

# A CLASSIFICATION OF GENUS 0 MODULAR CURVES WITH RATIONAL POINTS

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POINTS

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Let  $E$  be a non-CM elliptic curve defined over  $\mathbb{Q}$ . Fix an algebraic closure  $\overline{\mathbb{Q}}$  of  $\mathbb{Q}$ .  
We get a Galois representation

$$\rho_E: \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \rightarrow \text{GL}_2(\hat{\mathbb{Z}})$$

associated to  $E$  by choosing a compatible bases for the  $N$ -torsion subgroups of  $E(\overline{\mathbb{Q}})$ .  
Associated to an open subgroup  $G$  of  $\text{GL}_2(\hat{\mathbb{Z}})$  satisfying  $-I \in G$  and  $\det(G) = \hat{\mathbb{Z}}^\times$ , we  
have the modular curve  $(X_G, \pi_G)$  over  $\mathbb{Q}$  which loosely parametrises elliptic curves  
 $E$  such that the image of  $\rho_E$  is conjugate to a subgroup of  $G'$ . In this article we give  
a complete classification of all such genus 0 modular curves that have a rational  
point. This classification is given in finitely many families. Moreover, each such  
modular curve can be explicitly computed.

## **BIOGRAPHICAL SKETCH**

The author, Rakvi was born on August 6, 1993 in Darbhanga, Bihar. She is an Indian citizen. She was raised in Delhi and completed her schooling from Cambridge Foundation School, New Delhi. She received her Integrated M.Sc. degree in Mathematics from UM-DAE Centre for Excellence in Basic Sciences, Mumbai in June 2015. She has been pursuing Ph.D. in Mathematics from Cornell University, USA since August 2015.

This is dedicated to my parents for teaching me the values of sheer hard work and determination.

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All the computations that were done to complete this project were performed with Magma [Bos97].

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CHAPTER 1  
INTRODUCTION

Let  $E$  be a non-CM elliptic curve defined over  $\mathbb{Q}$ . Fix an algebraic closure  $\overline{\mathbb{Q}}$  of  $\mathbb{Q}$ . For a positive integer  $N$ , let  $E[N]$  denote the  $N$ -torsion subgroup of  $E(\overline{\mathbb{Q}})$ . As a  $\mathbb{Z}/N\mathbb{Z}$ -module  $E[N]$  is free of rank 2. Choosing a basis of  $E[N]$ , the natural action of  $\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$  on  $E[N]$  gives a representation,

$$\rho_{E,N}: \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \rightarrow \text{Aut}(E[N]) \simeq \text{GL}_2(\mathbb{Z}/N\mathbb{Z}).$$

By choosing compatible bases for  $E[N]$  with  $N \geq 1$ , we get a representation

$$\rho_E: \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \rightarrow \text{GL}_2(\hat{\mathbb{Z}})$$

such that the composition of  $\rho_E$  with the reduction modulo  $N$  map  $\text{GL}_2(\hat{\mathbb{Z}}) \rightarrow \text{GL}_2(\mathbb{Z}/N\mathbb{Z})$  is  $\rho_{E,N}$ .

The images of  $\rho_E$  and  $\rho_{E,N}$  are uniquely determined up to conjugacy in  $\text{GL}_2(\hat{\mathbb{Z}})$  and  $\text{GL}_2(\mathbb{Z}/N\mathbb{Z})$ , respectively. Serre's open image theorem [Ser72] says that the group  $\rho_E(\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}))$  is open and hence of finite index in  $\text{GL}_2(\hat{\mathbb{Z}})$ . We have  $\det(\rho_E(\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}))) = \hat{\mathbb{Z}}^\times$  since  $\det \circ \rho_E$  agrees with the cyclotomic character  $\chi_{\text{cyc}}: \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \rightarrow \hat{\mathbb{Z}}^\times$ . Recall that  $\chi_{\text{cyc}}$  satisfies the property  $\sigma(\zeta) = \zeta^{\chi_{\text{cyc}}(\sigma) \bmod n}$  for any integer  $n \geq 1$ ,  $n$ -th root of unity  $\zeta \in \overline{\mathbb{Q}}$  and  $\sigma \in \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ .

The following is a difficult open problem in number theory that serves as a moti-

vation for the classification problems related to modular curves, see section 1 of [Maz77].

**Mazur's Program B.** Given an open subgroup  $G$  of  $\mathrm{GL}_2(\hat{\mathbb{Z}})$ , classify all the elliptic curves  $E$  defined over  $\mathbb{Q}$  such that the image of  $\rho_E$  is conjugate in  $\mathrm{GL}_2(\hat{\mathbb{Z}})$  to a subgroup of  $G$ .

Let  $G$  be an open subgroup of  $\mathrm{GL}_2(\hat{\mathbb{Z}})$  satisfying  $\det(G) = \hat{\mathbb{Z}}^\times$  and  $-I \in G$ . Associated to  $G$ , there is a modular curve  $X_G$  which is a smooth, projective and geometrically connected curve defined over  $\mathbb{Q}$ . See chapter 2 for background on modular curves. The curve  $X_G$  loosely parametrises elliptic curves  $E$  for which the image of  $\rho_E$  is conjugate in  $\mathrm{GL}_2(\hat{\mathbb{Z}})$  to a subgroup of  $G'$  (here  $G'$  is the transpose of  $G$ ). An inclusion  $G \subseteq G'$  induces a natural morphism  $X_G \rightarrow X_{G'}$ . In particular, every  $X_G$  has an associated morphism  $\pi_G: X_G \rightarrow X_{\mathrm{GL}_2(\hat{\mathbb{Z}})} \simeq \mathbb{P}_{\mathbb{Q}}^1$ . Throughout this article, we will use the notation  $(X_G, \pi_G)$  to denote a modular curve with its morphism to the  $j$ -line.

In this article we provide a complete classification of all genus 0 modular curves  $(X_G, \pi_G)$  where  $X_G$  is isomorphic to  $\mathbb{P}_{\mathbb{Q}}^1$ .

Let  $X$  be a nice curve over  $\mathbb{Q}$  with a nonconstant morphism  $\pi_X: X \rightarrow \mathbb{P}_{\mathbb{Q}}^1$ . Let  $K$  be a Galois extension of  $\mathbb{Q}$ .

**Definition 1.0.1.** A  $K$ -twist of  $(X, \pi_X)$  is a pair  $(Y, \pi_Y)$  where  $Y$  is a curve over  $\mathbb{Q}$  and  $\pi_Y$  is a morphism  $Y \rightarrow \mathbb{P}_{\mathbb{Q}}^1$  such that there exists an isomorphism  $f: X_K \rightarrow Y_K$  which satisfies  $\pi_Y \circ f = \pi_X$ . We say that  $(X, \pi_X)$  and  $(Y, \pi_Y)$  are isomorphic if  $(Y, \pi_Y)$  is a

$\mathbb{Q}$ -twist of  $(X, \pi_X)$ .

Let  $\text{Aut}_K(X, \pi_X)$  be the subgroup of  $\text{Aut}(X_K)$  consisting of  $f$  such that  $\pi_X \circ f = \pi_X$ . For  $K = \mathbb{Q}$ , we will use the notation  $\text{Aut}(X, \pi_X)$  instead of  $\text{Aut}_{\mathbb{Q}}(X, \pi_X)$ .

Classifying genus 0 modular curves  $(X_G, \pi_G)$  up to isomorphism that have a rational point is equivalent to classifying open subgroups  $G \subseteq \text{GL}_2(\hat{\mathbb{Z}})$  up to conjugation that satisfy  $-I \in G$ ,  $\det(G) = \hat{\mathbb{Z}}^\times$ , for which there are infinitely many elliptic curves  $E$  over  $\mathbb{Q}$ , with distinct  $j$ -invariants, such that  $\rho_E(\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}))$  is conjugate to a subgroup of  $G$ . In this article we compute the modular curves  $(X_G, \pi_G)$  by computing their function fields and give a method to compute the generators for  $G$ . If  $X_G$  is isomorphic to  $\mathbb{P}_{\mathbb{Q}}^1$ , then the isomorphism class of  $(X_G, \pi_G)$  determines  $G \subseteq \text{GL}_2(\hat{\mathbb{Z}})$  up to conjugacy. See Lemma 2.3.4 in chapter 2 for a proof.

There are infinitely many subgroups  $G \subseteq \text{GL}_2(\hat{\mathbb{Z}})$  for which  $X_G$  has genus 0. Let us see an example of an infinite family of such groups  $G$ .

**Example 1.** Let  $d$  be a squarefree integer. Let  $G \subseteq \text{GL}_2(\hat{\mathbb{Z}})$  be the subgroup consisting of matrices whose image modulo 2 lies in the unique index 2 subgroup of  $\text{GL}_2(\mathbb{Z}/2\mathbb{Z})$ . Let  $\gamma: \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \rightarrow \{\pm 1\} \simeq \text{GL}_2(\hat{\mathbb{Z}})/G$  be the quadratic character corresponding to the extension  $\mathbb{Q}(\sqrt{d})/\mathbb{Q}$ . There is a unique homomorphism  $\phi: \hat{\mathbb{Z}}^\times \rightarrow \text{GL}_2(\hat{\mathbb{Z}})/G$  such that

$$\gamma = \phi \circ \chi_{\text{cyc}}.$$

Define

$$G_d := \{g \in \text{GL}_2(\hat{\mathbb{Z}}) \mid gG = \phi(\det g)\};$$

it is an open subgroup of  $\mathrm{GL}_2(\hat{\mathbb{Z}})$  of index 2 that has full determinant and contains  $-I$ . The group  $G_d \cap \mathrm{SL}_2(\mathbb{Z})$  does not depend on  $d$ ; i.e.,  $G_d \cap \mathrm{SL}_2(\mathbb{Z}) = G \cap \mathrm{SL}_2(\mathbb{Z})$  for every  $d$ .

The modular curve  $X_{G_d}$  is isomorphic to  $\mathbb{P}_{\mathbb{Q}}^1$  with the associated morphism  $\pi_{G_d}: X_{G_d} \rightarrow \mathbb{P}_{\mathbb{Q}}^1$  described by the rational function  $\pi_{G_d}(t) = dt^2 + 1728$ . The curves  $\{(X_{G_d}, \pi_{G_d})\}_d$  are  $\mathbb{Q}^{\mathrm{ab}}$ -twists of each other. Moreover,  $(X_{G_d}, \pi_{G_d})$  is a  $\mathbb{Q}(\sqrt{d})$ -twist of  $(X_{G_1}, \pi_{G_1})$ .

*Remark 1.0.1.* Any rational number is of the form  $dt^2 + 1728$  for some squarefree  $d$  and  $t \in \mathbb{Q}$ . Therefore, for every elliptic curve  $E$  over  $\mathbb{Q}$  with  $j(E) \notin \{0, 1728\}$ , the image of  $\rho_E$  is contained in  $G_d$  for some squarefree  $d$ .

Serre was the first to observe that  $\rho_E: \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \rightarrow \mathrm{GL}_2(\hat{\mathbb{Z}})$  is not surjective for all elliptic curves  $E/\mathbb{Q}$ , see Proposition 22 of [Ser72].

The following theorem states that we can classify all the genus 0 modular curves that have a rational point via finitely many of them.

**Theorem 1.0.2.** *There are nonconstant  $\pi_1, \dots, \pi_r \in \mathbb{Q}(t)$  and finite abelian subgroups  $\mathcal{A}_i \subseteq \mathrm{Aut}(\mathbb{P}_{\mathbb{Q}}^1, \pi_i) \subseteq \mathrm{PGL}_2(\mathbb{Q})$  such that the following hold:*

- For each  $i \in \{1, \dots, r\}$ ,  $(\mathbb{P}_{\mathbb{Q}}^1, \pi_i)$  is isomorphic to some modular curve  $(X_G, \pi_G)$ .
- For any modular curve  $X_G$  isomorphic to  $\mathbb{P}_{\mathbb{Q}}^1$ , there is an  $i \in \{1, \dots, r\}$ , such that  $(X_G, \pi_G)$  is a  $\mathbb{Q}^{\mathrm{ab}}$ -twist of  $(\mathbb{P}_{\mathbb{Q}}^1, \pi_i)$  via a cocycle  $\gamma: \mathrm{Gal}(\mathbb{Q}^{\mathrm{ab}}/\mathbb{Q}) \rightarrow \mathcal{A}_i \subseteq \mathrm{Aut}(\mathbb{P}_{\mathbb{Q}}^1, \pi_i)$ .

Moreover, we can explicitly compute  $\pi_i$  and  $\mathcal{A}_i$ .

Fix a pair  $(\pi_i, \mathcal{A}_i)$  as in Theorem 1.0.2. One can show that the twist of  $(\mathbb{P}_{\mathbb{Q}}^1, \pi_i)$  by a cocycle  $\gamma: \text{Gal}(\mathbb{Q}^{\text{ab}}/\mathbb{Q}) \rightarrow \mathcal{A}_i$  is isomorphic to some modular curve  $(X_G, \pi_G)$ . The curve  $X_G$  has genus 0 but it need not have a rational point. To give a classification we need to describe the twists with rational points.

We choose a  $u_i \in \mathbb{Q}(t)$  such that  $\mathbb{Q}(t)^{\mathcal{A}_i} = \mathbb{Q}(u_i)$ . Moreover,  $\text{Aut}(\mathbb{P}_{\mathbb{Q}}^1, u_i) = \mathcal{A}_i$ . We have  $\pi_i = J_i \circ u_i$  for a unique rational function  $J_i \in \mathbb{Q}(t)$ .

Take any  $v \in \mathbb{Q}$  with  $J_i(v) \notin \{0, 1728, \infty\}$ . Fix a point  $P \in u_i^{-1}(v) \subseteq \mathbb{P}^1(\overline{\mathbb{Q}})$ . Then, there exists a unique cocycle  $\gamma: \text{Gal}(\mathbb{Q}^{\text{ab}}/\mathbb{Q}) \rightarrow \mathcal{A}_i$  such that  $\sigma(P) = \gamma_\sigma(P)$  for all  $\sigma \in \text{Gal}(\mathbb{Q}^{\text{ab}}/\mathbb{Q})$ . We will construct a rational function  $\pi_{\mathcal{A}_i, v} \in \mathbb{Q}(t)$  such that  $(\mathbb{P}_{\mathbb{Q}}^1, J_i \circ \pi_{\mathcal{A}_i, v})$  is a twist of  $(\mathbb{P}_{\mathbb{Q}}^1, \pi_i)$  via  $\gamma$ . We will give an explicit  $\pi_{\mathcal{A}_i, v}$  in Theorem 1.0.3 for the  $\mathcal{A}_i \subseteq \text{PGL}_2(\mathbb{Q})$ , up to conjugacy, that occur in Theorem 1.0.2.

**Theorem 1.0.3.** *Let  $G$  be an open subgroup of  $\text{GL}_2(\widehat{\mathbb{Z}})$  with full determinant containing  $-I$ . If  $X_G$  is isomorphic to  $\mathbb{P}_{\mathbb{Q}}^1$ , then  $(X_G, \pi_G)$  is isomorphic to  $(\mathbb{P}_{\mathbb{Q}}^1, J_i \circ \pi_{\mathcal{A}_i, v})$  for some  $i \in \{1, \dots, r\}$  and  $v \in \mathbb{Q}$  with  $J_i(v) \notin \{0, 1728, \infty\}$ . Conversely, any such pair  $(\mathbb{P}_{\mathbb{Q}}^1, J_i \circ \pi_{\mathcal{A}_i, v})$  is isomorphic to a modular curve  $(X_G, \pi_G)$ . We list a function  $\pi_{\mathcal{A}_i, v}$  below for various cases of  $\mathcal{A}$  that occur.*

- For  $\mathcal{A} = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right\}$ , define  $\pi_{\mathcal{A}, v}(t) = t$ .
- For  $\mathcal{A} = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \right\} \simeq \mathbb{Z}/2\mathbb{Z}$ , define  $\pi_{\mathcal{A}, v}(t) = vt^2$ .
- For  $\mathcal{A} = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & \alpha \\ 1 & 0 \end{pmatrix} \right\} \simeq \mathbb{Z}/2\mathbb{Z}$  with  $\alpha$  a non-zero rational number which is not a square, define

$$\pi_{\mathcal{A}, v}(t) = \frac{vt^2 - 4\alpha t + \alpha v}{-t^2 + vt - \alpha}.$$

- For  $\mathcal{A} = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix}, \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix} \right\} \simeq \mathbb{Z}/3\mathbb{Z}$ , define  $\pi_{\mathcal{A},v}(t)$  as

$$\frac{(-v+3)t^3 + (-3v^2+9v-9)t^2 + (-3v^3+9v^2-15v)t + (-v^4+3v^3-6v^2-v+3)}{t^3 + 2vt^2 + (v^2+v-3)t + (v^2-3v+1)}.$$

- For  $\mathcal{A} = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} -1 & -1 \\ 1 & -1 \end{pmatrix}, \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \right\} \simeq \mathbb{Z}/4\mathbb{Z}$ , define  $\pi_{\mathcal{A},v}(t)$  as

$$\frac{-vt^4 + (8v+16)t^3 + (-18v-96)t^2 + (8v+176)t + (7v-96)}{t^4 + (v-8)t^3 + (-6v+18)t^2 + (11v-8)t + (-6v-7)}.$$

- For  $\mathcal{A} = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & \alpha \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & -\alpha \\ 1 & 0 \end{pmatrix} \right\} \simeq \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$  with  $\alpha$  a non-zero rational number, define  $\pi_{\mathcal{A},v}(t)$  as

$$\frac{(-3v\alpha^2 + v^3)\alpha^2 t^4 + (8\alpha^2 - 4v^2)\alpha^2 t^3 + 6v\alpha^2 t^2 - 8\alpha^2 t + v}{\alpha^4 t^4 - 2v\alpha^2 t^3 + (2\alpha^2 + v^2)t^2 - 2vt + 1}.$$

- For  $\mathcal{A} = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, \begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix} \right\} \simeq \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ , define  $\pi_{\mathcal{A},v}(t)$  as

$$\frac{(-25v^3 + 160v^2 - 256v)t^4 + (40v^3 - 208v^2 + 256v)t^3 + (-26v^3 + 96v^2 - 64v)t^2 + (8v^3 - 16v^2)t - v^3}{(6v^3 - 37v^2 + 64v - 64)t^4 + (-11v^3 + 56v^2 - 32v)t^3 + (6v^3 - 30v^2)t^2 + (-v^3 + 8v^2)t - v^2}.$$

- For  $\mathcal{A} = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & -5 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} -1 & 5 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 5 & 5 \\ 1 & -5 \end{pmatrix} \right\} \simeq \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ , define  $\pi_{\mathcal{A},v}(t)$  as  $P_1(t)/Q_1(t)$

where

$$\begin{aligned}
P_1(t) = & (-v^5 + 32v^4 - 336v^3 + 1280v^2 - 1600v)t^4 \\
& + (-4v^6 + 148v^5 - 1744v^4 + 5600v^3 + 16000v^2 - 64000v)t^3 \\
& + (-6v^7 + 252v^6 - 3376v^5 + 10880v^4 + 69600v^3 - 256000v^2 - 640000v)t^2 \\
& + (-4v^8 + 188v^7 - 2864v^6 + 10680v^5 + 82400v^4 - 368000v^3 - 1280000v^2)t \\
& + (-v^9 + 52v^8 - 896v^7 + 4120v^6 + 29500v^5 - 176000v^4 - 640000v^3)
\end{aligned}$$

and

$$\begin{aligned}
Q_1(t) = & (v^4 - 36v^3 + 240v^2 + 1600v - 14400)t^4 \\
& + (3v^5 - 128v^4 + 1320v^3 + 1920v^2 - 51200v)t^3 \\
& + (3v^6 - 149v^5 + 2056v^4 - 3600v^3 - 63200v^2 + 64000v)t^2 \\
& + (v^7 - 58v^6 + 1002v^5 - 4400v^4 - 20800v^3 + 96000v^2)t \\
& + (-v^7 + 26v^6 - 280v^5 + 3500v^4 - 16000v^3 - 160000v^2).
\end{aligned}$$

- For  $\mathcal{A} = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} -2 & 2 \\ 1 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ 1 & -1 \end{pmatrix}, \begin{pmatrix} 0 & -2 \\ 1 & 0 \end{pmatrix} \right\} \simeq \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ , define  $\pi_{\mathcal{A},v}(t)$  as  $P_2(t)/Q_2(t)$

where

$$\begin{aligned}
P_2(t) = & vt^4 + (2v^3 + 4v^2 - 24v)t^3 + (3v^5/2 + 6v^4 - 29v^3 - 68v^2 + 184v)t^2 \\
& + (v^7/2 + 3v^6 - 11v^5 - 62v^4 + 108v^3 + 320v^2 - 480v)t \\
& + (v^9/16 + v^8/2 - 5v^7/4 - 14v^6 + 41v^5/4 + 130v^4 - 80v^3 - 400v^2 + 400v)
\end{aligned}$$

and

$$\begin{aligned}
Q_2(t) = & t^4 + (3v^2/2 + 2v - 8)t^3 + (3v^4/4 + 7v^3/4 - 5v^2 - 8v + 24)t^2 \\
& + (v^6/8 + v^5/4 + v^4/4 + 3v^3 - 10v^2 + 8v - 32)t \\
& + (-v^7/16 + 3v^6/8 + 7v^5/2 - 29v^4/4 - 31v^3 + 64v^2 + 16).
\end{aligned}$$

## 1.0.1 Overview

In § 1.1 we discuss an overview of proofs of Theorem 1.0.2 and Theorem 1.0.3. In chapter 2, we discuss preliminaries and background about modular curves and modular functions. In chapter 3, we discuss the computation of hauptmoduls for genus 0 congruence subgroups. In chapter 4, we discuss some background material about cocycles and twists and give a method to determine if certain cocycles are coboundaries. In chapter 5, we give a description of the groups that show up in family of twists. In chapter 6, we discuss the parametrization of finite Galois extensions  $K$  of  $\mathbb{Q}$  and give a unique homomorphism  $\gamma: \text{Gal}(K/\mathbb{Q}) \rightarrow \text{Aut}(X_G, \pi_G)$  such that  $\gamma$  is a coboundary. In § 7.1, we give a proof of Theorem 1.0.2 and in § 7.2 we give a proof of Theorem 1.0.3. In chapter 8, we give a procedure to determine generators for an open subgroup  $G$  of  $\text{GL}_2(\hat{\mathbb{Z}})$  such that  $X_G$  is isomorphic to  $\mathbb{P}_{\mathbb{Q}}^1$ . We also give a method to determine  $X_G$  explicitly as a conic from a set of generators for  $G$ . Finally, in appendix A, we give a guide to go through the tables and give our tables in the end.

## 1.0.2 Notations

We will use  $\mathbb{Q}^{\text{ab}}$  to denote the maximal abelian extension of  $\mathbb{Q}$ . We say that a curve is nice if it is smooth, projective and geometrically irreducible. We will denote the  $j$ -invariant of an elliptic curve  $E$  by  $j(E)$ . Let  $N$  be a positive integer. We will use  $\zeta_N$  to denote  $e^{2\pi i/N} \in \mathbb{C}$ . We will denote the  $N$ -th cyclotomic field  $\mathbb{Q}(\zeta_N)$  by  $K_N$ . Let  $\mathbb{Q}^{\text{cyc}}$  be the compositum of cyclotomic fields  $K_N$  with  $N \geq 1$ . By Kronecker-Weber theorem  $\mathbb{Q}^{\text{ab}}$  is equal to the field  $\mathbb{Q}^{\text{cyc}}$ . For a prime  $p$ , we will use  $\mathbb{Q}_p$  to denote the field of  $p$ -adic numbers. All the topological groups have profinite topology.

## 1.1 Overview of Proof of Theorem 1.0.2 and Theorem 1.0.3

Let us give a brief description of the procedure we use to classify all the modular curves  $(X_G, \pi_G)$ , up to isomorphism such that  $X_G$  is isomorphic to  $\mathbb{P}_{\mathbb{Q}}^1$ .

For an open group  $G \subseteq \text{GL}_2(\hat{\mathbb{Z}})$  satisfying  $-I \in G$  and  $\det(G) = \hat{\mathbb{Z}}^\times$  define the congruence subgroup  $\Gamma := G \cap \text{SL}_2(\mathbb{Z})$  of  $\text{SL}_2(\mathbb{Z})$ . As a Riemann surface  $X_G(\mathbb{C})$  is isomorphic to the compact Riemann surface  $X(\Gamma)$  obtained after adding cusps to the quotient  $\Gamma \backslash \mathcal{H}$ . In particular, the genus of  $X_G$  is equal to the genus of  $X(\Gamma)$  which we call the genus of  $\Gamma$ . See chapter 2 for more details. The connection of these modular curves with congruence subgroups is a key point which we exploit to give a classification.

Fix a genus 0 congruence subgroup  $\Gamma \subseteq \text{SL}_2(\mathbb{Z})$  containing  $-I$  of level  $N$  see § 2.1

for a definition of level of congruence subgroups. Our goal is to compute all the pairs  $(X_G, \pi_G)$  such that  $X_G$  is  $\mathbb{P}_{\mathbb{Q}}^1$  and  $G \cap \mathrm{SL}_2(\mathbb{Z}) = \Gamma$ .

We compute an explicit function  $h$  such that  $\mathbb{C}(X_{\Gamma}) = \mathbb{C}(h)$ , see definition 2.3.3 for a definition of  $X_{\Gamma}$ . We compute the function  $\pi_{\Gamma} \in K_N(t)$  which satisfies  $\pi_{\Gamma}(h) = j$ . This function describes the morphism  $\pi_{\Gamma}: X_{\Gamma} \rightarrow \mathbb{P}_{K_N}^1$ .

We then search for a genus 0 modular curve  $(X_{G_1}, \pi_{G_1})$  satisfying the property that the modular curve  $(X_{G_1}, \pi_{G_1})$  is isomorphic to  $(X_{\Gamma}, \pi_{\Gamma})$  over some  $K_M$ , where  $M$  is a multiple of  $N$  and is isomorphic to  $\mathbb{P}_{\mathbb{Q}}^1$ . We either succeed in finding such a pair or we observe that there exists no such modular curve  $(X_G, \pi_G)$  such that  $G \cap \mathrm{SL}_2(\mathbb{Z}) = \Gamma$ .

Then, we search for all the genus 0 modular curves  $(X_G, \pi_G)$  with a rational point which are isomorphic to  $(X_{G_1}, \pi_{G_1})$  over  $\mathbb{Q}^{\mathrm{ab}}$ . After computing finite number of twists we end up considering homomorphisms  $\phi: \mathrm{Gal}(\mathbb{Q}^{\mathrm{ab}}/\mathbb{Q}) \rightarrow \mathcal{A}$ , where  $\mathcal{A}$  is a finite subgroup of  $\mathrm{PGL}_2(\mathbb{Q})$ . In the example 1,  $\mathcal{A}$  is  $\left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \right\}$ . Therefore, we end up having an infinite family  $\{X_{G_D}\}$  such that two of them become isomorphic over a quadratic extension of  $\mathbb{Q}$ .

