

# The Valuation of Convertible Bonds With Credit Risk

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## Abstract

Convertible bonds are typically issued by firms which have both relatively high growth and quite high risk. Convertibles can be difficult to value, given their hybrid nature of containing elements of both debt and equity. Further complications arise due to the frequent presence of additional options such as callability and puttability, and contractual complexities such as trigger prices and “soft call” provisions, in which the ability of the issuing firm to exercise its option to call is dependent upon the history of its stock price.

This paper explores the valuation of convertible bonds subject to credit risk using an approach based on the numerical solution of a system of coupled linear complementarity problems. We argue that many of the existing models, such as that of Tsiveriotis and Fernandes (1998), are unsatisfactory in that they do not explicitly specify what happens in the event of a default by the issuing firm. In fact, many of the differences between existing models appear to arise from varying implicit assumptions about this. In existing models it is assumed that upon a default either nothing happens to the firm’s stock price or else it instantly jumps to zero. Neither of these alternatives seems to be entirely appealing: while it is a significant event, implying that there will be some market reaction to the news of a default, a sudden and complete collapse is rare. Consequently, we propose a model where the firm’s stock price falls by some specified percentage between 0% and 100% (which includes the limiting cases implicit in existing models).

We also present a detailed description of our numerical algorithm, which uses a partially implicit method to decouple the system of linear complementarity problems at each timestep. Numerical examples illustrating the convergence properties of the algorithm are provided.

**Keywords:** Convertible bonds, credit risk, linear complementarity

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# 1 Introduction

The market for convertible bonds has been expanding rapidly. In the U.S., over \$105 billion of new convertibles were issued in 2001, as compared with just over \$60 billion in 2000. As of this writing, there are about \$270 billion of convertibles outstanding, more than double the level of five years ago. The global market for convertibles currently exceeds \$500 billion.<sup>1</sup> Moreover, in the past couple of decades there has been considerable innovation in the contractual features of convertibles. Examples include liquid yield option notes (McConnell and Schwartz, 1986), mandatory convertibles (Arzac, 1997), “death spiral” convertibles (Hillion and Vermaelen, 2001), and cross-currency convertibles (Yigitbasioglu, 2001). It is now common for convertibles to feature exotic and complicated features, such as trigger prices and “soft call” provisions. These preclude the issuer from exercising its call option unless the firm’s stock price is either above some specified level, has remained above a level for a specified period of time (e.g. 30 days), or has been above a level for some specified fraction of time (e.g. 20 out of the last 30 days).

The modern academic literature on the valuation of convertibles began with the papers of Ingersoll (1977) and Brennan and Schwartz (1977, 1980). These authors build on the “structural” approach for valuing risky non-convertible debt (e.g. Merton, 1974; Black and Cox, 1976; Longstaff and Schwartz, 1995). In this approach, the basic underlying state variable is the value of the issuing firm. The firm’s debt and equity are claims contingent on the firm’s value, and options on its debt and equity are compound options on this variable. In general terms, default occurs when the firm’s value becomes sufficiently low that it is unable to meet its financial obligations.<sup>2</sup> An overview of this type of model is provided in Nyborg (1996). While in principle this is an attractive framework, it is subject to the same criticisms that have been applied to the valuation of risky debt (Jarrow and Turnbull, 1995). In particular, because the value of the firm is not a traded asset, parameter estimation is difficult. Also, any other liabilities which are more senior than the convertible must be simultaneously valued.

To circumvent these problems, some authors have proposed models of convertible bonds where the basic underlying factor is the issuing firm’s stock price (augmented in some cases with additional random variables such as an interest rate). As this is a traded asset, parameter estimation is simplified (compared to the structural approach). Moreover, there is no need to estimate the values of all other more senior claims. An early example of this approach is McConnell and Schwartz (1986). The basic problem here is that the model ignores the possibility of bankruptcy. McConnell and Schwartz address this in an *ad hoc* manner by simply using a risky discount rate rather than the risk free rate in their valuation equation. More recent papers which similarly include a risky discount rate in a somewhat arbitrary fashion are those of Cheung and Nelken (1994) and Ho and Pfeffer (1996).

An additional complication which arises in the case of a convertible bond (as opposed to risky debt) is that different components of the instrument are subject to different default risks. This is noted by Tsiveriotis and Fernandes (1998), who argue that “the equity upside has zero default risk since the issuer can always deliver its own stock [whereas] coupon and principal payments and any put provisions . . . depend on the issuer’s timely access to the required cash amounts, and thus introduce credit risk” (p. 95). To handle this, Tsiveriotis and Fernandes propose splitting convertible bonds into two components: a “cash-only” part, which is subject to credit risk, and an equity part, which is not. This leads to a pair of coupled partial differential equations that can be solved to value convertibles. A simple description of this model in the binomial context may be found in Hull (2000). Yigitbasioglu (2001) extends this framework by adding an interest rate factor and, in the case of cross-currency convertibles, a foreign exchange risk factor.

Recently, another alternative has emerged. This is known as the “reduced-form” approach. This builds on developments in the literature on the pricing of risky debt (see, e.g. Jarrow and Turnbull, 1995; Duffie

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<sup>1</sup>See A. Schultz, “In These Convertibles, a Smoother Route to Stocks”, *The New York Times*, April 7, 2002.

<sup>2</sup>There are some variations across these models in terms of the precise specification of default. For example, Merton (1974) considers zero-coupon debt and assumes that default occurs if the value of the firm is lower than the face value of the debt at its maturity. On the other hand, Longstaff and Schwartz (1995) assume that default occurs when the firm value first reaches a specified default level, much like a barrier option.

and Singleton, 1999; Madan and Unal, 2000). In contrast to the structural approach, in this setting default is exogenous, the “consequence of a single jump loss event that drives the equity value to zero and requires cash outlays that cannot be externally financed” (Madan and Unal, 2000, p. 44). The probability of default over the next short time interval is determined by a specified hazard rate. When default occurs, some portion of the bond (either its market value immediately prior to default, or its par value, or the market value of a default-free bond with the same terms) is assumed to be recovered. Authors who have used this approach in the convertible bond context include Davis and Lischka (1999) and Takahashi et al. (2001). As in models such as that of Tsiveriotis and Fernandes (1998), the basic underlying state variable is the firm’s stock price (though Davis and Lischka also incorporate an interest rate factor and a random hazard rate).

While this approach is quite appealing, the assumption that the stock price instantly jumps to zero in the event of a default is highly questionable. While it may be a reasonable approximation in some circumstances, it is clearly not in others. For instance, Clark and Weinstein (1983) report that shares in firms filing for bankruptcy in the U.S. had average cumulative abnormal returns of -65% during the three years prior to a bankruptcy announcement, and had abnormal returns of about -30% around the announcement. Beneish and Press (1995) find average cumulative abnormal returns of -62% for the three hundred trading days prior to a Chapter 11 filing, and a drop of 30% upon the filing announcement. The corresponding figures for a debt service default are -39% leading up to the announcement and -10% at the announcement. This clearly indicates that the assumption of an instantaneous jump to zero is extreme. In most cases, default is better characterized as involving a gradual erosion of the stock price prior to the event, followed by a significant (but much less than 100%) decline upon the announcement (even in the most severe case of a bankruptcy filing).

However, as we shall see below, in some models it is at least implicitly assumed that a default has no impact on the firm’s stock price. This may also be viewed as unsatisfactory. To address this, we propose a model where the firm’s stock price drops by a specified percentage (between 0% and 100%) upon a default. This effectively extends the reduced-form approach which, in the case of risky debt, specifies a fractional loss in market value for a bond, to the case of convertibles by similarly specifying a fractional decline in the issuing firm’s stock price.

The outline of the article is as follows. Section 2 outlines the convertible bond valuation problem in the absence of credit risk. Section 3 reviews credit risk in the case of a simple coupon bearing bond. Section 4 presents our model for convertible bonds. Section 5 then describes some aspects of previous models, with particular emphasis on why we believe the Tsiveriotis and Fernandes (1998) model has some undesirable features. Motivated by this, Section 6 provides a new decomposition of the convertible bond into bond and equity components. Section 7 describes our numerical methods. In some cases a system of coupled linear complementarity problems must be solved. We discuss various numerical approaches for timestepping so that the problems become decoupled. Section 8 provides some numerical illustrations, and Section 9 presents conclusions.

