symmetric in their arguments.⁵ The U term, contributing to $G_{\text{ret}}(x, y)$ [$G_{\text{adv}}(x, y)$] when x is *on* the future [past] lightcone of y , is known as the 'direct' part. The V term, contributing when x is *inside* the future [past] lightcone of y , is known as the 'tail' term. These functions obey the important reciprocity relation

$$G_{\text{ret}}(x, y) = G_{\text{adv}}(y, x). \quad (2.3.4)$$

Detweiler and Whiting [67] showed that the part of the physical retarded self-field which *does not* affect the charge's motion can be constructed locally from a singular Green's function $G_{\text{S}}(x, y)$. This function is defined by its solving the sourced wave equation (2.3.1), being symmetric in its arguments, and vanishing when x and y are timelike-separated. It is only uniquely defined in the normal convex neighborhood, where it is given by

$$G_{\text{S}}(x, y) = \frac{1}{2}U(x, y)\delta(\sigma) - \frac{1}{2}V(x, y)\theta(\sigma), \quad (2.3.5)$$

which has support on the future and past lightcones and at spacelike separations.

Removing G_{S} from G_{ret} yields the Detweiler-Whiting regular Green's function,

$$G_{\text{R}}(x, y) = G_{\text{ret}}(x, y) - G_{\text{S}}(x, y), \quad (2.3.6)$$

⁵In Eq. (2.3.3), $\sigma(x, y)$ is Synge's world function, giving half the squared interval along the geodesic connecting x and y , which is $-\tau^2/2$ for timelike geodesics, zero for null geodesics, and $s^2/2$ for spacelike geodesics (s being proper distance). The distribution $\delta_+(\sigma)$ [$\delta_-(\sigma)$] is equal to the usual Dirac delta $\delta(\sigma)$ (a delta function on the lightcone) when x is in the future [past] of y and zero otherwise. The step function $\theta_+(-\sigma)$ [$\theta_-(-\sigma)$] is one when x is inside the future [past] lightcone of y and zero otherwise. The function $U(x, y)$ equals the square root of the van Vleck determinant. The tail function $V(x, y)$ satisfies the homogeneous wave equation, with certain boundary data given on the lightcone, as detailed in Poisson's review [25].

which is nonzero everywhere except when x is inside the past lightcone of y , and satisfies the homogeneous wave equation. The regular field ψ_R constructed from G_R via Eq. (2.3.2) is continuous and differentiable on the worldline $z(\lambda)$. The full regularized self-force on the scalar charge is determined by

$$D_\tau(mu_\mu) = q^2 \nabla_\mu \psi_R(z) \quad (2.3.7)$$

which replaces the ill-defined EoM (2.2.5) with the retarded field, and which is equivalent to Quinn's EoM (1.8.10).

The regular Green's function G_R can be split into its radiative (or dissipative) and conservative parts, which are defined as its anti-symmetric and symmetric parts under argument interchange,

$$G_{\text{rad}}(x, y) = \frac{1}{2} [G_R(x, y) - G_R(y, x)], \quad G_{\text{cons}} = \frac{1}{2} [G_R(x, y) + G_R(y, x)], \quad (2.3.8)$$

which, from Eqs. (2.3.5, 2.3.6, 2.3.4), are equivalent to

$$G_{\text{rad}} = \frac{1}{2} [G_{\text{ret}} - G_{\text{adv}}], \quad G_{\text{cons}} = \frac{1}{2} [G_{\text{ret}} + G_{\text{adv}}] - G_S, \quad (2.3.9)$$

where all the arguments are (x, y) . Both of these satisfy the homogeneous wave equation and are continuous and differentiable on the worldline. Our interest below will be in the conservative Green's function, which, from Eqs. (2.3.3, 2.3.5), is given in the normal convex neighborhood by

$$G_{\text{cons}}(x, y) = \frac{1}{2} V(x, y) \quad (2.3.10)$$

with all delta and step functions cancelled, leaving only the smooth tail function V with support everywhere. While the form (2.3.10) is only valid in the normal convex neighborhood because G_S and V are only uniquely defined there, note that there is no problem in extending the definition of G_{cons} to arbitrarily distant *timelike*-separated points. When x is inside the future [past] lightcone of y , G_{cons} must agree with $\frac{1}{2}G_{\text{ret}}$ [$\frac{1}{2}G_{\text{adv}}$], which is defined globally.

2.4 The nonlocal action principle for the conservative self-force

From this section onward, we will consider only the conservative part of the dynamics of the point particle, with the radiative part of the self-force ‘turned off’. We consider the following nonlocal worldline EoM which uses the conservative part of the regular self-field $\psi[z](x) = q \int d\tau' G(x, z')$,

$$\begin{aligned} D_\tau(mu_\mu) &= q^2 \nabla_\mu \psiz, & \psi &\equiv \psi_{\text{cons}}, \\ &= q^2 \int d\tau' \nabla_\mu G(z, z'), & G &\equiv G_{\text{cons}}, \end{aligned} \quad (2.4.1)$$

where $z = z(\lambda) = z^\mu$ is the point on the worldline where we are calculating the self-force (the field point), and $z' = z(\lambda') = z^{\mu'}$ is the source point, on the same worldline, which is integrated over the entire past and future history of the worldline.⁶ The dynamical mass is given by $m = m_0 - q\psiz$, and $d\tau'$ is defined in terms of $z(\lambda')$ just as with $d\tau$ and $z(\lambda)$ in Eq. (2.1.5).

This EoM can be derived from the nonlocal worldline action

$$S_{\text{nl}}[z] = -m_0 \int d\tau + \frac{q^2}{2} \int d\tau \int d\tau' G(z, z'), \quad (2.4.2)$$

at which one can arrive via (hand-wavy) formal manipulations of the full action $S[\psi, z]$ (2.2.1), by ‘integrating out’ the scalar field.⁷ Such an action could only ever yield the conservative part of the self-force; the symmetry in the double integral will pick out the symmetric part of the Green’s function.⁸

⁶The Green’s function $G(z, z')$ is a *bi-scalar*, i.e. a function of two spacetime points z and z' which is a spacetime scalar w.r.t. operations at both points. The derivatives of such a function are the simplest examples of *bi-tensors*, which can have spacetime indices associated with one or both points. Indices at z are unprimed and indices at z' are primed: $\nabla_\mu G(z, z')$ is the derivative of $G(z, z')$ w.r.t. z and is a 1-form at z while being a scalar w.r.t. operations at z' . Similarly, $\nabla_{\mu'} G(z, z')$ is the derivative of $G(z, z')$ w.r.t. z' , being a scalar at z and a 1-form at z' .

⁷Ignoring the fact that the physical field ψ diverges on the worldline, we have

$$-\frac{q^2}{8\pi} \int d^4x \sqrt{-g} (\nabla\psi)^2 = \frac{q^2}{8\pi} \int d^4x \sqrt{-g} \psi \nabla^2 \psi = -\frac{q^2}{2} \int d^4x \sqrt{-g} \psi \int d\tau \delta_4(x, z) = -\frac{q^2}{2} \int d\tau \psi(z),$$

where the first equality integrates by parts, and the second uses the field equation (2.2.3) for ψ . When this is substituted into the full action $S[\psi, z]$ (2.2.1), and when ψ is replaced by its (regular) conservative part, we arrive at Eq. (2.4.2).

⁸Alternate formulations of worldline action principles that can incorporate the radiative or dissipative part of the self-force have been presented by Galley [76].

To see that the action (2.4.2) yields the EoM (2.4.1), we vary it using the techniques of Sec. 2.1:

$$\begin{aligned}
\delta S_{\text{nl}} &= \delta \left\{ -m_0 \int d\tau + \frac{q^2}{2} \int d\tau \int d\tau' G(z, z') \right\} \\
&= m_0 \int d\tau u_\mu D_\tau \delta z^\mu + \frac{q^2}{2} \int d\tau \int d\tau' \left[-u_\mu D_\tau \delta z^\mu - u_{\mu'} D_{\tau'} \delta z^{\mu'} + \delta z^\mu \nabla_\mu + \delta z^{\mu'} \nabla_{\mu'} \right] G(z, z') \\
&= m_0 \int d\tau u_\mu D_\tau \delta z^\mu + q^2 \int d\tau \int d\tau' \left[-u_\mu D_\tau \delta z^\mu + \delta z^\mu \nabla_\mu \right] G(z, z') \\
&= \int d\tau \left\{ \left[m_0 - q^2 \int d\tau' G(z, z') \right] u_\mu D_\tau \delta z^\mu + q^2 \int d\tau' \nabla_\mu G(z, z') \delta z^\mu \right\} \\
&= \int d\tau \left(-D_\tau \left\{ \left[m_0 - q^2 \int d\tau' G(z, z') \right] u_\mu \right\} + q^2 \int d\tau' \nabla_\mu G(z, z') \right) \delta z^\mu. \tag{2.4.3}
\end{aligned}$$

Requiring that this variation vanish for all δz implies the EoM (2.4.1). The third line uses the symmetry of $G(z, z')$ and of the double integral to relabel $z \leftrightarrow z'$ in the two ‘primed terms’ of the second line, which then give contributions equal to the two unprimed terms. The fourth line simply rearranges, and the fifth line integrates by parts, dropping the boundary term.

It is interesting that the conservative self-force EoM (2.4.1) can be derived from this nonlocal action principle, but we cannot yet see any *direct* use for it in practical computations. The standard techniques for formulating a Hamiltonian system or for deriving conserved quantities from symmetries (applying Noether’s theorem) apply only to *local* actions, of the form $S = \int d\tau \mathcal{L}(z, u)$, which the action (2.4.2) is not, and we have yet to find suitable generalizations of those procedures that would be useful here. We turn then to consider an alternate *local* form of the conservative self-force EoM.

2.5 The osculating geodesic method: Existence of a local action?

When the scalar charge q is small, the motion will deviate only slightly from geodesic motion. Because the self-force (2.4.1) scales as q^2 , the true self-forced worldline z will be related to a geodesic x_g by $z = x_g + O(q^2)$ [at least on time scales $\sim O(q^0)$]. For this reason, if we wish to work only to $O(q^2)$, we can use the following ‘reduction of order’ technique: instead of using the actual worldline z as the source for the self-field, we can use a geodesic x_g , which is tangent to the worldline at a certain point, as the source for calculating the self-field at that point.

We define the ‘geodesic function’ $x_g(z, u, \tau)$ to give the point x_g reached by traveling a proper time τ along the geodesic that issues from the point z with velocity u . We then consider the

following alternative form of the conservative self-force EoM (2.4.1):

$$D_\tau(mu_\mu) = q^2 \int d\tau' \nabla_\mu G(z, x_g(z, u, \tau')) + O(q^4), \quad (2.5.1)$$

where z , u , and m are all functions of τ , and $x_g(z, u, \tau')$ is the ‘osculating’ (‘kissing’) geodesic, which continually changes with τ to remain tangent to the self-forced worldline at $z(\tau)$. This is a *local* EoM, with the force at a given point z on the worldline depending only z and the velocity u at z (though depending highly nonlocally on the background spacetime metric). Note that the local EoM (2.5.1) is formally of the same order of accuracy [$O(q^2)$] as the nonlocal EoM (2.4.1), even though the former is obtained from the latter via a reduction of order.

We would like to determine if the local EoM (2.5.1), widely used in practical computations of the self-force, can be derived from some local action principle (alternately, from some Hamiltonian system).

An obvious guess for a local action principle would be to apply a similar reduction of order to the nonlocal action (2.4.2) that yields the nonlocal self-force. Replacing the actual worldline z with the osculating geodesic x_g as the source for the self-field, defining

$$\psi_g(x; z, u) = \int d\tau' G(x, x_g(z, u, \tau')), \quad (2.5.2)$$

the reduced-order version of the nonlocal action (2.4.2) is

$$S_g[z] = \int d\tau \left[-m_0 + \frac{q^2}{2} \psi_g(z; z, u) \right]. \quad (2.5.3)$$

The variation of this conjectured local action, à la Sec. 2.1, is

$$\delta S_g = \int d\tau \left\{ \left[\left(m_0 - \frac{q^2}{2} \psi_g \right) u_\mu + \frac{q^2}{2} P_\mu^\nu \frac{\partial \psi_g}{\partial u^\nu} \right] D_\tau \delta z^\mu + \frac{q^2}{2} \delta z^\mu \tilde{\nabla}_\mu \psi_g \right\}, \quad (2.5.4)$$

where $\psi_g = \psi_g(z; z, u)$. If we work consistently to $O(q^2)$, noting that $a^\mu = O(q^2)$, this yields the EoM

$$m_0 a_\mu = \frac{q^2}{2} P_\mu^\nu \left(\tilde{\nabla}_\nu - u^\rho \tilde{\nabla}_\rho \frac{\partial}{\partial u^\nu} \right) \psi_g(z; z, u) + O(q^4). \quad (2.5.5)$$

We now ask, as the main questions of this investigation,

(A) Is this conjectured EoM (2.5.5) equivalent to the correct local EoM (2.5.1), which when consistently order-reduced, can be written as

$$m_0 a_\mu = q^2 P_\mu^\nu \left[\nabla_\nu^{(x)} \psi_g(x; z, u) \right]_{x=z} + O(q^4), \quad (2.5.6)$$

(B) And if the conjectured action (2.5.3) does not work, then does there exist any local action principle from which the local EoM (2.5.6) can be derived? Alternately, can the EoM (2.5.6) be encoded in a Hamiltonian system?

While we have the two expressions for the correct local EoM (2.5.6) and the conjectured local EoM (2.5.5), we have not been able to show analytically that they coincide (as discussed further below). We thus resort to looking at special cases. We begin in the next section, 2.6, by giving simple but instructive examples (in flat spacetime with a boundary, and in the linearized Schwarzschild spacetime) in which the conjectured action does in fact yield the correct EoM.

We have also carried out numerical calculations of the two EoMs for mildly eccentric orbits in the Schwarzschild spacetime. The methods we adopted for calculating the self-field and self-force for Schwarzschild geodesics are outlined in Sec. 2.7. Our preliminary results indicate that the answer to question (A) is no for Schwarzschild.

We have also attempted to address Question (A) via manipulations of Eq. (2.5.5) valid in an arbitrary spacetime, using general properties of the geodesic function. These efforts, outlined in Sec. 2.9, have yet to yield any answers (to corroborate the numerical calculations from Schwarzschild, or perhaps to point to a modification of the conjectured action), though some interesting simplifications have been found. The properties of geodesic function from Sec. 2.9 prove useful in the discussion of deviation vectors along fiducial geodesics in Sec. 2.10.

Question (B) is addressed in Sec. 2.8 using the geometric formulation of Hamiltonian mechanics. We present a proof that there does exist a Hamiltonian system which encodes the (osculating-geodesic) conservative self-force EoM (2.5.6), for the case of the Schwarzschild spacetime. Our results do not yet yield a sufficiently explicit specification of the Hamiltonian system to be useful, nor do they immediately generalize to the Kerr spacetime, but it is possible that further investigation may remedy these drawbacks.

2.6 Simple examples

We now present an explicit example, followed by a claim about another example, where the EoM (2.5.5) resulting from the conjectured action (2.5.3) is indeed equivalent to the correct osculating-geodesic EoM (2.5.6). The first example concerns a scalar charge moving in flat spacetime with an infinite plane boundary on which we impose the boundary condition $\psi = 0$. The boundary reflects the scalar field and gives rise to a tail contribution to the charge's self-field, resulting in a nonzero self-force.

We find this example sufficiently nontrivial for it to be intriguing that the conjecture works in this case (mostly because we don't understand why it does). It also provides a concrete demonstration the questions discussed in the previous section.

First, consider the solution to the flat-space wave equation (with no boundary) sourced by a flat-space geodesic. The retarded and advanced Green's functions, solving $\nabla_\mu \nabla^\mu G(x, y) = -4\pi \delta^4(x - y)$, are given by

$$G_{\text{ret/adv}}(x, y) = \frac{\delta(x^0 - y^0 \mp |\mathbf{x} - \mathbf{y}|)}{|\mathbf{x} - \mathbf{y}|},$$

with the upper/lower sign for ret/adv. The flat-space geodesic function $x_g(z, u, \tau)$ is given by

$$x_g^\mu = z^\mu + \tau u^\mu.$$

Thus, the solutions for the field sourced by this geodesic are

$$\psi_g^{\text{ret/adv}}(x; z, u) = \int d\tau' G_{\text{ret/adv}}(x, x_g(z, u, \tau')) = \int d\tau' \frac{\delta(x^0 - z^0 - \tau' u^0 \mp |\mathbf{x} - \mathbf{z} - \tau' \mathbf{u}|)}{|\mathbf{x} - \mathbf{z} - \tau' \mathbf{u}|}.$$

In the rest frame of the geodesic (where $u^0 = 1$, $\mathbf{u} = 0$) this evaluates to $1/|\mathbf{x} - \mathbf{z}|$, for both the retarded and advanced cases. The quantity $|\mathbf{x} - \mathbf{z}|_{\text{rest frame}}$ can be written covariantly as the length of the projection of $(x - z)^\mu$ orthogonal to u^μ :

$$|\mathbf{x} - \mathbf{z}|_{\text{rest frame}} = \sqrt{P_{\mu\nu}(x - z)^\mu (x - z)^\nu} = \sqrt{(x - z)^2 + [u \cdot (x - z)]^2}$$

We thus have

$$(\psi_g^{\text{ret}} = \psi_g^{\text{adv}} = \psi_g^{\text{S}})(x; z, u) = \frac{1}{\sqrt{(x - z)^2 + [u \cdot (x - z)]^2}}. \quad (2.6.1)$$

The radiative and conservative fields, $\psi^{\text{rad}} = \frac{1}{2}(\psi^{\text{ret}} - \psi^{\text{adv}})$ and $\psi^{\text{cons}} = \frac{1}{2}(\psi^{\text{ret}} + \psi^{\text{adv}}) - \psi^{\text{S}}$, both vanish. There is no self-force on a scalar charge in (otherwise empty) flat spacetime because there are no tail terms in its self-field.

We can introduce a tail term, giving rise to a nonzero self-force, by imposing the scalar field boundary condition $\psi = 0$ on the plane $x^1 = 0$, with the scalar charge moving in the region $x^1 > 0$. The method of images tells us that the solution for the field in this case is the solution given above plus the field of a negative mirror image charge moving on the geodesic

$$x_g(z_m, u_m, \tau), \quad z_m^\mu = (z^0, -z_1, z_2, z_3), \quad u_m^\mu = (u^0, -u_1, u_2, u_3),$$

where m is for mirror, and we have inconsequentially lowered spatial indices. By the same arguments as above, this adds to the direct field (2.6.1) (which is now $= \psi_g^S$ only) the tail contribution

$$\psi_g^{\text{cons}}(x; z, u) = \frac{-1}{\sqrt{(x - z_m)^2 + [u_m \cdot (x - z_m)]^2}} \equiv \frac{-1}{R} \quad (2.6.2)$$

which is regular at the charge's location $x = z$. The radiative field still vanishes.

This gives rise to a conservative self-force on the charge at z with velocity u , given by the osculating-geodesic EoM (2.5.6),

$$m_0 a_\mu = q^2 P_\mu^\nu \left[\nabla_\nu^{(x)} \psi_g^{\text{cons}}(x; z, u) \right]_{x=z} + O(q^4). \quad (2.6.3)$$

From the derivative of the field (2.6.2) w.r.t. x ,

$$\begin{aligned} \nabla_\mu^{(x)} \psi_g^{\text{cons}}(x; z, u) &= \frac{1}{R^3} \{ (x - z_m)_\mu + u_{m\mu} [u_m \cdot (x - z_m)] \}, \\ \left[\nabla_\mu^{(x)} \psi_g^{\text{cons}}(x; z, u) \right]_{x=z} &= \frac{2z_1}{R_0^3} (\delta_\mu^1 - u_1 u_{m\mu}), \quad R_0 = R_{x=z} = 2z_1 \sqrt{1 + u_1^2}, \end{aligned}$$

using the identities $u_m^\mu = u^\mu - 2u_1 \delta_1^\mu$ and $P_\mu^\nu u_{m\nu} = -2u_1 P_\mu^1$, the EoM (2.6.3) gives the self-force as

$$m_0 a_\mu = q^2 \frac{1 + 2u_1^2}{4z_1^2 (1 + u_1^2)^{3/2}} P_\mu^1. \quad (2.6.4)$$

Now we can compare this to the EoM (2.5.5) resulting from the conjectured action (2.5.3),

$$m_0 a_\mu = \frac{q^2}{2} P_\mu^\nu \left(\tilde{\nabla}_\nu - u^\rho \tilde{\nabla}_\rho \frac{\partial}{\partial u^\nu} \right) \psi_g^{\text{cons}}(z; z, u) + O(q^4), \quad (2.6.5)$$

with $\psi_g^{\text{cons}}(z; z, u) = -1/R_0$. Noting that $\tilde{\nabla}_\mu = \partial/\partial z^\mu$ in flat space, the derivatives are

$$\tilde{\nabla}_\mu \psi_g^{\text{cons}} = \frac{\delta_\mu^1}{2z_1^2 (1 + u_1^2)^{1/2}}, \quad \frac{\partial}{\partial u_\mu} \psi_g^{\text{cons}} = \frac{u_1 \delta_\mu^1}{2z_1 (1 + u_1^2)^{3/2}}, \quad u^\nu \tilde{\nabla}_\nu \frac{\partial}{\partial u_\mu} \psi_g^{\text{cons}} = \frac{-u_1^2 \delta_\mu^1}{2z_1^2 (1 + u_1^2)^{3/2}},$$

where $\psi_g^{\text{cons}} = \psi_g^{\text{cons}}(z; z, u)$. Plugging these into the conjectured EoM (2.6.5), we get

$$m_0 a_\mu = \frac{q^2}{2} P_\mu^\nu \left(\tilde{\nabla}_\nu - u^\rho \tilde{\nabla}_\rho \frac{\partial}{\partial u^\nu} \right) \psi_g^{\text{cons}} = q^2 \frac{1 + 2u_1^2}{4z_1^2 (1 + u_1^2)^{3/2}} P_\mu^1,$$

which does indeed match the correct self-force (2.6.4).

Thus, the conjecture that the osculating-geodesic action (2.5.3) encodes the conservative self-force EoM (2.5.6) is correct for this simple case of flat spacetime with a boundary. We have also found other similar examples where this is true.

As an example (somewhat) more closely related to the EMRI problem, take as a background spacetime the linearized Schwarzschild solution (first-post-Minkowskian GR with a static point mass source). A scalar field living on this background will scatter off the point mass, giving rise to a tail term in a point charge's self field. Working consistently to first-post-Minkowskian order, the charge must move on a flat-space geodesic at zeroth order in q^2 (so that we don't get bound orbits at zeroth order in q^2 , as in the real EMRI problem). One can analytically solve for the self-field and self-force in this situation, and one again finds that the conjectured action principle gives the correct osculating-geodesic conservative self-force.

We have also tested the conjectured action numerically for the case of a point scalar charge on a bound orbit in the Schwarzschild spacetime. The general methods we used are described in the following section. Our results (if they are to be trusted) indicate that the conjecture is not true in that case.

We find the former examples interesting mostly because we don't understand why the conjectured action works in those cases (apart from carrying out the explicit calculations). It's possible that understanding this could lend insight into why it does not work in Schwarzschild, or how it might be modified to work there.

In Secs. 2.9.I and 2.9.II below, we explore manipulations of the conjectured EoM using properties of the geodesic function valid in an arbitrary spacetime, finding some useful simplifications but no real answers. We have found that such manipulations are equally unfruitful when one specializes to flat-space geodesics, as in the two examples mentioned here. It seems that one would also need to use some properties of the Green's function to show (using manipulations along the lines of Sec. 2.9.II) that the conjecture works in those cases.

2.7 The scalar self-force in the Schwarzschild spacetime

This section reviews the explicit computation of the scalar self-force for eccentric orbits in the Schwarzschild spacetime, using a mode-sum solution to the scalar wave equation and mode-sum regularization, as in the seminal work of Barack and Ori [68] and later (using the Detweiler-Whiting decomposition) Detweiler, Messaritaki, and Whiting [69]. The methods we present here are assembled from Ref. [69], from the work of Barack, Ori, and Sago [77] on ‘extended homogeneous solutions’, and from the work of Drasco, Flanagan, and Hughes [78] on the scalar self-force in Kerr. We have used the methods outlined in this section to perform a numerical test of the conjectured action of Sec. 2.5, and found that it does not yield the correct self-force in Schwarzschild.

We begin in Sec. 2.7.I by reviewing some properties of geodesic motion in Schwarzschild. Section 2.7.II presents the mode-sum solution to the wave equation, and Sec. 2.7.III describes the regularization procedure using mode-sum regularization parameters (which have been provided to very high orders by Heffernan, Ottewill, and Wardell [79]). Section 2.7.IV discusses the properties of the self-force under a certain ‘reflection’ of a geodesic orbit, which are used in Sec. 2.7.V to prove that the conservative part of the self-force causes no average change in the geodesic ‘constants’ of motion (the energy and angular momentum). This proof is the Schwarzschild analog of important results in Kerr derived by Gal’tsov [71] and extended by Mino [70] (also discussed by Ref. [78]), and is an important ingredient in our proof in Sec. 2.8.III that the conservative self-force dynamics in Schwarzschild can be encoded in a Hamiltonian system. Finally, Sec. 2.7.VI summarizes the results of our numerical calculations which indicate that the conjectured local action of Sec. 2.5 fails to yield the correct local self-force in Schwarzschild.

I Geodesic motion in Schwarzschild and symmetry properties

The Schwarzschild geometry, in units where the black hole mass is $M = 1$, is given in Schwarzschild coordinates $x^\mu = (t, r, \theta, \phi)$ by the metric

$$ds^2 = -w dt^2 + \frac{1}{w} dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2), \quad w = 1 - \frac{2}{r}. \quad (2.7.1)$$

Thanks to the spherical symmetry, the motion of an orbiting particle (even one experiencing a self-force) will be confined to a fixed plane, which we can take to be the equatorial plane $\theta = \frac{\pi}{2}$. A

particle worldline $z(\tau)$ can be described by the coordinates $z^\mu(\tau) = (t(\tau), r(\tau), \frac{\pi}{2}, \phi(\tau))$, with the velocity $u^\mu = \dot{z}^\mu = (u^t, u^r, 0, u^\phi) = (\dot{t}, \dot{r}, 0, \dot{\phi})$ where dots are derivatives w.r.t. the proper time τ .

The coordinate basis vectors ∂_t and ∂_ϕ are Killing vectors of this spacetime corresponding to time translation and axial rotation symmetries. Their contractions with the velocity yield the energy E and angular momentum L (both per unit mass) which are conserved for geodesic motion:

$$E = -u \cdot \partial_t = w\dot{t}, \quad L = u \cdot \partial_\phi = r^2\dot{\phi}. \quad (2.7.2)$$

From the velocity normalization $u^2 = -1$, we have the third conserved quantity

$$-1 = -w\dot{t}^2 + \frac{\dot{r}^2}{w} + r^2\dot{\phi}^2 = -\frac{E^2}{w} + \frac{\dot{r}^2}{w} + \frac{L^2}{r^2},$$

which can be rewritten in terms of an effective potential V_r for the radial motion as

$$\dot{r}^2 + V_r = E^2, \quad V_r(r, L) = w \left(1 + \frac{L^2}{r^2} \right). \quad (2.7.3)$$

Equations (2.7.2) and (2.7.3) completely describe geodesic motion in Schwarzschild.

We will be primarily interested in stable bound (eccentric) orbits. For these, the radial motion is oscillatory between a maximum radius r_+ and minimum radius r_- [functions of (E, L)], which are roots of $V_r(r, L) = E^2$. A resultant precessing elliptical orbit is shown in Figure 3.

Instead of using (t, r, ϕ) as functions of τ , it is useful to work with a parametrization of the bound geodesics due to Darwin [80]. With the semi-latus rectum p and eccentricity e defined in terms of E and L by

$$r_\pm(E, L) = \frac{p}{1 \mp e} \quad \Leftrightarrow \quad E^2 = \frac{(p-2)^2 - 4e^2}{p(p-3-e^2)}, \quad L^2 = \frac{p^2}{p-3-e^2}, \quad (2.7.4)$$

we make a change of variables from the radius r to the ‘radial phase’ χ via

$$r = \frac{p}{1 + e \cos \chi}. \quad (2.7.5)$$

As χ increases from 0 to π , and then to 2π , r increases from r_- to r_+ , and then decreases back to r_- .⁹

⁹Note that while χ uniquely determines r , given e and p , the converse is not true, since χ and $2\pi - \chi$ give the same r . However, r and the sign of u^r do uniquely determine χ ; positive u^r corresponds to $0 \leq \chi \leq \pi$. A bound geodesic and a point on it are uniquely determined (up to t - and ϕ -translations) by e , p , and χ , or equivalently by r , u^t and u^r .

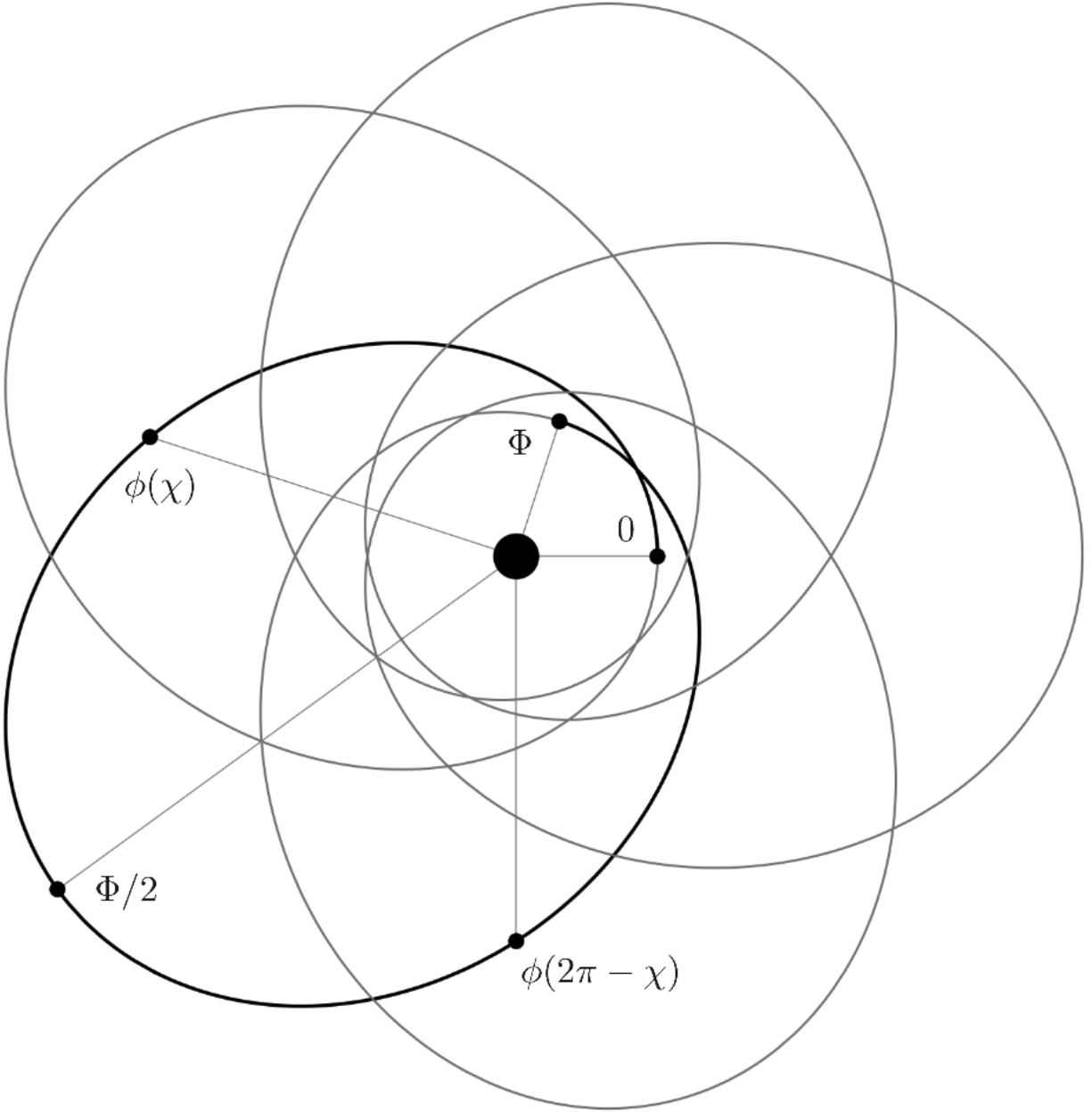


Figure 3: A precessing elliptical orbit in Schwarzschild, with one radial period in bold, and with lines from the center marking the stated angles $\phi(\chi)$. The minimum radius r_- occurs at $\phi(0) = 0$ and $\phi(2\pi) = \Phi$, the beginning and end of the period, and the maximum radius r_+ occurs at $\phi(\pi) = \Phi/2$, halfway through. A generic point $\phi(\chi)$ is shown along with its reflection $\phi(2\pi - \chi) = \Phi - \phi(\chi)$ about the $\Phi/2$ line. This orbit has $p = 19.8$, $e = 0.6$, $r_- = 12.3$, $r_+ = 49.4$, and has been specially chosen to make $\Phi = 12\pi/5$, so that the orbit closes after five radial periods (a very non-generic feature, having nothing to do with the reflection property, just for fun). The disk in the center shows the Schwarzschild radius $r = 2$.

It is convenient to use χ as the parameter for geodesic motion, writing $z^\mu(\chi) = (t(\chi), r(\chi), \frac{\pi}{2}, \phi(\chi))$.

With $r(\chi)$ given by Eq. (2.7.5), $t(\chi)$ and $\phi(\chi)$ are determined by integrating

$$\frac{dt}{d\chi} = \frac{p^2}{(p-2-2e\cos\chi)(1+e\cos\chi)^2} \sqrt{\frac{(p-2)^2-4e^2}{p-6-2e\cos\chi}}, \quad \frac{d\phi}{d\chi} = \sqrt{\frac{p}{p-6-2e\cos\chi}},$$

which follow from Eqs. (2.7.2–2.7.5). We will use the boundary conditions $t = \phi = 0$ at $\chi = 0$.

The radius function $r(\chi)$ in Eq. (2.7.5) has the periodicity and reflection properties

$$r(\chi + 2\pi) = r(\chi), \quad r(2\pi - \chi) = r(\chi). \quad (2.7.6)$$

The transformation $\chi \rightarrow 2\pi - \chi$ corresponds to reflecting the orbit about the radial line through the point of maximum radius ($r = r_+$, $\chi = \pi$), as shown in Figure 3. While $t(\chi)$ and $\phi(\chi)$ are not periodic, their first derivatives are, which means that they are each a periodic function plus a linear function:

$$t(\chi) = \tilde{t}(\chi) + T \frac{\chi}{2\pi}, \quad \phi(\chi) = \tilde{\phi}(\chi) + \Phi \frac{\chi}{2\pi}, \quad (2.7.7)$$

where T and Φ are the time elapsed and angle swept during one period of the radial motion, and the functions $\tilde{t}(\chi)$ and $\tilde{\phi}(\chi)$ are periodic:

$$T = \int_0^{2\pi} d\chi \frac{dt}{d\chi}, \quad \Phi = \int_0^{2\pi} d\chi \frac{d\phi}{d\chi}, \quad \tilde{t}(\chi + 2\pi) = \tilde{t}(\chi), \quad \tilde{\phi}(\chi + 2\pi) = \tilde{\phi}(\chi). \quad (2.7.8)$$

From these and the fact that $t(\chi)$ and $\phi(\chi)$ are odd in χ (because their derivatives are even), one can derive the reflection properties

$$t(2\pi - \chi) = T - t(\chi), \quad \phi(2\pi - \chi) = \Phi - \phi(\chi). \quad (2.7.9)$$

The velocity components $u^\mu = dz^\mu/d\tau$, as functions of χ , are all periodic and have the reflection properties

$$\chi \rightarrow 2\pi - \chi \quad \Rightarrow \quad u^r \rightarrow -u^r, \quad u^t \rightarrow u^t, \quad u^\phi \rightarrow u^\phi. \quad (2.7.10)$$

II Mode sum solutions to the wave equation

We wish to solve the scalar field wave equation (2.2.3),

$$\nabla^2 \psi = -4\pi \int d\tau \delta_4(x, z(\tau)), \quad (2.7.11)$$

in the Schwarzschild spacetime, where the source particle's worldline $z(\tau)$ is the bound geodesic described in the previous section.

The retarded solution can be written as a sum of modes, indexed by the usual ℓ and m of the spherical harmonics $Y_{\ell m}(\theta, \phi)$ and by an integer $k = -\infty, \dots, \infty$ from a discrete Fourier transform w.r.t. time:

$$\psi^{\text{ret}}(x) = \psi^{\text{ret}}(t, r, \theta, \phi) = \sum_{k\ell m} e^{-i\omega_{km}t} Y_{\ell m}(\theta, \phi) \begin{cases} Z_{k\ell m}^{\text{H}} u_{k\ell m}^{\text{H}}(r), & r < r(t), \\ Z_{k\ell m}^{\infty} u_{k\ell m}^{\infty}(r), & r > r(t). \end{cases} \quad (2.7.12)$$

The frequencies are given by $\omega_{km} = k(2\pi/T) + m(\Phi/T)$, with T and Φ from Eq. (2.7.8). The radial mode functions $u_{k\ell m}^{\text{H}}(r)$ (H for horizon) and $u_{k\ell m}^{\infty}(r)$ are the solutions to the ODE

$$\left[\partial_r^2 + \frac{2(r-1)}{r(r-2)} \partial_r + \frac{\omega_{km}^2 r^2}{(r-2)^2} - \frac{\ell(\ell+1)}{r(r-2)} \right] u_{k\ell m}^{H/\infty}(r) = 0 \quad (2.7.13)$$

which give respectively ingoing waves near the horizon and outgoing waves at infinity:

$$u_{k\ell m}^{\text{H}}(r) \rightarrow \frac{e^{-i\omega_{km}r_*}}{r} \text{ as } r \rightarrow 2, \quad u_{k\ell m}^{\infty}(r) \rightarrow \frac{e^{i\omega_{km}r_*}}{r} \text{ as } r \rightarrow \infty, \quad (2.7.14)$$

where $r_* = r + 2 \log(r/2 - 1)$ is the tortoise coordinate ($t \pm r_* = \text{constant}$ for radial null geodesics).

The amplitudes $Z_{k\ell m}^{H/\infty}$ are given by an integral over one period of the source particle's motion:

$$Z_{k\ell m}^{H/\infty} = -\frac{4\pi Y_{\ell m}(\frac{\pi}{2}, 0)}{ET} \int_0^{2\pi} d\chi \frac{dt}{d\chi}(\chi) \cos[\omega_{km}t(\chi) - m\phi(\chi)] \frac{u_{k\ell m}^{\infty/H}(r(\chi))}{W_{k\ell m}} \quad (2.7.15)$$

where $t(\chi)$, $r(\chi)$, and $\phi(\chi)$ describe the geodesic of the last section and $W = r^2(u_{,r}^{\text{H}} u^{\infty} - u^{\infty} u_{,r}^{\text{H}})$.

The advanced solution is found simply by complex conjugating the radial mode functions $u_{k\ell m}^{H/\infty}$ [changing Eqs. (2.7.14) to outgoing waves at the horizon and ingoing waves at infinity], giving

$$\psi^{\text{adv}}(x) = \sum_{k\ell m} e^{-i\omega_{km}t} Y_{\ell m}(\theta, \phi) \left[Z_{k\ell m}^{H/\infty} u_{k\ell m}^{H/\infty}(r) \right]^*. \quad (2.7.16)$$

On a given $t = \text{constant}$ slice, the mode functions are finite but have a kink (are continuous but not differentiable) across $r = r(t)$, the source particle's radius at that t -slice, where $Z_{k\ell m}^{\text{H}} u_{k\ell m}^{\text{H}}(r)$ switches to $Z_{k\ell m}^{\infty} u_{k\ell m}^{\infty}(r)$. The mode sum (ψ^{ret} or ψ^{adv}) diverges along the source worldline.

III Mode sum regularization

The retarded field ψ^{ret} can be regularized along the worldline by subtracting off the singular field ψ^{S} , yielding the regular field $\psi^{\text{R}} = \psi^{\text{ret}} - \psi^{\text{S}}$. Both ψ^{ret} and ψ^{S} are divergent, but their mode functions

are all finite, so that the regularization can be done mode by mode (most conveniently w.r.t. ℓ). For the field and its gradient, we can write

$$\psi^R = \sum_{\ell} (\psi_{\ell}^{\text{ret}} - \psi_{\ell}^{\text{S}}), \quad \psi_{,\mu}^R = \sum_{\ell} (\psi_{\ell,\mu}^{\text{ret}} - \psi_{\ell,\mu}^{\text{S}}). \quad (2.7.17)$$

All these ℓ -modes are functions only of (p, e, χ) [or (E, L, χ)] when evaluated on the worldline. The retarded modes are found from the solution (2.7.12) for $\psi^{\text{ret}}(x)$ evaluated at the worldline point $x = z(\chi)$:

$$\psi_{\ell}^{\text{ret}} = \sum_{km} Z_{k\ell m}^{\text{H}/\infty} \left(u_{k\ell m}^{\text{H}/\infty} Y_{\ell m} e^{-i\omega_{km}t} \right)_{x=z(\chi)}, \quad \psi_{\ell,\mu}^{\text{ret}} = \sum_{km} Z_{k\ell m}^{\text{H}/\infty} \left[\nabla_{\mu} \left(u_{k\ell m}^{\text{H}/\infty} Y_{\ell m} e^{-i\omega_{km}t} \right) \right]_{x=z(\chi)}. \quad (2.7.18)$$

The H and ∞ [$r < r(t)$ and $r > r(t)$] versions agree for ψ_{ℓ}^{ret} but disagree for $\psi_{\ell,\mu}^{\text{ret}}$, but this discontinuity in the gradient is reproduced by the singular field. The ℓ -modes of ψ^{S} can be found from a local expansion, as in the foundational work of Detweiler, Messaritaki, and Whiting [69] and Barack and Ori [68], yielding¹⁰

$$\begin{aligned} \psi_{\ell}^{\text{S}} &= B + \frac{D}{(2\ell-1)(2\ell+3)} + \frac{F}{(2\ell-3)(2\ell-1)(2\ell+3)(2\ell+5)} + \dots \\ \psi_{\ell,\mu}^{\text{S}} &= \pm(2\ell+1)A_{\mu} + B_{\mu} + \frac{D_{\mu}}{(2\ell-1)(2\ell+3)} + \frac{F_{\mu}}{(2\ell-3)(2\ell-1)(2\ell+3)(2\ell+5)} + \dots \end{aligned} \quad (2.7.19)$$

where the \pm sign gives the gradient discontinuity. These expansions have been carried to very high orders (to one higher order than shown here) by Heffernan, Ottewill, and Wardell [79], who give the coefficients B , A_{μ} , etc. as explicit functions of E , L , r and $u^r = dr/d\tau$.

The only information contained in u^r not already given by (E, L, r) is its sign, and it is found that $\psi_{\ell,t}^{\text{S}}$ and $\psi_{\ell,\phi}^{\text{S}}$ are simply proportional to $\text{sign}(u^r)$, while ψ_{ℓ}^{S} and $\psi_{\ell,r}^{\text{S}}$ are independent of it. Using the properties (2.7.6,2.7.10) of $r(\chi)$ and $u^r(\chi)$, this gives the reflection properties under $\chi \rightarrow 2\pi - \chi$

$$\psi_{\ell}^{\text{S}} \rightarrow \psi_{\ell}^{\text{S}} \quad \psi_{\ell,r}^{\text{S}} \rightarrow \psi_{\ell,r}^{\text{S}}, \quad \psi_{\ell,t}^{\text{S}} \rightarrow -\psi_{\ell,t}^{\text{S}}, \quad \psi_{\ell,\phi}^{\text{S}} \rightarrow -\psi_{\ell,\phi}^{\text{S}}, \quad (2.7.20)$$

for the ℓ -modes of singular field and its gradient evaluated at the worldline point $z(\chi)$. The next section derives similar reflection properties relating the ℓ -modes of the retarded and advanced fields.

¹⁰The B , A_{μ} , and B_{μ} terms are clearly responsible for the divergence of the ℓ -sums. The D and D_{μ} terms actually give zero under $\sum_{\ell=0}^{\infty}$, and similarly for all the higher order terms. But including them mode by mode dramatically increases the rate of convergence of the ℓ -sum, as discussed e.g. by Ref. [69].

IV Reflection properties of the retarded and advanced fields

The ℓ -modes of ψ^{ret} and ψ^{adv} , at the worldline point $x^\mu = z^\mu(\chi)$, are given by¹¹

$$\begin{aligned}\psi_\ell^{\text{ret}}(\chi) &= \sum_{km} Z_{k\ell m} u_{k\ell m}(r(\chi)) Y_{\ell m}^0 e^{-i\omega_{km}t(\chi)+im\phi(\chi)}, \\ \psi_\ell^{\text{adv}}(\chi) &= \sum_{km} Z_{k\ell m}^* u_{k\ell m}^*(r(\chi)) Y_{\ell m}^0 e^{-i\omega_{km}t(\chi)+im\phi(\chi)},\end{aligned}\tag{2.7.21}$$

and similarly for the field gradients, as in Eqs. (2.7.16,2.7.18), with $k = -\infty, \dots, \infty$, $m = -\ell, \dots, \ell$, and $Y_{\ell m}^0 \equiv Y_{\ell m}(\frac{\pi}{2}, 0)$. We now will consider the properties of the mode function factors under the reflection $\chi \rightarrow 2\pi - \chi$ and under the related interchange $(k, m) \leftrightarrow (-k, -m)$.

From the properties (2.7.6,2.7.9) of $z^\mu(\chi)$ under $\chi \rightarrow 2\pi - \chi$, namely, $r \rightarrow r$, $t \rightarrow T - t$, $\phi \rightarrow \Phi - \phi$, we see that the argument of the exponentials in Eq. (2.7.21) undergoes

$$-i\omega_{km}t + im\phi \rightarrow -i\left(k\frac{2\pi}{T} + m\frac{\Phi}{T}\right)(T - t) + im(\Phi - \phi) = i\omega_{km}t - im\phi - 2\pi ik,\tag{2.7.22}$$

which is a sign change mod $2\pi i$, giving $e^{-i\omega_{km}t+im\phi} \rightarrow e^{i\omega_{km}t-im\phi}$, which is equivalent to complex conjugation or to the interchange $(k, m) \leftrightarrow (-k, -m)$, since $\omega_{(-k)(-m)} = -\omega_{km}$. While $u_{k\ell m}(r(\chi))$ is unchanged by the reflection $\chi \rightarrow \chi - 2\pi$ [as, of course, are $Z_{k\ell m}$ and $Y_{\ell m}^0$], we have the $(k, m) \leftrightarrow (-k, -m)$ properties

$$Y_{\ell(-m)}^0 = (-1)^m Y_{\ell m}^0, \quad u_{(-k)\ell(-m)} = u_{k\ell m}^*, \quad Z_{(-k)\ell(-m)} = (-1)^m Z_{k\ell m}^*,\tag{2.7.23}$$

where the second follows from Eqs. (2.7.14) and (2.7.13) and the last follows from the first two and Eq. (2.7.15).

