SCHOOL OF OPERATIONS RESEARCH
AND INDUSTRIAL ENGINEERING
COLLEGE OF ENGINEERING
CORNELL UNIVERSITY
ITHACA, NEW YORK 14853

TECHNICAL REPORT NO. 904

May 1990

DISTRIBUTIONS ARISING IN THE MODELLING OF ENVIRONMENTAL PROCESSES

by

Svetlozar T. Rachev
University of California at Santa Barbara

Gennady Samorodnitsky
Cornell University

Keywords and phrases: environmental processes, limit theorem, infinitely divisible distribution, class L, stable distribution, geometric stable distribution, rate of convergence

*The second author was supported by the ONR grant N00014-90-J-1287

**AMS 1980 subject classification. Primary 60E07, 60F05. Secondary 60B12, 60K30.

ABSTRACT

We study a probabilistic model of effect of environmental changes on crop production proposed by Todorovic and Gani (1987). We discuss the limit behavior of total and maximal crop yields, describe the limit distributions arising in certain special cases and estimate the rate of convergence to the limit.

INTRODUCTION

The following probabilistic model of effect of environmental changes on crop production is considered. Let X_n denote the annual crop yield for the nth year for a given area. The sequence of crop yields can be modelled as

$$X_n = Y_n \prod_{i=1}^{n} Z_i, \; n = 1, 2, \ldots, \tag{1.1}$$

where Y's are nonnegative r.v.'s and Z's are i.i.d. independent of Y's. $\{Y_n\}$ is treated as a sequence of "ideal" annual yields (i.e. in the case when the environmental effect is not taken into account). In (1.1), Z_i is the proportion of crop yield maintained in year i as the result of the environmental effect from the previous year due to a "bad" year ($Z_i < 1$) or "good" year ($Z_i \geq 1$). In the case of $Z_i < 1$ (with Z_i treated as the effect of soil errosion during the year i - 1) Todorovic and Gani (1987) showed that the total crop yield over an n-year period,

$$S_n = \sum_{i=1}^{n} X_i = \sum_{i=1}^{n} Y_i \prod_{j=1}^{i} Z_j, \tag{1.2}$$

has a strong limit S under the regularity condition $\mathbb{E}\, Y_1 < \infty$. They also found the distribution of S for some special cases of d.f.'s F_{Z_1} and F_{Y_1}. Puri (1987) obtained a criterion for a.s. convergence of S_n.

Similarly to (1.2), the maximum

$$M_n = \bigvee_{i=1}^{n} X_i = \bigvee_{i=1}^{n} Y_i \prod_{j=1}^{i} Z_j \tag{1.3}$$

is viewed as the maximum crop yield for a period of n years. For $0 \leq Z \leq 1$ Todorovich (1987) showed the existence of a weak limit for M_n.

Suppose that in each year a disastrous event may occur with probability $p \in (0, 1)$. As a result, of the disastrous event (for example, drought) the whole production could be devastated. The "disastrous year" $\tau = \tau(p)$ is assumed to be geometrically distributed:

$$P(\tau(p) = k) = (1 - p)p^{k-1},\ k = 1, 2, \ldots, \tag{1.4}$$

and independent of Y's and Z's. Assuming that Y_i's are iid the above implies that the total crop yield (until the year τ) is

$$S_\tau = \sum_{i=1}^{\tau} X_i \overset{d}{=} \sum_{i=1}^{1+\delta\tau} X_i \tag{1.5}$$

(δ being Bernoulli with success probability $(1 - p)$ and independent of the rest r.v.'s)

$$= X_1 + \delta \sum_{k=2}^{1+\tau} X_k \overset{d}{=} (Y + \delta\, S_\tau)\, Z,$$

where $Y(\overset{d}{=} Y_1)$, $Z\ (\overset{d}{=} Z_1)$, S_τ and δ are assumed to be independent. (See Rachev and Todorovich (1990) for some examples of distributions of S_τ; if $Z \equiv 1$, S_τ is said to be geometrically infinitely divisible r.v., see Klebanov, Maniya, Melamed (1989).)

The disastrous event could be caused by the cumulative effect of a large number of "bad" events with high success probability, $p \approx 1$. This leads to a negative binomial model for the disastrous year $\tau = \tau_1 + \ldots + \tau_r$ where τ_i's are i.i.d. geometric with mean $1/p$. For $p \to 1$, $r \to \infty$, $r(1 - p) \to \lambda > 0$ we have the Poisson approximation $P(\tau = n + r) \approx e^{-\lambda}\lambda^n/n!$ This, in turn, leads to a Poisson model for the disastrous year. Assuming that N_i are i.i.d. $\text{Poiss}(\lambda_i)$ and $T_i = N_1 + \ldots + N_i$ is viewed as the ith disastrous year, of interest are the distributions of

$$S_{T_1} = \sum_{i=0}^{T_1} X_i,\quad S_{T_{k+1}} - S_{T_k} = \sum_{i=T_k+1}^{T_{k+1}} X_i. \tag{1.6}$$

For the sake of convenience we start the process in (1.6) at year 0, i.e. we consider the sequences $\{Y_n\}_{n\ge 0}$, $\{Z_n\}_{n\ge 0}$ and $X_n = Y_n \prod_{i=0}^{n} Z_i$, $n \ge 0$.

Similarly to (1.5) and (1.6) we shall be interested in the laws of

$$M_\tau = \bigvee_{i=1}^{\tau} X_i \tag{1.7}$$

and

(1.8) $$M_{T_1} = \bigvee_{i=0}^{T_1} X_i, \quad \bigvee_{i=T_k+1}^{T_k} X_i, \quad k= 1, 2,\dots .$$

Our results regarding the functionals (1.2), (1.3), (1.5)-(1.8) fall into three main categories. First, we prove (in Section 2) limit theorems for the above functionals. Second (in Section 3) we provide a characterization of some classes of distributions arising as limits in (1.1) and (1.3). Third (in Section 4) we consider the multidimensional setting. Given a family of crop years $\{ A_t, t \in T\}$ we will be interested in the limit behaviors of the random fields

$$\{S_n(t) := \sum_{i=1}^{n} Y_i(t) \prod_{j=1}^{i} Z_j(t), \; t \in T\},$$

and $M_n(\cdot)$ where n may be random. This approach is an example of how one can consider the problems discussed in sections 2 and 3 in a more general setting.

2. CENTRAL LIMIT THEOREMS

Let $\{Y_n\}_{n\geq 0}$ be a sequence of nonnegative i.i.d. r.v.'s, $P(Y_0 > 0) > 0$, and $\{ Z_n\}_{n\geq 0}$ is an independent of it sequence of positive i.i.d. r.v.'s. Set X_n, S_n and M_n as in (1.1) -(1.3). Denote $\xi_j = \log Z_j$, $\nu = \mathbb{E}\, \xi_0$ (when exists).

Lemma 2.1 Suppose that $\mathbb{E} \log (1 + Z_0) < \infty$.

(a) If $\nu \geq 0$ then w.p.1, X_n does not converge to 0 and thus $S_n \to \infty$. Moreover, $M_n \to \infty$ if $P(Z_0 = 1) \neq 1$.

(b) If $-\infty < \nu < 0$ the following are equivalent as $n \to \infty$:

(bi) $X_n \to 0$ a.s.;

(bii) S_n converges to a finite limit a.s.;

(biii) M_n converges to a finite limit a.s;

(biv) $0 \leq \mathbb{E} \log (1 + Y_0) < \infty$.

Moreover (biv) implies (bi)-(biii) even if $\nu = -\infty$.

