TRADING STRATEGIES AND PORTFOLIO OPTIMIZATION IN THE PRESENCE OF ASSET PRICE BUBBLES

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In this work, we aim to provide mathematical characterizations of effect of the existence of asset price bubbles to the financial market from various perspective, leveraging the arbitrage pricing framework and mathematical theory of financial bubbles. We analyze bubble's effect to several sets of trading strategies, including both widely employed practical strategies and those proven to replicate or super-replicate the risky asset's payoffs over the model's horizon.

Additionally, we delve into the portfolio optimization problem in the presence of asset price bubbles in multiple facets. A comparative analysis of optimal portfolios is conducted in diverse settings, including complete and incomplete markets, and with or without trading constraints.

Finally, we introduce a novel approach for simulating sophisticated stochastic processes that is free from space discretization. This chapter serves as a foundational framework for conducting advanced numerical and empirical research on challenges related to sophisticated stochastic dynamics. The scope of this research extends beyond, but is not confined to, the simulation of asset price bubbles.

BIOGRAPHICAL SKETCH

Yuxuan Liu was born in Taiyuan, China. He earned his Bachelor of Science degree in Mathematics from Peking University in 2018. Subsequently, in that same year, he started his doctoral journey by joining the Department of Operations Research and Information Engineering at Cornell University. In May 2022, he successfully earned his Master of Science in Operations Research during the course of his PhD pursuit.

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CHAPTER 1

INTRODUCTION

1.1 Background and Overview of Results

The results of this thesis span several topics in mathematical finance: asset price bubbles, dominating trading strategies, portfolio optimization, simulation of related stochastic processes and numerical methods for related PDEs.

1.2 Wealth Preserving and Dominating Trading Strategies

The second chapter, *Asset Price Bubbles, Wealth Preserving, Dominating, and Replicating Trading Strategies*, studies an arbitrage-free, competitive and frictionless market with trading in a single risky asset and money market account, where the risky asset exhibits a price bubble. We analyze two sets of self-financing and admissible trading strategies in this market. The first are simple trading strategies, and the second are trading strategies that replicate or super-replicate the risky asset's payoffs at the model's horizon. We show that in both sets there exist wealth preserving trading strategies, i.e. trading strategies whose initial value equals the present value of its future cash flows. And, in the second set, we show that there are wealth preserving replicating trading strategies that dominate buying and holding the risky asset. The practical applications of these insights are discussed.

1.3 Portfolio Optimization

The third chapter, *Asset Price Bubbles, Wealth Preserving, Dominating, and Replicating Trading Strategies*, studies portfolio optimization over terminal wealth in a finite horizon, arbitrage-free, competitive and frictionless market with a single risky asset and money market account, where the risky asset exhibits a price bubble. We show that in a complete market, the existence of a price bubble does not change an investor's welfare relative to an otherwise identical mar- ket with no bubble. The reason is that the optimal trading strategy enables the investor to avoid the losses due to a bubble and still obtain the risky asset's fundamental value. In an incomplete market, we provide a sufficient condition for the same conclusion to apply. This sufficient condition holds for a large class of Markov diffusion processes. An example is provided to show that bubbles can increase an investor's welfare if the sufficient condition is violated. We further show that, given certain trading constraints (e.g. collateral requirements), asset price bubbles can also decrease an investor's welfare.

1.4 Isospectral Methods for Stochastic Processes and PDEs

The fourth chapter, *Isospectral schemes for exact algorithms for stochastic processes and PDE's*, studies simulation of certain stochastic processes and solving related PDE based on some spetral ideas, which would be useful for many application of mathematical finance such as option pricing, portfolio optimization, especially in the presence of asset price bubbles. In this chapter, we propose a

procedure that identifies exactly these averages as averages of a random phenomena defined on a lattice whose time evolution is perfectly randomized. Both time and state sets are thus amenable to exact computation resulting in the reduction of the error to an inevitable random approximation which is easily analyzed. Our approach also extends to the computation of solutions of parabolic problems and enables to identify exact discretization schemes for solving such PDEs. These identifications are called gateway relations as they relate two seemingly disconnected worlds, the continuum and the lattice. We also resort to interweaving relations, a refinement of such gateway relations. This novel concept in numerical analysis turns out to be very efficient, in terms of both accuracy and computational cost. It is also scientifically grounded as it stems from the fundamental and well-established idea that linear operators are classified, independent of their state space, by a single object: their spectrum. We show promising numerical examples leveraging the gateway relationship between the semigroups of squared Bessel and CIR processes, in $\mathbb{R}^+$ and in its Weyl chamber, demonstrating the benefits of gateway-interweaving inspired simulation compared to traditional methods

CHAPTER 2

ASSET PRICE BUBBLES, WEALTH PRESERVING, DOMINATING, AND REPLICATING TRADING STRATEGIES

2.1 Introduction

In an arbitrage-free, competitive and frictionless market, asset price bubbles can theoretically exist (see [31] for a discussion of the theoretical models). And, there is mounting empirical evidence that asset price bubbles exist in reality as well (see [29], [51], and [33]). In light of these facts, we study how to trade an asset that exhibits a price bubble. Should you short the asset in hopes of covering the position after the bubble bursts? Or, should you buy the asset in hopes of selling it before the bubble bursts? Furthermore, suppose you want the buy the asset today in order to obtain the asset's payoff at some future time T. Is there a trading strategy that can be used to avoid the losses that occur if the bubble bursts on or before T?

Understanding the answers to these questions are a necessary first step to trading assets with price bubbles. And, they are essential to characterizing an investor's optimal portfolio in the presence of asset price bubbles. To state our answers to these questions, we need some definitions.

Consider a market consisting of a single risky asset with no cash flows and a money market with time horizon $[0, T]$. A wealth preserving trading strategy is one where the present value of the trading strategy's cash flows is equal to its initial investment. A dominating trading strategy is one that satisfies three conditions: (i) the trading strategy's initial cost is less than or equal to the cost of buying the risky asset, (ii) its payoff at time T is never less than the payoff to the risky asset, and (iii) with positive probability its payoff at time T is strictly greater. Such a trading strategy dominates a buy and hold position in the risky asset, hence, the terminology.

Given these definitions, we prove the following results in an arbitrage-free market with bubbles. These results provide the answers to the three questions previously posed.

- Shorting the asset in hopes of covering the short before time T is infeasible, because with strictly positive probability, it implies possible borrowing before time T that is greater than any fixed amount. Hence, this trading strategy is impossible given realistic borrowing constraints.
- Buying and holding the risky asset until time T is strictly wealth decreasing, and therefore a suboptimal trading strategy.
- Simple buy and sell trading strategies over $[0, \tau]$, with $\tau < T$ a stopping time, exist that are wealth preserving, i.e. maintain one's initial wealth. An example of such a trading strategy is to sell the risky asset the first time the risky asset price hits a fixed and predetermined level $k > S_0$ where S_0 is the risky asset's time 0 price. Hence, buying an asset with a bubble and selling it at an optimally determined future date is a reasonable trading strategy.
- In a complete market there exist self-financing trading strategies that exactly replicate the asset's time T payoff, which are both wealth preserving and dominating. These replicating trading strategies are dynamic and involve holding the risky asset in non-zero quantities over the entire time horizon $[0, T]$. For diffusion processes, it can be shown that these replicating strategies holds less than one unit of the risky asset at all times prior to T. Hence

there exist replicating trading strategies that obtain the asset's time T payoff and avoid the losses that occur if the bubble bursts on or before T.

- In an incomplete market, we provide a characterization of all wealth preserving and dominating self-financing trading strategies that super-replicate the asset's time T payoff. Analogous to a complete market, these dynamic trading strategies avoid the losses that occur if the bubble bursts on or before T, and provide at least the asset's payoff at time T.

The importance of these are insights are two-fold. First, in a complete or incomplete market with asset price bubbles, these results show that it possible to replicate an index's payoff using a wealth preserving trading strategy at lower cost than via a buy and hold position in the asset. This implies that in a market with bubbles, financial institutions that provide indexed funds for clients (e.g. a S&P 500 index fund) can create an index at less cost than by using the traditional method of buying and holding the assets in the index. Second, these results can be used to facilitate the characterization of the solution to an investor's portfolio optimization problem in a market with bubbles.

This paper lies within the literature studying the local martingale theory of bubbles, recently developed by [44], [11], [23], and [57] and [37]. For a review of the local martingale theory of bubble mathematics see [4]. The papers by [45], [28], and [21] include, as part of their content, a solution to an investor's optimal portfolio when asset prices exhibit bubbles. One can view our paper as studying the properties of an investor's feasible set in these portfolio optimization problems. In this regard, the insights of this paper can been used to understand and characterize and investor's optimal trading strategy in the presence of price bubbles, see [35].

An outline for this paper is as follows. Section 2 presents the model, and section 3 discusses optimal stopping times that characterize the asset's market price. Section 4 explores simple trading strategies and section analyzes replicating and super-replicating admissible self-financing trading strategies. Section 6 concludes the paper.

2.2 The Model

This section briefly reviews the local martingale theory of bubbles.

2.2.1 The Set-Up

We consider a continuous time, continuous trading model on a finite horizon $[0, T]$. The randomness is represented by $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$, which is a filtered complete probability space where the filtration $\mathbb{F} = (\mathcal{F}_t)_{t \geq 0}$ satisfies the usual hypothesis with $\mathcal{F}_T = \mathcal{F}$. The market is assumed to be competitive and frictionless. Competitive means that traders act as price takers, believing their trades have no quantity impact on the market price. Frictionless means that there are no transaction costs and no trading constraints.

Traded in the economy are a risky asset and a money market account (mma) that is locally riskless. The normalized market price of the risky asset is given by a non-negative semimartingale $(S_t)_{0 \leq t \leq T}$ adapted to $\mathbb{F}$. Without loss of generality, we assume that no cash flows are paid to the risky asset.

Trading strategies are defined to be holdings in the risky asset that are predictable (i.e. depending on only current and past information) and holdings in the mma that are optional. To exclude doubling strategies from discussion, we only consider trading strategies that are admissible (the value of the trading strategy is bounded below). A trading strategy is self-financing if it requires no cash inflows or outflows except at times 0 and T. Specifically, denote by $\mathscr{O}$ the optional σ-algebra and $\mathscr{L}(S)$ the set of predictable processes for which the stochastic integral with respect to S exists.

An admissible self financing trading strategy (s.f.t.s) with initial wealth x and wealth process X is the tuple of stochastic processes $(\alpha_0, \alpha) \in (\mathscr{O}, \mathscr{L}(S))$ such that there exists some constant c where

$$X_t = \alpha_0 + \alpha_t \cdot S_t = x + \int_0^t \alpha_u dS_u \geq c, \forall t \in [0, T].$$

The first equality represents the trading strategy's wealth at time t, which equals the number of units of the mma plus the number of shares of the risky asset times its market price. The second equality is the self-financing condition, which states that the trading strategy's time t wealth equals the initial wealth plus the accumulated capital gains from the trading strategy over $[0, T]$. The third inequality is the admissibility condition, which represents a uniform lower bound on the wealth process. This lower bound represents a borrowing constraint. We denote by $\mathcal{A}(x)$ the set of admissible s.f.t.s. $(\alpha_0, \alpha) \in (\mathscr{O}, \mathscr{L}(S))$ given an initial wealth x.

A simple arbitrage opportunity is an admissible s.f.t.s. $(\alpha_0, \alpha) \in \mathcal{A}(x)$ with initial wealth $x = 0$ and wealth process X such that

$$\mathbb{P}(X_T \geq 0) = 1, \qquad \text{and}$$
$$\mathbb{P}(X_T > 0) > 0.$$

A *Free Lunch with Vanishing Risk* (FLVR) is an admissible s.f.t.s. that is an extension of a simple arbitrage opportunity that includes (the limits of) approximate simple arbitrage opportunities. We say the market satisfies *No Free Lunch with Vanishing Risk* (NFLVR) if there exists no FLVR.

We invoke the following assumption.

Assumption 2.2.1. The market satisfies NFLVR.

An equivalent local martingale measure $\mathbb{Q}$ is any probability measure on $(\Omega, \mathcal{F})$ such that for $A \in \mathcal{F}$, $\mathbb{Q}(A) = 0$ iff $\mathbb{P}(A) = 0$ (in symbols $\mathbb{Q} \sim \mathbb{P}$) and S is a $\mathbb{Q}$ local martingale. This implies that for all $t \in [0, T]$, there exists a sequence of stopping time $\tau_n \to T$ a.s. such that

$$\mathbb{E}^{\mathbb{Q}}[S_{T \wedge \tau_n} | \mathcal{F}_t] = S_t$$

where $\mathbb{E}^{\mathbb{Q}}[\cdot]$ denotes expectation under $\mathbb{Q}$.

Define $\mathcal{M}$ to be the set of *equivalent local martingale measures* (ELMM). By the first fundamental theorem of asset pricing, the assumption of NFLVR implies that there exists an *ELMM*, i.e. $\mathcal{M} \neq \emptyset$.

A market is defined to be complete with respect to some $\mathbb{Q} \in \mathcal{M}$ if for any non-negative payoff $Z_T \in L^1(\mathbb{Q})$ at time T, there exists a $x \geq 0$ and $(\alpha_0, \alpha) \in \mathscr{A}(x)$ such that

$$x + \int_0^T \alpha_u dS_u = Z_T,$$

and the wealth process

$$X_t = \alpha_0 + \alpha_t \cdot S_t = x + \int_0^t \alpha_u dS_u$$

is a $\mathbb{Q}$ martingale.

By the second fundamental theorem of asset pricing, in a complete market satisfying NFLVR, the ELMM is unique. In a complete market, $\mathbb{E}^{\mathbb{Q}}[\cdot]$ gives the unique present value operator to determine the arbitrage-free price of any $Z \in L^1(\mathbb{Q})$ at time t, which is $\mathbb{E}^{\mathbb{Q}}[Z_T|\mathcal{F}_t]$.

In an incomplete market, $\mathbb{E}^{\mathbb{Q}}[\cdot]$ represents only one of the infinite number of present value operators consistent with NFLVR. We will study both complete and incomplete markets below. For more discussion on these topics, see [32], Chapter 2.

2.2.2 Wealth Preserving and Dominating Trading Strategies

Under NFLVR, because the wealth process X of any admissible s.f.t.s. in $\mathcal{A}(x)$ is uniformly bounded below, it is a $\mathbb{Q}$ local martingale. And, by Fatou's lemma, it is a $\mathbb{Q}$ supermartingale. We record this fact as a lemma.

Lemma 2.2.2. *(Wealth Processes)*

Assume NFLVR. Then, the wealth process X of any admissible s.f.t.s. in $\mathcal{A}(x)$ is a $\mathbb{Q}$ supermartingale, i.e. for all t $\in [0,T]$,

$$\mathbb{E}^{\mathbb{Q}}[X_T|\mathcal{F}_t] \leq X_t$$

where $\mathbb{E}^{\mathbb{Q}}[\cdot]$ denotes expectation under $\mathbb{Q}$.

In particular, this implies

$$\mathbb{E}^{\mathbb{Q}}[X_T] \leq X_0 = x.$$

Because $\mathbb{E}^{\mathbb{Q}}[\cdot]$ represents a present value operator, this lemma states that all admissible s.f.t.s.'s are wealth non-increasing. This is consistent with the market not admitting any arbitrage opportunities, which would be wealth increasing. We say that an admissible s.f.t.s. with wealth process X is wealth preserving if X is a $\mathbb{Q}$ martingale, i.e. $\mathbb{E}^{\mathbb{Q}}[X_T] = X_0 = x$.

An admissible s.f.t.s. with wealth process X is said to be a dominating if there exists an admissible s.f.t.s $(\alpha_0, \alpha) \in \mathscr{A}(x)$ such that $x < S_0$ and

$$x + \int_0^T \alpha_u dS_u = S_T \qquad \text{a.s.}$$

The market is said to satisfy No Dominance (ND) if there exist no such dominating s.f.t.s.

Not all admissible s.f.t.s. are wealth preserving. Indeed, there exists a class of admissible s.f.t.s. called suicide strategies, which are wealth decreasing. A suicide strategy is any admissible s.f.t.s. $(\alpha_0, \alpha) \in \mathcal{A}(x)$ with initial wealth $x = 0$ and wealth process X such that

$$\begin{aligned} \mathbb{P}(X_t \geq -k) = 1, &\qquad \text{for all t} \in [0, \text{T}], \text{ and} \\ \mathbb{P}(X_T = -k) = 1 &\qquad \text{for } k > 0. \end{aligned}$$

This trading strategy loses k units of the mma for sure over $[0, T]$ and is wealth decreasing, i.e. $\mathbb{E}^{\mathbb{Q}}[X_T] = -k < X_0 = 0$. For more details, see [32], Chapter 2.

If our market is populated by investors $i \in \mathbb{I}$ with beliefs represented by a probability measure $\mathbb{P}_i \sim \mathbb{P}$ and preferences such that they prefer more wealth to less, then when selecting admissible s.f.t.s., they would restrict consideration to wealth preserving and dominating admissible s.f.t.s. This insight motivates our study of such trading strategies below.

2.2.3 The Local Martingale Theory of Bubbles

Given the above set-up, we can now introduce the notion of an asset price bubble. Given the market satisfies NFLVR, $\mathcal{M} \neq \emptyset$. To define the fundamental value, we need to choose an element of $\mathcal{M}$. If the market is complete, then $\mathcal{M}$ is the singleton set, and we choose that element. If the market is incomplete, then there exists an infinite number of ELMMs.

To identify a unique equivalent local martingale measure in an incomplete market, one assumes that the market studied is embedded in a larger market that includes the trading of derivatives on the risky asset (e.g. call and put options with different strikes and maturities). There are two possible cases in the extended market. One, is that these traded derivatives complete the market and therefore the equivalent local martingale measure is again uniquely determined by the second fundamental theorem of asset pricing. The papers by [14], [12], and [60] provide models with markets where this is the case. Two, is that even with the traded derivatives, the extended market is incomplete. However, in this case under certain conditions, the traded derivatives' price processes may provide enough information to identify a unique local martingale measure consistent with these prices. The papers by [26] and [61] study such extended but incomplete markets. For subsequent use, we assume that the unique ELMM chosen by the market is determined by either of these two alternatives.

Given the $\mathbb{Q} \in \mathcal{M}$ chosen by the market, we define the fundamental value of the risky asset under $\mathbb{Q}$ by

$$\tilde{S}_t = \mathbb{E}^{\mathbb{Q}}[S_T|\mathcal{F}_t].$$

This represents the time t present value of buying and holding the asset until time T. As such, this definition is consistent with the standard definition of an asset's fundamental value used in the classic economics literature (see [31]).

Next, we define the risky asset's bubble to be

$$\beta_t = S_t - \tilde{S}_t,$$

which represents the deviation of the risky asset's market price from its fundamental value.

Given these definitions, it follows that

1. $\beta_t \geq 0$ for all $t \in [0, T]$
2. β is not constantly zero if and only if S is a strict local martingale under the chosen ELMM.
3. If $\beta_s = 0$ for some $s \in [0, T]$, then $\beta_t \equiv 0$ for $t \in [s, T]$.

Indeed, since S is a local martingale bounded from below, then by Fatou's lemma it is a supermartingale. Hence, we have $S_t \geq \mathbb{E}^{\mathbb{Q}}[S_T|\mathcal{F}_t]$ and β is non-negative. S is a true martingale if and only if $\beta_t \equiv 0$.

The importance of these properties for the subsequent analysis is that they imply that an asset price bubble bursts on or before time T with probability one. And, if it burst before time T, it cannot be "reborn."

Moreover, we have the following characterization of asset price bubbles.

Theorem 2.2.3. *(Bubble Characterization)*

For S a local martingale measure $\mathbb{Q}$, (1) - (3) are equivalent. If S has bounded jumps, then they are all equivalent to (4).

1. *S exhibits a bubble.*
2. *S is a strict supermartingale under $\mathbb{Q}$.*
3. *$\mathbb{E}^{\mathbb{Q}}[S_T] < S_0$.*
4. *$\limsup_{x\to\infty} x\mathbb{Q}(\sup_{t\in[0,T]} S_t > x) > 0$.*

Proof. By definition, S has a bubble if 2 holds. Furthermore, if S is a non-negative local martingale then by Fatou's lemma S is a supermartingale. 3 is a necessary and sufficient condition for S to be a strict supermartingale.

Define $\zeta_x := \inf\{t \in [0,T] : S_t \geq x\}$. For each $x > S_0$, given S has bounded jumps, S^{ζ_x} is a bounded local martingale hence a true martingale so that by optional sampling theorem

$$S_0 = \mathbb{E}^{\mathbb{Q}} S_{\zeta_x \wedge T} = \mathbb{E}^{\mathbb{Q}}[S_T \mathbb{1}_{\zeta_x \geq T} + x\mathbb{1}_{\zeta_x < T}].$$

Letting $x \to +\infty$ we have by dominated convergence that

$$\underset{x\to\infty}{lim} \mathbb{E}^{\mathbb{Q}}[S_T \mathbb{1}_{\zeta_x \geq T}] = \mathbb{E}^{\mathbb{Q}}[\underset{x\to\infty}{lim} S_T \mathbb{1}_{\zeta_x \geq T}] = \mathbb{E}^{\mathbb{Q}}[S_T]$$

and, therefore,

$$S_0 - \mathbb{E}^{\mathbb{Q}}[S_T] = \limsup_{x\to\infty} x\mathbb{Q}(\zeta_x < T) = \limsup_{x\to\infty} x\mathbb{Q}(\sup_{t\in[0,T]} S_t > x).$$

Hence 3 is equivalent to 4. □

Condition (4) is very revealing. It states that an asset price bubble exists if and only if for any given future asset price x, the "expected value" of the price process exceeding x is non-zero as x approaches infinity. Here we provide several examples of asset price processes with bubbles.

Example 2.2.4. (Local Volatility Model)

Suppose that under $\mathbb{Q}$, S is given by the following local volatility model

$$dS_t = \sigma(S_t)dW_t$$

where W is a $\mathbb{Q}$ standard Brownian motion. Then S is a strict local martingale if and only if

$$\int_{\epsilon}^{\infty} \frac{1}{\sigma(x)^2}dx < \infty$$

See [49]. An example is the CEV process with $\sigma(x) = x^{\delta+1}$ for some $\delta > 0$.

Example 2.2.5. (A Bubble Construction)

A risky asset price process with a bubble can be obtained by constructing the fundamental value and the bubble processes separately. Suppose that $\tilde{S}$ is a uniformly integrable martingale on $[0, T]$ and β is a nonnegative local martingale such that $\beta_0 > 0$ and $\beta_T = 0 \quad a.s.$ Define $S_t = \tilde{S}_t + \beta_t$, which is a strict local martingale. One example of such a β is given by

$$d\beta_t = \frac{\beta_t}{\sqrt{T-t}}dW_t \tag{2.1}$$

where W is a Brownian motion. Then, β is a time-changed Brownian motion such that $\lim_{t\to T} \beta_t = 0,\ a.s.$

Alternatively, we could let β be given by the simple single jump process

$$\beta_t = \frac{1}{T-t}\mathbb{1}_{t<\tau}$$

where τ is the bubble's crash time, which is uniformly distributed. Then it can be verified that β is a martingale on $[0, T)$ but a strict local martingale on $[0, T]$ since β converges to 0 almost surely.

Moreover, one can construct single jump strict local martingales for any distribution of crash time τ with support $[0, T)$. For example, let h_t be a deterministic process such that $h_0 = 0$ but $\lim_{t\to T} h_t = +\infty$. And we define $\tau = \inf\{t : \tilde{S}_t \leq h_t\}$ for some continuous true martingale $\tilde{S}$. Then the corresponding process β is a bubble process, the crash time of which depends on the evolution of the fundamental value.

2.2.4 Asset Return Explosion Times

This section presents an alternative asset price bubble characterization theorem. For this section, assume that S is a continuous process and $\mathbb{F} = (\mathcal{F}_t)_{t\geq 0}$ is the filtration generated by S.

First we need the following lemma.

Lemma 2.2.6. *(Exponential Martingales)*

For any positive continuous local martingale S, there exists a continuous local martingale M with $M_0 = 1$ such that $S = S_0\mathcal{E}(M)$ where $\mathcal{E}(\cdot)$ is the stochastic exponential.

Given the stochastic exponential representation of the asset price as
$S_t = S_0\mathcal{E}(M_t)$, M_t corresponds to the asset's cumulative return over $[0, T]$.

Now, define the sequence of stopping times $\zeta_n := \inf\{t \in [0, T] : [M]_t \geq n\}$ and the limit $\zeta := \underset{n\to\infty}{lim} \zeta_n$, which exists since ζ_n is a non-decreasing sequence. Then, because $[M, M]_{t\wedge\zeta_n}$ is bounded by n, $S_{t\wedge\zeta_n}$ is a true martingale by Novikov's criterion.

Define the Radon-Nikodym density process $Z_{t\wedge\zeta_n} := \frac{S_{t\wedge\zeta_n}}{S_0}$. Note that $Z_0 = Z_{0\wedge\zeta_n} = 1$. Define the associated measure $\mathbb{R}_n$ by

$$\mathbb{R}_n(A) := \int_A Z_{T\wedge\zeta_n} d\mathbb{Q}$$

for $A \in \mathcal{F}_{\zeta_n}$. It is a probability measure because the optional sampling theorem and the martingale property imply that $\mathbb{E}^{\mathbb{R}_n}[\mathbb{1}_\Omega] = \mathbb{E}^{\mathbb{Q}}[Z_{\zeta_n}] = 1$.

This construction gives a consistent family of probability measures $\{\mathbb{R}_n\}_{n\geq 1}$, that is

$$\mathbb{R}_{n+1} \mid_{\mathcal{F}_{\zeta_n}} (A) = \mathbb{R}_n(A)$$

for $A \in \mathcal{F}_{\zeta_n}$.

Proof. Note that

$$\mathbb{R}_{n+1} \mid_{\mathcal{F}_{\zeta_n}} (A) = \mathbb{E}^{\mathbb{Q}}[Z_{\zeta_{n+1}}\mathbb{1}_A] = \mathbb{E}^{\mathbb{Q}}\left[\mathbb{E}^{\mathbb{Q}}\left[Z_{\zeta_{n+1}}\mathbb{1}_A|\mathcal{F}_{\zeta_n}\right]\right]$$

$$= \mathbb{E}^{\mathbb{Q}}\left[\mathbb{1}_A\mathbb{E}^{\mathbb{Q}}\left[Z_{\zeta_{n+1}}|\mathcal{F}_{\zeta_n}\right]\right] = \mathbb{E}^{\mathbb{Q}}[Z_{\zeta_n}\mathbb{1}_A] = \mathbb{R}_n(A).$$

The second to last equality holds because $\mathbb{E}^{\mathbb{Q}}[Z_{\zeta_{n+1}}|\mathcal{F}_{\zeta_n}] = Z_{\zeta_n}$. □

Hence, there exists extended probability measure $\mathbb{R}$ on $\mathcal{F}_{\zeta-} := \sigma\left(\bigcup_n \mathcal{F}_{\zeta_n}\right)$ such that for all n and $A \in \mathcal{F}_{\zeta_n}$ (see [9]),

$$\mathbb{R} \mid_{\mathcal{F}_{\zeta_n}} (A) = \mathbb{R}_n(A).$$

Note that this construction implies that if $\mathbb{Q} \mid_{\mathcal{F}_{\zeta-}} (A) = 0$ then $\mathbb{R}(A) = 0$ for $A \in \mathcal{F}_{\zeta-}$, that is $\mathbb{R}$ is absolutely continuous with respect to $\mathbb{Q}$ on $\mathcal{F}_{\zeta-}$, in symbols $\mathbb{Q} \mid_{\mathcal{F}_{\zeta-}} \gg \mathbb{R}$.

