COMPUTING FORMULAE FOR ANALYZING AUGMENTED RANDOMIZED COMPLETE BLOCK DESIGNS

BU-207-M S. R. Searle[1] December, 1965

ABSTRACT

Straightforward computing formulae are established for the analysis of variance of augmented randomized complete block designs.

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The augmented randomized complete blocks design is a randomized complete blocks design with supplementary treatments added to the blocks, usually with no such treatment occurring more than once in any block. Federer (1961) uses this as an example of more general augmented designs, giving most of the expressions pertaining to the analysis of variance, together with a numerical illustration. In this paper explicit formulae for the augmented randomized complete blocks analysis are given, in a form considered well-suited to facilitating the necessary calculations. The formulae are, of course, equivalent to those shown or implied by Federer (1961), but they appear to be more suitable from the computing point of view.

We shall refer to the treatments in the regular randomized complete blocks design as the standard treatments, and those used to augment the design will be called test treatments.

Randomized complete blocks analysis

First consider the customary analysis of variance for a randomized complete blocks design of r blocks each of s treatments - the standards. Let

$$x_{ij} = \text{observation on (standard) treatment } i \text{ in block } j ,$$

and let the usual summation and dot notation apply:

$$x_{i.} = \sum_{j=1}^{r} x_{ij} , \quad x_{.j} = \sum_{i=1}^{s} x_{ij} , \quad x_{..} = \sum_{i=1}^{s} \sum_{j=1}^{r} x_{ij} .$$

Writing down the analysis of variance shown in Table 1, establishes definitions of T_B, T_T, T_E, and T_O, the sums of squares for blocks, treatment, error and

(Show Table 1)

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total respectively, with $T_0 = T_B + T_T + T_E$ as usual.

Augmented analysis

Suppose that the design is now augmented by the addition to the j'th block of n_j test treatments. Let

$$y_{kj} = \text{observation on (test) treatment } k \text{ in block } j$$

with $k = 1,2,\ldots,n_j$, and with the usual dot notation applying again:

$$y_{\cdot j} = \sum_{k=1}^{n_j} y_{kj}\ , \quad y_{\cdot\cdot} = \sum_{j=1}^{r} y_{\cdot j} \quad \text{and} \quad n_{\cdot} = \sum_{j=1}^{r} n_j\ .$$

The experiment must now be analyzed as a randomized blocks design of unequal block size. One possible analysis would be as follows.

Analysis of Variance

Term	D.F.	Sum of Squares
Blocks (ignoring treatments)	$r - 1$	T'_B
Treatments (eliminating blocks)	$n_{\cdot} + s - 1$	$T'_T = T'_0 - T'_B - T'_E$
Error	$(r - 1)(s - 1)$	T'_E
Total	$n_{\cdot} + rs - 1$	T'_0

To facilitate this analysis we compute the three expressions shown in Table 2. Then, in the analysis of variance, the first term is clearly that defined as (1) in Table 2; the second, T'_T, is obtained by differencing as indicated; T'_E is yet to be defined; and T'_0 is as given in (2) of Table 2.

The above analysis can be completed by establishing the value of T'_E. It is "obviously" the same as the error term in the analysis of the non-augmented design in Table 1; nevertheless it is instructive to so demonstrate. We do this by noting that the second term of the above analysis, the sum of squares due to treatments (eliminating blocks), T'_T, must be the same as the total of the following three sums of squares.

Sum of Squares	d.f.	Value
Among standards	$s - 1$	T_T (as in Table 1)
Among tests within blocks	$n_{.} - r$	T_W (as in (3) of Table 2)
Standards v. tests, within blocks	r	$T_C = \sum_{j=1}^{r} \left[\frac{x^2_{.j}}{s} + \frac{y^2_{.j}}{n_j} - \frac{(x_{.j} + y_{.j})^2}{s + n_j} \right]$

Hence

$$T'_T \equiv T'_O - T'_B - T'_E = T_T + T_W + T_C$$

and so

$$T'_E = T'_O - T'_B - T_T - T_W - T_C$$

and on substituting from Table 2 this gives

$$\begin{aligned} T'_E &= T_O + x^2_{..}/rs - \sum_j (x_{.j} + y_{.j})^2/(s + n_j) + \sum_j y^2_{.j}/n_j - T_T - T_C \\ &= T_O + x^2_{..}/rs + (T_C - \sum_j x^2_{.j}/s) - T_T - T_C \\ &= T_O - T_T - (\sum_j x^2_{.j}/s - x^2_{..}/rs) \\ &= T_O - T_T - T_B \\ &= T_E . \end{aligned}$$

Although introduced in establishing this result T_C is not needed in the calculations. The complete analysis, shown in Table 3, can be derived from the terms of Table 1 and 2.