Proof If $\nu > -\infty$, then

(2.1) $$X_n = Y_n e^{n\nu} \exp\{\sum_{j=1}^{n} (\xi_j - \nu)\}.$$

(a) Let first $\nu > 0$. By SSLN, for n large, $|\sum_{j=1}^{n} (\xi_j - \nu)| \le n\,\nu/2$ and thus $X_n \ge Y_n e^{n\nu/2}$, proving (a) in this case. Let now $\nu = 0$. Using (2.1) again and the fact that $\sum_{j=1}^{n} \xi_j$ does not converge to $-\infty$ w.p.1, it is clear that X_n w.p.1 does not converge to 0 as is claimed in (a). If $Z_j \ne 1$ then $X_n = Y_n \exp\{\sum_{j=1}^{n} \xi_j\}$ has with probability 1, infinite maximum as $n \to \infty$.

(b) If $\nu > -\infty$, then our statement follows from a simple modification (accounting for $P(Z_j > 1) > 0$, in general) of Theorems 1 and 2 of Puri (1987).
For $\nu = -\infty$ choose a finite negative K with $-\infty < \mathbb{E}(\xi_1 \vee K) < 0$. Set $W_j = Z_j \vee e^K$. Thus, as before, $\sum_{n=1}^{\infty} Y_n \prod_{j=1}^{n} W_j < \infty$ a.s., and since $Z_j \le W_j$, (bii) holds. □

Our next theorem is the CLT for the "total crop yield" S_n in (1.2). We assume that $\xi_j = \log Z_j$ belongs to the domain of attraction of an α-stable r.v. η_α $(1 < \alpha \le 2)$, i.e. there exist $a_n > 0$ and $b_n \in \mathbb{R}$ such that

(2.2) $$a_n \sum_{i=1}^{n} \xi_i + b_n \overset{\mathfrak{D}}{\Rightarrow} \eta_\alpha, \qquad a_n = n^{-1/\alpha} L(n),$$

$L(n)$ is a slowly varying function.

Theorem 2.1 Suppose that $\mathbb{E} \log (1 + Z_0) < \infty$.

(a) If $\nu > 0$ then $a_n \log S_n + b_n \overset{\mathfrak{D}}{\Rightarrow} \eta_\alpha$.

(b) If $\nu < 0$ then $a_n \log (S - S_n) + b_n \overset{\mathfrak{D}}{\Rightarrow} \eta_\alpha$.

(c) Let $\nu = 0$, and assume (without loss of generality), that $b_n \equiv 0$. Suppose also that

(2.3) $$P(\log Y_1 > 1/a_n) = o(n^{-1}), \; n \to \infty.$$

Then, as $n \to \infty$,

(2.4) $$a_n \log S_n \overset{\mathfrak{D}}{\Rightarrow} \sup_{o \le t \le 1} \mathcal{L}(t),$$

where $\mathcal{L}$ is a Levy stable motion on [0,1] with $\mathcal{L}(1) \overset{d}{=} \eta_\alpha$.

Proof (a) From Lemma 2.1 (a), $S_n \to \infty$ a.s. Let U_n stand for the first X_k, $k > n$, different from 0, i.e. $U_n = X_{R_n}$ where $R_n = \min\{k > n, X_k \neq 0\}$.

Now (2.2) implies the weak convergence

(2.5) $$a_n \log U_n + b_n = a_n \log Y_{R_n} + a_n \sum_{j=n+1}^{R_n} \xi_j + a_n \sum_{j=1}^{n} \xi_j + b_n \overset{\mathfrak{D}}{\Rightarrow} \eta_\alpha.$$

On the other hand for any $\epsilon > 0$,

(2.6) $$\lim_{n\to\infty} P(a_n \log (S_n/U_n) > \epsilon) = \lim_{n\to\infty} P(a_n \log (S_{R_n}/X_{R_n}) > \epsilon)$$

$$= \lim_{n\to\infty} P\Big(\sum_{i=1}^{n-1} Y_i \prod_{j=i+1}^{n} Z_j^{-1} > (e^{\epsilon/a_n} - 1)\, Y_{R_n} \prod_{j=n+1}^{R_n} Z_j\Big) = 0.$$

The last equality follows from Lemma 2.1 (b), as $\sum_{i=1}^{n} Y_i \prod_{j=1}^{i} Z_j^{-1}$ converges to a finite limit.

(b) The proof is similar to that of part (a).

(c) Let $m_n = a_n \max_{m=1,\dots,n} \sum_{j=1}^{m} \xi_j$, $n = 1, 2, \dots$ Then for any $\lambda > 0$

(2.7) $$P(S_n > \lambda) \le P((Y_1 + \dots + Y_n)\exp\{a_n^{-1} m_n\} > \lambda).$$

Define the processes $\mathcal{L}_n(\cdot) = a_n \sum_{j=1}^{[n\cdot]} \xi_j$ in $\mathfrak{D}[0, 1]$. Then by (2.2), $\mathcal{L}_n \overset{w}{\Rightarrow} \mathcal{L}$ as $n \to \infty$, in the Skorokhod topology. Thus, as $n \to \infty$, $m_n \vee 0 = \sup_{0 \le t \le 1} \mathcal{L}_n(t) \overset{\mathfrak{D}}{\Rightarrow} \sup_{0 \le t \le 1} \mathcal{L}(t)$.

Observe that $P(m_n \ge 0) \to 1$ and, therefore,

(2.8) $$m_n \overset{\mathfrak{D}}{\Rightarrow} \sup_{0 \le t \le 1} \mathcal{L}(t).$$

Invoking (2.7), we obtain

(2.9) $$P(a_n \log S_n > u) \leq P(a_n \log (Y_1+...+ Y_n) + m_n > u).$$

Next let us show that $a_n \log(Y_1+...+Y_n) \xrightarrow{P} 0$ provided that (2.3) holds: for any $\epsilon > 0$, $\delta > 0$ and n large,

$$P(a_n \log (Y_1+...+Y_n) > \epsilon) \leq P(\max_{i=1,...,n} Y_i > \tfrac{1}{n} \exp\{\epsilon/a_n\})$$

$$\leq 1 - (1 - P(\log Y_1 > \tfrac{1}{2} \epsilon/a_n))^n \leq 1 - (1 - \delta n^{-1})^n \approx 1 - e^{-\delta}.$$

Letting $\delta \to 0$, we obtain $a_n \log(Y_1+...+Y_n) \xrightarrow{P} 0$. Thus

(2.10) $$\overline{\lim_{n\to\infty}} \frac{P(a_n \log S_n > u)}{P(\sup_{0\leq t\leq 1} \mathcal{L}(t) > u)} \leq 1.$$

The opposite inequality is trivial in the case $P(Y_1 > 0) = 1$. Indeed, in this case we can use

$$P(S_n > \lambda) \geq P(Y_1 \exp\{a_n^{-1} m_n\} > \lambda),$$

which gives the right estimate in the same way as above. The general case $0 < P(Y_1 > 0) < 1$ is much more involved.