CHAPTER 2  
MODULAR CURVES AND MODULAR FUNCTIONS

## 2.1 Congruence subgroups

Let  $\Gamma$  be a congruence subgroup of  $\mathrm{SL}_2(\mathbb{Z})$ . The level of  $\Gamma$  is the smallest positive integer  $N$  for which  $\Gamma$  contains

$$\Gamma(N) := \{A \in \mathrm{SL}_2(\mathbb{Z}) \mid A \equiv I \pmod{N}\}.$$

The group  $\mathrm{SL}_2(\mathbb{Z})$  acts on the complex upper half plane  $\mathcal{H}$  by linear fractional transformations.

We define the extended upper half plane  $\mathcal{H}^*$  as  $\mathcal{H} \cup \mathbb{Q} \cup \{\infty\}$ . The action of  $\mathrm{SL}_2(\mathbb{Z})$  on  $\mathcal{H}$  naturally extends to the action on  $\mathcal{H}^*$ . A cusp of  $\Gamma$  is a  $\Gamma$ -orbit of  $\mathbb{Q} \cup \{\infty\}$ . After adding cusps to the quotient  $\Gamma \backslash \mathcal{H}$ , we get a smooth compact Riemann surface which we denote by  $X(\Gamma)$ . We will use the notation  $X_N$  to denote  $X(\Gamma(N))$ , where  $N$  is a positive integer.

## 2.2 Modular functions

Let  $\Gamma$  be a congruence subgroup of  $\mathrm{SL}_2(\mathbb{Z})$ . A modular function  $f$  for  $\Gamma$  is a meromorphic function of  $X(\Gamma)$  i.e.,  $f$  is a meromorphic function on the upper half plane

$\mathcal{H}$  that satisfies  $f(\gamma\tau) = f(\tau)$  for all  $\gamma \in \Gamma$  and is meromorphic at the cusps.

The width of the cusp at  $\infty$  is the smallest positive integer  $w$  such that  $\begin{pmatrix} 1 & w \\ 0 & 1 \end{pmatrix} \in \Gamma$ . Define  $q := e^{2\pi i\tau}$  and  $q^{1/w} := e^{2\pi i\tau/w}$ , where  $\tau \in \mathcal{H}$ . If  $f$  is a modular function for  $\Gamma$ , then  $f$  has a unique  $q$ -expansion  $f(\tau) = \sum_{n \in \mathbb{Z}} c_n q^{n/w}$  with  $c_n \in \mathbb{C}$  and  $c_n = 0$  for all but finitely many  $n$  less than 0.

Fix a positive integer  $N$ . Let  $\mathcal{F}_N$  denote the field of meromorphic functions of the Riemann surface  $X_N$  whose  $q$ -expansion has coefficients in  $K_N$ . For  $N = 1$ , we have  $\mathcal{F}_1 = \mathbb{Q}(j)$  where  $j$  is the modular  $j$ -invariant. The first few terms of  $q$ -expansion of  $j$  are  $q^{-1} + 744 + 196884q + 21493760q^2 + \dots$ . If  $N'$  is a divisor of  $N$ , then  $\mathcal{F}_{N'} \subseteq \mathcal{F}_N$ . In particular, we have that  $\mathcal{F}_1 \subseteq \mathcal{F}_N$ .

Take  $g \in \mathrm{GL}_2(\mathbb{Z}/N\mathbb{Z})$  and let  $d = \det(g)$ . Then,  $g = \begin{pmatrix} 1 & 0 \\ 0 & d \end{pmatrix} g'$  for some  $g' \in \mathrm{SL}_2(\mathbb{Z}/N\mathbb{Z})$ . There is a unique right action  $*$  of  $\mathrm{GL}_2(\mathbb{Z}/N\mathbb{Z})$  on  $\mathcal{F}_N$  that satisfies the following properties (see Chapter 2, section 2 in [KL81] for details):

- If  $g \in \mathrm{SL}_2(\mathbb{Z}/N\mathbb{Z})$ , then

$$f * g := f|_\gamma,$$

where  $\gamma \in \mathrm{SL}_2(\mathbb{Z})$  is congruent to  $g$  modulo  $N$  and  $f|_\gamma(\tau) = f(\gamma\tau)$ .

- If  $g = \begin{pmatrix} 1 & 0 \\ 0 & d \end{pmatrix}$ , then

$$f * g := \sigma_d(f) := \sum_{n \in \mathbb{Z}} \sigma_d(c_n) q^{n/w},$$

where  $f = \sum_{n \in \mathbb{Z}} c_n q^{n/w}$  and  $\sigma_d$  is the automorphism of  $K_N$  that satisfies  $\sigma_d(\zeta_N) = \zeta_N^d$ .

**Proposition 2.2.1.**

- (i)  $-I$  acts trivially on  $\mathcal{F}_N$  and the action  $*$  of  $\mathrm{GL}_2(\mathbb{Z}/N\mathbb{Z})/\{\pm I\}$  on  $\mathcal{F}_N$  is faithful.
- (ii) We have  $\mathcal{F}_N^{\mathrm{GL}_2(\mathbb{Z}/N\mathbb{Z})/\{\pm I\}} = \mathbb{Q}(j)$ .
- (iii) The extension  $\mathcal{F}_N$  over  $\mathbb{Q}(j)$  is Galois and the action  $*$  gives an isomorphism

$$\mathrm{GL}_2(\mathbb{Z}/N\mathbb{Z})/\{\pm I\} \rightarrow \mathrm{Gal}(\mathcal{F}_N/\mathbb{Q}(j))^{op},$$

where  $\mathrm{Gal}(\mathcal{F}_N/\mathbb{Q}(j))^{op}$  is the opposite group of  $\mathrm{Gal}(\mathcal{F}_N/\mathbb{Q}(j))$  (i.e., the same underlying set with the group law reversed.)

- (iv) The algebraic closure of  $\mathbb{Q}$  in  $\mathcal{F}_N$  is  $K_N$ . We have  $\mathcal{F}_N^{\mathrm{SL}_2(\mathbb{Z}/N\mathbb{Z})/\{\pm I\}} = K_N(j)$ .

*Proof.* Refer to Theorem 6.6, Chapter 6 in [Shi94]. □

*Remark 2.2.1.* We require the opposite in Proposition 2.2.1, (iii) since  $*$  is a right action.

## 2.3 Modular curves

Let  $G$  be an open subgroup of  $\mathrm{GL}_2(\hat{\mathbb{Z}})$  satisfying  $-I \in G$  and  $\det(G) = \hat{\mathbb{Z}}^\times$ . We define the level of  $G$  as the smallest positive integer  $M$  such that  $G$  is the inverse image of its image under the reduction modulo  $M$  map  $\pi_M: \mathrm{GL}_2(\hat{\mathbb{Z}}) \rightarrow \mathrm{GL}_2(\mathbb{Z}/M\mathbb{Z})$ .

Let  $M$  be the level of  $G$ . Let  $G_M \subseteq \mathrm{GL}_2(\mathbb{Z}/M\mathbb{Z})$  be the image of  $G$  under  $\pi_M$ . Since  $G_M$  has full determinant, it will follow that  $\mathbb{Q}$  is algebraically closed in  $\mathcal{F}_M^{G_M}$  by Proposition 2.2.1, (iii). The field  $\mathcal{F}_M^{G_M}$  has transcendence degree 1 over  $\mathbb{Q}$  since it is a finite extension of  $\mathbb{Q}(j)$ .

**Definition 2.3.1.** The modular curve  $X_G$  is the nice curve over  $\mathbb{Q}$  whose function field is  $\mathcal{F}_M^{G_M}$ . The map  $\pi_G: X_G \rightarrow \mathbb{P}_{\mathbb{Q}}^1$  is the nonconstant morphism corresponding to the inclusion  $\mathbb{Q}(j) \subseteq \mathcal{F}_M^{G_M}$ .

The following lemma partially explains how  $(X_G, \pi_G)$  parametrises elliptic curves.

**Lemma 2.3.2.** *For an elliptic curve  $E$  defined over  $\mathbb{Q}$  with  $j(E) \notin \{0, 1728\}$ , the group  $\rho_E(\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}))$  is conjugate in  $\mathrm{GL}_2(\hat{\mathbb{Z}})$  to a subgroup of  $G^t$  if and only if  $j(E) \in \pi_G(X_G(\mathbb{Q}))$ .*

*Proof.* Refer to section 3.5 of [Zyw15]. The transpose shows up because  $*$  is a right action. □

*Remark 2.3.1.* One could also define  $X_G$  as the generic fiber of the coarse space for the stack  $M_G$  which parametrizes elliptic curves with  $G$ -level structure (refer to [DR73] chapter 4, section 3 for details).

Let  $\Gamma$  be a congruence subgroup of level  $N$  that contains  $-I$ . Let  $\Gamma_N \subseteq \mathrm{SL}_2(\mathbb{Z}/N\mathbb{Z})$  be the image of  $\Gamma$  under reduction modulo  $N$ .

**Definition 2.3.3.** The algebraic curve  $X_{\Gamma}$  is the nice curve over  $K_N$  whose function field is  $\mathcal{F}_N^{\Gamma_N}$ . The map  $\pi_{\Gamma}: X_{\Gamma} \rightarrow \mathbb{P}_{K_N}^1$  is the morphism corresponding to the inclusion

$$K_N(j) \subseteq \mathcal{F}_N^{\Gamma_N}.$$

Using the inclusion  $K_N \subsetneq \mathbb{C}$ , we can identify  $X_\Gamma(\mathbb{C})$  with the Riemann surface  $X(\Gamma)$ .

Using Proposition 2.2.1, (iv) we know that the field  $K_M(X_G)$  which is the function field of the base extension of  $X_G$  to  $K_M$  is the function field  $K_M(X_\Gamma)$ , where  $\Gamma$  is the congruence subgroup  $G \cap \mathrm{SL}_2(\mathbb{Z})$ . It implies that there exists an isomorphism  $f: (X_\Gamma)_{K_M} \rightarrow (X_G)_{K_M}$  that satisfies  $\pi_G \circ f = \pi_\Gamma$ .

In the following lemma we prove that classifying modular curves  $(X_G, \pi_G)$  up to isomorphism is equivalent to classifying open subgroups  $G$  of  $\mathrm{GL}_2(\hat{\mathbb{Z}})$  up to conjugation in  $\mathrm{GL}_2(\hat{\mathbb{Z}})$ .

**Lemma 2.3.4.** *Let  $G$  and  $G'$  be two open subgroups of  $\mathrm{GL}_2(\hat{\mathbb{Z}})$  with full determinant and containing  $-I$ . Then,  $G$  and  $G'$  are conjugate in  $\mathrm{GL}_2(\hat{\mathbb{Z}})$  if and only if  $(X_G, \pi_G)$  and  $(X_{G'}, \pi_{G'})$  are isomorphic.*

*Proof.* Let us assume that  $G$  and  $G'$  are conjugate in  $\mathrm{GL}_2(\hat{\mathbb{Z}})$ , i.e.,  $G' = gGg^{-1}$  for some  $g \in \mathrm{GL}_2(\hat{\mathbb{Z}})$ . It is easy to check that  $G$  and  $G'$  have the same level. Let  $M$  be that level. Then, the map  $\mathcal{F}_M^{G_M} \rightarrow \mathcal{F}_M^{G'_M}$  that sends  $f$  to  $f * g^{-1}$  is an isomorphism that fixes  $\mathbb{Q}(j)$ . Therefore, we get a corresponding isomorphism between  $(X_G, \pi_G)$  and  $(X_{G'}, \pi_{G'})$ .

Let us assume that  $(X_G, \pi_G)$  and  $(X_{G'}, \pi_{G'})$  are isomorphic, i.e., there exists an isomorphism  $\bar{\phi}: \mathbb{Q}(X_G) \rightarrow \mathbb{Q}(X_{G'})$  that fixes  $\mathbb{Q}(j)$ . Let  $M$  and  $M'$  be respectively

the levels of  $G$  and  $G'$ . We will use  $G$  and  $G'$  to denote their image modulo  $M$  and  $M'$  respectively. We have the inclusions  $\mathbb{Q}(j) \subseteq \mathbb{Q}(X_G) \subseteq \mathcal{F}_{MM'}$  and  $\mathbb{Q}(j) \subseteq \mathbb{Q}(X_{G'}) \subseteq \mathcal{F}_{MM'}$ . There exists an element  $\phi \in \text{Gal}(\mathcal{F}_{MM'}/\mathbb{Q}(j))$  such that  $\phi|_{\mathbb{Q}(X_{G'})} = \bar{\phi}$ . Let  $g \in \text{GL}_2(\mathbb{Z}/MM'\mathbb{Z})$  be the corresponding element from Proposition 2.2.1, (iii), i.e., the map  $\phi: \mathbb{Q}(X_G) \rightarrow \mathbb{Q}(X_{G'})$  sends  $f$  to  $f * g$ . We will now show that  $gG'g^{-1} \subseteq G$ . Let  $a \in gG'g^{-1}$ , i.e.,  $a = ga'g^{-1}$  for some  $a' \in G'$ . For any  $f \in \mathbb{Q}(X_G)$ ,  $(f * g) * a' = f * g$ . Therefore,  $f * (ga'g^{-1}) = f$ . Hence,  $a = ga'g^{-1} \in G$ . Since, the degree of  $\pi_G$  is equal to the degree of  $\pi_{G'}$  we can conclude that  $gG'g^{-1} = G$ .

□

CHAPTER 3  
COMPUTING HAUPTMODULS

Let  $\Gamma$  be any genus 0 congruence subgroup of  $\mathrm{SL}_2(\mathbb{Z})$  of level  $N$ . Since  $\Gamma$  has genus 0, the function field  $\mathbb{C}(X(\Gamma))$  over  $\mathbb{C}$  is generated by a single modular function  $h$ . We can choose  $h$  so that it has a simple pole at any fixed point of  $X(\Gamma)$ .

A hauptmodul of  $\Gamma$  is a meromorphic function  $h$  on  $X(\Gamma)$  with a unique pole at the cusp  $\infty$  and this pole is simple. We have  $\mathbb{C}(X(\Gamma)) = \mathbb{C}(h)$ .

Fix a hauptmodul  $h$  of  $\Gamma$ . Every other hauptmodul of  $\Gamma$  is of the form  $ah + b$  for a unique  $a \in \mathbb{C}^\times$  and  $b \in \mathbb{C}$ . We say that a hauptmodul is normalized if the leading coefficient of its  $q$ -expansion is 1 and the constant coefficient is 0. A normalized hauptmodul of  $\Gamma$  is unique and all the coefficients of its  $q$ -expansion lie in  $K_N$ . This can be proven directly; it will also follow from our computations. To get a normalized hauptmodul one can rescale a given hauptmodul such that the constant coefficient is 0 and the leading coefficient is 1.

In this section, we will discuss computation of hauptmoduls. Our goal is to describe how to express the normalized hauptmodul of  $\Gamma$  in terms of explicit Siegel functions. This explicit description allows us to compute arbitrarily many terms of the  $q$ -expansion of the hauptmodul at each cusp of  $\Gamma$ .

Suppose that  $h$  is the normalized hauptmodul of  $\Gamma$ . Let  $\gamma\Gamma\gamma^{-1}$  be a conjugate of  $\Gamma$ , where  $\gamma \in \mathrm{SL}_2(\mathbb{Z})$ . Then,  $h' := h|_{\gamma^{-1}}$  is a meromorphic function on  $X(\gamma\Gamma\gamma^{-1})$ . If the pole of  $h'$  is at  $\infty$ , then  $h'$  is a hauptmodul. Otherwise,  $1/(h' - a_0)$  is a meromorphic

function on  $X(\gamma\Gamma\gamma^{-1})$  with the pole at  $\infty$ , where  $a_0$  is the constant coefficient of  $h'$ . Therefore, it suffices to compute the hauptmodul for representatives of the conjugacy classes of genus 0 congruence subgroups in  $\mathrm{SL}_2(\mathbb{Z})$ .

There are only finitely many congruence subgroups of genus 0. Moreover, the genus 0 congruence subgroups of  $\mathrm{SL}_2(\mathbb{Z})$  containing  $-I$  are listed, up to conjugation in  $\mathrm{GL}_2(\mathbb{Z})$ , by Cummins and Pauli in [CP03]. To obtain a list up to conjugation of genus 0 congruence subgroups containing  $-I$  in  $\mathrm{SL}_2(\mathbb{Z})$  we can do the following. Let  $A = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$  for the rest of this section. Fix a genus 0 congruence subgroup  $\Gamma$  in [CP03]. Let  $\Gamma' := A\Gamma A^{-1}$ . It can be checked that  $\Gamma$  and  $\Gamma'$  represent at most two conjugacy classes in  $\mathrm{SL}_2(\mathbb{Z})$ .

We will use the labels from [CP03] to denote their explicit genus 0 congruence subgroups whenever we compute the hauptmodul with respect to the set of generators listed in [CP03].

### 3.1 Siegel functions

We will briefly discuss the definition and key properties of Siegel functions.

Take  $a = (a_1, a_2) \in \mathbb{Q}^2 \setminus \mathbb{Z}^2$ . Define  $q_z := e^{2\pi iz}$  and recall that  $q = e^{2\pi i\tau}$ , where  $\tau \in \mathcal{H}$  and  $z = a_1\tau + a_2$ . The Siegel function  $g_a$  is the holomorphic function  $\mathcal{H} \rightarrow \mathbb{C}^\times$  given by

$$g_a(\tau) = -q^{(1/2)B_2(a_1)} e^{2\pi i a_2(a_1-1)/2} (1 - q_z) \prod_{n=1}^{\infty} (1 - q^n q_z)(1 - q^n / q_z)$$

where  $B_2(X) = X^2 - X + 1/6$  is the second Bernoulli polynomial.

**Lemma 3.1.1.** *Take  $a = (a_1, a_2) \in \mathbb{Q}^2 \setminus \mathbb{Z}^2$  and  $b = (b_1, b_2) \in \mathbb{Z}^2$ . The following properties are satisfied by Siegel functions:*

(i)  $g_{-a} = -g_a$ .

(ii)  $g_{a+b}(\tau) = \epsilon(a, b)g_a(\tau)$ , where  $\epsilon(a, b) = (-1)^{b_1+b_2+b_1b_2} e^{2\pi i(b_2a_1 - a_2b_1)/2}$

(iii) For any  $\gamma \in \mathrm{SL}_2(\mathbb{Z})$ , we have  $g_a(\gamma\tau) = \varepsilon(\gamma)g_{a\gamma}(\tau)$ , where  $\varepsilon: \mathrm{SL}_2(\mathbb{Z}) \rightarrow \mathbb{C}^\times$  is the group homomorphism satisfying  $\varepsilon\left(\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\right) = \zeta_{12}$  and  $\varepsilon\left(\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\right) = \zeta_4$ .

*Proof.* Property (ii) follows from K2 in Section 1, Chapter 2 of [KL81]. Property (iii) follows using K0 and K1 of Section 1, Chapter 2 of [KL81] and the following transformation property of Dedekind Eta function. Recall that the Dedekind Eta function is a holomorphic function on  $\mathcal{H}$  given by  $\eta(\tau) = q^{1/24} \prod_{n=1}^{\infty} (1 - q^n)$  and for  $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$ ,  $\eta(\gamma\tau)^2 = \varepsilon(\gamma)(c\tau + d)\eta(\tau)^2$ . Property (i) follows from property (iii) by taking  $\gamma = -I$ .

□

Let  $A_N$  be the set of elements in  $(1/N)\mathbb{Z}^2 \setminus \mathbb{Z}^2$  satisfying one of the following conditions.

- $0 < a_1 < 1/2$  and  $0 \leq a_2 < 1$
- $a_1 = 0$  and  $0 < a_2 \leq 1/2$

- $a_1 = 1/2$  and  $0 \leq a_2 \leq 1/2$

For each  $a \in N^{-1}\mathbb{Z}^2 \setminus \mathbb{Z}^2$ , there is a unique element  $a' \in A_N$  such that  $a = \pm a' + m$  for some  $m$  in  $\mathbb{Z}^2$ . Using properties (i) and (ii) of Lemma 3.1.1 we get that  $g_a = \zeta g_{a'}$ , where  $\zeta$  is an explicit  $2N$ -th root of unity.

The map  $(N^{-1}\mathbb{Z}^2 \setminus \mathbb{Z}^2) \times \mathrm{SL}_2(\mathbb{Z}) \rightarrow N^{-1}\mathbb{Z}^2 \setminus \mathbb{Z}^2$  that sends  $(a, \gamma) \mapsto a\gamma$  is a right action of  $\mathrm{SL}_2(\mathbb{Z})$  on  $N^{-1}\mathbb{Z}^2 \setminus \mathbb{Z}^2$ . This induces a right action of  $\mathrm{SL}_2(\mathbb{Z})$  on  $A_N$  given by  $a * \gamma := (a\gamma)'$ .

The following is Lemma 4.2 from section 4 of [SZ17] which gives us an explicit modular function for  $\Gamma$  which we use later to construct hauptmoduls.

**Lemma 3.1.2.** *Let  $O$  be a  $\Gamma$ -orbit of  $A_N$  and let  $g_O$  be the holomorphic function  $\mathcal{H} \rightarrow \mathbb{C}^\times$  which is defined as  $\prod_{a \in O} g_a$ . Then,  $g_O^{12N}$  is a modular function for  $\Gamma$ . Every pole and zero of  $g_O^{12N}$  on  $X(\Gamma)$  is a cusp.*

## 3.2 Multiple cusps

Assume that  $\Gamma$  has more than one cusp. In this part, we will discuss how to compute a hauptmodul for  $\Gamma$ .

Let  $O_1, \dots, O_n$  be the distinct  $\Gamma$ -orbits of  $A_N$ . For each  $O_i$ , let  $D_i$  be the divisor of  $g_{O_i}^{12N}$  on  $X(\Gamma)$ . The divisor  $D_i$  is supported on the cusps of  $\Gamma$ . Let  $P_1, \dots, P_r$  be the distinct cusps of  $\Gamma$  and let  $w_j$  be the width of the cusp  $P_j$ . For each  $1 \leq j \leq r$ , let  $A_j$

be a matrix in  $\mathrm{SL}_2(\mathbb{Z})$  such that  $A_{j,\infty} = s_j$ , where  $s_j \in \mathbb{Q} \cup \{\infty\}$  is a representative of the cusp  $P_j$ . Lemma 4.3 of [SZ17] states that

$$D_i = \sum_{j=1}^r \left( 6Nw_j \sum_{a \in O_i} B_2(\{(aA_j)_1\}) \right) \cdot P_j,$$

where for  $x \in \mathbb{R}$ ,  $\{x\}$  satisfies  $0 \leq \{x\} < 1$  and  $x - \{x\} \in \mathbb{Z}$ .

The following is Lemma 4.4 from section 4 of [SZ17]; we use it to explicitly construct hauptmoduls when  $\Gamma$  has multiple cusps.

**Lemma 3.2.1.** *Suppose there is an  $m \in \mathbb{Z}^n$  such that  $\sum_{i=1}^n m_i D_i = -12N \cdot P_1 + 12N \cdot P_2$  where  $P_1$  is the cusp at infinity and  $P_2$  is any other cusp of  $\Gamma$ . Let  $0 \leq e < 2N^2$  be an integer satisfying*

$$e \equiv \sum_{i=1}^n m_i \sum_{a \in O_i} Na_2(N - Na_1) \pmod{2N^2}.$$

*Then  $h := (\zeta_{2N^2})^e \prod_{i=1}^n g_{O_i}^{m_i}$  is a hauptmodul for  $\Gamma$  whose  $q$ -expansion has coefficients in  $K_N$ . On  $X(\Gamma)$ , we have  $\mathrm{div}(h) = -P_1 + P_2$ .*

To verify the existence of an  $n$ -tuple  $m$  as described in the previous lemma we used the generators for  $\Gamma$  as described in [CP03] and the set of generators obtained after conjugation with  $A$  (if  $\Gamma$  and  $\Gamma'$  are not conjugate in  $\mathrm{SL}_2(\mathbb{Z})$ ). The existence of  $n$ -tuple was verified using Magma for all genus 0 congruence subgroups with multiple cusps except those with labels  $10E^0$ ,  $12F^0$  and  $12H^0$ . It can be checked that the congruence subgroups with labels  $10E^0$ ,  $12F^0$  and  $12H^0$  are listed up to conjugation in  $\mathrm{SL}_2(\mathbb{Z})$ . We describe hauptmoduls for  $10E^0$  and  $12F^0$  later in the

section. For the genus 0 congruence subgroup with label  $12H^0$ , the  $n$ -tuple exists when we first conjugate the generators with  $\begin{pmatrix} 1 & 0 \\ 4 & 1 \end{pmatrix}$ .

We describe the hauptmoduls for congruence subgroups with label  $10E^0$  and  $12F^0$  below. These hauptmoduls are computed with respect to a set of generators modulo  $N$  denoted by  $S_\Gamma$ . Here  $N$  is the level of  $\Gamma$ .

*Remark 3.2.1.* In the following cases the function  $h$  comes from section 5.3 of [CLY04]. Unfortunately, their tables are no longer available. They compute modular functions in terms of "generalized Dedekind eta functions". We take their function  $h$  and then essentially adjust the constants to get a modular function for  $\Gamma$ .

- (i) Consider the group  $\Gamma$  with label  $12F^0$ . Here  $g_{(a_1, a_2)}$  denotes the Siegel function  $g_{(a_1/12, a_2/12)}(\tau)$ .

The function  $h$  listed below is fixed by  $\Gamma$ . After normalizing the  $q$ -expansion of  $h$  we find that the coefficients lie in  $K_{12}$  and it has a unique simple pole at the cusp  $\infty$ .

$$h = g_{(3,11)}g_{(3,8)}g_{(6,5)}g_{(6,1)}g_{(3,7)}g_{(3,4)}g_{(1,8)}g_{(1,5)}g_{(5,4)}g_{(5,1)}g_{(2,1)}g_{(2,7)} \\ + \zeta_{24}^3 g_{(5,2)}g_{(1,3)}g_{(1,6)}g_{(5,11)}g_{(4,3)}g_{(4,7)}g_{(4,9)}g_{(4,1)}g_{(5,3)}g_{(5,6)}g_{(1,7)}g_{(1,10)}.$$

- (ii) Consider the group  $\Gamma$  with label  $10E^0$ . Here  $g_{(a_1, a_2)}$  denotes the Siegel function  $g_{(a_1/10, a_2/10)}(\tau)$ .

The function  $h$  listed below is fixed by  $\Gamma$ . After normalizing the  $q$ -expansion of  $h$  we find that the coefficients lie in  $K_{10}$  and it has a unique simple pole at the cusp  $\infty$ .

$$\begin{aligned} h &= g_{(1,4)}g_{(5,2)}g_{(5,4)}g_{(3,2)}(g_{(1,3)}g_{(3,9)}g_{(1,5)}g_{(3,5)})^{-1} \\ &+ \zeta_{10}^3 g_{(1,7)}g_{(1,1)}g_{(3,1)}g_{(3,3)}(g_{(0,1)}g_{(2,3)}g_{(4,1)}g_{(0,3)})^{-1} \\ &+ \zeta_{10}^2 g_{(2,5)}g_{(4,5)}g_{(2,1)}g_{(4,7)}(g_{(1,2)}g_{(1,6)}g_{(3,6)}g_{(3,8)})^{-1}. \end{aligned}$$

### 3.3 Single cusp

In this part, we will assume that  $\Gamma$  has a single cusp. In this case, we cannot express hauptmodul of  $\Gamma$  as a product of Siegel functions (since such functions are modular units, i.e., their zeroes and poles lie at the cusps).

