IDENTIFYING THE PHASES OF GAUGE THEORIES WITH ANOMALY-MEDIATED SUPERSYMMETRY BREAKING

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Andrew Wesley Gomes

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IDENTIFYING THE PHASES OF GAUGE THEORIES WITH ANOMALY-MEDIATED SUPERSYMMETRY BREAKING

Andrew Wesley Gomes, Ph.D.

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We discuss recent developments in the understanding of non-supersymmetric gauge theories that have come from deforming supersymmetric (SUSY) gauge theories with anomaly-mediated supersymmetry breaking (AMSB). While the phases of SUSY gauge theories have been well understood since the 1990's, until recently there has been difficulty applying this knowledge to the non-supersymmetric versions of these theories. This dissertation reports significant steps in this direction. First, and with most relevance to real-world Quantum Chromodynamics (QCD), AMSB is applied to $SU(N_c)$ theories with fundamental quarks, where we discuss the stability of novel chiral symmetry breaking minima. Considerable discussion is devoted to the presence or absence of baryonic runaway directions in the various theories. Second, we analyze $SO(N_c)$ theories with vector-like quarks. In the particular case of $N_f = N_c - 2$ quarks, the theory has monopoles which condense upon the application of AMSB, leading to confinement via the dual Meissner effect. With a series of dualities and mass deformations, we extend this result to show that a wide class of $SO(N_c)$ theories lie in the confining phase. Again, we demonstrate the occurrence of chiral symmetry breaking in all of these theories.

BIOGRAPHICAL SKETCH

Andrew Gomes was born in Montreal, QC, Canada in 1996, and grew up in Calgary, AB. He first met the subatomic world at the age of 12 in a library book about quantum physics. While he was always interested in math and the sciences, from that moment his path was determined.

In 2014 he attended the University of Toronto, completing the Mathematics and Physics Specialist (B.Sc.) program in 2018. In Toronto he enjoyed living in the "big city" and pursuing his passions full time. It was during this time that I first got to know him.

In 2018 Andrew began the Physics Ph.D. program at Cornell University, receiving an M.Sc. in 2021. Here I learned of his love of languages and travel, and that he thinks best on long walks in the summertime.

Looking to the future, Andrew will spend three years doing postdoctoral research at the École polytechnique fédérale de Lausanne (EPFL). I asked him where he will go after that. He hasn't a clue.

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Tout y parlerait

À l'âme en secret

Sa douce langue natale.

– Charles Baudelaire, *L'Invitation au Voyage*

TABLE OF CONTENTS

CHAPTER 1
INTRODUCTION

One of the greatest challenges facing particle physics and quantum field theory (QFT) is to establish the phase structure of strongly coupled gauge theories. Of particular interest is the behaviors of ordinary Quantum Chromodynamics (QCD), the theory of the strong nuclear force. The high-energy (UV) description of such theories is quite simple. In the case of QCD the particle content consists only of massless gluons and massless[1] quarks. However, due to asymptotic freedom the effective gauge coupling increases as one probes the theory at lower energies. The scale at which the gauge coupling is so large that perturbation theory breaks down is called Λ_{QCD}. At the low energies (IR) below this scale, the theory takes on an entirely different character. A plethora of new particles (hadrons) emerge, including mesons and baryons.

These particles can be thought of as quark bound states, and are uncharged under the UV gauge group $SU(N_c)$. This phenomenon, known as "color confinement", is the most striking feature of QCD, and indeed of many gauge theories. Almost as surprising, a group of pseudoscalar mesons (which we shall call pions) are massless. The existence of such Goldstone bosons points to a spontaneously broken symmetry, in this case the chiral part of the flavor symmetry. This second important phenomenon is known as "chiral symmetry breaking". These two features of QCD (and QCD-like theories) will be the focus of this dissertation.

So how do we formulate the IR description of QCD? The answer comes from an important tenet of physics: "symmetry constrains dynamics". Knowing that

[1] The quarks of our world are of course massive, however, the changes to the theory due to quark masses are completely under control. For simplicity, we will therefore focus exclusively on the case of massless quarks.

QCD breaks chiral symmetry, we introduce a fictitious "spurion" field U that is a bifundamental under $SU(N_f)_L \times SU(N_f)_R$ flavor symmetry. The non-vanishing vacuum expectation value (VEV) of U breaks the flavor symmetry to the diagonal group $SU(N_f)_D$. The fluctuations of U around its VEV are the pions[2]. The "chiral Lagrangian" for the pions is then formed by writing down all terms compatible with the full UV flavor symmetry. However, the coefficients of these terms are theoretically undetermined and must come from experiment.

The problem then becomes to determine the details of the IR description from the UV theory. Even before trying to derive the coefficients of the chiral Lagrangian, we would like a "0^{th} order" approximation that demonstrates essential features such as color confinement and chiral symmetry breaking. Most directly one can numerically compute the path integral using lattice techniques. This field has experienced continued progress due to increasing computational power and improved algorithms, but still has a ways to go. Additionally, an analytic understanding of the UV-IR matching of QCD is preferable. Various ways of modeling QCD have been proposed. For example the Nambu–Jona-Lasinio (NJL) model [1], which replaces the gluons with a 4-Fermi interaction of the quarks. This model experiences chiral symmetry breaking but not confinement. In holographic QCD [2, 3], an application of the AdS/CFT correspondence, one obtains QCD-like towers of mesonic resonances if chiral symmetry breaking is imposed by hand. However, the masses of these resonances scale like a quantum mechanical infinite well ($m_n \sim n$) as opposed to the "stringy" scaling of QCD ($m_n^2 \sim n$). To capture this feature of QCD, one can instead model hadrons with confining flux tubes. These are just a few of the many models that capture some aspects of QCD. However, no model captures all of QCD's phenomena with satisfactory precision.

[2]In the large N_c limit one can additionally think of the baryons as "skyrmions", spatial twists in the chiral condensate U. We will make no use of this here.

Another direction, along which this dissertation reports further progress, is to employ supersymmetry (SUSY). This approach takes "symmetry constrains dynamics" even further. Supersymmetry is a symmetry relating bosons and fermions, originally exciting for its potential to explain the "lightness" of the Higgs boson mass with respect to the Plank scale. Instead we use it as a theoretical tool as follows. The imposition of SUSY introduces superpartners for the gluons and quarks: gluinos (fermions) and squarks (bosons). The extra symmetry allows for the derivation of exact results via the "power of holomorphy" [4–6]. In particular, the IR phases and superpotentials[3] of these supersymmetric gauge theories are exactly determined. The game is then to add small SUSY breaking perturbations (soft breaking) that lift the masses of the superpartners, returning the non-SUSY version of the theory [7–23]. The exact mapping of SUSY breaking perturbations from the UV theory to its IR manifestation has been done by linking the SUSY breaking either to holomorphic quantities [18], or to conserved and anomalous currents [20–23]. While being successful in mapping UV supersymmetry breaking to the IR, in many previous attempts at studying the vacuum structure of softly broken supersymmetric QCD (SQCD) the eventual IR phase was incalculable due to runaways and/or dependence on unknown Kähler terms. For this reason, they were of limited predictivity.

A systematic study of the phases of SUSY $SU(N_c)$ gauge theories perturbed via anomaly-mediated supersymmetry breaking (AMSB) was initiated in [24]. Many new results using this method have been obtained, including novel symmetry breaking patterns for chiral gauge theories [25–27]. This dissertation will focus on the application of AMSB to vector-like (that is, QCD-like) SUSY gauge theories.

[3]The superpotential is one component of a supersymmetric Lagrangian, the other being the Kähler potential. Very roughly, the superpotential gives the scalar potential while the Kähler potential gives the kinetic terms. In this dissertation we will assume a familiarity with SUSY.

The organization of this dissertation is as follows. We will begin in Chapter 2 by reviewing the AMSB technique. In Chapter 3 we present a careful study of the vacuum structure of AMSB-deformed SQCD. In particular we discuss the stability of chiral symmetry breaking minima and potential baryonic runaway directions. In Chapter 4 we use AMSB to study the phase structure of QCD-like non-SUSY $SO(N_c)$ theories. For a wide class of theories we give the first demonstration of continuous chiral symmetry breaking and confinement (via the dual Meissner effect) in a non-supersymmetric gauge theory. Unless otherwise stated, the content of the latter two chapters reflects the original work of [28–30][4].

[4]Collaborators on some or all of these works: Csaba Csáki, Hitoshi Murayama, Ofri Telem, Bea Noether, and Digvijay Roy Varier.

CHAPTER 2

ANOMALY-MEDIATED SUPERSYMMETRY BREAKING

As mentioned in the introduction, it is not enough to simply break SUSY in the UV. One must understand how to translate the breaking into the IR. For this reason anomaly-mediated supersymmetry breaking (AMSB) [31, 32] (see also [20, 33] for earlier work containing some important aspects of AMSB) is particularly well motivated. To see why, first consider that gravity couples universally to matter at all scales. If we could spontaneously break SUSY in one sector of a theory and mediate that breaking only with gravity to the physics that we care about, we could be confident in applying the same formulae to the UV and IR. This is exactly what happens in AMSB, which we now explain.

The union of SUSY and gravity necessitates supergravity (SUGRA). In fact, we will start from an even more "perfect" theory, superconformal gravity, in order to use the conformal compensator (or Weyl compensator) formalism [34]. Consider the following variant of the usual Einstein-Hilbert action for gravity,

$$S = \frac{M_{\mathrm{Pl}}^2}{2}\int d^4x\, e\, \sigma^*\sigma\, R, \tag{2.1}$$

Where M_{Pl} is the Plank mass, e is the determinant of the vierbein, and R is the Ricci scalar. The only new element is the conformal spurion σ, which is defined to have conformal weight 1 so that the action is scale invariant (the conformal weight of x^μ is -1). It is a spurion because if it takes on the VEV $\langle\sigma\rangle = 1$, conformal symmetry is broken and we are left with ordinary gravity.

To arrive at superconformal gravity, we need to add SUSY. Besides introducing the gravitino, σ gets promoted to a chiral superfield Σ, called the conformal (or Weyl) compensator. It must be stressed that Σ is not a dynamical field - its only role is to restore conformal invariance. To couple this field to matter we need only cancel the conformal weights[1] of the usual supergravity action,

$$\mathcal{L} = -3M_{\rm Pl}^2 \int d^4\theta\, \Sigma^\dagger \Sigma\, e^{-K/3M_{\rm Pl}} + \int d^2\theta \left(\Sigma^3 W + \frac{\tau}{16\pi i} W^\alpha W_\alpha \right) + h.c. \qquad (2.2)$$

Where $S = \int d^4x\, e\, \mathcal{L}$, K is the Kähler potential, W is the superpotential, and τ and W^α represent the holomorphic gauge couplings and field strengths of any gauge groups present in the theory.

To break SUSY our goal is to give an F-component to the conformal compensator so that

$$\langle \Sigma \rangle = 1 + m\,\theta^2 \qquad (2.3)$$

Where m is the scale of SUSY breaking. To this end we add a "hidden sector" to the theory, whose sole purpose is to spontaneously break SUSY and who only communicates to the "visible sector" (the physics we care about) via the conformal compensator. In their original paper, Randall and Sundrum came up with such a minimal sector [31],

$$K = \Lambda_H^2 \log\left(1 + \frac{X^\dagger X}{\Lambda_H^2}\right), \quad W = \Lambda_H^2 (X + c), \qquad (2.4)$$

Where c is tuned so that the cosmological constant vanishes. Using (2.2) one finds $m = \sqrt{3}\Lambda_H^2/M_{\rm Pl}$.

Now let's return to the visible sector, which we take to be far below the Plank scale so that we can take just the leading term in the exponent of (2.2). We have

$$\mathcal{L} = \int d^4\theta\, \Sigma^\dagger \Sigma\, K + \int d^2\theta \left(\Sigma^3 W + \frac{\tau}{16\pi i} W^\alpha W_\alpha \right) + h.c. \qquad (2.5)$$

We can make the leading term in the Kähler potential canonical with field rescalings $\Phi_i \to \Phi_i/\Sigma$. Factors of Σ will remain only in higher-order terms in the Kähler

[1] The conformal weight of θ is $-1/2$ and, as a Grassmann variable, $d\theta$ therefore has weight $1/2$. The definition of W^α includes 3 superderivatives so that its weight is $3/2$.

potential, or in non-marginal (non-cubic) terms of the superpotential. In fact, we will always find Σ paired with dimensionful parameters Λ as $\Sigma\Lambda$.

Plugging in auxiliary (F-component) field equations of motion as usual, we get a tree level scalar potential

$$\begin{aligned} V_{\text{tree}} &= \partial_i W g^{ij^*} \partial_j^* W^* + m^* m \left(\partial_i K g^{ij^*} \partial_j^* K - K \right) \\ &\quad + m \left(\partial_i W g^{ij^*} \partial_j^* K - 3W \right) + c.c. \end{aligned} \tag{2.6}$$

where g^{ij} is the inverse of the Kähler metric $g_{ij} = \partial_i \partial_j^* K$. For simplicity we will always take m to be real. The first term is the usual SUSY potential while the latter two terms are the AMSB effects. Note that (2.6) breaks the $U(1)_R$ symmetry explicitly. When the Kähler potential is canonical, this reduces to the more familiar

$$\delta V_{\text{tree}} = m \left(\varphi_i \frac{\partial W}{\partial \varphi_i} - 3W \right) + c.c. \tag{2.7}$$

When the superpotential does not include dimensionful parameters, this expression identically vanishes. Note that SQCD has no superpotential in the UV description so that this contribution vanishes.