Now we can show that, under $\chi \rightarrow 2\pi - \chi$,

$$\begin{aligned}\psi_\ell^{\text{ret}} &= \sum_{km} Z_{k\ell m} u_{k\ell m}(r) Y_{\ell m}^0 e^{-i\omega_{km}t+im\phi} \\ &\rightarrow \sum_{km} Z_{k\ell m} u_{k\ell m}(r) Y_{\ell m}^0 e^{i\omega_{km}t-im\phi} \\ &= \sum_{km} Z_{(-k)\ell(-m)} u_{(-k)\ell(-m)}(r) Y_{\ell(-m)}^0 e^{-i\omega_{km}t+im\phi} \\ &= \sum_{km} Z_{k\ell m}^* u_{k\ell m}^*(r) Y_{\ell m}^0 e^{-i\omega_{km}t+im\phi} = \psi_\ell^{\text{adv}},\end{aligned}\tag{2.7.24}$$

¹¹We have dropped the superscripts H/∞ with the understanding that $\psi_{\ell,\mu}^{\text{ret}}$, $\psi_{\ell,\mu}^{\text{adv}}$, and $\psi_{\ell,\mu}^{\text{S}}$ have different values approached from $r < r(t)$ and $r > r(t)$, while $\psi_{\ell,\mu}^{\text{R}}$ and $\psi_{\ell,\mu}^{\text{cons}}$ do not. We have also used $Y_{\ell m}(\theta, \phi) = Y_{\ell m}(\theta, 0)e^{im\phi}$.

where the first and fourth lines use Eqs. (2.7.21), the second uses Eq. (2.7.22), the third interchanges $(k, m) \leftrightarrow (-k, -m)$ leaving the km -sum unchanged, and the fourth uses Eqs. (2.7.23). Precisely analogous manipulations yield $\psi_{\ell,r}^{\text{ret}} \rightarrow \psi_{\ell,r}^{\text{adv}}$. For the t -derivatives, we have, under $\chi \rightarrow 2\pi - \chi$,

$$\begin{aligned}
\psi_{\ell,t}^{\text{ret}} &= \sum_{km} (-i\omega_{km}) Z_{k\ell m} u_{k\ell m}(r) Y_{\ell m}^0 e^{-i\omega_{km}t+im\phi} \\
&\rightarrow \sum_{km} (-i\omega_{km}) Z_{k\ell m} u_{k\ell m}(r) Y_{\ell m}^0 e^{i\omega_{km}t-im\phi} \\
&= \sum_{km} (i\omega_{km}) Z_{(-k)\ell(-m)} u_{(-k)\ell(-m)}(r) Y_{\ell(-m)}^0 e^{-i\omega_{km}t+im\phi} \\
&= \sum_{km} (i\omega_{km}) Z_{k\ell m}^* u_{k\ell m}^*(r) Y_{\ell m}^0 e^{-i\omega_{km}t+im\phi} = -\psi_{\ell,t}^{\text{adv}}.
\end{aligned} \tag{2.7.25}$$

Similarly, with im in place of $-i\omega_{km}$, one finds $\psi_{\ell,\phi}^{\text{ret}} \rightarrow -\psi_{\ell,\phi}^{\text{adv}}$. We have thus shown that

$$\begin{aligned}
\psi_{\ell}^{\text{ret}}(2\pi - \chi) &= \psi_{\ell}^{\text{adv}}(\chi), & \psi_{\ell,t}^{\text{ret}}(2\pi - \chi) &= -\psi_{\ell,t}^{\text{adv}}(\chi), \\
\psi_{\ell,r}^{\text{ret}}(2\pi - \chi) &= \psi_{\ell,r}^{\text{adv}}(\chi), & \psi_{\ell,\phi}^{\text{ret}}(2\pi - \chi) &= -\psi_{\ell,\phi}^{\text{adv}}(\chi).
\end{aligned} \tag{2.7.26}$$

These are clearly also true with $\text{ret} \leftrightarrow \text{adv}$.

V The conservative self-force averages to zero

The full scalar self-force on the particle, to linear order in q^2 , is given by the EoM $ma_{\mu} = q^2 P_{\mu}^{\nu} \psi_{,\nu}^{\text{R}}$, with the regular field given by Eq. (2.7.17). Similarly, the conservative part is given by the EoM

$$ma_{\mu} = q^2 P_{\mu}^{\nu} \psi_{,\nu}^{\text{cons}}, \quad \psi^{\text{cons}} = \sum_{\ell} \left(\frac{\psi_{\ell}^{\text{ret}} + \psi_{\ell}^{\text{adv}}}{2} - \psi_{\ell}^{\text{S}} \right). \tag{2.7.27}$$

From the latter and the reflection properties (2.7.26) of ψ^{ret} and ψ^{adv} and those (2.7.20) of ψ^{S} , we have

$$\begin{aligned}
\psi^{\text{cons}}(2\pi - \chi) &= \psi^{\text{cons}}(\chi), & \psi_{,t}^{\text{cons}}(2\pi - \chi) &= -\psi_{,t}^{\text{cons}}(\chi), \\
\psi_{,r}^{\text{cons}}(2\pi - \chi) &= \psi_{,r}^{\text{cons}}(\chi), & \psi_{,\phi}^{\text{cons}}(2\pi - \chi) &= -\psi_{,\phi}^{\text{cons}}(\chi),
\end{aligned} \tag{2.7.28}$$

where we have summed over ℓ . (The corresponding properties for the radiative field ψ^{rad} would have all the right-hand sides negated relative to these.)

The rate of change (w.r.t. proper time) of the energy E (2.7.2) caused by the conservative

self-force is given by

$$\begin{aligned}\dot{E}^{\text{cons}} &= D_\tau[(\partial_t)^\mu u_\mu] = (\partial_t)^\mu a_\mu + u_\mu u_\nu \nabla^\nu (\partial_t)^\mu \\ &= a_t = \frac{1}{m} P_t^\mu \psi_{,\mu}^{\text{cons}} = \frac{1}{m} \left[(1 + u_t u^t) \psi_{,t}^{\text{cons}} + u_t \left(u^\phi \psi_{,\phi}^{\text{cons}} + u^r \psi_{,r}^{\text{cons}} \right) \right],\end{aligned}\tag{2.7.29}$$

where the last term of the first line vanishes by the Killing equation $\nabla^{(\mu}(\partial_t)^{\nu)} = 0$. Considering the reflection properties (2.7.28) of $\psi_{,\mu}^{\text{cons}}$ and those (2.7.10) of u^μ (recalling that r and thus the metric are reflection-invariant), we find that under $\chi \rightarrow 2\pi - \chi$,

$$(1 + u_t u^t) \psi_{,t}^{\text{cons}} + u_t \left(u^\phi \psi_{,\phi}^{\text{cons}} + u^r \psi_{,r}^{\text{cons}} \right) \rightarrow (1 + u_t u^t) (-\psi_{,t}^{\text{cons}}) + u_t \left(u^\phi (-\psi_{,\phi}^{\text{cons}}) + (-u^r) \psi_{,r}^{\text{cons}} \right),$$

which is a total sign flip. Along with precisely analogous manipulations for the angular momentum L , this yields the properties

$$\dot{E}^{\text{cons}}(2\pi - \chi) = -\dot{E}^{\text{cons}}(\chi), \quad \dot{L}^{\text{cons}}(2\pi - \chi) = -\dot{L}^{\text{cons}}(\chi).$$

These then imply that the average of \dot{E} or \dot{L} over one period of the motion is zero:

$$\langle \dot{E}^{\text{cons}} \rangle = \frac{1}{2\pi} \int_0^{2\pi} d\chi \dot{E}^{\text{cons}}(\chi) = 0,\tag{2.7.30}$$

and similarly for L . Note that this also holds when the derivative $\dot{E} = dE/d\tau$ is replaced with one w.r.t. t or χ , or (independently) when the averaging measure $d\chi$ is replaced with dt or $d\tau$; this is because the cancellation occurs between the two reflected halves of the orbital period, and the ratios $dt : d\tau : d\chi$ are all reflection-invariant. For the same reasons, the average (2.7.30) will also be zero if \dot{E} is replaced by the parameter-derivative of any function of only E and L .

To summarize, at linear order (using the osculating geodesic approximation), the conservative self-force in Schwarzschild causes no average change in the orbital constants of motion. This fact, which underlies the usefulness of the adiabatic approximation, is the Schwarzschild analog of results in Kerr derived by Galtsov [71] and Mino [70]. This fact is also a crucial ingredient in our proof that there exists a Hamiltonian system encoding the conservative self-force dynamics, in Sec. 2.8.III.

VI Numerical test of the conjectured action

Using the mode-sum methods outlined in Secs. 2.7.II and 2.7.III, we have numerically calculated the self-field and self-force for mildly eccentric orbits in Schwarzschild, each split into their conservative and radiative parts. Our calculations of the self-force have been compared with results from Warburton et al. [81], with additional data graciously provided by Niels Warburton, yielding excellent agreement. We have used our results to test the conjectured action (2.5.3) for the conservative self-force in Schwarzschild.

The correct conservative self-force components F_μ^{cons} are constructed from the field-point derivatives $\psi_{,\mu}^{\text{cons}}$ according to

$$F_\mu^{\text{cons}} = q^2 P_\mu^\nu \psi_{,\nu}^{\text{cons}}.$$

These can then be compared with the conjectured self-force components $\tilde{F}_\mu^{\text{cons}}$ resulting from the conjectured EoM (2.5.5),

$$\tilde{F}_\mu^{\text{cons}} = q^2 P_\mu^\nu \left(\tilde{\nabla}_\nu - u^\rho \tilde{\nabla}_\rho \frac{\partial}{\partial u^\nu} \right) \psi^{\text{cons}}. \quad (2.7.31)$$

The conservative self-field ψ^{cons} , evaluated on the worldline, is calculated numerically as a function of the orbital parameters $(e, p, \chi) \equiv P^a$. The derivatives appearing in Eq. (2.7.31) are then evaluated as follows. The first and second partial derivatives $\partial\psi^{\text{cons}}/\partial P^a$ and $\partial^2\psi^{\text{cons}}/\partial P^a\partial P^b$ are calculated numerically from ψ^{cons} evaluated on a $5 \times 5 \times 5$ grid in P^a -space, using second-order finite difference techniques. These are then converted into partial derivatives w.r.t. the coordinate and velocity components $Y^A \equiv (r, u^t, u^r)$ using the transformation $P^a \leftrightarrow Y^A$ resulting from the formulae of Sec. 2.7.I. Finally, the partial derivatives w.r.t. Y^A are converted into the horizontal covariant derivatives appearing in Eq. (2.7.31) using the definitions of Sec. 2.1, with the result

$$\tilde{F}_\mu^{\text{cons}} = q^2 P_\mu^\nu \left(\frac{\partial}{\partial z^\nu} - u^\rho \frac{\partial^2}{\partial z^\rho \partial u^\nu} + u^\rho u^\sigma \Gamma_{\rho\sigma}^\lambda \frac{\partial^2}{\partial u^\lambda \partial u^\nu} \right) \psi^{\text{cons}},$$

where the only nonzero contributions come from $z^\mu \rightarrow r$ and $u^\mu \rightarrow u^t, u^r$, $\Gamma_{\rho\sigma}^\lambda$ are the Schwarzschild connection coefficients, and the projection tensor and velocity components are found from the formulae of Sec. 2.7.I.

Our results for the case of a geodesic orbit with $e = 0.02$ and $p = 10$ are shown in Figures 4-6. We find that the correct self-force F_μ and the conjectured self-force \tilde{F}_μ do not coincide.

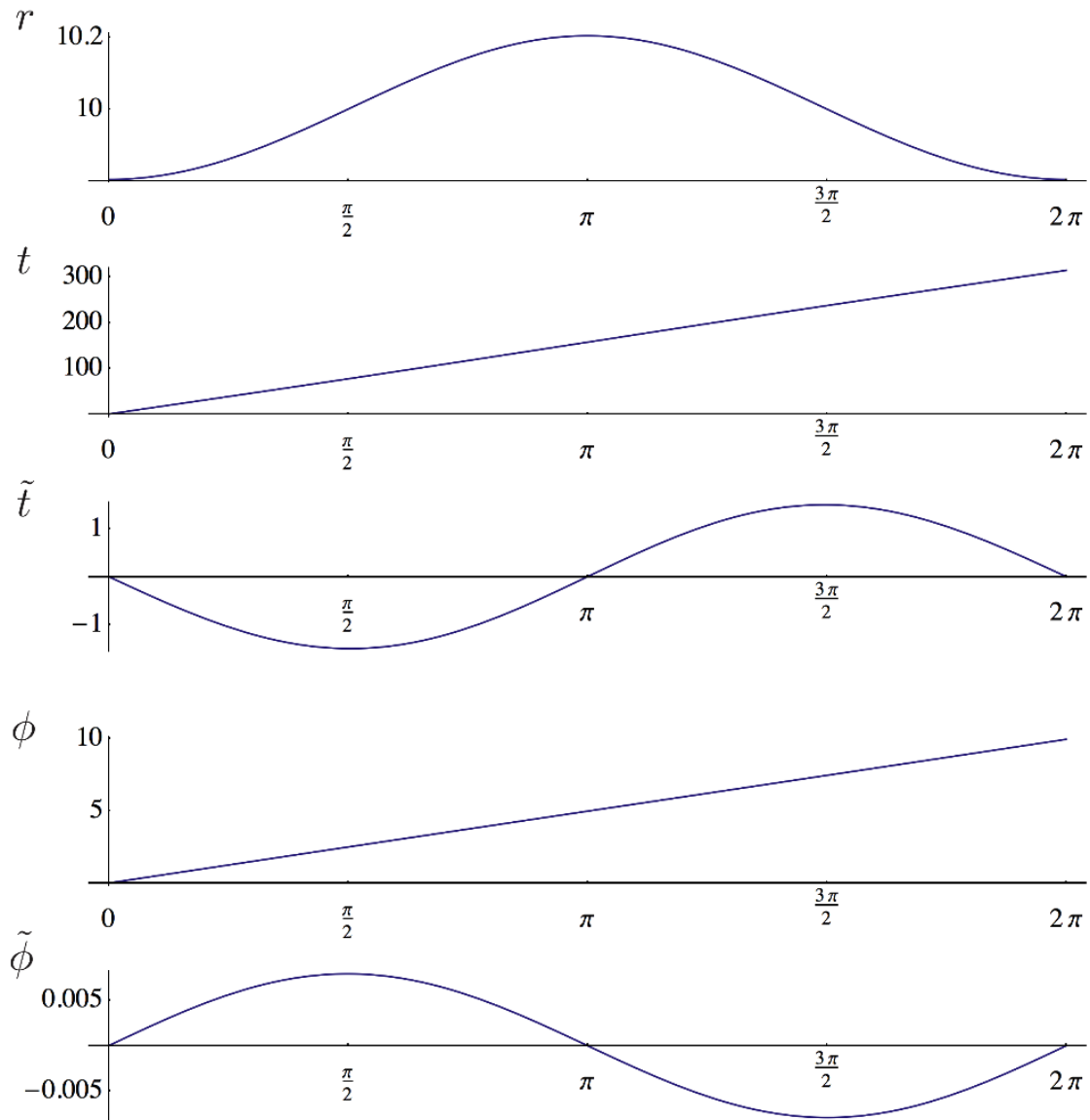


Figure 4: The coordinates r , t and ϕ [along with the oscillatory parts \tilde{t} and $\tilde{\phi}$ of Eqs. (2.7.7)], all as functions of the orbital phase χ , for a Schwarzschild orbit with eccentricity $e = 0.02$ and semi-latus rectum $p = 10$, the orbit used to calculate the conservative self-fields of Figure 5 and the correct and conjectured conservative self-forces of Figure 6. Recall that we are using units where $G = c = M = 1$, where M is the mass of the black hole.

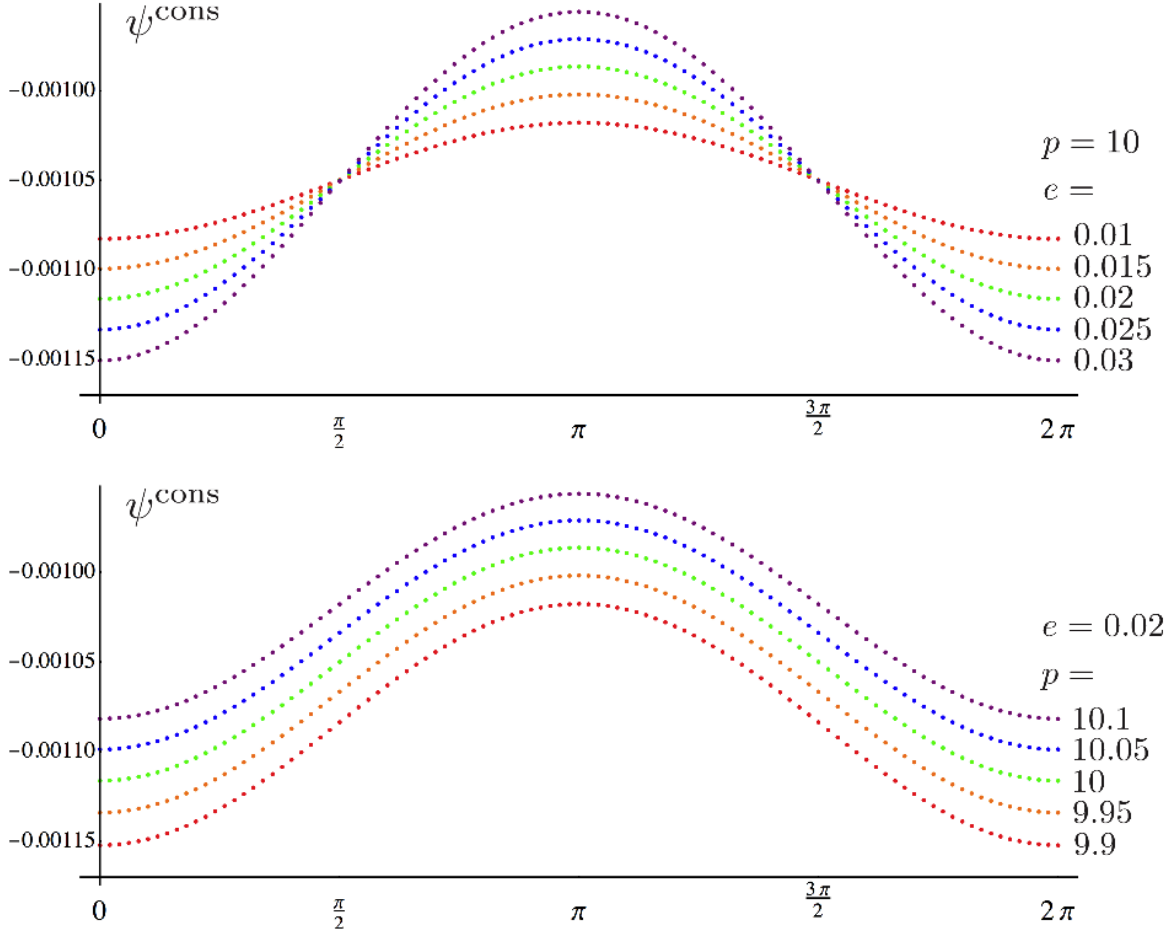


Figure 5: The conservative self-field ψ^{cons} , as a function of the radial phase χ , for various values of e and p ([some of] those used to calculate the self-forces of Figure 6). In this figure and the next, we have set the scalar charge q equal to one. Note that all these field values are negative, so that the fields of greater magnitude are towards the bottom of the figure, which can be seen to coincide with the particle being closer to the black hole. The green curves in the two panels are the same curve, corresponding to the orbits of Figures 4 and 6.

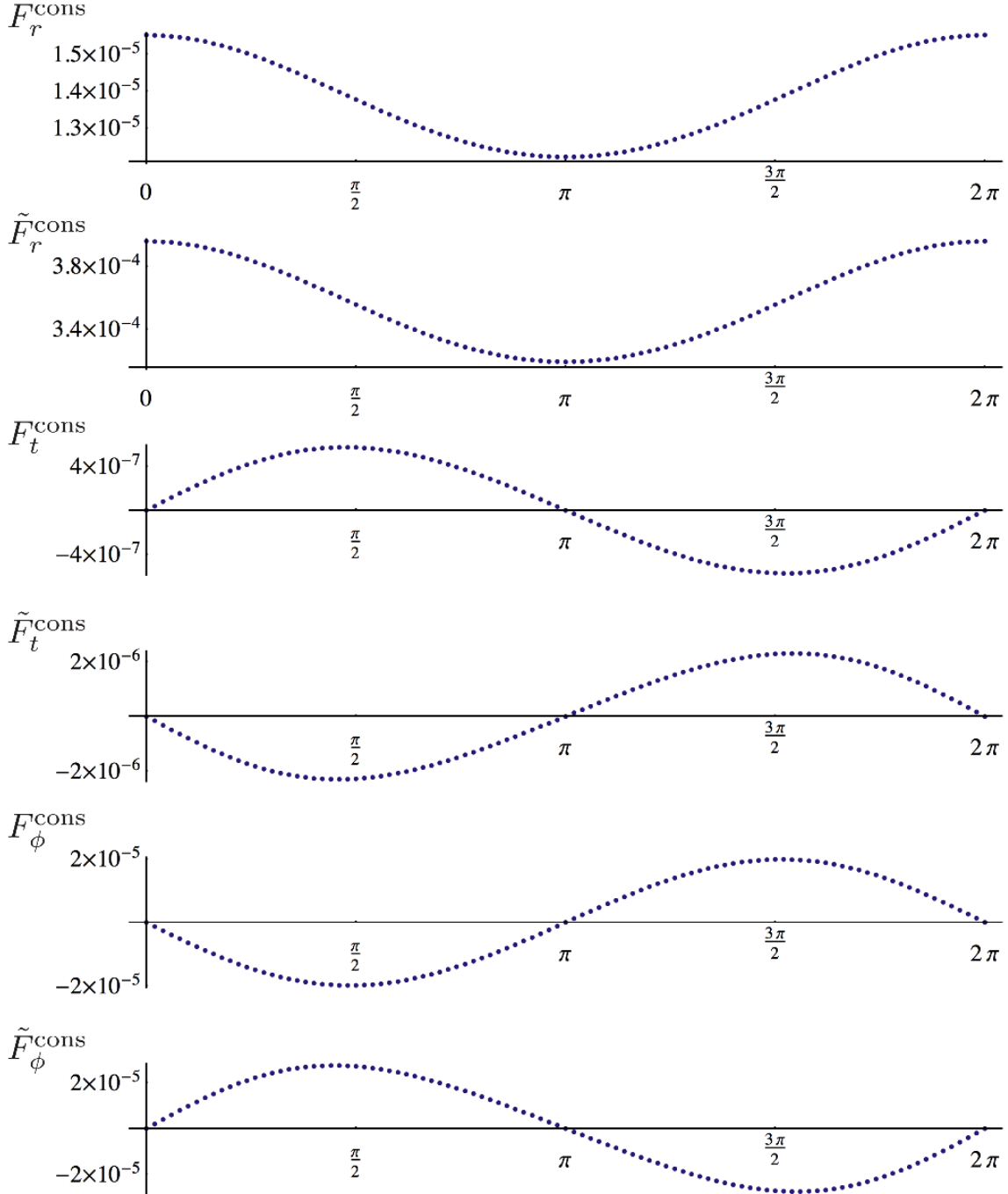


Figure 6: The correct conservative self-force components F_r^{cons} , F_t^{cons} , and F_ϕ^{cons} and the conjectured conservative self-force components $\tilde{F}_r^{\text{cons}}$, $\tilde{F}_t^{\text{cons}}$, and $\tilde{F}_\phi^{\text{cons}}$, all as functions of the orbital phase χ , for the Schwarzschild orbit with $e = 0.02$ and $p = 10$ of Figures 4 and 5. We see that the two sets of components do not coincide, meaning that the conjectured action of Sec. 2.5 does not yield the correct self-force in Schwarzschild. Though the conjectured components have roughly the right shapes (up to some minus signs), this is actually required by the reflection properties discussed above and does not suggest any correlation with the correct components. The r -, t -, and ϕ -components differ by (not actually constant) factors of roughly 25, -4 , and -1.5 , respectively.

2.8 Hamiltonian systems and perturbations

This section uses the geometric formulation of Hamiltonian mechanics to address whether or not the (osculating-geodesic) conservative self-force EoM (2.5.6) can be realized as a Hamiltonian system. Section 2.8.I gives a brief review the description of a (reparametrization-invariant) Hamiltonian system in terms of the (pre)symplectic geometry of phase space (see e.g. Arnold's text [82]), and its application to geodesic motion. Section 2.8.II then considers when a given perturbed Hamiltonian system is itself a Hamiltonian system. Finally, Section 2.8.III considers the case of the conservative self-force in the Schwarzschild and Kerr spacetimes.

I (Pre)Symplectic geometry in phase space and geodesic motion

We consider a $2n$ -dimensional phase space manifold \mathcal{P} , with coordinates Y^A ($A = 1 \dots 2n$). The symplectic geometry of \mathcal{P} is determined by a non-degenerate 2-form Ω_{AB} , the symplectic form, which is closed, $\Omega_{[AB,C]} = 0$, where commas denote partial derivatives w.r.t. Y (or any connection on \mathcal{P}). In canonical coordinates $Y^A = (q^\alpha, p_\alpha)$ ($\alpha = 1 \dots n$), the symplectic form can be written as $\Omega = \mathbf{d}q^\alpha \wedge \mathbf{d}p_\alpha$. We define its inverse Ω^{AB} by $\Omega^{AB}\Omega_{BC} = \delta_C^A$. Given a function on \mathcal{P} , a Hamiltonian $H(Y)$, the integral curves $Y^A(\lambda)$ of the vector field $\Omega^{AB}H_{,B}$ define the Hamiltonian flow:

$$\dot{Y}^A = \Omega^{AB}H_{,B}. \quad (2.8.1)$$

In canonical coordinates, this reads

$$\dot{q}^\alpha = \frac{\partial H}{\partial p_\alpha}, \quad \dot{p}_\alpha = -\frac{\partial H}{\partial q^\alpha}, \quad (2.8.2)$$

which we recognize as Hamilton's equations. Given a Lagrangian system, with an action $S = \int d\lambda L(q, \dot{q})$, one can (Legendre) transform to a Hamiltonian system in canonical coordinates by defining the canonical momenta and Hamiltonian by

$$p_\alpha = \frac{\partial L}{\partial \dot{q}^\alpha}, \quad H = p_\alpha \dot{q}^\alpha - L, \quad (2.8.3)$$

assuming that the first of these can be solved for $\dot{q}(q, p)$.

Now consider applying this to the reparametrization-invariant geodesic action for a worldline $z^\mu(\lambda)$, defining the momenta p_μ :

$$S = \int d\lambda L(z, \dot{z}) = -m \int d\lambda \sqrt{-\dot{z}^2} \quad \Rightarrow \quad p_\mu = \frac{\partial L}{\partial \dot{z}^\mu} = \frac{m\dot{z}_\mu}{\sqrt{-\dot{z}^2}}.$$

The latter cannot be solved for $\dot{z}(z, p)$, due to its reparametrization invariance. Instead, it implies the constraint $p^2 = -m^2$ on phase space, which we'll write for later convenience as

$$K(z, p) \equiv \frac{p^2 + m^2}{2m} = 0.$$

We thus cannot realize reparametrization-invariant geodesic motion as a standard Hamiltonian system in the symplectic geometry of the full (8D) phase space \mathcal{P} . However, this dynamical system can be described by the ‘presymplectic’ geometry on the (7D) constraint surface \mathcal{P}_K defined by $K = 0$. As detailed below,¹² the correct orbits $Y^A(\lambda)$ [i.e. $z^\mu(\lambda), p_\mu(\lambda)$] within the constraint surface \mathcal{P}_K are defined by the EoM

$$\dot{Y}^A = \eta \Omega^{AB} K_{,B}, \quad (2.8.6)$$

where $\eta(Y)$ is an arbitrary (nonzero) function on \mathcal{P} . This EoM is practically identical to the Hamiltonian EoM (2.8.1), with the constraint function K playing the role of the Hamiltonian, apart from the arbitrary function η , whose choice corresponds to reparametrization freedom. In terms of z^μ and p_μ , using a spacetime-covariant rearrangement, Eq. (2.8.6) reads

$$\dot{z}^\mu = \eta \frac{\partial K}{\partial p_\mu}, \quad D_\lambda p_\mu = -\eta \tilde{\nabla}_\mu K, \quad (2.8.7)$$

¹²In the full phase space \mathcal{P} , with coordinates $Y^A = (z^\mu, p_\mu)$, a constraint $K(Y) = 0$ defines a submanifold \mathcal{P}_K , say, with coordinates y^a . Given the embedding $Y(y) : \mathcal{P}_K \rightarrow \mathcal{P}$, and the usual symplectic form $\Omega = \mathbf{d}z^\mu \wedge \mathbf{d}p_\mu$ on \mathcal{P} , we define the presymplectic form ω on \mathcal{P}_K to be the pullback of Ω :

$$\omega_{ab} = \frac{\partial Y^A}{\partial y^a} \frac{\partial Y^B}{\partial y^b} \Omega_{AB}. \quad (2.8.4)$$

The dynamical system can then be defined by the integral curves $y^a(\lambda)$ of the null vector field of ω :

$$\omega_{ab} \dot{y}^b = 0. \quad (2.8.5)$$

This defines \dot{y}^a only up to multiplication by an arbitrary (nonzero) function on \mathcal{P}_K , which corresponds precisely to reparametrization freedom.

This can also be rewritten in terms of the full phase space coordinates. The vector \dot{y}^a can be pushed forward to $\dot{Y}^A = (\partial Y^A / \partial y^a) \dot{y}^a$, and then, using Eq. (2.8.4), the presymplectic EoM (2.8.5) can be written as

$$\frac{\partial Y^A}{\partial y^a} \Omega_{AB} \dot{Y}^B = 0.$$

This says that the projection (pullback) of the 1-form $\Omega_{AB} \dot{Y}^B$ into \mathcal{P}_K is zero, which means that $\Omega_{AB} \dot{Y}^B$ must be proportional to the normal to \mathcal{P}_K , i.e. $\Omega_{AB} \dot{Y}^B = \eta K_{,A}$, or

$$\dot{Y}^A = \eta \Omega^{AB} K_{,B},$$

which is Eq. (2.8.6). Note that evolution under this EoM preserves the value of K . See, e.g., Rovelli [83] for a discussion of presymplectic mechanics.

for general K , or with $K = (p^2 + m^2)/2m$ for geodesic motion,

$$\dot{z}^\mu = \eta \frac{p^\mu}{m}, \quad D_\lambda p_\mu = 0. \quad (2.8.8)$$

The first of these, along with the constraint, tells us that $\eta = \sqrt{-\dot{z}^2}$, and then together these imply the geodesic equation $D_\lambda(\dot{z}^\mu/\sqrt{-\dot{z}^2}) = 0$.

II Conservative self-force perturbations to geodesic motion

Given a zeroth-order reparametrization-invariant Hamiltonian system, with the symplectic form Ω_0 , the constraint $K_0 = 0$, and the EoM $\dot{Y}^A = \eta \Omega_0^{AB} K_{0,B}$, a general (reparameterization-invariant) linear perturbation to the system can be written as

$$\dot{Y}^A = \eta (\Omega_0^{AB} K_{0,B} + \epsilon V^A) + O(\epsilon^2) \quad (2.8.9)$$

where ϵ is a small parameter and V^A is an arbitrary vector field on phase space. We would like to know when such a system, with a given V^A , is itself Hamiltonian, i.e. when it is equivalent to

$$\dot{Y}^A = \eta \Omega^{AB} K_{,B}, \quad K = K_0 + \epsilon K_1, \quad \Omega^{AB} = \Omega_0^{AB} + \epsilon (\mathcal{L}_X \Omega_0)^{AB}, \quad (2.8.10)$$

where K_1 is a perturbation to the constraint, and the perturbation to the (inverse) symplectic form is written as the Lie derivative $(\mathcal{L}_X \Omega_0)^{AB}$ of the zeroth-order form w.r.t. a ‘gauge’ vector field X^A .¹³ This means that the vector field V^A must be given by

$$V^A = \Omega_0^{AB} K_{1,B} + (\mathcal{L}_X \Omega_0)^{AB} K_{0,B}. \quad (2.8.11)$$

The perturbed system (2.8.9) will be Hamiltonian if, given V^A , we can find a function K_1 and vector field X^A which satisfy Eq. (2.8.11).

An important necessary condition, implied by but weaker than Eq. (2.8.11), can be found by contracting it with $\eta K_{0,A}$, using the antisymmetry of Ω^{AB} and $(\mathcal{L}_X \Omega_0)^{AB}$, and $\dot{Y}_0^A = \eta \Omega_0^{AB} K_{0,B}$,

$$\eta K_{0,A} V^A = \eta K_{0,A} \Omega_0^{AB} K_{1,B} = -\dot{Y}_0^B K_{1,B} = -\left(\frac{d}{d\lambda}\right)_0 K_1, \quad (2.8.12)$$

¹³A general linear perturbation to the symplectic form Ω_0 form can be written locally as $\Omega = \Omega_0 + \epsilon \mathcal{L}_X \Omega_0$ for the following reasons. Because the perturbed form must also be closed and non-degenerate, there exist (at least locally) coordinates $(\bar{q}^\alpha, \bar{p}_\alpha)$ in which the perturbed form takes the canonical form $\Omega = \mathbf{d}\bar{q}^\alpha \wedge \mathbf{d}\bar{p}_\alpha$. There exists then a linear coordinate transformation between the barred coordinates and the original (zeroth-order) canonical coordinates, which can be parametrized by a gauge vector X . The pullback of the perturbed form into the original coordinates then takes the form (2.8.10).

where $(d/d\lambda)_0$ is the parameter derivative along the zeroth-order integral curves. By integrating this around a closed zeroth-order orbit, we obtain a condition for V^A which is independent of K_1 :

$$\oint_0 d\lambda \eta K_{0,A} V^A = 0. \quad (2.8.13)$$

Now consider the case where the zeroth order system is geodesic motion, given by Eqs. (2.8.8), which are $\dot{Y}^A = \eta \Omega_0^{AB} K_{0,B}$ with $Y^A = (z^\mu, p_\mu)$, $\Omega_0 = \mathbf{d}z^\mu \wedge \mathbf{d}p_\mu$, and $K_0 = (p^2 + m^2)/2m$. We consider a perturbation to this system given by

$$\dot{z}^\mu = \eta \left(\frac{p^\mu}{m} + \epsilon f^\mu \right), \quad D_\lambda p_\mu = \epsilon \eta F_\mu, \quad (2.8.14)$$

which is Eq. (2.8.9) with the perturbing vector field written as $\vec{V} = f^\mu \tilde{\nabla}_\mu + F_\mu (\partial/\partial p_\mu)$. Writing the gauge vector as $\vec{X} = \chi^\mu \tilde{\nabla}_\mu + \zeta_\mu (\partial/\partial p_\mu)$, Eq. (2.8.11) can be written as

$$\begin{aligned} f^\mu &= \frac{\partial K_1}{\partial p_\mu} - \frac{p^\nu}{m} \left(\tilde{\nabla}_\nu \chi^\mu + \frac{\partial \zeta_\nu}{\partial p_\mu} \right), \\ F_\mu &= -\tilde{\nabla}_\mu K_1 + \frac{p^\nu}{m} \left(\tilde{\nabla}_\mu \zeta_\nu - \tilde{\nabla}_\nu \zeta_\mu \right). \end{aligned} \quad (2.8.15)$$

The perturbed system will be Hamiltonian if, given f^μ and F_μ , we can find K_1 , χ^μ , and ζ_μ which satisfy these equations.

Now, the specific perturbation to geodesic motion we wish to consider is that produced by the conservative self-force, sourced by the osculating geodesic, as in the EoM (2.5.6). We define the dynamical system on phase space corresponding to this EoM by demanding that p_μ be related to \dot{z}^μ by $p_\mu = m \dot{z}_\mu / \sqrt{-\dot{z}^2}$, just as for geodesic motion. Identifying $\epsilon = q^2$ as the small parameter, this fixes the components of the perturbing vector field in Eq. (2.8.14) to be

$$f^\mu = 0, \quad F_\mu = P_\mu^\nu \left[\nabla_\nu^{(x)} \psi_g(x; z, p/m) \right]_{x=z}, \quad (2.8.16)$$

where ψ_g is the conservative self-field sourced by the osculating geodesic, given by Eq. (2.5.2), and $P_\mu^\nu = \delta_\mu^\nu + p_\mu p^\nu / m^2$ is the projector perpendicular to p^μ (or u^μ). Now, given this f^μ and F_μ , can we find K_1 , χ^μ , and ζ_μ which satisfy Eqs. (2.8.15)?

We will address this question for the specific cases of the Schwarzschild and Kerr spacetimes in the following subsection, using action-angle variables for geodesic motion in those spacetimes as phase space coordinates. But first, we note one property valid in an arbitrary spacetime. Applying the necessary condition (2.8.12) to the perturbed geodesic motion system, we find

$$-\left(\frac{d}{d\lambda} \right)_0 K_1 = \eta K_{0,A} V^A = \eta \frac{p^\mu}{m} F_\mu = 0, \quad (2.8.17)$$

where the last equality follows from Eq. (2.8.16) and $p^\mu P_\mu = 0$. This means that the perturbation K_1 to the constraint must be constant along the zeroth-order (geodesic) orbits. Note that this also shows that the system satisfies the loop-integral condition (2.8.13). Also note that these results, while derived using $Y^A = (z^\mu, p_\mu)$, are independent of the choice of phase space coordinates.

III The Schwarzschild and Kerr spacetimes and action-angle variables

Bound geodesic motion in the Schwarzschild and Kerr spacetimes is completely integrable, and thus can be described in terms of (generalized) action-angle variables, as discussed by Hinderer and Flanagan [23]. These are canonical coordinates $Y^A = (q^\alpha, J_\alpha)$ on phase space such that the Hamiltonian system for geodesic motion takes the form

$$\begin{aligned} \dot{Y}^A &= \eta \Omega_0^{AB} K_{0,B}, & \Omega_0 &= \mathbf{d}q^\alpha \wedge \mathbf{d}J_\alpha, & K_0 &= K_0(J), & \eta &= 1. \\ \Rightarrow & \dot{q}^\alpha &= \frac{\partial K_0}{\partial J_\alpha} \equiv \omega^\alpha(J), & \dot{J}_\alpha &= 0. \end{aligned} \quad (2.8.18)$$

The constraint (Hamiltonian) K_0 is a function only of the action variables J_α , which are all conserved along the geodesic orbits [and still $K_0 = (p^2 + m^2)/2m$, just expressed in terms of the new variables]. The angle variables q^α then increase linearly with λ (having chosen $\eta = 1$), at the constant frequencies ω^α .¹⁴ For the Schwarzschild and Kerr spacetimes, there is one action variable and one angle variable for each of the Schwarzschild or Boyer-Lindquist coordinates, $q^\alpha = (q^t, q^r, q^\theta, q^\phi)$ and $J_\alpha = (J_t, J_r, J_\theta, J_\phi)$. The (true) angle variables q^r , q^θ , and q^ϕ are periodic with period 2π , while q^t has an infinite range. In Schwarzschild, we can specialize to motion in the equatorial plane $\theta = \frac{\pi}{2}$ and ignore q^θ and J_θ without loss of generality; in Kerr, we cannot.

We consider the conservative self-force perturbation to this system which is the same as that given by Eqs. (2.8.14) and (2.8.16) above, simply translated into the action-angle coordinates. Writing the perturbing vector as $\vec{V} = f^\alpha(\partial/\partial q^\alpha) + F_\alpha(\partial/\partial J_\alpha)$, we have

$$\dot{q}^\alpha = \omega^\alpha(J) + \epsilon f^\alpha(q, J), \quad \dot{J}_\alpha = \epsilon F_\alpha(q, J). \quad (2.8.19)$$

¹⁴The parameter λ here could be any of the proper time τ , the coordinate time t , Mino time [70], or any other parameter, but the differing choices will change the definitions of (q^α, J_α) in terms of (z^μ, p_μ) . Following Hinderer and Flanagan [23], we choose the parameter to be $\lambda = \tau/m$, where m is the particle rest mass, with all dots in this section being $d/d\lambda = m d/d\tau$. This choice makes geodesic motion well-defined as a 4-dimensional action-angle system, with the rest mass as an additional conserved quantity; see Ref. [23] for further details and for the explicit construction of the action-angle variables for Kerr.

Due to the symmetries of Schwarzschild and Kerr under t -translations and ϕ -rotations, the functions f^α and F_α have the important property that they only depend on the angle variable q^r for Schwarzschild, and only on q^r and q^θ for Kerr, and not on the other angle variables (while depending on all of the action variables).

This perturbed system will be Hamiltonian if we can find a function K_1 and a vector $\vec{X} = \chi^\alpha(\partial/\partial q^\alpha) + \zeta_\alpha(\partial/\partial J_\alpha)$ which satisfy Eq. (2.8.12), which now reads

$$\begin{aligned} f^\alpha &= \frac{\partial K_1}{\partial J_\alpha} - \omega^\beta \left(\frac{\partial \chi^\alpha}{\partial q^\beta} + \frac{\partial \zeta_\beta}{\partial J_\alpha} \right), \\ F_\alpha &= -\frac{\partial K_1}{\partial q^\alpha} + \omega^\beta \left(\frac{\partial \zeta_\beta}{\partial q^\alpha} - \frac{\partial \zeta_\alpha}{\partial q^\beta} \right). \end{aligned} \quad (2.8.20)$$

We can attempt to address the solvability of these equations by carrying out a discrete Fourier transform for the dependence of the relevant functions on the angle variables. We make the (reasonable, though not strictly necessary) ansatz that the functions K_1 , χ^α , and ζ_α , like f^α and F_α , depend only on q^r for Schwarzschild and only on q^r and q^θ for Kerr, not on the other angle variables. For clarity, we will distinguish the two sets of angle variables with different index types: α, β , etc. are split into C, D , etc. and Γ, Δ , etc.,

$$\begin{aligned} q^\alpha &= (q^C, q^\Gamma), & q^C &= (q^r), & q^\Gamma &= (q^t, q^\theta, q^\phi) & \text{for Schw.}, \\ & & q^C &= (q^r, q^\theta), & q^\Gamma &= (q^t, q^\phi) & \text{for Kerr.} \end{aligned} \quad (2.8.21)$$

All functions involved depend only on q^C and not on q^Γ . We then Fourier transform all functions w.r.t. q^C , according to

$$\begin{aligned} f^\alpha(q, J) &= \sum_k \bar{f}^\alpha(k, J) e^{ik_C q^C}, \\ \sum_k &= \sum_{k_r=-\infty}^{\infty} \quad \text{for Schw.}, & \sum_k &= \sum_{k_r=-\infty}^{\infty} \sum_{k_\theta=-\infty}^{\infty} \quad \text{for Kerr.} \end{aligned}$$

and similarly for F_α , K_1 , χ^α , and ζ_α as with f^α . The k_C are vectors of integers because q^r and q^θ are periodic with period 2π . With this, the Fourier transforms of Eqs. (2.8.20) are

$$\bar{f}^\alpha = \frac{\partial \bar{K}_1}{\partial J_\alpha} - \omega^\beta \left(ik_\beta \bar{\chi}^\alpha + \frac{\partial \bar{\zeta}_\beta}{\partial J_\alpha} \right) = \frac{\partial}{\partial J_\alpha} \left(\bar{K}_1 - \omega^\beta \bar{\zeta}_\beta \right) + \frac{\partial \omega^\beta}{\partial J_\alpha} \bar{\zeta}_\beta - i\omega^\beta k_\beta \bar{\chi}^\alpha, \quad (2.8.22)$$

$$\bar{F}_\alpha = -ik_\alpha \bar{K}_1 + i\omega^\beta (k_\alpha \bar{\zeta}_\beta - k_\beta \bar{\zeta}_\alpha) = -ik_\alpha \left(\bar{K}_1 - \omega^\beta \bar{\zeta}_\beta \right) - i\omega^\beta k_\beta \bar{\zeta}_\alpha, \quad (2.8.23)$$

before splitting indices, where the second equalities simply rearrange, and after splitting (some)

indices,

$$\bar{f}^C = \frac{\partial}{\partial J_C} \left(\bar{K}_1 - \omega^\beta \bar{\zeta}_\beta \right) + \frac{\partial \omega^\beta}{\partial J_C} \bar{\zeta}_\beta - i\omega^D k_D \bar{\chi}^C, \quad (2.8.24)$$

$$\bar{f}^\Gamma = \frac{\partial}{\partial J_\Gamma} \left(\bar{K}_1 - \omega^\beta \bar{\zeta}_\beta \right) + \frac{\partial \omega^\beta}{\partial J_\Gamma} \bar{\zeta}_\beta - i\omega^D k_D \bar{\chi}^\Gamma, \quad (2.8.25)$$

$$\bar{F}_C = -ik_C \left(\bar{K}_1 - \omega^\beta \bar{\zeta}_\beta \right) - i\omega^D k_D \bar{\zeta}_C, \quad (2.8.26)$$

$$\bar{F}_\Gamma = -i\omega^D k_D \bar{\zeta}_\Gamma. \quad (2.8.27)$$

From the coordinate-independent result (2.8.17), we know that K_1 must be constant along the zeroth-order orbits. Attempting to find any solution to Eqs. (2.8.24-2.8.27), we make the ansatz

$$K_1 = 0.$$

The same result (2.8.17) also tells us that $K_{0,A} V^A = 0$, which in action-angle coordinates implies that $\omega^\alpha F_\alpha = 0$, and thus $\omega^\alpha \bar{F}_\alpha = 0$. We then see that Eqs. (2.8.26) and (2.8.27) will be solved if $\bar{\zeta}_\alpha$ satisfies (both for $\alpha = C$ and $\alpha = \Gamma$)

$$\bar{F}_\alpha = -i\omega^D k_D \bar{\zeta}_\alpha, \quad (2.8.28)$$

which is obvious for $\alpha = \Gamma$, and for $\alpha = C$ requires noting that this equation along with $\omega^\alpha \bar{F}_\alpha = 0$ implies that $\omega^\alpha \bar{\zeta}_\alpha = 0$, if $\omega^D k_D \neq 0$. This equation can be easily solved for $\bar{\zeta}_\alpha$, given \bar{F}_α , whenever $\omega^D k_D \neq 0$. Then, given this solution for $\bar{\zeta}_\alpha$, with $\bar{K}_1 = \omega^\beta \bar{\zeta}_\beta = 0$, Eqs. (2.8.24, 2.8.25) simplify to

$$\bar{f}^\alpha = \frac{\partial \omega^\beta}{\partial J_\alpha} \bar{\zeta}_\beta - i\omega^D k_D \bar{\chi}^\alpha, \quad (2.8.29)$$

for both $\alpha = C, \Gamma$, which can be easily solved for $\bar{\chi}^\alpha$, as long as $\omega^D k_D \neq 0$. Thus, for the case $\omega^D k_D \neq 0$, we have shown that Eqs. (2.8.24-2.8.27) can be solved.

The case $\omega^D k_D = 0$ can occur in Schwarzschild (where k_D has only one component, k_r) only when $k_D = 0$. In Kerr, however, $\omega^D k_D = \omega^r k_r + \omega^\theta k_\theta = 0$ can occur whenever ω^r / ω^θ is rational, which corresponds to resonances in the r and θ motions [84]. In that case, our solution would require the nontrivial condition that $\bar{F}_\alpha = 0$ on resonance.¹⁵ Without further investigation of this condition, we cannot make any claims about the Kerr case.