An alternative analysis is shown in Table 4. The sums of squares there are established by observing that the sum of squares for treatments (ignoring blocks) is

$$\sum_j x^2_{i.}/r + \sum_{j=1}^{r} \sum_{k=1}^{n_j} y^2_{kj} - (x_{..} + y_{..})^2/(rs + n_{.}) ,$$

and from Tables 1 and 2 this is

$$T_T + x_{..}^2/rs + T_0' - T_0 - x_{..}^2/rs$$

$$= T_0' - (T_0 - T_T)$$

$$= T_0' - (T_B + T_E).$$

Then, because the first two terms in Table 4 must have the same sum as the first two in Table 3, namely $T_0' - T_E$, the second term in Table 4, the sum of squares for blocks (eliminating treatments), must be $T_0' - T_E - [T_0' - (T_B + T_E)] = T_B$, as shown.

Treatment effects

The formulae given in Tables 3 and 4 are, of course, equivalent to those shown in Federer (1961). He also gives formulae for estimating treatment effects. Simplified forms of these are shown in Table 5. And for the sake of completeness the variances of estimated treatment differences are shown in Table 6.

Example

We use for our example the one given in Federer (1961):

Example

Standard treatments	Blocks 1	2	3	Totals
A	9	6	12	27
B	5	6	10	21
C	7	6	11	24
Totals	21	18	33	72
Test treatments	13(D)	10(E)	-	23
Totals	34	28	33	95

The sums of squares for the analysis of the standards are, as in Table 1,

$$T_B = (21^2 + 18^2 + 33^2)/3 - 72^2/9 = 618 - 576 = 42$$

$$T_T = (27^2 + 21^2 + 24^2)/3 - 72^2/9 = 582 - 576 = 6$$

$$T_E = 52 - 42 - 6 = 4$$

$$T_0 = 9^2 + 6^2 + 12^2 + 5^2 + \ldots + 10^2 - 72^2/9 = 628 - 576 = 52$$

The expressions given in Table 2 are

$$T'_B = 34^2/4 + 28^2/4 + 33^2/3 - 95^2/11 = 848 - 820.\dot{4}\dot{5} = 27.\dot{5}\dot{4}$$

$$T'_0 = 628 + 13^2 + 10^2 - 95^2/11 = 897 - 820.\dot{4}\dot{5} = 76.\dot{5}\dot{4}$$

$$T_W = 13^2 - 13^2 + 10^2 - 10^2 = 0$$

Hence, from Table 3, the analysis of variance is

Analysis of Variance

Term	D.F.	Sum of Squares
Blocks (ignoring treatments)	2	$27.\dot{5}\dot{4}$
Treatments (eliminating blocks)	4	45
Standards	2	6
Tests within blocks	0	0
Standards v. tests within blocks	2	39
Error	4	4
Total	10	$76.\dot{5}\dot{4}$

The first two terms of the alternative analysis shown in Table 4 would be

Treatments (ignoring blocks), 4 d.f. : $76.\dot{5}\dot{4} - 42 - 4 = 30.54$

Blocks (eliminating treatments), 2 d.f. : 42.00

And from Table 5 the estimated effects are:

$$\hat{b}_1 = 21/3 - 72/9 = -1$$
$$\hat{b}_2 = 18/3 - 72/9 = -2$$
$$\hat{b}_3 = 33/3 - 72/9 = 3$$

$$\hat{\mu} = [23 + 72/3 - 1(-1) - 1(-2) - 0(3)]/(3 + 2) = 10$$

$$\hat{\tau}_1 = 27/3 - 10 = -1$$
$$\hat{\tau}_2 = 21/3 - 10 = -3$$
$$\hat{\tau}_3 = 24/3 - 10 = -2$$

$$\hat{t}_1 = 13 - 10 - (-1) = 4$$
$$\hat{t}_2 = 10 - 10 - (-2) = 2$$

Reference

Federer, W. T. (1961). Augmented designs with one-way elimination of heterogeneity. Biometrics, 17, 447-473.