In this case, there are the loop-level supersymmetry breaking effects from the superconformal anomaly [34]. They lead to tri-linear scalar couplings, scalar masses, and gaugino masses,

$$A_{ijk}(\mu) = -\frac{1}{2}(\gamma_i + \gamma_j + \gamma_k)(\mu)\, m \tag{2.8}$$

$$m_i^2(\mu) = -\frac{1}{4}\dot{\gamma}_i(\mu)\, m^2 \tag{2.9}$$

$$m_\lambda(\mu) = -\frac{\beta(g^2)}{2g^2}(\mu)\, m. \tag{2.10}$$

Here, $\gamma_i = \mu \frac{d}{d\mu} \ln Z_i(\mu)$, $\dot{\gamma} = \mu \frac{d}{d\mu}\gamma_i$, and $\beta(g^2) = \mu \frac{d}{d\mu} g^2$. The first two equations can be traced to renormalization considerations when we did the field rescaling, while the last equation comes from the secret dependence of τ on a dynamical scale (dimensional transmutation).

Notice that when the gauge theory is asymptotically free (for example with an $SU(N_c)$ gauge group and N_f flavors such that $N_f < 3N_c$), $m_i^2 > 0$, stabilizing the theory against run-away behavior. Therefore, AMSB prepares exactly the state we are looking for: the squarks and gauginos become massive while the massless degrees of freedom are those of non-SUSY QCD. By the UV insensitive nature of AMSB, the expressions above can be reliably used in the dual (IR) description of the theory to determine the low-energy phase.

Here we present some expressions that will be useful later on. Suppose we have a $SU(\widetilde{N}_c)$ gauge theory with gauge coupling g and a superpotential $W = \lambda \, \mathrm{Tr}\, q_i M_{ij} \bar{q}_j$, where the q_i ($\bar{q}_j$) are N_f flavors of (anti-)quarks and M_{ij} is a gauge-singlet flavor-bifundamental meson. The anomalous dimensions are

$$\gamma_q = \frac{C_F g^2}{4\pi^2} - \frac{N_f \lambda^2}{8\pi^2} \tag{2.11}$$

$$\gamma_M = -\frac{\widetilde{N}_c \lambda^2}{8\pi^2} \tag{2.12}$$

where $C_F = (\widetilde{N}_c^2 - 1)/(2\widetilde{N}_c)$ is the quadratic Casimir of the dual gauge group. For the 1-loop beta functions one has

$$\beta(g^2) = -\frac{\widetilde{b} g^4}{8\pi^2} \tag{2.13}$$

$$\beta(\lambda^2) = -(\gamma_M + 2\gamma_q)\lambda^2 \tag{2.14}$$

where $\widetilde{b} = 3\widetilde{N}_c - N_f$.

CHAPTER 3

A GUIDE TO AMSB QCD

3.1 Introduction

In this chapter we apply AMSB to SQCD (a supersymmetric $SU(N_c)$ gauge theory with N_f quark flavors). This study was initiated by Murayama in [24] (and extended to the conformal window in [35]), who found a QCD-like vacuum with a chiral symmetry breaking pattern of the form $SU(N_f)_L \times SU(N_f)_R \to SU(N_f)_D$, and which appears to be the global minimum for at least $N_f < N_c$. However, the $N_f \geq N_c$ cases are complicated by the appearance of baryonic directions, which in many cases appear to cause a runaway behavior.

The aim of this chapter is to carefully examine the $SU(N_c)$ theories for $N_f \geq N_c$ and, in particular, the fate of the baryonic directions, thus establishing the phase structure of the $SU(N_c)$ theories, when it is calculable. We will show that for the special case of $N_f = N_c+1$ the potential baryonic runaway is stabilized by a 2-loop AMSB effect, while for $N_f = N_c$ the theory is incalculable along these directions, and one can not conclusively decide if the baryonic runaways are lifted or not. The lower end $N_c + 1 < N_f \lesssim 1.43N_c$ of the free magnetic phase will again have the baryonic runaways lifted via 2-loop AMSB, and one ends up with stable, calculable vacuum with chiral symmetry breaking. Such a "QCD-like" vacuum with chiral symmetry breaking exists for any number of flavors as long as $N_f < 3N_c$ (with the possible exception of $N_f = N_c$ where its stability cannot be determined).

In contrast, for $N_f \gtrsim 1.43N_c$ the baryonic directions will indeed contain runaways. We stress that these runaways do not invalidate the theory since they are cured once the field vacuum expectation values (VEVs) are of $\mathcal{O}(\Lambda_{\rm QCD})$. Here the IR description breaks down and one must return to the UV description, where the theory is stabilized by AMSB. Instead, they merely signal that the global minimum lies in the incalculable region where field VEVs are of $\mathcal{O}(\Lambda_{\rm QCD})$. In addition, the QCD-like minimum will persist as a local minimum, and one expects that as the magnitude of SUSY breaking is increased it will indeed take over as the true vacuum. Note also that baryonic runaway does not occur in Sp or SO gauge theories.

3.2 $N_f < N_c$: ADS superpotential

For completeness we quickly review here the results of [24] for $N_f < N_c$. The dynamics is described in terms of the meson fields M_{ij} with the Affleck–Dine–Seiberg (ADS) superpotential

$$W = (N_c - N_f)\left(\frac{\Lambda^{3N_c-N_f}}{\det M}\right)^{1/(N_c-N_f)}. \tag{3.1}$$

In the SUSY limit, this produces a runaway potential and hence has no ground states. When $M \gg \Lambda^2$, $M_{ij} = M\delta_{ij}$ describes the D-flat direction

$$Q = \bar{Q} = \phi \begin{pmatrix} 1_{N_f\times N_f} \\ 0_{(N_c-N_f)\times N_f} \end{pmatrix}, \qquad M = \phi^2. \tag{3.2}$$

Here, Q and $\bar{Q}$ are the quark/anti-quark superfields. The upper part is an $N_f \times N_f$ block, while the lower part is $(N_c - N_f) \times N_f$. Since the Kähler potential

$$K = 2\,\mathrm{Tr}(M^\dagger M)^{1/2} \tag{3.3}$$

is canonical in the variable ϕ, one can use (2.7) to obtain

$$V = \left| 2N_f \frac{1}{\phi} \left(\frac{\Lambda^{3N_c-N_f}}{\phi^{2N_f}} \right)^{1/(N_c-N_f)} \right|^2 - (3N_c - N_f) m \left(\frac{\Lambda^{3N_c-N_f}}{\phi^{2N_f}} \right)^{1/(N_c-N_f)} + c.c. \tag{3.4}$$

Note that there is now a minimum at

$$M_{ij} = \Lambda^2 \left(\frac{4N_f(N_c+N_f)}{3N_c - N_f} \frac{\Lambda}{m} \right)^{(N_c-N_f)/N_c} \delta_{ij}. \tag{3.5}$$

The minimum is indeed at $M \gg \Lambda^2$ which justifies the weakly-coupled analysis. The $SU(N_f)_L \times SU(N_f)_R$ flavor symmetry is dynamically broken to $SU(N_f)_V$. The case of non-homogeneous values for the diagonal entries of M was considered in [36]. There it was shown that the minimum is indeed found at $M_{ij} \propto \delta_{ij}$.

The massless particle spectrum consists of the Nambu–Goldstone bosons (pions) [1]. The scalar and fermion partners of the Nambu–Goldstone bosons (NGBs) have masses that grow with m. Naively increasing m beyond Λ, the only remaining degrees of freedom are massless NGBs. This matches the expectations of QCD with a small number of flavors. There is no sign of a phase transition and the two limits are likely continuously connected.

3.3 $N_f = N_c$: Quantum modified constraint

In this section we give a complete analysis of the case of the quantum modified constraint, finding that previous discussion requires modification.

[1]The case $N_f = 1$ is special as there is no non-anomalous flavor symmetry and hence the spectrum is gapped.

The low-energy degrees are meson fields M_{ij} and singlet baryon/anti-baryon fields B ($\bar{B}$), whose moduli space is subject to the quantum modified constraint

$$\det M - B\bar{B} = \Lambda^{2N_c}. \tag{3.6}$$

We first treat the general case $N_c > 2$, and treat the case $N_c = 2$ separately at the end.

There are two ways to frame the theory before the addition of AMSB. The first is to implement the constraint in the superpotential via a Lagrange multiplier field X. However due to the constraint (3.6), the fields have VEVs of $\mathcal{O}(\Lambda)$. Therefore, higher order terms in the Kähler potential are not suppressed relative to the canonical term and the formula (2.7) cannot be trusted.

Instead we should perform a non-linear analysis using the constraint (3.6). For simplicity, we will use units where $\Lambda = 1$. The moduli space contains two special points of enhanced symmetry: the meson point $M = \mathbf{1}, B = \bar{B} = 0$ with unbroken baryon number, and the baryon point $M = 0, B = -\bar{B} = 1$ with unbroken flavor symmetry. We perform AMSB around each of these points.

3.3.1 Meson point

To satisfy the constraint at the meson point we make the change of variables

$$M = (1 + B\bar{B})^{1/N_c} e^{\Pi} = \mathbf{1} + \frac{1}{N_c} B\bar{B} + \Pi + \frac{1}{2}\Pi^2 + \cdots \tag{3.7}$$

where Π is a traceless complex matrix. In what follows we will work to quadratic order. The Kähler potential is built out of flavor invariants, e.g. $\operatorname{Tr} M^\dagger M$, $(\operatorname{Tr} M^\dagger M)^2$, $\operatorname{Tr} M^\dagger M M^\dagger M$, etc. Notice that they will all contribute at quadratic order in the hadron superfields. Let's examine the Π contribution of the first term:

$$\mathrm{Tr}\, M^\dagger M \supset \mathrm{Tr}\, \Pi^\dagger \Pi + \frac{1}{2}\,\mathrm{Tr}\, \Pi^2 + \frac{1}{2}\,\mathrm{Tr}\, {\Pi^\dagger}^2. \tag{3.8}$$

A useful formula going forward will be the tree level AMSB potential corresponding to $K = \varphi^\dagger\varphi + \alpha/2\,(\varphi^2 + {\varphi^\dagger}^2)$. Using the general formula (2.6), we get

$$\begin{aligned} V_{\mathrm{AMSB}} &= \alpha^2 m^2 \varphi^\dagger \varphi + \frac{\alpha}{2} m^2 (\varphi^2 + {\varphi^\dagger}^2) \qquad (3.9)\\ &= (\alpha^2 + \alpha) m^2 (\mathrm{Re}\,\varphi)^2 + (\alpha^2 - \alpha) m^2 (\mathrm{Im}\,\varphi)^2. \end{aligned}$$

Setting $\alpha = 1$ corresponds to the Kähler potential for each component of Π in (3.8), so that the $\mathrm{Im}\,\Pi$ are the massless pions, the Goldstone bosons of broken chiral flavor symmetry. Goldstone's theorem ensures that all meson flavor invariants of the Kähler potential will give contributions proportional to the right-hand-side of (3.8). Moreover, they will (in aggregate) come with a positive sign in order for the Π to have a physical kinetic term. Thus, the $\mathrm{Re}\,\Pi$ will have a positive mass, stabilizing this direction.

Turning to the baryons, things are not as clear. The most general form of the Kähler potential at quadratic order is

$$K \supset \alpha(B^\dagger B + \bar{B}^\dagger \bar{B}) + \frac{\beta}{2}(B\bar{B} + c.c.) \tag{3.10}$$

where this includes contributions (3.7) from meson field traces. We cannot know the ratio β/α and thus are unable to determine whether the meson point is stable with respect to baryonic runaway to an incalculable minimum.

3.3.2 Baryon point

Here we parameterize the baryon and anti-baryon with a single complex field b:

$$B = (1 - \det M)^{1/2} e^{b} \tag{3.11}$$

$$\bar{B} = -(1 - \det M)^{1/2} e^{-b}. \tag{3.12}$$

Like at the meson point, we expect to find a Goldstone boson, now from spontaneously broken baryon number. Consider for example the Kähler potential terms

$$B^\dagger B + \bar{B}^\dagger \bar{B} = 2 + (b + b^\dagger)^2 + \cdots . \tag{3.13}$$

Again using (3.9), we identify $\mathrm{Im}\, b$ as the Goldstone boson, while $\mathrm{Re}\, b$ has positive mass. Regarding the mesons however, only the quadratic term must come with a positive sign (to give positive kinetic term). The coefficients of all higher order flavor invariants in the Kähler potential are unknown. With the application of (2.6), these will ultimately determine if the baryon point is stable once AMSB is turned on.

In summary, we can say very little about the behavior of AMSB-deformed QCD in the singular case when $N_f = N_c$. Neither global nor local minima can be identified, though based on the behavior of theories with more or fewer flavors we can conjecture a chiral symmetry breaking minimum at the meson point. This ambiguity can be traced to the quantum modified constraint, making the theory inherently strongly coupled.

3.3.3 $N_c = 2$

When $N_c = 2$, the quarks and anti-quarks belong to the same representation of the gauge group. Thus, the flavor symmetry is enhanced to $SU(4)$, with the meson M transforming in the anti-symmetric representation. This meson can be decomposed into the meson, baryon, and anti-baryon of the unenhanced flavor symmetry. The quantum modified constraint becomes $M^a M^a = 1$, with $a = 1, \ldots, 6$, meaning the moduli space has 5 complex dimensions. The constraint breaks the flavor symmetry to $Sp(4)$, leading to 5 Goldstone modes. Due to the kinetic term positivity arguments made above, their scalar partners have positive mass.

Thus, the enhanced symmetry causes the chiral symmetry breaking minimum to be stable in the case of $N_c = 2$. Similar results were found in [21]. Note that $N_c = 2$ is a special case of the Sp gauge theories that will be discussed elsewhere.