Let us first assume that there exists a proper genus 0 congruence subgroup  $\Gamma' \subseteq \Gamma$  containing  $-I$  of level  $N$  such that  $[\Gamma : \Gamma']$  is equal to the number of cusps of  $\Gamma'$ . From the previous subsection, we can construct a hauptmodul  $h'$  of  $\Gamma'$ .

**Lemma 3.3.1.** *The modular function  $h := \sum_{\alpha \in \Gamma' \setminus \Gamma} h'|_{\alpha}$  is a hauptmodul for  $\Gamma$  with coefficients in  $K_N$ .*

*Proof.* Refer to [SZ17], section 4.3.2. □

### 3.4 Remaining cases

In this part, we express the hauptmodul for remaining single cusp cases in terms of Siegel functions. In each of these cases there exists a proper congruence subgroup  $\Gamma' \subseteq \Gamma$  containing  $-I$  of level  $N$  (not necessarily of genus 0) such that  $[\Gamma : \Gamma']$  is equal to the number of cusps of  $\Gamma'$ . We take a modular function  $f$  for  $\Gamma'$ . The function  $f$  comes from section 5.3 of [CLY04]. Then, we compute the function  $h := \sum_{\alpha \in \Gamma' \backslash \Gamma} f|_{\alpha}$ . It is a modular function for  $\Gamma$  whose unique simple pole lies at infinity. This follows from section 4.3.2 of [SZ17]. After normalizing the  $q$ -expansion of  $h$ , we obtain a hauptmodul for  $\Gamma$ . We give  $f$ , a set  $S$  of representatives of  $\Gamma' \backslash \Gamma$  and the hauptmodul  $h$ .

- (i) Consider the group  $\Gamma$  with label  $14A^0$ . Here  $g_{(a_1, a_2)}$  denotes the Siegel function  $g_{(a_1/14, a_2/14)}(\tau)$ . The set  $S = \{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 5 \\ 11 & 7 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 13 & 13 \end{pmatrix}, \begin{pmatrix} 0 & 3 \\ 9 & 13 \end{pmatrix} \}$ ,  
 $f := (\zeta_{14}^5 - \zeta_{14}^4 + \zeta_{14}^3 - \zeta_{14}^2 + \zeta_{14} - 1)g_{(2,0)}g_{(2,10)}g_{(4,2)}g_{(4,4)}g_{(6,4)}g_{(6,12)}$ . The resulting hauptmodul is

$$\begin{aligned} h &= g_{(0,2)}g_{(2,8)}g_{(2,12)}g_{(4,0)}g_{(4,6)}g_{(4,12)} \\ &\quad - \zeta_{14}^4 g_{(0,4)}g_{(2,2)}g_{(2,4)}g_{(2,6)}g_{(6,2)}g_{(6,8)} \\ &\quad + \zeta_{14} g_{(0,6)}g_{(4,8)}g_{(4,10)}g_{(6,0)}g_{(6,6)}g_{(6,10)} \\ &\quad + (\zeta_{14}^5 - \zeta_{14}^4 + \zeta_{14}^3 - \zeta_{14}^2 + \zeta_{14} - 1)g_{(2,0)}g_{(2,10)}g_{(4,2)}g_{(4,4)}g_{(6,4)}g_{(6,12)}. \end{aligned}$$

- (ii) Consider the group  $\Gamma$  when it is the conjugate of the congruence subgroup

with label  $14A^0$  by A. Here  $g_{(a_1, a_2)}$  denotes the Siegel function  $g_{(a_1/14, a_2/14)}(\tau)$ . The set  $S = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & -5 \\ -11 & 7 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ -13 & 13 \end{pmatrix}, \begin{pmatrix} 0 & -3 \\ -9 & 13 \end{pmatrix} \right\}$ ,  $f := \zeta_{14}^3 g_{(2,0)} g_{(2,4)} g_{(4,12)} g_{(4,10)} g_{(6,10)} g_{(6,2)}$ . The resulting hauptmodul is

$$\begin{aligned} h &= -g_{(0,2)} g_{(2,6)} g_{(2,2)} g_{(4,0)} g_{(4,8)} g_{(4,2)} \\ &+ g_{(0,4)} g_{(2,12)} g_{(2,10)} g_{(2,8)} g_{(6,12)} g_{(6,6)} \\ &+ \zeta_{14}^2 g_{(0,6)} g_{(4,6)} g_{(4,4)} g_{(6,0)} g_{(6,8)} g_{(6,4)} \\ &- \zeta_{14}^3 g_{(2,0)} g_{(2,4)} g_{(4,12)} g_{(4,10)} g_{(6,10)} g_{(6,2)}. \end{aligned}$$

(iii) Consider the group  $\Gamma$  with label  $15A^0$ . Here  $g_{(a_1, a_2)}$  denotes the Siegel function  $g_{(a_1/15, a_2/15)}(\tau)$ . The set  $S = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 14 & 14 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 14 & 0 \end{pmatrix} \right\}$ ,  $f := g_{(0,3)} g_{(0,6)} g_{(3,12)} g_{(6,9)}$ . The resulting hauptmodul is

$$\begin{aligned} h &= g_{(0,3)} g_{(0,6)} g_{(3,12)} g_{(6,9)} \\ &- \zeta_{15}^2 g_{(3,6)} g_{(3,3)} g_{(6,12)} g_{(6,6)} \\ &+ (\zeta_{15}^5 + 1) g_{(3,9)} g_{(3,0)} g_{(6,0)} g_{(6,3)}. \end{aligned}$$

(iv) Consider the group  $\Gamma$  when it is the conjugate of the congruence subgroup with label  $15A^0$  by A. Here  $g_{(a_1, a_2)}$  denotes the Siegel function  $g_{(a_1/15, a_2/15)}(\tau)$ . The set  $S = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ -14 & 14 \end{pmatrix}, \begin{pmatrix} 1 & -1 \\ -14 & 0 \end{pmatrix} \right\}$ ,  $f := g_{(0,3)} g_{(0,6)} g_{(3,3)} g_{(6,6)}$ . The resulting haupt-

modul is

$$\begin{aligned}
h &= g_{(0,3)}g_{(0,6)}g_{(3,3)}g_{(6,6)} \\
&+ \zeta_{15}g_{(3,9)}g_{(3,12)}g_{(6,9)}g_{(6,3)} \\
&- \zeta_{15}^5g_{(3,6)}g_{(3,0)}g_{(6,0)}g_{(6,12)}.
\end{aligned}$$

(v) Consider the group  $\Gamma$  with label  $21A^0$ . Here  $g_{(a_1, a_2)}$  denotes the Siegel function  $g_{(a_1/21, a_2/21)}(\tau)$ . The set  $S = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 20 & 6 \end{pmatrix}, \begin{pmatrix} 0 & 2 \\ 10 & 15 \end{pmatrix} \right\}$ ,  $f := g_{(0,6)}g_{(3,18)}g_{(3,3)}g_{(3,9)}g_{(3,12)}g_{(6,9)}g_{(6,12)}g_{(9,0)}$ . The resulting hauptmodul is

$$\begin{aligned}
h &= -g_{(0,3)}g_{(3,15)}g_{(6,15)}g_{(6,18)}g_{(6,0)}g_{(6,3)}g_{(9,18)}g_{(9,9)} \\
&+ g_{(0,6)}g_{(3,18)}g_{(3,3)}g_{(3,9)}g_{(3,12)}g_{(6,9)}g_{(6,12)}g_{(9,0)} \\
&- \zeta_{21}^6g_{(0,9)}g_{(3,0)}g_{(3,6)}g_{(6,6)}g_{(9,12)}g_{(9,15)}g_{(9,3)}g_{(9,6)}.
\end{aligned}$$

(vi) Consider the group  $\Gamma$  when it is the conjugate of the congruence subgroup with label  $21A^0$  by  $A$ . Here  $g_{(a_1, a_2)}$  denotes the Siegel function  $g_{(a_1/21, a_2/21)}(\tau)$ . The set  $S = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ -20 & 6 \end{pmatrix}, \begin{pmatrix} 0 & -2 \\ -10 & 15 \end{pmatrix} \right\}$ ,  $f := (-\zeta_{21}^{11} - \zeta_{21}^4)g_{(0,6)}g_{(3,3)}g_{(3,18)}g_{(3,12)}g_{(3,9)}g_{(6,12)}g_{(6,9)}g_{(9,0)}$ . The resulting hauptmodul is

$$\begin{aligned}
h &= g_{(0,3)}g_{(3,6)}g_{(6,6)}g_{(6,3)}g_{(6,0)}g_{(6,18)}g_{(9,3)}g_{(9,12)} \\
&+ (-\zeta_{21}^{11} - \zeta_{21}^4)g_{(0,6)}g_{(3,3)}g_{(3,18)}g_{(3,12)}g_{(3,9)}g_{(6,12)}g_{(6,9)}g_{(9,0)} \\
&+ (\zeta_{21}^{11} - \zeta_{21}^9 + \zeta_{21}^8 - \zeta_{21}^6 + \zeta_{21}^4 - \zeta_{21}^3 + \zeta_{21} - 1)g_{(0,9)}g_{(3,0)}g_{(3,15)}g_{(6,15)}g_{(9,9)}g_{(9,6)}g_{(9,18)}g_{(9,15)}.
\end{aligned}$$

(vii) When  $\Gamma$  has level 11. A hauptmodul in terms of Siegel functions for any genus 0 congruence subgroup of level 11 is described in section 4.3.3 of [SZ17].

## CHAPTER 4

### TWISTS

#### 4.1 Background and preliminaries

For background in Galois cohomology see section 5.1, Chapter 1 of [Ser97]. We recall the definitions and concepts that we need for reading this article.

Let  $G$  be a topological group and  $A$  be a  $G$ -group i.e.,  $A$  is a topological group with the discrete topology, with a continuous  $G$ -action that respects the group law of  $A$ .

**Definition 4.1.1.** A cocycle  $\zeta: G \rightarrow A$ , is a continuous function satisfying the cocycle property, i.e.,  $\zeta(\sigma\tau) = \zeta(\sigma) \cdot \sigma(\zeta(\tau))$  for every  $\sigma, \tau \in G$ . We say that two cocycles  $\zeta_1$  and  $\zeta_2$  are cohomologous if there is an  $a \in A$  such that  $\zeta_1(\sigma) = a \cdot \zeta_2(\sigma) \cdot \sigma(a^{-1})$  for every  $\sigma \in G$ .

It is easy to check that being cohomologous is an equivalence relation. We will denote the set of cocycles modulo this relation by  $H^1(G, A)$ . Let  $e: G \rightarrow A$  be the cocycle which sends  $g$  to 1 for every  $g \in G$ , where 1 is the identity element of  $A$ . We say that a cocycle  $\zeta: G \rightarrow A$  is a coboundary if  $\zeta$  is cohomologous to  $e$ .

Let  $K$  and  $K'$  be Galois extensions of  $\mathbb{Q}$  such that  $K \cap K' = \mathbb{Q}$ . Let  $KK'$  denote the compositum of  $K$  and  $K'$ . Let  $i: \text{Gal}(K/\mathbb{Q}) \times \text{Gal}(K'/\mathbb{Q}) \rightarrow \text{Gal}(KK'/\mathbb{Q})$  be the isomorphism whose inverse is  $g \mapsto (g|_K, g|_{K'})$ . Let  $A$  be a  $\text{Gal}(KK'/\mathbb{Q})$ -group for

which  $\text{Gal}(KK'/K)$  acts trivially. There is a natural action of  $\text{Gal}(K/\mathbb{Q})$  on  $A$ . We let  $\text{Gal}(K'/\mathbb{Q})$  act trivially on  $A$ .

Let  $\phi: \text{Gal}(KK'/\mathbb{Q}) \rightarrow A$  be a cocycle. We have assumed that the group  $\text{Gal}(KK'/\mathbb{Q})$  acts via  $\text{Gal}(K/\mathbb{Q})$ , i.e.,  $i((\sigma, \tau))(a) = \sigma(a)$  for every  $(\sigma, \tau) \in \text{Gal}(K/\mathbb{Q}) \times \text{Gal}(K'/\mathbb{Q})$  and  $a \in A$ .

**Lemma 4.1.2.** *Let  $\zeta: \text{Gal}(K/\mathbb{Q}) \rightarrow A$  be defined as  $\zeta(\sigma) = \phi(i(\sigma, 1))$ . Let  $\gamma: \text{Gal}(K'/\mathbb{Q}) \rightarrow A$  be defined as  $\gamma(\tau) = \phi(i(1, \tau))$ . The following properties hold:*

(i) *The maps  $\zeta$  and  $\gamma$  are cocycles. In particular,  $\gamma$  is a homomorphism. Further, the image of  $\gamma$  is contained in the set*

$$\{a \in A \mid \sigma(a) = \zeta(\sigma)^{-1} \cdot a \cdot \zeta(\sigma) \forall \sigma \in \text{Gal}(K/\mathbb{Q})\}.$$

(ii) *Conversely, if  $\zeta: \text{Gal}(K/\mathbb{Q}) \rightarrow A$  and  $\gamma: \text{Gal}(K'/\mathbb{Q}) \rightarrow A$  are two cocycles such that the image of  $\gamma$  is contained in the set*

$$\{a \in A \mid \sigma(a) = \zeta(\sigma)^{-1} \cdot a \cdot \zeta(\sigma) \forall \sigma \in \text{Gal}(K/\mathbb{Q})\},$$

*then the map  $\phi: \text{Gal}(K/\mathbb{Q}) \times \text{Gal}(K'/\mathbb{Q}) \rightarrow A$  defined as  $\phi((\sigma, \tau)) = \zeta(\sigma) \cdot \sigma(\gamma(\tau))$  is a cocycle.*

*Proof.*

(i) We will show that  $\zeta$  satisfies the cocycle property. Using cocycle property of  $\phi$  we have that for any  $\sigma_1, \sigma_2 \in \text{Gal}(K/\mathbb{Q})$ ,  $\zeta(\sigma_1\sigma_2) = \phi(i(\sigma_1\sigma_2, 1)) = \phi(i(\sigma_1, 1)i(\sigma_2, 1)) = \phi(i(\sigma_1, 1))\sigma_1(\phi(i(\sigma_2, 1))) = \zeta(\sigma_1)\sigma_1(\zeta(\sigma_2))$ .

Similarly, we can easily show from the definition of  $\gamma$  and the cocycle property of  $\phi$  that  $\gamma$  is a cocycle. It is a homomorphism because the group  $\text{Gal}(KK'/\mathbb{Q})$  acts via  $\text{Gal}(K/\mathbb{Q})$ .

We know that

$$\phi((\sigma, \tau)) = \phi((\sigma, 1)) \circ \sigma(\phi(1, \tau)) = \zeta(\sigma) \circ \sigma(\gamma(\tau))$$

and

$$\phi((\sigma, \tau)) = \phi((1, \tau))\phi((\sigma, 1)) = \gamma(\tau)\zeta(\sigma).$$

Combining these, we get that the image of  $\gamma$  is contained in the set

$$\{a \in A \mid \sigma(a) = \zeta(\sigma)^{-1} \cdot a \cdot \zeta(\sigma) \forall \sigma \in \text{Gal}(K/\mathbb{Q})\}.$$

(ii) We will show that  $\phi$  satisfies the cocycle property. Let  $(\sigma_1, \tau_1)$  and  $(\sigma_2, \tau_2) \in \text{Gal}(K/\mathbb{Q}) \times \text{Gal}(K'/\mathbb{Q})$ . Then,

$$\begin{aligned}
\phi((\sigma_1, \tau_1)(\sigma_2, \tau_2)) &= \zeta(\sigma_1\sigma_2) \circ (\sigma_1\sigma_2)(\gamma(\tau_1\tau_2)) \\
&= \zeta(\sigma_1) \circ \sigma_1(\zeta(\sigma_2) \circ \sigma_2(\gamma(\tau_1\tau_2))) \\
&= \zeta(\sigma_1) \circ \sigma_1(\gamma(\tau_1)\gamma(\tau_2) \circ \zeta(\sigma_2)) \\
&= \zeta(\sigma_1) \circ \sigma_1(\gamma(\tau_1)) \circ \sigma_1(\gamma(\tau_2) \circ \zeta(\sigma_2)) \\
&= \phi((\sigma_1, \tau_1)) \circ (\sigma_1, \tau_1)(\phi((\sigma_2, \tau_2))).
\end{aligned}$$

□

Fix a nice curve  $X$  over  $\mathbb{Q}$  with a nonconstant morphism  $\pi_X: X \rightarrow \mathbb{P}_{\mathbb{Q}}^1$ . Let  $K$  be a Galois extension of  $\mathbb{Q}$ . Since  $\pi_X$  is nonconstant,  $\text{Aut}_K(X, \pi_X)$  is a finite  $\text{Gal}(K/\mathbb{Q})$ -group.

We will now define a map  $\theta$  between the set of  $K$ -twists of  $(X, \pi_X)$  up to  $\mathbb{Q}$ -isomorphism and  $H^1(\text{Gal}(K/\mathbb{Q}), \text{Aut}_K(X, \pi_X))$ .

Let  $(Y, \pi_Y)$  be a  $K$ -twist of  $(X, \pi_X)$ . So, there is an isomorphism  $f: X_K \rightarrow Y_K$  such that  $\pi_Y \circ f = \pi_X$ . The map  $\xi: \text{Gal}(K/\mathbb{Q}) \rightarrow \text{Aut}_K(X, \pi_X)$  defined as  $\xi(\sigma) = f^{-1} \circ \sigma(f)$  is a cocycle. Using that  $\pi_Y$  and  $\pi_X$  are defined over  $\mathbb{Q}$ , we get that  $\pi_X \circ \xi(\sigma) = \pi_X$  for every  $\sigma \in \text{Gal}(K/\mathbb{Q})$ . We define  $\theta((Y, \pi_Y)) = [\xi] \in H^1(\text{Gal}(K/\mathbb{Q}), \text{Aut}_K(X, \pi_X))$ .

**Proposition 4.1.3.** *The map  $\theta$  defined above is well-defined and is a bijection between the set of  $K$ -twists of  $(X, \pi_X)$  up to  $\mathbb{Q}$ -isomorphism and  $H^1(\text{Gal}(K/\mathbb{Q}), \text{Aut}_K(X, \pi_X))$ .*

*Proof.* We first show that  $\theta$  is well defined. Let  $f_1: X_K \rightarrow Y_K$  and  $f_2: X_K \rightarrow Y_K$  be two isomorphisms such that  $\pi_Y \circ f_i = \pi_X$  for  $i \in \{1, 2\}$ . For  $i \in \{1, 2\}$ , let  $\xi_i: \text{Gal}(K/\mathbb{Q}) \rightarrow$

$\text{Aut}_K(X, \pi_X)$  be the cocycle  $\xi_i(\sigma) = f_i^{-1} \circ \sigma(f_i)$ . Then, it is easy to check that  $\xi_1$  and  $\xi_2$  are cohomologous with  $a = f_1^{-1} \circ f_2$ . Therefore,  $\theta((Y, \pi_Y)) \in H^1(\text{Gal}(K/\mathbb{Q}), \text{Aut}_K(X, \pi_X))$  and does not depend on the choice of  $f$ .

Suppose  $(Y, \pi_Y)$  and  $(Y', \pi_{Y'})$  are  $\mathbb{Q}$ -isomorphic. There is an isomorphism  $h: Y \rightarrow Y'$  such that  $\pi_{Y'} \circ h = \pi_Y$ . Fix an isomorphism  $f: X_K \rightarrow Y_K$  that satisfies  $\pi_Y \circ f = \pi_X$ . Then  $h \circ f: X_K \rightarrow Y'_K$  is an isomorphism satisfying  $\pi_{Y'} \circ h \circ f = \pi_X$ . Since,  $(h \circ f)^{-1} \circ \sigma(h \circ f) = f^{-1} \circ h^{-1} \circ \sigma(h) \circ \sigma(f) = f^{-1} \circ \sigma(f)$ , we have  $\theta((Y, \pi_Y)) = \theta((Y', \pi_{Y'}))$ . Therefore, given a  $K$ -twist of  $X$ , we get a class  $[\xi]$  in  $H^1(\text{Gal}(K/\mathbb{Q}), \text{Aut}_K(X, \pi_X))$  independent of  $\mathbb{Q}$ -isomorphism class.

Let us show that the map is surjective. Consider any class  $[\xi]$  in  $H^1(\text{Gal}(K/\mathbb{Q}), \text{Aut}_K(X, \pi_X))$ . Using Proposition 1, section 1, Chapter 3 in [Ser97] we get a nice curve  $Y$  over  $\mathbb{Q}$  with an isomorphism  $f: X_K \rightarrow Y_K$  such that the cocycle  $f^{-1} \circ \sigma(f)$  is cohomologous to  $[\xi]$  in  $H^1(\text{Gal}(K/\mathbb{Q}), \text{Aut}(X_K))$  i.e., there exists  $A \in \text{Aut}(X_K)$  such that  $f^{-1} \circ \sigma(f) = A^{-1} \circ \xi(\sigma) \circ \sigma(A)$ . Rearranging the terms we get  $(f \circ A^{-1})^{-1} \circ \sigma(f \circ A^{-1}) = \xi(\sigma)$ . Since  $\pi_X \circ \xi(\sigma) = \pi_X$ , the morphism  $\pi_Y := \pi_X \circ (f \circ A^{-1})^{-1}$  is defined over  $\mathbb{Q}$  because  $\pi_Y = \sigma(\pi_Y)$  for every  $\sigma$  in  $\text{Gal}(K/\mathbb{Q})$ . Therefore  $(Y, \pi_Y)$  is a  $K$ -twist of  $(X, \pi_X)$  that maps to  $[\xi]$ .

Finally, let us show that the map is injective. Let  $\xi_1$  and  $\xi_2$  are two cocycles corresponding to  $(Y, \pi_Y)$  and  $(Y', \pi_{Y'})$  respectively, such that  $\xi_1(\sigma) = A^{-1} \circ \xi_2(\sigma) \circ \sigma(A)$  with  $A \in \text{Aut}_K(X, \pi_X)$ . Let  $f: X_K \rightarrow Y_K$  be an isomorphism associated to  $Y$  and  $g: X_K \rightarrow Y'_K$  be an isomorphism associated to  $Y'$ , then  $g \circ A \circ f^{-1}: Y \rightarrow Y'$  is an isomorphism defined over  $\mathbb{Q}$  such that  $\pi_{Y'} \circ g \circ A \circ f^{-1} = \pi_Y$ .  $\square$

Let  $(Y, \pi_Y)$  be a  $K$ -twist of  $(X, \pi_X)$ . Fix an isomorphism  $f: X_K \rightarrow Y_K$  that satisfies  $\pi_Y \circ f = \pi_X$ . The cocycle  $\zeta: \text{Gal}(K/\mathbb{Q}) \rightarrow \text{Aut}_K(X, \pi_X)$ ,  $\sigma \mapsto f^{-1} \circ \sigma(f)$  is a representative of  $\theta((Y, \pi_Y))$ .

**Lemma 4.1.4.** *We have  $\text{Aut}_K(Y, \pi_Y) = \{f \circ g \circ f^{-1} \mid g \in \text{Aut}_K(X, \pi_X)\}$ .*

*Proof.* Let  $h \in \text{Aut}_K(Y, \pi_Y)$ . We can rewrite  $h$  as  $f \circ f^{-1} \circ h \circ f \circ f^{-1}$  and we have  $\pi_X \circ f^{-1} \circ h \circ f = \pi_Y \circ h \circ f = \pi_Y \circ f = \pi_X$ . Therefore,  $h = f \circ g \circ f^{-1}$  for some  $g \in \text{Aut}_K(X, \pi_X)$ . We proved that  $\text{Aut}_K(Y, \pi_Y) \subseteq \{f \circ g \circ f^{-1} \mid g \in \text{Aut}_K(X, \pi_X)\}$ .

If  $g \in \text{Aut}_K(X, \pi_X)$ , then  $\pi_Y \circ f \circ g \circ f^{-1} = \pi_X \circ g \circ f^{-1} = \pi_X \circ f^{-1} = \pi_Y$ . Therefore,  $\{f \circ g \circ f^{-1} \mid g \in \text{Aut}_K(X, \pi_X)\} \subseteq \text{Aut}_K(Y, \pi_Y)$ .  $\square$

**Lemma 4.1.5.** *We have*

$$\text{Aut}(Y, \pi_Y) = \{f \circ g \circ f^{-1} \mid g \in \text{Aut}_K(X, \pi_X), \sigma(g) = \zeta(\sigma)^{-1} \circ g \circ \zeta(\sigma) \forall \sigma \in \text{Gal}(K/\mathbb{Q})\}.$$

*Proof.* An element  $h \in \text{Aut}_K(Y, \pi_Y)$  belongs to the group  $\text{Aut}(Y, \pi_Y)$  if and only if  $\sigma(h) = h$  for all  $\sigma \in \text{Gal}(K/\mathbb{Q})$ . The lemma now follows using Lemma 4.1.4.  $\square$

Now assume that the curve  $X$  is isomorphic to  $\mathbb{P}_{\mathbb{Q}}^1$ . Let  $(Y, \pi_Y)$  be a  $K$ -twist of  $(X, \pi_X)$ . Fix an isomorphism  $f: X_K \rightarrow Y_K$  and let  $\zeta: \text{Gal}(K/\mathbb{Q}) \rightarrow \text{Aut}_K(X, \pi_X)$  be the cocycle  $\zeta(\sigma) = f^{-1} \circ \sigma(f)$ .

**Lemma 4.1.6.** *The curve  $Y$  is isomorphic to  $\mathbb{P}_{\mathbb{Q}}^1$  if and only if  $\zeta: \text{Gal}(K/\mathbb{Q}) \rightarrow \text{Aut}_K(X, \pi_X) \hookrightarrow \text{Aut}(X_K)$  is a coboundary in  $H^1(\text{Gal}(K/\mathbb{Q}), \text{Aut}_K(X))$ .*

*Proof.* Assume that  $Y$  is isomorphic to  $\mathbb{P}_{\mathbb{Q}}^1$ . Then there exists an isomorphism  $h: X \rightarrow Y$ . Since,  $h$  is defined over  $\mathbb{Q}$  we have  $\zeta(\sigma) = f^{-1} \circ \sigma(f) = (h^{-1} \circ f)^{-1} \circ \sigma(h^{-1} \circ f)$ . Since,  $h^{-1} \circ f \in \text{Aut}_K(X)$  the cocycle  $\zeta$  is a coboundary in  $H^1(\text{Gal}(K/\mathbb{Q}), \text{Aut}_K(X))$ .

