THE DISTRIBUTIONS AND MOMENTS OF SOME VARIANCE COMPONENT ESTIMATORS

Ying Y. Wang
Cornell University

ABSTRACT

This is a study of the density functions and moments of the random variables

$$Z = \alpha X - \beta Y ,$$

and

$$T = \alpha X - \beta Y , \quad \text{if } \alpha X > \beta Y$$
$$= 0 \quad , \quad \text{otherwise} ,$$

where X,Y are two independent chi-square variables, and $\alpha > 0$, $\beta > 0$ are constants. The problem arises out of a study of estimators for variance components, where the estimator Z is a linear difference of two mean squares, and T is a truncated non-negative estimator.

Paper No. BU-125 in the Biometrics Unit Series and No. 511 in the Plant Breeding Series.

THE DISTRIBUTIONS AND MOMENTS OF SOME VARIANCE COMPONENT ESTIMATORS

Ying Y. Wang
Cornell University

1. Introduction and Summary

This is a study of the density functions and moments of the random variables

$$Z = \alpha X - \beta Y ,$$

and

$$T = \alpha X - \beta Y , \quad \text{if } \alpha X > \beta Y$$
$$= 0 , \quad \text{otherwise} ,$$

where X,Y are two independent chi-square variables, and $\alpha > 0$, $\beta > 0$ are constants. The problem arises out of a study of estimators for variance components, where the estimator Z is a linear difference of two mean squares, and T is a truncated non-negative estimator.

2. Density Functions and Moments

For ease of computation we shall consider the case with even degrees of freedom. Let X,Y be independent chi-squares with 2n and 2m degrees of freedom respectively.

Their joint density function is

$$f_{XY}(x,y) = K_1 e^{-\frac{x+y}{2}} x^{n-1} y^{m-1} , \quad x,y > 0$$
$$K_1 = (2^{n+m}\Gamma(n)\Gamma(m))^{-1} .$$

Paper No. BU-125 in the Biometrics Unit Series and No. 511 in the Plant Breeding Series.

Let

$$Z = \alpha X - \beta Y \ , \quad \alpha,\beta > 0$$

$$W = X + Y \quad ,$$

the joint density function for Z,W is

$$g_{ZW}(z,w) = K_2 e^{-\frac{w}{2}}\left(w + \frac{z}{\beta}\right)^{n-1}\left(w - \frac{z}{\alpha}\right)^{m-1} \ , \text{ where } w > \frac{z}{\alpha} \ , \quad \infty > z \geq 0$$

$$w > -\frac{z}{\beta}, \ -\infty < z < 0$$

$$K_2 = K_1 \alpha^{m-1} \beta^{n-1} (\alpha + \beta)^{1-n-m} \ .$$

The marginal density function for Z can be obtained by binomial expansion and integration,

$$f_{Z+}(z) = \int_{z/\alpha}^{\infty} g_{ZW}(z,w)dw = K_2 e^{-\frac{z}{2\alpha}} \int_0^{\infty} t^{m-1}\left(t + z\left(\frac{1}{\alpha} + \frac{1}{\beta}\right)\right)^{n-1} dt$$

$$= \left(\frac{\alpha}{\alpha+\beta}\right)^{m-1}\left(\frac{\beta}{\alpha+\beta}\right)^{n-1} \frac{1}{2(\alpha+\beta)} e^{-\frac{z}{2\alpha}} \sum_{j=0}^{n-1} \left(\frac{z}{2\alpha}\right)^{j}\left(\frac{\alpha+\beta}{\beta}\right)^{j} \frac{(m+n-2-j)!}{j!(n-1-j)!(m-1)!} \ , \quad z \geq 0$$

$$f_{Z-}(z) = \int_{-z/\beta}^{\infty} f_{ZW}(z,w)dw = K_2 e^{\frac{z}{2\beta}} \int_0^{\infty} t^{n-1}\left(t - z\left(\frac{1}{\alpha} + \frac{1}{\beta}\right)\right)^{m-1} dt$$

$$= \left(\frac{\alpha}{\alpha+\beta}\right)^{m-1}\left(\frac{\beta}{\alpha+\beta}\right)^{n-1} \frac{1}{2(\alpha+\beta)} e^{\frac{z}{2\beta}} \sum_{j=0}^{m-1} \left(\frac{-z}{2\beta}\right)^{j}\left(\frac{\alpha+\beta}{\alpha}\right)^{j} \frac{(m+n-2-j)!}{j!(m-1-j)!(n-1)!} \ , \quad z < 0 \ .$$

So each f_{Z+}, f_{Z-} is in the form of a product of exponential of z with a polynomial of z of degree $n-1$ or $m-1$ respectively. The method used and the results obtained are similar to those given by Gurland (1955) for $\alpha = \beta = 1$.