3.4 $N_f = N_c + 1$: S-Confinement

For this case we find a stable chiral symmetry breaking minimum, and demonstrate that there are no runaway directions. At the leading order we take a canonical Kähler potential for low energy fields B, $\bar{B}$ and M, which is justified when $B, \bar{B}, M \ll \Lambda$ where the theory is weakly coupled. The superpotential is

$$W = \alpha B M \bar{B} - \beta \det M \tag{3.14}$$

where we are again working in $\Lambda = 1$ units and α and β are unknown order one numbers used to make the Kähler canonical. The potential obtained is

$$V_{\text{SUSY}} = \alpha^2(|(M\bar{B})_a|^2 + |(BM)_a|^2) + |\alpha\bar{B}_a B_b - \beta \det M(M^{-1})_{ab}|^2 \tag{3.15}$$

$$V_{\text{AMSB}} = -(N_c - 2)\beta m \det M + c.c. \tag{3.16}$$

Seeking the minimum of this potential, we look along the direction

$$B = \begin{pmatrix} b \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \bar{B} = \begin{pmatrix} \bar{b} \\ 0 \\ \vdots \\ 0 \end{pmatrix}, M = \begin{pmatrix} x & & & \\ & v & & \\ & & \ddots & \\ & & & v \end{pmatrix}. \tag{3.17}$$

Using flavor rotations the baryon and anti-baryon take this form without loss of generality. They break the flavor symmetry to $SU(N_c)_L \times SU(N_c)_R$, justifying the inhomogeneous diagonal VEVs of M. For fixed $\det M$, any off-diagonal terms would simply increase V_{SUSY}, justifying their omission. Finally, given that we are taking m real, it is enough to look for minima with all fields real.

Using the fact that for fixed $b\bar{b}$, the quantity $b^2 + \bar{b}^2$ is minimized when $b = \bar{b}$, the potential is

$$V = 2\alpha^2 x^2 b^2 + (\alpha b^2 - \beta v^{N_c})^2 + N_c \beta^2 x^2 v^{2(N_c-1)} - 2(N_c - 2)\beta m x v^{N_c}. \tag{3.18}$$

Again we treat the general case $N_c > 2$ first, and the case $N_c = 2$ separately afterwards.

The crucial observation implicit above is that the baryon fields do not acquire tree-level SUSY breaking whose mass originates from AMSB and they do not induce threshold corrections when they are integrated out, called "non-decoupling effects" in [34].

3.4.1 Baryon number conserving direction, $b = 0$

For the baryon number conserving direction $b = 0$, one finds a minimum

$$v = x = \left(\frac{(N_c - 2)m}{N_c \beta}\right)^{\frac{1}{N_c - 1}}, V_{\rm min} = -\mathcal{O}(m^{2N_c/(N_c-1)}). \tag{3.19}$$

This is the chiral symmetry breaking minimum that we hope to be continuously connected to that of non-SUSY QCD. First we must check that it is not disturbed by loop effects coming from the marginal Yukawa term in (3.14). The baryons acquire a mass αv, and integrating them out and using (2.9) yields a 2-loop mass for the meson

$$m_M^2 = \frac{(2N_c + 3)\alpha(v)^4 m^2}{(16\pi^2)^2}. \tag{3.20}$$

Along the direction we are considering, this gives a potential

$$V_{2-\rm loop} = \frac{(N_c + 1)(2N_c + 3)\alpha(v)^4}{(16\pi^2)^2} m^2 v^2. \tag{3.21}$$

Notice that at the point (3.19), this is also $\mathcal{O}(m^{2N_c/(N_c-1)})$. However, since it is 2-loop suppressed, it does not destabilize the chiral symmetry breaking minimum.

We should finally check the effects of higher order terms in the Kähler potential, the leading ones being $(\mathrm{Tr}\, M^\dagger M)^2$ and $\mathrm{Tr}\, M^\dagger M M^\dagger M$ with unknown coefficients (including signs). Using (2.6), we find that these give potential terms $\sim m^2 v^4$. At the point (3.19), these are higher order in m and can be neglected.

3.4.2 Baryon number breaking direction, $b \neq 0$

In general one can minimize (3.18) with respect to b and x, finding

$$b^2 = \frac{\beta}{\alpha} v^{N_c} - 2x^2 \tag{3.22}$$

$$x = \frac{(N_c - 2)m}{2\alpha}. \tag{3.23}$$

Plugging these in we find the runaway potential found in [36]

$$V|_{b,x} = -\frac{(N_c - 2)^2 \beta}{2\alpha} m^2 v^{N_c}. \tag{3.24}$$

However, we must account for loop corrections. The bottom N_c components of B and $\bar{B}$ acquire a mass αv, so we integrate them out. This gives, to all but the upper-left component M_{11}, the 2-loop mass (3.20). At this point, the remaining superpotential is simply $W = \alpha B_1 M_{11} \bar{B}_1$. M_{11} then obtains a mass at the lower scale $\sqrt{2}\alpha b$. Integrating it out results in 2-loop AMSB masses for B_1 and $\bar{B}_1$

$$m_b^2 = \frac{3\alpha(b)^4 m^2}{(16\pi^2)^2}. \tag{3.25}$$

Adding up these contributions along the direction we are considering, this gives a potential

$$V_{2-\text{loop}} = \frac{m^2}{(16\pi^2)^2}[N_c(2N_c + 3)\alpha(v)^4 v^2 + 6\alpha(b)^4 b^2]. \tag{3.26}$$

Clearly the first term is dominant. Importantly however, this is the same order in m as the tree level runaway (3.24) and lower order in v since $N_c > 2$. While it is loop (logarithmically) suppressed, this is a smaller effect than the power suppression of (3.24). Therefore, around the origin where $v \ll 1$, the loop effects stabilize the tree level runaway!

In this case there is also a tri-linear AMSB term coming from (2.8) that goes as $\sim mb^2x$ with 1-loop suppression. Like the second term in (3.26), this is subdominant. Finally, subleading terms in the Kähler lead to power suppressed potential terms that can be neglected.

What we have shown is remarkable: the chiral symmetry breaking point for small m is stable and the AMSB loops effects play a subleading role. However, when we consider a possible runaway direction, the loops come in to save the day. While we cannot be sure of what happens when the fields are $\mathcal{O}(\Lambda)$, there are no runaways from the origin and the chiral symmetry breaking point stands a good chance of being the global minimum.

3.4.3 $N_c = 2$

In this case tree-level AMSB vanishes because the superpotential is marginal. Due to the positive 2-loop masses, the meson and baryon fields are pushed to the origin of moduli space, where the theory experiences confinement without chiral symmetry breaking. This does not match expectations of non-SUSY QCD and we expect a different global minimum to emerge in the large SUSY breaking limit. A similar phenomenon was seen for a Standard-Model-like chiral $SU(5)$ gauge theory in [37].

3.5 $N_c + 1 < N_f \leq 3/2N_c$: Free magnetic phase

For this range of flavors the SUSY theory is in the free magnetic phase and the IR is described by an $SU(\widetilde{N}_c)$ ($\widetilde{N}_c = N_f - N_c$) gauge theory with quarks and anti-quarks in representations $q_i(\bar{\Box}, \mathbf{1})$ and $\bar{q}_j(\mathbf{1}, \Box)$ of the $SU(N_f) \times SU(N_f)$ flavor group, respectively. Additionally, the magnetic theory has a gauge-singlet meson M_{ij} in the $(\Box, \bar{\Box})$ of the flavor symmetry. The superpotential is given by

$$W = \lambda \, \mathrm{Tr} \, q_i M_{ij} \bar{q}_j \tag{3.27}$$

where all fields have already been normalized to have canonical Kähler potentials. Importantly, only the deep IR behavior of the theory is specified and we do not have control over the relative strengths of the gauge interaction and the Yukawa interaction λ in Eq. (3.27).

The case of the free magnetic phase is very subtle, and so far has not been properly analyzed. In fact, this phase is expected to be beset by baryonic runaway directions, so that no useful information can be obtained. We show that for the majority of the free magnetic phase ($N_c + 1 < N_f \lesssim 1.43 N_c$) the baryonic runaway directions are lifted, and the chiral symmetry breaking minimum is stable and likely the global minimum of the theory. The analysis itself is quite involved, as one has to examine several branches, which we will present below.

We proceed by first analyzing the baryonic direction, where the entire dual gauge group is Higgsed. As mentioned, the free magnetic phase for $N_f \lesssim 1.43 N_c$ is free of runaways in this direction. We next exhibit the chiral symmetry breaking minimum along the mesonic direction. Finally, we check the mixed directions, where only some meson VEVs are turned on, to ensure that they contain no runaways.

3.5.1 RG analysis and baryonic branches

In a small neighborhood of the origin of moduli space, the theory is allowed to run into the deep IR. As suggested by the name, the theory is IR free, with both the gauge coupling g and Yukawa coupling running to zero. However, their coupled beta functions make them run asymptotically to the IR attractor given by

$$0 = \frac{d}{d\log\mu}\frac{g^2}{\lambda^2}. \tag{3.28}$$

This allows λ to be written in terms of g, and we can use (2.9) to find the 2-loop masses of the dual squarks and the mesons

$$m_q^2 = \frac{(-\widetilde{b})g^4}{(16\pi^2)^2}\frac{N_f^2 - 3N_f\widetilde{N}_c - \widetilde{N}_c^2 + 1}{2N_f + \widetilde{N}_c}m^2 \tag{3.29}$$

$$m_M^2 = \frac{(-\widetilde{b})\widetilde{N}_c\lambda^2 g^2}{(16\pi^2)^2}m^2 \tag{3.30}$$

where $\widetilde{b} = 3\widetilde{N}_c - N_f$ is negative. The mesons maintain a positive mass throughout the free magnetic window, as do the dual squarks for most of the window. However, at the upper end $N_f \gtrsim 1.43N_c$ (in the large N_c limit), the dual squark mass turns negative and we expect a baryonic runaway towards an incalculable minimum.

Concretely, for $N_f \gtrsim 1.43N_c$ we consider giving D-flat VEVs to the dual squark

$$q = \widetilde{B}\begin{pmatrix} 1_{\widetilde{N}_c\times\widetilde{N}_c} \\ 0_{\widetilde{N}_c\times N_c}\end{pmatrix}. \tag{3.31}$$

The effect of this is to Higgs the dual gauge group at the scale $\widetilde{B}$, and to give masses to the dual anti-quarks and some of the mesons. Substituting their equations of motion eliminates the superpotential. Eq. (3.29) then translates into a tachyonic mass for $\widetilde{B}$, where the gauge coupling is evaluated at the scale $\widetilde{B}$.

The first detailed exploration of baryonic runaways with SUSY breaking applied consistently between the UV and IR was undertaken in [18] (see also the more recent [22]). In both of these works, which used different mechanisms to break SUSY, baryonic runaways were present throughout the free-magnetic phase. It is encouraging that AMSB, while not eliminating them, lifts these directions for most of the phase.

3.5.2 Mesonic branch

In this section we give the meson a VEV with full rank, repeating the analysis of [24]. This gives masses to the dual quarks and anti-quarks. Without their effects, the beta function of the gauge theory flips sign, allowing the theory to generate a new IR dynamical scale given by

$$\Lambda_L^{3\widetilde{N}_c} = \widetilde{\Lambda}^{3\widetilde{N}_c - N_f} \det M. \tag{3.32}$$

The usual superpotential of pure SYM is generated:

$$W = \widetilde{N}_c \Lambda_L^3 = \widetilde{N}_c (\det M)^{1/\widetilde{N}_c} \tag{3.33}$$

where as usual we have set $\widetilde{\Lambda} = 1$. Upon adding tree level AMSB, the minimum can be found along the homogeneous direction $M = v\mathbf{1}$ with the potential

$$V = N_f |v^{N_f/\widetilde{N}_c - 1}|^2 + (N_f - 3\widetilde{N}_c) m v^{N_f/\widetilde{N}_c} + c.c. \tag{3.34}$$

at the point

$$v = \left(\frac{(3\widetilde{N}_c - N_f)m}{N_f - \widetilde{N}_c} \right)^{\frac{\widetilde{N}_c}{N_f - 2\widetilde{N}_c}}, V_{\min} = -\mathcal{O}\left(m^{2\frac{N_f - \widetilde{N}_c}{N_f - 2\widetilde{N}_c}} \right). \tag{3.35}$$

The 2-loop potential from (3.30) contributes at the same order in m, however it is loop suppressed. We find that the chiral symmetry breaking minimum is stable.

3.5.3 Mixed branches

Instead of turning on all of the meson VEVs, we can choose to turn on only some of them. These will reveal tree level AMSB contributions within the free magnetic phase with tree level runaways. However, as in the case of s-confinement, the AMSB loop effects will stabilize these directions.

We begin by writing the meson matrix as

$$M = \begin{pmatrix} \widetilde{M}_{R_f \times R_f} & 0 \\ 0 & \widehat{M}_{(N_f - R_f)\times(N_f - R_f)} \end{pmatrix} \tag{3.36}$$

and without loss of generality we look for minima at diagonal M. We then give the lower component $\widehat{M}$ a VEV. This gives masses to $N_f - R_f$ flavors of quarks, leaving an $SU(\widetilde{N}_c)$ gauge theory with R_f massless flavors and a new dynamical scale

$$\Lambda_L^{3\widetilde{N}_c - R_f} = \widetilde{\Lambda}^{3\widetilde{N}_c - N_f} \det \widehat{M} \tag{3.37}$$

with $\widetilde{\Lambda}$ the Landau pole of the dual theory. In what follows we will set $\widetilde{\Lambda} = 1$. Finally, we assume that $\widetilde{M}$ remains small compared to both $\widehat{M}$ and the generated scale Λ_L.

For $1 \leq R_f < \widetilde{N}_c$, the remaining theory is of ADS-type and has the superpotential

$$W = (\widetilde{N}_c - R_f) \left(\frac{\Lambda_L^{3\widetilde{N}_c - R_f}}{\det N} \right)^{\frac{1}{\widetilde{N}_c - R_f}} + \mathrm{Tr}\, \widetilde{M} N \tag{3.38}$$

where N is the meson formed by the remaining massless dual-quarks. We have ignored λ as it will be irrelevant for this discussion. The second term comes from the Yukawa of the dual theory.

The SUSY equation of motion (EOM) for $\widetilde{M}$ sets $N = 0$. Evidently, the EOM for N is singular at this point and to compensate we must have $\widetilde{M} \to \infty$. However, this violates the assumption of small $\widetilde{M}$. Therefore, even before a small AMSB deformation can be applied, this branch collapses back to the mesonic branch already considered.