Since our main interest in this article is the modelling of default risk, we will restrict attention to models where the interest rate is assumed to be a known function of time, and the stock price is stochastic. We can easily extend the models in this paper to handle the case where either or both of the risk free rate and the hazard rate are stochastic. However, this would detract us from our main goal of determining how to incorporate the hazard rate into the basic convertible pricing model. We also note that practitioners often regard a convertible bond primarily as an equity instrument, where the main risk factor is the stock price, and the random nature of the risk free rate is of second order importance.<sup>3</sup> For ease of exposition, we also ignore various contractual complications such as call notice periods, soft call provisions, trigger prices, dilution, etc.

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<sup>3</sup>Note that this is consistent with the results of Brennan and Schwartz (1980), who conclude that “for a reasonable range of interest rates the errors from the [non-stochastic] interest rate model are likely to be slight” (p. 926).

## 2 Convertible Bonds: No Credit Risk

We begin by reviewing the valuation of convertible bonds under the assumption that there is no default risk. We assume that interest rates are known functions of time, and that the stock price is stochastic. We assume that

$$dS = \mu S dt + \sigma S dz, \quad (2.1)$$

where  $S$  is the stock price,  $\mu$  is its drift rate,  $\sigma$  is its volatility, and  $dz$  is the increment of a Wiener process. Following the usual arguments, the no-arbitrage value  $V(S, t)$  of any claim contingent on  $S$  is given by

$$V_t + \left( \frac{\sigma^2}{2} S^2 V_{SS} + (r(t) - q) S V_S - r(t) V \right) = 0, \quad (2.2)$$

where  $r(t)$  is the known interest rate and  $q$  is the dividend rate.

We assume that a convertible bond has the following contractual features:

- A continuous (time-dependent) put provision (with an exercise price of  $B_p$ ).
- A continuous (time-dependent) conversion provision. At any time, the bond can be converted to  $\kappa$  shares.
- A continuous (time-dependent) call provision. At any time, the issuer can call the bond for price  $B_c > B_p$ . However, the holder can convert the bond if it is called.

Note that option features which are only exercisable at certain times (rather than continuously) can easily be handled by simply enforcing the relevant constraints at those times.

Let

$$\mathcal{L}V \equiv -V_t - \left( \frac{\sigma^2}{2} S^2 V_{SS} + (r(t) - q) S V_S - r(t) V \right). \quad (2.3)$$

We will consider the points in the solution domain where  $\kappa S \geq B_c$  and  $\kappa S < B_c$  separately.

- $B_c > \kappa S$ . In this case, we can write the convertible bond pricing problem as a linear complementarity problem

$$\left( \begin{array}{l} \mathcal{L}V = 0 \\ (V - \max(B_p, \kappa S)) \geq 0 \\ (V - B_c) \leq 0 \end{array} \right) \vee \left( \begin{array}{l} \mathcal{L}V \geq 0 \\ (V - \max(B_p, \kappa S)) = 0 \\ (V - B_c) < 0 \end{array} \right) \vee \left( \begin{array}{l} \mathcal{L}V \leq 0 \\ (V - \max(B_p, \kappa S)) \geq 0 \\ (V - B_c) = 0 \end{array} \right) \quad (2.4)$$

where the notation  $(x = 0) \vee (y = 0) \vee (z = 0)$  is to be interpreted as at least one of  $x = 0$ ,  $y = 0$ ,  $z = 0$  holds at each point in the solution domain.

- $B_c \leq \kappa S$ . In this case, the convertible value is simply

$$V = \kappa S \quad (2.5)$$

since the holder would choose to convert immediately.

Equation (2.4) is a precise mathematical formulation of the following intuition. The value of the convertible bond is given by the solution to  $\mathcal{L}V = 0$ , subject to the constraints

$$\begin{aligned} V &\geq \max(B_p, \kappa S) \\ V &\leq \max(B_c, \kappa S). \end{aligned} \quad (2.6)$$

As far as boundary conditions are concerned, we merely alter the operator  $\mathcal{L}V$  at  $S = 0, \infty$ . At  $S = 0$ ,  $\mathcal{L}V$  becomes

$$\mathcal{L}V \equiv -(V_t - r(t)V); \quad S \rightarrow 0, \quad (2.7)$$

while as  $S \rightarrow \infty$  we assume that the unconstrained solution is linear in  $S$

$$\mathcal{L}V \equiv V_{SS}; \quad S \rightarrow \infty. \quad (2.8)$$

The terminal condition is given by

$$V(S, t = T) = \max(F, \kappa S) \quad (2.9)$$

where  $F$  is the face value of the bond.

Equation (2.4) has been derived by many authors (though not using the formal and precise linear complementarity formulation). However, in practice, corporate bonds are not risk free. To highlight the modelling issues, we will consider a simplified model of risky corporate debt in the next section.

### 3 A Risky Bond

To motivate our discussion of credit risk, consider the valuation of a simple coupon bearing bond which has been issued by a corporation having a non-zero default risk. The ideas are quite similar to some of those presented in Duffie and Singleton (1999). However, we rely only on simple hedging arguments, and we assume that the risk free rate is a known deterministic function. Let the probability of default in the time period  $t$  to  $t + dt$  be  $p(S, t)dt$ , where  $p(S, t)$  is a deterministic hazard rate.

Let  $B(S, t)$  denote the price of a risky corporate bond. Construct the standard hedging portfolio

$$\Pi = B - \alpha S. \quad (3.1)$$

In the absence of default, if we choose  $\alpha = B_S$ , the usual arguments give

$$d\Pi = \left[ B_t + \frac{\sigma^2 S^2}{2} B_{SS} \right] dt + o(dt), \quad (3.2)$$

where  $o(dt)$  denotes terms that go to zero faster than  $dt$ . Assume that:

- The probability of default in  $t \rightarrow t + dt$  is  $p dt$ .
- The value of the bond immediately after default is  $RB$  where  $0 \leq R \leq 1$  is the recovery factor (and  $B$  is its value immediately prior to default).
- The stock price  $S$  is unchanged on default.

Then equation (3.2) becomes

$$\begin{aligned} d\Pi &= (1 - p dt) \left[ B_t + \frac{\sigma^2 S^2}{2} B_{SS} \right] dt - p dt(B(1 - R)) + o(dt) \\ &= \left[ B_t + \frac{\sigma^2 S^2}{2} B_{SS} \right] dt - p dt(B(1 - R)) + o(dt). \end{aligned} \quad (3.3)$$

We now make the assumption that default risk is diversifiable, so that

$$E(d\Pi) = r(t)\Pi dt \quad (3.4)$$

where  $E$  is the expectation operator. Taking the expectation of equation (3.3) and equating it to equation (3.4) gives

$$B_t + r(t)SB_S + \frac{\sigma^2 S^2}{2}B_{SS} - (r(t) + (1 - R)p)B = 0. \quad (3.5)$$

Note that if  $p = p(t)$ , then the solution to equation (3.5) for a zero coupon bond with face value  $F$  payable at  $t = T$  is

$$B = F \exp \left[ - \int_t^T (r(u) + p(u)(1 - R)) du \right] \quad (3.6)$$

which corresponds to the intuitive idea of a spread  $s = p(1 - R)$ .<sup>4</sup>

We can alter the above assumptions about the stock price in the event of default. If we assume that the stock price  $S$  jumps to zero in the case of default, then equation (3.3) becomes

$$\begin{aligned} d\Pi &= (1 - p dt) \left[ B_t + \frac{\sigma^2 S^2}{2}B_{SS} \right] dt - p dt(B(1 - R) - \alpha S) + o(dt) \\ &= \left[ B_t + \frac{\sigma^2 S^2}{2}B_{SS} \right] dt - p dt(B(1 - R) - \alpha S) + o(dt). \end{aligned} \quad (3.7)$$

Following the same steps as above with  $\alpha = B_S$ , we obtain

$$B_t + (r(t) + p)SB_S + \frac{\sigma^2 S^2}{2}B_{SS} - (r(t) + p)B + pRB = 0. \quad (3.8)$$

Note that in this case  $p$  appears in the drift term as well as in the discounting term. Even in this relatively simple case of a risky corporate bond, different assumptions about the behavior of the stock price in the event of default will change our valuation. While this is perhaps an obvious point, it is worth remembering that in some popular existing models for convertible bonds no explicit assumptions are made regarding what happens to the stock price upon default.