The characteristic function for Z is

$$\varphi_Z(u) = (1 - 2i\alpha u)^{-n}(1 + 2i\beta u)^{-m}$$

and the k^{th} moment of Z is then,

$$\mu_Z^{(k)} = (2\beta)^k \frac{k!}{(m-1)!(n-1)!} \sum_{j=0}^{k} \frac{(k+m-j-1)!(n+j-1)!(-1)^{k-j}}{(k-j)!j!} \left(\frac{\alpha}{\beta}\right)^j .$$

In particular, the mean and variance of Z are,

$$\mu_Z = 2(n\alpha - m\beta)$$

$$\sigma_Z^2 = 4\beta^2[m(m+1) - 2mn(\frac{\alpha}{\beta}) + n(n+1)(\frac{\alpha}{\beta})^2] - 4(n\alpha - m\beta)^2$$

$$= 4[n\alpha^2 + m\beta^2]$$

For the truncated random variable T, the density function is

$$h_T(t) = f_{Z+}(t) = \left(\frac{\alpha}{\alpha+\beta}\right)^{m-1}\left(\frac{\beta}{\alpha+\beta}\right)^{n-1} \frac{1}{2(\alpha+\beta)} e^{-\frac{t}{2\alpha}} \sum_{j=0}^{n-1} \left(\frac{z}{2\alpha}\right)^j \left(\frac{\alpha+\beta}{\beta}\right)^j \frac{(m+n-2-j)!}{j!(n-1-j)!(m-1)!}$$

$$\text{if } t > 0 ,$$

$$= \int_{-\infty}^{0} f_{Z-}(t)dt = \left(\frac{\beta}{\alpha+\beta}\right)^n \sum_{j=0}^{m-1} \left(\frac{\alpha}{\alpha+\beta}\right)^j \frac{(n-1+j)!}{j!(n-1)!} , \quad \text{if } t = 0 .$$

The characteristic function of T is

$$\varphi_T(u) = h_T(0) + \left(\frac{\alpha}{\alpha+\beta}\right)^m \sum_{j=0}^{n-1} \left(\frac{\beta}{\alpha+\beta}\right)^j \frac{(m-1+j)!}{(m-1)!j!} (1 - 2i\alpha u)^{n-j} .$$

The k^{th} moment of T is

$$\mu_T^{(k)} = (2\alpha)^k \left(\frac{\alpha}{\alpha+\beta}\right)^m \left(\frac{\beta}{\alpha+\beta}\right)^{n-1} \sum_{j=0}^{n-1} \left(\frac{\alpha+\beta}{\beta}\right)^j \frac{(m+n-2-j)!(k+j)!}{(m-1)!j!(n-1-j)!} .$$

In particular, the mean and variance are

$$\mu_T = (2\alpha)\left(\frac{\alpha}{\alpha+\beta}\right)^m \left(\frac{\beta}{\alpha+\beta}\right)^{n-1} \sum_{j=0}^{n-1} \left(\frac{\alpha+\beta}{\beta}\right)^j \frac{(m+n-2-j)!(j+1)!}{(m-1)!j!(n-1-j)!}$$

$$\sigma_T^2 = (2\alpha)^2\left(\frac{\alpha}{\alpha+\beta}\right)^m \left(\frac{\beta}{\alpha+\beta}\right)^{n-1} \left\{ \sum_{j=0}^{n-1} \left(\frac{\alpha+\beta}{\beta}\right)^j \frac{(m-1+j)!(j+2)!}{(m-1)!j!(n-1-j)!} - \left(\frac{\alpha}{\alpha+\beta}\right)^m \left(\frac{\beta}{\alpha+\beta}\right)^{n-1} \left[\sum_{j=0}^{n-1} \left(\frac{\alpha+\beta}{\beta}\right)^j \frac{(m+n-2-j)!}{(m-1)!j!(n-1-j)!} \right]^2 \right\}$$