Using this measure, we can prove the following theorem.

Proposition 2.2.7. *(Bubble Characterization for Continuous Price Processes)*

$$\mathbb{E}^{\mathbb{Q}}[S_t] = S_0 \mathbb{R}(\zeta > t)$$

Hence, S is a strict local martingale and exhibits a bubble if and only if $\mathbb{R}(\zeta \leq T) > 0$.

Proof. When M is continuous, $Z_t := \mathcal{E}(M_t)$ satisfies

$$Z_t = 0 \quad if \quad \zeta \leq t \leq T.$$

Now,

$$Z_{t\wedge\zeta_n} = Z_{t\wedge\zeta_n} 1_{t<\zeta_n} + Z_{t\wedge\zeta_n} 1_{\zeta_n \leq t} = Z_t 1_{t<\zeta_n} + Z_{\zeta_n} 1_{\zeta_n \leq t}.$$

Taking expectations,

$$1 = \mathbb{E}^{\mathbb{Q}}\left[Z_{t\wedge\zeta_n}\right] = \mathbb{E}^{\mathbb{Q}}\left[Z_t 1_{t<\zeta_n}\right] + \mathbb{E}^{\mathbb{Q}}\left[Z_{\zeta_n} 1_{\zeta_n \leq t}\right]$$

$$= \mathbb{E}^{\mathbb{Q}}\left[Z_t 1_{t<\zeta_n}\right] + \mathbb{E}^{\mathbb{Q}}\left[\frac{d\mathbb{R}_n}{d\mathbb{Q}} 1_{\zeta_n \leq t}\right] = \mathbb{E}^{\mathbb{Q}}\left[Z_t 1_{t<\zeta_n}\right] + \mathbb{R}_n\left(\zeta_n \leq t\right).$$

Using the definition of $\mathbb{R}$ gives

$$1 = \mathbb{E}^{\mathbb{Q}}\left[Z_t 1_{t<\zeta_n}\right] + \mathbb{R}\left(\zeta_n \leq t\right).$$

Taking limits we get

$$1 = \underset{n\to\infty}{lim} \mathbb{E}^{\mathbb{Q}}\left[Z_t 1_{t<\zeta_n}\right] + \underset{n\to\infty}{lim} \mathbb{R}\left(\zeta_n \leq t\right).$$

For the first term on the right side, note that $Z_t 1_{t<\zeta_n} \leq Z_t$ and $\mathbb{E}^{\mathbb{Q}}\left[Z_t\right] \leq 1$. By dominated convergence we get $\underset{n\to\infty}{lim} \mathbb{E}^{\mathbb{Q}}\left[Z_t 1_{t<\zeta_n}\right] = \mathbb{E}^{\mathbb{Q}}\left[Z_t 1_{t<\zeta}\right]$ because $\underset{n\to\infty}{lim} 1_{t<\zeta_n} = 1_{t<\zeta}$ a.s. $\mathbb{Q}$. But, $Z_t 1_{\zeta\leq t} = 0$. Hence,

$$\mathbb{E}^{\mathbb{Q}}\left[Z_t 1_{t<\zeta}\right] = \mathbb{E}^{\mathbb{Q}}\left[Z_t 1_{t<\zeta}\right] + \mathbb{E}^{\mathbb{Q}}\left[Z_t 1_{\zeta\leq t}\right] = \mathbb{E}^{\mathbb{Q}}\left[Z_t\right]$$

For the second term,

$$\underset{n\to\infty}{lim}\mathbb{R}\left(\zeta_n \leq t\right) = \mathbb{R}\left(\underset{n\to\infty}{lim}\zeta_n \leq t\right) = \mathbb{R}\left(\zeta \leq t\right).$$

Substitution yields

$$1 = \mathbb{E}^{\mathbb{Q}}\left[Z_t\right] + \mathbb{R}\left(\zeta \leq t\right).$$

Using the definition of $\frac{S_t}{S_0} = Z_t$, we get

$$\mathbb{E}^{\mathbb{Q}}\left[S_t\right] = S_0\mathbb{R}\left(t < \zeta\right).$$

$\square$

This proposition states that a bubble exists if the quadratic variation of the cumulative return process $[M]$ explodes under $\mathbb{R}$ before or at time T with positive probability.

We can use this characterization to generate additional examples of asset price bubble processes.

Example 2.2.8. (Stochastic Volatility)

Suppose that we have the following dynamics of the risky asset:

$$dS_t = S_t\sigma(Y_t)dW_t$$

where state variable Y is given by

$$dY_t = \mu^Y(t, Y_t)dt + \sigma^Y(t, Y_t)dW_t.$$

Then by Proposition 2.2.7 it follows that S is a strict local martingale if and only if

$$\mathbb{R}(\zeta > T) < 1$$

where Y is given by, under $\mathbb{R}$,

$$dY_t = (\mu^Y(t, Y_t) + \sigma(t, Y_t)\sigma^Y(t, Y_t))dt + \sigma^Y(t, Y_t)d\tilde{W}.$$

Example 2.2.9. (Return Process Explosion Time)

Suppose h is a continuous $\mathbb{R}$ martingale such that $h_t = \frac{d\mathbb{Q}}{d\mathbb{R}}|_{\mathcal{F}_t}$ and f is a continuous nonnegative $\mathbb{R}$ martingale. Define τ_0 the first hitting time of zero of process h. Then under $\mathbb{Q}$, the process $S_t = \frac{f_t}{h_t}\mathbb{1}_{\tau_0 > t}$ is a strict local martingale if $\mathbb{R}(\tau_0 \leq T) > 0$.

2.3 Optimal Stopping Times

This section studies a characterization of the risky asset's market price in a market with price bubbles. In this regard, we have the following proposition.

Proposition 2.3.1. *(The Risky Asset Market Price)*

$$S_t = ess \sup_{\tau \in \mathcal{T}(t,T)} \mathbb{E}^{\mathbb{Q}}[S_\tau|\mathcal{F}_t]$$

where $\mathcal{T}(t, T)$ is the collection of stopping times that are bounded between t and T.

Proof. Since S_t is a nonnegative local martingale under $\mathbb{Q}$, it is a supermartingale. This implies that $S_t \geq E^{\mathbb{Q}}[S_\tau\,|\mathcal{F}_t]$ for any stopping time τ. Hence, $S_t \geq \underset{\tau\in[t,T]}{ess\,sup}\, E^{\mathbb{Q}}\left[S_\tau\,|\mathcal{F}_t\right]$.

Conversely, consider a localizing sequence of stopping times $\tau^n \to T$ with $\tau^n \leq T$ for all n. Under this sequence, $S_{\tau^n \wedge t}$ is a martingale under $\mathbb{Q}$ for all n, hence $S_{\tau^n\wedge s} = E^{\mathbb{Q}}\left[S_{\tau^n\wedge t}\,|\mathcal{F}_s\right]$ for $s \leq t \leq T$. Assume $s < \tau^n \wedge t$, then $S_s = E^{\mathbb{Q}}\left[S_{\tau^n\wedge t}\,|\mathcal{F}_s\right]$.

For $t = T$, we have $S_s = E^{\mathbb{Q}}\left[S_{\tau^n\wedge T}\,|\mathcal{F}_s\right] = E^{\mathbb{Q}}\left[S_{\tau^n}\,|\mathcal{F}_s\right]$. Hence, for the stopping time $\tau^n \in [t, T]$ we have $S_t = E^{\mathbb{Q}}\left[S_{\tau^n}\,|\mathcal{F}_t\right]$. This implies $\underset{\tau\in[t,T]}{ess\,sup}\, E^{\mathbb{Q}}\left[S_\tau\,|\mathcal{F}_t\right] \geq S_t$, which completes the proof. $\square$

We call any stopping time that attains the essential supremum an optimal stopping time. If S is a true martingale under $\mathbb{Q}$, then for any stopping time τ, by the optional sampling theorem, we have $\mathbb{E}^{\mathbb{Q}}[S_\tau|\mathcal{F}_t] = S_t$. When there are no bubbles, every stopping time is optimal. However, if S is a strict local martingale, then there exists strictly sub-optimal stopping times. Indeed, an example is the deterministic stopping time $\tau \equiv T$. In this case, $S_t > E^{\mathbb{Q}}\left[S_T\,|\mathcal{F}_t\right]$ because S is a strict local martingale.

The proof of the previous theorem generates the following corollary.

Corollary 2.3.2. *(Localizing Sequence)*

If τ_n is a localizing sequence of stopping times, then τ_n is an optimal stopping time for all n.

We can identify a useful localizing sequence. Define a sequence of stopping times on the time domain $[0, T]$ as

$$\pi_n := \inf\{t \in [0,T] : |S_t| \geq n\}.$$

This sequence is strictly increasing if S_t is continuous or S_t has bounded jumps.

Proof. If S_t is continuous, then S_t cannot jump to $n+1$ before it hits n, so $\tau_{n+1} > \tau_n$.

If S_t has bounded jumps, then S_t can possibly jump to $n+1$ before it hits n, so $\tau_{n+1} = \tau_n$ is possible.

But, because the size of a jump is bounded by K, then $S_{\pi_n} + \Delta S_{\pi_n} \leq s_{\pi_n} + K$, so $\pi_{n+K+1} = \inf\{t \in [0,T] : |s_t| \geq n + K + 1\} > \pi_n$. Redefine the sequence on this subsequence to make it strictly increasing. This completes the proof. $\square$

Hence, because it is a non-decreasing sequence,

$$\pi_\infty := \lim_{n\to\infty} \pi_n \quad a.s. \quad \mathbb{Q}.$$

We have $S_{\pi_\infty} = lim_n S_{\pi_n} = \infty$ a.s. $\mathbb{Q}$. The process S_t is said to explode if $\pi_\infty \leq T$. But, S_t is a $\mathbb{Q}$ supermartingale and any stopping time $\pi_\infty \leq T$, by the optional sampling theorem, satisfies

$$lim_n E\left[S_{\pi_n}\right] = E\left[S_{\pi_\infty}\right] \leq S_0.$$

Hence, S_t does not explode under $\mathbb{Q}$, and

$$\lim_{n\to\infty} \pi_n = \infty \quad a.s. \quad \mathbb{Q}.$$

This implies that π_n is a localizing sequence for the local martingale S_t. By the corollary, the τ_n are optimal stopping times.

If S is continuous, proposition 2.2.7 provides a characterization of optimal stopping times.

Proposition 2.3.3. *(Characterization Optimal Stopping Times)*

Let S be a continuous process and $\mathbb{F}$ the filtration generated by S.

A stopping time $\tau \in \mathcal{T}(t,T)$ is optimal if and only if

$$\mathbb{R}(\zeta > \tau) = 1.$$

Recall the stopping time $\zeta_n := \inf\{t \in [0,T] : [M]_t \geq n\}$ with respect to the quadratic variation of the cumulative return process M. Under the hypothesis of this proposition, M is continuous with $M_0 = 1$. This implies that $\zeta_n < \zeta$ a.s. $\mathbb{Q}$ for all n, which also implies that $\zeta_n < \zeta$ a.s. $\mathbb{R}$, because $\{\zeta_n < \zeta\} \in \mathcal{F}_{\zeta-}$ and $\mathbb{Q}\mid_{\mathcal{F}_{\zeta-}} \gg \mathbb{R}$. Hence, the ζ_n are optimal stopping times.

2.4 Simple Trading Strategies

This section studies simple trading strategies, where a simple trading strategy is defined to be a buy and sell over $[0, \tau]$ where $\tau \leq T$ is a stopping time. More formally, a simple trading strategy is given by

$$\alpha_t := \mathbb{1}_{[0,\tau)}(t) \qquad \text{and}$$

$$\alpha_0(t) := S_{t\wedge\tau} - S_t \mathbb{1}_{[0,\tau)}(t)$$

with the wealth process $X_t = S_{t\wedge\tau}$ for $t \in [0, T]$, where the initial wealth invested is $x = S_0$. As defined, we see that this trading strategy is self-financing and admissible, its lower bound is zero because $S_t \geq 0$.

We note that the subsequent analysis applies equally well to a finite number of buys and sells over the time period.

Remark 2.4.1. (Naked Short Selling)

Simple trading strategies involving a short and cover are not admissible and excluded from the preceding definition. Indeed, if S is a strict local martingale, then it is unbounded above. Hence, $-S$ is unbounded below, and the s.f.t.s. $(\alpha_t := -\mathbb{1}_{[0,\tau)}(t),\ \alpha_0(t) := S_{t\wedge\tau} - S_t \mathbb{1}_{[0,\tau)}(t))$ with wealth process $X_t = -S_{t\wedge\tau}$ is not admissible.

An example of such a simple trading strategy is shorting the stock and covering the position at time T, after the bubble bursts. Because it violates admissibility, it is not a feasible trading strategy in practice since it implies unlimited borrowing with positive probability. The violation of admissibility explains why this short and cover trading strategy over $[0, T]$, in an arbitrage-free market, cannot remove asset price bubbles.

2.4.1 No Bubbles

If S has no bubbles and is a $\mathbb{Q}$ martingale, then all such admissible simple trading strategies, by the optional sampling theorem, are $\mathbb{Q}$ martingales, and hence wealth preserving. There can exist no dominating s.f.t.s. because the existence of an equivalent martingale measure implies that the market satisfies ND by the third fundamental theorem of asset pricing (see [34]).

2.4.2 Bubbles

This section explores admissible simple trading strategies when the risky asset exhibits a bubble. First, it is easy to see that buying and holding the risky asset over $[0, T]$ is wealth decreasing. Indeed, the initial cost is S_0, and the present value of the trading strategy is equal to $\mathbb{E}^{\mathbb{Q}}[S_T] < S_0$ because the asset is a $\mathbb{Q}$ strict supermartingale. As such, it is a suboptimal trading strategy.

However, we now show that there exist simple trading strategies that are wealth preserving, and in the process, this demonstration provides an economic interpretation of the optimal stopping times in Section 2.3 that generate a localizing sequence for the $\mathbb{Q}$ local martingale S.

With respect to the simple trading strategies just defined, when a "sell" occurs at an optimal stopping time that is an element of a localizing sequence, the trading strategy is wealth preserving. This follows because on the time interval $[0, \tau_n]$, the risky asset is a $\mathbb{Q}$ local martingale, so that the wealth process X satisfies

$$\mathbb{E}^{\mathbb{Q}}[X_T] = \mathbb{E}^{\mathbb{Q}}[S_{\tau_n}] = S_0 = x.$$

Consequently, the wealth process of such a simple trading strategy is a $\mathbb{Q}$ martingale. Interestingly, however, it is not a dominating trading strategy because the time T payoff, $X_T = S_{\tau_n}$, may not be greater than or equal to S_T at time T with $\mathbb{Q}$ probability one.

For risky asset processes that are continuous or have bounded jumps, one such wealth preserving simple trading strategy is to buy the risky asset at time t and sell it the first time the risky asset price hits the level n, i.e. at the stopping time $\pi_n := \inf\{t \in [0,T] : |S_t| \geq n\}$.

Another wealth preserving simple trading strategy for continuous price processes is to buy the risky asset at time t and sell it the first time the quadratic variation of the cumulative return process hits the level n, i.e. at the stopping time $\zeta_n := \inf\{t \in [0,T] : [M]_t \geq n\}$. Note that for all of these simple trading strategies, the selling time must occur strictly before time T, and consequently, they are not dominating.

2.5 Replicating and Super-replicating Admissible S.F.T.S.

This section studies wealth preserving and dominating admissible s.f.t.s. that replicate or super-replicate the risky asset's time T payoffs. In a market with no bubbles, a buy and hold trading strategy over $[0,T]$ replicates the asset's time T payoffs, and is wealth preserving. And, as is well known, in a market with no bubbles, there are no dominating admissible s.f.t.s.

In contrast, as shown in the previous section, this is not true in a market with bubbles. When the asset exhibits a bubble, the simple buy and hold trading strategy that generates the asset's time T payoff is wealth decreasing. This section shows that there exist dominating trading strategies that replicate or super-replicate the asset's time T payoffs.

For this section we assume that S exhibits a price bubble and is a $\mathbb{Q}$ strict local martingale for some $\mathbb{Q} \in \mathcal{M}$. There are two cases to study: complete and incomplete markets.

2.5.1 Complete Markets

We assume that the market is complete with respect to some $\mathbb{Q} \in \mathcal{M}$, hence, by the second fundamental theorem of asset pricing the ELMM is unique. In such a market, No Dominance (ND) is violated because by market completeness there exists an admissible s.f.t.s $(\alpha_0, \alpha) \in \mathscr{A}(x)$ with initial investment x where

$$x + \int_0^T \alpha_u dS_u = S_T \tag{2.2}$$

and the wealth process

$$X_t = \alpha_0(t) + \alpha_t S_t = x + \int_0^t \alpha_u dS_u \tag{2.3}$$

is a $\mathbb{Q}$ martingale, which implies that $x = \mathbb{E}^{\mathbb{Q}}[S_T] = \tilde{S}_0$. Hence, the trading strategy is wealth preserving. Since $\mathbb{Q}$ is a strict local martingale, we have that $x = \mathbb{E}^{\mathbb{Q}}[S_T] = \tilde{S}_0 < S_0$, which shows that it is also a dominating trading strategy.

The remainder of this section characterizes the standard buy and hold trading strategy. Because $\mathbb{Q} \in \mathcal{M}$ is unique, the fundamental value $\tilde{S}_t$ and the bubble β_t are uniquely defined, where $\tilde{S}_t = \mathbb{E}^{\mathbb{Q}}[S_T \mid \mathcal{F}_t]$ and $\beta_t = S_t - \tilde{S}_t \geq 0$. First, note that this implies $X_t = \tilde{S}_t$, hence we can rewrite expression (2.3) as

$$\tilde{S}_t = \tilde{S}_0 + \int_0^t \alpha_u dS_u = \alpha_0(t) + \alpha_t S_t. \tag{2.4}$$

Therefore, the bubble process can be written as

$$\begin{aligned}\beta_t = S_t - \tilde{S}_t &= \beta_0 + \int_0^t (1-\alpha_u) dS_u \\ &= \beta_0 + \int_0^t \alpha_u^\beta dS_u = \alpha_0^\beta(t) + \alpha_t^\beta S_t\end{aligned}$$

where $\alpha^\beta \coloneqq 1 - \alpha$ and $\alpha_0^\beta \coloneqq -\alpha_0$.[1]

This implies that the buy-and-hold trading strategy in the risky asset can be decomposed into two parts:

- the position that generates the fundamental value $(\alpha_0, \alpha) \in \mathscr{A}(\tilde{S}_0)$, and
- the position that generates the bubble $(\alpha_0^\beta, \alpha^\beta) \in \mathscr{A}(\beta_0)$. Notice that $\beta_T = 0$ a.s. but $\beta_0 > 0$, so α^β corresponds to a suicide strategy.

We can use this observation to generate bubble processes that depend on the fundamental value process.

Example 2.5.1. (Suicide Strategies and Bubbles).

Bubble processes can be constructed that depend on the fundamental value in the following manner. Assume that $\tilde{S}$ follows a geometric Brownian motion, which is a true martingale. Additionally, assume that the bubble process is of the form

$$\beta_t = \int_0^t \tilde{\alpha}_u^\beta d\tilde{S}_u.$$

We can construct an admissible s.f.t.s. $\tilde{\alpha}^\beta$ that is a suicide strategy as follows. First, partition $(0,T) = \bigcup_{i=1}^\infty (T_{i-1}, T_i]$ where $T_i = \left(1 - \frac{1}{2^i}\right) T$. For each of these time intervals, choose $\tilde{\alpha}_i^\beta$ such that the probability of the wealth process β_t hitting zero is $p_i > 0$. Then, we have the following result.

---

[1] This follows from $\beta_t + \tilde{S}_t = S_t$ and $\alpha^\beta \coloneqq 1 - \alpha$.

The process $\tilde{\alpha}_i^{\beta}$ is a suicide strategy if and only if

$$\prod_i (1 - p_i) = 0.$$

Hence,

$$S_t = \tilde{S}_t + \int_0^t \tilde{\alpha}_u^{\beta} d\tilde{S}_u$$

is a risky asset process with a bubble.

The following corollary characterizes this replicating trading strategy for continuous Markov processes. It is a direct consequence of expression (2.4), recognizing that $\tilde{S}_t = \mathbb{E}^{\mathbb{Q}}[S_T|\mathcal{F}_t]$.

Corollary 2.5.2. *(Characterization of the Replicating Trading Strategy)*

If S is a continuous Markov process, and $\mathbb{E}^{\mathbb{Q}}[S_T|S_t]$ is *a $C^{1,2}$ function of time t and risky asset price S_t, then*

$$\alpha_t = \frac{\partial \mathbb{E}^{\mathbb{Q}}[S_T|S_t]}{\partial S_t}.$$

Proof. If S is a Markov process, then $\tilde{S}_t = \mathbb{E}^{\mathbb{Q}}[S_T|\mathcal{F}_t] = \mathbb{E}^{\mathbb{Q}}[S_T|S_t]$. Ito's lemma gives

$$d\tilde{S}_t = \left(\frac{\partial \tilde{S}_t}{\partial t} + \frac{1}{2} \frac{\partial^2 \tilde{S}_t}{\partial S_t^2} \right) d\left\langle S, S \right\rangle_t + \frac{\partial \tilde{S}_t}{\partial S_t} dS_t.$$

Equating the integrand of dS_t in expression (2.4) completes the proof. □

As shown, for a continuous Markov process, the replicating trading strategy is the partial derivative of the asset's conditional expected value with respect to the asset's price at time t. The next question we study is whether the replicating trading strategy $(\alpha_0, \alpha) \in \mathscr{A}(\tilde{S}_0)$ in expression (2.2) has a non-zero position in the underlying asset for all times, i.e. it is not a simple wealth preserving trading strategy as discussed in Section 2.4. In this regard, we have the following result.

Theorem 2.5.3. *(Non-zero Holdings in the Asset)*

Assume $\tilde{S}$ is a $\mathcal{H}^1$ martingale. Then,

$$\alpha_t \neq 0, \mathbb{Q} \times \lambda[0, T]$$

when $[\tilde{S}]_t$ increases, and the ratio $\frac{\alpha^\beta}{\alpha}$ is unbounded where λ is the Lebesgue measure.

Proof. By associativity of stochastic integration, one can verify that

$$dS_t = \frac{1}{\alpha_t} d\tilde{S}_t.$$

Because S_t is finite a.e., this implies $\alpha_t \neq 0, \mathbb{Q} \times \lambda[0, T]$ when $[\tilde{S}]_t$ increases.

Next, we have

$$\beta_t = \int_0^t \frac{\alpha_u^\beta}{\alpha_u} d\tilde{S}_u.$$

By definition, we have

$$\mathbb{E}^{\mathbb{Q}}[[\tilde{S}]_t^{1/2}] < \infty, \quad \forall t$$

since $\tilde{S}$ is a $\mathcal{H}^1$ martingale. Suppose the integrand $\frac{\alpha^\beta}{\alpha}$ is bounded by $K > 0$, we have from Burkholder-Davis-Gundy inequalities that there exists $C_1 > 0$ such that

$$\begin{aligned}\mathbb{E}^{\mathbb{Q}}[\sup_{0 \leq u \leq t} |\beta_u|] &\leq C_1 \mathbb{E}^{\mathbb{Q}}[[\beta_t]^{1/2}] \\ &= C_1 \mathbb{E}^{\mathbb{Q}}[[\int \frac{\alpha^\beta}{\alpha} d\tilde{S}]^{1/2}] \\ &\leq C_1 K \mathbb{E}[[\tilde{S}_t]^{1/2}] < \infty\end{aligned}$$

Hence β is a local martingale of class (DL), so is a proper martingale, contradiction. Therefore, process $\frac{\alpha^\beta}{\alpha}$ is unbounded. □

Hence, these replicating trading strategies do not include the simple wealth preserving trading strategies in Section 2.4, which have a zero holding in the risky asset after time τ and before T. The result that the ratio $\frac{\alpha^\beta}{\alpha}$ is unbounded implies, recalling that $S_t = S_t(\alpha_t^\beta + \alpha_t)$, that the proportion of the asset's price represented by the bubble is unbounded above.

From the construction of the suicide strategy, we see that α^β could be negative or positive. As a result, the replicating trading strategy's position in the risky asset may be greater than or smaller than 1 share, and sometimes it is a negative holding in the risky asset! The latter case happens if the bubble is larger than the fundamental value. However, for a class of diffusion models we can show that this replicating trading strategy's holding in the risky asset is positive and less than one.

Theorem 2.5.4. *(Time-Homogeneous Diffusion)*

Suppose that the risky asset S with state-space $\mathbb{R}^+$ is governed by the following SDE

$$dS_t = \sigma(S_t)dW_t,\ S_0 \in [0, \infty)$$

$$S_t \equiv 0,\ t \geq \tau_0 := \inf\{t \in [0, \infty) : S_t = 0\}$$

where W_t is a Brownian motion and $\sigma : \mathbb{R}^+ \mapsto \mathbb{R}^+$ is a continuous function such that $\sigma(x) > 0, \forall x \in (0, \infty)$.

Assume further that

$$\tilde{S}_t = \mathbb{E}^{\mathbb{Q}}[S_T | \mathcal{F}_t] := f(t, S_t)$$

is a $C^{1,2}$ function of time t and risky asset price S_t. Then,

$$0 \leq \alpha_t \leq 1 \ a.s.$$

for all $t \in [0, T]$.

Proof. Since the process is continuous and Markov, by Corollary 2.5.2, it suffices to show that

$$0 \leq \frac{\partial}{\partial x} f(t,x) \leq 1.$$

To show $\frac{\partial}{\partial x} f(t,x) \geq 0$, notice that by the comparison theorem (see [38], p. 293) we have $f(t,x)$ is increasing in x for any fixed t.

To show $\frac{\partial}{\partial x} f(t,x) \leq 1$, it suffices to show that $f(t,x)$ is concave in x for any fixed t. The time-homogenous diffusion possesses a non-exploding solution (see [38], p. 332). And, the Feynman-Kac theorem is valid in such case (see the Appendix). Note that

$$f(t,x) = \mathbb{E}^{\mathbb{Q}}[S_T | S_t = x]$$

where f is the minimal solution of the following partial differential equation (the solution to which is not unique if S is a strict local martingale)

$$\partial_t f(t,x) + \frac{\sigma^2(x)}{2} \partial_{xx} f(t,x) = 0.$$

Because stochastic differential equation is homogeneous, we have $f(t,x) = \mathbb{E}[S_T | S_t = x] = \mathbb{E}[S_{T-t} | S_0 = x]$. By supermartingale property of S, $f(t,x)$ is non-decreasing in t for all x. Thus $\partial_t f(t,x) \geq 0$. The Feynman-Kac partial differential equation implies that $\partial_{xx} f(t,x) \leq 0$ and thus f is concave and

$$\partial_x f(t,x) \leq 1$$

since $0 \leq f(t,x) \leq x$. This completes the proof. □

Example 2.5.5. (CEV Process)

The CEV process evolution $dS_t = S_t^{\delta+1} dW_t$ with $\delta > 0$ is a strict local martingale. From [43], we have that

$$f(t,x) = \mathbb{E}^{\mathbb{Q}}[S_T | S_t = x] = xG\left(\upsilon, \frac{x^{-2\delta}}{2\delta^2(T-t)}\right)$$

where $\upsilon = \frac{1}{2\delta}$ and

$$G(\upsilon, x) = \frac{\int_0^y u^{\upsilon-1} e^{-u} du}{\Gamma(\upsilon)}.$$

The trading strategy can be explicitly computed as

$$\alpha_t = \frac{\partial}{\partial x} f(t,x) = xG\left(\upsilon, \frac{x^{-2\delta}}{2\delta^2(T-t)}\right) - \frac{x^{\alpha} e^{-\frac{x^{-2\delta}}{2\delta^2(T-t)}}}{\Gamma(\upsilon)\delta(T-t)} < 1$$

where $\alpha = \frac{2\delta-1}{2\delta^2(T-t)} - 2\delta$.