Next consider the case of $R_f = \widetilde{N}_c$, which will have emergent meson and baryon degrees of freedom with a quantum modified constraint. Furthermore, the superpotential $W = \mathrm{Tr}\,\widetilde{M}N$ fixes $\widetilde{M} = N = 0$. We thus find ourselves at the baryon point where as before the baryons are stable, but this time with the emergent meson directions stabilized by a superpotential. The only question that remains is the $\widehat{M}$ dependence. For simplicity consider $\widehat{M} = v\mathbf{1}$. The new dynamics will generate at leading order the Kähler potential term

$$K \supset a\Lambda_L^2 = av^{2C} \tag{3.39}$$

where a is an $\mathcal{O}(1)$ number of unknown sign and $C = (N_f - R_f)/(3\widetilde{N}_c - R_f) > 1$. This will give rise to a tree level AMSB potential of $\mathcal{O}(m^2 v^{2C})$. However, as before the 2-loop AMSB mass for the meson will give a positive contribution at $\mathcal{O}(m^2 v^2)$, stabilizing this direction.

For $\widetilde{N}_c + 1 \leq R_f < 3\widetilde{N}_c$, the IR dynamics of the remaining theory are described by a magnetic dual with gauge group $SU(R_f - \widetilde{N}_c)$ (except for $R_f = \widetilde{N}_c + 1$ where the theory is s-confining). The superpotential is

$$W_L = \mathrm{Tr}\, b_i N_{ij} \bar{b}_j + \mathrm{Tr}\,\widetilde{M}N. \tag{3.40}$$

The N, b, and $\bar{b}$ are dual mesons, quarks (baryons), and anti-quarks (anti-baryons) formed by the massless dual quarks. Again the superpotential term (3.27) has transformed to enforce $N = 0$ in the supersymmetric limit. This means when we introduce tree-level AMSB, $N = \mathcal{O}(m)$, and we were justified in ignoring the s-confining $\det N$ term as a high power of m (assuming N is even full rank). We rescale the fields by appropriate factors of Λ_L to make them canonical. Ignoring order one factors we have

$$W_L = \mathrm{Tr}\, b_i N_{ij} \bar{b}_j + \Lambda_L \,\mathrm{Tr}\, \widetilde{M} N. \tag{3.41}$$

Finally we substitute the value of Λ_L (and set $\widetilde{\Lambda} = 1$) to arrive at

$$W_L = \mathrm{Tr}\, b_i N_{ij} \bar{b}_j + (\det \widehat{M})^{1/(3\widetilde{N}_c - R_f)}\, \mathrm{Tr}\, \widetilde{M} N. \tag{3.42}$$

Let all fields be real and consider the direction given by $N_{ii} = n_i$, $\widetilde{M}_{ii} = x_i$, $b_{ii} = -\bar{b}_{ii} = y_i$, for $i = 1, \ldots, (R_f - \widetilde{N}_c)$ and with all other entries 0. Finally let $\widehat{M} = v\mathbf{1}$.

The potential is

$$\begin{aligned} V = \sum_i \big(& 2y_i^2 n_i^2 + (v^C x_i - y_i^2)^2 + v^{2C} n_i^2 \\ & + 2(C-1) m v^C n_i x_i\big) + \frac{C}{3\widetilde{N}_c - R_f} v^{2C-2} \left(\sum_i n_i x_i\right)^2 \end{aligned} \tag{3.43}$$

where C is defined as before and remains greater than 1. Notice that the final term is smaller than the third term in the first sum by a factor of $x^2/v^2 \ll 1$. Therefore, we can neglect this term and the potential splits into $R_f - \widetilde{N}_c$ identical parts. In what follows, we suppress the index i. Substituting the y and n equations of motion, and using $n, x \ll \Lambda_L = v^C$ along the way, we get

$$V|_{y,n} = -(C-1)^2 m^2 x^2. \tag{3.44}$$

As long we keep $x \ll v^C$, we can let $x, v \to 1$, signaling a tree level minimum of $-\mathcal{O}(m^2)$ in the incalculable region where field VEVs are $\mathcal{O}(\Lambda)$. Note that in this direction all fields, baryonic and mesonic, are turned on.

However, as we saw for the s-confining runaway, the loop effects must be considered. While this tree-level runaway is power suppressed as $\mathcal{O}(x^2) \ll \mathcal{O}(v^{2C})$, the 2-loop potential gives a positive contribution with $\mathcal{O}(v^2)$. Therefore, there is again no runaway.

When $R_f \geq 3\widetilde{N}_c$, the theory remains IR free and there are no tree level runaways. As long as $N_f \lesssim 1.43 N_c$ the dual quarks will have positive 2-loop AMSB mass.

In summary, we have demonstrated that there is a stable chiral symmetry breaking minimum and that for $N_f \lesssim 1.43 N_c$ there are no runaways.

3.6 $3/2 N_c < N_f < 3N_c$: Conformal window

In the conformal window, the magnetic description is no longer IR free. Rather, it has a non-trivial fixed point, which is weakly coupled at the lower end of the window. We will first analyze the behavior of AMSB in this region and find baryonic runaways to incalculable minima. Then, we will turn to the upper end of the window where the electric theory has a weakly coupled fixed point. As demonstrated in [35], AMSB makes a relevant deformation and destroys the superconformal phase. We can only conjecture about the intermediate region where both descriptions are strongly coupled. Finally, we demonstrate local chiral symmetry breaking minima throughout the window.

3.6.1 Lower conformal window

We begin by considering $N_f = 3\widetilde{N}_c/(1+\epsilon)$ where $\epsilon \ll 1$, and will work in the large $\widetilde{N}_c$ limit and leading non-trivial order in ϵ for simplicity. For notational convenience, we define

$$x \equiv \frac{\tilde{N}_c}{8\pi^2}\lambda^2, \qquad y \equiv \frac{\tilde{N}_c}{8\pi^2}g^2. \tag{3.45}$$

The beta functions of the magnetic theory, including the 2-loop contribution for y, are

$$\beta(x) = x(-2y + 7x), \tag{3.46}$$

$$\beta(y) = -3y^2(\epsilon - y + 3x). \tag{3.47}$$

They admit a BZ fixed point at $(x_0, y_0) = (2\epsilon, 7\epsilon)$. As the theory flows to the IR, x and y will approach this point from above, along the trajectory specified by (3.28). Define $\delta x = x - x_0$ and $\delta y = y - y_0$. Close to the fixed point this yields

$$\delta x = \frac{2}{7}\left(1 + \frac{3}{2}\epsilon\right)\delta y. \tag{3.48}$$

The RG flow is

$$\beta(y) = 21\epsilon^2 \delta y \tag{3.49}$$

yielding

$$\delta y \sim \mu^{21\epsilon^2}. \tag{3.50}$$

Using (2.9), the meson and dual squark masses are

$$m_M^2 = \frac{3}{2}\epsilon^2 \delta y\, m^2 \tag{3.51}$$

$$m_q^2 = -\frac{3}{4}\epsilon^2 \delta y\, m^2. \tag{3.52}$$

Thus in the lower conformal window the dual squarks are tachyonic and there is a runaway to an incalculable minimum.

3.6.2 Upper conformal window

We now examine the upper conformal window via the electric description, reviewing the results of [35]. Now $N_f = 3N_c/(1+\epsilon)$, and we use all conventions of the previous section. The beta function at 2-loop is

$$\beta(y) = -3y^2(\epsilon - y) \tag{3.53}$$

where the BZ fixed point $y_0 = \epsilon$ is now approached from below as

$$(-\delta y) \sim \mu^{3\epsilon^2}. \tag{3.54}$$

From (2.9) and (2.10) we obtain the squark and gluino masses

$$m_Q^2 = \frac{3}{4}\epsilon^2(-\delta y)m^2 \tag{3.55}$$

$$m_\lambda = \frac{3}{2}(-\delta y)m. \tag{3.56}$$

As expected the squark mass is positive. As long as $3\epsilon^2 < 1$ (this bound is outside of our small ϵ limit and should be taken with a grain of salt), at some point in the RG flow the squark and gluino masses will exceed the renormalization scale. At this point the superpartners can be integrated out and the superconformal phase is destroyed. What remains is non-SUSY QCD and must be analyzed from the (albeit strongly-coupled) magnetic description.

3.6.3 Chiral symmetry breaking minimum

We have shown that AMSB, at both the top and bottom of the conformal window, destroys the superconformal phase. It is reasonable to assume this is the case throughout the window. Furthermore, we demonstrated that at the bottom of the window the theory has a runaway to an incalculable minimum.

Looking instead for local minima, we examine the mesonic branch. Just as in the free magnetic phase, this gives masses to the dual quarks and generates a new dynamical scale. The superpotential is given by (3.33). However, unlike the free magnetic phase where the Kähler receives logarithmic wave-function renormalization (which we ignored), in the conformal window we have

$$Z_M(\mu) \sim \mu^{1-3\widetilde{N}_c/N_f} \tag{3.57}$$

which is evaluated at $\mu = v$, where $M = v\mathbf{1}$. The result is that the scaling of the local chiral symmetry minimum is modified to [35]

$$V = -\mathcal{O}(m^\sigma), \quad \sigma = 1 + \frac{N_f^2}{N_f^2 - 3N_f\widetilde{N}_c + 3\widetilde{N}_c^2}. \tag{3.58}$$

Note that σ goes from 4 ($N_f = \frac{3}{2}N_c$) to 5 ($N_f = 2N_c$) back to 4 ($N_f = 3N_c$).

3.7 $N_f \geq 3N_c$: Free electric phase

For large number of flavors, the 2-loop squark mass from AMSB is negative, leading to true runaway behavior. AMSB cannot be used to understand the non-SUSY theory in this case.

3.8 Conclusions

We carefully analyzed the behavior of $SU(N_c)$ gauge theories with N_f flavors upon the application of AMSB, focusing on the chiral symmetry breaking minima and potential baryonic runaway directions. For $N_c + 1 \leq N_f \leq 3/2N_c$ we found that naive tree level runaways are power suppressed in comparison to loop effects, which stabilize these directions. However, a true loop level runaway was found for the upper end of the free magnetic phase, $N_f \gtrsim 1.43N_c$. This baryonic runaway continued into the lower end of the conformal window, and we cannot discount such runaways throughout the window. Such runaways point to the existence of some non-calculable minimum at large field values of $\mathcal{O}(\Lambda)$, which may or may not correspond to the global minimum of the theory.

The case of $N_f = N_c$ required particular care due to the inherently strongly coupled nature of the quantum modified moduli space. We found that the theory is best analyzed after implementing the quantum constraint. Upon application of AMSB the stability of the chiral symmetry breaking point cannot be determined. This is not due to a problem with the AMSB method, but rather because the Kähler potential terms that are critical to this determination are incalculable.

In summary we found (with the exception of the cases $N_f = N_c$ for $N_c > 2$ and $N_f = N_c + 1$ for $N_c = 2$) that stable chiral symmetry breaking minima are present for $N_f < 3N_c$ upon application of AMSB in the small SUSY-breaking limit. Furthermore, the theories with $N_f \lesssim 1.43N_c$ are protected from runaways to incalculable minima. This does not prove that there are no deeper minima with fields of $\mathcal{O}(\Lambda)$, however we take it to be strong evidence for the conjecture that in these cases the chiral symmetry breaking minima are in fact global.

Our analysis was performed in the $m \ll \Lambda$ limit, and the question remains about the behavior in the non-supersymmetric limit of $m \gg \Lambda$. The existence of the chiral symmetry breaking minima for all flavors is indicative that these are continuously connected to the true vacua of non-SUSY QCD. Irrespective of the potential appearance of a phase transition between these two limits (see arguments based on holomorphy in [25, 26], and also see [38, 39]), these are the vacua that are of phenomenological interest for the study of real-world QCD.

CHAPTER 4

THE PHASES OF NON-SUPERSYMMETRIC GAUGE THEORIES: THE $SO(N_c)$ CASE STUDY

4.1 Introduction

While in Chapter 3 we discussed chiral symmetry breaking, we did not discuss confinement. This is because in $SU(N_c)$ gauge theories with matter in the fundamental representation, there is no order parameter for confinement. In pure gauge theories, one usually examines the Euclidean spacetime scaling behavior of Wilson loops $W[\mathcal{C}] = \exp(i \oint_{\mathcal{C}} A)$,

$$\langle W[C]\rangle \sim e^{-\text{Area}[\mathcal{C}]} \qquad \text{(confined)} \tag{4.1}$$

$$\langle W[C]\rangle \sim e^{-\text{Perimeter}[\mathcal{C}]} \qquad \text{(deconfined)} \tag{4.2}$$

However, when there are dynamical quarks in the fundamental representation they can pair-produce to break Wilson lines, screening gauge charges and always leading to a perimeter law. Consequently, the chiral symmetry breaking took place in a screened/Higgs phase, rather than a genuine confining phase. This will no longer be the case for gauge theories with Lie algebra $\mathfrak{so}(N_c)$ (more specifically $Spin(N_c)$) and matter in the vector representation, where the Wilson loop in the spinor representation cannot be screened. This makes the appearance of an area law possible – which is the strict definition of a confining phase.

