APPLICATIONS OF COMMUTATIVE ALGEBRA TO SPLINE THEORY AND STRING THEORY

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APPLICATIONS OF COMMUTATIVE ALGEBRA TO SPLINE THEORY AND STRING THEORY

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In this thesis, we study two problems: (1) the dimension problem on splines, and (2) Gorenstein Calabi-Yau varieties with regularity 4 and codimension 4. They come from approximation theory and physics, respectively, but can be studied with commutative algebra.

Splines play an important role in approximation theory, geometric modeling, and numerical analysis. One key problem in spline theory is to determine the dimension of spline spaces. The Schenck-Stiller "$2r+1$" conjecture is a conjecture on this problem. We present a counter-example to this conjecture and prove it with the spline complex. We also conjecture a new bound for the first homology of the spline complex.

Calabi-Yau varieties, especially Calabi-Yau threefolds, play a central role in string theory. A first example of a Calabi-Yau threefold is a quintic hypersurface in $\mathbb{P}^7$. Generalizing this construction, we may consider complete intersection Calabi-Yaus (CICY), or more generally Gorenstein Calabi-Yaus (GoCY). In 2016, Coughlan, Gołebiowski, Kapustka and Kapustka found 11 families of Gorenstein Calabi-Yau threefolds in $\mathbb{P}^7$ and they ask if it is a complete list. We consider the Artinian reduction and find there are 8 Betti diagrams for these GoCYs. There are another 8 Betti diagrams corresponding to Artinian Gorenstein rings of regularity 4 and codimension 4. We prove they cannot be Betti diagrams of Gorenstein threefolds in $\mathbb{P}^7$. Our result can be viewed as a step towards answering the CGKK question.

These two topics may seem to be unrelated at first sight. However, Macaulay's inverse systems provide a unifying theme. We discuss some of these topics as future directions of research.

BIOGRAPHICAL SKETCH

Beihui was born on October 28, 1986, in Urumqi, Xinjiang, People's Republic of China. She graduated from Tsinghua University with a B.E. in June 2009. After that, she decided to switch to mathematics. She entered graduate school to study mathematics in 2011 and got a M.S. from Tsinghua 3 years later.

She began studies at Cornell University in August 2014. She completed the Ph.D. thesis in 2021, under the supervision of Mike Stillman and Hal Schenck.

To my grandfather Daohua, I miss you.

ACKNOWLEDGEMENTS

I would like to thank my advisors, Mike Stillman and Hal Schenck. At the early stage of my graduate study, I once had difficulty in finding a appropriate topic to work on. It was Mike who suggested the topic of splines to me, which interested me and finally worked out. He guided me through my hardest time with great patience and sound advice. Hal has an keen eye on finding a good research direction. His suggestion on studying the Gorenstein Calabi-Yau varieties does not only result in the writing of the third chapter of this thesis, but also gets us the chance collaborating internationally with a group of people on a large project. Hal is not only a good mathematician, but also an expert on cheering people up and attracting the audience in a talk. Both of them were very supportive, always there when I looked for help and advice. I must be the luckiest person in the world to have them being my advisors.

I learned commutative algebra and homological algebra, which are the main tools I use in this thesis, from Irena Peeva and Yuri Berest. Until today, Irena's *Graded Syzygies* is still a frequently used handy reference when I am working. Yuri inspired me with suggesting a relation between certain kind of splines and quasi-invariants of Coxeter groups.

I also would like to express my gratitude to our Graduate Field Coordinators, Melissa Totman and Jim Utz. During the seven years, they helped me a lot on contacting with graduate school. Every time I faced some complicated situation which could cause trouble, they stood by my side and found solutions effectively.

The time I spent with my fellow graduate students are precious to me. Ayah Almousa was undoubtedly the leader in our small commutative algebra group. She organized all kinds of activities for us, from commutative algebra lunch/coffee to hiking. She is also a good friend to me, who always got me informed on conferences and job chances. Dylan Peifer and Mahrud Sayrafi helped me with setting up and learning Macaulay2. During COVID, we were unable to meet in person. Together with Guanyu Li, Yun Liu and Daoji Huang, we kept meeting regularly online and giving seminar talks to each other. Finally, I would like

to tell Lihai Qian, Zaoli Chen and Xiudi Tang that I really enjoyed meals we had together.

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INTRODUCTION

1.1 Problems

In this thesis, we study two problems: (1) the dimension problem on spline spaces, and (2) finding Gorenstein Calabi-Yau varieties. They come from approximation theory and physics, respectively, but can be studied with commutative algebra.

1.1.1 The dimension problem on spline spaces

To approximate a function over a region in $\mathbb{R}^n$, we may consider a subdivision of the region and then approximate the function by a piecewise polynomial. A C^r-differentiable piecewise polynomial function over a subdivision Δ of a region in $\mathbb{R}^n$ is called a *spline*.

Splines are fundamental objects in numerical analysis and approximation theory, where they are used in the finite element method to solve PDE, as well as in shape modeling of complicated objects.

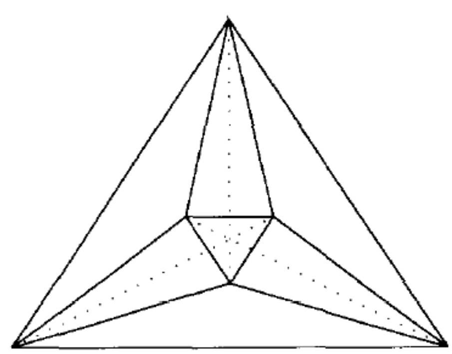

Figure 1.1: The Morgan-Scott configuration

In practice, there can be many different types of subdivisions for a given region. However, in this thesis, we only consider the cases when the subdivision is a triangulation. For example, the Morgan-Scott configuration in Figure 1.1, which is reproduced from [31, Example 6.2], is a triangulation of a planar region. Therefore, Δ has a natural structure of a simplicial complex. Assume Δ is a simplicial complex with pure dimension n. The set of i-faces of Δ is denoted by Δ_i.

Definition. $C^r(\Delta)$ is the set of functions $f : \Delta \to \mathbb{R}$ satisfying the following two properties:

- f is differentiable of order r, and
- if σ is a facet of Δ, then $f|_\sigma$ is a polynomial.

Once we have this definition, there are some immediate conclusions:

1. $C^r(\Delta)$ is an $\mathbb{R}$-vector space.
2. $C^r(\Delta)$ has an $\mathbb{R}$-algebra structure by taking multiplication of two elements pointwisely.
3. $C^\infty(\Delta)$ can be identified with $\mathbb{R}[x_1, \ldots, x_n]$.
4. With $C^\infty(\Delta) \simeq \mathbb{R}[x_1, \ldots, x_n]$, $C^r(\Delta)$ can be viewed as an $\mathbb{R}[x_1, \ldots, x_n]$-module.

There are interesting questions and results on each of these structures. For example, in [6] Billera shows that the $\mathbb{R}$-algebra structure of $C^0(\Delta)$ can be identified with a Stanley-Reisner ring.

In this thesis, we focus on the dimension problem. For each non-negative integer d, we define

$$C^r_d(\Delta) = \{f \in C^r(\Delta) : \deg(f|_\sigma) \leq d, \text{ for all } \sigma \in \Delta_n\}. \tag{1.1}$$

This is a finite dimensional $\mathbb{R}$-vector space. One of the key problems in spline theory is the determination of the dimensions of $C^r_d(\Delta)$ for all d. However, even for planar regions, it is still an open problem to find an explicit formula that works for all Δ, r and d. On the other hand, the planar cases were studied a lot and there is a formula that gives a lower bound for $\dim C_d^r(\Delta)$, thanks to Schumaker[34].

$$L(\Delta, r, d) = \binom{d+2}{2} + \binom{d-r+1}{2} f_1^\circ - (\binom{d+2}{2} - \binom{r+2}{2}) f_0^\circ + \sum \sigma_i, \tag{1.2}$$

where f_1° is the number of interior edges and f_0° is that of interior vertices in Δ, and

$$\sigma_i = \sum_{j=0}^{\infty} \max\{(r+1+j)(1-n(v_i)), 0\} \tag{1.3}$$

with $n(v_i)$ the number of distinct slopes at an interior vertex v_i. In other words, for any triangulation Δ, smoothness r and degree d, we always have the inequality

$$\dim C_d^r(\Delta) \geq L(\Delta, r, d). \tag{1.4}$$

Hence, it is natural to ask when does equality in (1.4) hold. First of all, it is known that formula $L(\Delta, r, d)$ does not always give the correct value of $\dim C_d^r(\Delta)$. For example, it fails for the Morgan-Scott configuration in Figure 1.1 when $(r, d) = (1, 2)$.

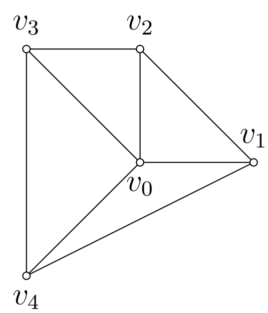

Figure 1.2: Δ is a star of v_0

Assume the support of Δ has genus 0. If there is only one interior vertex v in Δ, then we say Δ is a *star* of v. In [34], Schumaker also proves that $L(\Delta, r, d) = \dim C_d^r(\Delta)$ for all (r, d) if Δ is a star of some vertex v.

It is also known by [1] that formula $L(\Delta, r, d)$ gives $\dim C_d^r(\Delta)$ for any triangulation Δ when $d \geq 4r + 1$:

Theorem 1.1 (Alfeld-Schumaker,1987)**.** *If $d \geq 4r+1$, then*

$$\dim C_d^r(\Delta) = L(\Delta, r, d). \tag{1.5}$$

They also proved that the equality holds for generic triangulations when $r \geq 3r+1$ in a later paper[2]. In [13], Dong proves that (1.5) holds for all triangulation when $d \geq 3r+2$.

Schenck and Stiller conjectured in [29]:

Conjecture 1 (Schenck-Stiller, 2002)**.** *The equality (1.5) holds for all triangulations when $d \geq 2r+1$.*

They call it the "$2r+1$" conjecture.

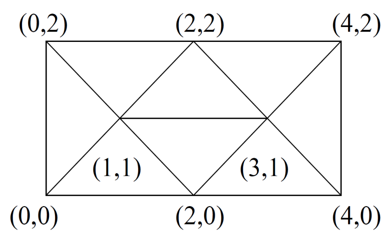

Figure 1.3: An example that (1.5) fails for $d = 2r$

Work of Tohăneanu[36] shows Conjecture 1 is optimal by showing a configuration in Figure 1.3, reproduced from [29, Example 2.4], such that equality (1.5) fails for $d = 2r$.

There is also a conjecture by Alfeld and Manni that the equality (1.5) holds for all Δ when $(r, d) = (1, 3)$. This conjecture appears earlier than the Schenck-Stiller conjecture.

Throughout this thesis, we are going to use the following notations:

- Δ° denotes the set of interior faces Δ.
- Δ_i° denotes the set of i-dimensional interior faces.
- Δ_i^∂ denotes the set of i-dimensional boundary faces.

- $f_i(\Delta)$, $f_i^\circ(\Delta)$ and $f_i^\partial(\Delta)$ denote the cardinality of Δ°, Δ_i° and Δ_i^∂, respectively. Sometimes, Δ will be omitted if it is clear which Δ we are referring to by the context.

1.1.2 Finding Gorenstein Calabi-Yau varieties

No.	degree	$h^{1,1}$	$h^{1,2}$	Description
1	14	2	86	$(2,4)$ divisor in $\mathbb{P}^1 \times \mathbb{P}^1$
2	15	1	76	$G(2,5) \cap X_3 \cap \mathbb{P}^7$
3	16	1	65	c.i. $(2,2,2,2)$
4	17	1	55	bilinked on c.i. $(2,2,2)$ to $\mathbb{P}^3$
5	17	2	58	2×2 minors of a 3×3 matrix
6	17	2	54	codim 2 in cubic roll
7	18	1	46	bilinked on c.i. $(2,2,3)$ to F_1
8	18	1	45	bilinked on c.i. $(2,2,3)$ to F_2
9	19	2	37	bilinked on special Pf_{13} to F_1
10	19	2	36	bilinked on special Pf_{13} to F_2
11	20	2	34	3×3 minors of 4×4 matrix with linear forms

Table 1.1: The CGKK list of families of GoCY threefolds in $\mathbb{P}^7$

Let $X \subseteq \mathbb{P}^N$ be a nonsingular projective variety of dimension n. Let $\Omega^p(X)$ denote the sheaf of regular p-forms over X.

Definition. The *Hodge number* $h^{p,q}(X)$ is defined as

$$h^{p,q}(X) = \dim H^q(X, \Omega^p(X)). \tag{1.6}$$

A nonsingular projective variety X is *Calabi-Yau* if $\Omega^n(X) \simeq \mathcal{O}_X$ and for all $i > 0$,

$$h^{0,i}(X) = 0. \tag{1.7}$$

Remark 1. There are several equivalent definitions of Calabi-Yau. Because all compact Calabi-Yau manifolds satisfy the criterion of Kodaira embedding Theorem, they can be viewed as subvarieties of $\mathbb{P}^N$.

In their 1985 paper [10], Candelas-Horowitz-Strominger-Witten showed that Calabi-Yau threefolds play a central role in string theory. According to [39], "hidden dimension" of our universe is a Calabi-Yau threefold. Therefore, finding and studying Calabi-Yau varieties, especially Calabi-Yau threefolds, are of great interest to both physicists and mathematicians.

A first example of a Calabi-Yau threefold is a quintic hypersurface in $\mathbb{P}^4$. Generalizing the hypersurface case, when X is a complete intersection (CI) of type $\{d_1, \ldots, d_{n-3}\} \subseteq \mathbb{P}^n$ we have

$$\Omega^n(X) \simeq \mathcal{O}_X(-n-1+\sum d_i). \tag{1.8}$$

So a complete intersection Calabi-Yau (CICY) threefold in $\mathbb{P}^n$ must have $\{d_1, \ldots, d_{n-3}\}$ satisfying

$$\begin{array}{ccc} \{5\} & \text{in} & \mathbb{P}^4 \\ \{2,4\} & \text{in} & \mathbb{P}^5 \\ \{3,3\} & \text{in} & \mathbb{P}^5 \\ \{2,2,3\} & \text{in} & \mathbb{P}^6 \\ \{2,2,2,2\} & \text{in} & \mathbb{P}^7 \end{array}$$

Green-Hübsch-Lütken characterize complete intersection Calabi-Yau threefolds $X \subset \prod_{i=1}^m \mathbb{P}^{n_i}$ in [19]. A complete intersection is the first avatar of a Gorenstein ring; a Gorenstein ideal of height two is a complete intersection, and Buchsbaum-Eisenbud[9] show that a height three Gorenstein ideal is generated by the Pfaffians of a skew-symmetric matrix. From the Calabi-Yau perspective, this is investigated in [26], [37] and subsequent papers. The codimension four case was first studied systematically by Bertin in [4]; in [11], Coughlan-Golebiowski-Kapustka-Kapustka list 11 arithmetically Gorenstein Calabi-Yau (GoCY) threefolds in $\mathbb{P}^7$ as in Table 1.1, where $\mathrm{Pf}_{13} \subseteq \mathbb{P}^7$ is a codimension 3 manifold defined by Pfaffians and F_i's are some del Pezzo threefolds. See [11] for a detailed description of these manifolds.

Conjecture 2 (Coughlan-Gołebiowski-Kapustka-Kapustka, 2016)**.** *The CGKK list (Table 1.1) is a complete classification of families of Gorenstein Calabi-Yau threefolds in* $\mathbb{P}^7$*.*

We also call this conjecture the *CGKK conjecture*.

1.2 The organization of this thesis

The organization of this chapter is as follows: in Section 1.3, we recall some basic knowledge of commutative algebra that we use. They can be found in [25], [15] and [8]. We also fix notations which we use throughout the thesis. In Section 1.4, we introduce a duality between two polynomial rings, which relates the two problems introduced in Section 1.1 to each other, and also to many other topics.

In Chapter 2, we study the dimension problem on spline spaces using the spline complex $S_\bullet/J_\bullet$, which was invented by Schenck and Stillman in [30]. In particular, we present a configuration Δ_Y which makes a counter-example to Conjecture 1. We prove it is a counter-example by performing some explicit computation on $H_1(S_\bullet/J_\bullet)$ of Δ_Y. Then we prove that the regularity of $H_1(S_\bullet/J_\bullet)$ of Δ_Y is greater than $2.2r$. That means if we would like to make a new conjecture, it cannot be of the form "$2r + c$" for any constant c. The coefficient of r has to be no less than 2.2.

In Chapter 3, we study Artinian Gorenstein rings. We are particularly interested in those with regularity 4 and codimension 4, because they are related to Conjecture 2. The main results are Table 3.1 and 3.2. They are all possible Betti diagrams for Artinian Gorenstein rings with regularity 4 and codimension 4. Those in Table 3.2 cannot be Betti diagrams of Gorenstein Calabi-Yau threefolds. We prove these results using a case-by-case argument.

We close this thesis with a discussion on directions of future research in Chapter 4.

1.3 Some notions and definitions in commutative algebra

Throughout the thesis, we assume $S = \Bbbk[x_0, \dots, x_n]$ is a standard graded polynomial ring with $\mathrm{char}(\Bbbk) = 0$. The ideal $\mathfrak{m} = \langle x_0, \dots, x_n \rangle$ is called the *irrelevant ideal*.

1.3.1 On dimensions

If $J \subseteq S$ is a homogeneous ideal and $R = S/J$, $\dim R$ refers to the Krull dimension of R, that is, the supreme of lengths of chains of prime ideals of R. Here the length of the chain $P_0 \subsetneq \cdots \subsetneq P_{r-1} \subsetneq P_r$ is taken to be r. We only talk about codimension of R when R is a quotient ring S/J, and in this case codim $R = \dim S - \dim R$. The *height* of J is the supreme of lengths of chains of primes descending from J.

We write $\operatorname{ann}(M)$ for the *annihilator* of M, that is,

$$\operatorname{ann}(M) = \{f \in S \mid fM = 0\}. \tag{1.9}$$

The dimension of M is defined to be the dimension of $S/\operatorname{ann}(M)$ and codimension of M is defined to be the height of $\operatorname{ann}(M)$.

A sequence $\mathbf{f} = f_1, \dots, f_n$ of elements of S is said to be a *regular sequence* (or S-sequence) if, for each i, the f_i is neither a zero divisor nor a unit to $S/\langle f_1, \dots, f_{i-1}\rangle$. Similarly, $\mathbf{f}$ is a regular sequence (or M-sequence) if $\langle f_1, \dots, f_n\rangle M \neq M$, and, for each i, the f_i is not a zero divisor to $M/\langle f_1, \dots, f_{i-1}\rangle M$. If J is generated by a regular sequence, then we call $V(J)$ a *complete intersection*.

If $J \subseteq S$ is an ideal such that $JM \neq M$, then the common length of the maximal M-sequences in J is called the *grade* of J on M, denoted by $\operatorname{grade}(J, M)$. The grade of J is defined to be $\operatorname{grade}(J, S)$. The *depth* of M is defined to be $\operatorname{grade}(\mathfrak{m}, M)$ and is denoted by depth M. In general, depth $M \leq \dim M$ and $\operatorname{grade}(J) \leq \operatorname{ht}(J)$, see [8, Proposition 1.2.12.].

Definition. Let $S = \Bbbk[x_0, \dots, x_n]$ be the standard graded polynomial ring. A finite generated graded S-module M is arithmetically *Cohen-Macaulay* if depth $M = \dim M$.

Remark 2. If $J \subseteq S$ is a homogeneous ideal, then $\operatorname{grade} J = \operatorname{ht} J$. See [8, Corollary 2.1.4]

1.3.2 On chain complexes

If M is a finitely generated graded S-module, the *Hilbert function* of M is

$$\mathrm{HF}(M,d) = \dim M_d. \tag{1.10}$$

Sometimes, we also use $h_d(M) = \mathrm{HF}(M,d)$ and call

$$\mathbf{h}(M) = (h_0(M), \ldots, h_d(M), \ldots) \tag{1.11}$$

the *h-vector* of M.

For $p \in \mathbb{Z}$ denoted by $M(-p)$ the graded S-module such that $M(-p)_d = M_{-p+d}$. We say $M(-p)$ is the module M *shifted* p degrees, and call p the *shift*. It is clear from the definition that

$$\mathrm{Hom}_S(S(p), S(q)) \simeq S(q-p). \tag{1.12}$$

Definition. A *chain complex* $F_\bullet$ over S is a sequence of homomorphisms of S-modules

$$F_\bullet : \cdots \to F_i \xrightarrow{d_i} F_{i-1} \to F_1 \xrightarrow{d_1} F_0 \to \ldots \tag{1.13}$$

such that $d_{i-1}d_i = 0$ for $i \in \mathbb{Z}$. The collection of maps $d_\bullet = \{d_i\}$ is called the *differential* of $F_\bullet$. Sometimes the chain complex is denoted $(F_\bullet, d_\bullet)$. The i-th *homology* of a chain complex $F_\bullet$ is defined by

$$H_i(F_\bullet) = \ker(d_i)/\mathrm{Im}(d_{i+1}). \tag{1.14}$$

The chain complex is *exact* at F_i (or at *step* i) if $H_i(F_\bullet) = 0$. The chain complex is *exact* if $H_i(F_\bullet) = 0$ for all i.