## 4 Convertible Bonds With Credit Risk

We now consider adding credit risk to the convertible bond model described in Section 2, using the approach discussed in Section 3 for incorporating credit risk. We follow the same general line of reasoning described in Ayache (2001). Let the value of the convertible bond be denoted by  $V(S, t)$ . To avoid complications at this stage, we assume that there are no put or call features and that conversion is only allowed at the terminal time or in the event of default. Let  $S^+$  be the stock price immediately after default, and  $S^-$  be the stock price right before default. We will assume that

$$S^+ = S^-(1 - \eta), \quad (4.1)$$

where  $0 \leq \eta \leq 1$ . We will refer to the case where  $\eta = 1$  as the “total default” case (the stock price jumps to zero), and we will call the case where  $\eta = 0$  the “partial default” case (the issuing firm defaults but the stock price does not jump anywhere).

As usual, we construct the hedging portfolio

$$\Pi = V - \alpha S. \quad (4.2)$$

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<sup>4</sup>This is analogous to the results of Duffie and Singleton (1999) in the stochastic interest rate context.

If there was no credit risk, i.e.  $p = 0$ , then choosing  $\alpha = V_S$  and applying standard arguments gives

$$d\Pi = \left[ V_t + \frac{\sigma^2 S^2}{2} V_{SS} \right] dt + o(dt). \quad (4.3)$$

Now, consider the case where the hazard rate  $p$  is nonzero. We make the following assumptions:

- Upon default, the stock price jumps according to equation (4.1).
- Upon default, the convertible bond holders have the option of receiving
  - (a) the amount  $RB$ , where  $0 \leq R \leq 1$  is the recovery factor, and  $B$  is the bond component of the convertible security (we will discuss models for  $B$  below in subsequent sections); or
  - (b) shares worth  $\kappa S(1 - \eta)$ .

Under these assumptions, the change in value of the hedging portfolio during  $t \rightarrow t + dt$  is

$$\begin{aligned} d\Pi &= (1 - p dt) \left[ V_t + \frac{\sigma^2 S^2}{2} V_{SS} \right] dt - p dt(V - \alpha S \eta) + p \max(\kappa S(1 - \eta), RB) + o(dt) \\ &= \left[ V_t + \frac{\sigma^2 S^2}{2} V_{SS} \right] dt - p dt(V - V_S S \eta) + p \max(\kappa S(1 - \eta), RB) + o(dt). \end{aligned} \quad (4.4)$$

Assuming the expected return on the portfolio is given by equation (3.4) and equating this with the expectation of equation (4.4), we obtain

$$r[V - SV_S] dt = \left[ V_t + \frac{\sigma^2 S^2}{2} V_{SS} \right] dt - p[V - V_S S \eta] dt + p[\max(\kappa S(1 - \eta), RB)] dt + o(dt). \quad (4.5)$$

This implies

$$V_t + (r(t) + p\eta)SV_S + \frac{\sigma^2 S^2}{2} V_{SS} - (r(t) + p)V + p \max(\kappa S(1 - \eta), RB) = 0. \quad (4.6)$$

Note that  $r(t) + p\eta$  appears in the drift term and  $r(t) + p$  appears in the discounting term in equation (4.6). In the case that  $R = 0$ ,  $\eta = 1$ , which is the total default model with no recovery, the final result is especially simple: we simply solve the full convertible bond problem (2.4), with  $r(t)$  replaced by  $r(t) + p$ . There is no need to solve an additional equation for the bond component. This has been noted in Takahashi et al. (2001).

Defining

$$\mathcal{M}V \equiv -V_t - \left( \frac{\sigma^2}{2} S^2 V_{SS} + (r(t) + p\eta - q)SV_S - (r(t) + p)V \right), \quad (4.7)$$

we can write equation (4.6) for the case where the stock pays a proportional dividend  $q$  as

$$\mathcal{M}V - p \max(\kappa S(1 - \eta), RB) = 0. \quad (4.8)$$

We are now in a position to consider the complete problem for convertible bonds with risky debt. We can generalize problem (2.4), using equation (4.8).

- $B_c > \kappa S$

$$\begin{aligned}
& \left( \begin{array}{l} \mathcal{M}V - p \max(\kappa S(1 - \eta), RB) = 0 \\ (V - \max(B_p, \kappa S)) \geq 0 \\ (V - B_c) \leq 0 \end{array} \right) \\
\vee & \left( \begin{array}{l} \mathcal{M}V - p \max(\kappa S(1 - \eta), RB) \geq 0 \\ (V - \max(B_p, \kappa S)) = 0 \\ (V - B_c) < 0 \end{array} \right) \\
\vee & \left( \begin{array}{l} \mathcal{M}V - p \max(\kappa S(1 - \eta), RB) \leq 0 \\ (V - \max(B_p, \kappa S)) \geq 0 \\ (V - B_c) = 0 \end{array} \right) \tag{4.9}
\end{aligned}$$

- $B_c \leq \kappa S$

$$V = \kappa S. \tag{4.10}$$

Although equations (4.9)-(4.10) appear formidable, the basic concept is easy to understand. The value of the convertible bond is given by

$$\mathcal{M}V - p \max(\kappa S(1 - \eta), RB) = 0, \tag{4.11}$$

subject to the constraints

$$\begin{aligned}
V & \geq \max(B_p, \kappa S) \\
V & \leq \max(B_c, \kappa S). \tag{4.12}
\end{aligned}$$

## 5 Previous Work

There have been various attempts to specify the bond component of a convertible bond. This is required in order to model the credit risk of the bond component.

An early effort is described in a research note published in 1994 by Goldman Sachs. In this article, the probability of conversion is estimated, and the discount rate is a weighted average of the risk free rate and the risk free rate plus spread, where the weighting factor is the probability of conversion.

More recently, the model described in Tsiveriotis and Fernandes (1998) has become popular. In the following, we will refer to it as the TF model. This model is outlined in the latest edition of Hull's standard text, and has been adopted by several software vendors. We will discuss this model in some detail.

### 5.1 TF Model

Although not stated in Tsiveriotis and Fernandes (1998), we deduce that the model described therein is a partial default model (stock price does not jump upon default) since the equity part of the convertible is discounted at the risk free rate.<sup>5</sup>

In Tsiveriotis and Fernandes (1998), the following variational inequality is proposed for the total convertible bond value  $V$

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<sup>5</sup>Actually, the drift rate in Tsiveriotis and Fernandes (1998) is "the growth rate of the stock", which is not precisely defined, so this is not entirely clear. Following the description of this model in Hull (2000), we will assume here that the "growth rate of the stock" is the risk free rate.