We start with the case $\alpha = 2$ in (2.2). For $1 \leq k \leq n$ let $m_n^{(k)}$ be the kth biggest value among $a_n \sum_{j=1}^{m} \xi_j$, $m=1,...,n$ ($m_n^{(1)} = m_n$). Then, if $n > 2k$,

(2.11) $$|m_n - m_n^{(k)}| \leq (k - 1)a_n^{-1}| \max_{i=1,...,n} \xi_i| \xrightarrow[n\to\infty]{P} 0.$$

Therefore, for any $k = 1, 2, ...,$ and $n > 2k$

$$P(S_n > \lambda) \geq P(\bigcup_{i=1}^{k} \{Y_i \exp\{a_n^{-1} m_n^{(i)}\} > \lambda\}),$$

$$\geq \sum_{i=1}^{k} p(1 - p)^{i-1} P(\tilde{Y}_1 \exp\{a_n^{-1} m_n^{(i)}\}>\lambda),$$

where $p = P(Y_1 > 0)$ and where $\tilde{Y}_1$ has the conditional distribution of Y_1 given $Y_1 > 0$. Therefore, by (2.8) and (2.11)

$$P(a_n \log S_n > u) \geq \sum_{i=1}^{k} p(1-p)^{i-1} P(\tilde{Y}_1 \exp\{a_n^{-1} m_n^{(i)}\} > \exp\{a_n^{-1} u\})$$

$$= \sum_{i=1}^{k} p(1-p)^{i-1} P(a_n \log \tilde{Y}_1 + m_n^{(i)} > u)$$

$$\to (1-(1-p)^k)\, P(\sup_{0 \leq t \leq 1} \mathcal{L}(t) > u).$$

Since this is true for every $k = 1, 2, \ldots,$ we obtain the required counterpart to (2.10).

The case $1 < \alpha < 2$ requires a different argument, as (2.11) fails in this case. Let $T_0 = 0$, $T_m = \inf\{k > T_{m-1}: Y_k > 0\}$, $m = 1, 2\ldots,$ and let $W_m = \sum_{i=T_{m-1}+1}^{T_m} \xi_i$, $m = 1, 2, \ldots$. Then W_1, $W_2, \ldots$ is a sequence of i.i.d. random variables such that

$$\lim_{\lambda\to\infty} \frac{P(W_1 > \lambda)}{P(\xi_1 > \lambda)} = \lim_{\lambda\to\infty} \frac{P(W_1 < -\lambda)}{P(\xi_1 < -\lambda)} = \frac{1}{p}.$$

Let now $\tilde{\mathcal{L}}_n(t) = a_n \sum_{j=1}^{[nt]} W_j$, $0 \leq t \leq 1$. Then $\tilde{\mathcal{L}}_n \overset{w}{\Rightarrow} p^{-1/\alpha} \mathcal{L}$ as $n \to \infty$ in the Skorokhod topology of $\mathcal{D}[0,1]$. For $n = 1, 2, \ldots$ define

$$N_n = \# \text{ of } T_i\text{'s such that } T_i \leq n.$$

By the SLLN,

$$\frac{N_n}{n} \underset{n\to\infty}{\to} p \text{ a.s.} \tag{2.12}$$

We have

$$P(S_n > \lambda) = P(\sum_{m=1}^{N_n} Y_{T_m} \exp\{\sum_{j=1}^{m} W_j > \lambda\})$$

$$\geq P(\max_{i=1,\ldots,n} Y_i > 0)\, P(Y_{T_1} e^{V_n} > \lambda) = (1-(1-p)^n)\, P(Y_{T_1} e^{V_n} > \lambda),$$

where $V_n = \max_{m=1,\ldots,N_n} \sum_{j=1}^{m} W_j$. Let $\approx{\mathcal{L}}_n(t) = a_{N_n} \sum_{j=1}^{[N_n t]} W_j$, $0 \leq t \leq 1$. It follows from (2.12) and Theorem 17.1 of Billingsley (1968) (easily modifiable to our case) that $\approx{\mathcal{L}}_n \overset{w}{\Rightarrow} p^{-1/\alpha} \mathcal{L}$ as $n \to \infty$ is $\mathcal{D}[0, 1]$. Therefore, $a_{N_n} V_n \overset{w}{\Rightarrow} p^{-1/\alpha} \sup_{0\leq t\leq 1} \mathcal{L}(t)$. Note also that by (12.12), $a_{N_n}/a_n \to p^{1/\alpha}$ a.s. as $n \to \infty$. Therefore,

$$P(a_n \log S_n > u) \ge (1 - (1 - p)^n)P\,(Y_{T_1} e^{V_n} > \exp\{a_n^{-1} u\})$$
$$= (1 - (1 - p)^n)P(a_n \log Y_{T_1} + a_n V_n > u) \underset{n\to\infty}{\to} P(\sup_{0\le t\le 1} \mathcal{L}(t) > \lambda).$$

This completes the proof of the theorem. □

As far as the maximal yield (1.3) for n years is concerned we have the following analogue of Theorem 2.1.

Theorem 2.2. Under the assumptions of Theorem 2.1 the following holds:

(a) If $\nu \ge 0$ then $a_n \log M_n + b_n \overset{\mathfrak{D}}{\Rightarrow} \eta_\alpha$;

(b) If $\nu < 0$ then $a_n \log(\bigvee_{j>n} X_j) + b_n \overset{\mathfrak{D}}{\Rightarrow} \eta_\alpha$;

(c) If $\nu = 0$ and (2.4) holds then $a_n \log M_n \overset{\mathfrak{D}}{\Rightarrow} \sup_{0\le t\le 1} \mathcal{L}(t)$.

Proof (a) and (c) follow from the corresponding assertions in Theorem 2.1 observing that $\frac{S_n}{n} \le M_n \le S_n$, and the proof of (b) uses arguments similar to those in Theorem 2.1. □

Next we examine the geometric random sum S_τ given by (1.5) and (1.4). We say that $\xi_0 = \log Z_0$ belongs to the domain of attraction of geometric α-stable r.v. G_α if there exist functions $a = a(p) > 0$ and $b = b(p)$ on [0, 1] such that

$$a \sum_{i=1}^{\tau} (\xi_i + b) \overset{\mathfrak{D}}{\Rightarrow} G_\alpha \text{ as } p \to 0. \tag{2.13}$$

Here $a(p) = p^{1/\alpha} L(1/p)$, where L is slowly varying function. The ch.f. of G_α admits the representation $f_{G_\alpha}(t) = 1/(1 - \log \phi_\alpha (t))$, where ϕ_α is the ch.f. of an α-stable r.v. (Klebanov, Maniya and Melamed (1984)). Similarly, $f_{\xi_0} = 1/(1 - \log \psi)$, where ψ is the ch.f. of a distribution in the domain of attraction of α-stable r.v. with ch.f. ϕ_α (Mittnik and Rachev (1990)). Examples of geometric α-stable distribution are the exponential law ($\alpha = 1$) and the Laplace law ($\alpha = 2$).

Theorem 2.3. Suppose that $\mathbb{E} \log (1 + Z_0) < \infty$ and (2.13) holds.

(a) If $\nu > 0$ then $a(\log S_\tau - \tau b) \overset{\mathfrak{D}}{\Rightarrow} G_\alpha$ as $p \to 0$;

(b) If $\nu < 0$ then $a(\log \sum_{j\ge\tau+1} X_j - \tau b) \overset{\mathfrak{D}}{\Rightarrow} G_\alpha$ as $p \to 0$;

(c) If $\nu = 0$ and

$$P(\log Y_1 > n^{1/\alpha} L(n)^{-1}) = o(n^{-1}), \quad n \to \infty,$$

then

$$a \log S_\tau \overset{\mathcal{D}}{\Rightarrow} \sup_{0 \le t \le 1} \mathcal{G}(t) \quad \text{as } p \to o. \tag{2.14}$$

Here $\mathcal{G}$ is a "geometric Levy stable motion", i.e. is the weak limit in $\mathcal{D}[0,1]$ of

$$\mathcal{G}_p(t) = a \sum_{j=1}^{[\tau t]} \xi_j, \quad 0 \le t \le 1.$$

The proof is similar to that of Theorem 2.1 and thus omitted.

