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Abstract

The classical Economic Order Quantity (EOQ) model of Harris and Wilson exhibits a convex objective function. The Dynamic Lot Size (DLS) model of Wagner and Whitin exhibits a concave objective function. In this paper, we reconcile these two results by showing that a reformulation of the DLS model along the lines of the EOQ model exhibits a convex objective function provided only that unit production costs are stationary. A counterexample is provided to show the necessity of the condition.

Keywords: production planning, lot sizing, convexity.

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The classical Economic Order Quantity (EOQ) model of Harris[Ha13] and Wilson[Wi34] exhibits a convex objective function. The Dynamic Lot Size (DLS) model of Wagner and Whitin[WW58] exhibits a concave objective function. In this paper, we reconcile these two results by showing that a reformulation of the DLS model along the lines of the EOQ model exhibits a convex objective function provided only that unit production costs are stationary. A counterexample is provided to show the necessity of the condition.

Both of these fundamental models have been studied extensively in the literature. See Maxwell and Muckstadt[MM85] for references to many papers related to the EOQ model and see Hax and Candea[HC84], and Aggarwal and Park[AP90] for references to the DLS. Aggarwal and Park[AP90] propose very fast algorithms for the DLS model and some variants of it.

We first consider a reorder interval formulation of the EOQ model. Let d, c, K and h denote, respectively, the demand per period, the unit production cost, the set-up or fixed order cost, and the unit holding cost per period, all assumed non-negative and stationary. Let T denote the number of periods between orders, a decision variable. The average cost per unit time is given by $C(T)$:

$$C(T) = \frac{K}{T} + \frac{hd}{2}T + (c - \frac{h}{2})d.$$

The EOQ problem is to find an integer T such that $C(T)$ is minimized. The problem is interesting only if K, h, and d are positive; in which case, it is well-known that $C(T)$ is strictly convex on the set of positive integers $\mathbb{N}$. (A function f mapping a countable subset of $\mathbb{R}$ onto a subset of $\mathbb{R}$ is said to be strictly convex if the epigraph of the piecewise-linear version of f is a convex set and all the target points are extreme points of the convex hull of the set. E.g. see Figure 1.)

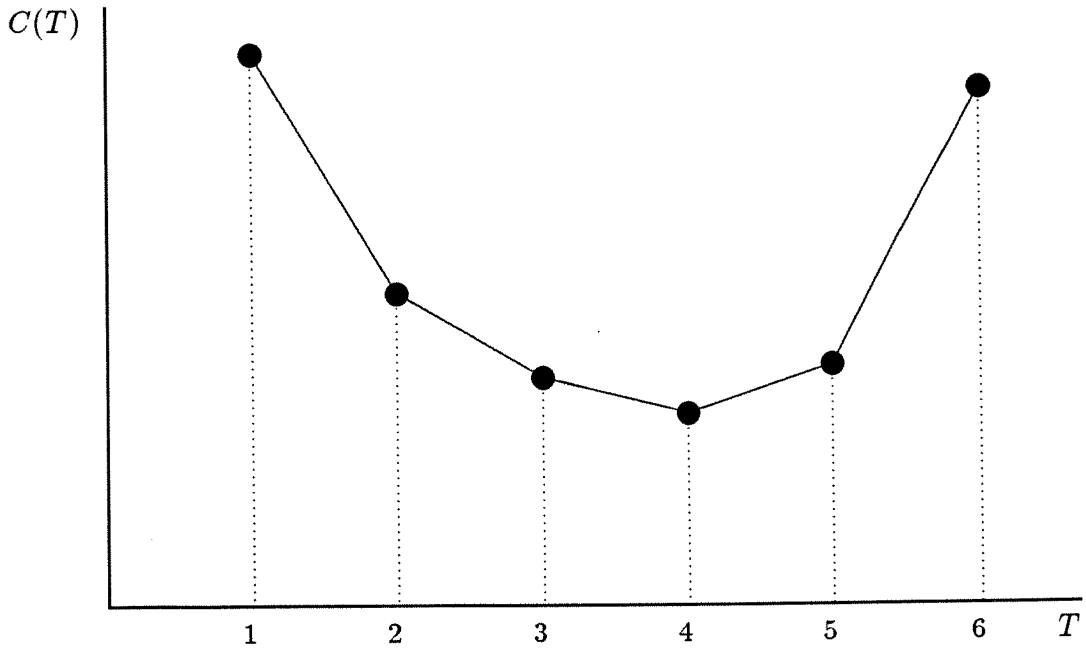

Figure 1. Graph $(T, C(T))$

To clarify the relationship with the DLS model, we perform a simple change of variable: let $M = T^{-1}$, the frequency of ordering. Let $Z(M)$ denote the average cost per unit time:

$$Z(M) = KM + \frac{hd}{2}.\frac{1}{M} + (c - \frac{h}{2})d.$$

Assuming positive K, h, and d, $Z(M)$ is strictly convex on the set of reals corresponding to $M^{-1} \in \mathbb{N}$.

Next, we consider a standard formulation of the DLS model. Given a schedule of time-dated nonnegative demands, the problem is to determine a schedule of production and inventory levels such that total cost is minimized over a finite planning horizon. Let n be the number of production periods in the planning horizon and d_j, c_j, K_j, and h_j denote, respectively, the demand, unit production cost, set-up or fixed order cost, and unit holding cost per period for period j, $j \in \{1, 2, \ldots, n\}$. The DLS model can be written as follows:

$$\min \sum_{j=1}^{n} [f_j(P_j) + h_j I_j]$$

$$\begin{aligned} \text{subject to} \quad & I_0 = 0; \\ & I_j = I_{j-1} + P_j - d_j, \forall j \in \{1, \ldots, n\}; \\ & I_j \geq 0, P_j \geq 0, \forall j \in \{1, \ldots, n\}; \end{aligned}$$

where P_j and I_j are production and end-of-period inventory levels in period j, and

$$f_j(P_j) = \begin{cases} K_j + c_j P_j & \text{if } P_j > 0; \\ 0 & \text{if } P_j = 0. \end{cases}$$

The objective function for this formulation is concave. Since the feasible region is defined by a set of linear inequalities, there is an extreme point optimum whenever the problem admits an optimal solution. Wagner and Whitin[WW58] observed that an extreme point solution of the constraint set is characterized by the condition

$$I_{j-1} P_j = 0, \forall j \in \{1, \ldots, n\}.$$

Consequently, the DLS problem reduces to the combinatorial problem of determining the periods for which $P_j > 0$ such that the total cost is minimized. They proposed a dynamic programming algorithm to identify the optimal production periods.

Our interest is in relating the DLS model to the EOQ model. To this end, let $Y_n(m)$ denote the minimum n-period cost of a feasible production plan that uses exactly m set-ups, $m \geq 1$. We develop an expression for $Y_n(m)$ as follows. We assume that $d_1 > 0$, so that an order must be placed for period 1. Let S denote any set of distinct production periods other than period 1 (i.e. $S \subseteq \{2, 3, \ldots, n\}$) such that $|S| = m - 1$. There are three components to the cost function: the fixed order cost, the holding cost, and the variable production cost. We treat each in turn.

Let $\mathcal{K}_S$ denote the fixed order cost implied by the production period schedule S:

$$\mathcal{K}_S := K_1 + \sum_{j \in S} K_j.$$

Define Φ_S to be the set of inter-order intervals as follows: if $S = \{j_1, \ldots, j_k\}$ then

$$\Phi_S = \{\{2, 3, \ldots, j_1 - 1\}, \{j_1 + 1, \ldots, j_2 - 1\}, \ldots, \{j_k + 1, \ldots, n\}\},$$

and if $S = \emptyset$ then $\Phi_S = \{\{2, 3, \ldots, n\}\}$.

Let $\mathcal{H}_S$ denote the holding cost implied by the production period schedule, S. It is easily shown that

$$\mathcal{H}_S = \sum_{J \in \Phi_S} \sum_{i \in J} d_i \left(\sum_{k=l_J-1}^{i-1} h_k \right),$$

where $l_J := \min\{j : j \in J\}$.

Let $\mathcal{C}_S$ denote the variable production cost implied by the production period schedule, S:

$$\mathcal{C}_S = c_1 d_1 + \sum_{j \in S} c_j d_j + \sum_{J \in \Phi_S} c_{l_J - 1} \left(\sum_{k=l_J}^{u_J} d_k \right),$$

where $u_J := \max\{j : j \in J\}$.