- $B_c > \kappa S$

$$\left( \begin{array}{l} \mathcal{L}V + p(1-R)B = 0 \\ (V - \max(B_p, \kappa S)) \geq 0 \\ (V - B_c) \leq 0 \end{array} \right) \vee \left( \begin{array}{l} \mathcal{L}V + p(1-R)B \geq 0 \\ (V - \max(B_p, \kappa S)) = 0 \\ (V - B_c) < 0 \end{array} \right) \vee \left( \begin{array}{l} \mathcal{L}V + p(1-R)B \leq 0 \\ (V - \max(B_p, \kappa S)) \geq 0 \\ (V - B_c) = 0 \end{array} \right) \quad (5.1)$$

- $B_c \leq \kappa S$

$$V = \kappa S. \quad (5.2)$$

It is convenient to describe the decomposition of the total convertible price as  $V = B + C$ , where  $B$  is the bond component, and  $C$  is the equity component. In general, we can express the solution for  $\{V, B, C\}$  in terms of a coupled set of equations. Assuming that equations (5.1)-(5.2) are also being solved for  $V$ , then we can specify  $\{B, C\}$ . In the TF model, the following decomposition is suggested

- $B_p > \kappa S$

$$\begin{array}{lll} \mathcal{L}C = 0; & \mathcal{L}B + p(1-R)B = 0 & \text{if } V \neq B_p; \quad V \neq B_c \\ & B = B_p; \quad C = 0 & \text{if } V = B_p \\ & B = 0; \quad C = B_c & \text{if } V = B_c \end{array} \quad (5.3)$$

- $B_p \leq \kappa S$

$$\begin{array}{lll} \mathcal{L}C = 0; & \mathcal{L}B + p(1-R)B = 0 & \text{if } V \neq \max(\kappa S, B_c) \\ & C = \max(\kappa S, B_c); \quad B = 0 & \text{if } V = \max(\kappa S, B_c). \end{array} \quad (5.4)$$

It is easy to verify that the sum of equations (5.3)-(5.4) gives equations (5.1)-(5.2), noting that  $V = B + C$ . The terminal conditions for the TF decomposition are

$$\begin{array}{l} C(S, t = T) = H(\kappa S - F)(\max(\kappa S - F, 0) + F) \\ B(S, t = T) = H(F - \kappa S)F, \end{array} \quad (5.5)$$

where

$$H(x) = \begin{cases} 1 & \text{if } x \geq 0 \\ 0 & \text{if } x < 0. \end{cases} \quad (5.6)$$

However, the splitting in equations (5.3)-(5.4) does not seem to be based on theoretical arguments which require specifying precisely what happens in the case of default. Tsiveriotis and Fernandes (1998) provide no discussion of the actual events in the case of default, and how this would affect the hedging portfolio. There is no clear statement in their paper as to what happens to the stock price in the event of default.

Figure 1 illustrates the decomposition of the convertible bond using equation (5.5). Note that the convertible bond payoff is split into two discontinuous components, a digital bond and an asset-or-nothing call. The splitting occurs at the conversion boundary. This can be expected to cause some difficulties for a numerical scheme, as we have to solve for a problem with a discontinuity which moves over time (as the conversion boundary moves).

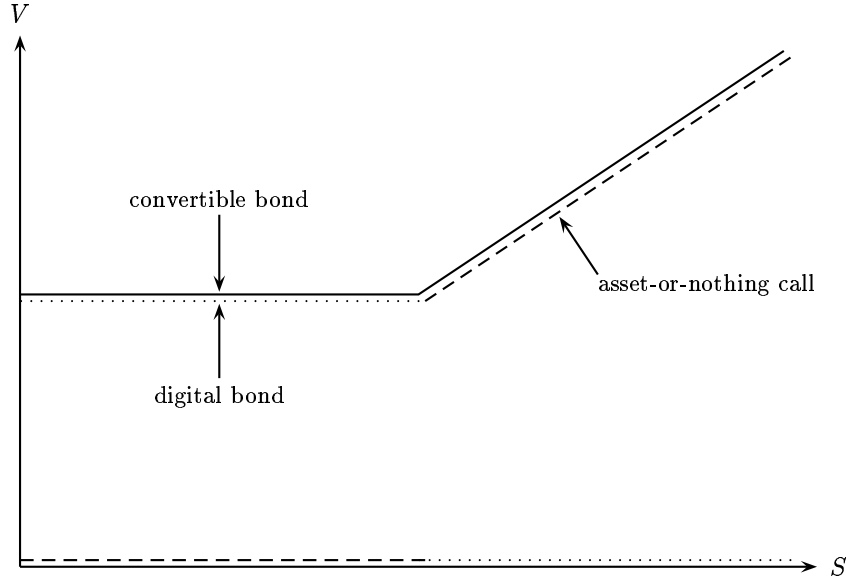


FIGURE 1: *Illustration of the TF method for decomposing a convertible bond into a digital bond plus an asset-or-nothing call.*

## 5.2 TF Splitting: Call Just Before Expiry

Consider a case where there are no put provisions, there are no coupons,  $\kappa = 1$ , conversion is allowed only at the terminal time (or at the call time), and the bond can only be called the instant before maturity, at  $t = T^-$ . The call price  $B_c = F - \epsilon$ ,  $\epsilon > 0$ ,  $\epsilon \ll 1$ .

Suppose that the bond is called at  $t = T^-$ . From equations (5.4) and (5.5), we conclude that we end up effectively solving the original problem with the altered payoff at  $t = T^-$

$$\begin{aligned}
 \mathcal{L}V + pB &= 0 \\
 V(S, T^-) &= \max(S, F - \epsilon) \\
 \mathcal{L}B + pB &= 0 \\
 B(S, T^-) &= 0.
 \end{aligned} \tag{5.7}$$

Note that the condition on  $B$  at  $t = T^-$  is due to the boundary condition (5.4). Now, since the solution of equation (5.7) for  $B$  (with  $B = 0$  initially) is  $B \equiv 0$  for all  $t < T^-$ , the equation for the convertible bond is simply

$$\begin{aligned}
 \mathcal{L}V &= 0 \\
 V(S, T^-) &= \max(S, F - \epsilon).
 \end{aligned} \tag{5.8}$$

In other words, there is no effect of the hazard rate in this case. This peculiar situation comes about because the TF model requires that the bond value be zero if  $V = B_c$ , even if the effect of the call on the total convertible bond value at the instant of the call is infinitesimally small. In the philosophy of the TF model, this boundary condition is required for consistency, since normally  $V = B_c$  only near where conversion takes place, and  $B = 0$  at the conversion boundary.

### 5.3 TF Splitting: General Case

Looking at the most general case (equations (5.4) and (5.5)), we can see there are some problems with the boundary conditions when  $B = B_p$  and when  $V = \kappa S$ . Consider a case where at some time  $t \ll T$ , the bond can be put for  $B_p$ , where  $B_p > V$  for  $S < S^*$ . In this case, from equations (5.4) and (5.5), we have that  $B = V, C = 0$  for  $S < S^*$ . In the event of default (with  $R = 0$ ), this implies that the total value of the convertible is zero for  $S < S^*$  (since by our assumption, none of the bond value is recovered). However, since the convertible is a contingent claim which can be converted into  $\kappa S$  shares at any time, it should be worth at least  $\kappa S$  (assuming that  $S$  does not jump to zero on default). These difficulties appear to arise from the fact that the precise events which occur on default are not specified in this model.

## 6 A New General Decomposition of the Convertible Bond

Assume that the total convertible bond value is given by equations (4.9)-(4.10). We will now devise a splitting of the convertible bond into two components, such that  $V = B + C$ , where  $B$  is the bond component and  $C$  is the equity component. The bond component, in the case where there are no put/call provisions, should satisfy an equation similar to equation (3.8).

In this case, we propose the following decomposition, which we will refer to as general splitting below:

$$\begin{aligned} & \left( \begin{array}{l} \mathcal{M}C - p \max(\kappa S(1 - \eta) - RB, 0) = 0 \\ (C - (\max(B_c, \kappa S) - B)) \leq 0 \\ (C - (\kappa S - B)) \geq 0 \end{array} \right) \\ & \vee \left( \begin{array}{l} \mathcal{M}C - p \max(\kappa S(1 - \eta) - RB, 0) \leq 0 \\ C = \max(B_c, \kappa S) - B \end{array} \right) \\ & \vee \left( \begin{array}{l} \mathcal{M}C - p \max(\kappa S(1 - \eta) - RB, 0) \geq 0 \\ C = \kappa S - B \end{array} \right), \end{aligned} \quad (6.1)$$

$$\left( \begin{array}{l} \mathcal{M}B - RpB = 0 \\ B - B_c \leq 0 \\ B - (B_p - C) \geq 0 \end{array} \right) \vee \left( \begin{array}{l} \mathcal{M}B - RpB \leq 0 \\ B = B_c \end{array} \right) \vee \left( \begin{array}{l} \mathcal{M}B - RpB \geq 0 \\ B = B_p - C \end{array} \right). \quad (6.2)$$

Adding together equations (6.1)-(6.2), and recalling that  $V = B + C$ , it is easy to see that equations (4.9)-(4.10) are satisfied. We informally rewrite equations (6.1) as

$$\mathcal{M}C - p \max(\kappa S(1 - \eta) - RB, 0) = 0, \quad (6.3)$$

subject to the constraints

$$\begin{aligned} B + C &\leq \max(B_c, \kappa S) \\ B + C &\geq \kappa S. \end{aligned} \quad (6.4)$$

We can also rewrite equations (6.2) as

$$\mathcal{M}B - RpB = 0, \quad (6.5)$$

subject to the constraints

$$\begin{aligned} B &\leq B_c \\ B + C &\geq B_p. \end{aligned} \quad (6.6)$$

Note that the constraints (6.4)-(6.6) make no assumptions about the behaviour of the individual  $B$  and  $C$  components, other than the fact that  $(B + C) = V$ , and that  $V$  has constraints, with the one exception that we require that  $B \leq B_c$ . This contrasts with the TF splitting, where  $B = 0$  when  $V = \kappa S$ , and  $C = 0$  when  $V = B_p$ , which leads to inconsistencies as described in Sections 5.2-5.3.