Suppose  $\zeta$  is a coboundary in  $H^1(\text{Gal}(K/\mathbb{Q}), \text{Aut}_K(X))$  i.e.,  $\zeta(\sigma) = f^{-1} \circ \sigma(f) = A^{-1}\sigma(A)$  for some  $A \in \text{Aut}_K(X)$ . Therefore,  $f \circ A^{-1}: X \rightarrow Y$ . Hence,  $Y$  is isomorphic to  $\mathbb{P}_{\mathbb{Q}}^1$ .  $\square$

*Remark 4.1.1.* In our application, given a nice curve  $(X, \pi_X)$  where  $X$  is  $\mathbb{P}_{\mathbb{Q}}^1$  we need to find  $K$ -twists  $(Y, \pi_Y)$  such that  $Y$  is isomorphic to  $\mathbb{P}_{\mathbb{Q}}^1$ . We do this as follows. We consider the cocycles from  $\text{Gal}(K/\mathbb{Q})$  to  $\text{Aut}_K(X, \pi_X)$ . There are finitely many if  $\text{Gal}(K/\mathbb{Q})$  is finite. We check if a cocycle

$$\zeta: \text{Gal}(K/\mathbb{Q}) \rightarrow \text{Aut}_K(X, \pi_X) \hookrightarrow \text{Aut}_K(X) = \text{PGL}_2(K)$$

is a coboundary in  $H^1(\text{Gal}(K/\mathbb{Q}), \text{Aut}_K(X))$ . If  $\zeta$  is a coboundary in  $H^1(\text{Gal}(K/\mathbb{Q}), \text{PGL}_2(K))$ , then there is an  $A \in \text{PGL}_2(K)$  such that  $\zeta(\sigma) = A^{-1}\sigma(A)$  for every  $\sigma \in \text{Gal}(K/\mathbb{Q})$ ;  $A$  is an isomorphism between  $X_K \rightarrow \mathbb{P}_{\mathbb{Q}}^1$  and using  $\pi_X \circ \zeta(\sigma) = \pi_X$  for every  $\sigma \in \text{Gal}(K/\mathbb{Q})$  we observe that  $\pi_X \circ A^{-1}: \mathbb{P}_{\mathbb{Q}}^1 \rightarrow \mathbb{P}_{\mathbb{Q}}^1$ . The pair  $(\mathbb{P}_{\mathbb{Q}}^1, \pi_X \circ A^{-1})$  is twist of  $(X, \pi_X)$ .

## 4.2 Determining if a cocycle is a coboundary

Let  $K$  be a finite Galois extension of  $\mathbb{Q}$  and let

$$\zeta: \text{Gal}(K/\mathbb{Q}) \rightarrow \text{Aut}(\mathbb{P}_K^1) = \text{PGL}_2(K)$$

be a cocycle. The cocycle  $\zeta$  determines a twist  $C$  of  $\mathbb{P}_{\mathbb{Q}}^1$  up to isomorphism. In this section, we explain how to explicitly compute  $C$  as a conic in  $\mathbb{P}_{\mathbb{Q}}^2$ . The Hasse principle can then be used to determine if  $C(\mathbb{Q})$  is empty or not. If  $C(\mathbb{Q})$  is nonempty, then it is isomorphic to  $\mathbb{P}_{\mathbb{Q}}^1$  and equivalently  $\zeta$  is a coboundary. When  $\zeta$  is a coboundary, we will explain how to compute an  $A \in \text{PGL}_2(K)$  such that  $\zeta(\sigma) = A^{-1}\sigma(A)$  for every  $\sigma \in \text{Gal}(K/\mathbb{Q})$ .

Let  $Q_0$  be the quadratic form  $y^2 - xz$ . Let  $C_0$  be the conic defined by  $Q_0 = 0$  in  $\mathbb{P}_{\mathbb{Q}}^2$ .

Let  $\phi: \text{PGL}_2(K) \rightarrow \text{GL}_3(K)$  be the map

$$\left[ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \right] \mapsto \frac{1}{ad - bc} \begin{pmatrix} a^2 & 2ab & b^2 \\ ac & ad+bc & bd \\ c^2 & 2cd & d^2 \end{pmatrix}.$$

The map  $\phi$  comes from remark 3.3 in [LLG19]. It is an injective group homomorphism that respects the  $\text{Gal}(K/\mathbb{Q})$ -action. The image of  $\phi$  gives automorphisms of  $\mathbb{P}_K^2$  that stabilizes  $(C_0)_K$ . The map  $\phi$  induces an isomorphism  $\bar{\phi}: \text{Aut}(\mathbb{P}_K^1) = \text{PGL}_2(K) \rightarrow \text{Aut}((C_0)_K)$  that respects the  $\text{Gal}(K/\mathbb{Q})$ -action. One can check that  $\bar{\phi}$  is induced by the isomorphism  $\mathbb{P}_{\mathbb{Q}}^1 \rightarrow C_0$  that sends  $[x : y]$  to  $[x^2 : xy : y^2]$ .

Define the map

$$\bar{\zeta} := \phi \circ \zeta: \text{Gal}(K/\mathbb{Q}) \rightarrow \text{GL}_3(K).$$

Since  $\phi$  is a group homomorphism and respects the  $\text{Gal}(K/\mathbb{Q})$ -action,  $\bar{\zeta} : \text{Gal}(K/\mathbb{Q}) \rightarrow \text{GL}_3(K)$  is a cocycle. By Hilbert 90, there is an  $M \in \text{GL}_3(K)$  such that  $\bar{\zeta}(\sigma) = M^{-1}\sigma(M)$  for every  $\sigma \in \text{Gal}(K/\mathbb{Q})$ . We explain how to compute  $M$  in § 4.2.1.

**Lemma 4.2.1.** *The quadratic form  $Q := Q_0(M^{-1}(x, y, z)^T)$  has coefficients in  $\mathbb{Q}$  and the conic  $C \subseteq \mathbb{P}_{\mathbb{Q}}^2$  defined by  $Q = 0$  is isomorphic to the twist of  $\mathbb{P}_{\mathbb{Q}}^1$  by  $\zeta$ .*

*Proof.* Since the cocycle  $\bar{\zeta}(\sigma)$  preserves the equation for  $Q_0$  for every  $\sigma \in \text{Gal}(K/\mathbb{Q})$ , it follows that  $Q$  has coefficients in  $\mathbb{Q}$ . □

From Lemma 4.2.1,  $M$  gives a conic  $C \subseteq \mathbb{P}_{\mathbb{Q}}^2$  which is a twist of  $C_0$  by  $\zeta$ . The cocycle  $\zeta$  is a coboundary if and only if  $C$  has a rational point. We use Magma [Bos97] to check this. Suppose that the conic  $C$  has a rational point. There is a change of variable matrix  $B \in \text{GL}_3(\mathbb{Q})$  from  $C$  to  $C_0$ . Observe that  $BM \in \text{Aut}_K(C_0)$  and  $\bar{\zeta}(\sigma) = M^{-1}\sigma(M) = (BM)^{-1}\sigma(BM)$ . Define  $A := (\bar{\phi})^{-1}(BM) \in \text{PGL}_2(K)$ . One can then check that  $\zeta(\sigma) = A^{-1}\sigma(A)$  for every  $\sigma \in \text{Gal}(K/\mathbb{Q})$ .

## 4.2.1 Computing Hilbert 90 matrices

Let  $K$  be a finite Galois extension of  $\mathbb{Q}$  and define  $G := \text{Gal}(K/\mathbb{Q})$ . Hilbert 90 states that  $H^1(G, \text{GL}_n(K)) = 1$ . Equivalently, given a cocycle  $\psi: G \rightarrow \text{GL}_n(K)$  there exists a matrix  $A \in \text{GL}_n(K)$  such that  $\psi(\sigma) = A^{-1}\sigma(A)$  for every  $\sigma \in G$ . We now describe how to find  $A$ .

- Define the  $K$ -vector space  $V := K^n$ . Define a new  $G$ -action on  $V$  by  $\sigma * v = \psi(\sigma)\sigma(v)$ ; it acts  $\mathbb{Q}$ -linearly. Let  $W$  be the  $\mathbb{Q}$ -subspace of  $V$  fixed by this  $G$ -action. Using Proposition 7(A.V.63) of [Bou03], the natural map  $K \otimes_{\mathbb{Q}} W \rightarrow V$  of  $K$ -vector spaces is an isomorphism. In particular,  $W$  has dimension  $n$  over  $\mathbb{Q}$ .
- Fix a basis  $\mathfrak{A}$  for  $K$  considered as a vector space over  $\mathbb{Q}$  and a basis  $\mathfrak{B}$  for  $V$  considered as a vector space over  $K$ . The set  $B = \{b.v \mid b \in \mathfrak{A}, v \in \mathfrak{B}\}$  is a basis of  $V$  over  $\mathbb{Q}$ .
- The  $\mathbb{Q}$ -linear map  $V \rightarrow W$  that sends  $v$  to  $1/|G|(\sum_{g \in G} g * v)$  is a projection. Therefore, the set  $C = \{1/|G|(\sum_{g \in G} g * u) : u \in B\}$  spans  $W$  over  $\mathbb{Q}$ .
- Choose  $n$  linearly independent vectors from  $C$ ; they form a basis for  $W$  and hence also a basis of  $V$  over  $K$ . Let  $A \in \text{GL}_n(K)$  be the change of basis matrix from our basis in  $C$  to the standard basis of  $K^n$ . We have  $\psi(\sigma) = A^{-1}\sigma(A)$  for all  $\sigma \in G$ .

## CHAPTER 5

### A DESCRIPTION OF GROUPS ASSOCIATED TO ABELIAN FAMILY OF TWISTS

Let  $(X_G, \pi_G)$  be a modular curve. Let  $\mathcal{A}$  be a finite abelian subgroup of  $\text{Aut}(X_G, \pi_G)$ . Given a cocycle  $\gamma : \text{Gal}(\mathbb{Q}^{\text{ab}}/\mathbb{Q}) \simeq \mathcal{A}$  we get a  $\mathbb{Q}^{\text{ab}}$ -twist  $(X_{G_\gamma}, \pi_{G_\gamma})$  of  $(X_G, \pi_G)$ . In this section we give a description of the group  $G_\gamma$ . Let  $\Gamma$  be  $G \cap \text{SL}_2(\mathbb{Z})$ .

The morphism  $\pi_G : X_G \rightarrow \mathbb{P}_{\mathbb{Q}}^1$  corresponds to the inclusion of fields  $\mathbb{Q}(j) \subseteq \mathbb{Q}(X_G)$ . The group  $\mathcal{A}$  acts on the function field  $\mathbb{Q}(X_G)$  on the right and fixes  $\mathbb{Q}(j)$ . Let  $\mathbb{Q}(X_G)^{\mathcal{A}}$  be the subfield of  $\mathbb{Q}(X_G)$  fixed by  $\mathcal{A}$  and  $X_G^{\mathcal{A}}$  be the nice curve over  $\mathbb{Q}$  whose function field is  $\mathbb{Q}(X_G)^{\mathcal{A}}$ . The inclusion of fields  $\mathbb{Q}(j) \subseteq \mathbb{Q}(X_G)^{\mathcal{A}} \subseteq \mathbb{Q}(X_G)$  corresponds to morphisms  $\pi : X_G \rightarrow X_G^{\mathcal{A}}$  and  $\pi'_G : X_G^{\mathcal{A}} \rightarrow \mathbb{P}_{\mathbb{Q}}^1$  satisfying  $\pi'_G \circ \pi = \pi_G$ . Then,  $(X_G^{\mathcal{A}}, \pi'_G)$  is a modular curve  $(X_{G_0}, \pi_{G_0})$  for an open subgroup  $G_0$  of  $\text{GL}_2(\hat{\mathbb{Z}})$  satisfying  $-I \in G_0$  and  $\det(G_0) = \hat{\mathbb{Z}}^\times$ .

The extension  $\mathbb{Q}(X_G)^{\mathcal{A}} \subseteq \mathbb{Q}(X_G)$  is Galois with  $\mathcal{A}^{\text{op}}$  as Galois group. So,  $\text{Aut}(X_G, \pi) = \mathcal{A}$ . Since,  $\pi$  is a Galois cover  $G$  is a normal subgroup of  $G_0$  and we have a natural isomorphism  $G_0/G \simeq \mathcal{A}$ . We also observe that  $\pi_{G_\gamma}$  factors through  $X_{G_0}$ . We will give a description of the group  $G_\gamma$  in terms of  $G_0$  and  $\gamma$ .

Let us define a new action of  $\text{Gal}(\mathbb{Q}^{\text{cyc}}/\mathbb{Q})$  on  $\mathbb{Q}^{\text{cyc}}(X_G)$  as follows. For  $\sigma \in \text{Gal}(\mathbb{Q}^{\text{cyc}}/\mathbb{Q})$  and  $f \in \mathbb{Q}^{\text{cyc}}(X_G)$ , let  $\sigma \bullet f = \sigma(f) \circ \gamma_\sigma^{-1}$ . Then,

$$\mathbb{Q}(X_{G_\gamma}) = \{f \in \mathbb{Q}^{\text{cyc}}(X_G) \mid \sigma \bullet f = f \ \forall \sigma \in \text{Gal}(\mathbb{Q}^{\text{cyc}}/\mathbb{Q})\}.$$

Since the cyclotomic character  $\chi_{\text{cyc}}: \text{Gal}(\mathbb{Q}^{\text{ab}}/\mathbb{Q}) \rightarrow \hat{\mathbb{Z}}^\times$  is an isomorphism, there is an unique homomorphism  $\phi_\gamma$  that makes the following diagram commutative.

$$\begin{array}{ccc} \text{Gal}(\mathbb{Q}^{\text{ab}}/\mathbb{Q}) & \xrightarrow{\chi_{\text{cyc}}} & \hat{\mathbb{Z}}^\times \\ & \searrow \gamma^{-1} & \swarrow \phi_\gamma \\ & & G_0/G \end{array}$$

Define

$$S := \{g \in G_0 \mid gG = \phi_\gamma(\det g)\}.$$

**Lemma 5.0.1.** *The group  $G_\gamma = S$ .*

*Proof.* Let us show that  $G_\gamma \subseteq S$ . Let  $g \in G_\gamma$ . Let  $\sigma \in \text{Gal}(\mathbb{Q}^{\text{cyc}}/\mathbb{Q})$  satisfy  $\chi_{\text{cyc}}(\sigma) = \det(g)$ . For every  $f \in \mathbb{Q}(X_G)$ ,  $f * g = \sigma \bullet f = \sigma(f) \circ \gamma_\sigma^{-1} = f \circ \gamma_\sigma^{-1}$ . Therefore,  $gG = \phi_\gamma(\det g)$ .

We will now show that  $S \subseteq G_\gamma$ . Let  $g \in S$ , i.e.,  $gG = \phi_\gamma(\det g)$ . For any  $f \in \mathbb{Q}(X_{G_\gamma})$  we have  $f = \sigma(f) \circ \gamma_\sigma^{-1} = f * g$ . Therefore,  $g \in G_\gamma$ .

□

## CHAPTER 6

### ABELIAN FAMILIES OF TWISTS

Fix a finite abelian subgroup  $\mathcal{A} \subseteq \mathrm{PGL}_2(\mathbb{Q}) = \mathrm{Aut}(\mathbb{P}_{\mathbb{Q}}^1)$ . We choose a function  $\pi(t) \in \mathbb{Q}(t)$  satisfying  $\mathbb{Q}(t)^{\mathcal{A}} = \mathbb{Q}(\pi(t))$ . The inclusion  $\mathbb{Q}(\pi(t)) \subseteq \mathbb{Q}(t)$  gives a morphism  $\pi : \mathbb{P}_{\mathbb{Q}}^1 \rightarrow \mathbb{P}_{\mathbb{Q}}^1$  satisfying  $\mathrm{Aut}_{\overline{\mathbb{Q}}}(\mathbb{P}_{\mathbb{Q}}^1, \pi) = \mathcal{A}$ .

Fix a variable  $v$  and let  $x$  be a root of the polynomial  $\pi(T) - v \in \mathbb{Q}(v)[T]$ . Let  $K := \mathbb{Q}(x)$ . The field extension  $\mathbb{Q}(v) \subseteq K$  is Galois.

For each  $\sigma \in \mathrm{Gal}(K/\mathbb{Q}(v))$ , there is a unique  $\gamma_{\sigma} \in \mathcal{A}$  satisfying

$$\sigma(x) = \gamma_{\sigma}(x).$$

It can be checked that  $\gamma : \mathrm{Gal}(K/\mathbb{Q}(v)) \xrightarrow{\sim} \mathcal{A} \subseteq \mathrm{PGL}_2(\mathbb{Q}) \subseteq \mathrm{PGL}_2(K)$  is a homomorphism.

In the cases we consider, we find a matrix  $A \in \mathrm{GL}_2(K)$  such that  $\gamma_{\sigma} = A^{-1}\sigma(A)$  for every  $\sigma \in \mathrm{Gal}(K/\mathbb{Q}(v))$ . This matrix  $A$  describes an isomorphism  $f : \mathbb{P}_K^1 \rightarrow \mathbb{P}_K^1$  such that  $\pi \circ f^{-1}$  is stable under  $\mathrm{Gal}(K/\mathbb{Q}(v))$ -action. Define  $\pi_{\mathcal{A},v} := \pi \circ f^{-1} \in \mathbb{Q}(v)(u)$ .

## 6.1 Specialization

Here  $v$  is a variable that can be specialized at rational values. More precisely, we will take  $v \in \mathbb{Q} \subseteq \mathbb{Q} \cup \{\infty\} \subseteq \mathbb{P}^1(\mathbb{Q})$  for which  $\pi$  is unramified at  $v$ . The cocycle  $\gamma$  gives

a  $K$ -twist of  $(\mathbb{P}_{\mathbb{Q}}^1, \pi)$  that has a rational point. Moreover, any  $\mathbb{Q}^{\text{ab}}$ -twist of  $(\mathbb{P}_{\mathbb{Q}}^1, \pi)$  that has a rational point arises from such a  $v \in \mathbb{Q}$ .

Given  $v \in \mathbb{Q}$ , we find the corresponding twist  $(\mathbb{P}_{\mathbb{Q}}^1, \pi_{\mathcal{A},v})$  where we can take  $\pi_{\mathcal{A},v} \in \mathbb{Q}(u)$ .

We are going to limit our discussion to those subgroups  $\mathcal{A}$  that show up when  $(\mathbb{P}_{\mathbb{Q}}^1, \pi)$  is a modular curve. As a remark, we point out that there are finite abelian subgroups of  $\text{PGL}_2(\mathbb{Q})$  which do not show up in our context for example, cyclic group of order 6.

For each  $\mathcal{A}$  we give the function  $\pi$ , a matrix  $A$  and resulting  $\pi_{\mathcal{A},v}$ .

- **Case 0** For  $\mathcal{A} = \{(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix})\}$ , the function  $\pi(T) = T$ . A matrix  $A = (\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix})$  and  $\pi_{\mathcal{A},v}(u) = u$ .
- **Case 1** For  $\mathcal{A} = \{(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}), (\begin{smallmatrix} -1 & 0 \\ 0 & 1 \end{smallmatrix})\}$ , the function  $\pi(T) = T^2$ . A matrix  $A = (\begin{smallmatrix} 1 & 0 \\ 0 & x \end{smallmatrix})$  and  $\pi_{\mathcal{A},v}(u) = vu^2$ .
- **Case 2** For  $\mathcal{A} = \{(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}), (\begin{smallmatrix} 0 & \alpha \\ 1 & 0 \end{smallmatrix})\}$ , where  $\alpha$  is a non-zero rational number the function  $\pi(T) = T + \alpha/T$ . A matrix  $A = (\begin{smallmatrix} x & \alpha \\ 1 & x \end{smallmatrix})$  and  $\pi_{\mathcal{A},v}(u) = (vu^2 - 4\alpha u + \alpha v)/(-u^2 + vu - \alpha)$ .
- **Case 3** For  $\mathcal{A} = \{(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}), (\begin{smallmatrix} 0 & -1 \\ 1 & -1 \end{smallmatrix}), (\begin{smallmatrix} 1 & -1 \\ 1 & 0 \end{smallmatrix})\}$ , the function  $\pi(T) = (T^3 - 3T + 1)/(T^2 - T)$ .

A matrix

$$A = \begin{pmatrix} (-6v+3)x^2+(6v^2-6v+6)x+(-3v^2+12v-6) & (3v-6)x^2+(-3v^2+3v-3)x+(3v^2-9v+12) \\ 6x^2+(-6v+3)x+(3v-12) & -3x^2+(3v+3)x+(-3v+6) \end{pmatrix}$$

and  $\pi_{\mathcal{A},v}(u) = ((-v+3)u^3 + (-3v^2+9v-9)u^2 + (-3v^3+9v^2-15v)u + (-v^4+3v^3-6v^2-v+3))/(u^3+2vu^2+(v^2+v-3)u+(v^2-3v+1))$ .

- **Case 4** For  $\mathcal{A} = \{(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}), (\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}), (\begin{smallmatrix} -1 & -1 \\ 1 & -1 \end{smallmatrix}), (\begin{smallmatrix} 1 & -1 \\ 1 & 1 \end{smallmatrix})\}$ , the function

$$\pi(T) = (T^4 - 6T^2 + 1)/(T^3 - T).$$

A matrix

$$A = \begin{pmatrix} (-2v+24)x^3+(2v^2-22v)x^2+(-2v^2+10v-136)x+(10v+16) & (4v-8)x^3+(-4v^2+4v)x^2+(4v^2-20v+72)x-32 \\ 8x^3-(8v)x^2-40x+4v & (2v-8)x^3+(-2v^2+6v)x^2+(2v^2-10v+56)x+(-2v-16) \end{pmatrix}$$

and  $\pi_{\mathcal{A},v}(u) = (-vu^4 + (8v+16)u^3 + (-18v-96)u^2 + (8v+176)u + (7v-96))/(u^4 + (v-8)u^3 + (-6v+18)u^2 + (11v-8)u + (-6v-7))$ .

- **Case 5** For  $\mathcal{A} = \{(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}), (\begin{smallmatrix} 0 & \alpha \\ 1 & 0 \end{smallmatrix}), (\begin{smallmatrix} -1 & 0 \\ 0 & 1 \end{smallmatrix}), (\begin{smallmatrix} 0 & -\alpha \\ 1 & 0 \end{smallmatrix})\}$ , where  $\alpha$  is a non-zero rational number the function  $\pi(T) = T^2 + \alpha^2/T^2$ .

A matrix  $A = (\begin{smallmatrix} 1 & x \\ x^2 & \alpha^2/x \end{smallmatrix})$  and  $\pi_{\mathcal{A},v}(u) = ((-3v\alpha^2+v^3)\alpha^2u^4 + (8\alpha^2-4v^2)\alpha^2u^3 + 6v\alpha^2u^2 - 8\alpha^2u + v)/(\alpha^4u^4 - 2v\alpha^2u^3 + (2\alpha^2+v^2)u^2 - 2vu + 1)$ .

- **Case 6** For  $\mathcal{A} = \{(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}), (\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}), (\begin{smallmatrix} 1 & 1 \\ 1 & -1 \end{smallmatrix}), (\begin{smallmatrix} -1 & 1 \\ 1 & 1 \end{smallmatrix})\}$ , the rational function

$$\pi(T) = (T^4 + 2T^2 + 1)/(T^3 - T).$$

A matrix  $A = (\begin{smallmatrix} a_1 & a_2 \\ a_3 & a_4 \end{smallmatrix})$ , where  $a_1 = 8x^3/(v^2-16) - 8vx^2/(v^2-16) + 24x/(v^2-16) + 4v/(v^2-16)$ ,

$$a_2 = -2x^3/(v-4) + (2v^2+10v)x^2/(v^2-16) + (-2v^2-6v-8)x/(v^2-16) + -6v-16/(v^2-16),$$

$$a_3 = (-2v+16)x^3/(v^2-16) + (2v^3-14v^2-32)x^2/(v^3-16v) + (-2v^2-6v+80)x/(v^2-16) + (10v^2-16v-32)/(v^3-16v) \text{ and}$$

$$a_4 = (-4v+8)x^3/(v^2-4v) + (4v^2+12v-40)x^2/(v^2-16) + (-4v^3-4v^2+8v+32)x/(v^3-16v) + -8v-8/(v^2-16) \text{ and}$$

$\pi_{\mathcal{A},v}(u)$  is

$$\frac{(-25v^3 + 160v^2 - 256v)u^4 + (40v^3 - 208v^2 + 256v)u^3 + (-26v^3 + 96v^2 - 64v)u^2 + (8v^3 - 16v^2)u - v^3}{(6v^3 - 37v^2 + 64v - 64)u^4 + (-11v^3 + 56v^2 - 32v)u^3 + (6v^3 - 30v^2)u^2 + (-v^3 + 8v^2)u - v^2}.$$

- **Case 7** For  $\mathcal{A} = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & -5 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} -1 & 5 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 5 & 5 \\ 1 & -5 \end{pmatrix} \right\}$ , the function

$$\pi(T) = (T^4 + 10T^2 + 25)/(T^3 - 4T^2 - 5T).$$

$$\text{A matrix } A = \begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix},$$

$$\text{where } a_1 = \frac{(22v^2/15 - 84v/5 - 224/3)x^3}{(v^3 - 36v^2 + 240v + 1600)} + \frac{(-22v^3/15 + 242v^2/15 + 248v/3 + 80/3)x^2}{(v^3 - 36v^2 + 240v + 1600)}$$

$$+ \frac{(98v^3/15 - 898v^2/15 - 1516v/3 - 2080/3)x}{(v^3 - 36v^2 + 240v + 1600)} + \frac{(4v^3 - 142v^2/3 - 200v + 400/3)}{(v^3 - 36v^2 + 240v + 1600)},$$

$$a_2 = \frac{(2v^2/3 - 8v - 80/3)x^3}{(v^3 - 36v^2 + 240v + 1600)} + \frac{(-2v^3/3 + 22v^2/3 + 92v/3 + 320/3)x^2}{(v^3 - 36v^2 + 240v + 1600)} +$$

$$\frac{(10v^3/3 - 98v^2/3 - 760v/3 - 400/3)x}{(v^3 - 36v^2 + 240v + 1600)} + \frac{(50v^2/3 - 60v - 3200/3)}{(v^3 - 36v^2 + 240v + 1600)},$$

$$a_3 = \frac{(-22v^2/15 + 152v/15 + 32)x^3}{(v^4 - 36v^3 + 240v^2 + 1600v)} + \frac{(22v^2/15 - 142v/15 - 112/3)x^2}{(v^3 - 36v^2 + 240v + 1600)} +$$

$$\frac{(-98v^3/15 + 458v^2/15 + 728v/3 + 160)x}{(v^4 - 36v^3 + 240v^2 + 1600v)} + \frac{(-4v^2 + 82v/3 + 400/3)}{(v^3 - 36v^2 + 240v + 1600)} \text{ and}$$

$$a_4 = \frac{(-2v/3 + 16/3)x^3}{(v^3 - 36v^2 + 240v + 1600)} + \frac{(2v^3/3 - 14v^2/3 + 8v/3 - 160)x^2}{(v^4 - 36v^3 + 240v^2 + 1600v)}$$

$$+ \frac{(-10v^2/3 + 46v/3 + 560/3)x}{(v^3 - 36v^2 + 240v + 1600)} + \frac{(10v^2/3 - 1400v/3 - 800)}{(v^4 - 36v^3 + 240v^2 + 1600v)}$$

$$\text{and } \pi_{\mathcal{A},v}(u) \text{ is } ((-v^5 + 32v^4 - 336v^3 + 1280v^2 - 1600v)u^4 + (-4v^6 + 148v^5 - 1744v^4 + 5600v^3 + 16000v^2 - 64000v)u^3 + (-6v^7 + 252v^6 - 3376v^5 + 10880v^4 + 69600v^3 - 256000v^2 - 640000v)u^2 + (-4v^8 + 188v^7 - 2864v^6 + 10680v^5 + 82400v^4 - 368000v^3 - 1280000v^2)u + (-v^9 + 52v^8 - 896v^7 + 4120v^6 + 29500v^5 - 176000v^4 - 640000v^3)) / ((v^4 - 36v^3 + 240v^2 + 1600v - 14400)u^4 + (3v^5 - 128v^4 + 1320v^3 + 1920v^2 - 51200v)u^3 + (3v^6 - 149v^5 + 2056v^4 - 3600v^3 - 63200v^2 + 64000v)u^2 + (v^7 - 58v^6 + 1002v^5 - 4400v^4 - 20800v^3 + 96000v^2)u + (-v^7 + 26v^6 - 280v^5 + 3500v^4 - 16000v^3 - 160000v^2)).$$

- **Case 8** For  $\mathcal{A} = \{(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}), (\begin{smallmatrix} -2 & 2 \\ 1 & 2 \end{smallmatrix}), (\begin{smallmatrix} 1 & 2 \\ 1 & -1 \end{smallmatrix}), (\begin{smallmatrix} 0 & -2 \\ 1 & 0 \end{smallmatrix})\}$ , the rational function  $\pi(T) = (T^4 + 4T^2 + 4)/(T^3 + T^2 - 2T)$ .