Definition. A *free resolution* of a finitely generated S-module M is a sequence of homomorphisms of S-modules

$$F_\bullet : \cdots \to F_i \xrightarrow{d_i} F_{i-1} \to F_1 \xrightarrow{d_1} F_0 \tag{1.15}$$

such that

(1) $F_\bullet$ is a chain complex of finitely generated free S-modules F_i,

(2) $F_\bullet$ is exact except at F_0, and

(3) M is isomorphic to cokernel of d_1.

A resolution is *graded* if M is graded, $F_\bullet$ is a graded complex and $M \simeq \text{coker } d_1$ is of degree 0. $\ker d_{i-1}$ is called the *i-th syzygy* of M. A graded free resolution is *minimal* if

$$d_{i+1}(F_{i+1}) \subseteq \mathfrak{m}F_i \text{ for all } i \geq 0. \tag{1.16}$$

Definition. For a graded S-module M, assume $F_\bullet$ is a minimal free resolution of M. The *total Betti numbers* are

$$b_i = \text{rank } F_i. \tag{1.17}$$

The *graded Betti numbers* are

$$b_{ij} = \dim_{\Bbbk} \text{Tor}_i(M, \Bbbk)_j. \tag{1.18}$$

A *Betti diagram* has the form

	b_0	b_1	b_2	$\ldots$
0	$b_{0,0}$	$b_{1,1}$	$b_{2,2}$	$\ldots$
1	$b_{0,1}$	$b_{1,2}$	$b_{2,3}$	$\ldots$
2	$b_{0,2}$	$b_{1,3}$	$b_{2,4}$	$\ldots$
3	$b_{0,3}$	$b_{1,4}$	$b_{2,5}$	$\ldots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$

The *projective dimension* of M is defined by

$$\text{proj.dim}(M) = \sup\{i \mid b_i \neq 0\}. \tag{1.19}$$

The *Castelnuovo-Mumford regularity* of M is defined by

$$\text{reg }(M) = \sup\{j \mid b_{i,i+j} \neq 0\}. \tag{1.20}$$

Assume $\alpha : F \to G$ is a homomorphism where $F \simeq S^f$ and $G \simeq S^g$ are two free modules, with rank $F \geq$ rank G. By choosing basis $(u_s)_{1\leq s\leq g}$ for G and $(v_t)_{1\leq t\leq f}$ for F, α may be written as

$$\alpha = \sum_{s,t} \alpha_{s,t} u_s \otimes v_t^*, \tag{1.21}$$

where $\alpha_{s,t} \in S$. The *Eagon-Northcott* complex of α is a complex

$$\begin{aligned}\mathbf{EN}(\alpha):\ 0 \to (\mathrm{Sym}_{f-g} G)^* \otimes \wedge^f F \xrightarrow{d_{f-g+1}} (\mathrm{Sym}_{f-g-1} G)^* \otimes \wedge^{f-1} F \xrightarrow{d_{f-g}} \\ \ldots \xrightarrow{d_4} (\mathrm{Sym}_2 G)^* \otimes \wedge^{g+2} F \xrightarrow{d_3} G^* \otimes \wedge^{g+1} F \xrightarrow{d_2} \wedge^g F \xrightarrow{\wedge^g \alpha} \wedge^g G,\end{aligned}$$

where

$$d_k : (\mathrm{Sym}_{k-1} G)^* \otimes \wedge^{g+k-1} F \to (\mathrm{Sym}_{k-2} G)^* \otimes \wedge^{g+k-2} F,$$

is defined by

$$\begin{aligned}& d_k((u_{j_1} \cdot \cdots \cdot u_{j_{k-1}})^* \otimes v_{i_1} \wedge \cdots \wedge v_{i_{g+k-1}}) \\ &= \sum_{s,t} \alpha_{s,t} (u_{j_1} \ldots \hat{u}_{j_s} \ldots u_{j_{k-1}})^* \otimes (-1)^t v_{i_1} \wedge \ldots \hat{v}_{i_t} \cdots \wedge v_{i_{g+k-1}}.\end{aligned}$$

The *Buchsbaum-Rim* complex of of α is a complex

$$\begin{aligned}\mathbf{BR}(\alpha):\ 0 \to (\mathrm{Sym}_{f-g-1} G)^* \otimes \wedge^f F \xrightarrow{d_{f-g+1}} (\mathrm{Sym}_{f-g-2} G)^* \otimes \wedge^{f-1} F \xrightarrow{d_{f-g}} \\ \ldots \xrightarrow{d_4} G^* \otimes \wedge^{g+2} F \xrightarrow{d_3} \wedge^{g+1} F \xrightarrow{\varepsilon} F \xrightarrow{\alpha} G,\end{aligned}$$

where

$$d_k : (\mathrm{Sym}_{k-2} G)^* \otimes \wedge^{g+k-1} F \to (\mathrm{Sym}_{k-3} G)^* \otimes \wedge^{g+k-2} F,$$

is defined by

$$\begin{aligned}& d_k((u_{j_1} \cdot \cdots \cdot u_{j_{k-2}})^* \otimes v_{i_1} \wedge \cdots \wedge v_{i_{g+k-1}}) \\ &= \sum_{s,t} \alpha_{s,t} (u_{j_1} \ldots \hat{u}_{j_s} \ldots u_{j_{k-2}})^* \otimes (-1)^t v_{i_1} \wedge \ldots \hat{v}_{i_t} \cdots \wedge v_{i_{g+k-1}}.\end{aligned}$$

The map $\varepsilon : \wedge^{g+1} F \to F$ in Buchsbaum-Rim complex is defined by

$$\varepsilon(\wedge^I v) = \sum_{J \subset I, |J|=g} \operatorname{sgn}(J \subset I)(\det \alpha_J) \wedge^{I-J} v^*, \tag{1.22}$$

where α_J is the $g \times g$ submatrix of α with columns corresponding to the basis elements indexed by J, $\operatorname{sgn}(J \subset I)$ is the sign of permutation of I that puts the elements of J into first g positions.

1.3.3 Miscellaneous

Throughout the thesis, we use $\mathbf{Gr}(k, V)$ to denote the Grassmannian, that is, the set of all k-dimensional subspace of a vector space V.

1.4 Macaulay's inverse system and Apolarity

Assume $\operatorname{char}(\Bbbk) = 0$. Let $T = \Bbbk[y_0, \dots, y_n]$ and $S = \Bbbk[x_0, \dots, x_n]$. Define the action of T on S by

$$\begin{aligned} T \times S &\to S, \\ (g, f) &\mapsto g(\frac{\partial}{\partial x_0}, \dots, \frac{\partial}{\partial x_n}) \cdot f. \end{aligned}$$

This map is bilinear, hence factors through $T \otimes S \to S$. This makes S a graded T-module.

Remark 3. The grading of S as a ring is opposite to that as an T-module.

This map can be restricted to a perfect pairing

$$T_d \times S_d \to \Bbbk.$$

Therefore, it defines a duality between finite dimensional vector spaces T_d and S_d for $d \in \mathbb{N}$. This duality connects several notions together.

First of all, in particular, it defines a duality between T_1 and S_1,

$$\varphi : \mathbf{Gr}(n, T_1) \to \mathbf{Gr}(1, S_1),$$
$$V \mapsto V^{\perp}.$$

Note that $\mathbf{Gr}(n, T_1)$ is equivalent to the set of closed points in $\mathrm{Proj}(T)$ and $\mathbf{Gr}(1, S_1)$ is just $\mathbb{P}(S_1)$. If $P \in \mathbb{P}^n = \mathrm{Proj}(T)$ is a closed point and $[L_P] = \varphi(P)$ for some $L_P \in S_1$, then L_P is determined by P up to a scalar. Hence we call L_P the *corresponding linear form* of P. So this duality sets a connection between closed points in one space and linear forms over the dual space. Next, we define for an ideal $I \subset T$ *Macaulay's inverse system* of I to be

$$I^{-1} = \{f \in S \mid g(f) = 0 \text{ for all } g \in I\}. \tag{1.23}$$

I^{-1} is a T-module, but not an S-module in general. In fact, by [17, Proposition 2.5], we have

$$(I^{-1})_d = I_d^{\perp}. \tag{1.24}$$

This correspondence relates powers of linear forms to fat points. See Section 4.2 for a discussion on this topic.

Remark 4. There are some equivalent definitions for Macaulay's inverse system. See [15, Section A2.4] and [21, Appendix A].

Definition. The *canonical module* ω_T of T is defined to be $\omega_T = T(-n-1)$. If I is a homogeneous ideal in T such that $U = T/I$ is arithmetically Cohen-Macaulay and $\dim T - \dim U = t$, then the *canonical module* ω_U is defined to be $\omega_U = \mathrm{Ext}^t_T(U, \omega_T)$. U is *arithmetically Gorenstein* if ω_U can be generated by one element as an U-module.

We also define *Macaulay's inverse system* of an ideal $J \subseteq S$ to be

$$J^{-1} = \{g \in T \mid g(f) = 0 \text{ for all } f \in J\}. \tag{1.25}$$

J^{-1} is an ideal in T. If $J = \langle f \rangle$ is a principal ideal generated by $f \in S_d$, then J^{-1} is also denoted as I_f and elements in I_f are also called polynomials *apolar* to f. Moreover, it is an Artinian Gorenstein ideal, which we study in Chapter 3. Conversely, any Artinian Gorenstein ideal $I \subset T$ can be obtained as Macaulay's inverse system of a principal ideal $\langle f \rangle \in S$ [15, Theorem 21.6].

Theorem 1.2 (Macaulay, 1916)**.** *The map $f \mapsto A_f$ is a bijection between degree d forms $f \in S$ and graded Artinian Gorenstein quotient rings $A_f = T/I$ of T with regularity d.*

CHAPTER 2

ON THE DIMENSION CONJECTURES OF SPLINE SPACES

We study the dimension problem of spline spaces in this chapter. In Section 2.1, we introduce the spline complex $S_\bullet/J_\bullet$, which is the main tool we use to study $C^r(\Delta)$. In particular, we will see $C^r(\hat{\Delta})$ can be identified with the top homology of $S_\bullet/J_\bullet$. Section 2.2 reviews results obtained by the local data of Δ. In Section 2.3, we focus on the planar cases. We review Schenck and Stillman's results on how homologies of $S_\bullet/J_\bullet$ affect $\dim C^r_d(\Delta)$ when Δ is planar. With their results, we can translate the conjecture on $\dim C^r_d(\Delta)$ to one on regularity of the first homology of $S_\bullet/J_\bullet$. We state our main results, a counter-example to Conjecture 2 and a new bound for $H_1(S_\bullet/J_\bullet)$, in 2.4-2.7.[1]

2.1 Homological methods for studying spline problems

Since we know that $C^r(\Delta)$ is an $\mathbb{R}[x_1,\dots,x_n]$-module, we want to use commutative algebra tools to study it. In order to use these notions, it is helpful to introduce the concept of homogenized spline modules. Let $\hat{\Delta}$ be the cone of Δ in $\mathbb{R}^{n+1}$. To be precise, suppose Δ is in $\mathbb{R}^n$ with coordinates $x_1,\dots,x_n$. Then $\hat{\Delta}$ corresponds to embedding Δ in hyperplane $x_0 = 1$, and forming a new simplicial complex $\hat{\Delta}$ by joining each simplex in Δ to the origin in $\mathbb{R}^{n+1}$. We call $C^r(\hat{\Delta})$ the *homogenized spline module* of Δ. Assume that $S = \mathbb{R}[x_0,\dots,x_n]$. Then $C^r(\hat{\Delta})$ is a graded S-module. Recall that $\mathrm{HF}(M,d)$ denotes the Hilbert function of a graded S-module M. Hence,

$$\dim_{\mathbb{R}} C^r_d(\Delta) = \dim_{\mathbb{R}} C^r(\hat{\Delta})_d = \mathrm{HF}(C^r(\hat{\Delta}), d). \tag{2.1}$$

[1]Portions of this chapter are reprinted from [35] and [33].

The Hilbert function is *additive*, meaning that if there is an short exact sequence of graded S-modules

$$0 \to U \to V \to W \to 0, \tag{2.2}$$

then

$$\mathrm{HF}(U, d) + \mathrm{HF}(W, d) = \mathrm{HF}(V, d). \tag{2.3}$$

As a graded S-module, $C^r(\hat{\Delta})$ can be identified with the kernel of a homomorphism ϕ : $S^{f_n} \oplus S^{f^\circ_{n-1}}(-r-1) \to S^{f^\circ_{n-1}}$ which will be defined later. Before we give the definition, we want to illustrate what ϕ is by a concrete example.

Example 1 ("A star of a vertex")**.** Figure 1.2 is a planar Δ which is the star of a single interior vertex v_0 at the origin. Let l_{0i} be a linear form vanishing on the edge $[v_0 v_i]$ for $i = 1, 2, 3, 4$. In this case, (h_1, h_2, h_3, h_4) is an element of $C^r(\hat{\Delta})$ if and only if

$$h_1 - h_2 = a_1 l_{01}^{r+1}$$

$$h_2 - h_3 = a_2 l_{02}^{r+1}$$

$$h_3 - h_4 = a_3 l_{03}^{r+1}$$

$$h_4 - h_1 = a_4 l_{04}^{r+1}.$$

And the homomorphism $\phi : S^{f_2} \oplus S^{f^\circ_1}(-r-1) \to S^{f^\circ_1}$ is given by

$$\phi = \begin{bmatrix} 1 & -1 & 0 & 0 & l_{01}^{r+1} & & & \\ 0 & 1 & -1 & 0 & & l_{02}^{r+1} & & \\ 0 & 0 & 1 & -1 & & & l_{03}^{r+1} & \\ -1 & 0 & 0 & 1 & & & & l_{04}^{r+1} \end{bmatrix}$$

Note that this matrix has two blocks. The left one is

$$\begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 \\ -1 & 0 & 0 & 1 \end{bmatrix},$$

which can be identified with the boundary map ∂_2 of the relative simplicial complex $C_\bullet(\Delta, \partial\Delta)$ with coefficients in S. The right block is a diagonal matrix $\mathrm{diag}(l_{01}^{r+1}, \dots, l_{04}^{r+1})$.

Therefore, $\phi : S^{f_n} \oplus S^{f_{n-1}^\circ}(-r-1) \to S^{f_{n-1}^\circ}$ is defined by a matrix with two blocks $\begin{bmatrix} \partial_n & \mathrm{diag}(l_\tau^{r+1}) \end{bmatrix}$. One block ∂_n is the top boundary map of $C_\bullet(\Delta, \partial\Delta)$ with coefficients in S. The other block $\mathrm{diag}(l_\tau^{r+1})$ is a diagonal matrix with entries l_τ^{r+1}, where τ runs over all $(n-1)$-faces of Δ and l_τ is a linear form vanishing on τ. Billera and Rose proved that $C^r(\hat{\Delta}) \simeq \ker(\phi)$ in [7].

Billera introduced the use of homological algebra in spline theory in [5]. Following this path, Schenck and Stillman[31] defined a chain complex $J_\bullet$ to deal with the problem of freeness of $C^r(\Delta)$. This is the main tool we use to study the splines.

Let $S_\bullet$ be the relative simplicial complex $C_\bullet(\Delta, \partial\Delta)$ with coefficients in S. Let l_τ be a linear form which is obtained by homogenizing a degree one polynomial that vanishes on $\tau \in \Delta_{n-1}$. The authors of [31] define the *ideal complex* $J_\bullet$ to be

$$J_\bullet : 0 \to \bigoplus_{\sigma \in \Delta_n^\circ} J(\sigma) \xrightarrow{\partial} \dots \xrightarrow{\partial} \bigoplus_{v \in \Delta_0^\circ} J(v) \to 0, \tag{2.4}$$

where

$$J(\sigma) = 0, \text{ for } \sigma \in \Delta_n$$
$$J(\tau) = \langle l_\tau^{r+1} \rangle, \text{ for } \tau \in \Delta_{n-1}$$
$$J(\zeta) = \sum_{\zeta \in \tau} J(\tau), \text{ for } \zeta \in \Delta_{n-2}$$
$$\vdots$$
$$J(v) = \sum_{v \in \tau} J(\tau), \text{ for } v \in \Delta_0$$

are ideals of S, and ∂ is induced by the boundary map in $S_\bullet$. Hence $J_\bullet$ is a subcomplex of $S_\bullet$, and we may consider the quotient complex $S_\bullet / J_\bullet$. We call $S_\bullet / J_\bullet$ the *spline complex*. It encodes the different types of data of Δ in different levels:

- the *combinatorial* (or *topological*) data is encoded by $C_\bullet(\Delta, \partial\Delta)$, and hence by $S_\bullet$.
- the *local geometric* data around a vertex $v \in \Delta_0^\circ$ is captured by $J(v)$, as discussed in Section 2.2.
- the *global geometric* data is determined by both the combinatorial data and the actual positions of vertices $v \in \Delta_0$ in $\mathbb{R}^n$. Some properties of $C^r(\hat{\Delta})$ affected by this type of data can be analyzed using homologies of $S_\bullet/J_\bullet$. We introduce Schenck and Stillman's analysis in Section 2.3.

Recall the Schumaker formula $L(\Delta, d, r)$ defined in (1.2) is a lower bound of $\dim C_d^r(\Delta)$. In fact, $L(\Delta, d, r)$ uses only the combinatorial and local geometric data of Δ. On the other hand, the actual value of $\dim C_d^r(\Delta)$ also depends on the global geometry. This is why there is a discrepancy $\dim C_d^r(\Delta) - L(\Delta, d, r)$.

The top homology of the spline complex can be identified with $C^r(\hat{\Delta})$.

Theorem 2.1 (Billera, 1988)**.** *The spline module $C^r(\Delta)$ is isomorphic to the top homology of the spline complex $H_n(S_\bullet/J_\bullet)$.*

Remark 5. This theorem is equivalent to Theorem 3.2 in [5]. The original statement is that $C^r(\Delta) \simeq H_n(S_\bullet/I_\bullet)$ for another chain complex $S_\bullet/I_\bullet$. However, the top two terms of $S_\bullet/I_\bullet$ are the same with those of $S_\bullet/J_\bullet$. Hence $H_n(S_\bullet/J_\bullet) = H_n(S_\bullet/I_\bullet)$ and these two statements are equivalent.

2.2 The ideal $J(v)$, local geometric data of Δ

Fix an interior vertex $v \in \Delta_0^\circ$. Recall from Section 2.1 that

$$J(v) = \langle l_\tau^{r+1} \mid \tau \in \Delta_{n-1},\ v \in \tau \rangle \tag{2.5}$$

where l_τ is the homogenization of a degree one polynomial vanishing on τ. If we choose a coordinate $(x_1, \dots, x_n)$ of $\mathbb{R}^n$ such that v is the origin, then l_τ only involves variables $x_1, \dots, x_n$. Because l_τ is determined by τ up to a scalar, we may consider the set $\Gamma = \{[l_\tau] \mid \tau \in \Delta_{n-1},\ v \in \tau\}$ as a finite subset of $\mathbf{Gr}(1, \mathbb{R}^n)$. We call Γ the *local geometric data* of Δ at v.

If $n = 1$, then Γ is trivial, because $\mathbf{Gr}(1, \mathbb{R}^1)$ has only one point.

If $n = 2$, then $J(v)$ is an ideal in 2 variables, and since each vertex we have at least two edges with different slopes, so $S/J(v)$ has projective dimension 2. In [34], Schumaker gives a free resolution of $S/J(v)$.

Theorem 2.2 (Schumaker, 1979)**.** *A free resolution of $S/J(v)$ is given by*

$$S(-r-1-\alpha(v))^{a_1} \oplus S(-r-2-\alpha(v))^{a_2} \xrightarrow{\mathrm{Syz}_1(v)} S(-r-1)^{k(v)} \xrightarrow{\mathrm{Syz}_0(v)} S \to S/J(v) \to 0, \quad (2.6)$$

where $\alpha(v) = \lfloor (r+1)/(k(v)-1) \rfloor$, $a_1 = (k(v)-1)\alpha(v) + k(v) - r - 2$ and $a_2 = k(v) - 1 - a_1 = r + 1 - (k(v) - 1)\alpha(v)$.

Remark 6. Using this theorem, we may compute the Hilbert function of $S/J(v)$.

For $n \geq 3$, we do not have a formula computing Hilbert function of $S/J(v)$ that works in general. This is one reason that computing $\dim C^r_d(\Delta)$ for $n \geq 3$ is much more difficult than in the planar case. There are formulas working for some special cases. For example, in [28], Schenck proves that the Foucart-Sorokina formula holds for Alfeld split of a simplex Δ_n.

Computing the Hilbert function of $S/J(v)$ is also related to the dimension problem on fat points spaces, which is discussed in Section 4.2.