¹⁵If we replace the ansatz $K_1 = 0$ with $K_1 = K_1(J)$, then the resonant cases would require only $\bar{F}_C = -ik_C \bar{K}_1$, not that \bar{F}_C vanishes. Still we would need $\bar{F}_\Gamma = 0$ on resonance.

In the Schwarzschild case, Eqs. (2.8.26, 2.8.27) shows that our solution requires that $\bar{F}_\alpha = 0$ when $k_D = 0$ (when $k_r = 0$). Using the inverse Fourier transform, this condition is

$$\bar{F}_\alpha(0, J) = \frac{1}{2\pi} \int_0^{2\pi} dq^r F_\alpha(q, J) = 0. \quad (2.8.30)$$

Recalling that $F_\alpha(q, J)$ depends only on q^r (and all the J_α), and that the J_α are constant for a given geodesic orbit, this integral over one period of q^r holding the J_α fixed is an average over one period of the geodesic motion. Recall from Eq. (2.8.19) [and footnote 14] that $F_\alpha = \dot{J}_\alpha = dJ_\alpha/d\lambda = m dJ_\alpha/d\tau$ are the rates of change of the action variables. The action variables J_α are functions only of the geodesic ‘constants’ of motion, the energy E and angular momentum L of Eq. (2.7.2) [and the rest mass m] for the Schwarzschild case. We showed in Sec. 2.7.V that, under the influence of the conservative self-force, the average over one geodesic orbital period of the τ -derivative of any function of only E and L is zero [see in particular the discussion immediately following Eq. (2.7.30)]. Thus, the condition (2.8.30) is satisfied in Schwarzschild.

To summarize, we have shown that the linear-order conservative dynamics of a scalar charge on a bound orbit in Schwarzschild can be described by the following Hamiltonian system. With phase space coordinates $Y^A = (q^\alpha, J_\alpha)$ being the action-angle variables for geodesic motion, the system is defined by

$$\dot{Y}^A = (\Omega_0 + \epsilon \mathcal{L}_X \Omega_0)^{AB} (K_0 + \epsilon K_1)_{,B} + O(\epsilon^2),$$

where Ω_0^{AB} and K_0 are as in Eqs. (2.8.18), $K_1 = 0$, and the gauge vector $\vec{X} = \chi^\alpha(\partial/\partial q^\alpha) + \zeta_\alpha(\partial/\partial J_\alpha)$ is given by Eqs. (2.8.28, 2.8.29) [via an inverse Fourier transform] with the functions f^α and F_α determined by the translation of Eqs. (2.8.14, 2.8.16) into Eqs. (2.8.19) via the coordinate transformation $(z^\mu, p^\mu) \leftrightarrow (q^\alpha, J_\alpha)$. This result easily generalizes from the scalar case to the EM and gravitational cases [because those self-forces share the same orbit reflection properties discussed in Sec. 2.7.V].

We have thus shown that the conservative self-force dynamics in Schwarzschild *is* ‘Hamiltonian’ in the above sense. As it stands, our result verifies the existence of the Hamiltonian system but does not represent a practical computational scheme; it is hoped that future work will remedy this drawback. To generalize the result to Kerr will require further investigation of the condition that $\bar{F}_\alpha = 0$ when $\omega^\alpha k_\alpha = 0$ (for the case of orbital resonances).

2.9 The geodesic function

This section considers properties of the geodesic function $x_g^{\mu'}(x^\mu, u^\mu, \tau)$ which gives the point $x_g^{\mu'}$ reached by traveling an interval τ along the geodesic leaving the point x^μ with tangent u^μ . We begin in Sec. 2.9.I by relating this function to Synge's world function $\sigma(x, x')$ and its derivatives, using well-known properties of the latter to derive properties of the former. Section 2.9.II shows how some of these properties can be used to (partially) simplify the conjectured local EoM (2.5.5). Section 2.9.III then develops further identities for the derivatives of the geodesic function. Finally, Sec. 2.9.IV discusses the parallel propagator and the expansion of bi-tensors near coincidence, applied to the derivatives of the geodesic function. These results will be useful for the discussion of geodesic deviation in Sec. 2.10.

For the treatment of Synge's world function, and more generally of bi-tensors, we draw heavily from Part I of the excellent review by Poisson et al. [25], as well as from the seminal work of DeWitt and Brehme [62] and the textbook by Synge [85].

I Relation to Synge's world function

Given a spacetime point x and a (not necessarily normalized) timelike tangent vector v at that point, there is a unique geodesic passing through x with tangent $\propto v$. Given also a proper time interval τ , we can find the point x' , or x'_g , reached by traveling a proper time $|\tau|$ along the geodesic in the direction of $\text{sign}(\tau)v$. This defines the geodesic function $x'_g(x, v, \tau)$, or $x_g^{\mu'}(x^\mu, v^\mu, \tau)$, used above, which has the property $x'_g(x, v, \tau) = x'_g(x, v/\sqrt{-v^2}, \tau)$.

Conversely, the two points x and x' uniquely determine a geodesic connecting them (if they are in each other's normal convex neighborhoods), which then determines the proper time τ along it and the tangent direction at either endpoint.¹⁶ The function that gives half the squared geodesic interval ($\sigma = -\tau^2/2$ for the timelike case) along the geodesic connecting x and x' is known as Synge's world function $\sigma(x, x')$. The world function is a bi-scalar which is symmetric in its arguments. One of its basic properties is that its gradients give vectors that are tangent to the geodesic at the endpoints,

¹⁶The discussion here focuses on timelike geodesics, setting sign and normalization conventions appropriate to that case, but this is all easily generalized to the null and spacelike cases. All relations below involving only the functions $\sigma(x, x')$ and $x'(x, \sigma^\mu)$ of Eq. (2.9.4) [excluding those relating them to τ , u^μ , $u^{\mu'}$ and $x'_g(x, u, \tau)$] are valid for all three cases.

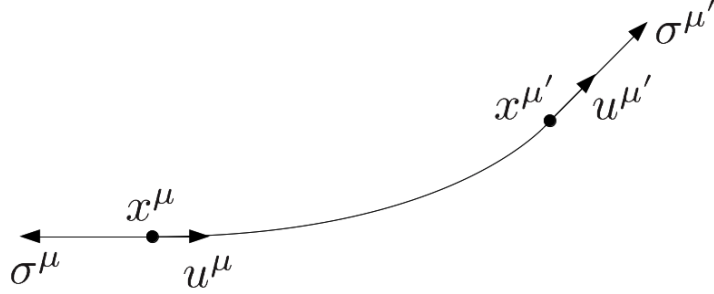


Figure 7: The timelike geodesic (with the future to the right) connecting the points x^μ and $x^{\mu'}$, with the velocities u^μ and $u^{\mu'}$ both future-pointing, with $\sigma^\mu = -\tau u^\mu$ and $\sigma^{\mu'} = \tau u^{\mu'}$ both pointing outward from the geodesic segment, and with τ being the proper time interval along the geodesic from x^μ to $x^{\mu'}$. Note that $x^{\mu'}$ is sometimes called $x_g^{\mu'}$.

with their norms being the proper time:

$$\sigma^\mu \equiv \nabla^\mu \sigma = -\tau u^\mu, \quad \sigma^{\mu'} \equiv \nabla^{\mu'} \sigma = \tau u^{\mu'}, \quad \tau = \sqrt{-2\sigma}. \quad (2.9.1)$$

Here, ∇_μ and $\nabla_{\mu'}$ are covariant derivatives w.r.t. x and x' , and u^μ and $u^{\mu'}$ are the normalized ($u^2 = -1$) tangents to the geodesics at x and x' . We use here the convention that τ is positive if x' is in the future of x , which fixes the signs in Eqs. (2.9.1) with both u s being future-pointing; both σ -vectors point outward from the geodesic segment. The relations (2.9.1) are equivalent to

$$\sigma_\mu \sigma^\mu = \sigma_{\mu'} \sigma^{\mu'} = 2\sigma, \quad (2.9.2)$$

which, along with $\sigma \rightarrow 0$ as $x \rightarrow x'$, can be taken as the definition of $\sigma(x, x')$. Differentiating Eqs. (2.9.3) gives

$$\sigma^\nu \nabla_\nu \sigma^\mu = \sigma^\mu, \quad \sigma^{\mu'} \nabla_{\mu'} \sigma^{\mu'} = \sigma^{\mu'}, \quad (2.9.3)$$

which are geodesic equations (equivalent to $D_\tau(\sigma^\mu/\sqrt{-\sigma}) = 0$) for the vector fields σ^μ and $\sigma^{\mu'}$. We will later use the notation $\sigma_{\mu\mu'} = \nabla_{\mu'} \sigma_\mu = \nabla_{\mu'} \nabla_\mu \sigma$, $\sigma_{\mu\nu\rho} = \nabla_\rho \nabla_\nu \nabla_\mu \sigma$, etc., for multiple derivatives of $\sigma(x, x')$.

So, given the world function $\sigma(x, x')$, we can differentiate at x to obtain the function $\sigma^\mu(x, x')$. This σ^μ encodes both the proper time interval τ along the geodesic from x to x' and the velocity u^μ at the x -end, since $\sigma^\mu = -\tau u^\mu$. As we've seen, x , u^μ and τ uniquely determine x' , and so x and σ^μ uniquely determine x' . Thus, we can invert the function $\sigma^\mu(x, x')$, solving for x' , defining the function $x'(x, \sigma^\mu)$:

$$\sigma^\mu(x, x') \longleftrightarrow x'(x, \sigma^\mu). \quad (2.9.4)$$

From $\sigma^\mu = -\tau u^\mu$, it's clear that this $x'(x, \sigma^\mu)$ is related to the function $x'_g(x, v, \tau)$ defined above, with v^μ being an arbitrary tangent, by

$$x'_g(x, v, \tau) \equiv x' \left(x, \frac{-\tau v^\mu}{\sqrt{-v^2}} \right) \quad (2.9.5)$$

$$= x'_g(x, u, \tau) = x'(x, -\tau u^\mu). \quad (2.9.6)$$

We will tend to write x'_g as a function of a normalized tangent u , while still taking the function x'_g to have the normalization-invariant definition (2.9.5) with $v \rightarrow u$. The relations (2.9.5) and (2.9.1) imply the relations

$$u^{\mu'} = \frac{\partial x_g^{\mu'}}{\partial \tau} = -u^\mu \frac{\partial x^{\mu'}}{\partial \sigma^\mu}, \quad (2.9.7)$$

$$\frac{\partial x_g^{\mu'}}{\partial u^\mu} = -\tau P_\mu^\nu \frac{\partial x^{\mu'}}{\partial \sigma^\nu}, \quad P_\mu^\nu = \delta_\mu^\nu + u_\mu u^\nu. \quad (2.9.8)$$

The derivatives of $x_g^{\mu'}$ w.r.t. τ and w.r.t. u^μ , in the first and second lines, are determined by the components of $\partial x^{\mu'}/\partial \sigma^\mu$ parallel and orthogonal to u^μ , respectively.¹⁷ In the first equality, we have noted that the derivative of $x_g^{\mu'}$ w.r.t. to τ , holding x and u fixed, must be the velocity at the x' -end.

The functions $\sigma^\mu(x, x')$ and $x'(x, \sigma^\mu)$ satisfy

$$\sigma^\mu(x, x'(x, \sigma^\mu)) = \sigma^\mu, \quad x'(x, \sigma^\mu(x, x')) = x'. \quad (2.9.9)$$

The first equation says, given a point x and a vector σ^μ there, find from the function $x'(x, \sigma^\mu)$ the point x' that is reached by the corresponding geodesic, feed the same x and that x' to the function $\sigma^\mu(x, x')$, and you should get back the original σ^μ . The second equation is the converse statement. By differentiating these relations, we can obtain important identities for the derivatives of these functions. For example, acting on the second relation with the partial derivative $\partial/\partial x^\mu$ yields

$$\begin{aligned} 0 &= \frac{\partial x^{\mu'}}{\partial x^\mu} + \frac{\partial x^{\mu'}}{\partial \sigma^\nu} \frac{\partial \sigma^\nu}{\partial x^\mu} = \frac{\partial x^{\mu'}}{\partial x^\mu} + \frac{\partial x^{\mu'}}{\partial \sigma^\nu} (\nabla_\mu \sigma^\nu - \Gamma_{\mu\rho}^\nu \sigma^\rho) = \left(\frac{\partial x^{\mu'}}{\partial x^\mu} - \sigma^\rho \Gamma_{\mu\rho}^\nu \frac{\partial x^{\mu'}}{\partial \sigma^\nu} \right) + \frac{\partial x^{\mu'}}{\partial \sigma^\nu} \nabla_\mu \sigma^\nu \\ &= \tilde{\nabla}_\mu x^{\mu'} + \frac{\partial x^{\mu'}}{\partial \sigma^\nu} \sigma^\nu_{;\mu}, \end{aligned} \quad (2.9.10)$$

where the second equality uses the definition of $\nabla_\mu \sigma^\nu$, the third just rearranges, and the fourth uses the notation $\sigma^\nu_{;\mu} = \nabla_\mu \sigma^\nu$ and the definition of the horizontal covariant derivative $\tilde{\nabla}_\mu$ w.r.t. x while

¹⁷In deriving Eq. (2.9.8) from Eq. (2.9.6), we have used the relation $\partial u^\nu/\partial u^\mu = P_\mu^\nu$ (not δ_μ^ν), where P_μ^ν is the projection operator orthogonal to u^μ , as in Eq. (2.9.8). This relation follows from the identity $u^\mu = u^\mu/\sqrt{-u^2}$ and reflects the fact that the u -derivative is really 3-dimensional (because $u^2 = -1$); it is *always* projected into the vector subspace orthogonal to u^μ .

parallel propagating σ^μ . Similarly, differentiating the second relation (2.9.9) w.r.t. x' , and the first w.r.t. x and w.r.t. σ^μ , yields the other three of these important identities:

$$\frac{\partial x^{\mu'}}{\partial \sigma^\mu} \sigma^\mu{}_{\nu'} = \delta^{\mu'}{}_{\nu'}, \quad \sigma^\mu{}_{\mu'} \frac{\partial x^{\mu'}}{\partial \sigma^\nu} = \delta^\mu{}_\nu, \quad (2.9.11)$$

$$\frac{\partial x^{\mu'}}{\partial \sigma^\mu} \sigma^\mu{}_\nu = -\tilde{\nabla}_\nu x^{\mu'}, \quad \sigma^\mu{}_{\mu'} \tilde{\nabla}_\nu x^{\mu'} = -\sigma^\mu{}_\nu. \quad (2.9.12)$$

Here, the derivatives of σ are functions of (x, x') , and the derivatives of x' are functions of (x, σ^μ) , but all of these identities (as with many to come) hold either with $(x, x') \rightarrow ((x, x'(x, \sigma^\mu)))$ or with $(x, \sigma^\mu) \rightarrow (x, \sigma^\mu(x, x'))$.

The identities (2.9.11) state that $\sigma^\mu{}_{\mu'}$ and $\partial x^{\mu'}/\partial \sigma^\mu$ are inverses; they are the Jacobians of the transformation $\sigma^\mu \leftrightarrow x'$, with x held fixed. This can be used to derive each of the two identities (2.9.12) for $\tilde{\nabla}_\mu x^{\mu'}$ from each other. These identities will prove invaluable in elucidating the properties of the geodesic function $x'(x, \sigma^\mu)$ and its derivatives $\tilde{\nabla}_\mu x^{\mu'}$ and $\partial x^{\mu'}/\partial \sigma^\mu$, using well-known properties of Synge's world function and its derivatives.

For example, the geodesic equations (2.9.3) and considering $\sigma^{\mu'} = \nabla^{\mu'}(\sigma = \frac{1}{2}\sigma_\mu\sigma^\mu)$ give us

$$\sigma^\mu \sigma^\nu{}_\mu = \sigma^\nu, \quad \sigma^\mu \sigma^\mu{}_{\mu'} = \sigma^{\mu'}, \quad (2.9.13)$$

and similarly with primed \leftrightarrow unprimed. Using these with Eqs. (2.9.11) and (2.9.12) then yields

$$\sigma^\mu \frac{\partial x^{\mu'}}{\partial \sigma^\mu} = \sigma^{\mu'}, \quad \sigma_{\mu'} \frac{\partial x^{\mu'}}{\partial \sigma^\mu} = \sigma_\mu, \quad \sigma^\mu \tilde{\nabla}_\mu x^{\mu'} = -\sigma^{\mu'}, \quad \sigma_{\mu'} \tilde{\nabla}_\mu x^{\mu'} = -\sigma_\mu, \quad (2.9.14)$$

the first of which is equivalent to the already noted Eq. (2.9.7), via Eqs. (2.9.1).

These also translate into properties of the geodesic function $x'_g(x, u, \tau)$, which is just $x'(x, \sigma^\mu)$ with $\sigma^\mu = -\tau u^\mu$. We will henceforth often drop the subscript g on x'_g , letting arguments or derivatives discriminate between the two functions, writing $x'(x, u, \tau) = x'(x, \sigma^\mu)$ with $\sigma^\mu = -\tau u^\mu$. The derivatives w.r.t. x , $\tilde{\nabla}_\mu x^{\mu'}$, are the same for both functions, and the derivatives w.r.t. u^μ and τ are related to those w.r.t. σ^μ , from Eqs. (2.9.7) and (2.9.8), by

$$\frac{\partial x^{\mu'}}{\partial u^\mu} = -\tau P^\nu{}_\mu \frac{\partial x^{\mu'}}{\partial \sigma^\nu}, \quad \frac{\partial x^{\mu'}}{\partial \tau} = -u^\mu \frac{\partial x^{\mu'}}{\partial \sigma^\mu} = u^{\mu'}. \quad (2.9.15)$$

Combining Eqs. (2.9.1) and (2.9.14) yields the useful relations

$$u^\mu \tilde{\nabla}_\mu x^{\mu'} = u^{\mu'}, \quad u_{\mu'} \tilde{\nabla}_\mu x^{\mu'} = u_\mu. \quad (2.9.16)$$

We now pause in our development of properties of the geodesic functions to note how the first of the identities (2.9.16) can be used to make some (ultimately unfruitful) manipulations of the EoM (2.5.3) which results from the conjectured local action principle (2.5.5).

II Some integration by parts in the conjectured equation of motion

Recall that the conjectured action (2.5.3),

$$S_g[z] = \int d\tau \left[-m_0 + \frac{q^2}{2} \psi(z; z, u) \right], \quad \psi(x; z, u) = \int d\tau' G(x, x'(z, u, \tau')), \quad (2.9.17)$$

leads to the EoM (2.5.5),

$$ma_\mu = \frac{q^2}{2} P_\mu^\nu \left(\tilde{\nabla}_\nu - u^\rho \tilde{\nabla}_\rho \frac{\partial}{\partial u^\nu} \right) \psi(z; z, u) + O(q^4) \equiv F_\mu^A, \quad (2.9.18)$$

and we would like to know if this is equivalent to the correct local EoM (2.5.6),

$$ma_\mu = q^2 P_\mu^\nu \left[\nabla_\nu^{(x)} \psi(x; z, u) \right]_{x=z} + O(q^4) \equiv F_\mu^B. \quad (2.9.19)$$

The latter can be expanded out as

$$F_\mu^B = q^2 P_\mu^\nu \int d\tau' \nabla_\nu G, \quad (2.9.20)$$

while the former has half this term plus four others:¹⁸

$$F_\mu^A = \frac{q^2}{2} P_\mu^\nu \int d\tau' \left[\nabla_\nu G + \tilde{\nabla}_\nu x^{\mu'} \nabla_{\mu'} G - \left(u^\rho \tilde{\nabla}_\rho \frac{\partial x^{\mu'}}{\partial u^\nu} \right) \nabla_{\mu'} G - \frac{\partial x^{\mu'}}{\partial u^\nu} u^\rho \left(\nabla_\rho \nabla_{\mu'} G + \tilde{\nabla}_\rho x^{\nu'} \nabla_{\nu'} \nabla_{\mu'} G \right) \right], \quad (2.9.22)$$

where all the derivatives of G are functions of $(z, x'(z, u, \tau'))$.

¹⁸In obtaining Eq. (2.9.22) from Eq. (2.9.18), we have used, with $G = G(z, x'(z, u, \tau'))$,

$$\begin{aligned} \tilde{\nabla}_\nu G &= \nabla_\nu G + \tilde{\nabla}_\nu x^{\mu'} \nabla_{\mu'} G, & \frac{\partial}{\partial u^\nu} G &= \frac{\partial x^{\mu'}}{\partial u^\nu} \nabla_{\mu'} G, \\ u^\rho \tilde{\nabla}_\rho \frac{\partial}{\partial u^\nu} G &= \left(u^\rho \tilde{\nabla}_\rho \frac{\partial x^{\mu'}}{\partial u^\nu} \right) \nabla_{\mu'} G + \frac{\partial x^{\mu'}}{\partial u^\nu} u^\rho \left(\nabla_\rho \nabla_{\mu'} G + \tilde{\nabla}_\rho x^{\nu'} \nabla_{\nu'} \nabla_{\mu'} G \right). \end{aligned} \quad (2.9.21)$$

Recall that $\tilde{\nabla}_\mu$ is the horizontal covariant derivative w.r.t. z , parallel propagating u , holding τ' fixed. The normal covariant derivatives ∇_μ and $\nabla_{\mu'}$ act on $G(z, x')$, differentiating w.r.t. z and x' each while holding the other fixed. The identities of the last subsection apply here with the notation changes $x \rightarrow z$, $\tau \rightarrow \tau'$, with u the velocity at z , and with x' unchanged.

We have not been able to show that these two EoMs are equivalent (and our numerical results from Schwarzschild indicate they are not), but we can make some interesting simplifications which reduce the number of terms in F_μ^A .

First, we note the result of commuting the operators $\partial/\partial u^\nu$ and $u^\rho \tilde{\nabla}_\rho$ of Eq. (2.9.18),

$$u^\rho \tilde{\nabla}_\rho \frac{\partial}{\partial u^\nu} = \frac{\partial}{\partial u^\nu} u^\rho \tilde{\nabla}_\rho - P_\nu^\rho \tilde{\nabla}_\rho,$$

keeping in mind $\partial u^\rho/\partial u^\nu = P_\nu^\rho$ from footnote 17 and the fact that $\partial/\partial u^\mu$ and $\tilde{\nabla}_\nu$ commute. From this, the conjectured self-force (2.9.18) can be rewritten as

$$F_\mu^A = \frac{q^2}{2} P_\mu^\nu \left(2\tilde{\nabla}_\nu - \frac{\partial}{\partial u^\nu} u^\rho \tilde{\nabla}_\rho \right) \psi, \quad (2.9.23)$$

where $\psi = \psi(z, z, u)$. Now consider expanding out $u^\rho \tilde{\nabla}_\rho \psi$:

$$\begin{aligned} u^\rho \tilde{\nabla}_\rho \psi &= \int d\tau' \left(u^\rho \nabla_\rho + u^\rho \tilde{\nabla}_\rho x^{\mu'} \nabla_{\mu'} \right) G(z, x'(z, u, \tau')) \\ &= \int d\tau' \left(u^\rho \nabla_\rho + u^{\mu'} \nabla_{\mu'} \right) G(z, x'(z, u, \tau')) \\ &= \int d\tau' \left(u^\rho \nabla_\rho + \frac{\partial}{\partial \tau'} \right) G(z, x'(z, u, \tau')). \end{aligned}$$

The second line uses the first of the identities (2.9.16), and the third uses the first equality of Eqs. (2.9.7) and the chain rule. The second term in the integrand is now a total derivative, and so, with the [very] reasonable assumption that $G(z, x'(z, u, \tau')) \rightarrow 0$ as $\tau' \rightarrow \pm\infty$, this term vanishes. Acting on this with $\partial/\partial u^\nu$ then gives

$$\frac{\partial}{\partial u^\nu} u^\rho \tilde{\nabla}_\rho \psi = \frac{\partial}{\partial u^\nu} \int d\tau' u^\rho \nabla_\rho G = \int d\tau' \left(P_\nu^\rho \nabla_\rho G + u^\rho \frac{\partial x^{\mu'}}{\partial u^\nu} \nabla_{\mu'} \nabla_\rho G \right).$$

Using this along with Eq. (2.9.21a) in Eq. (2.9.23) yields our simplified result for F_μ^A ,

$$F_\mu^A = \frac{q^2}{2} P_\mu^\nu \int d\tau' \left(\nabla_\nu G + 2\tilde{\nabla}_\nu x^{\mu'} \nabla_{\mu'} G - u^\rho \frac{\partial x^{\mu'}}{\partial u^\nu} \nabla_{\mu'} \nabla_\rho G \right),$$

which has whittled the five terms of the conjectured local EoM Eq. (2.9.18) down to three.

This seems a promising simplification, and it's conceivable that further manipulations along similar lines could yield insights into why the conjectured action does or does not work in various cases (in particular, the simple examples of Sec. 2.6 where it does work, or the case of Schwarzschild where our numerical results indicate it does not). It seems that one would need to use, as we have not done here, certain properties of the conservative Green's function, e.g. its symmetry or the fact that it satisfies the homogeneous wave equation.

III More derivatives of the geodesic function

In preparation for our discussion of geodesic deviation in Sec. 2.10, we present here identities for further derivatives of the geodesic function $x^{\mu'}(x^\mu, u^\mu, \tau) = x^{\mu'}(x^\mu, -\tau u^\mu) = x^{\mu'}(x^\mu, \sigma^\mu)$.

Throughout this section (and the next), we will consider this function and its derivatives w.r.t. x^μ , σ^μ , u^μ , and τ ,

$$\tilde{\nabla}_\mu x^{\mu'}, \quad \frac{\partial x^{\mu'}}{\partial \sigma^\mu}, \quad \frac{\partial x^{\mu'}}{\partial u^\mu} = -\tau P_\mu^\nu \frac{\partial x^{\mu'}}{\partial \sigma^\nu}, \quad u^{\mu'} = \frac{\partial x^{\mu'}}{\partial \tau} = -u^\mu \frac{\partial x^{\mu'}}{\partial \sigma^\mu}, \quad (2.9.24)$$

respectively, which are all defined as functions of (x^μ, σ^μ) [or (x^μ, u^μ, τ)], to be functions of the two points $(x^\mu, x^{\mu'})$, via the replacement $(x^\mu, \sigma^\mu) \rightarrow (x^\mu, \sigma^\mu(x^\mu, x^{\mu'}))$.

Our starting point will be the identities (2.9.11, 2.9.12),

$$\frac{\partial x^{\mu'}}{\partial \sigma^\mu} \sigma^\mu_{\nu'} = \delta^{\mu'}_{\nu'}, \quad \sigma^\mu_{\mu'} \frac{\partial x^{\mu'}}{\partial \sigma^\nu} = \delta^\mu_\nu, \quad (2.9.25)$$

$$\frac{\partial x^{\mu'}}{\partial \sigma^\mu} \sigma^\mu_\nu = -\tilde{\nabla}_\nu x^{\mu'}, \quad \sigma^\mu_{\mu'} \tilde{\nabla}_\nu x^{\mu'} = -\sigma^\mu_\nu. \quad (2.9.26)$$

which define $\tilde{\nabla}_\mu x^{\mu'}$ and $\partial x^{\mu'}/\partial \sigma^\mu$ in terms of the derivatives of the world function. We will be interested in the action on these functions of the derivatives

$$D \equiv u^\mu \nabla_\mu = -\frac{1}{\tau} \sigma^\mu \nabla_\mu, \quad D' \equiv u^{\mu'} \nabla_{\mu'} = \frac{1}{\tau} \sigma^{\mu'} \nabla_{\mu'},$$

which are effectively derivatives w.r.t. the interval τ along the geodesic connecting x and x' , the first while holding x' fixed and moving x in the direction of u , and the second while holding x fixed and moving x' in the direction of u' . Our ultimate goal will be to calculate the results of acting with D and D' on the functions $\tilde{\nabla}_\mu x^{\mu'}$ and $\partial x^{\mu'}/\partial u^\mu$ (2.9.24).

First, toward evaluating $D(\partial x^{\mu'}/\partial u^\mu)$, consider the result of acting with $\sigma^\rho \nabla_\rho$ on Eq. (2.9.25b):

$$\sigma^\mu_{\mu'} \sigma^\rho \nabla_\rho \frac{\partial x^{\mu'}}{\partial \sigma^\nu} = -\frac{\partial x^{\mu'}}{\partial \sigma^\nu} \left[\sigma^\rho \sigma^\mu_{\mu' \rho} = \nabla_{\mu'} (\sigma^\rho \sigma^\mu_\rho) - \sigma^\rho_{\mu'} \sigma^\mu_\rho = \sigma^\mu_{\mu'} - \sigma^\rho_{\mu'} \sigma^\mu_\rho \right] = -\delta^\mu_\nu + \sigma^\mu_\nu,$$

where we have used Eqs. (2.9.13) and (2.9.25b). Contracting this with $\partial x^{\nu'}/\partial \sigma^\mu$ and using Eqs. (2.9.25, 2.9.26) yields

$$\sigma^\rho \nabla_\rho \frac{\partial x^{\nu'}}{\partial \sigma^\nu} = -\frac{\partial x^{\nu'}}{\partial \sigma^\nu} - \tilde{\nabla}_\nu x^{\nu'}.$$

Then, noting from Eqs. (2.9.1, 2.9.3) that $\sigma^\rho \nabla_\rho \tau = \tau$ and $\sigma^\rho \nabla_\rho u^\mu = 0 = \sigma^\rho \nabla_\rho P_\mu^\nu$, we have

$$D \frac{\partial x^{\mu'}}{\partial u^\mu} = -\frac{1}{\tau} \sigma^\rho \nabla_\rho \left(-\tau P_\mu^\nu \frac{\partial x^{\mu'}}{\partial \sigma^\nu} \right) = P_\mu^\nu (1 + \sigma^\rho \nabla_\rho) \frac{\partial x^{\mu'}}{\partial \sigma^\nu} = -P_\mu^\nu \tilde{\nabla}_\nu x^{\mu'}.$$

Evaluating $D(\tilde{\nabla}_\mu x^{\mu'})$ proceeds along similar lines, starting from Eq. (2.9.26a) and using

$$\sigma^\rho \sigma^\nu_{\mu\rho} = \sigma^\rho \left(\nabla_\rho \nabla_\mu \sigma^\nu = \nabla_\mu \nabla_\rho \sigma^\nu - R_{\rho\mu\lambda}{}^\nu \sigma^\lambda \right) = \left(\sigma^\rho \sigma^\nu_{\rho\mu} = \sigma^\nu_{\mu} - \sigma^\rho_{\mu} \sigma^\nu_{\rho} \right) - R_{\rho\mu\lambda}{}^\nu \sigma^\rho \sigma^\lambda,$$

which has used the Ricci identity, with the Riemann tensor evaluated at x .

Suppressing further details of the derivations, we note the relations

$$\sigma^\rho \nabla_\rho \sigma^\mu_{\mu'} = \sigma^\mu_{\mu'} - \sigma^\rho_{\rho} \sigma^\mu_{\mu'}, \quad (2.9.27)$$

$$\sigma^\rho \nabla_\rho \sigma^\nu_{\mu} = \sigma^\nu_{\mu} - \sigma^\nu_{\rho} \sigma^\rho_{\mu} - R_{\rho\mu\lambda}{}^\nu \sigma^\rho \sigma^\lambda, \quad (2.9.28)$$

$$\sigma^\rho \nabla_\rho \frac{\partial x^{\mu'}}{\partial \sigma^\mu} = -\frac{\partial x^{\mu'}}{\partial \sigma^\mu} - \tilde{\nabla}_\mu x^{\mu'}, \quad (2.9.29)$$

$$\sigma^\rho \nabla_\rho \tilde{\nabla}_\mu x^{\mu'} = -\tilde{\nabla}_\mu x^{\mu'} + \sigma^\nu_{\mu} \tilde{\nabla}_\nu x^{\mu'} - R_{\rho\mu\lambda}{}^\nu \sigma^\rho \sigma^\lambda \frac{\partial x^{\mu'}}{\partial \sigma^\nu}, \quad (2.9.30)$$

which (along with others from above) lead to the results

$$D \frac{\partial x^{\mu'}}{\partial u^\mu} = -P_\mu^\nu \tilde{\nabla}_\nu x^{\mu'}, \quad D \tilde{\nabla}_\mu x^{\mu'} = R_{\rho\mu\lambda}{}^\nu u^\rho u^\lambda \frac{\partial x^{\mu'}}{\partial u^\nu}, \quad (2.9.31)$$

of acting with $D = u^\mu \nabla_\mu = -(\sigma^\mu/\tau) \nabla_\mu$ on $\partial x^{\mu'}/\partial u^\mu = -\tau P_\mu^\nu (\partial x^{\mu'}/\partial \sigma^\nu)$ and $\tilde{\nabla}_\mu x^{\mu'}$.

The results of acting with $D' = u^{\mu'} \nabla_{\mu'} = (\sigma^{\mu'}/\tau) \nabla_{\mu'}$ are found by analogous methods. Some useful identities are

$$\sigma^{\rho'} \nabla_{\rho'} \sigma^\nu_{\mu} = \sigma^\nu_{\mu} - \sigma^{\nu'}_{\mu'} \sigma^{\mu'}_{\mu}, \quad (2.9.32)$$

$$\sigma^{\rho'} \nabla_{\rho'} \sigma^{\mu'}_{\mu} = \sigma^{\mu'}_{\mu} - \sigma^{\mu'}_{\nu'} \sigma^{\nu'}_{\mu}, \quad (2.9.33)$$

$$\sigma^{\rho'} \nabla_{\rho'} \sigma^{\mu'}_{\nu'} = \sigma^{\mu'}_{\nu'} - \sigma^{\mu'}_{\rho'} \sigma^{\rho'}_{\nu'} - R_{\rho'\nu'\lambda'}{}^{\mu'} \sigma^{\rho'} \sigma^{\lambda'}, \quad (2.9.34)$$

$$\sigma^{\rho'} \nabla_{\rho'} \frac{\partial x^{\mu'}}{\partial \sigma^\mu} = -\frac{\partial x^{\mu'}}{\partial \sigma^\mu} + \sigma^{\mu'}_{\nu'} \frac{\partial x^{\nu'}}{\partial \sigma^\mu}, \quad (2.9.35)$$

$$\sigma^{\rho'} \nabla_{\rho'} \tilde{\nabla}_\mu x^{\mu'} = \sigma^{\mu'}_{\nu'} \tilde{\nabla}_\mu x^{\nu'} + \sigma^{\mu'}_{\mu}, \quad (2.9.36)$$

where the Riemann tensor is evaluated at x' . These lead to the first derivatives

$$D' \frac{\partial x^{\mu'}}{\partial u^\mu} = \frac{1}{\tau} \sigma^{\mu'}_{\nu'} \frac{\partial x^{\nu'}}{\partial u^\mu}, \quad D' \tilde{\nabla}_\mu x^{\mu'} = \frac{1}{\tau} \left(\sigma^{\mu'}_{\nu'} \tilde{\nabla}_\mu x^{\nu'} + \sigma^{\mu'}_{\mu} \right), \quad (2.9.37)$$

and the second derivatives

$$(D')^2 \frac{\partial x^{\mu'}}{\partial u^\mu} = -R_{\rho'\nu'\lambda'}{}^{\mu'} u^{\rho'} u^{\lambda'} \frac{\partial x^{\nu'}}{\partial u^\mu}, \quad (D')^2 \tilde{\nabla}_\mu x^{\mu'} = -R_{\rho'\nu'\lambda'}{}^{\mu'} u^{\rho'} u^{\lambda'} \tilde{\nabla}_\mu x^{\nu'}. \quad (2.9.38)$$

We recognize these last two equations as forms of the geodesic deviation equation for $\tilde{\nabla}_\mu x^{\mu'}$ and $\partial x^{\mu'}/\partial u^\mu$ (ignoring the μ index). They will be useful in formulating the general solution to the geodesic deviation equation in Sec. 2.10.

IV The parallel propagator and expansions near coincidence

Use of Synge's world function $\sigma(x, x')$ and its derivatives provides a handy covariant method to 'expand bi-tensors near coincidence', i.e. to expand bi-tensors defined w.r.t. two points x and x' in the limit where x' approaches x . This procedure makes use of a basic bi-tensor we have not yet mentioned, the 'parallel propagator' $g^{\mu'}_{\mu}(x, x')$. After discussing $g^{\mu'}_{\mu}$ and the expansion method, following Part I of Poisson et al. [25], we present the expansions of the derivatives $\tilde{\nabla}_{\mu}x^{\mu'}$ and $\partial x^{\mu'}/\partial\sigma^{\mu}$ of the geodesic function.

Note that all results in this section (unlike the last) are valid when x and x' are timelike-, spacelike-, or null-separated. Our use of this section's results in Sec. 2.10 will actually be for the spacelike case, considering two spacelike-separated points on neighboring timelike worldlines.

Given two points x and x' with a unique geodesic linking them, and a vector A^{μ} at x , consider parallel propagating the vector along the geodesic, obtaining the vector $A^{\mu'}$ at x' . This is a linear map from vectors at x to vectors at x' , $A^{\mu'} = g^{\mu'}_{\mu}A^{\mu}$, which defines the bi-tensor $g^{\mu'}_{\mu}(x, x')$ known as the parallel propagator. Its inverse $g^{\mu}_{\mu'}$ (satisfying $g^{\mu'}_{\mu}g^{\mu}_{\nu'} = \delta^{\mu'}_{\nu'}$ and $g^{\mu}_{\mu'}g^{\nu}_{\nu'} = \delta^{\nu}_{\nu'}$) is simply $g^{\mu}_{\mu'} = g_{\mu'}^{\mu} = g^{\mu\nu}g_{\mu'\nu'}g^{\nu'}_{\nu}$, so that index ordering and up/down placement don't really matter.

The parallel propagator can also be defined by $\sigma^{\nu}\nabla_{\nu}g^{\mu'}_{\mu} = \sigma^{\nu'}\nabla_{\nu'}g^{\mu'}_{\mu} = 0$, and similarly with primed \leftrightarrow unprimed, along with the condition that the 'coincidence limit' of $g^{\mu'}_{\nu}$ is δ^{μ}_{ν} , written $[g^{\mu'}_{\nu}] = \delta^{\mu}_{\nu}$. The coincidence limit of a bi-tensor of type $T_{\mu\nu\dots}(x, x')$ is the normal tensor of type $T_{\mu\nu\dots}(x)$ obtained from the limit $x' \rightarrow x$. We note the coincidence limits results

$$[\sigma] = [\sigma^{\mu}] = [\sigma^{\mu'}] = 0, \quad [\sigma^{\mu'}_{\nu'}] = [\sigma^{\mu}_{\nu}] = [g^{\mu'}_{\nu}] = [g^{\mu}_{\nu'}] = \delta^{\mu}_{\nu}, \quad [\sigma^{\mu'}_{\nu}] = [\sigma^{\mu}_{\nu'}] = -\delta^{\mu}_{\nu},$$

and the transport results $g^{\mu'}_{\mu}\sigma^{\mu} = -\sigma^{\mu'}$, $g^{\mu}_{\mu'}\sigma^{\mu'} = -\sigma^{\mu}$, which are equivalent to Eqs. (2.9.1) and the fact that a geodesic parallel transports its tangent.

We have seen how the information in the two points x and x' is equivalent to the information in the point x and the vector σ^{μ} at x : the point x' is reached by leaving x on the geodesic with tangent $\propto -\sigma^{\mu}$, going an interval $\sqrt{|\sigma^{\mu}\sigma_{\mu}|}$ (recall $\sigma^{\mu} = -\tau u^{\mu}$ for the timelike case). The vector σ^{μ} at x is like a covariant version of a displacement vector from x to x' (but with a minus sign because σ^{μ} points outward from the geodesic segment).

The covariant expansion of bi-tensors near coincidence (as $x' \rightarrow x$) expands in powers of the vector $\sigma^{\mu}(x, x')$ in analogy to how a normal Taylor expansion in flat space expands in powers of a

displacement vector. A (smooth) bi-tensor with indices only at x , e.g. $T_{\mu\nu}(x, x')$, is expanded as

$$T_{\mu\nu}(x, x') = A_{\mu\nu} + A_{\mu\nu\rho}\sigma^\rho + \frac{1}{2}A_{\mu\nu\rho\lambda}\sigma^\rho\sigma^\lambda + O(\sigma^\mu)^3,$$

where the $A = A(x)$ coefficients are determined by coincidence limits and derivatives thereof of T and its derivatives (e.g. $A_{\mu\nu} = [T_{\mu\nu}]$), as detailed in Sec. 6 of Poisson et al. [25]. For bi-tensors with indices at x' , one factor of the parallel propagator is needed for each such index, to turn a tensor expansion at x into a bi-tensor of the proper index structure, e.g.,

$$S_{\mu\nu'\rho'}(x, x') = g^\nu{}_{\nu'}g^\rho{}_{\rho'} \left(B_{\mu\nu\rho} + B_{\mu\nu\rho\lambda}\sigma^\lambda + \frac{1}{2}B_{\mu\nu\rho\lambda\xi}\sigma^\lambda\sigma^\xi \right) + O(\sigma^\mu)^3.$$

Some important expansions are those of the second derivatives of the world function and the first derivatives of the parallel propagator:

$$\sigma_{\mu\nu} = g_{\mu\nu} - \frac{1}{3}R_{\mu\rho\nu\lambda}\sigma^\rho\sigma^\lambda + O(\sigma^\mu)^3, \quad (2.9.39)$$

$$\sigma_{\mu\nu'} = -g^\nu{}_{\nu'} \left(g_{\mu\nu} + \frac{1}{6}R_{\mu\rho\nu\lambda}\sigma^\rho\sigma^\lambda \right) + O(\sigma^\mu)^3, \quad (2.9.40)$$

$$\sigma_{\mu'\nu'} = g^\mu{}_{\mu'}g^\nu{}_{\nu'} \left(g_{\mu\nu} - \frac{1}{3}R_{\mu\rho\nu\lambda}\sigma^\rho\sigma^\lambda \right) + O(\sigma^\mu)^3, \quad (2.9.41)$$

$$\nabla_\rho g^{\mu'}{}_\nu = \frac{1}{2}g^{\mu'}{}_\mu R^\mu{}_{\nu\rho\lambda}\sigma^\lambda + O(\sigma^\mu)^2, \quad (2.9.42)$$

$$\nabla_{\rho'} g^{\mu'}{}_\nu = \frac{1}{2}g^{\mu'}{}_\mu g^\rho{}_{\rho'} R^\mu{}_{\nu\rho\lambda}\sigma^\lambda + O(\sigma^\mu)^2. \quad (2.9.43)$$

From the first two of these and Eqs. (2.9.11b, 2.9.12a),

$$\sigma^\mu{}_{\rho'} \frac{\partial x^{\rho'}}{\partial \sigma^\nu} = \delta^\mu{}_\nu, \quad \tilde{\nabla}_\nu x^{\mu'} = -\frac{\partial x^{\mu'}}{\partial \sigma^\rho} \sigma^\rho{}_\nu,$$

it is straightforward to derive the expansions

$$\frac{\partial x^{\mu'}}{\partial \sigma^\nu} = -g^{\mu'}{}_\mu \left(\delta^\mu{}_\nu - \frac{1}{6}R^\mu{}_{\rho\nu\lambda}\sigma^\rho\sigma^\lambda \right) + O(\sigma^\mu)^3, \quad (2.9.44)$$

$$\tilde{\nabla}_\nu x^{\mu'} = g^{\mu'}{}_\mu \left(\delta^\mu{}_\nu - \frac{1}{2}R^\mu{}_{\rho\nu\lambda}\sigma^\rho\sigma^\lambda \right) + O(\sigma^\mu)^3, \quad (2.9.45)$$

of the derivatives of the geodesic function.

2.10 Geodesic deviation

This section considers how the dynamics of a worldline can be expanded about a fixed (fiducial) geodesic in terms of a deviation vector field along the geodesic. We begin in Sec. 2.10.I by discussing how a vector along a geodesic can determine a neighboring worldline, first to all orders in the deviation vector, and then covariantly expanding relevant quantities to quadratic order in the deviation vector. We use these results to derive the geodesic deviation equation for the deviation vector as well as an action from which it can be derived. In Sec. 2.10.II we discuss a form of the general solution to the geodesic deviation equation. Finally, Sec. 2.10.III considers how the use of deviation vectors along fiducial geodesics might be useful in investigations of actions for the conservative self-force.

I A covariant derivation of an action for the geodesic deviation equation

Given a fixed geodesic worldline $z^\mu(\tau)$, the fiducial geodesic, with velocity u^μ , a neighboring worldline can be specified by a deviation vector field $\xi^\mu(\tau)$ along the geodesic. To linear order in ξ , the coordinates x^μ of points on the neighboring worldline can be defined by

$$x^\mu(\tau) = z^\mu(\tau) + \xi^\mu(\tau) + O(\xi^2). \quad (2.10.1)$$

The correspondence between points on the two worldlines can be made unique by demanding that ξ is orthogonal to the geodesic's tangent, $u_\mu \xi^\mu = 0$. We choose the parameter τ to be the proper time along the fiducial geodesic $z^\mu(\tau)$, and we will also use this τ as the parameter for the neighboring worldline $x^\mu(\tau)$.

The requirement that the worldline $x^\mu(\tau)$ be a geodesic [to $O(\xi)$] leads to the geodesic deviation equation for the vector $\xi^\mu(\tau)$:

$$D^2 \xi^\mu + R^\mu{}_{\rho\nu\lambda} u^\rho u^\lambda \xi^\nu = 0,$$

where $D \equiv D/d\tau = u^\mu \nabla_\mu$. We present here derivations of this equation and of an action principle from which it can be derived, using the formalism of bi-tensors and the geodesic function discussed in Sec. 2.9. The derivation of the action principle requires extending the relation (2.10.1) to quadratic order in the deviation vector ξ .

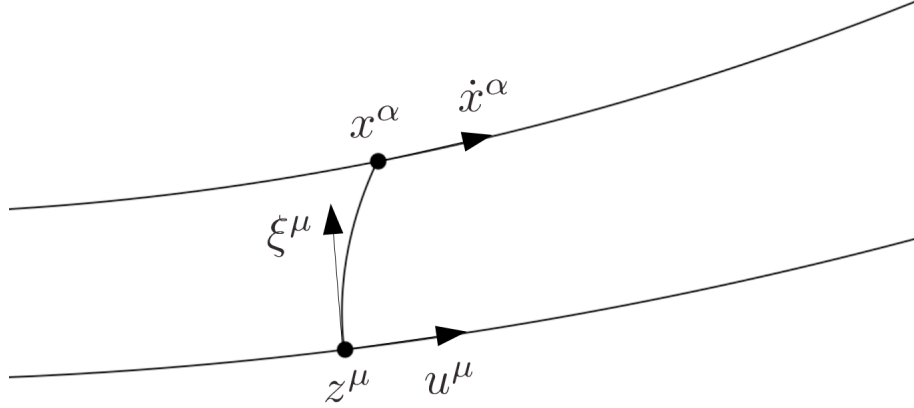


Figure 8: The fiducial geodesic $z^\mu(\tau)$ and the neighboring worldline $x^\alpha(\tau)$. At the point z^μ , the normalized tangent to the fiducial geodesic is u^μ . At the point x^α , the (unnormalized) tangent to the neighboring worldline is \dot{x}^α . The deviation vector ξ^μ at z^μ is tangent to the geodesic segment linking z^μ to x^α , and the norm $\sqrt{\xi^\mu \xi_\mu}$ is the proper distance along the segment.