Note that as $t \uparrow T$, $\lim_{t\to T} \frac{\partial}{\partial x} f(t,x) = 1$ for all $x > 0$.

We can also obtain the following corollary.

Corollary 2.5.6. *(Time-Homogeneous Diffusions)*

There does not exist a time-homogeneous diffusion bubble process β.

Proof. Suppose β is a nonnegative, strict local martingale such that $\beta_T = 0$ *a.s.* and $\beta_0 > 0$, then we have for any $t \in (0, T)$,

$$\mathbb{E}[\beta_t | \beta_0] = \mathbb{E}[\beta_T | \beta_{T-t} = \beta_0] = 0.$$

Hence, $\beta_t = 0$ *a.s.* $\forall t > 0$, contradicting the continuity of diffusion process. □

This means that either the market price or the fundamental value process has to be a time-inhomogeneous diffusion. We can extend Theorem 2.5.4 to a time-inhomogeneous diffusion using the explosion time characterization of a strict local martingale.

Theorem 2.5.7. *(Trading Strategy within [0,1])*

Let

$$dS_t = S_t \sigma(t, S_t) dW_t$$

where $\sigma(t,x)$ is a positive $C^{1,2}$ function such that for any given t, $\sigma(t,x)$ is an increasing function in x. Let $\mathbb{F}$ be the filtration generated by S. Define

$$R(t,x) \coloneqq \mathbb{R}(\zeta > T | S_t = x).$$

where from Proposition 2.2.7

$$\tilde{S}_t = S_t R(t, S_t).$$

Assume $R(t,\cdot) \in C^1$ for all t. Then,

$$\alpha_t \in [0,1], \forall t \in [0,T].$$

Proof. By the same proof as in Theorem 2.5.4, we have $\alpha_t \geq 0$. To show $\alpha_t \leq 1$, since

$$\alpha_t = \frac{\partial(xR(t,x))}{\partial x}|_{x=S_t} = R(t,S_t)z + S_t \partial_x R(t,S_t),$$

it suffices to show that $R(t,\cdot)$ is a decreasing function. Abusing notation, we define the process $\sigma_t = \sigma(t,S_t)$. By Ito's formula and Girsanov's theorem, under the measure $\mathbb{R}$,

$$d\sigma_t = \left(\partial_t \sigma(t,S_t) + \frac{1}{2} \left(\partial_{xx}\sigma(t,S_t)\right) \sigma^2(t,S_t) \right) dt + \left(\partial_x \sigma(t,S_t)\right) \sigma(t,S_t) dW_t^{\mathbb{R}} \quad (2.5)$$

where $W^{\mathbb{R}}$ is a $\mathbb{R}$-Brownian motion. For any fixed t and $x_1 \leq x_2$, by a comparison theorem for stochastic differential equations (see p. 293 in [38]) we have

$$\sigma_u^{t,y_1} \geq \sigma_u^{t,y_2}, \; \forall u \geq t$$

where $\{\sigma_u^{t,y_i}\}_{u \in [t,T]}$ is the solution to the stochastic differential equation 2.5 with initial value at time t, $y_i = \sigma(t,x_i)$. Because σ_t is a continuous process, $\zeta = \lim_n \{u : \sigma_u \geq n\}$. Hence, we have $R(t,x_1) \geq R(t,x_2)$ since by the comparison result, the process σ^{t,y_1} explodes before σ^{t,y_2}. □

Example 2.5.8. (Complete Market with Stochastic Volatility)

Consider the price evolution

$$dS_t = S_t \sigma_t^3 dW_t$$

$$d\sigma_t = \sigma_t dW_t$$

By the Markov property,

$$\tilde{S}_t = \mathbb{E}[S_T | \mathcal{F}_t] = g(t, S_t, \sigma_t)$$

where $g(t,x,v) = \mathbb{E}[S_t | S_t = x, \sigma_t = v]$. Then, by Theorem 2.5.7 we have

$$g(t,x,v) = xR(T-t, v)$$

where $R(t,v) = \mathbb{R}(\zeta > t | \sigma_0 = v)$ and σ_t follows

$$d\sigma_t = \sigma_t^3 dt + \sigma_t dW_t^{\mathbb{R}}$$

with $W^{\mathbb{R}}$ a $\mathbb{R}$-Brownian motion.[2] It can be shown that

$$R(t,v) = 2N(1/v\sqrt{t}) - 1$$

where $N(\cdot)$ is the cumulative distribution function of a standard normal distribution.

Finally, by Ito's formula and the fact that $\tilde{S}$ is a local martingale (so the dt term is zero) we have

$$d\tilde{S} = R(T-t, \sigma_t) dS_t + S_t \frac{\partial_v R(T-t, \sigma_t)}{\partial \sigma_t} d\sigma_t$$

Substituting $d\sigma_t = \sigma_t dW_t = \sigma_t \left(\frac{dS_t}{S_t \sigma_t^3} \right)$ into the above expression we have:

$$d\tilde{S}_t = R(T-t, \sigma_t) dS_t + \frac{\partial_v R(T-t, \sigma_t)}{\sigma_t^2} dS_t$$

---

[2] The conditional probability of explosion of the quadratic variation, $R(t,v)$, only depends on the volatility because the evolution of the volatility does not depend on the asset's price.

so

$$\alpha_t = R(T-t,\sigma_t) + \frac{\partial_v R(T-t,\sigma_t)}{\sigma_t^2}.$$

The above results applies to only Markov processes, but for a large class of non-Markov models we could also specify the wealth-preserving strategy by use of Malliavin calculus.

Example 2.5.9. (The Dominating Trading Strategy)

Consider the following model in a complete financial market,

$$d\tilde{S}_t = \tilde{S}_t \sigma(\tilde{S}_t)\left(\theta(\beta_t,\tilde{S}_t)dt + dW_t\right)$$

$$d\beta_t = \sigma^{\beta}(t,\beta_t)(\theta(\beta_t,\tilde{S}_t) + dW_t)$$

where β is the bubble process such that $\beta_T = 0,\ a.s.$ and θ is the market price of risk. In this model, the market price of risk depends on both the fundamental value and the bubble.

Then, we have using the Clark-Ocone formula that

$$S_T = \mathbb{E}^{\mathbb{Q}}[S_T] + \int_0^T \mathbb{E}^{\mathbb{Q}}[\mathcal{D}_t S_T - S_T \int_t^T \mathcal{D}_t \theta_s d\tilde{W}_s | \mathcal{F}_t] d\tilde{W}_t$$

where for any t, $\mathcal{D}_t$ is the Malliavin derivative operator (see for example [50]). By the chain rule of Malliavin calculus we have that

$$\mathcal{D}_t \theta_s = \theta'(\beta_s)\mathcal{D}_t \beta_s.$$

And, $\mathcal{D}_t \beta_s$ has the following SDE:

$$d(\mathcal{D}_t \beta_s) = \partial_x \sigma^{\beta}(t,\beta_t)(\mathcal{D}_t \beta_s)dW_s.$$

In this model, β is the state variable that drives the evolution of the market.

Finally, in this case, we have

$$\alpha_t = \frac{1}{\tilde{S}_t\sigma(\tilde{S}_t) + \sigma^\beta(t,\beta_t)}\left[\mathbb{E}^{\mathbb{Q}}[\mathcal{D}_t S_T - S_T\int_t^T \mathcal{D}_t\theta_s d\tilde{W}_s|\mathcal{F}_t]\right].$$

Example 2.5.10. (Counter Examples to $\alpha_t \in [0,1]$)

1. ($\alpha_t < 0$) Suppose that the fundamental value and bubble are given by the following processes

$$d\beta_t = \sigma_t^\beta dW_t^{\mathbb{Q}}$$

$$d\tilde{S}_t = \sigma_t^{\tilde{S}} dW_t^{\mathbb{Q}}$$

where

$$\sigma_t^\beta = -2\sqrt{T}\frac{(1+\tilde{S}_t)\beta_t}{\sqrt{T-t}}$$

and

$$\sigma_t^{\tilde{S}} = \tilde{S}_t(\beta_t \wedge 1).$$

By Novikov's condition, $\tilde{S}$ is a proper positive martingale. And, β is a local martingale. Furthermore, $\beta_T = 0$ *a.s.* so it is a strict local martingale, hence a bubble process. Note that β is also a positive proper martingale on $[0,T)$, since $\tilde{S}$ does not explode on $[0,T]$.

Both σ^β and $\sigma^{\tilde{S}}$ are non-zero and have $\frac{\sigma^\beta}{\sigma^{\tilde{S}}} < -2, \forall t \in [0,T)$. This shows that

$$dS_t = \left(\frac{\sigma_t^\beta}{\sigma_t^{\tilde{S}}} + 1\right) d\tilde{S}_t,$$

which implies that

$$d\tilde{S}_t = \frac{1}{\frac{\sigma_t^\beta}{\sigma_t^{\tilde{S}}} + 1} dS_t.$$

Hence, $\alpha_t = \frac{1}{\frac{\sigma_t^\beta}{\sigma_t^{\tilde{S}}}+1}$, which is strictly negative throughout the whole period, i.e. the wealth preserving and dominating trading strategy has a negative position in the risky asset over $[0,T)$.

2. ($\alpha_t > 1$) Suppose that $\tilde{S}$ is a geometric Brownian motion with unit volatility and $S_0 = 1$. Define the bubble process β as follows:

$$d\beta_t = -(\beta_t \wedge \frac{1}{3})dW_t^{\mathbb{Q}}, t \in [0, (T/2) \wedge \tau_{1/2}^{\tilde{S}})$$

$$d\beta_t = \frac{\beta_t}{\sqrt{T-t}}dW_t^{\mathbb{Q}}, t \in [(T/2) \wedge \tau_{1/2}^{\tilde{S}}, T]$$

where $\tau_{1/2}^{\tilde{S}} = \inf\{t : \tilde{S}_t \leq 1/2\}$. The choice of $\tau_{1/2}^{\tilde{S}} = \inf\{t : \tilde{S}_t \leq 1/2\}$ guarantees that the volatility of S is never zero so the market is complete. By construction, the volatility of S is smaller than $\tilde{S}$, which implies using expression (2.4) that $\alpha > 1$ on $[0, (T/2) \wedge \tau_{1/2}^{\tilde{S}})$.

This completes the counter examples.

Let S be a continuous Markov process with $\mathbb{F}$ the filtration generated by S, and define

$$R(t, x) := \mathbb{R}(\zeta > T | S_t = x).$$

Recall from Proposition 2.2.7 that

$$\tilde{S}_t = x\mathbb{R}(\tau > T | S_t = x) = \alpha_0(t) + \alpha_t x.$$

We can prove the following result.

Proposition 2.5.11. *(Rate of Change of Percentage Holdings in the Risky Asset)*

Assume $R(t, x)$ is differential in x for all $t \in [0, T]$ a.s. $\mathbb{Q}$. Then,

$$\frac{\alpha_t S_t}{\tilde{S}_t} = 1 - S_t \frac{d \log R(t, S_t)}{dS_t}.$$

Proof. Given $x\mathbb{R}(\tau > T | S_t = x) = \alpha_0(t) + \alpha_t x$, taking a derivative with respect to x yields

$$\alpha_t = R(t, x) - x\frac{dR(t, x)}{dx}.$$

But $\alpha_t S_t = \alpha_t x = xR(t,x) - x^2 \frac{dR(t,x)}{dx}$. Hence,

$$\frac{\alpha_t S_t}{\tilde{S}_t} = \frac{xR(t,x) - x^2 \frac{dR(t,x)}{dx}}{xR(t,x)} = 1 - x\frac{d \log R(t,x)}{dx}.$$

$\square$

This proposition implies that as the market price $S_t = x$ increases, if $\frac{d \log R(t,x)}{dx}$ increases, then the holdings in the risky asset as a percent of the fundamental value, within the replicating trading strategy, decreases. But, $\frac{d \log R(t,x)}{dx}$ measures the percentage change in the size of the bubble, so as the bubble percentage increases, the holdings in the risky asset as a percent of the fundamental value decline.

2.5.2 Incomplete Markets

This section studies super-replicating admissible s.f.t.s. in an incomplete market. Because the market is incomplete, the set of ELMMs contains an infinite number of elements. We assume that the market selects a particular ELMM, $\tilde{\mathbb{Q}} \in \mathcal{M}$, for use in defining the asset's fundamental value. The asset's price bubble is defined with respect to $\tilde{\mathbb{Q}} \in \mathcal{M}$.

In this incomplete market, for any $\mathbb{Q} \in \mathcal{M}$, the risky asset price process S is a non-negative $\mathbb{Q}$ local martingale, hence a supermartingale. This implies $\mathbb{E}^{\mathbb{Q}}[S_T] \leq S_0$ for all $\mathbb{Q} \in \mathcal{M}$, hence, $\sup_{\mathbb{Q}\in\mathcal{M}} \mathbb{E}^{\mathbb{Q}}[S_T] \leq S_0$. Define the process

$$Y_t := \operatorname*{ess\,sup}_{\mathbb{Q}\in\mathcal{M}} \mathbb{E}^{\mathbb{Q}}[S_T|\mathcal{F}_t]$$

where $Y_0 = \sup_{\mathbb{Q}\in\mathcal{M}} \mathbb{E}^{\mathbb{Q}}[S_T] \leq S_0$ and $Y_T = S_T$. Y is a supermartingale for all $\mathbb{Q} \in \mathcal{M}$. By the Optional Decomposition Theorem (see [17]), there exists a non-decreasing cadlag adapted process C with $C_0 = 0$ and an $\alpha \in \mathscr{L}(S)$ such that

$$Y_t = Y_0 + \int_0^t \alpha dS_t - C_t \tag{2.6}$$

where $\int_0^t \alpha dS_t$ is a $\mathbb{Q}$ local martingale for all $\mathbb{Q} \in \mathcal{M}$.

Consider the trading strategy represented by $(\alpha_0(t), \alpha_t)$ with α as given in expression (2.6), $\alpha_0(t) \coloneqq \tilde{Y}_t - \alpha_t S_t$, and the wealth process

$$\tilde{Y}_t = \sup_{\mathbb{Q} \in \mathcal{M}} \mathbb{E}^{\mathbb{Q}}[S_T] + \int_0^t \alpha dS_t = \alpha_0(t) + \alpha_t S_t. \tag{2.7}$$

This trading strategy $(\alpha_0(t), \alpha_t) \in \mathcal{A}(\sup_{\mathbb{Q} \in \mathcal{M}_l} \mathbb{E}^{\mathbb{Q}}[S_T])$ because it is self-financing by construction, and it is admissible since $\tilde{Y}_t = Y_t + C_t \geq 0$ a.s. For this admissible s.f.t.s. we have

$$Y_t = \tilde{Y}_t - C_t$$

which implies that Y represents the wealth process of an admissible trading strategy with a cumulative cash outflow of C_t over $[0, t]$. The trading strategy $(\alpha_0(t), \alpha_t) \in \mathcal{A}(\sup_{\mathbb{Q} \in \mathcal{M}} \mathbb{E}^{\mathbb{Q}}[S_T])$ is said to super-replicate the risky asset's time T payoff because

$$\tilde{Y}_T = S_T + C_T \geq S_T \quad a.s.$$

(see [32], Chapter 8 for more details on super-replication trading strategies).

Hence, the trading strategy $(\alpha_0(t), \alpha_t) \in \mathcal{A}(\sup_{\mathbb{Q} \in \mathcal{M}} \mathbb{E}^{\mathbb{Q}}[S_T])$ is a dominating trading strategy if $\tilde{\mathbb{Q}}(\tilde{Y}_T = Y_T + C_T > Y_T) > 0$ because its initial cost $\tilde{Y}_0 = \sup_{\mathbb{Q} \in \mathcal{M}} \mathbb{E}^{\mathbb{Q}}[S_T] \leq S_0$ and $\tilde{\mathbb{Q}}(\tilde{Y}_T = Y_T + C_T \geq Y_T) = 1$ where $\tilde{\mathbb{Q}} \in \mathcal{M}$ is the ELMM chosen by the market under which S is a strict local martingale. It is wealth preserving if $\tilde{Y}_t$ is a $\tilde{\mathbb{Q}}$ martingale. These insights are formalized in the following proposition.

Proposition 2.5.12. *(Wealth Preserving and Dominating s.f.t.s.)*

The super-replicating admissible s.f.t.s. $(\alpha_0(t), \alpha_t) \in \mathcal{A}(\sup_{\mathbb{Q}\in\mathcal{M}} \mathbb{E}^{\mathbb{Q}}[S_T])$ *from expression (2.7) with wealth process*

$$\tilde{Y}_t = \sup_{\mathbb{Q}\in\mathcal{M}} \mathbb{E}^{\mathbb{Q}}[S_T] + \int_0^t \alpha dS_t = \alpha_0(t) + \alpha_t S_t$$

(1) is dominating iff $\tilde{\mathbb{Q}}(\tilde{Y}_T > S_T) > 0$*, and*

(2) is wealth preserving iff $\tilde{Y}_t$ *is a* $\tilde{\mathbb{Q}}$ *martingale*

where $\tilde{\mathbb{Q}} \in \mathcal{M}$ *is the ELMM chosen by the market under which* S *is a strict local martingale.*

The following corollary characterizes this super-replicating trading strategy for continuous Markov processes.

Corollary 2.5.13. *(Characterization of the Super-replicating Trading Strategy)*

If S *is a continuous Markov process, and* $\operatorname{ess\,sup}_{\mathbb{Q}\in\mathcal{M}} \mathbb{E}^{\mathbb{Q}}[S_T|S_t]$ is *a* $C^{1,2}$ *function of time* t *and risky asset price* S_t*, then*

$$\alpha_t = \frac{\partial \left(ess \sup_{\mathbb{Q}\in\mathcal{M}} \mathbb{E}^{\mathbb{Q}}[S_T|S_t]\right)}{\partial S_t}.$$

Proof. If S is a Markov process, then $\mathbb{E}^{\mathbb{Q}}[S_T|\mathcal{F}_t] = \mathbb{E}^{\mathbb{Q}}[S_T|S_t]$.

Ito's lemma on expression $Y_t := \operatorname{ess\,sup}_{\mathbb{Q}\in\mathcal{M}_l} \mathbb{E}^{\mathbb{Q}}[S_T|S_t]$ gives

$$dY_t = \left(\frac{\partial Y_t}{\partial t} + \frac{1}{2}\frac{\partial^2 Y_t}{\partial S_t^2}\right) d\left\langle S, S\right\rangle_t + \frac{\partial Y_t}{\partial S_t} dS_t.$$

Equating the integrand of dS_t in expression (2.6) completes the proof. □

As shown, for a continuous Markov process, the super replicating trading strategy is just the partial derivative of the essential supremum of the conditional expected value with respect to the asset's price at time t.

Remark 2.5.14. (Essential Supremum Attained)

In general, if the set of uniformly integrable martingale measures is non-empty, then it is always true that there exists a $\mathbb{Q} \in \mathcal{M}$ such that

$$Y_0 := \sup_{\mathbb{Q} \in \mathcal{M}} \mathbb{E}^{\mathbb{Q}}[S_T] = S_0.$$

Indeed, clearly we have $Y_0 \leq S_0$ by the supermartingale property. If $\mathbb{Q}_{UI}$ is a uniformly integrable martingale measure, then we have $Y_0 = \sup_{\mathbb{Q} \in \mathcal{M}} \mathbb{E}^{\mathbb{Q}}[S_T] \geq \mathbb{E}^{\mathbb{Q}_{UI}}[S_T] = S_0$. Note that it is possible that $Y_0 = S_0$ and there is no true martingale measure (see Example 3.7 in [22]).

We next give an example where there exists a uniformly integrable martingale measure, the essential supremum is obtained, and the super-replicating trading strategy $\alpha \equiv 1$. Here, the trading strategy is neither a dominating strategy nor a wealth preserving strategy.[3]

Example 2.5.15. (Super-replicating Trading Strategy)

Suppose that the market contains two independent $\mathbb{Q}$ - Brownian motions B^1 and B^2. Define $\sigma := \inf\{t : \mathcal{E}(B^2)_{A_t} = k\} \wedge T$ for some fixed $k > 0$ where $\mathcal{E}$ denotes the stochastic exponential and A_t is a strictly increasing process such that $\lim_{t \to T} A_t = \infty$. Define the risky asset price to be

$$S_t = \mathcal{E}(B^1)_{A_{t \wedge \sigma}}.$$

Define $Z = \mathcal{E}(B^2)_{A_{t \wedge \sigma}}$. Then, we have

- S is a strict $\mathbb{Q}$ local martingale,

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[3] Note that when a martingale measure exists, this implies that the economy satisfies no dominance. Hence, the buy and hold trading strategy generating the asset's time T payoff cannot be otherwise attained.

- Z is a positive martingale so that $\mathbb{Q}^* \sim \mathbb{Q}$ where $Z_T = \frac{d\mathbb{Q}^*}{d\mathbb{Q}}$.
- Under $\mathbb{Q}^*$, S is a true martingale so

$$\operatorname*{ess\,sup}_{\mathbb{Q}\in\mathcal{M}_l} \mathbb{E}^{\mathbb{Q}}[S_T|S_t] = \mathbb{E}^{\mathbb{Q}^*}[S_T|\mathcal{F}_t] = S_t.$$

This implies

$$\alpha_t = \frac{\partial\left(\mathbb{E}^{\mathbb{Q}^*}[S_T|\mathcal{F}_t]\right)}{\partial S_t} = 1, \forall t \in [0,T].$$

2.6 Conclusion

This paper studies an arbitrage-free, frictionless and competitive market with trading in a single risky asset and a money market account, where the risky asset exhibits a price bubble. We study various self-financing and admissible trading strategies that are wealth preserving and take a positive holding in the risky asset. Included in this class are dominating trading strategies that replicate or super-replicate the payoffs to the risky asset at the model's horizon. The insights of this paper can be used to understand and characterize an investor's optimal trading strategy in the presence of price bubbles, in this regard see [35].

CHAPTER 3

PORTFOLIO OPTIMIZATION IN THE PRESENCE OF ASSET PRICE BUBBLES

3.1 Introduction

Asset price bubbles have had a long history in market folklore and in the academic literature (see [31]). Recently, there is mounting empirical evidence that asset price bubbles exist and can be described using the local martingale theory of asset price bubbles (see [29], [51], [33], [10]). The local martingale theory of bubbles was developed in a sequence of papers by [44], [11], [23], and [57], [37]. For a review of the mathematics underlying the local martingale theory of bubble see [4].

Existence of an optimal portfolio and rational equilibrium with bubbles has been well studied in the literature, for example see the papers by [45], [25], and [28]. Our paper extends the insights from these papers to study the optimal trading strategy for an investor in a market with price bubbles. In a standard finite horizon, arbitrage-free, competitive and frictionless market with a single risky asset and money market account, we assume sufficient conditions to guarantee the existence of an optimal portfolio for an investor maximizing his preferences over terminal wealth.

We are interested in the following problem. Given a market with bubbles, how does the optimal trading strategy change versus that used in an identical market without bubbles. And, whether the investor's welfare (value function) declines. In this regard, our paper is similar to [21], but we have a more general risky asset process (a nonnegative semimartingale) and investor preferences (state dependent, increasing, and concave). Our major conclusions are three:

- In a complete market, the optimal trading strategy with a bubble is to: (i) first trade to replicate the risky asset's fundamental value, then (ii) use the optimal trading strategy employed in a market without bubbles. Because of the ability to replicate the asset's fundamental value, the investor's value function is the same with and without bubbles.
- In an incomplete market, if there exists a trading strategy that replicates the risky asset's fundamental value, then the same conclusion holds as in a complete market. We characterize the conditions under which this is true, which is that the set of local martingale deflators must be the same across the two different economies - with and without bubbles.
- If the sufficient condition is violated, then the investor's welfare can be increased or decreased by the existence of price bubbles. We provide an example where an investor's welfare is increased.

To illustrate the previous results and to show their general applicability, we apply them to a general Markov diffusion process for the risky asset price process that is well studied in the asset price bubble literature (see [4]).

In a similar setting, a companion paper by [36] studies the properties of admissible self-financing trading strategies (s.f.t.s.) that replicate an asset's fundamental value process in the presence of price bubbles. Using these properties in both complete and incomplete markets, [36] characterize those admissible s.f.t.s. that are wealth preserving and replicate an asset's terminal payoff.[1]

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[1] Roughly speaking, an admissible s.f.t.s is wealth preserving if the trading strategy's initial value equals the present value of the trading strategy's terminal payoff.

In contrast, the current paper provides a comparative analysis of both an investor's optimal trading strategy and their value function in markets with and without price bubbles. The set of wealth preserving, admissible s.f.t.s analyzed in [36] are those from which an investor's optimal admissible s.f.t.s. are selected. Consequently, many of the results from [36] are used herein to help characterize the investor's optimal admissible s.f.t.s.

In the portfolio optimization literature in the presence of price bubbles, the optimal trading strategy for an investor holding a positive position in the asset is often described as "riding the bubble" (e.g. [21]). We add some additional insights into why an investor rides a bubble. In a complete market, the trader creates a dynamic self-financing trading strategy to hold the risky asset and avoid the losses due to the bubble. In an incomplete market, the same interpretation applies if the fundamental value can be synthetically constructed with a self-financing trading strategy. We provide a characterization of the conditions under which this condition is satisfied.

The rest of the paper is organized as follows. Section 2 provides the setup and notation, an introduction to the local martingale theory of financial asset bubbles and some characterizations of the bubble process which is helpful in understanding the optimal portfolio. Section 3 studies portfolio optimization in a complete market, while Section 4 studies it in an incomplete market. Section 5 concludes the paper.

3.2 The Model

This section briefly reviews the local martingale theory of bubbles.

3.2.1 The Set-Up

We consider a continuous time model on a finite horizon $[0, T]$. The randomness is represented by $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$, which is a filtered complete probability space where the filtration $\mathbb{F} = (\mathcal{F})_{0 \leq t \leq T}$ satisfies the usual hypothesis with $\mathcal{F}_T = \mathcal{F}$. In the subsequent portfolio optimization problem, the probability measure $\mathbb{P}$ and the filtration $\mathbb{F}$ should be interpreted as the beliefs and information set of an investor, respectively, where $\mathbb{P}$ is equivalent to the statistical probability measure.

To abstract from portfolio diversification considerations when considering multiple risky assets, we consider an economy with trading in a single risky asset and a money market account (mma) that is locally riskless. Without loss of generality, we assume that no cash flows are paid to the risky asset and the mma's value is equal to unity for all times. The (normalized) market price of the risky asset is given by a non-negative semimartingale $(S_t)_{0 \leq t \leq T}$ adapted to $\mathbb{F}$.

