Schwarz-Christoffel Mapping in the 1980's*

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Note

One day I hope to write a book on numerical Schwarz-Christoffel mapping. In the meantime, this document is the closest I have come to preparing a survey of the field. It consists of the transparencies (slightly edited) from my talk "Schwarz-Christoffel mapping in the 1980's" delivered at the Conference on Computational Aspects of Complex Analysis organized by Al Marden and Burt Rodin in Phoenix, Arizona, 11–14 January 1989. The material is based on ten years of experience with solving conformal mapping problems, often brought to me by users or would-be users of my Fortran package SCPACK.

My chief purpose here is to outline the wide range of variations on the theme of Schwarz-Christoffel mapping that arise in practical problems — for only rarely does one encounter precisely the "standard" problem of mapping a disk or half-plane onto a prescribed polygon. Details are omitted, but can be found in the references. My emphasis is entirely on algorithmic and numerical matters, and the references are heavily biased in that direction. Undoubtedly they are also biased towards my own contributions, and I apologize to others whom I may have accidentally slighted.

I would like to highlight two themes that arise repeatedly in Schwarz-Christoffel mapping:

- *Modified Schwarz-Christoffel integrals.* The standard Schwarz-Christoffel integrand is a product $f' = \prod f_k$, where each f_k is an elementary conformal map $(z-z_k)^{-\beta_k}$. By modifying the choice of f_k in appropriate ways, this same prescription can be adapted to many different mapping problems, including exterior polygons, polygonal Riemann surfaces, doubly-connected polygons, maps of an infinite strip, and the maps that arise in ideal 2D free-streamline flows.

- *Generalized parameter problems.* Almost any Schwarz-Christoffel map requires the solution of a parameter problem; in the simplest case one has to determine unknown "prevertices" $\{z_k\}$ such that the corresponding vertices $\{w_k\}$ are separated by the correct side lengths $|w_{k+1} - w_k|$, and this amounts to a nonlinear system of equations to be solved numerically. Surprisingly often, however, this paradigm needs to be enlarged to a "generalized parameter problem" — which, fortunately, is often no harder to solve. For example, in an inverse problem one may wish to delete one of the side length conditions and replace it by a global condition involving the conformal module (see p. 25). To put it generally: not all the conditions that define a Schwarz-Christoffel map need be geometric. Often the geometry is only incompletely specified, or specified partly in the domain and partly in the range; and indeed, sometimes polygons do not enter into the problem at all except implicitly.

My work on Schwarz-Christoffel mapping began in 1978 at the suggestion of Peter Henrici, who died, too soon, in 1987.

Nick Trefethen
20 January 1989

SCHWARZ-CHRISTOFFEL MAPPING IN THE 1980's

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1. Fundamentals of S-C Mapping

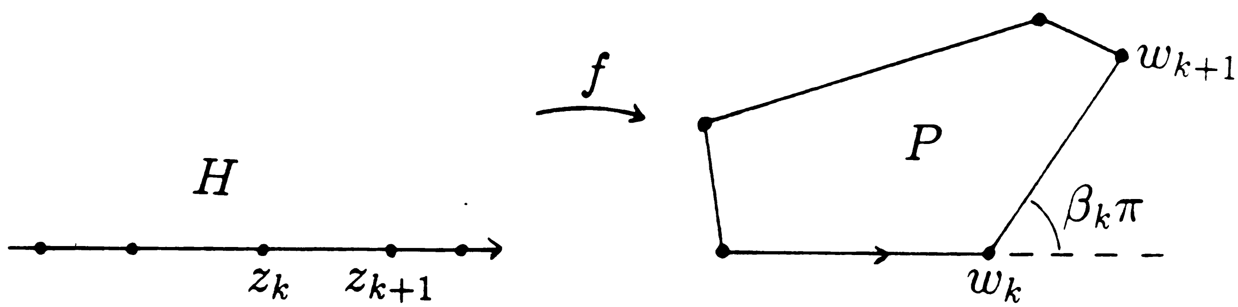
The Schwarz-Christoffel Idea

H = upper half-plane

P = polygonal region with n vertices

f = conformal map from H to P

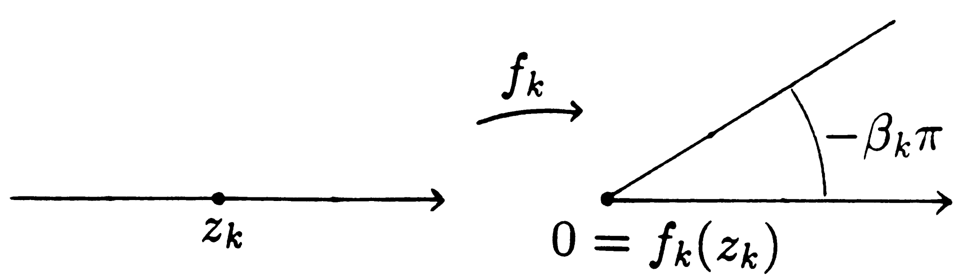
$\{w_k\}$ = vertices of P, $\{z_k\}$ = "prevertices" $z_k = f^{-1}(w_k)$



Idea: arg f' is piecewise constant on the real axis.

Therefore f' can be written as a product of functions f_k:

$$f'(z) = C \prod_{k=1}^{n} f_k(z), \qquad f_k(z) = (z-z_k)^{-\beta_k}.$$



Each f_k introduces a jump in arg f' at z_k.

The Schwarz-Christoffel Formula

Integration gives

$$f(z) = C \int^{z} \prod_{k=1}^{n} (s - z_k)^{-\beta_k} \, ds$$

Reminder:

$\beta_k \pi$ = turning angle at kth vertex w_k – *known*

z_k = kth prevertex – *unknown*

The same formula works for a disk as well as a half-plane, and vertices at ∞ are permitted.

For maps of the half-plane, one prevertex may lie at ∞ and is then omitted from the S-C product (e.g. $z_n = \infty$).

Refs: Z. Nehari, *Conformal Mapping*, 1952.
W. Koppenfels & F. Stallmann, *Praxis der konf. Abbildung*, 1959.
P. Henrici, *Applied & Computational Complex Analysis I*, 1974.

Numerical S-C Mapping

To apply the S-C formula, two obstacles must be overcome:

"PARAMETER PROBLEM"

Given vertices $\{w_k\}$, the prevertices $\{z_k\}$ are unknown.

1) Formulate the problem as a system of constrained nonlinear equations involving the side lengths $|w_{k+1} - w_k|$;

2) Change variables to eliminate the constraints $z_k < z_{k+1}$;

3) Solve the system iteratively by standard optimization software (e.g. NS01A or MINPACK);

NUMERICAL INTEGRATION

The S-C integral cannot be evaluated analytically.

1) Gauss-Jacobi quadrature (for endpoint singularities);

2) Adaptive subdivision (to combat "crowding").

Refs: K. Reppe, *Siemens Forsch. u. Entwickl. Ber.*, 1979.
R.T. Davis, *4th AIAA Comp. Fluid Dynamics Conf.*, 1979.
L.N.T., *SIAM J. Sci. Stat. Comp.*, 1980.