2.3 The planar case

In this section, we briefly review Schenck and Stillman's analysis of homologies of $S_\bullet/J_\bullet$ for $n = 2$ as in [30] and [31]. Throughout this section, we assume the Δ is on the $z = 1$ plane in $\mathbb{R}^3$ with coordinates (x, y, z) and fix the polynomial ring $S = \mathbb{R}[x, y, z]$.

The ideal complex $J_\bullet$ has only two non-zero terms:

$$J_\bullet : 0 \to \bigoplus_{\tau \in \Delta_1^\circ} J(\tau) \xrightarrow{\partial} \bigoplus_{v \in \Delta_0^\circ} J(v) \to 0, \tag{2.7}$$

where

$$J(\sigma) = 0, \text{ for } \sigma \in \Delta_2$$
$$J(\tau) = \langle l_\tau \rangle^{r+1}, \text{ for } \tau \in \Delta_1$$
$$J(v) = \sum_{v \in \tau} J(\tau), \text{ for } v \in \Delta_0$$

Recall in Section 2.1, we defined a homomorphism $\phi : S^{f_n} \oplus S^{f_{n-1}^\circ}(-r-1) \to S^{f_{n-1}^\circ}$ and by [7] $C^r(\Delta)$ can be identified with $\ker \phi$. In particular, for $n = 2$, we have the following theorem:

Theorem 2.3 (Billera-Rose, 1991). *Let ∂_2 be the second boundary map in $S_\bullet$. There is an exact sequence of graded S-modules*

$$0 \to C^r(\hat{\Delta}) \to S^{f_2} \oplus S^{f_1^\circ}(-r-1) \xrightarrow{\phi} \oplus S^{f_1^\circ} \to M \to 0, \tag{2.8}$$

where

$$\phi = \left(\begin{array}{c|ccc} & l_{\epsilon_1}^{r+1} & & \\ \partial_2 & & \ddots & \\ & & & l_{\epsilon_{f_1^\circ}}^{r+1} \end{array} \right) \tag{2.9}$$

By Theorem 2.1, $C^r(\hat{\Delta}) \simeq H_2(S_\bullet/J_\bullet)$. The short exact sequence of chain complexes

$$0 \to J_\bullet \to S_\bullet \to S_\bullet/J_\bullet \to 0, \tag{2.10}$$

induces the long exact sequence of their homologies:

$$\begin{aligned}0 \to H_2(J_\bullet) \to H_2(S_\bullet) \to H_2(S_\bullet/J_\bullet)\\ \to H_1(J_\bullet) \to H_1(S_\bullet) \to H_1(S_\bullet/J_\bullet)\\ \to H_0(J_\bullet) \to H_0(S_\bullet) \to H_0(S_\bullet/J_\bullet) \to 0.\end{aligned}$$

Among these homologies,

$$H_2(J_\bullet) = H_0(S_\bullet) = H_0(S_\bullet/J_\bullet) = 0$$

and

$$H_2(S_\bullet) \simeq S.$$

If the genus of Δ is 0, then $H_1(S_\bullet) = 0$. Therefore, the long exact sequence breaks into two short exact sequences:

$$0 \to H_2(S_\bullet) \to H_2(S_\bullet/J_\bullet) \to H_1(J_\bullet) \to 0 \tag{2.11}$$

and

$$0 \to H_1(S_\bullet/J_\bullet) \to H_0(J_\bullet) \to 0. \tag{2.12}$$

Therefore,

$$H_1(S_\bullet/J_\bullet) \simeq H_0(J_\bullet). \tag{2.13}$$

From (2.11) and the exact sequence

$$0 \to H_1(J_\bullet) \to \bigoplus_{\tau\in\Delta_1^\circ} J(\tau) \xrightarrow{\partial} \bigoplus_{v\in\Delta_0^\circ} J(v) \to H_0(J_\bullet) \to 0, \tag{2.14}$$

it follows that

$$\begin{aligned}\dim C_d^r(\Delta) &= \mathrm{HF}(S,d) + \dim H_1(J_\bullet)_d\\ &= \mathrm{HF}(S,d) + \dim \bigoplus_{\tau\in\Delta_1^\circ} J(\tau)_d - \dim \bigoplus_{v\in\Delta_0^\circ} J(v)_d + \dim H_0(J_\bullet)_d.\end{aligned}$$

Theorem 2.4 (Schenck-Stillman, 1997)**.** *The Schumaker formula*

$$L(\Delta, r, d) = \mathrm{HF}(S, d) + \dim \bigoplus_{\tau \in \Delta_1^\circ} J(\tau)_d - \dim \bigoplus_{v \in \Delta_0^\circ} J(v)_d \tag{2.15}$$

Hence,

$$\dim C_d^r(\Delta) = L(\Delta, r, d) + \dim H_0(J_\bullet)_d. \tag{2.16}$$

Remark 7. From (2.15), we can see $L(\Delta, r, d)$ only uses combinatorial and local geometric data of Δ.

Therefore, the discrepancy $\dim C_d^r(\Delta) - L(\Delta, r, d)$ is just $\dim H_0(J_\bullet)_d$. Schenck and Stillman's analysis on $H_0(J_\bullet)$ shows that $\dim H_0(J_\bullet)_d = 0$ for $d >> 0$:

Theorem 2.5 (Schenck-Stillman, 1997)**.** *The S-module $H_0(J_\bullet)$ has finite length.*

If N is an S-module of finite length, then

$$\mathrm{reg}\ \ N = \max\{d \geq 0 \mid N_d \neq 0\}. \tag{2.17}$$

By Theorem 2.4, $\dim C_d^r(\Delta) = L(\Delta, r, d)$ if and only if $d > \mathrm{reg}\ \ H_0(J_\bullet)$. Therefore, Conjecture 1 can be translated into a conjecture that $\mathrm{reg}\ \ H_0(J_\bullet) \leq 2r$.

Recall from [15, Section A3.12] and [25, Section 27] that if $\beta : (F_\bullet, \varphi_\bullet) \to (G_\bullet, \psi_\bullet)$ is a map of complexes, then the *Mapping cone* $P_\bullet$ of β is the complex such that $P_i = F_{i-1} \oplus G_i$, with differential

$$F_i \oplus G_{i+1} \xrightarrow{\begin{bmatrix} -\varphi_i & 0 \\ \beta_i & \psi_{i+1} \end{bmatrix}} F_{i-1} \oplus G_i. \tag{2.18}$$

Clearly, $G_\bullet$ is a subcomplex of $P_\bullet$. The quotient $P_\bullet / G_\bullet$ is isomorphic to $F_\bullet[-1]$. In other words, there is a short exact sequence of complexes

$$0 \to G_\bullet \to P_\bullet \to F_\bullet[-1] \to 0, \tag{2.19}$$

inducing a long exact sequence of homologies

$$\cdots \to H_i(G_\bullet) \to H_i(P_\bullet) \to H_{i-1}(F_\bullet) \to H_{i-1}(G_\bullet) \to \dots$$

In particular, if $H_i(F_\bullet) = H_i(G_\bullet) = 0$ for $i > 0$, then there is an exact sequence

$$0 \to H_1(P_\bullet) \to H_0(F_\bullet) \to H_0(G_\bullet) \to H_0(P_\bullet) \to 0, \tag{2.20}$$

and

$$H_i(P_\bullet) = 0, \text{ for } i \geq 2. \tag{2.21}$$

With the notion of mapping cone, we prove the following lemma on $H_0(J_\bullet)$.

Lemma 2.6. *Let $(F_\bullet, \varphi_\bullet)$ and $(G_\bullet, \psi_\bullet)$ be free resolutions of $\bigoplus_{\tau\in\Delta_1^\circ} J(\tau)$ and $\bigoplus_{v\in\Delta_0^\circ} J(v)$, respectively. Assume*

$$\beta : F_\bullet \to G_\bullet. \tag{2.22}$$

is the lift of $\partial : \bigoplus J(\tau) \to \bigoplus J(v)$. Then $H_0(J_\bullet)$ is isomorphic to the cokernel of the homomorphism

$$F_0 \oplus G_1 \xrightarrow{\begin{bmatrix}\beta_1 & \psi_1\end{bmatrix}} G_0. \tag{2.23}$$

Proof. Note that we have the identifications

$$H_0(F_\bullet) = \bigoplus_{\tau\in\Delta_1^\circ} J(\tau) \text{ and } H_0(G_\bullet) = \bigoplus_{v\in\Delta_0^\circ} J(v) \tag{2.24}$$

and the map $H_0(F_\bullet) \to H_0(G_\bullet)$ in (2.20) is just ∂. Hence the mapping cone $P_\bullet$ of β has

$$H_0(P_\bullet) = H_0(J_\bullet). \tag{2.25}$$

The differential $P_1 \to P_0$ is

$$F_0 \oplus G_1 \xrightarrow{\begin{bmatrix}\beta_1 & \psi_1\end{bmatrix}} G_0, \tag{2.26}$$

Hence $H_0(J_\bullet)$ is the cokernel of this map. □

Remark 8. This lemma can also be viewed as a corollary of [31, Lemma 3.8].

2.4 A counter-example to the "$2r+1$" conjecture

In this section, we present a counter-example to Conjecture 1. Let Δ_Y be the configuration as shown in Figure 2.1.

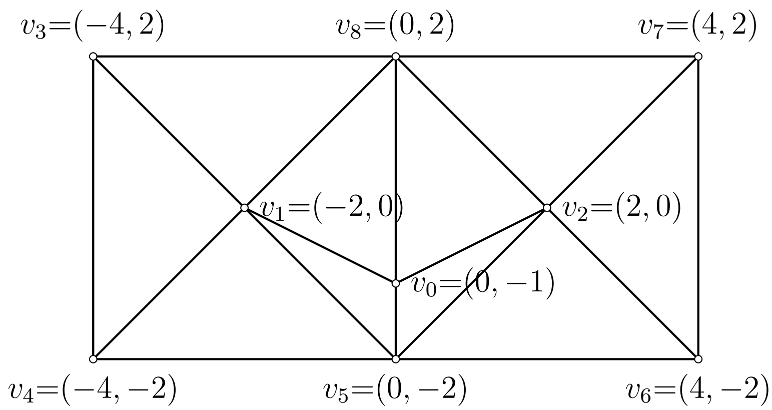

Figure 2.1: Δ_Y, a counter-example to the "$2r+1$" conjecture

Note that the genus of Δ_Y is 0, hence by the analysis in Section 2.3, we only need to show $\operatorname{reg}\ H_0(J_\bullet) > 2r$ for some r. In fact, $\operatorname{reg}\ H_0(J_\bullet) > 4$ for $r=2$:

Theorem 2.7. *Let $r=2$ and Δ be as in Figure 2.1. Then*

$$(H_0(J_\bullet))_{d=5} \neq 0.$$

This means that $\operatorname{reg}\ H_0(J_\bullet) \geq 5$*. So Conjecture 1 fails in this case.*

In order to prove it, first, we find a presentation of $H_0(J_\bullet)$, that is, we write $H_0(J_\bullet)$ as a cokernel of a map between free modules. Then we specify to case $r=2$ and compute that $H_0(J_\bullet)_{d=5} \neq 0$, so it makes a counter-example to Conjecture 1.

2.5 A presentation of $H_0(J_\bullet)$

We use Lemma 2.6 to obtain a presentation of $H_0(J_\bullet)$.

	ε_{13}	ε_{15}	ε_{14}	ε_{18}	ε_{01}	ε_{05}	ε_{08}	ε_{02}	ε_{25}	ε_{27}	ε_{26}	ε_{28}
$e_{1,13}$	1	1	0	0	0	0	0	0	0	0	0	0
$e_{1,14}$	0	0	1	1	0	0	0	0	0	0	0	0
$e_{1,01}$	0	0	0	0	-1	0	0	0	0	0	0	0
$e_{0,01}$	0	0	0	0	1	0	0	0	0	0	0	0
$e_{0,05}$	0	0	0	0	0	1	1	0	0	0	0	0
$e_{0,02}$	0	0	0	0	0	0	0	1	0	0	0	0
$e_{2,02}$	0	0	0	0	0	0	0	-1	0	0	0	0
$e_{2,25}$	0	0	0	0	0	0	0	0	1	1	0	0
$e_{2,26}$	0	0	0	0	0	0	0	0	0	0	1	1

Table 2.1: Matrix β_1

Note that in Δ_Y, there are 3 interior vertices v_0, v_1 and v_2. Using Theorem 2.2, we obtain free resolutions of $J(v_i)$ for $i = 0, 1, 2$:

$$G(v_i)_\bullet : G_1(v_i) \xrightarrow{\mathrm{Syz}_1(v_i)} G_0(v_i) \xrightarrow{\mathrm{Syz}_0(v_i)} J(v_i). \tag{2.27}$$

and for $\bigoplus_{\varepsilon\in\Delta_1^\circ} J(\varepsilon)$:

$$F_\bullet : F_0 \to \bigoplus_{\varepsilon\in\Delta_1^\circ} J(\varepsilon) \tag{2.28}$$

Because each of v_i has 3 incident edges $\varepsilon_{i,j}$ with distinct slopes, so rank $G_0(v_i) = 3$ and rank $G_1(v_i) = 2$. According to analysis in Section 2.2, we may choose a basis $\{e_{i,[l_\varepsilon]} \mid \varepsilon \in \Delta_1^\circ,\ v_i \in \varepsilon\}$ for $G_0(v_i)$ and $\{\varepsilon \mid \varepsilon \in \Delta_1^\circ\}$ for F_0. Now if we fix bases

$$\{e_{1,13}, e_{1,14}, e_{1,01}\} \text{ for } G_0(v_1),$$

$$\{e_{0,01}, e_{0,05}, e_{0,02}\} \text{ for } G_0(v_0),$$

$$\{e_{2,02}, e_{2,25}, e_{2,26}\} \text{ for } G_0(v_2),$$

and

$$\{\varepsilon_{13}, \varepsilon_{15}, \varepsilon_{14}, \varepsilon_{18}, \varepsilon_{01}, \varepsilon_{05}, \varepsilon_{08}, \varepsilon_{02}, \varepsilon_{25}, \varepsilon_{27}, \varepsilon_{26}, \varepsilon_{28}\} \text{ for } F_0.$$

With these bases, the map $\beta_1 : F_0 \to G_0$ induced by the boundary map in (2.4) can be written in a 9×12 matrix given by Table 2.1.

	$\eta_{1,1}$	$\eta_{1,2}$	$\eta_{0,1}$	$\eta_{0,2}$	$\eta_{2,1}$	$\eta_{2,2}$
$e_{1,13}$	$A_{1,1}$	$A_{1,2}$	0	0	0	0
$e_{1,14}$	$B_{1,1}$	$B_{1,2}$	0	0	0	0
$e_{1,01}$	$C_{1,1}$	$C_{1,2}$	0	0	0	0
$e_{0,01}$	0	0	$A_{0,1}$	$A_{0,2}$	0	0
$e_{0,05}$	0	0	$B_{0,1}$	$B_{0,2}$	0	0
$e_{0,02}$	0	0	$C_{0,1}$	$C_{0,2}$	0	0
$e_{2,02}$	0	0	0	0	$A_{2,1}$	$A_{2,2}$
$e_{2,25}$	0	0	0	0	$B_{2,1}$	$B_{2,2}$
$e_{2,26}$	0	0	0	0	$C_{2,1}$	$C_{2,2}$

Table 2.2: Matrix ψ_1

In addition, if we fix some bases $\{\eta_{i,1}, \eta_{i,2}\}$ of $G_1(v_i)$ for $i = 0, 1, 2$, then by Theorem 2.2, $\mathrm{Syz}_1(v_i)$ in (2.27) can be written in the form of

$$\mathrm{Syz}_1(v_i) = \begin{bmatrix} A_{i,1} & A_{i,2} \\ B_{i,1} & B_{i,2} \\ C_{i,1} & C_{i,2} \end{bmatrix}. \tag{2.29}$$

If r is even, then

$$\begin{cases} \deg A_{i,1} = \deg B_{i,1} = \deg C_{i,1} = \frac{r}{2}, \\ \deg A_{i,2} = \deg B_{i,2} = \deg C_{i,2} = \frac{r}{2} + 1. \end{cases}$$

If r is odd, then

$$\deg A_{i,1} = \deg B_{i,1} = \deg C_{i,1} = \deg A_{i,2} = \deg B_{i,2} = \deg C_{i,2} = \frac{r+1}{2}.$$

With these notations and fixed bases, ψ_1 can be written as a 9×6 matrix given by Table 2.2.

Now consider the matrix $\begin{bmatrix} \beta_1 & \psi_1 \end{bmatrix}$. Note that if entries of j-th column are in $\mathbb{R}$ and the (i, j)-entry is the only non-zero entry in this column, then by deleting the i-th row and j-th column, we still have the same cokernel. Therefore, the cokernel of $\begin{bmatrix} \beta_1 & \psi_1 \end{bmatrix}$ is isomorphic to the cokernel of

$$\begin{bmatrix} -1 & 0 & C_{1,1} & C_{1,2} & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & A_{0,1} & A_{0,2} & 0 & 0 \\ 0 & 1 & 0 & 0 & C_{0,1} & C_{0,2} & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & A_{2,1} & A_{2,2} \end{bmatrix} \tag{2.30}$$

Let

$$\eta'_{1,1} = \eta_{1,1} + C_{1,1}\varepsilon_{0,1},$$

$$\eta'_{1,2} = \eta_{1,2} + C_{1,2}\varepsilon_{0,1},$$

$$\eta'_{2,1} = \eta_{2,1} + A_{2,1}\varepsilon_{0,1},$$

$$\eta'_{2,2} = \eta_{2,2} + A_{2,2}\varepsilon_{0,1},$$

$$\eta'_{0,1} = \eta_{0,1}, \text{ and } \eta'_{0,2} = \eta_{0,2}.$$

Using the basis $\{\eta'_{i,j}\}$ for G_1, the map corresponding to (2.30) can with written as

$$\begin{bmatrix} -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & C_{1,1} & C_{1,2} & A_{0,1} & A_{0,2} & 0 & 0 \\ 0 & 1 & 0 & 0 & C_{0,1} & C_{0,2} & A_{2,1} & A_{2,2} \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{2.31}$$

which has the same cokernel as

$$\begin{bmatrix} C_{1,1} & C_{1,2} & A_{0,1} & A_{0,2} & 0 & 0 \\ 0 & 0 & C_{0,1} & C_{0,2} & A_{2,1} & A_{2,2} \end{bmatrix}. \tag{2.32}$$

Assume I and J are ideals of S.

Definition. The *colon ideal* (or *ideal quotient*) is defined as

$$I : J = \{f \in S \mid f \cdot g \in I \text{ for all } g \in J\}.$$

There is a nice connection of colon ideals to syzygies: if $I = \langle f_1, \ldots, f_k \rangle$ and

$$\sum_{i=1}^{k} a_i f_i = 0, \tag{2.33}$$

is a syzygy on I, then $a_k \in \langle f_1, \ldots, f_{k-1} \rangle : \langle f_k \rangle$. Therefore, if l_{ij} is a linear form vanishing on the edge ε_{ij}, then

$$\langle C_{1,1}, C_{1,2} \rangle = \langle l_{13}^{r+1}, l_{14}^{r+1} \rangle : \langle l_{01}^{r+1} \rangle \tag{2.34}$$

and

$$\langle A_{2,1}, A_{2,2} \rangle = \langle l_{25}^{r+1}, l_{26}^{r+1} \rangle : \langle l_{02}^{r+1} \rangle \tag{2.35}$$

With the above analysis, we have proved the following lemma:

Lemma 2.8. *Let Δ be as in figure 2.1. Then $H_0(J_\bullet)$ is isomorphic to the cokernel of*

$$S(-r-1-\frac{r}{2})^3 \oplus S(-r-2-\frac{r}{2})^3 \xrightarrow{\mathrm{Syz}_1} S(-r-1)^2, \text{ if } r \text{ is even,}$$

and

$$S(-r-1-\frac{r+1}{2})^6 \xrightarrow{\mathrm{Syz}_1} S(-r-1)^2, \text{ if } r \text{ is odd,}$$

where Syz_1 is a matrix of the form

$$\begin{bmatrix} C_{1,1} & C_{1,2} & A_{0,1} & A_{0,2} & 0 & 0 \\ 0 & 0 & C_{0,1} & C_{0,2} & A_{2,1} & A_{2,2} \end{bmatrix}.$$

The non-zero entries of Syz_1 can be obtained from the first differential in $G_\bullet$.