A matrix  $A = \begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix}$ , where  $a_1 = (-v^3/9 + 10v^2/9 + 28v/9 - 64/9)x^3/(v^3 + 4v^2 - 32v) +$

$$(v^4/9 - 14v^3/9 - 14v^2/3 + 104v/9 + 32/9)x^2/(v^3 + 4v^2 - 32v) + (5v^4/9 + 2v^3/9 - 16v^2/3 + 88v/9 - 128/9)x/(v^3 + 4v^2 - 32v) + (28v^3/9 + 44v^2/9 - 352v/9 + 64/9)/(v^3 +$$

$$a_2 = (4v^3/9 + 14v^2/9 - 40v/9 - 32/9)x^3/(v^3 + 4v^2 - 32v) + (-4v^4/9 - 16v^3/9 + 20v^2/3 + 88v/9 - 128/9)x^2/(v^3 + 4v^2 - 32v) +$$

$$(-2v^3/9 + 2v^2/9 + 92v/9 + 16/9)x/(v^2 + 8v) + (-4v^3/9 + 40v^2/9 + 112v/9 - 256/9)/(v^3 + 4v^2 - 32v),$$

$$a_3 = (2v/9 - 32/9)x^3/(v^3 + 4v^2 - 32v) + (-2v^2/9 + 40v/9 + 16/9)x^2/(v^3 + 4v^2 - 32v) + (-10v^2/9 + 20v/9 - 64/9)x/(v^3 + 4v^2 - 32v) +$$

$$(-56v/9 + 32/9)/(v^3 + 4v^2 - 32v) \text{ and}$$

$$a_4 = (-8v/9 - 16/9)x^3/(v^3 + 4v^2 - 32v) + (8v^2/9 + 20v/9 - 64/9)x^2/(v^3 + 4v^2 - 32v) + (4v/9 + 8/9)x/(v^2 + 8v) +$$

$$(8v/9 - 128/9)/(v^3 + 4v^2 - 32v)$$

and  $\pi_{\mathcal{A},v}(u)$  is  $(vu^4 + (2v^3 + 4v^2 - 24v)u^3 + (3v^5/2 + 6v^4 - 29v^3 - 68v^2 + 184v)u^2 + (v^7/2 + 3v^6 - 11v^5 - 62v^4 + 108v^3 + 320v^2 - 480v)u + (v^9/16 + v^8/2 - 5v^7/4 - 14v^6 + 41v^5/4 + 130v^4 - 80v^3 - 400v^2 + 400v))/(u^4 + (3v^2/2 + 2v - 8)u^3 + (3v^4/4 + 7v^3/4 - 5v^2 - 8v + 24)u^2 + (v^6/8 + v^5/4 + v^4/4 + 3v^3 - 10v^2 + 8v - 32)u + (-v^7/16 + 3v^6/8 + 7v^5/2 - 29v^4/4 - 31v^3 + 64v^2 + 16))$ .

## CHAPTER 7

### PROOFS OF Theorem 1.0.2 AND Theorem 1.0.3

#### 7.1 Proof of Theorem 1.0.2

Take a genus 0 congruence subgroup  $\Gamma$  of  $\mathrm{SL}_2(\mathbb{Z})$  that contains  $-I$ . Let  $N$  be the level of  $\Gamma$ . In this section, we will prove that there are nonconstant  $\pi_1, \dots, \pi_l \in \mathbb{Q}(t)$  and finite abelian subgroups  $\mathcal{A}_i \subseteq \mathrm{Aut}(\mathbb{P}_{\mathbb{Q}}^1, \pi_i) \subseteq \mathrm{PGL}_2(\mathbb{Q})$  such that the following hold:

- (i) For each  $i \in \{1, \dots, l\}$ ,  $(\mathbb{P}_{\mathbb{Q}}^1, \pi_i)$  is isomorphic to some modular curve  $(X_G, \pi_G)$  where  $G$  is an open subgroup of  $\mathrm{GL}_2(\hat{\mathbb{Z}})$  satisfying  $-I \in G$ ,  $\det(G) = \hat{\mathbb{Z}}^\times$  and  $G \cap \mathrm{SL}_2(\mathbb{Z}) = \Gamma$ .
- (ii) For any modular curve  $X_G$  isomorphic to  $\mathbb{P}_{\mathbb{Q}}^1$  where  $-I \in G$ ,  $\det(G) = \hat{\mathbb{Z}}^\times$  and  $G \cap \mathrm{SL}_2(\mathbb{Z}) = \Gamma$ , there is an  $i \in \{1, \dots, l\}$ , such that  $(X_G, \pi_G)$  is a  $\mathbb{Q}^{\mathrm{ab}}$ -twist of  $(\mathbb{P}_{\mathbb{Q}}^1, \pi_i)$  via a cocycle  $\gamma: \mathrm{Gal}(\mathbb{Q}^{\mathrm{ab}}/\mathbb{Q}) \rightarrow \mathcal{A}_i \subseteq \mathrm{Aut}(\mathbb{P}_{\mathbb{Q}}^1, \pi_i)$ .

It suffices to prove Theorem 1.0.2 for a fixed  $\Gamma$  because there are only finitely many genus 0 congruence subgroups up to conjugation in  $\mathrm{GL}_2(\mathbb{Z})$ .

##### 7.1.1 Computing a pair $(\mathbb{P}_{\mathbb{Q}}^1, \pi_1)$

In this section, we will discuss our computation of a pair  $(\mathbb{P}_{\mathbb{Q}}^1, \pi_1)$  that is isomorphic to a modular curve  $(X_{G_1}, \pi_{G_1})$  for some open subgroup  $G_1$  of  $\mathrm{GL}_2(\hat{\mathbb{Z}})$  satisfying

$-I \in G_1$ ,  $\det(G_1) = \hat{\mathbb{Z}}^\times$  and  $G_1 \cap \mathrm{SL}_2(\mathbb{Z}) = \Gamma$ . We either find one such pair  $(\mathbb{P}_{\mathbb{Q}}^1, \pi_1)$  or show that there does not exist any.

*Remark 7.1.1.* The point of computing such a pair is that to compute all the other modular curves  $(X_G, \pi_G)$  such that  $G \cap \mathrm{SL}_2(\mathbb{Z})$  is  $\Gamma$  and the curve  $X_G$  is isomorphic to  $\mathbb{P}_{\mathbb{Q}}^1$  we have to consider the cocycles  $\phi: \mathrm{Gal}(\mathbb{Q}^{\mathrm{ab}}/\mathbb{Q}) \rightarrow \mathrm{Aut}_{\mathbb{Q}^{\mathrm{ab}}}(X_{G_1}, \pi_{G_1})$ .

Let  $h$  be the normalized hauptmodul of  $\Gamma$ . Since the coefficients of  $q$ -expansion of  $h$  lie in  $K_N$ , the function field of  $X_\Gamma$  is  $K_N(h)$ . There is a unique  $\pi_\Gamma$  in  $K_N(t)$  such that  $\pi_\Gamma(h) = j$  where  $j$  is the modular  $j$ -invariant. To compute  $\pi_\Gamma$  we use the method described in section 4.4 of [SZ17]; this uses that from chapter 3, we can compute arbitrarily many terms of the  $q$ -expansion of  $h$ . The function  $\pi_\Gamma$  describes a morphism  $\pi_\Gamma: X_\Gamma \rightarrow \mathbb{P}_{K_N}^1$ .

Suppose there is an open subgroup  $G$  of  $\mathrm{GL}_2(\hat{\mathbb{Z}})$  containing  $-I$  with full determinant for which  $G \cap \mathrm{SL}_2(\mathbb{Z})$  is  $\Gamma$ . Let  $M$  be the level of  $G$ . We know that  $N$  divides  $M$ . There is an isomorphism  $f: (X_\Gamma)_{K_M} \rightarrow (X_G)_{K_M}$  satisfying  $\pi_G \circ f = \pi_\Gamma$ . We observe that  $\mathbb{Q}(h)$  is a function field for a model of  $X_\Gamma$ , isomorphic to  $\mathbb{P}_{\mathbb{Q}}^1$ . Since we have models for  $X_\Gamma$  and  $X_G$  over  $\mathbb{Q}$  there is a natural action of  $\mathrm{Gal}(K_M/\mathbb{Q})$  on  $f$ . Using that  $\pi_G$  is defined over  $\mathbb{Q}$  we see that  $f$  satisfies the following condition

$$\sigma(\pi_\Gamma) = \pi_\Gamma \circ f^{-1} \circ \sigma(f) \tag{7.1.1}$$

for every  $\sigma \in \mathrm{Gal}(K_M/\mathbb{Q})$ .

equation (7.1.1) gives us a condition on the image of the cocycle

$$\zeta: \text{Gal}(K_M/\mathbb{Q}) \rightarrow \text{Aut}((X_\Gamma)_{K_M}) = \text{PGL}_2(K_M)$$

that sends  $\sigma$  to  $f^{-1} \circ \sigma(f)$ .

For a fixed  $M$  divisible by  $N$ , there are only finitely many cocycles from  $\text{Gal}(K_M/\mathbb{Q})$  to  $\text{PGL}_2(K_M)$  that satisfy equation (7.1.1). We compute the cocycles on a set of generators of  $\text{Gal}(K_M/\mathbb{Q})$  and extend to the whole group using the cocycle property. Let  $\zeta$  be one such cocycle. From Lemma 4.1.6 and Remark 4.1.1, we know that if  $\zeta$  is a coboundary in  $H^1(\text{Gal}(K_M/\mathbb{Q}), \text{PGL}_2(K_M))$ , then  $\zeta$  gives a pair  $(\mathbb{P}_\mathbb{Q}^1, \pi_1)$  which is isomorphic to a modular curve  $(X_G, \pi_G)$ .

For congruence subgroups  $\Gamma$  with label  $7A^0, 7C^0, 7G^0, 8A^0, 8M^0, 11A^0, 14A^0, 15A^0, 16A^0, 18A^0$  and  $21A^0$ ; we verify using Magma that there exists a  $\sigma \in \text{Gal}(K_N/\mathbb{Q})$  for which there is no degree 1 function  $g \in \mathbb{Q}^{\text{ab}}(t)$  satisfying the equation  $\sigma(\pi_\Gamma) = \pi_\Gamma \circ g$ . Since  $\pi_\Gamma \in K_N(t)$  the action of  $\text{Gal}(K_M/\mathbb{Q})$  for any multiple  $M$  of  $N$  factors through  $\text{Gal}(K_N/\mathbb{Q})$ . Therefore, for any multiple  $M$  of  $N$  there exists a  $\sigma \in \text{Gal}(K_M/\mathbb{Q})$  such that there is no function  $g$  satisfying the equation  $\sigma(\pi_\Gamma) = \pi_\Gamma \circ g$ . Thus, there exists no pair  $(X_G, \pi_G)$  such that  $G \cap \text{SL}_2(\hat{\mathbb{Z}})$  is equal to  $\Gamma$ . In these cases, Theorem 1.0.2 is vacuously true.

For congruence subgroup  $\Gamma$  with label  $5F^0$ , we find a pair  $(\mathbb{P}_\mathbb{Q}^1, \pi_1)$  when  $M = 15$ .

For congruence subgroups with labels  $8E^0$  and  $8K^0$ , we find a pair  $(\mathbb{P}_\mathbb{Q}^1, \pi_1)$  when

$M = 16$ .

For congruence subgroup  $\Gamma$  with label  $16F^0$ , we find a pair  $(\mathbb{P}_{\mathbb{Q}}^1, \pi_1)$  when  $M = 32$ .

For all the other congruence subgroups  $\Gamma$ , we find a pair  $(\mathbb{P}_{\mathbb{Q}}^1, \pi_1)$  when  $M = N$  where  $N$  is the level of  $\Gamma$ .

## 7.1.2 Breaking up the cocycles

Let us denote the group  $\text{Aut}_{\mathbb{Q}^{\text{ab}}}(X_{G_1}, \pi_{G_1})$  obtained in § 7.1.1 as  $A_{G_1}$ . We would like to mention at this point that for every pair  $(X_{G_1}, \pi_{G_1})$  which we compute we have that  $\text{Aut}_{\mathbb{Q}^{\text{ab}}}(X_{G_1}, \pi_{G_1})$  is defined over  $K_N$ , i.e.,  $A_{G_1}$  equals  $\text{Aut}_{K_N}(X_{G_1}, \pi_{G_1})$ , where  $N$  is the level of corresponding congruence subgroup  $\Gamma$ .

Let  $K := K_{N^\infty}$  be the compositum of all fields  $K_{N^e}$  with  $e \geq 1$ . Let  $K'$  be the compositum of all  $K_m$  with  $m$  relatively prime to  $N$ . Then,  $KK' = \mathbb{Q}^{\text{ab}}$  and  $K \cap K' = \mathbb{Q}$ .

Let  $\phi: \text{Gal}(\mathbb{Q}^{\text{ab}}/\mathbb{Q}) \rightarrow A_{G_1}$  be a cocycle. Since,  $A_{G_1}$  is defined over  $K_N$  the group  $\text{Gal}(\mathbb{Q}^{\text{ab}}/\mathbb{Q})$  acts via  $\text{Gal}(K/\mathbb{Q})$ .

The following lemma shows that there are finitely many cocycles from  $\text{Gal}(K/\mathbb{Q})$  to  $A_{G_1}$ .

**Lemma 7.1.1.** *Let  $b$  be the least common multiple of orders of elements  $a \in A_{G_1}$  that divide some power of  $N$ . If  $N$  is congruent to 2 modulo 4, then any cocycle  $\zeta: \text{Gal}(K/\mathbb{Q}) \rightarrow A_{G_1}$  factors through  $\text{Gal}(K_{2Nb}/\mathbb{Q})$ . If  $N$  is not congruent to 2 modulo 4, then any cocycle*

$\zeta: \text{Gal}(K/\mathbb{Q}) \rightarrow A_{G_1}$  factors through  $\text{Gal}(K_{Nb}/\mathbb{Q})$ .

*Proof.* Any cocycle  $\zeta: \text{Gal}(K/\mathbb{Q}) \rightarrow A_{G_1}$  factors through  $\text{Gal}(K_M/\mathbb{Q})$  where  $M$  is a multiple of  $N$  with the same prime divisors as  $N$ . Define  $G := \text{Gal}(K_M/K_N)$  if  $N$  is not congruent to 2 modulo 4 and  $G := \text{Gal}(K_M/K_{2N})$  if  $N$  is congruent to 2 modulo 4. The Galois group  $G$  is cyclic and its cardinality has the same prime factors as  $N$ . Let  $g$  be a generator of  $G$ . Let  $a := \zeta(g)$ . Then by our choice of  $b$  we get that  $a^b = 1$ . Also observe that  $\zeta$  is a homomorphism on  $G$ , since  $A_{G_1}$  is defined over  $K_N$ . Therefore,  $\zeta(g^b) = 1$ . It follows that  $\zeta$  factors through  $\text{Gal}(K_{Nb}/\mathbb{Q})$  if  $N$  is not congruent to 2 modulo 4 and  $\zeta$  factors through  $\text{Gal}(K_{2Nb}/\mathbb{Q})$  if  $N$  is congruent to 2 modulo 4.  $\square$

Let  $\{\zeta_i\}_{i=1}^l$  be the finite set of cocycles up to coboundaries from  $\text{Gal}(K/\mathbb{Q})$  to  $A_{G_1}$ . They can be computed using Lemma 7.1.1. Fix a cocycle

$$\zeta: \text{Gal}(K/\mathbb{Q}) \rightarrow A_{G_1}.$$

Fix  $(X_{G'}, \pi_{G'})$  as a  $K$ -twist of  $(X_{G_1}, \pi_{G_1})$  with respect to  $\zeta$ . There is an isomorphism  $f: (X_{G_1})_K \rightarrow (X_{G'})_K$  such that  $\zeta(\sigma) = f^{-1} \circ \sigma(f)$  is a representative of  $\theta((X_{G'}, \pi_{G'}))$ .

**Lemma 7.1.2.** *Given a cocycle  $\phi: \text{Gal}(\mathbb{Q}^{\text{ab}}/\mathbb{Q}) \rightarrow A_{G_1}$  satisfying  $\phi(i(\sigma, 1)) = \zeta(\sigma)$  for every  $\sigma \in \text{Gal}(K/\mathbb{Q})$ , there exists a cocycle  $\gamma': \text{Gal}(K'/\mathbb{Q}) \rightarrow \text{Aut}(X_{G'}, \pi_{G'})$ , where  $\text{Gal}(K'/\mathbb{Q})$  acts trivially on  $\text{Aut}(X_{G'}, \pi_{G'})$  satisfying  $\gamma'(\tau) = f \circ \phi(i(1, \tau)) \circ f^{-1}$  for every  $\tau \in \text{Gal}(K'/\mathbb{Q})$ . Conversely, given a cocycle  $\gamma': \text{Gal}(K'/\mathbb{Q}) \rightarrow \text{Aut}(X_{G'}, \pi_{G'})$ , there exists a cocycle  $\phi: \text{Gal}(\mathbb{Q}^{\text{ab}}/\mathbb{Q}) \rightarrow A_{G_1}$  satisfying  $\phi(i(\sigma, 1)) = \zeta(\sigma)$  for every  $\sigma \in \text{Gal}(K/\mathbb{Q})$  and*

$\phi(i(1, \tau)) = f^{-1} \circ \gamma'(\tau) \circ f$  for every  $\tau \in \text{Gal}(K'/\mathbb{Q})$ .

*Proof.* Let  $\phi: \text{Gal}(\mathbb{Q}^{\text{ab}}/\mathbb{Q}) \rightarrow A_{G_1}$  be a cocycle satisfying  $\phi(i(\sigma, 1)) = \zeta(\sigma)$  for every  $\sigma \in \text{Gal}(K/\mathbb{Q})$ . Let  $\gamma: \text{Gal}(K'/\mathbb{Q}) \rightarrow A_{G_1}$  be defined as  $\gamma(\tau) = \phi(i(1, \tau))$ . Using Lemma 4.1.5,  $\gamma(\tau) = f^{-1} \circ h_\tau \circ f$  for some  $h_\tau \in \text{Aut}(X_{G'}, \pi_{G'})$ . Let  $\gamma': \text{Gal}(K'/\mathbb{Q}) \rightarrow \text{Aut}(X_{G'}, \pi_{G'})$  defined as  $\gamma'(\tau) = h_\tau$ . Since,  $\gamma$  is a homomorphism,  $\gamma'$  is also a homomorphism.

We will now show the converse. Define  $\gamma: \text{Gal}(K'/\mathbb{Q}) \rightarrow A_{G_1}$  as  $\gamma(\tau) = f^{-1} \circ \gamma'(\tau) \circ f$ . Define

$$\phi: \text{Gal}(\mathbb{Q}^{\text{ab}}/\mathbb{Q}) \rightarrow A_{G_1}$$

as  $\phi(i(\sigma, \tau)) = \gamma(\tau)\zeta(\sigma)$ . Using Lemma 4.1.2, it follows that  $\phi$  is a cocycle.  $\square$

**Lemma 7.1.3.** *Assume that  $\zeta$  is a coboundary in  $H^1(\text{Gal}(K/\mathbb{Q}), \text{PGL}_2(K))$ . A twist of  $(X_{G_1}, \pi_{G_1})$  by a cocycle  $\phi: \text{Gal}(\mathbb{Q}^{\text{ab}}/\mathbb{Q}) \rightarrow A_{G_1}$  satisfying  $\phi(i(\sigma, 1)) = \zeta(\sigma)$  for every  $\sigma \in \text{Gal}(K/\mathbb{Q})$  is isomorphic to  $\mathbb{P}_{\mathbb{Q}}^1$  if and only if twist of  $(X_{G'}, \pi_{G'})$  by the corresponding cocycle  $\gamma'$  is isomorphic to  $\mathbb{P}_{\mathbb{Q}}^1$ .*

*Proof.* Since  $\zeta$  is a coboundary in  $H^1(\text{Gal}(K/\mathbb{Q}), \text{PGL}_2(K))$ ,  $\zeta(\sigma) = A^{-1}\sigma(A)$  for every  $\sigma$  in  $\text{Gal}(K/\mathbb{Q})$  and for some  $A \in \text{PGL}_2(K)$ . Explicitly,  $\gamma(\tau) = A^{-1}\gamma'(\tau)A$ , where  $\gamma'(\tau) \in \text{Aut}(X_{G'}, \pi_{G'}) \subsetneq \text{PGL}_2(\mathbb{Q})$ . The inclusion  $\text{Aut}(X_{G'}, \pi_{G'}) \subsetneq \text{PGL}_2(\mathbb{Q})$  uses the assumption that  $X_{G'}$  is isomorphic to  $\mathbb{P}_{\mathbb{Q}}^1$ . The cocycle  $\phi: \text{Gal}(\mathbb{Q}^{\text{ab}}/\mathbb{Q}) \rightarrow A_{G_1}$  which is given by  $\phi(\sigma, \tau) = \zeta(\sigma)\sigma(\gamma(\tau))$  becomes  $A^{-1}\sigma(A)\sigma(A^{-1}\gamma'(\tau)A) = A^{-1}\gamma'(\tau)\sigma(A)$ . The two cocycles  $\phi$  and  $\gamma'$  are cohomologous. Therefore, the lemma holds.

□

*Remark 7.1.2.* Lemma 7.1.3 holds for any pair of fields  $(K, K')$  as long as they satisfy the two conditions;  $K \cap K' = \mathbb{Q}$  and  $KK' = \mathbb{Q}^{\text{ab}}$ .

Lemma 7.1.3 shows that if  $X_{G'}$  is isomorphic to  $\mathbb{P}_{\mathbb{Q}}^1$ , then twists of  $(X_{G_1}, \pi_{G_1})$  by cocycles  $\phi: \text{Gal}(\mathbb{Q}^{\text{ab}}/\mathbb{Q}) \rightarrow A_{G_1}$  that satisfy  $\phi(i(\sigma, 1)) = \zeta(\sigma)$  for every  $\sigma \in \text{Gal}(K/\mathbb{Q})$  and are isomorphic to  $\mathbb{P}_{\mathbb{Q}}^1$  can be obtained via homomorphisms from  $\text{Gal}(K'/\mathbb{Q}) \rightarrow \text{Aut}(X_{G'}, \pi_{G'}) \rightarrow \text{PGL}_2(\mathbb{Q})$  that are coboundaries.

Let us now assume that  $\zeta$  is not a coboundary in  $H^1(\text{Gal}(K/\mathbb{Q}), \text{PGL}_2(K))$ . We want to classify the cocycles  $\gamma': \text{Gal}(K'/\mathbb{Q}) \rightarrow \text{Aut}(X_{G'}, \pi_{G'})$  that give us  $\mathbb{P}_{\mathbb{Q}}^1$ . Since  $\text{Gal}(K'/\mathbb{Q})$  acts trivially on  $\text{Aut}(X_{G'}, \pi_{G'})$ , the image of  $\gamma'$  is an abelian subgroup of  $\text{Aut}(X_{G'}, \pi_{G'})$ . Let us denote it by  $\mathcal{A}$ .

In the following two lemmas we describe the cases where there exists no cocycle from  $\text{Gal}(K'/\mathbb{Q})$  to  $\mathcal{A}$  that gives rise to a  $\mathbb{P}_{\mathbb{Q}}^1$ .

**Lemma 7.1.4.** *If  $\mathcal{A}$  is a cyclic group of order 3, then twist of  $(X_{G'}, \pi_{G'})$  by any cocycle  $\gamma': \text{Gal}(K'/\mathbb{Q}) \rightarrow \mathcal{A}$  is not isomorphic to  $\mathbb{P}_{\mathbb{Q}}^1$ .*

*Proof.* Assume that there exists a cocycle  $\gamma': \text{Gal}(K'/\mathbb{Q}) \rightarrow \mathcal{A}$  such that twist of  $(X_{G'}, \pi_{G'})$  by  $\gamma'$  is isomorphic to  $\mathbb{P}_{\mathbb{Q}}^1$ . We are given that  $\mathcal{A}$  is of order 3. Therefore,  $\gamma'$  factors through  $\text{Gal}(L/\mathbb{Q})$ , where  $L$  is a degree 3 Galois extension of  $\mathbb{Q}$ . This would imply that  $X_{G'}$  becomes isomorphic to  $\mathbb{P}_{\mathbb{Q}}^1$  over an extension of degree 3 over  $\mathbb{Q}$  which is a contradiction. □

**Lemma 7.1.5.** *Let  $X_{G'}$  be the genus 0 curve explicitly given as a conic  $C : X^2 + Y^2 + Z^2 = 0 \subseteq \mathbb{P}_{\mathbb{Q}}^2$ . Let  $Q$  denote a Klein-4 twist of  $C$  described by  $[X : Y : Z] \rightarrow [-X : -Y : Z]$  and  $[X : Y : Z] \rightarrow [X : -Y : -Z]$  over  $K'$ , here  $K' = \bigcup_{(m,2)=1} K_m$ . Then,  $Q$  has no rational points.*

*Proof.* A Klein-4 twist  $Q$  of  $C$  is given by  $dX^2 + Y^2 + eZ^2 = 0$  for squarefree integers  $d, e$ . Further, since we are twisting over  $K'$  we can assume that  $d$  and  $e$  are congruent to 1 modulo 4 and hence are squares in  $\mathbb{Q}_2$ . It is now easy to verify that  $Q$  has no 2-adic points. Hence, the lemma holds.  $\square$

In the cases where we get a curve  $X_{G'}$  which is not isomorphic to  $\mathbb{P}_{\mathbb{Q}}^1$ , we either face the obstructions discussed in Lemma 7.1.4 and Lemma 7.1.5 or we find a cocycle  $\gamma' : \text{Gal}(K'/\mathbb{Q}) \rightarrow \mathcal{A}$  such that twist of  $(X_{G'}, \pi_{G'})$  by  $\gamma'$  is  $\mathbb{P}_{\mathbb{Q}}^1$ ; it factors through  $\text{Gal}(L_0/\mathbb{Q})$ , where  $L_0 \subseteq K_n$  and  $n$  is 3, 5 or 15. We compute all the pairs  $(\mathbb{P}_{\mathbb{Q}}^1, \pi_i)$  that come from coboundaries  $\phi_i : \text{Gal}(KK_n/\mathbb{Q}) \rightarrow A_{G_1} \rightarrow \text{PGL}_2(KK_n)$  which satisfy the condition  $\phi_i(i(\sigma, 1)) = \zeta(\sigma)$  for every  $\sigma \in \text{Gal}(K/\mathbb{Q})$ .