We can define the neighboring worldline x in terms of the vector ξ along z in a covariant and *exact* manner, to all orders in ξ , as follows. First, note that [though it didn't matter at $O(\xi)$] we should use distinct types of indices for the tangent spaces on the geodesic z and the on neighboring worldline x . Instead of using primes here,¹⁹ we will use $\mu, \nu, \rho, \lambda, \dots$ indices on the fiducial geodesic $z^\mu(\tau)$ while using $\alpha, \beta, \gamma, \delta, \dots$ indices on the neighboring worldline $x^\alpha(\tau)$. We then use the geodesic function $x^\alpha(z^\mu, \sigma^\mu)$ of Sec. 2.9 to define the neighboring worldline in terms of ξ by

$$x^\alpha(\tau) = x^\alpha(z^\mu(\tau), -\xi^\mu(\tau)),$$

so that x^α is the point reached by leaving z^μ on the (spacelike) geodesic with tangent $\propto \xi^\mu$, traveling an interval $\sqrt{\xi^\mu \xi_\mu}$. The minus sign appears because ξ points from z toward x , while the world function derivative σ^μ [in terms of which the geodesic function $x^\alpha(z^\mu, \sigma^\mu)$ was defined] points in the opposite direction, outward from the segment linking z and x .²⁰

¹⁹This is because, in following sections, we will be considering four points simultaneously, using indices of types μ, μ', α and α' .

²⁰The translation in notation, Secs. 2.9.I and 2.9.IV \rightarrow here, is accomplished by $x^\mu \rightarrow z^\mu$ and $x^{\mu'} \rightarrow x^\alpha$. The geodesic function $x^\alpha(z^\mu, \sigma^\mu)$ here is defined just as $x^{\mu'}(x^\mu, \sigma^\mu)$ there; its vector argument here is $\sigma^\mu = -\xi^\mu$. Note that the $u^\mu = -\sigma^\mu/\tau$ of Secs. 2.9.I and 2.9.III, which concerned timelike cases of the geodesic function, is unrelated to the u^μ in this section, which is the tangent to the fiducial geodesic $z^\mu(\tau)$. The vectors $\sigma^\mu = -\xi^\mu$ here are spacelike, and we don't use normalized versions.

With the neighboring worldline defined by $x^\alpha(\tau) = x^\alpha(z^\mu(\tau), -\xi^\mu(\tau))$, consider its tangent $\dot{x}^\alpha(\tau)$. [Since τ is the proper time along the fiducial geodesic $z^\mu(\tau)$, it is a non-affine parameter along $x^\alpha(\tau)$.] Applying the ('horizontal') chain rule yields

$$\dot{x}^\alpha = u^\mu \tilde{\nabla}_\mu x^\alpha(z, -\xi) - D\xi^\mu \frac{\partial x^\alpha}{\partial \sigma^\mu}(z, -\xi), \quad (2.10.2)$$

where $\tilde{\nabla}_\mu x^\alpha$ is the horizontal covariant derivative of the geodesic function $x^\alpha(z^\mu, \sigma^\mu)$ w.r.t. z^μ , and $\partial x^\alpha / \partial \sigma^\mu$ its partial derivative w.r.t. σ^μ , both evaluated at $(z^\mu, \sigma^\mu) = (z^\mu(\tau), -\xi^\mu(\tau)) = (z, -\xi)$.

Now, an expansion in powers of the deviation vector ξ^μ matches up with the coincidence expansion as $x \rightarrow z$ in powers of $\sigma^\mu(z, x) = -\xi^\mu$, as in Sec. 2.9.IV. Using the expansions (2.9.44, 2.9.45) of the geodesic function derivatives,

$$\frac{\partial x^\alpha}{\partial \sigma^\mu}(z, -\xi) = -g^\alpha{}_\mu + O(\xi^2), \quad (2.10.3)$$

$$\tilde{\nabla}_\mu x^\alpha(z, -\xi) = g^\alpha{}_\nu \left(\delta^\nu{}_\mu - \frac{1}{2} R^\nu{}_{\rho\mu\lambda} \xi^\rho \xi^\lambda \right) + O(\xi^3), \quad (2.10.4)$$

the neighboring worldline's tangent vector \dot{x}^α (2.10.2) has the expansion

$$\dot{x}^\alpha = g^\alpha{}_\mu \left(u^\mu + D\xi^\mu - \frac{1}{2} R^\mu{}_{\rho\nu\lambda} u^\nu \xi^\rho \xi^\lambda \right) + O(\xi^3) + D\xi \cdot O(\xi^2). \quad (2.10.5)$$

The parallel propagator is evaluated as $g^\alpha{}_\mu = g^\alpha{}_\mu(z, x) = g^\alpha{}_\mu(z, x(z, -\xi))$, thus having a hidden ξ -dependence which will come into play later. For the order counting here, we note $g^\alpha{}_\mu = O(\xi^0)$.

We count will orders of ξ , $D\xi$, and $D^2\xi$ separately for the following reasons. It is consistent in some circumstances to take $O(\xi) \sim O(D\xi)$ and to work to $O(\xi, D\xi)^n$, meaning dropping all terms with $n+1$ factors of ξ and/or $D\xi$. However, one *cannot* consistently use $D^2\xi = O(\xi, D\xi)$, because $D^2\xi$ is constrained by the equation of motion. For geodesic motion, one will have $D^2\xi = O(\xi, D\xi)$, but in the presence of an external force (or a self-force), $D^2\xi$ will have an $O(\xi, D\xi)^0$ contribution. Note that this means, generally, $D[O(\xi, D\xi)^n] = O(\xi, D\xi)^{n-1}$, so that one must be careful when differentiating expanded expressions; note however that $D[O(\xi^n)] = O(\xi, D\xi)^n$.

From the expansion (2.10.5) of \dot{x}^α , we can calculate the proper time τ_x along $x^\alpha(\tau)$:

$$\frac{d\tau_x}{d\tau} = \sqrt{-\dot{x}^2} = 1 - \frac{1}{2}(D\xi)^2 + \frac{1}{2}R_{u\xi u\xi} + O(\xi, D\xi)^3, \quad (2.10.6)$$

where $(D\xi)^2 = D\xi_\mu D\xi^\mu$ and $R_{u\xi u\xi} = R_{\mu\rho\nu\lambda} u^\mu u^\nu \xi^\rho \xi^\lambda$. Note we have used $u_\mu \xi^\mu = 0 \Rightarrow u_\mu D\xi^\mu = 0$ (since $Du^\mu = 0$), and $g_{\alpha\beta} g^\alpha{}_\mu g^\beta{}_\nu = g_{\mu\nu}$.

We can now write down the action for the geodesic deviation equation for $\xi(\tau)$, from the geodesic action for the worldline $x(\tau)$, $S[x] = -m\tau_x = -m \int d\tau \sqrt{-\dot{x}^2}$, using Eq. (2.10.6),

$$S[\xi] = -m \int d\tau \left[1 - \frac{1}{2}(D\xi)^2 + \frac{1}{2}R_{u\xi u\xi} \right] + O(\xi, D\xi)^3, \quad (2.10.7)$$

It is easy to verify that varying this w.r.t. ξ yields the geodesic deviation equation $D^2\xi^\mu + R_{u\xi u}{}^\mu = 0$, where $R_{u\xi u}{}^\mu = R_{\nu\rho\lambda}{}^\mu u^\nu \xi^\rho u^\lambda$.

We can also directly calculate the acceleration a^α of the worldline $x^\alpha(\tau)$, using Eqs. (2.10.5) and (2.10.6),

$$a^\alpha = \frac{1}{\sqrt{-\dot{x}^2}} D \left(\frac{\dot{x}^\alpha}{\sqrt{-\dot{x}^2}} \right) \quad (2.10.8)$$

$$\begin{aligned} &= D \left[g^\alpha{}_\mu \left(u^\mu + D\xi^\mu + \frac{1}{2}(D\xi)^2 u^\mu \right) + O(\xi^2) + O(\xi, D\xi)^3 \right] + O(\xi, D\xi)^2 \\ &= g^\alpha{}_\mu \left[D^2\xi^\mu + (D\xi \cdot D^2\xi) u^\mu \right] + (u^\mu + D\xi^\mu) Dg^\alpha{}_\mu + O(\xi, D\xi)^2. \end{aligned} \quad (2.10.9)$$

Recalling $g^\alpha{}_\mu = g^\alpha{}_\mu(z, x) = g^\alpha{}_\mu(z, x(z, -\xi))$, the covariant τ -derivative of the parallel propagator is

$$Dg^\alpha{}_\mu = u^\nu \nabla_\nu g^\alpha{}_\mu + \dot{x}^\beta \nabla_\beta g^\alpha{}_\mu = -g^\alpha{}_\rho R^\rho{}_{\mu\nu\lambda} u^\nu \xi^\lambda + O(\xi, D\xi)^2, \quad (2.10.10)$$

where the second equality has used $\dot{x}^\beta = g^\beta{}_\nu u^\nu + O(\xi, D\xi)$ from Eq. (2.10.5) and the expansions (2.9.42, 2.9.43) of the parallel propagator derivatives,

$$\nabla_\nu g^\alpha{}_\mu = -\frac{1}{2} g^\alpha{}_\rho R^\rho{}_{\mu\nu\lambda} \xi^\lambda + O(\xi)^2, \quad \nabla_\beta g^\alpha{}_\mu = -\frac{1}{2} g^\alpha{}_\rho g^\nu{}_\beta R^\rho{}_{\mu\nu\lambda} \xi^\lambda + O(\xi)^2.$$

Plugging Eq. (2.10.10) into Eq. (2.10.8) gives

$$a^\alpha = g^\alpha{}_\mu \left[D^2\xi^\mu + R_{u\xi u}{}^\mu + (D\xi \cdot D^2\xi) u^\mu \right] + O(\xi, D\xi)^2 \quad (2.10.11)$$

for the acceleration of the neighboring worldline $x^\alpha(\tau)$.²¹

If we now impose the geodesic equation $a^\alpha = 0$ for the worldline, this implies that the quantity in square brackets in Eq. (2.10.11) vanishes (since $g^\alpha{}_\mu$ is invertible). Contracting this with u_μ (which kills the first two terms due to $u \cdot \xi = 0$ and Riemann symmetries) yields $D^2\xi \cdot D\xi = 0$, which when back-substituted yields the geodesic deviation equation:

$$a^\alpha = 0 \quad \Rightarrow \quad D^2\xi^\mu + R_{u\xi u}{}^\mu = 0. \quad (2.10.12)$$

²¹Note that without the term $(D\xi \cdot D^2\xi) u^\mu$ in Eq. (2.10.11), the acceleration a^α would not be orthogonal [to $O(\xi, D\xi)$] to the tangent \dot{x}^α of Eq. (2.10.5), in cases where $D^2\xi = O(\xi, D\xi)^0$.

II The general solution to the geodesic deviation equation

Given a fiducial geodesic $z^\mu(\tau)$, a deviation vector ξ^μ at a single point z^μ and a value for the derivative $D\xi^\mu$ at that point will uniquely determine a solution to the geodesic deviation equation (2.10.12). We can find an explicit formula for the solution $\xi^{\mu'}$ at a different point $z^{\mu'}(\tau')$, in terms of ξ^μ and $D\xi^\mu$ at the initial point $z^\mu(\tau)$, and in terms of bi-tensors between z^μ and $z^{\mu'}$, as follows.

First, we recall some results from Sec. 2.9.III. There, we considered the geodesic function $x^{\mu'}(x^\mu, u^\mu, \tau)$; here, we use that same function in the form $z^{\mu'}(z^\mu, u^\mu, \tau' - \tau)$, which in the context of the previous paragraph gives the second point $z^{\mu'}(\tau')$ on the fiducial geodesic. As discussed surrounding Eq. (2.9.24), the derivatives $\tilde{\nabla}_\mu z^{\mu'}$ and $\partial z^{\mu'}/\partial u^\mu$ of the geodesic function $z^{\mu'}(z^\mu, u^\mu, \tau' - \tau)$ w.r.t. z^μ and u^μ can be considered as (bi-tensor) functions of z^μ and $z^{\mu'}$ (abbr. z and z'). On the functions $\tilde{\nabla}_\mu z^{\mu'}(z, z')$ and $\partial z^{\mu'}/\partial u^\mu(z, z')$, we found the results (2.9.31, 2.9.37, 2.9.38) for the action of the derivatives $D = u^\mu \nabla_\mu$ and $D' = u^{\mu'} \nabla_{\mu'}$:

$$D \frac{\partial z^{\mu'}}{\partial u^\mu} = -P_\mu^\nu \tilde{\nabla}_\nu z^{\mu'}, \quad D \tilde{\nabla}_\mu z^{\mu'} = R_{u\mu u}{}^\nu \frac{\partial z^{\mu'}}{\partial u^\nu}, \quad (2.10.13)$$

$$D' \frac{\partial z^{\mu'}}{\partial u^\mu} = \frac{1}{\tau' - \tau} \sigma^{\mu'}{}_{\nu'} \frac{\partial z^{\nu'}}{\partial u^\mu}, \quad D' \tilde{\nabla}_\mu z^{\mu'} = \frac{1}{\tau' - \tau} \left(\sigma^{\mu'}{}_{\nu'} \tilde{\nabla}_\mu z^{\nu'} + \sigma^{\mu'}{}_\mu \right), \quad (2.10.14)$$

$$(D')^2 \frac{\partial z^{\mu'}}{\partial u^\mu} = -R_{u'\nu' u'^{\mu'}} \frac{\partial z^{\nu'}}{\partial u^\mu}, \quad (D')^2 \tilde{\nabla}_\mu z^{\mu'} = -R_{u'\nu' u'^{\mu'}} \tilde{\nabla}_\mu z^{\nu'}. \quad (2.10.15)$$

The results (2.10.15) show that

$$\xi^{\mu'} = A^\mu \tilde{\nabla}_\mu z^{\mu'} + B^\mu \frac{\partial z^{\mu'}}{\partial u^\mu}, \quad (2.10.16)$$

where A^μ and B^μ only depend on $z^\mu(\tau)$, is a solution to the geodesic deviation equation

$$(D')^2 \xi^{\mu'} + R_{\nu'\rho'\lambda'^{\mu'}} u^{\nu'} u^{\lambda'} \xi^{\rho'} = 0. \quad (2.10.17)$$

The fact that $\tilde{\nabla}_\mu z^{\mu'}$ and $\partial z^{\mu'}/\partial u^\mu$ are linearly independent [as shown in the next paragraph] means that Eq. (2.10.16) is a general solution to the geodesic deviation equation.

To relate the coefficients A^μ and B^μ to the initial data ξ^μ and $D\xi^\mu$ at z^μ , we can take coincidence limits as $z^{\mu'} \rightarrow z^\mu$ of the solution (2.10.16). From the coincidence expansions (2.9.45, 2.9.44) and from $\partial z^{\mu'}/\partial u^\mu = -(\tau' - \tau) P_\mu^\nu \partial z^{\mu'}/\partial \sigma^\nu$ [as in Eq. (2.9.24)], we have the coincidence limits

$$\left[\tilde{\nabla}_\nu z^{\mu'} \right] = \delta_\nu^\mu, \quad \left[\frac{\partial z^{\mu'}}{\partial u^\nu} \right] = 0, \quad (2.10.18)$$

which, along with the condition $[\xi^{\mu'}] = \xi^\mu$ applied to the solution (2.10.16), tell us that $A^\mu = \xi^\mu$.

[Note that Eqs. (2.10.18) and the fact that $\partial z^{\mu'}/\partial u^{\mu}$ is not zero show that $\tilde{\nabla}_{\mu} z^{\mu'}$ and $\partial z^{\mu'}/\partial u^{\mu}$ are linearly independent.] For the D' derivatives (2.10.14), using Eq. (2.9.24) and the coincidence expansions (2.9.40, 2.9.41, 2.9.44, 2.9.45), we have the coincidence limits

$$\left[D' \tilde{\nabla}_{\nu} z^{\mu'} \right] = 0, \quad \left[D' \frac{\partial z^{\mu'}}{\partial u^{\nu}} \right] = \delta_{\nu}^{\mu},$$

which, along with the condition $[D' \xi^{\mu}] = D \xi^{\mu}$ applied to the solution (2.10.16), tell us that $B^{\mu} = D \xi^{\mu}$.

Thus, the solution at $z^{\mu'}(\tau')$ to the geodesic deviation equation (2.10.17), with the initial data ξ^{μ} and $D \xi^{\mu}$ at $z^{\mu}(\tau)$, is given by

$$\xi^{\mu'} = \xi^{\mu} \tilde{\nabla}_{\mu} z^{\mu'} + D \xi^{\mu} \frac{\partial z^{\mu'}}{\partial u^{\mu}}. \quad (2.10.19)$$

Recall that the geodesic function derivatives can be expressed in terms of world function derivatives, as in Eqs. (2.9.24, 2.9.25, 2.9.26).

It is also nice to note the result of acting on the solution (2.10.19) for $\xi^{\mu'}$ with the derivative D at the initial point $z^{\mu}(\tau)$. Using Eqs. (2.10.13) we find

$$D \xi^{\mu'} = (D^2 \xi^{\mu} + R_{u \xi u}{}^{\mu}) \frac{\partial z^{\mu'}}{\partial u^{\mu}}, \quad (2.10.20)$$

which vanishes if the initial data also satisfies the geodesic deviation equation.

III Geodesic deviation and the self-force

Section 2.10.I showed how one can take the geodesic action and the geodesic equation for a worldline $x^{\alpha}(\tau)$, expand them in powers of a deviation vector $\xi^{\mu}(\tau)$ along a fiducial geodesic $z^{\mu}(\tau)$, and obtain an action and EoM (the geodesic deviation equation) for the vector ξ^{μ} . We now consider how the dynamics of a particle experiencing a conservative scalar self-force can be expanded about a fiducial geodesic.

For a worldline $x^{\alpha}(\tau)$ [τ being an arbitrary parameter] with tangent \dot{x}^{α} and acceleration a^{α} , recall the *nonlocal* conservative self-force EoM (2.4.1),

$$m a^{\alpha} = q^2 P^{\alpha\beta} \int d\tau' \sqrt{-\dot{x}^2(\tau')} \nabla_{\beta} G(x(\tau), x(\tau')) + O(q^4), \quad (2.10.21)$$

and the *local* EoM (2.5.6) [with the osculating geodesic],

$$ma^\alpha = q^2 P^{\alpha\beta} \int d\tau'' \nabla_\beta G\left(x(\tau), x_g(x(\tau), \dot{x}(\tau), \tau'')\right) + O(q^4), \quad (2.10.22)$$

both adapted slightly for our use here.²²

In expanding the worldline x^α about a fiducial geodesic z^μ in powers of the deviation vector ξ^μ , we must make a choice as to how the expansions in small charge q and in small deviation ξ are related to one another. Since the self-force causing the worldline to deviate from a geodesic scales as q^2 , it first might seem natural to use the relation $O(\xi, D\xi) = O(q^2)$.

If we go with $O(\xi, D\xi) = O(q^2)$, then the two EoMs (2.10.21) and (2.10.22) will give the same EoM for ξ^μ at $O(q^2)$. Since the self-force is $O(q^2)$, both the actual worldline $x(\tau')$ and the osculating geodesic $x_g(x, \dot{x}, \tau')$ which source the force can be replaced with the fiducial geodesic at $O(q^2)$ accuracy, because they both differ from the fiducial geodesic only at $O(q^2)$; and similarly for the field points as with the source points. Using the conventions and results of the Sec. 2.10.I, we find that the $O(\xi, D\xi) = O(q^2)$ expansions of both the nonlocal and local EoMs for x^α yield the following local EoM for ξ^μ :

$$D^2\xi^\mu + R_{u\xi u}{}^\mu = q^2 P^{\mu\nu} \int d\tau' \nabla_\nu G(z(\tau), z(\tau')) + O(q^4). \quad (2.10.23)$$

Here, the self-force is sourced by and evaluated on the fiducial geodesic. This is (the conservative part of) the self-force EoM for a deviation vector field discussed by Gralla and Wald [86].

It is easy to write down a local action [extending the geodesic deviation action (2.10.7)] from which the self-force ξ -EoM (2.10.23) can be derived [and this action can be obtained from the $O(\xi, D\xi) = O(q^2)$ expansion of the nonlocal action (2.4.2), dropping ξ -independent terms]:

$$S[\xi] = \int d\tau \left[\frac{m}{2} (D\xi)^2 - \frac{m}{2} R_{u\xi u}{}^\xi + q^2 \xi^\mu \int d\tau' \nabla_\mu G(z(\tau), z(\tau')) \right] + O(q^6), \quad (2.10.24)$$

where the ξ s and u s are evaluated at τ . Note that the radiative part of the self-force could be straightforwardly incorporated into this action.

The action (2.10.24) and resultant EoM (2.10.23) for ξ^μ , though of the same accuracy as the worldline EoMs (2.10.21) and (2.10.22) in naively counting orders in q^2 , could only be trusted over

²²In the nonlocal worldline EoM (2.10.21), τ' is a different value of the same (arbitrary) parameter τ used for $x^\alpha(\tau)$. In the local worldline EoM (2.10.22), τ'' is the proper time interval along the osculating geodesic $x_g(x, \dot{x}, \tau'')$. In the ξ -EoM (2.10.23), and in all other equations below, τ' is (a different) proper time along the fiducial geodesic.

relatively short time scales, since they completely ignore the deviation from the fiducial geodesic in calculating the self-force. While it is clear that a better understanding of the error scalings is needed here, along with a clearer statement of what ends we are working towards, it might be useful to re-expand the local and nonlocal worldline EoMs without using the assumption $O(\xi, D\xi) = O(q^2)$.

Consider expanding the self-force EoMs about a fiducial geodesic while working to $O(\xi, D\xi)$ and to $O(q^2)$ independently. The nonlocal worldline EoM (2.10.21) expands to a nonlocal EoM for ξ^μ :

$$\begin{aligned} D^2\xi^\mu + R_{u\xi}u^\mu &= q^2 (P^{\mu\nu} + D\xi^\mu u^\nu) \int d\tau' \left(\nabla_\nu + \xi^\rho \nabla_\rho \nabla_\nu + \xi^{\rho'} \nabla_{\rho'} \nabla_\nu \right) G(z, z') \\ &+ O(\xi, D\xi)^2 + O(q^4). \end{aligned} \quad (2.10.25)$$

²³This EoM can be obtained from the nonlocal action [found by similarly expanding the nonlocal worldline action (2.4.2)]

$$\begin{aligned} S[\xi] &= \int d\tau \left(1 - \frac{1}{2}(D\xi)^2 + \frac{1}{2}R_{u\xi}u\xi \right) \left[-m + \frac{q^2}{2} \int d\tau' \left(1 - \frac{1}{2}(D'\xi')^2 + \frac{1}{2}R_{u'\xi'u'\xi'} \right) \right. \\ &\quad \left. \times \left(1 + \xi^\mu \nabla_\mu + \xi^{\mu'} \nabla_{\mu'} + \frac{1}{2}\xi^\mu \xi^\nu \nabla_\mu \nabla_\nu + \frac{1}{2}\xi^{\mu'} \xi^{\nu'} \nabla_{\mu'} \nabla_{\nu'} + \xi^\mu \xi^{\mu'} \nabla_\mu \nabla_{\mu'} \right) G(z, z') \right]. \end{aligned}$$

Now consider expansion of the local EoM (2.10.22), making use of the solution to the geodesic deviation equation from Sec. 2.10.II. The osculating geodesic $x_g^{\alpha'}(x^\alpha, \dot{x}^\alpha, \tau'')$ can be specified by the deviation vector field $\xi_g^{\mu'}$ at points $z^{\mu'}(\tau')$, with $x_g^{\alpha'} = x^{\alpha'}(z^{\mu'}, -\xi_g^{\mu'})$, where $\xi_g^{\mu'}$ is the solution (2.10.19) to the geodesic deviation equation given initial data ξ^μ and $D\xi^\mu$ (which determine x^α and \dot{x}^α) at $z^\mu(\tau)$. The expansion of the local worldline EoM (2.10.22) then yields the following local EoM for ξ^μ :

$$\begin{aligned} D^2\xi^\mu + R_{u\xi}u^\mu &= q^2 (P^{\mu\nu} + D\xi^\mu u^\nu) \int d\tau' \left(\nabla_\nu + \xi^\rho \nabla_\rho \nabla_\nu \right. \\ &\quad \left. + \left\{ \xi^\rho \tilde{\nabla}_\rho z^{\rho'} + D\xi^\rho \frac{\partial z^{\rho'}}{\partial u^\rho} \right\} \nabla_{\rho'} \nabla_\nu \right) G(z, z') + O(\xi, D\xi)^2 + O(q^4), \end{aligned} \quad (2.10.26)$$

where the quantity in curly brackets is $\xi_g^{\mu'}$. It seems possible that one could construct or test the existence of an action for this EoM, with some guesswork and the machinery of Secs. 2.9 and 2.10, which might help address the questions of Sec. 2.5.

²³Here, z, u, ξ , etc. are evaluated at τ and their primed versions are evaluated at τ' .

Note that the projection tensor $P^{\alpha\beta} = g^{\alpha\beta} + \dot{x}^\alpha \dot{x}^\beta / \sqrt{-\dot{x}^2}$ of Eq. (2.10.21), when parallel propagated from x^α to z^μ , would result in the projector $P^{\mu\nu} + D\xi^\mu u^\nu + [u^\mu D\xi^\nu]$, where $P^{\mu\nu} = g^{\mu\nu} + u^\mu u^\nu$. The contribution to the EoM (2.10.25) from the bracketed term in this projector is cancelled off with the contribution from the $(D\xi \cdot D^2\xi)u^\mu$ term in the worldline acceleration a^α from Eq. (2.10.11), when the latter is order-reduced [and similarly for the EoM (2.10.26)].

CHAPTER 3

FIRST-POST-NEWTONIAN QUADRUPOLE TIDAL INTERACTIONS IN BINARY SYSTEMS

This chapter treats orbital-tidal coupling in a binary stellar system to first-post-Newtonian order. We derive the orbital equations of motion for bodies with spins and mass quadrupole moments and show that they conserve the total linear momentum of the binary. We note that spin-orbit coupling must be included in a 1PN treatment of tidal interactions in order to maintain consistency; inclusion of 1PN quadrupolar tidal effects while omitting spin effects would lead to a failure of momentum conservation for generic evolution of the quadrupoles. We use momentum conservation to specialize our analysis to the system's center-of-mass-energy frame; we find the binary's relative equation of motion in this frame and also present a generalized Lagrangian from which it can be derived. We then specialize to the case in which the quadrupole moment is adiabatically induced by the tidal field (in which case it is consistent to ignore spin effects). We show how the adiabatic dynamics for the quadrupole can be incorporated into our action principle and present the simplified orbital equations of motion and conserved energy for the adiabatic case. These results are relevant to the gravitational wave signal of inspiralling binary neutron stars.

3.1 Introduction and summary

I Background and motivation

Inspiralling and coalescing compact binaries present one of the most promising sources for ground-based gravitational wave (GW) detectors [87]. A primary goal in the measurement of GW signals from neutron star-neutron star (NSNS) and black hole-neutron star (BHNS) binaries is to probe

the neutron star matter equation of state (EoS), which is currently only loosely constrained by electromagnetic observations in the relevant density range $\rho \sim 2\text{--}8 \times 10^{14}$ g/cm³ [14]. The EoS will leave its imprint on the GW signal via the effects of tidal coupling, as the neutron star is distorted by the non-uniform field of its companion. Many recent studies of the effects of the neutron star EoS on binary GW signals have been based on numerical simulations of the fully relativistic hydrodynamical evolution of NSNS and BHNS binaries (see e.g. the reviews [19, 20]). These simulations have largely focused on the binaries' last few orbits and merger, at GW frequencies $\gtrsim 500$ Hz, and have investigated constraining neutron star structure via (for example) the GW energy spectrum [88], effective cutoff frequencies at merger [89], and tidal disruption signals [90].

As recently investigated by Flanagan and Hinderer [91], neutron star internal structure may also have a measurable influence on the GW signal from the earlier inspiral stage of a binary's orbital evolution, at GW frequencies $\lesssim 500$ Hz. While tidal coupling will produce only a small perturbation to the GW signal in this low-frequency adiabatic regime, the tidal signal should be relatively clean, depending (at leading order) on a single parameter pertaining to the neutron star structure. This tidal deformability parameter λ is the proportionality constant between the applied tidal field and the star's induced quadrupole moment and is sensitive to the neutron star EoS. The measurement scheme proposed in Ref. [91] is based on an analytical model for the tidal contribution to the GW signal, giving a linear perturbation to the GW phase proportional to λ . At GW frequencies $\lesssim 400$ Hz, the model should be sufficiently accurate to constrain λ to $\sim 10\%$, with the largest source of error being first-post-Newtonian (1PN) corrections to the tidal-orbital coupling [91, 35]. Recent work in Ref. [92] suggests that the effective one body (EOB) formalism (discussed further below) can be used to extend the description of the GW phasing up to merger, allowing Advanced LIGO to detect and measure tidal polarizations of neutron stars.

The modeling of tidal effects in GW signals from neutron star binaries can be divided into three separate problems: (i) to calculate the deformation response of each neutron star to the tidal field generated by its companion, (ii) to calculate the influence of the tidal deformations on the system's (conservative) orbital dynamics, and (iii) to calculate the gravitational waveform emitted by the system and incorporate the corresponding radiation reaction effects in the orbital dynamics. While this paper is focused on solving problem (ii) to 1PN accuracy, we will briefly mention work on each of these problems.

Tidal Deformability Computations: While computing tidal deformations of stars is a well studied problem in Newtonian gravity [93], it has recently been re-examined in the context of fully relativistic stars by Hinderer et al. [34, 35], Damour and Nagar [37], and Binington and Poisson [36]. These authors used fully relativistic models for the star’s interior, with various candidate equations of state, to calculate the perturbations to the star’s equilibrium configuration produced by a static external tidal field. In Refs. [34, 35], the results were used to determine the (electric-type) quadrupolar tidal deformability λ , while Refs. [37, 36] corrected computational errors in Ref. [35] and extended the analysis and included all higher-multipolar electric-type response coefficients (for the mass multipole moments) as well as their magnetic-type analogues (for the current multipole moments). While these results concern a star’s response to a static tidal field, they are still applicable for inspiralling binaries at sufficiently low orbital frequencies; a star will approximately maintain equilibrium with the instantaneous tidal field if the field changes adiabatically.

Orbital-tidal conservative dynamics: In their treatment of tidal effects in binary GW signals in Ref. [91], Flanagan and Hinderer used Newtonian gravity to treat (the conservative part of) the system’s orbital dynamics; they estimated that 1PN corrections to the orbits would modify the calculated tidal signal by $\sim 10\%$. A general formalism for calculating 1PN corrections to the orbital dynamics of extended bodies has been developed in a series of papers [29, 33, 47, 48] by Damour, Soffel, and Xu (DSX) and later extended by Racine and Flanagan [31]. In this paper, we apply that formalism to determine and analyze the explicit 1PN translational equations of motion for binary systems with quadrupolar tidal interactions. While such equations of motion have previously been presented in Refs. [50, 51], we find that those results differ from ours. We also extend previous results by analyzing 1PN momentum conservation for the binary system (which serves as a strong consistency check for our equations of motion, and one not satisfied by the results of Refs. [50, 51]), by specializing the equations of motion to the system’s center-of-mass-energy frame, and by formulating an action principle for the orbital dynamics.

Tidal effects in the conservative dynamics to 1PN order have also been investigated recently by Damour and Nagar (DN) [94] who considered circular orbits, and (since the first appearance of this work) by Bini, Damour and Faye (BDF) [52], who considered generic orbits and have in fact carried the analysis to 2PN order. These works incorporate their results into an EOB description of the dynamics. In Appendix 3.8, we demonstrate that our results agree with those of DN and with (the

1PN restriction of) those of BDF.

Gravitational Wave Emission: To calculate the emitted GW signal and radiation reaction effects, Flanagan and Hinderer [91] used the quadrupole radiation approximation and associated 2.5PN radiation reaction forces. To consistently generalize their calculation in Ref. [91] to (relative) 1PN order, it is necessary to compute the 3.5PN corrections to the GW generation, in addition to the 1PN orbital corrections considered in this paper. These calculations have been carried out by Vines, Hinderer and Flanagan in Ref. [95], which builds off of the work presented here. The issue of tidal effects in inspiral-stage NSNS binary GW signals has also been recently addressed by means of numerical relativity simulations in Refs. [15, 96].

We now turn to a more detailed overview of the problem of the conservative orbital-tidal dynamics, at Newtonian order in Sec. 3.1.II and at 1PN order in Sec. 3.1.III, before summarizing the results of this paper in Sec. 3.1.IV.

II Newtonian tidal coupling

Gravitational tidal coupling arises from the interaction of the non-spherical components of a body's matter distribution with a non-uniform gravitational field. The non-sphericity is characterized at leading order by the body's mass quadrupole moment,

$$Q^{ij}(t) = \int d^3x \rho(\bar{x}^i \bar{x}^j - \frac{1}{3} \delta^{ij} |\bar{\mathbf{x}}|^2), \quad (3.1.1)$$

with $\rho(t, \mathbf{x})$ being the mass density and $\bar{x}^i(t) = x^i - z^i(t)$ being the displacement from the body's center of mass position $x^i = z^i(t)$. The quadrupole is (at most) on the order of $Q^{ij} \sim MR^2$, with M being the body's mass and R its radius. Higher-order deformations are described by the octupole, $Q^{ijk} \sim \int d^3x \rho \bar{x}^i \bar{x}^j \bar{x}^k \sim MR^3$, and higher-order multipole moments. The non-uniform field can be characterized by derivatives of an external Newtonian potential $\phi_{\text{ext}}(t, \mathbf{x})$; in a binary system, the leading-order potential is $\phi_{\text{ext}} = -GM/r$, where G is Newton's constant, M is the mass of the companion, and r the distance between the body and its companion.

The non-uniform field of the companion produces tidal forces on the non-spherical body (in

addition to the usual $1/r^2$ force) according to

$$\begin{aligned}
M\ddot{z}^i &= - \left(M\partial_i + \frac{1}{2}Q^{jk}\partial_{ijk} + \frac{1}{6}Q^{jkl}\partial_{ijkl} + \dots \right) \phi_{\text{ext}} \\
&\sim GM \left[\frac{M}{r^2} + \frac{|Q^{ij}|}{r^4} + \frac{|Q^{ijk}|}{r^5} + \dots \right] \\
&\sim \frac{GM^2}{r^2} \left[1 + \frac{R^2}{r^2} + O\left(\frac{R^3}{r^3}\right) \right],
\end{aligned} \tag{3.1.2}$$

where M is the mass of the body or its companion, assumed here to be roughly equal, and the derivatives of the external potential are evaluated at the body's center of mass, $x^i = z^i(t)$. The contributions to the net force from the quadrupole and higher-order multipoles are thus seen to take the form of an expansion in R/r , the ratio of the size of the body to the orbital separation. When this finite-size parameter is small, as in the early stages of binary inspiral, the tidal force is well approximated by the quadrupolar term alone. A more detailed account of Newtonian tidal forces and related results is given in Sec. 3.2 below.

In neutron star binaries, a quadrupole is induced by differential forces resulting from the non-uniform field of the companion. In the adiabatic limit, when the response time scale of the body is much less than the time scale on which the tidal field changes, the induced quadrupole will be given (to linear order in the tidal field) by

$$Q^{ij}(t) = -\lambda\partial_i\partial_j\phi_{\text{ext}}(t, \mathbf{z}), \tag{3.1.3}$$

with the constant λ being the tidal deformability. This is related to the more often used dimensionless Love number k_2 by $\lambda = 2k_2R^5/3G$, where R is the body's radius [93].

Using Newtonian gravity to describe the orbital dynamics and the adiabatic approximation to model the stars' induced quadrupoles, Flanagan and Hinderer [91] calculated the effect of tidal interactions on the phase of the gravitational waveform emitted by an inspiralling neutron star binary and analyzed the measurability of the tidal effects. They found that Advanced LIGO should be able to constrain the neutron stars' tidal deformability to $\lambda \leq (2.0 \times 10^{37} \text{ g cm}^2 \text{ s}^2)(D/50 \text{ Mpc})$ with 90% confidence, for a binary of two $1.4 M_\odot$ neutron stars at a distance D from the detector, using only the portion of the signal with GW frequencies less than 400 Hz. The calculations of λ for a $1.4 M_\odot$ neutron star in Refs. [34, 35, 37, 36], using several different equations of state, give values in the range $0.03\text{--}1.0 \times 10^{37} \text{ g cm}^2 \text{ s}^2$, so that nearby events may allow Advanced LIGO to place useful

constraints on candidate equations of state. Reference [92] discusses how the EOB formalism can be used to extend the range of validity of analytic waveforms up to merger, and argues that Advanced LIGO should be able to detect and measure the tidal deformability of neutron stars.

Refs. [91, 35] estimate the fractional corrections to the tidal signal due to several effects neglected by the model of the GW phasing used in Ref. [91], namely, non-adiabaticity ($\lesssim 1\%$), higher-multipolar tidal coupling ($\lesssim 0.7\%$), nonlinear hydrodynamic effects ($\lesssim 0.1\%$), spin effects ($\lesssim 0.3\%$), nonlinear response to the tidal field ($\lesssim 3\%$), viscous dissipation (negligible), and post-Newtonian effects ($\lesssim 10\%$). The largest corrections, from post-Newtonian effects in the orbital dynamics and GW emission, will depend on the neutron star physics only through the same tidal deformability parameter λ used in the Newtonian treatment and thus can be easily incorporated into the data analysis methods outlined in Refs. [91, 35].

III First-post-Newtonian corrections

For inspiralling neutron star binaries with a total mass of $\sim 3M_\odot$ at orbital frequencies of ~ 200 Hz (GW frequencies of ~ 400 Hz), the post-Newtonian expansion parameter $v^2/c^2 \sim GM/c^2 r$ is ~ 0.1 , so the 1PN approximation is well suited to describing relativistic corrections to the binary orbit. As discussed in depth in Sec. 3.3 below, the 1PN orbital dynamics of a binary system with tidal coupling can be described by translational equations of motion (EoMs) similar in form to the Newtonian equations schematically represented in Eq. (3.1.2), giving the center-of-mass acceleration of each constituent body in terms of their positions and multipole moments. The 1PN equations of motion add order $1/c^2$ correction terms to (3.1.2) which depend not only on the bodies' positions and mass multipole moments, M , Q^{ij} , Q^{ijk} , etc., but also on their velocities and current multipole moments, S^i , S^{ij} , etc. Expanding in both the post-Newtonian parameter v^2/c^2 and the finite size parameter R/r , the 1PN equations of motion can be written schematically as

$$\begin{aligned}
M\ddot{z}^i \sim & GM \left[\frac{M}{r^2} + \frac{v^2}{c^2} \frac{M}{r^2} + O\left(\frac{v^4}{c^4} \frac{M}{r^2}\right) \right. \\
& + \frac{|Q^{ij}|}{r^4} + \frac{v^2}{c^2} \frac{|Q^{ij}|}{r^4} + O\left(\frac{|Q^{ijk}|}{r^5} \sim \frac{R^3}{r^3} \frac{M}{r^2}\right) \\
& \left. + \frac{v|S^i|}{c^2 r^3} + O\left(\frac{v|S^{ij}|}{c^2 r^4} \sim \frac{v^2}{c^2} \frac{R^2}{r^2} \frac{M}{r^2}\right) \right] \quad (3.1.4)
\end{aligned}$$

In the top line, the first term gives the usual point-particle (monopole) force of Newtonian gravity, and the second term represents its 1PN corrections. The last term of the top line denotes 2PN and higher post-Newtonian order corrections, which we neglect in this paper. Point-particle EoMs are in fact currently known up through 3PN order [97].

In the second line of Eq. (3.1.4), we have first the Newtonian quadrupole-tidal term, followed by its 1PN corrections (which are the subject of this paper), and finally contributions from the octupole and higher mass multipoles (and their post-Newtonian corrections) which are suppressed by higher powers of R/r and which we neglect in our analysis [31].

The first term in the bottom line represents the 1PN spin-orbit coupling [33], which will be included in our analysis, and the final term of the bottom line denotes 1PN contributions from the bodies' current quadrupoles and higher current multipoles, which will not be included.

The DSX treatment of 1PN celestial mechanics [29, 33, 47, 48] provides a framework for calculating the orbital EoMs for bodies with arbitrarily high-order mass and current multipole moments. In Ref. [33], DSX applied their formalism to rederive the explicit 1PN EoMs for bodies with mass monopoles and current dipoles (spins). The calculation was then extended to include the effects of the bodies' mass quadrupoles by Xu et al [50, 51]. Racine and Flanagan (RF) [31] later reworked the DSX formalism and presented explicit 1PN EoMs for bodies with arbitrarily high-order multipoles, which can be specialized to the case of bodies with only spins and mass quadrupoles. We have found, however, that the final results of Xu et al and those of RF are in disagreement with each other and with our recent calculations. Some typos and omissions leading to errors in RF have been identified and are outlined in an upcoming erratum; the results given in this paper agree with the corrected results of RF. In Sec. 3.3 below, we review the essential ideas of the DSX formalism, following the notations and conventions of RF, and we outline the full procedure by which our results for the 1PN EoMs are derived.

Though it would be convenient to be able to specialize to the case of non-spinning bodies when studying tidal interactions, this would lead to inconsistencies at 1PN order when considering generic behavior of the quadrupole moments. A body with a quadrupole in a tidal field will generically experience tidal torques [according to Eq. (3.1.7) below] which will spin up the body even if it started with no spin; this is a Newtonian-order effect. The resultant spin affects the orbital dynamics via the 1PN spin-orbit coupling. For this reason, if one were to work through the DSX formalism and

simply drop all spin terms while keeping mass quadrupole terms, one would arrive at inconsistencies. In particular, one would find that momentum is not conserved at 1PN order (see Sec. 3.4.III below). However, for the special case of adiabatically induced quadrupoles, the tidal torques vanish, and it is then consistent to ignore all spin terms.

The relation between the work of DN [94] and BDF [52] on the 1PN conservative orbital-tidal dynamics and the results of this paper is described in the next subsection along with the summary of our results, and in more detail in Appendix 3.8. We also refer the reader to Refs. [92, 98, 40, 99, 100, 101, 102, 103, 104, 105] for other recent work on tidal effects in inspiralling binaries.

IV Summary of results

IV.a The M_1 - M_2 - S_2 - Q_2 system

Our results concern the 1PN gravitational interactions in a system of two bodies, which we label “1” and “2”. We model body 1 as an effective point particle, with a mass monopole moment only, while we take body 2 to have additionally a spin and a mass quadrupole moment. We consistently work to linear order in the spin and quadrupole, and our results can thus be easily generalized to the case of two spinning, deformable bodies by interchanging particle labels. We initially assume nothing about the bodies’ internal structure or dynamics. Our primary assumption is the validity of the 1PN approximation to general relativity in a vacuum region surrounding the bodies.

The system’s orbital dynamics can be formulated in terms of the bodies’ center-of-mass worldlines $z_1^i(t)$ and $z_2^i(t)$ and their multipole moments: the mass monopoles $M_1(t)$ and $M_2(t)$, the spin $S_2^i(t)$, and the mass quadrupole $Q_2^{ij}(t)$.

The worldlines $x^i = z_1^i(t)$ and $x^i = z_2^i(t)$ parametrize the bodies’ positions to 1PN accuracy in a (conformally Cartesian and harmonic) ‘global’ coordinate system (t, x^i) , which tends to an inertial coordinate system in Minkowski spacetime as $|\mathbf{x}| \rightarrow \infty$. The global coordinates and center-of-mass worldlines are defined more precisely in Sec. 3.3.IV. We use the following notation for the relative position and velocity:

$$\begin{aligned} z^i &= z_2^i - z_1^i, & r &= |\mathbf{z}| = \sqrt{\delta_{ij} z^i z^j}, & n^i &= z^i/r, \\ v_1^i &= \dot{z}_1^i, & v_2^i &= \dot{z}_2^i, & v^i &= v_2^i - v_1^i, & \dot{r} &= v^i n^i, \end{aligned}$$

with dots denoting derivatives with respect to the global frame time coordinate t .

The multipole moments— $M_1(t)$, $M_2(t)$, $S_2^i(t)$, and $Q_2^{ij}(t)$ for our truncated system—are defined in Secs. 3.3.II and 3.3.V via a multipole expansion of the 1PN metric in a vacuum region surrounding each body [31]. In the case of weakly self-gravitating bodies, these moments can be defined as integrals of the stress-energy tensor over the volume of the bodies, as in Eqs. (3.3.12); these definitions coincide with those of the Blanchet-Damour multipole moments introduced in Ref. [49] and used by DSX [29, 33, 47, 48]. The mass multipole moments (like M_1 , M_2 , and Q_2^{ij}) are defined with 1PN accuracy, while the current multipole moments (like S_2^i) appear only in 1PN-order terms and thus need only be defined with Newtonian accuracy. We will often denote the spin and quadrupole of body 2 by S_i and Q^{ij} , dropping the “2” labels.

IV.b General equations of motion and orbital Lagrangian

The equations of motion for the monopoles $M_1(t)$ and $M_2(t)$, the spin $S^i(t)$, and the worldlines $z_1^i(t)$ and $z_2^i(t)$ are determined by Einstein’s equations alone, while that for the quadrupole $Q^{ij}(t)$ will depend on the details of body 2’s internal dynamics and can initially be left unspecified. The mass monopole of body 1, the effective point particle, is found to be conserved to 1PN order:

$$\dot{M}_1 = O(c^{-4}), \tag{3.1.5}$$

while that of body 2 is not. As discussed in Sec. 3.3.VI, the 1PN-accurate mass monopole M_2 can be decomposed according to

$$M_2 = {}^nM_2 + c^{-2} (E_2^{\text{int}} + 3U_Q) + O(c^{-4}). \tag{3.1.6a}$$

Here, nM_2 is the Newtonian-order (rest mass) contribution, which is conserved:

$${}^n\dot{M}_2 = 0. \tag{3.1.6b}$$

The 1PN contributions to (3.1.6a) involve the Newtonian potential energy of the quadrupole-tidal interaction,

$$U_Q = -\frac{3M_1}{2r^3} Q^{ij} n^i n^j, \tag{3.1.6c}$$

and the Newtonian internal energy of body 2, E_2^{int} , whose evolution is governed by the rate at which the tidal field does work on the body [106]:

$$\dot{E}_2^{\text{int}} = \frac{3M_A}{2r^3} \dot{Q}^{ij} n^i n^j, \tag{3.1.6d}$$

(cf. Sec. 3.2.VI). Equations (3.1.6) ensure that M_2 satisfies the 1PN evolution equation (3.3.28,3.3.29). The decomposition of the monopole M_2 in (3.1.6a) is essential for properly formulating an action principle for the orbital dynamics. The evolution of the spin S^i is determined by the (Newtonian-order) tidal torque formula:

$$\dot{S}^i = \frac{3M_1}{r^3} \epsilon^{ijk} Q^{ja} n^a n^k + O(c^{-2}), \quad (3.1.7)$$

as in (3.2.45). Finally, the 1PN translational equations of motion, which govern the evolution of the worldlines z_1^i and z_2^i , are of the form

$$\begin{aligned} \ddot{z}_1^i &= \mathcal{F}_1^i(z^j, v_1^j, v_2^j, M_1, M_2, S^j, Q^{jk}, \dot{Q}^{jk}, \ddot{Q}^{jk}), \\ \ddot{z}_2^i &= \mathcal{F}_2^i(z^j, v_1^j, v_2^j, M_1, M_2, S^j, Q^{jk}, \dot{Q}^{jk}), \end{aligned}$$

and are given explicitly by Eqs. (3.3.33).

