NONPERTURBATIVE APPROACH TO GRAVITATIONAL SELF-FORCE AND HAMILTONIAN FORMULATION OF ITS CONSERVATIVE DYNAMICS

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NONPERTURBATIVE APPROACH TO GRAVITATIONAL SELF-FORCE AND HAMILTONIAN FORMULATION OF ITS CONSERVATIVE DYNAMICS

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The two-body problem in general relativity has been the focus of intense observational and theoretical interest in recent years, especially as gravitational-wave astronomy has begun delivering on its long promised potential. A wide variety of approaches, valid in different regimes, have been used to understand the dynamics of binaries: numerical relativity , the post-Newtonian approximation, the post-Minkowskian approximation, the small mass ratio approximation, and the effective one-body framework, which synthesizes information from the other approaches. The success of future gravitational wave observations (both future ground and space based) depends on our ability to model the fundamental physics of two-body systems and produce accurate waveform templates that are used in the matched filtering techniques that these observatories employ.

An issue that arises in the study of two-body systems is whether one can define dissipative and conservative sectors of the dynamics for which the conservative sector admits a Hamiltonian description. Such a description would unlock the full power of Hamiltonian methods to study integrability and chaotic motion, obtain new gauge invariant quantities and better understand the effect of resonant orbits. In this thesis, we show that such dissipative and conservative sectors can be defined in the limit where one of the bodies is much smaller than the other, and derive a Hamiltonian description to linear order in the mass and spin of the secondary. We also extend this result to second order, in the context of a scalar toy model of the gravitational interaction.

In deriving these results, we develop two useful theoretical tools. First, we show that a broad class of dynamical systems, defined by non-local in time action principles, can be recast as local Hamiltonian systems to all orders in the non-locality. Second, we develop a reformulation of the dynamics of bodies with strong self-interactions which shows that their motion is equivalent to that of bodies with negligible self-interactions, albeit in a renormalized or effective external field.

BIOGRAPHICAL SKETCH

Francisco Martín Blanco was born in 1994 in Buenos Aires, Argentina. He attended Universidad de Buenos Aires starting in 2012, graduating with a Licenciatura in physical sciences in 2018. The Licenciatura is a Bachelor and Master equivalent, which culminated with a master thesis under the supervision of Prof. Daniel Cabra, investigating the properties of quantum ferromagnets using a novel modification of the Holstein-Primakoff transformation. In 2018, he started his doctorate at Cornell University, where he joined Prof. Eanna Flanagan's group at the start of 2020. In addition to his passion for physics, he enjoys music, astronomy, watching movies and spending time with friends and family.

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Contents

1 Preliminaries **1**
- 1.1 Gravitational wave astronomy . . . 4
- 1.2 Motion of self-gravitating bodies in Newtonian mechanics . . . 9
- 1.3 The two-body problem in General Relativity . . . 12

2 Review of the Self-Force Formalism **16**
- 2.1 Context . . . 16
- 2.2 Electromagnetic self-force in flat spacetime . . . 18
- 2.3 Generalization to curved spacetimes . . . 22
- 2.4 Gravitational self-force . . . 27
- 2.5 Detweiler-Whiting regularization scheme . . . 30
- 2.6 Adiabatic inspirals . . . 37

3 Nonperturbative approach to self-force: Scalar case (Adapted from [1]) **47**
- 3.1 Context . . . 47
- 3.2 General framework . . . 49
- 3.3 Field redefinition . . . 52
- 3.4 Weak nonlinearity expansion . . . 65
- 3.5 Point-particle limit . . . 81
- 3.6 Application to nonlinear scalar field theory in flat spacetime . . . 86

4 Hamiltonian formulation of conservative nonlocal in time dynamical systems (Adapted from [2]) **89**
- 4.1 Introduction . . . 89
- 4.2 Dynamical systems described by non-local action principles . . . 92
- 4.3 Local Hamiltonian description . . . 96
- 4.4 Alternative formulation of local Hamiltonian system . . . 102

4.5 Application: Binary systems in the post-Newtonian approximation 110

5 The conservative sector of the dynamics of binary systems (Adapted from [3, 4]) **112**
5.1 Context 112
5.2 Pseudo-Hamiltonian description of dynamical systems 114
5.3 Gravitational case: Linear order in the mass ratio 115
5.4 Gravitational case: Linear-in-spin conservative effects 126
5.5 Scalar case: Second order in the scalar charge 139

6 Conclusion **144**

Appendix A General Theory of bitensors **147**
A.1 Synge's Worldfunction 148
A.2 Hadamard 2-point functions 149
A.3 Hadamard Expansion and Parametrices 151

Appendix B Derivation of equations of motion **155**
B.1 Covariant techniques for phase-space variations 155
B.2 Geodesic Motion 158
B.3 First-order gravitational self-force in Hamiltonian formulation 161
B.4 Second-order scalar self-force in Hamiltonian formulation 164

Appendix C Alternative description of non-local dynamical systems **166**
C.1 Relation to the work of Llosa and Vives 166

Appendix D Strong causality condition **168**

References **177**

In the present small treatise, I set forth some matters of great interest for all observers of natural phenomena to look at and consider. They are of great interest, I think, first, from their intrinsic excellence; secondly, from their absolute novelty; and lastly, also on account of the instrument by the aid of which they have been presented to my apprehension.

Galileo Galilei

1 Preliminaries

Newtonian mechanics can solve the two-body problem but not the three-body problem; general relativity can solve the one-body problem but not the two-body problem; and quantum gravity cannot even solve the one-body problem*. It is the goal of this thesis to get us one step closer to ruining the joke by allowing general relativity to describe two-body dynamics in all

*I thank Professor Avery Broderick for telling me this joke.

its glory, as once Kepler and Newton did.

The two-body problem concerns the motion of two compact objects under their mutual gravitational attraction. Although we have been watching two-body systems move in the night sky since humanity first put its eyes on the celestial movements of the sun, moon, and planets, the first observations that led to their detailed understanding were made in the 17th century. Around 1609, Galileo Galilei observed the Moon's mountains and Jupiter's four biggest satellites with the aid of the recently developed refracting telescope. Before these observations, the prevalent model for the cosmos was the Ptolemaic model, where every celestial object rotates around the earth in complex orbits. The observation of moons orbiting other planets, as well as the phases of Venus and the retrograde motion of inner and outer planets were the first conclusive observations that began to settle the heliocentric debate. At almost the same time, Johannes Kepler was writing his *Astronomia Nova*, where he first presented two of his laws of planetary motion. With the aid of Tycho Brahe's observations, he was the first one to describe the shapes of planetary orbits and to provide accurate calculations for their periods. But it was not until Isaac Newton's *Principia*, published in 1687, that a full solution to the problem was derived. In Newtonian mechanics, the two-body problem is elegantly solved by applying Newton's laws of motion and his law of universal gravitation. In modern treatments, the dynamics of the two-body system are simplified by utilizing its symmetries, effectively reducing it to a single-body problem. This reduction leads to closed-form solutions, such as circular or elliptical orbits, which obey the planetary laws of motion previously set forth by Kepler. Despite their apparent simplicity, two-body systems are the archetypal problem in both classical and quantum physics. After all, the atomic model that kickstarted quantum mechanics is nothing more than the quantum (and electromagnetic)

analog of the classical two-body problem.

However, the advent of general relativity introduced new complexities to the two-body problem, such as nonlinear interactions and energy loss due to gravitational radiation. Because of these complexities, the problem cannot be solved analytically in full generality. Numerical relativity has made great strides in recent years and is now a key tool for integrating gravitational wave signals from binary systems [5]. However, numerical relativity is limited in its ability to simulate binaries in regimes where the number of orbits becomes large, such as the regime of wide orbital separations or the limit where one body is much smaller than the other. Therefore, there is strong motivation for improving our analytic understanding of these regimes, which are based on a variety of approximations.

The first approximation schemes were already present in the work of Einstein, who expanded his equations to linear order in the inverse speed of light squared to calculate the perihelion precession of Mercury's orbit [6]. Although this constituted the first use of the post-Newtonian approximation, its foundations were more rigorously laid by Subrahmanyan Chandrasekhar and his collaborators [7–11] in the 1960s. This work also had its own limitations due to the inability of the approximation method used to describe radiating fields in the far zone. A more modern treatment of the post-Newtonian expansion was later developed ([12] and references therein), where the approximation is matched onto a post-Minkowskian expansion in powers of Newton's constant G at large distances from the sources.

The two-body problem in general relativity has received intense observational and theoretical interest in recent years, as gravitational-wave astronomy has begun to deliver on its long-promised potential [13–15]. The first direct detection of gravitational waves in 2015 by the Laser Interferometer Gravitational-Wave Observatory (LIGO) marked the beginning

of gravitational wave astronomy. The LIGO project was conceived in the 80's but it was not until 2002 that the first runs started. It took thirteen more years of technical improvements to get a detection. LIGO, along with the Virgo detector in Italy and KAGRA in Japan, focuses on high-frequency gravitational waves ($10 - 1000$ Hz) from stellar-mass compact objects. The upcoming Laser Interferometer Space Antenna (LISA) will instead target low-frequency gravitational waves ($10^{-4} - 10^{-1}$ Hz) from supermassive black holes, as well as other sources [16, 17]. The success of future gravitational wave observations (both future runs of LIGO/Virgo/ KAGRA and the first runs of LISA) depends on our ability to model the fundamental physics of two-body systems and produce accurate waveform templates that are used in the matched filtering techniques that these observatories employ.

* * *

This chapter's goal is to provide the context necessary for the remainder of this thesis. In Section 1.1, we provide order-of-magnitude estimates of the intensity and frequency of gravitational waves coming from different types of sources and review the method of matched filtering. In Section 1.2, we begin our exploration of two-body dynamics by studying the effect of self-fields in Newtonian gravity. In Section 1.3, we discuss the obstacles that arise in general relativity when trying to solve the two-body problem. We end this chapter with a discussion of the questions addressed by this thesis.

1.1 Gravitational wave astronomy

The intensity of a gravitational wave coming from a distant binary system can be estimated using Newtonian gravity supplemented by the quadrupole formula for the leading order grav-

itational wave emission. The first order metric perturbation then scales as

$$h \propto \frac{G}{c^4}\frac{\ddot{Q}}{r} \tag{1.1}$$

where $\ddot{Q}$ is the second time derivative of the source's quadrupole moment and r is the distance from the source. The quadrupole moment Q can be approximated by

$$\ddot{Q} \approx E_{\text{non-sph,kin}}, \tag{1.2}$$

where $E_{\text{non-sph,kin}}$ is the non-spherical piece of the kinetic energy. By the virial theorem, the kinetic energy must be comparable to the gravitational energy of the system so that

$$E \propto \frac{Gm_1m_2}{L}, \tag{1.3}$$

where m_1 and m_2 are the masses of each object in the two-body system. The size L of the system will get smaller as the orbit radiates energy away. We can estimate the smallest separation as the sum of the Schwarzschild radii of both objects

$$L_{\text{closest}} \propto \frac{2GM_{\text{tot}}}{c^2}. \tag{1.4}$$

Combining equations (1.1), (1.2), (1.3) and (1.4) we get that the metric perturbation is

$$h \propto \frac{G}{c^2}\frac{\mu}{r}. \tag{1.5}$$

where μ is the reduced mass of the system

$$\mu = \frac{m_1 m_2}{m_1 + m_2},$$

For systems with a reduced mass close to a solar mass $M_\odot$, the metric perturbation is

$$h \sim 10^{-22} \text{ for Hubble distances } r \approx 3000\text{Mpc},$$
$$h \sim 10^{-17} \text{ for outer region of Milky Way } r \approx 200\text{kpc}.$$

Next, we move on to the frequency range of gravitational waves. For binary systems with orbital frequency f_{orb}, emitted gravitational waves contain all integer multiples of f_{orb}. The greatest contribution comes from $f \approx 2f_{\text{orb}}$, because most of the radiation comes from the quadrupole moment of the system. Using Kepler's law, the orbital period will approximately be

$$T^2 \sim \frac{L^3}{GM_{\text{tot}}}. \tag{1.6}$$

This means the orbital frequency will be

$$f \sim \frac{\left(GM_{\text{tot}}\right)^{1/2}}{L^{3/2}}. \tag{1.7}$$

The strongest signals will be emitted at the closest approach $L_{\text{closest}} = \frac{2GM_{\text{tot}}}{c^2}$ and therefore have a frequency of

$$f_{\text{max}} \sim \frac{c^3}{GM_{\text{tot}}} \sim 10^4 \text{Hz} \frac{M_\odot}{M_{\text{tot}}}. \tag{1.8}$$

Hence, the frequency of the gravitational waves will be inversely proportional to the total

mass M_{tot} of the binary system, as opposed to their amplitude, which depends on the reduced mass μ.

1.1.1 Experimental frequency ranges

High-frequency sources $1\text{Hz} < f < 10^4\text{Hz}$

Gravitational waves have a natural upper bound at approximately $f \sim 10^4$Hz. From equation (1.8), we know that the maximum frequency is inversely proportional to the total mass. The most intense gravitational waves are emitted by compact systems so detectable signals must come from systems more massive than the Chandrasekhar limit (approximately $1.4M_\odot$) which sets the upper limit

$$f \leq 10^4\text{Hz} \tag{1.9}$$

The lower bound at 1Hz is not related to the physics of the gravitational waves, but to the sources of noise that affect the experiments. Ground-based experiments face an insurmountable amount of noise coming from gradients in Earth's gravitational potential and vibrational noise. Using eq. (1.8) we see that the typical sources that fall in the high-frequency band are

- Coalescence of binary neutron star systems.
- Coalescence of Stellar-mass ($M \leq 1000M_\odot$) black holes.
- Pulsars.

Low-frequency sources $10^{-4}\text{Hz} < f < 1\text{Hz}$

This frequency range is defined entirely by the limitations of the gravitational-wave experiments we can do. As we explained in the previous subsection, ground-based experiments

won't be able to detect gravitational waves with frequencies lower than 1Hz, which is why the low-frequency range is the domain of detectors flown in space, either in Earth orbit or interplanetary orbit. The lower bound at 10^{-4}Hz is defined by expected difficulties in isolating the spacecraft from cosmic rays, solar winds and solar radiation pressure. Typical sources in this range of frequencies are

- Supermassive black hole binaries.
- Extreme mass-ratio inspirals: stellar-mass black holes into supermassive black hole.

There are two more frequency bands accessible to observations: The very-low-frequency band range at $10^{-7}\text{Hz} < f < 10^{-4}\text{Hz}$ in which gravitational waves have recently been detected by pulsar timing observations [18] and the extreme-low-frequency range at $10^{-18}\text{Hz} < f < 10^{-15}\text{Hz}$, which is the target of indirect search for gravitational waves with the cosmic microwave background. In this thesis, we will focus on the low-frequency band -the domain of the LISA mission- and the high-frequency band- the domain of ground-based detectors like LIGO.

1.1.2 Waveform templates and matched filtering

In most gravitational-wave detectors, the amplitude of most expected signals is near or even below the detector's instrumental noise threshold. Despite this, these signals can still be reliably identified if their waveform corresponds closely to a predicted or modeled waveform [19]. To achieve this identification, a technique known as matched filtering is employed, which involves comparing the incoming data with a library of known gravitational wave templates. A straightforward example of this type of filter is the Fourier transform, which

acts as a matched filter for a constant-frequency signal. In this process, the noise power in the data is distributed across the spectrum, whereas the signal's power is focused at a single frequency. This concentration enhances the signal's detectability. The end goal of a good portion of present day research in the two-body problem is to produce waveforms that are accurate enough to detect gravitational waves and extract information from them.

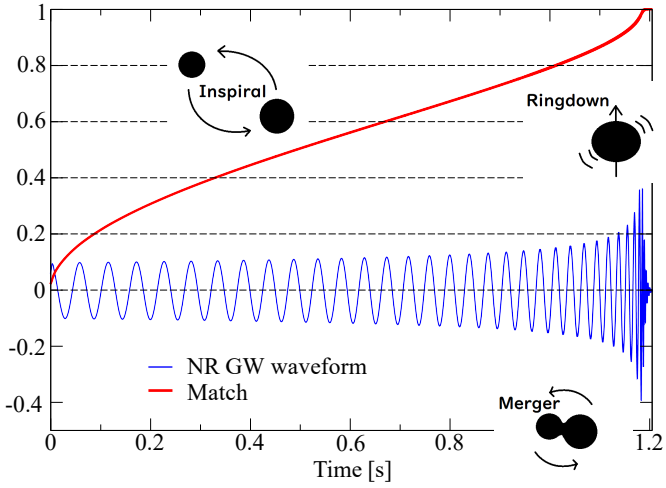

Figure 1.1: Example of a Inspiral-merger-ringdown waveform from the numerical-relativity simulation SXS:BBH:0305 [20] which models the detected gravitational wave signal GW150914. The horizontal axis is the real time. The blue thin curve is a gravitational wave template for the "+" mode. The red thick curve shows the "match", i.e. the accumulation of the (normalized) signal-to-noise ratio. (Plot taken from [21])

1.2 Motion of self-gravitating bodies in Newtonian mechanics

We commence our study of the physics of two-body systems by considering the motion of self-gravitating extended objects in Newtonian gravity. Our observations in this simple scenario will set the scene for most of the work in this thesis.

Following [22], we consider an inertial frame (t, x^i) and study the dynamics of a massive object which is extended but has a finite size, so that at any instant t, its mass density $\rho(t, x)$ has support on a compact region $\Sigma_t \subset \mathcal{M}$. The object's center-of-mass acceleration a^i_{cm} obeys Newton's equation

$$a^i_{\text{cm}}(t) = -\int \rho(t,x)\nabla^i\varphi(t,x)d^3x \tag{1.10}$$

The gravitational field $\varphi(x)$ is the sum of some external field φ_{ext} plus the object's self-field φ_{self}, which is sourced by ρ

$$\nabla^2\varphi_{\text{self}} = 4\pi\rho \tag{1.11}$$

and determined by equation (1.11) plus suitable boundary conditions. The external field is possibly sourced by some other mass distribution but it is a vacuum solution on the object's compact support

$$\nabla^2\varphi_{\text{ext}}\big|_{\Sigma_t} = 0 \tag{1.12}$$

Equation (1.11) can be solved using a Green function

$$G(x, x') = \frac{1}{|\mathbf{x} - \mathbf{x}'|}. \tag{1.13}$$

This Green function satisfies two properties that are important in later chapters: It is fully symmetric and it vanishes when the points are infinitely separated.

We are interested in the effect of the self-field on the equations of motion. Plugging equa-

tion (1.13) into (1.10) we get

$$a^i_{\rm cm} = -\int \rho(t,x)\nabla^i \varphi_{\rm ext}(x) d^3x - \int \rho(t,x)\rho(t,x')\nabla^i G(x,x') d^3x d^3x'. \tag{1.14}$$

In units where $G_N = 1$, the first term on the right hand side of equation (1.14) -the external force- must be proportional to $mM/L^2_{\rm ext}$, where M and $L_{\rm ext}$ are the mass and distance associated with the external field and m is the mass of the object. The second term, which represents the self-force, must be proportional to m^2/L^2, where L is the lengthscale of the object. We are interested in the regime where the details of the body's composition decouple from its equations of motion. Physically, this happens when the size of the object L is small compared to all other lengthscales in the system. If this is the case, we can do a multipole expansion of the mass density ρ, where we replace it by a "skeleton", made up of multipole moments living on the object's worldline z_s

$$\rho(x) \to m\delta(x - z_s) + \sum_{n=1}^{\infty} I^{a_1 \dots a_n} \nabla_{a_1} \dots \nabla_{a_n} \delta(x - z_s). \tag{1.15}$$

The nth multipole moment scales as mL^n, so they tend to zero as we take $L \to 0$. However, the second term in equation (1.14) does not appear to be well-behaved in this limit, since it scales as m^2/L^2. Luckily, it vanishes before taking any limits: The product $\rho(t,x) \times \rho(t,x')$ is symmetric under exchange of x and x' but the gradient of the Green function is antisymmetric. Alternatively, this can also be thought of as a consequence of Newton's law of action and reaction. We conclude that motion is determined by an effective field

$$\hat{\varphi}(t,x) = \varphi(t,x) - \int G(x,x')\rho(t,x')dV' \tag{1.16}$$

which is simply the external field φ_{ext}. The effective field is a renormalized field which is obtained by subtracting the self-field of the compact object from its retarded field. Because it is a homogeneous solution (cf. Equation (1.12)), it is independent of L and therefore finite in the limit $L \to 0$.

We have proven that there exists an effective external field that governs the motion of extended objects in Newtonian physics. In the context of general relativity, a new dimensionless parameter, Gm/Lc, measures the compactness of the object, while the parameter L/L_{ext} governs the strength of extended-object or tidal effects, as we described in the preceding paragraphs. It is possible to obtain a finite self-force by taking both m and L to zero in a self-similar manner that keeps Gm/Lc fixed [23].

The result derived here, in the context of Newtonian gravity, encapsulates the spirit of the framework we develop in Chapter 3. In that context, the effects of the self-field do not vanish, but can be absorbed into the renormalization of the compact object's mass density and stress-energy tensor. The statement becomes: There exists a renormalized description of the compact object in which it moves under the influence of an effective external field which is a vacuum solution.

1.3 The two-body problem in General Relativity

There are a series of obstacles that arise when one tries to extend the results of the previous section to general relativity. We list the most apparent complications below:

- Due to the nonlinearity of Einstein's equations, the gravitational field cannot be split into a piece sourced by each body. This makes it difficult to identify the self-field of the

body of interest, which is the piece we subtracted in Section 1.2 to obtain an effective gravitational field with a finite point-particle limit.

- Another consequence of the nonlinear nature of gravity is that point particles are ill-defined objects in general relativity. After all, a point object with finite mass should collapse into a black hole. This makes a naive perturbative analysis of gravity in the point-particle limit useless past linear order.