2.6 Case $r = 2$

Using Lemma 2.8, we can compute $H_0(J_\bullet)$ explicitly with the coordinates in figure 2.1 for $r = 2$. Let l_{ij} be a linear form vanishing on the edge ε_{ij}. Then

$$\begin{cases} l_{01} = x + 2y + 2z \\ l_{02} = -x + 2y + 2z \\ l_{13} = l_{15} = x + y + 2z \\ l_{14} = l_{18} = x - y + 2z \\ l_{26} = l_{28} = -x - y + 2z \\ l_{25} = l_{27} = -x + y + 2z \\ l_{05} = l_{08} = x \end{cases}$$

For $G(v_0)_\bullet$,

$$\mathrm{Syz}_0(v_0) = \begin{bmatrix} l_{01}^3 & l_{05}^3 & l_{02}^3 \end{bmatrix}, \tag{2.36}$$

and

$$\mathrm{Syz}_1(v_0) = \begin{bmatrix} 3x - 2y - 2z & -9x^2 + 6xy - 2y^2 + 6xz - 4yz - 2z^2 \\ -32y - 32z & 9x^2 + 48xy + 76y^2 + 48xz + 152yz + 76z^2 \\ 3x + 2y + 2z & 2y^2 + 4yz + 2z^2 \end{bmatrix}, \tag{2.37}$$

In particular,

$$\begin{bmatrix} A_{0,1} & A_{0,2} \\ C_{0,1} & C_{0,2} \end{bmatrix} = \begin{bmatrix} 3x - 2y - 2z & -9x^2 + 6xy - 2y^2 + 6xz - 4yz - 2z^2 \\ 3x + 2y + 2z & 2y^2 + 4yz + 2z^2 \end{bmatrix}. \tag{2.38}$$

By (2.34) and (2.35),

$$\langle C_{1,1}, C_{1,2} \rangle = \langle 2x + y + 4z,\ y^2 \rangle \tag{2.39}$$

and

$$\langle A_{2,1}, A_{2,2} \rangle = \langle -2x + y + 4z,\ y^2 \rangle \tag{2.40}$$

By Lemma 2.8, $H_0(J_\bullet)$ is isomorphic to the cokernel of

$$\bigoplus_{0\le i\le 2} S(-4)\eta_{i,1}\oplus S(-5)\eta_{i,2} \xrightarrow{\mathrm{Syz}_1} S(-3)\mathbf{e}_1\oplus S(-3)\mathbf{e}_2 \tag{2.41}$$

where $\deg \mathbf{e}_1 = \deg \mathbf{e}_2 = 3$, $\deg \eta_{i,1} = 4$, $\deg \eta_{i,2} = 5$ and Syz_1 can be written as a matrix

$$\begin{bmatrix} 2x+y+4z & y^2 & 3x-2y-2z & -3(3x-2y-2z)x-2(y+z)^2 & 0 & 0 \\ 0 & 0 & 3x+2y+2z & 2(y+z)^2 & -2x+y+4z & y^2 \end{bmatrix}$$

In other words, the image of Syz_1 is generated by

$$\begin{cases} f_1 = (2x+y+4z)\mathbf{e}_1 \\ f_2 = y^2\mathbf{e}_1 \\ f_3 = (3x-2y-2z)\mathbf{e}_1 + (3x+2y+2z)\mathbf{e}_2 \\ f_4 = -3(3x-2y-2z)x-2(y+z)^2\mathbf{e}_1 + 2(y+z)^2\mathbf{e}_2 \\ f_5 = (-2x+y+4z)\mathbf{e}_2 \\ f_6 = y^2\mathbf{e}_2 \end{cases} \tag{2.42}$$

Recall that a *monomial order* on S is a total order $\succ$ on monomials on S such that for any monomials $x^\alpha, x^\beta, x^\gamma \in S$ and any scalar k,

- $x^\alpha \succ k$ if m_0 is not a scalar, and
- $x^\alpha \succ x^\beta$ implies $x^\alpha x^\gamma \succ x^\beta x^\gamma$ if $x^\gamma \neq 0$.

Since S is a graded polynomial ring, we may define the *homogeneous lexicographic order* $\succ_{\mathrm{hlex}}$ on S: $x^\alpha \succ_{\mathrm{hlex}} x^\beta$ if $\deg x^\alpha > \deg x^\beta$, or if $\deg x^\alpha = \deg x^\beta$ and $\alpha_i > \beta_i$ for the first index i such that $\alpha_i \neq \beta_i$.

Assume N is a free S-module with basis $\{\mathbf{e}_1,\dots,\mathbf{e}_s\}$. A monomial order on N is a total order $\succ$ on elements of the form $x^\alpha\mathbf{e}_i$ for monomials $x^\alpha \in S$. Fix a monomial order $\succ$ on S. The *position-over-term* (POT) order $\succ_{\mathrm{POT}}$ on N induced by $\succ$ is defined as $x^\alpha\mathbf{e}_i \succ_{\mathrm{POT}} x^\beta\mathbf{e}_j$

- if $i > j$, or
- if $i = j$ and $x^\alpha \succ x^\beta$.

Fix a monomial order $\succ$ on a free S-module N. Then for any $f \in N$, we deine the *initial term* of f, written $\mathrm{In}_\succ(f)$, to be the greatest term of f with respect to the order $\succ$. If M is a submodule of N, we define $\mathrm{In}_\succ(M)$ to be the monomial submodule generated by the elements $\mathrm{In}_\succ(f)$ for all $f \in M$. The following lemma can be viewed as a corollary of Macaulay's Theorem [15, Theorem 15.3]:

Lemma 2.9. *M and $\mathrm{In}_\succ(M)$ have the same Hilbert function.*

In particular, since $H_0(J_\bullet)$ has finite length, we may conclude that

$$\mathrm{reg}\ \ H_0(J_\bullet) = \mathrm{reg}\ \ \mathrm{In}_\succ(H_0(J_\bullet)). \tag{2.43}$$

Recall that a *Gröbner* basis of M is a basis $\{g_1, \dots, g_t\}$ of M such that $\{\mathrm{In}_\succ(g_1), \dots, \mathrm{In}_\succ(g_t)\}$ generates $\mathrm{In}_\succ(M)$. Starting with a generating set $\{f_1, \dots, f_s\}$ of M, we may use *Buchberger's Algorithm* to compute a Gröbner basis of M. See Appendix A.1 for the detail. After the computation, we get a Gröbner basis of the image of Syz_1

$$\begin{cases} g_1 = (2x + y + 4z)\mathbf{e}_1 \\ g_2 = -(7y + 16z)\mathbf{e}_1 + (7y + 16z)\mathbf{e}_2 \\ g_3 = z^2\mathbf{e}_1 - z^2\mathbf{e}_2 \\ g_4 = (-2x + y + 4z)\mathbf{e}_2 \\ g_5 = y^2\mathbf{e}_2 \\ g_6 = (11yz + 16z^2)\mathbf{e}_2 \\ g_7 = z^3\mathbf{e}_2 \end{cases} \tag{2.44}$$

Therefore, $z^2\mathbf{e}_2 \neq 0$ is in the cokernel of Syz_1, and $\deg z^2\mathbf{e}_2 = 2 + (2 + 1) = 5$. Hence,

$$H_0(J_\bullet)_{d=5} \neq 0,$$

which means reg $H_0(J_\bullet) > 4$. Thus, Conjecture 1 fails for Δ_Y when $(r,d) = (2,5)$.

2.7 A new bound for smooth spline spaces

In Section 2.6, we proved that the configuration Δ_Y has the property $\dim C^r_d(\Delta_Y) > L(\Delta_Y, r, d)$ for $(r,d) = (2,5)$. This proof does not eliminate the possibility that $\dim C^r_d(\Delta) = L(\Delta, r, d)$ for every triangulation if $d \geq 2r+2$. In this section, we want to show that this is impossible:

Theorem 2.10. *There is no constant c so that $\dim C^r_d(\Delta_Y) > L(\Delta_Y, r, d)$ for all Δ and all $d \geq 2r + c$. In particular, there exists a planar simplicial complex Δ for which*

$$\dim H_0(J_\bullet)_d \neq 0 \text{ for all } d \leq \frac{22r+7}{10}. \tag{2.45}$$

This shows there exists a simplicial complex Δ such that $\dim C^r_d(\Delta) > L(\Delta, r, d)$ for all $d \leq \frac{22r+7}{10}$. For $L(\Delta, r, d)$ to be equal to $\dim C^r_d(\Delta)$ for every triangulation Δ, we must have

$$d > \frac{22r+7}{10} > 2.2r. \tag{2.46}$$

To prove it, we have to use some properties of complete intersection ideals. If I is a complete intersection, then the Koszul complex $\mathcal{K}(f_1, \ldots, f_k)$ gives a minimal free resolution of S/I. In particular, the projective dimension equals k, the minimal number of generators of I. If the projective dimension of S/I is $(k-1)$, then I is said to an *almost complete intersection.*

Define

$$I_1 = \langle C_{1,1}, C_{1,2} \rangle, \;\; I_2 = \langle A_{2,1}, A_{2,2} \rangle \tag{2.47}$$

and

$$\phi = \begin{bmatrix} A_{0,1} & A_{0,2} \\ C_{0,1} & C_{0,2} \end{bmatrix} \tag{2.48}$$

where $A_{i,j}$'s and $C_{i,j}$'s are the entries in (2.32). By Theorem 2.2, $\deg A_{i,1} = \deg C_{i,1} = \lfloor \frac{r+1}{2} \rfloor$ and $\deg A_{i,2} = \deg C_{i,2} = \lceil \frac{r+1}{2} \rceil$ for $i = 0, 1, 2$.

Lemma 2.11. *The ideals I_1 and I_2 are complete intersections.*

Proof. An ideal with two generators f, g is a complete intersection when f and g are relatively prime, or equivalently when the unique minimal syzygy on f, g is given by

$$f \cdot g - g \cdot f = 0.$$

By (2.34) and (2.35), both I_1 and I_2 can be written as colon ideals of the form $\langle l_1^{r+1}, l_2^{r+1} \rangle : \langle l_3^{r+1} \rangle$ for some linear forms l_1, l_2, l_3 in two variables. The ideal $\langle l_1^{r+1}, l_2^{r+1}, l_3^{r+1} \rangle$ is an almost complete intersection, which means that two generators, say $\{l_1^{r+1}, l_2^{r+1}\}$ are a complete intersection. Proposition 5.2 in [9] proves that an almost complete intersection is directly linked to a Gorenstein ideal. In this case the linked ideal is

$$\langle l_1^{r+1}, l_2^{r+1} \rangle : \langle l_3^{r+1} \rangle = \langle s_{11}, s_{12} \rangle. \tag{2.49}$$

A homogeneous Gorenstein ideal in two variables is a complete intersection, so the result follows. □

For simplicity, we denote $\lfloor \frac{r+1}{2} \rfloor$ by κ_1 and $\lceil \frac{r+1}{2} \rceil$ by κ_2, respectively. With the same coordinates as in Section 2.5, $H_0(J_\bullet)$ may be presented as the cokernel of

$$S(-r-1-\kappa_1) \oplus S(-r-1-\kappa_2) \xrightarrow{\phi} S(-r-1)/I_1 \oplus S(-r-1)/I_2 \tag{2.50}$$

Hence,

$$\begin{aligned} \mathrm{HF}(H_0(J_\bullet), d) \geq \sum_{i=1,2} \mathrm{HF}(S/I_i, d-r-1) - \mathrm{HF}(S, d-r-1-\kappa_1) \\ - \mathrm{HF}(S, d-r-1-\kappa_2) \end{aligned} \tag{2.51}$$

Since I_1 and I_2 are complete intersections, we may obtain their free resolutions by Koszul complexes. Hence, there are exact sequences:

$$0 \to S(-\kappa_1 - \kappa_2) \to S(-\kappa_1) \oplus S(-\kappa_2) \to S \to S/I_i \to 0. \tag{2.52}$$

Therefore,

$$\begin{aligned} \mathrm{HF}(S/I_i, d-r-1) = {} & \mathrm{HF}(S, d-r-1) + \mathrm{HF}(S, d-r-1-\kappa_1-\kappa_2) \\ & - \mathrm{HF}(S, d-r-1-\kappa_1) - \mathrm{HF}(S, d-r-1-\kappa_2) \end{aligned} \tag{2.53}$$

Putting (2.51) and (2.53) together,

$$\begin{aligned} \mathrm{HF}(H_0(J_\bullet), d) \geq {} & 2\,\mathrm{HF}(S, d-r-1) + 2\,\mathrm{HF}(S, d-r-1-\kappa_1-\kappa_2) \\ & - 3\,\mathrm{HF}(S, d-r-1-\kappa_1) - 3\,\mathrm{HF}(S, d-r-1-\kappa_2) \end{aligned} \tag{2.54}$$

If $d < r+1$, then the right hand side of (2.54) is 0, so we assume $d \geq r+1$ and let $d' = d-r-1$. Then the right hand side of (2.54) equals

$$2\,\mathrm{HF}(S, d') + 2\,\mathrm{HF}(S, d'-\kappa_1-\kappa_2) - 3(\mathrm{HF}(S, d'-\kappa_1) + \mathrm{HF}(S, d'-\kappa_2)) \tag{2.55}$$

Assume $d' \geq \kappa_1 + \kappa_2 = r+1$. Then (2.55) equals

$$2\binom{d'+2}{2} + 2\binom{d'-\kappa_1-\kappa_2+2}{2} - 3\binom{d'-\kappa_1+2}{2} - 3\binom{d'-\kappa_2+2}{2}. \tag{2.56}$$

If $r+1$ is even, then $\kappa_1 = \kappa_2 = \frac{r+1}{2}$ and (2.56) equals

$$-d'^2 + (2\kappa_1 - 3)d' + \kappa_1^2 + 3\kappa_1 - 2, \tag{2.57}$$

which has two real roots, the larger at

$$d' = \frac{2\kappa_1 - 3 + \sqrt{8\kappa_1^2 + 1}}{2} > (1+\sqrt{2})\kappa_1 - \frac{3}{2} > 1.2r - 1.5, \tag{2.58}$$

and the smaller root is negative. This means $\mathrm{HF}(H_0(J_\bullet), d) > 0$ for $2r+2 \leq d \leq 2.2r + 0.7$.

If $r+1$ is odd, then $\kappa_1 = \frac{r}{2}$ and $\kappa_2 = \frac{r}{2} + 1$ and (2.56) equals

$$-d'^2 + (2\kappa_1 - 2)d' + \kappa_1^2 + 4\kappa_1 - 1, \tag{2.59}$$

which has two real roots, the larger at

$$d' = \kappa_1 - 1 + \sqrt{2\kappa_1^2 + \kappa_1} > 1.2r - 1 \tag{2.60}$$

and the smaller root is negative. This means $\mathrm{HF}(H_0(J_\bullet), d) > 0$ for $2r + 2 \leq d \leq 2.2r + 1$. Therefore, we have proved Theorem 2.10 with the assumption that $[2r + 2,\ 2.2r + 0.7]$ is non-empty. For $r \geq 7$, this assumption holds. For $r \in \{2, \ldots, 6\}$, a direct computation verifies that $\mathrm{coker}(\phi)$ is non-zero at degree $d = \lfloor 2.2r + 0.7 \rfloor$.

CHAPTER 3

ON GORENSTEIN RINGS OF CODIMENSION 4 AND REGULARITY 4

3.1 Preliminaries

Assume $\tilde{T} = \mathbb{C}[y_0, \ldots, y_7]$. Let $\mathbb{P}^7 = \text{Proj}(\tilde{T})$. Assume $I_X \subseteq \tilde{T}$ is a reduced irreducible ideal such that $U = \tilde{T}/I_X$ is a arithmetically Cohen-Macaulay ring, $X = \text{Proj}(U)$ is nonsingular and $\dim X = 3$.

Recall from §1.1.2 that X is a Calabi-Yau threefold if $\Omega^3(X) \simeq \mathcal{O}_X$ and $h^{0,i}(X) = 0$ for all $i > 0$. Also recall from §1.4 that $\omega_U = \text{Ext}^4_{\tilde{T}}(U, \tilde{T}(-8))$, and that U is arithmetically Gorenstein if $\omega_U \simeq U(a)$ for some $a \in \mathbb{Z}$.

Definition. A *Gorenstein Calabi-Yau variety (GoCY)* is a Calabi-Yau variety whose homogeneous coordinate ring is arithmetically Gorenstein.

Remark 9. If X is arithmetically Cohen-Macaulay, then $\Omega^{\dim X}(X) \simeq \mathcal{O}_X$ implies $\omega_U \simeq U$ by [20, III Corollary 7.12]. Hence, for Calabi-Yau threefolds, arithmetically Cohen-Macaulay and arithmetically Gorenstein are equivalent conditions.

In general, when U is Gorenstein, we have

$$\omega_U \simeq U(-8 + \text{reg } (U) + \text{codim } (U)). \tag{3.1}$$

In particular, if $X = \text{Proj}(U)$ is GoCY threefold, then U is Gorenstein and $\omega_U \simeq U$. Therefore,

$$\omega_U \simeq U \iff -4 + \text{reg } (U) = 0 \iff \text{reg } (U) = 4. \tag{3.2}$$

For U Gorenstein, we may quotient by a regular sequence of linear forms, reducing to an Artinian Gorenstein ring A with the same homological behavior. We call A an *Artinian* *reduction* of U. Since we are interested in the homological behavior of U in this chapter, we focus on its Artinian reduction and study Artinian Gorenstein rings. By choosing coordinates, we may assume $A \simeq T/I$ for $T = \mathbb{C}[y_0, \ldots, y_3]$.

```
CGKK 1
       1 9 16 9 1
    0: 1 .  . . .
    1: . 6  8 3 .
    2: . .  . . .
    3: . 3  8 6 .
    4: . .  . . 1

CGKK 2
       1 6 10 6 1
    0: 1 .  . . .
    1: . 5  5 . .
    2: . 1  . 1 .
    3: . .  5 5 .
    4: . .  . . 1

CGKK 3
       1 4 6 4 1
    0: 1 . . . .
    1: . 4 . . .
    2: . . 6 . .
    3: . . . 4 .
    4: . . . . 1

CGKK 4
       1 7 12 7 1
    0: 1 .  . . .
    1: . 3  . . .
    2: . 4 12 4 .
    3: . .  . 3 .
    4: . .  . . 1

CGKK 5,6
       1 9 16 9 1
    0: 1 . .  . .
    1: . 3 2  . .
    2: . 6 12 6 .
    3: . . 2  3 .
    4: . . .  . 1

CGKK 7,8
       1 10 18 10 1
    0: 1  .  .  . .
    1: .  2  .  . .
    2: .  8 18  8 .
    3: .  .  .  2 .
    4: .  .  .  . 1

CGKK 9,10
       1 13 24 13 1
    0: 1  .  .  . .
    1: .  1  .  . .
    2: . 12 24 12 .
    3: .  .  .  1 .
    4: .  .  .  . 1

CGKK 11
       1 16 30 16 1
    0: 1  .  .  . .
    1: .  .  .  . .
    2: . 16 30 16 .
    3: .  .  .  . .
    4: .  .  .  . 1
```

Table 3.1: Betti diagrams for GoCY's in Table 1.1

Our main results in this chapter are

Theorem 3.1. *An Artinian Gorenstein ring* $A = T/I$ *with* $\operatorname{reg}(A) = 4 = \operatorname{codim}(A)$ *and* I *nondegenerate has one of the 16 Betti diagrams as in Table 3.1 and 3.2. Table 3.1 corresponds to the 11 classes of GoCY in Table 1.1, and Table 3.2 to the remaining classes.*

```
Type 2.1            1 11 20 11 1
                 0: 1  .  .  .  .
                 1: .  2  1  .  .
                 2: .  9 18  9  .
                 3: .  .  1  2  .
                 4: .  .  .  .  1
```

```
Type 2.5            1 11 20 11 1
                 0: 1  .  .  .  .
                 1: .  3  3  1  .
                 2: .  7 14  7  .
                 3: .  1  3  3  .
                 4: .  .  .  .  1
```

```
Type 2.2            1 8 14 8 1
                 0: 1 .  . . .
                 1: . 3  1 . .
                 2: . 5 12 5 .
                 3: . .  1 3 .
                 4: . .  . . 1
```

```
Type 2.6            1 9 16 9 1
                 0: 1 .  . . .
                 1: . 4  4 1 .
                 2: . 4  8 4 .
                 3: . 1  4 4 .
                 4: . .  . . 1
```

```
Type 2.3            1 7 12 7 1
                 0: 1 .  . . .
                 1: . 4  3 . .
                 2: . 3  6 3 .
                 3: . .  3 4 .
                 4: . .  . . 1
```

```
Type 2.7            1 7 12 7 1
                 0: 1 .  . . .
                 1: . 5  5 1 .
                 2: . 1  2 1 .
                 3: . 1  5 5 .
                 4: . .  . . 1
```

```
Type 2.4            1 6 10 6 1
                 0: 1 .  . . .
                 1: . 4  2 . .
                 2: . 2  6 2 .
                 3: . .  2 4 .
                 4: . .  . . 1
```

```
Type 2.8            1 9 16 9 1
                 0: 1 .  . . .
                 1: . 5  6 2 .
                 2: . 2  4 2 .
                 3: . 2  6 5 .
                 4: . .  . . 1
```

Table 3.2: Betti diagrams for the remaining 8 Artinian Gorenstein rings.

and

Theorem 3.2. *There does not exist a smooth irreducible GoCY $X \subseteq \mathbb{P}^7$ such that the Betti diagram of $\tilde{T}/I_X$ is one in Table 3.2.*

3.2 Organization of the chapter

To prove Theorem 3.1, we have to exclude all impossible cases and find an example for each case we claim existing. We prove it in §3.3: Note that in Theorem 3.1, we consider the Artinian reduction $A = T/I$ of $U = \tilde{T}/I_X$, with $T = \mathbb{C}[y_0, \ldots, y_3]$. First, we decide the range for Hilbert function of A by the range of $\deg(X)$. Second, for a fixed Hilbert function, the graded Betti number $b_{ij}(A)$ is bounded up by those of the monomial ideals. Therefore, there are only finite number of possible Betti diagrams. We analyze them case-by-case.