In this chapter we study the low-energy dynamics of the $SO(N_c)$ gauge theories with N_f Weyl fermions in the vector representation, obtained by perturbing the corresponding $\mathcal{N} = 1$ $SO(N_c)$ theory via AMSB. We will also include a discussion of the global properties of these theories, which is necessary to distinguish the truly confining theories with an area law for the Wilson loop from those with a perimeter law. Thus, as will be explained later, we will have to distinguish between $Spin(N_c)$ and the $SO(N_c)_\pm$ groups. The $N_f = N_c - 2$ case will be particularly interesting: the SUSY theory in this case corresponds to a pure Coulomb branch, where a pair of monopoles becomes massless at one point, together with a full multiplet of dyons becoming massless at the origin. We will show that the AMSB perturbation of this theory leads to the condensation of monopoles, along with (partial) breaking of the chiral global symmetries at the true ground state. Such monopole (or dyon) condensation was conjectured by Mandelstam [40] and 't Hooft [41] long ago to be the dynamics leading to confinement via the dual Meissner effect. This has indeed been found to be the case by Seiberg and Witten when perturbing pure $\mathcal{N} = 2$ to pure $\mathcal{N} = 1$ supersymmetric Yang–Mills (SYM) [42], and also by [8, 9, 11, 14] who studied the small non-SUSY perturbations of the Seiberg-Witten theory. Our results provide the first example of (true) confinement with continuous chiral symmetry breaking in a non-SUSY gauge theory.

By considering mass deformations to the $N_f = N_c - 2$ case, we will be able to show that all of the cases with $N_f < 3(N_c - 2)$ exhibit the same behavior as for $N_f = N_c - 2$, *i.e.*, electric confinement via monopole condensation, together with chiral symmetry breaking. In other words, with the AMSB perturbation the many phases present in the SUSY case collapse down to just the confining phase. The Coulomb, free magnetic, and superconformal phases do not survive the breaking of supersymmetry.

First we present a short summary of the moduli space and symmetries of $\mathcal{N} = 1$ SUSY $SO(N_c)$ theories with N_f vectors. The quantum vacuum structure of the entire SUSY series has been worked out in a beautiful paper by Intriligator and

Seiberg in [43], which will be the basis of our analysis for the AMSB perturbations. We begin our discussion of the various cases with the most novel $N_f = N_c - 2$ case, which is the only one known example so far giving rise to monopole condensation with chiral symmetry breaking in a non-SUSY theory. We then show that whenever "non-trivial" (spinorial) Wilson lines exist in the theory with $N_f = N_c - 2$, they exhibit an area law, signaling true electric confinement. By considering mass deformations to the $N_f = N_c - 2$ case, we demonstrate this fact, as well as chiral symmetry breaking, for all $SO(N_c)$ theories with $1 < N_f \leq N_c - 2$. While there are several special cases to examine following the analysis of [43], in the end the global minimum of the theory for $1 < N_f \leq N_c - 2$ is always the one with the $SU(N_f) \to SO(N_f)$ chiral symmetry breaking pattern.

We then continue on to consider the cases with a larger number of flavors. We find results similar to the case of SUSY QCD. For $N_c - 2 < N_f < 3(N_c - 2)$ we again find a global minimum where the chiral symmetry is broken to $SO(N_f)$. In this case monopole condensation does not directly appear in the description, but non-trivial Wilson loops still exhibit an area law. To see this, note that at a generic point in the moduli space of the dual theory, the dual quarks are integrated out, leaving us with pure $SO(N_f - N_c + 4)$ SYM. Consequently, whenever the dual gauge group allows for dyonic ('t Hooft) loops, these exhibit an area (perimeter) law. By the duality of [44], whenever the original theory allows for non-trivial Wilson ('t Hooft) loops, they exhibit an area (perimeter) law.

Finally, for $3(N_c - 2) < N_f$, the theory with AMSB has tachyonic squarks and hence no ground state, and is not continuously connected to the non-supersymmetric $SO(N_c)$ theory.

4.2 $SO(N_c)$ with N_f Fundamentals

We consider a supersymmetric gauge theory with gauge group $SO(N_c)$, N_f vectors, and no tree-level superpotential. This theory has been thoroughly studied in *e.g.*, [43, 45]. The anomaly-free global symmetry of the theory is $\left(SU(N_f) \times U(1)_R \times \mathbb{Z}_{2N_f} \times \mathbb{Z}_2/\mathbb{Z}_{N_f}\right)$, where the $\mathbb{Z}_2$ is charge conjugation [1]. Under the continuous part of the global symmetry, the matter fields transform as $Q(\Box)_{\frac{N_f-N_c+2}{N_f}}$.

For $N_f < N_c$, the D-flat directions of the theory are given, up to gauge and global transformations, by

$$Q = \left(\begin{array}{ccc|c} \varphi_1 & & & \\ & \ddots & & 0 \\ & & \varphi_{N_f} & \end{array}\right) , \tag{4.3}$$

For $N_f < N_c$ D-flat directions are conveniently parameterized by the $\frac{1}{2}N_f(N_f+1)$ gauge invariant "meson" operators $M^{ij} = Q^i Q^j$. Along the D-flat directions, they are given by

$$M = \mathrm{diag}\left(\varphi_1^2, \ldots, \varphi_{N_f}^2\right) . \tag{4.4}$$

For $N_f \geq N_c$, the D-flat direction is given by

$$Q = \left(\begin{array}{ccc} \varphi_1 & & \\ & \ddots & \\ & & \varphi_{N_c} \\ \hline & 0 & \end{array}\right) \tag{4.5}$$

It is conveniently parameterized in terms of meson operators, as well as the baryon operators $B^{[i_1,\ldots i_{N_c}]} = Q^{[i_1} \ldots Q^{i_{N_c}]}$, where the i are flavor indices and we take the

[1] For $N_f = 3$ the discrete part of the global symmetry is enhanced from Z_{2N_f} to Z_{4N_f}

Range	SUSY	+AMSB
$N_f = 1$	run-away	confinement
$1 < N_f < N_c - 4$	run-away	confinement+χSB
$N_f = N_c - 4$	2 branches	confinement+χSB
$N_f = N_c - 3$	2 branches	confinement+χSB
$N_f = N_c - 2$	Coulomb	confinement+χSB
$N_f = N_c - 1$	free magnetic 2 branches	confinement+χSB
$N_f = N_c$	free magnetic 2 branches	confinement+χSB
$N_c + 1 \leq N_f \leq \frac{3}{2}(N_c - 2)$	free magnetic	confinement +χSB
$\frac{3}{2}(N_c - 2) < N_f \leq 3(N_c - 2)$	CFT	confinement +χSB
$3(N_c - 2) < N_f$	IR free	run-away

Table 4.1: Summary of IR Behavior of $SO(N_c)$ theories with N_f fundamentals with AMSB. χSB stands for chiral symmetry breaking. For $N_f = N_c - 1$ and N_c, two branches appear along the flat direction of the maximum rank of the meson M^{ij}, yet the AMSB chooses one over the other, resulting in the χSB.

gauge singlet out of the tensor product of N_c fundamentals of $SO(N_c)$. The gauge invariant operators are given, up to global transformations, by

$$M = \left(\begin{array}{c|c} \text{diag}\left(\varphi_1^2, \ldots, \varphi_{N_c}^2\right) & 0 \\ \hline 0 & 0_{N_f - N_c \times N_f - N_c} \end{array}\right)$$

$$B^{1,\ldots N_c} = \varphi_1 \ldots \varphi_{N_c} \,. \tag{4.6}$$

The IR behavior of the theory strongly depends on the relative magnitudes of N_c and N_f and is summarized in Table 4.1. Below we will show that adding AMSB to the theory leads to chiral symmetry breaking for all $1 < N_f \leq \frac{3}{2}(N_c - 2)$. Furthermore, in this range the theory confines; below we give an exact meaning to this statement in terms of the loop operators of the theory. We assume throughout that $N_c > 3$, and leave the $N_c = 3$ to future work.

4.3 Phases of Gauge Theories

One often hears the word "confinement" describing the situation in which colored degrees of freedom are bound into color-singlet states, even if a linear potential is lacking due to screening from quark-antiquark production. We will be more careful with the word and, following [44], only use it in the narrow context of a particular Wilson/'t Hooft/dyonic loop operator [2] exhibiting an area law, in which case we will say that the given loop operator confines. We are especially interested in the confinement of non-trivial loop operators - the ones which transform non-trivially under the center of the gauge group. Note that in some of the literature, *e.g.*, [44], these closed loops are referred to as line operators - we will use the more conventional name "loop operators" to stress their gauge invariance.

The allowed non-trivial loop operators in the theory depend on the choice of the global properties of the gauge group – for example, with the Lie algebra $\mathfrak{so}(N_c)$ the gauge group can be $Spin(N_c)$, $SO(N_c)$, and so on. Depending on the choice of gauge group, the allowed non-trivial loop operators can be Wilson, 't Hooft, or dyonic loops. Whatever the choice may be, these loops exhibit either a perimeter or an area law, depending on the local physics, which is in and of itself insensitive to the global properties of the gauge group.

One possible choice of the gauge group is $Spin(N_c)$, which is the universal cover of all Lie groups that share the Lie algebra $\mathfrak{so}(N_c)$. In this case the non-trivial loop operators are Wilson loops, while others are forbidden by Dirac quantization. An area law for non-trivial Wilson loops indicates the confinement of the electric degrees of freedom associated with it. Other choices for the global structure are

[2] By a loop operator we mean a closed line around which we take a test charge, monopole or dyon in some representation allowed by the gauge group.

obtained by modding out the $Spin(N_c)$ by subgroups of its center, which is $\mathbb{Z}_2$ for odd N_c and larger for even N_c [44]. Here we only consider $SO(N_c) = Spin(N_c)/\mathbb{Z}_2$. In this case the non-trivial loop operators are either purely magnetic 't Hooft loops, a choice denoted by $SO(N_c)_+$, or dyonic loops, a choice denoted by $SO(N_c)_-$. In each case, other non-trivial loop operators are forbidden by Dirac quantization. Below, when we speak of "the loop" in a particular theory, we will be referring to the single non-trivial loop that the theory admits, whether electric, magnetic, or dyonic. Additionally, when we are only concerned with the local physics we will simply refer to the gauge group as $SO(N_c)$.

While $Spin(N_c)$ and $SO(N_c)_\pm$ exist on equal footing as possible gauge groups, we will be particularly interested in $Spin(N_c)$, as it can provide what eludes us for $SU(N)$ gauge theories with fundamental matter: an order parameter for electric confinement. Whereas the fundamental Wilson loop of $SU(N)$ (which is its own universal cover) can be screened by the fundamental matter, the spinorial Wilson loop of $Spin(N_c)$ cannot be screened by the vectorial quarks.

The main objective of our study is to determine the phases of supersymmetric $SO(N_c)$ gauge theories when perturbed by AMSB. We will be able to determine the local behavior of the theory – be it chiral symmetry breaking, monopole condensation, etc. As a result, we will be able to establish the behavior of the allowed non-trivial loop operators in the theory, and whether they exhibit an area or a perimeter law. Our final results about the phase structure (along with the corresponding SUSY phases) are summarized in Tab. 4.1. We can see that in the non-SUSY theory the only surviving phase is that of confinement (with chiral symmetry breaking). The Abelian Coulomb, free magnetic, and superconformal phases do not survive the AMSB perturbation, and they all collapse to the generic confining+χSB phase.

4.4 Confinement with Chiral Symmetry Breaking for $1 < N_f < 3(N_c-2)$

Next we present a detailed analysis of the vacuum structure of $SO(N_c)$ with N_f flavors in the presence of AMSB. For $N_f = N_c - 2$ we show monopole condensation with chiral symmetry breaking – leading to the confinement of non-trivial Wilson lines for $Spin(N_c)$ [3]. For $N_f < N_c - 2$ we can still explicitly see chiral symmetry breaking, while monopole condensation and thus electric confinement is established by adding mass deformations to the $N_f = N_c - 2$ case. Finally, theories with $N_c - 2 < N_f < 3(N_c - 2)$ exhibit chiral symmetry breaking, while confinement is demonstrated by finding a "hidden" monopole condensate in the dual theory.

4.4.1 $N_f = N_c - 2$

In this case the supersymmetric theory is in an Abelian Coulomb phase [43]. Since the M^{ij} are not charged under $U(1)_R$, there is no superpotential even at the quantum level, and hence the theory has a quantum moduli space. On this moduli space, the gauge symmetry is higgsed to a $SO(2) \simeq U(1)$, namely, the theory is on the Coulomb branch. On the moduli space, the gauge coupling $\tau = \frac{\theta}{2\pi} + \frac{i8\pi}{g^2}$ is given only as a function of the $SU(N_f)$ invariant $U \equiv \det M$. It is singular at two points $U = 0$ ($U = U_1 \equiv 16\Lambda^{2N_f}$), where the dyons $q_i^{\pm}$ (monopoles $E^{\pm}$) of the $U(1)$

[3] Alternatively, for $SO(N_c)_+$ the non-trivial 't Hooft line exhibit a perimeter law, while for $SO(N_c)_-$, the non-trivial dyonic line exhibits an area law.

	$SO(N_c)$	$SU(N_f)$	$U(1)_R$
Q^i	$\square$	$\square$	0
λ	$\begin{smallmatrix}\square\\\square\end{smallmatrix}$	$\mathbf{1}$	1
M_{ij}	$\mathbf{1}$	$\square\square$	0
$q_i^\pm$	–	$\overline{\square}$	1
λ_{mag}	–	$\mathbf{1}$	1

Table 4.2: DOF of the $SO(N_c)$ theory with $N_f = N_c - 2$ near $M = 0$. λ are the $SO(N_c)$ gauginos, while λ_{mag} are the photinos of the unbroken (magnetic) $U(1)$ in the IR. For the supersymmetric theory at the origin, the full global symmetry is unbroken. With AMSB there is a local minimum, where the global symmetry is broken to $SU(N_f - 2)$.

gauge symmetry become massless. In the original paper [43], the authors chose to label the particles condensing at $U = 0$ and $U = U_1$ as monopoles and dyons, respectively. Our opposite labeling leads to line behaviors consistent with those in [44] for all confining theories, and is also consistent with the finding in [46].