- Orbiting masses radiate gravitational waves and lose energy over time. Because of this, the dynamics of the system is wildly different at different moments in the inspiral process. This feature makes it difficult to have one closed form equation describe the complete evolution of the system and is the reason why we need different approximation schemes for each stage of the inspiral.

These obstacles are related to many open questions in the study of the dynamics of binary systems, some of which are the focus of this thesis.

Questions addressed in this thesis

Can the results of Section 1.2 on the effect of self-fields on the dynamics of self-gravitating objects be generalized to curved spacetimes?

This question is related to the existence of a generalized equivalence principle. Originally, the equivalence principle states that test point particles move along geodesics of the external spacetime geometry, regardless of their internal properties. This principle doesn't hold for test extended objects, since they experience tidal forces that perturb their motion in ways that depend on their particular inner composition. However, it remains true that the tidal forces

are caused by the external gravitational field, since test objects don't produce self-fields. An even stronger statement, which is sometimes called *generalized equivalence principle*, is that self-gravitating extended objects move as test extended objects in an effective external field. In other words, there's a way to absorb their self-fields into their stress-energy tensor, such that their renormalized multipolar momenta evolve under the influence of an effective external field. This is the main result of chapter 3. It holds to any order in a perturbation expansion in powers of the scalar charge density ρ. However, it does depend on the existence of a series of n-point functions whose existence hasn't been proved past linear order, except in a few simplified toy models.

* * *

Can the dynamics of binary systems of self-gravitating objects be split into conservative and dissipative sectors to any order in perturbation theory?

This question will have different answers depending on the approximation scheme used. Newtonian gravity, for example, is purely conservative, so this question is trivial. The post-Newtonian expansion starts to produce dissipative effects at $O(1/c^5)$. At this order, there is still a clear split and a Hamiltonian describing the conservative dynamics can be found. The question is more subtle at $O(1/c^8)$ because the interaction becomes nonlocal in time. This means that the force depends not only on the position of the objects locally, but also on their complete past history. Then, even if a conservative/dissipative split is possible, it is not clear that a local Hamiltonian for the conservative dynamics exists.

These issues are addressed in chapter 4, where we prove the existence of a local Hamiltonian description for a general class of theories with nonlocal interactions. In particular, we apply this result to obtain a local Hamiltonian description of the conservative post-Newtonian

dynamics up to $O(1/c^8)$. In chapter 5, we apply the proof of chapter 4 to the small mass-ratio expansion and obtain the local Hamiltonian description of the conservative piece of the two-body dynamics up to first order in the mass ratio and spin of the small object. We then obtain a local Hamiltonian description of the conservative piece of the dynamics in a scalar toy model, to second order in the scalar charge.

Dos cuerpos frente a frente
son a veces dos olas
y la noche es océano.

Dos cuerpos frente a frente
son a veces dos piedras
y la noche desierto.

Dos cuerpos frente a frente
son a veces raíces
en la noche enlazadas.

Dos cuerpos frente a frente
son a veces navajas
y la noche relámpago.

Dos cuerpos frente a frente
son dos astros que caen
en un cielo vacío.

Octavio Paz

2

Review of the Self-Force Formalism

2.1 Context

This chapter is concerned with the dynamics of extreme mass-ratio inspirals. These describe the orbits of two-body systems with a small object of mass m (*the secondary*) orbiting a more massive object of mass M (*the primary*). They abound in nature, in the form of stellar-mass

black holes or neutron stars that have been captured into inspiral orbits around super massive black holes with masses between 10^6 and $10^9 M_\odot$ [24]. The starting point of the self-force formalism is to analyze the dynamics perturbatively in the mass ratio $\varepsilon = m/M$. At zeroth order, the secondary behaves as a test particle moving along a geodesic of the background metric determined by the primary. At first order, we include the effect of secondary's own mass m, so its gravitational field acts as a perturbation of the background geometry. The interaction between the secondary and its own gravitational field gives rise to an acceleration with respect to the background geometry, described by the gravitational self-force.

The self-force formalism was first developed in the context of flat spacetime electromagnetism. The orbits of test charges are straightforwardly derived from Maxwell's equations, but it wasn't until the work of Lorentz, Abrahams, Poincaré, and Dirac [25] that the motion of self-interacting charges was understood. The interaction of electric charges with their own electromagnetic fields is responsible for the radiation-reaction force, which turns the bound orbits of test particles into slowly radiating inspirals. In 1960, DeWitt and Brehme [26] generalized Dirac's result to curved spacetimes, and their calculation was corrected by Hobbs [27] several years later. The gravitational self-force was first calculated in 1997, when Mino, Sasaki, and Tanaka [28] derived an expression for the acceleration of self-gravitating masses to linear order in the mass of the secondary (which vanishes in the limit of test particles); the same equations of motion were later obtained by Quinn and Wald [29] using an axiomatic approach. This result is now known as the MiSaTaQuWa equation and constitutes a foundational part of the self-force formalism [30]. The second-order contribution to the self-force was later studied and derived in [31–36], where more advanced techniques were developed to deal with some problems that arise at that order, as we will see later in this chapter.

This chapter is organized as follows. In Section 2.2, we review the derivation of the electromagnetic self-force in flat spacetimes. In Section 2.3, we generalize these results to curved spacetimes. We review the Detweiler-Whiting treatment of the linear self-force, where a singular solution is subtracted from the retarded solution to create an effective field. The effective field satisfies the vacuum field equations so the divergence of the retarded solution along the geodesic is avoided. In Section 2.4, we move on to the gravitational self-force and describe the downfalls of using naive perturbation theory for point particles. We later explore the role that self-fields play in these complications. In Section 2.5, we derive the first-order equations in Lorentz gauge and find a solution for the metric perturbation in terms of a retarded Green function. Finally, in Section 2.6, we review the two-timescale method and the effect of resonances on the dynamics of extreme mass-ratio inspirals.

2.2 Electromagnetic self-force in flat spacetime

Consider a charged point particle moving along a geodesic $z(\tau)$ with 4-velocity $u^\alpha \equiv \frac{dz^\alpha}{d\tau}$. Maxwell equations are

$$\nabla^\nu F_{\mu\nu} = 4\pi J_\mu \tag{2.1a}$$

$$\nabla_{[\mu} F_{\nu\rho]} = 0 \tag{2.1b}$$

where J^μ is the current density of the charged particle. Its dynamics are governed by the Lorentz force

$$m\dot{u}_\mu = f^{\text{ext}}_\mu + qF_{\alpha\beta}u^\beta \tag{2.2}$$

where $\dot{u}^\mu$ is the 4-acceleration, $F_{\alpha\beta} = \partial_\alpha A_\beta - \partial_\beta A_\alpha$ is the Maxwell tensor and f^{ext}_μ is an external force. The 4-potential A_μ is constructed from the retarded Green's function as

$$A_\mu(x; [z(\tau)]) = \int G^+_{\mu\nu'}\left[x, z(\tau')\right] u^{\nu'} d\tau'. \tag{2.3}$$

Here, we use primed indices to distinguish tensors evaluated at $z(\tau')$. Just as we saw with self-fields on Newtonian gravity, if we evaluate $A_\mu(x)$ along particle's worldline then the Green function gets evaluated at coincidence and the self-force diverges. However, the particle's interaction with it's own field is what causes it to radiate energy away, which is the phenomenon we are interested in. An alternative would be to consider extended objects instead, but then we would need to model the internal dynamics as well. The question we face is: How do we regularize the 4-potential so that it still describes the radiation-reaction process without diverging along the worldline? Dirac's solution [25] was to calculate energy and momentum fluxes through a worldtube around the charge's particle and relate them to its worldline. Instead of following that derivation, we will explain a prescription for obtaining the same result. Given retarded and advanced Green functions $G^\pm_{\mu\nu'}(x, x')$ it is easy to construct a time symmetric Green function as

$$G^C_{\mu\nu'} = \frac{1}{2}\left(G^+_{\mu\nu'} + G^-_{\mu\nu'}\right) \tag{2.4a}$$

Here we use C for *conservative*, which we will justify below. The retarded solution describes a situation with no incoming radiation, i.e., a solution that vanishes at $\mathcal{I}^-$. The advanced solution describes a situation with no outgoing radiation, i.e., a solution that vanishes at $\mathcal{I}^+$. The conservative solution, on the other hand, describes equal amounts of incoming and outgoing

radiation. It follows from this picture that the conservative Green function does not exert a radiation-reaction force on the particle. This can alternatively be understood by virtue of $G^C_{\mu\nu'}$ being invariant under time reversals. Nonetheless, retarded, advanced and conservative fields are all sourced by the charged particle and are divergent in the same way. It is possible then to construct a regular 2-point function* by subtracting the conservative solution from the retarded one

$$\begin{aligned} G^D_{\mu\nu'} =& G^+_{\mu\nu'} - G^C_{\mu\nu'} \\ =& \frac{1}{2}\left(G^+_{\mu\nu'} - G^-_{\mu\nu'}\right). \end{aligned} \tag{2.5}$$

The regular 2-point function is nothing but the *dissipative* 2-point function; the antisymmetric combination of retarded and advanced solutions (which is why we use a D instead of the probably more intuitive R for *regular*). The dissipative field A^D_μ is a homogeneous solution and behaves smoothly as we get close to the worldline.

We are presented with four different 2-point functions, retarded, advanced, conservative (or Coulomb/singular) and dissipative (or radiative/regular). When we generalize this approach to curved spacetimes, there will be a distinction between conservative and singular fields. Similarly, dissipative and regular fields will not be the same either. This is just a feature of flat spacetimes.

The domain of each 2-point function is represented on figure 2.1. The retarded Green function $G^+(x,x')$ vanishes unless the source is at a position x' on the past lightcone of x. The advanced Green function $G^-(x,x')$ is the exact opposite: It vanishes unless x' lies on the future lightcone of x. The conservative and dissipative 2-point functions have support on

*Notice that we restrict the use of the term "Green function" to solutions to the field equations sourced by a delta function. More general constructions can satisfy the homogeneous equation instead, in which case we use the term "2-point function", which encompasses both scenarios.

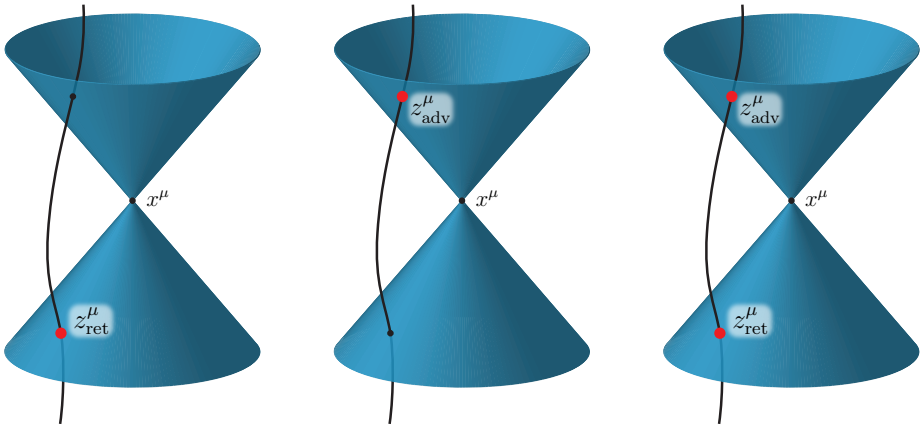

Figure 2.1: Relevant points for the retarded field A^+_μ (left), advanced field A^-_μ (middle), and conservative and dissipative fields A^C_μ and A^D_μ in flat spacetime. A^+_μ at the point x^μ depends on the state of the particle at the retarded point $z^\mu_{\text{ret}} = z^\mu(\tau_{\text{ret}})$, where the particle's worldline intersects x^μ's past light cone. A^-_μ at x^μ depends on the state of the particle at the advanced point $z^\mu_{\text{adv}} = z^\mu(\tau_{\text{adv}})$, where the particle's worldline intersects x^μ's future light cone. Both A^C_μ and A^D_μ each depend on the state of the particle at both z^μ_{ret} and z^μ_{adv} (Image taken from [37]).

both the future and past ligthcones of x. From this picture, it is clear that the conservative and dissipative solutions are not causal: The retarded (advanced) Green function has support on the past (future) lightcone, but the conservative and dissipative fields will depend on the position of the particle at both the retarded $z^\mu(\tau_{\text{ret}})$ and advanced $z^\mu(\tau_{\text{adv}})$ times. It is only when we evaluate the them along the worldline that both points converge at the present position of the particle, thus making the field causal.

If we replace the retarded field (2.3) in equation (2.2) by the dissipative field

$$A^D_\mu(x) = \int G^D_{\mu\nu'}\left[x, z(\tau')\right] u^{\nu'} d\tau'. \tag{2.6}$$

we get regularized equations of motion

$$m\dot{u}_\mu = f^{\text{ext}}_\mu + qF^D_{\alpha\beta}u^\beta. \tag{2.7}$$

This equation is now well defined along the worldline, since F^D is constructed from the regular 4-potential A^D. Furthermore, it can be shown [25] that the radiation-reaction force in Equation (2.7) can be expressed as

$$m\dot{u}_\mu = f^{\text{ext}}_{\mu\nu} + \frac{2q^2}{3}\left(\eta_{\mu\nu} + u_\mu u_\nu\right)\ddot{u}_\mu \tag{2.8}$$

which is known as the Abraham-Lorentz-Dirac equation. This equation has some problematic behavior, such as runaway solutions, due to the dependence on the first derivative of the acceleration. A solution [30] is to apply a *reduction-of-order procedure* where $\ddot{u}^\mu$ is replaced by $m^{-1}\frac{df^\mu_{\text{ext}}}{d\tau}$ to the lowest order in q. The equation becomes

$$m\dot{u}_\mu = f^{\text{ext}}_{\mu\nu} + \frac{2q^2}{3m}\left(\eta_{\mu\nu} + u_\mu u_\nu\right)\dot{f}^{ext}_{\mu} + O(q^4) \tag{2.9}$$

which is now well-behaved.

2.3 Generalization to curved spacetimes

A similar approach to that used by Dirac was later developed by DeWitt and Brehme for curved spacetimes [26]. However, the prescription used in the previous section, where we replace the retarded field by the symmetric half retarded plus advanced field no longer works. The reason is that in a curved background, electromagnetic waves can scatter off the curvature of spacetime and effectively travel slower than the speed of light so the retarded (advanced) Green function now has support inside the past (future) lightcone, as well as on it (See figure 2.2).

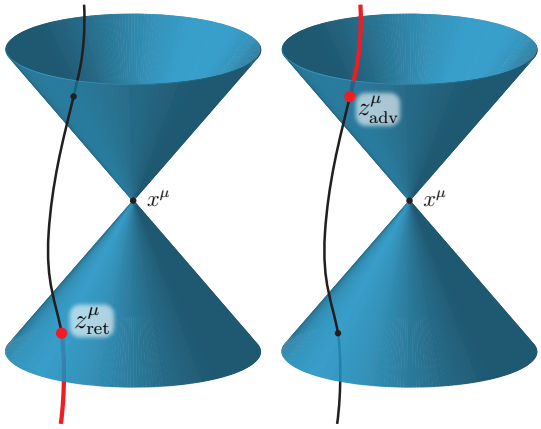

Figure 2.2: Relevant points for the retarded and advanced fields A^+_μ and A^-_μ in curved spacetime. A^+_μ at the point x^μ depends not just on the state of the particle at the retarded point on x^μ's past light cone, but also on the particle's state at all points *within* the past light cone. Analogously, A^-_μ depends on the state of the particle at all points on and within x^μ's future light cone.

This means that the conservative and dissipative fields we constructed in the previous section now depend on the position of the particle arbitrarily far into the past and future light-cones. In consequence, they are not causal even at the coincidence limit. A new prescription for deriving the electromagnetic self-force in curved spacetimes was developed by Detweiller and Whiting [38], who proposed the existence of a new 2-point function $H_{\mu\nu'}$ which satisfies the following properties:

- $H_{\mu\nu'}(x,x')$ is a homogeneous solution,
- It is symmetric under exchange of arguments: $H_{\mu\nu'}(x,x') = H_{\nu'\mu}(x',x)$,
- It agrees with the retarded Green function when x lies in the chronological future of x': $H_{\mu\nu'}(x,x') = G^+_{\mu\nu'}(x,x')$ for $x \in I^+(x')$.
- By reciprocity, it must also be true that $H_{\mu\nu'}(x,x')$ agrees with the advanced Green function when x lies in the chronological past of x': $H_{\mu\nu'}(x,x') = G^-_{\mu\nu'}(x,x')$ for $x \in$

$I^-(x')$.

We then define the singular Green function

$$G^S_{\mu\nu'} = \frac{1}{2}\left(G^+_{\mu\nu'} + G^-_{\mu\nu'} - H_{\mu\nu'}\right). \tag{2.10}$$

The last two properties of $H_{\mu\nu'}$ guarantee that the singular field vanishes for timelike separations, since $H_{\mu\nu'}$ will match either the retarded or advanced Green function, depending on the ordering of x and x'. The singular Green function satisfies the following properties.

- It satisfies the field equations $\Box G^{\mu}_{S\nu'}(x, x') = -4\pi\delta^{\mu}_{\nu'}\delta^4(x - x')$,
- It is symmetric under exchange of arguments,
- It vanishes for timelike separations.

We can now build the regular retarded/advanced 2-point function

$$\begin{aligned} G^{\pm,R}_{\mu\nu'} &= G^{\pm}_{\mu\nu'} - G^S_{\mu\nu'} \\ &= \frac{\pm\left[G^+_{\mu\nu'} - G^-_{\mu\nu'}\right] + H_{\mu\nu'}}{2}. \end{aligned} \tag{2.11}$$

The addition of $H_{\mu\nu'}$ cancels the advanced solution to make the regular retarded field depend on the history of the particle only in the past of z_{adv}, which becomes causal in the coincidence limit, as can be seen in figure 2.3.

The equations of motion for a point charge can now be derived following the following prescription, now known as the Detweiler-Whiting regularization scheme. Start with a charged point particle and derive its equations of motion. Then regularize the retarded

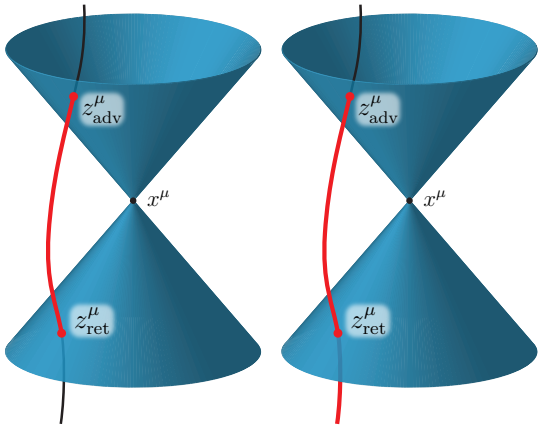

Figure 2.3: Relevant points for the singular and regular fields A^S_μ and A^R_μ in curved spacetime. A^S_μ depends on the state of the particle at all points on and outside x^μ's past and future light cones. A^R_μ depends on the state of the particle at the advanced point $z^\mu_{\rm adv} = z^\mu(\tau_{\rm adv})$ and at all prior points $z(\tau < \tau_{\rm adv}$.

solution by subtracting the singular field. The resulting equations are

$$m\frac{D}{d\tau}u_\mu = f^{\rm ext}_\mu + qF^{+,R}_{\mu\nu}u^\nu, \tag{2.12}$$

where $F^{+,R}_{\mu\nu}$ is derived from the 4-potential $A^{+,R}_\mu$ constructed with the regularized retarded 2-point function (2.11). By solving for the regularized retarded field [30], the equations of motion can be expressed as

$$m\frac{D}{d\tau}u_\mu(\tau) = f^{\rm ext}_\mu + q^2\left(g_{\mu\nu}+u_\mu u_\nu\right)\left(\frac{2}{3m}\frac{Df^\nu_{\rm ext}}{d\tau} + \frac{1}{3}R^\nu{}_\lambda u^\lambda\right) + 2q^2u^\mu\int_{-\infty}^{\tau^-}\nabla_{[\mu}G_{\nu]\lambda'}\left[z(\tau),z(\tau')\right]u^{\lambda'}d\tau'. \tag{2.13}$$

Here, $R^\mu{}_\nu$ is the Ricci tensor and we used the same reduction-of-order procedure to get the first derivative of the external force in the second term on the right hand side. The last term is the *tail integral*, which goes up $\tau^- \equiv \tau - \varepsilon$ to avoid the singular behavior of the retarded Green's function at coincidence.

In flat spacetime the Ricci tensor is zero and the tail term vanishes since the Green function is zero in the domain of integration, so we recover Equation 2.7. In curved spacetimes, the self-force does not vanish even in the absence of an external force. The tail integral tells us that the charged particle interacts with radiation emitted earlier and coming back after scattering off spacetime curvature. This means that the self-force is fundamentally nonlocal in time, since its value at present time depends on the complete past history of the particle's worldine. We will go back to this idea again and again in this thesis. Although the self-force is nonlocal in time, the leading-order interaction in q can be simplified by replacing the worldline $z(\tau')$ in the tail integral by the geodesic going through that position at each time. This effectively makes the self-force local in time, up to higher-order corrections.