Trading strategies are a tuple of stochastic processes $(\alpha_0, \alpha) \in (\mathscr{O}, \mathscr{L}(S))$ where $\mathscr{O}$ is the optional σ-algebra and $\mathscr{L}(S)$ is the set of predictable processes for which the stochastic integral with respect to S exists. To exclude doubling strategies from discussion, we only consider trading strategies that are admissible (the value of the trading strategy is bounded below). A trading strategy is self-financing if it requires no cash inflows or outflows except at times 0 and T.

An admissible self financing trading strategy (s.f.t.s) with initial wealth x and wealth process X is a trading strategy $(\alpha_0, \alpha) \in (\mathscr{O}, \mathscr{L}(S))$ such that there exists some constant c where

$$X_t = \alpha_0 + \alpha_t S_t = x + \int_0^t \alpha_u dS_u \geq c, \forall t \in [0, T].$$

The first equality represents the trading strategy's wealth at time t, which equals the number of units of the mma plus the number of shares of the risky asset times its market price. The second equality is the self-financing condition, which states that the trading strategy's time t wealth equals the initial wealth plus the accumulated capital gains from the trading strategy over $[0, T]$. The third inequality is the admissibility condition, which represents a uniform lower bound on the wealth process. This lower bound represents a borrowing constraint. We denote by $\mathcal{A}(x)$ the set of admissible s.f.t.s. $(\alpha_0, \alpha) \in (\mathscr{O}, \mathscr{L}(S))$ given an initial wealth x.

A simple arbitrage opportunity is an admissible s.f.t.s. $(\alpha_0, \alpha) \in \mathcal{A}(x)$ with initial wealth $x = 0$ and wealth process X such that

$$\begin{aligned} \mathbb{P}(X_T \geq 0) = 1, \qquad &\text{and} \\ \mathbb{P}(X_T > 0) > 0. \end{aligned}$$

A *free lunch with vanishing risk*

A *free lunch with vanishing risk* (FLVR) is an admissible s.f.t.s. that is an extension of a simple arbitrage opportunity that includes (the limits of) approximate simple arbitrage opportunities. We say the market satisfies *no free lunch with vanishing risk*(NFLVR) if there exists no FLVR.

We invoke the following assumption.

assumption: (No Arbitrage)

The market satisfies NFLVR.

An equivalent local martingale measure $\mathbb{Q}$ is any probability measure on $(\Omega, \mathcal{F})$ such that for $A \in \mathcal{F}$, $\mathbb{Q}(A) = 0$ iff $\mathbb{P}(A) = 0$ (in symbols $\mathbb{Q} \sim \mathbb{P}$) and S is a $\mathbb{Q}$ local martingale. Define $\mathcal{M}_l$ to be the set of *equivalent local martingale measures*

(ELMM) and

$$M_l := \left\{ Y_T \in L^0_+ : \exists \mathbb{Q} \in \mathcal{M}_l,\ Y_T = \frac{d\mathbb{Q}}{d\mathbb{P}} \right\}$$

the set of local martingale deflators.

For subsequent use, we also define the set of supermartingale deflators

$$D_s := \{Y_T \in L^0_+ : Y_0 = 1,\ \exists (Z(T)_n)_{n\geq 1} \in M_l,\ Y_T \leq \lim_{n\to\infty} Z_n(T)\ a.s.\}.$$

The first fundamental theorem of asset pricing states that NFLVR is equivalent to the existence of an *ELMM*, i.e. $\mathcal{M}_l \neq \emptyset$.

For subsequent use, we say that an admissible s.f.t.s. $(\alpha_0, \alpha) \in \mathscr{A}(x)$ with wealth process X is wealth preserving with respect to a $\mathbb{Q} \in \mathcal{M}_l$ if

$$x = \mathbb{E}^{\mathbb{Q}}[X_T].$$

A market is defined to be complete with respect to some $\mathbb{Q} \in \mathcal{M}_l$ if for any non-negative payoff $Z \in L^1(\mathbb{Q})$ at time T, there exists a $x \geq 0$ and $(\alpha_0, \alpha) \in \mathscr{A}(x)$ such that

$$x + \int_0^T \alpha_u dS_u = Z_T,$$

and the wealth process

$$X_t = \alpha_0 + \alpha_t S_t = x + \int_0^t \alpha_u dS_u$$

is a $\mathbb{Q}$ martingale. The trading strategy $(\alpha_0, \alpha) \in \mathscr{A}(x)$ is said to replicate the payoff $Z \in L^1(\mathbb{Q})$. Note that this replicating trading strategy is wealth preserving because by the martingale property,

$$x = X_0 = \mathbb{E}^{\mathbb{Q}}[Z_T].$$

By the second fundamental theorem of asset pricing, in a complete market satisfying NFLVR, the ELMM is unique. In a complete market, expectation with respect to $\mathbb{Q} \in \mathcal{M}_l$ gives the unique present value operator to determine the arbitrage-free price of any $Z \in L^1(\mathbb{Q})$ at time t, which is

$$\mathbb{E}^{\mathbb{Q}}[Z_T|\mathcal{F}_t].$$

In an incomplete market, $\mathbb{E}^{\mathbb{Q}}[\cdot]$ represents only one of the infinite number of present value operators consistent with NFLVR. We will study both complete and incomplete markets below.

An ELMM $\mathbb{Q} \in \mathcal{M}_l$ is said to be an equivalent martingale measure if S is a $\mathbb{Q}$ martingale. Define $\mathcal{M}$ to be the set of equivalent martingale measures *(EMM)*.

An admissible s.f.t.s. with wealth process X is said to be dominating if there exists an admissible s.f.t.s $(\alpha_0, \alpha) \in \mathscr{A}(x)$ such that $x < S_0$ and

$$x + \int_0^T \alpha_u dS_u = S_T \qquad \text{a.s.}$$

The market is said to satisfy No Dominance (ND) if there exist no such dominating s.f.t.s. The third fundamental theorem of asset pricing states that NFLVR and ND is equivalent to the existence of an *EMM*, i.e. $\mathcal{M} \neq \emptyset$. For more discussion on these topics, see [32], Chapter 2.

3.2.2 The Local Martingale Theory for Bubbles

Given the above set-up, we can now introduce the notion of an asset price bubble. Given the market satisfies NFLVR, $\mathcal{M}_l \neq \emptyset$. To define a price bubble, we need to choose an element of $\mathcal{M}_l$. If the market is complete, then $\mathcal{M}_l$ is the singleton set, and we choose that element. If the market is incomplete, then there exists an infinite number of ELMMs. To identify a unique ELMM in this case, there are two possibilities. One is to consider the market as embedded in a larger economy that includes the trading of derivatives on the the risky asset (e.g. call and put options with different strikes and maturities). The second is to introduce a trader, and use the ELMM determined by the first order condition of the solution of the trader's optimal portfolio problem. We will return to the details of identifying this ELMM in a subsequent section. For the moment, we fix a unique $\mathbb{Q} \in \mathcal{M}_l$.

The fundamental value is defined to be the expected value of asset's time T payoff using this ELMM $\mathbb{Q}$, i.e.

$$S_t^h := \mathbb{E}^{\mathbb{Q}}[S_T|\mathcal{F}_t] \leq S_t.$$

The inequality follows by Fatou's lemma, because a local martingale bounded from below is a supermartingale.

Consistent with the economics literature (see [31]), we define the price bubble process to be

$$\beta_t = S_t - S_t^h \geq 0.$$

From the definition, the bubble is always nonnegative and there exists a nontrivial bubble (i.e. $\beta_0 > 0$) if and only if S is a strict local martingale under $\mathbb{Q}$.

If we further assume that the market is complete with respect to $\mathbb{Q} \in \mathcal{M}_l$, then by the definition, we have that there exists a wealth preserving trading strategy $(\alpha_0, \alpha) \in \mathscr{A}(\mathbb{E}^{\mathbb{Q}}[S_T])$ that replicates the asset's time T payoff, i.e.

$$S_T = \mathbb{E}^{\mathbb{Q}}[S_T] + \int_0^T \alpha_t dS_t,$$

and whose wealth process $S_t^h = \mathbb{E}^{\mathbb{Q}}[S_T|\mathcal{F}_t]$ is a $\mathbb{Q}$ martingale.

Here, if S is a martingale under the ELMM $\mathbb{Q}$, i.e. $\mathbb{Q} \in \mathcal{M}$, then $\mathbb{E}^{\mathbb{Q}}[S_T] = S_0$ and $(\alpha_0, \alpha) \in \mathscr{A}(\mathbb{E}^{\mathbb{Q}}[S_T])$ consists of buying one unit of the risky asset at time 0 and holding it until time T, i.e. $(\alpha_0, \alpha) := (0, 1)$. However, if the S exhibits a price bubble and is a strict local martingale under the ELMM $\mathbb{Q}$, then $\mathbb{E}^{\mathbb{Q}}[S_T] < S_0$ and $(\alpha_0, \alpha) \in \mathscr{A}(\mathbb{E}^{\mathbb{Q}}[S_T])$ is not the buy and hold position in the risky asset, but it is a dynamic s.f.t.s. In this case the replicating s.f.t.s. $(\alpha_0, \alpha) \in \mathscr{A}(\mathbb{E}^{\mathbb{Q}}[S_T])$ is a dominating trading strategy. Consequently, the existence of a price bubble in a complete market violates ND.

In [36], characterizations of this replicating s.f.t.s. $(\alpha_0, \alpha) \in \mathscr{A}(\mathbb{E}^{\mathbb{Q}}[S_T])$ for various price evolutions have been investigated. For later use, we note the following two propositions from [36].

Proposition 3.2.1. *(Time-Homogeneous Diffusion)*

Let

$$dS_t = S_t \sigma(S_t) dW_t$$

where $\sigma(\cdot) > 0$ *for all* $t \in [0, T]$ *a.s. is such that there is a unique strong solution to this expression with* W *a standard Brownian motion under* $\mathbb{Q}$ *initialized at* $W_0 = 0$. *Then,*

(a) there exists a bubble (S *is a strict local martingale under* $\mathbb{Q}$*) if and only if*

$$\int_{\epsilon}^{\infty} \frac{1}{s\sigma(s)^2} ds < \infty$$

for some $\epsilon > 0$.

(b) Suppose S *is a strict local martingale under* $\mathbb{Q}$, *then the wealth preserving admissible s.f.t.s.* $(\alpha_0, \alpha) \in \mathscr{A}(\mathbb{E}^{\mathbb{Q}}[S_T])$ *that replicates* S_T *satisfies*

$$\alpha_t \in [0,1]$$

for all $t \in [0,T]$ *a.s.*

Proposition 3.2.2. *(Time-Inhomogeneous Diffusion)*

Let

$$dS_t = S_t \sigma(t, S_t) dW_t$$

where $\sigma(t,\cdot) > 0$ *for all* $t \in [0,T]$ *a.s. is an increasing function such that there is a unique strong solution to this expression with* W *a standard Brownian motion under* $\mathbb{Q}$ *initialized at* $W_0 = 0$.

Suppose S *is a strict local martingale under* $\mathbb{Q}$, *then the wealth preserving admissible s.f.t.s.* $(\alpha, \alpha^0) \in \mathscr{A}(\mathbb{E}^{\mathbb{Q}}[S_T])$ *that replicates* S_T *satisfies*

$$\alpha_t \in [0,1]$$

for all $t \in [0,T]$ *a.s.*

Given an asset price bubble, the propositions state that the s.f.t.s. that replicates the asset's time T payoff, S_T, is a dynamic portfolio that holds a positive amount of the risky asset, but typically less than one share of the risky asset, except at time T.

These two propositions relate to asset price processes often used in the literature where necessary and sufficient conditions are known under which these processes exhibit price bubbles (see [4]). Because of these characterizations, we use these price processes below to illustrate the subsequent theorems. We also note that these evolutions imply that the markets underlying both of these propositions are complete.

3.2.3 The Investor's Preferences

We assume that the investor has a state dependent utility function $U : (0, +\infty) \times \Omega \to \mathbb{R}$ defined over terminal wealth, which is $\mathcal{B}(0, \infty) \times \mathcal{F}_T$-measurable and that satisfies the following conditions: for $\omega \in \Omega$, $\mathbb{P}$ - a.s.

(i) $U(x, \omega)$ is continuous and differentiable in x,

(ii) $U(x, \omega)$ is strictly increasing and strictly concave in x,

(iii) (Inada Conditions) $\lim_{x \to 0} \frac{dU(x,\omega)}{dx} = \infty$ and $\lim_{x \to \infty} \frac{dU(x,\omega)}{dx} = 0$, and

(iv) $AE(U) = \limsup_{x \to \infty} \frac{xU'(x,\omega)}{U(x,\omega)} < 1$.

Condition (ii) implies the investor prefers more wealth to less and is risk averse for independent gambles (see [30]). Conditions (iii) and (iv), in conjunction with an additional assumption (given later), imply the existence and uniqueness of an optimal terminal wealth.

3.3 Complete Markets

This section studies an investor's portfolio optimization problem when the risky asset exhibits a price bubble in a complete market. The purpose of which is to determine the investor's optimal trading strategy. Recall that we assume the market satisfies NFLVR and since it is complete, there exists a unique ELMM

$\mathbb{Q} \in \mathcal{M}_l$. This ELMM determines the asset's price bubble.

To characterize the optimal strategy, we consider two different economies. The first is the actual economy, that exhibits the price bubble. The second is a hypothetical economy, where the risky asset's price process equals its fundamental value. We determine the optimal trading strategy in both economies and contrast them to determine the impact of the price bubble in the actual economy. To avoid notational confusion, the superscript $*$ is used to denote optimality, the superscript a is for the actual economy, and the superscript h is for the hypothetical economy. This and the subsequent section relies heavily on the results in [32], Chapters 10 and 11.

3.3.1 The Actual Economy

In the actual economy, the price process S^a is assumed to be a strict local martingale under $\mathbb{Q} \in \mathcal{M}_l$, and hence exhibits a price bubble. Consider the investor's terminal wealth optimization problem in the actual economy.

$$v^a(x) := \underset{X_T \in \mathcal{C}^a(x)}{sup} E\left[U\left(X_T\right)\right] \qquad \text{where}$$

$$\mathcal{C}^a(x) := \{X_T \in L^0_+ : \ \exists(\alpha_0, \alpha) \in \mathcal{N}^a(x), \ x + (\alpha \cdot S^a)_T = X_T\}$$

$$\mathcal{N}^a(x) := \{(\alpha_0, \alpha) \in (\mathcal{O}, \mathscr{L}(S)) : \ X_t = \alpha_0(t) + (\alpha \cdot S^a)_t\,,$$

$$X_t = x + (\alpha \cdot S^a)_t \geq 0, \ \forall t \in [0, T]\}$$

where by the self-financing condition:

$$\alpha_0(T) + \alpha_T S^a_T = x + (\alpha \cdot S^a)_T$$

with $(\alpha \cdot S^a)_t := \int_0^t \alpha_t dS^a_t$ and where L^0_+ denotes the set of non-negative $\mathcal{F}_T$ measurable random variables.

Here, the investor chooses an optimal terminal wealth X_T that can be obtained by an admissible s.f.t.s. $(\alpha_0, \alpha) \in \mathcal{N}^a(x)$ that has a non-negative wealth process X_t for all times a.s. We add the following assumption that guarantees the existence of a unique solution X_T^* (see [32], Chapter 10).

assumption: (Existence)

There exists x such that $v^a(x) < \infty$.

For later comparison with the hypothetical economy, we rewrite this optimization problem using the following budget constraint equivalence proven in [32], Chapter 10, Theorem 49.

Proposition 3.3.1. *(Budget Equivalence)*

$$\mathcal{C}^a(x) = \{X_T \in L_+^0 : \mathbb{E}^{\mathbb{Q}}[X_T] \leq x\}$$

This equivalence states that the set of feasible terminal wealths is equivalent to those wealths whose time 0 values $\mathbb{E}^{\mathbb{Q}}[X_T]$ are affordable, i.e. less than or equal to x. This implies that the investor's optimization problem can be written as:

$$v^a(x) = \underset{X_T \in L_+^0}{sup} \mathbb{E}\left[U\left(X_T\right)\right] \qquad \text{where} \tag{3.1}$$

$$\mathbb{E}^{\mathbb{Q}}\left[X_T\right] \leq x.$$

3.3.2 The Hypothetical Economy

The hypothetical economy is identical to the actual economy with the exception that the market price of the risky asset is the risky asset's fundamental value, i.e.

$$S_t^h := \mathbb{E}^{\mathbb{Q}}[S_T^a|\mathcal{F}_t].$$

Because the actual risky asset price S_t^a is a strict local martingale under $\mathbb{Q}$, we have that

$$S_t^h \leq S_t^a = S_t^h + \beta_t$$

with $\beta_t \geq 0$ and $\beta_T = 0$.

As noted, the price process in the actual economy consists of the fundamental value and the bubble. To insure that the fundamental value in the hypothetical economy also generates a complete market, we add the following assumption.

assumption: (Complete Hypothetical Market)

The hypothetical market consisting of S_t^h and the mma is complete with respect to $\mathbb{Q}$.

In essence, this assumption states that the bubble process by itself, does not make an otherwise incomplete market, complete. If S_t^h follows a diffusion process with one Brownian motion generating the filtration, then this assumption is equivalent to the risky asset not being locally riskless on any non-trivial (random) subinterval over the horizon $[0, T]$. An equivalent characterization of this assumption is given in Remark 5 in the subsequent section. Examples of price processes that satisfy this assumption are given in propositions 3.2.1 and 3.2.2 above, and presented in section 3.3.4 below.

The investor's optimization problem for the hypothetical economy is

$$v^h(x) \coloneqq \underset{X_T \in \mathcal{C}^h(x)}{sup} E\left[U\left(X_T\right)\right] \qquad \text{where}$$

$$\mathcal{C}^h(x) \coloneqq \{X_T \in L^0_+ : \exists (\alpha_0, \alpha) \in \mathcal{N}^h(x),\ x + \left(\alpha \cdot S^h\right)_T = X_T\}$$

$$\mathcal{N}^h(x) \coloneqq \big\{(\alpha_0, \alpha) \in (\mathcal{O}, \mathscr{L}(S)) : \ X_t = \alpha_0(t) + \left(\alpha \cdot S^h\right)_t,$$
$$X_t = x + \left(\alpha \cdot S^h\right)_t \geq 0,\ \forall t \in [0, T]\big\}.$$

where by the self-financing condition:

$$\alpha_0(T) + \alpha_T S^h_T = x + \left(\alpha \cdot S^h\right)_T.$$

Similiar to the actual economy, we have an alternative characterization of the budget constraint, which implies that the optimization problem is equivalent to:

$$v^h(x) = \underset{X_T \in L^0_+}{sup} \mathbb{E}\left[U\left(X_T\right)\right] \qquad \text{where} \tag{3.2}$$
$$\mathbb{E}^{\mathbb{Q}}\left[X_T\right] \leq x.$$

3.3.3 The Comparison Result

Comparing the two problem statements (expressions (3.1) and (3.2), we see that the problems are identical, which implies the following corollary. For subsequent use, we note that in the solutions to problems (3.1) and (3.2), both v^a and v^h are increasing in x.

Corollary 3.3.2. *(Identical Value Functions and Optimal Terminal Wealths)*

$v^a(x) = v^h(x), \quad \forall x \in \mathbb{R}^+$ *and*

$X_T^* := (X_T^a)^* = \left(X_T^h\right)^*.$

This corollary proves that the existence of a price bubble in the actual economy does not reduce the investor's welfare. Indeed, both the value function and the optimal terminal wealths are the same across the two different economies. However, the optimal trading strategy that attains the optimal terminal wealth is different across the two economies. We now analyze their differences.

Let the optimal trading strategy in the actual economy be denoted (η_0^a, η^a) and the optimal trading strategy in the hypothetical economy be denoted (η_0^h, η^h). Since the actual market is complete, there exists an admissible s.f.t.s. $(\alpha_0, \alpha) \in \mathscr{A}(\mathbb{E}^{\mathbb{Q}}[S_T])$ in the actual market that replicates the risky asset's fundamental value S^h (see the discussion in Section 3.2.2 above), i.e.

$$S_t^h = x + (\alpha \cdot S^a)_t$$

for all $t \in [0, T]$ a.s.

Remark 3.3.3. (Complete Hypothetical Market)

The assumption of a complete hypothetical market is equivalent to assuming that $\alpha \neq 0$, a.s. $d[S, S]_T(\omega) \times d\mathbb{Q}$. This completes the remark.

From the optimality conditions,

$$X_T^* = x + (\eta^a \cdot S^a)_T$$

and

$$X_T^* = x + \left(\eta^h \cdot S^h\right)_T = x + \left(\eta^h \cdot (\alpha \cdot S^a)\right)_T = x + \left((\eta^h \alpha) \cdot S^a\right)_T$$

where the last equality uses the associativity of stochastic integration (see [58], Theorem 21, p. 165). Hence, by the uniqueness of the integrand of the stochastic integral, we have that

$$\eta_t^a = \eta_t^h \alpha_t \qquad a.s.\; d[S,S]_T(\omega) \times d\mathbb{Q}. \tag{3.3}$$

The optimal holdings of the risky asset in the actual economy η_t^a are some fraction α_t of the optimal holdings in the hypothetical economy η_t^h, where α_t represents the holdings necessary to reconstruct the fundamental value process. Alternatively stated, even though a bubble exists in the actual economy, the investor first reconstructs the fundamental value process synthetically and then buys the optimal holdings in the fundamental value. Using this wealth preserving and dominating trading strategy, the investor obtains the risky asset's time T payoffs but avoids the loss due to the bubble component.

Given this explanation, we see that in the actual economy where the price process exhibits a bubble, the investor holds the risky asset, but does not "buy the risky asset to sell it before it bursts." Instead, the investor avoids the bubble component in the risky asset price process by using a wealth preserving and dominating replicating trading strategy. This insight explains why the investor's value function for an initial wealth is the same across the actual and hypothetical economies, i.e. with and without bubbles.

3.3.4 Markov Diffusion Price Processes

As an application of the previous results, we consider the asset price process contained in proposition 3.2.2, rewritten here for convenience, i.e.

$$dS_t^a = S_t^a \sigma(t, S_t^a) dW_t$$

where $\sigma(t, S_t^a) > 0$ for all $t \in [0, T]$ a.s. By construction, the actual economy with this price process is complete.

For this diffusion process, the optimal trading strategy given in expression (3.3) can be characterized using proposition 3.2.2 where the replicating s.f.t.s. is shown to satisfy $\alpha_t \in [0, 1]$ for all $t \in [0, T]$ a.s., so that

$$0 \leq \eta_t^a = \eta_t^h \alpha_t \leq \eta_t^h.$$

That is, the optimal holdings in risky asset in the actual economy η_t^a are non-negative and less than or equal to the optimal portfolio held in an identical economy with no bubble.

We can obtain more insights into the relation between the two optimal holdings in the risky asset by explicitly considering the bubble process

$$\beta(t, S_t^a) = S_t^a - S_t^h.$$

Define $\pi^a := \frac{\eta_t^a S_t^a}{X_t^*}, \pi^h := \frac{\eta_t^h S_t^h}{X_t^*}$ to be the percentage holdings of risky asset relative to the portfolio's time t value in actual and hypothetical economies, respectively. And, define the bubble leverage process as $\gamma(t, S_t^a) := \frac{\beta(t, S_t^a)}{S_t^a}$. Then, we have

Proposition 3.3.4. *(Bubble's Leverage Process)*

$$\pi_t^a = \pi_t^h \left(1 + S_t^a \frac{\partial \log(1 - \gamma(t, S_t^a))}{\partial S_t^a} \right)$$

Proof. Given

$$S_t^h = (1 - \gamma(t, S_t^a)) S_t^a$$

and (using Corollary 14 in [36]):

$$\alpha_t = \frac{\partial\left((1-\gamma(t,S_t^a))S_t^a\right)}{\partial S_t^a} = 1-\gamma(t,S_t^a) - S_t^a\frac{\partial\gamma(t,S_t^a)}{\partial S_t^a}.$$

Hence,

$$\begin{aligned}
X_t^*\pi_t^a &= \eta_t^a S_t^a \\
&= \eta_t^h S_t^h \alpha_t \frac{S_t^a}{S_t^h} \\
&= X_t^*\pi_t^h\left(1 - \frac{S_t^a}{1-\gamma(t,S_t^a)}\frac{\partial\gamma(t,S_t^a)}{\partial S_t^a}\right) \\
&= (X_t^*\pi_t^h)\left(1 + S_t^a\frac{\partial\log(1-\gamma(t,S_t^a))}{\partial S_t^a}\right).
\end{aligned}$$

□

From the above multiplier we see that the bubble's leverage effect in the optimal trading strategy depends on the relative size of the bubble. For the time-homogeneous case of proposition 3.2.1 we have the following result:

Theorem 3.3.5. *(Time-Homogeneous Diffusion)*

Suppose that $\sigma(t,x) \equiv \sigma(x)$*, then*

$$\pi_t^a \leq \pi_t^h.$$

Proof. By Theorem 16 in [36], $f(t,x) := \mathbb{E}^{\mathbb{Q}}[S_T^a | S_t^a = x]$ is a concave function for any fixed $t \in [0, T)$. Hence, $S_t^h = f(t, S_t^a)$ and

$$\begin{aligned} X_t^* \pi_t^a &= \eta_t^a S_t^a \\ &= \eta_t^h \alpha_t S_t^a \\ &= \eta_t^h S_t^h \frac{\frac{\partial f(t, S_t^a}{\partial S_t^a}}{f(t, S_t^a)/S_t^a} \\ &= X_t^* \pi_t^h \frac{\frac{\partial f(t, S_t^a}{\partial S_t^a}}{f(t, S_t^a)/S_t^a}. \end{aligned}$$

To obtain the conclusion, it suffices to show that

$$\frac{\frac{\partial f(t, S_t^a}{\partial S_t^a}}{f(t, S_t^a)/S_t^a} \leq 1.$$

By the concavity of f and the intermediate value theorem we have that

$$\begin{aligned} \frac{f(t, S_t^a)}{S_t^a} &= \frac{f(t, S_t^a) - f(t, 0)}{S_t^a - 0} \\ &= \frac{\partial f(t, \xi)}{\partial \xi}, \text{for some } \xi \in [0, S_t^a] \\ &\leq \frac{\partial f(t, S_t^a)}{\partial S_t^a}. \end{aligned}$$

This completes the proof. □

As this theorem shows, for the time-homogeneous diffusion model, the percentage of the wealth optimally invested in the risky asset in an economy with bubbles is less than that in an economy without. For the time-inhomogeneous model, however, this need not be the case. A sufficient condition for an analogous result is given in the following theorem.

Theorem 3.3.6. *(Time-Inhomogeneous Diffusion)*

Suppose $\sigma(t, x) \in C^{1,2}([0, T] \times \mathbb{R}^+)$ *is strictly positive and increasing in* x *for all* $t \in [0, T]$*. Then,*

$$\pi_t^a \leq \pi_t^h.$$

Proof. From Proposition 3.3.4, it suffices to show that $\gamma(t, x)$ is an increasing function in x for any $t \in [0, T)$. By Theorem 19 in [36] in conjunction with above assumption, γ is an increasing function in S_t^a and the above result follow from Proposition 3.3.4. $\square$

Example 3.3.7. (A CEV Process Economy)

To give an explicit illustration of the previous results, we consider the following economy from [32], Chapter 11. Let the market consist of a single risky asset and a mma where the risky asset follows the constant elasticity of variance (CEV) process:

$$dS_t^a = (S_t^a)^{1+\delta}(\theta_t dt + dW_t)$$

where$\int_0^T \theta_t^2 dt < \infty$. This implies that the market is complete with respect to $\mathbb{Q}$ where

$$\frac{d\mathbb{Q}}{d\mathbb{P}} := e^{-\int_0^T \theta_t dW_t - \frac{1}{2}\int_0^T \theta_t^2 dt}$$

is the unique equivalent local martingale measure.