History

GAUSS (1820's) – idea of conformal mapping
RIEMANN (1851) – Riemann mapping theorem
CHRISTOFFEL (1868)
SCHWARZ (1869,1870) – independent

⋮

POLOZKII (1955)
KANTOROVICH & KRYLOV (1958)
SAVENKOV (1963,1964)
GAIER (1964) – book on numerical conformal mapping
FILCHAKOV (1961,1968,1969,1975)
HAEUSLER (1966)
LAWRENSON & GUPTA (1968)
BEIGEL (1969)
HOFFMAN (1971,1974)
HOWE (1973)
VECHESLAVOV, TOLSTOBROVA, KOKOULIN (1973,1974)
CHEREDNICHENKO & ZHELANKINA (1975)
SQUIRE (1975)
MEYER (1976,1979) – comparison of algorithms
NICOLAIDE (1978)
HOPKINS & ROBERTS (1979) – solution by Kufarev's method
BINNS, REES, & KAHAN (1979)
VOLKOV (1979)
REPPE (1979) – first fully robust algorithm
DAVIS, SRIDHAR (1979,1982,1983) – curved boundaries, channel maps
TREFETHEN (1980–1988) – Fortran package SCPACK
BROWN (1981)
TOZONI (1983)
PROCHAZKA (1978,1982,1983)
HOEKSTRA (1983,1986) – curved boundaries, annuli
FLORYAN, ZEMACH (1985–1988) – channel maps, periodic domains
BJØRSTAD & GROSSE (1987) – circular polygons
DIAS (1987–1989) – hydrodynamics applications
DÄPPEN (1988) – doubly-connected S-C
HOWELL (1989) – elongated polygons, circular polygons

SCPACK

Fortran package for S-C mapping (L.N.T., 1982)

Polygons may be unbounded (i.e., vertices permitted at ∞)

Available by tape or by e-mail (via "Netlib"):

~~mail netlib@anl-mcs.arpa~~ ⟶ mail netlib@research.att.com or mail netlib@ornl.gov

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send index from conformal
send scpdbl from conformal
send sclibdbl from conformal
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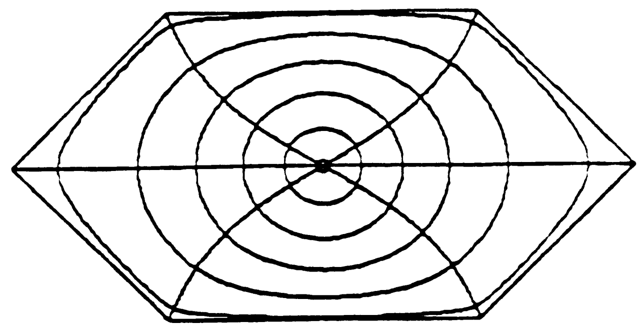
The User's Guide can be obtained by contacting L.N.T.

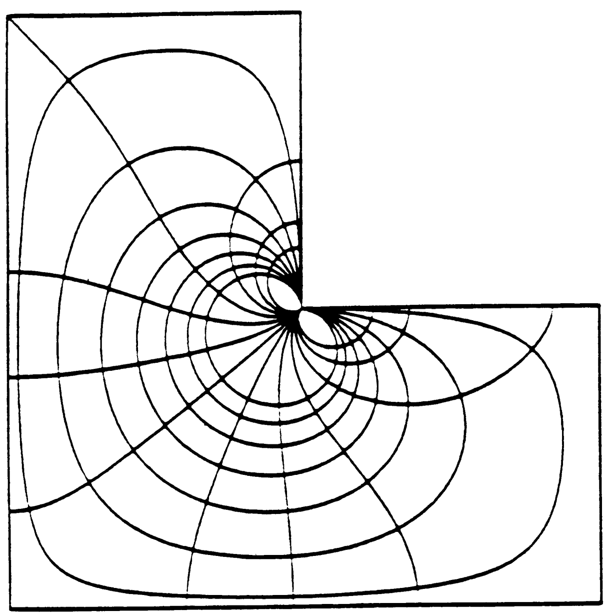
Refs: L.N.T., *SCPACK User's Guide*, M.I.T. report, 1989.
(Netlib:) J.J. Dongarra & E. Grosse, *Commun. ACM*, 1987.

Examples

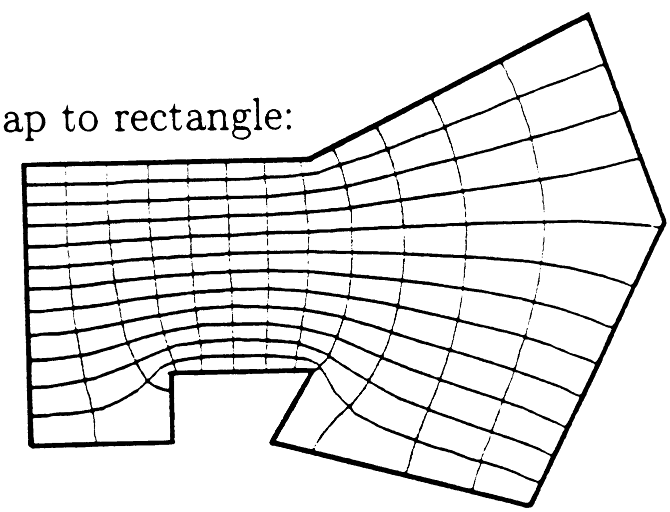
Map to disk:



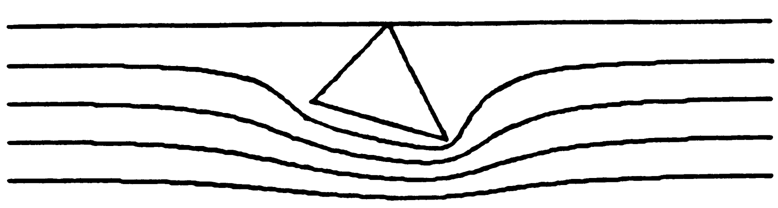
Map to half-plane:



Map to rectangle:



Map to infinite strip:



2. Generalizations of the S-C formula

Refs: Schwarz and Christoffel, 1868–1870.
J.M. Floryan & C. Zemach, *J. Comp. Phys.*, 1987.

Riemann Surfaces

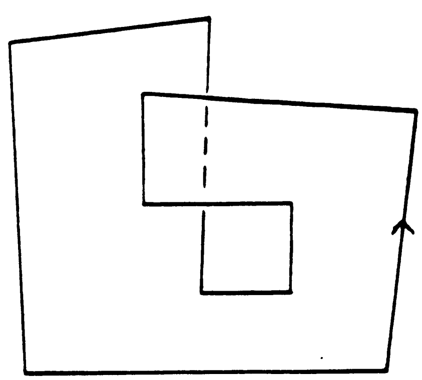
Idea: f' is still piecewise constant on the real axis, but may have zeros ζ_k in the upper half-plane.

Each ζ_k introduces a factor

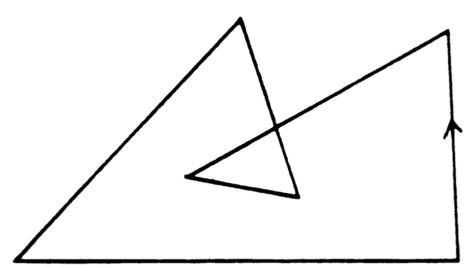
$$b_k(z) = (z-\zeta_k)(z-\overline{\zeta_k})$$

in the S-C formula:

$$\boxed{f(z) = C\int^z \prod_{k=1}^{n} (s-z_k)^{-\beta_k} \prod_{k=1}^{B} b_k(s)\, ds}$$



no branch points



one branch point

No numerical implementations as yet.