* * *

From the regular retarded and advanced 2-point functions, we can construct regular conservative and dissipative 2-point functions by adding or subtracting them

$$\begin{aligned} G^{C,R}_{\mu\nu'}(x,x') &= \frac{G^{+,R}_{\mu\nu'}(x,x') + G^{-,R}_{\mu\nu'}(x,x')}{2} \\ &= \frac{1}{2}H_{\mu\nu'}(x,x'), \qquad (2.14a) \\ G^{D,R}_{\mu\nu'}(x,x') &= \frac{G^{+,R}_{\mu\nu'}(x,x') - G^{-,R}_{\mu\nu'}(x,x')}{2} \\ &= \frac{G^{+}_{\mu\nu'}(x,x') - G^{-}_{\mu\nu'}(x,x')}{2}. \qquad (2.14b) \end{aligned}$$

Notice that the new 2-point function $H_{\mu\nu'}$ is exactly the regular conservative 2-point function. In flat spacetime, we set $H_{\mu\nu'} = 0$, which is why the regular retarded 2-point function is simply the dissipative 2-point function: There is no conservative piece of the regular 2-point

function. Similarly, notice that the regular dissipative 2-point function did not require a regularization at all, since the singular 2-point functions of the retarded and advanced Green functions cancel out! This is sometimes used as a "radiative" approximation where one neglects conservative effects and focuses on the dissipative self-force, in order to avoid having to deal with regularizations.

It is not clear whether the 2-point function $H_{\mu\nu'}$(x,x') exists for general curved spacetimes, although some solutions have been obtained for simplified examples and only in the convex neighborhood of x [30]. In chapter 3, we develop a different approach to regularize the retarded field and relax some conditions on the singular Green function.

2.4 Gravitational self-force

We now move on to the gravitational self-force and describe its conceptual differences with the electromagnetic case. The first difference is that general relativity is a nonlinear theory, which introduces the necessity for some type of approximation scheme. The second difference, which is related to the first, is that its nonlinearities make the concept of a point particle ill-defined. In the electromagnetic case, we could understand point charges as the limit where an extended object shrinks to zero size. In general relativity, however, this procedure will create a black hole before the size of the object reaches zero. As we will see in this section, point-particle sources still make sense in linearized gravity, but it is complicated to generalize those results to second order. The third difference is that, in the electromagnetic case, the field equations and the laws of motion for the particle could be formulated independently (i.e., the action functional could be varied independently with respect to the 4-potential or the particle's worldline). In general relativity, the laws of motion for the point mass follow

from energy-momentum conservation, which is itself a consequence of the field equations. In other words, once the Einstein equation is satisfied, applying a gradient to it shows that the divergence of the stress-energy tensor must also be conserved. The dynamics of a point mass must, therefore, be formulated with care. We will describe a formal approach to this problem in Subsection 2.5.2

A rigorous derivation of the first order gravitational self-force experienced by a point particle orbiting a black hole was first introduced by Gralla and Wald [23], which we summarize below. In order to understand their approach, let us first consider a stellar-mass extended object of mass m orbiting a supermassive black hole of mass M. The metric tensor $g_{\mu\nu}$ will satisfy the Einstein equation

$$G_{\mu\nu}[g] = \frac{8\pi G}{c^4} T_{\mu\nu}[g, \mathrm{m}] \tag{2.15}$$

where $G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu}$ is the Einstein tensor and we are using m to denote the matter degrees of freedom of the secondary. Equation (2.15) plus the equations for the matter degrees of freedom m will fully determine the evolution of the inspiral. There exist various definitions for the center-of-mass worldline of the secondary object [39], whose evolution will depend both on the external metric sourced by the primary object and the internal dynamics of the secondary. We are interested in a limit in which the evolution of the secondary's worldline decouples from its internal dynamics. As we discussed in Section 1.2, the dimensionless parameter that governs the strength of the extended-body effects relative to the external force is L/L_{ext}. There is also a parameter, Gm/cL, that measures the compactness of the secondary and the strength of the divergent (i.e. Coulomb-like) piece of its self-field. Thus, in order to neglect the internal dynamics of the secondary, while keeping the influence of its self-field

finite, we must take the limit where both its mass m and its characteristic length L go to zero at the same rate. This corresponds physically with a situation where the mass and size of the secondary are much smaller than the mass of the secondary and the distance to it, respectively.

Although it would seem sufficient to solve Einstein equations perturbatively in the mass of the secondary, this approximation does not work globally. The reason is that a metric perturbation sourced by the secondary will still be order unity in its vicinity. In other words, the gravitational field of a compact object cannot be treated as a perturbation, at least not close to that object. Instead, there will be a double expansion of the metric tensor: In the region far from the secondary, the metric will be dominated by the gravitational field of the primary plus a small perturbation sourced to the secondary. In the region close to the secondary, the metric tensor will be dominated by the gravitational field of the secondary plus a small perturbation sourced by the primary. This is known as a matched asymptotic expansion of the metric tensor [37].

Gralla and Wald considered a secondary object with mass and size that scale with a parameter λ. Then, there exists a one-parameter family of solutions $g_{\mu\nu}(\lambda)$ to the Einstein field equations. This family of solutions returns the external metric sourced by the primary at $\lambda = 0$, when we shrink the secondary to zero size and mass $g_{\mu\nu}(\lambda = 0) = g^{\text{ext}}_{\mu\nu}$. However, there exists a ε-dependent diffeomorphism $\psi(\varepsilon)$ that "zooms" into the secondary by rescaling distances by a factor of $1/\varepsilon$. The limit $\lambda \to 0$ at a fixed distance $\tilde{r} = r/\varepsilon$ will take the one-parameter family of metric tensors $g_{\mu\nu}(\lambda)$ to the internal metric sourced by the secondary. The limit of small λ is known as the outer expansion, while the rescaled limit is the inner expansion. Both expansions should agree in some buffer, intermediate region, which means they can be matched, order by order. To zeroth order in λ, the secondary moves on a geodesic of the ex-

ternal geometry. To leading order in λ, the deviation from geodesic motion is sourced by the first-order self-force.

In this thesis, we will not use this formal derivation of the gravitational self-force. Instead, we will follow a prescription to derive the same result, which is the gravitational analogous of the Detweiler-Whiting regularization scheme utilized in Section 2.3.

2.5 Detweiler-Whiting regularization scheme

We start by expanding the metric tensor in powers of the mass ratio $\varepsilon = m/M$

$$g_{\mu\nu}(x) = g^{(0)}_{\mu\nu}(x) + \varepsilon h^{(1)}_{\mu\nu}(x) + \varepsilon^2 h^{(2)}_{\mu\nu}(x) + \dots . \tag{2.16}$$

If we plug eq. (2.16) into the Einstein tensor, we get

$$G_{\mu\nu}\big[g\big] = G_{\mu\nu}\big[g^{(0)}\big] + \varepsilon \delta G_{\mu\nu}\big[h^{(1)}\big] + \varepsilon^2 \Big\{ \delta G_{\mu\nu}\big[h^{(2)}\big] + \delta^2 G_{\mu\nu}\big[h^{(1)}\big] \Big\} + \mathcal{O}(\varepsilon^3) \tag{2.17}$$

where $\delta G_{\mu\nu}$ represents the linear terms in the expansion while $\delta^2 G_{\mu\nu}$ represents quadratic terms that schematically look like $\partial h \partial h + h\partial^2 h$. Next, we'll assume that the stress-energy tensor of the secondary is that of a point particle

$$T^{\mu\nu}(x) = m \int u^\mu(\tau) u^\nu(\tau) \delta^4\big[x, z(\tau)\big] d\tau. \tag{2.18}$$

Here, $z^\mu(\tau)$ is the worldine of the particle and $u^\mu \equiv \frac{dz^\mu}{d\tau}$ is its 4-veocity. The covariant delta function is $\delta^4(x,y) \equiv \delta^4(x-y)/\sqrt{g}$. The stress-energy tensor can be expanded in powers of

ε through its dependence on the metric tensor

$$T_{\mu\nu} = \varepsilon T^{(1)}_{\mu\nu} + \varepsilon^2 T^{(2)}_{\mu\nu} + \dots \tag{2.19}$$

At zeroth order we get

$$G_{\mu\nu}\left[g^{(0)}\right] = 0 \tag{2.20}$$

which determines the background metric. At first order we get

$$\delta G_{\mu\nu}\left[h^{(1)}\right] = \frac{8\pi G}{c^4} T^{(1)}_{\mu\nu}. \tag{2.21}$$

This equation admits solutions that can be expressed as integrals of Green functions over the background metric.

2.5.1 Solution to linearized Einstein equations

The first-order field equations (2.21) present an immediate problem. The linearized Bianchi equation makes the gradient of the left hand side vanish, so we get $\nabla^\mu T^{(1)}_{\mu\nu} = 0$, Since ∇_μ is a covariant derivative respect to the background metric, this forces the worldline to be a geodesic of the unperturbed spacetime. Therefore, although we can get a first-order metric perturbation, the particle won't be affected by it!

One way to circumvent this problem [28–30, 38] is known as "gauge relaxation". First,

we impose the Lorenz gauge

$$L^{\mu}\left[h^{(1)}\right] = \left(g^{\mu\rho}_{(0)}g^{\sigma\gamma}_{(0)} - \frac{1}{2}g^{\gamma\mu}_{(0)}g^{\rho\sigma}_{(0)}\right)\nabla_{\gamma}h^{(1)}_{\rho\sigma} \tag{2.22}$$
$$= 0.$$

Using the trace reversed field $\bar{h}^{(1)}_{\mu\nu} = h^{(1)}_{\mu\nu} - \frac{1}{2}g^{(0)}_{\mu\nu}g^{\alpha\beta}_{(0)}h^{(1)}_{\alpha\beta}$, we get that Equation (2.21) splits into two equations

$$E^{\mu\nu}\left[\bar{h}^{(1)}\right] = -\frac{16\pi G}{c^4}T^{\mu\nu}_{(1)} \tag{2.23a}$$
$$\nabla^{\mu}\bar{h}^{(1)}_{\mu\nu} = 0 \tag{2.23b}$$

where the wave operator is

$$E^{\mu\nu}\left[\bar{h}^{(1)}\right] = \left(g^{\mu\rho}_{(0)}g^{\sigma\nu}_{(0)}\nabla^2 + 2R^{\mu\rho\nu\sigma}\right)\bar{h}^{(1)}_{\rho\sigma}. \tag{2.24}$$

Equation (2.23a) by itself has solutions for any source worldline. It is only when the Lorenz gauge condition (2.23b) is imposed, that the equations are equivalent to the linearized Einstein equations and geodesic motion is enforced. Therefore, if one solves the Lorenz-gauge form (2.23a) of the linearized Einstein equations while simply ignoring the Lorenz gauge condition that was used to derive them, one allows for the possibility of non-geodesic motion. The gauge relaxation procedure produces an equation inequivalent to the original. However, deviations from geodesic motion are expected to be small and so the Lorenz gauge violation should likewise be small. Solving Equation (2.23a), we get the metric perturbation as a functional of an unspecified worldline. At this point, we can apply the Lorenz gauge

condition and recover that the worldline is a geodesic, not on the background spacetime, but on the perturbed metric tensor. We should clarify that, while the gauge-relaxation method produces the correct answer, it is largely *ad hoc* and does not come with a clear justification. If one wanted a first-principles derivation of the self-force, the method of matched asymptotic expansions explained in the beginning of the Section provides a much clearer picture. However, Gralla and Wald's method expands not only the metric tensor, but the worldline itself. Therefore, the first-order self-force determines the evolution of a deviation vector that describes how the actual worldline deviates from geodesic motion. Therefore, while this approach is a much more rigorous derivation, it eventually breaks down as the deviation vector becomes large after a radiation-reaction time.

We now proceed with solving Equation (2.23a). The first-order stress-energy tensor is

$$T^{\mu\nu} = \int_z m u^\mu u^\nu \delta^4\big[x, z(\tau)\big]\, d\tau \bigg|_{g=g^{(0)}} . \tag{2.25}$$

That is: We evaluate the stress-energy tensor from equation (2.18) on the background solution to get its first-order piece. The solution to equation (2.23a) is

$$\bar{h}^{(1)}_{\mu\nu}(x; [z(\tau)]) = m \int_\gamma G^+_{\mu\nu\alpha'\beta'}\big[x, z(\tau')\big]\, u^{\alpha'} u^{\beta'}\, d\tau' \bigg|_{g=g^{(0)}} \tag{2.26}$$

where $G^+_{\mu\nu\alpha'\beta'}$ is the retarded Green function for the wave operator $E_{\mu\nu}$. Notice that the solution for the first-order metric perturbation is a function of a spacetime point x and a functional of an undefined worldline $z(\tau)$. This will lead to an integro-differential equation for $z(\tau)$ that should be solved self-consistently. The first-order metric perturbation diverges for $x = z(\tau)$ due to the singular nature of the retarded solution close to its source. This is a

problem because that is precisely where we intend to evaluate the metric perturbation once we write down the equations of motion for the point particle.

The Detweiler-Whiting prescription explained in Section 2.3 can be used in the case of linearized gravity by simply adding two more indices to all the 2-point functions and replacing the wave equation by the linearized Einstein equations. We assume the existence of a 2-point function $H_{\mu\nu\alpha'\beta'}(x,x')$ [38] satisfying

- $H_{\mu\nu\alpha'\beta'}(x,x')$ is a homogeneous solution,
- It is symmetric under exchange of arguments: $H_{\mu\nu\alpha'\beta'}(x,x') = H_{\alpha'\beta'\mu\nu}(x',x)$,
- It agrees with the retarded Green function when x lies in the chronological future of x': $H_{\mu\nu\alpha'\beta'}(x,x') = G^{+}_{\mu\nu'}(x,x')$ for $x \in I^{+}(x')$.
- By reciprocity, it must be also true that $H_{\mu\nu\alpha'\beta'}(x,x')$ agrees with the advanced Green function when x lies in the chronological past of x':$H_{\mu\nu\alpha'\beta'}(x,x') = G^{-}_{\mu\nu'}(x,x')$ for $x \in I^{-}(x')$.

We can then construct a singular Green function

$$G^{S}_{\mu\nu\alpha'\beta'}(x,x') = \frac{G^{+}_{\mu\nu\alpha'\beta'}(x,x') + G^{-}_{\mu\nu\alpha'\beta'}(x,x') - H_{\mu\nu\alpha'\beta'}(x,x')}{2} \tag{2.27}$$

and finally construct a regularized 2-point function

$$\begin{aligned} G^{+,R}_{\mu\nu\alpha'\beta'}(x,x') &= G^{+}_{\mu\nu\alpha'\beta'}(x,x') - G^{S}_{\mu\nu\alpha'\beta'}(x,x') \\ &= \frac{G^{+}_{\mu\nu\alpha'\beta'}(x,x') - G^{-}_{\mu\nu\alpha'\beta'}(x,x') + H_{\mu\nu\alpha'\beta'}(x,x')}{2}. \end{aligned} \tag{2.28}$$

The detweiler-Whiting regularization scheme consists in replacing the retarded Green function in in Equation (2.26) by the regularized retarded 2-point function. the regularized metric perturbation $h^{(1)}_R$ is now smooth along the worldline and is well-suited for a study of the dynamics of the linear self-force.

2.5.2 First-order equations of motion

With the regular Green function, we can now write down an expression for the first-order self-force experienced by a point particle of mass m. The particle's 4-velocity $u^\mu = \frac{dz^\mu}{d\tau}$ satisfies the geodesic equation for the effective metric $g_{(0)} + h^R_{(1)}$

$$\left.\frac{Du^\mu}{d\tau}\right|_{g=g_{(0)}+h^R_{(1)}} = 0 \tag{2.29}$$

where $\frac{D}{d\tau}\big|_{g=g_{(0)}+h^R_{(1)}} = u^\mu \nabla_\mu\big|_{g=g_{(0)}+h^R_{(1)}}$ is the covariant derivative with respect to the regularized metric. This equation can be expanded to linear order to obtain the covariant derivative respect to the background metric on the left hand side and the self-force on the right

$$\begin{aligned}\left.\frac{Du^\mu}{d\tau}\right|_{g=g_{(0)}} &= -\frac{1}{2}\left(h^\mu{}_{\nu;\lambda} + h^\mu{}_{\lambda;\nu} - h_{\lambda\nu}{}^{;\mu} + h_{\nu\lambda;\rho}u^\rho u^\mu\right)u^\nu u^\lambda + O(\varepsilon^2) \\ &= F^\mu(z,u;[z(\tau)]).\end{aligned} \tag{2.30}$$

Here, F^μ is the force per unit mass but we use the term force for short and every instance of h on the equation above is supposed to be the linear order retarded regularized metric perturbation $h^{(1)}_{+,R}$ but we omitted the labels for clarity.

The self-force, as presented in equation (2.30), is a function of the present position and 4-velocity and a functional of the particle's past worldline $z(\tau)$. This is because the metric

perturbation $h^R_{(1)}$ is sourced by the particle's own path (See equation (2.26)). This makes the equation of motion an integro-differential equation that must be solved self-consistently. The functional dependence can be simplified by a reduction-of-order procedure. The equations of motion are valid to linear order in the mass ratio, so any higher-order corrections to the particle's worldline will be second order. Therefore, we can order-reduce the metric perturbation and evaluate it on the zeroth-order worldline

$$h^R_{(1)\,\mu\nu}(x;z,u) = \int G^{+,R}_{\mu\nu\alpha'\beta'}\left[x, z_{(0)}(\tau';z,u)\right] u^{\alpha'}_{(0)}(\tau';z,u) u^{\beta'}_{(0)}(\tau';z,u) d\tau' \tag{2.31}$$

where $z_{(0)}(\tau';z,u)$ means the zeroth-order geodesic with initial conditions (z,u). The first-order metric perturbation is now a function of a space-time point x and a function of initial data (z,u), up to corrections quadratic in the mass ratio.

2.5.3 Problems with the Detweiler-Whiting regularization scheme at second order

The second-order Einstein equation is

$$\delta G_{\mu\nu}\left[h^{(2)}\right] = \frac{8\pi G}{c^4} T^{(2)}_{\mu\nu} - \delta^2 G_{\mu\nu}\left[h^{(1)}\right] \tag{2.32}$$

Physically, this means that the first-order correction $h^{(1)}$ acts as a source for the second-order correction $h^{(2)}$. The first-order metric perturbation $h^{(1)}$ is sourced by the small particle's motion and it will generally scale as $h^{(1)} \sim m/r$ near the particle's worldline. This means the first term on the right hand side is not well defined, since its integrand scales as $h^{(1)}(x)\delta^4(x)$ which diverges in the distributionally ill-defined manner $\frac{1}{r}\delta(r)$. Similarly, the second source

term on the right hand side will scale as $\delta^2 G_{\mu\nu}\left[h^{(1)}\right] \sim \partial h^{(1)} \partial h^{(1)} \sim m^2/r^4$. This divergence is too strong to be integrated as a source. Because of these issues, if we want to proceed to second-order corrections and beyond, we must deal with the finite size of the small object directly.

We should clarify that the method of matched asymptotic expansions explained in the beginning of this chapter does work at second order and has been successfully used to derive the second-order self-force. What we show here is that the "naive" prescription to derive the self-force starting from a perturbative expansion of the metric tensor and a point-particle source breaks down at second order. In Chapter 3, we will develop a formalism to derive the Detweiler-Whiting regularization scheme from first principles and extend it to any order in perturbation theory, in the context of a scalar toy model.

2.6 Adiabatic inspirals

The discussion in the previous sections was concerned with obtaining an expression for the instantaneous self-force experienced by a point-particle source. A different goal is to find the best way to model the complete inspiral of the two-body system, in a way that remains accurate after times longer than the radiation-reaction time. One such way is the two-timescale method, first applied to EMRIs by Flanagan and Hinderer in [40]. The key insight used in the two-timescale method is the realization that extreme mass-ratio inspirals have two very distinct timescales: The orbital time $t_{\rm orb} \propto M$ and the radiation-reaction time $t_{\rm rad} \propto M/\varepsilon$. The phase of the inspiral evolves on the short orbital timescale but the "conserved quantities" slowly radiate away in the much longer radiation-reaction time. The two-timescale method is then used to model the inspiral over long timescales. The correction to geodesic motion

created by the self-force, as derived in the previous section, cannot be treated as small over the entire inspiral time. Instead, the two-timescale method uses action-angle variables in order to obtain a set of coordinates in which one subset (the angle variables) evolves on the orbital timescale, and another subset (the action variables) evolves on the radiation reaction timescale.

Consider a small secondary with mass m orbiting a Kerr black hole of mass M. To zeroth order in m/M, the secondary follows a geodesic of the Kerr background. Geodesic motion in a Kerr spacetime is integrable, so we can use action-angle variables (q^α, J_μ). Here, $q^\alpha = (q^t, q^r, q^\theta, q^\varphi)$ is a set of angle variables associated with Boyler-Lindquist coordinates and the action variables $J_\mu = (E/m, L_z/m, Q/m^2, M, a)$ are the energy per unit mass, angular momentum per unit mass, Carter constant per unit mass squared, and mass and spin of the primary black hole. In this section, we use Greek letters from the beginning of the alphabet for angle variables (that is, $\alpha = 0, 1, 2, 3$) and Greek letters from the middle of the alphabet for action variables (that is, $\mu = 1, 2, 3, 4, 5$). The geodesic evolution of the action-angle variables is given by

$$\frac{dq^\alpha}{d\tau} = \omega^\alpha(J), \tag{2.33a}$$

$$\frac{dJ_\mu}{d\tau} = 0. \tag{2.33b}$$

Here, τ is proper time. Once we include the effects of the secondary's own gravitational field, we get a self-force on the right-hand side of equation (2.33). Schematically, it would look like

this

$$\frac{dq^\alpha}{d\tau} = \omega^\alpha(J) + \varepsilon g^\alpha_{(1)}(\mathbf{q},J) + \varepsilon^2 g^\alpha_{(2)}(\mathbf{q},J) + O(\varepsilon^3), \tag{2.34a}$$

$$\frac{dJ_\mu}{d\tau} = \varepsilon G^{(1)}_\mu(\mathbf{q},J) + \varepsilon^2 G^{(2)}_\mu(\mathbf{q},J) + O(\varepsilon^3). \tag{2.34b}$$

Here, we use uppercase latin indices for $\mathbf{q} = (q^r, q^\theta)$ to indicate that the self-force cannot depend on q^t or q^φ due to the time and axial Killing vectors on the Kerr background. We assume that the self-force terms $g_{(1)}$, $g_{(2)}$, $G^{(1)}$, and $G^{(2)}$ are known, although their specific form is not important here.