Let the trader's preferences be represented by a state independent logarithmic utility function

$$U(x) = \ln(x), x > 0.$$

We have

$$I(y) = \frac{1}{y} \qquad \text{and}$$

$$\tilde{U}(y) = -\ln(y) - 1$$

where I is the inverse of U' and $\tilde{U}$ is the convex conjugate of U. As shown in [32], Chapter 11, p. 236 - 237, the optimal wealth process is

$$X_t^* = xe^{\int_0^t \theta_s dW_s + \frac{1}{2}\int_0^t \theta_s^2 ds}$$

and the optimal trading strategy is

$$\frac{\alpha_t^a}{X_t^*} := \pi_t^a = \frac{\theta_t}{(S_t^a)^\delta}.$$

For the hypothetical economy we have that

$$f(t,x) := \mathbb{E}^{\mathbb{Q}}[S_T^a | S_t^a = x] = xG\left(\upsilon, \frac{x^{-2\delta}}{2\delta^2(T-t)}\right)$$

where $\upsilon = \frac{1}{2\delta}$ and

$$G(\upsilon, x) = \frac{\int_0^y u^{\upsilon-1} e^{-u} du}{\Gamma(\upsilon)}.$$

Then, given $S_t^h = f(t, S_t^a)$, Ito's formula implies that

$$\frac{dS_t^h}{S_t^h} = \frac{\alpha_t^a (S_t^a)^{1+\delta}}{S_t^h} \left(\theta_t dt + dW_t\right)$$

where

$$\alpha_t^a = \frac{\partial}{\partial x} f(t,x) = xG\left(\upsilon, \frac{x^{-2\delta}}{2\delta^2(T-t)}\right) - \frac{x^\alpha e^{-\frac{x^{-2\delta}}{2\delta^2(T-t)}}}{\Gamma(\upsilon)\delta(T-t)} < 1.$$

Hence, in the hypothetical economy, the optimal trading strategy is given by

$$\frac{\alpha_t^h}{X_t^*} := \pi_t^h = \frac{\theta_t}{\alpha_t^a (S_t^a)^{1+\delta} / S_t^h}.$$

To see $\pi_t^a \leq \pi_t^h$, it suffices to show that

$$(S_t^a)^\delta \geq \alpha_t^a (S_t^a)^{1+\delta} / S_t^h$$

which is equivalent to

$$\alpha_t^a \leq \frac{S_t^h}{S_t^a}.$$

And, it suffices to show that

$$\frac{\partial f(t,x)}{\partial x} \leq \frac{f(t,x)}{x}.$$

But, this follows because $f(t, \cdot)$ is concave and $f(t, 0) = 0$, which completes the example.

3.3.5 Collateral Constraints

The neutrality to bubbles of the trader's value function in a complete market depends on the underlying frictionless market assumption. If we relax this assumption, then bubbles can decrease an investor's welfare. The friction we investigate is a collateral requirement imposed on the trader's wealth process. In particular, the investor's budget constraint is modified to be the following set of G-admissible s.f.t.s. defined as

$$\begin{gathered}\mathscr{N}_G(x) \coloneqq \Big\{(\alpha_0, \alpha) \in (\mathscr{O}, \mathscr{L}(S)) : X_t = \alpha_0(t) + \alpha_t \cdot S_t \\ X_t = x + \textstyle\int_0^t \alpha_u \cdot dS_u \geq G(S_t), \forall t \in [0, T]\Big\}\end{gathered}$$

where $G : (0, \infty) \to (0, \infty)$ is such that

(i) G is convex, and

(ii) $\lim_{x\to\infty} \frac{G(x)}{x} = c \geq 0$ for c a constant.

The convexity assumption implies that as the risky asset price increases, the constraint becomes more binding, but at a decreasing rate. The asymptotic condition implies that for large risky asset prices, the constraint is approximately linear. The constant must be nonnegative, for otherwise for large enough risky asset prices, the wealth process would violate the admissibility condition (any uniform fixed lower bound on the wealth process). An example of a valid collateral constraint function is $G(S_t) = cS_t$.

For this section only, we add the assumption that the risky asset price process S_t has bounded jumps. Given this structure, we are interested in the following portfolio optimization problem:

$$v(x) = \sup_{(\alpha_0,\alpha)\in\mathscr{N}_G(x)} E\left[U\left(X_T\right)\right].$$

We can rewrite the above problem as:

$$v(x) = \sup_{X_T \in \mathscr{C}(x)} E\left[U\left(X_T\right)\right] \qquad \text{where}$$
$$\mathscr{C}_G(x) := \left\{X_T \geq G(S_T) : \exists\left(\alpha_0, \alpha\right) \in \mathscr{N}_G(x), x + \int_0^T \alpha_t \cdot dS_t \geq X_T\right\}$$

The following proposition gives an equivalent budget set.

Proposition 3.3.8. *(Budget Constraint Equivalence)*

$$\mathscr{C}_G(x) = \{X_T \geq G(S_T) : \mathbb{E}^{\mathbb{Q}}[X_T] + (c \vee 0)\beta_0 \leq x\}$$

Proof. Define

$$\mathscr{C}'_G(x) := \{X_T \geq G(S_T) : \mathbb{E}^{\mathbb{Q}}[X_T] + (c \vee 0)\beta_0 \leq x\}$$

For any $X_T \in \mathscr{C}_G(x)$, there exists a s.f.t.s such that $X_t = x + \int_0^t \alpha_u dS_u \geq G(S_t)$, which is a $\mathbb{Q}$ supermartingale. Now define $V_t = G(S_t)\mathbb{1}_{t<T} + X_T\mathbb{1}_{t=T}$. Thus,

$$x \geq \sup_{\tau \leq T} \mathbb{E}^{\mathbb{Q}}[X_\tau] \geq \sup_{\tau \leq T} \mathbb{E}^{\mathbb{Q}}[V_\tau].$$

By lemma B.1.1 in the Appendix,

$$x \geq \sup_{\tau \leq T} \mathbb{E}^{\mathbb{Q}}[V_\tau] = \mathbb{E}^{\mathbb{Q}}[X_T] + (c \vee 0)\beta_0$$

This implies $\mathscr{C}_G(x) \subseteq \mathscr{C}'_G(x)$.

For the opposite direction, suppose that $X_T \in \mathscr{C}'_G(x)$. We have $X_T \geq G(S_T)$ and $\mathbb{E}^{\mathbb{Q}}[X_T] + (c \vee 0)\beta_0 \leq x$. Define a supermartingale

$$V_t := \operatorname{ess} \sup_{t \leq \tau \leq T} \mathbb{E}^{\mathbb{Q}}[J(\tau)|\mathcal{F}_t]$$

where $J(t) = G(S_t)$ for $t < T$ and $J(T) = X_T$. By lemma B.1.1, we have $V_0 \leq x$ and $V_T = X_T$, $V_t \geq G(S_t)$ for all $t \in [0, T]$. By optional decomposition theorem (see [17]), there exists a s.f.t.s $(\alpha_0, \alpha) \in \mathscr{N}_G(V_0)$ and a non-decreasing process A with $A_0 = 0$ such that

$$V_t = V_0 + \int_0^t \alpha_u dS_u - A_t.$$

Hence,

$$x + \int_0^T \alpha_u dS_u \geq X_T.$$

Therefore, by the definition of $\mathscr{C}_G(x)$, $\mathscr{C}'_G(x) \subseteq \mathscr{C}_G(x)$. This completes the proof.

□

Hence, the investor's problem is

$$\begin{aligned} v(x) = \sup\nolimits_{X_T \in \mathscr{C}(x)} E\left[U\left(X_T\right)\right] \quad &\text{where} \\ \mathscr{C}_G(x) = \{X_T \geq G(S_T) : \mathbb{E}^{\mathbb{Q}}[X_T] + (c \vee 0)\beta_0 \leq x\}. \end{aligned}$$

This rewritten problem gives insight into why a bubble has a negative impact on the investor's optimal expected utility. Note that the original budget constraint $\mathscr{C}(x) = \{X_T : \mathbb{E}^{\mathbb{Q}}[X_T] \leq x\}$ is changed to $\{X_T : \mathbb{E}^{\mathbb{Q}}[X_T] \leq x - c\beta_0 \leq x\}$ where the present value of the feasible terminal wealths is smaller than with no bubble. To prove that the collateral constraint decreases the value function for an economy with a bubble, we construct a modified utility function:

$$\tilde{U}(X_T) := \begin{cases} U(X_T) & X_T \geq G(S_T) \\ -\infty & X_T < G(S_T) \end{cases}$$

which still satisfies the properties of a utility function given in section 3.2.3. Then, the above optimization problem is equivalent to

$$\begin{aligned} v(x - c\beta_0) = \sup_{X_T \in \widetilde{C}_G(x)} \widetilde{U}(X_T) \quad &\text{where} \\ \widetilde{C}_G(x) := \{X_T \geq 0, \mathbb{E}^{\mathbb{Q}}[X_T] \leq x - c\beta_0\}. \end{aligned}$$

As rewritten, we see that a bubble ($\beta_0 > 0$) strictly decreases the investor's welfare if $c > 0$ because $v(x) \geq v(x - c\beta_0)$ and v is increasing in x.

3.4 Incomplete Markets

This section studies an investor's portfolio optimization problem when the risky asset exhibits a price bubble in an incomplete market that satisfies NFLVR, i.e. $\mathcal{M}_l \neq \emptyset$ where $\mathcal{M}_l$ contains an infinite number of ELMMs.

3.4.1 Actual Economy

We consider the following optimization problem:

$$v^a(x) = \underset{X_T \in \mathcal{C}^a(x)}{sup} E\left[U\left(X_T\right)\right] \qquad \text{where}$$

$$\mathcal{C}^a(x) := \left\{X_T \in L^0_+ :\ \exists(\alpha_0, \alpha) \in \mathcal{N}^a(x),\ x + (\alpha \cdot S^a)_T \geq X_T\right\}$$

$$\mathcal{N}^a(x) := \{(\alpha_0, \alpha) \in (\mathcal{O}, \mathscr{L}(S)) :\ X_t = \alpha_0(t) + (\alpha \cdot S^a)_t\,,$$
$$X_t = x + (\alpha \cdot S^a)_t \geq 0,\ \forall t \in [0, T]\}\,.$$

where by the self-financing condition:

$$\alpha_0(T) + \alpha_T S^a_T = x + (\alpha \cdot S^a)_T\,.$$

To guarantee the existence of a unique optimal wealth process, we add the following assumption (see [32], Chapter 11, Theorem 53).

assumption: (Existence)

There exists x such that $v^a(x) < \infty$.

[32], Chapter 11, Theorem 52 proves the following budget set equivalence.

Proposition 3.4.1. *(Budget Equivalence)*

$$\mathcal{C}^a(x) = \{X_T \in L^0_+ : E\left[X_T Y_T\right] \leq x \text{ for all } Y_T \in M^a_l\}$$

Hence, the investor's optimization problem is

$$v^a(x) = \underset{X_T \in L^0_+}{sup} E\left[U\left(X_T\right)\right] \qquad \text{where} \tag{3.4}$$
$$E\left[X_T Y_T\right] \leq x \text{ for all } Y_T \in M^a_l.$$

[32], Chapter 11 shows that for an initial wealth $x \geq 0$, the optimal solution v^a is increasing in x with

$$\left(X^a_T\right)^* = I(yY^*_T) \quad \text{with} \quad I := (U')^{-1} = -\tilde{U}'$$

where

$$\tilde{U}(y, \omega) = \underset{x>0}{sup}\,[U(x, \omega) - xy], \quad y > 0$$

is the convex conjugate of U and $Y^*_T \in D^a_s$ is the solution to

$$\tilde{v}^a(y) = \underset{Y_T \in D^a_s}{inf} E[\tilde{U}(yY_T)].$$

Finally, the optimal wealth process when multiplied by the optimal supermartingale, $\left(X^a_t\right)^* Y^*_t \geq 0$, is a $\mathbb{P}$ martingale.

To facilitate a comparison with the hypothetical economy, we add the following assumption.

assumption: (The Trader's ELMM)

Assume $Y_T^* \in M_l^a$.

This assumption implies that the trader's optimal supermartingale deflator determines a unique ELMM $\mathbb{Q} \in \mathcal{M}_l$ defined by

$$\frac{d\mathbb{Q}}{d\mathbb{P}} := Y_T^*.$$

This is the ELMM that determines whether or not an asset's price bubble exists to the trader under consideration. Hence, the asset's fundamental value is defined relative to the trader's beliefs as

$$S_t^h := E^{\mathbb{Q}}\left[S_T^a \middle| \mathcal{F}_t \right].$$

For the next section, we assume that the actual economy exhibits a price bubble, i.e.

$$\beta_0 = S_0^a - S_0^h > 0.$$

3.4.2 Hypothetical Economy

As before, the difference between the actual and hypothetical economy is that the hypothetical economy's price is the fundamental value S^h. In this economy, the investor's optimization problem is

$$\begin{gathered} v^h(x) = \underset{X_T \in \mathcal{C}^h(x)}{sup} E\left[U\left(X_T\right)\right] \qquad \text{where} \\ \mathcal{C}^h(x) = \{X_T \in L^0_+ :\ \exists(\alpha_0, \alpha) \in \mathcal{N}^h(x),\ x + \left(\alpha \cdot S^h\right)_T \geq X_T\} \\ \mathcal{N}^h(x) = \left\{(\alpha_0, \alpha) \in (\mathcal{O}, \mathscr{L}(S)) :\ X_t = \alpha_0(t) + \left(\alpha \cdot S^h\right)_t, \right. \\ \left. X_t = x + \left(\alpha \cdot S^h\right)_t \geq 0,\ \forall t \in [0, T]\right\}. \end{gathered}$$

where by the self-financing condition:

$$\alpha_0(T) + \alpha_T S^h_T = x + \left(\alpha \cdot S^h\right)_T.$$

The same budget equivalence holds, yielding the equivalent optimization problem:

$$\begin{gathered} v^h(x) = \underset{X_T \in L^0_+}{sup} E\left[U\left(X_T\right)\right] \qquad \text{where} \\ E\left[X_T Y_T\right] \leq x \text{ for all } Y_T \in M^h_l. \end{gathered} \tag{3.5}$$

3.4.3 The Comparison Result

Comparing the investor's optimization problems (3.4) and (3.5) shows that their budget constraints are different. The actual economy is constrained by the local martingale deflator set M^a_l, which in general differs from the local martingale deflator set M_l^h that constrains the choices in the hypothetical economy. The two sets have non-empty intersection, since Y_T^* as defined above is in both sets.

When these two sets are equivalent ($M_l^h = M_l^a$), the investor's value functions in the actual and hypothetical economies are the same. And, as implied by the following characterization theorem, when this condition holds (condition (1) in the following theorem), the optimal trading strategy in the presence of a bubble is the same as that employed in a complete market. That is, the trader first replicates the risky asset's fundamental value, and then uses the optimal trading strategy as employed in a market without bubbles (condition (2) in the following theorem guarantees this outcome).

Theorem 3.4.2. *(Characterization Theorem - Incomplete Markets)*

Assume that $\alpha_t^a \neq 0$ *a.s.* and $\alpha_t^h \neq 0$ *a.s.*

Then, the following are equivalent:

(1) $M_l^h = M_l^a$.

(2) $S^h = \alpha \cdot S^a$ *for some predictable* α *such that* $\alpha \neq 0$.

The equivalent conditions (1) and (2) imply that the investor's value function is the same in both the actual and hypothetical economies.

Proof. (1) implies (2). Assume $M_l^h = M_l^a$. Then by budget constraint we have $v^h(x) = v^a(x)$ and

$$\left(X_T^h\right)^* = \left(X_T^a\right)^* := X_T^*.$$

And, there exists a nonnegative wealth s.f.t.s. $(\alpha_0^h, \alpha^h) \in \mathcal{N}^h(x)$ and $(\alpha_0^a, \alpha^a) \in$

$\mathcal{N}^a(x)$ such that

$$x + \left(\alpha^h \cdot S^h\right)_T = X_T^* = x + \left(\alpha^a \cdot S^a\right)_T .$$

From the optimality condition in the actual economy, we have that X_T^* is a $(\mathbb{Q}^a)^*$ martingale, implying

$$x + \left(\alpha^h \cdot S^h\right)_t = x + \left(\alpha^a \cdot S^a\right)_t .$$

Using the associativity of stochastic integration (see [58], Theorem 21, p. 165),

$$x + \left(\alpha^a \cdot S^a\right)_t = x + \left(\alpha^h \cdot \left(\frac{\alpha^a}{\alpha^h} \cdot S^a\right)\right)_t .$$

Finally, by uniqueness of the integrand of the stochastic integral $a.s.d[S^a, S^a]_T(\omega) \times d\mathbb{Q}$:

$$S^h = \frac{\alpha^a}{\alpha^h} \cdot S^a.$$

(2) implies (1). Fix a $Y_T^a = \frac{d\mathbb{Q}^a}{d\mathbb{P}} \in M_l^a$. Note that S^h is a $\mathbb{Q}^h$ local martingale for any $Y_T^h = \frac{d\mathbb{Q}^h}{d\mathbb{P}} \in M_l^h$, by the definition of M_l^h. But, S^a is a $\mathbb{Q}^a$ local martingale. Since the local martingale property is preserved by stochastic integration (see [58], Theorem 25, p. 170), we have that $S^h = \alpha \cdot S^a$ is a $\mathbb{Q}^a$ local martingale, which implies that $M_l^h \subset M_l^a$.

Since $\alpha \neq 0$, we can reverse the previous argument with $S^a = \frac{1}{\alpha} \cdot S^h$, hence $M_l^h \subset M_l^a$. This completes the proof. $\square$

This characterization theorem shows that in an incomplete market, bubbles have no impact on an investor's value function (welfare) if and only if there is an admissible s.f.t.s. that enables the investor to avoid any loss on the risky asset due to the bubble. This is condition (2) in the above theorem. Of course, condition (1) guarantees that the value function is unchanged with or without bubbles. The following section applies this characterization to an incomplete economy with a Markov diffusion process that generalizes that contained in section 3.3.4 above.

3.4.4 Markov Diffusion Price Processes

This section studies an incomplete actual economy where the asset price process follows the Markov diffusion process generalizing that contained section 3.3.4 above, i.e.

$$dS_t^a = S_t^a \sigma^a(t, S_t^a) \left(b(t, S_t^a, Y_t) + W_t^S \right)$$

$$dY_t = \mu_t^Y dt + \sigma_t^Y dW_t^Y$$

where b is the market price of risk and W^Y and W^S are two correlated Brownian motions initialized at zero.

Assuming NFLVR holds, under any $\mathbb{Q} \in \mathcal{M}_l$, we have that

$$dS_t^a = S_t^a \sigma^a(t, S_t^a) d\tilde{W}_t^S$$

by the martingale representation theorem where $d\tilde{W}_t^S$ is a Brownian motion under $\mathbb{Q}$.

The Markov property of this diffusion process implies that

$$S_t^h = f(t, S_t^a)$$

where

$$f(t, x) = \mathbb{E}^{\mathbb{Q}}[S_T^a | S_t^a = x].$$

By Ito's formula (notice again that dt term is zero),

$$dS_t^h = \frac{\partial f(t, S_t^a)}{\partial S_t^a} dS_t^a.$$

This expression is valid regardless of choice of $\mathbb{Q}$. Given this, a direct application of condition (2) in theorem 3.4.2 gives the following theorem.

Theorem 3.4.3. *(Markov Diffusion Process - Incomplete Market)*

The investor's value function is the same in both the actual and hypothetical economies if and only if

$$\frac{\partial f(t,x)}{\partial x} > 0$$

for all (t,x).

3.4.5 A Counter Example (Increasing Value Function with Bubbles)

This section gives an example of an incomplete market where the equivalent budget constraint condition (1) in the previous theorem is violated, and the investor in the actual economy has a larger value function with bubbles. The example is a special case of Section 11.8 in Chapter 11 of [32].

Suppose that the investor has logarithmic utility $U(x) = \log(x)$. We assume the risky asset in the actual economy $S^a := S^h + \beta$ with

$$dS_t^h = S_t^h V_t^h(\theta_t^1 dt + dW_t^1)$$

$$d\beta_t = \beta_t V_t^\beta(\theta_t^2 dt + dW_t^2)$$

where W^1, W^2 are independent Brownian motions with $W_0^1 = 0, W_0^2 = 0$ generating the filtration $\mathbb{F}$, $\theta_t^1 := \theta_t S_t^h V_t^h$ and $\theta_t^2 := \theta_t \beta_t V_t^\beta$ for some $\mathbb{F}$ progressive process θ under $\mathbb{P}$ where V_t^h and V_t^h are $\mathbb{F}$ adapted processes.

The dynamics of the actual price process S^a is

$$dS_t^a = S_t^a \left(b_t dt + \sigma_t^1 dW_t^1 + \sigma_t^2 dW_t^2 \right)$$

where $b_t := \frac{\theta_t^1 S_t^h V^h + \theta_t^2 \beta_t V_t^\beta}{S_t^a}$, $\sigma_t^1 := \frac{S_t^h V_t^h}{S_t^a}$, and $\sigma_t^2 := \frac{\beta_t V_t^\beta}{S_t^a}$. As shown in [32], p. 234, the optimal supermartingale deflator Y_T^a is given by

$$Y_T^a = \exp\left(-\int_0^T \theta^a \cdot dW - \frac{1}{2} \int_0^T \|\theta^a\|^2 dt \right)$$

where defining $\sigma_t := (\sigma_t^1, \sigma_t^2)^\top$, we have

$$\theta_t^a := \sigma_t^\top (\sigma_t \sigma_t^\top)^{-1} b_t = \frac{\theta_t^1 S_t^h V_t^h + \theta_t^2 \beta_t V_t^h}{(S_t^h V_t^h)^2 + (\beta_t V_t^\beta)^2} (S_t^h V_t^h, \beta_t V_t^h)^\top = \left(\theta_t^1, \theta_t^2 \right)^\top .$$

Here, we assume that σ_t is of full-rank and $\int_0^T \|\theta^a\|^2 dt < \infty$.

Assume further that $\mathbb{E}^{\mathbb{P}}[Y_T^a] = 1$, so that Y_T^a is a martingale deflator and a probability density with respect to $\mathbb{P}$. Then, $\mathbb{Q}^a$ defined by $Y_T^a = \frac{d\mathbb{Q}^a}{d\mathbb{P}}$ is an equivalent probability measure, and S^a, S^h and β are $\mathbb{Q}^a$ local martingales.

Finally, choose V^h and V^β so that S^h is a true $\mathbb{Q}^a$ martingale and β is a strict local $\mathbb{Q}^a$ martingale such that $\beta_T = 0$. For example, $V^h \equiv 1$ and $V_t^\beta = \frac{1}{\sqrt{T-t}}$ satisfies these conditions under suitable integrability conditions for θ.

Combined, this implies that

$$S_t^h = \mathbb{E}^{\mathbb{Q}^a}[S_T^h | \mathcal{F}_t] = \mathbb{E}^{\mathbb{Q}^a}[S_T^h + \beta_T | \mathcal{F}_t] = \mathbb{E}^{\mathbb{Q}^a}[S_T^a | \mathcal{F}_t].$$

Consequently, S^h is the risky asset's fundamental value and β is a price bubble.

In the hypothetical economy, the risky asset's price evolution is

$$dS_t^h = S_t^h V_t^h \left(\theta_t^1 dt + dW_t^1 \right)$$

and the corresponding optimal supermartingale deflator in the hypothetical economy is given by

$$Y_T^h = \exp\left(-\int_0^T \theta^h \cdot dW - \frac{1}{2} \int_0^T \|\theta^h\|^2 dt \right)$$

where $\theta_t^h := (\theta_t^1, 0)^\top$.

From [32], p. 237, the value functions in the two economies are:

$$v^a(x) = \log(x) + \frac{1}{2}\mathbb{E}[\int_0^T \|\theta_t^a\|^2 dt]$$

$$v^h(x) = \log(x) + \frac{1}{2}\mathbb{E}[\int_0^T \|\theta_t^h\|^2 dt].$$

Then, $v^a(x) > v^h(x)$ follows from the fact that

$$\|\theta_t^a\|^2 = (\theta_t^1)^2 + (\theta_t^2)^2 > (\theta_t^1)^2 = \|\theta^h\|^2.$$

We see that in this example, the existence of price bubble in the actual economy leads to a higher risk premium and a larger optimal logarithmic utility for the investor in the actual economy.

To understand this result, let π_t^a, π_t^h denote the optimal holdings in the risky asset for the actual and hypothetical markets as a proportion of wealth, respectively. Then, [32], p. 236, shows

$$\pi_t^a = \left(\sigma_t \sigma_t^\top\right)^{-1} b_t = \theta_t S_t^a$$

$$\pi_t^h = \theta_t S_t^h.$$

As seen, the investor's optimal trading strategy holds the same fraction θ_t of the risky asset's price, regardless of the existence of a price bubble. And, because $S_t^a > S_t^h$, $\pi_t^h < \pi_t^a$. Hence, the investor's increased welfare is due to a larger percentage holding in the risky asset when a bubble exists. The investor rides the bubble to earn a larger risk premium due to the bubble's existence.

Finally, note that in this example the set of martingale deflators differs across the actual and hypothetical economies, i.e. $\mathcal{M}^a \neq \mathcal{M}^h$, because the risky asset's price evolution in the actual market is driven by two Brownian motions (W^1, W^2)

while the risky asset's price in the hypothetical economy is only driven by one (W^1).

3.5 Conclusion

This paper studies the problem of portfolio optimization in a finite horizon, arbitrage-free, frictionless and competitive market with trading in a single risky asset and a money market account, where the risky asset exhibits a price bubble. We show that in a complete market, the existence of a price bubble does not change an investor's welfare relative to an otherwise identical market with no bubble. The reason is that the optimal trading strategy enables the investor to avoid the losses due to a bubble and still obtain the risky asset's fundamental value. In an incomplete market, we provide a sufficient condition for the same conclusion to apply. This sufficient condition holds for a large class of Markov diffusion processes. An example is provided to show that bubbles can increase an investor's welfare, if the sufficient condition is violated.

CHAPTER 4

ISOSPECTRAL SCHEMES FOR EXACT ALGORITHMS FOR STOCHASTIC PROCESSES AND PDE'S

4.1 Introduction

The understanding of the dynamics of random phenomena or the solution of deterministic evolution equations is fundamental to addressing a wide range of modern problems in various fields, including sciences, engineering, economics, and social sciences. However, due to the inherent complexity of these systems and their continuous nature, a vast amount of mathematical research has been devoted to establish effective methods for the numerical approximation of quantities of interest using computers. We refer to the excellent monographs [19, 6, 41, 18] that cover a wide range of fascinating topics for stochastic simulation that revolve around the classical idea of averaging random samples by the Monte Carlo method, and to [15, 63] for a clear exposition of the methods and applications used in the numerical solution of partial differential equations.

In this paper, we propose a novel paradigm to compute quantities of the form $P_t f(x), t \geq 0, x \in \mathrm{S}$, a continuum, where the family of linear operators $P = (P_t)_{t\geq 0} \subset \mathbf{B}(\mathrm{H})$, the unital algebra of bounded linear operators on the separable Hilbert space $\mathrm{H} = \mathrm{L}^2(\mathrm{S}, \mu)$, takes one of the two following forms.