Remark 2.1. Regarding the existence of the process $\mathcal{G}$ as the weak limit of $\mathcal{G}_p$ one can check the following:

(a) The finite dimensional distribution $(\mathcal{G}_p(t_1),\dots, \mathcal{G}_p(t_d))$ $(0 \le t_1 < \dots < t_d \le 1)$ converges to a "geometric strictly stable distribution" $(\mathcal{G}(t_1),\dots,\mathcal{G}(t_d))$ with ch.f. $g(\theta)$ of the form $1/(1 - \log \psi(\theta))$, where $\psi(\theta)$ is the ch.f. of a strictly α-stable random vector on $\mathbb{R}^d$;

(b) The set of laws of $\mathcal{G}_p(\cdot)$ $(o < p < 1)$ is tight.

Similarly to Theorems 2.2 and 2.3 we obtain the following limit theorem for the distribution of the maximal crop yield.

Theorem 2.4 Under the assumptions of Theorem 2.3 the following holds:

(a) If $\nu > 0$ then $a(\log M_\tau - \tau b) \overset{\mathcal{D}}{\Rightarrow} G_\alpha$ as $p \to 0$;

(b) If $\nu < 0$ then $a(\log \bigvee_{j=\tau+1} X_j - \tau b) \overset{\mathcal{D}}{\Rightarrow} G_\alpha$ as $p \to 0$;

(c) Suppose that the conditions of Theorem 2.3 (c) holds. Then, as $p \to 0$,

$$a \log M_\tau \overset{\mathcal{D}}{\Rightarrow} \sup_{0 \le t \le 1} \mathcal{G}(t).$$

Finally, let us consider the total crop yield until a Poisson (λ) random moment $T = T(\lambda)$. Suppose, as before, that Y_i's are i.i.d., Z_i's are i.i.d. and T is independent of Y_i's and Z_i's. Suppose also that the ch.f. f_{ξ_0} of $\xi_0 = \log Z_0$ satisfies

(2.15) $$\lim_{u\to 0} |u|^{-\alpha}(1 - f_{\xi_0 - a}(u)) = \mu$$

for some $\mu > 0$, real a and $0 < \alpha \leq 2$.

Theorem 2.5. Suppose that (2.15) holds and let

$$S_T = \sum_{i=0}^{T} X_i = \sum_{i=0}^{T} Y_i \prod_{j=0}^{i} Z_j.$$

(a) If $\nu = \mathbb{E}\xi_0 > 0$ then, as $\lambda \to \infty$,

(2.16) $$\lambda^{-1/\alpha} (\log S_T - aT) \overset{\mathfrak{D}}{\Rightarrow} Y_{(\alpha)},$$

where $Y_{(\alpha)}$ is symmetric stable r.v. with ch.f. $\exp\{-\mu|\theta|^{\alpha}\}$;

(b) If $\nu < 0$ then, as $\lambda \to \infty$,

(2.17) $$\lambda^{-1/\alpha}(\log \sum_{j=T}^{\infty} X_j - aT) \overset{\mathfrak{D}}{\Rightarrow} Y_{(\alpha)},$$

(2.18) $$\lambda^{-1/\alpha} (\log \sum_{j=T_1}^{T_k} X_j - aT_1) \overset{\mathfrak{D}}{\Rightarrow} Y_{(\alpha)},$$

where T_1, T_k are as in (1.6);

(c) Let $\nu = 0$ and $P(\log Y_1 > u) = o(u^{-\alpha})$ as $u \to \infty$. Then, as $\lambda \to \infty$,

$$\lambda^{-1/\alpha} \log S_T \overset{\mathfrak{D}}{\Rightarrow} \sup_{0\leq t\leq 1} \mathcal{L}(t),$$

where $\mathcal{L}(\cdot)$ is a Levy stable motion on [0, 1] with $\mathcal{L}(1) \overset{d}{=} Y_{(\alpha)}$.

Proof. For simplicity we assume that $P(Y_1 > 0) = 1$. The problems arising from $P(Y_1 = 0) > 0$ can be handled as in the proof of Theorem 2.1.

(a) Since $\nu > 0$ then by Lemma 2.1, $S_T \to \infty$ as $\lambda \to \infty$.

Claim. Under (2.15),

$$\lambda^{-1/\alpha} \sum_{i=0}^{T} \eta_i \overset{\mathfrak{D}}{\Rightarrow} Y_{(\alpha)}, \tag{2.19}$$

where $\eta_i = \xi_i - a$.

In fact, for the ch.f. of the left hand side of (2.19) we have

$$f_{\lambda^{-1/\alpha}\sum_{i=1}^{T} Y_i}(\theta) = \exp\{-\lambda(1 - f_{\eta_0}(\lambda^{-1/\alpha}\theta))\}$$

(setting $\lambda^{-1/\alpha}\,\theta = u$ and letting $u \to 0$)

$$= \exp\{-\mu\,|\theta|^{\alpha}\,\tfrac{1}{\mu}\,|u|^{-\alpha}(1 - f_{\eta_0}(u))\} \to \exp\{-\mu|\theta|^{\alpha}\}. \qquad \square$$

Using the claim we arrive at

$$\lambda^{-1/\alpha}(\log X_T - aT) = \lambda^{-1/\alpha}\log Y_T + \lambda^{-1/\alpha}\sum_{i=0}^{T}(\xi_i - a) \overset{\mathfrak{D}}{\Rightarrow} Y_{(\alpha)}.$$

Further, similarly to Theorem 2.1 (a), we conclude that for any $\epsilon > 0$,

$$P(\lambda^{-1/\alpha}(\log S_T - \log X_T) > \epsilon) \to 0 \text{ as } \lambda \to \infty.$$

The proof of (b) and (c) is handled as in Theorem 2.1 (b) and (c), respectively. $\square$

Analogous theorems can be established regarding the limit distributions of M_{T_1}, $\bigvee_{i=T_k+1}^{T_{k+1}} X_i$ (c.f. (1.8)).

3. CHARACTERIZATIONS OF THE DISTRIBUTIONS OF THE TOTAL AND MAXIMAL CROP YIELD.

The subject of this section is the characterizations of the limit laws of S_n (cf.(1.2)) and M_n (cf. (1.3)).

Invoking Lemma 2.1 (b) we conclude that for $Z_0 \in (0, 1)$, S_n (resp. M_n) converges to a finite limit S (resp. M) if (and only if, if $\mathbb{E} \log Z_0 > -\infty$)

$$0 \le \mathbb{E} \log (1 + Y_0) < \infty. \tag{3.1}$$

Given (3.1), the limits S and M satisfy the equations:

$$S \stackrel{d}{=} (S + Y) Z \tag{3.2}$$

and

$$M \stackrel{d}{=} (M \vee Y) Z, \tag{3.3}$$

where $Y \stackrel{d}{=} Y_0$, $Z \stackrel{d}{=} Z_0$ and all r.v.'s above are assumed to be mutually independent. Next we give a complete characterization of the class $\mathcal{S}_1$(resp. $\mathcal{M}_1$) of laws L(S) of S (resp. L(M)) such that for any $L(Z) \in \mathcal{Z}_1$ there exist $Y = Y(Z)$ such that (3.2) (resp. (3.3)) holds. The class $\mathcal{Z}_1$ of Z- laws L(Z) consists of distributions on (0, 1) with densities

$$f_\alpha(z) = (1 + \alpha)z^\alpha,\ 0 < z < 1,\ \alpha \ge 0. \tag{3.4}$$

The class $\mathcal{Z}_1$ was introduced by Todorovich and Gani (1987). Some particular examples of laws $L(S) \in \mathcal{S}_1$, $L(M) \in \mathcal{M}_1$ were studied by Todorovich and Gani (1987), Todorovich (1987), and Rachev and Todorovich (1990).