Around the singular point $U = 0$ the relevant light degrees of freedom are the dyons $q_i^\pm$ with magnetic charge ± 1, which transform under the UV global symmetry $SU(N_f) \times U(1)_R$ as $q_i^\pm(\overline{\square})_1$. These have a dynamically generated superpotential about $U = 0$ of

$$W_{\text{dyon}} = \frac{1}{\mu} f(t)\, M^{ij} q_i^+ q_j^- \,, \tag{4.7}$$

where μ is an effective mass scale, $t = U\Lambda^{4-2N_c}$, and $f(t)$ is a holomorphic function in the neighborhood of $t = 0$, normalized so that $f(0) = 1$. Expanding f to higher orders in t introduces tree level AMSB, but it is highly suppressed by powers of the meson VEV over Λ and results in no qualitative changes. Exactly at $U = 0$, 't Hooft anomaly matching is saturated by $q_i^\pm$, M^{ij}, and the photinos $\mathcal{W}_\alpha \sim W_\alpha Q^{N_c-2}$ [43], whose charges are given in Table 4.2.

Using the formulae for loop level AMSB Eq. (2.9), we can explore the local minima around the origin of moduli space. The IR free nature of the $U(1)$ gauge theory gives a tachyonic contribution to the dyon masses. However, the dyons also receive a positive mass-squared contribution from the Yukawa-like coupling to the meson field Eq. (4.7). The co-dependence of the Yukawa and gauge beta functions results in a flow to a fixed ratio between the two couplings. This ratio is such that the mass-squared due to loop AMSB is positive for both the meson and the dyons. Thus, the loop-level AMSB trilinear term in combination with the tree-level quartic potential gives a local minimum a distance $\mathcal{O}(\frac{m}{16\pi^2})$ from the origin. To understand the symmetry breaking pattern at this minimum, we must examine the form of the tree-level potential in terms of the $SU(N_f)$ matrix M and vectors $q^\pm$,

$$\begin{aligned} V &= \frac{1}{2}(q^+ \cdot q^{+*})(q^- \cdot q^{-*}) \\ &\quad +|Mq^+|^2 + |Mq^-|^2 + V_{\text{AMSB}} \end{aligned} \tag{4.8}$$

The dot product term is due to the symmetric nature of the meson matrix (*i.e.*, M^{ij} couples to $q_i^+ q_j^-$ and $q_j^+ q_i^-$). This term encourages the $q^\pm$ VEVs to point in different directions in flavor space. We find a minimum along the direction,

$$q^+ = \left(\begin{array}{c} 1 \\ 0 \\ \hline 0 \\ \vdots \\ 0 \end{array}\right)\alpha, \quad q^- = \left(\begin{array}{c} 0 \\ 1 \\ \hline 0 \\ \vdots \\ 0 \end{array}\right)\alpha, \tag{4.9}$$

$$M \propto \left(\begin{array}{cc|ccc} 0 & 1 & 0 & \cdots & 0 \\ 1 & 0 & 0 & \cdots & 0 \\ \hline 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{array}\right), \tag{4.10}$$

Breaking the global flavor symmetry to $SU(N_f-2)$. The vacuum energy of this minimum is $V=-\mathcal{O}(\frac{m}{16\pi^2})^4$.

In the vicinity of the singular point $U=U_1$, on the other hand, the light degrees of freedom are the monopoles $E^\pm\sim q_i^\pm Q^i$, whose magnetic charges are ± 1. These transform under the $SU(N_f)\times U(1)_R$ global symmetry of the UV theory as $E^\pm(\mathbf{1})_{\mathbf{1}}$. Since $\det M\equiv U\neq 0$, the global $SU(N_f)\times U(1)_R$ is spontaneously broken to $SO(N_f)\times U(1)_R$. In the neighborhood of $U=U_1$, the theory generates a dynamical superpotential

$$W_{\rm mon} \quad = \quad \tilde{f}\left(\frac{U-U_1}{\Lambda^{2N_f}}\right)\, E^+\, E^- \,. \tag{4.11}$$

Here $\tilde{f}(t)=t+\cdots$ is holomorphic near $t=0$. For all practical purposes, only the leading order in $\tilde{f}$ matters for the stabilization of the minimum. Using canonically normalized fields we have

$$W_{\rm mon} \quad = \quad \Lambda\left(\frac{\tilde{U}}{\Lambda^{N_f}}-16\right)\tilde{E}^+\,\tilde{E}^- \,, \tag{4.12}$$

where $\tilde{E}^\pm=E^\pm/\sqrt{\Lambda}$, $\tilde{U}=\det\tilde{M}$, and $\tilde{M}=M/\Lambda$ is the canonically normalized meson. Exactly at $\tilde{U}=\tilde{U}_1\equiv 16\Lambda^{N_f}$, 't Hooft anomaly matching is saturated by $E^\pm$, $\tilde{M}^{ij}$, and the photinos $\mathcal{W}_\alpha\sim W_\alpha Q^{N_c-2}$, whose charges are given in Table 4.3.

Explicitly,

$U(1)_R$ gravity^2 and $U(1)_R^3$:

$$(-1)N_fN_c+(1)\frac{N_c(N_c-1)}{2}=(1)+(-1)\frac{N_f(N_f+1)}{2}, \tag{4.13}$$

$U(1)_R\,SO(N_f)^2$:

$$(-1)(1)N_c=(-1)(N_f+2). \tag{4.14}$$

	$SO(N_c)$	$SU(N_f)$	$U(1)_R$	$U(1)_{\rm mag}$	$SO(N_f)$
Q^i	$\square$	$\square$	0	–	$\square$
λ	$\begin{smallmatrix}\square\\\square\end{smallmatrix}$	$\mathbf{1}$	1	–	$\mathbf{1}$
M_{ij}	$\mathbf{1}$	$\square\square$	0	–	$\mathbf{1}+\square\square$
$E^{\pm}$	–	$\mathbf{1}$	1	±1	$\mathbf{1}$
$\lambda_{\rm mag}$	–	$\mathbf{1}$	1	0	$\mathbf{1}$

Table 4.3: DOF of the $SO(N_c)$ theory with $N_f = N_c - 2$ near $M^{ij} \sim \delta^{ij}, U = U_1$. The unbroken global symmetry near $M^{ij} \sim \delta^{ij}, U = U_1$ is $SO(N_f) \times U(1)_R$, with $U(1)_R$ explicitly broken by AMSB.

Contrary to the point $\tilde{U} = 0$, here adding AMSB generates a tree-level contribution to the scalar potential from (2.7). This results in the global minimum at $\tilde{U} = \tilde{U}_1$. In particular, the scalar potential along $\tilde{M}^{ij} = \tilde{M}\delta^{ij}$ is given locally as

$$\begin{aligned} V_{\tilde{U}\sim\tilde{U}_1} \;&=\; \Lambda^2 \left| \left(\frac{\tilde{M}}{\Lambda}\right)^{N_f} - 16 \right|^2 \left(|\tilde{E}^+|^2 + |\tilde{E}^-|^2 \right) \\ &+ \frac{1}{kN_f} \left| N_f \left(\frac{\tilde{M}}{\Lambda}\right)^{N_f - 1} \right|^2 |\tilde{E}^+\tilde{E}^-|^2 \;+\; V_{\rm AMSB}\,. \end{aligned} \tag{4.15}$$

Note the $(kN_f)^{-1}$ factor in the second line, which comes from the Kähler term $kN_f\tilde{M}^\dagger\tilde{M}$ for $\tilde{M}$, where k is an unknown $\mathcal{O}(1)$ normalization factor. The tree-level AMSB contribution is given by (2.7), *i.e.*,

$$V_{\rm AMSB} = m\Lambda \left[16 + (N_f - 1)\left(\frac{\tilde{M}}{\Lambda}\right)^{N_f} \right] \tilde{E}^+\tilde{E}^- + \text{c.c.} \tag{4.16}$$

This potential has a minimum at

$$\begin{aligned} \tilde{M} \;&=\; 16^{\frac{1}{N_f}}\Lambda \quad , \quad |\tilde{E}^+||\tilde{E}^-| = 16^{\frac{2}{N_f}-1}\, km\Lambda \\ V_{\rm min} \;&=\; -16^{\frac{2}{N_f}} N_f\, km^2\Lambda^2\,. \end{aligned} \tag{4.17}$$

Since $\tilde{M}^{ij} = \tilde{M}\delta^{ij}$ in this minimum, the global symmetry is broken to $SO(N_f)$, and there are no 't Hooft anomalies to match. Because it is generated by a tree-level contribution from AMSB, it is lower than the local minimum near the origin $U \approx 0$, and so it is the global minimum of the theory. A similar phenomenon of AMSB leading to a global minimum generated by a tree-level contribution and a local minimum generated at loop-level was seen in [26]. It is easy to see that there are no minima with $M^{ij} \neq 0$, $M^{ij} \not\propto \delta^{ij}$.

Notably, the minimum (4.17) involves the condensation of monopoles $E^{\pm}$, which in turn leads to confinement [41, 47, 48], in the sense that non-trivial Wilson lines get an area law. Famously, monopole condensation has also been seen in the breaking of $\mathcal{N} = 2$ Seiberg-Witten theory to $\mathcal{N} = 1$ via a tree level superpotential for the matter field [42]. In [13, 14], it was shown in a non-supersymmetric theory by introducing soft SUSY breaking on top of the breaking to $\mathcal{N} = 1$. Here, on the other hand, monopole condensation and SUSY breaking emerge together as a result of AMSB. Furthermore, since the global $SU(N_f)$ symmetry is spontaneously broken to $SO(N_f)$, this is a demonstration of *confinement with chiral symmetry breaking* in a non-supersymmetric setting.

We can also connect the chiral symmetry breaking observed here to the familiar one due to fermion bilinears. To see this, note that the UV theory of quarks has no superpotential and their F-components vanish. Therefore the only contribution to the F-component of the *meson* superfield comes from fermion bilinears:

$$\langle \psi_i^* \psi_j^* \rangle = F_{M_{ij}}^* = 16\Lambda^2 M_{ij}^{-1} E^+ E^- \propto \delta_{ij} k m \Lambda^2 \neq 0. \tag{4.18}$$

In other words, our analysis demonstrates the condensation of fermion bilinears in a non-supersymmetric theory, in addition to the monopole condensate.

4.4.2 $N_f < N_c - 4$

At a generic point in the moduli space, the gauge group $SO(N_c)$ is higgsed down to $SO(N_c - N_f)$ pure SYM, whose gaugino condensation induces an Affleck–Dine–Seiberg (ADS) superpotential given by [49]

$$W_{\rm ADS} = \frac{N_c - N_f - 2}{2}\omega^k \left(\frac{16\Lambda^{3N_c - N_F - 6}}{\det M}\right)^{\frac{1}{N_c - N_F - 2}} , \tag{4.19}$$

where Λ is the strong scale of the theory and $\omega = e^{2\pi i/(N_c - N_f - 2)}$ with $k = 0, 1, \ldots, N_c - N_f - 3$.

The Kähler potential of M is singular at the origin and writing $M^{ij} = \varphi^2 \delta^{ij}$, we identify φ as the canonical DOF. Turning on AMSB stabilizes the runaway behavior of the superpotential via the tree-level scalar potential

$$V_{\rm AMSB} = -m\Lambda^3 \, \frac{3N_c - N_f - 6}{2} \left(\frac{16\Lambda^{2N_F}}{\varphi^{2N_f}}\right)^{\frac{1}{N_c - N_F - 2}} + \text{c.c.} , \tag{4.20}$$

which together with the scalar potential derived from the superpotential (4.19) gives a minimum

$$\begin{aligned} \varphi &= 2^{\frac{2}{N_c - 2}} \left(f_{N_f} \frac{\Lambda}{m}\right)^{\frac{N_c - N_f - 2}{2(N_c - 2)}} \Lambda \\ V_{\rm min} &= -2^{\frac{4}{N_c - 2}} \frac{N_c - 2}{f_{N_f}^2} \left(f_{N_f} \frac{\Lambda}{m}\right)^{\frac{N_c - N_f - 2}{N_c - 2}} m^2 \Lambda^2 \, . \end{aligned} \tag{4.21}$$

with $f_{N_f} = \frac{N_c + N_f - 2}{3N_c - N_f - 6}$. We see that the minimum is at $\varphi \gg \Lambda$, which justifies a weakly coupled analysis in an asymptotically free theory. Since $\tilde{M}^{ij} \propto \delta^{ij}$, in this minimum the global symmetry is broken to $SO(N_f)$. There are no minima with $\tilde{M}^{ij} \neq 0$, $\tilde{M}^{ij} \not\propto \delta^{ij}$. Since the $U(1)_R$ symmetry was explicitly broken by AMSB, there are no 't Hooft anomaly matching conditions to check in this scenario.

The non-trace components of $\tilde{M}^{ij}$ are split into massless Nambu–Goldstone bosons (NGBs), massive fermions, and massive scalar partners of the NGBs, where masses are $\mathcal{O}(m)$. The NGBs form the chiral Lagrangian on the $SU(N_f)/SO(N_f)$ coset space. Once the massive fermions are integrated out, the one-loop diagram [50] produces the Wess–Zumino–Witten (WZW) term [51, 52] because $\pi_5(SU(N_f)/SO(N_f)) = \mathbb{Z}$ ($N_f \geq 3$). For $N_f = 2$, there is no WZW term. To summarize, we establish that the $1 < N_f < N_c - 4$ case with AMSB has a global minimum in which the chiral symmetry is broken to $SO(N_f)$, similar to the $N_f = N_c - 2$ case.

The case $N_f = 1$ is an exception because the meson has only one component. There is no exact flavor symmetry, no massless NGB, and the theory is gapped.

4.4.3 $N_f = N_c - 4$

In this case the gauge symmetry is higgsed on the moduli space to $SO(4) \simeq SU(2)_L \times SU(2)_R$. The theory has two distinct branches corresponding to the gaugino condensates in $SU(2)_{L,R}$ having the same, or opposite signs. On the first branch with aligned condensates, the superpotential is of the same form as (4.19), while on the second branch, it vanishes. The second branch contains the point $M = 0$, at which there is confinement without chiral symmetry breaking. In [53], it was shown that on this branch there is also a VEV for the exotic baryon $S = W_\alpha W^\alpha Q^{N_f}$, which breaks the discrete global symmetry $\mathbb{Z}_{2F}$ down to $\mathbb{Z}_F$.