We can write the payoff of the convertible as

$$V(S, T) = F + \max(\kappa S - F, 0), \quad (6.7)$$

which suggests terminal conditions of

$$\begin{aligned} C(S, T) &= \max(\kappa S - F, 0) \\ B(S, T) &= F \end{aligned} \quad (6.8)$$

Note that equations (6.1)-(6.2) do not force  $B \rightarrow 0$  when conversion is optimal. Consider the case of a zero coupon bond where  $p = \text{const.}$ ,  $B < B_c$ ,  $B_p = 0$ . In this case, the solution for  $B$  is

$$B = F \exp \left[ - \int_t^T (r(u) + p(u)(1 - R)) du \right], \quad (6.9)$$

independent of  $S$ . It would seem clear that all holders of the convertible bond would be entitled to  $RB$  in the event of default (provided they have not converted), independent of  $S$ . Hence  $B$  should not be forced to zero at the conversion boundary, since this would penalize holders of the bond who have not converted near the exercise boundary. Of course, those who have converted would recover nothing of the bond component upon default, but non-zero  $B$  will not affect the value of the convertible once conversion has taken place (since it is forced to be  $\kappa S$ ).

## 6.1 Some Special Cases

If we assume that  $\eta = 0$  (i.e. the partial default case where the stock price does not jump if a default occurs), the recovery rate  $R = 0$ , and the bond is continuously convertible, then equations (6.1)-(6.2) become (in the continuation region)

$$\mathcal{M}V + p(V - \kappa S) = 0. \quad (6.10)$$

This has a simple intuitive interpretation. The convertible is discounted at the risk free rate plus spread when  $V \gg \kappa S$  and at the risk free rate when  $V \simeq \kappa S$ , with smooth interpolation between these values. Equation (6.10) was suggested in Ayache (2001). Note that in this case, we need only solve a single linear complementarity problem for the total convertible value  $V$ .

Making the assumptions that  $\eta = 1$  (i.e. the total default case where the stock price jumps to zero upon default) and that the recovery rate  $R = 0$ , equations (6.1)-(6.2) reduce (in the continuation region) to

$$\mathcal{M}V = 0, \quad (6.11)$$

which agrees with Takahashi et al. (2001). In this case, there is no need to split the convertible bond into equity and bond components. If the recovery rate is non-zero, our model is slightly different from that in Takahashi et al.. There it is assumed that upon default the holder recovers  $RV$ , compared to our assumption that the holder recovers  $RB$ . Consequently, for nonzero  $R$ , our approach requires the solution of the coupled set of linear complementarity problems (6.1)-(6.2), while the assumption in Takahashi et al. requires only the solution of a single linear complementarity problem. Since the total convertible bond value  $V$  includes a fixed income component and an option component, it seems more reasonable to us that in the event of total default (the assumption made in Takahashi et al. (2001)), the option component is by definition worthless

and only a fraction of the bond component can be recovered. The total default case also appears to be similar to the model suggested in Davis and Lischka (1999).

Note that if we assume that the stock price of a firm jumps to zero on default, then when pricing options on the firm's equity (e.g. vanilla calls/puts) we should replace the risk free rate  $r$  by  $r + p$  in both the drift term and the discounting term.

## 7 Numerical Method

Define  $\tau = T - t$ , so that the operator  $\mathcal{L}V$  becomes

$$\mathcal{L}V = V_\tau - \left( \frac{\sigma^2}{2} S^2 V_{SS} + (r(t) - q) S V_S - r(t) V \right), \quad (7.1)$$

and

$$\mathcal{M}V = V_\tau - \left( \frac{\sigma^2}{2} S^2 V_{SS} + (r(t) + p\eta - q) S V_S - (r(t) + p) V \right). \quad (7.2)$$

It is also convenient to define

$$\mathcal{H}V \equiv \left( \frac{\sigma^2}{2} S^2 V_{SS} + (r(t) - q) S V_S \right), \quad (7.3)$$

and

$$\mathcal{P}V \equiv \left( \frac{\sigma^2}{2} S^2 V_{SS} + (r(t) + p\eta - q) S V_S \right), \quad (7.4)$$

so that equation (7.1) can be written

$$\mathcal{L}V = V_\tau - (\mathcal{H}V - r(t)V), \quad (7.5)$$

and equation (7.2) becomes

$$\mathcal{M}V = V_\tau - (\mathcal{P}V - (r(t) + p)V). \quad (7.6)$$

The terms  $\mathcal{H}V$  and  $\mathcal{P}V$  are discretized using standard methods (see Zvan et al., 2001; Forsyth and Vetzal, 2001, 2002). Let  $V_i^n \equiv V(S_i, \tau^n)$ , and denote the discrete form of  $\mathcal{H}V$  at  $(S_i, \tau^n)$  by  $(\mathcal{H}V)_i^n$ , and the discrete form of  $\mathcal{M}V$  by  $(\mathcal{M}V)_i^n$ . In the following, for ease of exposition, we will describe the timestepping method for a fully implicit discretization of equation (7.1). In actual practice, we use Crank-Nicolson timestepping with the modification suggested in Rannacher (1984) to handle non-smooth initial conditions (which generally occur at each coupon payment). The reader should have no difficulty generalizing the equations to the Crank-Nicolson or BDF (Becker, 1998) case. We also suppress the dependence of  $r$  on time for notational convenience.

### 7.1 TF Model: Numerical Method

In this section, we describe a method which can be used to solve equations (5.3)-(5.4). We denote the total value of the convertible bond computed using explicit constraints by  $V^E$ . A corrected total convertible value, obtained by applying estimates for the constraints in implicit fashion, is denoted by  $V^I$ . Given initial values of  $(V^E)_i^n$ ,  $(V^I)_i^n$  and  $B_i^n$ , the timestepping proceeds as follows. First, the value of  $B_i^{n+1}$  is estimated, ignoring any constraints. We denote this estimate by  $(B^*)_i^{n+1}$ :

$$\frac{(B^*)_i^{n+1} - B_i^n}{\Delta\tau} = (\mathcal{H}B^*)_i^{n+1} - (r + p)_i^{n+1} (B^*)_i^{n+1}. \quad (7.7)$$

This value of  $(B^*)_i^{n+1}$  is then used to compute  $(V^E)_i^{n+1}$  from

$$\frac{(V^E)_i^{n+1} - (V^E)_i^n}{\Delta\tau} = (\mathcal{H}V^E)_i^{n+1} - (rV^E)_i^{n+1} - (pB^*)_i^{n+1}. \quad (7.8)$$

Then, we check the minimum value constraints:

```

For  $i = 1, \dots$ ,
   $B_i^{n+1} = (B^*)_i^{n+1}$ 
  If  $(B_p > \kappa S)$  then
    If  $((V^E)_i^{n+1} < B_p)$  then
       $(B)_i^{n+1} = B_p; \quad (V^E)_i^{n+1} = B_p$ 
    Endif
  Else
    If  $((V^E)_i^{n+1} < \kappa S)$  then
       $(B)_i^{n+1} = 0.0; \quad (V^E)_i^{n+1} = \kappa S$ 
    Endif
  Endif