We now discuss how can we get all the other cocycles  $\gamma' : \text{Gal}(K'/\mathbb{Q}) \rightarrow \mathcal{A}$  that give rise to  $\mathbb{P}_{\mathbb{Q}}^1$ . Suppose there is another cocycle  $\gamma'' : \text{Gal}(K'/\mathbb{Q}) \rightarrow \mathcal{A}$  that gives us a  $\mathbb{P}_{\mathbb{Q}}^1$  and factors through  $\text{Gal}(L_1/\mathbb{Q})$ , where  $L_1$  is a finite Galois extension of  $\mathbb{Q}$ . Without loss of generality, we can assume that  $L_1 \subseteq K_m$ , where  $m$  is coprime to  $n$ . Therefore, to obtain all the other coboundaries  $\phi : \text{Gal}(\mathbb{Q}^{\text{ab}}/\mathbb{Q}) \rightarrow A_{G_1} \rightarrow \text{PGL}_2(\mathbb{Q}^{\text{ab}})$  satisfying  $\phi(i(\sigma, 1)) = \zeta(\sigma)$  for every  $\sigma \in \text{Gal}(K/\mathbb{Q})$  we use Lemma 7.1.3 and Remark 7.1.2.

### 7.1.3 Concluding the proof

Let  $\{(X_{\mathfrak{G}_i}, \pi_{\mathfrak{G}_i})\}_{i=2}^s$  be the finite set of genus 0 modular curves isomorphic to  $\mathbb{P}_{\mathbb{Q}}^1$  such that either  $(X_{\mathfrak{G}_i}, \pi_{\mathfrak{G}_i})$  is a twist of  $(X_{G_1}, \pi_{G_1})$  with respect to a coboundary  $\zeta: \text{Gal}(K/\mathbb{Q}) \rightarrow \text{PGL}_2(K)$  or  $(X_{\mathfrak{G}_i}, \pi_{\mathfrak{G}_i})$  is a twist of  $(X_{G_1}, \pi_{G_1})$  with respect to a coboundary  $\phi: \text{Gal}(KK_n/\mathbb{Q}) \rightarrow \text{PGL}_2(KK_n)$  satisfying  $\phi(i(\sigma, 1)) = \zeta(\sigma)$  for every  $\sigma \in \text{Gal}(K/\mathbb{Q})$  and  $\zeta: \text{Gal}(K/\mathbb{Q}) \rightarrow \text{PGL}_2(K)$  is not a coboundary. Further, if  $(X_G, \pi_G)$  is any other  $\mathbb{Q}^{\text{ab}}$ -twist of  $(X_{G_1}, \pi_{G_1})$  isomorphic to  $\mathbb{P}_{\mathbb{Q}}^1$ , then using Lemma 7.1.3 and Remark 7.1.2  $(X_G, \pi_G)$  can be obtained as a  $\mathbb{Q}^{\text{ab}}$ -twist of some  $(X_{G_i}, \pi_{G_i})$  for  $i \in \{1, \dots, l\}$  by considering cocycles  $\gamma: \text{Gal}(\mathbb{Q}^{\text{ab}}/\mathbb{Q}) \rightarrow \text{Aut}(X_{G_i}, \pi_{G_i})$ .

Hence, Theorem 1.0.2 holds.

## 7.2 Proof of Theorem 1.0.3

From Theorem 1.0.2 we know that any modular curve  $X_G$  isomorphic to  $\mathbb{P}_{\mathbb{Q}}^1$  is a  $\mathbb{Q}^{\text{ab}}$ -twist of  $(\mathbb{P}_{\mathbb{Q}}^1, \pi_i)$  for some  $i \in \{1, \dots, r\}$  via a homomorphism  $\gamma: \text{Gal}(\mathbb{Q}^{\text{ab}}/\mathbb{Q}) \rightarrow \mathcal{A}_i \subseteq \text{Aut}(\mathbb{P}_{\mathbb{Q}}^1, \pi_i)$ .

Let  $(X_G, \pi_G)$  be a modular curve isomorphic to  $(\mathbb{P}_{\mathbb{Q}}^1, \pi_i)$  for some  $i \in \{1, \dots, r\}$ . Let  $\gamma: \text{Gal}(\mathbb{Q}^{\text{ab}}/\mathbb{Q}) \rightarrow \text{Aut}(X_G, \pi_G)$  be a homomorphism. Since  $\text{Gal}(\mathbb{Q}^{\text{ab}}/\mathbb{Q})$  is abelian, the image of  $\gamma$  denoted by  $\mathcal{A}$  is a finite abelian group. We compute all such possible finite abelian subgroups up to conjugation in  $\text{PGL}_2(\mathbb{Q})$ . We list these possibilities in chapter 6. We are interested in computing the cocycles  $\gamma: \text{Gal}(\mathbb{Q}^{\text{ab}}/\mathbb{Q}) \rightarrow \text{Aut}(X_G, \pi_G)$

such that the twist of  $(X_G, \pi_G)$  by  $\gamma$  is isomorphic to  $\mathbb{P}_{\mathbb{Q}}^1$ . This is described in chapter 6 for various cases of  $\mathcal{A}$  that occur. Furthermore, we can compute the morphism  $\pi'$  to j-line of the twist as follows. We factor  $\pi_G$  as  $J_i \circ u_i$ , where  $u_i: X_G \rightarrow \mathbb{P}_{\mathbb{Q}}^1$  is the function  $\pi$  described in the appropriate case of  $\mathcal{A}$  we are in. Then define  $\pi' := J_i \circ \pi_{\mathcal{A},v}$ , where  $\pi_{\mathcal{A},v}(u) \in \mathbb{Q}(u)$  is given explicitly in each case.

## CHAPTER 8

### COMPUTING GENERATORS FOR $G$ AND FINDING EQUATION OF $X_G$ AS A CONIC USING GENERATORS OF $G$

#### 8.1 Computing generators for $G$

Fix a genus 0 congruence subgroup  $\Gamma$  of level  $N$  containing  $-I$ . Let  $h$  be the normalized hauptmodul of  $\Gamma$ , see chapter 3 for computation of hauptmoduls. Fix a genus 0 modular curve  $(X_G, \pi_G)$  such that  $X_G$  is isomorphic to  $\mathbb{P}_\mathbb{Q}^1$  and  $G \cap \mathrm{SL}_2(\mathbb{Z}) = \Gamma$ . Let  $C$  be a matrix in  $\mathrm{PGL}_2(\mathbb{Q}^{\mathrm{ab}})$  such that  $\pi_G(C(h)) = j$ . Let  $M$  be the smallest positive integer satisfying  $N|M$  such that  $C \in \mathrm{PGL}_2(K_M)$ . Let  $H := \pi_M(\Gamma)$ . In this section we will describe a procedure to compute generators for  $G$ . Let us denote  $\mathrm{Aut}_{K_M}(X_G, \pi_G)$  by  $A_G$ . Observe that  $A_G$  is a finite subgroup of  $\mathrm{Aut}_{K_M}(X_G) \subsetneq \mathrm{PGL}_2(K_M)$ .

Let us denote  $C(h)$  as  $u$  for this section. Consider the set  $\Delta = \{\xi(u) \mid \xi \in A_G\}$ .

**Lemma 8.1.1.** *For  $A$  in  $N_{\mathrm{GL}_2(\mathbb{Z}/M\mathbb{Z})}(H)/(H)$  and  $\xi(u)$  in  $\Delta$ , define  $\xi(u).A = \sigma_d(\xi)(u * A)$  where  $d = \det(A)$ . This is an action of  $N_{\mathrm{GL}_2(\mathbb{Z}/M\mathbb{Z})}(H)/(H)$  on  $\Delta$ .*

*Proof.* Let  $A \in N_{\mathrm{GL}_2(\mathbb{Z}/M\mathbb{Z})}(H)$ . For every  $h_1$  in  $H$  we have  $(u * A) * h_1 = u * (Ah_1) = u * (h_2A) = (u * h_2) * A = u * A$  where  $h_2 \in H$ . Therefore,  $u * A$  lies in  $K_M(h)$ . Since,  $j * A = j$ ,  $(\pi_G(u)) * A = \pi_G(u * A) = j$ . Therefore,  $u * A = \phi(u)$  where  $\phi \in \mathrm{PGL}_2(K_M)$ . Since,  $\pi_G(u) = \pi_G(\phi(u)) = j$  we get  $\pi_G(\phi(C)) = \pi_G(C)$ , therefore,  $\pi_G(\phi) = \pi_G$ . Hence,  $\phi(u)$  lies in  $\Delta$ .

We know  $(\xi(u)).A = \sigma_d(\xi)(u * A)$ . Since,  $\pi_G(\sigma_d(\xi)) = \pi_G, (\xi(u)).A$  lies in  $\Delta$ .

Since,  $H$  acts trivially we get an action of the quotient group. □

Let  $\tilde{G}$  be the stabilizer of  $\{u\}$  under this action. Taking inverse image of  $\tilde{G}$  under quotient map (from  $N_{GL_2(\mathbb{Z}/M\mathbb{Z})}(H)$  to  $N_{GL_2(\mathbb{Z}/M\mathbb{Z})}(H)/(H)$ ) gives us our required subgroup  $G$ .

### 8.1.1 Alternate method to compute the generators for $G$

We compute the following set

$$S := \{B \subseteq N_{GL_2(\mathbb{Z}/M\mathbb{Z})}(H)/(H) \mid \det(B) \simeq (\mathbb{Z}/M\mathbb{Z})^\times\}.$$

After we compute  $S$ , we search for the element which fixes  $u$ , taking its inverse image of under quotient map gives us our required subgroup  $G$ .

## 8.2 Computing equation for the curve $X_G$ from $G$

Fix an open subgroup  $G$  of  $GL_2(\hat{\mathbb{Z}})$  such that  $\det(G) = \hat{\mathbb{Z}}^*$ ,  $-I \in G$  and  $X_G$  has genus 0. Let  $N$  be the level of  $G$ . Fix a set  $S$  of generators for  $\pi_N(G)$  in  $GL_2(\mathbb{Z}/N\mathbb{Z})$ . In this section we will explain how to compute an explicit model for  $X_G$ .

Let  $\Gamma$  be the genus 0 congruence subgroup  $G \cap \mathrm{SL}_2(\mathbb{Z})$  and let  $H := \pi_N(G) \cap \mathrm{SL}_2(\mathbb{Z}/N\mathbb{Z})$ . Let  $h$  be the normalized hauptmodul of  $\Gamma$ , see chapter 3 for computation of hauptmoduls. The quotient group  $G/H$  is isomorphic to the Galois group  $\mathrm{Gal}(K_N/\mathbb{Q})$  via the determinant function. Using the definition of  $H$  we have that  $G \subseteq N_{\mathrm{GL}_2(\mathbb{Z}/N\mathbb{Z})}(H)$ . Therefore,  $h * g \in K_N(h)$  for every  $g \in G$  where  $*$  is the right action of  $\mathrm{GL}_2(\mathbb{Z}/N\mathbb{Z})$  on  $\mathcal{F}_N$  as described in section 2. For  $g \in G$ , let  $d_g = \det(g)$ . From the equation  $\pi_\Gamma(h) = j$  we get that  $\sigma_{d_g}(\pi_\Gamma)(h * g) = j$  for every  $g \in G$ . Thus,  $h * g = \phi_{\sigma_{d_g}}(h)$  where  $\phi_{\sigma_{d_g}} \in \mathrm{PGL}_2(K_N)$ . Since  $H$  acts trivially on  $h$  we get that the quotient group  $G/H$  acts on  $h$ .

**Lemma 8.2.1.** *Let  $\zeta: \mathrm{Gal}(K_N/\mathbb{Q}) \rightarrow \mathrm{PGL}_2(K_N)$  be the map defined as  $\zeta(\sigma_g) = \phi_{\sigma_{d_g}}^{-1}$ . The map  $\zeta$  is a cocycle.*

*Proof.* We know that  $*$  is a right action. Therefore,  $(h * g_1) * g_2 = h * (g_1 g_2)$ . This gives us  $\sigma_{d_{g_2}}(\phi_{\sigma_{d_{g_1}}})\phi_{\sigma_{d_{g_2}}}(h) = \phi_{\sigma_{d_{g_1 g_2}}}(h)$ . Thus,  $\zeta(\sigma_{g_2} \sigma_{g_1}) = \zeta(\sigma_{g_1} \sigma_{g_2}) = \phi_{\sigma_{d_{g_1 g_2}}}^{-1} = \phi_{\sigma_{d_{g_2}}}^{-1} \sigma_{d_{g_2}}(\phi_{\sigma_{d_{g_1}}}^{-1}) = \zeta(\sigma_{g_2})\sigma_{d_{g_2}}(\zeta(\sigma_{g_1}))$ .  $\square$

Using the procedure described in chapter 4 we can now get an equation for the modular curve  $X_G$  as a conic in  $\mathbb{P}^2$ .

## APPENDIX A

### HOW TO READ THE TABLES

In tables [A.1](#) to [A.9](#) we use a label of the form  $NA_1 - MA_2$  to denote the groups  $G_i$  as in [Theorem 1.0.2](#). Here  $N, M$  are natural numbers and  $A_1, A_2$  are letters which need not be equal. The natural number  $M$  is the level of  $G$ . The part  $NA$  denotes the label of congruence subgroup  $\Gamma$  equal to  $G_i \cap \mathrm{SL}_2(\hat{\mathbb{Z}})$  from [\[CP03\]](#). The label 1A-1A denotes the  $j$ -line. The function given in the column " $\pi(t)$ " describes the morphism  $X_G \rightarrow X_{G'}$  where  $G'$  is a supergroup containing  $G$  whose label is given in the column "sup". Thus, to obtain the map  $\pi_G: X_G \rightarrow \mathbb{P}_{\mathbb{Q}}^1$  we can compose the functions  $X_G \rightarrow X_{G'} \rightarrow \mathbb{P}_{\mathbb{Q}}^1$ . In few cases we describe  $\pi(t)$  by a sequence of the form  $[f_1, f_2, \dots, f_r]$  where each  $f_i \in \mathbb{Q}(t)$  and the corresponding  $\pi(t)$  is  $f_r(\dots(f_2(f_1(t))))$ .

To determine the family described by  $(X_G, \pi_G)$  we use tables [A.10](#) to [A.16](#). The column "C" gives the matrix  $C$  such that a maximal abelian subgroup of  $\mathrm{Aut}(\mathbb{P}_{\mathbb{Q}}^1, \pi(C(t)))$  is equal to  $\mathcal{A}$  of the case mentioned in column "Case." The column "Case" refers to the case number in [chapter 6](#). The column " $\alpha$ " gives the value of  $\alpha$ , if Case is 2 or 5.

Let us go through an example.

**Example 2.** Define

$$\pi(t) := \frac{(1728t^6 - 5184t^5 + 10368t^4 - 12096t^3 + 10368t^2 - 5184t + 1728)}{(t^6 - 3t^5 - 3t^4/4 + 13t^3/2 - 3t^2/4 - 3t + 1)}.$$

The pair  $(\mathbb{P}_{\mathbb{Q}}^1, \pi)$  is labeled  $2C - 2A$  in table [A.1](#).

Let  $\mathcal{A}_1 = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \right\} \simeq \mathbb{Z}/2\mathbb{Z}$ ; it is  $\mathcal{A}$  given in Case 1 of chapter [6](#) and  $\mathcal{A}_2 = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix}, \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix} \right\} \simeq \mathbb{Z}/3\mathbb{Z}$ ; it is  $\mathcal{A}$  given in Case 3 of chapter [6](#).

Define the rational functions

$$J_1(t) := \frac{(t^3 + 576t^2 + 110592t + 7077888)}{(t^2 - 128t + 4096)}$$

and

$$J_2(t) := \frac{(1728t^2 - 5184t + 15552)}{(t^2 - 3t + 9/4)}.$$

Then, any genus 0 modular curve  $(X_G, \pi_G)$  such that  $X_G$  has a rational point and is a  $\mathbb{Q}^{\text{ab}}$ -twist of  $(\mathbb{P}_{\mathbb{Q}}^1, \pi)$  is isomorphic to  $(\mathbb{P}_{\mathbb{Q}}^1, J_i \circ \pi_{\mathcal{A}_i, \nu})$  for some  $i \in \{1, 2\}$ .

*Remark A.0.1.* The tables are automated. The maps can be chosen to make the columns look nicer. We can also include more information here if needed.

Label	$\pi(t)$	Sup
1A-1A		
2A-2A	$t^2 + 1728$	1A-1A
2B-2A	$(t^3 + 24t^2 + 192t + 512)/(t + 24)$	1A-1A
2C-2A	$(t^2 - 8t + 256)/(t - 8)$	2B-2A
3A-3A	$t^3$	1A-1A
3B-3A	$(3645t^4/4 + 243t^3/2 + 27t^2/8 - t/8 - 1/192)/(t^4 + t^3/24)$	1A-1A
3C-3A	$(-729t^6/8 - 243t^5/8 - 27t^4/32 + 7t^3/16 + t^2/128 - t/384 + 1/13824)/(t^6 - t^5/4 + t^4/48 - t^3/1728)$	1A-1A
3D-3A	$(t^4 + 216t)/(t^3 - 27)$	3A-3A
4A-4A	$(192t^4 - 128t^3 + 24t^2 - 1/4)/t^4$	1A-1A
4B-4A	$t^2 + 40$	2B-2A
4C-4A	$t^2 - 24$	2B-2A
4D-4A	$(t^2 - t/8)/(t^2 + t - 1/8)$	4A-4A
4E-4A	$t^2 + 8$	2C-2A
4F-4A	$(t^2 + 4t + 4)/(t + 4)$	4C-4A
4G-4A	$(t^2 - 4)/t$	4E-4A
4G-4B	$(t^3/8 + t^2/16 + t/64)/(t^3 + 5t^2/8 + t/16 + 1/64)$	4D-4A
4G-8A	$(-t^{24} + 672t^{20} - 151296t^{16} + 11583488t^{12} - 38731776t^8 + 44040192t^4 - 16777216)/(t^{20} + 64t^{16} + 1536t^{12} + 16384t^8 + 65536t^4)$	1A-1A
4G-8B	$(-17t^6/312 + t^5/52 + 55t^4/312 + 5t^3/26 + 25t^2/312 + t/52 + 1/312)/(t^6 + 24t^5/13 + 31t^4/13 + 24t^3/13 + 7t^2/13 - 1/39)$	4A-4A
4G-8C	$(4t^2 + 8t - 4)/(t^2 - 2t - 1)$	4E-4A
4G-8D	$(136t^4 + 224t^3 + 144t^2 + 32t + 8)/(t^4 - 4t^3 + 2t^2 + 4t + 1)$	2C-2A
4G-8E	$(-72t^4/7 + 96t^3/7 - 80t^2/7 + 32t/7 - 8/7)/(t^4 - 20t^3/7 + 6t^2/7 + 4t/7 - 1/7)$	4C-4A
4G-8F	$[t/(t^2 - 1), t/(t^2 + 1/4), (t^2 - 3t)/(t + 1), (1728t^3 - 8640t^2 + 14400t - 8000)/(t^3 + 19t^2 + 99t + 81)]$	1A-1A
4G-16A	$(8t^4 - 32t^3 - 48t^2 + 32t + 8)/(t^4 + 4t^3 - 6t^2 - 4t + 1)$	4B-4A
5A-5A	$-32t^5 - 240t^3 - 400t^2 - 210t - 36$	1A-1A
5B-5A	$(-32t^6 - 96t^5 + 360t^4 + 880t^3 - 1710t^2 - 2166t + 6859/2)/(t + 3)$	1A-1A
5C-5A	$(-3375t^{10} - 13500t^9 - 6750t^8 + 13500t^7 - 21375t^6 - 27000t^5 + 30500t^4 - 16000t^3 - 18000t^2 + 16000t - 8000)/(t^{10} - 5t^9 + 5t^8 + 10t^7 - 15t^6 - 11t^5 + 15t^4 + 10t^3 - 5t^2 - 5t - 1)$	1A-1A
5D-5A	$(2t^2 + 9t/2 - 2)/(t^2 + t - 1)$	5B-5A
5E-5A	$(t^3 - 3t^2/2 - 9t/4 + 15/8)/(t^2 - 3t/2 + 1/4)$	5A-5A
5F-15A	$(-2t^2 + 4t - 1/3)/(t^2 + 3t - 2/3)$	5C-5A
5G-5A	$[t^5/(t^4 + t^2/4 + 1/80), (-122023936t^6/161051 + 420687360t^5/161051 - 488064000t^4/161051 + 195795000t^3/161051 - 6150000t^2/161051 + 534375t/1288408 - 15625/10307264)/(t^6 + 25t^5/22 + 125t^4/242 + 625t^3/5324 + 3125t^2/234256 + 3125t/5153632)]$	1A-1A
5H-5A	$(t^{60} + 684t^{55} + 157434t^{50} + 12527460t^{45} + 77460495t^{40} + 130689144t^{35} - 33211924t^{30} - 130689144t^{25} + 77460495t^{20} - 12527460t^{15} + 157434t^{10} - 684t^5 + 1)/(t^{55} - 55t^{50} + 1205t^{45} - 13090t^{40} + 69585t^{35} - 134761t^{30} - 69585t^{25} - 13090t^{20} - 1205t^{15} - 55t^{10} - t^5)$	1A-1A
5H-15A	$[(t^2 - 1)/(t - 1/6), (t^2 + 2t/3)/(t + 3/2), (t^3 + 176t^2/57 + 256t/171)/(t^2 + 11t/12 + 19/144), (3792752640000t^5/28629151 - 12991841280000t^4/28629151 + 19126026240000t^3/28629151)/(t^5 - 40960t^4/1767 + 671088640t^3/3122289 - 5497558138880t^2/5517084663 + 22517998136852480t/9748688599521 - 36893488147419103232/17225932755353607)]$	1A-1A
6A-6A	$(2t^3 - 1/1728)/t^6$	1A-1A
6B-6A	$t^2 + 12$	3A-3A
6C-6A	$-t^2/24/(t^2 + 1/48)$	3B-3A

Table A.1: Table of  $\pi$  for groups with label 1A – 1A to 6C – 6A

Label	$\pi(t)$	Sup
6D-6A	$t^3 - 24$	2B-2A
6E-6A	$t^2/12/(t^2 + 1/72)$	3C-3A
6E-12A	$1/(2t^2 + 6)$	3C-3A
6E-24A	$t^2/(12t^2 + 1/36)$	3C-3A
6E-24B	$t^2/(12t^2 + 1/3)$	3C-3A
6F-6A	$(3t^3/88 - t^2/176)/(t^3 - 3t^2/22 - 3t/176 + 1/352)$	3B-3A
6G-6A	$(3t^3/44 - t^2/88)/(t^3 - 3t^2/22 - 3t/88 + 1/176)$	3C-3A
6H-6A	$(-24t^6 - 72t^5 - 96t^4 - 72t^3 - 32t^2 - 8t - 23/27)/(t^6 + 3t^5 + 4t^4 + 3t^3 + 4t^2/3 + t/3 + 1/27)$	2B-2A
6I-6A	$(t^2/8 + t/16)/(t^2 + t/8 + 1/16)$	6F-6A
6J-6A	$(-49t^6/1122 + 7t^5/374 - 41t^4/13464 + t^3/4488 - t^2/161568)/(t^6 - 75t^5/187 + 7t^4/132 - t^3/1122 - t^2/2448 + t/26928 - 1/969408)$	3B-3A
6K-6A	$(t^3 - 2)/t$	3D-3A
6K-24A	$(1892t^{12}/45 + 1648t^{11}/5 + 6392t^{10}/5 + 132496t^9/45 + 23388t^8/5 + 25824t^7/5 + 11120t^6/3 + 6816t^5/5 + 1212t^4/5 + 19504t^3/45 + 2168t^2/5 + 592t/5 + 68/45)/(t^{12} - 4t^{11}/15 - 586t^{10}/45 - 76t^9/3 + 23t^8 + 536t^7/3 + 1652t^6/5 + 1592t^5/5 + 541t^4/3 + 52t^3 - 14t^2/15 - 52t/9 - 5/3)$	3A-3A
6L-6A	$(t^4 + 4t^3 + 4t^2)/(t^3 + 4t^2 + 6t + 3)$	6D-6A
6L-12A	$(-4t^4 - 16t^3 - 16t^2)/(t^4 - 4t^2 - 8t - 4)$	6D-6A
6L-24A	$-8t^2/3/(t^4 + t^2/3 - 1/12)$	6D-6A
6L-24B	$-8t^2/(t^4 + t^2 - 3/4)$	6D-6A
7B-7A	$(-t^8 - 4t^7 + 14t^6 - 315t^4 + 1134t^2 - 1188t + 351)/(t + 4)$	1A-1A
7D-7A	$(550731776t^{21} + 9500123136t^{20} + 75278149632t^{19} + 364093940224t^{18} + 1204037345280t^{17} + 2888450482176t^{16} + 5203203027904t^{15} + 7186730209344t^{14} + 7705898708256t^{13} + 6455302073752t^{12} + 4230707519808t^{11} + 2161077942912t^{10} + 851643816597t^9 + 254098898445t^8 + 55620363099t^7 + 8489837559t^6 + 831946500t^5 + 45983952t^4 + 1075648t^3)/(t^{21} - 35t^{20} + 469t^{19} - 2702t^{18} + 2429t^{17} + 38486t^{16} - 88424t^{15} - 33859t^{14} + 630420t^{13} + 2604420t^{12} + 52437t^{11} - 10144155t^{10} - 19651177t^9 - 19107424t^8 - 11240167t^7 - 4245724t^6 - 1060598t^5 - 177149t^4 - 19593t^3 - 1379t^2 - 56t - 1)$	1A-1A
7E-7A	$(t^3 - 5t^2 + 4t + 5)/(t^2 - 5t + 6)$	7B-7A
7F-7A	$(-12288000t^{28} - 190668800t^{27} - 1395394560t^{26} - 6407299584t^{25} - 20722727424t^{24} - 50295974400t^{23} - 95542730304t^{22} - 147000575232t^{21} - 189679346976t^{20} - 213607407384t^{19} - 218793407896t^{18} - 209741587272t^{17} - 189011564091t^{16} - 158435094930t^{15} - 123017500155t^{14} - 89515425174t^{13} - 61956500270t^{12} - 40703065914t^{11} - 24917758131t^{10} - 14042095354t^9 - 7349216238t^8 - 3624251502t^7 - 1659184891t^6 - 667411290t^5 - 219538851t^4 - 54970832t^3 - 9674280t^2 - 1058400t - 54000)/(t^{28} - 14t^{27} + 35t^{26} + 273t^{25} - 938t^{24} - 3668t^{23} + 9142t^{22} + 40815t^{21} - 24500t^{20} - 294469t^{19} - 320705t^{18} + 828870t^{17} + 2814070t^{16} + 2612932t^{15} - 3122329t^{14} - 13515138t^{13} - 22861538t^{12} - 25648595t^{11} - 21325927t^{10} - 13713455t^9 - 6946926t^8 - 2788931t^7 - 884870t^6 - 219478t^5 - 41692t^4 - 5859t^3 - 574t^2 - 35t - 1)$	1A-1A
8B-8A	$t^2$	4C-4A
8B-8B	$(-24t^4 - 1/4)/t^4$	2B-2A
8B-8C	$1/(2t^2)$	4C-4A
8B-8D	$-t^4 - 24$	2B-2A
8C-8A	$-t^4 - 16t^2 - 24$	2B-2A
8D-8A	$t^2 - 8$	4C-4A
8E-16A	$(-t^4/8 + t^3/2 - t^2/4 - t/2 - 1/8)/(t^4 + 8t^3 - 2t^2 - 3)$	4A-4A
8E-16B	$(t^4/8 - t^2/8 + 1/32)/(t^4 + 2t^3 - t^2 + t + 1/4)$	4A-4A
8F-8A	$(t^4/20 - t^2/20 + 1/80)/(t^4 - 8t^3/5 + 3t^2/5 - 3/20)$	4A-4A
8G-8A	$(t^4 + 24t^2 + 16)/(t^3 + 4t)$	2C-2A
8G-8B	$(1728t^{24} + 44928t^{22} + 452448t^{20} + 2229344t^{18} + 5699796t^{16} + 7773168t^{14} + 5537396t^{12} + 1943292t^{10} + 1424949t^8/4 + 69667t^6/2 + 14139t^4/8 + 351t^2/8 + 27/64)/(t^{24} + 2t^{22} - 15t^{20}/2 - 3t^{18}/2 + 367t^{16}/16 - 127t^{14}/4 + 343t^{12}/16 - 127t^{10}/16 + 367t^8/256 - 3t^6/128 - 15t^4/512 + t^2/512 + 1/4096)$	1A-1A