Now, we provide an ansatz for the time-dependence of the action-angle variables. First, we define the slow time variable

$$\tilde{\tau} = \varepsilon\tau. \tag{2.35}$$

The slow-time variable $\tilde{\tau}$ increases by $O(1)$ after the proper time τ increases by a radiation-reaction time $\propto 1/\varepsilon$, hence its name. We'll also define a fast-parameter variable Ψ^α, which will be related to an orbital phase. The ansatz for the action-angle variables is

$$q^\alpha(\tau,\varepsilon) = \sum_{s=0}^{\infty} \varepsilon^s q^\alpha_{(s)}(\Psi,\tilde{\tau}), \tag{2.36a}$$

$$J_\mu(\tau,\varepsilon) = \sum_{s=0}^{\infty} \varepsilon^s J^{(s)}_\mu(\Psi,\tilde{\tau}). \tag{2.36b}$$

The order s in this expansion has its own naming convention, which differs from the naive perturbative expansion that we did in the beginning of this chapter. We will call the zeroth-order solutions the *adiabatic* solution. The following orders will be called *post-1-adiabatic* and so on. This reason behind this will become clear once we derive the adiabatic solution.

The derivative of the phase variable Ψ respect to proper time determines its own angular frequency, which we must obtain by solving the equations of motion

$$\begin{aligned}\frac{d\Psi^\alpha}{d\tau}(\tilde{\tau},\varepsilon) \equiv& \Omega^\alpha(\tilde{\tau},\varepsilon) \\ =& \sum_{s=0}^{\infty} \Omega^\alpha_{(s)}(\tilde{\tau})\end{aligned} \tag{2.37}$$

This equation can be integrated to get an expansion for the phase variable Ψ

$$\Psi^\alpha(\tilde{\tau},\varepsilon) = \frac{1}{\varepsilon}\psi^\alpha_{(0)}(\tilde{\tau}) + \psi^\alpha_{(1)}(\tilde{\tau}) + \varepsilon\psi^\alpha_{(2)}(\tilde{\tau}) + O(\varepsilon^2) \tag{2.38}$$

where we defined

$$\psi^\alpha_{(s)} \equiv \int \Omega^\alpha_{(s)}(\tilde{\tau})d\tilde{\tau}. \tag{2.39}$$

Note that we are doing an expansion in powers of the mass ratio at fixed *slow time*, and not at fixed time τ. This is the key of the two-timescale method, taking the limit $\varepsilon \to 0$ at fixed slow time $\tilde{\tau} = \varepsilon\tau$ pushes τ to infinity, which is why this approximation scheme applies to the full inspiral and not just a snapshot of it, as the Gralla/Wald approach.

Next, we want our ansatz to satisfy some properties. The phase variable should behave like a phase, which means that the angle variables J_α should be periodic on Ψ and the angle variables q^α should evolve 2π after the phase variable Ψ evolves by 2π. These conditions can

be expressed as

$$
\begin{aligned}
J^{(s)\alpha}(\Psi + 2\pi k, \tilde{\tau}) &= J^{(s)\alpha}(\Psi, \tilde{\tau}) && \text{(2.40a)}\\
q^{(0)\alpha}(\Psi + 2\pi k, \tilde{\tau}) &= 2\pi k^{\alpha} + q^{(0)\alpha}(\Psi, \tilde{\tau}) && \text{(2.40b)}\\
q^{(s)\alpha}(\Psi + 2\pi k, \tilde{\tau}) &= q^{(s)\alpha}(\Psi + 2\pi, \tilde{\tau}), \quad \text{for } s \geq 1. && \text{(2.40c)}
\end{aligned}
$$

Here, k^{α} is a vector of integers. The periodicity of the phase variable Ψ is useful to define average and secular pieces of the action-angle variables. We define the average of $J^{(s)}_{\alpha}$ as

$$
\begin{aligned}
\mathcal{J}^{(s)}(\tilde{\tau}) &= \langle J^{(s)}(\Psi, \tilde{\tau}) \rangle_{\Psi} \\
&= \frac{1}{(2\pi)^4} \int_0^{2\pi} d\Psi^0 \ldots \int_0^{2\pi} d\Psi^3 J^{(s)}(\Psi, \tilde{\tau}).
\end{aligned}
\tag{2.41}
$$

Then, we can split the solution for the action variables into secular and oscillatory pieces

$$
J^{(s)}_{\mu}(\Psi, \tilde{\tau}) = \mathcal{J}^{(s)}_{\mu}(\tilde{\tau}) + \hat{J}^{(s)}_{\mu}(\Psi, \tilde{\tau}) \tag{2.42}
$$

where $\hat{J}$ is the oscillatory piece, which averages to zero by definition. A similar procedure applied to the angle variables yields

$$
q^{\alpha}_{(s)}(\Psi, \tilde{\tau}) = \Psi^{\alpha} + \mathcal{Q}^{\alpha}_{(s)}(\tilde{\tau}) + \hat{q}^{\alpha}_{(s)}(\Psi, \tilde{\tau}) \tag{2.43}
$$

where $\mathcal{Q}$ is the average piece of q^{α} and $\hat{q}$ is its oscillatory piece. Finally, we will choose the integration constants that arise from equation (2.39) to fix

$$
q_{(s)}(0, \tilde{\tau}) = 0. \tag{2.44}
$$

2.6.1 Adiabatic solution

All that is left now is to plug our ansatz into equation (2.34). To zeroth order in ε, we get

$$\frac{dq^{\alpha}_{(0)}(\Psi,\tilde{\tau})}{d\Psi^{\beta}}\Omega^{\beta}_{(0)} = \omega\big[J_{(0)}\big], \tag{2.45a}$$

$$\frac{dJ^{(0)}_{\mu}(\Psi,\tilde{\tau})}{d\Psi^{\beta}}\Omega^{\beta}_{(0)} = 0. \tag{2.45b}$$

Equation (2.45b), can be solved using a Fourier expansion of the action variables

$$J^{(0)}_{\mu}(\Psi,\tilde{\tau}) = \sum_{\mathbf{k}} J^{(0)}_{\mu\,\mathbf{k}}(\tilde{\tau})e^{i\mathbf{k}\cdot\Psi}. \tag{2.46}$$

Plugging this into equation (2.45b), we get

$$\sum_{\mathbf{k}}\Big[i\Omega^{\beta}_{(0)}k_{\beta}\Big]J^{(0)}_{\mu\,\mathbf{k}}(\tilde{\tau})e^{i\mathbf{k}\cdot\Psi} = 0. \tag{2.47}$$

It follows that

$$J^{(0)}_{\mu\,\mathbf{k}}(\tilde{\tau}) = 0 \tag{2.48}$$

for all vectors of integrers $\mathbf{k}$ except for $\mathbf{k} = 0$ and for those non-zero $\mathbf{k}$ which satisfy the resonance condition

$$\Omega^{\beta}_{(0)}k_{\beta} = 0. \tag{2.49}$$

A resonance occurs in any multi-periodic orbit when the ratio of two frequencies becomes a small rational number. We will discuss more about resonances in Section 2.6.2. For now, let us assume that there are no resonances. Then, all $\mathbf{k} \neq 0$ coefficients of $J^{(0)}$ must vanish,

which means that

$$J^{(0)}(\Psi, \tilde{\tau}) = \mathcal{J}^{(0)}(\tilde{\tau}). \tag{2.50}$$

Moving on, now we insert the expansion (2.43) into equation (2.45b) and Fourier expand the oscillatory piece of $q_{(0)}$ to get

$$\Omega^{\alpha} + \sum_{\mathbf{k}} \left[i\Omega^{\beta}_{(0)} k_{\beta} \right] \hat{q}^{\alpha}_{(0)\,\mathbf{k}}(\tilde{\tau}) e^{i\mathbf{k}\cdot\Psi} = \omega\left[\mathcal{J}^{(0)}(\tilde{\tau})\right]. \tag{2.51}$$

The $k = 0$ Fourier mode of this equation gives

$$\Omega^{\alpha}_{(0)}(\tilde{\tau}) = \omega\left[\mathcal{J}^{(0)\alpha}(\tilde{\tau}\right]. \tag{2.52}$$

The $\mathbf{k} \neq 0$ Fourier modes imply that $\hat{q}_{(0)\mathbf{k}} = 0$ by an argument similar to the one used in Equation (2.47). By equation (2.43), we are left with

$$q_{(0)}(\Psi, \tilde{\tau}) = \Psi + \mathcal{Q}_{(0)}(\tilde{\tau}). \tag{2.53}$$

Finally, since we set $q_{(0)}(0, \tilde{\tau}) = 0$, we must have $\mathcal{Q}_{(0)}(\tilde{\tau}) = 0$.

Note that we still do not know how to determine the evolution of the averaged action variable $\mathcal{J}^{(0)}$. In order to do so, we must go to linear order. Plugging our ansatz for the action variables in equation (2.34b) to linear order in ε, we get

$$\Omega^{\beta}_{(0)} \frac{dJ^{(1)}_{\mu}(\Psi, \tilde{\tau})}{d\Psi^{\beta}} + \frac{d\mathcal{J}^{(0)}_{\mu}(\tilde{\tau})}{d\tilde{\tau}} = G^{(1)}_{\mu}(\Psi, \mathcal{J}^{(0)}) \tag{2.54}$$

Taking an average of this equation respect to Ψ, the first term vanishes and we get

$$\frac{d\mathcal{J}^{(0)}_{\mu}(\tilde{\tau})}{d\tilde{\tau}} = \langle G^{(1)}_{\mu}(\Psi, \mathcal{J}^{(0)})\rangle \tag{2.55}$$

which tells us that the evolution of the averaged action varaible is determined by the averaged self-force.

The adiabatic solution is then given by

$$\frac{dq^{\alpha}_{(0)}(\tau)}{d\tau} = \omega\big[\mathcal{J}^{(0)}(\tilde{\tau})\big] \tag{2.56a}$$

$$\frac{d\mathcal{J}^{(0)}_{\mu}(\tilde{\tau})}{d\tilde{\tau}} = \langle G^{(1)}_{\mu}(\Psi, \mathcal{J}^{(0)})\rangle \tag{2.56b}$$

Physically, we are letting the angle variables evolve as if moving along a geodesic, but we are letting the action variables that determine the geodesic evolve slowly under the averaged self-force. This is the reason why this is called the adiabatic order. An important insight coming from this analysis is that the adiabatic motion is determined by the dissipative piece of the self-force, but not its conservative piece. This is because the conservative self-force comes from a symmetric 2-point function, which averages to zero. This extends to the following orders as well: The adiabatic order needs only the averaged first-order dissipative self-force, the post-1-adiabatic order needs the averaged second-order dissipative self-force and the secular pieces of the first-order dissipative and conservative self-force, and so on. This analysis breaks down over resonances, which we discuss next.

2.6.2 Transient resonances

The 6D phase space of a test particle orbiting around a Kerr black hole is parametrized by three angle variables $(q^r, q^\theta, q^\varphi)$ and their corresponding action variables (E, L_z, Q), which are the energy, angular momentum and Carter constant, respectively. Phase space is foliated by 3-tori parametrized by the angle variables, with each tori labeled by the action variables. Different action variables J_μ produce different frequencies $\omega^\alpha(J)$ for each angle variable. Generic values for J_α will produce orbits that cover the torus ergodically. However, if the action variables are such that the ratio between any two frequencies becomes a rational number, then the orbits will loop back on themselves and never cover the torus. This feature dramatically affects the two-timescale method. The average of any function of q along a generic orbit will eventually cover the full torus and therefore be insensitive to initial conditions, which is a desirable property of averages. However, during a resonance, even the averaged variables are highly sensitive to the initial data. This can induce errors at an order in ε that is not accounted for if resonances are neglected. Because of this, the action variables can start depending on the fast phase variable Ψ before including post-1-adiabatic effects. In what follows, we provide an estimation of how big is the effect of resonances and paint a schematic picture of the physical process by which they arise.

Consider the Fourier series of the first-order self-force $G^{(1)}_\mu(q^A, J)$

$$G^{(1)}_\mu(q^A, J) = \sum_{k_A} G^{(1)}_{\mu\, k^A}(J) e^{i q^A k_A}, \tag{2.57}$$

where the index $A = r, \theta$. Now, let's expand the phase $q^A k_A$ around some fixed time τ_0

$$k_A q^A(\tau) = k_A q^A(\tau_0) + k_A \omega^A(\tau_0)\big(\tau - \tau_0\big) + k_A \dot{\omega}^A(\tau_0)\big(\tau - \tau_0\big)^2 + \dots \tag{2.58}$$

For generic orbits, the second term is non vanishing and dominates the exponential in Equation (2.57). This term increases by $O(1)$ after a time $\tau \propto \frac{1}{k_A \omega^A} \propto O(1)$. In other words, the exponential oscillates on a timescale much shorter than the radiation-reaction time and thus the modes $k^A \neq 0$ of the self-force average to zero. However, in a resonant mode, the second term vanishes and the third term dominates the exponential. The second term increases by $O(1)$ after a time $\tau \propto \frac{1}{\sqrt{k_A \dot{\omega}^A}} \propto 1/\sqrt{\varepsilon}$. This time is much longer than the orbital timescale and therefore the resonant modes dominate the exponential for a long time and do not average to zero. The change in the action variables will be $\Delta J \approx \varepsilon G^{(1)} \Delta t \approx \sqrt{\varepsilon}$. This means that:

- Resonances add half-integer powers to the ansatz (2.36). If we include resonant effects, there will be contributions to the action-angle variables that scale as $\varepsilon^{1/2}$, $\varepsilon^{3/2}$ and so on. These will have resonant modes as sources and will vanish outside of resonances.

- The conservative self-force can produce order $\sqrt{\varepsilon}$ corrections during resonances. The reason why we only need the dissipative self-force for the adiabatic solution is because the conservative piece of the self-force averages to zero. This is no longer true over a resonant orbit, and resonant modes of the conservative self-force can drive the evolution of the action variables at an order $\sqrt{\varepsilon}$ above the adiabatic order.

All Cretans are liars!

—Epimenides, a Cretan

3

Nonperturbative approach to self-force: Scalar case (Adapted from [1])

3.1 Context

As we discussed in Chapter 2, the point-particle limit of the self-force presents some complications beyond linearized gravity. Although these obstacles have been overcome through

the use of puncture schemes and matched asymptotic expansions [31–36], the Detweiler-Whiting regularization prescription to subtract a singular field from the retarded field does not have a generalization past linearized gravity. In this chapter, we present an alternative derivation of the second-order self-force that embraces a different philosophy. Instead of tackling point particles head on, we will describe the dynamics of extended objects. The use of finite sized objects avoids any difficulties associated with self-field divergences encountered in the limit of small objects. The key idea is to define appropriate renormalizations already for finite bodies, before taking any limits to zero size. We will define a nonperturbative field redefinition that takes the physical field to an effective one, which has a unique, well-defined point-particle limit. To linear order in the mass or charge of the body, this field redefinition is essentially equivalent to the subtraction of the singular Green function described in Equation (2.11). The field redefinition formalism was invented by Harte and has been extensively developed by him and collaborators [22, 41–43]. The goal of this chapter is to extend the field redefinition formalism to all orders in the mass or charge in the simplified context of a nonlinear scalar field model. In Section 3.2, we present the framework for a general class of theories with matter degrees of freedom coupled to a scalar field. We assume that the matter has extended but finite support on a worldtube and derive equations for the evolution of its energy, momentum, angular momentum, and center-of-mass charge. In Section 3.3, we define the field redefinition that takes the physical scalar field to an effective one, which is finite in the point-particle limit. In Section 3.4, we expand the field redefinition in powers of the scalar density to parametrize it in terms of a series of n-point functions. We then derive perturbative constraints for them, up to any order in the scalar density. In Section 3.5, we do a multipole expansion of the scalar density, where we replace it by a series of multi-

pole moments times derivatives of delta functions. This replaces the extended object by a "structured" point-particle: A world-line equipped with a series of multipole moments, to any order in the distance to the worldline. If we neglect all higher multipole moments, we recover the point-particle description of the dynamics, which is governed by an effective self-force. Finally, in section 3.6, we apply this formalism to a particular nonlinear scalar theory in flat spacetime.

3.2 General framework

We consider the motion of a finite-sized object coupled to a nonlinear scalar field in a four-dimensional spacetime with fixed metric g_{ab}. This system is a simplified toy model for the motion of a body under its gravitational self-force.

Such a system can be described by an action functional of the form

$$S = S_{\rm m}[\chi, \varphi, g_{ab}] + S_\varphi[\varphi, g_{ab}]. \tag{3.1}$$

Here S_φ is the action for the scalar field φ, and the matter action $S_{\rm m}$ depends on one or more matter fields $\chi = (\chi_1, \ldots, \chi_N)$, and contains any terms that couple the matter fields to the scalar field. We assume that the matter fields χ are nonzero only inside a worldtube $\mathcal{B}$ which is spatially compact, whereas the scalar field φ is long ranged. We will assume that both terms in the action (3.1) are individually covariant. In other words, if $\sigma : M \to M$ is any diffeomorphism on the spacetime manifold M, we assume that

$$S_{\rm m}[\sigma_*\chi, \sigma_*\varphi, \sigma_* g_{ab}] = S_{\rm m}[\chi, \varphi, g_{ab}], \tag{3.2}$$

where σ_* is the pullback, and similarly for S_φ. We use the mostly positive metric tensor signature.

3.2.1 Continuum equations of motion

Varying the action (3.1) with respect to φ gives the scalar field equation of motion, which we can write as

$$\mathcal{E}_\varphi[\varphi] = \rho, \tag{3.3}$$

where

$$\mathcal{E}_\varphi = \frac{1}{\sqrt{-g}}\frac{\delta S_\varphi}{\delta \varphi}, \quad \rho = -\frac{1}{\sqrt{-g}}\frac{\delta S_{\rm m}}{\delta \varphi}. \tag{3.4}$$

The quantity ρ is a function of the matter fields χ and of φ, and is the charge density for the scalar field. Similarly the matter field equations of motion obtained from the action (3.1) are

$$\frac{\delta S_{\rm m}}{\delta \chi} = 0. \tag{3.5}$$

Finally we define the matter and scalar field stress energy tensors in the usual way [*]

$$T^{\mu\nu}_{\rm m} = \frac{2}{\sqrt{-g}}\frac{\delta}{\delta g_{\mu\nu}}S_{\rm m}, \quad T^{\mu\nu}_{\varphi} = \frac{2}{\sqrt{-g}}\frac{\delta}{\delta g_{\mu\nu}}S_{\varphi}, \tag{3.6}$$

where g is the determinant of the metric $g_{\mu\nu}$. Diffeomorphism invariance then implies conservation of the total stress energy tensor

$$\nabla_\mu(T^{\mu\nu}_{\rm m} + T^{\mu\nu}_{\varphi}) = 0. \tag{3.7}$$

[*] We assume that the matter stress energy tensor $T^{\mu\nu}_{\rm m}$ vanishes when $\chi = 0$, so that it has compact spatial support. This assumption excludes certain kinds of coupling terms in the action $S_{\rm m}$.

We now argue that the stress energy conservation equation (3.7) can be rewritten in the form

$$\nabla_\mu T^{\mu\nu}_{\rm m} = -\rho\nabla^\nu\varphi. \tag{3.8}$$

This follows from the diffeomorphism invariance (3.2) of the matter action, specialized to a linearized diffeomorphism parameterized by a vector field ζ^μ:

$$0 = \delta S_{\rm m} = \int d^4x \left[\frac{\delta S_{\rm m}}{\delta\chi}\delta\chi + \frac{\delta S_{\rm m}}{\delta\varphi}\delta\varphi + \frac{\delta S_{\rm m}}{\delta g_{\mu\nu}}\delta g_{\mu\nu}\right]. \tag{3.9}$$

Using the equation of motion (3.5), the linearized field variations $\delta\varphi = \mathcal{L}_\zeta\varphi = \zeta^\mu\nabla_\mu\varphi$ and $\delta g_{\mu\nu} = \mathcal{L}_\zeta g_{\mu\nu} = \nabla_\mu\zeta_\nu + \nabla_\nu\zeta_\mu$ and the definitions (3.6) and (3.4) we get

$$0 = \int d^4x\sqrt{-g}\left[-\rho\zeta^\mu\nabla_\mu\varphi + T^{\mu\nu}_{\rm m}\nabla_\mu\zeta_\nu\right]. \tag{3.10}$$

Now integrating by parts, taking ζ_μ to be of compact support to enable throwing away the boundary term, and using the fact that ζ^μ is otherwise arbitrary yields the result (3.8).

3.2.2 Equation of motion for the body

We next define the bare linear and angular momenta of the body and their evolution equations. Renormalized versions of these quantities will be defined in Sec. 3.3 below.

We start by choosing a spacelike foliation of spacetime specified by a coordinate $s : M \to \mathbb{R}$, with leaves Σ_s. Specializing to flat spacetime, we define the generalized momentum to be

the linear map on Killing vector fields ξ^a

$$\mathcal{P}_\xi(s) = \int_{\Sigma_s} T^{\mu\nu}_{\rm m} \xi_\mu dS_\nu, \tag{3.11}$$

At fixed s, this is is a linear map from the space of Killing fields into $\mathbb{R}$, and may therefore be viewed as a vector in the ten-dimensional space dual to the space of Killing fields. Note that in curved spacetimes where there may not exist any Killing vector fields, there is a ten-dimensional space of generalized Killing vectors given a choice of representative worldline, which takes its place in these expressions [22, 44]. With this generalization, there are additional gravitational forces (at quadrupole and higher order) as well as scalar ones.

