GROUP TESTING - A COMBINATORIAL APPROACH[1]

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BU-473-M July, 1973

Abstract

An easy group testing procedure is given based on combinatorial considerations, where a unit is tested not more than 3 times. The use of incomplete block designs in group testing procedures, when the number of defective items is known, is also proposed.

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[1] This research work was supported by NIH Grant GM-05900.

[2] On leave from Punjab Agricultural University, India.

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(Preliminary Report)

1. Introduction. Let us consider the problem of classifying each of n given units into one of two disjoint categories called satisfactory and unsatisfactory (or, simply, good and bad or defective). The characteristic feature of group testing is that any number of units, say x, can be tested simultaneously, but the information obtained from a single test on x units, without any chance of error, is that either (i) all the x units are good, or (ii) at least one of the x units tested is bad, but it is unknown how many and which ones are bad. The problem is to devise a suitable method of classifying all the n units into good or bad categories with the least number of trials.

The first application of group testing in the literature was made by Dorfman (1943) in pooling blood samples in order to classify each one of a large group of people as to whether or not they have a particular disease. Sobel and his co-workers (1959, 1960, 1966, 1967, 1971) have devised various sequential procedures to classify the units and established the optimality of their results for large n . Lindström (1964, 1973) was interested in a slightly modified problem, in which each trial determines the exact number of defectives, and provided optimal procedures in a set-theoretic frame.

In this communication, we devise a simple method of group testing based on

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elementary combinatorial techniques. We call our method the R-F procedure and the simplicity of this technique is expected to make the new procedure readily adaptable by biologists.

2. Assumptions for the R-F Procedure. We want to make the following assumptions for developing this procedure:

(a) Each of the n units are to be classified into good or bad items.

(b) Each item is defective with an unknown probability p, which is very small and the items are defective independently. In other words, we are searching for rarely defective items.

(c) Any number of items can be included in the tests and the items do not interact when tested in combination with others. Further, there is no error in the results of each trial.

(d) Each test shows a positive result if there is at least on bad item included in the test.

(e) Each item cannot be tested more than 3 times.

3. R-F Group Testing Procedure. Let the n units be arranged in a staircase array as illustrated below for n=8,10, and 46:

	1	4			1	5
n=8,	2	5	7	or	2	6
	3	6	8		3	7
					4	8

	1		
	2	5	8
n=10,	3	9	9
	4	7	10

	1	6	11	16	21							
	2	7	12	17	22	26	30	34	38			
$n=46$,	3	8	13	18	23	27	31	35	39			
	4	9	14	19	24	28	32	36	40	42	44	
	5	10	15	20	25	29	33	37	41	43	45	46

Such an arrangement is possible in at most 4 steps as any number is a sum of at most 4 squares and the 4 squares can be put together as in the above illustration for $n=46$ to give the required staircase arrangement.

If n is arranged as indicated above and if the first step has a_1 rows and b_1 columns, the second step has a_2 rows and b_2 columns, the third step has a_3 rows and b_3 columns, and the fourth step has a_4 rows and b_4 columns; then we represent such an arrangement by

$$n = \left(a_1, b_1;\ a_2, b_2;\ a_3, b_3;\ a_4, b_4\right) \ .$$

If any a's are zero, we omit them in the representation for n . The examples for $n=8$ are $(3,2;2,1)$ and $(4,2)$ and the example for $n=46$ is $(5,5;4,4;2,2;1,1)$.

Let $\ell = a_1 + b_1 + b_2 + b_3 + b_4$. Then ℓ represents the length to be traversed from the top of the staircase to reach the bottom. Clearly the staircase representation for a given n is not unique. However, among all possible arrangements, we choose the one which has ℓ a minimum.