In the sequel for any $0 < \beta < 1$, and a r.v. Y, denote

$$Y_\beta := \begin{cases} 0 \text{ with probability } 1 - \beta, \\ Y \text{ with probability } \beta. \end{cases} \tag{3.5}$$

<u>Lemma 3.1</u> Suppose S is a solution of (3.2) for a given Y with $0 \le \mathbb{E} \log (1 + Y) < \infty$ and Z being uniform. Then S is also solution (3.2) with Z having density (3.4) and Y replaced by $Y_{1/(1+\alpha)}$.

<u>Proof</u> If Z is uniform then (3.2) implies the following expression for the Laplace transform ϕ_Y of Y:

$$\phi_Y(\theta) = 1 + \theta\, \phi_S'(\theta)/\phi_S(\theta).$$

Using (3.5) we observe that

$$\phi_{Y_{1/(1+\alpha)}}(\theta) = 1 + (1+\alpha)^{-1}\theta\, \phi_S'(\theta)/\phi_S(\theta).$$

Integrating the above equation gives

$$\phi_S(\theta) = \theta^{-\alpha} \int_0^\theta (1+\alpha)\, x^\alpha \phi_S(x)\phi_Y(x)dx = \mathbb{E}e^{-\theta(Z+Y_{1/(1+\alpha)})Z},$$

where Z has density given by (3.4). □

The above lemma allows us to restrict our considerations only to the case of Z being uniform, which is what we do in the sequel.

Lemma 3.2. Any Y with

(3.6) $$0 \le \mathbb{E} \log (1+Y) < \infty$$

solves (3.2) for S whose Laplace transform is given by

(3.7) $$\phi_S(\theta) = \exp\{ - \int_0^\theta \tfrac{1}{x}(1 - \phi_Y(x))dx\}.$$

Moreover, (3.6) is necessary for (3.7) to be a solution of (3.2).

Proof. Assuming, $S \overset{d}{=} (S+Y)Z$ and Z being uniform, (3.7) holds (cf. Todorovich and Gani (1987)). From (3.7), $\int_0^\theta \frac{1}{x}(1 - \phi_Y(x))dx < \infty$. Now,

(3.8) $$\tfrac{1}{\theta}(1 - \phi_Y(\theta)) = \int_0^\infty e^{-\theta x}\bar{F}_Y(x)\, dx, \quad \bar{F}_Y := 1 - F_Y,$$

and so

$$\infty > \int_0^1 \int_0^\infty e^{-xy} \bar{F}_Y(y)\, dy\, dx = \int_0^\infty \bar{F}_Y(y) \frac{1-e^{-y}}{y}\, dy.$$

Thus

$$\infty > \int_1^\infty \frac{1}{y} \bar{F}_Y(y)\, dy = \int_1^\infty \log y\, F_Y(dy).$$

i.e. (3.6) holds.

Note that by (3.8), $\frac{1}{\theta}(1 - \phi_Y(\theta))$ is a completely monotone function. Thus, by Feller (1971), Criterion XIII, 4.2, if (3.6) holds then $\exp\{-\int_0^\theta \frac{1}{x}(1 - \phi_Y(x))\, dx\}$ is also a completely monotone function. Therefore (3.7) defines a Laplace transform of a nonnegative r.v., which clearly satisfies (3.2). □

Lemma 3.3. Any r.v. S solving $S \stackrel{d}{=} (S + Y) Z$ is infinite divisible. More specifically, let Y correspond to S in (3.2), then for any $\beta \in (0, 1)$ the r.v. $S^{(\beta)}$ with Laplace transform ϕ_S^β satisfies $S^{(\beta)} \stackrel{d}{=} (S^{(\beta)} + Y_\beta) Z$, where Y_β is given by (3.5).

Proof. Using (3.7),

$$\phi_S^\beta(\theta) = \exp\{ - \int_0^\infty \frac{1}{\theta}[1 - (\beta\phi_Y(\theta) + 1 - \beta)]\, d\theta\}.$$

Note that $\beta\phi_Y + 1 - \beta$ is the Laplace transform of Y_β and, clearly, $\mathbb{E} \log (1 + Y_\beta) < \infty$. Now use Lemma 3.2. □

We move next to give a detailed description of the class $\mathcal{S}_1$.

Theorem 3.1. The class $\mathcal{S}_1$ of the laws $L(S)$ solving $S \stackrel{d}{=} (S + Y) Z$ consists of all nonnegative infinitely divisible r.v.'s, whose Levy measure M_S is of the following form:

$$M_S << \text{Leb} \quad \text{and} \quad M_S(dx) = H(x)dx, \tag{3.9}$$

where $H(0) \in [0,1]$, H is non-increasing on $[0,\infty)$ and vanishing at ∞. The corresponding Y has $1 - H$ as its distribution function.

Proof. By Lemma 2.3 and Theorem XIII, 7.2 of Feller (1971), there is a unique Levy measure M_S on $(0, \infty)$ such that $\int_0^\infty (1 \wedge x^{-1})\, M_S\, (dx) < \infty$ and $\phi_S(\theta) = \exp\{ - \int_0^\infty \frac{1}{x}(1 - e^{-\theta x})\, M_S(dx)\}$.

On the other hand, ϕ_S enjoys (3.7) and thus $\int_0^\infty \frac{1}{x}(1 - e^{-\theta x})M_S(dx) = \int_0^\theta \frac{1}{x}(1 - \phi_Y(x))dx$. Differentiating for $\theta > 0$ we obtain, by (3.6), that $\int_0^\infty e^{-\theta x}M_S(dx) = \frac{1}{\theta}(1 - \phi_Y(\theta)) = \int_0^\infty e^{-\theta x}\bar{F}_Y(x)dx$. By the uniqueness of M_S we conclude that $M_S(dx) = (1 - F_Y(x))\, dx,\ x > 0$. □

Our next task is the characterization of the class $\mathcal{M}_1$ of laws L(M) satisfying (3.3) for any $L(Z) \in \mathcal{Z}_1$.

Theorem 3.2. $\mathcal{M}_1$ consists all absolutely continuous laws L(M) with density f_M and c.d.f. F_M satisfying the following conditions:

(i) $f_M(x)$ is non-increasing on $(0, \infty)$;

(ii) $xf_M(x)/F_M(x)$ is non-increasing on $(0, \infty)$.

Proof. Suppose first that $L(M) \in \mathcal{M}_1$ and let $Z^{(\alpha)}$ have a density $f_{Z^{(\alpha)}}(z) = (1 + \alpha)z^\alpha$, $0 < z < 1$. Then $M \stackrel{d}{=} (M \vee Y)\, Z^{(\alpha)}$ is equivalent to

$$\bar{F}_Y(x) = \frac{1}{1+\alpha}\frac{xf_M(x)}{F_M(x)},\quad x > 0. \tag{3.10}$$

The latter implies the necessity of (ii) and

$$\begin{cases} \lim_{x\downarrow 0} xf_M(x)/F_M(x) \in (0, 1], \\ \lim_{x\to\infty} xf_M(x) = 0. \end{cases} \tag{3.11}$$

For $\alpha = 0$, by (3.10), we obtain $F_M(x) = \exp\{-\int_x^\infty \frac{1}{t}\bar{F}_Y(t)\, dt\}$ so that $f_M(x)$ is a non-increasing function. This proves the necessity part of the theorem.