Table A.2: Table of  $\pi$  for groups with label 6D – 6A to 8G – 8B

Label	$\pi(t)$	Sup
8G-8C	$(8t^4 - 8)/(t^4 + 1)$	2C-2A
8G-8D	$[t/(t^2 - 1), t/(t^2 + 1/4), (t^2 - 3t)/(t + 1), (1728t^3 + 60480t^2 + 705600t + 2744000)/(t^3 + 11t^2 + 19t + 9)]$	1A-1A
8G-8E	$(24t^4 - 32t^3 - 48t^2 - 32t - 8)/(5t^4 + 4t^3 + 6t^2 + 4t + 1)$	2C-2A
8G-8F	$[(t^2 + t)/(t - 1), t^2/(t - 1/6), (t^2 - 6t)/(t + 2/21), (97336000t^3/321489 + 1692800t^2/189 + 88320t + 290304)/(t^3 + 358t^2/21 + 844t/9 + 168)]$	1A-1A
8G-8G	$[(t^2 + 2t)/(t - 2), t/(t^2 + 4), t^2/(t - 1/8), (54000t^3 + 5400t^2 + 180t + 2)/(t^3 - t^2 + t/4)]$	1A-1A
8G-8H	$[t/(t^2 + 1/8), t/(t^2 + 2), (t^3 - t/8)/(t^2 - 1/72), (1728t^2 + 648)/t^2]$	1A-1A
8G-8I	$(-40t^4/3 - 32t^3/3 - 16t^2 - 32t/3 - 8/3)/(t^4 - 4t^3/3 - 2t^2 - 4t/3 - 1/3)$	2C-2A
8G-8J	$[t/(t^2 - 1), t/(t^2 + 1/4), (t^2 - 3t)/(t + 1), (8t^3 + 792t^2 + 26136t + 287496)/(t^2 + 2t + 1)]$	1A-1A
8G-8K	$[t/(t^2 + 1), t^2, t^2 - t/4, (16777216t^3 + 589824t^2 + 6912t + 27)/(t + 1/64)]$	1A-1A
8G-8L	$[t/(t^2 + 2), t^2, t^2 - t/8, (268435456t^3 + 786432t^2 + 768t + 1/4)/t]$	1A-1A
8G-16A	$[t/(t^2 - 1), (t^2 + 1/2)/(t - 1/2), t/(t^2 + 1/4), (3456t^3 + 3456t^2 + 1152t + 128)/(t^3 - t^2 - t + 1)]$	1A-1A
8H-8A	$(t^2 + 2)/t$	8D-8A
8I-8A	$(-24t^8 + 832t^6 + 448t^4 - 768t^2 - 384)/(t^8 + 8t^6 + 24t^4 + 32t^2 + 16)$	2B-2A
8J-8A	$(t^4 + 8t^2 + 32)/(t^2 + 4)$	2C-2A
8K-16A	$8t^4/(t^4 + 4t^2 + 2)$	4C-4A
8L-8A	$(-24t^8 - 192t^7 - 480t^6 - 192t^5 - 3472t^4 + 192t^3 - 480t^2 + 192t - 24)/(t^8 + 8t^7 + 20t^6 + 8t^5 - 26t^4 - 8t^3 + 20t^2 - 8t + 1)$	2B-2A
8N-8A	$(t^2 - 1)/t$	4G-4A
8N-8B	$(-8t^8 - 12t^4 - 1/2)/(t^8 - t^4/2 + 1/16)$	2C-2A
8N-8C	$(-t^8 - 4t^6 - 22t^4 - 4t^2 - 1)/(t^6 + 2t^4 + t^2)$	2C-2A
8N-8D	$(-10t^8 + 16t^7 - 72t^6 + 16t^5 + 4t^4 - 16t^3 - 72t^2 - 16t - 10)/(t^8 - 4t^7 + 4t^6 - 4t^5 + 6t^4 + 4t^3 + 4t^2 + 4t + 1)$	2C-2A
8N-8E	$(-2t^8 - 4t^6 - 11t^4 - t^2 - 1/8)/(t^6 + t^4 + t^2/4)$	2C-2A
8N-8F	$(2t^8 - 4t^6 + 11t^4 - t^2 + 1/8)/(t^6 - t^4 + t^2/4)$	2C-2A
8N-16A	$-16t^4/(t^8 + 1)$	4C-4A
8N-16B	$-4t^4/(t^8 + 1/16)$	4C-4A
8N-16C	$(t^2 - 1/2)/(t^2 - 2t - 1/2)$	4G-8E
8N-16D	$(-t^2 + 2t + 1)/(t^2 + 2t - 1)$	4G-8C
8N-16E	$(-8t^8 + 16t^6 - 12t^4 + 4t^2 - 1/2)/(t^8 + 14t^6 + 35t^4/2 + 7t^2/2 + 1/16)$	4C-4A
8N-16F	$(-8t^8 + 64t^7 - 160t^6 + 64t^5 + 208t^4 - 64t^3 - 160t^2 - 64t - 8)/(17t^8 + 56t^7 + 84t^6 + 56t^5 + 70t^4 - 56t^3 + 84t^2 - 56t + 17)$	4C-4A
8N-24A	$(-40t^8/3 - 32t^7/3 - 160t^6/3 - 80t^5/3 - 28t^4/3 + 40t^3/3 - 40t^2/3 + 4t/3 - 5/6)/(t^8 + 4t^7 + 4t^6 + 2t^5 + 7t^4/2 - t^3 + t^2 - t/2 + 1/16)$	2C-2A
8N-24B	$(-26t^8/3 - 32t^7/3 - 68t^6/3 + 16t^5/3 + t^4/3 - 8t^3/3 - 17t^2/3 + 4t/3 - 13/24)/(t^8 + 2t^7 + t^6 + t^5 + t^4/2 - t^3/2 + t^2/4 - t/4 + 1/16)$	2C-2A
8N-24C	$(t^2 + 2t - 1/2)/(t^2 - 4t - 1/2)$	4G-8D
8N-32A	$(-8t^8 + 64t^7 + 224t^6 - 448t^5 - 560t^4 + 448t^3 + 224t^2 - 64t - 8)/(t^8 + 8t^7 - 28t^6 - 56t^5 + 70t^4 + 56t^3 - 28t^2 - 8t + 1)$	4B-4A
8N-40A	$(-8t^4/11 - 16t^3 + 48t^2/11 + 16t - 8/11)/(t^4 - 8t^3/11 - 6t^2 + 8t/11 + 1)$	4E-4A
8N-80A	$(36t^4/13 - 16t^3 - 216t^2/13 + 16t + 36/13)/(t^4 + 36t^3/13 - 6t^2 - 36t/13 + 1)$	4E-4A
8N-120A	$(-5882t^8/135 - 14752t^7/135 - 35504t^6/135 - 30016t^5/135 - 6448t^4/27 - 41344t^3/135 - 31424t^2/135 - 11008t/135 - 1952/135)/(t^8 - 20t^7/3 + 28t^6/9 + 56t^5/3 + 40t^4 + 368t^3/9 + 80t^2/3 + 32t/3 + 16/9)$	2C-2A
8N-120B	$(-74t^8/3 + 16t^7/3 - 104t^6/3 + 272t^5/15 - 764t^4/75 - 272t^3/75 - 104t^2/75 - 16t/375 - 74/1875)/(t^8 - 4t^7 + 4t^6 - 4t^5/5 + 38t^4/25 + 4t^3/25 + 4t^2/25 + 4t/125 + 1/625)$	2C-2A
8N-120C	$(104t^8/5 - 128t^7/5 + 544t^6/25 + 896t^5/25 + 368t^4/125 - 896t^3/125 + 544t^2/625 + 128t/625 + 104/3125)/(t^8 + 16t^7/5 + 4t^6/25 - 112t^5/25 + 62t^4/125 + 112t^3/125 + 4t^2/625 - 16t/625 + 1/625)$	2C-2A

Table A.3: Table of  $\pi$  for groups with label 8G – 8C to 8N – 120C

Label	$\pi(t)$	Sup
8N-160A	$(584t^8/161 - 64t^7 - 2336t^6/23 + 448t^5 + 5840t^4/23 - 448t^3 - 2336t^2/23 + 64t + 584/161)/(t^8 + 584t^7/161 - 28t^6 - 584t^5/23 + 70t^4 + 584t^3/23 - 28t^2 - 584t/161 + 1)$	4B-4A
8N-160B	$[(t^2 + 5t/2)/(t+2), (t^2 + 25t/6)/(t+3/2), (t^2 - 92567t/20118)/(t+3353/438), (t^2 - 56164375t/8274246)/(t+210167/462714), (54842436196758752t^3/443722515625 + 212174785418442003223568t^2/58676449201234375 + 820863234555260888104550960312t/23277559080017925046875 + 3175761193846164977813377734244919108/27703351042874259734215828125)/(t^3 + 641238822368429196825t/1986351708161529604 + 348954094419137229280940234375/14184115733951620983918504)]$	1A-1A
8O-8A	$(-t^8 - 20t^6 + 26t^4 - 20t^2 - 1)/(t^7 + t^5 - t^3 - t)$	2C-2A
8P-8A	$(-3t^4/16 - t^3/2 - t^2/8 + 1/16)/(t^4 + t^3 + t^2 + t)$	8B-8C
8P-8B	$(t^3/2 - t^2/2 + t/4)/(t^4 - 2t^3 + t^2 - t + 1/4)$	8B-8C
9A-9A	$(-6t^3 - 9t^2/2 - 1/8)/t^3$	3A-3A
9B-9A	$(t^{12} - 72t^9 + 1728t^6 - 13824t^3)/(t^3 - 27)$	1A-1A
9C-9A	$(t^{12} + 36t^9 + 270t^6 + 756t^3 + 729)/t^3$	1A-1A
9D-9A	$(t^6 - 6t^3 - 27)/t^3$	3A-3A
9E-9A	$(t^6 - 18t^4 + 12t^3 + 81t^2 - 108t)/(t^3 - 9t + 9)$	3A-3A
9F-9A	$(15786448344t^{27} + 97104328713t^{26} + 211613665104t^{25} + 101393851464t^{24} - 346873745196t^{23} - 606838975362t^{22} - 185069381256t^{21} + 399055622058t^{20} + 390201775632t^{19} + 29491911513t^{18} - 69001275924t^{17} + 41883219504t^{16} + 42293754396t^{15} - 35457527007t^{14} - 46825227144t^{13} - 16282263114t^{12} - 5617685664t^{11} - 8106994674t^{10} - 5811182676t^9 - 1798317612t^8 - 475698744t^7 - 394283295t^6 - 214308504t^5 - 45428364t^4 - 4618944t^3 - 4723920t^2 - 2519424t - 419904)/(t^{27} + 54t^{26} + 1323t^{25} + 19431t^{24} + 190404t^{23} + 1310688t^{22} + 6497856t^{21} + 23385564t^{20} + 60714306t^{19} + 111061608t^{18} + 135371790t^{17} + 93896442t^{16} + 10254726t^{15} - 39544632t^{14} - 26334774t^{13} + 2870190t^{12} + 8392545t^{11} + 923346t^{10} - 1578201t^9 - 227637t^8 + 221238t^7 + 13572t^6 - 21978t^5 + 1890t^4 + 981t^3 - 270t^2 + 27t - 1)$	1A-1A
9G-9A	$[(t^9 + 4t^8 + 7t^7 + 7t^6 + 13t^5/4 - t^4/2 - 5t^3/4 - t^2/2 + t/4)/(t^8 + 55t^7/7 + 157t^6/7 + 1247t^5/56 + 73t^4/28 - 307t^3/56 - 17t^2/28 + 5t/8 - 1/14), (-884736000t^3 + 464486400t^2 - 81285120t + 4741632)/(t^3 + 42t^2 + 588t + 2744)]$	1A-1A
9H-9A	$(-3t^2 - 6t)/(t^2 + t + 1)$	9E-9A
9I-9A	$[(t^3 - 3t^2/4 + t/8)/(t^2 - t/2 + 1/18), t^3/(t^2 + 3t/4 + 3/16), (-1167051t^4/512 - 3597075t^3/512 - 19637073t^2/4096 - 10097379t/8192 - 14348907/131072)/(t^4 + t^3/4)]$	1A-1A
9J-9A	$[(t^2 + t)/(t^3 - 3t - 1), (t^3 + 10t^2/21 + 4t/63)/(t^2 + 5t/6 + 7/36), (1193859t^4/512 + 1381131t^3/224 + 4523445t^2/784 + 1486791t/686 + 1167051/4802)/(t^4 + 67t^3/42 + 65t^2/588 + 193t/74088 + 4/194481)]$	1A-1A
10A-10A	$-2t^2 - 2$	5A-5A
10B-10A	$-2t^2 - 3$	5B-5A
10C-10A	$(-2t^3 + t^2 + t - 3/2)/(t - 1/2)$	5B-5A
10D-10A	$(-t^2/2 - 1/2)/(t + 1/2)$	5C-5A
10E-10A	$(-10t^6 + 10t^5 - 7t^4/2 + t^3 - 19t^2/40 + t/8 - 9/800)/(t^4 - t^3 + 7t^2/20 - t/20 + 1/400)$	5A-5A
10F-10A	$(5t^2/2 + 5t/2)/(t^2 + 3t + 1)$	10C-10A
10G-10A	$(t^2 + t/2 + 1)/(t^2 + 3t + 1)$	10C-10A
12A-12A	$-t^3/(12t^3 + 1/2)$	4A-4A
12B-12A	$-1/(8t^2)$	6C-6A
12B-12B	$1/(8t^4 - 24)$	3B-3A
12B-12C	$-3t^2/2$	6C-6A
12B-12D	$1/(1152t^4 - 24)$	3B-3A
12B-24A	$1/(512t^4 - 24)$	3B-3A
12B-24B	$-1/(4t^2)$	6C-6A
12B-24C	$-3/(4t^2)$	6C-6A
12B-24D	$-t^4/(24t^4 - 24/108)$	3B-3A
12C-12A	$t^2$	6D-6A
12C-12B	$(-24t^6 - 1/1728)/t^6$	2B-2A
12C-12C	$12t^2$	6D-6A

Table A.4: Table of  $\pi$  for groups with label 8N – 160A to 12C – 12C

Label	$\pi(t)$	Sup
12C-12D	$(-24t^6 - 1/64)/t^6$	2B-2A
12C-24A	$(-24t^6 - 1/216)/t^6$	2B-2A
12C-24B	$6t^2$	6D-6A
12C-24C	$2t^2$	6D-6A
12C-24D	$-512t^6 - 24$	2B-2A
12D-12A	$-1728t^6 + 1728t^4 - 576t^2 + 40$	2B-2A
12E-12A	$1/(24t^2 + 8)$	6F-6A
12F-12A	$(-t^6/12 - t^5/2 - t^4/2 + 4t^3/3 + t^2 - 2t + 2/3)/(t^6 + 6t^5 + 6t^4 + 56t^3 - 12t^2 + 24t - 8)$	4A-4A
12G-12A	$(46656t^{12} - 7776t^{10} + 108t^8 + 4t^6 - t^4/4 - t^2/24 - 1/1728)/t^6$	2B-2A
12G-12B	$(-t^{12} - 6t^{10} - 3t^8 + 4t^6 + 9t^4 - 54t^2 + 27)/t^6$	2B-2A
12G-24A	$(10077696t^{12} - 279936t^{10} + 648t^8 + 4t^6 - t^4/24 - t^2/864 - 1/373248)/t^6$	2B-2A
12G-24B	$(5832t^{12} - 1944t^{10} + 54t^8 + 4t^6 - t^4/2 - t^2/6 - 1/216)/t^6$	2B-2A
12H-12A	$(t^2 + 1)/t$	12C-12A
12I-12A	$(-t^4/10 - 1/360)/(t^4 - 1/60)$	6F-6A
12I-12B	$(-t^2/6 - 1/18)/(t^2 - 1/9)$	6I-6A
12I-12C	$(-5t^4/34 - 2t^3/17 - 3t^2/17 - 2t/17 - 1/34)/(t^4 - 12t^3/17 - 18t^2/17 - 12t/17 - 3/17)$	6F-6A
12I-12D	$(-t^2 - 1)/(6t^2 - 2)$	6I-6A
12I-24A	$(-t^2/6 - 1/4)/(t^2 - 1/2)$	6I-6A
12I-24B	$(-t^4/10 - 1/90)/(t^4 - 1/15)$	6F-6A
12I-24C	$(-t^4 - 1)/(10t^4 - 6)$	6F-6A
12I-24D	$(-t^2/6 - 1/48)/(t^2 - 1/24)$	6I-6A
12J-12A	$(t^4/6 + 2t^3/3 + 4t^2/3 + 4t/3 + 2/3)/(t^4 + 20t^3/3 + 32t^2/3 + 8t + 4)$	6F-6A
13A-13A	$(-189t^{14}/2 - 837t^{13} - 5265t^{12}/2 - 3510t^{11} - 2340t^{10} - 2106t^9 - 1495t^8 - 650t^7 - 403t^6 - 182t^5 - 52t^4 - 26t^3 - 13t^2/2 - t - 1/2)/(t^{14} + t^{13}/2)$	1A-1A
13B-13A	$(-3t^2 - 5t - 1)/(t^2 - 7t - 4)$	13A-13A
13C-13A	$(5t^3/2 - 7t^2/2 - 4t + 5/2)/(t^3 + 31t^2/2 - 37t/2 + 1)$	13A-13A
14B-14A	$(7t^8 - 70t^6 - 63t^4 - 14t^2 - 1)/t^7$	2A-2A
14C-14A	$(t^3 + 2t^2 - t - 1)/(7t^2 + 7t)$	14B-14A
15B-15A	$(-3t^3 - 1/16)/t^3$	5B-5A
15C-15A	$(-131t^6/2 + 429t^5/2 - 555t^4/2 + 395t^3/2 - 90t^2 + 27t - 3)/(t^6 - 9t^5 + 30t^4 - 45t^3 + 30t^2 - 9t + 1)$	5B-5A
16B-16A	$t^2$	8B-8A
16B-16B	$-2t^2$	8B-8D
16B-16C	$2t^2$	8B-8A
16B-16D	$-1/t^2$	8B-8D
16C-16A	$-16t^8 + 64t^4 - 24$	2B-2A
16C-16B	$-t^2$	8C-8A
16C-16C	$-256t^8 + 256t^4 - 24$	2B-2A
16C-16D	$-2t^2$	8C-8A
16D-16A	$(-24t^8 - 128t^6 - 80t^4 - 16t^2 - 1)/t^8$	2B-2A

Table A.5: Table of  $\pi$  for groups with label 12C – 12D to 16D – 16A

Label	$\pi(t)$	Sup
16E-16A	$t^2 - 4$	8D-8A
16F-32A	$(-t^8/8 + t^6/4 - 3t^4/16 + t^2/16 - 1/128)/(t^8 + 12t^7 + 30t^6 + 42t^5 + 67t^4/2 + 21t^3 + 15t^2/2 + 3t/2 + 1/16)$	4A-4A
16F-32B	$(-t^8/552 - t^7/69 - 5t^6/138 - t^5/69 + 13t^4/276 + t^3/69 - 5t^2/138 + t/69 - 1/552)/(t^8 - 80t^7/23 + 316t^6/69 - 96t^5/23 + 166t^4/69 - 16t^3/23 - 20t^2/69 - 1/23)$	4A-4A
16G-16A	$(-t^2 + 1/2)/(t^2 - 2t - 1/2)$	8G-8F
16G-16B	$(t^2 + 1)/t$	8G-8A
16G-16C	$(-t^2 + 1)/(2t)$	8G-8K
16G-16D	$(t^2 - 1/2)/t$	8G-8L
16G-16E	$(t^2/2 + t - 1/2)/(t^2 + 1)$	8G-8B
16G-16F	$-t^2$	8G-8C
16G-16G	$t/(t^2 + 1/2)$	8G-8B
16G-16H	$-2t^2$	8G-8C
16G-16I	$t/(2t^2 + 1)$	8G-8H
16G-16J	$(-t^2/4 + t/2 + 1/4)/(t^2 + 1)$	8G-8H
16G-16K	$(-2t^2 - 1)/(t^2 - 1/2)$	8G-8A
16G-16L	$(-2t^2 + 4t + 2)/(t^2 + 2t - 1)$	8G-8G
16G-32A	$(-t^2 + 2t + 1)/(t^2 + 2t - 1)$	8G-16A
16G-48A	$[t/(t^2 - 1/2), (t^2 + 2t)/(t - 1/4), (t^2 - 16t/7)/(t - 14/5), (t^2 - 10t/7)/(t - 160/49), (1770025017602t^3/75 - 71162343172t^2/5 + 2861021192t - 575124560/3)/(t^3 - 101t^2/5 + 104t - 20)]$	1A-1A
16G-48B	$[t/(t^2 - 1/2), (t^2 - 2t)/(t + 1/4), (t^2 + 10t/7)/(t + 7/4), (t^2 + t)/(t + 16/7), (28756228t^3/3 + 1430510596t^2/7 + 71162343172t/49 + 3540050035204/1029)/(t^3 + 52t^2/7 + 101t/49 + 50/343)]$	1A-1A
16G-48C	$[(t^2 - t/2)/(t + 1), t/(t^2 + 1/4), (t^2 - t/3)/(t + 3/7), (t^2 - 49t/123)/(t + 287/267), (62161150998242t^3/1607445 - 8270096200046t^2/9415035 + 1100277103298t/165435615 - 1024687950818/61045741935)/(t^3 + 14507t^2/1435 + 213867t/8405 - 4934783/9304335)]$	1A-1A
16G-80A	$[t/(t^2 - 1), (t^2 + t)/(t - 1/4), t/(t^2 + 1), (t^2 - 3t/10)/(t - 5/14), (320013504t^3/30625 - 30878496t^2/21875 + 993168t/15625 - 74536/78125)/(t^3 - 107t^2/70 + 67t/100 - 63/1000)]$	1A-1A
16G-80B	$[t/(t^2 - 1), (t^2 + t/4)/(t - 1), t/(t^2 + 1/16), (t^2 + 6t/5)/(t - 10/13), (868327204t^3/5625 + 488343752t^2/3125 + 823927728t/15625 + 463373664/78125)/(t^3 - 442t^2/45 + 4732t/225 + 17576/1125)]$	1A-1A
16G-160A	$[t/(t^2 - 1), t/(t^2 - 1/4), (t^2 - 7t)/(t + 1/7), t/(t^2 + 49), (58436224t^3/31875 + 3957184t^2/10625 + 1875808t/74375 + 6224272/10933125)/(t^3 + 151t^2/714 + 149t/9996 + 1/2856)]$	1A-1A
16G-160B	$[t/(t^2 - 1), t/(t^2 - 1/4), (t^2 + 73t/161)/(t - 161/73), t/(t^2 + 5329/25921), (123136219490280064t^3/13957763671875 + 498751146993545152t^2/339638916015625 + 2020142470323384736t/24793640869140625 + 8182388401518971248/5429807350341796875)/(t^3 - 19267031t^2/8346966 + 1758195509t/1218657036 - 4173281/33387864)]$	1A-1A
16G-240A	$[(t^2 - t/5)/(t + 1), (t^2 + 11t/5)/(t + 1/55), t/(t^2 + 121/25), (t^2 + 1205t/22)/(t - 5/5258), (23855046548417282t^3/1785 - 263508037137917869t^2/119 + 320184385492107744655t/2618 - 4279559971266900784673975/1900668)/(t^3 - 429005t^2/2618 + 515517025t/57596 - 1706489875/10472)]$	1A-1A
16G-240B	$[(t^2 + t)/(t - 1/5), (t^2 - 7t/5)/(t - 5/7), t/(t^2 + 49/25), (t^2 - 73t/70)/(t + 125/322), (-27995042t^3/1171875 - 25970512583t^2/13671875 - 48184783850209/957031250t - 89400368486033207/200976562500)/(t^3 - 1081t^2/70 - 51313t/4900 - 12167/7000)]$	1A-1A