In Sec. 3.4, we define and calculate the 1PN-accurate mass dipole moment of the entire system $M_{\text{sys}}^i(t)$, given in Eq. (3.4.9). We find that the condition $\dot{M}_{\text{sys}}^i = O(c^{-4})$, required by Einstein's equations and reflecting the conservation of the system's total momentum, is satisfied as a consequence of the orbital EoMs (3.3.33); this serves as a non-trivial consistency check for our results. The conservation of momentum also allows us to specialize the EoMs to the system's center-of-mass(-energy) (CoM) frame, which can be defined by the condition $M_{\text{sys}}^i = 0$ as in Sec. 3.5.I. The two EoMs (3.3.33) for the worldlines z_1^i and z_2^i can then be traded for the single EoM for the CoM-frame relative position $z^i = z_2^i - z_1^i$, given in Eq. (3.5.9).

Our results can be most compactly summarized by giving a Lagrangian formulation of the CoM-frame orbital dynamics, as discussed in Sec. (3.5.II). We find that the CoM-frame orbital EoM (3.5.9) can be derived from the generalized Euler-Lagrange equation

$$\left(\frac{\partial}{\partial z^i} - \frac{d}{dt} \frac{\partial}{\partial v^i} + \frac{d^2}{dt^2} \frac{\partial}{\partial a^i} \right) \mathcal{L}_{\text{orb}} = 0, \quad (3.1.8)$$

with a generalized Lagrangian \mathcal{L}_{orb} given by

$$\mathcal{L}_{\text{orb}} = \mathcal{L}_M + \mathcal{L}_S + \mathcal{L}_Q, \quad (3.1.9a)$$

with the monopole part,

$$\mathcal{L}_M = \frac{\mu v^2}{2} + \frac{\mu M}{r} + \frac{\mu}{c^2} \left[\frac{1-3\eta}{8} v^4 + \frac{M}{2r} \left(\eta \dot{r}^2 + (3+\eta)v^2 - \frac{M}{r} \right) \right] + O(c^{-4}), \quad (3.1.9b)$$

the spin part,

$$\mathcal{L}_S = \frac{\chi_1}{c^2} \epsilon^{abc} S^a v^b \left[\frac{2M}{r^2} n^c + \frac{\chi_1}{2} a^c \right] + O(c^{-4}), \quad (3.1.9c)$$

and the quadrupole part,

$$\begin{aligned} \mathcal{L}_Q = & \frac{3M_1}{2r^3} Q^{ab} n^a n^b + \frac{1}{c^2} \left\{ \frac{M}{r^3} Q^{ab} \left[n^a n^b \left(A_1 v^2 + A_2 \dot{r}^2 + A_3 \frac{M}{r} \right) + A_4 v^a v^b + A_5 \dot{r} n^a v^b \right] \right. \\ & \left. + \frac{M}{r^2} \dot{Q}^{ab} \left[A_6 n^a v^b + A_7 \dot{r} n^a n^b \right] + E_2^{\text{int}} \left[A_8 v^2 + A_9 \frac{M}{r} \right] \right\} + O(c^{-4}). \end{aligned} \quad (3.1.9d)$$

We use here the notation

$$\begin{aligned} M &= M_1 + {}^n M_2, & \chi_1 &= M_1/M, & \chi_2 &= {}^n M_2/M, \\ \mu &= M_1 {}^n M_2/M, & \eta &= \chi_1 \chi_2 = \mu/M, \end{aligned}$$

with M being the total (conserved) Newtonian rest mass, μ the reduced mass, and η the symmetric mass ratio. The dimensionless coefficients A_1 – A_9 appearing in (3.1.9d) are functions only of the mass ratios χ_1 and χ_2 given by (3.5.10e).

IV.c Adiabatic approximation for the induced quadrupole

The above results concerning the orbital EoM and its Lagrangian formulation are valid regardless of the internal structure of body 2, i.e. for arbitrary evolution of its quadrupole $Q^{ij}(t)$. In Sec. 3.6, we discuss a simple adiabatic model for the evolution of Q^{ij} . In the adiabatic limit, the (body-frame) quadrupole responds to the instantaneous tidal field according to

$$Q^{ij}(t) = \lambda G_2^{ij}(t), \quad (3.1.10)$$

where $G_2^{ij}(t)$, given by Eq. (3.6.6c), is the (body-frame) quadrupolar gravito-electric tidal moment of body 2, a 1PN generalization of the derivatives of the Newtonian potential in Eq. (3.1.3), and λ is the (constant) tidal deformability. With the quadrupole given by (3.1.10), the tidal torque (3.1.7) vanishes (cf. Eq. (3.2.46)), so that the spin S^i is constant; thus, in the adiabatic limit, we can specialize to the case $S^i = 0$ without generating inconsistencies. In Sec. 3.6.I, we show that the adiabatic evolution for the quadrupole (3.1.10) can be derived from a Lagrangian that adds to the orbital Lagrangian an internal elastic potential energy term which is quadratic in the 1PN-accurate quadrupole:

$$\mathcal{L} = \mathcal{L}_{\text{orb}}[z^i, Q^{ij}] - \frac{1}{4\lambda} Q^{ab} Q^{ab} + O(c^{-4}), \quad (3.1.11)$$

with \mathcal{L}_{orb} given by Eq. (3.1.9) with $S^i = 0$, and with $E_2^{\text{int}} = (1/4\lambda)Q^{ab}Q^{ab} + O(c^{-2})$. (Note that any additional constant contribution to the internal energy E_2^{int} can be included as a 1PN contribution to the constant ${}^{\text{N}}M_2$ in Eq. 3.1.6a.) Substituting the solution (3.1.10) for $Q^{ij}(t)$ into this Lagrangian, we obtain a simplified Lagrangian involving only the CoM-frame relative position $z^i(t)$:

$$\begin{aligned} \mathcal{L}[z^i] = & \frac{\mu v^2}{2} + \frac{\mu M}{r} \left(1 + \frac{\Lambda}{r^5} \right) \\ & + \frac{\mu}{c^2} \left\{ \theta_0 v^4 + \frac{M}{r} \left[v^2 \left(\theta_1 + \xi_1 \frac{\Lambda}{r^5} \right) + \dot{r}^2 \left(\theta_2 + \xi_2 \frac{\Lambda}{r^5} \right) + \frac{M}{r} \left(\theta_3 + \xi_3 \frac{\Lambda}{r^5} \right) \right] \right\} \end{aligned} \quad (3.1.12)$$

with $\Lambda = (3\chi_1/2\chi_2)\lambda$, and with the dimensionless coefficients

$$\begin{aligned} \theta_0 &= (1 - 3\eta)/8, & \theta_1 &= (3 + \eta)/2, & \theta_2 &= \eta/2, & \theta_3 &= -1/2, \\ \xi_1 &= (\chi_1/2)(5 + \chi_2), & \xi_2 &= -3(1 - 6\chi_2 + \chi_2^2), & \xi_3 &= -7 + 5\chi_2. \end{aligned} \quad (3.1.13)$$

This Lagrangian represents one of the primary results of this paper. Since the first appearance of this work, an analogous Lagrangian (in a different gauge) has been derived by Bini, Damour and Faye in Ref. [52], using effective action techniques. In fact, BDF have greatly extended the analysis by carrying the calculation to 2PN order. Also (prior to this work), Damour and Nagar [94] presented a 1PN Hamiltonian valid for circular orbits. These works both incorporate their results into the EOB formalism. In Appendix 3.8, we demonstrate the complete equivalence of those results with ours at 1PN order.

The orbital EoM resulting from the Lagrangian (3.1.12), which can also be found by substituting the adiabatic solution (3.1.10) for $Q^{ij}(t)$ directly into the general orbital EoM (3.5.9), is given by Eq. (3.6.10). The conserved energy $E(\mathbf{z}, \mathbf{v})$ derived from the Lagrangian (3.1.12) is given by Eq. (3.6.13). In the case of circular orbits, we find the gauge-invariant energy-frequency relationship

$$E(\omega) = \mu(M\omega)^{2/3} \left[-\frac{1}{2} + \frac{9\chi_1}{2\chi_2} \frac{\lambda\omega^{10/3}}{M^{5/3}} + \frac{(9 + \eta)(M\omega)^{2/3}}{24c^2} + \frac{11\chi_1}{4\chi_2} (3 + 2\chi_2 + 3\chi_2^2) \frac{\lambda\omega^4}{Mc^2} \right]. \quad (3.1.14)$$

This result, along with others from this paper, are used in calculating the phasing of GW signals from inspiralling neutron star binaries in Ref. [95].

V Notation and Conventions

We use units where Newton's constant is $G = 1$, but retain factors of the speed of light c , with $1/c^2$ serving as the formal expansion parameter for the post-Newtonian expansion. We use lowercase Latin

letters a, b, i, j, \dots for indices of (three-dimensional) spatial tensors. Spatial indices are contracted with the Euclidean metric, $v^i w^i = \delta_{ij} v^i w^j$, with up or down placement of the indices having no meaning. We use uppercase Latin letters to denote multi-indices: L denotes the l indices $a_1 a_2 \dots a_l$, K denotes the k indices $b_1 b_2 \dots b_k$, etc. For a given vector v^i or for the partial derivative operator ∂_i , we use multi-indices or explicit sequences of indices to denote their tensorial powers:

$$\begin{aligned} v^L &= v^{a_1 a_2 \dots a_l} = v^{a_1} v^{a_2} \dots v^{a_l}, \\ \partial_K &= \partial_{b_1 b_2 \dots b_k} = \partial_{b_1} \partial_{b_2} \dots \partial_{b_k}, \end{aligned} \tag{3.1.15}$$

and, for example, $v^{ij} = v^i v^j$. We also use $v^2 = v^{ii}$ and $\nabla^2 = \partial_{ii}$. Multi-indices are also used for sets of distinct tensors of varying rank, $\{M, M^a, M^{ab}, \dots\}$, with $M^L = M^{a_1 a_2 \dots a_l}$ denoting the tensor of rank l . We use the Einstein summation convention for both individual indices and multi-indices. Derivatives with respect to a time coordinate t are denoted by ∂_t or by overdots.

We use angular brackets to denote the symmetric, trace-free (STF) projection of tensors [29]:

$$\begin{aligned} T^{<ab>} &= T^{(ab)} - \frac{1}{3} \delta^{ab} T^{cc}, \\ T^{<abc>} &= T^{(abc)} - \frac{3}{5} \delta^{(ab} T^{c)dd}, \end{aligned} \tag{3.1.16}$$

and so on, with parentheses denoting the symmetric projection. For a STF tensor $S^L = S^{<L>}$ and general tensor T^L , note that $S^L T^L = S^L T^{<L>}$.

3.2 Newtonian tidal interactions

In this section, we review the standard treatment of tidal coupling in Newtonian theory [93]; the first-post-Newtonian treatment given subsequent sections makes extensive use of these Newtonian-order results. We define the multipole moments and tidal moments of an extended object and use them to derive the orbital (or translational) equations of motion for systems of gravitating bodies. We consider in particular the case of a binary system containing a point particle (body 1) and an extended deformable star (body 2), working to quadrupolar order in the star's multipole series. We also discuss an action principle formulation of the orbital dynamics, the process of energy transfer between the gravitational field and the deformable body, and the evolution of the body's spin due to tidal torques. Finally, we discuss the evolution of the body's tidally induced quadrupole moment in the adiabatic limit.

I Field equations

In Newtonian physics, the scalar potential $\phi(t, \mathbf{x})$ obeys the Poisson equation,

$$\nabla^2 \phi = 4\pi\rho, \quad (3.2.1)$$

with $\rho(t, \mathbf{x})$ being the rest mass density of matter. The influence of the gravitational field on matter is described by the test particle acceleration $\ddot{x}^i = -\partial_i \phi$, or more generally, by Euler's equation supplemented by the continuity equation (the conservation of mass),

$$\partial_t(\rho v^i) + \partial_j(\rho v^i v^j + t^{ij}) = -\rho \partial_i \phi, \quad (3.2.2)$$

$$\dot{\rho} + \partial_i(\rho v^i) = 0, \quad (3.2.3)$$

with $v^i(t, \mathbf{x})$ being the matter's velocity field, and $t^{ij}(t, \mathbf{x})$ the material stress tensor. Together, Eqs. (3.2.1-3.2.3) provide a complete description of Newtonian gravitational interactions. However, they do not in general form a closed set of evolution equations for the fields ϕ , ρ , v^i , and t^{ij} ; one needs also to specify the matter's internal dynamics, in particular concerning the stress tensor t_{ij} . (In the simplest cases, one can fix t^{ij} by an algebraic equation, like $t^{ij} = 0$ for 'dust' or $t^{ij} = p\delta^{ij}$ with $p(\rho)$ being the pressure for an isentropic perfect fluid.) Still, one can derive many useful results, like the form of the translational equations of motion for a system of gravitating bodies, while leaving the matter's internal dynamics unspecified.

II Multipole moments

We consider N isolated celestial bodies, i.e. regions of space containing matter ($\rho \neq 0$) surrounded by regions of vacuum ($\rho = 0$), and label the bodies by an index A , with $1 \leq A \leq N$. The potential that is locally generated by body A , which we will call the internal potential ϕ_A^{int} , is given by the standard solution to (3.2.1) as an integral over the volume of the body:

$$\phi_A^{\text{int}}(t, \mathbf{x}) = - \int_A d^3x' \frac{\rho(t, \mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|}. \quad (3.2.4)$$

We can express this potential as a multipole series around a (moving) point $x^i = z_A^i(t)$ by using the Taylor series

$$\frac{1}{|\mathbf{x} - \mathbf{x}'|} = \sum_{l=0}^{\infty} \frac{(-1)^l}{l!} (x' - z_A)^L \partial_L \frac{1}{|\mathbf{x} - \mathbf{z}_A|}, \quad (3.2.5)$$

where $L = a_1 a_2 \dots a_l$ is a spatial multi-index, denoting here tensorial powers of the vector $(x' - z_A)^i$ and of the operator ∂_i (cf. Eq. (3.1.15)). Using the Taylor series (3.2.5) in (3.2.4) allows us to write the internal potential, for points x^i exterior to the body, in the form

$$\phi_A^{\text{int}}(t, \mathbf{x}) = - \sum_{l=0}^{\infty} \frac{(-1)^l}{l!} M_A^L(t) \partial_L \frac{1}{|\mathbf{x} - \mathbf{z}_A(t)|}, \quad (3.2.6)$$

with

$$M_A^L(t) = \int_A d^3x \rho(t, \mathbf{x}) [x - z_A(t)]^{<L>}. \quad (3.2.7)$$

The quantities M_A^L are the mass multipole moments of the body about the worldline $z_A^i(t)$. They are symmetric, trace-free spatial tensors of varying order l , $M_A^L = M_A^{a_1 a_2 \dots a_l}$. The STF property follows from the fact that $\partial_L |\mathbf{x}|^{-1}$ is an STF tensor, because partial derivatives commute, and because $\partial_{jj} |\mathbf{x}|^{-1} = \nabla^2 |\mathbf{x}|^{-1} = 0$. As the multipole moments M_A^L are contracted with the STF tensors $\partial_L |\mathbf{x} - \mathbf{z}^A|^{-1}$ in (3.2.6), only their STF parts will contribute to the potential ϕ_A^{int} ; this is why the STF projection (denoted by angular brackets) has been included in the definition of the multipole moments in (3.2.7).

The leading-order terms in the multipole series (3.2.6) can be written more explicitly as

$$\phi_A^{\text{int}} = - \frac{M_A}{|\bar{\mathbf{x}}|} - \frac{1}{2} Q_A^{ij} \partial_{ij} \frac{1}{|\bar{\mathbf{x}}|} + \dots \quad (3.2.8)$$

with

$$M_A = \int_A d^3x \rho, \quad (3.2.9a)$$

$$0 = M_A^i = \int_A d^3x \rho \bar{x}^i, \quad (3.2.9b)$$

$$Q_A^{ij} = M_A^{ij} = \int_A d^3x \rho \left(\bar{x}^i \bar{x}^j - \frac{1}{3} |\bar{\mathbf{x}}|^2 \delta^{ij} \right), \quad (3.2.9c)$$

and $\bar{x}^i = x^i - z_A^i$. First is the monopole term ($l = 0$), generated by the total mass of the body M_A (3.2.9a), and giving rise to a Coulomb-type potential in (3.2.8). We have omitted the dipole term ($l = 1$) in (3.2.8) because M_A^i can always be made to vanish by choosing the point $z_A^i(t)$ about which the multipole expansion is centered to be fixed to the body's center of mass position:

$$z_A^i(t) = \frac{1}{M_A} \int d^3x \rho(t, \mathbf{x}) x^i, \quad (3.2.10)$$

which is equivalent to (3.2.9b). Next comes the $l = 2$ term involving the body's mass quadrupole tensor $Q_A^{ij}(t)$ (3.2.9c), which we have renamed $M_A^{ij} \rightarrow Q_A^{ij}$ to accord with common convention. The

higher-order terms in the multipole series are suppressed by increasing powers of the finite size parameter $R/|\bar{\mathbf{x}}|$, where R is the size of the body and $|\bar{\mathbf{x}}|$ is a typical separation.

We note that the continuity equation (3.2.3) implies the constancy of the body's total mass, $\dot{M}_A = 0$. Similarly, the Euler equation (3.2.2) and the vanishing of the mass dipole (3.2.9b) determine the translational equation of motion for the body's worldline $\mathbf{z}_A(t)$, as discussed in Sec. 3.2.IV below. Equations (3.2.2) and (3.2.3) do not, however, fully determine the evolution of the quadrupole and higher multipoles, which will depend on the body's internal dynamics.

III Tidal moments

Having described the potential generated by an isolated body with its multipole moments, we can similarly describe the potential felt by the body with tidal moments. Given a collection of several bodies indexed by B , each giving rise its own intrinsic potential of the form (3.2.6), we define the external potential felt by a given body A to be the sum of the potentials due to all the other bodies:

$$\phi_A^{\text{ext}} = \sum_{B \neq A} \phi_B^{\text{int}} \quad (3.2.11)$$

The body's tidal moments¹ $G_{\text{g},A}^L(t)$ are then defined as coefficients in the Taylor expansion of the external potential about the center-of-mass position z_A^i :

$$\phi_A^{\text{ext}}(t, \mathbf{x}) = - \sum_{l=0}^{\infty} \frac{1}{l!} G_{\text{g},A}^L(t) [x - z_A(t)]^L, \quad (3.2.12)$$

$$G_{\text{g},A}^L(t) = - \partial_L \phi_A^{\text{ext}}(t, \mathbf{x}) \Big|_{\mathbf{x}=\mathbf{z}_A(t)}. \quad (3.2.13)$$

Like the multipole moments, the tidal moments are STF tensors, $G_{\text{g},A}^L = G_{\text{g},A}^{<L>}$, as can be seen from the definition (3.2.13) and the fact that $\nabla^2 \phi_A^{\text{ext}} = 0$ everywhere outside the bodies $B \neq A$. We see that $G_{\text{g},A}$ is simply (minus) the potential at the body's center, and $G_{\text{g},A}^i$ is the would-be test particle acceleration $-\partial_i \phi_A^{\text{ext}}$. For $l \geq 2$, the $G_{\text{g},A}^L$ are higher-order derivatives of the potential that will give rise to tidal forces on a non-spherical body.

¹ The subscript g, standing for 'global,' has been included here to avoid confusion with tidal moments introduced in our post-Newtonian treatment below. We introduce there a set of body-frame tidal moments G_A^L , defined in an accelerated reference frame attached to the body, and a set of global-frame tidal moments $G_{\text{g},A}^L$, defined in an (asymptotically) inertial frame. The tidal moments defined in (3.2.13) coincide with the latter at Newtonian order. While we have chosen to work exclusively in an inertial frame in our Newtonian treatment here, an analogous Newtonian treatment that uses accelerated frames can be found in Sec. III of Ref. [33].

The tidal moments of a body A can be expressed in terms of the multipole moments of the other bodies $B \neq A$ by combining (3.2.6), (3.2.11), and (3.2.13), with the result

$$\begin{aligned} G_{g,A}^L &= - \sum_{B \neq A} \partial_L \phi_B^{\text{int}}(t, \mathbf{x}) \Big|_{\mathbf{x}=\mathbf{z}_A(t)} \\ &= \sum_{B \neq A} \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} M_B^K \partial_{KL}^{(A)} \frac{1}{|\mathbf{z}_A - \mathbf{z}_B|}, \end{aligned} \quad (3.2.14)$$

where $\partial_i^{(A)} = \partial/\partial z_A^i$, and $\partial_{KL} = \partial_{b_1} \dots \partial_{b_k} \partial_{a_1} \dots \partial_{a_l}$.

IV Translational equations of motion

A primary advantage of the language of multipole and tidal moments is that it allows one to take the PDEs (3.2.1-3.2.3) governing the evolution of the fields ρ , v^i , t^{ij} , and ϕ and extract from them ODEs for the center-of-mass worldlines $z_A^i(t)$ of a collection of gravitating bodies. To this end, we consider a body A with multipole moments M_A^L defined by (3.2.7), in the presence of an external potential ϕ_A^{ext} generated by other bodies $B \neq A$ according to (3.2.12) and (3.2.14). The body's translational EoM can be found by applying two time derivatives to the definition of z_A^i in (3.2.10), using the Euler and continuity equations (3.2.3) and (3.2.2), and integrating by parts. The result is an expression for the body's center-of-mass acceleration,

$$M_A \ddot{z}_A^i = - \int d^3x \rho \partial_i \phi_A^{\text{ext}}, \quad (3.2.15)$$

which can be rewritten in terms of the body's multipole and tidal moments by using (3.2.12) and (3.2.7):

$$\begin{aligned} M_A \ddot{z}_A^i &= \sum_{l=0}^{\infty} \frac{1}{l!} M_A^L G_{g,A}^{iL} \\ &= M_A G_{g,A}^i + \frac{1}{2} Q_A^{jk} G_{g,A}^{ijk} + \dots \end{aligned} \quad (3.2.16)$$

The first term in the second line represents the force that would act on a freely falling test mass M_A , while the second term gives the leading-order tidal force.

To render the EoM (3.2.16) fully explicit, one can use the expressions for the tidal moments (3.2.14) in terms of the multipole moments and worldlines of the other bodies. Considering the case of a two-body system $A = 1, 2$, with body 1 having only a monopole moment M_1 , and body 2

having a monopole M_2 and a quadrupole $Q^{ij} \equiv Q_2^{ij}$, we find the following EoMs:

$$\begin{aligned} M_1 \ddot{z}_1^i &= M_1 M_2 \partial_i^{(1)} \frac{1}{|\mathbf{z}_1 - \mathbf{z}_2|} + \frac{1}{2} M_1 Q^{jk} \partial_{ijk}^{(1)} \frac{1}{|\mathbf{z}_1 - \mathbf{z}_2|} \\ M_2 \ddot{z}_2^i &= M_2 M_1 \partial_i^{(2)} \frac{1}{|\mathbf{z}_2 - \mathbf{z}_1|} + \frac{1}{2} Q^{jk} M_1 \partial_{ijk}^{(2)} \frac{1}{|\mathbf{z}_2 - \mathbf{z}_1|} \end{aligned}$$

Defining the radius and unit vector associated with the relative position,

$$z^i = z_2^i - z_1^i, \quad r = |\mathbf{z}|, \quad n^i = z^i / r, \quad (3.2.17)$$

and using the general identity,

$$\partial_L \frac{1}{r} = (-1)^l (2l - 1)!! \frac{n^{<L>}}{r^{l+1}}, \quad (3.2.18)$$

these EoMs can be written as

$$\begin{aligned} M_1 \ddot{z}_1^i &= -M_2 \ddot{z}_2^i \\ &= \frac{M_1 M_2}{r^2} n^i + \frac{15 M_1}{2 r^4} Q^{jk} n^{<ijk>} \end{aligned} \quad (3.2.19)$$

In this form, it is evident that the total momentum $p^i = M_1 \dot{z}_1^i + M_2 \dot{z}_2^i$ is conserved, and that these two EoMs for z_1^i and z_2^i can be traded for the EoM of the relative position $z^i = z_2^i - z_1^i$:

$$\begin{aligned} \ddot{z}^i &= -\frac{M}{r^2} n^i - \frac{15M}{2M_2 r^4} Q^{jk} n^{<ijk>} \\ &= -\frac{M}{r^2} n^i - \frac{3M}{2M_2 r^4} Q^{jk} (5n^{ijk} - 2\delta^{ij} n^k) \end{aligned} \quad (3.2.20)$$

where $M = M_1 + M_2$ is the total mass. In the second line, we have used Eq. (3.1.16) to expand $n^{<ijk>}$ and the fact that Q^{jk} is STF.

With Eq. (3.2.20), we have reduced the description of the binary system's translational dynamics to a single ODE. Still, we can only solve this ODE if we know the time evolution of the quadrupole moment $Q^{ij}(t)$, which requires a detailed model of body 2's interior. We will describe a simple adiabatic model for Q^{ij} in Sec. 3.2.VII below.

V Action principle

As for any Newtonian system, a Lagrangian for a collection of gravitating bodies can be constructed from $\mathcal{L} = T - U$, with T being the kinetic energy and U the potential energy. The total kinetic

energy receives separate contributions from each body, $T = \sum_A T_A$, and each T_A can be split into a contribution from the center-of-mass motion of the body and an internal contribution:

$$T_A = \frac{1}{2} \int_A d^3x \rho v^2 = \frac{1}{2} M_A \dot{z}_A^2 + T_A^{\text{int}}, \quad (3.2.21)$$

$$T_A^{\text{int}} = \frac{1}{2} \int_A d^3x \rho (v - \dot{z}_A)^2. \quad (3.2.22)$$

Here, we have used the fact that $\int_A d^3x \rho v^i = M_A \dot{z}_A^i$, implied by the continuity equation (3.2.3) and the definition of the center-of-mass position z_A^i in (3.2.10). The gravitational potential energy $U = \sum_A U_A$ can be similarly split into external and internal parts:

$$U_A = \frac{1}{2} \int_A d^3x \rho \phi = U_A^{\text{ext}} + U_A^{\text{int}}, \quad (3.2.23)$$

$$U_A^{\text{int}} = \frac{1}{2} \int_A d^3x \rho \phi_A^{\text{int}}, \quad (3.2.24)$$

$$U_A^{\text{ext}} = \frac{1}{2} \int_A d^3x \rho \phi_A^{\text{ext}} = -\frac{1}{2} \sum_{l=0}^{\infty} \frac{1}{l!} M_A^L G_{g,A}^L. \quad (3.2.25)$$

In the last line, we have used (3.2.12) to express ϕ_A^{ext} in terms of the tidal moments and (3.2.7) for the definition of the multipole moments.

While the system's total potential energy will also receive non-gravitational contributions from the internal structure of each body, we can lump these contributions, along with T_A^{int} and U_A^{int} as defined above, into an internal Lagrangian $\mathcal{L}_A^{\text{int}}$ for each body. We can then write the total Lagrangian for an N -body system as

$$\mathcal{L} = \sum_A \left(\frac{1}{2} M_A \dot{z}_A^2 + \frac{1}{2} \sum_{l=0}^{\infty} \frac{1}{l!} M_A^L G_{g,A}^L + \mathcal{L}_A^{\text{int}} \right). \quad (3.2.26)$$

The bodies' center-of-mass worldlines $z_A^i(t)$ enter this Lagrangian through the translational kinetic energy terms and through the external gravitational potential energy terms (via the tidal moments); the internal Lagrangians $\mathcal{L}_A^{\text{int}}$, however, are independent of the worldlines z_A^i , by construction. The $\mathcal{L}_A^{\text{int}}$ will be functions of some set of internal configuration variables q_A^α (and their time derivatives) for each body, which will include e.g. Euler angles for the orientation of the body, vibrational mode amplitudes, etc., depending on the model of the body's internal structure. (In full generality, the proper internal configuration variables q_A^α are the fields ρ , v^i and t^{ij} , subject to (3.2.10) as a constraint.) The bodies' multipole moments M_A^L , for $l \geq 2$, appearing in the gravitational potential energy terms in (3.2.26), will be functions of these same internal variables q_A^α . Together, the z_A^i and

q_A^α , for all bodies A , form a complete set of dynamical variables for the N -body system. Varying the action $S = \int \mathcal{L} dt$ with respect to the worldlines z_A^i reproduces the translational EoMs (3.2.16). Determining the evolution of the variables q_A^α , and hence the moments M_A^L for $l \geq 2$, will require a model for $\mathcal{L}_A^{\text{int}}(q_A^\alpha, \dot{q}_A^\alpha)$ and $M_A^L(q_A^\alpha)$.

Specializing to the two-body M_1 - M_2 - Q_2 case and using (3.2.14), (3.2.17), and (3.2.18), the Lagrangian (3.2.26) becomes

$$\mathcal{L} = \frac{1}{2}M_1\dot{z}_1^2 + \frac{1}{2}M_2\dot{z}_2^2 + \frac{M_1M_2}{r} - U_Q + \mathcal{L}_2^{\text{int}}. \quad (3.2.27)$$

where U_Q is the potential energy of the quadrupole-tidal interaction:

$$U_Q = -\frac{1}{2}Q^{ij}G_{\text{g},2}^{ij}, \quad G_{\text{g},2}^{ij} = \frac{3M_1}{r^3}n^{<ij>}. \quad (3.2.28)$$

(We have omitted the internal Lagrangian for body 1 as it is completely decoupled from the rest of the system.) Varying this action with respect to z_1^i and z_2^i leads to their EoMs found in (3.2.19). Alternately, varying with respect to their separation $z^i = z_2^i - z_1^i$ gives the relative EoM (3.2.20), while the system's center of mass $(M_1z_1^i + M_2z_2^i)/M$ is found to be cyclic. Specializing the Lagrangian (3.2.27) to the center-of-mass frame, where $M_1z_1^i + M_2z_2^i = 0$, gives

$$\mathcal{L} = \frac{\mu\dot{z}^2}{2} + \frac{\mu M}{r} - U_Q + \mathcal{L}_2^{\text{int}}, \quad (3.2.29)$$

with $\mu = M_1M_2/M$ being the reduced mass.

While leaving the functions $\mathcal{L}_2^{\text{int}}(q_2^\alpha, \dot{q}_2^\alpha)$ and $Q^{ij}(q_2^\alpha)$ unspecified, we can still use the Lagrangian (3.2.27) to write down EoMs for body 2's internal configuration variables q_2^α :

$$\frac{d}{dt} \frac{\partial \mathcal{L}_2^{\text{int}}}{\partial \dot{q}_2^\alpha} = \frac{\partial \mathcal{L}_2^{\text{int}}}{\partial q_2^\alpha} + \frac{1}{2}G_{\text{g},2}^{ij} \frac{\partial Q^{ij}}{\partial q_2^\alpha}. \quad (3.2.30)$$

which will be useful in the next subsection.

VI Energy

Continuing to specialize to the two-body M_1 - M_2 - Q_2 case, we can construct a conserved energy for the system from

$$E = T + U = \frac{\partial \mathcal{L}}{\partial \dot{z}_1^i} \dot{z}_1^i + \frac{\partial \mathcal{L}}{\partial \dot{z}_2^i} \dot{z}_2^i + \frac{\partial \mathcal{L}}{\partial \dot{q}_2^\alpha} \dot{q}_2^\alpha - \mathcal{L}, \quad (3.2.31)$$

with summation over α implied. Using the (CoM-frame) Lagrangian (3.2.29), we find

$$E = \frac{\mu \dot{z}^2}{2} - \frac{\mu M}{r} + U_Q + E_2^{\text{int}}, \quad (3.2.32)$$

where U_Q is given by (3.2.28), and the internal energy of body 2 is given by

$$E_2^{\text{int}}(q_2^\alpha, \dot{q}_2^\alpha) = \frac{\partial \mathcal{L}_2^{\text{int}}}{\partial \dot{q}_2^\alpha} \dot{q}_2^\alpha - \mathcal{L}_2^{\text{int}} \quad (3.2.33)$$

This internal energy will generally have several contributions: internal gravitational potential energy, rotational kinetic energy, vibrational kinetic and potential energy, thermal energy, etc. Nonetheless, the rate at which energy is exchanged between the interior of body 2 and its surroundings, via the gravitational tidal interaction, is a function only of the orbital separation, M_1 , and Q^{ij} . Differentiating (3.2.33) with respect to time and using the EoM (3.2.30) for the internal variables q_2^α , we find

$$\begin{aligned} \dot{E}_2^{\text{int}} &= \dot{q}_2^\alpha \left(\frac{d}{dt} \frac{\partial \mathcal{L}_2^{\text{int}}}{\partial \dot{q}_2^\alpha} - \frac{\partial \mathcal{L}_2^{\text{int}}}{\partial q_2^\alpha} \right) = \frac{1}{2} G_{g,2}^{ij} \frac{\partial Q^{ij}}{\partial q_2^\alpha} \dot{q}_2^\alpha \\ &= \frac{1}{2} G_{g,2}^{ij} \dot{Q}^{ij} = \frac{3M_1}{2r^3} n^{ij} \dot{Q}^{ij} \end{aligned} \quad (3.2.34)$$

The energy transfer described by (3.2.34) is often referred to as tidal heating (see e.g. [107]). This expression for the power delivered to the body is valid (in the quadrupolar approximation) regardless of the body's internal dynamics. Using (3.2.34) and the orbital EoM (3.2.20), one can confirm that the binary system's total energy (3.2.32) is conserved.

VII Adiabatic approximation

While we have thus far left unspecified the internal dynamics for the deformable body 2, which determines the evolution of the quadrupole, we now specialize our analysis to the case where $Q^{ij}(t)$ is adiabatically induced by the tidal field. This will lead to a closed system of evolution equations for the binary. In the adiabatic limit, when the body's internal dynamical time scales are much less than the orbital period, the quadrupole will respond to the instantaneous tidal field according to

$$Q^{ij}(t) = \lambda G_{g,2}^{ij}(t), \quad (3.2.35)$$

where λ is the tidal deformability, and $G_{g,2}^{ab}(t)$ is the tidal tensor given in Eq. (3.2.28). As discussed in Sec. 3.1.II and in more detail in Refs. [91, 35], the relation (3.2.35) should be valid to $\sim 1\%$ for neutron star binaries at GW frequencies $\lesssim 400$ Hz.

The adiabatic evolution of the quadrupole can be incorporated into our action principle (3.2.29) by taking $Q^{ij}(t)$ to be our lone internal configuration variable (q_2^α), and by taking the internal Lagrangian $\mathcal{L}_2^{\text{int}}$ to contain a simple quadratic potential energy cost for the quadrupole:

$$\begin{aligned}\mathcal{L}[z^i, Q^{ij}] &= \frac{\mu \dot{z}^2}{2} + \frac{\mu M}{r} - U_Q + \mathcal{L}_2^{\text{int}} \\ &= \frac{\mu \dot{z}^2}{2} + \frac{\mu M}{r} + \frac{1}{2} G_{\text{g},2}^{ab} Q^{ab} - \frac{1}{4\lambda} Q^{ab} Q^{ab}.\end{aligned}\quad (3.2.36)$$

Varying the action with respect to the orbital separation z^i still gives the orbital EoM (3.2.20), and varying with respect to the quadrupole Q^{ij} reproduces the adiabatic evolution equation (3.2.35). Using Eq. (3.2.35) to replace Q^{ij} and Eq. (3.2.28) for $G_{\text{g},2}^{ab}$, the Lagrangian can be written solely in terms of z^i as

$$\mathcal{L}[z^i] = \frac{\mu \dot{z}^2}{2} + \frac{\mu M}{r} \left(1 + \frac{3\lambda M_1}{2r^5 M_2} \right).\quad (3.2.37)$$

This Lagrangian leads to the orbital EoM (3.2.20) with Q^{ij} replaced by its adiabatic value (3.2.35),

$$a^i = -\frac{M n^i}{r^2} \left(1 + \frac{9\lambda M_1}{r^5 M_2} \right),\quad (3.2.38)$$

which shows that the tidal coupling results in an attractive force. For circular orbits, with $a^i = -r\omega^2 n^i$, we find the radius-frequency relationship

$$r(\omega) = \frac{M^{1/3}}{\omega^{2/3}} \left(1 - \frac{3\lambda M_1 \omega^{10/3}}{M_2 M^{5/3}} \right),\quad (3.2.39)$$

to linear order in the tidal deformability λ .

The internal energy E_2^{int} (3.2.33), in the adiabatic approximation, is given by

$$E_2^{\text{int}} = \frac{1}{4\lambda} Q^{ab} Q^{ab},\quad (3.2.40)$$

up to a constant, and satisfies the tidal heating equation (3.2.34) by virtue of Eq. (3.2.35). (We should note that there is actually no ‘heating’ taking place here, as this model neglects dissipative effects and is completely conservative.) Using Eq. (3.2.35), the binary system’s total energy E (3.2.32) can be written as

$$E = \frac{\mu \dot{z}^2}{2} - \frac{\mu M}{r} \left(1 + \frac{3\lambda M_1}{2r^5 M_2} \right),\quad (3.2.41)$$

in the adiabatic model, and is conserved by the orbital EoM (3.2.38). Then, using $\dot{z}^2 = r^2 \omega^2$ and Eq. (3.2.39), we find

$$E(\omega) = -\frac{\mu}{2} (M\omega)^{2/3} \left(1 - \frac{9\lambda M_1 \omega^{10/3}}{M_2 M^{5/3}} \right),\quad (3.2.42)$$

for the circular-orbit energy-frequency relationship.

VIII Spin

In anticipation of the 1PN treatment of the binary's orbital dynamics, in which a body's angular momentum (or spin) has a direct influence on the orbit, it will be useful to discuss the evolution of an extended body's spin at Newtonian order. The spin of a body A about its CoM worldline z_A^i is defined by

$$S_A^a(t) = \epsilon^{abc} \int d^3x \rho(t, \mathbf{x}) [x^b - z_A^b(t)] v^c(t, \mathbf{x}), \quad (3.2.43)$$

with ρ being the mass density and v^c the velocity field. Taking a time derivative of this equation, using the Euler and continuity equations (3.2.2) and (3.2.3) and the definition of z_A^i in (3.2.10), and integrating by parts, we find

$$\begin{aligned} \dot{S}_A^a &= -\epsilon^{abc} \int d^3x \rho (x^b - z_A^b) \partial_c \phi_A^{\text{ext}} \\ &= \epsilon^{abc} \sum_{l=0}^{\infty} \frac{1}{l!} M_A^{bL} G_{g,A}^{cL}. \end{aligned} \quad (3.2.44)$$

In the second line, we have used the definitions of M_A^L and $G_{g,A}^L$ in (3.2.7) and (3.2.13). This formula gives the torque on the body due to tidal forces. As it is not directly relevant to our purposes, we will not discuss a Lagrangian formulation of the Newtonian rotational dynamics.

Applying Eq. (3.2.44) to the M_1 - M_2 - Q_2 system, we find that the tidal torque on body 2 is given by

$$\dot{S}_2^a = \epsilon^{abc} Q^{bd} G_{g,2}^{cd} \quad (3.2.45)$$

with the tidal tensor $G_{g,2}^{cd}$ given by (3.2.28). Eq. (3.2.45) is valid (in the quadrupolar approximation) regardless of the internal dynamics of body 2. In the special case of an adiabatically induced quadrupole, as in Eq. (3.2.35), we find

$$\dot{S}^a = \lambda \epsilon^{abc} G_{g,2}^{bd} G_{g,2}^{cd} = 0, \quad (3.2.46)$$

so that the spin is conserved.

3.3 Post-Newtonian tidal interactions

I Overview

The Newtonian theory of gravity arises as a limiting case of general relativity (GR). In the limit of small source velocities and weak gravity, the spacetime metric of GR takes the form

$$ds^2 = - \left(1 + \frac{2\phi}{c^2} \right) c^2 dt^2 + \delta^{ij} dx^i dx^j + O(c^{-2}), \quad (3.3.1)$$

with $\phi(t, \mathbf{x})$ being the Newtonian potential. This expression represents a perturbation expansion of the theory with $1/c^2$ playing the role of a formal expansion parameter. At leading order in $1/c^2$, Einstein's equation and covariant stress-energy conservation for the metric (3.3.1) reproduce the Poisson, Euler, and continuity equations (3.2.1-3.2.3)—the basic equations of Newtonian gravity.

The first post-Newtonian (1PN) approximation to GR continues this perturbation expansion to next-to-leading order in $1/c^2$. The 1PN metric can be written as

$$ds^2 = - \left[1 + \frac{2\phi}{c^2} + \frac{2}{c^4}(\phi^2 + \psi) \right] c^2 dt^2 + \frac{2\zeta^i}{c^2} dt dx^i + \left(1 - \frac{2\phi}{c^2} \right) \delta^{ij} dx^i dx^j + O(c^{-4}), \quad (3.3.2)$$

(cf. Weinberg [41]), with two new degrees of freedom appearing: a 1PN (three-)vector potential $\zeta^i(t, \mathbf{x})$ (often called the gravito-magnetic potential), and a 1PN scalar potential $\psi(t, \mathbf{x})$. Following Refs. [49, 29] (apart from a change of sign) we shall work with the single scalar potential single scalar potential $\Phi(t, \mathbf{x})$ which has ϕ and ψ as its Newtonian- and 1PN-order parts (and hence a hidden c -dependence),

$$\Phi = \phi + c^{-2}\psi + O(c^{-4}), \quad (3.3.3)$$

so that the metric can be written as

$$ds^2 = - \left[1 + \frac{2\Phi}{c^2} + \frac{2\Phi^2}{c^4} \right] c^2 dt^2 + \frac{2\zeta^i}{c^2} dt dx^i + \left(1 - \frac{2\Phi}{c^2} \right) \delta^{ij} dx^i dx^j + O(c^{-4}). \quad (3.3.4)$$

We choose to work here in conformally Cartesian coordinates (see e.g. [29]), which is already implicit in the form (3.3.4) of the metric, and to adopt the harmonic gauge condition:

$$\partial_\mu(\sqrt{-g}g^{\mu\nu}) = 0 \quad \Leftrightarrow \quad 4\dot{\Phi} + \partial_i \zeta^i = O(c^{-2}). \quad (3.3.5)$$

In this gauge, one finds that Einstein's equation for the metric, at next-to-leading order in $1/c^2$, is equivalent to the following linear field equations for the potentials:

$$\nabla^2\Phi = 4\pi T^{tt} + c^{-2} \left(4\pi T^{ii} + \ddot{\Phi} \right) + O(c^{-4}), \quad (3.3.6a)$$

$$\nabla^2\zeta^i = 16\pi T^{ti} + O(c^{-2}), \quad (3.3.6b)$$

where $T^{\mu\nu}$ are the contravariant components of the stress-energy tensor in the (t, x^i) coordinate system. Note that the T^{tt} component must include both $O(c^0)$ Newtonian and $O(c^{-2})$ post-Newtonian contributions, while the components T^{ti} and T^{ij} and the quantity $\ddot{\Phi}$ are needed only to Newtonian order. The influence of the gravitational field on matter is governed by covariant stress-energy conservation: $\nabla_\mu T^{\mu\nu} = 0$; the 1PN-expanded form of this equation can be found in Appendix D of RF [31].

In the remainder of this section, we review the treatment of tidal interactions within this first-post-Newtonian framework. We employ a formalism, originally developed by DSX [29, 33] and later expounded upon by RF [31], that uses multiple coordinate systems to describe the global motion and local structure of extended bodies. We attempt to present here the primary ingredients and broad logical flow of this formalism, which are essential for properly interpreting the results stated in Sec. 3.1.IV above.

We begin in Sec. 3.3.II by presenting a general solution to the 1PN-order Einstein equations (3.3.6) which gives the spacetime metric in a vacuum region surrounding an astronomical body A . The solution (3.3.8) is parametrized by and defines the body's multipole moments and tidal moments. The mass and current multipole moments M_A^L and S_A^L characterize the body's internal structure, and the gravito-electric and -magnetic tidal moments G_A^L and H_A^L characterize the external gravitational fields felt by the body.

In Sec. 3.3.III we discuss the gauge freedom in the 1PN metric, summarized by the parametrization of a general 1PN coordinate transformation in Eq. (3.3.13). In 1PN celestial mechanics, it is advantageous to use and transform between two types of coordinate systems: global coordinates (t, x^i) used to describe the motion of multiple bodies, and body-adapted coordinates (s_A, y_A^i) used in the local description of a given body A . We discuss how to fix all 1PN coordinate freedom in the body-adapted coordinates (s_A, y_A^i) by enforcing the body-frame gauge conditions (3.3.15). The body-frame multipole moments $M_A^L(s^A)$ and $S_A^L(s^A)$ —the moments defined by the multipole

expansions in Sec. 3.3.II using the body-adapted coordinates—then become unique and meaningful descriptors of a body’s internal structure.

In Sec. 3.3.IV, we discuss the form of the metric in the global coordinates (t, x^i) . It is written in terms of a set of global-frame multipole moments $M_{g,A}^L(t)$ and $Z_{g,A}^{iL}(t)$ and tidal moments $G_{g,A}^L(t)$ and $Y_{g,A}^{iL}(t)$ for each body A , which differ from the body-frame moments. The relationship between the global- and body-frame moments is determined by the transformation (3.3.13) between the global and body-adapted coordinates; the moment transformation formulae are presented in full detail in Appendix 3.9. The functions parameterizing the coordinate transformation, or the worldline data \mathcal{D}_A (3.3.14), are seen to take on the role of configuration variables for body A with respect to the global frame. Among other things, they determine the body’s center-of-mass worldline, $x^i = z_A^i(t)$.

In Sec. 3.3.V, we discuss the 1PN single-body laws of motion (3.3.22) which govern the evolution of a body’s mass monopole M_A , mass dipole M_A^i , and current dipole (or spin) S_A^i . These laws reflect the conservation of energy, momentum, and angular momentum and can be derived from stress-energy conservation at 1PN order [33], or equivalently, from Einstein’s equation at 2PN order [31]. A body’s translational equation of motion—an ODE for its global-frame center-of-mass worldline $z_A^i(t)$ —can be deduced from the law of motion for its body-frame mass dipole. The result is an expression for the acceleration \ddot{z}_A^i , for each member A of an N -body system, written in terms of the body-frame multipole moments M_A^L and S_A^L and global-frame worldlines z_A^i of all the bodies A .

Finally, in Sec. 3.3.VI, we specialize our discussion to the case of a two-body system with a body 1 having only a mass monopole M_1 , and a body 2 having a mass monopole M_2 , a mass quadrupole $Q_2^{ij} \equiv Q^{ij}$, and a spin $S_2^i \equiv S^i$. We present and discuss the explicit forms of the evolution equations for the moments M_1 , M_2 , and S^i and the worldlines z_1^i and z_2^i , which depend only on these quantities and Q^{ij} .

II 1PN multipole and tidal moments

In Sec. 3.2, we defined the Newtonian mass multipole moments ${}^nM_A^L$ (called simply M_A^L there) as integrals over the body’s mass distribution (3.2.7). In so doing, we implicitly assumed that the Newtonian Poisson equation (3.2.1) was valid in all space, including the interior of the body. A

similar approach can be taken at 1PN order, defining 1PN-accurate multipole moments as integrals over a body's stress-energy distribution (as in (3.3.12) below), assuming that the 1PN field equations (3.3.6) are valid in all space. This was the approach taken in the original DSX formalism.