[A] With $X = (X_t)_{t\geq 0}$ a stochastic process issued from $X_0 = x \in \mathrm{S}$, its state space, we have

$$P_t f(x) = \mathbf{E}_x[f(X_t)] \tag{4.1}$$

[B] $P = (P_t = e^{tA})_{t\geq 0}$ is a $\mathcal{C}_0$-semigroup in H, that is, for $f \in \mathrm{Dom}(A) \subset \mathrm{H}$, the domain of A, the generator of P, the mapping $(t,x) \mapsto u_f(t,x) = P_t f(x)$ is the unique strong solution to the following Cauchy problem

$$\begin{cases} \frac{d}{dt}u_f(t,x) = Au_f(t,x) \quad (t,x) \in \mathbb{R}_+ \times \mathrm{S} \\ u_f(0,x) = f(x) \quad x \in \mathrm{S} \end{cases} \tag{4.2}$$

Note that when X in (A) is a nice Markov process, or, equivalently, P in (B) is positivity-preserving then two representations above coincide.

Our approach, which relies on isospectral schemes, enables to transfer, in an exact way, these two problems to a similar formulation on a lattice, which can be precisely computed. This departure from the classical notion of approximation sets our viewpoint apart and empowers us to design algorithms that are not only highly accurate but also computationally efficient. These two factors combined are pivotal nowadays, as they significantly contribute to reducing the total energy consumed by numerical computer-oriented algorithms in the field of engineering.

We also point out that, from an operator theoretical standpoint, isospectral relations between operators acting on Hilbert spaces with different underlying topologies are natural, in the sense that the spectrum of an operator, a set of points in the complex plane, is independent of the underlying geometry of the Hilbert space. Several fundamental isospectral schemes have been proposed in various areas of mathematics including group representation theory, operator theory and algebra, mathematical physics and probability theory, see the recent paper [52], and the references therein, where the theory of self-adjoint operators is revisited in this spirit. The purpose of this paper is to identify a context when such connections enable to design original and efficient numerical algorithms.

In Section 4.2, we present two types of isospectral schemes which are relevant for numerical purposes. One is based on the concept of intertwining, and the other one relies on interweaving relations which have been introduced recently by Miclo and Patie [47]. As an illustration, we also describe specific instances of our toy examples, the squared Bessel (for short BESQ) and the CIR semigroups of index 0, and their Dyson analogues, both families playing an important role in the theory and applications of Markov semigroups. In Section 4.3, we provide several exact simulation algorithms and numerical schemes that exploit gateway and interweaving relations and the results of their performance of several test cases. Finally, a general framework for lifting a family P satisfying (A) or P in (B) to the Weyl chamber is given in Section 4.4 and applied to develop the Dyson BESQ and CIR semigroups.

4.2 Isospectral numerical schemes

Throughout, unless stated, we assume that the family of linear operators $(P_t)_{t\geq 0} \subset \mathbf{B}(\mathrm{H})$ can be represented as in (A) or (B) above, and denote by $\mathbb{H} = \ell^2(\mathbb{S}, \mathbb{m})$, with $\mathbb{S}$ a lattice, a separable Hilbert space, and by $\mathbf{B}(\mathbb{H}, \mathrm{H})$ the set of bounded operators on $\mathbb{H}$ into H.

4.2.1 Gateway relations

Assume that there exist $\Lambda \in \mathbf{B}(\mathbb{H}, \mathrm{H})$ Markov and $(\mathbb{P}_t)_{t\geq 0} \subset \mathbf{B}(\mathbb{H})$ such that, for any $t \geq 0$ and any $\mathbb{f} \in \mathbb{H}$, the following gateway relation holds

$$P_t \Lambda \mathbb{f}(x) = \Lambda \mathbb{P}_t \mathbb{f}(x) \tag{4.3}$$

By Λ Markov, we mean that, for any fixed $x \in \mathrm{S}$, writing $\mathbb{d}_n(p) = \mathbb{I}_{\{n=p\}}$, the sequence $\mathbb{p}_x = (\Lambda \mathbb{d}_n(x))_{n \in \mathbb{S}}$ defines a probability measure on $\mathbb{S}$.

The relation (4.3) is in fact a specific instance of an intertwining relation. However, to underscore the bridging of operators that act on a continuum and a lattice, the term gateway was coined in [46].

The relation (4.3) relates isospectral operators in the following sense. In the case when the two operators P_t and $\mathbb{P}_t$ are normal, i.e. they commute with their adjoint $P_t^* P_t = P_t P_t^*$, Douglas [13] showed that the two operators are in fact unitarily equivalent implying that their spectrum, including multiplicities, are identical, see Remark 4.2.1. Otherwise, when one of the operator is non-normal, under some conditions on Λ, we have a weak notion of isospectrality, as the two operators share the same spectrum with possibly different multiplicities and its decomposition between point, continuous and residual parts may not be preserved, see [54, Chap. 11] for the case of point/residual spectrum, [53] for the case of point/continuous/residual spectrum, and [1] for a study of the spectral theory of quasi-hermitians operators used in quantum mechanics.

We indicate that there is a rich literature regarding gateway relations in different context. In statistical mechanics, under the duality terminology, this goes back to the work of Spitzer [62], see also [24, 27]. More recently, there have been a renewed interest in diverse areas, see [16] and the references therein for interacting particle models, [20] in biology, [48] for some non-self-adjoint Markov semigroups, [5] at the interplay of representation theory and combinatorics, [2, 7] for semigroups on the Weyl chambers, and, [52] for a spectral construction of such relations. Since the list of references is too large, we have created a website[1] collecting further in-

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[1] `https://sites.google.com/cornell.edu/gateways`

stances of gateway/duality relations. This work serves to highlight the relevance of such relations to numerical analysis and to encourage the development of a comprehensive theory of gateways relations from the perspective of spectral theory.

We now specify how the relation (4.3) translates in each of our two settings.

- [AG] Assuming first that both P and $\mathbb{P}$ fall in the setting (A), then the identity (4.3) reads, for any $t \geq 0$ and $f = \Lambda \mathbb{f} \in \mathrm{Ran}(\Lambda) \subset \mathrm{H}$, as

  $$\mathbf{E}_x[f(X_t)] = \mathbf{E}_{\mathbb{p}_x}[\mathbb{f}(\mathbb{X}_t)] \tag{4.4}$$

  where $\mathbb{P}_t \mathbb{f}(n) = \mathbf{E}_n[\mathbb{f}(\mathbb{X}_t)]$ with $\mathbb{X} = (\mathbb{X}_t)_{t \geq 0}$ a continuous-time stochastic process on the lattice $\mathbb{S}$, and, for any measure $\mathbb{p}$ on $\mathbb{S}$, $\mathbf{E}_{\mathbb{p}}[\mathbb{f}(\mathbb{X}_t)] = \sum_{n \in \mathbb{S}} \mathbf{E}_n[\mathbb{f}(\mathbb{X}_t)]\mathbb{p}(n)$.

  The identity (4.4) relates in an exact and simple manner two stochastic objects of different natures with the only constraint that $f \in \mathrm{Ran}(\Lambda)$ the range of Λ. For instance, when $\mathbb{X}$ is a continuous-time Markov chain on $\mathbb{S}$, it is possible to simulate exactly its trajectories as the holding time in any state is random, and, more specifically, exponentially distributed. Similar ideas can be developed in the case when $\mathbb{X}$ is merely semi-Markov, as the holding times are also random.

- [BG] Assuming now that both P and $\mathbb{P}$ fall in the setting (B), then the identity (4.4) yields an exact relation between the solution to the Cauchy problem (4.2) and a solution to the following lattice Cauchy problem

  $$\begin{cases} \frac{d}{dt}\mathbb{u}_{\mathbb{f}}(t,n) = \mathbb{A}\mathbb{u}_{\mathbb{f}}(t,n) \quad (t,n) \in \mathbb{R}_+ \times \mathbb{S} \\ \mathbb{u}_{\mathbb{f}}(0,n) = \mathbb{f}(n) \quad n \in \mathbb{S} \end{cases} \tag{4.5}$$

  where $\mathbb{u}_{\mathbb{f}}(t,n) = \mathbb{P}_t \mathbb{f}(n)$ with $\mathbb{f} \in \mathrm{Dom}(\mathbb{A})$. More specifically, for any $t \geq 0$

and $f = \Lambda\mathbb{f} \in \mathrm{Dom}(A) \cap \mathrm{Ran}(\Lambda) \subset \mathrm{H}$, we have

$$u_f(x,t) = \mathbf{E}\left[\mathbb{u}_{\mathbb{f}}(t, \mathbb{I}(x))\right] \tag{4.6}$$

where $\mathbb{I}(x)$ is a random variable on $\mathbb{S}$ whose law is $\mathbb{p}_x$. The previous identity enables, up to a specific choice of the initial condition and a randomization of the solution, to design an exact space-discretization scheme for the Cauchy problem (4.2) based on (4.5).

Note that if $\mathbb{P}$ is the semigroup of a Markov process $\mathbb{X}$ on $\mathbb{S}$, then one may use the stochastic approach described in (AG) to compute $\mathbb{u}_{\mathbb{f}}(t,n)$, and hence $u_f(x,t)$, the solution to the Cauchy problem (4.2), via (4.6).

As an illustration of the gateway relations (4.3) and (4.6), we now present two toy examples for the numerical experiments that will be discussed in Section 4.3.

Let $Q = (Q_t)_{t\geq 0}$ (resp. $(\mathbb{Q}_t)_{t\geq 0}$) be the Markov $\mathcal{C}_0$-semigroup in $\mathrm{H} = \mathrm{L}^2(\mathbb{R}_+)$ (resp. $\mathbb{H} = \ell^2(\mathbb{Z}_+)$) of the squared Bessel (BESQ) process X (resp. the discrete BESQ birth-death chain $\mathbb{X}$) of index $\nu = 0$, whose infinitesimal generator $A = B$ (resp. $\mathbb{A} = \mathbb{B}$) is given, for any $f \in \mathrm{Dom}(B) \cap \mathrm{C}^2(\mathbb{R}_+)$, the space of twice continuously differentiable functions on $\mathbb{R}_+$, (resp. $\mathbb{f} \in \mathrm{Dom}(\mathbb{B})$), by

$$Bf(x) = xf''(x) + f'(x) \quad (\text{resp. } \mathbb{B}\,\mathbb{f}(n) = (n+1)\mathbb{f}(n+1) + n\mathbb{f}(n-1) - (2n+1)\mathbb{f}(n)) \tag{4.7}$$

Next, the scaling property of Q entails that $K = (K_t)_{t\geq 0}$ defined, for any $t \geq 0$, by

$$K_t = Q_{e^t - 1} d_{e^{-t}} = d_{e^{-t}} Q_{1-e^{-t}} \tag{4.8}$$

is the $\mathcal{C}_0$-semigroup on $\mathrm{L}^2(\mathbb{R}_+, \gamma)$, where $\gamma(x) = e^{-x}$, of an ergodic Markov process Y, namely the Laguerre process or also known in mathematical finance as the

CIR model. Let us now define $\mathbb{K} = (\mathbb{K}_t)_{t\geq 0}$ as $\mathbb{K}_t = \mathbb{Q}_{e^t-1}\mathbb{D}_{e^{-t}}$, where, for any $0 \leq c \leq 1, n \in \mathbb{Z}_+$, $\mathbb{D}_c\mathbb{f}(n) = \mathbf{E}\left[\mathbb{f}(\mathrm{Bin}(n,c))\right]$ with $\mathrm{Bin}(n,c)$ a binomial random variable. $\mathbb{K}$ turns out to be the $\mathcal{C}_0$-semigroup in $\ell^2(\mathbb{Z}_+, \mathbb{g})$, where $\mathbb{g}(n) = 2^{-n-1}$, of an ergodic birth-death chain $\mathbb{Y}$. The infinitesimal generator $A = L$ (resp. $\mathbb{A} = \mathbb{L}$) of K (resp. $\mathbb{K}$) takes the form, for a function $f \in \mathrm{Dom}(L)$ (resp. $\mathbb{f} \in \mathrm{Dom}(\mathbb{L})$),

$$Lf(x) = Bf(x) - xf'(x) \quad (\text{resp. } \mathbb{L}\mathbb{f}(n) = \mathbb{B}\mathbb{f}(n) - n(\mathbb{f}(n-1) - \mathbb{f}(n))) \tag{4.9}$$

We mention that these two family of diffusions and birth-death chains have not only played a central role in the theory of Markov processes, see [40] but also have found applications in different fields of sciences, such as mathematical finance, biology, mathematical physics, queuing theory to name but a few.

Then, writing $\mathbb{pois}_x$ the distribution of $\mathbb{Pois}(x)$, a Poisson random variable of parameter x, and

$$\Lambda\mathbb{f}(x) = \mathbf{E}\left[\mathbb{f}(\mathbb{Pois}(x))\right] = \sum_{n\in\mathbb{Z}_+} \mathbb{f}(n)\frac{x^n}{n!}e^{-x} \tag{4.10}$$

we get, from Patie and Miclo [46], the following gateway relations. First, for any $t \geq 0$ and $f = \Lambda\mathbb{f} \in \mathrm{Ran}(\Lambda) \subset \mathrm{L}^2(\mathbb{R}_+)$, we have

$$\mathbf{E}_x[f(X_t)] = Q_t\Lambda\mathbb{f}(x) = \Lambda\mathbb{Q}_t\mathbb{f}(x) = \mathbf{E}_{\mathbb{pois}_x}[\mathbb{f}(\mathbb{X}_t)] \tag{4.11}$$

where $\Lambda \in \mathbf{B}(\ell^2(\mathbb{Z}_+), \mathrm{L}^2(\mathbb{R}_+))$ is Markov, and, if in addition $\mathbb{f} \in \mathrm{Dom}(\mathbb{B})$, then, writing $u_f(t,x) = \mathbf{E}_x[f(X_t)]$ and $\mathbb{u}_{\mathbb{f}}(t,n) = \mathbf{E}_n[\mathbb{f}(\mathbb{X}_t)]$,

$$u_f(t,x) = \mathbf{E}\left[\mathbb{u}_{\mathbb{f}}(t, \mathbb{Pois}(x))\right] \tag{4.12}$$

Moreover, we have, for any $t \geq 0$ and $f = \Lambda\mathbb{f} \in \mathrm{Ran}(\Lambda) \subset \mathrm{L}^2(\gamma)$,

$$\mathbf{E}_x[f(Y_t)] = K_t\Lambda\mathbb{f}(x) = \Lambda\mathbb{K}_t\mathbb{f}(x) = \mathbf{E}_{\mathbb{pois}_x}[\mathbb{f}(\mathbb{Y}_t)] \tag{4.13}$$

where $\Lambda \in \mathbf{B}(\ell^2(\mathbb{g}), \mathrm{L}^2(\gamma))$, and, if $\mathbb{f} \in \mathrm{Dom}(\mathbb{L})$, writing $v_f(t,x) = \mathbf{E}_x[f(Y_t)]$ and $\mathbb{v}_{\mathbb{f}}(t,n) = \mathbf{E}_n[\mathbb{f}(\mathbb{Y}_t)]$

$$v_f(t,x) = \mathbf{E}\left[\mathbb{v}_{\mathbb{f}}(t, \mathbb{Pois}(x))\right] \tag{4.14}$$

Remark 4.2.1. By a result of Douglas [13], since Λ is a quasi-affinity, i.e. one-to-one with dense range, the gateway relation (4.11) can be lifted to a unitarily equivalence between the two operators, yielding to the identity, for any $x \geq 0, t > 0$,

$$u_f(t, x) = G_0^* \mathbb{u}_{G_0 f}(t, .)(x) \tag{4.15}$$

where $G_0 : \mathrm{L}^2(\mathbb{R}_+) \mapsto \mathrm{L}^2(\mathbb{Z}_+)$ is an unitary operator, i.e. G_0 and its adjoint G_0^* are onto, and, its inverse $G_0^{-1} = G_0^*$, where $G_0 f(n) = \int_0^\infty f(x)\mathcal{L}_n(2x)e^{-x}dx$ and $G_0^* \mathbb{f}(x) = \sum_{n \in \mathbb{Z}_+} \mathbb{f}(n)\mathcal{L}_n(2x)e^{-x}$, where $(\mathcal{L}_n)_{n \geq 0}$ is the sequence of the classical Laguerre polynomials. Note that a similar unitarily equivalence holds for the CIR semigroups.

Remark 4.2.2. We note the gateway relation (4.11) (resp. (4.13)) and the interweaving relations below extend in full generality between two family of self-adjoint continuous and discrete processes on $\mathbb{R}_+$ and $\mathbb{Z}_+$, also called BESQ (resp. CIR) processes, indexed by $\nu > -1$. For clarity of presentation, we have stated the $\nu = 0$ case where the notation is slightly lighter.

4.2.2 Interweaving relations

We now provide a refinement of the gateway relations which has been recently proposed by Miclo and Patie [47], see also [7], in the setting (B), which we now assume. More specifically, suppose that, in addition to the gateway relation (4.3), there exists another $\overline{\Lambda} \in \mathbf{B}(\mathrm{H}, \mathbb{H})$ Markov and a positive random variable τ such that, for any $t \geq 0$ and $f \in \mathrm{H}$, the following hold

$$\mathbb{P}_t \overline{\Lambda} f(n) = \overline{\Lambda} P_t f(n) \text{ and } \Lambda\overline{\Lambda} f(x) = P_\tau f(x) \tag{4.16}$$

where $P_\tau = \int_0^\infty P_s \mathbf{P}(\tau \in ds)$. Several instances of such interweaving relationships are presented in the aforementioned reference, see also [56, 8, 7]. It is also shown that interweaving, up to the delay time τ, enables to transfer many deep properties from one semigroup $\mathbb{P}$ to P which includes spectral, hypercontractivity, ultracontractivity, hypercoercivity phenomena, entropy convergence to equilibrium and mixing properties.

Combining now (4.3) with (4.16), and since, by the semigroup property, $P_{t+\tau} = P_t \Lambda \overline{\Lambda} = \Lambda \mathbb{P}_t \overline{\Lambda}$, one gets, for all $f \in \mathrm{H}$ and $t \geq 0$, that

$$P_{t+\tau} f(x) = \Lambda \mathbb{P}_t \overline{\Lambda} f(x) \tag{4.17}$$

Then, using the notation used and introduced in (BG), the relation (4.17) provides the following exact relation between the solutions of the Cauchy problems (4.2) and (4.5). For any $t \geq 0$ and $f \in \mathrm{Dom}(A)$, observing that $\mathbb{f} = \overline{\Lambda} f \in \mathrm{Dom}(\mathbb{A}) \subset \mathbb{H}$,

$$\mathbf{E}_x[u_f(x, t+\tau)] = \mathbf{E}\left[\mathbb{u}_{\mathbb{f}}(t, \mathbb{I}(x))\right] \tag{4.18}$$

In particular, when $\tau = \delta_{\bar{t}}$, the degenerate distribution at $\bar{t} > 0$, the previous relation reduces to

$$u_f(x, t+\bar{t}) = \mathbf{E}\left[\mathbb{u}_{\mathbb{f}}(t, \mathbb{I}(x))\right] \tag{4.19}$$

The last two identities enable, up to a randomization of the initial condition and of the solution, to design an exact space-discretization scheme for the Cauchy problem (4.2) based on (4.5) which is, this time, valid for any $f \in \mathrm{Dom}(A)$ but the solution is obtained after a delay τ which may be deterministic. Note that when both semigroups P and $\mathbb{P}$ are Markovian, then, with the notation introduced in (4.4), in particular in such a case, $P_\tau f(x) = \mathbf{E}_x[f(X_\tau)]$, we have, for any $t \geq 0$ and $f \in \mathrm{H}$, writing $\mathbb{f} = \overline{\Lambda} f$,

$$\mathbf{E}_x[f(X_{t+\tau})] = \mathbf{E}_{\mathbb{P}_x}[\mathbb{f}(\mathbb{X}_t)] \tag{4.20}$$

The interweaving identity also relates in an exact manner the two stochastic objects, this time, for any $f \in \mathrm{H}$ but with some delay τ.

Coming back to our toy examples, the following interweaving relations have been proved in [46]. First, for any $f \in \mathrm{L}^2(\mathbb{R}_+)$ (or $f \in \mathrm{Dom}(B)$ in the case of the solution of the Cauchy problem) and $t > 0$,

$$u_f(t+1, x) = \mathbf{E}_x[f(X_{t+1})] = \mathbf{E}_{\mathbb{p}_x}[\Lambda^* f(\mathbb{X}_t)] = \mathbf{E}\left[\mathbb{u}_{\Lambda^* f}(t, \mathbb{Pois}(x))\right] \tag{4.21}$$

where, with $\mathrm{Gam}(a)$ a gamma random variable of parameter a, $\Lambda^* f(n) = \mathbf{E}\left[f(\mathrm{Gam}(n+1))\right] \in \mathbf{B}(\mathrm{L}^2(\mathbb{R}_+), \ell^2(\mathbb{Z}_+))$ is the adjoint of Λ. Moreover, we have, for any $f \in \mathrm{L}^2(\gamma)$ (or $f \in \mathrm{Dom}(L)$ in the case of the solution of the Cauchy problem) and $t > 0$,

$$v_f(t + \ln 2, x) = \mathbf{E}_x[f(Y_{t+\ln 2})] = \mathbf{E}_{\mathbb{p}_x}[\widehat{\Lambda} f(\mathbb{Y}_t)] = \mathbf{E}\left[\mathbb{v}_{\widehat{\Lambda} f}(t, \mathbb{Pois}(x))\right] \tag{4.22}$$

where $\widehat{\Lambda} f(n) = \mathbf{E}\left[f(\mathrm{Gam}(n+1)/2)\right] \in \mathbf{B}(\mathrm{L}^2(\gamma), \ell^2(\mathbb{g}))$ is the adjoint of Λ.

The identities (4.12) and (4.21) (resp. (4.14) and (4.22)) offer an exact space-discretization scheme for the partial differential equation (4.2) with $A = B$ (resp. with $A = L$) based on (4.5) with $\mathbb{A} = \mathbb{B}$ (resp. with $\mathbb{A} = \mathbb{L}$). It means that from the finite-difference scheme based on the generator of the birth-death chain, we get, up to a Poissonization, an exact solution of the PDE associated to the diffusion which is in contrast with the classical approximation scheme based on finite-difference methods. There are two notable differences between our exact scheme and the classical finite-difference approximation ones. First, the initial condition for our discrete scheme is not a discrete approximation of the initial condition of the PDE but is defined by means of one of the gateway operators. Moreover, the solution of our discrete scheme needs to be Poissonized to give the exact solution of the PDE. The description of the corresponding algorithm is also given in Section 4.3, with numerical outcomes illustrating that our approaches are highly efficient.

An additional significant attribute of the gateway and interweaving relations, such as (4.3), is their inherent stability under various transformations, such as n-fold tensorization, (pseudo)-unitary conjugations, see Section 4.4 for an illustration of a combination of these, but, also by orthogonal direct sums and stochastic-time change; see [52] for more details. For the latter, consider a non-decreasing and non-negative stochastic process $\mathcal{T} = (\mathcal{T}_t)_{t\geq 0}$. Then, it is easy to show that, for any $t \geq 0$ and $f = \Lambda \mathbb{f} \in \mathrm{Ran}(\Lambda) \subset \mathrm{H}$,

$$\mathbf{E}_x[f(X_{\mathcal{T}_t})] = \mathbf{E}_{\mathbb{p}_x}[\mathbb{f}(\mathbb{X}_{\mathcal{T}_t})] \tag{4.23}$$

This includes the classical setting of subordination sense of Bochner when $\mathcal{T}$ is a subordinator where the processes $(X_{\mathcal{T}_t})_{t\geq 0}$ and $(\mathbb{X}_{\mathcal{T}_t})_{t\geq 0}$ are again Markov. On the other hand, it also includes the case when $\mathcal{T}$ is the inverse of a subordinator or of an increasing self-similar Markov process, where in general, there is an interesting transformation of Markovian dynamics into subdiffusive ones, i.e. non-Markovian ones, and, from the functional analysis viewpoint, a parabolic problem into a non-local one, i.e. the time-first-order partial derivative in (4.2) is replaced by a non-local operator, see e.g. [64, 55].

4.3 Exact Isospectral Algorithms and Numerical Results

In this section, we present exact simulation algorithms and finite-difference schemes to compute statistics of the form (4.1) and solve for solutions to the Cauchy problem (4.2) based on gateway and interweaving relations. These algorithms are stated for concreteness in the context of BESQ and CIR processes, but are straightforward to adapt to other examples. All the experiments in this paper are run on an Apple 8-Core 2.50 Ghz CPU with 16GB RAM using a single thread.

4.3.1 Exact Algorithms for the BESQ and CIR processes

Consider the BESQ process X (resp. CIR process Y) of index 0 and a function $f = \Lambda\mathbb{f}$ where $\mathbb{f} \in \ell^2(\mathbb{Z}_+)$ (resp. $\mathbb{f} \in \ell^2(\mathbb{g})$). The following exact simulation algorithm computes the quantity

$$u_f(t,x) = \mathbf{E}_x[f(X_t)] \quad (\text{resp. } v_f(t,x) = \mathbf{E}_x[f(Y_t)]) \tag{4.24}$$

for $x \in \mathbb{R}_+$, based on the gateway relations (4.11), (4.13).

Algorithm 1 Gateway Simulation Algorithm for BESQ (resp. CIR) Processes

```
Input: test function $f = \Lambda\mathbb{f}$, test time $t > 0$, initial value $x \in \mathbb{R}_+$, number of paths $\mathtt{N}$
for i=1:N do
    Generate $\mathbb{n}^{(i)} \sim \mathbb{Pois}(x)$ independently
    Simulate birth-death process $\mathbb{X}^{(i)}$ (resp. $\mathbb{Y}^{(i)}$) with initial value $\mathbb{n}^{(i)}$ and generator given by $\mathbb{B}$ (resp. $\mathbb{L}$) until time $t$
end for
Output $u_f(t,x) \simeq \frac{1}{\mathtt{N}}\sum_{i=1}^{\mathtt{N}} \mathbb{f}(\mathbb{X}_t^{(i)}) \left(\text{resp. } v_f(t,x) \simeq \frac{1}{\mathtt{N}}\sum_{i=1}^{\mathtt{N}} \mathbb{f}(\mathbb{Y}_t^{(i)})\right)$
```

Note the restriction in Algorithm 1 of the domain $f \in \mathrm{Ran}(\Lambda)$. The following exact simulation algorithm, based on interweaving relations (4.21), (4.22), trades a larger domain, namely all of $\mathrm{L}^2(\mathbb{R}_+)$ (resp. $\mathrm{L}^2(\gamma)$) for the time restriction $t \geq \tau$, where $\tau = 1$ for BESQ processes (resp. $\tau = \ln 2$ for CIR processes).