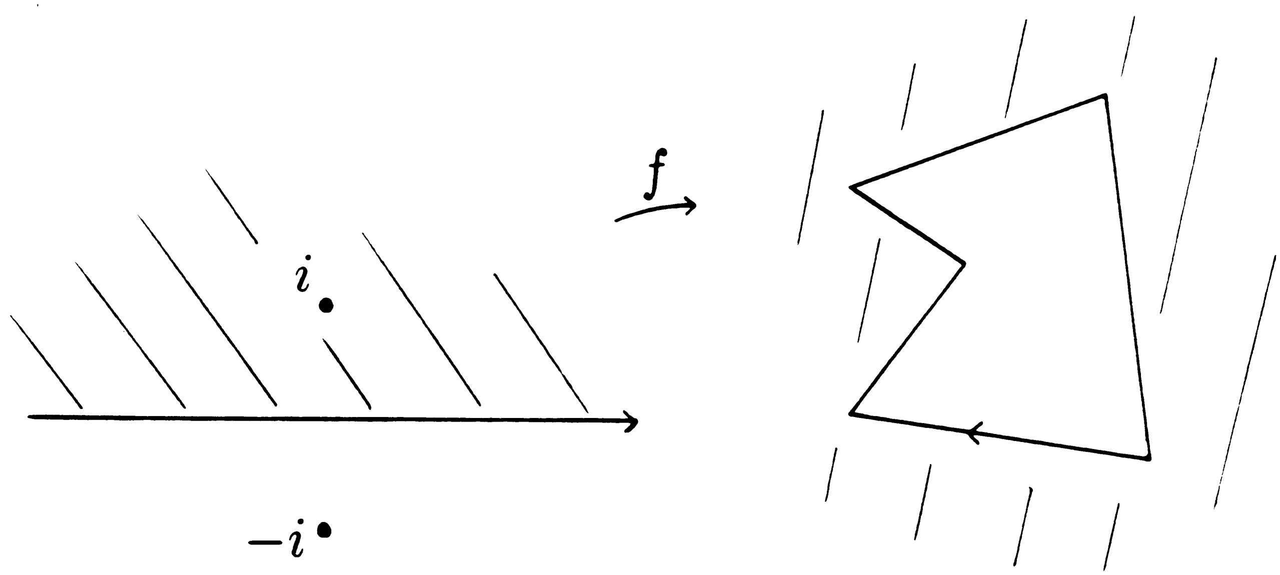
Refs: D. Gilbarg, *Proc. National Academy of Sciences*, 1949.
A. W. Goodman, *Trans. Amer. Math. Soc.*, 1950.

Exterior polygons

Problem: map the upper half-plane to the exterior of a polygon, with $f(i) = \infty$.

Solution: analogous to Riemann surface case. $f'(z)$ is piecewise constant on the real axis, with a double pole at $z = i$:

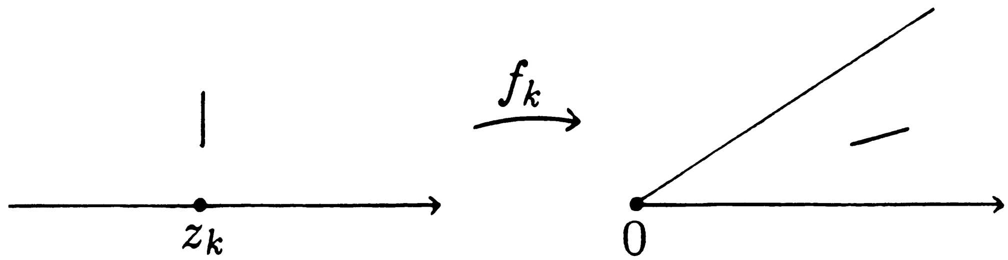
$$f(z) = C \int^z \prod_{k=1}^{n} (s-z_k)^{-\beta_k} (s^2+1)^{-2} \, ds$$



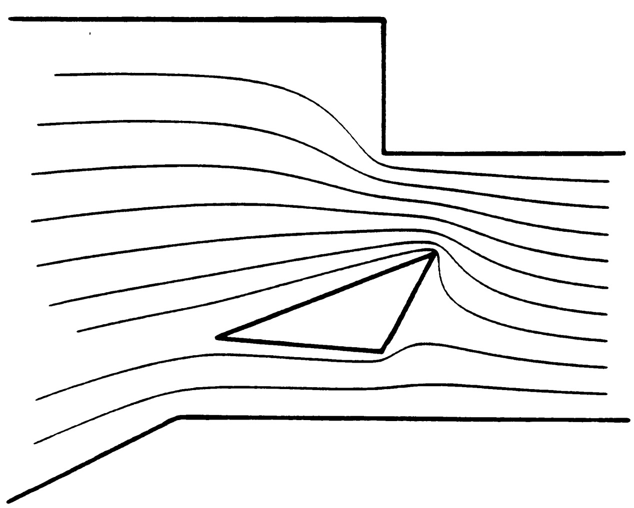
Ref: L.N.T., unpublished memo, 1987.

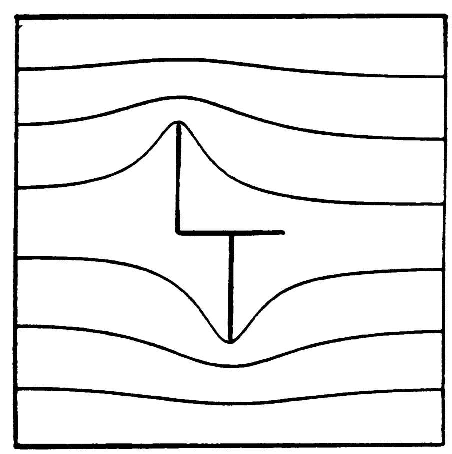
Doubly-Connected Polygons

Idea: write $f' = C \prod_k f_k$, with



f_k can be expressed in terms of theta functions.





Regions with higher connectivity: no S-C methods exist.

Refs: P. Henrici, *Applied & Computational Complex Analysis III*. 1986.
M. Hoekstra, in *Numerical Grid Generation*, 1986.
H.D. Däppen, PhD thesis, ETH Zurich, 1988.

Free boundaries with $|f'| =$ const.

Standard S-C:

arg $f' =$ piecewise constant on real axis

S-C for free-boundary problems:

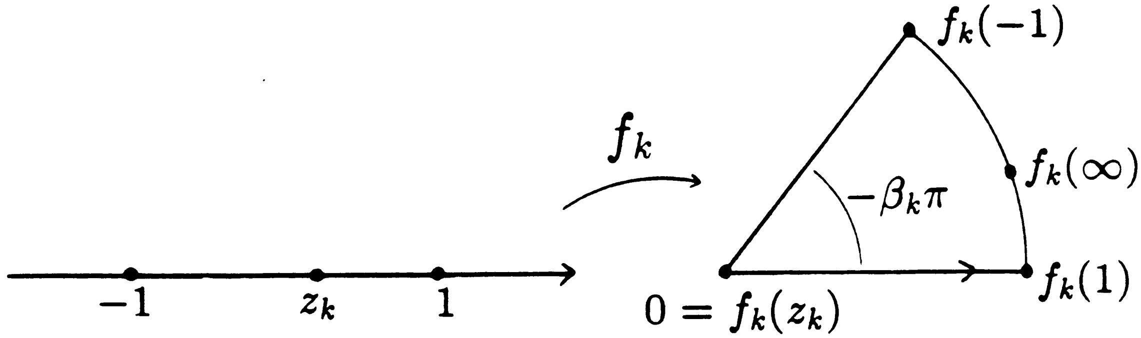
arg $f' =$ piecewise constant on $[-1, 1]$,

$|f'| =$ const. elsewhere on real axis

Applications: wakes, jets, cavities; area minimization (see p. 30)

Idea: write $f' = C \prod_k f_k$, with

$$f_k(z) \;=\; \left(\frac{z - z_k}{1 - z_k z + \sqrt{(1-z^2)(1-z_k)^2}} \right)^{-\beta_k}$$



Refs: A.R. Elcrat & L.N.T., *J. Comp. Appl. Math.*, 1986.

D. Gaier, *Results in Mathematics*, 1986.

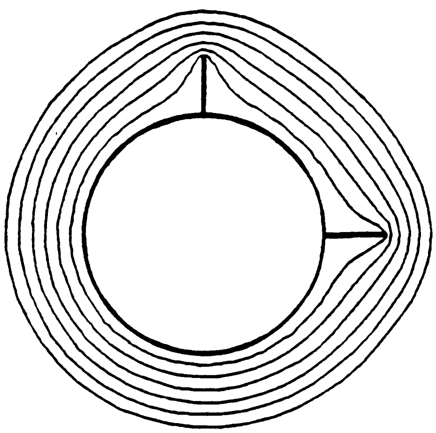
F. Dias, A.R. Elcrat & L.N.T., *J. Fluid Mech.*, 1987.

