ON THE NECESSITY OF REGULARITY CONDITIONS

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Abstract

The purpose of the present note is to draw attention to the necessity of regularity conditions in using mathematical results by looking at some results involving expectations.

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Often in deriving results in statistics, a number of regularity conditions are used. However, in transition from theory to applications these conditions are frequently ignored or forgotten.

Two examples are given to demonstrate the importance of regularity conditions in equations involving expectations of random variables.

Lemma: <u>In order that results involving expectations of random variables be true, it is necessary that ALL expectations in the result exist.</u>

Often one looks at one side of the equation and if it exists, it is assumed that the other side also exists. The following examples show that this is not always the case.

<u>Example 1.</u> Let X,Y be random variables, possibly dependent. Then,

$$E(X) = E\{E(X|Y)\} \quad \underline{\text{only if}} \quad E(X),\ E(X|Y) \text{ and } E\{E(X|Y)\} \text{ exist.}$$

Suppose $Y \sim \Gamma(\frac{n}{2},\frac{n}{2})$ with

$$f_Y(y) = \begin{cases} (\frac{n}{2})^{\frac{n}{2}} e^{-\frac{n}{2}y} y^{\frac{n}{2}-1} \big/ \Gamma(\frac{n}{2}) ; & y > 0 \\ 0 & \text{otherwise} \end{cases}$$

Also suppose the conditional distribution of X given Y = y is $N(0,\frac{1}{y})$

i.e.
$$g_{X|Y}(x|Y=y) = \frac{y^{\frac{1}{2}}}{\sqrt{2\pi}} e^{-\frac{1}{2}yx^2} ; \quad -\infty < x < \infty$$

Then the marginal distribution of X is t_n

i.e. $$g_X(x) = \frac{\Gamma(\frac{n}{2}+1)}{\Gamma(\frac{n}{2})(\pi n)^{\frac{1}{2}}} (1 + \frac{x^2}{n})^{-\frac{n+1}{2}} \; ; \quad -\infty < x < \infty$$

$\therefore$ $E\{E(X|Y)\} = 0$ for all n .

But for $n = 1$, $X \sim$ Cauchy $(0,1) = t_1$.

$\therefore$ $E(X)$ does not exist for $n = 1$.

Example 2. Let X and Y be two random variables, distributed independently. Then $E(XY) = E(X) \cdot E(Y)$ only if $E(X)$, $E(Y)$ and $E(XY)$ exist.

Suppose $X \sim N(0,1)$ and $Z \sim \chi^2_n$; X and Z independent. Then $E(X) \cdot E(\frac{\sqrt{n}}{\sqrt{Z}}) = 0$ for all n.

Also $\frac{X}{\sqrt{Z}} \sqrt{n} \sim t_n$

i.e. In particular, for $n = 1$,

$$\frac{X}{\sqrt{Z}} \sim t_1 = \text{Cauchy } (0,1).$$

$\therefore$ $E(\frac{X}{\sqrt{Z}})$ does not exist; whereas $E(X) \cdot E(\frac{1}{\sqrt{Z}}) = 0$.

We could easily construct similar examples for many other results involving expectations.

Reference

1. Cramer, H. (1962). Mathematical Methods of Statistics. Asia Publishing House.