Suppose now that (i) and (ii) hold. Then (3.11) holds as well. Then the right-hand side of (3.10) is the distribution tail of a nonnegative random variable which satisfies $M \stackrel{d}{=} (M \vee Y)\, Z^{(\alpha)}$. □

Remark 3.1. Note that by (3.10), for any $L(M) \in \mathcal{M}_1$ and $0 < \alpha < 1$,

$$M \stackrel{d}{=} (M \vee Y_\alpha)Z^{(\alpha)} \Leftrightarrow M \stackrel{d}{=} (M \vee Y_0)Z^{(0)},$$

where Y_α is determined by (3.5). The last relation is paralel to the corresponding relation in the scheme of summation (cf. Lemma 3.1).

Remark 3.2. Note that Gamma (p, λ) distributions with $0 < p \le 1$ belong to $\mathcal{M}_1$ while those with $p > 1$ do not.

Next we consider the class $\mathcal{S}_2$ (resp. $\mathcal{M}_2$) of laws L(S) (resp. L(M)) such that (3.2) (resp. (3.3)) holds for any $L(Z) \in \mathcal{Z}_2 \equiv \{\delta_z, 0 < z < 1\}$ and suitably chosen $Y = Y(Z)$.

The class $\mathcal{S}_2$ coincides with class L of Khinchine (cf. Feller (1971), Sect. 8, Chapter XVII) of nonnegative r.v.'s. We state here a more refined description of $\mathcal{S}_2$ that in Feller (1971), Theorem XVII.8.

Theorem 3.3. The class $\mathcal{S}_2$ coincides with the family of all nonnegative infinitely divisible r.v.'s with the Laplace transform

$$\phi_S(t) = \exp\{- at - \int_0^\infty \tfrac{1}{x} (1 - e^{-tx})\, M_S(dx)\} \tag{3.11}$$

with $a \ge 0$ and Levy measure M_S : $M_S << \lambda$, (λ is the Lebesgue measure on $[0, \infty)$), and $\frac{d\, M_S}{d\lambda}$ has a nonincreasing version.

For any $S \in \mathcal{S}_2$ and $z \in (0, 1)$, the corresponding Y in the equation

$$S \overset{d}{=} (S + Y)\, z \tag{3.12}$$

is nonnegative infinitely divisible with Laplace transform

$$\phi_Y(t) = \exp\{ - \frac{at(1 - z)}{z} - \int_0^\infty \tfrac{1}{x}(1 - e^{-tx})\, M_Y(dx)\}, \tag{3.13}$$

where $M_Y << \lambda$ and

$$\frac{dM_Y}{d\lambda}(x) = \frac{dM_S}{d\lambda}(zx) - \frac{dM_S}{d\lambda}(x). \tag{3.14}$$

Proof. Clearly,

$$\phi_Y(t) = \phi_S(t/z)/\phi_S(t). \tag{3.15}$$

It follows from Lemma XVII.8 of Feller (1971) that S and Y in (3.12) must be infinitely divisible. Moreover, for any Borel set A in $(0, \infty)$,

$$\tfrac{1}{z} M_S(zA) = M_S(A) + M_Y(A) \ge M_S(A). \tag{3.16}$$

Let $F(x) = M_S((0, x])$, $x > 0$. From (3.16) it follows that $M_S << \lambda$ with a nonincreasing Radon-Nikodim derivative. The fact that $a \geq 0$ in (3.13) follows from Section XVII.4.(b) of Feller(1971). The description of ϕ_Y now follows from (3.15).

Conversely, if S has a distribution described by (3.11), then for any $z \in (0, 1)$ choose Y as in (3.13). By (3.15), S and Y solve (3.12). □

Remark 3.3 It is interesting to note that $\mathcal{S}_1 \subset \mathcal{S}_2$.

The class $\mathcal{M}_2$ coincides with the class of the laws of max self-decomposable r.v.'s M, that are solutions of $M \stackrel{d}{=} \alpha M \vee W$ $(0 < \alpha < 1)$ (see Balkema, de Haan and Karandikar (1990) and the references there). The next theorem is similar to the Mejzler (1956) charaterization of the class of max self-decomposable r.v.'s. The Mejzler (1956) result is based on a characterization of the weak limits of the normalized maxima $a_n\{\max(X_1, X_2, \ldots, X_n) - b_n\}$ when X_i's are independent nonidentically distributed.

Theorem 3.4. $\mathcal{M}_2$ consists of the laws of positive absolutely continuous r.v.'s M such tht $xf_M(x)/F_M(x)$ is a non-increasing function on $(0, \infty)$.

Proof. For $z \in (0, 1)$, $M \stackrel{d}{=} (M \vee Y)\, z$ implies

(a) $F_M(xz)/F_M(x)$ is a nondecreasing function on $(0, \infty)$ for every $0 < z < 1$.

Let $H(x) = -\log F_M(x)$. Then (a) is equivalent to

(b) $H(x/z) - H(x)$ is a nonincreasing function on $(0,\infty)$ for every $0 < z < 1$.

This is equivalent to $H(e^x)$ being convex, then H, and thus F_M as well, are absolutely continuous. Then (b) is equivalent to

(c) $yH'(y)$ is a nondecreasing function on $(0, \infty)$.

Since $H'(y) = -f_M(y)/F_M(y)$, the proof is now complete. □

4. THE RATE OF CONVERGENCE PROBLEM IN A GENERAL SETTING

Throughout this section, $(\mathbb{B}, ||\cdot||)$ is the separable Banach space $C(T)$ of continuous mappings $x: T \to \mathbb{R}$ where T is compact and $||\cdot||$ is the usual supremum norm in $C(T)$. For any $x, y \in \mathbb{B}$ we denote $(x\cdot y)(t) := x(t)\cdot y(t)$, $(x \vee y)(t) = x(t) \vee y(t)$, $t \in T$.

Given a nonatomic probability space, let $\mathfrak{X}(\mathbb{B})$ be the space of all random fields (r.f.'s) X,

$\mathbb{B}$-valued random variables, and let $\mathfrak{L}(\mathbb{B})$ be the space of all laws P_X. Suppose $Y_1, Y_2...$ are i.i.d. r.f.'s, $Z_1, Z_2...$ are independent of Y_i's i.i.d. r.f.'s and define

(4.1) $$S_n := \sum_{i=1}^{n} X_i, \quad X_i := Y_i \prod_{j=1}^{i} Z_j.$$

The r.f. S_n can be interpreted as the total crop yields of crop producing areas $t \in T = T(U)$ for a period of n years. Here U is a compact metric space, $U = (U, \rho)$, and $T(U)$ is the set of all closed subsets (crop producing areas) t endowed with the Hansdorff metric

$$h(t_1, t_2) = \inf \{\epsilon > 0: t_1 \subset t_2^{\epsilon}, t_2 \subset t_1^{\epsilon}\},$$

where t^{ϵ} stands for the ϵ-neighborhood of t (Hansdorff (1969), Sect 29). Then (T, h) is a compact metric space (cf. Hansdorff (1969), Sect. 29; Kuratowski (1966) § 21; Kuratowski (1969) § 31). Similarly we define the maximal crop yields

(4.2) $$M_n = \bigvee_{i=1}^{n} X_i.$$

Next we are interested in the conditions providing exponential rate of convergence of S_n and M_n to a finite limits S and M respectively. The obtained theorem improves on the existing reults in this area (cf. Rachev and Todorovich (1990)).