Theorem 3.2 is proved in §3.4. We apply Schenck-Stillman's Theorem (Theorem 3.6) to prove that a smooth irreducible 3-fold X does not have Betti diagram of either Type 2.1-2.3, or Type 2.5-2.7. To exclude Type 2.8, we have to use Buchsbaum-Rim resolution, which is introduced in §1.3.2. In §3.4.5 we apply results of [38] to prove a structure theorem for any irreducible nondegenerate threefold in $\mathbb{P}^7$ with Betti diagram of Type 2.4, and show the resulting variety cannot be smooth.

In Appendix B, we present explicit examples for each of the 16 Betti diagrams.

3.3 Proof of Theorem 3.1

For T/I Artinian Gorenstein of regularity 4, the h-vector of T/I is

$$\mathbf{h}(T/I) = \Big(1, 4, h_2, 4, 1\Big), \text{ with } h_2 \leq 10. \tag{3.3}$$

By [11, Lemma 2.1], a GoCY $X \subseteq \mathbb{P}^7$ has $14 \leq \deg(X) \leq 20$. If $A = T/I$ is the Artinian reduction of $U = \tilde{T}/I_X$, then

$$\deg(X) = \sum_d h_d(T/I) = 10 + h_2. \tag{3.4}$$

Hence $4 \leq h_2 \leq 10$.

For the proof of Theorem 3.1, we will need the theorems of Macaulay and Gotzmann [27]: For a graded algebra T/I with Hilbert function h_i, write

$$h_i = \binom{a_i}{i} + \binom{a_{i-1}}{i-1} + \cdots \tag{3.5}$$

and

$$h_i^{\langle i \rangle} = \binom{a_i + 1}{i+1} + \binom{a_{i-1}+1}{i} + \cdots, \text{ with } a_i > a_{i-1} > \cdots \tag{3.6}$$

Theorem 3.3 (Macaulay's Theorem)**.**

$$h_{i+1} \leq h_i^{\langle i \rangle}. \tag{3.7}$$

Theorem 3.4 (Gotzmann's Persistence Theorem)**.** *If I is generated in a single degree t and equality holds in Macaulay's formula in the first degree t, then*

$$h_{t+j} = \binom{a_t + j}{t+j} + \binom{a_{t-1} + j - 1}{t+j-1} + \cdots \tag{3.8}$$

We also need the following lemma to preclude some Betti diagrams.

Lemma 3.5. *Let I_2 be the subideal of I generated by the quadrics in I, and let $v = (b_{23}, b_{24})$, where $b_{i,j} = b_{i,j}(T/I)$ are graded Betti numbers of T/I. Then*

(a) $b_{45}(T/I_2) = b_{46}(T/I_2) = 0$.

(b) $v \neq (2,1)$.

(c) if $a = b_{12} \geq 4$, *then* $v \neq (3,1)$.

Proof. We prove (a) first: Because $b_{i,i+1}(T/I_2) = b_{i,i+1}(T/I)$ for all $i \geq 1$, so $b_{45}(T/I_2) = 0$. To prove $b_{46}(T/I_2) = 0$, note that $b_{46}(T/I) = 0$ and that adding additional generators to I_2 cannot force cancellation: for a cubic F, we have the short exact sequence

$$0 \longrightarrow T(-3)/I_2 : F \longrightarrow T/I_2 \longrightarrow T/I_2 + F \longrightarrow 0 \tag{3.9}$$

and the associated long exact sequence gives exact sequence of vector spaces

$$0 \to \mathrm{Tor}_4(T(-3)/I_2 : F, \mathbb{C})_6 \to \mathrm{Tor}_4(T/I_2, \mathbb{C})_6 \to \mathrm{Tor}_4(T/I_2 + F, \mathbb{C})_6. \tag{3.10}$$

Note that

$$\mathrm{Tor}_4(T(-3)/I_2 : F, \mathbb{C})_6 = \mathrm{Tor}_4(T/I_2 : F, \mathbb{C})_3 = 0. \tag{3.11}$$

Hence $\mathrm{Tor}_4(T/I_2, \mathbb{C})_6 \neq 0$ implies $\mathrm{Tor}_4(T/I_2+F, \mathbb{C})_6 \neq 0$. Therefore, we conclude $b_{46}(T/I_2) = 0$.

Next we prove (b): To see that $v = (2, 1)$ cannot occur, observe that if it did then there would be a unique relation $L_1 \cdot V_1 + L_2 \cdot V_2 = 0$ where L_1, L_2 are linear forms, and V_i are vectors of linear first syzygies. Changing variables so $L_1 = y_1$ and $L_2 = y_2$, we have that $y_1 \cdot V_{i1} + y_2 \cdot V_{i2} = 0$ for all i, implying V_1 is $y_2 \cdot C$ and V_2 is $-y_1 \cdot C$, with C a vector of constants, a contradiction. So $v = (2, 1)$ is impossible.

To prove part (c), the key point is that $v = (3, 1)$ implies that I_2 contains $\{Ly_1, Ly_2, Ly_3\}$ with L a linear form. If $v = (3, 1)$ then the unique linear second syzygy S must have rank 3, otherwise the argument showing that $v = (2, 1)$ is impossible applies. After change of variables, we may write S as below, with a_i, b_i, c_i linear forms:

$$\begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \\ a_4 & b_4 & c_4 \end{bmatrix} \cdot \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = 0 \tag{3.12}$$

So the rows of the matrix of linear first syzygies on I_2 are Koszul syzygies on $[y_1, y_2, y_3]^t$, that is to say

$$\begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \\ a_4 & b_4 & c_4 \end{bmatrix} = C \begin{bmatrix} y_2 & -y_1 & 0 \\ -y_3 & 0 & y_1 \\ 0 & y_3 & -y_2 \end{bmatrix} \tag{3.13}$$

where C is a full rank 4×3 scalar matrix. This forces I_2 to contain $\{Ly_1, Ly_2, Ly_3\}$.

When $a \geq 4$ the mapping cone construction implies I_2 is *inconsistent with the Gorenstein* *hypothesis (IGH)*. If $a \geq 4$, I_2 must contain a quadric Q which is a nonzero divisor on $\{Ly_1, Ly_2, Ly_3\}$. To see this, note that if $Q \in \langle L \rangle$ then $\mathrm{ht}(I_2) = 1$. After a change of variables I_2 consists of a linear form times a subset of the variables, so that I_2 has a Koszul resolution, hence $b_{45}(T/I_2) \neq 0$ which is IGH by (a); if $Q \in \langle y_1, y_2, y_3 \rangle$ then there is at least one additional linear first syzygy, so $b \geq 4$. Now we know Q must be a non-zero divisor on $\{Ly_1, Ly_2, Ly_3\}$. This implies that if $v = (3, 1)$, then I_2 has mapping cone Betti diagram

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 4 & 3 & 1 & 0 \\ 0 & 0 & 3 & 3 & 1 \end{bmatrix}.$$

This is IGH by (a), because $b_{46}(T/I_2) \neq 0$. Therefore we conclude $v = (3, 1)$ is IGH. □

Remark 10. When $a = 3$, $v = (3, 1)$ occurs.

We use the Hilbert function to establish the possible shape of the Betti diagram, combined with an analysis of the structure of the subideal I_2 generated by the quadrics in I and subideal C_3 generated by the quadrics and cubics in I. Let $a = b_{12}(T/I)$ be the number of quadratic generators of $I \subseteq T = \mathbb{C}[y_0, \ldots, y_3]$, and let $v = (b_{23}(T/I), b_{34}(T/I)) = (b, c)$. Note that $b_{45}(T/I_2) \neq 0$ cannot occur by Lemma 3.5.

For an Artinian Gorenstein ideal I with $\mathrm{ht}(I) = 4 = \mathrm{reg}\,(T/I)$, its Hilbert series is determined by b_{12}, and $(b_{12}, b_{23}, b_{34}) = (a, b, c)$ determines the entire Betti diagram. When $a \in \{0, 1, 2\}$ the analysis is straightforward, so we begin with $a = 3$.

3.3.1 Case $a = 3$

When $a = 3$, the Hilbert function is $(1, 4, 7, 4, 1)$ and a computation shows the Betti diagram must be (dropping the 1 in upper left and lower right corners)

$$\begin{bmatrix} 3 & b & c \\ b+4 & 2c+12 & b+4 \\ c & b & 3 \end{bmatrix}.$$

By Macaulay's theorem

$$h_2(T/I_2) = 7 = \binom{4}{2} + \binom{1}{1}, \text{ so } h_2^{\langle 2\rangle}(T/I_2) = 11 \geq h_3(T/I_2) = 20 - 3\cdot 4 + b, \text{ so } b \leq 3.$$

A direct computation shows that for an ideal generated by three quadratic monomials in T, $v \in \{(0,0),(1,0),(2,0),(3,1)\}$, all of which occur in Tables 3.1 and 3.2. By uppersemicontinuity, I_2 must have $v = (b,c) \leq (b',c')$ for (b',c') in the list above, so we need only show that $v \in \{(3,0),(2,1)\}$ do not occur. If $b = 3$ then we are in the situation where Gotzmann's theorem applies, that is, $h_2^{\langle 2\rangle} = h_3$ implies $h_3^{\langle 3\rangle} = h_4$, and we compute

$$h_3^{\langle 3\rangle}(T/I_2) = 16 \tag{3.14}$$

and

$$h_4(T/I_2) = 35 - 3\cdot 10 + 3\cdot 4 - c + b_{24}(T/I_2). \tag{3.15}$$

In particular, $c = 1 + b_{24}(T/I_2) \geq 1$, so $c \geq 1$ and $v = (3,0)$ does not occur. By Lemma 3.5, $v = (2,1)$ is impossible.

When $a \geq 4$, the set of Betti diagrams possible for quadratic monomial ideals has an element that is so large that a similar analysis via the initial ideal becomes cumbersome.

3.3.2 Case $a = 4$

When $a = 4$, the Hilbert function is $(1,4,6,4,1)$ and the Betti diagram is:

$$\begin{bmatrix} 4 & b & c \\ b & 2c+6 & b \\ c & b & 4 \end{bmatrix}.$$

Values for v which actually occur are $v \in \{(0,0),(2,0),(3,0),(4,1)\}$. Applying Macaulay's theorem to the ideal I_2 generated by the quadrics in I shows $b \leq 6$. Now let C_3 denote the ideal generated by the quadrics and cubics in I.

$$h_3(T/C_3) = h_3(T/I) = 4 = \binom{4}{3} \tag{3.16}$$

so

$$h_3^{\langle 3 \rangle}(T/C_3) = 5 \geq h_4(T/C_3) = c + 1. \tag{3.17}$$

Hence $c \leq 4$.

The case $b = 6$ is extremal, and applying Gotzmann's theorem we find

$$h_4(T/I_2) = 35 - 4 \cdot 10 + 6 \cdot 4 + b_{24}(T/I_2) - c = 15, \tag{3.18}$$

so

$$c = 4 + b_{24}(T/I_2). \tag{3.19}$$

Combined with our work above, this shows $b = 6 \implies c = 4$. As $h_4(T/C_3) = h_4(T/I) + c = 5$, we have

$$h_4^{\langle 4 \rangle}(T/C_3) = 6 \geq h_5(T/C_3) = 56 - 80 + 40 + b_{25}(T/C_3) - 6, \tag{3.20}$$

we conclude $b_{25}(T/C_3) \leq -4$, which is impossible. Thus, $b \in \{0, \ldots, 5\}$.

If $b \in \{0,1\}$ then $c = 0$; clearly $v = (0,0)$ yields a complete intersection, which occurs, while $v = (1,0)$ leads to an almost complete intersection (ACI), and by [22] there are no Gorenstein ACI's. Henceforth we assume $b \in \{2,3,4,5\}$. We saw above that $c \leq 4$; we now show that $c \in \{2,3,4\}$ is IGH.

$$h_5(T/C_3) = 56 - 80 + 4(c+6) + b_{25}(T/C_3) - b. \tag{3.21}$$

So

$$\begin{aligned}
c = 2 &\implies h_4(T/C_3) = 3 \implies h_4^{\langle 4 \rangle}(T/C_3) = 3 \;\geq\; h_5(T/C_3) = 8 + b_{25}(T/C_3) - b \\
c = 3 &\implies h_4(T/C_3) = 4 \implies h_4^{\langle 4 \rangle}(T/C_3) = 4 \;\geq\; h_5(T/C_3) = 12 + b_{25}(T/C_3) - b \\
c = 4 &\implies h_4(T/C_3) = 5 \implies h_4^{\langle 4 \rangle}(T/C_3) = 6 \;\geq\; h_5(T/C_3) = 16 + b_{25}(T/C_3) - b
\end{aligned}$$

As $b \leq 5$, only the case $b = 5, c = 2, b_{25}(T/C_3) = 0$ is possible; this has Betti diagram

$$\begin{bmatrix} 4 & 5 & 2 \\ 5 & 10 & 5 \\ 2 & 5 & 4 \end{bmatrix}.$$

Computing, we find that in this situation $h_5(T/C_3) = 3$, so

$$h_5^{\langle 5 \rangle}(T/C_3) = 3 \geq h_6(T/C_3) = 4 + b_{26}(T/C_3) - b_{36}(T/C_3). \tag{3.22}$$

In particular, $b_{36}(T/C_3) \geq 1 + b_{26}(T/C_3) \geq 1$, which means the 5×4 submatrix M of d_3 representing the "bottom right corner" of Betti diagram for T/I, one of the four columns of M is zero. By symmetry of the free resolution this means that one of the four rows of the matrix M^t of linear first syzygies on I_2 is zero. Hence the five linear first syzygies on I_2 only involve a subideal $Q \subseteq I_2$ generated by 3 quadrics, which is impossible.

It remains to deal with $c \in \{0, 1\}$. When $c = 0$, we know $v \in \{(0,0), (2,0), (3,0)\}$ occur, and we have already shown that $v = (1,0)$ is IGH. As $b \leq 5$, we need to show $v \in \{(4,0), (5,0)\}$ are IGH. To do this, we use the ideal I_2 of four quadrics; $h_3(T/I_2) = 20 - 16 + b = 4 + b$, so we have

- For $b = 4$, $h_3(T/I_2) = 8$. Hence

$$h_3^{\langle 3 \rangle}(T/I_2) = 10 \geq h_4(T/I_2) = 35 - 40 + 16 + b_{24}(T/I_2) = 11 + b_{24}(T/I_2). \tag{3.23}$$

- For $b = 5$, $h_3(T/I_2) = 9$. Hence

$$h_3^{\langle 3 \rangle}(T/I_2) = 12 \geq h_4(T/I_2) = 35 - 40 + 20 + b_{24}(T/I_2) = 15 + b_{24}(T/I_2), \tag{3.24}$$

Both force $b_{24}(I_2) \leq -1$, which is impossible. When $c = 1$, the only change to the second equation above is to subtract one (because $c = 1$) from the right hand side, so $h_4(I_2) = 14 + b_{24}(I_2)$, forcing $b_{24} \leq -2$, which is impossible.

3.3.3 Case $a = 5$

When $a = 5$, the Hilbert function is $(1, 4, 5, 4, 1)$, so the Betti diagram is

$$\begin{bmatrix} 5 & b & c \\ b-4 & 2c & b-4 \\ c & b & 5 \end{bmatrix}.$$

Note that $h_3(T/I_2) = 20 - 5 \cdot 4 + b = b$. By Macaulay's theorem

$$h_2(T/I_2) = 5 = \binom{3}{2} + \binom{2}{1}, \text{ so } h_2^{\langle 2 \rangle}(T/I_2) = 7 \geq h_3(T/I_2) = b, \text{ so } 7 \geq b.$$

1. Case 1: Suppose $b = 4$. This means there are no cubics in the ideal, and

$$h_3(T/I_2) = 4 = \binom{4}{3}. \tag{3.25}$$

Theorem 3.3 shows

$$h_3^{\langle 3 \rangle}(T/I_2) = 5 \geq h_4(T/I_2) = 35 - 5 \cdot 10 + 4 \cdot 4 + c. \tag{3.26}$$

We conclude $c \leq 4$. We can immediately rule out $c = 0$, as then I would be an ACI, which is IGH. The possibilities $c \in \{2, 3, 4\}$ are also ruled out by Macaulay; we illustrate for $c = 2$:

$$h_4(T/I_2) = 35 - 5 \cdot 10 + 4 \cdot 4 + 2 = 3, \tag{3.27}$$

so

$$h_4^{\langle 4 \rangle}(T/I_2) = 3 \geq h_5(T/I_2) = 4 + b_{25}(T/I_2), \tag{3.28}$$

which would force $b_{25}(T/I_2) \leq -1$.

Finally, suppose $c = 1$, so $I = I_2 + g$ for a single quartic g. Since $I_2 + g$ has height four, the height of I_2 must be three or four, and if $\mathrm{ht}(I_2) = 4$ then I_2 is an almost complete intersection, containing a complete intersection C. We claim this is impossible: write

$I_2 = C + f$ with $f \in I_2 \setminus C$. Since $b_{23}(T/C) = 0$ the fact that $b_{23}(T/I_2) = 4$ means that $C : f = \langle y_0, y_1, y_2, y_3 \rangle$, whose mapping cone is inconsistent with the Betti diagram for I_2. Hence $\mathrm{ht}(I_2) = 3$, and g is a nonzero divisor on the height three associated primes of I_2. Since $h_4(T/I_2) = 2$, Macaulay's theorem implies the degree of I_2 is one or two. Observe that the rank of the linear second syzygy Syz_2 cannot be 4; if it was then $\mathrm{Syz}_2 = [y_0, y_1, y_2, y_3]^t$. By the symmetry of the differentials in the free resolution, this means that $I_2 : g = \langle y_0, \ldots, y_3 \rangle$. By additivity of the Hilbert polynomials on the short exact sequence

$$0 \longrightarrow T(-4)/(I_2 : q) \longrightarrow T/I_2 \longrightarrow T/I \longrightarrow 0,$$

this is impossible. Hence rank $(\mathrm{Syz}_2) = 3$, and as in the proof that $v = (3, 1)$ is impossible for $a = 4$, I_2 is generated by, after a change of variables, $\{L \cdot y_1, L \cdot y_2, L \cdot y_3, q_4, q_5\}$ for a linear form L and two quadrics q_4 and q_5. Since $\mathrm{ht}(I_2) = 3$, this forces (L, q_4, q_5) to be a regular sequence. In particular, $\deg(I_2) = 4$, a contradiction.

2. Case 2: Suppose $b = 5$. The cases $v \in \{(5,0), (5,1)\}$ do occur.

$$h_3(T/I_2) = 5 = \binom{4}{3} + \binom{2}{2},$$

Macaulay's theorem shows

$$h_3^{\langle 3 \rangle}(T/I_2) = 6 \geq h_4(T/I_2) = 35 - 5 \cdot 10 + 4 \cdot 5 + b_{24}(T/I_2) - c. \tag{3.29}$$

So $c + 1 \geq b_{24}(T/I_2)$. Let C_3 denote the subideal of I generated in degrees two and three.

$$h_3(T/C_3) = 4, \tag{3.30}$$

so

$$h_3^{\langle 3 \rangle}(T/C_3) = 5, \tag{3.31}$$

thus

$$5 \geq h_4(T/C_3) = 35 - 50 + 16 + c, \tag{3.32}$$

implying $c \leq 4$. Since $c \in \{0, 1\}$ does occur, we need to rule out $c \in \{2, 3, 4\}$. Computing values for $h_4(T/C_3)$, we find

$$\begin{array}{lll} c = 2 \text{ implies} & h_4(T/C_3) = 3 & \text{hence } h_5(T/C_3) \leq 3 \\ c = 3 \text{ implies} & h_4(T/C_3) = 4 & \text{hence } h_5(T/C_3) \leq 4 \\ c = 4 \text{ implies} & h_4(T/C_3) = 5 & \text{hence } h_5(T/C_3) \leq 6 \end{array}$$

Since $h_5(T/C_3) = 56 - 100 + 40 + 4c - 1 + b_{25}(T/C_3)$, combining this with the above shows

$$\begin{array}{lll} c = 2 \text{ implies} & h_5 = 3 + b_{25}(T/C_3) & \leq 3 \\ c = 3 \text{ implies} & h_5 = 7 + b_{25}(T/C_3) & \leq 3 \\ c = 4 \text{ implies} & h_5 = 11 + b_{25}(T/C_3) & \leq 6 \end{array}$$

This rules out $c \in \{3, 4\}$, and shows if $c = 2$ then $b_{25}(T/C_3) = 0$. So in this case $h_5(T/C_3) = 3$, and

$$h_5^{\langle 5 \rangle}(T/C_3) = 3 \geq h_6(T/C_3) = 5 + b_{26}(T/C_3) - b_{36}(T/C_3). \tag{3.33}$$

Hence $b_{36}(T/C_3) \geq 2$, so the Betti diagram for T/C_3 is at least

$$\begin{bmatrix} 5 & 5 & 2 \\ 1 & 4 & 1 \\ 0 & 0 & 2 \end{bmatrix}.$$

Hence in the 5×5 submatrix M of d_3 representing the "bottom right corner" of the table for I, two of the five columns of M are zero, which by symmetry of the Betti diagram means that two of the five rows of the matrix M^t of linear first syzygies on I_2 are zero. Hence the five linear first syzygies on I_2 only involve a subideal $J \subseteq I_2$ generated by 3 quadrics, which is impossible.