Let $A_m := \{S \subseteq \{2, 3, \ldots, n\} : |S| = m - 1\}$. Hence, we can write $Y_n(m)$ as

$$Y_n(m) = \min_{S \in A_m} \{\mathcal{K}_S + \mathcal{H}_S + \mathcal{C}_S\}.$$

Under this reformulation, the DLS problem can be described as the problem of choosing $m \in \{1, \ldots, n\}$ to minimize $Y_n(m)$. The relationship of this new DLS objective and the EOQ objective can now be clarified.

Proposition 1. *If all the cost parameters of the DLS model are stationary(i.e. $K_j = K$, $d_j = d$, $h_j = h$, and $c_j = c\ \forall j$) and if $n = mT$ where m and T are positive integers, then*

$$\frac{1}{n}Y_n(m) = Z(\frac{1}{T}).$$

Proof: By stationarity,

$$Y_n(m) = mK + d \min_{S\in A_m}\left\{h\left[\sum_{J\in\Phi_S}\sum_{i\in J}(i - l_J + 1)\right] + cn\right\}$$

$$= mK + dcn + \frac{dh}{2}(n-m) + \frac{dh}{2}\min_{S\in A_m}\left\{\sum_{J\in\Phi_S}|J|^2\right\}.$$

Since n is an integral multiple of an integral T, it is easily shown that the latter minimum is achieved by equal reorder intervals

$(|J| = T - 1)$. Hence,

$$Y_n(m) = m\left[K + \frac{hdT^2}{2} + (c - \frac{h}{2})dT\right].$$

□

Corollary 1. *If all the cost parameters of the DLS model are stationary, then* $\frac{1}{n}Y_n(\lceil nM\rceil) \to Z(M)$ *as* $n \to \infty$.

Proof: In the second equality of the proof substitute $m = \lceil nM\rceil$ (the smallest integer greater than or equal to nM) and note that the minimum will be achieved by "almost" equal intervals (see appendix). Then, dividing by n and taking limits as $n \to \infty$ gives the desired result. □

Having established a relationship between the objectives of the EOQ and DLS models, at least for stationary parameters, our interest is now in showing that the objectives exhibit similar behavior. The reformulated DLS objective is not necessarily a convex function as the following counterexample shows.

Example: We take $n = 3$, $d_1 = d_2 = d_3 = 1$, $K_1 = K_2 = K_3 = 0$, $h_1 = h_2 = 1$, $c_1 = 1$, $c_2 = 4$, $c_3 = 6$. Then $Y_3(1) = 6$, $Y_3(2) = \min\{10, 9\} = 9$, and $Y_3(3) = 11$. As is clear from Figure 2, $Y_n(m)$ is not convex.