With AMSB, the theory on the first branch develops a minimum identical to (4.21), breaking the global symmetry down to $SO(N_f)$. This is the global minimum of the theory. As for the second branch, the identically zero superpotential means we need to consider the more general tree level AMSB formula (2.6) accounting for a non-canonical Kähler potential. Higher order terms in the Kähler potential will give rise to irrelevant interactions, and as the theory is IR free, we expect the effects of AMSB to be highly suppressed by the dynamical scale. To estimate these effects, consider the leading corrections to the canonical Kähler potential for M,

$$\begin{aligned} K &= \mathrm{Tr} M^\dagger M + \frac{a}{\Lambda^2}\left(\mathrm{Tr} M^\dagger M\right)^2 \\ &\quad + \frac{b}{\Lambda^2}\mathrm{Tr} M^\dagger M M^\dagger M\,, \end{aligned} \tag{4.22}$$

where a, b are order one numbers. Note that cubic terms are forbidden because M is in the symmetric of $SU(N_f)$.

In this case, only the second term in Eq. (2.6) will contribute, and at leading order gives,

$$V \sim \pm\frac{m^2}{\Lambda^2}|M|^4 \tag{4.23}$$

The potential in this theory arises exclusively from AMSB. Clearly the power series expansion makes sense only up to $M \sim \Lambda$, and the maximum contribution of the higher dimension terms to the potential is $O(m^2\Lambda^2)$. Note that the minimum we obtained in Eq. (4.21) (and is also relevant for the branch with nonzero superpotential) is parametrically enhanced by $(\Lambda/m)^{2/(N_c-2)}$. Therefore the branch with $W = 0$ does not yield the global minimum, which instead arises from the branch with the ADS-type superpotential.

4.4.4 $N_f = N_c - 3$

Here the gauge symmetry on the moduli space is higgsed down to $SO(3)$. As in [43], it is useful to first turn on the VEVs of $N_f - 1$ of the fundamentals, in which case the theory is higgsed to $SU(2)_L \times SU(2)_R$. Then, the VEV of the last fundamental higgses $SU(2)_L \times SU(2)_R$ to the diagonal $SO(3)$. The superpotential for this theory is dynamically generated by a combination of gaugino condensation in the unhiggsed $SO(3)$ and instantons in the broken $SU(2)_L \times SU(2)_R/SO(3)$. The theory again has two branches: on the first, the gaugino contribution is aligned with the instanton contributions, and the superpotential is of the form (4.19). On the second branch, the contributions cancel out, and the ADS-type superpotential vanishes. However, it was shown via a mass deformation to the $N_f = N_c - 4$ case that the second branch has a dynamically generated superpotential:

$$W_{\text{dyn}} = \frac{1}{2\mu} f(t)\, M^{ij} q_i q_j \,, \tag{4.24}$$

where $q_i = \left(W_\alpha W^\alpha Q^{N_f-1}\right)_i / \Lambda^{N_f+1}$ is the exotic baryon of the theory. Here μ is an effective mass scale, $t = \frac{1}{\Lambda^{2N_f+4}} \det M\, M^{ij} q_i q_j$, and $f(t)$ is a holomorphic function in the neighborhood of $t = 0$, normalized so that $f(0) = 1$. We can canonically normalize the superpotential yielding

$$W_{\text{dyn}} = \frac{1}{2} f(t)\, \tilde{M}^{ij} q_i q_j \,, \tag{4.25}$$

where $\tilde{M}$ is the canonically normalized meson.

As usual, adding AMSB to the theory generates for the first branch the minimum (4.21) and breaks the global symmetry down to $SO(N_f)$. This is again the global minimum of the theory. On the second branch, the tree-level AMSB contribution vanishes at $\mathcal{O}(t^0)$, while the loop-level contribution (2.9) generates a minimum in the neighborhood of $\tilde{M}^{ij} = 0$, with

$$V \approx -\left(\frac{m}{16\pi^2}\right)^4 \,, \tag{4.26}$$

Again the $\mathcal{O}(t)$ corrections give a sub-leading contribution. Around the origin loop-level AMSB is again the dominant perturbation, which will clearly not be the global minimum of the theory, since its (negative) height is loop suppressed.

4.4.5 $N_c - 1 \le N_f \le 3/2(N_c - 2)$

In this case the correct IR description of the theory is in terms of its IR free Seiberg dual $SO(N_f - N_c + 4)$ with N_f fundamentals q_i and $\frac{1}{2}N_f(N_f+1)$ singlets M^{ij} in the $q_i(\bar{\square})_{\frac{N_c-2}{N_f}}$ and $M^{ij}(\square\square)_{\frac{2(N_f-N_c+2)}{N_f}}$ representations of the global $SU(N_f)\times U(1)_R$, respectively. First we will focus on the case when $N_f \ge N_c + 1$, leaving the $N_f = N_c - 1$ and $N_f = N_c$ special cases to the end of the section. For $N_f \ge N_c + 1$, the dual theory has a superpotential

$$W_{\rm dual} \;=\; \frac{1}{2\mu}\, M^{ij} q_i q_j \,. \tag{4.27}$$

The scales of the original and dual theories are related by

$$2^8 \Lambda^{3(N_c-2)-N_f} \tilde{\Lambda}^{2N_f-3(N_c-2)} \;=\; (-1)^{N_f-N_c} \mu^{N_f} \,. \tag{4.28}$$

For later convenience, we switch to canonically normalized fields $\tilde{M}$,

$$W_{\rm dual} \;=\; \frac{1}{2}\, \tilde{M}^{ij} q_i q_j \,, \tag{4.29}$$

with $\tilde{M} = M/\mu$. When we turn on AMSB, the situation is similar to the one encountered in [26] and to the $N_f = N_c - 2$ case in the present work. Near $M = 0$ there is a local minimum generated by the loop level AMSB contribution. The tree-level contribution vanishes as usual because the superpotential (4.29) is marginal. Again we expect only a local minimum with $V = -\mathcal{O}(\frac{m}{16\pi^2})^4$ along the direction,

$$q \propto \left(\begin{array}{ccc} 1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 1 \\ \hline 0 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 0 \end{array}\right), \quad M \propto \left(\begin{array}{ccc|ccc} 1 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 1 & 0 & \cdots & 0 \\ \hline 0 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & 0 & \cdots & 0 \end{array}\right), \tag{4.30}$$

where q is a $N_f \times (N_f - N_c + 4)$ matrix.

At nonzero M with rank$(M) = N_f$ on the moduli space for $N_f \geq N_c + 1$, the dual quarks q_i are integrated out, and we are left with pure $SO(N_f - N_c + 4)$ SYM, with a scale

$$\tilde{\Lambda}_L^{3N_f - 3(N_c-2)} \quad = \quad \det(\tilde{M}) \; \tilde{\Lambda}^{2N_f - 3(N_c-2)} \tag{4.31}$$

Gaugino condensation in the dual theory now generates a dynamical superpotential

$$W_{\lambda\lambda} \;=\; 2^{-\frac{N_f-(N_c-2)-4}{N_f-(N_c-2)}} \epsilon_{N_f-(N_c-2)} \; \tilde{\Lambda}_L^3 \tag{4.32}$$

With AMSB this superpotential leads to the tree-level SUSY breaking scalar potential

$$\begin{aligned} V_{\mathrm{AMSB}} = \quad & -2m\tilde{\Lambda}^3 \, \frac{\frac{3}{2}(N_c-2) - N_f}{N_f - (N_c-2)} \times \\ & \left(\frac{16^{\frac{1}{N_f}} \det(\tilde{M})}{\tilde{\Lambda}^{N_F}} \right)^{\frac{1}{N_F - (N_c-2)}} \quad + c.c. \end{aligned} \tag{4.33}$$

For $N_f > N_c$, this potential, together with the usual scalar potential from the superpotential, is minimized at

$$\begin{aligned} \tilde{M}^{ij} \quad &\sim \quad 4^{\frac{N_f-(N_c-2)-2}{2(N_c-2)-N_f}} \left(f\frac{m}{\Lambda} \right)^{\frac{N_f-(N_c-2)}{2(N_c-2)-N_f}} \; \Lambda \, \delta^{ij} \\ V_{\min} \quad &\sim \quad -4^{\frac{N_f-4}{2((N_c-2)-N_f}} \, \frac{2(N_c-2)-N_f}{[N_f-(N_c-2)]^2} \times \\ & \left(f\frac{m}{\Lambda} \right)^{\frac{2(N_c-2)}{2(N_c-2)-N_f}} \; \Lambda^4 \, , \end{aligned} \tag{4.34}$$

where $f = \frac{1}{N_c-2} \left[N_f - (N_c-2) \right] \left[\frac{3}{2}(N_c-2) - N_f \right]$. Noting that $N_c - 2 < N_f \leq \frac{3}{2}(N_c-2)$ from (4.34), this minimum satisfies $\tilde{\Lambda}_L \ll \tilde{M} \ll \tilde{\Lambda}$, below the Landau pole of the UV magnetic theory and above the scale of gaugino condensation in the IR pure SYM. This justifies our weakly coupled analysis. Again the global symmetry at the minimum is broken down to $SO(N_f)$.

We now comment on the $N_f = N_c - 1$ and $N_f = N_c$ cases. For $N_f = N_c - 1$, the dual gauge group is $SO(3)$. The superpotential of the dual now includes a contribution from instantons:

$$W_{\text{dual},N_c-1} \;=\; \frac{1}{2\mu}\, M^{ij} q_i q_j \;-\; \frac{\det M}{64\tilde{\Lambda}^{2N_c-5}}\,. \tag{4.35}$$

The extra contribution can be seen by deforming the dual for $N_f = N_c$ by a mass term for the last flavor, which leads to instantons in the broken $SU(2)_L \times SU(2)_R/SO(3)$ in the magnetic theory [54]. Here the scale matching is given by

$$2^{14}\left(\Lambda^{2N_c-5}\,\tilde{\Lambda}^{4-N_c}\right)^2 \;=\; \left(\mu^{N_c-1}\right)^2\,. \tag{4.36}$$

Interestingly, this relation looks like the square of the usual relation (4.28) - for more details, see the original [43]. When M has full rank on the moduli space, the q_i become massive and the gaugino condensation in the IR $SO(3)$ SYM generates a superpotential

$$W_{\lambda\lambda,\text{dual},N_c-1} \;=\; \epsilon_2\,\frac{\det M}{64\Lambda^{2N_c-5}}\,. \tag{4.37}$$

Note that this superpotential has the same magnitude as the instanton contribution in (4.35), and so the theory again has two branches: one in which the two contributions add, leading to an AMSB minimum of the form (4.34), and another with vanishing superpotential. On the second branch there is no superpotential, and we can repeat the arguments concerning the second branch in the case of $N_f = N_c - 4$. Again, any minimum produced by AMSB and a non-canonical Kähler is parametrically suppressed in comparison with the first branch.

For $N_f = N_c$, the magnetic gauge group is $SU(2)_L \times SU(2)_R$. There is no instanton contribution to the tree-level superpotential, but when M is given full rank, there are again two branches: one with aligned gaugino condensates in the IR pure $SU(2)_L \times SU(2)_R$ SYM, and one with opposite sign condensates and vanishing superpotential. As usual, with AMSB the first branch leads to a global minimum of the form (4.34), while any minimum in the second branch is again subdominant.

4.4.6 $3/2(N_c - 2) < N_f < 3(N_c - 2)$

In this section we will simply describe the symmetry breaking behavior as the details are very much along the lines of Section 3.6. The SUSY theory approaches an interacting fixed point in the IR. Upon introducing a full rank meson VEV, one can integrate out the dual quarks. The remaining pure $SO(N_f - N_c + 4)$ confines, generating a superpotential just as in the previous section. As in that case, the application of AMSB leads to a chiral symmetry breaking minimum. Thus, the superconformal phase has been replaced by a confining phase with chiral symmetry breaking.

4.4.7 Monopole Condensation for $N_f < N_c - 2$ via Mass Deformations

When discussing the theory with $N_f = N_c - 2$, the non-supersymmetric vacuum of the theory explicitly involved monopole condensation. In the next section, this will enable us to determine the behavior of the loop operators in the theory, and in particular establish confinement of non-trivial Wilson loops for $Spin(N_c)$. In this section we wish to make contact between the $N_f = N_c - 2$ case and the cases with fewer flavors, by treating the latter as the $N_f = N_c - 2$ deformed by a supersymmetric mass μ, with $\mu \gg \Lambda$.

We begin by considering the $N_f = N_c - 2$ theory in the supersymmetric limit, with just one mass term for the last flavor,

$$W = \Lambda \left(\frac{\det \tilde{M}}{\Lambda^{N_f}} - 16 \right) \tilde{E}^+ \tilde{E}^- + \frac{1}{2} \mu \Lambda \tilde{M}_{N_f N_f} \tag{4.38}$$

The equation of motion for $\tilde{M}_{N_f N_f}$ gives

$$\tilde{E}^+ \tilde{E}^- = -\frac{1}{2} \frac{\mu \Lambda^{N_f}}{\det \tilde{M}'}, \tag{4.39}$$

where $\tilde{M}'$ is the matrix of the remaining mesons. As already demonstrated, tree level AMSB corrections to the ADS superpotential stabilize the runaway, and the finite VEV of M' ensures that a non-vanishing monopole condensate persists. In Fig. 4.1 we show this explicitly by studying the minimum of the mass-deformed theory (4.38) in the presence of AMSB with $m < \mu$. Since we are ultimately interested in the infinite μ limit, this does not interfere with our extrapolation to the non-SUSY limit with large m. As can be seen in the plot, the VEV of the first $N_c - 3$ flavors interpolates between the minimum (4.17) for $\mu = 0$, and the ADS+AMSB minimum (4.21) with $N_f = N_c - 3$ and $\Lambda \to \Lambda_{N_f = N_c - 3}$ in the large μ limit. We can see that the monopole condensate persists in the large μ limit.