3. Case 3: Suppose $b = 6$; the only case that actually occurs is $v = (6, 2)$.

$$h_3(T/I_2) = 6 = \binom{4}{3} + \binom{2}{2} + \binom{1}{1},$$

Macaulay's theorem shows

$$h_3^{\langle 3\rangle}(T/I_2) = 7 \geq h_4(T/I_2) = 35 - 5\cdot 10 + 4\cdot 6 + b_{24}(T/I_2) - c. \tag{3.34}$$

So $c \geq b_{24}(T/I_2) + 2$. Let C_3 denote the subideal of I generated in degrees two and three.

$$h_3(T/C_3) = 4 = \binom{4}{3} \tag{3.35}$$

so

$$5 \geq h_4 = 35 - 50 + 16 + c \tag{3.36}$$

Thus, $c \leq 4$. To show that $c \in \{3, 4\}$ do not occur, we compute

$$\text{If } c = 4, \text{ then } \quad h_4(T/C_3) = 5 \text{ and } \quad h_5(T/C_3) \leq 6$$
$$\text{If } c = 3, \text{ then } \quad h_4(T/C_3) = 4 \text{ and } \quad h_5(T/C_3) \leq 4$$

Since $h_5(T/C_3) = 56 - 100 + 40 + 4c + b_{25}(T/C_3) - 2$, we see that

$$\text{If } c = 4 \quad \text{then } h_5(T/C_3) = 10 + b_{25}(T/C_3) \quad \leq 6, \text{ so } b_{25}(T/C_3) \leq -4$$
$$\text{If } c = 3 \quad \text{then } h_5(T/C_3) = 6 + b_{25}(T/C_3) \quad \leq 4, \text{ so } b_{25}(T/C_3) \leq -2$$

We have shown that when $b = 6$, the only value possible for v is $(6, 2)$.

4. If $b = 7$, applying Gotzmann's theorem gives $c = b_{24}(T/I_2) + 4$. Let C_3 denote the subideal of I generated in degrees two and three; applying Macaulay's theorem to $h_3(T/C_3) = 4$ yields

$$5 \geq h_4(T/C_3) = 35 - 5\cdot 10 + 4\cdot 4 + c, \tag{3.37}$$

so $c \leq 4$; combined with $c = b_{24}(T/I_2) + 4$ this forces $c = 4$. Since $h_4^{\langle 4\rangle}(T/C_3) = 6$, we find

$$6 \geq h_5(T/C_3) = 9 + b_{25}(T/C_3). \tag{3.38}$$

This shows $b_{25}(T/C_3) \leq -3$, hence $b = 7$ is IGH.

3.3.4 Case $a = 6$

When $a = 6$, the Hilbert function is $(1, 4, 4, 4, 1)$ so the Betti diagram is

$$\begin{bmatrix} 6 & b & c \\ b-8 & 2c-6 & b-8 \\ c & b & 6 \end{bmatrix}.$$

As

$$h_2(T/I_2) = 4 = \binom{3}{2} + \binom{1}{1}. \tag{3.39}$$

Theorem 3.3 shows

$$h_2^{\langle 2 \rangle}(T/I_2) = 5 \geq h_3(T/I_2) = 20 - 6 \cdot 4 + b. \tag{3.40}$$

So $b \leq 9$. If $b = 9$ there is a unique cubic $F \in I$; since $b = 9$ is extremal we may apply Gotzmann's Persistence Theorem to conclude that $h_4(T/I_2) = 6$, so

$$(b_{24}(T/I_2) - c) + \binom{3+1}{1} \cdot 9 - \binom{3+2}{2} \cdot 6 + \binom{3+4}{4} = 6, \tag{3.41}$$

which implies $b_{24}(T/I_2) = c - 5$. Since $(2c-6) - (c-5) = c - 1$ and $c \geq 5$, this means there are always at least four independent syzygies which are linear on F and quadratic on elements of I_2. Hence $I_2 : F = \langle y_0, \dots y_3 \rangle$ and the mapping cone arising from short exact sequence

$$0 \longrightarrow T(-3)/I_2 : F \longrightarrow T/I_2 \longrightarrow T/I_2 + F \longrightarrow 0, \tag{3.42}$$

gives a resolution of T/I. The top row of the mapping cone is simply the Koszul complex on the variables, and a check of the degrees shows the second syzygies involve a summand $T^6(-5)$ which cannot cancel. This would imply $b_{35}(T/I) = b - 8 \geq 6$, which is impossible since $b = 9$.

Finally, we need to show that when $b = 8$ we must have $c = 3$. From the Hilbert function constraint on the Betti diagram, $c \geq 3$. When $b = 8$, there are no cubics in I; this means

$$b_{24}(T/I_2) - c = c - 6. \tag{3.43}$$

We compute

$$h_3(T/I_2) = 4 = \binom{4}{3}, \tag{3.44}$$

Theorem 3.3 shows

$$h_3^{\langle 3 \rangle}(T/I_2) = 5 \geq h_4(T/I_2) = 35 - 6 \cdot 10 + 8 \cdot 4 + c - 6, \tag{3.45}$$

hence $c \leq 4$. Finally, if $c = 4$, then $h_4(T/I_2) = 5$ and $h_4^{\langle 4 \rangle}(T/I_2) = 6$. So

$$6 \geq h_5(T/I_2) = 56 - 6 \cdot 20 + 8 \cdot 10 - 2 \cdot 4 + b_{25}(T/I_2).$$

This would force $b_{25}(T/I_2) \leq -2$. We have shown that the only Betti diagram possible for $a = 6$ is

$$\begin{bmatrix} 6 & 8 & 3 \\ 0 & 0 & 0 \\ 3 & 8 & 6 \end{bmatrix}.$$

Hence there are 16 Betti diagrams for an Artinian Gorenstein algebra A with reg $(A) = 4 =$ codim (A). All diagrams in Table 3.1 and Table 3.2 do occur, which can be checked via a `Macaulay2` search. See Appendix B.

3.4 Proof of Theorem 3.2

In this section, we consider $\tilde{T} = \mathbb{C}[y_0, \ldots, y_7]$. We would like to show if $U = \tilde{T}/I_X$ has one of the Betti diagram in Table 3.2, then $X = \text{Proj}(U)$ must be either reducible or singular. In order to do this, we need to introduce Schenck-Stillman's Theorem.

A matrix of linear forms is *1-generic* if no entry can be reduced to zero by (scalar) row or column operations; a linear n-th syzygy is an element f of $\text{Tor}_{n+1}^{\tilde{T}}(U, \mathbb{C})_{n+2}$. The *rank* of f is the dimension of smallest vector space V such that the diagram below commutes:

$$\begin{array}{ccc}
\mathrm{Tor}_n^{\tilde{T}}(U,\mathbb{C})_{n+1} \otimes \tilde{T}(-n-1) & \longleftarrow & \mathrm{Tor}_{n+1}^{\tilde{T}}(U,\mathbb{C})_{n+2} \otimes \tilde{T}(-n-2) \\
\uparrow & & \uparrow \\
V \otimes \tilde{T}(-n-1) & \longleftarrow & f \otimes \tilde{T}(-n-2)
\end{array}$$

Theorem 1.7 of [32] shows:

Theorem 3.6 (Schenck-Stillman, 2012)**.** *For a nondegenerate prime ideal P,*

(1) P cannot have a linear n^{th} syzygy of rank $\leq n+1$, or P is not prime.

(2) If P has a linear n^{th} syzygy of rank $n+2$, then P contains the 2×2 minors of a 1-generic $2\times(n+2)$ matrix.

(3) If P has a linear n^{th} syzygy of rank $n+3$, then P contains the 4×4 Pfaffians of a skew-symmetric 1-generic $(n+4)\times(n+4)$ matrix.

3.4.1 Type 2.1 and 2.2

```
Type 2.1          1 11 20 11 1
               0: 1  .  .  .  .
               1: .  2 1   .  .
               2: .  9 18 9   .
               3: .  . 1   2  .
               4: .  . .   .  1
```

```
Type 2.2          1 8 14 8 1
               0: 1 . .  . .
               1: . 3 1  . .
               2: . 5 12 5 .
               3: . . 1  3 .
               4: . . .  . 1
```

By Theorem 3.6, a Betti diagram of Type 2.1 is ruled out by (1), and a Betti diagram of Type 2.2 is ruled out by (2), since the 2×2 minors of a 2×3 matrix have two independent linear syzygies.

```
Type 2.5:
   1 11 20 11 1
0: 1 .   .  .  .
1: . 3   3  1  .
2: . 7   14 7  .
3: . 1   3  3  .
4: . .   .  .  1
```

```
Type 2.6:
   1 9 16 9 1
0: 1 . .  . .
1: . 4 4  1 .
2: . 4 8  4 .
3: . 1 4  4 .
4: . . .  . 1
```

```
Type 2.7:
   1 7 12 7 1
0: 1 .  . . .
1: . 5  5 1 .
2: . 1  2 1 .
3: . 1  5 5 .
4: . .  . . 1
```

3.4.2 Type 2.5-2.7

For the three Betti diagrams having top row of the form $(a, a, 1)$, we argue as follows.
When $a = 3$ (Type 2.5), the linear second syzygy can have rank at most 3, since it involves the 3 first syzygies. Hence by (1), the ideal cannot be prime.
When $a = 4$ (Type 2.6), the linear second syzygy can have rank at most 4, and in this case by (2) it contains the 2×2 minors of a 1-generic 2×4 matrix, which would yield a top row of the Betti diagram with entries $(6, 8, 3)$.
When $a = 5$ (Type 2.7), (3) implies that P contains the Pfaffians, and since there are only five quadrics, the quadratic part of the idea is exactly the Pfaffians, which do not have a linear second syzygy.

3.4.3 Type 2.3

```
Type 2.3          1 7 12 7 1
               0: 1 .  . . .
               1: . 4  3 . .
               2: . 3  6 3 .
               3: . .  3 4 .
               4: . .  . . 1
```

For Type 2.3, we will show that a prime non-degenerate ideal P cannot have top row of the Betti diagram equal to $(4, 3, 0)$. Let I_2 be the subideal of P generated by quadrics in P. By (1) and (3) the first syzygies all have rank three; take a subideal $Q \subseteq I_2$ consisting of three elements, which by (2) is generated by the 2×2 minors of a 2×3 matrix and has Betti diagram

$$\begin{matrix} 1 & - & - \\ - & 3 & 2 \end{matrix}. \qquad (3.46)$$

In particular, Q is Cohen-Macaulay, $\operatorname{ht} Q = 2$ and $\deg Q = 3$. Let F denote the remaining quadric, so $I_2 = Q + \langle F \rangle$. Consider the mapping cone resolution of $\tilde{T}/I_2$ from the short exact sequence

$$0 \longrightarrow \tilde{T}(-2)/Q : F \longrightarrow \tilde{T}/Q \longrightarrow \tilde{T}/I_2 \longrightarrow 0. \qquad (3.47)$$

It follows that $Q : F$ must have a linear generator L, so $LF \in Q$. If Q is prime, then either $L \in Q$ or $F \in Q$, a contradiction.

So suppose Q is not prime, and take a primary decomposition

$$Q = \cap_{i=1}^{m} Q_i, \text{ with } \sqrt{Q_i} = P_i \text{ all height two.} \qquad (3.48)$$

Since Q is height two and Cohen-Macaulay and $\deg(Q) = 3$, we must have $m \leq 3$.

1. Case 1: $m = 3$. Then $Q_i = P_i$ and $Q = \cap_{i=1}^{3} P_i$ with P_i generated by two linear forms.
2. Case 2: $m = 2$. Then $\deg(Q_1) = 1$, $\deg(Q_2) = 2$, so $Q_1 = P_1$ is generated by two linear forms.
3. Case 3: $m = 1$. Then $\sqrt{Q_1} = P_1$, with $\deg(P_1) \in \{1, 2, 3\}$. If $\deg(P_1) = 3$, then $Q = P_1$ is prime, and if $\deg(P_1) = 1$ or 2, P_1 contains a linear form.

In particular, we see that P is degenerate.

3.4.4 Type 2.8

For Type 2.8, there are two second linear syzygies. If either of them have rank less than 6, the we would be in one the cases (1), (2), (3) of Schenck-Stillman's Theorem, all of which

```
Type 2.8         1 9 16 9 1
            0: 1 . .  . .
            1: . 5 6  2 .
            2: . 2 4  2 .
            3: . 2 6  5 .
            4: . . .  . 1
```

are inconsistent with a Betti diagram having top row $(5, 6, 2)$.

Hence, both second syzygies must have rank 6. Let Syz_2 denote the corresponding 6×2 matrix of linear second syzygies; Syz_2 is 1-generic: if not, there is a second syzygy of rank ≤ 5, a contradiction. By [14], since Syz_2 is 1-generic, the 2×2 minors are Cohen-Macaulay with an Eagon-Northcott resolution; in particular $\text{grade}(I_2(\text{Syz}_2)) = 5$. By [15, Theorem A2.10], the Buchsbaum-Rim complex is a resolution for coker (Syz_2^t), because the ideal generated by 2×2 minors of Syz_2 has grade $6 - 2 + 1 = 5$. This means

$$\wedge^3 \tilde{T}(-3)^6 \xrightarrow{\varepsilon} \tilde{T}(-1)^6 \xrightarrow{\text{Syz}_2^t} \tilde{T}^2 \tag{3.49}$$

is exact, where ε is defined by equation (1.22). In particular, Syz_2^t has no linear first syzygies. We conclude that there are no linear second syzygies on coker (Syz_2^t), hence no linear first syzygies on I_X, a contradiction.

3.4.5 Type 2.4

```
Type 2.4          1 6 10 6 1
              0: 1 .  . . .
              1: . 4  2 . .
              2: . 2  6 2 .
              3: . .  2 4 .
              4: . .  . . 1
```

We now show that the Betti diagram of Type 2.4 corresponds to a mapping cone, and that any nondegenerate irreducible GoCY in $\mathbb{P}^7$ with Betti diagram of Type 2.4 must be singular.

A key tool in our analysis is a result of Vasconcelos-Villereal [38, Theorem 1.2], which shows that if T is a Gorenstein local ring and $2 \in T$ is a unit, then if I is a Gorenstein ideal of height 4 and deviation two, such that I is a generic complete intersection (the localization at all minimal primes is a complete intersection), then I is a hypersurface section of a Gorenstein ideal of height 3.

Theorem 3.7 (Vasconcelos-Villereal, 1986)**.** *Let T be a Gorenstein local ring in which 2 is a unit. Let I be a Gorenstein ideal of height four and deviation two. If I is a generic complete intersection, then I is a hypersurface section of a Gorenstein ideal of height three. That is , $I = (I', f)$, where I' is the ideal generated by 4×4 Pfaffians of an alternating 5×5 matrix and f is a regular element on T/I'.*

We start with several preparatory lemmas. Note that a Betti diagram of Type 2.4 cannot arise as the mapping cone of a cubic, so will arise from quotienting the Pfaffians by a quadric.

Lemma 3.8. *There is a prime subideal $Q \subseteq I_2$ generated by three quadrics, such that Q consists of the 2×2 minors of a 1-generic 2×3 matrix $\mathrm{Syz}_1(Q)$, and the quadric $q_4 \in I_2 \setminus Q$ is a nonzero divisor on T/Q.*

Proof. By Theorem 3.6, a linear first syzygy on I_2 of rank four would imply that I_2 contains the Pfaffians of a 5×5 skew matrix of linear forms, while if there was a linear first syzygy on I_2 of rank two, I would not be prime. So Theorem 3.6 implies that I_2 contains a subideal Q of 2×2 minors of a 1-generic 2×3 matrix of linear forms. The ideal Q must be prime, for if not, it would have a primary decomposition into components of degrees one or two, which would force I to be degenerate. Finally, q_4 is regular on Q, for if not, then $\mathrm{ht}(I_2) = 2$ and degree one or two; the two cubics in I must be nonzero divisors on the height two primary component, because $\mathrm{ht}(I) = 4$. But this would imply that $\deg(I)$ is 9 or 18, contradicting the fact that $\deg(I) = 16$. □

In what follows, we use the notation of Lemma 3.8, so Q is the ideal of 2×2 minors of the one-generic matrix $\mathrm{Syz}_1(Q)$. The entries of $\mathrm{Syz}_1(Q)$ are linear forms, because Q is prime the linear forms span a space of dimension $\{4, 5, 6\}$. This means $V(Q)$ is a cone, with singular locus of dimension (respectively) $\{3, 2, 1\}$. Let C be the ideal generated by q_4 and the two cubic generators of I; intersecting $V(Q)$ with $V(C)$ drops the dimension by two, so if the linear forms of $\mathrm{Syz}_1(Q)$ span a space of dimension four or five, $V(I)$ is singular. It remains to deal with the case that the span of the linear forms has dimension six; after a change of variables we may assume

$$\mathrm{Syz}_1(Q) = \begin{bmatrix} y_1 & y_2 & y_3 \\ y_4 & y_5 & y_6 \end{bmatrix} \tag{3.50}$$

Lemma 3.9. *Let I be a height four Gorenstein prime ideal with Betti diagram Type 2.4. If I_2 contains an ideal Q consisting of the 2×2 minors of $\mathrm{Syz}_1(Q)$ as above, then $I = I' + \langle F \rangle$, with $\mathrm{codim}\,(I') = 3$ and I' Gorenstein, and F a nonzero divisor on T/I'. Hence T/I has a mapping cone resolution.*

Proof. Because the two linear first syzygies on I_2 are of the form $[y_1, y_2, y_3]^t$ and $[y_4, y_5, y_6]^t$ and I is nondegenerate, I contains no linear form, so $\{y_1, \ldots, y_6\}$ are all units when T/I is localized at I. Thus, in the localization, two of the generators for Q are redundant, and therefore I is a generic complete intersection, of deviation two, so the result of [38] applies. □

Lemma 3.10. *Assume Y is an arithmetically Gorenstein variety of codimension 3 and X is a nondegenerate hypersurface section of Y with Betti diagram of Type 2.4. Then Y must have Betti diagram:*

```
    1 5 5 1
0:  1 . . .
1:  . 3 2 .
2:  . 2 3 .
3:  . . . 1
```

Proof. The Hilbert series of X is

$$h_t(X) = \frac{1}{(1-t)^n}(1-t^2)^4. \tag{3.51}$$

Assume $h_t(Y) = \frac{1}{(1-t)^n} f(t)$. Then

$$f(t)(1-t^d) = (1-t^2)^4. \tag{3.52}$$

So $d \in \{1, 2\}$. But X does not lie in any hyperplane. Therefore d must be 2 and Y has the desired Betti diagram. □

Proposition 3.11. *Let $V(I_X)$ be GoCY in $\mathbb{P}^7$ with Betti diagram of Type 2.4. If the linear forms of the matrix $\mathrm{Syz}_1(Q)$ span a space of dimension six, then up to a change of basis, I_X is generated by the Pfaffians of a 5×5 skew symmetric matrix $\mathrm{Syz}_1(I')$ as below, along with a quadric q_4 which is a nonzero divisor on $\tilde{T}/I_X$. Denote $\mathrm{Pf}(\mathrm{Syz}_1(I'))$ by I'. The ideal I' is singular along a $\mathbb{P}^1$, and so $V(I_X)$ has at least two singular points.*

$$\mathrm{Syz}_1(I') = \begin{bmatrix} 0 & y_1 & y_2 & y_3 & 0 \\ -y_1 & 0 & q_1 & q_2 & y_4 \\ -y_2 & -q_1 & 0 & q_3 & y_5 \\ -y_3 & -q_2 & -q_3 & 0 & y_6 \\ 0 & -y_4 & -y_5 & -y_6 & 0 \end{bmatrix} \tag{3.53}$$

where the q_j's are quadrics.