The R-F procedure now consists of testing all the n units in the first instance and if the test is positive, then a_1 tests corresponding to the rows and $b_1 + b_2 + b_3 + b_4$ tests corresponding to the columns will be made. If a test corresponding to one row, say i, and columns, say $j_1, j_2, \cdots, j_s$, are positive, then the units in the cells (i,j_1), $(i,j_2), \cdots, (i,j_s)$ are defective while all others are good. Analogously if a test corresponding to one column, say j, and rows, say $i_1, i_2, \cdots, i_r$, are positive, then the units in the cells $(i_1,j),(i_2,j), \cdots,(i_r,j)$ are bad while others are good. However, if the tests corresponding to $r(>1)$ rows, say $i_1, i_2, \cdots, i_r$, and $s(>1)$ columns, say $j_1, j_2, \cdots, j_s$ are positive, then rs more tests have to be made on the units in cells (i_α, j_β), $\alpha = 1,2,\cdots,r$; $\beta = 1,2,\cdots,s$ by taking the units individually.

To illustrate the proposed design for classifying n=10 units by the R-F procedure, we consider the staircase representation 10 = (4,1;3,2) illustrated above. Firstly, all 10 units will be tested and if the test is negative then all units are good. If this test is positive, then 7 more tests will be conducted as described below:

Test No.	Items Included
1	1
2	2,5,8
3	3,6,9
4	4,7,10
5	1,2,3,4
6	5,6,7
7	8,9,10

If tests, say, 2, 6, and 7 give positive results then items 5 and 8 are defective, while the rest are good.

It can be easily verified that the probability that tests corresponding to more than one row and more than one column give positive results is

$$\begin{aligned} p_1 = \Big\{ & 1 - q^n - (a_1-a_2)(1-q^{b_1})q^{n-b_1} - (a_2-a_3)(1-q^{b_2})q^{n-b_2} \\ & - (a_3-a_4)(1-q^{b_3})q^{n-b_3} - a_4(1-q^{b_4})q^{n-b_4} \Big\} \\ \times \Big\{ & 1 - q^n - b_1(1-q^{a_1})q^{n-a_1} - b_2(1-q^{a_2})q^{n-a_2} \\ & - b_3(1-q^{a_3})q^{n-a_3} - b_4(1-q^{a_4})q^{n-a_4} \Big\} , \end{aligned}$$

where $q=1-p$, and $p_1 \to 0$ as $n \to \infty$ as p is assumed to be very small. Thus with almost certainty all the n units will be classified in $1+\ell$ trials. The expected number of trials is approximately $1+\ell\left(1-q^n\right)$.

4. Multi-stage R-F Group Testing Procedure. When n is large, the R-F procedure described in Section 3 can be modified so that all the units will be classified in fewer trials by considering a multi-stage procedure.

We shall first discuss the two-stage procedure in detail. Let $n=n_1 n_2$ and let $n_1=\left(a_1,b_1;\ a_2,b_2;\ a_3,b_3;\ a_4,b_4\right)$ with the least $\ell_1 = a_1+b_1+b_2+b_3+b_4$ and $n_2=\left(a_1',b_1';\ a_2',b_2';\ a_3',b_3';\ a_4',b_4'\right)$ with the least $\ell_2 = a_1'+b_1'+b_2'+b_3'+b_4'$. Let us replace assumption (e) by (e') that each unit can be tested at most 6 times.

Now, in each of the cells of the staircase arranged with n_1 numbers, we consider n_2 units so that all the $n_1 n_2$ units are considered. In the first stage, the R-F procedure will be applied on the staircase of n_1 units to determine the defective groups of units and when the defective groups are located, in the second stage the R-F procedure will be applied on the n_2 units of each of the defective groups to determine the defective units.

Assuming that the probability that trials corresponding to more than one row and more than one column in each stage give positive results is negligible, we can show that the expected number of trials to classify the units is approximately:

$$E(R) \approx 1 + \ell_1\left(1-q^n\right) + 2\left(a_1b_1+a_2b_2+a_3b_3+a_4b_4\right) \times \left(1+\ell_2(1-q)\right)\left(1-q^{n_2}\right) .$$

The above expression for the number of trials indicates that in order to get fewer trials, n_1 should be kept small and n_2 large. To find the optimum size of n_2, one may compute $E(R)$ for various values of n_2 and choose that n_2 which gives least $E(R)$.