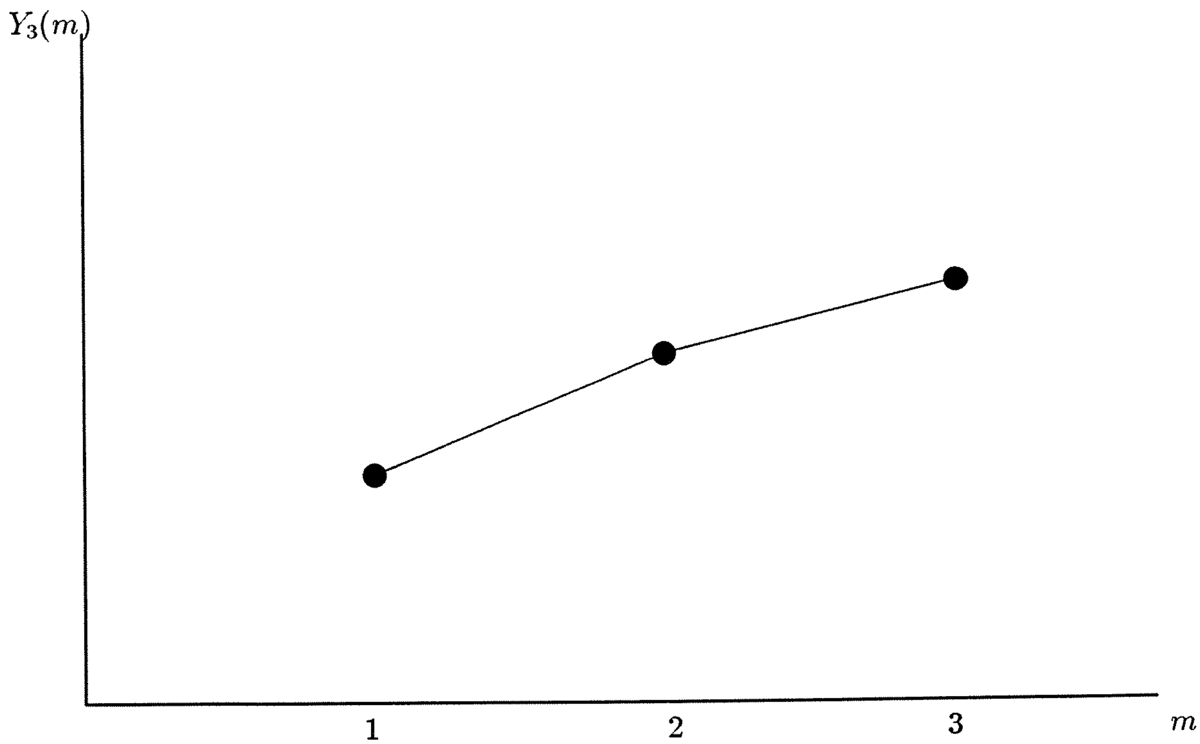

Figure 2. Graph of $(m, Y_n(m))$ for the counterexample

□

The essential element of the counterexample is the nonstationarity of the unit production costs. In the case of stationary unit production costs, the objective is convex. This is the central result of this paper.

Theorem 1. *If the unit production costs in the DLS model are stationary, then $Y_n(m)$ is convex on $\{1, 2, \ldots, n\}$.*

Proof: Note that $Y_n(m)$ is convex if and only if

$$Y_n(m-1) + Y_n(m+1) \geq 2Y_n(m),\ \forall m \in \{2, ..., n-1\}.$$

Since $c_j = c$, $\forall j$, we have $\mathcal{C}_S = c\sum_{j=1}^{n} d_j\ \forall S$. So, $2\mathcal{C}$ will be in both sides of the inequality (and therefore can be cancelled). Without loss of generality, we can assume that $d_1 > 0$, so $2K_1$ will be in both sides of the inequality(and therefore can be cancelled).

Then, $Y_n(m)$ is convex if and only if (1) holds for all $m \in \{2, ..., n-1\}$:

$$\min_{S_1 \subseteq A_{m-1}} \{\mathcal{K}_{S_1} + \mathcal{H}_{S_1}\} + \min_{S_2 \subseteq A_{m+1}} \{\mathcal{K}_{S_2} + \mathcal{H}_{S_2}\} \geq 2 \min_{S_3 \subseteq A_m} \{\mathcal{K}_{S_3} + \mathcal{H}_{S_3}\} \quad (1)$$

Suppose the minimums in the LHS of the inequality are achieved by $S_1^* = \{t_1, t_2, \ldots, t_{m-2}\}$ and $S_2^* = \{r_1, r_2, \ldots, r_m\}$, respectively. Suppose, for now, that $S_1^* \cap S_2^* = \emptyset$. Construct $S_1^* \cup S_2^*$ and sort the indices in $S_1^* \cup S_2^*$ in increasing order:

$$S_1^* \cup S_2^* = \{s_1, s_2, \ldots, s_{2m-2}\}.$$

Now, consider the following sets:
$S^- := \{s_1, s_3, \ldots, s_{2m-3}\}$ and $S^+ := \{s_2, s_4, \ldots, s_{2m-2}\}$.