As stressed by RF, one can also define a body's 1PN multipole moments without requiring the validity of the 1PN field equations in the interior of the body. Instead, one need only impose the field equations in a vacuum buffer region \mathcal{B}_A , a region of finite extent enclosed between two coordinate spheres centered on the body; the moments can then be defined through the multipole expansion of the 1PN metric in the region \mathcal{B}_A . This allows one to consider objects with strong internal gravity, like neutron stars and black holes, as long as there exists a region \mathcal{B}_A exterior to the object where gravity is sufficiently weak and quasi-static for the 1PN field equations to be valid.

Taking the latter approach, we assume the existence of a local coordinate system (s_A, y_A^i) , in the vicinity of the body A , having the following properties: (i) The range of the coordinates includes the product of the open ball $|\mathbf{y}_A| < r_2$, for some finite radius r_2 , with an open interval of time (s_A^1, s_A^2) . (ii) There exists a spatial region \mathcal{W}_A (the worldtube) of the form $|\mathbf{y}_A| < r_1$ that contains all the body's stress-energy and/or regions of strong gravity. (iii) In the buffer region \mathcal{B}_A ($r_1 < |\mathbf{y}_A| < r_2$), the coordinates (s_A, y_A^i) are conformally Cartesian and harmonic, and the metric takes the 1PN form (3.3.4), with potentials $\Phi_A(s_A, \mathbf{y}_A)$ and $\zeta_A^i(s_A, \mathbf{y}_A)$ satisfying the 1PN vacuum field equations:

$$\begin{aligned}\nabla^2 \Phi_A &= c^{-2} \ddot{\Phi} + O(c^{-4}), \\ \nabla^2 \zeta_A^i &= O(c^{-2}).\end{aligned}\tag{3.3.7}$$

Under these assumptions, RF showed that the general solution for the potentials in \mathcal{B}_A is of the form

$$\begin{aligned}\Phi_A(s_A, \mathbf{y}_A) &= - \sum_{l=0}^{\infty} \frac{1}{l!} \left\{ (-1)^l M_A^L(s_A) \partial_L \frac{1}{|\mathbf{y}_A|} + G_A^L(s_A) y_A^L \right. \\ &\quad + \frac{1}{c^2} \left[\frac{(-1)^l (2l+1)}{(l+1)(2l+3)} \dot{\mu}_A^L(s_A) \partial_L \frac{1}{|\mathbf{y}_A|} + \frac{(-1)^l}{2} \ddot{M}_A^L(s_A) \partial_L |\mathbf{y}_A| \right. \\ &\quad \left. \left. - \dot{\nu}_A^L(s_A) y_A^L + \frac{1}{2(2l+3)} \ddot{G}_A^L(s_A) y_A^{jjL} \right] \right\} + O(c^{-4}),\end{aligned}\tag{3.3.8a}$$

$$\zeta_A^i(s_A, \mathbf{y}_A) = - \sum_{l=0}^{\infty} \frac{1}{l!} \left\{ (-1)^l Z_A^{iL}(s_A) \partial_L \frac{1}{|\mathbf{y}_A|} + Y_A^{iL}(s_A) y_A^L \right\} + O(c^{-4}),\tag{3.3.8b}$$

with

$$Z_A^{iL}(s_A) = \frac{4}{l+1} \dot{M}_A^{iL}(s_A) - \frac{4l}{l+1} \epsilon^{ji < a_l} S_A^{L-1 > j}(s_A) + \frac{2l-1}{2l+1} \delta^{i < a_l} \mu_A^{L-1 >}(s_A) + O(c^{-4}), \quad (3.3.9a)$$

$$Y_A^{iL}(s_A) = \nu_A^{iL}(s_A) + \frac{l}{l+1} \epsilon^{ji < a_l} H_A^{L-1 > j}(s_A) - \frac{4(2l-1)}{2l+1} \dot{G}_A^{< L-1} \delta^{a_l > i}(s_A) + O(c^{-4}). \quad (3.3.9b)$$

The potentials are parametrized by the following sets of (multi-index) spatial tensors, which are STF on all their indices, and which are functions only of the time coordinate s_A . First, $M_A^L(s_A)$, with $l \geq 0$, are the body's mass multipole moments, which are defined with 1PN-accuracy. Next are the current multipole moments, $S_A^L(s_A)$, with $l \geq 1$, needed only to Newtonian accuracy. Together, the mass and current multipole moments contain all the information about the body's internal structure that is encoded in the gravitational field it produces (at 1PN order). They are associated with the contributions to the potentials that appear to diverge as $|\mathbf{y}_A| \rightarrow 0$, which can be referred to as the internal contributions. Also associated with such parts of the potentials are the internal gauge moments μ_A^L ($l \geq 0$), so called because they contain no gauge-invariant information about the body.

Associated with the parts of the the potentials that appear to diverge as $|\mathbf{y}_A| \rightarrow \infty$ (the external parts) are the tidal moments: the gravito-electric tidal moments, $G_A^L(s_A)$, with $l \geq 0$, are defined with 1PN accuracy (like M_A^L), and the gravito-magnetic tidal moments $H_A^L(s_A)$, with $l \geq 1$, are defined with Newtonian accuracy (like S_A^L). The tidal moments contain information about gravitational fields generated by external sources and about inertial effects associated with the motion of the local coordinate system. Finally, there are the tidal gauge moments ν_A^L , defined for $l \geq 1$.

The tensors $Z_A^{iL}(s_A)$ and $Y_A^{iL}(s_A)$ appearing in the gravito-magnetic potential (3.3.8b) have been defined as useful shorthands for the expressions in (3.3.9). Unlike all the other moments just introduced, they are not STF on all their indices, but they are STF on their last l indices (i.e. on all but the first index). Eqs. (3.3.9) in fact represent their unique decompositions in terms of fully STF tensors; the 'inverse' relations are

$$S_A^L = -\frac{1}{4} Z_A^{jk < L-1} \epsilon^{a_l > jk}, \quad (3.3.10a)$$

$$\mu_A^L = Z_A^{jjL}, \quad (3.3.10b)$$

$$\dot{M}_A^{iL} = -\frac{l+1}{4} Z_A^{< iL >}, \quad (3.3.10c)$$

and

$$H_A^L = Y_A^{jk<L-1}\epsilon^{a_l>jk}, \quad (3.3.11a)$$

$$\nu_A^L = Y_A^{<L>}, \quad (3.3.11b)$$

$$\dot{G}_A^L = -\frac{1}{4}Y_A^{jjL}. \quad (3.3.11c)$$

The relations (3.3.10c) and (3.3.11c) are implied by the harmonic gauge condition (3.3.5).

In the case where the 1PN field equations (3.3.6) are in fact valid in the interior of the body, the mass and current multipole moments can be defined by integrals over the stress-energy distribution in the volume of the body, as in DSX [33]:

$$M_A^L = \int_A d^3y_A \left\{ y_A^{<L>} T^{tt} + \frac{1}{c^2} \left[y_A^{<L>} T^{jj} + \frac{1}{2(2l+3)} y_A^{jj<L>} \dot{T}^{tt} - \frac{4(2l+1)}{(l+1)(2l+3)} y_A^{<jL>} \dot{T}^{tj} \right] \right\} + O(c^{-4}), \quad (3.3.12a)$$

$$S_A^L = \int_A d^3y_A \epsilon^{jk<a_l>j} y_A^{L-1>j} T^{tk} + O(c^{-2}). \quad (3.3.12b)$$

When considering a body with strong internal gravity, its interior cannot be modeled by a 1PN stress-energy distribution, and such integrals cannot be defined. Instead, we rely on the multipole expansion of the potentials (3.3.8) in the buffer region \mathcal{B}_A to define the multipole moments M_A^L and S_A^L , as well as the tidal moments G_A^L and H_A^L . Appendix E of RF [31] demonstrates the sufficiency of this method of definition by giving explicit formulae for the moments in terms of surface integrals of the potentials in \mathcal{B}_A .

III Coordinate transformations and body-frame gauge conditions

The 1PN metric (3.3.4) harbors residual coordinate freedom not fixed by the conformally Cartesian and harmonic gauge conditions. As a result, the multipole and tidal moments defined in the last section (not just the ‘gauge moments’ μ_A^L and ν_A^L , but rather all of the moments) are not unique and will vary with the choice of coordinates. To define a unique set of multipole moments for a given body, one must further specialize the body-frame coordinates. Thus we turn now to a discussion of 1PN coordinate transformations.

In RF [31], it was shown that the most general transformation between two harmonic coordinate

systems (s, y^i) and (t, x^i) in which the metric takes the 1PN form (3.3.4) can be written as

$$x^i(s, \mathbf{y}) = y^i + z^i(s) + \frac{1}{c^2} \left\{ \left[\frac{1}{2} \dot{z}^{kk}(s) \delta^{ij} - \dot{\alpha}(s) \delta^{ij} + \epsilon^{ijk} R^k(s) + \frac{1}{2} \dot{z}^{ij}(s) \right] y^j + \left[\frac{1}{2} \ddot{z}^i(s) \delta^{jk} - \dot{z}^k(s) \delta^{ij} \right] y^{jk} \right\} + O(c^{-4}), \quad (3.3.13a)$$

$$t(s, \mathbf{y}) = s + \frac{1}{c^2} [\alpha(s) + \dot{z}^j(s) y^j] + \frac{1}{c^4} \left[\beta(s, \mathbf{y}) + \frac{1}{6} \ddot{\alpha}(s) y^{jj} + \frac{1}{10} \ddot{z}_A^j(s) y^{jkk} \right] + O(c^{-6}), \quad (3.3.13b)$$

being parametrized by the following functions. The vector $z^i(s)$ provides a time-dependent translation between the spatial coordinates and is defined with 1PN accuracy. Each defined with Newtonian accuracy² are the rotation vector $R_i(s)$, and the functions $\alpha(s)$ and $\beta(s, \mathbf{y})$ which transform the time coordinate. All of these may be arbitrary functions of their arguments (within the bounds of their post-Newtonian scaling), except that $\beta(s, \mathbf{y})$ must be harmonic, $\nabla^2 \beta = 0$, in order to preserve the harmonic gauge condition.

In the treatment of the N -body problem, we will make use of one global coordinate system (t, x^i) and one body-adapted coordinate system (s_A, y_A^i) for each body A . The global coordinates (described further in the next section) are used to track the bulk motion of all the bodies, while the body-adapted coordinates are used in the local description of each body—in particular, to define their body-frame multipole and tidal moments. The transformation between the (t, x^i) and (s_A, y_A^i) coordinates will take the form (3.3.13), with different ‘worldline data’ functions,

$$\mathcal{D}_A = \{z_A^i(s_A), R_A^i(s_A), \alpha_A(s_A), \beta_A(s_A, y_A^i)\} \quad (3.3.14)$$

for each body A . These functions may be viewed as configuration variables for the body-adapted frame, specifying its position, orientation, etc. relative to the global frame.

In order to uniquely define the body-frame multipole and tidal moments, we must fix some of the remaining gauge freedom in the body-adapted coordinates (s_A, y_A^i) . Here, we will fix all remaining gauge freedom in the body-adapted coordinates (in the bodies’ buffer regions), which will also uniquely determine the gauge moments. It was shown in RF that this can be always be

² Though an $O(c^{-2})$ contribution to $\alpha(s)$ would contribute at $O(c^{-4})$ in Eq. (3.3.13b), this contribution can be absorbed into the function $\beta(s, \mathbf{y})$.

accomplished by imposing the following conditions, which define the body-adapted gauge:

$$M_A^i(s_A) = 0 \tag{3.3.15a}$$

$$R_A^i(s_A) = 0 \tag{3.3.15b}$$

$$G_A(s_A) = \mu_A(s_A) = 0 \tag{3.3.15c}$$

$$\mu_A^L(s_A) = \nu_A^L(s_A) = 0, \quad l \geq 1 \tag{3.3.15d}$$

Eq. (3.3.15a), setting the body-frame mass dipole M_A^i to zero, fixes the body’s center of mass-energy to the origin of the spatial coordinates $y_A^i = 0$ to 1PN order. Setting the rotation vector R_A^i to zero in Eq. (3.3.15b) fixes the orientation of the body-frame spatial axes to those of the global frame.³ If the extended body A were replaced by a freely falling observer at $y_A^i = 0$ (assuming an extension of the body-frame coordinates to $y_A^i = 0$), Eq. (3.3.15c) would ensure that the time coordinate s^A measures their proper time. Finally, the fact that all the gauge moments can always be set to zero by a coordinate transformation, as in (3.3.15d), shows that they are pure gauge degrees of freedom.

We can think of the body-adapted coordinates as defining the body’s local asymptotic rest frame [28], in which the effects of external gravitational fields and inertial effects have been removed as much as possible⁴. The body-frame moments—the multipole and tidal moments defined by (3.3.8) in the body-adapted coordinates—then take on the values that would be measured by a local comoving observer in the body frame. The body-frame multipole moments M_A^L and S_A^L are the quantities describing the bodies’ internal structure that will appear in the final form of the translational equation of motions for an N -body system.

IV The global frame

To treat the orbital dynamics of a collection of several bodies $A = 1 \dots N$, we consider $N + 1$ separate coordinate systems: one body-adapted coordinate system (s_A, y_A^i) for each body A , and one global coordinate system (t, x^i) . We take these coordinate systems to have the following properties: (i) For each body A , the body-adapted coordinates (s_A, y_A^i) cover the body’s buffer region \mathcal{B}_A and satisfy

³ In place of the condition (3.3.15b), RF chose to set the gravito-magnetic dipole tidal moment $H_A^i(s_A)$ to zero, which cancels leading-order Coriolis forces in the body-adapted frame and requires a non-zero value of the rotation vector $R_A^i(s_A)$. While this more completely effaces external gravitational and inertial effects in the body frame, Eq. (3.3.15b) leads to more simplifications in calculations. The effects of these differing gauge choices cancel in all final results.

all the assumptions and gauge conditions outlined in Secs. 3.3.II and 3.3.III. (ii) The bodies' buffer regions \mathcal{B}_A are non-overlapping. (iii) The global coordinates (t, x^i) cover the buffer regions of all the bodies as well as the intervening space; i.e. they cover the region $\mathcal{B}_g = \mathcal{M} \setminus \bigcup_A \mathcal{W}_A$, the entire spacetime manifold \mathcal{M} except for the worldtubes. (iv) In the region \mathcal{B}_g , the coordinates (t, x^i) are conformally Cartesian and harmonic, and the metric takes the form (3.3.4), with potentials $\Phi_g(t, \mathbf{x})$ and $\zeta_g^i(t, \mathbf{x})$ satisfying the PN vacuum field equations (Eqs. (3.3.7) with $A \rightarrow g$).

The final assumption allows us to write down the following multipole expansion of the global-frame potentials in \mathcal{B}_g :

$$\Phi_g(t, \mathbf{x}) = - \sum_{A=1}^N \sum_{l=0}^{\infty} \frac{(-1)^l}{l!} \left\{ M_{g,A}^L(t) \partial_L \frac{1}{|\mathbf{x} - \mathbf{z}_A(t)|} + \frac{1}{2c^2} \partial_t^2 \left[M_{g,A}^L(t) \partial_L |\mathbf{x} - \mathbf{z}_A(t)| \right] \right\} + O(c^{-4}), \quad (3.3.16a)$$

$$\zeta_g^i(t, \mathbf{x}) = - \sum_{A=1}^N \sum_{l=0}^{\infty} \frac{(-1)^l}{l!} Z_{g,A}^{iL}(t) \partial_L \frac{1}{|\mathbf{x} - \mathbf{z}_A(t)|} + O(c^{-2}), \quad (3.3.16b)$$

This expansion is analogous to that for the body-frame potentials (3.3.8) but has several important differences. Firstly, the potentials are written as a sum of contributions from each body A ; this is justified by the linearity of the field equations (3.3.6). Each such contribution is parametrized by the body's global-frame multipole moments: the mass multipole moments $M_{g,A}^L(t)$ are fully STF and 1PN-accurate, and the tensors $Z_{g,A}^{iL}(t)$ are STF on all but the first index and Newtonian-accurate. Both sets of tensors are defined for $l \geq 0$. As these global-frame moments will only appear in intermediate stages of our calculations, we will not bother decomposing the tensors $Z_{g,A}^{iL}$ in terms of fully STF current and gauge moments as in the body-frame case (3.3.9a). We have taken the moments $Z_{g,A}^{iL}$ to satisfy

$$Z_{g,A}^{<iL>} = -\frac{4}{l+1} \dot{M}_{g,A}^{iL}, \quad Z_{g,A}^{jjL} = 0. \quad (3.3.17)$$

The first of these is required by the harmonic gauge condition, and the second is equivalent to setting the would-be global-frame gauge moments $\mu_{g,A}^L$ to zero.⁴ The global-frame moments $M_{g,A}^L(t)$ and $Z_{g,A}^{iL}(t)$ are distinct from (though related to) the corresponding body-frame moments $M_A^L(s_A)$ and $Z_A^{iL}(s_A)$.

⁴ The second condition in (3.3.17), along with the fact that the global-frame potentials all vanish as $|\mathbf{x}| \rightarrow \infty$, reduces the residual gauge freedom in the global-frame metric to the group of post-Galilean transformations (the post-Newtonian Poincaré group) [31], which are the coordinate transformations given by (3.3.13) with ${}^n\dot{z}^i = \dot{h}^i = \dot{R}^i = \beta = 0$ and $\dot{\alpha} = {}^n\dot{z}^2/2$.

A second important difference with the body frame case is that the multipole expansions appearing here are centered not around the spatial origin $x^i = 0$ but around the worldlines $x^i = z_A^i(t)$. One can check that the potentials as written here still satisfy the harmonic-gauge 1PN field equations in \mathcal{B}_g for any choices of these worldlines. Below, we will identify the z_A^i with the bodies' center-of-mass worldlines, which appear as parameters in the transformations from body-adapted to global coordinates (3.3.13).

Finally, one can note that we have included, in each body's contributions to the potentials, only internal pieces (which appear to diverge as $|\mathbf{x} - \mathbf{z}^A| \rightarrow 0$) and not tidal pieces (which would appear to diverge as $|\mathbf{x} - \mathbf{z}^A| \rightarrow \infty$). This makes the global-frame metric tend to the Minkowski metric as $|\mathbf{x}| \rightarrow \infty$, thus eliminating any tidal or inertial forces on the N -body system as a whole. Each body will still experience local tidal fields, but they will arise from the contributions to the potentials generated by the other bodies.

We can introduce a set of global-frame tidal moments for each body A by rewriting the global-frame potentials, in the body's buffer region \mathcal{B}_A , as

$$\begin{aligned} \Phi_g(t, \mathbf{x}) = & - \sum_{l=0}^{\infty} \frac{1}{l!} \left\{ (-1)^l M_{g,A}^L(t) \partial_L \frac{1}{|\mathbf{x} - \mathbf{z}_A(t)|} + G_{g,A}^L(t) [x - z_A(t)]^L \right. \\ & \left. + \frac{1}{2c^2} \partial_t^2 \left[(-1)^l M_{g,A}^L(t) \partial_L |\mathbf{x} - \mathbf{z}_A(t)| + \frac{1}{2l+3} G_{g,A}^L(t) [x - z_A(t)]^{jjL} \right] \right\} + O(c^{-4}) \end{aligned} \quad (3.3.18a)$$

$$\zeta_g^i(t, \mathbf{x}) = - \sum_{l=0}^{\infty} \frac{1}{l!} \left[(-1)^l Z_{g,A}^{iL}(t) \partial_L \frac{1}{|\mathbf{x} - \mathbf{z}_A(t)|} + Y_{g,A}^{iL} [x - z_A(t)]^L \right] + O(c^{-2}), \quad (3.3.18b)$$

Here, we have absorbed the contributions to the potentials from the other bodies $B \neq A$ into tidal terms for body A . This defines the global-frame tidal moments $G_{g,A}^L(t)$ and $Y_{g,A}^{iL}(t)$. They can be expressed in terms of the global-frame multipole moments of the other bodies $B \neq A$ and the worldlines z_A^i of all the bodies A by equating the expressions for the potentials in (3.3.18) with those in (3.3.16); these relations are given in Appendix 3.9.II.

Now, as the global coordinates $x^\mu = (t, x^i)$ and the body-adapted coordinates $y_A^\mu = (s_A, y_A^i)$ are related by the coordinate transformation (3.3.13), the metrics in the global and body frames must be related by the tensor transformation law:

$$g_{\mu\nu}^A = \frac{\partial x^\rho}{\partial y_A^\mu} \frac{\partial x^\sigma}{\partial y_A^\nu} g_{\rho\sigma}^g \quad (3.3.19)$$

This requirement allows one to determine both the parameters of the coordinate transformation

between the two coordinate systems (3.3.13) and the relationship between the global- and body-frame multipole and tidal moments. Making use of the form (3.3.4) for the metric in terms of the potentials (in both coordinate systems) and the expressions for the body-frame potentials (3.3.8) and the global-frame potentials (3.3.18), as detailed in RF [31], Eq. (3.3.19) yields expressions for the body-frame moments in terms of the global-frame moments and the worldline data (or the inverse relations):

$$\begin{aligned} (M_A^L, S_A^L) &\xleftrightarrow{\mathcal{D}_A} (M_{g,A}^L, Z_{g,A}^{iL}) \\ (G_A^L, H_A^L) &\xleftrightarrow{\mathcal{D}_A} (G_{g,A}^L, Y_{g,A}^{iL}) \end{aligned} \quad (3.3.20)$$

These moment transformations are presented in full detail in Appendices 3.9.I and 3.9.III.

By combining the transformation formulae for the tidal moments with the body-frame gauge conditions (3.3.15), one can solve for and eliminate the worldline data functions $\alpha_A(s_A)$ and $\beta_A(s_A, y_A^j)$. The only remaining piece of the worldline data \mathcal{D}_A (3.3.14) is the translation vector $z_A^i(s_A)$. Recall that the body-frame gauge condition $M_A^i = 0$ (3.3.15a), setting the mass dipole to zero, fixes the body's center of mass-energy to the body-frame spatial origin $y_A^i = 0$. Setting $y_A^i = 0$ in the coordinate transformation (3.3.13) and eliminating s_A , we see that $x^i = z_A^i(t)$ encodes the body's global-frame center-of-mass worldline, where

$$z_A^i(t) = z_A^i(s_A) \Big|_{s_A=s_A^0(t)} \quad (3.3.21)$$

is the quantity $z_A^i(s_A)$ expressed as a function of t , with the function $s_A^0(t)$ found by setting $y_A^i = 0$ in Eq. (3.3.13b) (see Eq. (3.9.1) and discussion thereabouts). The translational equation of motion for a body A , discussed in the next subsection, can be written in the form of a second-order ODE for the global-frame CoM worldline $z_A^i(t)$.

V Single-body laws of motion and translational equations of motion

The single-body laws of motion are constraints on the lowest-order multipole moments of any body which govern the exchange of energy, momentum, and angular momentum between the body and the gravitational field. The laws of motion at 1PN order were first found by DSX, who derived them by using covariant stress-energy conservation at 1PN order in the interior of the body. The same laws of motion were later rederived by RF by using the 2PN (next-to-next-to-leading order in

$1/c^2$) vacuum Einstein equation in a buffer region surrounding the body, thus extending their range of validity to include bodies with strong internal gravity.

The laws of motion are written in terms of the body's multipole and tidal moments as defined by the expansion of the 1PN potentials (3.3.8) and are valid in any coordinate system in which the spacetime metric takes the form given by (3.3.4) and (3.3.8)—not just in body-adapted coordinates.

The results are

$$\dot{M}_A = -\frac{1}{c^2} \sum_{l=0}^{\infty} \frac{1}{l!} \left[(l+1) {}^n M_A^L {}^n \dot{G}_A^L + l {}^n \dot{M}_A^L {}^n G_A^L \right] + O(c^{-4}), \quad (3.3.22a)$$

$$\begin{aligned} \ddot{M}_A^i &= \sum_{l=0}^{\infty} \frac{1}{l!} \left\{ M_A^L G_A^L + \frac{1}{c^2} \left[\frac{1}{l+2} \epsilon_{ijk} M_A^{jL} \dot{H}_A^{kL} + \frac{1}{l+1} \epsilon_{ijk} \dot{M}_A^{jL} H_A^{kL} \right. \right. \\ &\quad - \frac{2l^3 + 7l^2 + 15l + 6}{(l+1)(2l+3)} M_A^{iL} \ddot{G}_A^L - \frac{2l^3 + 5l^2 + 12l + 5}{(l+1)^2} \dot{M}_A^{iL} \dot{G}_A^L - \frac{l^2 + l + 4}{l+1} \ddot{M}_A^{iL} G_A^L \\ &\quad \left. \left. + \frac{l}{l+1} S_A^L H_A^{iL} - \frac{4(l+1)}{(l+2)^2} \epsilon_{ijk} S_A^{jL} \dot{G}_A^{kL} - \frac{4}{l+2} \epsilon_{ijk} \dot{S}_A^{jL} G_A^{kL} \right] \right\} + O(c^{-4}). \quad (3.3.22b) \end{aligned}$$

$$\dot{S}_A^i = \sum_{l=0}^{\infty} \frac{1}{l!} \epsilon_{ijk} M_A^{jL} G_A^{kL} + O(c^{-2}), \quad (3.3.22c)$$

Eq. (3.3.22a) shows that the mass monopole M_A is conserved at Newtonian order ($O(c^0)$), but not at 1PN order. As discussed further in Sec. 3.3.VI below, M_A contains $O(c^{-2})$ contributions from the internal energy of the body, which can vary as tidal forces do work on the body. The law of motion (3.3.22c) for the spin S_A^i is the same Newtonian-order tidal torque formula found in Eq. (3.2.45).

The law of motion (3.3.22b) for the mass dipole M_A^i governs the evolution of the body's total linear momentum. The body's translational equation of motion can be derived by applying (3.3.22b) in the body frame, i.e. by applying it to the body-frame dipole moment, as follows.

At Newtonian order, the $O(c^0)$ part of \ddot{M}_A^i in (3.3.22b) gives the net force acting on the body, as \dot{M}_A^i is the body's total momentum (cf. (3.3.12a)). Since the body-adapted coordinates are chosen to be mass-centered ($M_A^i = 0$), this net force must vanish in the body frame. This apparent equilibrium in the body frame is achieved by the balancing of gravitational forces from the other bodies with *inertial* forces, which are due to the fact that the body frame is accelerating with respect to the (asymptotically) inertial global frame, along the worldline $z_A^i(t)$. Both of these effects are accounted

for by the body-frame tidal moments G_A^L ; from the $O(c^0)$ part of [ref], we have

$$\begin{aligned} G_A^i &= G_{g,A}^i - \ddot{z}_A^i + O(c^{-2}), \\ G_A^L &= G_{g,A}^L + O(c^{-2}), \quad (l \geq 2), \\ G_{g,A}^L &= \sum_{B \neq A} \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} M_B^K \partial_{KL}^{(A)} \frac{1}{|\mathbf{z}_A - \mathbf{z}_B|} + O(c^{-2}) \end{aligned}$$

Using these relations in (3.3.22b), the requirement of equilibrium in the body frame, $\ddot{M}_A^i = 0$, determines the equation of motion for the worldline $z_A^i(t)$:

$$\begin{aligned} M^A \ddot{z}_i^A &= \sum_{B \neq A} \sum_{l=0}^{\infty} \frac{1}{l!} M_L^A G_{iL}^{g,A} + O(c^{-2}) \\ &= \sum_{B \neq A} \sum_{k,l=0}^{\infty} \frac{1}{k!l!} M_L^A M_B^K \partial_{iKL}^{(A)} \frac{1}{|\mathbf{z}^A - \mathbf{z}^B|} + O(c^{-2}). \end{aligned} \tag{3.3.23}$$

This matches the Newtonian equation of motion found above in (3.2.16).

At 1PN order, the procedure is essentially the same, but more involved. One begins by setting $\ddot{M}_A^i = 0$ in the body frame, with \ddot{M}_A^i given by (3.3.22b). To arrive at a suitable form for the final equation of motion, one must then rewrite the body-frame tidal moments G_A^L and H_A^L of body A in terms of the body-frame multipole moments M_B^L and S_B^L of the other bodies $B \neq A$ and the worldline data \mathcal{D}_C for all the bodies C ; the details of this procedure are presented in Appendix 3.9.

In the end, one arrives at an expression for the acceleration $\ddot{z}_A^i(t)$ of the 1PN-accurate global-frame center-of-mass worldline $z_A^i(t)$, defined by (3.3.21). (As in the Newtonian case, the acceleration term, describing inertial forces in the body frame, emerges from the transformation laws for the body-frame tidal moments.) The expression depends only on the body-frame mass and current multipole moments $M_B^L(t)$ and $S_B^L(t)$,⁵ the global-frame worldlines $z_B^i(t)$, and the time derivatives of these quantities, for all bodies B :

$$\ddot{z}_A^i(t) = \mathcal{F}_A^i(z_B^i, \dot{z}_B^i, M_B^L, \dot{M}_B^L, \ddot{M}_B^L, S_B^L, \dot{S}_B^L). \tag{3.3.24a}$$

Similar (though simpler) manipulations applied to the laws of motion (3.3.22a) and (3.3.22c) allow one to write equations of motion for the mass monopole $M_A(t)$ and spin $S_A^i(t)$ in terms of the same

⁵ Here, and throughout, $M_A^L(t)$ and $S_A^L(t)$ are the body-frame moments $M_A^L(s_A)$ and $S_A^L(s_A)$ expressed as functions of t at $y_A^i = 0$, i.e. the same physical quantities expressed as functions of different variables; cf. Eq. (3.9.1) and surrounding discussion.

variables:

$$\dot{M}_A(t) = \mathcal{F}_A(z_B^i, \dot{z}_B^i, M_B^L, \dot{M}_B^L), \quad (3.3.24b)$$

$$\dot{S}_A^i(t) = \tilde{\mathcal{F}}_A^i(z_B^i, M_B^L). \quad (3.3.24c)$$

The explicit forms of the translational equations of motion (3.3.24a) and the mass and spin evolution equations (3.3.24b) and (3.3.24c) will be given for the M_1 - M_2 - S_2 - Q_2 case in Sec. 3.3.VI, and are given in the fully general case in RF [31] as corrected by an upcoming erratum.

To arrive at a closed set of evolution equations for the quantities $z_A^i(t)$, $M_A^L(t)$, and $S_A^L(t)$, for all bodies A , the equations of motion (3.3.24) must be supplemented by equations for the multipole moments $M_L^A(t)$ and $S_L^A(t)$ for $l \geq 2$. Finding such equations will require a model for the bodies' internal dynamics, which will be addressed in Sec. 3.6.

VI Monopole-spin-quadrupole truncation

We have presented above the formalism for treating the 1PN dynamics of a collection of many bodies, each with arbitrarily high-order multipole moments. Here, we apply that formalism to the two-body system discussed in Sec. 3.1.IV, with a body 1 having only a mass monopole moment M_1 , and a body 2 having a mass monopole M_2 , a current dipole, or spin, $S_2^i \equiv S^i$, and a mass quadrupole $M_2^{ij} \equiv Q_2^{ij} \equiv Q^{ij}$. More precisely, we truncate the internal parts of body-frame multipole series (3.3.8) for each body according to

$$\Phi_{1,\text{int}} = -\frac{M_1}{|\mathbf{y}_1|} + O(c^{-4}), \quad \zeta_{1,\text{int}}^i = O(c^{-2}),$$

neglecting the moments M_1^L for $l \geq 2$ and S_1^L for $l \geq 1$, and

$$\begin{aligned} \Phi_{2,\text{int}} &= -\frac{M_2}{|\mathbf{y}_2|} - \frac{1}{2} Q^{ij} \partial_{ij} \frac{1}{|\mathbf{y}_2|} - \frac{1}{4c^2} \ddot{Q}^{ij} \partial_{ij} |\mathbf{y}_2| + O(c^{-4}), \\ \zeta_{2,\text{int}}^i &= 2 \left(\dot{Q}^{ij} - \epsilon_{ijk} S^k \right) \partial_j \frac{1}{|\mathbf{y}_2|} + O(c^{-2}), \end{aligned}$$

neglecting the moments M_2^L for $l \geq 3$ and S_2^L for $l \geq 2$. (The external parts of these potentials will be just as in (3.3.8), with arbitrarily higher-order tidal moments.) These expressions for the body-frame potentials define the body-frame moments $M_1(s_1)$, $M_2(s_2)$, $Q^{ij}(s_2)$, and $S^i(s_2)$ as functions of the body-frame time coordinates s_1 and s_2 . These moments can be expressed as

functions of the global time coordinate, written $M_1(t)$, $M_2(t)$, $S^i(t)$ and $Q^{ij}(t)$, using the coordinate transformation (3.3.13) with $y_A^i = 0$ (cf. Eq. (3.9.1)). For the bodies' global-frame CoM worldlines $z_1^i(t)$ and $z_2^i(t)$, we will use the definitions

$$z^i = z_2^i - z_1^i, \quad r = |\mathbf{z}|, \quad n^i = z^i/r, \quad (3.3.25)$$

as similar to (3.2.17), except that the worldlines are now defined with 1PN accuracy, and

$$v_1^i = \dot{z}_1^i, \quad v_2^i = \dot{z}_2^i, \quad v^i = v_2^i - v_1^i. \quad (3.3.26)$$

With these conventions in place, we can apply the laws of motion presented in Sec. 3.3.V to find the evolution equations for the moments $M_1(t)$, $M_2(t)$, and $S^i(t)$ and the global-frame center-of-mass worldlines $z_1^i(t)$ and $z_2^i(t)$. The results involve only these quantities and the quadrupole $Q^{ij}(t)$.

As body 1 has no higher-order multipole moments, the law of motion (3.3.22a) requires that its mass monopole M_1 be constant in time:

$$\dot{M}_1 = O(c^{-4}). \quad (3.3.27)$$

The same law of motion applied to body 2 gives

$$\dot{M}_2 = -\frac{1}{c^2} \left(\frac{3}{2} Q^{ij} \dot{G}_2^{ij} - \dot{Q}^{ij} G_2^{ij} \right) + O(c^{-4}), \quad (3.3.28)$$

where the body-frame gravito-electric tidal moment G_2^{ij} is given by (3.9.7) as

$$G_2^{ij} = G_{g,2}^{ij} + O(c^{-2}) = \frac{3M^A}{r^3} n^{<ij>} + O(c^{-2}). \quad (3.3.29)$$

It is worth pausing here to compare this rate of change of the 1PN-accurate mass monopole M_2 with the rate of change Newtonian internal energy E_2^{int} discussed in Sec. 3.2.VI. From (3.2.34) and (3.3.28), we find that they are related by

$$\dot{M}_2 = c^{-2} \left(\dot{E}_2^{\text{int}} + 3\dot{U}_Q \right) + O(c^{-4}), \quad (3.3.30)$$

where U_Q is the Newtonian gravitational potential energy associated with the quadrupole-tidal interaction given by (3.2.28). The mass monopole M_2 thus contains contributions not only from body 2's internal energy but also from the tidal part of its external gravitational potential energy.

Though Newtonian internal energy is not a well-defined concept for strongly self-gravitating bodies, we will simply take the relation

$$M_2 = {}^nM_2 + c^{-2} (E_2^{\text{int}} + 3U_Q) + O(c^{-4}) \quad (3.3.31)$$

to define the quantity E_2^{int} in the 1PN context, with nM_2 being the conserved Newtonian-order rest-mass contribution. This partitioning of M_2 , which is fully consistent with the equation of motion (3.3.28), given (3.2.34) and (3.2.28), will be useful in our discussion of internal dynamics below. (A thorough discussion of the ambiguity in the total mass-energy of a body by an amount of the order of its tidal potential energy, and of tidal heating in GR, can be found in Ref. [107].)

The evolution equation for the spin (3.3.22c) gives the following tidal torque on body 2:

$$\dot{S}^i = \epsilon^{ijk} Q^{aj} G_2^{ka} + O(c^{-2}), \quad (3.3.32)$$

with G_2^{ka} being given by (3.3.29). This coupling between the spin and the quadrupole, which is a purely Newtonian effect, is the essential reason we cannot (in general) ignore the spin-orbit coupling terms in the 1PN translational equations of motion for bodies with quadrupole moments.

Finally, working from the laws of motion (3.3.22b) for the mass dipoles, we can apply the procedure outlined in Sec. 3.3.V and Appendix 3.9 to the M_1 - M_2 - S_2 - Q_2 system to find the translational equations of motion for the worldlines z_1^i and z_2^i . The results are

$$M_1 \ddot{z}_1^i(t) = F_{1,M}^i + F_{1,S}^i + F_{1,Q}^i, \quad (3.3.33a)$$

$$M_2 \ddot{z}_2^i(t) = F_{2,M}^i + F_{2,S}^i + F_{2,Q}^i, \quad (3.3.33b)$$

with the monopole contributions,

$$F_{1,M}^i = \frac{M_2}{r^2} n^i + \frac{1}{c^2} \frac{M_2}{r^2} \left\{ n^i \left[2v^2 - v_1^2 - \frac{3}{2} (n^a v_2^a)^2 - \frac{5M_1}{r} - \frac{4M_2}{r} \right] + v^i n^a (4v_1^a - 3v_2^a) \right\}, \quad (3.3.33c)$$

$$F_{2,M}^i = -\frac{M_1}{r^2} n^i - \frac{1}{c^2} \frac{M_1}{r^2} \left\{ n^i \left[2v^2 - v_2^2 - \frac{3}{2} (n^a v_1^a)^2 - \frac{4M_1}{r} - \frac{5M_2}{r} \right] + v^i n^a (4v_2^a - 3v_1^a) \right\}, \quad (3.3.33d)$$

the spin contributions,

$$F_{1,S}^i = \frac{1}{c^2} \frac{M_1}{r^3} \epsilon^{abc} S^c \left[\delta^{ai} (4v^b - 6n^{bd} v^d) - 6n^{ai} v^b \right], \quad (3.3.33e)$$

$$F_{2,S}^i = \frac{1}{c^2} \frac{M_1}{r^3} \epsilon^{abc} S^c \left[3\delta^{ai} (n^{bd} v^d - v^b) + 6n^{ai} v^b \right], \quad (3.3.33f)$$

and the quadrupole contributions,

$$\begin{aligned}
F_{1,Q}^i &= \frac{3M_1}{2r^4} Q^{ab} (5n^{abi} - 2n^a \delta^{bi}) + \frac{1}{c^2} \left(\frac{3M_1}{2r^4} Q^{ab} \left\{ 5n^{abi} \left[2v^2 - v_1^2 - \frac{7}{2}(n^c v_2^c)^2 - \frac{47M_1}{5r} - \frac{24M_2}{5r} \right] \right. \right. \\
&\quad - 2n^a \delta^{bi} \left[2v^2 - v_1^2 - \frac{5}{2}(n^c v_2^c)^2 - \frac{19M_1}{2r} - \frac{4M_2}{r} \right] + n^a v_2^{bi} + (5n^{ai} - \delta^{ai}) v_2^{bc} n^c \\
&\quad \left. \left. + v^i (5n^{abc} - 2n^a \delta^{bc})(4v_1^c - 3v_2^c) \right\} + \frac{3M_1}{2r^3} \dot{Q}^{ab} \left[n^{ab} (5v_2^c n^{ci} + 3v^i) - 4v^a n^{bi} - 2\delta^{ai} n^{bc} (2v_1^c - v_2^c) \right] \right. \\
&\quad \left. - \frac{3M_1}{4r^2} \ddot{Q}^{ab} (n^{abi} + 2n^a \delta^{bi}) \right), \tag{3.3.33g}
\end{aligned}$$

$$\begin{aligned}
F_{2,Q}^i &= -\frac{3M_1}{2r^4} Q^{ab} (5n^{abi} - 2n^a \delta^{bi}) + \frac{1}{c^2} \left(\frac{3M_1}{2r^4} Q^{ab} \left\{ -5n^{abi} \left[2v^2 - v_2^2 - \frac{7}{2}(n^c v_1^c)^2 - \frac{8M_1}{r} - \frac{6M_2}{r} \right] \right. \right. \\
&\quad + 2n^a \delta^{bi} \left[3v^2 - v_2^2 - 5(n^c v_1^c)^2 - \frac{5}{2}(n^c v_1^c)^2 - \frac{8M_1}{r} - \frac{11M_2}{2r} \right] + n^i v^{ab} + 5n^{aci} (2v^b v_1^c - v_2^{bc}) \\
&\quad \left. \left. + v^i (5n^{abc} - 2n^a \delta^{bc})(4v_2^c - 3v_1^c) + n^a v_2^b (v_2^i - 2v_1^i) + \delta^{bi} n^c [(5v_2^a - 4v_1^a) v_2^c - 6v^a v_1^c] \right\} \right. \\
&\quad \left. + \frac{3M_1}{r^3} \dot{Q}^{ab} \left[v^b (2n^{ai} - \delta^{ai}) + \delta^{ai} n^{bc} v^c - 2n^{ab} v^i \right] \right), \tag{3.3.33h}
\end{aligned}$$

all plus $O(c^{-4})$ corrections. (It should be noted that occurrences of \dot{S}^i in the equations of motion have been replaced by (3.3.32) and included in the quadrupole contributions.)

The monopole contributions (3.3.33c,3.3.33d) give the well-known Lorentz-Droste-Einstein-Infeld-Hoffmann accelerations, and the spin contributions (3.3.33e,3.3.33f) give the well-known 1PN spin-orbit terms [33]. The quadrupole contributions (3.3.33g,3.3.33h) have been derived previously by Xu, Wu, and Schafer [50], though our results disagree with theirs in several terms; we have not been able to pin down the source of the disagreement. Our results also disagree with the final results of RF [31], but agree with their corrected results given in an upcoming erratum. The strongest indication of the correctness of our expressions for the EoMs is the fact that, unlike the results in [50, 31], they are consistent with the conservation of the binary system's total linear momentum, as discussed in Sec. 3.4.III below. We can also note that the action derived below from these results agrees with the recent work (using a rather different method) by Damour and Nagar [94] and Bini, Damour, and Faye [52].

3.4 System multipole moments and conservation laws

In Sec. 3.3.II, we defined the multipole moments of a single body through the multipole expansion of the metric in a vacuum buffer region surrounding the body. The same procedure can be applied to a collection of several bodies to define multipole moments for the entire system. Applying the general laws of motion discussed in Sec. 3.3.V to these system multipole moments will allow us to formulate conservation laws for the energy, momentum, and angular momentum of an isolated N -body system, expressed as constraints on the worldlines and multipole moments of the constituent bodies. These conservation laws can serve both as a consistency check for the equations of motion given in Sec. 3.3.VI and as a means to specialize the equations of motion to the system's center-of-mass frame.

I General formulae

We have already discussed, in Sec. 3.3.IV, a form for the metric generated by a system of N bodies. Using the global coordinate system (t, x^i) , we expressed the potentials parameterizing the metric as a sum of multipole expansions for each body A , written in terms of the bodies' global-frame multipole moments $M_{g,A}^L$ and $Z_{g,A}^{iL}$ and Newtonian-order worldlines z_A^i :

$$\Phi_g = - \sum_A \sum_{l=0}^{\infty} \frac{(-1)^l}{l!} \left[M_{g,A}^L \partial_L \frac{1}{|\mathbf{x} - \mathbf{z}_A|} + \frac{1}{2c^2} \partial_t^2 (M_{g,A}^L \partial_L |\mathbf{x} - \mathbf{z}_A|) \right] + O(c^{-4}), \quad (3.4.1a)$$

$$\zeta_g^i = - \sum_A \sum_{l=0}^{\infty} \frac{(-1)^l}{l!} Z_{g,A}^{iL} \partial_L \frac{1}{|\mathbf{x} - \mathbf{z}_A|} + O(c^{-2}), \quad (3.4.1b)$$

with $Z_{g,A}^{iL}$ satisfying (3.3.17). This solution for the metric was constructed to be valid in the region \mathcal{B}_g , which extends out to spatial infinity.

In a region far outside the system, we can rewrite these expressions for the global-frame potentials to mirror the forms (3.3.8) used to define the multipole moments of a single body, with multipole expansions about the global-frame origin $x^i = 0$:

$$\Phi_g = - \sum_{l=0}^{\infty} \frac{(-1)^l}{l!} \left\{ M_{\text{sys}}^L \partial_L \frac{1}{|\mathbf{x}|} + \frac{1}{c^2} \left[\frac{(2l+1)}{(l+1)(2l+3)} \dot{\mu}_{\text{sys}}^L \partial_L \frac{1}{|\mathbf{x}|} + \frac{1}{2} {}^n \ddot{M}_{\text{sys}}^L \partial_L |\mathbf{x}| \right] \right\} + O(c^{-4}), \quad (3.4.2a)$$

$$\zeta_g^i = - \sum_{l=0}^{\infty} \frac{(-1)^l}{l!} Z_{\text{sys}}^{iL} \partial_L \frac{1}{|\mathbf{x}|} + O(c^{-2}), \quad (3.4.2b)$$

with

$$Z_{\text{sys}}^{iL} = \frac{4}{l+1} {}^n M_{\text{sys}}^{iL} - \frac{4l}{l+1} \epsilon^{ji < a_l} S_{\text{sys}}^{L-1 > j} + \frac{2l-1}{2l+1} \delta^{i < a_l} \mu_{\text{sys}}^{L-1 >} + O(c^{-2}). \quad (3.4.3)$$

These expansions define the system multipole moments M_{sys}^L and S_{sys}^L and the gauge moments μ_{sys}^L . The tidal terms present in (3.3.8) are absent here, as the global-frame potentials vanish as $|\mathbf{x}| \rightarrow \infty$ (cf. Eq. (3.4.1)). To compare the metric (3.4.2) here to the metric (3.4.1) above, we must express them in the same gauge. We have chosen the gauge that enforces $Z_{g,A}^{jjL} = 0$ in (3.4.1), which will result in nonzero values for the system gauge moments μ_{sys}^L in (3.4.2).

Since the potentials given by (3.4.1) and by (3.4.2) represent the same metric in the same gauge, they should be explicitly equal. This condition will allow us to solve for the system multipole moments appearing in (3.4.2) in terms of the individual bodies' global-frame multipole moments and worldlines appearing in (3.4.1).

Considering first the vector potential ζ_g^i , we can use the Taylor series

$$|\mathbf{x} - \mathbf{z}_A|^n = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} z_A^K \partial_K |\mathbf{x}|^n \quad (3.4.4)$$

to rewrite (3.4.1b) in the form

$$\begin{aligned} \zeta_g^i &= - \sum_A \sum_{l,k=0}^{\infty} \frac{(-1)^{l+k}}{l!k!} Z_{g,A}^{iL} z_A^K \partial_{LK} \frac{1}{|\mathbf{x}|} \\ &= - \sum_A \sum_{p=0}^{\infty} \sum_{k=0}^p \frac{(-1)^p}{p!} \frac{p!}{k!(p-k)!} Z_{g,A}^{<P-K} z_A^{>K} \partial_P \frac{1}{|\mathbf{x}|}. \end{aligned}$$

In the second line, we have relabeled the multi-indices according to $LK \rightarrow P$, adjusted the summations accordingly, and used the fact that $\partial_P |x|^{-1}$ is STF. Renaming $P \rightarrow L$ and comparing this with (3.4.2b) gives an expression for the tensors Z_{sys}^{iL} :

$$Z_{\text{sys}}^{iL} = \sum_A \sum_{k=0}^l \frac{l!}{(l-k)!k!} Z_{g,A}^{i < L-K} {}^n z_A^{>K}. \quad (3.4.5a)$$

The current moments and gauge moments can then be found from formulae analogous to (3.3.10a) and (3.3.10b):

$$S_{\text{sys}}^L = \frac{1}{4} Z_{\text{sys}}^{jk < L-1} \epsilon^{aj > kj} \quad (3.4.5b)$$

$$\mu_{\text{sys}}^L = Z_{\text{sys}}^{jjL} \quad (3.4.5c)$$

The scalar potential Φ_g can be manipulated in a similar manner, using the Taylor series (3.4.4), to find formulae for the system mass multipole moments M_{sys}^L . The details of this more involved procedure are given in Appendix 3.10. The result is

$$M_{\text{sys}}^L = \sum_A \sum_{k=0}^l \frac{l!}{k!(l-k)!} \left[M_{g,A}^{<L-K>K} z_A^K + \frac{1}{c^2} \frac{1}{2(2l+3)} \partial_t^2 \left(2M_{g,A}^{j<L-K>K} z_A^{K>j} + M_{g,A}^{<L-K>K} z_A^{K>jj} \right) \right] - \frac{1}{c^2} \frac{2l+1}{(l+1)(2l+3)} \dot{\mu}_{\text{sys}}^L + O(c^{-4}). \quad (3.4.6)$$

The gauge moments $\dot{\mu}_{\text{sys}}^L$ appearing here can be found from (3.4.5).