Algorithm 2 Interweaving Simulation Algorithm for BESQ (resp. CIR) Processes

```
Input: test function $f$, test time $t \geq 1$ (resp. $t \geq \ln 2$), initial value $x \in \mathbb{R}_+$, number of paths $\mathtt{N}$
for i=1:N do
    Generate $\mathbb{n}^{(i)} \sim \mathbb{Pois}(x)$ independently
    Simulate birth-death process $\mathbb{X}^{(i)}$ (resp. $\mathbb{Y}^{(i)}$) with initial value $\mathbb{n}^{(i)}$ and generator given by $\mathbb{B}$ (resp. $\mathbb{L}$) until time $t-1$
    Generate $g^{(i)} \sim \mathrm{Gam}(\mathbb{X}_{t-1}^{(i)}+1)$ (resp. $g^{(i)} \sim \frac{1}{2}\mathrm{Gam}(\mathbb{Y}_{t-1}^{(i)}+1)$) independently
end for
Output $u_f(t,x) \simeq \frac{1}{\mathtt{N}}\sum_{i=1}^{\mathtt{N}} f(g^{(i)}) \left(\text{resp. } v_f(t,x) \simeq \frac{1}{\mathtt{N}}\sum_{i=1}^{\mathtt{N}} \mathbb{f}(g^{(i)})\right)$
```

Remark 4.3.1. For the simulation of the discrete Bessel birth-death process, the expected of arrivals until time t is of order t^2. This can be made precise but heuristically this happens because $\mathbb{X}$ is transient and at large time t, $\mathbb{X}_t$ is order t and inter-arrival times are order t^{-1}. This is in contrast to simulation of the discrete CIR process whose time complexity is linear in t, which can be attributed to its stationarity. However, due to the self-similarity relation (4.8), we find an extraordinary variant of Algorithms 1 and 2 whose time complexity is order $\ln t$. This variant that simulates the discrete CIR process instead of its BESQ analog is based on the following reformulations of discrete analog of (4.8).

If there exists $\mathbb{f}_t \in \ell^2(\mathbb{m})$ such that $d_{t+1} f = \Lambda \mathbb{f}_t$, then

$$\mathbf{E}_x[f(X_t)] = \mathbf{E}_{\mathbb{pois}_x}[\mathbb{f}_t(\mathbb{Y}_{\ln(t+1)})] \tag{4.25}$$

Furthermore, if $t \geq 1$, then, for any $f \in \mathrm{L}^2(\mathbb{R}_+)$,

$$\mathbf{E}_x[f(X_t)] = \mathbf{E}_{\mathbb{pois}_x}\left[f\left(\frac{t}{2}\,\mathrm{Gam}(\mathbb{Y}_{\ln t} + 1)\right)\right] \tag{4.26}$$

We note that relationships between transient and stationary processes are plentiful and well-studied, e.g. Brownian motion and Ornstein-Uhlenbeck. Indeed, it is straightforward to observe the construction in (4.8) applies to generate a stationary process from any self-similar one. As such, the improvement described can be widely applied to dramatically the time complexity of simulation algorithms.

We present numerical results to demonstrate the efficiency and accuracy of the exact simulation algorithms using several experiments based on the BESQ and CIR processes. For all experiments below, we report our shifted confidence interval in the form of $(\hat{u} - u) \pm w$ where $\hat{u}$ is the simulation estimate, u is the exact value and w is the computed confidence interval width.

Numerical Results for the BESQ Process

We apply exact simulation algorithms to compute the quantity $u_f(t,x)$ where $f(x) = e^{-bx} = \Lambda \mathbb{f}(x)$ and $\mathbb{f}(n) = (1-b)^n$, $b > -\frac{1}{2(1+t)}$. In this case, one has an explicit and simple expression given, for any $x, t > 0$,

$$u_f(t,x) = \frac{1}{1+bt} e^{-\frac{bx}{1+bt}} \tag{4.27}$$

see e.g. [59]. This enables to evaluate the efficiency of the different algorithms in a very precise way. For all test cases, we choose $b = -0.005$, the number of Monte Carlo simulation is $\mathtt{N} = 10^6$. We present results for the following 4 simulation estimators: BESQ gateway estimator given by Algorithm 1, BESQ-CIR gateway estimator given by Algorithm 1 with the variation described in Remark 4.3.1, BESQ interweaving estimator given by Algorithm 2, BESQ-CIR interweaving estimator given by Algorithm 2 with the variation described in Remark 4.3.1.

t	x	gateway	CIR gateway	interweaving	CIR interweaving
3	2	-0.09 ± 0.53	-0.74 ± 0.67	$+0.10 \pm 0.48$	-0.06 ± 0.48
	10	-0.23 ± 0.99	$+0.67 \pm 1.21$	-0.34 ± 0.91	-0.40 ± 0.91
	50	$+0.02 \pm 2.55$	$+0.04 \pm 3.08$	-1.24 ± 2.35	$+1.28 \pm 2.36$
10	2	$+0.68 \pm 1.40$	-0.73 ± 1.87	$+0.28 \pm 1.34$	$+1.25 \pm 1.34$
	10	-1.98 ± 2.13	-1.00 ± 2.73	$+0.06 \pm 2.06$	-1.58 ± 2.06
	50	-4.96 ± 5.08	-0.85 ± 6.33	$+3.23 \pm 4.94$	-2.13 ± 4.94

Table 4.1: Error and 95% confidence interval of simulation error (units: 10^{-4}) of $u_f(t,x)$, $f = e^{-bx}$, $b = -0.005$, $\mathtt{N} = 10^6$

t	x	gateway	CIR gateway	interweaving	CIR interweaving
3	2	4.04	3.55	3.75	3.45
	10	5.55	4.15	4.78	3.87
	50	12.93	6.95	9.70	5.74
10	2	7.91	3.67	7.19	3.59
	10	12.98	4.41	11.74	4.30
	50	38.56	7.81	34.83	7.34

Table 4.2: Computational time (sec.) of simulation of $u_f(t,x)$, $f = e^{-bx}$, $b = -0.005$, $\mathtt{N} = 10^6$

As is shown in the table 4.1 and 4.2, for all 4 algorithms and all test cases, the exact values fall into the provided confidence intervals and the confidence interval width is consistent with the $\mathcal{O}(\frac{1}{\sqrt{\mathtt{N}}})$ Monte Carlo error. In terms of accuracy, all 4 algorithms shows comparable performance. In terms of computational time, as is discussed in Remark 4.3.1, for larger test time t, it is significantly faster to apply the variant that exploits the relation between the (discrete) BESQ and CIR processes. For example, for $(t,x) = (10,50)$, CIR-based simulation is 5 times faster to run than BESQ-based simulation. For the provided test cases, we also observe that interweaving algorithms are slightly faster to run that their gateway counterparts. This is mainly due to the fact that interweaving requires a slightly shorter time horizon to simulate.

Numerical Results for the CIR Process

We apply exact simulation algorithms to compute the quantity $\mathbf{E}_x[f(X_t)]$ where

$$f(x) = \mathcal{L}_n(x) = \Lambda\mathbb{M}_n(x)$$

with

$$\mathcal{L}_n(x) = \sum_{k=0}^{m} (-1)^k \binom{n}{k} \frac{x^k}{k!}$$

the Laguerre polynomial and

$$\mathbb{M}_n(m) = \sum_{k=0}^{m} (-1)^k \binom{m}{k} \binom{n}{k}$$

the Meixner polynomial, see [46]. For $t, x > 0$ and $n \in \mathbb{N}$, the solution is

$$v_f(t,x) = e^{-tn} \mathcal{L}_n(x) \tag{4.28}$$

see e.g. [54]. In the test cases, we consider $n = 2$ and the number of paths is $\mathtt{N} = 10^6$. We present results for the following simulation estimators: CIR gateway estimator given by Algorithm 1, CIR interweaving estimator given by Algorithm 2.

		gateway		interweaving	
		Error and CI (10^{-3})	Time (sec.)	Error and CI (10^{-3})	Time (sec.)
t	x				
3	2	-1.84 ± 4.26	3.78	-0.38 ± 2.18	3.73
	10	-2.14 ± 6.64	4.52	$+1.49 \pm 3.82$	4.49
	50	-10.94 ± 22.62	8.08	-3.11 ± 15.92	7.89
10	2	$+0.48 \pm 4.00$	4.64	-0.64 ± 1.98	4.65
	10	-2.02 ± 4.01	5.45	$+2.91 \pm 2.00$	5.43
	50	-1.72 ± 3.98	9.17	-1.34 ± 2.01	9.18

Table 4.3: Error and 95% confidence interval of simulation and computational time of $v_f(t,x)$, $f = \mathcal{L}_n$, $n = 2$, $\mathtt{N} = 10^6$

Similar to simulation for BESQ processes, for all test cases shown in the Table 4.3, the exact values fall into the provided confidence intervals, which have width consistent with the $\mathcal{O}(\frac{1}{\sqrt{\mathtt{N}}})$ Monte Carlo error.

4.3.2 Exact finite-difference schemes for PDE's

In this section, we present several numerical schemes to solve the Cauchy problem (4.2). As highlighted earlier, by exploiting gateway and interweaving relations, we reduce the complex problem of discretization in both space and time to a simpler problem of discretization only in time, thereby significantly reducing approximation error and avoiding common concerns such as stability and convergence.

The following algorithms take two parameters: a fixed time-step $\Delta t > 0$ and an upper boundary $\mathbb{N}_{max} \in \mathbb{Z}_+$. In application, the time-step Δt can be taken to be variable in time and tailored to each specific problem. The upper boundary $\mathbb{N}_{max}$ is chosen to be a large truncation threshold for the discrete Cauchy problem (4.5) such that the solution $\mathbb{u}_{\mathbb{f}}(t,n)$ is computed for $n \in \{0,1,...,\mathbb{N}_{max}\}$ subject to well-chosen boundary conditions at $n = \mathbb{N}_{max}+1$.

The first numerical scheme for the BESQ and CIR processes is based on gateway relations (4.12), (4.14) and applies for $f \in \mathrm{Ran}(\Lambda)$ and any time $t > 0$.

Algorithm 3 Gateway Numerical Scheme for BESQ (resp. CIR) Processes

```
Input: initial condition $f = \Lambda\mathbb{f}$, test time $t$, time discretization $\Delta t$, upper boundary $\mathbb{N}_{max} \in \mathbb{Z}^+$
Compute the solution $\mathbb{u}_{\mathbb{f}}(t,n)$ (resp. $\mathbb{v}_{\mathbb{f}}(t,n)$) to (4.5) on $\{0,...,\mathbb{N}_{max}\}$ where $\mathbb{A} = \mathbb{B}$ (resp. $\mathbb{A} = \mathbb{L}$); see Remark 4.3.2
for n=0:$\mathbb{N}_{max}$ do
    Compute $\mathbb{pois}_x(n) = \mathbf{P}(\mathbb{Pois}(x) = n)$
end for
Output $u_f(t,x) \simeq \sum_{n=0}^{\mathbb{N}_{max}} \mathbb{pois}_x(n)\mathbb{u}_{\mathbb{f}}(t,n)$ (resp. $v_f(t,y) \simeq \sum_{n=0}^{\mathbb{N}_{max}} \mathbb{pois}_x(n)\mathbb{v}_{\mathbb{f}}(t,n)$)
```

Remark 4.3.2. There are many classical and widely-applied approaches to computing the solution $\mathbb{u}_{\mathbb{f}}(t,n)$ to the discrete Cauchy problem (4.5), which include methods for exact computation and time-discretized forward approximation schemes. We refer to the text of Thomas [63] for a comprehensive discussion. For the sake of demonstration, we apply the following explicit scheme with Dirichlet boundary conditions in our test cases. Let $K\Delta t = t$. Then, $\mathbb{u}_{\mathbb{f}}(t,n) \simeq \hat{\mathbb{u}}^{(K)}(n)$ where $\hat{\mathbb{u}}^{(i)}$ is defined by $\hat{\mathbb{u}}^{(0)} = \mathbb{f}$ and the iteration

$$\hat{\mathbb{u}}^{(i)} = (I + \mathbb{A}_{\mathbb{N}_{max}}\Delta t)\hat{\mathbb{u}}^{(i-1)} \tag{4.29}$$

where $\mathbb{A}_{\mathbb{N}_{max}}$ is the restriction of $\mathbb{A}$ to $\{0, 1, ..., \mathbb{N}_{max}\}$. To ensure stability of the forward scheme, we require that $\Delta t < (2\mathbb{N}_{max}+1)^{-1}$. This condition is, in general, much less restrictive than stability conditions for classical algorithms to solve (4.2) as it is not constrained by space discretization.

The following is analogous numerical scheme applying interweaving relations (4.21), (4.22). Similar to the exact simulation algorithm, it reflects the trade-off between a larger domain and a time restriction.

Algorithm 4 Interweaving Numerical Scheme for BESQ (resp. CIR) Processes

```
Input: initial condition $f$, test time $t \geq 1$ (resp. $t \geq \ln 2$), time discretization $\Delta t$, upper boundary $\mathbb{N}_{max} \in \mathbb{Z}^+$
for n=0:$\mathbb{N}_{max}$ do
    Compute $\mathbb{f}(n) = \mathbf{E}[f(\mathrm{Gam}(n+1))]$ (resp. $\mathbb{f}(n) = \mathbf{E}[f(\mathrm{Gam}(n+1)/2)]$ )using a numerical method.
end for
Compute the solution $\mathbb{u}_{\mathbb{f}}(t-1,n)$ (resp. $\mathbb{v}_{\mathbb{f}}(t-\ln 2,n)$) to (4.5) on $\{0,...,\mathbb{N}_{max}\}$ where $\mathbb{A} = \mathbb{B}$ (resp. $\mathbb{A} = \mathbb{L}$); see Remark 4.3.2
for n=0:$\mathbb{N}_{max}$ do
    Compute $\mathbb{pois}_x(n) = P(\mathbb{Pois}(x) = n)$
end for
Output $u_f(t,x) \simeq \sum_{n=0}^{\mathbb{N}_{max}} \mathbb{pois}_x(n)\mathbb{u}_{\mathbb{f}}(t-1,n)$ (resp. $v_f(t,y) \simeq \sum_{n=0}^{\mathbb{N}_{max}} \mathbb{pois}_x(n)\mathbb{v}_{\mathbb{f}}(t-\ln 2,n)$)
```

Remark 4.3.3. We note that the final summation in Algorithms 3 and 4 and the computation of $\mathbb{f}$ in Algorithm 4 can be replaced by Monte Carlo simulation of Poisson and Gamma random variables, respectively.

Exact Simulation Numerical Results for PDE's

This section contains the results of Algorithms 3 and 4 applied to several test cases based on the BESQ processes. We compare their performance relative to classical explicit and implicit methods for solving the Cauchy problem (4.2), which entail discretization both in space and time. For each method M, we will present the computational error ϵ_M, time t_M and relative overall performance (ROP_M) metric, defined as follows:

$$\text{ROP}_M = 100 \times \frac{t_M \epsilon_M}{t_E \epsilon_E}$$

where t_E and ϵ_E are the time and error for the classical explicit scheme. This score measures the relative performance of method M (taking both accuracy and time into account) over the classical method, where a lower score implies better overall performance of the algorithm.

Consider the test case $f(x) = e^{-bx}$, $b = -0.005$, whose closed-form solution is given by (4.27). Moreover, for the interweaving numerical scheme 4, we have

$$\mathbb{f}(n) = \Lambda^* f(n) = \mathbf{E}\left[f(\text{Gam}(n+1))\right] = (1+b)^{-(n+1)} \tag{4.30}$$

The results are as follows.

		gateway scheme (Algorithm 3)			interweaving scheme (Algorithm 4)		
		Error (10^{-7})	Time (sec.)	ROP	Error (10^{-7})	Time (sec.)	ROP
t	x						
	2	0.070	0.144	0.088	0.031	0.129	0.035
3	10	0.062	0.128	0.072	0.027	0.129	0.032
	50	0.030	0.132	0.044	0.013	0.130	0.019
	2	0.210	0.428	0.076	0.170	0.426	0.061
10	10	0.187	0.425	0.072	0.151	0.424	0.058
	50	0.381	0.426	0.004	0.071	0.424	0.001

Table 4.4: Computational error, computational time and ROP for numerical solution to PDE (4.2) by gateway and interweaving solvers with initial condition $f(x) = e^{-bx}$, $\Delta t = 10^{-4}$, $\mathbb{N}_{max} = 300$

		explicit scheme ($\Delta x = 0.2$)			implicit scheme ($\Delta x = 0.01$)		
t	x	Error (10^{-7})	Time (sec.)	ROP	Error (10^{-7})	Time (sec.)	ROP
3	2	71.993	0.158	100.000	3.674	141.965	4,583.739
	10	69.217	0.159	100.000	3.527	142.016	4,560.582
	50	56.820	0.157	100.000	2.835	142.084	4,506.725
10	2	224.170	0.528	100.000	11.288	474.473	4,529.111
	10	208.723	0.529	100.000	3.815	472.571	1,633.437
	50	7,958.372	0.532	100.000	8,179.781	473.588	91,456.959

Table 4.5: Computational error, computational time and ROP for numerical solution to PDE (4.2) by benchmark solvers with initial condition $f(x) = e^{-bx}$, $\Delta t = 10^{-4}, \mathbb{N}_{max} = 300$

For all test cases shown in table 4.4 and 4.5, we observe improvement in accuracy and efficiency from applying gateway/interweaving solvers. In terms of computational time, gateway and interweaving solvers have speed consistent with the explicit solver, since in both the gateway and interweaving schemes we have applied an embedded explicit scheme. Furthermore, they are 100 times faster than the implicit scheme with chosen Δx. In terms of accuracy, the gateway and interweaving schemes benefited from the lack of space discretization error and we observe an improvement of $20 - 10000$ times over the classical methods. Although the time complexity is comparable to the explicit scheme, the stability condition needed for explicit scheme makes it impossible for the explicit solver to improve by reducing the space discretization step Δx without changing time discretization Δt.

4.3.3 Simulation on the Weyl chamber

We refer the reader to Section 4.4 for details and discussion on Dyson semigroups on the Weyl chamber and their gateway and interweaving relations. In this section, we consider the Dyson BESQ process $\mathbf{X}$ on $W_n(\mathbb{R}_+)$. Given $t \geq 0$, $\mathbf{x} \in W_n(\mathbb{R}_+)$ and a function $\mathbf{f} \in \mathrm{L}^2(\mathbf{m}^{\Delta})$, we aim to compute $u_{\mathbf{f}}(t, \mathbf{x}) = \mathbf{Q}_t\mathbf{f}(\mathbf{x})$. The gateway exact simulation algorithm for simulating the Dyson BESQ on the Weyl chamber is given as follows, where we assume $\mathbf{f} \in \mathrm{Ran}(\boldsymbol{\Lambda})$.

Algorithm 5 Gateway Simulation Algorithm for Dyson BESQ Processes

```
Input: test function $\mathbf{f} = \boldsymbol{\Lambda}\boldsymbol{\mathsf{f}}$, test time $t \geq 0$, initial value $\mathbf{x} = (x_i)_{i=1}^n \in W_n(\mathbb{R}_+)$, number of paths N
for i=1:N do
    generate $\boldsymbol{\mathsf{n}} \in W_n(\mathbb{Z}_+)$, independently, from the distribution $\boldsymbol{\Lambda}(\mathbf{x}, \cdot)$.
    With initial value $\boldsymbol{\mathsf{n}}$, simulate the random walk $\boldsymbol{\mathsf{X}}^{(i)}$ until time $t$ with generator kernel given by
    $$\boldsymbol{\mathsf{B}}(\mathbf{k}, \mathbf{k}\pm e_j) = \mathbb{B}(k_j, k_j\pm 1)\frac{\Delta(\mathbf{k}+e_j)}{\Delta(\mathbf{k})} = \mathbb{B}(k_j, k_j\pm 1)\prod_{i\neq j}\frac{k_j \pm 1 - k_i}{k_j - k_i}, \quad j = 1, \ldots, n \tag{4.31}$$
    where $e_j$ is the elementary basis vector in the $j$th element.
end for
Output $u_{\mathbf{f}}(t, \mathbf{x}) \simeq \frac{1}{N}\sum_{i=1}^{N}\boldsymbol{\mathsf{f}}(\boldsymbol{\mathsf{X}}_t^{(i)})$
```

Remark 4.3.4. In practice, the simulation of $\boldsymbol{\mathsf{n}}$ in Algorithm 5 can be restricted to a large finite domain depending on t that contains points comparable to $\mathbf{x}$. In high dimensional problems, the initial distribution $\boldsymbol{\mathsf{n}}$ may have exponential complexity to generate. Thus, we also provide an algorithm with importance sampling which only requires simulation of the marginal distribution of each dimension.

Algorithm 6 Gateway Simulation Algorithm for Dyson BESQ Processes via Importance Sampling

```
    Input: test function $\mathbf{f} = \mathbf{\Lambda}\mathbb{f}$, test time $t \geq 0$, initial value $x = (x_1, \dots, x_n) \in W_n(\mathbb{R}_+)$, number of paths $\mathtt{N}$
    for i=1:N do
        for $k = 1, \dots, n$, draw $\xi_k \sim \mathbb{Pois}(x_i)$ independently to form $\xi = (\xi_1, \dots, \xi_n)$
        for $k = 1, \dots, n$, simulate the birth-death process $\mathbb{X}^{(k)}$ with generator $\mathbb{B}$ and initial value $X_0^{(k)} = \xi_k$
        let $\boldsymbol{X}_t \in W_n(\mathbb{R}_+)$ be the order statistics of $(\mathbb{X}_t^{(k)})_{k=1}^n$ and $\omega = \frac{\Delta(\mathbb{X}_t^{(1)}, \dots, \mathbb{X}_t^{(n)})}{\Delta(\xi)}$
        let $f_i \leftarrow \omega \cdot \mathbf{f}(\mathbb{X}_t)$
    end for
    Output estimate by $u_{\mathbf{f}}(t, \mathbf{x}) \simeq \frac{1}{\mathtt{N}} \sum_{i=1}^{\mathtt{N}} f_i$
```

Remark 4.3.5. To simulate $\mathbf{E}[\mathbf{f}(\boldsymbol{X}_t^\dagger)]$ where $\boldsymbol{X}^\dagger$ is the dynamical determinantal point process associated to $\boldsymbol{X}$, the process of n independent squared Bessel processes killed upon coincidence, the above algorithm can be applied with ω replaced by 0 if $\boldsymbol{X}_t$ has any two identical elements and otherwise by $\operatorname{sign}(\sigma)$ where σ is the unique permutation such that $(\boldsymbol{X}_t)_\sigma = \boldsymbol{X}_t^\dagger$.

Again, there is an analogous algorithm based on interweaving which extends to all functions in $\mathrm{L}^2(\mathbf{m}^\Delta)$ but comes with a time restriction. We leave the reader to extrapolate the details from Algorithms 2 and 5. Moreover, the relation between BESQ and CIR processes means there is a variant of the exact simulation algorithms for Dyson BESQ processes that has poly-log time complexity via simulation of the discrete CIR Dyson process; see Remark 4.3.1.

Exact Simulation Numerical Results for the Dyson BESQ process

We present numerical results to demonstrate the performance of the Algorithm 6. Consider the function, for any $\mathbf{x} \in W_n(\mathbb{R}_+)$, $\mathbf{f}(\mathbf{x}) = \mathbf{\Lambda}\mathbb{f}(\mathbf{x}) = \prod_{i=1}^n e^{-bx_i}$ where, for any $\mathbf{k} \in W_n(\mathbb{Z}_+)$, $\mathbb{f}(\mathbf{k}) = \prod_{i=1}^n (1-b)^{k_i}, b \in (-(2(1+t))^{-1}, 1)$. With this initial condition, the exact solution is

$$\mathbf{u_f}(t,\mathbf{x}) = \prod_{i=1}^{n}(1+tb)^{-(2i-1)}e^{-\frac{bx_i}{1+tb}}$$

where for $b<0$, we used the fact, that as in the one-dimensional case, the domain of the isospectral identities, extend to an appropriate weighted Hilbert space, see [7] for more details. We have the following numerical results for Algorithm 6. For dimension n, we test for $\mathbf{x} = (1,2,\ldots,2^{n-1})$. We choose parameter $b = 0.005$ and number of paths $\mathtt{N} = 10^6$ for all test cases. The computational results show high accuracy independent of dimension and simulation time horizon. Moreover, note this algorithm is feasible in high dimensions due to its has polynomial time complexity.

n	t	Error and CI (10^{-6})	Time (sec.)
3	1	-1701.95 ± 7917.02	4.52
	5	$+1977.20 \pm 6709.98$	24.56
	20	-1899.66 ± 3689.08	268.29
10	1	-133.23 ± 139.86	428.80
	5	$+0.58 \pm 27.93$	2519.55
	20	$+0.01 \pm 0.06$	16003.26

Table 4.6: Error and 95% confidence interval (CI) of simulation error and computational time of $u_{\mathbf{f}}(t,\mathbf{x})$, $\mathtt{N} = 10^6$

4.4 Tensorized and Dyson versions of the gateway and interweaving relations

In this part, as discussed in Section 4.2, we present some transformations that preserve the structure of gateway and interweaving relations, allowing to design exact simulation algorithms involving operators acting in high-dimensional and more complex geometric structures. For instance, we construct, starting from the setup of the Section 4.2, gateway and interweaving relations for $\mathcal{C}_0$-semigroups on the Weyl chambers of S and $\mathbb{S}$ which are assumed to be ordered sets. As we shall see, this entails that their transition kernels, which may be signed, have a determinantal structure. These types of kernels play a central role in various contexts such as mathematical physics, random matrices and machine learning. We refer to the work of Chee and Patie [7] for more details, and where a comprehensive theory of $\mathcal{C}_0$-semigroups based on transformations of isospectral classes is established.

4.4.1 Tensorization

For $n \in \mathbb{N}$ and $\mathrm{S} \subseteq \mathbb{R}$, we denote by S^n the n-fold cartesian product of S, and, for a linear operator T (resp. a Hilbert space H), $T^{\otimes n}$ (resp. $\mathrm{H}^{\otimes n} \simeq \mathrm{L}^2(\mathrm{S}^n, \mu^n), \mu^n$ is the product measure) stands for its n-fold tensor product. The gateway relation (4.3) extends to the n-dimensional setting, for any $t \geq 0, \mathbf{x}_n \in \mathrm{S}^n$ and $\mathbf{f}_n = \Lambda^{\otimes n}\mathbb{f}_n \in \mathrm{Ran}(\Lambda^{\otimes n}) \subset \mathrm{H}^{\otimes n}$, as follows

$$P_t^{\otimes n}\Lambda^{\otimes n}\mathbb{f}_n(\mathbf{x}_n) = \Lambda^{\otimes n}\mathbb{P}_t^{\otimes n}\mathbb{f}_n(\mathbf{x}_n) \tag{4.32}$$

and the interweaving relation (4.17) reads, when $\tau = \delta_{\bar{t}}, \bar{t} > 0$, for any $t \geq 0, \mathbf{x}_n \in \mathrm{S}^n$ and $\mathbf{f}_n \in \mathrm{H}^{\otimes n}$, as

$$P_{t+\bar{t}}^{\otimes n}\mathbf{f}_n(\mathbf{x}_n) = \Lambda^{\otimes n}\mathbb{P}_t^{\otimes n}\overline{\Lambda}^{\otimes n}\mathbf{f}_n(\mathbf{x}_n) \tag{4.33}$$

For sake of clarity, we restrict here the presentation to the case when $\tau = \delta_{\bar{t}}$. Indeed, when τ is an arbitrary positive random variable, the lifting of interweaving relations by tensorization requires the introduction of additional notation and concepts, see [7], that are not necessary for our applications to BESQ and related semigroups. We mention that, for instance, when P is the semigroup of a Markov process X on $\mathrm{S} \subseteq \mathbb{R}$, then $P^{\otimes n}$ is the semigroup the Markov process $(X^{(i)})_{i=1}^{n}$ on S^n, where the $X^{(i)}$'s are n independent realizations of X. When P is a $\mathcal{C}_0$-semigroup, then $P^{\otimes n}$ is also a $\mathcal{C}_0$-semigroup with generator $A^{\oplus n}$, the n-fold direct sum of A, with domain the closure of $\mathrm{Dom}(A)^{\otimes n}$. With the notation introduced in (BG), the two previous identities can be expressed, for any $t \geq 0, \mathbf{x}_n \in \mathrm{S}^n$ and $\mathbf{f}_n = \Lambda^{\otimes n} \mathbb{f}_n \in \mathrm{Ran}(\Lambda^{\otimes n}) \subset \mathrm{H}^{\otimes n}$ (resp. $\mathbf{f}_n \in \mathrm{H}^{\otimes n}$), as follows

$$u_{\mathbf{f}_n}(t, \mathbf{x}_n) = \mathbf{E}\left[\mathbb{u}_{\mathbb{f}_n}(t, \mathbb{I}_n(\mathbf{x}_n))\right] \qquad \text{(resp. } u_{\mathbf{f}_n}(t + \bar{t}, \mathbf{x}_n) = \mathbf{E}[\mathbb{u}_{\overline{\Lambda}^{\otimes n} \mathbf{f}_n}(t, \mathbb{I}_n(\mathbf{x}_n))]) \tag{4.34}$$

where $\mathbb{I}_n$ is the n-dimensional version of the random variable $\mathbb{I}$. When considering the solutions to a Cauchy problem, one restricts the validity of these identities to the appropriate domains.