3. Variance Component Estimators

In the usual one way model II analysis of variance problem, let

$$y_{ij} = \mu + \alpha_i + \epsilon_{ij} ,$$

$$\alpha_i \sim N(0, \sigma_\beta^2) \qquad i = 1,\dots,g$$

$$\epsilon_{ij} \sim N(0, \sigma_\epsilon^2), \qquad j = 1,\dots,r .$$

Let the mean squares be denoted by

Source	M.S.	Degree of freedom	Expected values
Among groups	B	$g - 1$	$\sigma_\epsilon^2 + r\sigma_\beta^2$
Within	W	$g(r - 1)$	σ_ϵ^2
Total	T	$gr - 1$	

We wish to consider estimators for σ_β^2 in the form

$$Z = C_1B - C_2W ,$$

or the truncated non-negative form

$$T = C_1B - C_2W , \quad \text{if } C_1B \geq C_2W$$

$$= 0 \qquad \text{otherwise} ,$$

where C_1, C_2 are functions of g and r, as for example the estimators given by Herbach (1959) and Thompson (1962).

For g and r both odd, $g-1 = 2n$, $g(r-1) = 2m$, we can apply the results of section 2, with

$$\alpha = \frac{C_1(\sigma_\epsilon^2 + r\sigma_\beta^2)}{g-1}$$

$$\beta = \frac{C_2\sigma_\epsilon^2}{g(r-1)} ,$$

so the mean and variance of Z are

$$\mu_Z = (C_1 - C_2)\sigma_\epsilon^2 + rC_1\sigma_\beta^2$$

$$\sigma_Z^2 = 2\left\{ \left(\frac{C_1^2}{g-1} + \frac{C_2^2}{g(r-1)}\right)\sigma_\epsilon^2 + \left(\frac{2rC_1^2}{g-1}\right)\sigma_\epsilon^2\sigma_\beta^2 + \frac{C_1^2 r^2}{g-1}\sigma_\beta^4\right\} .$$

For the special case of $C_1 = C_2 = \frac{1}{r}$

$$\mu_2 = \sigma_\beta^2$$

$$\sigma_Z^2 = \frac{2\sigma_\beta^4}{(g-1)}\left\{ \frac{gr-1}{gr^2(r-1)}\left(\frac{\sigma_\epsilon^2}{\sigma_\beta^2}\right)^2 + \frac{2}{r}\left(\frac{\sigma_\epsilon^2}{\sigma_\beta^2}\right) + 1\right\} ,$$

when $r \to \infty$, $\alpha \to \sigma_\epsilon^2/(g-1)$, $\beta \to 0$

$$f_{Z+} \to e^{-\frac{z(g-1)}{2\sigma_\epsilon^2}} z^{n-1}\left(\frac{g-1}{2\sigma_\epsilon^2}\right)^n \frac{1}{(n-1)!} \qquad z \geq 0$$

$$f_{Z-} \to 0 \qquad z < 0$$

$$\text{so } Z \sim \left(\frac{\sigma_\epsilon^2}{g-1}\right)\chi^2_{(g-1)} \text{ as } r \to \infty$$

For the truncated variable T, as $r \to \infty$,

$$h_T(t) \to e^{-\frac{t(g-1)}{2\sigma_\epsilon^2}} \, t^{n-1} \left(\frac{g-1}{2\sigma_\epsilon^2} \right)^n \frac{1}{(n-1)!} \qquad t > 0$$

$$T \sim \left(\frac{\sigma_\epsilon^2}{g-1} \right) \chi^2_{(g-1)} \; .$$

References

Gurland, J. (1955). Distribution of definite and indefinite quadratic forms. Am. Math. Statist. 26, 122-127.

Herbach, L. H. (1959). Properties of Model II-type analysis of variance tests. Am. Math. Statist. 30, 939-959.

Thompson, W. A. (1962). The problem of negative estimates of variance components. Am. Math. Statist. 33, 273-289.