Table A.6: Table of  $\pi$  for groups with label 16E – 16A to 16G – 240B

Label	$\pi(t)$	Sup
16G-240C	$[(t^2 + 2t)/(t + 1/3), (t^2 - 3t)/(t - 4/3), (t^2 + t/3)/(t + 27), t/(t^2 + 1/9), (257924804t^3/32805 - 76862878t^2/3645 + 22905521t/1215 - 13651919/2430)/(t^3 + 3t^2/2 - 9t/4 - 27/8)]$	1A-1A
16H-16A	$(t^2 - 1/2)/t$	16C-16A
18B-18A	$t^2$	9C-9A
18B-18B	$(-5159780352t^{24} + 107495424t^{18} - 466560t^{12} + 756t^6 - 27/64)/t^6$	1A-1A
18B-36A	$1/(12t^2)$	9C-9A
18B-36B	$(-262144t^{24} + 147456t^{18} - 17280t^{12} + 756t^6 - 729/64)/t^6$	1A-1A
18B-72A	$(-157464t^{24} + 756t^{18} - 5t^{12}/4 + t^6/1296 - 1/10077696)/t^{18}$	1A-1A
18B-72B	$1/(2t^2)$	9C-9A
18B-72C	$6t^2$	9C-9A
18B-72D	$(-373248t^{24} + 756t^{18} - 135t^{12}/256 + 9t^6/65536 - 1/13421728)/t^{18}$	1A-1A
18C-18A	$(-128t^{21} - 116t^{18}/27 - 13t^{15}/216 - 85t^{12}/186624 - t^9/497664 - 5t^6/967458816 - t^3/139314069504 - 1/240734712102912)/(t^{24} + t^{21}/108 + t^{18}/46656)$	1A-1A
18D-18A	$t^2$	9D-9A
18D-36A	$1/(12t^2)$	9D-9A
18D-72A	$1/(2t^2)$	9D-9A
18D-72B	$6t^2$	9D-9A
18E-18A	$(t^3 - 2)/t$	9B-9A
20A-20A	$2t^2 + 1$	10C-10A
24A-24A	$(t^{12} + 16)/t^4$	3A-3A
24A-24B	$(-12230590464t^{36} + 768t^{24} - 1/62208t^{12} + 1/8916100448256)/t^{24}$	1A-1A
24A-24C	$(-191102976t^{36} + 768t^{24} - t^{12}/972 + 1/2176782336)/t^{24}$	1A-1A
24A-24D	$(1296t^{12} + 4/9)/t^4$	3A-3A
24A-24E	$(64t^{12} + 1/16)/t^8$	3A-3A
24A-24F	$(2304t^{12} + 1/20736)/t^8$	3A-3A
24A-24G	$(-16777216t^{36} + 768t^{24} - 3t^{12}/256 + 1/16777216)/t^{24}$	1A-1A
24A-24H	$(4096t^{36} - 3072t^{24} + 768t^{12} - 64)/t^{12}$	1A-1A
24B-24A	$(t^{48} - 36t^{44} + 1242t^{40} - 24084t^{36} + 407511t^{32} - 4671432t^{28} + 42908940t^{24} - 265011912t^{20} + 977319999t^{16} - 2085374484t^{12} + 2496709818t^8 - 1549681956t^4 + 387420489)/(t^{40} - 28t^{36} + 270t^{32} - 972t^{28} + 729t^{24})$	1A-1A
24B-24B	$(8916100448256t^{48} + 2229025112064t^{44} + 224450445312t^{40} + 11717001216t^{36} + 343201536t^{32} + 5816448t^{28} + 58860t^{24} + 801t^{20}/2 + 559t^{16}/256 + 223t^{12}/27648 + 23t^8/884736 + t^4/21233664 + 1/12230590464)/(t^{24} + t^{20}/12 + 5t^{16}/3456 + 7t^{12}/746496 + t^8/47775744)$	1A-1A
24B-24C	$(16t^{48} + 144t^{44} + 1242t^{40} + 6021t^{36} + 407511t^{32}/16 + 583929t^{28}/8 + 10727235t^{24}/64 + 33126489t^{20}/128 + 977319999t^{16}/4096 + 521343621t^{12}/4096 + 1248354909t^8/32768 + 387420489t^4/65536 + 387420489/1048576)/(t^{40} + 7t^{36} + 135t^{32}/8 + 243t^{28}/16 + 729t^{24}/256)$	1A-1A
24B-24D	$(1296t^{48} - 1296t^{44} + 1242t^{40} - 669t^{36} + 5031t^{32}/16 - 801t^{28}/8 + 1635t^{24}/64 - 561t^{20}/128 + 1839t^{16}/4096 - 109t^{12}/4096 + 29t^8/32768 - t^4/65536 + 1/9437184)/(t^{40} - 7t^{36}/9 + 5t^{32}/24 - t^{28}/48 + t^{24}/2304)$	1A-1A

Table A.7: Table of  $\pi$  for groups with label 16G – 240C to 24B – 24D

Label	$\pi(t)$	Sup
24B-24E	$(16t^{48} - 144t^{44} + 1242t^{40} - 602t^{36} + 407511t^{32}/16 - 583929t^{28}/8 + 10727235t^{24}/64 - 33126489t^{20}/128 + 977319999t^{16}/4096 - 521343621t^{12}/4096 + 1248354909t^8/32768 - 387420489t^4/65536 + 387420489/1048576)/(t^{40} - 7t^{36} + 135t^{32}/8 - 243t^{28}/16 + 729t^{24}/256)$	1A-1A
24B-24F	$(4738381338321616896t^{48} - 131621703842267136t^{44} + 1472619371692032t^{40} - 8541693886464t^{36} + 27799324416t^{32} - 52348032t^{28} + 58860t^{24} - 89t^{20}/2 + 559t^{16}/20736 - 223t^{12}/20155392 + 23t^8/5804752896 - t^4/1253826625536 + 1/6499837226778624)/(t^{24} - t^{20}/108 + 5t^{16}/279936 - 7t^{12}/544195584 + t^8/313456656384)$	1A-1A
24B-24G	$(4738381338321616896t^{48} + 131621703842267136t^{44} + 1472619371692032t^{40} + 8541693886464t^{36} + 27799324416t^{32} + 52348032t^{28} + 58860t^{24} + 89t^{20}/2 + 559t^{16}/20736 + 223t^{12}/20155392 + 23t^8/5804752896 + t^4/1253826625536 + 1/6499837226778624)/(t^{24} + t^{20}/108 + 5t^{16}/279936 + 7t^{12}/544195584 + t^8/313456656384)$	1A-1A
24B-24H	$(1156831381426176t^{48} + 128536820158464t^{44} + 5752419420672t^{40} + 133463966976t^{36} + 1737457776t^{32} + 13087008t^{28} + 58860t^{24} + 178t^{20} + 559t^{16}/1296 + 223t^{12}/314928 + 23t^8/22674816 + t^4/1224440064 + 1/1586874322944)/(t^{24} + t^{20}/27 + 5t^{16}/17496 + 7t^{12}/8503056 + t^8/1224440064)$	1A-1A
25A-25A	$16t^5 + 20t^3 + 5t + 5/2$	5B-5A
25B-25A	$(-t^2/2 - t + 1/8)/(t^2 - t/2 - 1/4)$	25A-25A
25B-25B	$(5t^{10}/2 + 10t^9 - 25t^8/2 + 120t^7 + 25t^6 + 252t^5 - 25t^4 + 120t^3 + 25t^2/2 + 10t - 5/2)/(t^{10} - 5t^8 + 10t^6 - 10t^4 + 5t^2 - 1)$	5B-5A
26A-26A	$-t^2/(2t^2 + 1)$	13A-13A
27A-27A	$(-216t^{36} + 756t^{33} - 3771t^{30}/4 + 7783t^{27}/16 - 47277t^{24}/512 + 4761t^{21}/512 - 4741t^{18}/8192 + 783t^{15}/32768 - 1413t^{12}/2097152 + 163t^9/12582912 - 11t^6/67108864 + t^3/805306368 - 1/231928233984)/(t^{33} - t^{30}/24 + t^{27}/1728)$	1A-1A
28A-28A	$1/t^2$	14B-14A
28A-28B	$(-16384t^{32} + 114688t^{28} - 329728t^{24} + 487424t^{20} - 378560t^{16} + 137984t^{12} - 16072t^8 + 748t^4 - 49/4)/t^4$	1A-1A
28A-28C	$7t^2$	14B-14A
28A-28D	$(-1111200682558016t^{32} + 1587429546508288t^{28} - 93139998902272t^{24} + 2809902362624t^{20} - 44537205440t^{16} + 331299584t^{12} - 787528t^8 + 748t^4 - 1/4)/t^4$	1A-1A
28A-56A	$(-49t^{32} + 748t^{28} - 4018t^{24} + 8624t^{20} - 5915t^{16} + 1904t^{12} - 322t^8 + 28t^4 - 1)/t^{28}$	1A-1A
28A-56B	$14t^2$	14B-14A
28A-56C	$1/(2t^2)$	14B-14A
28A-56D	$(-2401t^{32} + 748t^{28} - 82t^{24} + 176t^{20}/49 - 845t^{16}/16807 + 272t^{12}/823543 - 46t^8/40353607 + 4t^4/1977326743 - 1/678223072849)/t^{28}$	1A-1A
30A-30A	$(-3t^6 + 1/3456)/t^6$	5B-5A
30A-30B	$(-3t^6 - 1/746496000)/t^6$	5B-5A
30A-30C	$125t^6/54 - 3$	5B-5A
30A-30D	$1/(2t^3)$	10B-10A
30A-60A	$(-3t^6 - 1/27648000)/t^6$	5B-5A
30A-60B	$(-3t^6 - 1/221184)/t^6$	5B-5A
30A-60C	$(-3t^6 + 1/2)/t^6$	5B-5A
30A-60D	$125t^6/2 - 3$	5B-5A
30A-120A	$(-3t^6 - 1/3456000)/t^6$	5B-5A
30A-120B	$(-3t^6 - 1/27648)/t^6$	5B-5A
30A-120C	$125t^6/16 - 3$	5B-5A

Table A.8: Table of  $\pi$  for groups with label 24B – 24E to 30A – 120C

Label	$\pi(t)$	Sup
30A-120D	$(-3t^6 + 1/16)/t^6$	5B-5A
30A-120E	$(-3t^6 + 1/27648)/t^6$	5B-5A
30A-120F	$(-3t^6 + 1/3456000)/t^6$	5B-5A
30A-120G	$(-3t^6 - 1/746496)/t^6$	5B-5A
30A-120H	$(-3t^6 - 1/93312000)/t^6$	5B-5A
32A-32A	$-1/t^2$	16C-16B
32A-32B	$-t^{16} + 16t^8 - 24$	2B-2A
32A-32C	$-1/(2t^2)$	16C-16B
32A-32D	$-256t^{16} + 256t^8 - 24$	2B-2A
36A-36A	$t^2$	18B-18A
36A-36B	$(46656t^{24} - t^{12}/96 - 1/5159780352)/t^{18}$	2A-2A
36A-36C	$-4t^4$	9C-9A
36A-36D	$(-101559956668416t^{48} + 78364164096t^{36} - 12597120t^{24} + 756t^{12} - 1/64)/t^{12}$	1A-1A
36A-72A	$(-10077696t^{24} - 3888t^{12} + 1/8)/t^6$	2A-2A
36A-72B	$1/(2t^2)$	18B-18A
36A-72C	$-1/(16t^4)$	9C-9A
36A-72D	$(-2176782336t^{48} + 756t^{36} - 5t^{24}/55296 + t^{12}/247669456896 - 1/26623333280885243904)/t^{36}$	1A-1A
48A-48A	$t^2$	24A-24A
48A-48B	$6t^2$	24A-24G
48A-48C	$3t^2$	24A-24G
48A-48D	$12t^2$	24A-24A
48A-48E	$2t^2$	24A-24A
48A-48F	$6t^2$	24A-24A
48A-48G	$4t^2$	24A-24G
48A-48H	$1/(2t^2)$	24A-24G

Table A.9: Table of  $\pi$  for groups with label 30A – 120D to 48A – 48H

Label	Index	Case	C	$\alpha$
2A-2A	2	Case 1	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	
2B-2A	3	Case 0	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	
2C-2A	6	Case 1	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	
2C-2A	6	Case 3	$\begin{pmatrix} 1 & 0 \\ -1/24 & 1/12 \end{pmatrix}$	
3A-3A	3	Case 0	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	
3B-3A	4	Case 0	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	
3C-3A	6	Case 2	$\begin{pmatrix} 1 & 0 \\ 12 & 6 \end{pmatrix}$	$-1/3$
3D-3A	12	Case 2	$\begin{pmatrix} 1 & 0 \\ -1/6 & 1/2 \end{pmatrix}$	3
4A-4A	4	Case 0	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	
4B-4A	6	Case 1	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	
4C-4A	6	Case 1	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	
4D-4A	8	Case 2	$\begin{pmatrix} 1 & 0 \\ 8 & 2 \end{pmatrix}$	$-1/2$
4E-4A	12	Case 5	$\begin{pmatrix} 0 & 1 \\ -4 & -2 \end{pmatrix}$	1
4F-4A	12	Case 1	$\begin{pmatrix} -1/2 & 1/2 \\ 1/12 & -1/4 \end{pmatrix}$	
4G-4A	24	Case 5	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	4
4G-4A	24	Case 4	$\begin{pmatrix} 1 & 0 \\ 0 & 1/2 \end{pmatrix}$	
4G-4A	24	Case 6	$\begin{pmatrix} 1 & 0 \\ 0 & 1/2 \end{pmatrix}$	
4G-4B	24	Case 5	$\begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$	$-1/2$
4G-8A	24	Case 5	$\begin{pmatrix} 0 & 1 \\ -2 & -2 \end{pmatrix}$	1
4G-8B	24	Case 5	$\begin{pmatrix} 1 & -1 \\ 0 & 2 \end{pmatrix}$	2
4G-8C	24	Case 5	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	1
4G-8C	24	Case 4	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	
4G-8C	24	Case 6	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	
4G-8D	24	Case 5	$\begin{pmatrix} 1 & -1 \\ 0 & 2 \end{pmatrix}$	2
4G-8E	24	Case 5	$\begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix}$	2
4G-8F	24	Case 6	$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$	
4G-16A	24	Case 4	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	
5A-5A	5	Case 0	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	
5B-5A	6	Case 0	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	
5C-5A	10	Case 0	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	
5D-5A	12	Case 2	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	$-1$
5E-5A	15	Case 0	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	
5F-15A	20	Case 2	$\begin{pmatrix} 1 & 0 \\ 1 & 5 \end{pmatrix}$	$-5$
5G-5A	30	Case 2	$\begin{pmatrix} 1 & 0 \\ -2 & 10 \end{pmatrix}$	5
5H-5A	60	Case 2	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	$-1$
5H-15A	60	Case 2	$\begin{pmatrix} 1 & 0 \\ 1 & 5 \end{pmatrix}$	$-5$
6A-6A	6	Case 0	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	
6B-6A	6	Case 1	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	
6C-6A	8	Case 1	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	
6D-6A	9	Case 0	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	
6E-6A	12	Case 1	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	
6E-12A	12	Case 1	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	
6E-24A	12	Case 1	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	
6E-24B	12	Case 1	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	

Table A.10: Table of families for groups with label 2A – 2A to 6E – 24B

Label	Index	Case	C	$\alpha$
6F-6A	12	Case 0	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	
6G-6A	18	Case 2	$\begin{pmatrix} 1 & 0 \\ 4 & 6 \end{pmatrix}$	-3
6H-6A	18	Case 1	$\begin{pmatrix} -1/2 & 1/2 \\ 0 & -1 \end{pmatrix}$	
6I-6A	24	Case 1	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	
6I-6A	24	Case 3	$\begin{pmatrix} 1 & 0 \\ -2 & 4 \end{pmatrix}$	
6J-6A	24	Case 0	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	
6K-6A	36	Case 2	$\begin{pmatrix} 1 & 0 \\ 1/2 & 3/2 \end{pmatrix}$	3
6K-24A	36	Case 2	$\begin{pmatrix} 1 & 0 \\ -2 & -3 \end{pmatrix}$	3
6L-6A	36	Case 1	$\begin{pmatrix} 1 & 0 \\ -1/2 & 1/2 \end{pmatrix}$	
6L-12A	36	Case 1	$\begin{pmatrix} 1 & 0 \\ -1/2 & 1/2 \end{pmatrix}$	
6L-24A	36	Case 1	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	
6L-24B	36	Case 1	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	
7B-7A	8	Case 0	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	
7D-7A	21	Case 0	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	
7E-7A	24	Case 3	$\begin{pmatrix} 1 & 0 \\ 2/7 & 1/7 \end{pmatrix}$	
7F-7A	28	Case 0	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	
8B-8A	12	Case 1	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	
8B-8B	12	Case 1	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	
8B-8C	12	Case 1	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	
8B-8D	12	Case 1	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	
8C-8A	12	Case 1	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	
8D-8A	12	Case 1	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	
8E-16A	16	Case 2	$\begin{pmatrix} 1 & 0 \\ -1 & 2 \end{pmatrix}$	2
8E-16B	16	Case 2	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	1/2
8F-8A	16	Case 0	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	
8G-8A	24	Case 5	$\begin{pmatrix} 0 & 1 \\ -2 & -2 \end{pmatrix}$	1
8G-8B	24	Case 5	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	-1/2
8G-8C	24	Case 5	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	1
8G-8D	24	Case 6	$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$	
8G-8E	24	Case 5	$\begin{pmatrix} 1 & -1 \\ 0 & 2 \end{pmatrix}$	2
8G-8F	24	Case 5	$\begin{pmatrix} 1 & -1 \\ 0 & 2 \end{pmatrix}$	2
8G-8G	24	Case 5	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	4
8G-8G	24	Case 4	$\begin{pmatrix} 1 & 0 \\ 0 & 1/2 \end{pmatrix}$	
8G-8G	24	Case 6	$\begin{pmatrix} 1 & 0 \\ 0 & 1/2 \end{pmatrix}$	
8G-8H	24	Case 5	$\begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$	-1/2
8G-8I	24	Case 5	$\begin{pmatrix} 1 & -1 \\ 0 & 2 \end{pmatrix}$	2
8G-8J	24	Case 6	$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$	
8G-8K	24	Case 5	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	1
8G-8K	24	Case 4	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	
8G-8K	24	Case 6	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	
8G-8L	24	Case 5	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	2
8G-16A	24	Case 4	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	
8H-8A	24	Case 5	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	2
8I-8A	24	Case 1	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	
8J-8A	24	Case 1	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	

Table A.11: Table of families for groups with label 6F – 6A to 8J – 8A

Label	Index	Case	C	$\alpha$
8K-16A	24	Case 1	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	
8L-8A	24	Case 2	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	-1
8N-8A	48	Case 5	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	1
8N-8A	48	Case 4	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	
8N-8A	48	Case 6	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	
8N-8B	48	Case 5	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	-1/2
8N-8C	48	Case 5	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	1
8N-8D	48	Case 6	$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$	
8N-8E	48	Case 5	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	-1/2
8N-8F	48	Case 5	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	-1/2
8N-16A	48	Case 5	$\begin{pmatrix} 0 & 1 \\ -1 & -2 \end{pmatrix}$	1
8N-16B	48	Case 5	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	-1/2
8N-16C	48	Case 5	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	-1/2
8N-16D	48	Case 5	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	1
8N-16D	48	Case 4	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	
8N-16D	48	Case 6	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	
8N-16E	48	Case 5	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	-1/2
8N-16F	48	Case 6	$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$	
8N-24A	48	Case 8	$\begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$	
8N-24B	48	Case 8	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	
8N-24C	48	Case 8	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	
8N-32A	48	Case 4	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	
8N-40A	48	Case 4	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	
8N-80A	48	Case 4	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	
8N-120A	48	Case 7	$\begin{pmatrix} 1 & -1 \\ 0 & 5/2 \end{pmatrix}$	
8N-120B	48	Case 7	$\begin{pmatrix} 1 & 0 \\ 0 & 5 \end{pmatrix}$	
8N-120C	48	Case 7	$\begin{pmatrix} 1 & -1 \\ 0 & 5 \end{pmatrix}$	
8N-160A	48	Case 4	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	
8N-160B	48	Case 4	$\begin{pmatrix} 1 & 0 \\ -2/5 & 1/5 \end{pmatrix}$	
8O-8A	48	Case 5	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	1
8O-8A	48	Case 4	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	
8O-8A	48	Case 6	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	
8P-8A	48	Case 2	$\begin{pmatrix} 1 & 0 \\ -1 & 2 \end{pmatrix}$	2
8P-8B	48	Case 2	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	1/2
9A-9A	9	Case 0	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	
9B-9A	12	Case 0	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	
9C-9A	12	Case 0	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	
9D-9A	18	Case 2	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	-3
9E-9A	18	Case 0	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	
9F-9A	27	Case 0	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	
9G-9A	27	Case 0	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	
9H-9A	36	Case 2	$\begin{pmatrix} 1 & 0 \\ -2 & -3 \end{pmatrix}$	3
9H-9A	36	Case 3	$\begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}$	
9I-9A	36	Case 3	$\begin{pmatrix} 1 & 0 \\ 2 & 2 \end{pmatrix}$	
9J-9A	36	Case 3	$\begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}$	

Table A.12: Table of families for groups with label 8K – 16A to 9J – 9A

Label	Index	Case	C	$\alpha$
10A-10A	10	Case 1	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	
10B-10A	12	Case 1	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	
10C-10A	18	Case 0	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	
10D-10A	20	Case 2	$\begin{pmatrix} 1 & 0 \\ 1/2 & 5/2 \end{pmatrix}$	5
10E-10A	30	Case 0	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	
10F-10A	36	Case 2	$\begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}$	-1
10G-10A	36	Case 1	$\begin{pmatrix} -1/2 & 1/2 \\ 1/2 & 1/2 \end{pmatrix}$	
12A-12A	12	Case 0	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	
12B-12A	16	Case 1	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	
12B-12B	16	Case 1	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	
12B-12C	16	Case 1	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	
12B-12D	16	Case 1	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	
12B-24A	16	Case 1	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	
12B-24B	16	Case 1	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	
12B-24C	16	Case 1	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	
12B-24D	16	Case 1	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	
12C-12A	18	Case 1	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	
12C-12B	18	Case 1	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	
12C-12C	18	Case 1	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	
12C-12D	18	Case 1	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	
12C-24A	18	Case 1	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	
12C-24B	18	Case 1	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	
12C-24C	18	Case 1	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	
12C-24D	18	Case 1	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	
12D-12A	18	Case 1	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	
12E-12A	24	Case 1	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	
12F-12A	24	Case 2	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	-2
12G-12A	36	Case 1	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	
12G-12B	36	Case 1	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	
12G-24A	36	Case 1	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	
12G-24B	36	Case 1	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	
12H-12A	36	Case 5	$\begin{pmatrix} 0 & 1 \\ -1 & -2 \end{pmatrix}$	1
12I-12A	48	Case 1	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	
12I-12B	48	Case 1	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	
12I-12C	48	Case 1	$\begin{pmatrix} 1 & 0 \\ -1 & 1/2 \end{pmatrix}$	
12I-12D	48	Case 1	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	
12I-24A	48	Case 1	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	
12I-24B	48	Case 1	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	
12I-24C	48	Case 1	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	
12I-24D	48	Case 1	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	
12J-12A	48	Case 1	$\begin{pmatrix} 1 & 0 \\ -1/2 & 1/2 \end{pmatrix}$	
13A-13A	14	Case 0	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	
13B-13A	28	Case 2	$\begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}$	-1
13C-13A	42	Case 3	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	

Table A.13: Table of families for groups with label 10A – 10A to 13C – 13A

Label	Index	Case	C	$\alpha$
14B-14A	16	Case 1	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	
14C-14A	48	Case 3	$\begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}$	
15B-15A	18	Case 0	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	
15C-15A	36	Case 2	$\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$	-1
16B-16A	24	Case 1	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	
16B-16B	24	Case 1	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	
16B-16C	24	Case 1	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	
16B-16D	24	Case 1	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	
16C-16A	24	Case 1	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	
16C-16B	24	Case 1	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	
16C-16C	24	Case 1	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	
16C-16D	24	Case 1	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	
16D-16A	24	Case 1	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	
16E-16A	24	Case 1	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	
16F-32A	32	Case 2	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	1/2
16F-32B	32	Case 2	$\begin{pmatrix} 1 & 0 \\ 1 & 2 \end{pmatrix}$	2
16G-16A	48	Case 5	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	-1/2
16G-16B	48	Case 5	$\begin{pmatrix} 0 & 1 \\ -1 & -2 \end{pmatrix}$	1
16G-16C	48	Case 5	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	1
16G-16C	48	Case 4	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	
16G-16C	48	Case 6	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	
16G-16D	48	Case 5	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	-1/2
16G-16E	48	Case 6	$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$	
16G-16F	48	Case 5	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	1
16G-16G	48	Case 5	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	-1/2
16G-16H	48	Case 5	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	-1/2
16G-16I	48	Case 5	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	-1/2
16G-16J	48	Case 6	$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$	
16G-16K	48	Case 5	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	-1/2
16G-16L	48	Case 5	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	1
16G-16L	48	Case 4	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	
16G-16L	48	Case 6	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	
16G-32A	48	Case 4	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	
16G-48A	48	Case 8	$\begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$	
16G-48B	48	Case 8	$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$	
16G-48C	48	Case 8	$\begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$	
16G-80A	48	Case 4	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	
16G-80B	48	Case 4	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	
16G-160A	48	Case 4	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	
16G-160B	48	Case 4	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	
16G-240A	48	Case 7	$\begin{pmatrix} 1 & -1 \\ 0 & 5 \end{pmatrix}$	
16G-240B	48	Case 7	$\begin{pmatrix} 1 & -1 \\ 0 & 5 \end{pmatrix}$	
16G-240C	48	Case 7	$\begin{pmatrix} 1 & -1/2 \\ 0 & 5/2 \end{pmatrix}$	
16H-16A	48	Case 5	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	-1/2

Table A.14: Table of families for groups with label 14B – 14A to 16H – 16A

Label	Index	Case	C	$\alpha$
18B-18A	24	Case 1	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	
18B-18B	24	Case 1	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	
18B-36A	24	Case 1	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	
18B-36B	24	Case 1	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	
18B-72A	24	Case 1	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	
18B-72B	24	Case 1	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	
18B-72C	24	Case 1	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	
18B-72D	24	Case 1	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	
18C-18A	24	Case 0	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	
18D-18A	36	Case 1	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	
18D-36A	36	Case 1	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	
18D-72A	36	Case 1	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	
18D-72B	36	Case 1	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	
18E-18A	36	Case 0	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	
20A-20A	36	Case 1	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	
24A-24A	36	Case 1	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	
24A-24B	36	Case 1	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	
24A-24C	36	Case 1	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	
24A-24D	36	Case 1	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	
24A-24E	36	Case 1	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	
24A-24F	36	Case 1	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	
24A-24G	36	Case 1	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	
24A-24H	36	Case 1	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	
24B-24A	48	Case 1	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	
24B-24B	48	Case 1	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	
24B-24C	48	Case 1	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	
24B-24D	48	Case 1	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	
24B-24E	48	Case 1	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	
24B-24F	48	Case 1	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	
24B-24G	48	Case 1	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	
24B-24H	48	Case 1	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	
25A-25A	30	Case 0	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	
25B-25A	60	Case 2	$\begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$	-1
25B-25B	60	Case 2	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	-1
26A-26A	28	Case 1	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	
27A-27A	36	Case 0	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	
28A-28A	32	Case 1	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	
28A-28B	32	Case 1	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	
28A-28C	32	Case 1	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	
28A-28D	32	Case 1	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	
28A-56A	32	Case 1	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	
28A-56B	32	Case 1	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	
28A-56C	32	Case 1	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	

Table A.15: Table of families for groups with label 18B – 18A to 28A – 56C

Label	Index	Case	C	$\alpha$
28A-56D	32	Case 1	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	
30A-30A	36	Case 1	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	
30A-30B	36	Case 1	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	
30A-30C	36	Case 1	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	
30A-30D	36	Case 1	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	
30A-60A	36	Case 1	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	
30A-60B	36	Case 1	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	
30A-60C	36	Case 1	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	
30A-60D	36	Case 1	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	
30A-120A	36	Case 1	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	
30A-120B	36	Case 1	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	
30A-120C	36	Case 1	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	
30A-120D	36	Case 1	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	
30A-120E	36	Case 1	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	
30A-120F	36	Case 1	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	
30A-120G	36	Case 1	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	
30A-120H	36	Case 1	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	
32A-32A	48	Case 1	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	
32A-32B	48	Case 1	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	
32A-32C	48	Case 1	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	
32A-32D	48	Case 1	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	
36A-36A	48	Case 1	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	
36A-36B	48	Case 1	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	
36A-36C	48	Case 1	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	
36A-36D	48	Case 1	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	
36A-72A	48	Case 1	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	
36A-72B	48	Case 1	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	
36A-72C	48	Case 1	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	
36A-72D	48	Case 1	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	
48A-48A	72	Case 1	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	
48A-48B	72	Case 1	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	
48A-48C	72	Case 1	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	
48A-48D	72	Case 1	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	
48A-48E	72	Case 1	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	
48A-48F	72	Case 1	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	
48A-48G	72	Case 1	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	
48A-48H	72	Case 1	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	

Table A.16: Table of families for groups with label 28A – 56D to 48A – 48H

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