In summary, Eqs. (3.4.5) and (3.4.6) give the total multipole moments M_{sys}^L and S_{sys}^L of an N -body system, defined in the global frame (t, x^i) , in terms of the individual bodies' global-frame multipole moments $M_{g,A}^L$ and $Z_{g,A}^{iL}$ and worldlines z_A^i . In the following subsections, considering the two-body M_1 - M_2 - S_2 - Q_2 case, we will use these results along with the moment transformation formulae from Appendix 3.9 to write the system's mass monopole M_{sys} and mass dipole M_{sys}^i in terms of the body-frame moments (M_1, M_2, S^i, Q^{ij}) and the worldlines z_1^i and z_2^i . We note that a similar procedure can be applied to find $S_{\text{sys}}^i = \epsilon^{ijk} (M_1 z_1^j v_1^k + M_2 z_2^j v_2^k) + S_2^i$ for the system's total (Newtonian) angular momentum. The system's 1PN accurate mass quadrupole, which will be needed for the calculation of the gravitational wave signal from the binary system, can also be calculated from Eq. (3.4.6).

II System mass monopole

Specializing the general formula (3.4.6) for the system mass multipoles to the monopole ($l = 0$) case, and using the M_1 - M_2 - S_2 - Q_2 truncation, we find the binary system's total 1PN-accurate mass monopole to be

$$M_{\text{sys}} = M_{g,1} + M_{g,2} + \frac{1}{c^2} \left[-\frac{1}{3} \dot{\mu}_{\text{sys}} + \frac{1}{6} \frac{d^2}{dt^2} (M_{g,1} z_1^2 + M_{g,2} z_2^2) \right] + O(c^{-4}).$$

Using (3.4.5) for $\dot{\mu}_{\text{sys}}$ and the formulae in Appendix 3.9.I relating the global- and body-frame multipole moments, we can rewrite this expression in terms of the body-frame multipole moments and the CoM worldlines:

$$M_{\text{sys}} = M_1 + M_2 + \frac{1}{c^2} \left(\frac{M_1 v_1^2}{2} + \frac{M_2 v_2^2}{2} - \frac{M_1 M_2}{r} - 2U_Q \right) + O(c^{-4}), \quad (3.4.7)$$

where the tidal potential energy U_Q , as in (3.2.28), is

$$U_Q = -\frac{3M_1}{2r^3}Q^{ij}n^{ij}.$$

If we rewrite the mass monopole of body 2 as $M_2 = {}^nM_2 + c^{-2}(E_2^{\text{int}} + 3U_Q)$, as in (3.3.31), we find that the 1PN contribution to the system mass monopole is exactly the system's total Newtonian energy E given by (3.2.32):

$$\begin{aligned} M_{\text{sys}} &= M_1 + {}^nM_2 + c^{-2}E + O(c^{-4}) \\ E &= \frac{1}{2}M_1v_1^2 + \frac{1}{2}M_2v_2^2 - \frac{M_1M_2}{r} + U_Q + E_2^{\text{int}}. \end{aligned} \quad (3.4.8)$$

This is a further validation of the decomposition of the total mass monopole M_2 in Eq. (3.3.31). The constancy of M_{sys} , required by the law of motion (3.3.22a) as applied to the entire system in the global frame (for which there are no tidal moments), then follows from the constancy of E .

III System mass dipole

Taking the $l = 1$ case in the general formula (3.4.6) gives the system's 1PN-accurate mass dipole:

$$\begin{aligned} M_{\text{sys}}^i &= M_{g,1}z_1^i + M_{g,2}z_2^i + M_{g,1}^i + M_{g,2}^i + \frac{1}{c^2} \left[-\frac{3}{10}\dot{\mu}_{\text{sys}}^i \right. \\ &\quad \left. + \frac{1}{10}\partial_t^2 \left(2M_{g,2}^{ij}z_2^j + M_{g,1}z_1^{ijj} + M_{g,2}z_2^{ijj} \right) \right] + O(c^{-4}). \end{aligned}$$

Using (3.4.5) for $\dot{\mu}_{\text{sys}}^i$, (3.3.32) to replace an occurrence of \dot{S}^i , (3.3.31) to replace M_2 , (3.2.28) for U_Q , and the moment transformations from Appendix 3.9.I, this becomes

$$\begin{aligned} M_{\text{sys}}^i &= M_1z_1^i + {}^nM_2z_2^i + \frac{1}{c^2} \left[z_1^i \left(\frac{M_1v_1^2}{2} - \frac{M_1M_2}{2r} + \frac{U_Q}{2} \right) \right. \\ &\quad \left. + z_2^i \left(\frac{M_2v_2^2}{2} - \frac{M_1M_2}{2r} + \frac{U_Q}{2} + E_2^{\text{int}} \right) + \frac{3M_1}{2r^2}Q^{ij}n^j + \epsilon^{ijk}v_2^jS^k \right] + O(c^{-4}). \end{aligned} \quad (3.4.9)$$

From the law of motion (3.3.22b) as applied to the entire system in the global frame, for which there are no tidal moments, we see that \ddot{M}_{sys}^i should vanish; this is a statement of total momentum conservation. By differentiating (3.4.9), order reducing as appropriate, using the full 1PN translational equations of motion (3.3.33) in the Newtonian terms and their Newtonian parts in the 1PN terms, and also using (3.2.34) and (3.3.32) for \dot{E}_2^{int} and \dot{S}_i , we find that indeed $\ddot{M}_{\text{sys}}^i = 0$. This is an important check of the correctness of our expressions for the equations of motion and of the

consistency of the formalism (which is not satisfied by the EoMs given in Ref. [31, 50]). Note that the inclusion of the spin term in Eq. (3.4.9) is essential, reflecting the necessity of including spin terms when working with mass quadrupoles at 1PN order.

3.5 Orbital dynamics in the system's center-of-mass frame

I Equation of motion of the relative position

The conservation of momentum allows us to reduce the problem of solving for the two worldlines $z_1^i(t)$ and $z_2^i(t)$ to solving for just their separation $z^i(t) = z_2^i(t) - z_1^i(t)$ in the binary system's center-of-mass (CoM) frame. We can define the CoM frame to be that in which the 1PN-accurate mass dipole vanishes,

$$M_{\text{sys}}^i(t) = 0, \quad (3.5.1)$$

so that the system's center-of-mass(-energy) is at rest at the global-frame spatial origin. This fixes all remaining (post-Galilean) coordinate freedom in the global-frame metric. Using Eq. (3.4.9), this condition can be used to solve for the worldlines z_1^i and z_2^i in the global CoM frame in terms of the relative position z^i (working perturbatively in c^{-2}); one finds

$$z_1^i = -\chi_2 z^i + c^{-2} (\mathcal{P} z^i - \mathcal{D}^i) + O(c^{-4}), \quad (3.5.2)$$

$$z_2^i = \chi_1 z^i + c^{-2} (\mathcal{P} z^i - \mathcal{D}^i) + O(c^{-4}), \quad (3.5.3)$$

with

$$\begin{aligned} \mathcal{P} &= \eta(\chi_2 - \chi_1) \left(\frac{v^2}{2} - \frac{M}{2r} - \frac{3}{4\chi_2 r^3} Q^{ij} n^{ij} \right) - \frac{\chi_1}{M} E_2^{\text{int}}, \\ \mathcal{D}_i &= \frac{3\chi_i}{2r^2} Q^{ij} n_j + \frac{\chi_1}{M} \epsilon^{ijk} v^j S^k, \end{aligned} \quad (3.5.4)$$

and with the new notation

$$\begin{aligned} M &= M_1 + {}^n M_2, & \chi_1 &= M_1/M, & \chi_2 &= {}^n M_2/M, \\ \mu &= M_1 {}^n M_2/M, & \eta &= \chi_1 \chi_2 = \mu/M. \end{aligned} \quad (3.5.5)$$

To find the acceleration of the relative position in the CoM frame, we can simply subtract our above results (3.3.33) for the individual accelerations:

$$a^i \equiv \ddot{z}^i = \ddot{z}_2^i - \ddot{z}_1^i. \quad (3.5.6)$$

The resulting expression depends only on z^i , v_1^i , v_2^i , M_1 , M_2 , Q^{ij} , S^i , and E_2^{int} . As v_1^i and v_2^i appear only in 1PN terms, we can replace them with their Newtonian values in the CoM frame,

$$v_1^i = -\chi_2 v^i + O(c^{-2}), \quad v_2^i = \chi_1 v^i + O(c^{-2}), \quad (3.5.7)$$

from differentiating the $O(c^0)$ parts of (3.5.2,3.5.3). After defining one last shorthand,

$$\dot{r} = n^a v^a, \quad (3.5.8)$$

we can write our result for the 1PN-accurate CoM-frame relative acceleration as follows:

$$a^i = a_M^i + a_S^i + a_Q^i, \quad (3.5.9a)$$

with the monopole contribution,

$$a_M^i = -\frac{M}{r^2} n^i - \frac{1}{c^2} \frac{M}{r^2} \left\{ n^i \left[(1 + 3\eta) v^2 - \frac{3\eta}{2} \dot{r}^2 - 2(2 + \eta) \frac{M}{r} \right] - 2(2 - \eta) \dot{r} v^i \right\} + O(c^{-4}), \quad (3.5.9b)$$

the spin contribution,

$$a_S^i = \frac{\epsilon^{abc} S^c}{c^2 \chi_2 r^3} \left[(3 + \chi_2) v^a \delta^{bi} - 3(1 + \chi_2) \dot{r} n^a \delta^{bi} + 2n^{ai} v^b \right] + O(c^{-4}), \quad (3.5.9c)$$

and the quadrupole contribution,

$$\begin{aligned} a_Q^i = & -\frac{3Q^{ab}}{2\chi_2 r^4} \left[5n^{abi} - 2n^a \delta^{bi} \right] + \frac{1}{c^2} \left\{ \frac{Q^{ab}}{r^4} \left[n^{abi} \left(B_1 v^2 + B_2 \dot{r}^2 + B_3 \frac{M}{r} \right) \right. \right. \\ & + n^a \delta^{bi} \left(B_4 v^2 + B_5 \dot{r}^2 + B_6 \frac{M}{r} \right) + B_7 \dot{r} n^{ab} v^i + B_8 n^a v^{bi} + B_9 \dot{r} n^{ai} v^b + B_{10} v^{ab} n^i + B_{11} \dot{r} v^a \delta^{bi} \left. \right] \\ & + \frac{\dot{Q}^{ab}}{r^3} \left[B_{12} n^{ab} v^i + B_{13} \dot{r} n^{abi} + B_{14} n^{ai} v^b + B_{15} v^a \delta^{bi} + B_{16} \dot{r} n^a \delta^{bi} \right] + \frac{\ddot{Q}^{ab}}{r^2} \left[B_{17} n^{abi} + B_{18} n^a \delta^{bi} \right] \\ & \left. - B_{19} \frac{E_2^{\text{int}}}{r^2} n^i \right\} + O(c^{-4}), \end{aligned} \quad (3.5.9d)$$

with coefficients

$$\begin{aligned} B_1 &= -\frac{15}{2\chi_2} (1 + 3\eta), \quad B_2 = \frac{105\chi_1}{4}, \quad B_3 = \frac{12}{\chi_2} (5 - 2\chi_2^2), \quad B_4 = \frac{3}{\chi_2} (2 + 2\chi_2 - 3\chi_2^2), \\ B_5 &= -\frac{15}{2\chi_2} (2 - \chi_2 - \chi_2^2), \quad B_6 = -\frac{3}{\chi_2} (8 - \chi_2 - 3\chi_2^2), \quad B_7 = \frac{15}{\chi_2} (2 - \eta), \quad B_8 = -\frac{3}{2\chi_2} (7 - 2\chi_2 + 3\chi_2^2), \\ B_9 &= -\frac{15\chi_1}{2\chi_2} (1 + \chi_2), \quad B_{10} = \frac{3\chi_1}{2\chi_2}, \quad B_{11} = \frac{3}{2\chi_2} (5 - 4\chi_2 - \chi_2^2), \quad B_{12} = -\frac{3}{2\chi_2} (4 - \chi_2), \quad B_{13} = -\frac{15\chi_1}{2}, \\ B_{14} &= \frac{6}{\chi_2}, \quad B_{15} = -\frac{3\chi_1}{\chi_2}, \quad B_{16} = \frac{3}{\chi_2} (1 - 2\chi_2 - \chi_2^2), \quad B_{17} = \frac{3}{4}, \quad B_{18} = \frac{3}{2}, \quad B_{19} = 1. \end{aligned} \quad (3.5.9e)$$

In this form for the CoM-frame orbital EoM, we have used (3.3.31) to write the total 1PN-accurate mass monopole M_2 in terms of the (constant) Newtonian mass ${}^n M_2$, the internal energy E_2^{int} , and the tidal potential energy U_Q (giving a contribution to B_3). This decomposition is useful in formulating an action principle for the orbital dynamics (as in the next subsection), as E_2^{int} is independent of the orbital degrees of freedom, while M_2 is not.

II Generalized Lagrangian for the orbital dynamics

The monopole contributions (3.5.9b) to the 1PN CoM-frame orbital EoMs are known to be derivable from the Lagrangian

$$\mathcal{L}_M = \frac{\mu v^2}{2} + \frac{\mu M}{r} + \frac{\mu}{c^2} \left[\frac{1-3\eta}{8} v^4 + \frac{M}{2r} \left((3+\eta)v^2 + \eta \dot{r}^2 - \frac{M}{r} \right) \right] + O(c^{-4}), \quad (3.5.10a)$$

(see e.g. [108]). The spin contributions (3.5.9c) can also be derived from an action principle, but with a generalized Lagrangian (one depending not only on the relative position z^i and velocity $v^i = \dot{z}^i$, but also on the acceleration $a^i = \ddot{z}^i$) given by adding

$$\mathcal{L}_S = \frac{\chi_1}{c^2} \epsilon^{abc} S^a v^b \left[\frac{2M}{r^2} n^c + \frac{\chi_1}{2} a^c \right] + O(c^{-4}) \quad (3.5.10b)$$

to (3.5.10a) (see e.g. [109]). Applying the generalized Euler-Lagrange equation,

$$\left(\frac{\partial}{\partial z^i} - \frac{d}{dt} \frac{\partial}{\partial v^i} + \frac{d^2}{dt^2} \frac{\partial}{\partial a^i} \right) \mathcal{L} = 0, \quad (3.5.10c)$$

to $\mathcal{L} = \mathcal{L}_M + \mathcal{L}_S$, and using $\dot{M}_1 = \dot{M}_2 = \dot{S}^i = 0$ (which replaces (3.3.27,3.3.28,3.3.32) in the case with no quadrupole), one recovers the EoM $a^i = a_M^i + a_S^i$ from (3.5.9b,3.5.9c).

We have found that the quadrupole contributions to the orbital EoM (3.5.9d) can also be encoded in a generalized Lagrangian. To determine the necessary additions to the Lagrangian, one can proceed by guesswork, using the known Newtonian Lagrangian (3.2.27), and writing down all possible 1PN-order scalars that can be formed from the relative position z^i and velocity v^i , the total (Newtonian) mass M , and linear factors of the quadrupole Q^{ij} , its time derivative \dot{Q}^{ij} , and the internal energy E_2^{int} ; including dimensionless coefficients A_1 – A_9 for each such term, we have

$$\begin{aligned} \mathcal{L}_Q = & \frac{3M_1}{2r^3} Q^{ab} n^{ab} + \frac{1}{c^2} \left\{ \frac{M}{r^3} Q^{ab} \left[n^{ab} \left(A_1 v^2 + A_2 \dot{r}^2 + A_3 \frac{M}{r} \right) + A_4 v^{ab} + A_5 \dot{r} n^a v^b \right] \right. \\ & \left. + \frac{M}{r^2} \dot{Q}^{ab} \left[A_6 n^a v^b + A_7 \dot{r} n^{ab} \right] + E_2^{\text{int}} \left[A_8 v^2 + A_9 \frac{M}{r} \right] \right\} + O(c^{-4}). \end{aligned} \quad (3.5.10d)$$

Since the spin-orbit terms (3.5.10b) require the acceleration a^i , one might expect that terms with factors of a^i and also \ddot{Q}^{ij} should be included here; we find, however, that such terms are not necessary. The only further term allowed by general considerations but not included here is $E_2^{\text{int};2}$, as recovering the EoM (3.5.9) requires its coefficient to be zero.

By applying the Euler-Lagrange equation (3.5.10c) to the generalized Lagrangian $\mathcal{L} = \mathcal{L}_M + \mathcal{L}_S + \mathcal{L}_Q$, using the evolution equations (3.3.32) and (3.2.34) for time derivatives of S^i and E_2^{int} , one finds an EoM of the same form as (3.5.9) but with coefficients B_1 – B_{19} in (3.5.9d) given as functions of the Lagrangian coefficients A_1 – A_9 and the mass ratios χ_1 and χ_2 . Setting these coefficients equal to the values for B_1 – B_{19} given (3.5.9e) gives a system of 19 equations for the 9 unknowns A_1 – A_9 , which has the unique solution

$$\begin{aligned} A_1 &= \frac{3\chi_1}{4}(3 + \eta), & A_2 &= \frac{15\eta\chi_1}{4}, & A_3 &= -\frac{3\chi_1}{2}(1 + 3\chi_1), & A_4 &= \frac{3\chi_1^2}{2}, \\ A_5 &= -\frac{3\chi_1^2}{2}(3 + \chi_2), & A_6 &= -\frac{3\eta}{2}, & A_7 &= -\frac{3\eta}{4}, & A_8 &= \frac{\chi_1^2}{2}, & A_9 &= \chi_1. \end{aligned} \quad (3.5.10e)$$

Thus, the action principle (3.5.10) reproduces the 1PN CoM-frame equation of motion (3.5.9) for the relative position z^i —if we also make use of the evolution equations (3.3.32) and (3.2.34) for the spin S^i and internal energy E^{int} of body 2. In the next section, we discuss an action principle that leads to a closed set of evolution equations for the binary system in the adiabatic approximation.

3.6 Internal dynamics in the adiabatic approximation

I Euler-Lagrange equation for the quadrupole

We have just seen that the CoM-frame orbital EoM (3.5.9), the evolution equation for the binary’s 1PN-accurate relative position $z^i(t) = z_2^i(t) - z_1^i(t)$, can be derived from the action principle (3.5.10). We now seek to extend this action principle to incorporate the internal dynamics of the deformable body 2 in the case where the quadrupole moment is adiabatically induced by the tidal field.

In Sec. 3.2.VII, we saw how the adiabatic evolution of the quadrupole can be encoded in a Newtonian action principle; varying the action (3.2.36) with respect to the quadrupole Q^{ij} gives

$$Q^{ij} = \lambda^n G_2^{ij} + O(c^{-2}) = \lambda \frac{3M_1}{r^3} n^{<ij>} + O(c^2), \quad (3.6.1)$$

for the Newtonian-order quadrupole (cf. (3.2.35)). We also saw that, with the quadrupole given by (3.6.1), the spin evolution equation becomes $\dot{S}^i = O(c^{-2})$ (cf. Eq. 3.2.46), so that body 2 experiences no tidal torques. For this reason, in the adiabatic case (unlike in the general case), we can specialize our analysis to the case of zero spin without generating inconsistencies, which we will do for the remainder of this section.

We have found that the simple Newtonian Lagrangian (3.2.36) can be extended to govern the 1PN-accurate adiabatic evolution of Q^{ij} in a relatively straightforward manner. We consider the following Lagrangian:

$$\mathcal{L} = \mathcal{L}_{\text{orb}} + \mathcal{L}_2^{\text{int}} + O(c^{-4}),$$

$$\mathcal{L}_{\text{orb}} = \mathcal{L}_M + U^{ab}Q^{ab} + V^{ab}\dot{Q}^{ab} + WE^{\text{int}}, \quad (3.6.2a)$$

$$\mathcal{L}_2^{\text{int}} = -\frac{1}{4\lambda}Q^{ab}Q^{ab}, \quad (3.6.2b)$$

with $\mathcal{L}_M(z, v)$ given by (3.5.10a) and with $U^{ab}(z, v)$, $V^{ab}(z, v)$, and $W(z, v)$ being the coefficients appearing in \mathcal{L}_Q (3.5.10e). We have postulated (motivated by symmetry considerations) that the internal Lagrangian $\mathcal{L}_2^{\text{int}}$ can still be taken as a simple quadratic in Q^{ab} (3.6.2b), generalizing the Newtonian Lagrangian (3.2.36) only by using the 1PN-accurate value of Q^{ab} in place of its Newtonian value. (It is consistent to use here the fully relativistic value for the tidal deformability λ .) The internal energy E^{int} appearing in \mathcal{L}_{orb} (3.6.2a), which is needed only to Newtonian order, will still be given by

$$E^{\text{int}} = \frac{1}{4\lambda}Q^{ab}Q^{ab} + O(c^{-2}), \quad (3.6.3)$$

up to a constant, as in (3.2.40). To avoid explicitly introducing an additional constant contribution to the internal energy, we can absorb any such contribution into the constant ‘Newtonian’ mass monopole ${}^{\text{n}}M_2$ [cf. Eq. (3.3.31)].

Treating the relative position $z^i(t)$ and the quadrupole $Q^{ab}(t)$ as the independent dynamical variables in the Lagrangian (3.6.2), we still recover the orbital EoM (3.5.9) from the Euler-Lagrange equation for z^i (with $S^a = 0$), and that for Q^{ab} ,

$$\left(\frac{\partial}{\partial Q^{ab}} + \frac{d}{dt} \frac{\partial}{\partial \dot{Q}^{ab}} \right) \mathcal{L} = 0, \quad (3.6.4)$$

gives

$$Q^{ab} = 2\lambda(1 + W)(-U^{ab} + \dot{V}^{ab}) + O(c^{-4}). \quad (3.6.5)$$

Using the coefficients U^{ab} , V^{ab} , and W from the Lagrangian (3.6.2a,3.5.10), we find that

$$Q_{ab} = \lambda G_2^{ab} + O(c^{-4}) \quad (3.6.6a)$$

where G_2^{ab} is the body-frame gravito-electric tidal moment defined in Sec. 3.3.II and calculated in Appendix 3.9:

$$G_2^{ab} = \frac{3\chi_1 M}{r^3} n_{\langle ab \rangle} \quad (3.6.6b)$$

$$+ \frac{1}{c^2} \frac{3\chi_1 M}{r^3} \left[\left(2v^2 - \frac{5\chi_2^2}{2} \dot{r}^2 - \frac{5 + \chi_1}{2} \frac{M}{r} \right) n_{\langle ab \rangle} + v_{\langle ab \rangle} - (3 - \chi_2^2) \dot{r} n_{\langle a} v_{b \rangle} \right] + O(c^{-4}),$$

The relation (3.6.6a) between the body-frame tidal moment G_2^{ab} and the adiabatically induced quadrupole Q^{ab} , derived here from the Lagrangian (3.6.2), is the same relation typically used to define the adiabatic approximation (e.g. in [94]). The explicit expression for the tidal moment in (3.6.6c) matches those given in Refs. [47, 48, 110] when the latter are specialized to the CoM frame via (3.5.7).

II Reduced Lagrangian, equations of motion, and conserved energy

By substituting the solution (3.6.6) for the quadrupole into the Lagrangian (3.6.2), we find a reduced Lagrangian for the orbital dynamics involving only the CoM-frame orbital separation $z^i(t)$:

$$\mathcal{L}[z^i] = \frac{\mu v^2}{2} + \frac{\mu M}{r} \left(1 + \frac{\Lambda}{r^5} \right) \quad (3.6.7)$$

$$+ \frac{\mu}{c^2} \left\{ \theta_0 v^4 + \frac{M}{r} \left[v^2 \left(\theta_1 + \xi_1 \frac{\Lambda}{r^5} \right) \dot{r}^2 \left(\theta_2 + \xi_2 \frac{\Lambda}{r^5} \right) + \frac{M}{r} \left(\theta_3 + \xi_3 \frac{\Lambda}{r^5} \right) \right] \right\}$$

with

$$\Lambda = \frac{3\chi_1}{2\chi_2} \lambda, \quad (3.6.8)$$

and with the dimensionless coefficients

$$\theta_0 = (1 - 3\eta)/8, \quad \theta_1 = (3 + \eta)/2, \quad \theta_2 = \eta/2, \quad \theta_3 = -1/2,$$

$$\xi_1 = (\chi_1/2)(5 + \chi_2), \quad \xi_2 = -3(1 - 6\chi_2 + \chi_2^2), \quad \xi_3 = -7 + 5\chi_2. \quad (3.6.9)$$

While this form for the Lagrangian has been derived in harmonic gauge, we note that (some) other gauge choices lead to a Lagrangian with the same terms as in (3.6.7) but with different values of the θ and ξ coefficients. In particular, the Lagrangian derived by BDF [52] (when specialized

to 1PN accuracy and to the center-of-mass frame) has this form, as does that obtained (via a Legendre transformation) from the EOB Hamiltonian including 1PN tidal effects proposed by Damour and Nagar [94] (except that their work originally did not provide a value for the coefficient ξ_2). In Appendix 3.8, we derive the canonical transformation relating the EOB Hamiltonian to the harmonic-gauge Hamiltonian, which fixes the value of ξ_2 in the EOB Hamiltonian, and we present a separate transformation relating our results to those of BDF. We thus demonstrate the complete equivalence of all of these results at 1PN order.

The orbital EoM resulting from the Lagrangian (3.6.7) is given by

$$a^i = -\frac{Mn^i}{r} \left(1 + \frac{6\Lambda}{r^5} \right) + \frac{M}{c^2 r^2} \left[v^2 n^i \left(\phi_1 + \zeta_1 \frac{\Lambda}{r^5} \right) + \dot{r}^2 n^i \left(\phi_2 + \zeta_2 \frac{\Lambda}{r^5} \right) + \frac{M}{r} n^i \left(\phi_3 + \zeta_3 \frac{\Lambda}{r^5} \right) + \dot{r} v^i \left(\phi_4 + \zeta_4 \frac{\Lambda}{r^5} \right) \right], \quad (3.6.10)$$

with coefficients

$$\begin{aligned} \phi_1 &= 4\theta_0 - \theta_1 - 2\theta_2, & \phi_2 &= 3\theta_2, & \phi_3 &= 2(\theta_1 + \theta_2 - \theta_3), & \phi_4 &= 2(4\theta_0 + \theta_1), \\ \zeta_1 &= 2(12\theta_0 - 3\xi_1 - \xi_2), & \zeta_2 &= 8\xi_2, & \zeta_3 &= 12\theta_1 + 12\theta_2 + 2\xi_1 + 2\xi_2 - 7\xi_3, & \zeta_4 &= 12(4\theta_0 + \xi_1), \end{aligned} \quad (3.6.11)$$

for general values of the Lagrangian coefficients, and with

$$\begin{aligned} \phi_1 &= -1 - 3\eta, & \phi_2 &= 3\eta/2, & \phi_3 &= 2(2 + \eta), & \phi_4 &= 2(2 - \eta), \\ \zeta_1 &= -3(2 - \chi_2)(1 + 6\chi_2), & \zeta_2 &= 24(1 - 6\chi_2 + \chi_2^2), & \zeta_3 &= 66 + 9\chi_2 - 19\chi_2^2, \\ \zeta_4 &= 6(2 - \chi_2)(3 - 2\chi_2), \end{aligned} \quad (3.6.12)$$

in harmonic gauge.

Finally, from the Lagrangian (3.6.7), we can construct the conserved energy,

$$\begin{aligned} E &= v^i \partial \mathcal{L} / \partial v^i - \mathcal{L} \\ &= \frac{\mu v^2}{2} - \frac{\mu M}{r} \left(1 + \frac{\Lambda}{r^5} \right) \\ &\quad + \frac{\mu}{c^2} \left\{ 3\theta_0 v^4 + \frac{M}{r} \left[v^2 \left(\theta_1 + \xi_1 \frac{\Lambda}{r^5} \right) \dot{r}^2 \left(\theta_2 + \xi_2 \frac{\Lambda}{r^5} \right) - \frac{M}{r} \left(\theta_3 + \xi_3 \frac{\Lambda}{r^5} \right) \right] \right\}, \end{aligned} \quad (3.6.13)$$

which is a constant of motion of the orbital EoM (3.6.10).

As an application of these results, we can compute the gauge-invariant energy-frequency relationship for circular orbits. Using the relations $\dot{r} = 0$, $v^2 = r^2 \omega^2$, and $a^i = -r \omega^2 n^i$ for a circular

orbit, the orbital EoM (3.6.10) can be solved perturbatively, working to linear order both in the post-Newtonian parameter $1/c^2$ and in the tidal deformability parameter Λ (3.6.8), to find the radius r as a function of the orbital frequency ω . Combining this result with a similar treatment of the energy (3.6.13), we can eliminate r to find $E(\omega)$:

$$\begin{aligned}
E(\omega) &= \mu(M\omega)^{2/3} \left[-\frac{1}{2} + \frac{3\Lambda\omega^{10/3}}{M^{5/3}} + f_M \frac{(M\omega)^{2/3}}{c^2} + f_Q \frac{\Lambda\omega^4}{Mc^2} \right], \\
f_M &= \frac{1}{3}(\theta_0 + \theta_1 + \theta_3) = \frac{9 + \eta}{24}, \\
f_Q &= \frac{11}{3}(8\theta_0 + 2\theta_1 - 4\theta_3 + \xi_1 + \xi_3) = \frac{11}{6}(3 + 2\chi_2 + 3\chi_2^2).
\end{aligned}
\tag{3.6.14}$$

While the θ and ξ coefficients may take different values in different gauges, their combinations appearing here must be gauge-invariant. As $E(\omega)$ is independent of ξ_2 , we can check this result against those obtained from the EOB Hamiltonian of Damour and Nagar [94] (see Appendix 3.8), and we find that they agree.

3.7 Conclusion

We have derived the first-post-Newtonian orbital equations of motion for binary systems of bodies with spins and mass quadrupole moments, at linear order in the spin and quadrupole, and shown that they conserve the total linear momentum of the binary. After specializing these results to the binary's center-of-mass-energy frame, we have found an action principle from which the orbital equations of motion can be derived. Finally, we considered the case in which the quadrupole moment is adiabatically induced by the tidal field, giving a simplified Lagrangian and equation of motion for this case, as well as the conserved energy function and the energy-frequency relationship for circular orbits. These results are useful for the calculation of tidal effects in the gravitational wave signals from inspiralling neutron star binaries.

3.8 Appendix: Hamiltonian for the adiabatic orbital dynamics, canonical transformations, and comparison with Damour and Nagar and Bini, Damour, and Faye

Our aim here is to demonstrate the equivalence of the results for the orbital-tidal conservative dynamics given by Damour and Nagar [94], Bini, Damour and Faye [52], and the present work. We begin with a discussion of Hamiltonians and canonical transformations, then relate our results to the EOB Hamiltonian given by DN (also revisited and completed by BDF), and finish by relating the Lagrangian given by BDF to ours.

As in Sec. 3.6.II, we consider a Lagrangian for the CoM-frame orbital separation $z^i = z_2^i - z_1^i$ of the form

$$\begin{aligned} \hat{\mathcal{L}} &= \frac{v^2}{2} + \frac{1}{r} \left(1 + \frac{\Lambda}{r^5} \right) \\ &+ \frac{1}{c^2} \left[\theta_0 v^4 + \frac{v^2}{r} \left(\theta_1 + \xi_1 \frac{\Lambda}{r^5} \right) + \frac{\dot{r}^2}{r} \left(\theta_2 + \xi_2 \frac{\Lambda}{r^5} \right) + \frac{1}{r^2} \left(\theta_3 + \xi_3 \frac{\Lambda}{r^5} \right) \right] + O(c^{-4}), \end{aligned} \quad (3.8.1)$$

in units that set $M = 1$, and with the Lagrangian having been rescaled by the symmetric mass ratio, $\hat{\mathcal{L}} = \mathcal{L}/\eta$. We have shown that the harmonic-gauge values of the θ and ξ coefficients are given by Eq. (3.6.9).

The (rescaled) momentum canonically conjugate to z^i is

$$\begin{aligned} p^i = \frac{\partial \hat{\mathcal{L}}}{\partial v^i} &= v^i + \frac{1}{c^2} \left[4\theta_0 v^2 v^i + \frac{2v^i}{r} \left(\theta_1 + \xi_1 \frac{\Lambda}{r^5} \right) + \frac{2\dot{r}v^i}{r} \left(\theta_2 + \xi_2 \frac{\Lambda}{r^5} \right) \right] + O(c^{-4}) \\ &\equiv v^i + \frac{1}{c^2} \delta p^i(\mathbf{z}, \mathbf{v}) + O(c^{-4}). \end{aligned} \quad (3.8.2)$$

From a Legendre transformation of the Lagrangian $\hat{\mathcal{L}}(\mathbf{z}, \mathbf{v})$, we can construct the Hamiltonian:

$$\begin{aligned} \hat{H}(\mathbf{z}, \mathbf{p}) &= p^j v^j - \hat{\mathcal{L}}(\mathbf{z}, \mathbf{v}) \\ &= p^2 - \frac{1}{c^2} p^j \delta p^j - \hat{\mathcal{L}}(\mathbf{z}, \mathbf{p}) + \frac{1}{c^2} \frac{\partial \hat{\mathcal{L}}}{\partial v^j} \delta p^j + O(c^{-4}) \\ &= p^2 - \hat{\mathcal{L}}(\mathbf{z}, \mathbf{p}) + O(c^{-4}), \end{aligned} \quad (3.8.3)$$

having used $\delta p^i(\mathbf{z}, \mathbf{v}) = \delta p^i(\mathbf{z}, \mathbf{p}) + O(c^{-4})$. This gives

$$\begin{aligned} \hat{H} &= \frac{v^2}{2} - \frac{1}{r} \left(1 + \frac{\Lambda}{r^5} \right) \\ &- \frac{1}{c^2} \left[\theta_0 v^4 + \frac{v^2}{r} \left(\theta_1 + \xi_1 \frac{\Lambda}{r^5} \right) + \frac{\dot{r}^2}{r} \left(\theta_2 + \xi_2 \frac{\Lambda}{r^5} \right) + \frac{1}{r^2} \left(\theta_3 + \xi_3 \frac{\Lambda}{r^5} \right) \right] + O(c^{-4}), \end{aligned} \quad (3.8.4)$$

Note that the effect of the Legendre transformation has been simply to flip the sign of all terms except the first.

In Ref. [94], Damour and Nagar consider an EOB Hamiltonian of the form

$$\hat{H}_{\text{EOB}} = \frac{1}{\eta} \left[1 + 2\eta(\hat{H}_{\text{eff}} - 1) \right]^{1/2}, \quad (3.8.5)$$

with

$$\hat{H}_{\text{eff}} = \left[\frac{A}{B} \frac{p_r^2}{c^2} + A \left(1 + \frac{p_\phi^2}{c^2 r^2} \right) \right]^{1/2}, \quad (3.8.6)$$

Here, p_r and p_ϕ are the momenta conjugate to the polar coordinates (r, ϕ) in the plane of motion, related to the Cartesian momenta used above by $p_r = \mathbf{n} \cdot \mathbf{p}$ and $p_\phi^2/r^2 = p^2 - p_r^2$. The functions $A(r)$ and $B(r)$ are coefficients in the EOB effective metric [111]; $A(r)$ completely encodes the energetics of circular orbits, while $B(r)$ has effects only when $p_r \neq 0$. Damour and Nagar proposed to incorporate Newtonian and 1PN tidal effects into this EOB Hamiltonian by adding tidal terms to the radial potential A :

$$A(r) = 1 - \frac{2}{c^2 r} - \frac{2\Lambda}{c^2 r^6} \left(1 + \frac{\alpha_1}{c^2 r} \right) + O(c^{-6}), \quad (3.8.7)$$

for the quadrupole $l = 2$ case. They have computed the 1PN tidal coefficient to be

$$\alpha_1 = \frac{5}{2} \chi_2. \quad (3.8.8)$$

While they did not propose to modify the potential $B(r)$ from its point-particle value of $B = 1 + 2/c^2 r + O(c^{-4})$, we find that such a modification,

$$B(r) = 1 + \frac{2}{c^2 r} \left(1 + \beta_0 \frac{\Lambda}{r^5} \right) + O(c^{-4}) \quad (3.8.9)$$

for some coefficient β_0 , is necessary to match our results. Expanding the EOB Hamiltonian with these values for the potentials, we find a Hamiltonian of the form (3.8.4) with coefficients

$$\begin{aligned} \bar{\theta}_0 &= (1 + \eta)/8, & \bar{\theta}_1 &= (1 - \eta)/2, & \bar{\theta}_2 &= 1, & \bar{\theta}_3 &= (1 + \eta)/2, \\ \bar{\xi}_1 &= (1 - \eta)/2, & \bar{\xi}_2 &= \beta_0, & \bar{\xi}_3 &= 1 + \eta + \alpha_1, \end{aligned} \quad (3.8.10)$$

instead of the harmonic-gauge coefficients in Eq. (3.6.9).

Without tidal effects, the EOB Hamiltonian and the harmonic-gauge Hamiltonian (which coincides with the ADM Hamiltonian at 1PN order) are known to be related by a canonical transformation

[111]. Considering a 1PN-order canonical transformation with generating function G ,

$$z^i \rightarrow z^i + \frac{1}{c^2} \frac{\partial}{\partial p^i} G(\mathbf{z}, \mathbf{p}), \quad p^i \rightarrow p^i - \frac{1}{c^2} \frac{\partial}{\partial z^i} G(\mathbf{z}, \mathbf{p}), \quad (3.8.11)$$

the Hamiltonian changes by

$$\hat{H} \rightarrow \hat{H} + \frac{1}{c^2} \{\hat{H}, G\} + O(c^{-4}), \quad (3.8.12)$$

with only the Newtonian part of the Hamiltonian contributing in the Poisson bracket. We find that the most general generating function G that preserves the form of the Hamiltonian including tidal effects (3.8.4), changing its coefficients but adding no new terms, is of the form

$$G = (\mathbf{z} \cdot \mathbf{p}) \left(\gamma_1 p^2 + \gamma_2 \frac{1}{r} + \gamma_3 \frac{\Lambda}{r^6} \right) \quad (3.8.13)$$

with arbitrary constant γ coefficients. The changes in the Hamiltonian coefficients induced by the canonical transformation are

$$\begin{aligned} \Delta\theta_0 &= -\gamma_1, & \Delta\theta_1 &= \gamma_1 - \gamma_2, & \Delta\theta_2 &= 2\gamma_1 + \gamma_2, & \Delta\theta_3 &= \gamma_2, \\ \Delta\xi_1 &= 6\gamma_1 - \gamma_3, & \Delta\xi_2 &= 12\gamma_1 + 6\gamma_3, & \Delta\xi_3 &= 6\gamma_2 + \gamma_3, \end{aligned} \quad (3.8.14)$$

If we set the EOB Hamiltonian coefficients (3.8.10) equal to the harmonic Hamiltonian coefficients (3.6.9) plus the transformation parameters (3.8.14), we find that this (redundant) system of equations has a unique solution. The coefficients in the canonical transformation (3.8.13) must be

$$\gamma_1 = -\eta/2, \quad \gamma_2 = (2 + \eta)/2, \quad \gamma_3 = 2 - 9\chi_2/2 + 2\chi_2^2, \quad (3.8.15)$$

with the values for γ_1 and γ_2 matching those computed in Ref. [111], and the parameters in the EOB potentials must be

$$\alpha_1 = 5\chi_2/2, \quad \beta_0 = 3(3 - 5\eta). \quad (3.8.16)$$

The value for α_1 matches that given by Damour and Nagar, and the value for β_0 can be used to extend the range of validity of their EOB Hamiltonian to non-circular orbits.

We turn now to the more recent work of BDF [52]. They have derived a Lagrangian for the orbital-tidal conservative dynamics including terms up to 2PN order. Here, we restrict attention to the 0PN and 1PN terms, specialize their results to the center-of-mass frame, and translate their notation into ours. Their Lagrangian [whose tidal part can be found from their Eqs. (2.12), (4.3),

(4.4) and (4.10)] is of the form (3.8.1) with the same point-mass (θ) coefficients coefficients, but with tidal (ξ) coefficients

$$\tilde{\xi}_1 = -\chi_1^2/2 + 3 \quad \tilde{\xi}_2 = -3 - 3\chi_2^2 \quad \tilde{\xi}_3 = -7 + 2\chi_2. \quad (3.8.17)$$

The corresponding Hamiltonian is again given by Eq. (3.8.4). One can then verify that a canonical transformation of the form (3.8.13) with

$$\gamma_1 = \gamma_2 = 0, \quad \gamma_3 = 3\chi_2, \quad (3.8.18)$$

using (3.8.14), will transform their ξ coefficients from Eq. (3.8.17) into ours from Eq. (3.6.9). This demonstrates the complete equivalence of our respective results.

3.9 Appendix: Moment transformations and translational equations of motion

We present here the formulae that relate the body-frame multipole and tidal moments and the global-frame multipole and tidal moments. Such formulae were first derived by Damour, Soffel, and XU [29, 33, 47, 48] (see, in particular, Sec. VD of Ref. [33]). They are derived by requiring the equivalence of the body- and global-frame metrics in the body's buffer region \mathcal{B}^A . More specifically, one substitutes the expansions of the body-frame potentials (3.3.8) and the global-frame potentials (3.3.18), along with the coordinate transformation (3.3.13), into the tensor transformation law for the metric (3.3.19), using (3.3.4) to express the metrics in terms of the potentials in both coordinate systems. Matching coefficients of the resultant multipole expansions gives the moment transformation formulae. Having used the body-frame gauge conditions (3.3.15) to eliminate the worldline-data functions α_A and β_A , one finds that the transformation formulae involve only the various moments and the CoM worldlines z_A^i . A more detailed account of the procedure is given in RF [31] Sec. V.

After presenting the various moment transformation formulae in Secs. 3.9.I, 3.9.II, and 3.9.III, we show in Sec. (3.9.IV) how to use them to arrive at the translational equations of motion for an N -body system.

Note that *all* the quantities appearing below (moments and worldlines) are treated as functions of the global time coordinate t (with the argument suppressed). Any quantities originally defined as functions of the body-frame time coordinate s_A (like the body-frame moments or the worldlines z_A^i) can be converted to functions of t by using the coordinate transformation (3.3.13) at $y_A^i = 0$, which gives

$$s_A = s_A^0(t) = t - c^{-2}\alpha_A(s_A)\big|_{s_A=t} + O(c^{-4}). \quad (3.9.1)$$

This change of variables does affect the forms of any equations presented here (or elsewhere in the paper). It is important to note that our use of $f(s_A)$ and $f(t)$ to denote the same physical quantity differs from the convention of (e.g.) RF. There, symbols are used to denote functions of a particular variable, not physical quantities, and $f(s_A)$ and $f(t)$ would be different physical quantities.

I Body-frame multipole moments \rightarrow global-frame multipole moments

First, via the metric transformation law, the global-frame multipole moments $M_{g,A}^L$ and $Z_{g,A}^{iL}$ [defined by (3.3.16)] are given in terms of the body-frame moments M_A^L and S_A^L [defined by (3.3.8)] and the CoM worldlines z_A^i , by

$$M_{g,A}^L = M_A^L + \frac{1}{c^2} \left[\left(\frac{3}{2}v_A^2 - (l+1)G_{g,A} \right) M_A^L - \frac{2l^2 + 5l - 5}{(l+1)(2l+3)} v_A^j \dot{M}_A^{jL} - \frac{2l^3 + 7l^2 + 16l + 7}{(l+1)(2l+3)} a_A^j M_A^{jL} - \frac{2l^2 + 17l - 8}{2(2l+1)} v_A^{j < a_l} M_A^{L-1 > j} + \frac{4l}{l+1} v_A^j \epsilon^{jk < a_l} S_A^{L-1 > k} \right] + O(c^{-4}), \quad (3.9.2)$$

$$Z_{g,A}^{iL} = \frac{4}{l+1} \dot{M}_A^{iL} + 4v_A^i M_A^L - \frac{4(2l-1)}{2l+1} v_A^j M_A^{j < L-1 > i} \delta^{a_l > i} - \frac{4l}{l+1} \epsilon^{ji < a_l} S_A^{L-1 > j} + O(c^{-2}), \quad (3.9.3)$$

Recall that $v_A^i = \dot{z}_A^i$ and $a_A^i = \ddot{z}_A^i$; the accelerations a_A^i here (and anywhere they appear in 1PN-order terms) may be replaced here with their Newtonian values from (3.3.23). The global-frame monopole tidal moments $G_{g,A}$ appearing here (and needed only with Newtonian accuracy here) can be expressed in terms of the body-frame multipole moments of bodies $B \neq A$ and the bodies' worldlines by using Eq. (3.2.14).

The specific (nonzero) instances of these formulae needed in the M_1 - M_2 - S_2 - Q_2 system are as

follows (with $M_2^{ij} \equiv Q^{ij}$ and $S_2^i \equiv S^i$):

$$\begin{aligned}
M_{g,1} &= M_1 + \frac{M_1}{c^2} \left(\frac{3}{2}v_1^2 - \frac{M_2}{r} - \frac{3}{2r^3}n^{jk}Q^{jk} \right) + O(c^{-4}), \\
M_{g,2} &= M_2 + \frac{M_2}{c^2} \left(\frac{3}{2}v_2^2 - \frac{M_1}{r} \right) + O(c^{-4}), \\
M_{g,2}^i &= \frac{1}{c^2} \left(-\frac{1}{5}v_2^j\dot{Q}^{ij} + \frac{16M_1}{5r^2}n^jQ^{ij} + 2\epsilon^{ijk}v_2^jS^k \right) + O(c^{-4}), \\
M_{g,2}^{ij} &= Q^{ij} + \frac{1}{c^2} \left[\left(\frac{3}{2}v_2^2 - \frac{3M_1}{r} \right) Q^{ij} - \frac{17}{5}v_2^{k<i}Q^{j>k} \right] + O(c^{-4}),
\end{aligned} \tag{3.9.4}$$

and

$$\begin{aligned}
Z_{g,1}^i &= 4M_1v_1^i + O(c^{-2}), \\
Z_{g,2}^i &= 4M_2v_2^i + O(c^{-2}), \\
Z_{g,2}^{ij} &= 2\dot{Q}_2^{ij} - 2\epsilon^{ijk}S_2^k + O(c^{-2}), \\
Z_{g,2}^{ijk} &= 4v_2^iQ_2^{jk} - \frac{12}{5}v_2^aQ_2^{a<j}\delta^{k>i} + O(c^{-2}),
\end{aligned} \tag{3.9.5}$$

Here, $r = |\mathbf{z}_2 - \mathbf{z}_1|$ and $n^i = (z_2^i - z_1^i)/r$, as in (3.3.25).