The equation of motion for the body is obtained by differentiating the generalized momentum (3.11), which yields [22, 44]

$$\frac{d}{ds}\mathcal{P}_\xi(s) = \int_{\Sigma_s} \nabla_\mu (T^{\mu\nu}_{\rm m} \xi_\nu) dS = -\int_{\Sigma_s} \rho \mathcal{L}_\xi \varphi \, dS, \tag{3.12}$$

where $dS = t^\mu dS_\mu$ with t^μ any vector field with $t^\mu \nabla_\mu s = 1$. Here the second equality uses the conservation equation (3.8) and the Killing equation $\nabla_{(\mu}\xi_{\nu)} = 0$.

3.3 Field redefinition

For test bodies with negligible self-fields, it is possible to simplify the generalized force in Eq. (3.12) by Taylor expanding the field φ about some representative worldline, thus obtaining a multipole expansion. More generally, this is not possible, essentially due to the objects self-field varying over the object in a manner not well approximated by a Taylor expansion.

Moreover, it is generally difficult to take a point particle limit, as the self-field diverges in this limit. As explained in section 3.1, a general strategy for circumventing this difficulty is to demonstrate that the self-field does not contribute to the integral (3.12), and to subtract it off from the physical field, leaving an external or effective field. While this approach is standard in Newtonian gravity, it is more subtle to define an appropriate notion of effective field in relativistic theories. For linear theories, the details have been worked out in Refs. [22, 44], and the scheme is closely related to Detweiler-Whiting regularization. Here we aim to generalize this framework to nonlinear theories.

Specifically, we seek to define some effective fields $\hat{\rho}$ and $\hat{\varphi}$, which will in general be nonlocal nonlinear functionals of the physical fields[†] ρ and φ:

$$\hat{\rho}(x) = \hat{\rho}(x;\rho,\varphi], \quad \hat{\varphi}(x) = \hat{\varphi}(x;\rho,\varphi]. \tag{3.13}$$

The mapping (3.13) can be viewed as a change of coordinates on the off-shell field configuration space. In order for this transformation to be useful we will require it to satisfy four key properties:

1. The transformation should transform covariantly under diffeomorphisms.

2. The form (3.12) of the equation of motion for the body should be preserved.

3. Appropriate locality and causality conditions should be satisfied.

[†]More precisely, our coordinates on field configuration space are $(\chi_1, \ldots, \chi_N, \varphi)$, and ρ is the local function of these coordinates given by Eq. (3.4). However, assuming that ρ and φ are functionally independent, we can find new coordinates of the form $(\tilde{\chi}_1, \ldots, \tilde{\chi}_{N-1}, \rho, \varphi)$. In this sense, we can regard ρ as one of the independent field variables. The transformations we discuss in this section keep $(\tilde{\chi}_1, \ldots, \tilde{\chi}_{N-1})$ as well as g_{ab} fixed.

4. The effective scalar field $\hat{\varphi}$ should have a smooth point particle limit, defined in a suitable sense.

In the remainder of this section we discuss these four requirements in detail. We will demonstrate the existence of a large class of transformations that satisify properties 1, 2 and 3. We will also conjecture that transformations exist that additionally satisfy property 4, and present some circumstantial evidence in favor of this conjecture. In specific models this existence question reduces to the existence of n-point generalizations of the Detweiler-Whiting 2-point function that satisfy some conditions which we specify.

Transformations which satisfy properties 1 through 4 need not be unique in order to be useful. Non-uniqueness will not affect the form of point particle equations of motions, but can affect the form of corrections due to spin and tidal couplings (analogous to the choice of spin supplementary condition in the Mathisson-Papapetrou-Dixon equation of motion). See References [22, 41–44] for further discussion of this issue.

3.3.1 Covariance condition

Although the transformation will in general be nonlocal and nonlinear, we require that it satisfy

$$\hat{\rho}[\sigma_*\rho, \sigma_*\varphi] = \sigma_*\hat{\rho}[\rho, \varphi], \quad \hat{\varphi}[\sigma_*\rho, \sigma_*\varphi] = \sigma_*\hat{\varphi}[\rho, \varphi], \tag{3.14}$$

for any diffeomorphism $\sigma : M \to M$.

3.3.2 Preservation of the form of the equation of motion

We next describe a general prescription for transformations that preserve the form (3.12) of the equation of motion, albeit with a renormalized version of the momentum appearing on

the left hand side. The key idea is to make use of a modified splitting of the total action (3.1) into two terms. We define

$$\hat{S}_{\mathrm{m}} = S_{\mathrm{m}} + \tilde{W}, \quad \hat{S}_{\varphi} = S_{\varphi} - \tilde{W} \tag{3.15}$$

for some covariant functional $\tilde{W}$, so that

$$S = \hat{S}_{\mathrm{m}} + \hat{S}_{\varphi}. \tag{3.16}$$

We also rewrite the action in terms of the effective fields $\hat{\rho}$, $\hat{\varphi}$. If we can choose the field redefinitions and the new splitting in such a way that the new scalar action $\hat{S}_{\varphi}$ depends only on $\hat{\varphi}$ and not on $\hat{\rho}$,

$$\hat{S}_{\varphi} = \hat{S}_{\varphi}[\hat{\varphi}, g_{\mu\nu}], \tag{3.17}$$

then the derivation given in Sec. 3.2 of the equation of motion (3.12) starting from the action continues to be valid[‡] for the hatted variables. Explicitly we obtain by rewriting Eqs. (3.4), (3.6) and (3.12) that

$$\nabla_{\mu} \hat{T}^{\mu\nu} = -\hat{\rho} \nabla^{\nu} \hat{\varphi} \tag{3.18}$$

and

$$\frac{d}{ds} \hat{\mathcal{P}}_{\xi}(s) = -\int_{\Sigma_s} \hat{\rho} \mathfrak{L}_{\xi} \hat{\varphi} \, dS, \tag{3.19}$$

[‡]A key point is that even though the new actions $\hat{S}_{\mathrm{m}}$ and $\hat{S}_{\varphi}$ are nonlocal functionals of the fields $\hat{\rho}$ and $\hat{\varphi}$, locality is not used in the derivation of Sec. 3.2.

where the renormalized or effective variables are defined as

$$\hat{\mathcal{P}}_{\xi}(s) = \int_{\Sigma_s} \hat{T}^{\mu\nu}_{\mathrm{m}} \xi_{\mu} dS_{\nu}, \tag{3.20a}$$

$$\hat{T}^{\mu\nu}_{\mathrm{m}} = \frac{2}{\sqrt{-g}} \frac{\delta \hat{S}_{\mathrm{m}}}{\delta g_{\mu\nu}}\bigg|_{\hat{\varphi},\hat{\rho}}, \tag{3.20b}$$

$$\hat{\rho} = -\frac{1}{\sqrt{-g}} \frac{\delta \hat{S}_{\mathrm{m}}}{\delta \hat{\varphi}}\bigg|_{g_{\mu\nu},\hat{\rho}}, \tag{3.20c}$$

Next, we explain how to choose the functional $\tilde{W}$ and the field redefinitions (3.13) in order to satisfy the required conditions (3.17) and (3.20c). We take $\tilde{W}$ to be of the form

$$\tilde{W} = W[\rho, \hat{\varphi}, g_{\mu\nu}] - \int d^4x \sqrt{-g} \rho \varphi \tag{3.21}$$

for some covariant functional W. Substituting this into the second of Eqs. (3.15), taking a variation at fixed $g_{\mu\nu}$ and making use of the definition (3.4) to get rid of the variation respect to φ gives

$$\delta \hat{S}_{\varphi} = \int d^4x \sqrt{-g} \left[\left(\varphi - \frac{1}{\sqrt{-g}} \frac{\delta W}{\delta \rho} \right) \delta \rho - \frac{1}{\sqrt{-g}} \frac{\delta W}{\delta \hat{\varphi}} \delta \hat{\varphi} \right]. \tag{3.22}$$

We next define the field redefinition in terms of W by

$$\varphi = \frac{1}{\sqrt{-g}} \frac{\delta W}{\delta \rho}[\rho, \hat{\varphi}, g_{\mu\nu}]. \tag{3.23}$$

Here, derivatives of $W[\rho, \hat{\varphi}, g_{\mu\nu}]$ are taken while keeping all other arguments fixed. Inserting Equation (3.23) into Equation (3.22), we see that the requirement (3.17) is satisfied. Fur-

thermore, combining Equations (3.20c) and (3.22), yields the relation .

$$\hat{\rho} = \frac{1}{\sqrt{-g}} \frac{\delta W}{\delta \hat{\varphi}} [\rho, \hat{\varphi}, g_{\mu\nu}]. \tag{3.24}$$

Equations (3.23) and (3.24) together define a mapping from $(\rho, \hat{\varphi})$ to $(\hat{\rho}, \varphi)$, which can be inverted to obtain the mapping from physical variables (ρ, φ) to effective variables $(\hat{\rho}, \hat{\varphi})$. This inversion is carried out explicitly in Section 3.4 below in a perturbative expansion.

To summarize, we have shown that any covariant functional $W[\rho, \hat{\varphi}, g_{\mu\nu}]$ gives rise to a transformation which satisfies conditions 1 and 2 discussed above, given by Eqs. (3.23) and (3.24). The renormalized stress energy tensor is given by combining Eqs. (3.15), (3.20b) and (3.21) and is

$$\hat{T}^{\mu\nu}_m = T^{\mu\nu}_{\mathrm{m}} - \rho\varphi g^{\mu\nu} + \frac{2}{\sqrt{-g}} \frac{\delta W}{\delta g_{\mu\nu}}. \tag{3.25}$$

We note that our class of transformations include a subclass of local transformations of the form

$$\hat{\varphi} = h(\varphi), \tag{3.26a}$$

$$\hat{\rho} = \frac{\rho}{h'(\varphi)}, \tag{3.26b}$$

where h is a monotonic function. These special transformations are given by choosing

$$W = \int d^4x \sqrt{-g} \rho h^{-1}(\hat{\varphi}), \tag{3.27}$$

and leave the stress energy tensor and splitting (3.1) unchanged, from Eqs. (3.25), (3.15) and (3.21). Another important subclass is given by the condition that the transformation reduce

to the identity when $\rho = 0$, *i.e.* $\hat{\rho}[0, \varphi] = 0$ and $\hat{\varphi}[0, \varphi] = \varphi$. This condition is equivalent to

$$W[\rho, \hat{\varphi}, g_{\mu\nu}] = \int d^4x \sqrt{-g} \rho \hat{\varphi} + O(\rho^2). \tag{3.28}$$

We note that the transformation given by Equations (3.23) and (3.24) is a type II functional canonical transformation which preserves the symplectic form

$$\int d^4x \sqrt{-g} \delta\rho \wedge \delta\varphi \tag{3.29}$$

on the field configuration space. It is not related to the physical symplectic form which is defined on the on-shell subspace of the field configuration space. We can alternatively parametrize the transformation as a type III canonical transformation as follows. Instead of the form (3.21) of the functional $\hat{W}$, we choose

$$\tilde{W} = -\hat{W}[\hat{\rho}, \varphi, g_{\mu\nu}] + \int d^4x \sqrt{-g} \hat{\rho}\hat{\varphi}. \tag{3.30}$$

Inserting this definition into the second action functional in Equation (3.15) and taking a variation at fixed $g_{\mu\nu}$ gives

$$\delta\hat{S}_\varphi = \int d^4x \sqrt{-g} \left[\left(\frac{1}{\sqrt{g}} \frac{\delta S_\varphi}{\delta\varphi} + \frac{1}{\sqrt{g}} \frac{\delta\hat{W}}{\delta\varphi} \right) \delta\varphi - \left(\hat{\varphi} - \frac{\delta\hat{W}}{\delta\hat{\rho}} \right) \delta\hat{\rho} - \hat{\rho}\delta\hat{\varphi} \right] \tag{3.31}$$

We now define the field redefinition

$$\hat{\varphi} = \frac{1}{\sqrt{g}} \frac{\delta \hat{W}}{\delta \hat{\rho}} [\hat{\rho}, \varphi, g_{\mu\nu}], \tag{3.32a}$$

$$\rho = \frac{1}{\sqrt{g}} \frac{\delta \hat{W}}{\delta \varphi} [\hat{\rho}, \varphi, g_{\mu\nu}]. \tag{3.32b}$$

Inserting this into equation (3.31), we see that the second term vanishes identically, and the first term vanishes when we make use of the definition (3.4). The condition (3.17) is thus satisfied and the rest of the argument proceeds as previously. The type III generating functional $\hat{W}$ and the type II generating functional W are related by

$$W = -\hat{W} + \int d^4x \sqrt{-g} \left[\rho\varphi + \hat{\rho}\hat{\varphi}\right] \tag{3.33}$$

from Equations (3.21) and (3.30).

3.3.3 Locality condition

We now discuss the conditions we impose on the field redefinition (3.13) in order to make the description in terms of the effective field have certain locality properties. These properties are weaker than the usual criterion that the Lagrangian is a local functional of the fields.

Let $\mathcal{B}$ be the worldtube in spacetime where the stress-energy tensor $T^{\mu\nu}_{\mathrm{m}}$ and the density ρ are nonvanishing. For two points x and y, we denote by $\gamma(x, y)$ the geodesic segment that connects x to y (or segments if there is more than one geodesic connecting them). We then define the convex geodesic hull $\mathcal{C}(R)$ of any region R to be the smallest region containing R for which any two points $x, y \in \mathcal{C}(R)$, the geodesic $\gamma(x, y)$ is also contained in $\mathcal{C}(R)$. Then,

we express the locality condition as

$$\hat{T}^{\mu\nu}_{\mathrm{m}}(x) = 0 \text{ for } x \notin \mathcal{C}(\mathcal{B}). \tag{3.34}$$

In other words, the effective stress energy tensor must vanish outside the convex geodesic hull of the support of the physical stress-energy tensor. This condition guarantees that the effective description of the system will remain localized to a compact spatial neighborhood of the physical system. Also, the equations of motion for the effective variables (3.18) guarantee that if $\hat{T}^{\mu\nu}_{\mathrm{m}}$ has compact support, then the effective charge density $\hat{\rho}$ must have the same support

$$\hat{\rho}(x) = 0 \text{ for } x \notin \mathcal{C}(\mathcal{B}), \tag{3.35}$$

assuming the gradient of $\hat{\varphi}$ is nonvanishing. In section 3.4.1, we prove that the condition (3.35) implies that

$$\left(\frac{\delta\hat{\varphi}(x)}{\delta\varphi(y)}\right)_{\rho} = 0 \text{ for } x \in \mathcal{B} \text{ and } y \notin \mathcal{C}(\mathcal{B}). \tag{3.36}$$

This means that the effective field $\hat{\varphi}$ on the worldtube is constructed from the physical field φ in the vicinity of the worldtube.

3.3.4 Causality condition

In this section, we discuss the conditions that we impose on the field redefinition (3.13), which are motivated by considerations of causality. While the underlying theory is manifestly causal, the dynamics in the effective variables can have an apparent acausality due to the nonlocality in the field redefinition. We would like to ensure that this apparent acausality is suppressed in the limit when the size of the body becomes small.

The causality condition can be expressed as follows. Take n variational derivatives of the generating functional $W[\rho, \hat{\varphi}, g_{\mu\nu}]$ with respect to the charge density ρ evaluated at different points $x_1, x_3, \ldots, x_n$. Then, the causality condition is

$$\left.\frac{\delta^n W[\rho, \hat{\varphi}, g_{\mu\nu}]}{\delta\rho(x_1)\ldots\delta\rho(x_n)}\right|_{\rho=0} = 0 \quad \begin{array}{l}\text{if any subset of the points } \{x_1, \ldots, x_n\} \\ \text{is timelike to its complement.}\end{array} \tag{3.37}$$

In other words, if the set of points $\{x_1, \ldots, x_n\}$ can be partitioned into two subsets such that the separation between any point in the first subset and any point in the second subset is timelike, then the left hand side of equation (3.37) must vanish. This is a perturbative statement and should be applied to all $2 \leq n \leq N$ with N being the order in ρ in which we are expanding the dynamics. In section 3.4.2 below, we will show that when the locality condition 3.34 is assumed, the causality condition 3.37 on the type II generating functional is equivalent to a similar condition for the type III generating functional $\hat{W}[\hat{\rho}, \varphi, g_{\mu\nu}]$

$$\left.\frac{\delta^n \hat{W}[\hat{\rho}, \varphi, g_{\mu\nu}]}{\delta\hat{\rho}(x_1)\ldots\delta\hat{\rho}(x_n)}\right|_{\rho=0} = 0 \quad \begin{array}{l}\text{if any subset of the points } \{x_1, \ldots, x_n\} \\ \text{is timelike to its complement.}\end{array} \tag{3.38}$$

We also show that these conditions imply a constraint on the dependence of the effective field $\hat{\varphi}$ on the physical charge ρ at fixed physical field φ, namely

$$\left.\frac{\delta^{n-1}\hat{\varphi}(x_1; \rho, \varphi, g_{\mu\nu}]}{\delta\rho(x_2)\ldots\delta\rho(x_n)}\right|_{\rho=0} = 0 \quad \begin{array}{l}\text{if any subset of the points } \{x_1, \ldots, x_n\} \\ \text{is timelike to its complement.}\end{array} \tag{3.39}$$

Equation (3.39) is easier to understand physically, so now we turn to discuss the interpretation and motivation for this condition. First, note that the on-shell physical field φ satisfying Equation (3.3) will be a functional of the scalar density ρ, satisfying some boundary condi-

tions (i.e., retarded, advanced, etc.). We are interested in field redefinitions under which the effective field $\hat{\varphi}$ preserves said causality in the point-particle limit. Importantly, we will allow the effective field $\hat{\varphi}$ to depend acausally on the scalar density ρ as long as that dependence is suppressed in the point-particle limit.

Second, the effective field $\hat{\varphi}$ can always be expressed as

$$\hat{\varphi} = \varphi - \varphi_S, \tag{3.40}$$

where the S field φ_S is defined as the difference between the physical and effective fields. The name S-field is chosen to show the fact that in a point particle limit this field will have a singular limit, like the physical field φ, while the effective field $\hat{\varphi}$ will be finite[§]. It is clear from Equation (3.40) that the S-field will be responsible for any acausalities in the effective field $\hat{\varphi}$. In other words, if the S-field were exactly local in ρ, then the on-shell effective field $\hat{\varphi}$ would automatically have the same causality as the on-shell physical field φ. This explains why the derivatives in condition (3.39) are taken at fixed physical field φ: They are equivalent to taking derivatives with respect to the S-field, constraining its acausality.

In Section 3.4.2 below, we show that in a perturbative expansion in powers of the scalar density ρ, the condition (3.37) implies that up to $O(\rho^n)$ the variational derivative

$$\frac{\delta\hat{\varphi}(x;\rho,\varphi]}{\delta\rho(y)} = 0 \text{ whenever } x \text{ and } y \text{ are separated by more than } n \text{ light-crossing times.} \tag{3.41}$$

This forces $\hat{\varphi}$ to be independent of ρ in the distant past/future to any finite order in perturba-

[§]The "S" can also stand for *Self-field*, since the field redefinition's goal is to generalize the Detweiler-Whiting renormalization scheme, where the self-field of a point-particle is subtracted from its physical field.

tion order. Furthermore, the point-particle limit will collapse the worldtube $\mathcal{B}$ of the object into a worldline $z(\tau)$, making the light-crossing time vanish. Therefore, the effective field $\hat{\varphi}$ at any position $z(\tau_0)$ in the worldline, will not have acausal dependence in ρ, at fixed φ. Requirements of this general kind have previously been imposed on the definition of effective fields and effective 4-momenta in References [41, 43].

3.3.5 Source-free condition for effective field

We would like to impose that the effective scalar field $\hat{\varphi}$ be source free, as has normally been done in previous work [38]. This would entail specializing the field redefinition to ensure that

$$\mathcal{E}_\varphi[\varphi]\big|_{\varphi=\hat{\varphi}[\rho,\varphi]} = \mathcal{E}_\varphi[\varphi] - \rho. \tag{3.42}$$

This condition is imposed off-shell, so that ρ and φ are allowed to vary independently. When we go on-shell and impose the equation of motion (3.3), the condition reduces to

$$\mathcal{E}_\varphi[\varphi]\big|_{\varphi=\hat{\varphi}[\rho,\varphi]} = 0. \tag{3.43}$$

Thus the effective field $\hat{\varphi}$ satisfies the same equation of motion as the original field, but with a vanishing source. This ensures that the effective field varies slowly over the object, and remains finite in the point particle limit, enabling a simple derivation of the point particle self-force.

Rather than imposing the source-free condition (3.42), we shall, in fact, impose the weaker condition

$$\mathcal{E}_\varphi[\varphi]\big|_{\varphi=\hat{\varphi}[\rho,\varphi]} - \mathcal{E}_\varphi[\varphi] + \rho = \mathcal{S}[\rho]. \tag{3.44}$$

Here, $\mathcal{S}$ is a smooth source satisfying certain properties which we discuss further below. The on-shell version of this condition is

$$\mathcal{E}_\varphi[\varphi]\big|_{\varphi=\hat{\varphi}[\rho,\varphi]} = \mathcal{S}[\rho]. \tag{3.45}$$

The reason for imposing this weaker condition is as follows. For linear theories, in even spacetime dimensions, it can be shown that the source-free condition (3.43), as well as our locality and causality conditions can be satisfied if one chooses the field redefinition to be

$$\hat{\varphi}(x;\rho,\varphi] = \varphi(x) - \int dV_y G_{\rm DW}(x,y)\rho(y). \tag{3.46}$$

In addition, this two-point function $G_{\rm DW}(x,y)$ must satisfy $G_{\rm DW}(x,y) = G_{\rm DW}(y,x)$, $G_{\rm DW}(x,y) = 0$ for timelike separations and it must be a Green function for the equation of motion with a delta function source. Two-point functions that satisfy these conditions are called Detweiler-Whiting Green functions [30, 38]. They are known to exist in spacetimes where the metric is analytic [26]. A general procedure to compute them using a kind of series expansion called the Hadamard series is known (See Appendix A), and when this series converges, a DW Green function will exist [26]. However, the series does not always converge, and in general a DW Green function may not exist [45].