Clearly, $\mathcal{K}_{S^+} + \mathcal{K}_{S^-} = \mathcal{K}_{S_1^*} + \mathcal{K}_{S_2^*}$. Since $S^+, S^- \subseteq A_m$, to show (1) it suffices to show

$$\mathcal{H}_{S_1^*} + \mathcal{H}_{S_2^*} \geq \mathcal{H}_{S^+} + \mathcal{H}_{S^-} \quad (2)$$

For a given production period schedule, S,and any period, k, let $s_S(k)$ denote the largest element of S that is less than or equal to k, or 1, if no such element exists. An alternative expression for the holding cost of schedule S is given by

$$\mathcal{H}_S = \sum_{k=1}^{n} d_k \left(\sum_{i=s_S(k)}^{k-1} h_i \right)$$

where any summation is null if the lower limit exceeds the upper limit. Written in this form, the left hand side of (2) becomes

$$LHS = \sum_{k=1}^{n} d_k \left(\sum_{i=s_{S_1^*}(k)}^{k-1} h_i + \sum_{i=s_{S_2^*}(k)}^{k-1} h_i \right),$$

and the right hand side becomes

$$RHS = \sum_{k=1}^{n} d_k \left(\sum_{i=s_{S^+}(k)}^{k-1} h_i + \sum_{i=s_{S^-}(k)}^{k-1} h_i \right).$$

For any k it is easily seen that

$$\max\{s_{S^+}(k), s_{S^-}(k)\} = \max\{s_{S_1^*}(k), s_{S_2^*}(k)\}$$

and

$$\min\{s_{S^+}(k), s_{S^-}(k)\} \geq \min\{s_{S_1^*}(k), s_{S_2^*}(k)\}.$$

Hence, $LHS \geq RHS$ and (2) holds. To handle the case in which $S_1^* \cap S_2^* \neq \emptyset$, simply insert the elements of $S_1^* \cap S_2^*$ into both sets S^- and S^+. The same results carry through. □

Corollary 2. *If $h_j > 0$ and $d_j > 0$ $\forall j$, then $Y_n(m)$ is strictly convex on* $\{1, 2, \ldots, n\}$.

Proof: In the proof of Theorem 1, note that $|S_2^*| = |S_1^*| + 2$. For this reason, there must exist at least one value of k such that

$$\min\{s_{S^+}(k), s_{S^-}(k)\} > \min\{s_{S_1^*}(k), s_{S_2^*}(k)\}.$$

It follows under the assumption of the corollary, that the inequality in (2), and hence in (1), is strict. □

The proof of the theorem is didactical. Given two minimum cost production plans with $(m-1)$ and $(m+1)$ set-ups, the proof shows a way of getting two production plans with m set-ups each such that average cost of the latter two plans less than or equal to the average cost of the former two plans. More importantly, we have reconciled the EOQ model with the DLS model by showing that the DLS objective function, after suitable reformulation, has the same convexity property as the EOQ objective function, provided unit production costs are stationary. Convexity of the EOQ objective, $Z(M)$, permits the development of efficient algorithms to compute its minimum. Convexity of the DLS objective, $Y_n(m)$, is less useful because a combinatorial problem still remains (selection of S from A_m). Nevertheless, this result may open the way to alternative algorithms for the DLS.

Appendix

Details in the proof of Corollary 1

Substituting $m = \lceil nM \rceil$ we get

$$\frac{1}{n} Y_n(\lceil nM \rceil) = \frac{1}{n}\left[\lceil nM \rceil K + dcn + \frac{dh}{2}(n - \lceil nM \rceil) + \frac{dh}{2} \min_{S \in A_{\lceil nM \rceil}} \left\{ \sum_{J \in \Phi_S} |J|^2 \right\}\right] \quad (3)$$

Finding the minimum in the RHS is equivalent to solving

$$\min v^* = \sum_{i=1}^{\lceil nM \rceil} x_i^2$$

$$\sum_{i=1}^{\lceil nM \rceil} x_i = n - \lceil nM \rceil$$

$$x_i \geq 0 \text{ and integer}, \forall i$$

Clearly the optimum solution of the continuous version is given by $x_i^* = \frac{n - \lceil nM \rceil}{\lceil nM \rceil}\ \forall i$. The same solution is optimal in the discrete version if n is divisible by $\lceil nM \rceil$. It is also clear that for any optimal solution we have the following properties:

- $|x_i - x_j| \leq 1, \forall i, j$;
- $|x_i - \frac{n - \lceil nM \rceil}{\lceil nM \rceil}| < 1, \forall i$.

So, as $n \to \infty$, we have $\frac{v^*}{n} \to M(\frac{1}{M} - 1)^2$. Then taking limits in (3) term by term yields $\frac{1}{n} Y_n(\lceil nM \rceil) \to Z(M)$ as $n \to \infty$.

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