Gearlike Domains

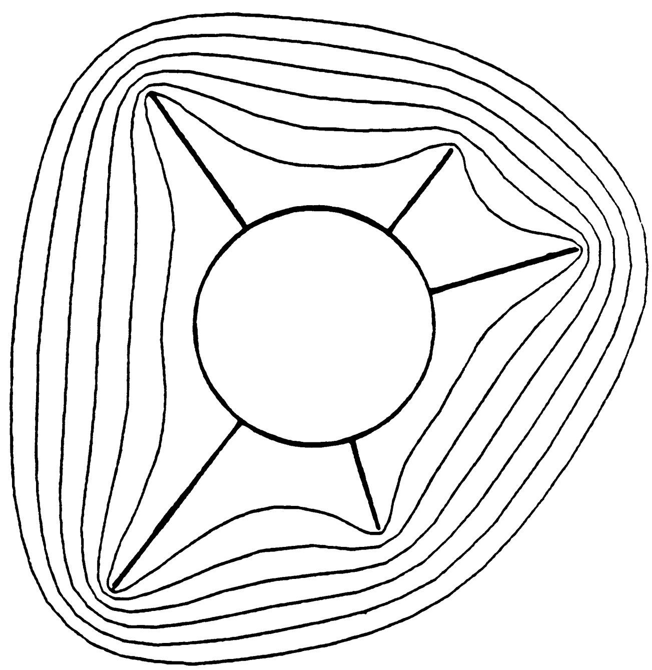
"Gearlike" domain G: bounded by radial line segments and concentric circular arcs

Idea: $P = \log G$ is a polygon (possibly periodic) with horizontal and vertical sides

Apply S-C type formula to P



capacity = 1.08184



capacity = 1.72442

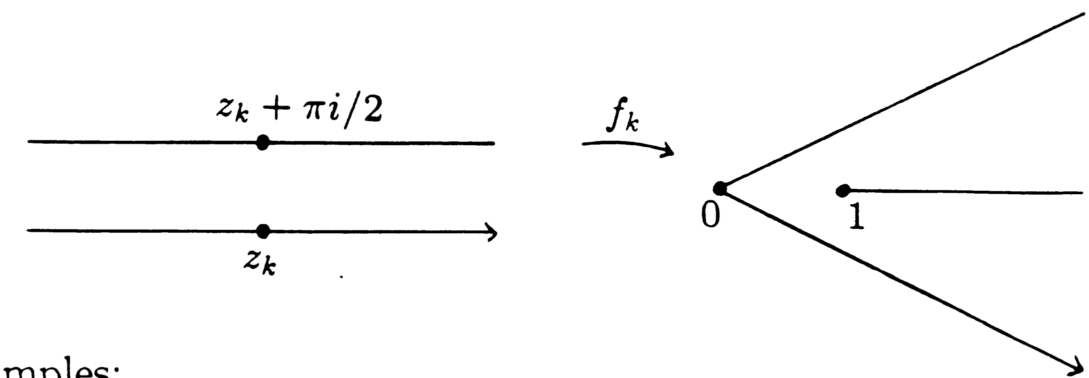
Refs: A.W. Goodman, *Univ. Nat. Tucuman*, 1960.
L.N.T., unpublished memo, 1983.
K.P. Jackson & J.C. Mason, in *Algorithms for Approximation*, 1987.
K. Pearce, ~~to appear.~~ SIAM J. Sci. Stat. Comput., 1991.

Channels and Elongated Polygons

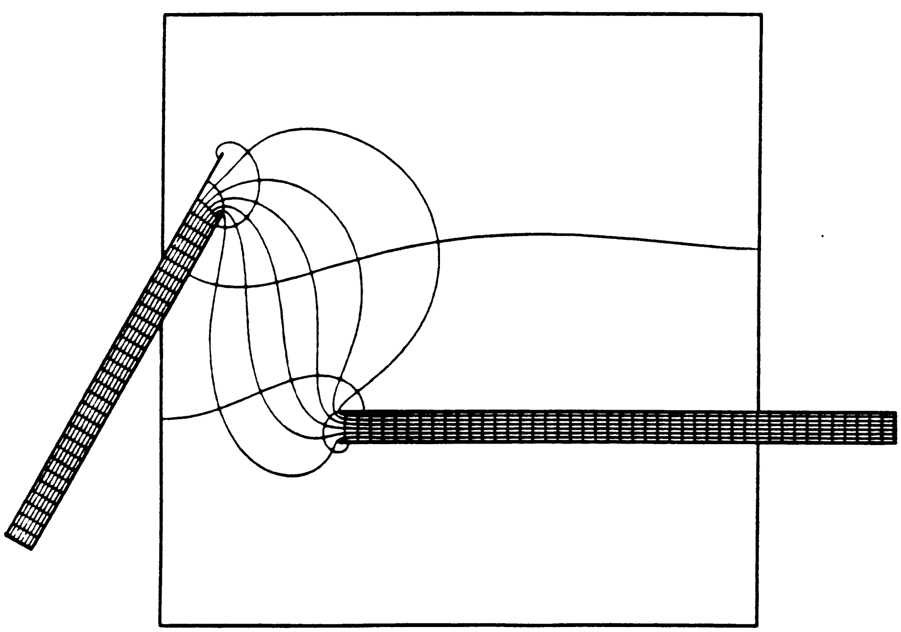
Mapping elongated regions via a disk or half-plane is too ill-conditioned to be feasible (the "crowding phenomenon").

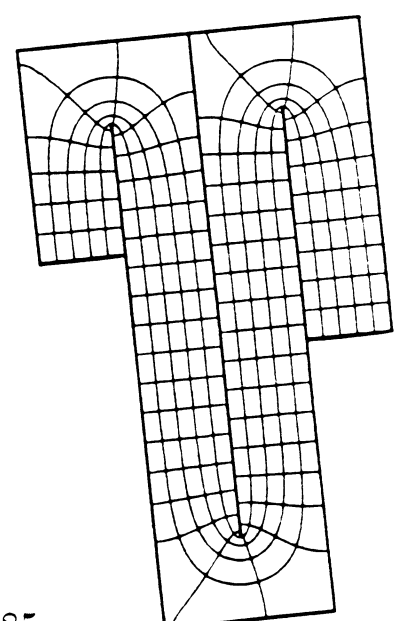
Idea: use an infinite strip instead as fundamental domain.

$$f'(z) = C \prod_{k=1}^{n} f_k(z), \qquad f_k(z) = (-i \sinh(z - z_k))^{-\beta_k}.$$



Examples:





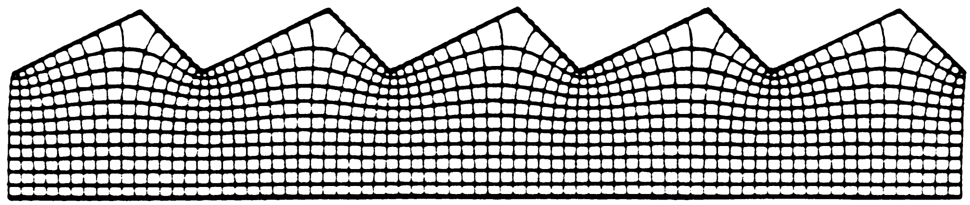
Refs: K.P. Sridhar & R.T. Davis, *J. Fluids Engr.*, 1985.

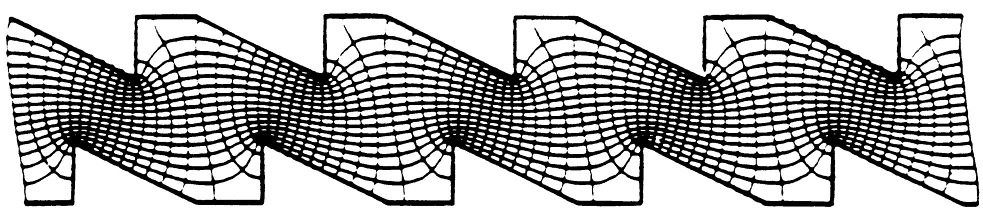
J.M. Floryan, *J. Comp. Phys.*, 1985.