To correctly reproduce the ADS+AMSB minimum, we had to interpolate the Kähler potential between the neighborhood of $\det\tilde{M} \sim \tilde{U}_1$, where it is canonical in $\tilde{M}$, to large $\det\tilde{M}$, where the Kähler potential is canonical in $\varphi \sim \sqrt{\tilde{M}\Lambda}$. More specifically, we used the following interpolating Kähler potential in the numerical study:

$$K_{\text{interp.}} = \Lambda^2 \sqrt{1 + \frac{\tilde{M}\tilde{M}^\dagger}{\Lambda^2}}. \tag{4.40}$$

For $\mu < m$, the UV theory has a runaway at $E^+E^- = 0$ and $M_i \to \infty$. This is a consequence of the mass term in (4.38) in the presence of AMSB. In this regime we follow the local minimum which goes over to the global minimum for $\mu > m$.

Importantly, the condensation of monopoles in the large μ limit is independent of this subtlety.

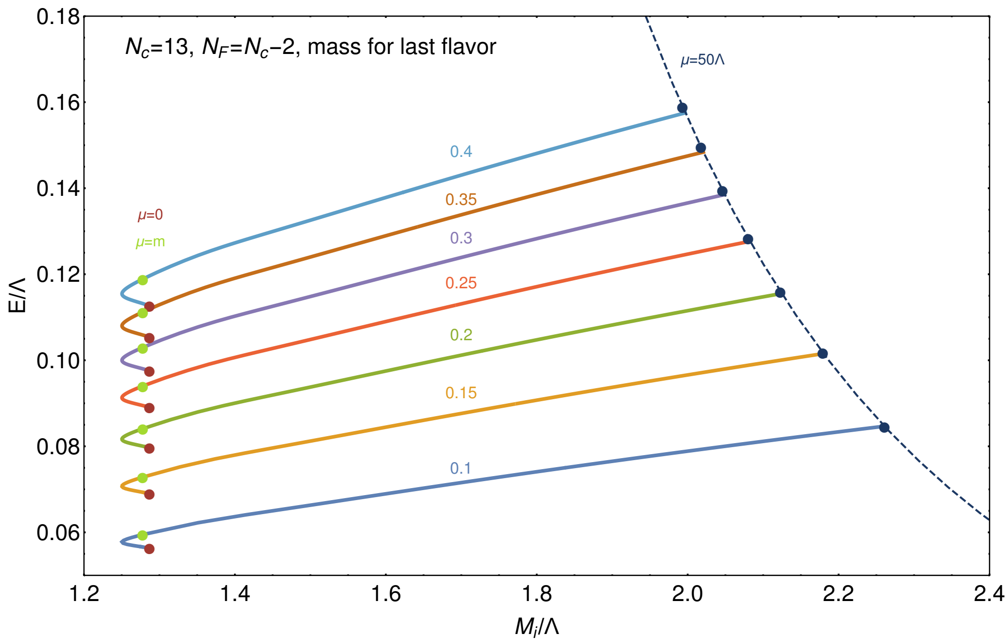

Figure 4.1: The supersymmetry breaking minimum for the theory with AMSB and $N_f = N_c - 2$, with the extra mass term $\frac{1}{2}\mu M^{N_f N_f}$. E is the VEV of the monopoles $\tilde{E}^{\pm}$, while M_i is the VEV of the first $N_c - 3$ flavors. The labeling on the different curves indicates different values of m/Λ. For $\mu = 0$, the curves are at the $N_c - 2$ minimum (4.17) . As μ grows, the VEV of M_i initially decreases, but then starts increasing as μ passes m. For large $\mu/\Lambda \to \infty$, the vacuum of the theory goes over to Eq. (4.21) with $N_f = N_c - 3$, while the monopole condensate persists. The relation (4.39) is shown in the dashed line for $\mu = 50\Lambda$. We chose $N_c = 13$ for this plot.

Similarly, we can give a mass term to any number of flavors in the $N_f = N_c - 2$ theory and show that monopole condensation persists. In Fig. 4.2 we present the case where all of the flavors get the same mass term, resulting in a pure SYM theory with monopole condensation. In the small μ limit, the minimum is given by the $N_f = N_c - 2$ vacuum (4.17). The $\mu \gg m$ case can be fully understood in the supersymmetric limit – the monopoles get a VEV

$$\tilde{E}^{+}\tilde{E}^{-} = -\frac{\mu\Lambda}{2}\left(\frac{\Lambda}{\tilde{M}}\right)^{N_f - 1}, \tag{4.41}$$

where $\tilde{M}$ is the common VEV of all of the flavors. This generates an ADS superpotential for the $\tilde{M}$, which is balanced by the μ term and leads to an overall minimum at

$$\begin{aligned} \tilde{M} &= 16^{\frac{1}{N_f}} \Lambda \,, \\ \tilde{E}^+ \tilde{E}^- &= -2^{\frac{4}{N_f}-5} \mu \Lambda \,. \end{aligned} \tag{4.42}$$

Notably, the VEV of $\tilde{M}$ in this case is equal to the pure $N_f = N_c - 2$ case. This vacuum is the one depicted by the dashed line in Fig. 4.2.

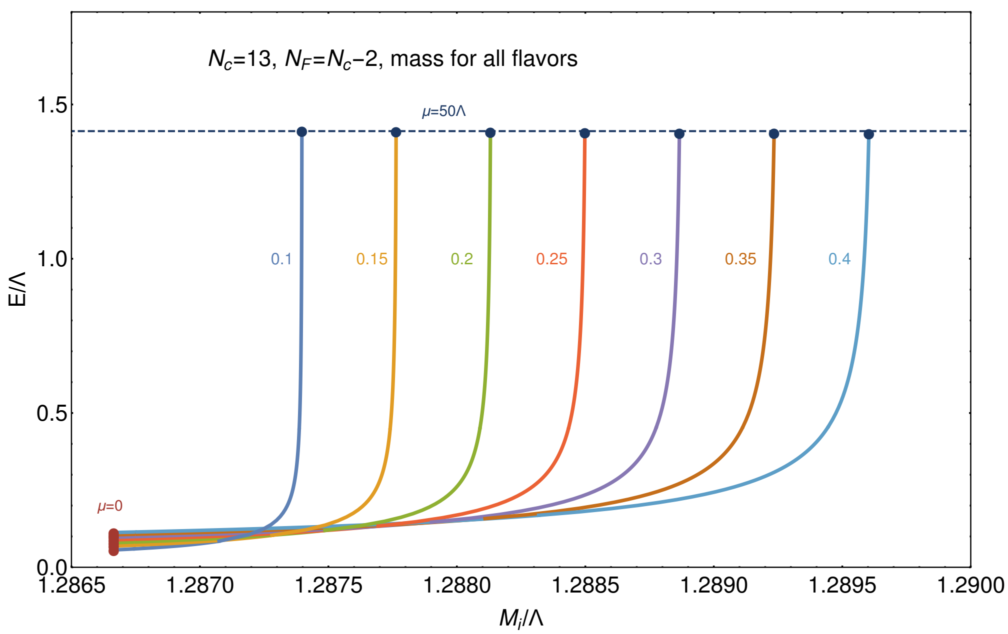

Figure 4.2: Location of the minimum in the theory with AMSB and $N_f = N_c - 2$, deformed by a universal mass term $\frac{1}{2}\mu M^{ii}$ for all flavors. E is the VEV of the condensed monopoles, while M_i is the common VEV for all of the flavors. The different curves are labeled by the value of m/Λ. The curves start at the $N_c - 2$ minimum (4.17) for $\mu = 0$. As $\mu/\Lambda \to \infty$, the theory goes over to pure SYM, while the VEV of the monopoles E is given by (4.42), represented by the dashed line. We have again chosen set $N_c = 11$ for this plot.

Though we don't show this explicitly, the same conclusion persists for any number of flavors that are integrated out from the $N_f = N_c - 2$ theory, and so we explicitly see monopole condensation for the entire range $0 \leq N_f \leq N_c - 2$.

4.4.8 Loop Operators and Confinement

The observed monopole condensation for $N_f \leq N_c - 2$ implies the confinement of electric and dyonic loop operators and a perimeter law for magnetic 't Hooft loops. This is in agreement with arguments made in [44]. For $N_f < N_c - 4$, the VEV of the meson field Higgses the gauge group to pure YM with more than four colors, in which Wilson and dyonic loops confine while 't Hooft loops do not.

With $N_c - 4$ flavors, the situation is a bit more subtle because the unbroken $SO(4) \simeq SU(2)_L \times SU(2)_R$ gauge theory forms a gaugino condensate for each $SU(2)$ factor. The branch where the gaugino condensates are aligned is connected by a mass deformation to the $N_c - 2$ case, and so by the argument of the previous section it involves monopole condensation. Consequently, magnetic loops acquire a perimeter law while dyonic loops acquire an area law. The other branch with anti-aligned condensates is related to the first one by a shift $\theta_1 \to \theta_1 + 2\pi$ [44], and so by the Witten effect, on this branch it is the dyons that condense. On this branch, dyonic loops acquires a perimeter law while magnetic loops acquires an area law. In both cases the electric Wilson loop is confined.

In the case of $N_c - 3$ flavors, the unbroken gauge group is again special because $SO(3)_+$ is in fact related to $SO(3)_-$ by a shift of the vacuum angle $\theta \to \theta + 2\pi$ [44]. Such a shift permutes the magnetic and dyonic loops, however the shift also exchanges the two orientations of the gaugino condensate, thus exchanging the ADS and exotic baryon branches. Again the global minimum exhibits the same loop operator behavior as of all other values of $N_f \leq N_c - 2$. We are free to interpret this case as either monopoles condensing in $SO(3)_-$ or dyons condensing in $SO(3)_+$.

For $N_c - 1 \leq N_f < 3(N_c - 2)$, the global minimum of the dual theory finds the meson with non-vanishing VEV. Integrating out the dual quarks leaves behind pure YM, for which we already demonstrated monopole condensation in the previous section. Thus, the electric and dyonic loops of the *dual theory* confine. Matching the behavior of the single non-trivial loop in each of the dual theories, the correspondence of gauge groups is [44, 55],

$$\begin{aligned} Spin(N_c) &\longleftrightarrow SO(N_f - N_c + 4)_- \\ SO(N_c)_+ &\longleftrightarrow SO(N_f - N_c + 4)_+ \\ SO(N_c)_- &\longleftrightarrow Spin(N_f - N_c + 4) \end{aligned} \tag{4.43}$$

Put concisely, the duality exchanges the electric and dyonic loops. The duality (4.43) therefore implies the same loop operator behavior as in the $N_f \leq N_c - 2$ case.

As first noted in [43] and elaborated in [44], the cases $N_f = N_c - 1$, N_c are special in the sense that there are extra dual descriptions for the same theory. Let us first focus on the $N_f = N_c - 1$ case, and take the original theory to be $Spin(N_c)$. In that case the dual is $SO(3)_-$ with $N_f = N_c - 1$ flavors and a superpotential (4.35). But we know that this theory is equivalent to $SO(3)_+$ with θ shifted by 2π, which results in an exchange of the ADS and the exotic baryon branch. Dualizing back, we find another IR description of the theory in terms of $SO(N_c)_+$ and a superpotential $W_{\text{2nd dual}} = -\frac{\det M}{32\Lambda^{2N_c-5}}$. By the arguments above, at the AMSB minimum the dyonic loop of the $SO(3)_-$ description and the Wilson loop of the $Spin(N_c)$ description confine, while the 't Hooft loop of the $SO(N_c)_+$ description has a perimeter law. A similar logic applies if we choose the original theory to be $SO(N_c)_-$, in which case the first dual is $Spin(3)$ and the second dual is again $SO(N_c)_-$. Here both the Wilson loop of $Spin(3)$ and the dyonic loop of $SO(N_c)_-$

confine. For $N_f = N_c$ there are again two dual descriptions of the original theory, albeit with a different superpotential for the second dual. Since the loop behavior is identical to the $N_f = N_c - 1$ case, we do not repeat the analysis here.

In summary, theories with $N_f < 3(N_c - 2)$ and AMSB all experience monopole condensation and the same behavior for their loop operators. In particular, the non-trivial Wilson loop has an area law, signaling electric confinement.

4.5 Conclusions

We have examined the low-energy phase structure of $SO(N_c)$ gauge theories with N_f Weyl fermions in the vector representation, obtained by perturbing the SUSY version of this theory via AMSB. We found that the intricate phase structure of the SUSY theory does not survive the non-SUSY perturbation. Instead the phase structure is very simple: for $N_f < 3(N_c - 2)$ the theory is confining with chiral symmetry breaking. This suggests that the conformal, free magnetic, and Abelian Coulomb phases are rather special to supersymmetry, and are lifted as soon as SUSY is broken. We have also paid special attention to the loop operators of the theory that can be used as proper order parameters. In the most interesting case of $Spin(N_c)$ (in which case the electric Wilson loop in the spinor can not be screened by the dynamical matter fields) we indeed find an area law behavior corresponding to true confinement for all $N_f < 3(N_c - 2)$, while the $SU(N_f)$ global symmetry is broken to $SO(N_f)$. The dynamics leading to confinement is monopole condensation. This is most clearly seen for the $N_f = N_c - 2$ special case, where massless monopoles (and dyons) indeed appear at special points in the moduli space. With AMSB, we indeed find a non-vanishing monopole condensate in accordance with the original conjecture by Mandelstam and 't Hooft. By considering mass deformations to the $N_f = N_c - 2$ case, we have numerically verified that the monopole condensate persists for all $N_f \leq N_c - 2$. For the $N_c - 2 < N_f < 3(N_c - 2)$, the AMSB vacuum is obtained when the quarks of the dual theory are integrated out and the dual theory becomes pure YM, for which we already established monopole condensation as the special case $N_f = 0$.

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