Nevertheless, one can proceed by using instead of $G_{\rm DW}$ an approximate version of it obtained by truncating the Hadamard series to some finite order. Such a truncated 2-point function will yield effective fields that give the correct self-force in the point-particle limit, and so they are all that are needed in practice. They satisfy inhomogeneous equations of motion with additional source terms that vanish in the coincidence limit $x \to y$. In the non-

linear context, our condition (3.44) is a natural generalization to the condition satisfied by the truncated version of the DW Green functions in linear theories.

3.4 Weak nonlinearity expansion

In the remainder of this chapter, we will restrict attention to weakly nonlinear systems, that is, systems for which effects that are nonperturbative in the charge density ρ can be neglected. We will model these systems using expansions in powers of ρ. In this section, we will consider such expansions of the field redefinition (3.13) off-shell, without imposing the equations of motion. In later sections we will specialize to on-shell configurations.

We make the assumption (3.28) that the field transformation reduces to the identity when $\rho \to 0$, from which it follows that the generating functional W can be expanded as the functional Taylor series

$$W[\rho, \hat{\varphi}, g_{ab}] = \int dV \rho \hat{\varphi} + \sum_{n=2}^{\infty} \frac{1}{n} \int dV_1 \dots dV_n \, \mathcal{G}_n(x_1, \dots, x_n; \hat{\varphi}, g_{\mu\nu}] \rho_1 \dots \rho_n. \quad (3.47)$$

Here, $dV_j = d^4x_j \sqrt{-g(x_j)}$, $\rho_j = \rho(x_j)$ and $\mathcal{G}_n$ are some symmetric n-point functions which are functionals of the effective field $\hat{\varphi}$ and metric $g_{\mu\nu}$. For brevity we will not show explicitly from now on the dependence of $\mathcal{G}_n$ on $\hat{\varphi}$ and $g_{\mu\nu}$. Substituting into the canonical transfor-

mation formulae Equations (3.23) and (3.24), and (3.25) we obtain

$$\begin{aligned}
\varphi(x) &= \hat{\varphi}(x) + \sum_{n=2}^{\infty} \int dV_1 \dots dV_{n-1}\, \mathcal{G}_n(x, x_1, \dots, x_{n-1}) \rho_1 \dots \rho_{n-1}, && \text{(3.48a)} \\
\hat{\rho}(x) &= \rho(x) + \sum_{n=2}^{\infty} \frac{1}{n} \int dV_1 \dots dV_n\, \mathcal{G}_{n,1}(x_1, \dots, x_n; x) \rho_1 \dots \rho_n, && \text{(3.48b)} \\
\hat{T}^{\mu\nu}_{\mathrm{m}}(x) &= T^{\mu\nu}_{\mathrm{m}}(x) + \sum_{n=2}^{\infty} \frac{1}{n} \int dV_1 \dots dV_n\, \mathcal{J}^{\mu\nu}_{n,1}(x_1, \dots, x_n; x) \rho_1 \dots \rho_n, && \text{(3.48c)}
\end{aligned}$$

where we have defined

$$\begin{aligned}
\mathcal{G}_{n,m}(x_1, \dots, x_n; y_1, \dots, y_m) &= \frac{1}{\sqrt{-g(y_1)}} \frac{\delta}{\delta \hat{\varphi}(y_1)} \dots \frac{1}{\sqrt{-g(y_m)}} \frac{\delta}{\delta \hat{\varphi}(y_m)} \mathcal{G}_n(x_1, \dots, x_n), && \text{(3.49a)} \\
\mathcal{J}^{\mu_1\nu_1 \dots \mu_m\nu_m}_{n,m}(x_1, \dots, x_n; y_1, \dots, y_m) &= \frac{1}{\sqrt{-g(y_1)}} \frac{\delta}{\delta g_{\mu_1\nu_1}(y_1)} \dots \frac{1}{\sqrt{-g(y_m)}} \frac{\delta}{\delta g_{\mu_m\nu_m}(y_m)} \mathcal{G}_n(x_1, \dots, x_n). && \text{(3.49b)}
\end{aligned}$$

We can also compute the field redefinition perturbatively starting from the type III generating functional $\hat{W}$. We assume an expansion of the form

$$\hat{W}[\hat{\rho}, \varphi, g_{\mu\nu}] = \int dV_y \hat{\rho} \varphi - \sum_{n=2}^{\infty} \frac{1}{n} \int dV_1 \dots dV_n \hat{\mathcal{G}}_n\big((x_1, \dots, x_n; \varphi, g_{\mu\nu}\big] \hat{\rho}_1 \dots \hat{\rho}_n, \quad \text{(3.50)}$$

for some symmetric n-point functions $\hat{\mathcal{G}}_n$, functionals of φ and $g_{\mu\nu}$. Substituting these generating functional into the canonical transformation formulae (3.32) we obtain

$$\begin{aligned}
\hat{\varphi}(x) &= \varphi(x) - \sum_{n=2}^{\infty} \int dV_1 \dots dV_{n-1}\, \hat{\mathcal{G}}_n(x, x_1, \dots, x_{n-1}; \varphi, g_{\mu\nu}] \hat{\rho}_1 \dots \hat{\rho}_{n-1}, && \text{(3.51a)} \\
\rho(x) &= \hat{\rho} - \sum_{n=2}^{\infty} \frac{1}{n} \int dV_1 \dots dV_n \hat{\mathcal{G}}_{n,1}(x_1, \dots, x_n; x) \hat{\rho}_1 \dots \hat{\rho}_n. && \text{(3.51b)}
\end{aligned}$$

where $\hat{\mathcal{G}}_{n,1}$ is defined by an equation analogous to Equation (3.49a). By comparing the expansions (3.47) and (3.50) for the type II and III generating functionals and expanding perturbatively in the density, we can relate the sets of n-point functions $\mathcal{G}_n$ and $\hat{\mathcal{G}}_n$. The first few relations are

$$\begin{aligned}
\hat{\mathcal{G}}_2(x_1, x_2)[\varphi] &= \mathcal{G}_2(x_1, x_2)|_{\hat{\varphi}=\varphi}\,, && \text{(3.52a)}\\
\hat{\mathcal{G}}_3(x_1, x_2, x_3)[\varphi] &= \mathcal{G}_3(x_1, x_2, x_3)|_{\hat{\varphi}=\varphi} && \text{(3.52b)}\\
&- \frac{1}{2}\int dV'\; \mathcal{G}_{2,1}(x_1, x_2; x')\mathcal{G}_2(x', x_3)|_{\hat{\varphi}=\varphi}\\
&- \frac{1}{2}\int dV'\; \mathcal{G}_{2,1}(x_1, x_3; x')\mathcal{G}_2(x', x_2)|_{\hat{\varphi}=\varphi}\\
&- \frac{1}{2}\int dV'\; \mathcal{G}_{2,1}(x_2, x_3; x')\mathcal{G}_2(x', x_1)|_{\hat{\varphi}=\varphi}\,.
\end{aligned}$$

We can also solve perturbatively for the effective fields in terms of the physical fields by expanding in powers of the density. Starting from Equation (3.48) this yields

$$\hat{\varphi}(x) = \varphi(x) - \sum_{n=2}^{\infty} dV_1 \ldots dV_n \tilde{G}_n(x, x_1, \ldots, x_{n-1}; \varphi, g_{\mu\nu}]\rho_1 \ldots \rho_{n-1}. \qquad \text{(3.53)}$$

Here, the n-point functions $\tilde{\mathcal{G}}_n$ are symmetric in their last $n-1$ arguments but not in all n

arguments. The first few of these n-point functions are

$$\tilde{\mathcal{G}}_2(x_1, x_2)[\varphi] = \mathcal{G}_2(x_1, x_2)|_{\hat{\varphi}=\varphi}\,, \tag{3.54a}$$

$$\begin{aligned}\tilde{\mathcal{G}}_3(x_1, x_2, x_3)[\varphi] &= \mathcal{G}_3(x_1, x_2, x_3)|_{\hat{\varphi}=\varphi} \\ &- \frac{1}{2}\int dV'\, \mathcal{G}_{2,1}(x_1, x_2; x')\mathcal{G}_2(x', x_3)|_{\hat{\varphi}=\varphi} \\ &- \frac{1}{2}\int dV'\, \mathcal{G}_{2,1}(x_1, x_3; x')\mathcal{G}_2(x', x_2)|_{\hat{\varphi}=\varphi}\,.\end{aligned} \tag{3.54b}$$

3.4.1 Locality constraint on the n-point functions

We now turn to evaluating the constraints on the n-point functions $\mathcal{G}_n$ that arise from imposing the locality condition (3.34). From equation (3.48c), it is clear that the effective stress-energy tensor $\hat{T}^{\mu\nu}_{\mathrm{m}}$ will vanish outside the convex geodesic hull of $\mathcal{B}$ if we impose

$$\mathcal{J}^{\mu\nu}_{n,1}(x_1, \ldots, x_n; y) = 0 \text{ when } y \notin \mathcal{C}\big(\{x_1, \ldots, x_n\}\big). \tag{3.55}$$

Here, $\mathcal{C}\big(\{x_1, \ldots, x_n\}\big)$ is the convex geodesic hull of the set of points $\{x_1, \ldots, x_n\}$. As explained in Section 3.3.3, the locality condition (3.34) for $\hat{T}^{\mu\nu}_{\mathrm{m}}$ guarantees the locality of $\hat{\rho}$, by the effective equations of motion $\nabla_\mu \hat{T}^{\mu\nu}_{\mathrm{m}} = -\hat{\rho}\nabla^\nu\hat{\varphi}$. Therefore, using equation (3.48b) we get

$$\mathcal{G}_{n,1}(x_1, \ldots, x_n; y) = 0 \text{ when } y \notin \mathcal{C}\big(\{x_1, \ldots, x_n\}\big). \tag{3.56}$$

In other words, the n-point functions $\mathcal{G}_n$ must be *quasi-local* functionals [¶] of the metric and the effective field. Combining Equation (3.51a) and the relation (3.52), we see that the

[¶] A function f that takes n spacetime points $x_1, \ldots, x_n$ as arguments is a quasi-local functional of another function g if its dependence on g is restricted to a region locally constructed from its arguments.

variational derivative of $\hat{\varphi}$ with respect to φ at fixed $\hat{\rho}$, is constructed from $\mathcal{G}_{n,1}$. It is then straightforward to show that (3.56) implies (3.36), using the expansion (3.52).

3.4.2 Causality constraint on the n-point functions

By inserting the expansion (3.21) into the causality condition (3.37), we find that the condition constraints the n-point functions

$$\mathcal{G}_n(x_1, \ldots, x_n) = 0 \quad \begin{array}{l} \text{if any subset of the points } \{x_1, \ldots, x_n\} \\ \text{is timelike to its compliment.} \end{array} \tag{3.57}$$

Similarly, the causality condition (3.38) is equivalent to

$$\hat{\mathcal{G}}_n(x_1, \ldots, x_n) = 0 \quad \begin{array}{l} \text{if any subset of the points } \{x_1, \ldots, x_n\} \\ \text{is timelike to its compliment.} \end{array} \tag{3.58}$$

Finally, the condition (3.39) is equivalent to

$$\tilde{\mathcal{G}}_n(x_1, \ldots, x_n) = 0 \quad \begin{array}{l} \text{if any subset of the points } \{x_1, \ldots, x_n\} \\ \text{is timelike to its compliment.} \end{array} \tag{3.59}$$

In this section we show that conditions (3.37), (3.38) and (3.39) are equivalent, with the aid of graph theory.

First, we define *causality graphs*. Recall that a graph is a finite set of vertices, each pair of which may or may not be connected by an edge. Given n points $\{x_1, \ldots, x_n\}$ in spacetime, we define the associated causality graph $\mathfrak{G}(x_1, \ldots, x_n)$ by taking the vertices to be the spacetime points and by linking with edges any pairs of points that are timelike separated. This graph is simple (at most one edge between any two vertices) and undirected (the edges have no sense

of direction).

The condition (3.57) on the n-point function $\mathcal{G}_n$ is equivalent to the requirement that the vertices of the causality graph $\mathfrak{G}(x_1, \ldots, x_n)$ can be partitioned into two subsets for which every vertex in the first subset is linked to every vertex on the second. Simple undirected graphs with this property are called joined graphs [46]. Joined graphs can be characterized as follows. A spanning subgraph of a graph is a subgraph which contains all the vertices of the original graph. A bipartite graph is a graph where vertices can be divided into two disjoint sets, such that each edge connects an element of the first set to an element of the second set. A complete bipartite graph is one where every element of the first set is linked to every element of the second set. Finally, it can be shown that a simple undirected graph is joined if and only if it contains a spanning complete bipartite subgraph. We can therefore rewrite the causality condition (3.57) as

$$\mathcal{G}_n(x_1, \ldots, x_n) = 0 \quad \text{if } \mathfrak{G}(x_1, \ldots, x_n) \text{ contains a spanning complete bipartite graph.} \tag{3.60}$$

The conditions (3.58) and (3.59) can be rewritten similarly. All complete bipartite graphs with up to five vertices are shown in Table 3.1. The causality graph $\mathfrak{G}(x_1, \ldots, x_n)$ contains a spanning complete bipartite graph when there is a subset of points located "so far away" that they become timelike to all others. In this sense, the causality condition prevents the dependence of the effective field $\hat{\varphi}$ on the charge density being too spread out, although it allows greater separations as we go to higher orders in the scalar density. In Appendix D, we discuss a stronger causality condition, which is nonperturbative but also creates constraints potentially too restrictive to be satisfied by the field redefinition.

Number of vertices	Complete bipartite graphs
$n = 2$	
$n = 3$	
$n = 4$	
$n = 5$	

Table 3.1: List of complete bipartite graphs up to $n = 5$ vertices.

We now describe how this condition applies to the n-point functions with $n - 2, 3, 4$. At each order n, we need to enumerate the number of nontrivial ways that the set of points $\{x_1, \ldots, x_n\}$ can be partitioned into disjoint subsets. For $n = 2$ there is only one way, $2 = 1 + 1$ and the causality condition reduces to

$$\mathcal{G}_2(x_1, x_2) = 0 \text{ if } x_1, x_2 \text{ are timelike separated.} \tag{3.61}$$

For $n = 3$, there is also only one way $3 = 1 + 2$, and the condition reduces to

$$\mathcal{G}_3(x_1, x_2, x_3) = 0 \text{ if at least two points are timelike separated.} \tag{3.62}$$

For $n - 4$, there are two ways, $4 = 1 + 3$ and $4 = 2 + 2$, and the condition is

$$\mathcal{G}_4(x_1, x_2, x_3, x_4) = 0 \quad \begin{array}{l} \text{if either there is one point timelike} \\ \text{to the other three, or there is a pair} \\ \text{of points timelike to the other pair.} \end{array} \tag{3.63}$$

We now turn to the derivation of the result described in Section 3.3.4, which stated that the conditions (3.37), (3.38) and (3.39) are equivalent when we assume the locality condition (3.34). Given an n-point function $\mathcal{G}_n$ that satisfies the constraint (3.57), let us investigate what the corresponding constraint for $\hat{\mathcal{G}}_n$ is. From the relations (3.52) that relate the two sets of n-point functions, it is sufficient to show that the correction terms in those equations vanish when the graph $\mathfrak{G}(x_1, \ldots, x_n)$ satisfies the properties discussed above.

Let us look at the term

$$\frac{1}{2} \int \mathcal{G}_{2,1}(x_1, x_2; y) \mathcal{G}_2(y, x_3) dV_y. \tag{3.64}$$

If the points x_1, x_2 and x_3 form a causality graph $\mathfrak{G}(x_1, x_2, x_3)$ which is joined, then that graph must have a spanning complete bipartite graph. It follows that there must be two timelike pairs (See table 3.1). Any causality graph containing the pair (x_1, x_2) will make the first factor in (3.64) vanish due to the causality constraint in $\mathcal{G}_{2,1}$. Now, consider the only other option, where (x_1, x_3) and (x_2, x_3) are timelike pairs. Since x_3 is timelike to both x_1 and x_2, x_3 must also be timelike to y, since the locality condition (3.56) forces y to lie on the geodesic connecting x_1 and x_2. This makes the second factor $\mathcal{G}_2(y, x_3)$ vanish. The same argument can be used for the other two correction terms in equation (3.52). It follows that whenever there are two timelike pairs, both $\hat{\mathcal{G}}_3(x_1, x_2, x_3)$ and $\mathcal{G}_3(x_1, x_2, x_3)$ vanish. Furthermore, similar arguments can be made for all the correction terms relating $\hat{\mathcal{G}}_n$ to $\mathcal{G}_n$ at any order n. The key is that the causality condition involves bipartite complete graphs. Then, if the arguments on the first term in the integrand are not timelike (which would make it vanish), then they must be timelike to the arguments of the other term, thus making that one vanish by the locality condition. This argument can also be applied to the correction terms that relate the n-point

functions $\tilde{\mathcal{G}}_n$ defined by (3.50) to $\mathcal{G}_n$. Therefore, the constraint (3.57) guarantees that the causality condition (3.39) holds.

Finally, we turn to the derivation of the result discussed in Section 3.3.4 above, that the causality conditions (3.37) or (3.38) imply that the quantity

$$\frac{\delta\hat{\varphi}(x;\rho,\varphi]}{\delta\rho(y)}$$

vanishes when x and y are inside the support $\mathcal{B}$ of ρ and when x and y are separated in time by more than n light-crossing times, when working to order $O(\rho^n)$. From Equation (3.54), we find

$$\frac{\delta\hat{\varphi}(x;\rho,\varphi]}{\delta\rho(y)} = -\sum_{n=2}^{\infty}\int dV_1\ldots dV_{n-2}\tilde{G}_n(x,y,x_1,\ldots,x_{n-2})\rho_1\ldots\rho_{n-2} \tag{3.65}$$

We now make use of the fact that the n-point function $\tilde{G}_n$ satisfies the condition (3.59). We assume that x and y lie in the worldtube $\mathcal{B}$ and we focus on the term involving $\tilde{G}_n(x,y,x_1,\ldots,x_{n-2})$. Without loss of generality, we can assume that the points $x_1,\ldots,x_{n-2}$ lie in $\mathcal{B}$, otherwise the term vanishes. Consider now the partition of $\{x,y,x_1,\ldots,x_{n-2}\}$ into $\{y\}$ and $\{x,,x_1,\ldots,x_{n-2}\}$. By our assumption, y must be spacelike related to at least one of $\{x,,x_1,\ldots,x_{n-2}\}$, otherwise we have a spanning complete bipartite subgraph, which forces $\tilde{G}_n$ to vanish. If y is spacelike related to x, then x and y are separated by at least one light-crossing time and the result follows. If instead y is spacelike separated from one of $\{x_1,\ldots,x_{n-2}\}$, we relabel the points to call the point x_1, so y is spacelike related to x_1. Consider next the partition of $\{x,y,x_1,\ldots,x_{n-2}\}$ into $\{y,x_1\}$ and $\{x,x_2,\ldots,x_{n-2}\}$. In order to avoid a complete spanning bipartite subgraph, we need either a) y is spacelike separation from one of $\{x,x_2,\ldots,x_{n-2}\}$, or b) x_1 is spacelike

separated from one of $\{x_2, \ldots, x_{n-2}\}$. In either case, we relabel the points to make the new point x_2, and we consider the partition $\{y, x_1, x_2\}$ and $\{x, x_3, \ldots, x_{n-2}\}$. Proceeding in this way, the end result that maximizes the allowed temporal displacement between x and y has y spacelike separated from x_1, x_1 spacelike separated from x_2,... x_{n-3} spacelike separated from x_{n-2} and x_{n-2} spacelike separated from x. This corresponds to x and y being separated by at most $n-1$ light-crossing times, when we use the expression for $\hat{\varphi}$ that includes terms up to $O(\rho^{n-1})$.