Table 1

Analysis of Variance for
Randomized Complete Blocks

Term	D.F.	Sum of Squares
Blocks	$r - 1$	$T_B = \sum_1^r x_{.j}^2/s - x_{..}^2/rs$
Treatments	$s - 1$	$T_T = \sum_1^s x_{i.}^2/r - x_{..}^2/rs$
Error	$(r - 1)(s - 1)$	$T_E = T_0 - T_B - T_T$
Total	$rs - 1$	$T_0 = \sum_{11}^{sr}\sum x_{ij}^2 - x_{..}^2/rs$

Table 2

Additional terms to compute for
analyzing augmented designs

(1) Block sum of squares over <u>all</u> treatments, standards and tests:

$$T_B' = \sum_{j=1}^{r} \frac{(x_{.j} + y_{.j})^2}{s + n_j} - \frac{(x_{..} + y_{..})^2}{rs + n_.}$$

(2) Total sum of squares for the whole experiment

$$T_0' = \sum_{i=1}^{s} \sum_{j=1}^{r} x_{ij}^2 + \sum_{j=1}^{r} \sum_{k=1}^{n_j} y_{kj}^2 - \frac{(x_{..} + y_{..})^2}{rs + n_.}$$

$$= T_0 + \sum_{j=1}^{n} \sum_{k=1}^{n_j} y_{kj}^2 + \frac{x_{..}^2}{rs} - \frac{(x_{..} + y_{..})^2}{rs + n_.}$$

(3) Within-block sum of squares of the test treatments

$$T_W = \sum_{j=1}^{r} \left(\sum_{k=1}^{n_j} y_{kj}^2 - y_{.j}^2/n_j \right) .$$

Table 3

Analysis of variance for
augmented randomized complete blocks

Term	D.F.	Sum of Squares
Blocks (ignoring treatments)	$r-1$	T'_B
Treatments (eliminating blocks)	$n_{\cdot}+s-1$	$T'_O-T'_B-T_E$
Standards	$s-1$	T_T
Tests within blocks	$n_{\cdot}-r$	T_W
Standards v. tests within blocks	$r-1$	$T'_O-T'_B-T_E-T_T-T_W$
Error	$(r-1)(s-1)$	T_E
Total	$n_{\cdot}+rs-1$	T'_O

Table 4

Alternative analysis of augmented
randomized complete blocks design

Term	D.F.	
Treatments (ignoring blocks)	$n_{\cdot} + s - 1$	$T'_O - (T_B + T_E)$
Blocks (eliminating treatments)	$r - 1$	T_B
Error	$(r - 1)(s - 1)$	T_E
Total	$n_{\cdot} + rs - 1$	T'_O

Table 5

Estimation of effects

Term estimated	Estimator
Block constant	$\hat{b}_j = \bar{x}_{.j} - \bar{x}_{..}$
Grand mean	$\hat{\mu} = (y_{..} + x_{..}/r - \sum_{j=1}^{r} n_j \hat{b}_j)/(s + n_.)$
Treatment constant	
Standard treatment	$\hat{\tau}_i = \bar{x}_{i.} - \hat{\mu}$
Test treatment	$\hat{t}_{kj} = y_{kj} - \hat{\mu} - \hat{b}_j$

Table 6

Variances of differences

between estimated treatment effects

Difference between			Variance of difference $[M_E = T_E/(r-1)(s-1)]$
2 standards:	$\hat{\tau}_i - \hat{\tau}_{i'}$	$(i \neq i')$	$2M_E/r$
2 tests, in same block:	$\hat{t}_{kj} - \hat{t}_{k'j}$	$(k \neq k')$	$2M_E$
2 tests, in different blocks:	$\hat{t}_{kj} - \hat{t}_{k'j'}$	$(k \neq k',\ j \neq j')$	$2M_E(1+1/s)$
a standard and a test:	$\hat{\tau}_i - \hat{t}_{kj}$		$M_E(1+1/r+1/s-1/rs)$