Proof. Combining Lemmas 3.8, 3.9, and 3.10 and the results of [38] shows that I_X is of the form above. To see that the singular locus is as claimed, we compute that

$$I' = Q + \langle y_3q_1 - y_2q_2 + y_1q_3, y_6q_1 - y_5q_2 + y_4q_3\rangle, \tag{3.54}$$

where Q is the ideal of the minors of the matrix $\mathrm{Syz}_1(Q)$ above. In particular,

$$V(y_1, \dots y_6) \simeq \mathbb{P}^1 \subseteq V(I'), \tag{3.55}$$

and $V(I')$ is singular along this $\mathbb{P}^1$, because the Jacobian matrix of I' is

$$\operatorname{Jac}(I') = \begin{bmatrix} y_5 & y_6 & 0 & * & * \\ -y_4 & 0 & y_6 & * & * \\ 0 & -y_4 & -y_5 & * & * \\ -y_2 & -y_3 & 0 & * & * \\ y_1 & 0 & -y_3 & * & * \\ 0 & y_1 & y_2 & * & * \\ 0 & 0 & 0 & * & * \\ 0 & 0 & 0 & * & * \end{bmatrix} \tag{3.56}$$

where $*$ are quadrics. Hence when $\{y_1, \ldots, y_6\}$ vanish, $\operatorname{Jac}(I')$ has rank ≤ 2, so is singular along the $\mathbb{P}^1$. Intersecting $V(I')$ with the hypersurface $V(q_4)$, we find that $V(I_X)$ must be singular (at least) at a degree two zero scheme. $\square$

CHAPTER 4

SUMMARY AND FUTURE RESEARCH

In this thesis, we studied the dimension problem on spline spaces and the Betti diagrams of Artinian Gorenstein rings.

For the dimension problem on spline spaces, we proved there is a counter-example to the $2r+1$ conjecture by analyzing that example with Billera's spline complex. We also found a new bound for reg $H_1(S_\bullet/J_\bullet)$.

For Artinian Gorenstein rings, we find all possible Betti diagrams corresponded to Artinian Gorenstein rings with regularity 4 and codimension 4. We proved what we found is a complete list of such Betti diagrams and those in Table 3.2 cannot be Betti diagrams of Gorenstein Calabi-Yau threefolds in $\mathbb{P}^7$. A case-by-case analysis of 2-linear strand for each Betti diagram is crucial to our proof.

The study on both problems does not end with this thesis. In fact, there are several topics interest us. We discuss these directions for future research in Section 4.1-4.5.

4.1 On dimension conjectures of spline spaces

Before Schenck and Stiller made the "$2r+1$" conjecture, Alfeld and Manni have conjectured for case $(r,d)=(1,3)$:

Conjecture 3 (Alfeld-Manni). *The equality (1.5) holds for all triangulation when* $(r,d)=(1,3)$.

Note that Δ_Y in Chapter 2 does not make a counter-example for Alfeld-Manni conjecture, because $H_0(J_\bullet)_{d=3}=0$ for $r=1$.

If we fix the combinatorial data of Δ, then global geometric data of Δ is determined by the actual positions of all $v \in \Delta_0$. The space of positions of vertices is a Zariski open subset $\mathfrak{U}$ of $\mathbb{R}^n \times \cdots \times \mathbb{R}^n$. We have already seen that reg $H_0(J_\bullet)$ depends on the global geometric data. In fact, reg $H_0(J_\bullet)$ remains constant over a Zariski open subset of $\mathfrak{U}$. We say Δ is *generic* if it falls into this open dense subset. Intuitively, this means that jiggling the position of any vertex of Δ does not change reg $H_0(J_\bullet)$. In particular, our counter-example is not generic.

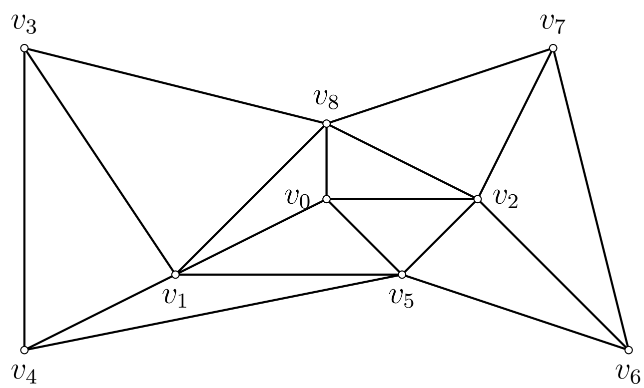

Figure 4.1: A generic Δ having the same combinatorial data as the counter-example

Example 2. Figure 4.1 is an example of generic Δ with the same combinatorial data as in the counter-example.

Alfeld and Schumaker proved in [2] that the "$3r+1$" conjecture holds for generic Δ.

Conjecture 4. *For generic Δ, the "$2r+1$" Conjecture holds.*

From the computation results of Macaulay 2, we also notice that in our case, for r from 1 to 20, reg $H_0(J_\bullet) = \lfloor \frac{9r+2}{4} \rfloor$.

Conjecture 5 (Schenck-Yuan)**.** *For Δ_Y, there exists constants c_1, c_2, such that*

$$\frac{9r}{4} + c_1 \leq \text{reg } H_0(J_\bullet) \leq \frac{9r}{4} + c_2.$$

We also propose the following open problem:

Open Problem. For a given Δ, find both upper and lower bounds for reg $H_0(J_\bullet)$.

4.2 Powers of linear forms, the inverse system and fat points

In this section, we start with a dimension problem on fat points ideals, and show it is related to the dimension problem of spline spaces.

Definition. Assume $P_i \in \mathbb{P}^n$ for $i = 1, \dots, s$. Let $\mathfrak{p}_i = I(P_i) \subseteq T$ be the ideal defining P_i. A *fat points ideal* is an ideal of the form $I = \cap_{i=1}^m \mathfrak{p}_i^{k_i}$ for $k_i \geq 1$.

Problem. If $P_1, \dots, P_s$ are sufficiently general points of $\mathbb{P}^n$ with corresponding prime ideals $\mathfrak{p}_1, \dots, \mathfrak{p}_s$. Let $I = \cap_{i=1}^m \mathfrak{p}_i^{k_i}$ for $k_i \geq 1$ be a fat points ideal. What is the Hilbert function of T/I?

Note that $(I)_d$ is the space of d-forms that has zero at P_i with multiplicity at least k_i. We denote this space by $\mathcal{L}_d(-\sum_{i=1}^s k_i P_i)$. If $s = 1$, there is a single point and there is $\binom{k_1+n-1}{n}$ linearly independent conditions posed on d-forms, so $\dim T_d - \dim(I)_d = \binom{k_1+n-1}{n}$. A naive guess is that s points with given multiplicities would pose $\min\{\sum_{i=1}^s \binom{k_i+n-1}{n}, \binom{d+n-1}{n}\}$ linearly independent conditions. However, this fails for $(n, s, d) = (2, 5, 4)$ and $k_1 = \cdots = k_5 = 2$. See Miranda's paper[24] for more examples on which the expected dimension fails. In the same paper, there is also a conjecture made by Segre-Harbourne-Gimigliano-Hirschowitz saying the fat points ideals have the expected Hilbert function under some conditions. This conjecture is unsolved yet. Therefore, the form of Hilbert function for fat points is still unknown.

Recall in Section 1.4, we define the corresponded linear form $L_P \in S_1$ for a reduced point $P \in \mathrm{Proj}(T)$. In [16], Ensalem and Iarrobino have the following theorem for fat points ideals.

Theorem 4.1 (Ensalem-Iarrobino, 1995). *Assume that* $I = \mathfrak{p}_1^{k_1+1} \cap \cdots \cap \mathfrak{p}_s^{k_s+1}$, *then*

$$(I^{-1})_d = \begin{cases} S_d, & \text{for } d \le \max\{k_i\}, \\ S_{k_1} \cdot L_{P_1}^{d-k_1} + \cdots + S_{k_s} \cdot L_{P_s}^{d-k_s}, & \text{for } d > \max\{k_i\}, \end{cases} \tag{4.1}$$

where L_{P_i} *is the corresponded linear form of* P_i, *and*

$$\mathrm{HF}(I^{-1}, d) = \mathrm{HF}(T/I, d). \tag{4.2}$$

Note that the ideal $J(v)$ defined in Section 2.2 is generated by powers of linear forms. In fact, if $J = \langle L_{P_1}^{r+1}, \ldots, L_{P_s}^{r+1} \rangle$, then for $d \ge r+1$,

$$(J)_d = L_{P_1}^{r+1} \cdot S_{d-r-1} + \cdots + L_{P_s}^{r+1} \cdot S_{d-r-1}. \tag{4.3}$$

Then by Theorem 4.1, $(J)_d = (I^{-1})_d$ where

$$I = \mathfrak{p}_1^{d-r} \cap \cdots \cap \mathfrak{p}_s^{d-r}. \tag{4.4}$$

Therefore, questions about Hilbert function of fat points on $\mathbb{P}^n$ can be translated into questions about ideals generated by powers of linear forms in $(n+1)$ variables. In [18], the authors use this correspondence to compute dimension of spline spaces with mixed smoothness. Since the form of Hilbert function of fat points on $\mathbb{P}^2$ is unknown, there is also no known form of the Hilbert function for the local data $J(v)$ when Δ is 3-dimensional.

4.3 On quaternary quartic forms

Let $S = \mathbb{C}[x_0, \ldots, x_n]$ and $T = \mathbb{C}[y_0, \ldots, y_n] = \mathbb{C}[\frac{\partial}{\partial x_0}, \ldots, \frac{\partial}{\partial x_n}]$ be the ring of differential operators on S. We say a homogeneous polynomial $f \in S$ has a *length s power sum decomposition* if

$$f = l_1^d + \cdots + l_s^d. \tag{4.5}$$

Definition. The *Waring rank* of a homogeneous polynomial $f \in S$ is the least number r such that f has a length r power sum decomposition. We denote the Waring rank of f by $\operatorname{rank}(f)$.

Question (Waring's problem)**.** What is $\operatorname{rank}(f)$ for a given f?

For a generic form of degree d in $n+1$ variables, we have the following theorem:

Theorem 4.2 (Alexander-Hirschowitz)**.** *A generic form f of degree d in $n+1$ variables in a sum of $\lceil \frac{1}{n+1}\binom{n+d}{n} \rceil$ powers of linear forms, unless*

- $d = 2$, *where $s = n+1$ instead of $\lceil \frac{n+2}{2} \rceil$, or*
- $d = 4$ *and* $n = 2, 3, 4$, *where* $s = 6, 10, 15$ *instead of* $5, 9, 14$ *respectively, or*
- $d = 3$ *and* $n = 4$, *where* $s = 8$ *instead of* 7.

However, finding out $\operatorname{rank}(f)$ for every f is still an open problem.

By Theorem 1.2, any Artinian Gorenstein ideal with regularity 4 in T is Macaulay's inverse system I_f of a principal ideal $\langle f \rangle \subset S$ with $\deg f = 4$. By our results in Chapter 3, if $n = 3$, then the Betti diagram of $A_f = T/I_f$ must be one of the 16 tables in Table 3.1 and 3.2. With the Apolarity Lemma described below, we would like to classify the forms f in terms of power sum decomposition according to the Betti diagram of $A_f = T/I_f$. We call a subscheme $\Gamma \subset \operatorname{Proj}(T)$ *apolar* to f, if $I(\Gamma) \subset I_f \subset T$.

Lemma 4.3 (Apolarity Lemma)**.** *Let $\Gamma \subset \{P_1, \ldots, P_s\}$ be a set of points in* $\operatorname{Proj}(T)$. *Then $f = L_1^d + \cdots + L_s^d$ if and only if Γ is apolar to f.*

We are also interested in giving a complete description of the relation between the Betti diagram of A_f and the geometry of Γ.

4.4 Extension of Artinian Gorenstein to higher dimension

In Chapter 3, we studied Artinian Gorenstein rings, especially those with regularity 4 and codimension 4. Those with Betti diagram in Table 3.1 can be viewed as Artinian reduction of GoCY threefolds in Table 1.1. The authors of [11] ask if Table 1.1 is a complete list of families of GoCY threefolds in $\mathbb{P}^7$. Since we already have a complete list of Artinian reduction for all codimension 4 GoCYs, we would like to lift them to higher dimension. In particular, if we are able to obtain all possible extensions of Artinian GoCYs to threefolds, then we may answer their question. On the other hand, we have proved that GoCY's with Betti diagram in Table 3.2 cannot be threefolds, but their possible extensions to higher dimension are still interesting to us.

Let $\mathcal{F}_B$ be the set of forms f for which the apolar Artinian Gorenstein ring A_f has Betti diagram B. The parameter space of all quaternary quartic forms is isomorphic to $\mathbb{P}^{34}$. In fact, every family $\mathcal{F}_B$ with B in Table 3.1 and 3.2 is a quasi-projective algebraic set in $\mathbb{P}^{34}$. If $\mathcal{F}_B$ is irreducible, then it makes sense to talk about a *general element* $f_B \in \mathcal{F}_B$. We may try to obtain all possible extensions of A_B corresponding to f_B. If $\mathcal{F}_B$ is not irreducible, then we can study their irreducible components and look for extensions corresponding to the general element for each of these components.

4.5 Calabi-Yau varieties in toric spaces

One generalization of complete intersection Calabi-Yau varieties in projective spaces is GoCY. Another way is to consider Calabi-Yau varieties embedded in toric varieties, because projective space is the simplest complete toric variety. In [3], Batyrev shows how to obtain Calabi-Yau varieties as hypersurfaces in toric varieties corresponding to reflexive polytopes. Using Batyrev's construction, one gets a pair of Calabi-Yau varieties (X, X') such that $h^{1,1}(X) = h^{1,2}(X')$, so they are potentially *mirror symmetric pairs*, as introduced in [12]. What's more, their Hodge numbers can be obtained from the toric varieties they lie in.

To study such Calabi-Yau varieties, we are not only interested in their Hodge numbers, but also in all cohomologies $H^q(X, \Omega^p(d))$. Assume Calabi-Yau variety X is obtained as a degree k hypersurface of a toric variety V. In [23], Maclagan and Smith introduce a method to compute $H^q(V, \mathcal{O}_V(d))$ for all q and d. Since the short exact sequence of sheaves

$$0 \to \mathcal{O}_V(-k+d) \to \mathcal{O}_V(d) \to \mathcal{O}_X(d) \to 0 \tag{4.6}$$

induces the long exact sequence of cohomologies

$$\cdots \to H^q(V, \mathcal{O}_V(-k+d)) \to H^q(V, \mathcal{O}_V(d)) \to H^q(X, \mathcal{O}_X(d)) \to \ldots, \tag{4.7}$$

we may obtain information about $H^q(X, \mathcal{O}_X(d))$ from it. This is a topic we would like to investigate further.

APPENDIX A

COMPUTATION ON SPLINE PROBLEMS

A.1 Buchberger's Algorithm and computation of Gröbner basis

The introduction to Buchberger's Algorithm can be found in [25, section 39] and [15, Chapter 15]. We briefly recall the algorithm here. Fix a monomial order $\succ$ on a free S-module F with basis $\mathbf{e}_1, \ldots, \mathbf{e}_t$. For $f, g \in F$, set

$$\tau_{f,g} = \frac{\mathrm{In}(g)}{\gcd(\mathrm{In}(f), \mathrm{In}(g))} f - \frac{\mathrm{In}(f)}{\gcd(\mathrm{In}(f), \mathrm{In}(g))} g \tag{A.1}$$

Assume M is a submodule of F generated by $\{g_1, \ldots, g_s\}$. For each pair (g_i, g_j), there is an expression

$$\tau_{ij} = \sum a_k g_k + h_{ij} \tag{A.2}$$

such that either $h_{ij} = 0$ or $\mathrm{In}(h_{ij}) \succ \mathrm{In}(g_k)$ for all k. We call h_{ij} the remainders/

Theorem A.1 (Buchberger's Criterion)**.** *The elements $g_1, \ldots, g_s$ form a Gröbner basis if and only if $h_{ij} = 0$ for all i and j.*

Buchberger's Algorithm: In the situation of Theorem A.1, suppose that M is a submodule of F, and let $g_1, \ldots, g_s$ be a set of generators of M. Compute the remainders h_{ij}. If all the $h_{ij} = 0$, then $\{g_1, \ldots, g_s\}$ forms a Gröbner basis for M. If some $h_{ij} \neq 0$, then replace $g_1, \ldots, g_s$ with $g_1, \ldots, g_s, h_{ij}$, and repeat the process. This process must terminate, see [15].

Next, we use Buchberger's Algorithm to compute a Gröbner basis of the image of Syz_1

with respect to the P.O.T. order, where $\mathbf{e}_1 \prec \mathbf{e}_2$. Starting with

$$\begin{cases} f_1 = (2x+y+4z)\mathbf{e}_1 \\ f_2 = y^2\mathbf{e}_1 \\ f_3 = (3x-2y-2z)\mathbf{e}_1 + (3x+2y+2z)\mathbf{e}_2 \\ f_4 = -3(3x-2y-2z)x - 2(y+z)^2\mathbf{e}_1 + 2(y+z)^2\mathbf{e}_2 \\ f_5 = (-2x+y+4z)\mathbf{e}_2 \\ f_6 = y^2\mathbf{e}_2 \end{cases},$$

we may compute the remainder $h_{12} = 0$ and $h_{13} = -(7y+16z)\mathbf{e}_1 + (7y+16z)\mathbf{e}_2$. Replace f_3 with h_{13}.

$$\begin{cases} f_1^{(1)} = (2x+y+4z)\mathbf{e}_1 \\ f_2^{(1)} = y^2\mathbf{e}_1 \\ f_3^{(1)} = -(7y+16z)\mathbf{e}_1 + (7y+16z)\mathbf{e}_2 \\ f_4^{(1)} = -3(3x-2y-2z)x - 2(y+z)^2\mathbf{e}_1 + 2(y+z)^2\mathbf{e}_2 \\ f_5^{(1)} = (-2x+y+4z)\mathbf{e}_2 \\ f_6^{(1)} = y^2\mathbf{e}_2 \end{cases}$$

is still a basis of $\mathrm{Im}(Syz_1)$. Now $h_{12} = h_{13} = 0$, $h_{23} = -z^2\mathbf{e}_1 + z^2\mathbf{e}_1$. Replace $f_2^{(1)}$ with h_{23} and re-index the generators with respect to $\succ$.

$$\begin{cases} f_1^{(2)} = (2x+y+4z)\mathbf{e}_1 \\ f_2^{(2)} = -(7y+16z)\mathbf{e}_1 + (7y+16z)\mathbf{e}_2 \\ f_3^{(2)} = -z^2\mathbf{e}_1 + z^2\mathbf{e}_1 \\ f_4^{(2)} = -3(3x-2y-2z)x - 2(y+z)^2\mathbf{e}_1 + 2(y+z)^2\mathbf{e}_2 \\ f_5^{(2)} = (-2x+y+4z)\mathbf{e}_2 \\ f_6^{(2)} = y^2\mathbf{e}_2 \end{cases}$$

Now $h_{ij} = 0$ for $1 \leq i < j \leq 3$ and $h_{14} = (11yz + 16z^2)\mathbf{e}_2$. Replace $f_4^{(2)}$ with h_{14} and re-index the generators with respect to $\succ$.

$$\begin{cases} f_1^{(3)} = (2x + y + 4z)\mathbf{e}_1 \\ f_2^{(3)} = -(7y + 16z)\mathbf{e}_1 + (7y + 16z)\mathbf{e}_2 \\ f_3^{(3)} = -z^2\mathbf{e}_1 + z^2\mathbf{e}_1 \\ f_4^{(3)} = (-2x + y + 4z)\mathbf{e}_2 \\ f_5^{(3)} = y^2\mathbf{e}_2 \\ f_6^{(3)} = (11yz + 16z^2)\mathbf{e}_2 \end{cases}$$

Now $h_{ij} = 0$ for $1 \leq i < j \leq 5$ and $h_{i6} = 0$ for $1 \leq i \leq 4$. $h_{56} = z^3\mathbf{e}_2$. Add h_{56} to the generating set.

$$\begin{cases} f_1^{(4)} = (2x + y + 4z)\mathbf{e}_1 \\ f_2^{(4)} = -(7y + 16z)\mathbf{e}_1 + (7y + 16z)\mathbf{e}_2 \\ f_3^{(4)} = -z^2\mathbf{e}_1 + z^2\mathbf{e}_1 \\ f_4^{(4)} = (-2x + y + 4z)\mathbf{e}_2 \\ f_5^{(4)} = y^2\mathbf{e}_2 \\ f_6^{(4)} = (11yz + 16z^2)\mathbf{e}_2 \\ f_7^{(4)} = z^3\mathbf{e}_2 \end{cases}$$

Now $h_{ij} = 0$ for $1 \leq i < j \leq 7$, so it is a Gröbner basis of $\mathrm{Im}(\mathrm{Syz}_1)$.

A.2 Compute $\mathrm{reg}\ H_0(J_\bullet)$ for $r \leq 20$ using `Macaulay2`

We write `Macaulay2` codes to investigate the counter-example in Chapter 2. The command standardGraph records Δ_Y. It expects no input and the output is (V, E), where V is the list of coordinates of vertices of Δ_Y and E is a list of interior edges.