3.4.3 Differential equations for the n-point functions $\mathcal{G}_n$

In this subsection we use the source-free condition (3.44) to obtain a set of differential equations that the n-point functions $\mathcal{G}_n$ must satisfy. It will be convenient to rewrite that condition treating $\hat{\varphi}$ and ρ as the independent variables

$$\mathcal{E}_\varphi[\hat{\varphi}] - \mathcal{E}_\varphi[\varphi[\hat{\varphi},\rho]] + \rho = \mathcal{S}[\rho]. \tag{3.66}$$

Next, we recall the definition of the S-field φ_S from Equation (3.40). The expansion of the self-field in powers of density is

$$\varphi_S[\hat{\varphi},\rho](x) = \sum_{n=2}^{\infty} \int dV_1 \ldots dV_{n-1} \mathcal{G}_n(x, x_1, \ldots, x_{n-1}; \hat{\varphi}]\rho_1 \ldots \rho_{n-1}. \tag{3.67}$$

We parametrize the expansion in powers of density of the source $\mathcal{S}[\rho]$ as

$$\mathcal{S}(x;\rho] = \sum_{n=2}^{\infty} \int dV_2 \ldots dV_n \mathfrak{s}_n(x, x_2, \ldots, x_n; g_{\mu\nu}]\rho_2 \ldots \rho_n. \tag{3.68}$$

We now insert the definitions and expansions (3.40), (3.67) and (3.68) into the condition (3.66), expand in powers of ρ at fixed $\hat{\varphi}$ and equate the coefficients of successive powers of ρ. In expanding the functional $\mathcal{E}_\varphi$, we make use of the definition

$$\mathcal{E}^{(n)}_\varphi(x;x_1,\ldots,x_n l\varphi) \equiv \frac{1}{\sqrt{-g(x_1)}\ldots\sqrt{-g(x_n)}}\frac{\delta^n\mathcal{E}_\varphi[\varphi(x)]}{\delta\varphi(x_1)\ldots\delta\varphi(x_n)}. \tag{3.69}$$

This results in the series of equations for the n-point functions

$$\begin{aligned}
\int \mathcal{E}^{(1)}_\varphi(x;x_1)\mathcal{G}_2(x_1,y)dV_1 &= \delta^4(x,y) - \mathfrak{d}_2(x,y), && \text{(3.70a)}\\
\int \mathcal{E}^{(1)}_\varphi(x;x_1)\mathcal{G}_3(x_1,y,z)dV_1 &= -\frac{1}{2}\int dV_1 dV_2 \mathcal{E}^{(2)}_\varphi(x;x_1,x_2)\times\\
&\times\ \mathcal{G}_2(x_1,y)\mathcal{G}_2(x_2,z) - \mathfrak{d}_3(x,y,z), && \text{(3.70b)}\\
&\vdots
\end{aligned}$$

where $\delta^4(x,y) \equiv \delta^4(x-y)/\sqrt{-g(x)}$. Note that although the definition (3.69) of the functionals $\mathcal{E}^{(n)}_\varphi$ involves only the physical field φ, they are evaluated at $\varphi = \hat{\varphi}$ in Equation (3.70). Explicit versions of these equations for a specific model is given in Section 3.2 below.

We now discuss the existence and uniqueness of n-point functions $\mathcal{G}_n$ that satisfy the Equations (3.70) as well as the locality and causality conditions (3.37) and (3.56). Consider first Equation (3.70a) in the case with vanishing source $\mathfrak{d}_2 = 0$. This equation says that $\mathcal{G}_2$ is a Green function for linearized solutions. It must also satisfy the symmetry condition $\mathcal{G}_2(x,y) = \mathcal{G}_2(y,x)$ from the definition (3.21) together with the causality condition that it vanishes for timelike separations. As discussed in Section 3.3.5 above, these three conditions on a 2-point function are called the Detweiler-Whiting conditions [30]. In certain cases, the

three conditions determine a unique 2-point function, called the Detweiler-Whiting Green function. In certain other cases, when the Hadamard solution fails to converge, it is not known whether any 2-point function satisfying the conditions exists. For that reason, one can use instead as a 2-point function the quantity obtained from a version of the Hadamard expansion procedure, truncated at a finite order. In this case, there is a non-zero smooth source term $\mathfrak{d}_2$ on the right hand side of Equation (3.70a), a specific local functional of the background fields $\hat{\varphi}$ and $g_{\mu\nu}$ [43] [‖]. Such truncated versions of the Detweiler-Whiting Green function are often used in practice to compute self-forces [37]. They also automatically satisfy our locality constraint (3.56) because of the nature of the Hadamard construction [42].

Turn next to the corresponding situation for the 3-point function $\mathcal{G}_3$, which is much less well understood currently. We would like to impose on $\mathcal{G}_3$ the following conditions

- Symmetry under interchange of any pair of arguments (From the definition (3.21)).
- Our locality condition (3.56).
- Our causality condition which for $n = 3$ reduces to the requirement that $\mathcal{G}_3(x, y, z)$ should vanish whenever any two pairs of its arguments are timelike separated.
- Satisfy Equation (3.70b).

We conjecture that for analytic background fields $\hat{\varphi}$ and $g_{\mu\nu}$, unique 3-point functions exist which satisfy the four conditions, just as for $n = 2$ with Detweiler-Whiting Green functions. One piece of circumstantial evidence in support of this conjecture is that in Minkowski space-

[‖] 2-point functions which obey linearized equations of motion with a source consisting of a sum of a delta function and a smooth source are called *parametrices*, instead of Green functions. Thus, truncated Detweiler-Whiting 2-point functions are parametrices.

times, 3-point functions do exist which satisfy the first three requirements and the equation

$$\Box \mathcal{G}_3(x,y,z) = \mathcal{G}_2(x,y)\mathcal{G}_2(x,z) \tag{3.71}$$

similar to Equation (3.70b). Perhaps some generalization of the Hadamard construction can be found to define this 3-point functions. More generally, for arbitrtary smooth $g_{\mu\nu}$ and $\hat{\varphi}$, it seems clear that one must allow a source like $\mathfrak{d}_3$ on the right hand side of Equation (3.70b), since they are already necessary for $n = 2$.

In the remainder of this chapter, we will assume the existence of a 3-point function satisfying all four requirements and will express our results in terms of this 3-point function.

3.4.4 On-shell expansion of the physical field

Our discussion so far in this chapter has been off-shell, in the sense that the scalar field φ and the matter density ρ were treated as independent variables. Now, we will consider retarded solutions to the field equations, so that φ becomes a functional of ρ. Inserting the retarded solution into the self-force expression 3.19 will enable us to compute the self-force solely in terms of the effective density and to then take the point-particle limit (See Section 3.5 below). We start by writing the physical density as

$$\rho = \varepsilon \bar{\rho} \tag{3.72}$$

with $\varepsilon \ll 1$, where ε is a formal expansion parameter and $\bar{\rho}$ is independent of ε. Then, the physical and effective fields are expanded as

$$\varphi(x) = \bar{\varphi}(x) + \sum_{n=1}^{\infty} \varphi^{(n)} \varepsilon^n \tag{3.73a}$$

$$\hat{\varphi}(x) = \bar{\varphi}(x) + \sum_{n=1}^{\infty} \hat{\varphi}^{(n)} \varepsilon^n \tag{3.73b}$$

where $\bar{\varphi}$ is a background scalar field, independent of the matter density. This means that $\bar{\varphi}$ is a homogeneous solution, at least in the vicinity of the object, although it might be sourced by some other charge density far away.

Furthermore, we introduce the following notation for derivatives of the n-point functions $\mathcal{G}_n$ evaluated on the background field $\bar{\varphi}$

$$G_{n,m}(x_1, \dots, x_n; y_1, \dots, y_m) \equiv \frac{1}{\sqrt{-g(y_1)} \dots \sqrt{-g(y_m)}} \left. \frac{\delta^m \mathcal{G}_n(x_1, \dots, x_n)}{\delta\hat{\varphi}(y_1) \dots \delta\hat{\varphi}(y_m)} \right|_{\hat{\varphi}=\bar{\varphi}} . \tag{3.74}$$

Similarly, we expand the functional dependence of the source terms $\mathfrak{s}_n$ with respect to $\hat{\varphi}$ as

$$s_{n,m}(x_1, \dots, x_n; y_1, \dots, y_m) = \frac{1}{\sqrt{-g(y_1} \dots \sqrt{-g(y_m)}} \left. \frac{\delta^m \mathfrak{s}_n(x_1, \dots, x_n)}{\delta\hat{\varphi}(y_1) \dots \delta\hat{\varphi}(y_n)} \right|_{\hat{\varphi}=\bar{\varphi}} \tag{3.75}$$

The equations of motion get a similar expansion

$$E^{(n)}_{\bar{\varphi}}(x; x_1, \dots, x_n) \equiv \frac{1}{\sqrt{-g(x_1)} \dots \sqrt{-g(x_n)}} \left. \frac{\delta^n \mathcal{E}_{\varphi}[\varphi(x)](x)}{\delta\varphi(x_1) \dots \delta\varphi(x_n)} \right|_{\varphi=\bar{\varphi}} . \tag{3.76}$$

Substituting the expansion (3.73a) into the equations of motion (3.4) and making use of the

definition (3.69) shows that the physical fields $\bar{\varphi}$, $\varphi^{(1)}$, $\varphi^{(2)}$, ... obey

$$\begin{aligned} E_{\bar{\varphi}}[\bar{\varphi}] &= 0 && \text{(3.77a)} \\ \int E^{(1)}_{\bar{\varphi}}(x;y)\varphi^{(1)}(y)dV_y &= \bar{\rho}(x) && \text{(3.77b)} \\ \int E^{(1)}_{\bar{\varphi}}(x;y)\varphi^{(2)}(y) &= -\frac{1}{2}\int dV_y dV_z E^{(2)}_{\bar{\varphi}}(x;y,z)\varphi^{(1)}(y)\varphi^{(1)}(z) && \text{(3.77c)} \\ &\vdots \end{aligned}$$

We write the retarded solutions of these equations for the physical field as

$$\varphi(x) = \bar{\varphi}(x) + \varepsilon \int G_2^+(x,x')\bar{\rho}'dV' + \varepsilon^2 \int G_3^+(x,x',x'')\bar{\rho}'\bar{\rho}''dV'dV'' + \ldots \qquad (3.78)$$

Here, $G_2^+(x,x')$ is the retarded Green function for the differential operator $E_{\bar{\varphi}}$ and G_3^+ is a retarded 3-point function that can be written in terms of G_2^+ and $E^{(2)}_{\bar{\varphi}}$ using Equation (3.77c) as

$$G_3^+(x,y,z) = \int dV'dV''dV'''G_2^+(x,x')E^{(2)}_{\bar{\varphi}}(x';x'',x''')G_2^+(x'',y)G_2^+(x''',z) \qquad (3.79)$$

We also expand the equations (3.70) for the n-point functions $\mathcal{G}_n$ around $\bar{\varphi}$ to obtain similar

equations for the background n-point functions $G_{n.m}$

$$\begin{aligned}
\int E^{(1)}_{\bar{\varphi}}(x;x_1)G_{2,0}(x_1,y)dV_1 &= \delta^4(x,y) - s_{2,0}(x,y), && \text{(3.80a)}\\
\int E^{(1)}_{\bar{\varphi}}(x;x_1)G_{3,0}(x_1,y,z)dV_1 &= -\frac{1}{2}\int dV_1 dV_2 E^{(2)}_{\bar{\varphi}}(x;x_1,x_2)G_{2,0}(x_1,y)G_{2,0}(x_2,z)\\
&- s_{3,0}(x,y,z), && \text{(3.80b)}\\
\int E^{(1)}_{\bar{\varphi}}(x;x_1)G_{2,1}(x_1,y,z)dV_1 &= -\int dV_1 E^{(2)}(x;x_1,z)G_{2,0}(x_1,y) - s_{2,1}(x,y,z) && \text{(3.80c)}\\
&\vdots
\end{aligned}$$

Unlike equations (3.70), equations (3.80) are evaluated on a fixed background field $\bar{\varphi}$ and therefore can be solved order by order.

We now take the expansion (3.51a) of the effective field $\hat{\varphi}$ in terms of φ and $\hat{\rho}$, rewrite the n-point functions $\hat{\mathcal{G}}_n$ in terms of $\mathcal{G}_n$ using Equation (3.54), write the physical field φ in terms of the physical density ρ using the retarded solution (3.78), and finally write ρ in terms of the effective density $\hat{\rho}$ using Equation (3.48b). The result is

$$\hat{\varphi}(x;\hat{\rho}] = \bar{\varphi}(x) + \int G_2^{+,R}(x,x')\hat{\rho}'dV' + \int G_3^{+,R}(x,x',x'')\hat{\rho}'\hat{\rho}''dV'dV'' + \dots \qquad \text{(3.81)}$$

where we have defined the on-shell retarded regularized n-point functions

$$
\begin{aligned}
G_2^{+,R}(x_1,x_2) &= G_2^{\text{ret}}(x_1,x_2) - G_{2,0}(x_1,x_2) && \text{(3.82a)}\\
G_3^{+,R}(x_1,x_2,x_3) &= G_3^{+}(x_1,x_2,x_3) - G_{3,0}(x_1,x_2,x_3) && \text{(3.82b)}\\
&- \frac{1}{2}\int G_2^{+,R}(x_1,y)G_{2,1}(x_2,x_3,y)dV_y\\
&- \frac{1}{2}\int G_2^{+,R}(y,x_2)G_{2,1}(x_1,x_3,y)dV_y\\
&- \frac{1}{2}\int G_2^{+,R}(y,x_3)G_{2,1}(x_1,x_2,y)dV_y
\end{aligned}
$$

Here, R stands for regularized. The effective 2-point function $G_2^{+,R}$ is a generalization of the Detweiler-Whiting regularized 2-point function that we obtained in Section 2.5 after subtracting the singular field from the retarded Green function. As discussed in Section 3.3.5, $G_2^{+,R}$ is the 2-point function one would obtain by truncating the Hadamard expansion used to solve the source-free equations of motion (3.44). It instead solves the Equation (3.43). Similarly, the effective 3-point function $G_3^{+,R}$ is a generalization of the Detweiler-Whiting prescription to second order in the charge density. We assume that the 3-point function $G_{3,0}(x,y,z)$ satisfying our symmetry, causality, locality and source-free conditions exists such that $G_3^{+,R}$ exists and determines the self-force up to second order in the density.

3.5 Point-particle limit

In this section, we return to the equation of motion of the body in terms of the effective fields $\hat{\rho}$ and $\hat{\varphi}$. Since these fields remain finite in the point-particle limit where the size of the body becomes small, we can use multipole expansions [41] to derive the general form of

the equations of motion of the body. As discussed in Chapter 1 of this thesis, this form is identical to an equation of motion of a test body with no self-interactions. We then take the point-particle limit (discarding all the body's mass and spin multipole moments except for the monopole) to derive a point-particle equation of motion in terms of the n-point functions defined in Section 3.4.4 above.

We start by picking a representative worldline γ_s^μ within the body, with tangent vector $\dot{\gamma}_s^\mu$. This worldline is often chosen to be a suitably-defined center-of-mass worldline, but for our purposes it can be left unspecified. Next, we note that the effective generalized momentum $\hat{\mathcal{P}}_\xi(s)$ of the body given by Equation (3.20a) is a linear function of the Killing vector ξ^μ. Since Killing vector fields are determined by their value and their anti-symmetrized first derivative at any point, we can define quantities $\hat{p}^\mu(s)$ and $\hat{S}^{\mu\nu} = \hat{S}^{[\mu\nu]}(s)$ by the relation

$$\hat{\mathcal{P}}_\xi(s) = \hat{p}^\mu(\gamma_s)\xi_\mu(\gamma_s) + \frac{1}{2}\hat{S}^{\mu\nu}(\gamma_s)\nabla_\mu\xi_\nu(\gamma_s). \tag{3.83}$$

These quantities are the effective momentum and angular momentum of the body with respect to the worldline γ_s. The time derivative of Equation (3.83) is

$$\frac{d}{ds}\hat{\mathcal{P}}_\xi(s) = \xi_\mu\Big(\frac{D\hat{p}^\mu}{ds} - \frac{1}{2}R_{\alpha\beta\nu}{}^\mu\hat{S}^{\alpha\beta}\dot{\gamma}_s^\nu\Big) + \nabla_\mu\xi_\nu\Big(\frac{1}{2}\frac{D}{ds}S^{\mu\nu} - \hat{p}^{[\mu}\dot{\gamma}_s^{\nu]}\Big). \tag{3.84}$$

Here, we used the Killing equation $\nabla_{(\mu}\xi_{\nu)} = 0$ and $\nabla_\alpha\nabla_\mu\xi_\nu = -R_{\mu\nu\alpha}{}^\beta\xi_\beta$. If the right hand side of Equation (3.19) were zero, corresponding to no net force, then Equation (3.84) would give the Mathison-Papapertrou-Dixon equations for a spinning object with 4-momentum $\hat{p}^a$

and spin tensor S^{ab}

$$\frac{D\hat{p}^\mu}{ds} = \frac{1}{2}R_{\alpha\beta\nu}{}^{\mu}\hat{S}^{\alpha\beta}\dot{\gamma}_s^\nu, \tag{3.85a}$$

$$\frac{D}{ds}\hat{S}^{\mu\nu} = 2\hat{p}^{[\mu}\dot{\gamma}_s^{\nu]}. \tag{3.85b}$$

These equations must be supplemented with a spin supplementary condition [47] to fix choice of representative worldline. This amounts to fixing the freedom in picking the reference frame from which we measure angular momentum. More generally, if there is a non-zero effective generalized force $\hat{\mathcal{F}}_\xi(s) = \frac{d}{ds}\hat{\mathcal{P}}_\xi(s)$, then we can define an effective force $\hat{F}^\mu(s)$ and effective torque $\hat{N}^{\mu\nu}(s) = \hat{N}^{[\mu\nu]}(s)$ by a relation analogous to Equation (3.83)

$$\hat{\mathcal{F}}_\xi(s) = \hat{F}^\mu(s)\xi_\mu(\gamma_s) + \frac{1}{2}\hat{N}^{\mu\nu}(s)\nabla_\mu\xi_\nu(\gamma_s) \tag{3.86}$$

Comparing with Equation (3.84) now gives the general equation of motion

$$\frac{D\hat{p}^\mu}{ds} = \frac{1}{2}R_{\alpha\beta\nu}{}^{\mu}\hat{S}^{\alpha\beta}\dot{\gamma}_s^\nu + \hat{F}^\mu, \tag{3.87a}$$

$$\frac{D}{ds}\hat{S}^{\mu\nu} = 2\hat{p}^{[\mu}\dot{\gamma}_s^{\nu]} + \hat{N}^{\mu\nu}. \tag{3.87b}$$

We next move on to obtaining an expression for $\hat{F}^\mu$ in the point-particle, where we neglect the spin $\hat{S}^{\mu\nu}$ and torque $\hat{N}^{\mu\nu}$ from the right hand side of Equation (3.87). We use the on-shell expansion of the effective field, given by equation (3.81). Plugging the expansion (3.81) into

the equations of motion (3.19) we obtain

$$\frac{d}{ds}\hat{\mathcal{P}}_\xi(s) = -\int_{\Sigma_s} dS\hat{\rho}\mathcal{L}_\xi\bar{\varphi} - \int_{\Sigma_s}\int dV'\hat{\rho}\hat{\rho}'\mathcal{L}_\xi G_2^{+,R}(x,x') - \int_{\Sigma_s}\int dV'dV''\hat{\rho}\hat{\rho}'\hat{\rho}''\mathcal{L}_\xi G_3^{+,R}(x,x',x'') + O(\hat{\rho}^3) \tag{3.88}$$

We now perform covariant Taylor expansions of the Green functions $G_2^{+,R}$, $G_3^{+,R}$ and background field $\bar{\varphi}$ about γ_s, and express the results in terms of the charge multipoles defined in (3.90) below. Recall that in a curved spacetime, the covariant form of the Taylor expansion of a function $f(x)$ around a point x_0 is

$$f(x) = \sum_{n=0}^{\infty} \frac{\sigma^{\alpha_1}(x_0,x)\ldots\sigma^{\alpha_n}(x_0,x)}{n!}\,\nabla_{(\alpha_1}\ldots\nabla_{\alpha_n)}f(x)\big|_{x=x_0}\,. \tag{3.89}$$

Here, $\sigma(x_1,x_2)$ is Synge's worldfunction (See Appendix A), and σ^a is the derivative respect to its first argument, which turns out to be proportional to the tangent vector to the geodesic connecting x_1 and x_2. We now define the effective scalar charge multipoles of the object at position γ_s along the worldline by

$$\hat{I}^{\alpha_1\ldots\alpha_n}(s) = \int_{\Sigma_s} \frac{\sigma^{\alpha_1}(\gamma_s,x)\ldots\sigma^{\alpha_n}(\gamma_s,x)}{n!}\hat{\rho}(x)dS. \tag{3.90}$$

We also define the monopole as

$$\hat{q}_s = \int_{\Sigma_s}\hat{\rho}(x)dS. \tag{3.91}$$

For generic scalar models, the scalar charge might not be conserved. However, in the spirit of using this as a toy model for the gravitational self-force, we assume that $\hat{q}_s$ is independent of s.

In the point-particle limit, the contribution of all the multipoles will be suppressed rela-

tive to that of the monopole (3.91), and so we drop all these higher-order multipoles. This approximation is equivalent to simply evaluating the expression with $\hat{\rho}$ chosen to be a delta function concentrated on the representative worldline. Comparing with the decomposition (3.86), we obtain for the effective force

$$F^{\mu}(x;\gamma] = -\hat{q}\nabla^{\mu}\bar{\varphi} - \hat{q}^2\nabla^{\mu}\int G_2^{+,R}(x,\gamma_s)ds - \hat{q}^3\nabla^{\mu}\int G_3^{+,R}(x,\gamma_s.\gamma_{s'})dsds' + \dots \quad (3.92)$$

Here, the covariant derivatives only act on the first argument of each n-point function. In the point-particle limit, the 4-momentum becomes parallel to the tangent vector to the worldline, and specializing the parameter s to be the proper time gives $\hat{p}^{\mu} = m\dot{\gamma}^{\mu}$, yielding from Equation (3.87) setting $\hat{S}^{\alpha\beta} = 0$

$$\frac{D}{ds}\left(m\dot{\gamma}^{\mu}\right) = F^{\mu}. \quad (3.93)$$

Equation (3.93) can be simplified in the following way. First, apply the covariant time derivative to the mass m and the 4-velocity $\dot{\gamma}^{\mu}$. Second, contract the equation with another 4-velocity and use the fact that 4-acceleration and 4-velocity are perpendicular. We then get

$$\frac{D}{ds}m = -\dot{\gamma}_{\mu}\hat{F}^{\mu}. \quad (3.94)$$

Plugging this equation back into Equation (3.93) we get the equation for the 4-acceleration

$$m\frac{D}{ds}\dot{\gamma}^{\mu} = \left(g^{\mu\nu} + \dot{\gamma}^{\mu}\dot{\gamma}^{\nu}\right)\hat{F}_{\nu}. \quad (3.95)$$

Equation (3.95) constitutes the main result of this chapter: A finite, point-particle, effective self-force that describes the motion of self-interacting objects to third order on their effective

charge $\hat{q}$. This method can be extended to any order in $\hat{q}$. Notice that the equations of motion (3.95) are nonlocal in time. In other words, the self-force that drives the evolution of the path γ_s is itself a functional of it. In Chapter 4 we study the dynamics of nonlocal systems such as these.