```

Endfor

Next, the maximum value constraints are applied:

```

For  $i = 1, \dots$ ,
  If  $((V^E)_i^{n+1} > \max(B_c, \kappa S))$  then
     $(B)_i^{n+1} = 0; \quad (V^E)_i^{n+1} = \max(B_c, \kappa S)$ 
  Endif

```

Endfor

In principle, we could simply go on to the next timestep at this point using  $B_i^{n+1}$  and  $(V^E)_i^{n+1}$ . However, we have found that convergence (as the timestep size is reduced) is enhanced and the delta and gamma values are smoother if we add the following steps. Let

$$(\mathcal{Q}V)_i^{n+1} \equiv \left( \frac{(V^I)_i^{n+1} - (V^I)_i^n}{\Delta\tau} \right) - ((\mathcal{H}V^I)_i^{n+1} - (rV^I)_i^{n+1} - (pB)_i^{n+1}). \quad (7.9)$$

Then if  $B_c > \kappa S$ ,  $(V^I)_i^{n+1}$  is determined by solving the discrete linear complementary problem

$$\begin{pmatrix} (\mathcal{Q}V^I)^{n+1} = 0 \\ ((V^I)^{n+1} - \max(B_p, \kappa S)) \geq 0 \\ ((V^I)^{n+1} - B_c) \leq 0 \end{pmatrix} \quad (7.10)$$

$$\vee \begin{pmatrix} (\mathcal{Q}V^I)^{n+1} \geq 0 \\ ((V^I)^{n+1} - \max(B_p, \kappa S)) = 0 \\ ((V^I)^{n+1} - B_c) < 0 \end{pmatrix} \quad (7.11)$$

$$\vee \begin{pmatrix} (\mathcal{Q}V^I)^{n+1} \leq 0 \\ ((V^I)^{n+1} - \max(B_p, \kappa S)) \geq 0 \\ ((V^I)^{n+1} - B_c) = 0 \end{pmatrix}, \quad (7.12)$$

while if  $B_c \leq \kappa S$ , we apply the Dirichlet conditions

$$(V^I)_i^{n+1} = \kappa S_i. \quad (7.13)$$

A penalty method (Forsyth and Vetzal, 2002) is used to solve the discrete complementarity problem (7.12). Finally, we set

$$\begin{aligned} (V^E)^{n+1} &= (V^I)^{n+1} \\ B_i^{n+1} &= \min(B_i^{n+1}, (V^I)^{n+1}). \end{aligned} \quad (7.14)$$

The above algorithm essentially decouples the system of linear complementarity problems for  $B$  and  $V$  by applying the constraints in a partially explicit fashion. However, we apply the constraints as implicitly as possible, without having to solve the fully coupled linear complementarity problem. Consequently, we can only expect first order convergence (in the timestep size  $\Delta\tau$ ), even if Crank-Nicolson timestepping is used. However, this approach makes it comparatively straightforward to experiment with different convertible bond models. As well, it is unlikely that the overhead of the fully coupled approach will result in lower computational cost compared to the decoupled method above (at least for practical convergence tolerances).

## 7.2 General Splitting Model: Numerical Method

In this section, we describe the numerical method used to solve discrete forms of (4.9)-(4.10) and (6.1)-(6.2). Given initial values of  $C_i^n$  and  $B_i^n$ , and the total value  $V_i^{n+1}$ , the timestepping proceeds as follows. First, the value of  $B_i^{n+1}$  is estimated, ignoring any constraints. We denote this estimate by  $(B^*)_i^{n+1}$ :

$$\frac{(B^*)_i^{n+1} - B_i^n}{\Delta\tau} = (\mathcal{P}B^*)_i^{n+1} - (r+p)_i^{n+1}(B^*)_i^{n+1} + (pRB^*)_i^{n+1}. \quad (7.15)$$

Then,  $C^{n+1}$  is estimated, also ignoring constraints. We denote this estimate by  $(C^*)_i^{n+1}$ :

$$\frac{(C^*)_i^{n+1} - C_i^n}{\Delta\tau} = (\mathcal{P}C)_i^{n+1} - ((r+p)C^*)_i^{n+1} + p \max(\kappa S(1-\eta) - R(B^*)_i^{n+1}, 0). \quad (7.16)$$

Then, we check the minimum value constraints

```

For  $i = 1, \dots$ ,
   $B_i^{n+1} = \min(B_c, (B^*)_i^{n+1})$  // overriding maximum constraint
   $C_i^{n+1} = (C^*)_i^{n+1}$ 
  If  $(B_p > \kappa S_i)$  then
     $B_i^{n+1} = \max(B_i^{n+1}, B_p - C_i^{n+1})$ 
  Else
     $C_i^{n+1} = \max(\kappa S_i - B_i^{n+1}, C_i^{n+1})$ 
  Endif

```

Endfor

Then, the maximum value constraints are applied:

```

For  $i = 1, \dots$ ,
   $C_i^{n+1} = \min(C_i^{n+1}, \max(\kappa S_i, B_c) - B_i^{n+1})$ 
Endfor

```

In principle, we could simply continue on to the next timestep, setting  $V_i^{n+1} = B_i^{n+1} + C_i^{n+1}$ . However, as for the case of TF splitting, convergence is enhanced if we carry out the following additional steps. Let

$$(\mathcal{T}V)_i^{n+1} \equiv \left( \frac{V_i^{n+1} - V_i^n}{\Delta t} \right) - ((\mathcal{P}V)_i^{n+1} - ((r+p)V)_i^{n+1} + p \max(\kappa S_i(1-\eta), (RB)_i^{n+1})). \quad (7.17)$$

Then  $V_i^{n+1}$  (the total convertible value) is determined by solving the discrete linear complementary problem

$$\left( \begin{array}{l} (\mathcal{T}V)^{n+1} = 0 \\ ((V)^{n+1} - \max(B_p, \kappa S)) \geq 0 \\ ((V)^{n+1} - B_c) \leq 0 \end{array} \right) \vee \left( \begin{array}{l} (\mathcal{T}V^I)^{n+1} \geq 0 \\ (V^{n+1} - \max(B_p, \kappa S)) = 0 \\ (V^{n+1} - B_c) < 0 \end{array} \right) \vee \left( \begin{array}{l} (\mathcal{T}V^I)^{n+1} \leq 0 \\ (V^{n+1} - \max(B_p, \kappa S)) \geq 0 \\ (V^{n+1} - B_c) = 0 \end{array} \right) \quad (7.18)$$

if  $B_c > \kappa S$ , while if  $B_c \leq \kappa S$  we apply the Dirichlet conditions

$$V_i^{n+1} = \kappa S_i. \quad (7.19)$$

As with the TF case, a penalty method is used to solve the discrete complementarity problem (7.12). Finally, we set

$$\begin{aligned} B_i^{n+1} &= \min(B_i^{n+1}, V_i^{n+1}) \\ C_i^{n+1} &= V_i^{n+1} - B_i^{n+1}, \end{aligned} \tag{7.20}$$

which ensures that

$$V_i^{n+1} = C_i^{n+1} + B_i^{n+1}. \tag{7.21}$$

## 8 Numerical Example

In order to be precise about the way put and call provisions are handled, we will describe the method used to compute the effect of accrued interest and the coupon payments in some detail.

The payoff condition for the convertible bond is (at  $t = T$ )

$$V(S, T) = \max(\kappa S, F + K_{last}), \tag{8.1}$$

where  $K_{last}$  is the last coupon payment.

Let  $t$  be the current time in the forward direction,  $t_p$  the time of the previous coupon payment, and  $t_n$  be the time of the next pending coupon payment, i.e.  $t_p \leq t \leq t_n$ . Then, define the accrued interest on the pending coupon payment as

$$\text{AccI}(t) = K_n \frac{t - t_p}{t_n - t_p} \tag{8.2}$$

where  $K_n$  is the coupon payment at  $t = t_n$ .