II Global-frame multipole moments \rightarrow global-frame tidal moments

Next, one can express the global-frame tidal moments $G_{g,A}^L(t)$ and $Y_{g,A}^{iL}(t)$ (def) a body A in terms of the global-frame multipole moments $M_{g,A}^L$ and $Z_{g,A}^{iL}$ (def) of all the other bodies $B \neq A$ and the all the bodies' worldlines z_C^i . For this purpose, rather than working directly with $G_{g,A}^L$, it is easier to work with the tensors $F_{g,A}^L$ defined by

$$\begin{aligned}
\Phi_{g,\text{ext}} &= - \sum_{l=0}^{\infty} \frac{1}{l!} \left\{ G_{g,A}^L(x - z^A)^L + \frac{1}{2(2l+3)c^2} \partial_t^2 [G_{g,A}^L(x - z_A)^{jjL}] \right\} + O(c^{-4}) \\
&= - \sum_{l=0}^{\infty} \frac{1}{l!} \left\{ F_{g,A}^L(x - z^A)^L + \frac{1}{2(2l+3)c^2} J_{g,A}^L(x - z_A)^{jjL} \right\} + O(c^{-4}),
\end{aligned} \tag{3.9.6}$$

cf. (3.3.18). The tensor $J_{g,A}^L$ will not be needed in our calculations (and contains no extra information).

Note that $F_{g,A}^L$ and $G_{g,A}^L$ agree at Newtonian order,

$$F_{g,A}^L = G_{g,A}^L + O(c^{-2}).$$

The Newtonian part of $G_{g,A}^L$ is also given in Eq. (3.2.14).

By equating the Eq. (3.9.6) and the external part of Eq. (3.3.16), one finds that $F_{g,A}^L$ is given by

$$F_{g,A}^L = \sum_{B \neq A} \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \left[N_{g,B}^{KL} \partial_{KL}^{(A)} \frac{1}{|\mathbf{z}_A - \mathbf{z}_B|} + \frac{1}{2c^2} P_{g,B}^{KL} \partial_{K \langle L}^{(A)} |\mathbf{z}_A - \mathbf{z}_B| \right] + O(c^{-4}), \quad (3.9.7)$$

where the tensors $N_{g,A}^L$ and $P_{g,A}^L$ are given by

$$\begin{aligned} N_{g,A}^L &= M_{g,A}^L + \frac{1}{(2l+3)c^2} \left[v_A^2 M_A^L + 2v_A^j \dot{M}_A^{jL} + 2lv_A^{j \langle a_l} M_A^{L-1 \rangle j} + a_A^j M_A^{jL} \right] + O(c^{-4}), \\ P_{g,A}^L &= \ddot{M}_A^L + 2lv_A^{\langle a_l} \dot{M}_A^{L-1 \rangle} + la_A^{\langle a_l} M_A^{L-1 \rangle} + l(l-1)v_A^{\langle a_l a_{l-1}} M_A^{L-2 \rangle} + O(c^{-2}). \end{aligned}$$

Similarly, in the gravito-magnetic sector, one finds

$$Y_{g,A}^{iL} = \sum_{B \neq A} \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} Z_{g,B}^{iK} \partial_{KL}^{(A)} \frac{1}{|\mathbf{z}_A - \mathbf{z}_B|}. \quad (3.9.8)$$

For the M_1 - M_2 - S_2 - Q_2 system, using the non-zero global-frame multipole moments from the last subsection, one must compute the $l = 0, 1, 2, 3$ (resp. $l = 0, 1$) cases of (3.9.7) and the $l = 0, 1, 2$ (resp. $l = 0$) cases of (3.9.8) for $A = 2, B = 1$ (resp. $A = 1, B = 2$). The derivatives appearing here can be easily expressed in terms of $r = |\mathbf{z}_2 - \mathbf{z}_1|$ and $n^i = (z_2^i - z_1^i)/r$ via the identities

$$\begin{aligned} \partial_L^{(1)} \frac{1}{r} &= (-1)^l \partial_L^{(2)} \frac{1}{r} = \frac{1}{(2l+1)!!} \frac{n^{\langle L \rangle}}{r^{l+1}}, \\ \partial_L r &= \frac{r^2}{2l-1} \partial_L \frac{1}{r} + \frac{l(l-1)}{2l-1} \delta_{(a_l a_{l-1}} \partial_{L-2)} \frac{1}{r}. \end{aligned}$$

III Global-frame tidal moments \rightarrow body-frame tidal moments

Finally, the metric transformation law gives the body-frame tidal moments G_A^L and H_A^L (defined by (3.3.8)) in terms of the global-frame tidal moments $F_{g,A}^L$ and $Y_{g,A}^L$ (defined by (3.9.6),(3.3.18)). First, the gravito-magnetic tidal moments can be found from

$$H_A^L = Y_A^{jk \langle L-1} \epsilon^{a_l \rangle jk},$$

with the tensors Y_A^{iL} given by

$$Y_A^{iL} = (\delta_{jK}^{iL} - \delta_{jK}^{\langle iL \rangle})(Y_{g,A}^{jK} - 4v_A^j {}^n G_{g,A}^{K} - l! \Lambda_{\zeta}^{jK}). \quad (3.9.9)$$

Here, δ_{jK}^{iL} is the multi-Kronecker delta ($\delta_{jK}^{iL} T^{jK} = T^{iL}$), and $\delta_{jK}^{<iL>}$ is the STF projector ($\delta_{jK}^{<iL>} T^{jK} = T^{<iL>}$), and the nonzero 'inertial moments' Λ_ζ^{iL} are

$$\begin{aligned}\Lambda_\zeta^i &= -2G_{g,A} v_A^i, \\ \Lambda_\zeta^{ij} &= -\frac{3}{2} v_A^{[i} a_A^{j]} - 2v_A^{<i} a_A^{j>} - \frac{4}{3} \dot{G}_{g,A} \delta^{ij}, \\ \Lambda_\zeta^{ijk} &= -\frac{6}{5} \delta^{i<j} \dot{a}_A^{k>}.\end{aligned}$$

Then, the gravito-electric tidal moments are given by

$$\begin{aligned}{}^{\text{pn}}G_A^L &= F_{g,A}^L + -l! \Lambda_\Phi^L + \frac{1}{c^2} \left[\dot{Y}_{g,A}^{<L>} - v_A^j Y_{g,A}^{jL} + (2v_A^2 - lG_{g,A}) G_{g,A}^L - (l/2) v_A^{j<ai} G_{g,A}^{L-1>j} \right. \\ &\quad \left. + (l-4) v_A^{<ai} \dot{G}_{g,A}^{L-1>} - (l^2 - l + 4) a_A^{<ai} G_{g,A}^{L-1>} - (l-1)! \dot{\Lambda}_\zeta^{<L>} \right] + O(c^{-4}),\end{aligned}\quad (3.9.10)$$

for $l \geq 1$ and by $G_A = 0$ for $l = 0$ (cf. (3.3.15c)). The non-zero inertial moments Λ_Φ^L needed here are

$$\begin{aligned}\Lambda_\Phi^i &= a_A^i + \frac{1}{c^2} \left[(v_A^2 + G_{g,A}) a_A^i + \frac{1}{2} v_A^{ij} a_A^j + 2\dot{G}_{g,A} v_A^i \right], \\ \Lambda_\Phi^{ij} &= \frac{1}{c^2} \left(-\frac{1}{2} a_A^{<ij>} + v_A^{<i} \dot{a}_A^{j>} \right).\end{aligned}$$

IV Translational equations of motion

The results of Secs. 3.9.I, 3.9.II, and 3.9.III allow one to express the body-frame tidal moments G_A^L and H_A^L for a given body A in terms of the body-frame multipole moments M_B^L and S_B^L and CoM worldlines z_B^i of all bodies B in an N -body system (eliminating all reference to the global frame moments). With this done, one can find the body's translational equation of motion, written only in terms of the M_B^L , S_B^L , and z_B^i , by using the law of motion (3.3.22b) for the body-frame mass dipole M_A^i .

As the body-frame gauge condition (3.3.15a) requires $M_A^i = 0$ (fixing the body's center of mass-energy to the body-frame origin), one proceeds by setting the right-hand side of (3.3.22b) to zero. This yields an expression for the acceleration $a_A^i = \ddot{z}_A^i$ of the body's 1PN-accurate global-frame CoM worldline $z_A^i(t)$. To see this more clearly, we can explicitly evaluate the $l = 0$ case of the first term on the RHS of (3.3.22b) using the moment transformation formulae presented above. The

result is

$$\begin{aligned}
M_A a_A^i &= M_A F_{g,A} + \sum_{l=2}^{\infty} \frac{1}{l!} M_A^L G_A^L + \frac{1}{c^2} g_A^i + \frac{1}{c^2} \left[\dot{Y}_{g,A}^i - v_A^j Y_{g,A}^{ji} + (2v_A^2 - G_{g,A}) G_{g,A}^i \right. \\
&\quad \left. - \frac{1}{2} v_A^{ij} G_{g,A}^j - (v_A^2 + 3G_{g,A}) a_A^i - \frac{1}{2} v_A^{ij} a_A^j - 3\dot{G}_{g,A} v_A^i \right] + O(c^{-4}), \tag{3.9.11}
\end{aligned}$$

where g_A^i represents the terms on the RHS of Eq. (3.3.22b) except for the first.

3.10 Appendix: Derivation of system mass multipole moment formulae

We derive here Eq. (3.4.6), which gives the mass multipole moments M_{sys}^L of an N -body system in terms of its global frame multipole moments $M_{g,A}^L$ and $Z_{g,A}^L$. This parallels the derivation given in Sec. (3.4.I) for the moments Z_{sys}^L . We begin by using the Taylor series (3.4.4) to rewrite the global-frame scalar potential Φ_g as given in Eq. (3.4.1a) in the form

$$\begin{aligned}
\Phi_g &= - \sum_A \sum_{l,k=0}^{\infty} \frac{(-1)^{l+k}}{l!k!} \left[M_{g,A}^L z_A^K \partial_{LK} \frac{1}{|\mathbf{x}|} + \frac{1}{2c^2} \partial_t^2 (M_{g,A}^L z_A^K \partial_{LK} |\mathbf{x}|) \right] \tag{3.10.1} \\
&= - \sum_A \sum_{p=0}^{\infty} \sum_{k=0}^p \frac{(-1)^p}{p!} \frac{p!}{k!(p-k)!} \left[M_{g,A}^{\langle P-K} z_A^K \rangle \partial_P \frac{1}{|\mathbf{x}|} + \frac{1}{2c^2} \partial_t^2 \left(M_{g,A}^{(P-K} z_A^K) \right) \partial_P |\mathbf{x}| \right].
\end{aligned}$$

To bring this into the form (3.4.2a), which gives Φ_g in terms of the system moments, we must decompose $\partial_P |\mathbf{x}|$ into its STF and trace parts. Using the identity

$$\partial_{ijL} |x| = \partial_{\langle ijL \rangle} |x| + \frac{(l+1)(l+2)}{2l+3} \delta_{(ij} \partial_L) \frac{1}{|x|}, \tag{3.10.2}$$

and making the index change $P \rightarrow ijL$, but only in the term resulting from the trace part of $\partial_P |x|$, we find

$$\begin{aligned}
\Phi_g &= - \sum_A \sum_{p=0}^{\infty} \sum_{k=0}^p \frac{(-1)^p}{p!} \frac{p!}{k!(p-k)!} \left[M_{g,A}^{\langle P-K} z_A^K \rangle \partial_P \frac{1}{|\mathbf{x}|} + \frac{1}{2c^2} \partial_t^2 \left(M_{g,A}^{\langle P-K} z_A^K \rangle \right) \partial_{\langle P \rangle} |x| \right] \\
&\quad - \frac{1}{2c^2} \sum_A \sum_{l=0}^{\infty} \sum_{k=0}^{l+2} \frac{(-1)^l}{(l+2)!} \frac{(l+2)!}{k!(l+2-k)!} \partial_t^2 \left(M_{g,A}^{(ijL-K} z_A^K) \right) \frac{(l+1)(l+2)}{2l+3} \delta_{ij} \partial_L \frac{1}{|x|}. \tag{3.10.3}
\end{aligned}$$

Though the sum over l should start at $l = -2$ after $P \rightarrow ijL$, the $l = -1, -2$ terms are killed by the $(l+1)(l+2)$ factor. In the $\sum_{k=0}^{l+2}$ term, the k indices K are to be chosen from the $l+2$ indices

ijL . This term can be simplified by explicitly performing the symmetrization over all $l + 2$ indices; using the fact that $M_{g,A}^L$ and $\partial_L |\mathbf{x}|^{-1}$ are STF, we have

$$M_{g,A}^{(ijL-K} z_A^K) \delta^{ij} \partial_L \frac{1}{|\mathbf{x}|} = \left[\frac{2k(l+2-k)}{(l+2)(l+1)} M_{g,A}^{j<L-(K-1)} z_A^{K-1>j} + \frac{k(k-1)}{(l+2)(l+1)} M_{g,A}^{<L-(K-2)} z_A^{K-2>jj} \right] \partial_L \frac{1}{|\mathbf{x}|}. \quad (3.10.4)$$

Using this identity, relabeling $K - 1 \rightarrow K$ in the first term and $K - 2 \rightarrow K$ in the second term, and adjusting summations appropriately, we find that the second line of (3.10.3) can be written as

$$- \frac{1}{2(2l+3)c^2} \sum_A \sum_{l=0}^{\infty} \sum_{k=0}^l \frac{(-1)^l}{l!} \frac{l!}{k!(l-k)!} \partial_t^2 \left(2M_{g,A}^{j<L-K} z_A^{K>j} + M_{g,A}^{<L-K} z_A^{K>jj} \right) \partial_L \frac{1}{|x|}. \quad (3.10.5)$$

Finally, we can compare (3.10.3) (with the second line replaced by (3.10.5)) to the expression for Φ_g given in (3.4.2a); we see that the system's 1PN-accurate mass multipoles must be given by

$$M_{\text{sys}}^L = \sum_A \sum_{k=0}^l \frac{l!}{k!(l-k)!} \left[M_{g,A}^{<L-K} z_A^{K>} + \frac{1}{c^2} \frac{1}{2(2l+3)} \partial_t^2 \left(2M_{g,A}^{j<L-K} z_A^{K>j} + M_{g,A}^{<L-K} z_A^{K>jj} \right) \right] - \frac{1}{c^2} \frac{2l+1}{(l+1)(2l+3)} \dot{\mu}_{\text{sys}}^L + O(c^{-4}),$$

as in Eq. (3.4.6).

CHAPTER 4

FIRST-POST-NEWTONIAN TIDAL EFFECTS IN THE GRAVITATIONAL WAVEFORM FROM BINARY INSPIRALS

The gravitational wave signal from an inspiralling binary neutron star system will contain detailed information about tidal coupling in the system, and thus, about the internal physics of the neutron stars. To extract this information will require highly accurate models for the gravitational waveform. We present here a calculation of the gravitational wave signal from a binary with quadrupolar tidal interactions which includes all post-1-Newtonian-order effects in both the conservative dynamics and wave generation. We consider stars with adiabatically induced quadrupoles moving in circular orbits, and work to linear order in the stars' quadrupole moments. We find that post-1-Newtonian corrections increase the tidal signal by approximately 20% at gravitational wave frequencies of 400 Hz.

4.1 Introduction

I Background and motivation

Inspiralling and coalescing binary neutron stars are key sources for ground-based gravitational wave (GW) detectors [59]. An important science goal in the detection of such sources is to obtain robust information on the highly uncertain equation of state (EoS) of neutron star matter [14]. The effects of the EoS on the GW signal are largest during the late inspiral and merger stages of binary evolution, at GW frequencies $\gtrsim 500$ Hz, and the strong gravity and complex hydrodynamics

involved in these regimes require the use of fully relativistic numerical simulations for their study (see e.g. Refs. [19, 15] and references therein). A small but clean EoS signature will also be present in the early inspiral waveform, at frequencies $\lesssim 500$ Hz within LIGO’s most sensitive band, arising from the effects of tidal coupling [91]. The relative weakness of orbital gravity in this regime makes it possible to construct good approximate waveforms using post-Newtonian-based analytic models [21].

For point-particle models of binary inspiral, analytic gravitational waveforms have been computed to 3PN accuracy [97], and spin effects have been computed to 2PN accuracy [112].¹ More recent efforts to improve the analytic description of neutron star binary GW signals by including tidal effects began with Refs. [91, 34], which used a leading-order model of the tidal coupling and GW emission to demonstrate the potential feasibility of measuring EoS effects in inspiralling neutron stars in the low frequency ($\lesssim 400$ Hz) regime with Advanced LIGO. (See also Refs. [113, 93] for earlier analyses of tidal effects in inspiralling binaries.) The tidal contribution to the GW signal computed in Ref. [91] depends on a single tidal deformability parameter λ , which characterizes the star’s deformation response to a static (or adiabatically changing) tidal field and which is sensitive to the star’s EoS.

The quadrupolar tidal deformability λ was defined in a fully relativistic context and calculated for a variety of EoS models in Refs. [91, 34, 35, 37, 36], and Refs. [37, 36] extended the analysis to include higher-multipolar tidal responses of both electric- and magnetic-type. It was found in Ref. [35] that Advanced LIGO should be able to constrain the neutron stars’ tidal deformability to $\lambda \lesssim (1.2 \times 10^{37} \text{ g cm}^2 \text{ s}^2)(D/100 \text{ Mpc})$ with 95% confidence, for a binary of two $1.4 M_{\odot}$ neutron stars at a distance D from the detector, using only the portion of the signal with GW frequencies less than 400 Hz. The calculations of λ for a $1.4 M_{\odot}$ neutron star in Refs. [34, 35, 37, 36], using several different equations of state, give values in the range $0.03\text{--}1.0 \times 10^{37} \text{ g cm}^2 \text{ s}^2$, so nearby events may allow Advanced LIGO to place useful constraints on candidate equations of state.

To detect or constrain the tidal deformability λ will require models for the tidal contribution to the GW signal that are accurate to $\lesssim 10\%$, much less than the current uncertainty in λ . References [91, 35] estimate the fractional corrections to the tidal signal at GW frequencies below 400 Hz due to

¹ The shorthand n PN, for post- n -Newtonian, is used to describe corrections of order c^{-2n} relative to Newtonian gravity, where c is the speed of light.

several effects neglected by the model of the GW phasing used in Ref. [91], namely, non-adiabaticity ($\lesssim 1\%$), higher multipolar tidal coupling ($\lesssim 0.7\%$), nonlinear hydrodynamic effects ($\lesssim 0.1\%$), spin effects ($\lesssim 0.3\%$), nonlinear response to the tidal field ($\lesssim 3\%$), viscous dissipation (negligible), and post-Newtonian effects ($\lesssim 10\%$). The largest expected corrections, from post-Newtonian effects in the orbital dynamics and GW emission, are thus essential for an accurate analysis of the tidal signal. These corrections will depend on the neutron star physics only through the same tidal deformability parameter λ used in the Newtonian treatment and thus can be easily incorporated into the same data analysis methods used in the Newtonian (tidal) case.

The extension of the tidal signal calculation to 1PN order was recently discussed in Ref. [94] by Damour and Nagar (DN). Working within the framework of the effective-one-body (EOB) formalism, DN gave a complete description of the 1PN conservative dynamics of tidally interacting binaries in circular orbits, parametrized the forms of further 1PN corrections to the GW emission, and made comparisons with numerical simulations (see also Ref. [96]). The 1PN conservative dynamics has also been recently studied in Ref. [114] by Vines and Flanagan (VF). Working from the formalism for 1PN celestial mechanics developed in Refs. [29, 33] and extended by Ref. [31], VF found the explicit equations of motion and action principle for generic orbits and generic evolution of the bodies' quadrupoles. Specializing to adiabatically induced quadrupoles and circular orbits, the results of VF agree with those of DN for the 1PN conservative dynamics. The construction of the 1PN metric given by VF also allows for explicit computation of the binary system's 1PN-accurate mass multipole moments.

In the present paper, we use the results of VF [114] to derive the 1PN-accurate GW signal from an inspiralling binary with quadrupolar tidal interactions. Working to linear order in the stars' quadrupole moments, and using adiabatically induced quadrupoles and circular orbits, we compute the binary's binding energy and GW energy flux and use them to determine the phase evolution of the emitted GW signal in the stationary phase approximation. The results presented here can be used to extend the validity of analytic GW signals to higher frequencies, and to provide useful information for hybrid schemes that attempt to bridge the gap in frequencies between analytic inspiral models and the start of numerical simulations, such as the EOB formalism of Ref. [94]. Our expressions for the orbital equations of motion and binding energy may also be useful for the construction of quasi-equilibrium initial data for numerical simulations [115]. We note that the 1PN

corrections calculated here slightly improve the prospects for detection of tidal effects in binary GW signals, as they increase the tidal signal by $\sim 20\%$ at GW frequencies of 400 Hz. The studies in Refs. [99, 96] have recently made use of our results in making assessments of the measurability of tidal effects and in calibrating analytic models against numerical simulations.

II Organization

The organization of this paper is as follows. In Sec. 4.2, we briefly state the key results of Ref. [114] for the 1PN conservative dynamics of a binary in which one member has a mass quadrupole moment. We specialize to the adiabatic limit and circular orbits and compute the gauge-invariant binding energy as a function of orbital frequency. In Sec. 4.3, we consider the gravitational radiation and obtain the 1PN tidal corrections to the radiated energy flux. We then compute the resulting 1PN tidal corrections to the phase of the Fourier transform of the waveform in the stationary phase approximation and conclude in Sec. 4.4 with a short discussion of the results.

III Notation and conventions

We use units where Newton's constant is $G = 1$, but retain factors of the speed of light c , with $1/c^2$ serving as the formal expansion parameter for the post-Newtonian expansion. We use lowercase latin letters a, b, i, j, \dots for indices of spatial tensors. Spatial indices are contracted with the Euclidean metric, $v^i w^i = \delta_{ij} v^i w^j$, with up or down placement of the indices having no meaning. We use angular brackets to denote the symmetric, trace-free projection of tensors, e.g. $T^{<ab>} = T^{(ab)} - \frac{1}{3} \delta^{ab} T^{cc}$. We also use multiple indices to denote tensorial powers of vectors, e.g. $v^{ij} = v^i v^j$, $n^{<ijk>} = n^{<i} n^j n^{k>}$, etc.

4.2 Conservative dynamics in the adiabatic limit

In this section we briefly review the key results of VF [114] concerning the 1PN conservative dynamics of a binary system with quadrupolar tidal coupling. For simplicity, we consider a binary composed of one point-mass (body 1) and one deformable star (body 2). Since we consistently work to linear order in the quadrupole, our results can be easily generalized to the case of two deformable

bodies by interchanging body labels. The binary's orbital dynamics can be formulated in terms of the separation (three-)vector $z^i = z_2^i - z_1^i$ between the bodies, the bodies' masses M_1 and M_2 , and the quadrupole moment Q_2^{ij} of body 2.

The 1PN-accurate worldlines $x^i = z_1^i(t)$ and $x^i = z_2^i(t)$ of the bodies' centers of mass-energy and their separation $z^i(t) = z_2^i(t) - z_1^i(t)$ are defined in a 'global' 1PN coordinate system (t, x^i) . The global coordinates are conformally Cartesian and harmonic, and they tend to inertial coordinates in Minkowski spacetime as $|\mathbf{x}| \rightarrow \infty$. Also, the binary system's center of mass-energy is taken to be at rest at the origin $x^i = 0$ (the system's 1PN-accurate mass dipole moment is set to zero), so that the (t, x^i) coordinates correspond to the center-of-mass-energy frame of the system. We use the following notation for the relative position, velocity, and acceleration:

$$z^i = z_2^i - z_1^i, \quad r = |\mathbf{z}| = \sqrt{\delta_{ij} z^i z^j}, \quad n^i = z^i/r,$$

$$v^i = \dot{z}^i, \quad \dot{r} = v^i n^i, \quad a^i = \ddot{z}^i,$$

with dots denoting derivatives with respect t .

We take M_1 and M_2 to be the bodies' conserved rest masses,² and we define the total mass M , mass fractions χ_1, χ_2 , reduced mass μ , and symmetric mass ratio η by

$$M = M_1 + M_2, \quad \chi_1 = M_1/M,$$

$$\chi_2 = M_2/M, \quad \mu = \eta M = \chi_1 \chi_2 M. \quad (4.2.1)$$

Note that there are only two independent parameters among these quantities; we will tend to express our results in terms of the total mass M and the mass fraction χ_2 of the deformable body, unless factorizations make it more convenient to use $\chi_1 = 1 - \chi_2$ or $\eta = \chi_1 \chi_2$.

The tidal deformation of body 2 is described by its 1PN-accurate Blanchet-Damour [49] mass quadrupole moment $Q_2^{ij}(t)$. We will work in the limit where the quadrupole is adiabatically induced by the tidal field; i.e. we assume that the quadrupole responds to the instantaneous tidal field according to

$$Q_2^{ij}(t) = \lambda G_2^{ij}(t). \quad (4.2.2a)$$

²Note that the mass M_2 used here is not the 1PN-accurate Blanchet-Damour [49] mass monopole moment (which was called M_2 in VF [114]); rather, the M_2 used here is the conserved part of the BD mass monopole (called ${}^{\text{c}}M_2$ in VF [114]). The full 1PN-accurate monopole also receives contributions from the body's internal elastic energy (and from the tidal gravitational potential energy), which for a deformable body, will vary as tidal forces do work on the body. The effects of these time-dependent contributions to the monopole have been separately accounted for in the Lagrangian (4.2.3), and the mass M_2 appearing there is constant.

Here, the constant λ is the tidal deformability,³ and $G_2^{ij}(t)$ is the quadrupolar gravito-electric DSX [33] tidal moment of body 2 which encodes the leading order ($l = 2$) tidal field felt by body 2. For the binary system under consideration, the tidal moment is given by

$$G_2^{ij} = \frac{3\chi_1 M}{r^3} n^{<ij>} + \frac{1}{c^2} \frac{3\chi_1 M}{r^3} \left[\left(2v^2 - \frac{5\chi_2^2}{2} r^2 - \frac{6 - \chi_2}{2} \frac{M}{r} \right) n^{<ij>} + v^{<ij>} - (3 - \chi_2^2) \dot{r} n^{<ij>} \right] + O(c^{-4}) + O(\lambda). \quad (4.2.2b)$$

With the quadrupole given by Eqs. (4.2.2) in the adiabatic limit, the only independent degree of freedom is the binary's relative position $z^i(t)$. It was shown by VF [114] that the evolution of $z^i(t)$ is governed by the Lagrangian

$$\mathcal{L}[z^i] = \frac{\mu v^2}{2} + \frac{\mu M}{r} \left(1 + \frac{\Lambda}{r^5} \right) + \frac{\mu}{c^2} \left\{ \theta_0 v^4 + \frac{M}{r} \left[v^2 \left(\theta_1 + \xi_1 \frac{\Lambda}{r^5} \right) + \dot{r}^2 \left(\theta_2 + \xi_2 \frac{\Lambda}{r^5} \right) + \frac{M}{r} \left(\theta_3 + \xi_3 \frac{\Lambda}{r^5} \right) \right] \right\} + O(c^{-4}) + O(\lambda^2), \quad (4.2.3)$$

with $\Lambda = (3\chi_1/2\chi_2)\lambda$, and with the dimensionless coefficients

$$\begin{aligned} \theta_0 &= (1 - 3\eta)/8, & \theta_1 &= (3 + \eta)/2, & \theta_2 &= \eta/2, & \theta_3 &= -1/2, \\ \xi_1 &= (\chi_1/2)(5 + \chi_2), & \xi_2 &= -3(1 - 6\chi_2 + \chi_2^2), & \xi_3 &= -7 + 5\chi_2. \end{aligned} \quad (4.2.4)$$

The orbital equation of motion resulting from this Lagrangian, via $(d/dt)(\partial\mathcal{L}/\partial v^i) = \partial\mathcal{L}/\partial z^i$, is given by

$$a^i = -\frac{Mn^i}{r} \left(1 + \frac{6\Lambda}{r^5} \right) + \frac{M}{c^2 r^2} \left[v^2 n^i \left(\phi_1 + \zeta_1 \frac{\Lambda}{r^5} \right) + \dot{r}^2 n^i \left(\phi_2 + \zeta_2 \frac{\Lambda}{r^5} \right) + \frac{M}{r} n^i \left(\phi_3 + \zeta_3 \frac{\Lambda}{r^5} \right) + \dot{r} v^i \left(\phi_4 + \zeta_4 \frac{\Lambda}{r^5} \right) \right] + O(c^{-4}) + O(\lambda^2), \quad (4.2.5)$$

with coefficients

$$\begin{aligned} \phi_1 &= 4\theta_0 - \theta_1 - 2\theta_2, & \phi_2 &= 3\theta_2, & \phi_3 &= 2(\theta_1 + \theta_2 - \theta_3), & \phi_4 &= 2(4\theta_0 + \theta_1), \\ \zeta_1 &= 2(12\theta_0 - 3\xi_1 - \xi_2), & \zeta_2 &= 8\xi_2, & \zeta_3 &= 12\theta_1 + 12\theta_2 + 2\xi_1 + 2\xi_2 - 7\xi_3, & \zeta_4 &= 12(4\theta_0 + \xi_1), \end{aligned} \quad (4.2.6)$$

The conserved energy constructed from the Lagrangian (4.2.3) is

$$\begin{aligned} E &= v^i \partial\mathcal{L}/\partial v^i - \mathcal{L} \\ &= \frac{\mu v^2}{2} - \frac{\mu M}{r} \left(1 + \frac{\Lambda}{r^5} \right) + \frac{\mu}{c^2} \left\{ 3\theta_0 v^4 + \frac{M}{r} \left[v^2 \left(\theta_1 + \xi_1 \frac{\Lambda}{r^5} \right) + \dot{r}^2 \left(\theta_2 + \xi_2 \frac{\Lambda}{r^5} \right) - \frac{M}{r} \left(\theta_3 + \xi_3 \frac{\Lambda}{r^5} \right) \right] \right\} + O(c^{-4}) + O(\lambda^2), \end{aligned} \quad (4.2.7)$$

³The tidal deformability is related to the dimensionless order-unity Love number k_2 [93] and the star's areal radius R by $\lambda = 2k_2 R^5/3$.

which is a constant of motion of the equation of motion (4.2.5).

The orbital equation of motion (4.2.5) admits solutions of the form

$$z^i(t) = rn^i(t) = r(\cos(\omega t), \sin(\omega t), 0), \quad (4.2.8a)$$

with $\dot{r} = 0$, $v^2 = r^2\omega^2$ and $a^i = -r\omega^2 n^i$, corresponding to circular orbits in the x - y plane with frequency ω . For later convenience, we introduce the unit vector ϕ^i in the direction of the velocity v^i , which satisfies

$$\dot{z}^i = v^i = r\omega\phi^i, \quad \dot{n}^i = \omega\phi^i, \quad \dot{\phi}^i = -\omega n^i, \quad n^i\phi^i = 0, \quad (4.2.8b)$$

for circular orbits. Working to linear order both in the post-Newtonian parameter c^{-2} and in the tidal deformability parameter λ , Eqs. (4.2.5) and (4.2.8) yield the radius-frequency relationship

$$r(\omega) = \frac{M^{1/3}}{\omega^{2/3}} \left[1 + \frac{3\chi_1}{\chi_2} \hat{\lambda} + \frac{\eta - 3}{3} x - \frac{\chi_1}{2\chi_2} (6 - 26\chi_2 + \chi_2^2) x \hat{\lambda} \right] + O(c^{-4}) + O(\lambda^2). \quad (4.2.9)$$

Here, we have introduced the ω -dependent dimensionless quantities

$$\hat{\lambda} \equiv \frac{\lambda\omega^{10/3}}{M^{5/3}}, \quad x \equiv \frac{(M\omega)^{2/3}}{c^2}, \quad (4.2.10)$$

which characterize the fractional corrections due to tidal effects and to post-Newtonian effects. We note that the parameter $\hat{\lambda}$ takes values in the range $\sim 1 - 8 \times 10^{-5}$ for two $1.4M_\odot$ stars at orbital frequencies of 200 Hz, given the range of λ values from Ref. [35], while $x \sim 0.07$. Using Eqs. (4.2.7), (4.2.8) and (4.2.9), we can also find the gauge-invariant energy-frequency relationship for circular orbits:

$$E(\omega) = \mu(M\omega)^{2/3} \left[-\frac{1}{2} + \frac{9\chi_1}{2\chi_2} \hat{\lambda} + \frac{9 + \eta}{24} x + \frac{11\chi_1}{4\chi_2} (3 + 2\chi_2 + 3\chi_2^2) x \hat{\lambda} \right] + O(c^{-4}) + O(\lambda^2). \quad (4.2.11)$$

This expression for the binding energy can be directly compared with Eqs. (37,38,50-57) of DN [94], and indicates that their parameter $\bar{\alpha}'_1$ giving the 1PN tidal contribution to the binding energy should have the value $\bar{\alpha}'_1 = (11/18)(3 + 2\chi_2 + 3\chi_2^2)$ instead of $55\chi_2/18$ (note the the quantity denoted here by χ_2 is denoted by X_A in DN [94]). For the case of equal masses ($\chi_1 = \chi_2 = 1/2$, $\eta = 1/4$), the binding energy (4.2.11) simplifies to

$$E(\omega) = -\frac{M^{5/3}\omega^{2/3}}{8} \left[1 - \frac{37}{48} x - 18\hat{\lambda} \left(1 + \frac{209}{72} x \right) \right].$$

For orbital frequencies of 200 Hz (GW frequencies of 400 Hz) and total mass $M = 2.8M_\odot$, the 1PN fractional correction to the Newtonian tidal term in the binding energy is $(209/72)x \approx 19\%$.

4.3 Gravitational radiation

The energy flux from the binary due to gravitational radiation is determined by the time variation of the binary system's multipole moments [21]. The flux \dot{E} to 3.5PN-order (or to 1PN-order relative to the leading 2.5PN flux) is given in terms of the total system's mass quadrupole moment $Q_{\text{sys}}^{ij}(t)$, current quadrupole moment $S_{\text{sys}}^{ij}(t)$, and mass octupole moment $Q_{\text{sys}}^{ijk}(t)$ by

$$\begin{aligned} \dot{E} = & -\frac{1}{5c^5}(\partial_t^3 Q_{\text{sys}}^{ij})^2 - \frac{1}{189c^7}(\partial_t^4 Q_{\text{sys}}^{ijk})^2 \\ & + \frac{16}{45c^7}(\partial_t^3 S_{\text{sys}}^{ij})^2 + O(c^{-8}), \end{aligned} \quad (4.3.1)$$

cf. Eq. (223) of Ref. [21].

The binary system's multipole moments can be computed from the asymptotic form of the global metric, as in Sec. IV of Ref. [114]. The mass quadrupole Q_{sys}^{ij} , which is needed to 1PN accuracy in the flux formula (4.3.1), can be found from Eqs. (4.6,4.5,B4,B5,6.1) of VF [114]; the result is

$$\begin{aligned} Q_{\text{sys}}^{ij} = & Q_2^{ij} + \mu r^2 n^{<ij>} + \frac{\mu r^2}{c^2} \left\{ n^{<ij>} \left[v^2 \left(\tau_1 + \sigma_1 \frac{\lambda}{r^5} \right) + \dot{r}^2 \left(\tau_2 + \sigma_2 \frac{\lambda}{r^5} \right) + \frac{M}{r} \left(\tau_3 + \sigma_3 \frac{\lambda}{r^5} \right) \right] \right. \\ & \left. + v^{<ij>} \left(\tau_4 + \sigma_4 \frac{\lambda}{r^5} \right) + \dot{r} n^{<i} v^{j>} \left(\tau_5 + \sigma_5 \frac{\lambda}{r^5} \right) \right\} + O(c^{-4}) + O(\lambda^2), \end{aligned} \quad (4.3.2a)$$

where the 1PN-accurate body quadrupole Q_2^{ij} is given by Eqs. (4.2.2) above and the dimensionless τ and σ coefficients are given by

$$\begin{aligned} \tau_1 = (29/42)(1 - 3\eta), \quad \tau_2 = 0, \quad \tau_3 = (1/7)(8\eta - 5), \quad \tau_4 = (11/21)(1 - 3\eta), \\ \tau_5 = (4/7)(3\eta - 1), \quad \sigma_1 = 13\chi_1^2/7\chi_2, \quad \sigma_2 = 185\chi_1^2/14\chi_2, \\ \sigma_3 = -(3\chi_1/14\chi_2)(8 + 23\chi_2 + 13\chi_2^2), \quad \sigma_4 = 38\chi_1^2/7\chi_2, \quad \sigma_5 = -151\chi_1^2/7\chi_2. \end{aligned} \quad (4.3.2b)$$

This result holds for generic orbits (in a binary where body 2 has an adiabatically induced quadrupole). Using Eqs. (4.2.2) for the body quadrupole, Eqs. (4.2.8) to specialize to circular orbits, and the radius-frequency relationship (4.2.9), the system quadrupole simplifies to

$$Q_{\text{sys}}^{ij} \frac{\eta M^{5/3}}{\omega^{4/3}} \left[n^{<ij>} \left(1 + \sigma_0 \hat{\lambda} + \tau_6 x + \sigma_6 x \hat{\lambda} \right) + \phi^{<ij>} \left(\tau_4 x + \sigma_7 x \hat{\lambda} \right) \right] + O(c^{-4}) + O(\lambda^2), \quad (4.3.3a)$$

with τ_4 as in Eq. (4.3.2b), and with

$$\begin{aligned} \sigma_0 = (3/\chi_2)(3 - 2\chi_2), \quad \tau_6 = -(85 + 11\eta)/42, \\ \sigma_6 = (1/14\chi_2)(4 + 56\chi_2 + 264\chi_2^2 - 219\chi_2^3), \\ \sigma_7 = (1/7\chi_2)(103 - 252\chi_2 + 302\chi_2^2 - 132\chi_2^3). \end{aligned}$$

The expression (4.3.3a) for the total quadrupole determines the unknown 1PN correction coefficient introduced in Eq. (71) of DN [94].⁴

Similarly, the system's mass octupole and current quadrupole, which are needed only to Newtonian order, are given by

$$\begin{aligned} Q_{\text{sys}}^{ijk} &= \mu r^3 n^{<ijk>} \left[(\chi_1 - \chi_2) + \frac{9\chi_1 \lambda}{\chi_2 r^5} \right] \\ &= \frac{\eta M^2}{\omega^2} n^{<ijk>} \left[(\chi_1 - \chi_2) + \frac{18\chi_1^2 \hat{\lambda}}{\chi_2} \right], \end{aligned} \quad (4.3.4)$$

and

$$\begin{aligned} S_{\text{sys}}^{ij} &= \mu r^2 \epsilon^{kl<i} n^{j>k} v^l \left[(\chi_1 - \chi_2) + \frac{9\chi_1 \lambda}{2\chi_2 r^5} \right] \\ &= \frac{\eta M^2}{\omega} \epsilon^{kl<i} n^{j>k} \phi^l \left[(\chi_1 - \chi_2) + \frac{9\chi_1(3 - 4\chi_2) \hat{\lambda}}{2\chi_2} \right]. \end{aligned} \quad (4.3.5)$$

Here, the first equalities hold for generic orbits, and the second equalities hold for circular orbits; also, both Q_{sys}^{ijk} and S_{sys}^{ij} are subject to $O(c^{-2}) + O(\lambda^2)$ corrections.

Having gathered the expressions (4.3.3a), (4.3.4) and (4.3.5) for the system multipole moments, we can insert them into the flux formula (4.3.1). Using also Eqs. (4.2.8) to for the time derivatives of n^i and ϕ^i (which are the only time-dependent quantities in the final expressions for the multipoles), and working out the STF projections and contractions (e.g. $n^{<ijk>} n^{<ijk>} = 2/5$) using the STF identities from (e.g.) Ref. [31], we find the GW energy flux from the binary to be given by

$$\begin{aligned} \dot{E}(\omega) &= -\frac{32}{5} \eta^2 x^{5/2} \left[1 - \left(\frac{1247}{336} + \frac{35\eta}{12} \right) x + \frac{6(3 - 2\chi_2)}{\chi_2} \hat{\lambda} \right. \\ &\quad \left. + \frac{1}{28\chi_2} \left(-704 - 1803\chi_2 + 4501\chi_2^2 - 2170\chi_2^3 \right) x \hat{\lambda} + O(c^{-3}) + O(\lambda^2) \right]. \end{aligned} \quad (4.3.6)$$

The coefficients for the 1PN point-mass (second) and Newtonian tidal (third) terms match those given in Refs. [21, 91].

Using energy balance and the stationary phase approximation [59, 60], the Fourier transform of the gravitational waveform can be written as $h = \mathcal{A} e^{i\psi}$, with the phase $\psi(\omega)$ determined from the

⁴The parametrization of the tidal contribution to the system quadrupole given in Eqs. (68-71) of DN [94] does not quite match the form given in Eq. (4.3.3a) here, as no $\phi^{<ij>}$ term is included. Also, their parametrization leaves some dependence on the radius r , while ours eliminates r in favor of the gauge invariant quantity ω . Still, as the coefficients of $x \hat{\lambda} n^{<ij>}$ and $x \hat{\lambda} \phi^{<ij>}$ in our Eq. (4.3.3a) end up additively combined in the final contribution to the energy flux, one could in principle determine an effective value for the coefficient β_1 in Eq. (71) of DN [94] that would lead to the correct flux \dot{E} .

binding energy $E(\omega)$ and flux $\dot{E}(\omega)$ as functions of the orbital frequency ω by the relation

$$\frac{d^2\psi}{d\omega^2} = \frac{2}{\dot{E}} \frac{dE}{d\omega}. \quad (4.3.7)$$

Taking \dot{E} from Eq. (4.3.6), finding $dE/d\omega$ from a derivative of Eq. (4.2.11), and integrating twice (dropping unimportant integration constants) yields the phase:

$$\begin{aligned} \psi(\omega) &= \frac{3}{128\eta x^{5/2}} \left[1 + \psi_{0,1}\hat{\lambda} + \psi_{1,0}x + \psi_{1,1}x\hat{\lambda} + O(c^{-3}) + O(\lambda^2) \right] \\ &= \frac{3c^5}{128\eta(M\omega)^{5/3}} \left[1 + \psi_{0,1} \frac{\lambda\omega^{10/3}}{M^{5/3}} + \psi_{1,0} \frac{(M\omega)^{2/3}}{c^2} + \psi_{1,1} \frac{\lambda\omega^4}{Mc^2} + O(c^{-3}) + O(\lambda^2) \right], \end{aligned} \quad (4.3.8a)$$

with coefficients

$$\begin{aligned} \psi_{0,1} &= -\frac{24}{\chi_2}(1 + 11\chi_1), & \psi_{1,0} &= \frac{20}{9} \left(\frac{743}{336} + \frac{11\eta}{4} \right), \\ \psi_{1,1} &= -\frac{5}{28\chi_2} (3179 - 919\chi_2 - 2286\chi_2^2 + 260\chi_2^3). \end{aligned} \quad (4.3.8b)$$

The above results concern a binary where only one body (body 2) develops a tidally induced quadrupole, with quadrupolar tidal deformability $\lambda_2 = \lambda$. For the case of two deformable bodies, the contribution to the tidal signal from the other body (body 1) can simply be added to the phase by interchanging body labels ($1 \leftrightarrow 2$) in the tidal terms. For the case of equal masses and identical equations of state, $M_1 = M_2 = M/2$ and $\lambda_1 = \lambda_2 = \lambda$, the phase correction is

$$\psi(\omega) = \frac{3}{32x^{5/2}} \left[1 - 624\hat{\lambda} + \frac{2435}{378}x - \frac{3115}{2}x\hat{\lambda} \right].$$

The 1PN correction increases the tidal signal by $\approx 17\%$ at gravitational wave frequencies of 400Hz for $M = 2.8M_\odot$.

From the expressions (4.3.6) and (4.2.11) for the gravitational wave luminosity $\dot{E}(\omega)$ and the binding energy $E(\omega)$, it is straightforward to construct the phase $\varphi(t)$ of the time-domain gravitational waveform based on the various PN Taylor approximants used in several approaches to interfacing analytical and numerical relativity [116]. We provide here the explicit expressions for the Taylor T4 approximant, in which the function $\mathcal{F} \equiv \dot{E}/(dE/dx)$ is expanded in a Taylor series and the differential equations

$$\frac{dx}{dt} = \mathcal{F}, \quad \frac{d\varphi}{dt} = 2x^{3/2}/M, \quad (4.3.9)$$

are integrated numerically [with x as in (4.2.10)]. The tidal contribution to the function \mathcal{F}^{T4} adds linearly to the 3.5PN point mass terms and is given to 1PN order by

$$\mathcal{F}_{\text{tidal}}^{\text{T4}} = \frac{32\chi_1\lambda_2}{5M^6} \left[12(1 + 11\chi_1)x^{10} + \left(\frac{4421}{28} - \frac{12263}{28}\chi_2 + \frac{1893}{2}\chi_2^2 - 661\chi_2^3 \right) x^{11} \right] + (1 \leftrightarrow 2). \quad (4.3.10)$$

We note that the Newtonian and 1PN tidal terms in the expressions (4.3.8) and (4.3.10) for the waveform phasing are formally of 5PN and 6PN orders, having the same frequency dependencies as x^5 and x^6 . However, the coefficients of the tidal terms are not of order unity, but rather scale as $(R/M)^5 \sim 10^5$, where R and M are the stellar mass and radius (see footnote 3). As discussed in more detail in Refs. [93, 94, 99, 96], this makes the magnitudes of the tidal terms comparable to those of 3PN-order and 3.5PN-order point-mass terms, which have also been computed analytically and are now routinely included in waveform templates [97]. We also note that including the 1PN tidal terms presented here in waveform templates will (slightly) change the constraints on λ achievable by Advanced LIGO quoted in the introduction. We refer the reader to Refs. [99, 96] for analyses of the measurability of tidal effects in inspiral waveforms including 1PN tidal terms.

4.4 Discussion and Conclusions

We have provided the 1PN accurate description of quasi-circular binary inspiral with quadrupolar tidal coupling and obtained the 1PN tidal contributions to the phasing of the emitted gravitational radiation in the low-frequency, adiabatic limit. Our results show that 1PN effects increase the tidal corrections by approximately 20% at gravitational wave frequencies of 400 Hz in the case of two $1.4M_\odot$ stars. These results should be of use in constructing GW measurement templates and can be easily be incorporated into the EOB formalism as discussed by DN [94]; the unknown coefficients introduced by DN pertaining to 1PN quadrupolar tidal effects have been determined here. Our results for the 1PN tidal corrections to GW generation have been used in Refs. [99, 96] in making comparisons between numerical simulations and analytic models of the inspiral waveform and in assessing the measurability of tidal effects.

While we have restricted attention here to the case of circular orbits, the results necessary to compute the GW signal for generic orbits can all be found in this paper. This work could also be extended to consider 1PN tidal coupling at higher multipolar orders; the necessary machinery (and

the template of the quadrupolar case) is fully contained in VF [114].

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