L.H. Howell and L.N.T., *SIAM J. Sci. Stat. Comp.*, ~~to appear~~. 1990

L. Greengard, "Potential flow in channels," ~~to appear~~.
SIAM J. Sci. Stat. Comp., 1990

Periodic Domains





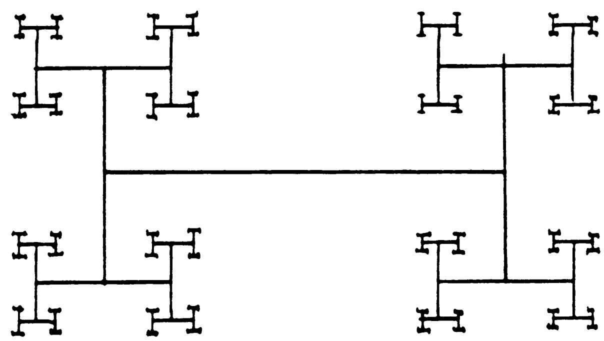
Ref: J. M. Floryan, *J. Comp. Phys.*, 1986.

Fractals

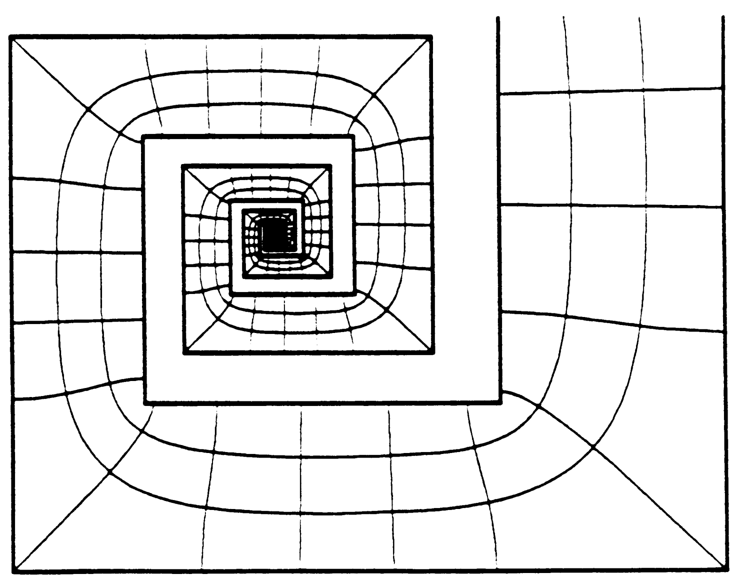
Polygonal fractal $\leftrightarrow$ S-C map with $n = \infty$

Applications in diffusion-limited aggregation, etc.?

Algorithms not yet developed



Example: (self-similar; map onto infinite strip)



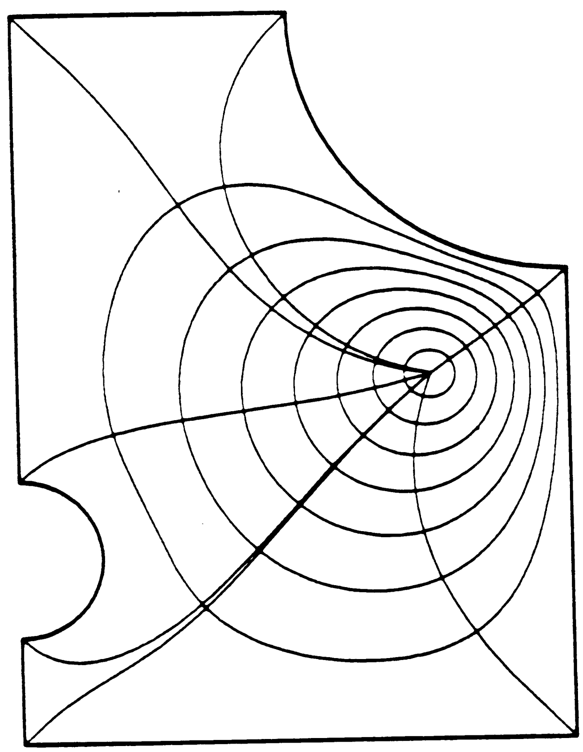
Ref: L. H. Howell & L.N.T., *SIAM J. Sci. Stat. Comp.*, ~~to appear.~~ 1990

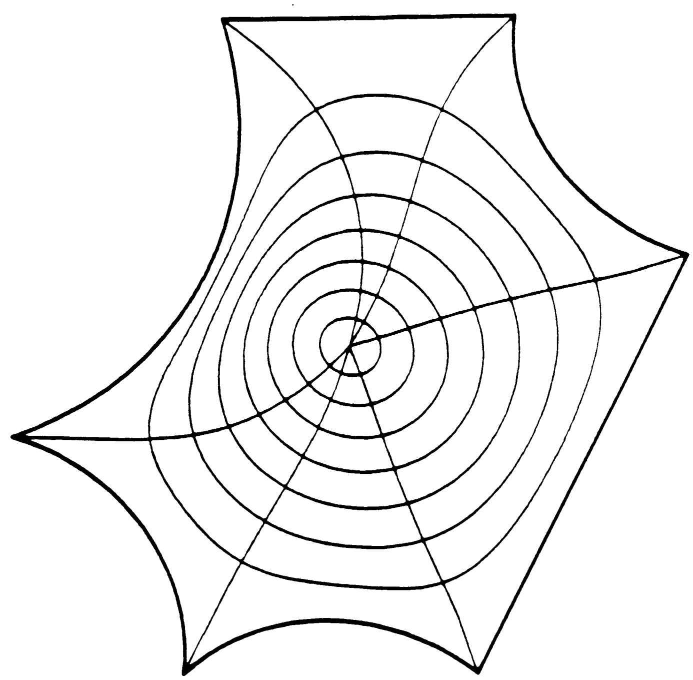
Circular Polygons

"Circular polygon": bounded by circular (or straight) arcs

The S-C integral becomes a 3rd-order o.d.e.

Fully robust implementations not yet available.





Refs: P. Bjørstad & E. Grosse, *SIAM J. Sci. Stat. Comp.*, 1987.

L.H. Howell, ~~in preparation.~~ J. Comp. Appl. Math., 1993

General Curved Boundaries

Standard S-C — turning angle $\pi\beta_k$ at vertex z_k:

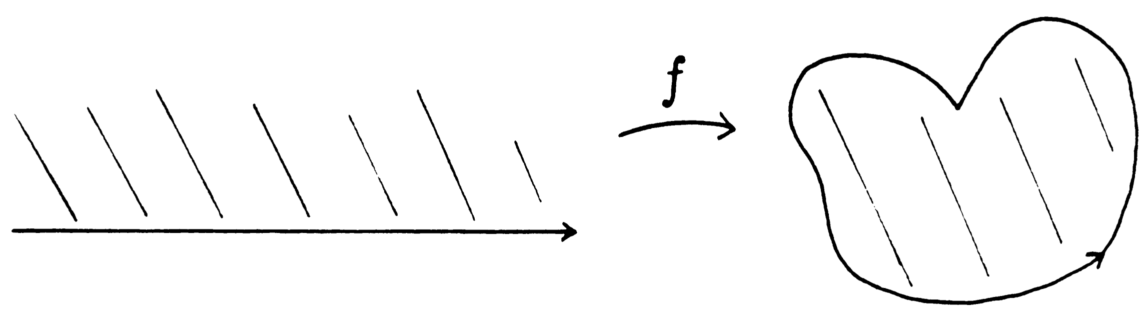
$$f'(z) = C \prod_{k=1}^{n} (z - z_k)^{-\beta_k} = C \exp[-\sum_{k=1}^{n} \beta_k \log(z - z_k)].$$

Continuous S-C — turning density function $\pi\beta(z)$:

$$f'(z) = C \exp\left[-\int_{-\infty}^{\infty} \beta(t) \log(z - t)\, dt\right].$$

Integration gives:

$$\boxed{f(z) = C \int^{z} \exp\left[-\int_{-\infty}^{\infty} \beta(t) \log(s - t)\, dt\right] ds} \qquad (*)$$



There are dozens of integral equations for numerical conformal mapping besides $(*)$, most of them simpler. Nevertheless, $(*)$ has proved very useful for some problems.