The above-discussed two-stage R-F group testing procedure can be easily generalized, without any loss of generality to multi-stage procedures by suitably modifying the assumption (e). With three stages, if $n=n_1n_2n_3$ and

$$n_1=(a_1,b_1;\ a_2,b_2;\ a_3,b_3;\ a_4,b_4),\quad \ell_1=a_1+b_1+b_2+b_3+b_4$$
$$n_2=(a_1',b_1';\ a_2',b_2';\ a_3',b_3';\ a_4',b_4'),\quad \ell_2=a_1'+b_1'+b_2'+b_3'+b_4'$$
$$n_3=(a_1'',b_1'';\ a_2'',b_2'';\ a_3'',b_3'';\ a_4'',b_4''),\quad \ell_3=a_1''+b_1''+b_2''+b_3''+b_4''\ ,$$

the expected number of trials is approximately

$$E(R) \approx 1 + \ell_1(1-q^n) + 2(a_1b_1 + a_2b_2 + a_3b_3 + a_4b_4)(1 + \ell_2(1-q^{n_3}))(1-q^{n_2})$$
$$+2(a_1'b_1' + a_2'b_2' + a_3'b_3' + a_4'b_4')(1 + \ell_3(1-q))(1-q^{n_3})\ ,$$

and in order to get minimum $E(R)$, n_1 and n_2 must be chosen to be small and n_3 large.

5. Use of Incomplete Block Designs in Group Testing. An incomplete block design, D, is an arrangement of v symbols in b sets of sizes $k_1,k_2,\cdots,k_b$ ($k_i < v$ for $i=1,2,\cdots,b$) such that the i^{th} symbol occurs in r_i sets and every symbol occurs in r_i sets and every symbol occurs at most once in a set. A design D is said to have the R-F property on x symbols, if given any x symbols, the sets of D not containing any of the x symbols contain between themselves all the other v-x symbols. The design with v=7=b given below

$$(0,1,3);\ (1,2,4);\ (2,3,5);\ (3,4,6);$$
$$(4,5,0);\ (5,6,1);\ (6,0,2)$$

has the R-F property on 2 symbols and the design

$$(0,1,3,6);\ (1,2,4,0);\ (2,3,5,1);\ (3,4,6,2);$$
$$(4,5,0,3);\ (5,6,1,4);\ (6,0,2,5)$$

has the R-F property on 1 symbol, but not on 2 symbols.

Given a design D with R-F property on x symbols, where v=n are the number of units to be tested, if it is known that there are exactly x defective units and if any number of trials can be made on each of the n units, then one can make trials corresponding to the sets of D and when the tests showing negative results contain between themselves exactly v-x symbols, then those distinct units will be classified as good, while the other x units are bad or defective.

For example, with 7 units had the tests been made with the design described above having R-F property on 2 symbols, the testing could have been stopped when 2 tests had given negative results and the conclusions could have been as follows:

Test numbers giving negative results	Good items	Bad items
1,2	0,1,2,3,4	5,6
1,3	0,1,2,3,5	4,6
2,3	1,2,3,4,5	0,6
1,4	0,1,3,4,6	2,5
2,4	1,2,3,4,6	0,5
3,4	2,3,4,5,6	0,1
1,5	0,1,3,4,5	2,6
2,5	0,1,2,4,5	3,6
3,5	0,2,3,4,5	1,6
4,5	0,3,4,5,6	1,2
1,6	0,1,3,5,6	2,4
2,6	1,2,4,5,6	0,3
3,6	1,2,3,5,6	0,4
4,6	1,3,4,5,6	0,2
5,6	0,1,4,5,6	2,3
1,7	0,1,2,3,6	4,5
2,7	0,1,2,4,6	3,5
3,7	0,2,3,5,6	1,4
4,7	0,2,3,4,6	1,5
5,7	0,2,4,5,6	1,3
6,7	0,1,2,5,6	3,4

The expected number of trials in which the classification of 7 units by this procedure will be made is 5.3.

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