The command H0matrix expects three parameters (r, V, E), where r is the smoothness, V is the list of coordinates of vertices of Δ and E is a list of interior edges. The V and E has some restrictions: the first 3 vertices in V are the interior ones, in a line, and the rest are the vertices which provide exactly 3 edges from each of these vertices. If not, H0matrix will give an error. The output is the matrix (2.32).

The command computeCounterexample expects three parameters (r, V, E), where r is the smoothness, V is the list of coordinates of vertices of Δ_Y and E is a list of interior edges. The output includes the expected regularity of $H_0(J_\bullet)$ and the actual regularity. For example,

```
For r = 2, the top degree of HH_0(J) should = 5
  In degree 5: (#rows,#cols,rank)=(12, 12, 11)
  In degree 6: (#rows,#cols,rank)=(20, 27, 20)
```

means for $r = 2$, the expected regularity of $H_0(J_\bullet)$ is $\lfloor \frac{9r+2}{4} \rfloor = 5$. In degree 5, #rows is 12 and rank is 11, so they are not equal. In degree 6, #rows is 20 and rank is 20, so they are equal. Therefore, the actual regularity is 5.

With these commands, we compute the actual regularity of $H_0(J_\bullet)$ for $r \leq 20$. The computation result shows that the actual regularity is the same as the expected regularity $\frac{9r+2}{4}$.

```
i2 : R=QQ[x,y,z]

o2 = R

o2 : PolynomialRing

i3 : (V,E) = standardGraph()
```

```
o3 = ({{-2, 1}, {0, 0}, {2, 1}, {0, -1}, {0, 3}, {0, 3}, {0, 3}, {0, -1}}, {{0,
     --------------------------------------------------------------------------
     1}, {1, 2}, {0, 3}, {0, 4}, {1, 5}, {2, 6}, {2, 7}})

o3 : Sequence

i4 : for r from 1 to 20 do computeCounterexample(r,V,E)
For r = 1, the top degree of HH_0(J) should = 2
  In degree 2: (#rows,#cols,rank)=(2, 0, 0)
  In degree 3: (#rows,#cols,rank)=(6, 6, 6)
For r = 2, the top degree of HH_0(J) should = 5
  In degree 5: (#rows,#cols,rank)=(12, 12, 11)
  In degree 6: (#rows,#cols,rank)=(20, 27, 20)
For r = 3, the top degree of HH_0(J) should = 7
  In degree 7: (#rows,#cols,rank)=(20, 18, 18)
  In degree 8: (#rows,#cols,rank)=(30, 36, 30)
For r = 4, the top degree of HH_0(J) should = 9
  In degree 9: (#rows,#cols,rank)=(30, 27, 27)
  In degree 10: (#rows,#cols,rank)=(42, 48, 42)
For r = 5, the top degree of HH_0(J) should = 11
  In degree 11: (#rows,#cols,rank)=(42, 36, 36)
  In degree 12: (#rows,#cols,rank)=(56, 60, 56)
For r = 6, the top degree of HH_0(J) should = 14
  In degree 14: (#rows,#cols,rank)=(72, 75, 70)
  In degree 15: (#rows,#cols,rank)=(90, 108, 90)
For r = 7, the top degree of HH_0(J) should = 16
```

```
  In degree 16: (#rows,#cols,rank)=(90, 90, 88)
  In degree 17: (#rows,#cols,rank)=(110, 126, 110)
For r = 8, the top degree of HH_0(J) should = 18
  In degree 18: (#rows,#cols,rank)=(110, 108, 105)
  In degree 19: (#rows,#cols,rank)=(132, 147, 132)
For r = 9, the top degree of HH_0(J) should = 20
  In degree 20: (#rows,#cols,rank)=(132, 126, 124)
  In degree 21: (#rows,#cols,rank)=(156, 168, 156)
For r = 10, the top degree of HH_0(J) should = 23
  In degree 23: (#rows,#cols,rank)=(182, 192, 180)
  In degree 24: (#rows,#cols,rank)=(210, 243, 210)
For r = 11, the top degree of HH_0(J) should = 25
  In degree 25: (#rows,#cols,rank)=(210, 216, 208)
  In degree 26: (#rows,#cols,rank)=(240, 270, 240)
For r = 12, the top degree of HH_0(J) should = 27
  In degree 27: (#rows,#cols,rank)=(240, 243, 234)
  In degree 28: (#rows,#cols,rank)=(272, 300, 272)
For r = 13, the top degree of HH_0(J) should = 29
  In degree 29: (#rows,#cols,rank)=(272, 270, 264)
  In degree 30: (#rows,#cols,rank)=(306, 330, 306)
For r = 14, the top degree of HH_0(J) should = 32
  In degree 32: (#rows,#cols,rank)=(342, 363, 340)
  In degree 33: (#rows,#cols,rank)=(380, 432, 380)
For r = 15, the top degree of HH_0(J) should = 34
  In degree 34: (#rows,#cols,rank)=(380, 396, 378)
  In degree 35: (#rows,#cols,rank)=(420, 468, 420)
For r = 16, the top degree of HH_0(J) should = 36
```

```
  In degree 36: (#rows,#cols,rank)=(420, 432, 414)
  In degree 37: (#rows,#cols,rank)=(462, 507, 462)
For r = 17, the top degree of HH_0(J) should = 38
  In degree 38: (#rows,#cols,rank)=(462, 468, 454)
  In degree 39: (#rows,#cols,rank)=(506, 546, 506)
For r = 18, the top degree of HH_0(J) should = 41
  In degree 41: (#rows,#cols,rank)=(552, 588, 550)
  In degree 42: (#rows,#cols,rank)=(600, 675, 600)
For r = 19, the top degree of HH_0(J) should = 43
  In degree 43: (#rows,#cols,rank)=(600, 630, 598)
  In degree 44: (#rows,#cols,rank)=(650, 720, 650)
For r = 20, the top degree of HH_0(J) should = 45
  In degree 45: (#rows,#cols,rank)=(650, 675, 644)
  In degree 46: (#rows,#cols,rank)=(702, 768, 702)
```

The codes for those commands are listed below:

```
standardGraph = () -> (
    -- return a (V, E) pair
    V := {
        {-2,1}, {0,0}, {2,1},  -- interior vertices, in line order
        {0,-1}, {0,3}, -- connect to 0th vertex
        {0,3}, -- connect to vertex #1
        {0,3}, {0,-1} -- connect to vertex #2
        };
    E := {{0,1}, {1,2}, -- interior edges
        {0,3}, {0,4}, -- connect to vertex #0
        {1,5}, -- connect to vertex #1
```

```
        {2,6}, {2,7} -- connect to vertex #2
        };
    (V,E)
    )

containsVertex = method()
-- given an index 'v' into the list of vertices V, find the edges in E incident
to v.
containsVertex(ZZ, List) := (v, E) -> positions(E, e -> member(v, e))

linearForm = method(Options => {Ring => QQ[getSymbol "x", getSymbol "y",
getSymbol "z"]})
linearForm(List, List) := opts -> (e, V) -> (
    -- e is a list of 2 indices into V, generally an element of E
    -- returns a linear form in the ring opts.Ring
    R := opts#Ring;
    x := R_0;
    y := R_1;
    z := R_2;
    f := det matrix{{x,y,z},append(V_(e_0), 1), append(V_(e_1), 1)};
    (trim ideal f)_0
    )

H0matrix = method(Options => options linearForm)
H0matrix(ZZ, List, List) := opts -> (r,V,E) -> (
    -- assumptions:
    --  (1) the first 3 vertices of V are the interior ones, in a line
```

```
    --  (2) each of these vertices is connected to precisely 3 others.
    S := opts#Ring;
    e0 := containsVertex(0, E);
    e1 := containsVertex(1, E);
    e2 := containsVertex(2, E);
    if #e0 =!= 3 or #e1 =!= 3 or #e2 =!= 3 then error "expected 3 edges from
    the first three vertices";
    linforms := for e in E list linearForm(e, V, opts);
    Z0 := syz matrix{(linforms_e0)/(f -> f^(r+1))};
    Z1 := syz matrix{(linforms_e1)/(f -> f^(r+1))};
    Z2 := syz matrix{(linforms_e2)/(f -> f^(r+1))};
    M := (submatrix(Z0, {0}, ) || matrix{{0,0}})
         | submatrix(Z1, {0,1}, )
         | (matrix{{0,0}} || submatrix(Z2, {0}, ));
    map(S^{2: -r-1},,M)
    )

computeCounterexample = method()
computeCounterexample (ZZ,List,List) := (r,V,E) -> (
  deg1 := floor((9*r+2)/4);
  << "For r = " << r << ", the top degree of HH_0(J) should = " << deg1 <<
  endl;
  --(V,E) := standardGraph();
  f := H0matrix(r, V, E);
  Rp := ZZ/32003[gens ring f];
  fp := sub(f, Rp);
  d1 := degreePart(deg1,fp);
```

```
<< "  In degree " << deg1 << ": (#rows,#cols,rank)=" << (numrows d1, numcols
d1, rank d1) << endl;
d2 := degreePart(deg1+1,fp);
<< "  In degree " << deg1+1 << ": (#rows,#cols,rank)=" << (numrows d2,
numcols d2, rank d2) << endl;
)
```

APPENDIX B

EXPLICIT EXAMPLES OF ARTINIAN GORENSTEIN RINGS WITH GIVEN BETTI DIAGRAM

In this section, we would like to find an explicit example for each Betti diagram in Table 3.1 and 3.2.

Recall that a subscheme $\Gamma \subset \mathbb{P}^n$ *apolar* to F, if the homogeneous ideal $I_\Gamma \subset F^\perp \subset T$. The following lemma is well-known and can be found in [21, Lemma 1.15].

Lemma B.1 (Apolarity Lemma). *Let* $\Gamma = \{V(l_1), \dots, V(l_s)\} \subset \mathbb{P}(S_1) = \mathbb{P}^n$ *be a collection of* s *distinct points. Then*

$$F = \lambda_1 l_1^d + \cdots + \lambda_s l_s^d \tag{B.1}$$

if and only if

$$I_\Gamma \subset F^\perp \subset T.$$

With Lemma B.1, we find explicit examples of Artinian Gorenstein rings with given Betti diagram $B = (\beta_{i,j})$ by the following steps:

- Step 1: Find a point set $\Gamma \subseteq \mathbb{P}^3$ such that the defining ideal T/I_Γ has the same top row Betti numbers with B.
- Step 2: If $\Gamma = \{V(l_1), \dots, V(l_s)\} \subset \mathbb{P}(S_1) = \mathbb{P}^n$, we take

$$f = l_1^d + \cdots + l_s^d$$

  to be the dual socle generator of A_f.
- Step 3: Verify that A_f has the given Betti diagram B.

We perform these steps by `Macaulay2`.

```
i1 : kk=QQ

o1 = QQ

o1 : Ring

i2 : T = kk[x,y,z,w]

o2 = T

o2 : PolynomialRing

i3 : linearForm = pt -> sum for i from 0 to 3 list pt#i * T_i

o3 = linearForm

o3 : FunctionClosure

i4 : quartic = (pts) -> sum for p in pts list (linearForm p)^4

o4 = quartic

o4 : FunctionClosure

i5 : randomPoint = nvars -> for i from 1 to nvars list random kk

o5 = randomPoint

o5 : FunctionClosure

i6 : randomPoints = (d, nvars) -> for i from 1 to d list randomPoint nvars

o6 = randomPoints

o6 : FunctionClosure
```

The command quartic expects one parameter pts, which is a list of coordinates of points in $\mathbb{P}^3$. The output is a quartic form as in equation (B.1).

The command randomPoints expects two parameter (s, n), where s and n are positive integers. The output is a list of coordinates of random s points in $\mathbb{P}^{n-1}$.

Using these commands, we obtain a quartic form f such that A_f has given Betti diagram in each of the following case, where Γ denotes an apolar point set to f as in Lemma B.1.

B.1 CGKK1

For CGKK1, the apolar point set Γ contains four points in general position.

```
i7 : --CGKK1
     F = quartic {{1,0,0,0},{0,1,0,0},{0,0,1,0},{0,0,0,1}};

i8 : I = inverseSystem F;

o8 : Ideal of T

i9 : betti res I

            0 1  2 3 4
o9 = total: 1 9 16 9 1
         0: 1 .  . . .
         1: . 6  8 3 .
         2: . .  . . .
         3: . 3  8 6 .
         4: . .  . . 1

o9 : BettiTally
```

B.2 CGKK2

For CGKK2, the apolar point set Γ contains five points in general position.

```
i10 : F = quartic {{1,0,0,0},{0,1,0,0},{0,0,1,0},{0,0,0,1},{1,1,1,1}};

i11 : I = inverseSystem F;

o11 : Ideal of T

i12 : betti res I

             0 1  2 3 4
o12 = total: 1 6 10 6 1
          0: 1 .  . . .
          1: . 5  5 . .
          2: . 1  . 1 .
          3: . .  5 5 .
          4: . .  . . 1

o12 : BettiTally
```

B.3 CGKK3

CGKK3 is a complete intersection of four quadrics.

```
i13 : I = ideal(x^2,y^2,z^2,w^2)

              2   2   2   2
o13 = ideal (x , y , z , w )

o13 : Ideal of T

i14 : betti res I

             0 1 2 3 4
o14 = total: 1 4 6 4 1
          0: 1 . . . .
          1: . 4 . . .
          2: . . 6 . .
          3: . . . 4 .
          4: . . . . 1

o14 : BettiTally
```

B.4 CGKK4

For CGKK4, the apolar point set Γ contains 7 points in general position.

```
i15 : F = quartic randomPoints(7,4);

i16 : I = inverseSystem F;

o16 : Ideal of T

i17 : betti res I

               0 1  2 3 4
o17 = total: 1 7 12 7 1
          0: 1 .  . . .
          1: . 3  . . .
          2: . 4 12 4 .
          3: . .  . 3 .
          4: . .  . . 1

o17 : BettiTally
```

B.5 CGKK5,6

For CGKK5,6, the apolar point set Γ contains 7 points on a twisted cubic curve.

```
i18 : --CGKK5,6
      pts = for i from 0 to 6 list for j from 0 to 3 list i^j;

i19 : F = quartic pts;

i20 : I = inverseSystem F;

o20 : Ideal of T

i21 : betti res I

             0 1  2 3 4
o21 = total: 1 9 16 9 1
          0: 1 .  . . .
          1: . 3  2 . .
          2: . 6 12 6 .
          3: . .  2 3 .
          4: . .  . . 1

o21 : BettiTally
```

B.6 CGKK7,8

For CGKK7,8, the apolar point set Γ contains 8 points in general position.

```
i22 : --CGKK7,8
      F = quartic randomPoints(8,4);

i23 : I = inverseSystem F;

o23 : Ideal of T

i24 : betti res I

             0  1  2  3 4
o24 = total: 1 10 18 10 1
          0: 1  .  .  . .
          1: .  2  .  . .
          2: .  8 18  8 .
          3: .  .  .  2 .
          4: .  .  .  . 1

o24 : BettiTally
```

B.7 CGKK9,10

For CGKK9,10, the apolar point set Γ contains 9 points in general position.

```
i25 : F = quartic randomPoints(9,4);

i26 : I = inverseSystem F;

o26 : Ideal of T

i27 : betti res I

             0  1  2  3 4
o27 = total: 1 13 24 13 1
          0: 1  .  .  . .
          1: .  1  .  . .
          2: . 12 24 12 .
          3: .  .  .  1 .
          4: .  .  .  . 1

o27 : BettiTally
```

B.8 CGKK11

For CGKK11, the apolar point set Γ contains 10 points in general position.

```
i28 : F = quartic randomPoints(10,4);

i29 : I = inverseSystem F;

o29 : Ideal of T

i30 : betti res I

             0  1  2  3 4
o30 = total: 1 16 30 16 1
          0: 1  .  .  . .
          1: .  .  .  . .
          2: . 16 30 16 .
          3: .  .  .  . .
          4: .  .  .  . 1

o30 : BettiTally
```

B.9 Type 2.1

For Type 2.1, the apolar point set Γ contains 8 points such that 6 of them lie on a $\mathbb{P}^2$.

```
i31 : --Type 2.1
      F = quartic {{1,0,0,0}, {0,1,0,0}, {0,0,1,0}, {1,1,1,0},
      {2,3,7,0}, {17,2,31,0}, {-2,7,4,10}, {4,8,20,50}};

i32 : I = inverseSystem F;

o32 : Ideal of T

i33 : betti res I

             0  1  2  3 4
o33 = total: 1 11 20 11 1
          0: 1  .  .  . .
          1: .  2  1  . .
          2: .  9 18  9 .
          3: .  .  1  2 .
          4: .  .  .  . 1

o33 : BettiTally
```

B.10 Type 2.2

For Type 2.2, the apolar point set Γ contains 7 points such that 5 of them lie on a $\mathbb{P}^2$.

```
i34 : F = quartic {{1,0,0,0}, {0,1,0,0}, {0,0,1,0}, {1,1,1,0},
      {2,3,7,0}, {-2,7,4,10}, {13,-2,17,30}};

i35 : I = inverseSystem F;

o35 : Ideal of T

i36 : betti res I

             0 1  2 3 4
o36 = total: 1 8 14 8 1
          0: 1 .  . . .
          1: . 3  1 . .
          2: . 5 12 5 .
          3: . .  1 3 .
          4: . .  . . 1

o36 : BettiTally
```

B.11 Type 2.3

For Type 2.3, the apolar point set Γ contains 6 points such that 3 of them lie on a $\mathbb{P}^1$.

```
i37 : F = quartic {{1,0,0,0}, {0,1,0,0}, {1,1,0,0}, {1,0,1,0},
      {0,0,0,1}, {0,0,1,1}};

i38 : I = inverseSystem F;

o38 : Ideal of T

i39 : betti res I

             0 1  2 3 4
o39 = total: 1 7 12 7 1
          0: 1 .  . . .
          1: . 4  3 . .
          2: . 3  6 3 .
          3: . .  3 4 .
          4: . .  . . 1

o39 : BettiTally
```

B.12 Type 2.4

For Type 2.4, the apolar point set Γ contains 6 points in general position.

```
i40 : F = quartic randomPoints(6, 4);

i41 : I = inverseSystem F;

o41 : Ideal of T

i42 : betti res I

             0 1  2 3 4
o42 = total: 1 6 10 6 1
          0: 1 .  . . .
          1: . 4  2 . .
          2: . 2  6 2 .
          3: . .  2 4 .
          4: . .  . . 1

o42 : BettiTally
```

B.13 Type 2.5

For Type 2.5, the apolar point set Γ contains 7 points such that 6 of them lie on a $\mathbb{P}^2$.

```
i43 : F = quartic {{1,0,0,0}, {0,1,0,0}, {0,0,1,0}, {1,1,1,0},
      {2,3,7,0}, {17,2,31,0},{-2,7,4,10}};

i44 : I = inverseSystem F;

o44 : Ideal of T

i45 : betti res I

             0  1  2  3 4
o45 = total: 1 11 20 11 1
          0: 1  .  .  . .
          1: .  3  3  1 .
          2: .  7 14  7 .
          3: .  1  3  3 .
          4: .  .  .  . 1

o45 : BettiTally
```

B.14 Type 2.6

For Type 2.6, we may choose the apolar point set Γ to be a set of 6 points such that they lie on two skew lines.

```
i46 : F = quartic {{1,0,0,0}, {0,1,0,0}, {1,1,0,0}, {0,0,1,0},
      {0,0,0,1}, {0,0,1,1}};

i47 : I = inverseSystem F;

o47 : Ideal of T

i48 : betti res I

             0 1  2 3 4
o48 = total: 1 9 16 9 1
          0: 1 .  . . .
          1: . 4  4 1 .
          2: . 4  8 4 .
          3: . 1  4 4 .
          4: . .  . . 1

o48 : BettiTally
```

Another way to obtain this Betti diagram is to choose Γ to be a set of 6 points such that 5 of them are on a $\mathbb{P}^2$.

B.15 Type 2.7

For Type 2.7, the apolar point set Γ contains 5 points such that 4 of them lie on a $\mathbb{P}^2$.

```
i49 : --Type 2.7
      F = quartic {{1,0,0,0},{0,1,0,0},{0,0,1,0},{1,1,1,0},{0,0,0,1}};

i50 : I = inverseSystem F;

o50 : Ideal of T

i51 : betti res I

             0 1  2 3 4
o51 = total: 1 7 12 7 1
          0: 1 .  . . .
          1: . 5  5 1 .
          2: . 1  2 1 .
          3: . 1  5 5 .
          4: . .  . . 1

o51 : BettiTally
```

B.16 Type 2.8

For Type 2.8, the apolar point set Γ contains 5 points such that 3 of them lie on a $\mathbb{P}^1$.

```
i52 : F = quartic {{1,0,0,0},{0,1,0,0},{1,1,0,0},{0,0,1,0},{0,0,0,1}};

i53 : I = inverseSystem F;

o53 : Ideal of T

i54 : betti res I

             0 1  2 3 4
o54 = total: 1 9 16 9 1
          0: 1 .  . . .
          1: . 5  6 2 .
          2: . 2  4 2 .
          3: . 2  6 5 .
          4: . .  . . 1

o54 : BettiTally
```

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