Refs: L.C. Woods, *The Theory of Subsonic Plane Flow*, 1961.
R.T. Davis, *4th AIAA Comp. Fluid Dynamics Conf.*, 1978.
J.M. Floryan, *J. Comp. Phys.*, 1985.
M. Hoekstra, in *Numerical Grid Generation*, 1986.
P. Henrici, *Applied & Computational Complex Analysis III*, 1986.
L.N.T., ed., *Numerical Conformal Mapping*, 1986.

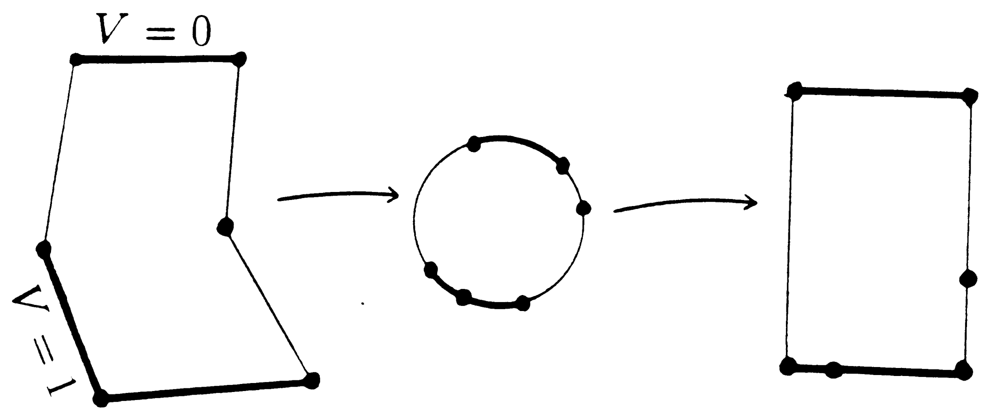
3. Applications

Electrical Resistance

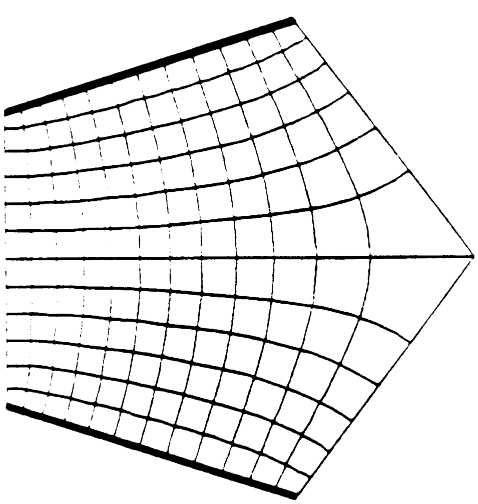
Problem: find resistance (conformal module) of a "quadrilateral"

1) Map the resistor conformally onto a rectangle

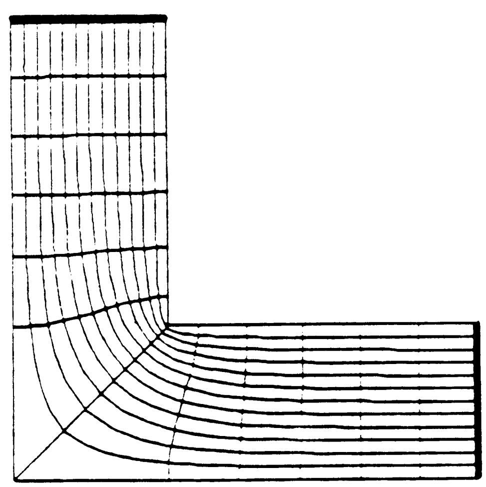
2) Resistance = Length/Width



Examples:



$R = 1.11575250227$



$R = 4.55872841596$

Refs: D. Gaier, *Numer. Math.*, 1972.
L.N.T., *Z. Angew. Math. Phys.*, 1984.

Inverse Problems, Side Conditions

Standard S-C:

geometry fully specified (e.g., n side lengths)

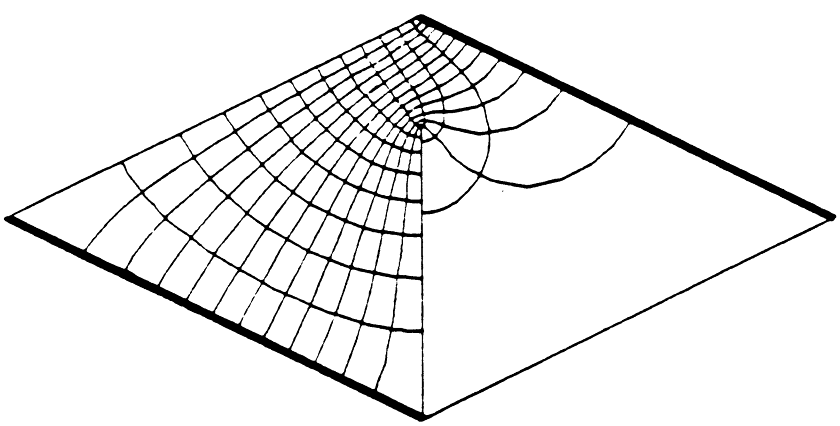
S-C with side conditions:

geometry partly specified (e.g., $n-1$ side lengths)

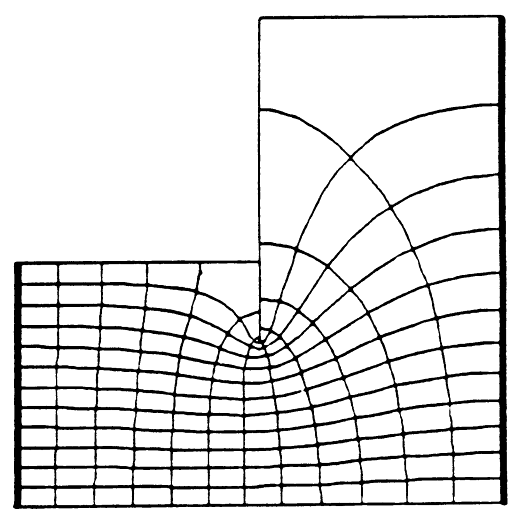
+ additional constraints (e.g., specified conformal module)

⟶ "Generalized parameter problem"

Example — slit resistors with $R = 2$:



slit length = 0.727775151589



slit length = 0.330164529748

Ref: L.N.T., *Z. Angew. Math. Phys.*, 1984.