4.4.2 The Weyl chamber

We now introduce the n-dimensional Weyl chamber $W_n(\mathrm{S})$ associated to an ordered set S, which is the space of strictly increasing S-valued n-tuples, i.e.

$$W_n(\mathrm{S}) = \{\mathbf{x} = (x_1, \ldots, x_n) \in \mathrm{S}^n : x_1 < x_2 < \cdots < x_n\} \tag{4.35}$$

A similar definition holds for $W_n(\mathbb{S})$ the Weyl chamber associated to the lattice ordered set $\mathbb{S}$. Let us write the Hilbert space $\mathrm{L}^2(W_n(\mathrm{S}), \boldsymbol{\mu}^{W_n}) = \mathrm{H}^{W_n}$ where $\boldsymbol{\mu}^{W_n} = \mu^n_{|W_n(\mathrm{S})}$ is the restriction to the Weyl chamber of the product measure, and, denote by Sym_n the symmetric group. We define the alternating transform $\mathcal{A} : \mathrm{H}^{W_n} \to A_0(\mathrm{S}^n) \subset \mathrm{H}^{\otimes n}$, where $A_0(\mathrm{S}^n)$ is the set of alternating functions in $\mathrm{H}^{\otimes n}$, by

$$\mathcal{A}\mathbf{f}(\mathbf{x}) = \sum_{\sigma \in \mathrm{Sym}_n} \mathrm{sign}(\sigma) \mathbf{f}(\mathbf{x}_\sigma) \mathbb{I}_{\{\mathbf{x}_\sigma \in W_n(\mathrm{S})\}} \tag{4.36}$$

where σ is the unique permutation in Sym_n such that $\mathbf{x}_\sigma \in W_n(\mathrm{S})$. $\mathcal{A}$ is one-to-one and onto on $A_0(\mathrm{S}^n) \subset \mathrm{H}^{\otimes n}$, that is $\mathcal{A}$ is a pseudo-unitary operator from

$\mathrm{H}^{W_n} = \mathrm{L}^2(W_n(\mathrm{S}), \boldsymbol{\mu}^{W_n})$ into $\mathrm{H}^{\otimes n}$, and we denote its inverse by $\mathcal{A}^{-1} : A_0(\mathrm{S}^n) \mapsto \mathrm{L}^2(W_n(\mathrm{S}), \boldsymbol{\mu}^{W_n})$ which is the restriction map to the Weyl chamber. Then, for a linear operator $T \in \mathbf{B}(\mathrm{H})$, we define $\boldsymbol{T}^\dagger \in \mathbf{B}(\mathrm{H}^{W_n})$ as

$$\boldsymbol{T}^\dagger \mathbf{f}(\mathbf{x}) = \mathcal{A}^{-1} T^{\otimes n} \mathcal{A} \mathbf{f}(\mathbf{x}) \tag{4.37}$$

where det stands for the determinant. When $Tf(x) = \int_{\mathrm{S}} f(y)\, T(x, dy)$ is an integral operator, the Cauchy-Binet formula entails that

$$\boldsymbol{T}^\dagger \mathbf{f}(\mathbf{x}) = \int_{\mathbf{y} \in W_n(\mathrm{S})} \mathbf{f}(\mathbf{y}) \det \left[T(x_j, dy_j)\right]_{i,j=1}^n \tag{4.38}$$

Moreover, if $\mathbf{f}$ takes the determinantal form $\mathbf{f}(\mathbf{x}) = \det\left[f_i(x_j)\right]_{i,j=1}^n$, then

$$\boldsymbol{T}^\dagger \mathbf{f}(\mathbf{x}) = \det\left[Tf_i(x_j)\right]_{i,j=1}^n \tag{4.39}$$

Coming back to our setting, we set, for any $t \geq 0$,

$$\boldsymbol{P}_t^\dagger \mathbf{f}(\mathbf{x}) = \mathcal{A}^{-1} P_t^{\otimes n} \mathcal{A} \mathbf{f}(\mathbf{x}) \tag{4.40}$$

which, when P is a $\mathcal{C}_0$-semigroup in H, according to [7], defines a $\mathcal{C}_0$-semigroup in H^{W_n}. When P is in addition the $\mathcal{C}_0$-semigroup of a diffusion (resp. a birth-death process) X on $\mathrm{S} \subseteq \mathbb{R}$, a celebrated Theorem of Karlin-McGregor [39] states that the family $\mathbf{P}^\dagger = (\mathbf{P}_t^\dagger)_{t \geq 0}$ defines a sub-Markov $\mathcal{C}_0$-semigroup corresponding to the process $(X^{(i)})_{i=1}^n$ of n independent diffusion (resp. birth-death) processes with semigroup P that is killed when any two coincide, i.e. at the random coincidence time $\inf\{t > 0;\ X_t^{(i)} = X_t^{(j)} \text{ for any } i \neq j\}$. Note that it is shown in [7] that the condition that P is the semigroup of a diffusion for $\mathbf{P}^\dagger$ to be a Markov semigroup is also necessary. With this notation, we deduce, combining (4.32) with (4.40) that, for any $t \geq 0, \mathbf{x} \in W_n(\mathrm{S})$, and $\mathbb{f} \in \mathbb{H}^{W_n} = \ell^2(W_n(\mathbb{S}), \mathbb{m}^{W_n})$

$$\mathbf{P}_t^\dagger \boldsymbol{\Lambda}^\dagger \mathbb{f}(\mathbf{x}) = \boldsymbol{\Lambda}^\dagger \mathbb{P}_t^\dagger \mathbb{f}(\mathbf{x}) \tag{4.41}$$

where $\mathbb{P}^\dagger_t = (\mathbb{P}^\dagger_t)_{t\geq 0}$ is the $\mathcal{C}_0$-semigroup in $\mathbb{H}^{W_n}$, defined from the $\mathcal{C}_0$-semigroup $\mathbb{P}$ in $\mathbb{H}$ by means of (4.40). Moreover, using now (4.33), for any $\mathbf{f} \in \mathrm{H}^{W_n}$,

$$\boldsymbol{P}^\dagger_{t+\bar{t}}\mathbf{f}(\mathbf{x}) = \boldsymbol{\Lambda}^\dagger \mathbb{P}^\dagger_t \overline{\boldsymbol{\Lambda}}^\dagger \mathbf{f}(\mathbf{x}) \tag{4.42}$$

As above, and under the conditions that $\boldsymbol{\Lambda}^\dagger$ is (sub-)Markov, which is equivalent to the kernel of Λ being totally positive of order n, see e.g. [39], these relations can be expressed, for any $t \geq 0, \mathbf{x} \in W_n(\mathrm{S})$ and $\mathbf{f} \in \boldsymbol{\Lambda}^\dagger \mathbb{f} \in \mathrm{Ran}(\boldsymbol{\Lambda}^\dagger) \subset \mathrm{H}^{W_n}$ (resp. $\mathbf{f} \in \mathrm{H}^{W_n}$), as follows

$$u_{\mathbf{f}}(t,\mathbf{x}) = \mathbf{E}\left[\mathbb{u}_{\mathbb{f}}(t, \mathbb{I}^\dagger_n(\mathbf{x}))\right] \qquad (\text{resp. } u_{\mathbf{f}}(t+\bar{t},\mathbf{x}) = \mathbf{E}[\mathbb{u}_{\overline{\boldsymbol{\Lambda}}^\dagger \mathbf{f}}(t, \mathbb{I}^\dagger_n(\mathbf{x}))]) \tag{4.43}$$

where $\mathbb{I}^\dagger_n(\mathbf{x})$ is a random variable with distribution $\boldsymbol{\Lambda}^\dagger(\mathbf{x}, \cdot)$.

4.4.3 Normalization on the Weyl chamber

Under mild conditions, see again [7], there exist $\lambda_{\mathbf{P}} \geq 0$ and a positive measurable function $\Delta_{\mathbf{P}}$ on $W_n(\mathrm{S})$ such that, for all $t \geq 0$, $\mathbf{P}^\dagger_t \Delta_{\mathbf{P}} = e^{-\lambda_{\mathbf{P}} t}\Delta_{\mathbf{P}}$. Writing $M_{\Delta_{\mathbf{P}}}\mathbf{f}(\mathbf{x}) = \Delta_{\mathbf{P}}(\mathbf{x})\mathbf{f}(\mathbf{x})$ for the multiplication operator, it follows that $\mathbf{P} = (\mathbf{P}_t)_{t\geq 0}$ where

$$\mathbf{P}_t\mathbf{f}(\mathbf{x}) = e^{\lambda_{\mathbf{P}} t} M^{-1}_{\Delta_{\mathbf{P}}} \mathbf{P}^\dagger_t M_{\Delta_{\mathbf{P}}}\mathbf{f}(\mathbf{x}) \tag{4.44}$$

defines a $\mathcal{C}_0$-semigroup in H^{W_n}. By analogy to the semigroup of the Dyson Brownian motion, which can be constructed this way with P being the semigroup of the Brownian motion, we call $\mathbf{P}$ the (n-dimensional) Dyson (determinantal) semigroup associated to P. When P is the semigroup of a diffusion on $\mathrm{S} \subseteq \mathbb{R}$, $\mathbf{P}$ defines a conservative (i.e. mass-preserving) Markov semigroup of a diffusion (resp. birth-death) process on $W_n(\mathrm{S})$, its realization as a Markov process can formally be thought of as the process $(X^{(i)})_{i=1}^n$ conditioned such that no two of its components are ever coincident. Next, from (4.41), assuming that $\Delta_{\mathbf{P}} \in \mathrm{Ran}(\mathbf{\Lambda}^{\dagger})$, we deduce that there exists a positive measurable function $\Delta_{\mathbb{P}}$ on $W_n(\mathbb{S})$ such that $\mathbf{\Lambda}^{\dagger}\Delta_{\mathbb{P}}(\mathbf{x}) = \Delta_{\mathbf{P}}(\mathbf{x})$ and $\mathbb{P}_t^{\dagger}\Delta_{\mathbb{P}} = e^{-\lambda_{\mathbf{P}}t}\Delta_{\mathbb{P}}$. Note that $\lambda_{\mathbf{P}} = \lambda_{\mathbb{P}}$. Then, define the linear operator $\mathbf{\Lambda}$ for $\mathbb{f} \in \mathbb{H}^{\Delta_{\mathbb{P}}} = \ell^2(W_n(\mathbb{S}), \mathbb{m}^{\Delta_{\mathbb{P}}})$, where $\mathbb{m}^{\Delta_{\mathbb{P}}} = \Delta_{\mathbb{P}}^2 \mathbb{m}^{W_n}$, by

$$\mathbf{\Lambda}\mathbb{f}(\mathbf{x}) = M_{\Delta_{\mathbf{P}}}\mathbf{\Lambda}^{\dagger}M^{-1}_{\Delta_{\mathbb{P}}}\mathbb{f}(\mathbf{x}) \tag{4.45}$$

With this notation, we deduce, combining (4.32) with (4.40), for any $t \geq 0, \mathbf{x} \in W_n(\mathrm{S})$, and, $\mathbb{f} \in \mathbb{H}^{\Delta_{\mathbb{P}}}$,

$$\mathbf{P}_t\mathbf{\Lambda}\mathbb{f}(\mathbf{x}) = \mathbf{\Lambda}\mathbb{P}_t\mathbb{f}(\mathbf{x}) \tag{4.46}$$

where $\mathbb{P}_t = (\mathbb{P}_t)_{t\geq 0}$ is the $\mathcal{C}_0$-semigroup in $\mathbb{H}^{\Delta_{\mathbb{P}}}$, defined from $\mathbb{P}$ by means of (4.44). Moreover, using now (4.42), for any $\mathbf{f} \in \mathrm{H}^{\Delta_{\mathbf{P}}} = \mathrm{L}^2(W_n(\mathrm{S}), \boldsymbol{\mu}^{\Delta_{\mathbf{P}}})$, where $\boldsymbol{\mu}^{\Delta_{\mathbf{P}}}(d\mathbf{x}) = \Delta_{\mathbf{P}}^2(\mathbf{x})\boldsymbol{\mu}^{W_n}(d\mathbf{x})$

$$\boldsymbol{P}_{t+\bar{t}}\mathbf{f}(\mathbf{x}) = \mathbf{\Lambda}\mathbb{P}_t\overline{\mathbf{\Lambda}}\mathbf{f}(\mathbf{x}) \tag{4.47}$$

where $\overline{\mathbf{\Lambda}}$ is defined from $\overline{\mathbf{\Lambda}}^{\dagger}$ by a similar construction than the one used for $\mathbf{\Lambda}$ in (4.45). As above, these relations can be expressed, for any $t \geq 0, \mathbf{x} \in W_n(\mathrm{S})$ and $\mathbf{f} \in \mathbf{\Lambda}\mathbb{f} \in \mathrm{Ran}(\mathbf{\Lambda}) \subset \mathrm{H}^{\Delta_{\mathbf{P}}}$ (resp. $\mathbf{f} \in \mathrm{H}^{\Delta_{\mathbf{P}}}$), as follows

$$u_{\mathbf{f}}(t,\mathbf{x}) = \mathbf{E}\left[\mathbb{u}_{\mathbb{f}}(t, \mathbb{I}_n(\mathbf{x}))\right] \qquad (\text{resp. } u_{\mathbf{f}}(t+\bar{t},\mathbf{x}) = \mathbf{E}[\mathbb{u}_{\overline{\mathbf{\Lambda}}\mathbf{f}}(t, \mathbb{I}_n(\mathbf{x}))]) \tag{4.48}$$

where $\mathbb{I}_n(\mathbf{x})$ is a random variable with distribution $\mathbf{\Lambda}(\mathbf{x}, \cdot)$ on $W_n(\mathbb{Z}_+)$.

4.4.4 Gateway and interweaving relations for the Dyson BESQ semigroup

We now apply the construction proposed above to the BESQ semigroup for which $\mathrm{S} = \mathbb{R}_+$ (resp. $\mathbb{S} = \mathbb{Z}_+$), and, for $n \in \mathbb{N}$, $\mathrm{S}^n = \mathbb{R}_+^n$ (resp. $\mathbb{S}^n = \mathbb{Z}_+^n$). Note that similar results hold for BESQ semigroups of any index $\nu > -1$ and their associated CIR semigroups, see [7] for more details. First, with $\Lambda^{\otimes n}$ denoting the n-dimensional version of the Poisson kernel defined in (4.10), the gateway relation (4.20) extends to the n-dimensional setting, for any $t \geq 0, \mathbf{x}_n \in \mathbb{R}_+^n$ and $\mathbf{f}_n = \Lambda^{\otimes n}\mathbb{f}_n \in \mathrm{Ran}(\Lambda^{\otimes n}) \subset \mathrm{H}^{\otimes n} = \mathrm{L}^2(\mathbb{R}_+^n)$, as follows

$$Q_t^{\otimes n}\Lambda^{\otimes n}\mathbb{f}_n(\mathbf{x}_n) = \Lambda^{\otimes n}\mathbb{Q}_t^{\otimes n}\mathbb{f}_n(\mathbf{x}_n) \tag{4.49}$$

and the interweaving relation (4.21) extends, for any $t \geq 0, \mathbf{x}_n \in \mathbb{R}_+^n$ and $\mathbf{f}_n \in \mathrm{L}^2(\mathbb{R}_+^n)$, as

$$Q_{t+1}^{\otimes n}\mathbf{f}_n(\mathbf{x}_n) = \Lambda^{\otimes n}\mathbb{Q}_t^{\otimes n}\overline{\Lambda}^{\otimes n}\mathbf{f}_n(\mathbf{x}_n) \tag{4.50}$$

Let now $W_n(\mathbb{R}_+)$ and $W_n(\mathbb{Z}_+)$ denote the n-dimensional Weyl chamber associated to $\mathbb{R}_+$ and $\mathbb{Z}_+$ respectively, and recall, from (4.36), that $\mathcal{A} : \mathrm{L}^2(W_n(\mathbb{R}_+)) \to A_0(\mathbb{R}_+^n) \subset \mathrm{L}^2(\mathbb{R}_+^n)$, $A_0(\mathbb{R}_+^n)$ being the set of alternating functions in $\mathrm{L}^2(\mathbb{R}_+^n)$, is the alternating transform. With this notation, we have, for any $t \geq 0, \mathbf{x} \in W_n(\mathbb{R}_+)$, and $\boldsymbol{\mathbb{f}} \in \ell^2(W_n(\mathbb{Z}_+))$

$$\boldsymbol{Q}_t^{\dagger}\boldsymbol{\Lambda}^{\dagger}\boldsymbol{\mathbb{f}}(\mathbf{x}) = \boldsymbol{\Lambda}^{\dagger}\boldsymbol{\mathbb{Q}}_t^{\dagger}\boldsymbol{\mathbb{f}}(\mathbf{x}) \tag{4.51}$$

and, for any $\mathbf{f} \in \mathrm{L}^2(W_n(\mathbb{R}_+))$,

$$\boldsymbol{Q}_{t+1}^{\dagger}\mathbf{f}(\mathbf{x}) = \boldsymbol{\Lambda}^{\dagger}\boldsymbol{\mathbb{Q}}_t^{\dagger}\boldsymbol{\Lambda}^{*\dagger}\mathbf{f}(\mathbf{x}) \tag{4.52}$$

Assiotis [2] showed that $\boldsymbol{\Lambda}^{\dagger}$ and $\boldsymbol{\Lambda}^{*\dagger}$ defined by (4.45) and properly normalized are Markov operators, which are easily seen to extend to $\mathbf{B}(\ell^2(W_n(\mathbb{Z}_+), \mathrm{L}^2(W_n(\mathbb{R}_+))$ and $\mathbf{B}(\mathrm{L}^2(W_n(\mathbb{R}_+), \ell^2(W_n(\mathbb{Z}_+))$ respectively. Next, writing Δ the Vandermonde determinant, i.e. for any $\mathbf{x} = (x_i)_{i=1}^n \in W_n(\mathbb{Z}_+)$ or $W_n(\mathbb{R}_+)$,

$$\Delta(\mathbf{x}) = \det\left[x_j^{i-1}\right]_{i,j=1}^n = \prod_{1\leq i<j\leq n}(x_j - x_i) \tag{4.53}$$

it is shown by König and O'Connell [42], see also Assiotis et al. [3] that $\Delta_{\mathbf{Q}} = \Delta$ on $W_n(\mathbb{R}_+)$ and $\Delta_{\boldsymbol{\mathbb{Q}}} = \Delta$ on $W_n(\mathbb{Z}_+)$. Furthermore, $\lambda_{\mathbf{Q}} = \lambda_{\boldsymbol{\mathbb{Q}}} = 0$. Let $\mathbf{m}^{\Delta} = \Delta^2$

on $W_n(\mathbb{R}_+)$ and $\mathbb{m}^\Delta$ be analogously defined on $W_n(\mathbb{Z}_+)$. Furthermore, $\mathbf{\Lambda}$ defined by (4.45) and properly normalized is again a Markov operator in $\mathbf{B}(\ell^2(\mathbb{m}^\Delta), \mathrm{L}^2(\mathbf{m}^\Delta))$. Let $\mathbf{\Lambda}^* \in \mathbf{B}(\mathrm{L}^2(\mathbf{m}^\Delta), \ell^2(\mathbb{m}^\Delta))$ be similarly defined, which are also Markov operators.

For any $t \geq 0$, $\mathbf{x} \in W_n(\mathbb{R}_+)$ and $\mathbb{f} \in \ell^2(\mathbb{m}^\Delta)$, the gateway relation (4.11) extends to the Weyl chamber as follows

$$\mathbf{Q}_t \mathbf{\Lambda} \mathbb{f}(\mathbf{x}) = \mathbf{\Lambda} \mathbb{Q}_t \mathbb{f}(\mathbf{x}) \tag{4.54}$$

and, for $\mathbf{f} \in \mathrm{L}^2(\mathbf{m}^\Delta)$, the interweaving relation (4.21) extends to

$$\mathbf{Q}_{t+1} \mathbf{f}(\mathbf{x}) = \mathbf{\Lambda} \mathbb{Q}_t \mathbf{\Lambda}^* \mathbf{f}(\mathbf{x}) \tag{4.55}$$

This reads, in terms of the solution of the associated Cauchy problems, for any $t \geq 0$, $\mathbf{x} \in W_n(\mathbb{R}_+)$ and $\mathbf{f} = \mathbf{\Lambda}\mathbb{f}$, as follows

$$u_{\mathbf{f}}(t, \mathbf{x}) = \mathbf{E}\left[\mathbb{u}_{\mathbb{f}}(t, \mathbb{l}_n(\mathbf{x}))\right] \tag{4.56}$$

where $\mathbb{l}_n(\mathbf{x})$ is a random variable on $W(\mathbb{Z}_+)$ with distribution $\mathbf{\Lambda}(\mathbf{x}, \cdot)$ and can be thought of independent Poisson variables with increasing parameters $\mathbf{x}$ conditioned to be strictly increasing, and, for any $\mathbf{f} \in \mathrm{L}^2(\mathbf{m}^\Delta)$,

$$u_{\mathbf{f}}(t+1, \mathbf{x}) = \mathbf{E}\left[\mathbb{u}_{\mathbf{\Lambda}^*\mathbf{f}}(t, \mathbb{l}_n(\mathbf{x}_n))\right] \tag{4.57}$$

We note that the self-similarity property of BESQ process and the relation (4.8) between the BESQ and CIR semigroups extends to their Dyson semigroup counterparts, where dilation is applied entrywise. We point out that these gateway and interweaving relations between the Dyson semigroups have been derived initially by Assiotis [2] by relating directly the transition kernels of the associated Markov processes, and, further examples of such relations between $\mathcal{C}_0$-semigroups on Weyl chambers can be found in [7].

APPENDIX A

CHAPTER 2 OF APPENDIX

A.1 Appendix

Lemma A.1.1. *Suppose that*

$$dS_t = \sigma(S_t)dW_t$$

where $\mathbb{P}(S_t \in (a,b)) > 0$ *for all* $a < b \in \mathbb{R}$*, and define* $f(t,x) = \mathbb{E}[S_T|S_t = x]$ *for* $x \in [0,\infty)$*.*

And, U *is any nonnegative classical solution to PDE:*

$$\partial_t u + \frac{\sigma^2}{2}\partial_{xx}u = 0, (t,x) \in [0,T] \times \mathbb{R}^+$$

$$u(T,x) = x, x \in \mathbb{R}^+$$

First, the process $f(t,S_t)$ *is a true martingale since we have* $f(t,S_t) = \mathbb{E}[S_T|\mathcal{F}_t]$*. And,* $f(T,S_T) = S_T$*. Second, we have* $U(t,S_t)$ *is a local martingale since it is driftless by Ito's formula. Finally, it is a supermartingale since it is bounded from below. Then, we have*

$$\begin{aligned} U(t,S_t) &\geq \mathbb{E}[U(T,S_T)|\mathcal{F}_t] \\ &= \mathbb{E}[S_T|\mathcal{F}_t] \\ &= f(t,S_t). \end{aligned}$$

In particular, we have $f(t,x) \leq U(t,x), \forall (t,x) \in [0,T] \times \mathbb{R}^+$ *.*

APPENDIX B

CHAPTER 3 OF APPENDIX

B.1 Appendix

Lemma B.1.1. *Let* $G : \mathbb{R}^+ \mapsto \mathbb{R}$ *be a convex function with* $\limsup_{x \to +\infty} \frac{G(x)}{x} = c \geq 0$*. Let* $X_T \in \mathcal{F}_T$ *be a random variable such that* $X_T \geq G(S_T)$*. Define*

$$J(t) = \mathbb{1}_{t<T} G(S_t) + X_T \mathbb{1}_{t=T}.$$

Then,

$$\sup_{\tau \leq T} \mathbb{E}^{\mathbb{Q}}[J(\tau)] = \mathbb{E}^{\mathbb{Q}}[X_T] + (c \vee 0)\beta_0.$$

Proof. Define $\tau_n = \inf\{t : S_t \geq n\}$, which is an increasing sequence of stopping times because S has bounded jumps. Hence,

$$\begin{aligned} \sup_{\tau \leq T} \mathbb{E}^{\mathbb{Q}}[J(\tau)] &\geq \mathbb{E}^{\mathbb{Q}}[J(\tau_n)] \\ &= \mathbb{E}^{\mathbb{Q}}[X_T \mathbb{1}_{\tau_n = T}] + G(n)\mathbb{Q}[\tau_n < T]. \end{aligned}$$

Taking the limit and using the fact that $\beta_0 = \lim_n n\mathbb{Q}(\tau_n < T)$, we have

$$\sup_{\tau \leq T} \mathbb{E}^{\mathbb{Q}}[J(\tau)] \geq \mathbb{E}^{\mathbb{Q}}[X_T] + c\beta_0.$$

For the reverse inequality, define the martingale

$$Y_t = \mathbb{E}^{\mathbb{Q}}[G(S_T)|\mathcal{F}_t].$$

We first show that

$$J(t) \leq Y_t + c\beta_t.$$

Define the function $G^c(x) = G(x) - cx$. Because $c \geq 0$, G^c is a decreasing convex function. Then, by Jensen's inequality we have

$$\mathbb{E}^{\mathbb{Q}}[G^c(S_T)|\mathcal{F}_t] \geq G^c(\mathbb{E}^{\mathbb{Q}}[S_T|\mathcal{F}_t]) \geq G^c(S_t)$$

which implies

$$J(t) \leq Y_t + c\beta_t, \forall t \in [0,T).$$

Therefore, we have

$$J(t) \leq \mathbb{E}^{\mathbb{Q}}[X_T|\mathcal{F}_t] + c\beta_t, \forall t \in [0,T]$$

and

$$\sup_{\tau} \mathbb{E}^{\mathbb{Q}}[J(\tau)] \leq \mathbb{E}^{\mathbb{Q}}\left[\mathbb{E}^{\mathbb{Q}}[X_T|\mathcal{F}_\tau]\right] + c\mathbb{E}^{\mathbb{Q}}\left[\beta_\tau\right] \leq \mathbb{E}^{\mathbb{Q}}[X_T] + c\beta_0.$$

$\square$

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