The dirty call price  $B_c$  and the dirty put price  $B_p$ , which are used in equations (6.1)-(6.2) and equations (5.3)-(5.4), are given by

$$\begin{aligned} B_c(t) &= B_c^{cl}(t) + \text{AccI}(t) \\ B_p(t) &= B_p^{cl}(t) + \text{AccI}(t), \end{aligned} \tag{8.3}$$

where  $B_c^{cl}$  and  $B_p^{cl}$  are the clean prices.

Let  $t_i^+$  be the forward time the instant after a coupon payment, and  $t_i^-$  be the forward time the instant before a coupon payment. If  $K_i$  is the coupon payment at  $t = t_i$ , then the discrete coupon payments are handled by setting

$$\begin{aligned} V(S, t_i^-) &= V(S, t_i^+) + K_i \\ B(S, t_i^-) &= B(S, t_i^+) + K_i \\ C(S, t_i^-) &= C(S, t_i^+), \end{aligned} \tag{8.4}$$

where  $V$  is the total convertible value and  $B$  is the bond component. The coupon payments are modelled in the same way for both TF and general splitting models.

The data used for the numerical examples is given in Table 1, which is similar to the data used in Tsiveriotis and Fernandes (1998) (except that some data, such as the volatility of the stock, was not provided in that paper). We will confine our numerical examples to the two common assumptions of total default ( $\eta = 1.0$ ) or partial default ( $\eta = 0.0$ ) (see equation (4.1)).

Table 2 demonstrates the convergence of the numerical methods for both models. It is interesting to note that the general splitting partial and total default models appear to give solutions correct to \$.01 with



$T$	5 years
Clean call price	110 in years 2 – 5 0 in years 0 – 2
Clean put price	105 at 3 years
$r$	.05
$p$	.02
$\sigma$	.20
Conversion ratio	1.0
Recovery factor $R$	0.0
Face value of bond	100
Coupon dates	.5, 1.0, 1.5, . . . , 5.0
Coupon payments	4.0
Total default	$\eta = 1.0$
Partial default	$\eta = 0.0$

TABLE 1: Data for numerical example. Partial and total default cases defined by equation (4.1).

Nodes	Timesteps	General Splitting (Partial Default)	General Splitting (Total Default)	TF
200	200	124.9158	122.7341	124.0025
400	400	124.9175	122.7333	123.9916
800	800	124.9178	122.7325	123.9821
1600	1600	124.9178	122.7319	123.9754
3200	3200	124.9178	122.7316	123.9714

TABLE 2: Comparison of general splitting (partial and total default) and TF models. Value at  $t = 0$ ,  $S = 100$ . Data given in Table 1. For the TF model, partially implicit application of constraints. Total default ( $\eta = 1.0$ ) and partial default ( $\eta = 0.0$ ) defined in equation (4.1).

coarse grids/timesteps, while considerably finer grids/timesteps are required to achieve this level of accuracy for the TF model. This reflects our earlier comment (at the end of Section 5.1) that the bond component of the TF model effectively involves a time dependent knock-out barrier, which is difficult to solve accurately. Note that the general splitting partial default model gives a price which is about \$1.00 higher than the TF price. In contrast, the total default model is about \$1.00 less than the TF price.

The results in Table 2 should be contrasted with the results in Table 3, where the hazard rate  $p$  is set to zero. In this case, the value of the bond is about \$2.00 more than for the TF model, and about \$1.00 more than the for general splitting partial default model.

Table 4 shows the convergence of the TF model, where the last fully implicit solution of the total bond value in equations (7.12)-(7.14) is omitted. In this case, we can compare the results in Table 4 to those in Table 2. We observe that the extra implicit solve (equations (7.12)-(7.14)) does indeed speed up convergence as the grid is refined and the timestep size is reduced.

Figure 2 provides a plot for the cases of no default, the TF model, and the two general splitting models (partial and total default). For high enough levels of the underlying stock price, the bond will be converted and all of the models converge to the same value. Similarly, although it is not shown in the figure, as  $S \rightarrow 0$  all of the models (except for the no default case) converge to the same value as the valuation equation becomes an ordinary differential equation which is independent of  $\eta$  (though not of  $p$ ). Between these two extremes, the graph reflects the behavior shown in Table 2, with the general splitting partial default value above the TF model which is in turn above the general splitting total default value. The figure also shows

Nodes	Timesteps	Value ( $p = 0$ )
200	200	125.9500
400	400	125.9523
800	800	125.9528
1600	1600	125.9529
3200	3200	125.9529

TABLE 3: Value of convertible bond at  $t = 0, S = 100$ . Data given in Table 1, except that the hazard rate  $p = 0$ . In this case, both the TF and the general splitting models give the same result.

Nodes	Timesteps	TF (Table 2) (partially implicit constraints)	TF (explicit constraints)
200	200	124.00249	124.09519
400	400	123.99160	124.05384
800	800	123.98210	124.02508
1600	1600	123.97538	124.00798
3200	3200	123.97141	123.99433
6400	6400	123.97050	123.98531

TABLE 4: TF model value at  $t = 0, S = 100$ . Data given in Table 1. Comparison of partially implicit constraints (use equations (7.12)-(7.14)) and explicit application of constraints (omit equations (7.12)-(7.14)).

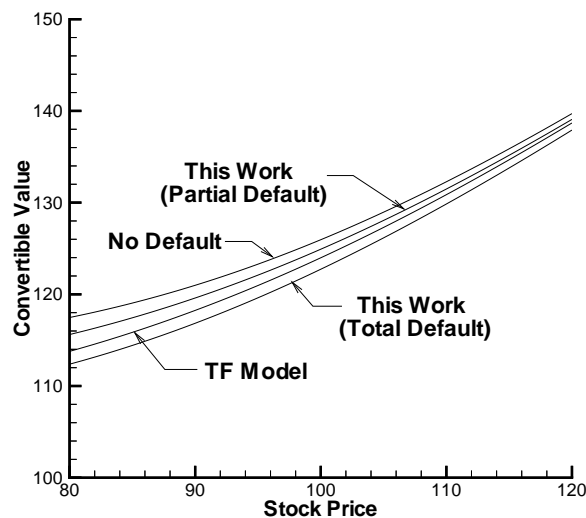


FIGURE 2: Convertible bond values at  $t = 0$ , showing the results for no default, the TF model, and the general splitting (partial default ( $\eta = 0.0$ ) and total default ( $\eta = 1.0$ ) models. (see equation (4.1)). Data as in Table 1.

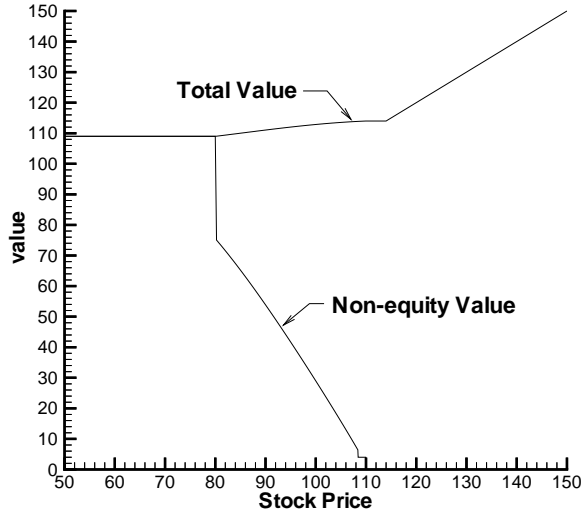


FIGURE 3: *TF model, total and non-equity (bond) component at  $t = 3$  years, just before coupon payment and put provision. Data as in Table 1.*

the additional intuitive feature not documented in the table that the case of no default yields higher values than any of the models with default.

It is interesting to see the behavior of the TF bond component and the TF total convertible value an instant before  $t = 3$  years. Recall from Table 1 the bond is puttable at  $t = 3$ , and there is a pending coupon payment as well. Figure 3 shows the discontinuous behavior of the bond component near the put price for the TF model. Since  $V = B + C$ , the call component also has a discontinuity.