Piecewise Constant B.C.'s

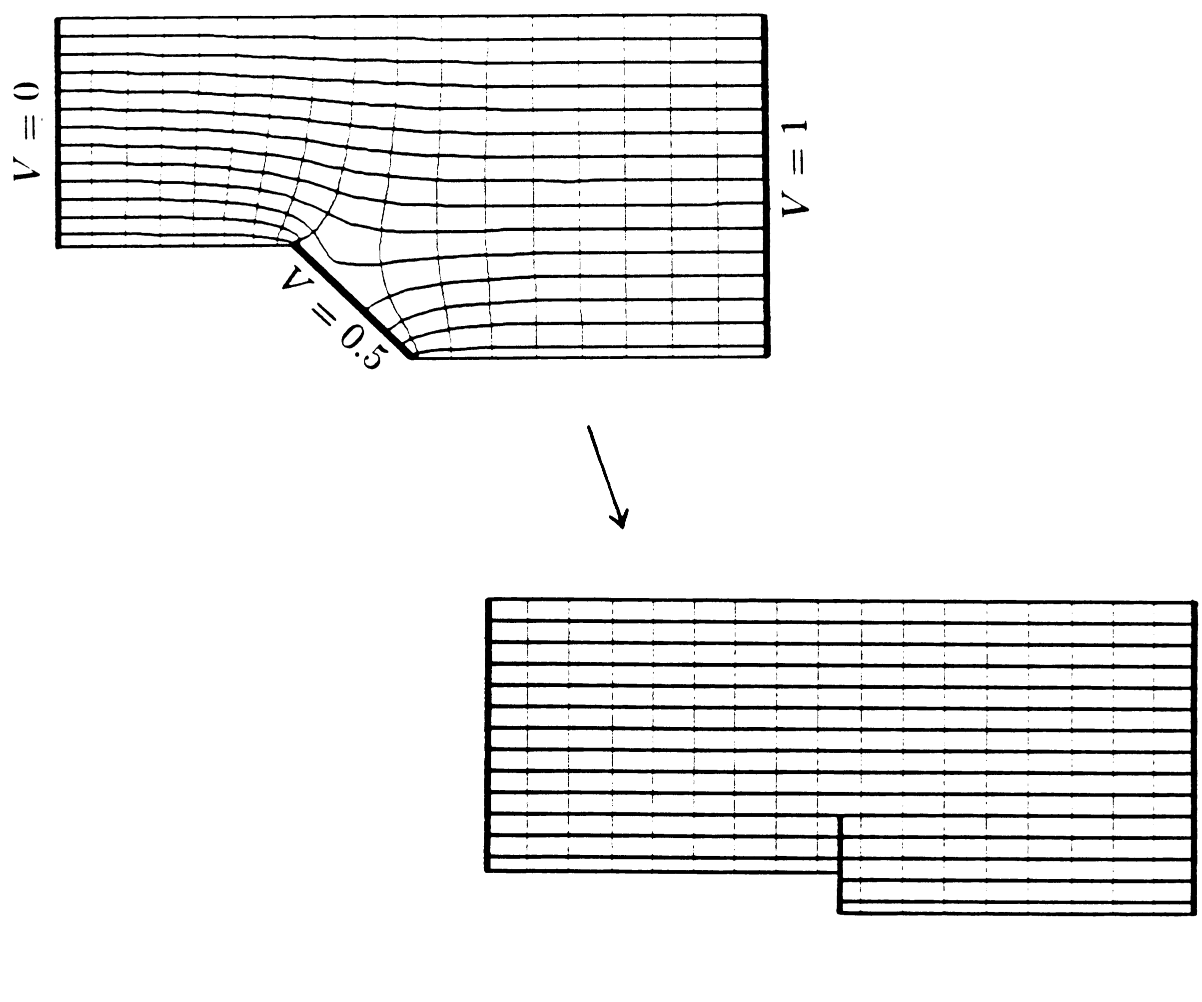
Resistance problem (p. 24) → rectangle of unknown aspect ratio

Laplace problems with a larger number of piecewise constant b.c.'s

→ rectilinear domain with slits of unknown dimensions

→ generalized parameter problem (linear, hence easy)

Example:



Refs: R.F. Wick., *J. Appl. Phys.*, 1954.

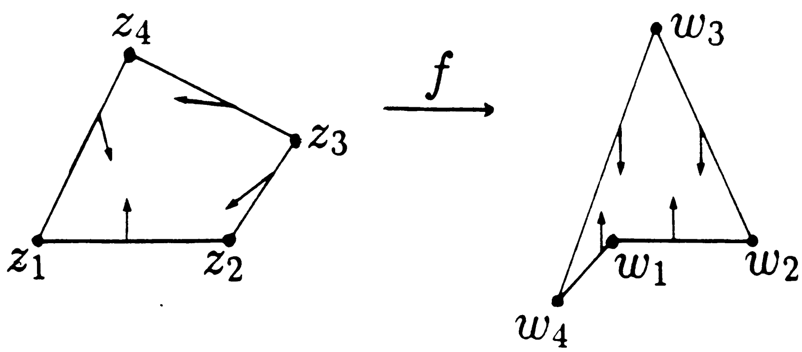
L.N.T. & R.J. Williams, *J. Comp. Appl. Math.*, 1986.

Oblique Derivative B.C.'s

Problem: $\nabla^2 u = 0$ on a polygon, oblique Neumann b.c.'s

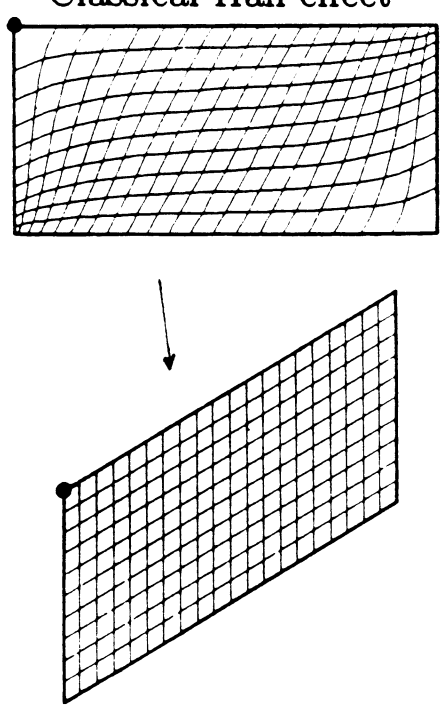
1) Find map f to another polygon with "vertical b.c.'s";

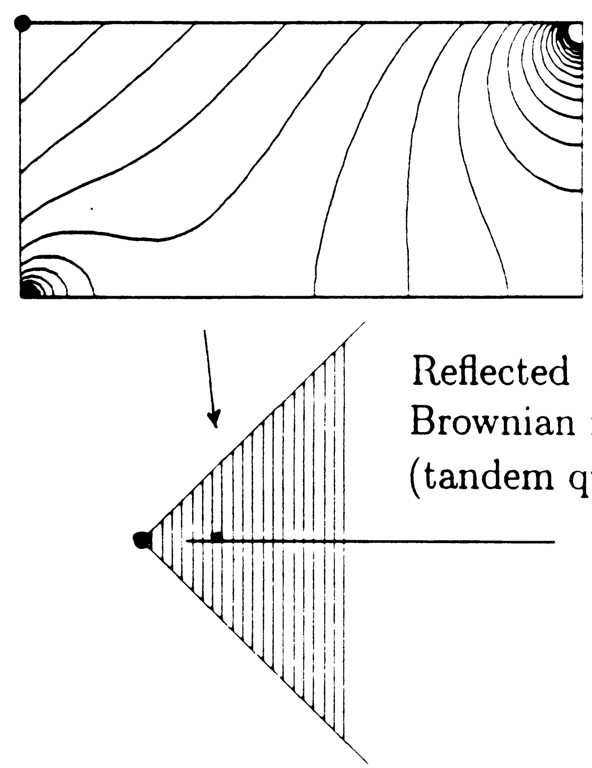
2) $u := \operatorname{Re} f$.



Examples:

Classical Hall effect





Reflected
Brownian motion
(tandem queues)

Refs: R.F. Wick., *J. Appl. Phys.*, 1954.

L.N.T. & R.J. Williams, *J. Comp. Appl. Math.*, 1986.

Vortex Methods in Fluid Dynamics

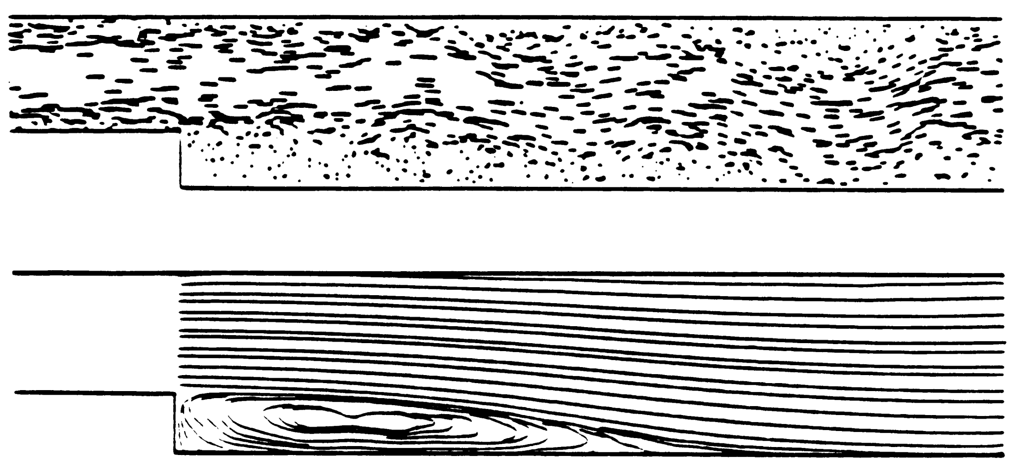
Viscous flow at high Reynolds number

Simulation via point (or "blob") vortices

Boundary conditions imposed by method of images

Conformal map to half-plane ensures one image per vortex

Example:



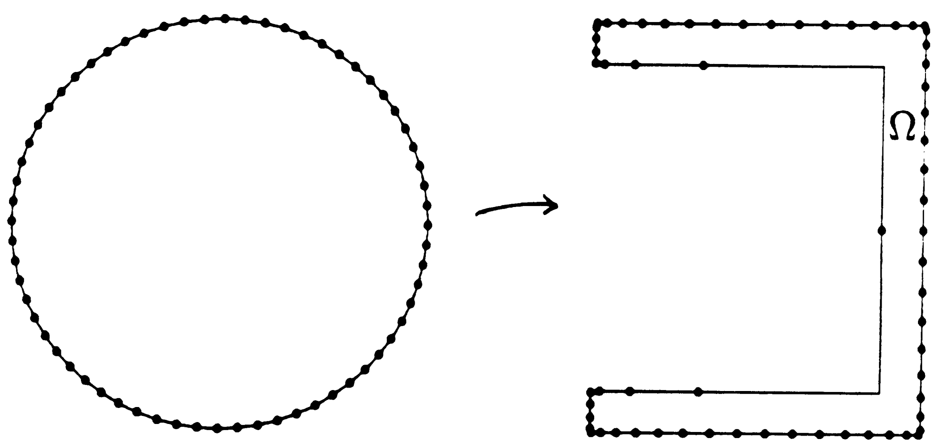
Re = 191

Ref: A.F. Ghoniem & Y. Gagnon, *J. Comp. Phys.*, 1987.

Complex Approximation; Soution of Ax=b

Given: domain Ω with boundary Γ

Fejér Points on Γ: images of roots of unity under conformal map of exterior of unit disk to exterior of Ω. "Uniformly distributed," hence good for polynomial interpolation.



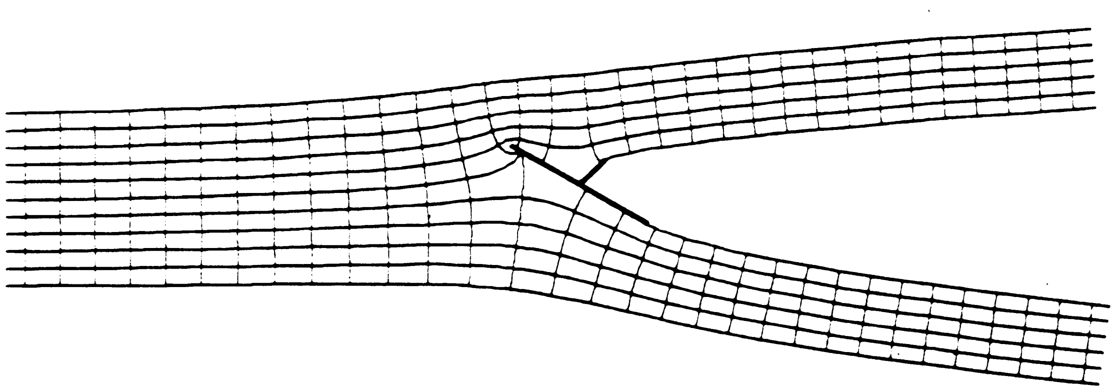
Application to iterative solution of $Ax = b$ (nonsymmetric):

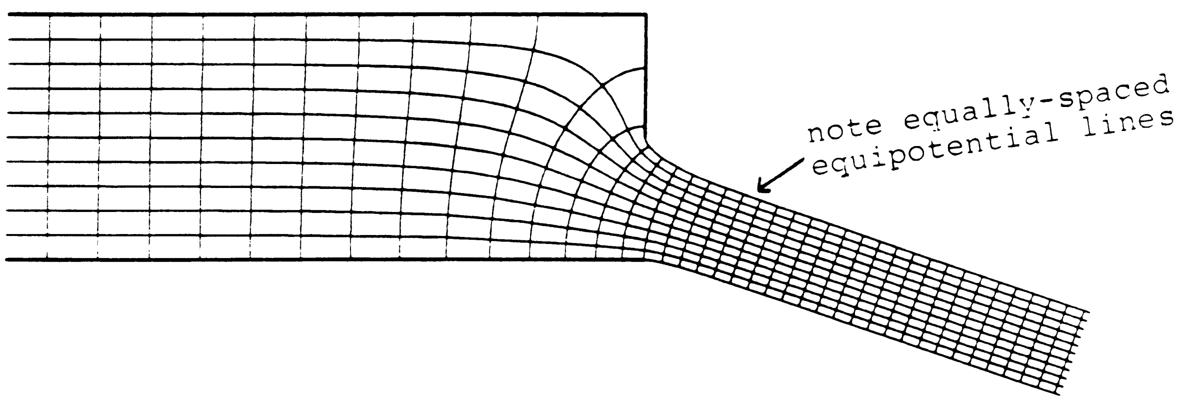
1) Determine estimate Ω of spectrum of A
2) Calculate Fejér points for Ω via conformal map
3) Construct iteration based on interpolation of z^{-1}

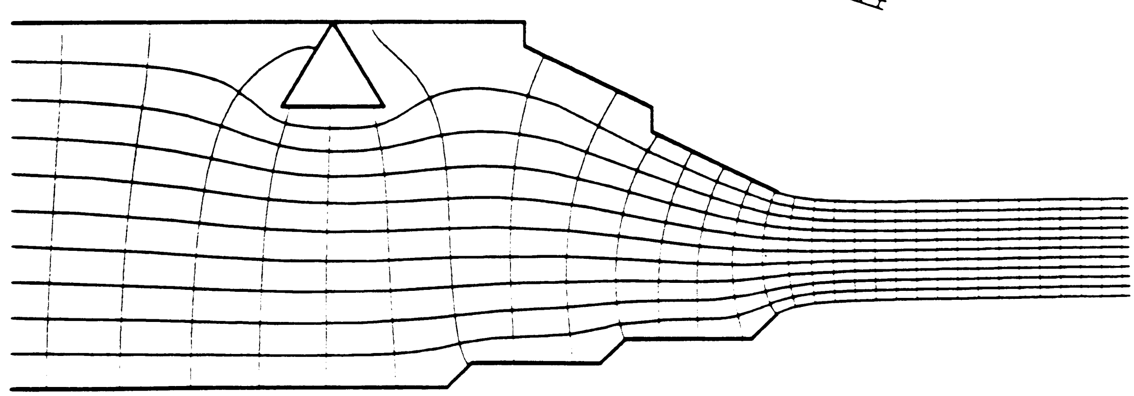
Refs: J.L. Walsh, *Interpolation and Approximation...*, 1935.
D. Gaier, *Lectures on Complex Approximation*, 1987.
B. Fischer & L. Reichel, *Numer. Math.*, ~~to appear~~. 1988
H. Tal-Ezer, *SIAM J. Sci. Stat. Comp.*, submitted.
L.N.T., *Algorithms for Approximation II*, ~~to appear.~~ 1990

Jets, Wakes, and Cavities

(Ideal 2D flow, no gravity... see p. 15)







Refs: A.R. Elcrat & L.N.T., *J. Comp. Appl. Math.*, 1986.
F. Dias, A.R. Elcrat & L.N.T., *J. Fluid Mech.*, 1987.

Summary

- Most S-C problems can be solved to full machine precision in seconds or minutes (work = $O(n^3)$)
- S-C variants $\longrightarrow$ "modified S-C integrands" $f' = \prod_k f_k$
- Often not all conditions are geometric $\longrightarrow$ "generalized parameter problems"
- All exactly-solvable conformal mapping problems are S-C ! (well, *almost* all... sometimes in disguise)