The rate of convergence of the laws P_{X_n} to P_X will be expressed, as usual in the Banach space setting, in terms of the Prokhorov metric

(4.3) $$\pi(X, Y) := \pi(P_X, P_Y)$$

$$:= \inf\{\epsilon > 0 : P(X \in A) \leq P(Y \in A^{\epsilon}) + \epsilon \text{ for all Borel subsets } A \text{ of } \mathbb{B}\},$$

where A^{ϵ} is the open ϵ - neighborhood of A. Further, we shall use also the following metrics and functions in $\mathfrak{X}(\mathbb{B})$ and $\mathfrak{L}(\mathbb{B})$:

(i) χ_p - metric in $\mathfrak{X}(\mathbb{B})$ (cf. Pisier and Zinn (1977)):

(4.4) $$\chi_p(X, Y) = [\sup_{t>0} t^P P(||X - Y|| > t)]^{\frac{1}{1+p}}, \quad p > 0;$$

(ii) $\hat{\chi}_p$ - minimal metric in $\mathfrak{L}(\mathbb{B})$ (cf. de Haan and Rachev (1989)):

(4.5) $$\hat{\chi}_p\,(X, Y) := \hat{\chi}_p\,(P_X, P_Y)$$

$$= \inf\{\chi_p(\tilde{X}, \tilde{Y}) : \tilde{X}, \tilde{Y} \in \mathfrak{X}\,(\mathbb{B}), \tilde{X} \overset{d}{=} X, \tilde{Y} \overset{d}{=} Y\},\ p > 0;$$

(iii) $$\omega_{p,N}(X)^{p+1} := \sup_{t>N} t^p P(||X|| > t),$$

$$\omega_p(X) := \omega_{p,0}(X),$$

$$N_p(X) := \{\mathbb{E}\,||X||^p\}^{1/(1+p)} \quad p > 0.$$

Note that $\hat{\chi}_p$ is a metric in $\mathcal{L}(\mathbb{B})$, $\hat{\chi}_p \geq \pi$ and the following convergence criterion holds: if

(4.6) $$\omega_{p,N}(X_n) + \omega_{p,N}\,(X) \to 0 \text{ as } N \to \infty \text{ for any } n = 1, 2, \ldots,$$

then

(4.7) $$\hat{\chi}_p(X_n, X) \underset{n\to\infty}{\to} 0 \text{ iff } X_n \overset{\mathfrak{D}}{\underset{n\to\infty}{\to}} X \text{ and } \lim_{N\to\infty} \overline{\lim_{n\to\infty}}\ \omega_{p,N}(X_n) = 0.$$

Similarly, if (4.6) holds then

(4.8) $$\chi_p(X_n, X) \underset{n\to\infty}{\to} 0 \text{ iff } X_n \overset{P}{\underset{n\to\infty}{\to}} X \text{ and } \lim_{N\to\infty} \overline{\lim_{n\to\infty}}\ \omega_{p,N}\,(X_n) = 0.$$

<u>Theorem 4.1</u>.

a) If for some $p > 0$, $N_p\,(Z_1) < 1$ and $\omega_p(Y_1) < \infty$ then S_n converges in probability to a P -a.e. finite limit $S = \sum_{i=1}^{\infty} X_i$, and

(4.9) $$\pi(S_n, S) \leq \chi_p\,(S_n, S) \leq \phi_n(Z_1)\,\omega_p(Y_1),$$

where $\phi_n(Z_1) := N_p(Z_1)^{n+1} / (1 - N_p(Z_1))$.

b) Under the above conditions, if Y_i^*'s are i.i.d. with $\chi_p(Y_1, Y_1^*) < \infty$ then

$$S_n^* := \sum_{i=1}^{n} Y_i^* \prod_{j=1}^{i} Z_j \xrightarrow{P} S^*. \tag{4.10}$$

Moreover, suppose that $\tilde{Z}_i$'s are independent of Z_i's and Y_i's i.i.d. copies of Z_1. Then, as $n \to \infty$,

$$\pi(S^* - S_n^*, \tilde{Z}_1 \cdot ... \cdot \tilde{Z}_n \cdot S) \le \phi_n(Z_1)\, \hat{\chi}_p(Y_1, Y_1^*) \to 0. \tag{4.11}$$

Remark 4.1. Clearly, in (4.11) the factor $\tilde{Z}_1 ... \tilde{Z}_n$ plays the same role as the normalizing scaling in the usual CLT.

Proof.

a) Since $\chi_p \ge \kappa$ where κ is the distance in probability (the Ky Fan metric):

$$\kappa(X, Y) = \inf\{ u > 0: P(||X - Y|| > u) < u \}, \tag{4.12}$$

then to prove $S_n \xrightarrow{P} S$ it is enough to show that S_n is χ_p- fundamental. In fact, for any $k = 1, 2, ...,$

$$\begin{aligned} \chi_p(S_{n+k}, S_n) &= \omega_p\Big(\sum_{i=n+1}^{n+k} X_i\Big) \le \sum_{i=n+1}^{n+k} \omega_p(X_i) \\ &\le \sum_{i=n+1}^{n+k} \{\mathbb{E}_{Z_1=z_1,...,Z_n=z_n}\ \omega_p^{p+1}(Y_i \prod_{j=1}^{i} z_j)\}^{\frac{1}{1+p}} \\ &= \sum_{i=n+1}^{n+k} [\mathbb{E}_{Z_1=z_1,...,Z_n=z_n}(\prod_{j=1}^{i} ||z_j||)^p]^{\frac{1}{1+p}}\ \omega_p(Y_1) \\ &\le \sum_{n>i} N_p(Z_1)^i\ \omega_p(Y_1) \\ &= N_p(Z_1)^{n+1} \frac{1}{1 - N_p(Z_1)}\ \omega_p(Y_1) \to 0 \text{ as } n \to \infty. \end{aligned} \tag{4.13}$$

Next, (4.9) follows by the same arguments as (4.13).

b) By (4.5), $\hat{\chi}_p$ is a minimal metric with respect to χ_p and thus, for any joint distribution of Y_1 and Y_1^*,

$$\hat{\chi}_p(S^* - S_n^*, \tilde{Z}_1 \cdot ... \cdot \tilde{Z}_n \cdot S) \le \chi_p\Big(\sum_{i>n} Y_i^* \prod_{j=1}^{i} Z_j, \sum_{i>n} Y_i \prod_{j=1}^{i} Z_j\Big).$$

Now proceed as in (4.13) to obtain

$$\leq \chi_p\,(Y_1, Y_1^*)\; N_p(Z_1)^{n+1}\, \frac{1}{1-N_p(Z_1)}.$$

Taking the infimum is the last inequality over of all joint distributions of (Y_1, Y_1^*) with fixed marginals, we use $\hat{\chi}_p \geq \pi$ to complete the proof of (4.11). □

Theorem 4.2. Suppose Y_i's and Z_i's are nonnegative r.f.'s. Then under the assumptions of Theorem 4.1 (a), $M_n \xrightarrow{P} M$ and

$$\pi(M_n, M) \leq \phi_n(Z_1)\, \omega_p(Y_1) \to 0, \text{ as } n \to \infty.$$

Moreover if $\chi_p(Y_1, Y_1^*) < \infty$ then

$$\pi\Big(\bigvee_{i=n+1}^{\infty} Y_i \prod_{j=1}^{i} Z_j,\;\; \tilde{Z}_1 \cdot \ldots \cdot \tilde{Z}_n \cdot M\Big) \leq \phi_n\,(Z_1)\; \hat{\chi}_p(Y_1, Y_1^*) \to 0.$$

Remark 4.2. If N is an integer valued r.v. independent of Y_i's and Z_i's then under the conditions of Theorem 4.1,

$$\pi(S_N, S) \leq \psi_N(Z_1)\; \omega_p\,(Z_1),$$

where $\psi_N(Z_1) = (\mathbb{E}\, N_p\,(Z_1)^N) \,/\, (1 - N_p(Z_1))$, and moreover,

$$\pi(S^* - S_N^*,\, \tilde{Z}_1 \cdot \ldots \cdot \tilde{Z}_N \cdot S) \leq \psi_N(Z_1)\; \hat{\chi}_p(Y_1, Y_1^*).$$

Similar results regarding the limit behavior of the distribution of the maximum $\bigvee_{i=1}^{N} X_i$ can be obtained as a corollary to Theorem 4.1.