Figure 4 shows results for the general splitting total default model with different recovery factors  $R$  (equation (4.9)). We also show the case with no default risk ( $p = 0$ ) for comparison. Note the rather curious fact that for the admittedly unrealistic case of  $R = 100\%$ , the value of the convertible bond is above the value with no default risk. This can be explained with reference to the hedging portfolio (4.2). Note that the portfolio is long the bond and short the stock. If there is a default, and the recovery factor is high, the hedger obtains a windfall profit, since there is a gain on the short position, and a very small loss on the bond position.

The previous examples used a constant hazard rate (Table 1). However, it is more realistic to model the hazard rate as increasing as the stock price decreases. A parsimonious model of the hazard rate is given by

$$p(S) = p_0 \left( \frac{S}{S_0} \right)^\alpha \quad (8.5)$$

where  $p_0$  is the estimated hazard rate at  $S = S_0$ . In Muromachi (1999), a function of the form of equation (8.5) was observed to be a reasonable fit to bonds rated BB+ and below in the Japanese market. Typical values for  $\alpha$  are in the range from -1.2 to -2.0 (Muromachi, 1999).

In Figure 5 we compare the value of the general splitting total default model for constant  $p(S)$  as well as for  $p(S)$  given by equation (8.5). The data are as in Table 1, except that for the non-constant  $p(S)$  cases, we use equation (8.5) with  $p_0 = .02$ ,  $S_0 = 100$ . Figures 6 and 7 show the corresponding delta and gamma values.

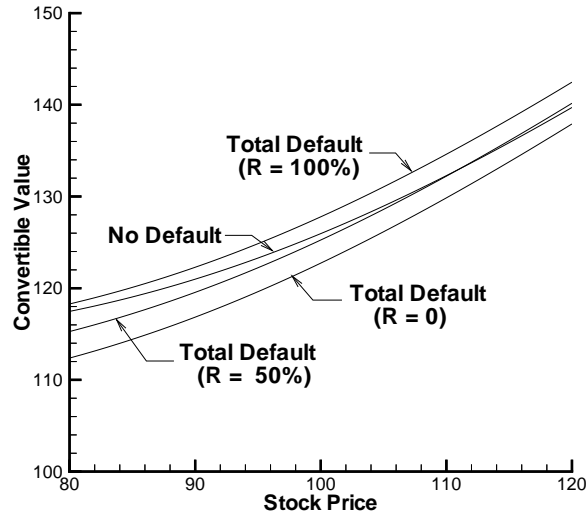


FIGURE 4: *General splitting total default model with different recovery rates. Data as in Table 1.*

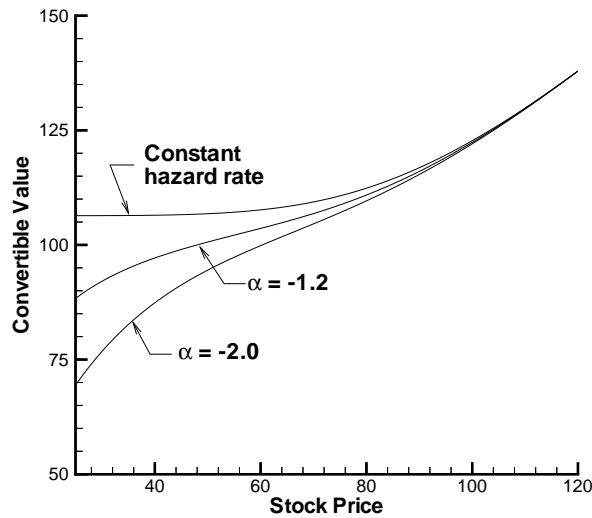


FIGURE 5: *General splitting total default models with constant hazard rate and non-constant hazard rate with different exponents in equation (8.5).  $S_0 = 100$ ,  $p_0 = .02$ ; other data as in Table 1.*

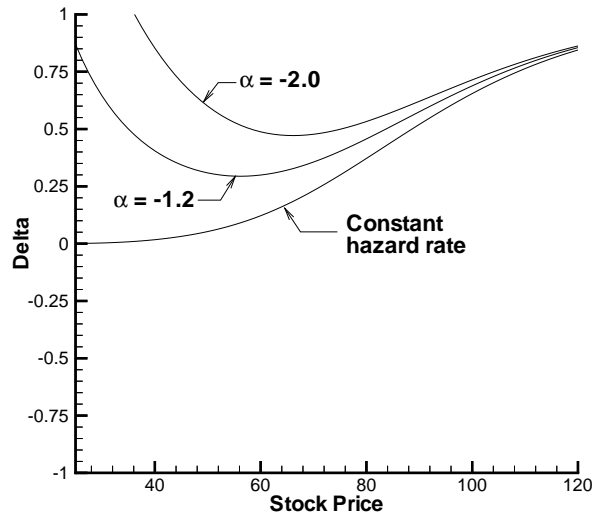


FIGURE 6: *Delta for general splitting total default models with constant hazard rate and non-constant hazard rate with different exponents in equation (8.5).  $S_0 = 100$ ,  $p_0 = .02$ ; other data as in Table 1.*

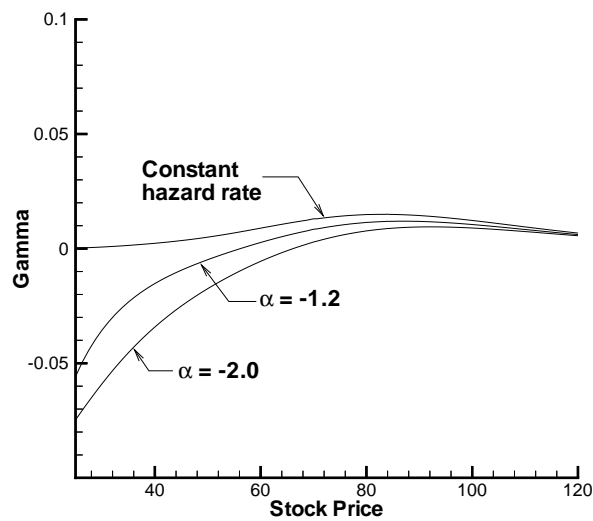


FIGURE 7: *Gamma for general splitting total default models with constant hazard rate and non-constant hazard rate with different exponents in equation (8.5).  $S_0 = 100$ ,  $p_0 = .02$ ; other data as in Table 1.*

## 9 Conclusions

Even in the simple case where the single risk factor is the stock price (interest rates being deterministic), there have been several models proposed for default risk involving convertible bonds. In order to value convertible bonds with credit risk, it is necessary to specify precisely what happens to the components of the hedging portfolio in the event of a default.

In this work, we consider a continuum of possibilities for the value of the stock price after default. Two special cases are:

- Partial default: the stock price is unchanged upon default. The holder of the convertible bond can elect to
  - (a) Receive a recovery factor times the bond component value.
  - (b) Convert the bond to shares.
- Total default: the stock price jumps to zero upon default. The equity component of the convertible bond is, by definition, zero. A fraction of the bond value of the convertible is recovered.

In the case of total default with a recovery factor of zero, our model agrees with that in Takahashi et al. (2001). In this situation, there is no need to split the convertible bond into equity and bond components. In the case of non-zero recovery, our model is slightly different from that in Takahashi et al. (2001). This would appear to be due to a different definition of the term *recovery factor*.

In the partial default case, the model developed in this work uses a different splitting (i.e. bond and equity components) than that used in Tsiveriotis and Fernandes (1998). We have presented several arguments as to why we think their model is somewhat inconsistent.

It is possible to make other assumptions about the behavior of the stock price on default. As well, there may be limits on conversion rights on default. The approach described in this paper can handle a variety of these assumptions.

Convertible bond pricing generally results in a complex coupled system of linear complementarity problems. We have used a partially implicit method to decouple the system of linear complementarity problems at each timestep. The final value of the convertible bond is computed by solving a full linear complementarity problem (but with explicitly computed source terms), which gives good convergence as the mesh and timestep are reduced, and also results in smooth delta and gamma values.

It is clear that the value of a convertible bond depends on the precise behavior assumed when the issuer goes into default. Given any particular assumption, it is straightforward to model these effects in the framework presented in this paper. A decision concerning which assumptions are appropriate requires an extensive empirical study for different classes of corporate debt.

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