5. DISCUSSION OF THE LITERATURE

The equivalence of (bii) and (biv) in Lemma 2.1 has been proved by Vervaat (1979) (Lemmas 1.7, 1.8 and Corollary 1.9). Vervaat (1979) (Theorem 6.1) proved weak convergence of

$$S^{(n)} \stackrel{d}{=} Z^{1/n} S^{(n)} + B, \quad (Z, B) \text{ and } S^{(n)} \text{ independent} \tag{5.1}$$

to the normal law under conditions similar to that of Theorem 2.1(b) ($\alpha = 2$). The equation $S \stackrel{d}{=} (S + Y) Z$ with $Z \in \mathcal{Z}_1$, has been studied by Chandresekhar and Munch (1950), Takacz (1953), Weiss (1973), Yeo (1974), Vervaat (1974), Chamayou and Schorr (1975) who proved particular cases of Theorem 3.1. A weaker version of Lemma 3.2 was proved by Vervaat (1979) p. 764.

The class $\mathcal{I}_2$ is also known as the class of self-decomposable r.v.'s, see Lukacz (1970, Section 5.11). The random sums

$$S_n = \sum_{k=1}^{n} \Big(\prod_{j=k+1}^{n} Z_j \Big) Y_k + \Big(\prod_{j=1}^{n} Z_j \Big) Y_0 \tag{5.2}$$

where Y_0 is independent of the R^2-valued (Y_i, Z_i) are of interest in economics (see de Haan, Resnick, Rootzen and Vries (1989)), physics (Chamayou (1973)), environmental processes (Solomon (1975)), sociology (Cavalli-Sforza and Feldman (1973)). Excellent reviews are given by Kesten (1973) and Vervaat (1979), more references can be found in de Haan, Resnick, Rootzen and Vries (1979) and de Haan and Karandikar (1979). It will be of future interest to obtain a set of limit results as in Section 2 for the sums (5.2) under mild condition on the joint distribution of (Y_1, Z_1).

REFERENCES

1. Balkema, A. A., de Haan, L. and Karandikar, R., (1990). The maximum of n independent stochastic processes. Preprint, Erasmus University, Rotterdam.

2. Billingsley, P., (1968). Convergence of Probability Measures, John Wiley & Sons, New York.

3. Cavalli-Sforza, L. and Feldman, M. W., (1973). Models for cultural inheritance I. Group mean and within group variation. Theoret. Popn. Biol. 4, 42-55.

4. Chamayou, J.M.F., (1973). Volterra's functional integral equations of the statistical theory of damage. J. Computational Phys. 13, 70-93.

5. Chamayou, J.M.F. and Schorr, B., (1975). On a class of random variables arising in atomic cascade models. Report, European Orgainization for Nuclear Research, Geneva.

6. Chandrasekhar, S. and Munch, G., (1950). The theory of the fluctuations in brightness of the Milky Way. I and II. Astrophys. J. 112, 380-398.

7. de Haan, L., and Rachev, S.T., (1989). Estimates of the rate of convergence for max-stable processes. Ann. Probability, 17, 651-677.

8. de Haan, L., Resnick, S.I., Rootzen, H. and Vries, C.G., (1985). Extremal behavior of solutions to a stochastic difference equation with applications to ARCH processes. Stoch. Processes and Appl. 32, 213-224.

9. de Haan, L. and Karandikar, R.L., (1989). Embedding a stochastic difference equation into a continuous-time process. Stoch. Processes and Appl. 32, 225-235.

10. Feller, W., (1971). An Introduction to Probability Theory and Its Applications. Vol. II, Second edition, John Wiley, New York.

11. Hansdorff, F., (1949). Set. Theory, Dover.

12. Kesten, H., (1973). Random difference equations and renewal theory for products of random matrices. Acta Math. 131, 207-248.

13. Klebanov, L.B., Maniya, G.M. and Melamed, I.A., (1984). A problem of Zolotarev and analogs of infinitely divisible and stable distributions in a scheme for summing a random number of random variables. Theor. Prob. Appl., 29, 791-794.

14. Kuratowski, K., (1966). Topology, Vol. I, Academic Press, New York.

15. Kuratowski, K., (1969). Topology, Vol. II, Academic Press, New York.

16. Lukacs, E., (1970). Characteristic Functions, 2nd Ed., Griffin, London.

17. Mejzler, D., (1956). On the problem of the limit distributions for the maximal term of a variational series. Lvov Politechn. Inst. Naucn. Zap. Ser. Fiz.-Mat. 38, 90-109 (Russian).

18. Mittnik, S. and Rachev, S.T., (1990). Alternative multivariate stable distributions and their applications to financial modeling. Proceedings of MSI Workshop on Stable Processes and Related Topics, Cornell University, January 1990, (to appear).

19. Paulson, A.S. and Uppuluri, V.R.R., (1972). Limit laws of a sequence determined by a random difference equation governing a one-compartment system. Math. Biosci. 13, 325-333.

20. Pisier, G. and Zinn, J., (1977). On limit theorems for random variables with values in the spaces L^p. Z. Wahrsch. Verw. Geb., 41, 286-305.

21. Puri, P.S., (1987). On almost sure convergence of an erosion process due to Todorovic and Gani. J. Appl. Prob., 24, 1001-1005.

22. Solomon, F., (1975). Random walks in a random environment. Ann. Prob., 3, 1-31.

23. Takacs, L., (1954). On secondary processes generated by a Poisson process and their applications in physics. Acta Math. Acad. Sci. Hung. 5, 203-236.

24. Todorovic, P., (1987). An extremal problem arising in soil erosion modeling. In Applied Probability, Stochastic Processes and Sampling Theory, eds. Ian B. MacNeil and Gary J. Umphrey, Reidel Dordrecht, 65-73.

25. Todorovic, P. and Gani, J., (1987). Modeling of the effect of erosion on crop production. J. Appl. Prob., 24, 787-797.

26. Vervaat, W., (1974). On records, maxima and a stochastic difference equation. Report 73-11, Department of Mathematics, University of Amsterdam.

27. Vervaat, W., (1979). On a stochastic difference equation and a representation of non-negative infinitely divisible random variables. Adv. in Appl. Probab., 11, 750-783.

28. Weiss, G., (1973). Filtered Poisson processes as models for daily streamflow data. Thesis, University of London.

29. Yeo, G.F., (1974). A finite dam with exponential release. J. Appl. Prob., 11, 122-133.