This framework is also well suited for a simplified study of tidal effects for self-interacting extended objects. We leave this for future work. In the remainder of this thesis, all scalar charge multipoles are set to zero, although we go back to analyze the effect of the spin tensor in the conservative dynamics in Chapter 5.

3.6 Application to nonlinear scalar field theory in flat spacetime

In this section, we specialize the discussion so far presented in the chapter to a nonlinear scalar theory described by the action principle

$$S[\rho, \varphi] = S_\rho[\rho] + \int d^4x\sqrt{-g}\left[-\frac{1}{2}g^{\mu\nu}\nabla_\mu\varphi\nabla_\nu\varphi - V(\varphi) - \rho\varphi\right]. \tag{3.96}$$

The field operator (3.4) that determines the field equations is

$$\mathcal{E}_\varphi[\varphi] = \Box\varphi - V'(\varphi) \tag{3.97}$$

so the physical field φ and the effective field $\hat{\varphi}$ satisfy

$$\Box\varphi - V'(\varphi) = \rho, \tag{3.98a}$$

$$\Box\hat{\varphi} - V'(\hat{\varphi}) = \mathcal{S}(x). \tag{3.98b}$$

We obtain the solution for the retarded physical field in terms of retarded two and three point functions, as defined in Equation (3.78). Equation (3.79) reduces to

$$G_3^+(x,x',x'') = \frac{1}{2}\int V^{(3)}\left[\bar{\varphi}(y)\right] G_2^+(x,y) G_2^+(y,x') G_2^+(y,x'') dV_y \tag{3.99}$$

Similarly, the differential equations (3.70) for the n-point functions $\mathcal{G}_n$ reduce to

$$\hat{D}_x \mathcal{G}_2(x,y) = \delta^4(x,y) - \mathfrak{d}_2(x,y), \tag{3.100a}$$

$$\hat{D}_x \mathcal{G}_3(x,y,z) = \frac{V^{(3)}\left[\hat{\varphi}(x)\right]}{2}\mathcal{G}_2(x,y)\mathcal{G}_2(x,z) - \mathfrak{d}_3(x,y,z) \tag{3.100b}$$

$$\vdots$$

where we defined $\hat{D}_x = \Box_x - V''\left[\hat{\varphi}(x)\right]$. Equation (3.80) becomes

$$\bar{D}_x G_{2,0}(x,y) = \delta^4(x,y) - \mathfrak{d}_2(x,y), \tag{3.101a}$$

$$\bar{D}_x G_{3,0}(x,y,z) = \frac{V^{(3)}\left[\bar{\varphi}(x)\right]}{2} G_{2,0}(x,y) G_{2,0}(x,z) - \mathfrak{d}_3(x,y,z) \tag{3.101b}$$

$$\bar{D}_x G_{2,1}(x,y;z) = V^{(3)}\left[\bar{\varphi}(x)\right] G_{2,0}(x,y)\delta^4(x,z) \tag{3.101c}$$

$$\vdots$$

where we defined $\bar{D}_x = \Box_x - V''\left[\bar{\varphi}(x)\right]$. The functions $G_{2,0}$ and $G_{2,1}$ are determined if we use the Hadamard construction truncated at second order. If a solution to Equation (3.101b) satisfying all other conditions can be found, then Equation (5.54) determines the regularized second-order self-force for a nonlinear scalar field theory in flat spacetime.

El tiempo es la sustancia de que estoy hecho.

El tiempo es un río que me arrebata, pero yo soy el río;

es un tigre que me destroza, pero yo soy el tigre;

es un fuego que me consume, pero yo soy el fuego.

Jorge Luis Borges

4

Hamiltonian formulation of conservative nonlocal in time dynamical systems (Adapted from [2])

4.1 Introduction

In this chapter, we study a class of finite-dimensional dynamical systems with non-local-in-time interactions. Such systems can be described in terms of action functionals of paths in

phase space, where the action contains multiple integrals with respect to time. Their equations of motion are integro-differential equations as opposed to the ordinary differential equations characteristic of Hamiltonian dynamical systems. A simple example of such an integro-differential equation is

$$\ddot{x}(t) = f(x,t) + \int_{-\infty}^{\infty} K(t,t')x(t')dt'. \tag{4.1}$$

Here $f(x,t)$ is the local piece of the force and the integral is a non-local-in-time force that is a functional of the position $x(t')$. The equations of motion (3.95), encountered in the previous chapter, are an example of this type of dynamics.

Non-local-in-time interactions generally arise when one "integrates out" some of the degrees of freedom of a system, giving rise to a non-local interaction between the remaining degrees of freedom. Our motivation for studying such systems is to understand the conservative/dissipative split of the self-force dynamics obtained in Equation (3.95) in the previous chapter. Non-local interactions are not unique to the small mass-ratio approximation, they appear at 4PN order in post-Newtonian approximations of two-body dynamics as well [12, 48, 49]. They are also useful for the description of cracks and other non-local deformations on materials [50]. Non-local-in-time interactions are sometimes parametrized in terms of frequency dependent coefficients, such as the electric permittivity and susceptibility [51]. They also appear in Fokker-Wheeler-Feynman electrodynamics [52].

In the case of ordinary differential equations obtained from a Hamiltonian system, standard existence and uniqueness theorems [53] state that the space of solutions can be parametrized by initial data, i.e. points in phase space. When non-local-in-time interactions are included, however, it is not clear how to obtain a simple parametrization of the space of solutions [54].

However, as is well known, when non-local-in-time interactions are treated perturbatively, the resulting dynamics can be cast as a local dynamical system, order by order. However, it is less well known under what circumstances this local dynamical system admits a Hamiltonian description at each order. In this chapter, we derive the existence of such Hamiltonian description for a broad class of non-local-in-time action principles (Equation (4.8) below).

This chapter is organized as follows: In section 4.2, we introduce a broad class of nonlocal in time dynamical systems and derive their equations of motion. We then treat the non-localities perturbatively to obtain local equations of motion order by order. In Section 4.3, we prove that the local dynamics admits a local Hamiltonian description to any order in the perturbations. We provide explicit expressions for the Hamiltonian and symplectic form up to $\mathcal{N}^{\text{th}}$ order in terms of the $(\mathcal{N}-1)^{\text{th}}$ Hamiltonian flow. In Section 4.4, we prove that, up to any order in perturbation theory, there exists a diffeomorphism on phase space that puts the symplectic form into canonical form. We then apply this result up to second order and give explicit expressions for the diffeomorphism and the resulting Hamiltonian. Section 4.5 specializes to the dynamics of binary systems in the post-Newtonian approximation, where non-local effects start at fourth order [48, 49]. In previous work by the author of this thesis [3, 4] (See Section 5.3), the conservative piece of the dynamics of a binary system in the small mass ratio regime was recast as a local Hamiltonian system to first order in mass-ratio. The results in this chapter generalize the methods used in Refs. [3, 4] to a more general class of non-local systems and to arbitrary high orders in perturbation theory.

The formalism of this chapter is an extension of work done by Llosa and Vives in [54]. The relation between their work and the results of this chapter is discussed in Appendix C.

4.2 Dynamical systems described by non-local action principles

We start this section by reviewing the description of phase-space flows. Consider a phase space Γ with coordinates

$$Q^A = (q^\mu, p_\mu), \tag{4.2}$$

and a symplectic form $\Omega_0 = \delta p_\mu \wedge \delta q^\mu$. We define a flow on phase space $X_s(Q) : \mathbb{R} \times \Gamma \to \Gamma$ which takes any point $Q \in \Gamma$ into $X_s(Q) \in \Gamma$. The flow is required to be the identity map at $s = 0$

$$X_0(Q) = Q, \tag{4.3}$$

and to satisfy the composition rule

$$X_s(X_{s'}(Q)) = X_{s+s'}(Q), \tag{4.4}$$

for all $s, s' \in \mathbb{R}$. A flow $X_s(Q)$ on phase space will be determined by a vector field $\boldsymbol{V} = V^A \partial_A$ according to

$$\frac{dX_s^A(Q)}{ds} = V^A[X_s(Q)]. \tag{4.5}$$

If we specialize equation (4.5) to $s = 0$ we get

$$\left.\frac{dX_s^A(Q)}{ds}\right|_{s=0} = V^A(Q), \tag{4.6}$$

so the flow is determined by its derivative at $s = 0$. Throughout this chapter, we will parametrize and characterize flows by their derivatives (4.6) at $s = 0$ with the understanding that the full flows are obtained by solving equation (4.5).

We will consider dynamical systems described by non-local action functionals of paths X_s of the form

$$S[X] = \int p_\mu dq^\mu - \int H_0(X_s)ds + S_{nl}[X]. \tag{4.7}$$

Here, $H_0(Q)$ is a local Hamiltonian function on phase space and the non-local piece of the action is

$$S_{nl}[X] = -\sum_{n=2}^{N} \frac{\varepsilon_n}{n} \int ds_1 \dots ds_n \mathcal{G}_n(X_{s_1}, \dots, X_{s_n}; s_2 - s_1, \dots, s_n - s_1), \tag{4.8}$$

where $\mathcal{G}_n$ is some n-point function $\mathcal{G}_n : \Gamma^n \times \mathbb{R}^{n-1} \to \mathbb{R}$. Here ε_n is a formal expansion parameter used to keep track of orders in the non-local action and N is a finite but otherwise arbitrary positive integer. Note that because the n-point function $\mathcal{G}_n$ is integrated n times, the non-local action will automatically pick out its fully symmetric piece, so that without loss of generality we can assume that $\mathcal{G}_n$ satisfies

$$\mathcal{G}_n(X_{s_1}, \dots, X_{s_n}; \sigma_{12}, \dots, \sigma_{1n}) = \mathcal{G}_n(X_{s_{p_1}}, \dots, X_{s_{p_n}}; \sigma_{p_1p_2}, \dots, \sigma_{p_1p_n}) \tag{4.9}$$

for all $(s_1, \dots, s_n)$. Here $\sigma_{ij} = s_j - s_i$ for short and $\{p_i\}$ is any permutation of the integers from 1 to n. We will also assume that the n-point functions satisfy asymptotic fall-off conditions given in detail in equation (4.15) below.

We will write the equations of motion in terms of a function $\Phi(Q, Q', [X])$ which is a local

function of two points Q and Q' in phase space in its first two arguments and a functional of a trajectory X_s which passes through Q' at $s = 0$ in its last argument. The definition of Φ is

$$\Phi(Q, Q', [X]) = \sum_{n=2}^{N} \varepsilon_n \int ds_2 \dots ds_n \mathcal{G}_n\big(Q, X_{s_2}(Q'), \dots, X_{s_n}(Q'); s_2, \dots, s_n\big). \tag{4.10}$$

The equations of motion are obtained by varying the action functional (4.7) with respect to the trajectory X. The variation of the nth term in the non-local piece will give n contributions with derivatives acting on each of the first n arguments of $\mathcal{G}_n$. From the property (4.9), it follows that all these contributions coincide, so we can add them up. The final result is a factor of n times the derivative with respect to the first argument of Φ. The resulting equations of motion are

$$\Omega^0_{AB} \frac{dX^B_s}{ds} = \left[\frac{\partial}{\partial Q^A} H_0(Q) + \frac{\partial}{\partial Q^A}\Phi(Q, Q', [X])\right]_{Q'=Q=X_s}. \tag{4.11}$$

Here, the notation $Q' = Q = X_s$ means that the derivative of Φ with respect to Q^A is taken, and then the result is evaluated at $Q = Q' = X_s$, and the third argument of Φ is evaluated on the path $X_{s'}$ that starts at $s' = s$.

4.2.1 Local dynamical systems obtained by treating non-localities perturbatively

Equation (4.11) is an integro-differential system of equations for the trajectories X_s on phase space, as opposed to a differential system of equations that depend locally on a point Q, as is the case for Hamilton's equations derived from action principles without non-localities. Because of this property, solutions will generally not be parametrized by initial data Q. In

fact, the space of initial data required to determine solutions of integro-differential systems of equations can be, in general, infinite dimensional and require derivatives of x^μ and p_μ with respect to time of all orders [54].

However, if we take the non-local contribution to the action to be small we can treat the problem perturbatively. We define a sequence of phase space flows $\bar{X}_s^{(\mathcal{N})}(Q)$ by induction as follows. The zeroth order flow $\bar{X}_s^{(0)}(Q)$ is generated by the Hamiltonian H_0, with all non-local terms in equation (4.11) dropped. Then, we can evaluate the functional dependence of Φ in equation (4.11) on the zeroth order flow and define the first order flow $\bar{X}_s^{(1)}(Q)$ by

$$\Omega^0_{AB}\left.\frac{d\bar{X}_s^{(1)B}(Q)}{ds}\right|_{s=0} = \frac{\partial}{\partial Q^A}H_0 + \left[\frac{\partial}{\partial Q^A}\Phi(Q,Q',[\bar{X}^{(0)}])\right]_{Q'=Q}. \tag{4.12}$$

This process can be repeated to any desired order to define the $\mathcal{N}^{\text{th}}$ order flow in terms of the $(\mathcal{N}-1)^{\text{th}}$ flow as

$$\Omega^0_{AB}\left.\frac{d\bar{X}_s^{(\mathcal{N})B}(Q)}{ds}\right|_{s=0} = \frac{\partial}{\partial Q^A}H_0 + \left[\frac{\partial}{\partial Q^A}\Phi(Q,Q',[\bar{X}^{(\mathcal{N}-1)}])\right]_{Q'=Q}. \tag{4.13}$$

Equation (4.13) is a set of ordinary differential equation which determines the $\mathcal{N}^{\text{th}}$ flow, once the $(\mathcal{N}-1)^{\text{th}}$ flow is specified. Hence all the flows are determined by induction*.

The flow determined by equation (4.13) agrees with the exact flow determined by equation (4.11) up to corrections of order $O(\varepsilon_1^{q_1}\times\varepsilon_2^{q_2}\times\cdots\times\varepsilon_N^{q_N})$ with $\sum_{n=1}^N q_i = \mathcal{N}+1$. For simplicity, when we expand the Hamiltonian and symplectic form explicitly bellow, we will introduce a

*As is well known, perturbative expansions of this form can break down after long timescales when there are dissipative effects present. Here, we are concerned only with conservative dynamics and so we can neglect this issue.

formal expansion parameter ε such that

$$O(\varepsilon^{\mathcal{N}+1}) \equiv O(\varepsilon_1^{q_1} \times \varepsilon_2^{q_2} \times \cdots \times \varepsilon_N^{q_N}) \,, \ \sum_{n=1}^{N} q_i = \mathcal{N}+1. \tag{4.14}$$

We will also assume that the sequence of flows $\bar{X}_s^{(\mathcal{N})}(Q)$ are such that the n-point functions $\mathcal{G}_n$ introduced in the non-local action principle in equation (4.8) satisfy the following property: For any $j \in \{1, 2, \ldots, n\}$ and with all s_k with $k \neq j$ fixed, the limit when $s_j \to \pm\infty$ of the n-point function $\mathcal{G}_n$ evaluated on the flow $\bar{X}_s^{(\mathcal{N})}$ is zero

$$\lim_{s_j \to \pm\infty} \mathcal{G}_n\big(\bar{X}_{s_1}^{(\mathcal{N})}, \ldots, \bar{X}_{s_j}^{(\mathcal{N})}, \ldots, \bar{X}_{s_n}^{(\mathcal{N})}; s_2 - s_1, \ldots, s_j - s_1, \ldots, s_n - s_1\big) = 0. \tag{4.15}$$

4.3 Local Hamiltonian description

In this section, we will obtain a local Hamiltonian description for the $\mathcal{N}^{\text{th}}$ order flow, in terms of the $(\mathcal{N}-1)^{\text{th}}$ order flow, which we assume to be known. We define the Hamiltonian and symplectic form in this subsection and derive their equivalence to the system (4.13) in the next subsection.

Given a phase space flow $X_s(Q)$ and a point Q in Γ, we define a function

$$\Psi(Q,[X]) = \frac{1}{2}\sum_{n=2}^{N} \varepsilon_n \int ds_1 \ldots ds_n \chi(s_1, \ldots, s_n) \frac{\partial}{\partial s_1} \mathcal{G}(X_{s_1}(Q) \ldots, X_{s_n}(Q); s_2 - s_1, \ldots, s_n - s_1) \tag{4.16}$$

where

$$\chi(s_1, \ldots, s_n) = \frac{\operatorname{sgn}(s_1) - \operatorname{sgn}(s_2) - \cdots - \operatorname{sgn}(s_n)}{2}. \tag{4.17}$$

Here the partial derivative notation $\partial/\partial s_1$ indicates that the derivative acts only on the explicit dependence of the n-point function on s_1 in its last $n-1$ arguments and not on the implicit dependence that arises through X_{s_1}.

We now define the local Hamiltonian function in terms of the $(\mathcal{N}-1)^{\text{th}}$ flow as

$$H^{(\mathcal{N})}(Q) = H_0(Q) + \Phi^{(\mathcal{N})}(Q) + \Psi^{(\mathcal{N})}(Q), \tag{4.18}$$

where

$$\Phi^{(\mathcal{N})}(Q) = \Phi(Q, Q, [\bar{X}^{(\mathcal{N}-1)}]), \tag{4.19a}$$

$$\Psi^{(\mathcal{N})}(Q) = \Psi(Q, [\bar{X}^{(\mathcal{N}-1)}]). \tag{4.19b}$$

We also define a new function of n points on phase space as

$$K_n^{(\mathcal{N})}(Q_1, \ldots, Q_n) = \varepsilon_n \int ds_1 \ldots ds_n \chi(s_1, \ldots, s_n) \mathcal{G}_n\big(\bar{X}_{s_1}^{(\mathcal{N}-1)}(Q_1), \ldots, \bar{X}_{s_n}^{(\mathcal{N}-1)}(Q_n); s_2 - s_1, \ldots, s_n - s_1\big). \tag{4.20}$$

Note that the subscript n labels the number of arguments in the n-point function $\mathcal{G}_n$ while the superscript $(\mathcal{N})$ denotes an object constructed from the $(\mathcal{N}-1)^{\text{th}}$ order flow and contains terms of order $O(\varepsilon^{\mathcal{N}})$ and lower. Using the definition (4.20) we define the local sym-

plectic form

$$\Omega^{(\mathcal{N})} = \Omega_0 + \Delta\Omega^{(\mathcal{N})}, \tag{4.21a}$$

$$\Omega_0 = \delta p_\mu \wedge \delta q^\mu, \tag{4.21b}$$

$$\Delta\Omega^{(\mathcal{N})}_{AB}(Q) = \left[\sum_{n=2}^{N}\sum_{m=2}^{n} \frac{\partial^2}{\partial Q_1^{[A}\partial Q_m^{B]}} K_n^{(\mathcal{N})}(Q_1,\ldots,Q_n)\right]_{\{Q_j\}=Q} \tag{4.21c}$$

where $\{Q_j\} = Q$ means that we evaluate at coincidence $Q_1 = Q_2 = \cdots = Q_n = Q$. Here, brackets denote antisymmetrization $\Omega_{[AB]} = \frac{1}{2}\big(\Omega_{AB} - \Omega_{BA}\big)$.

Both $H^{(\mathcal{N})}$ and $\Omega^{(\mathcal{N})}$ can be expanded perturbatively using the formal expansion parameter (4.14) as

$$H^{(\mathcal{N})} = H_0 + \sum_{r=1}^{\mathcal{N}} \varepsilon^r H^{[r]} \tag{4.22a}$$

$$\Omega^{(\mathcal{N})} = \Omega_0 + \sum_{r=1}^{\mathcal{N}} \varepsilon^r \Delta\Omega^{[r]} \tag{4.22b}$$

where a superscript $[r]$ indicates a term that is exclusively $O(\varepsilon^r)$, as opposed to a superscript $(\mathcal{N})$ which indicates a term that contains contributions of order $O(\varepsilon^{\mathcal{N}})$ and lower.

4.3.1 Derivation of Hamilton formulation

In this subsection we will prove that the Hamiltonian function (4.18) equipped with the symplectic form (4.21) reproduces the perturbative local dynamical system (4.13) up to corrections of order $O(\varepsilon^{\mathcal{N}+1})$.

The Hamiltonian function (4.18) equipped with the symplectic form (4.21) determines

the flow

$$\left[\Omega^0_{AB} + \Delta\Omega^{(\mathcal{N})}_{AB}\right] \left.\frac{d\bar{X}^{(\mathcal{N})}_s}{ds}\right|_{s=0} = \frac{\partial}{\partial Q^A}\Big[H_0 + \Phi^{(\mathcal{N})} + \Psi^{(\mathcal{N})}\Big]. \tag{4.23}$$

First, note that since we want the equations of motion to be accurate up to corrections of order $O(\varepsilon^{\mathcal{N}+1})$, we can drop higher order corrections in the second term in the left side of equation (4.23)

$$\Delta\Omega^{(\mathcal{N})}_{AB} \left.\frac{d\bar{X}^{(\mathcal{N})}_s}{ds}\right|_{s=0} = \Delta\Omega^{(\mathcal{N})}_{AB} \left.\frac{d\bar{X}^{(\mathcal{N}-1)}_s}{ds}\right|_{s=0} + O(\varepsilon^{N+1}) \tag{4.24}$$

where we replaced $\bar{X}^{(\mathcal{N})}$ with $\bar{X}^{(\mathcal{N}-1)}$ since $\Delta\Omega^{(\mathcal{N})}$ is $O(\varepsilon)$. We will calculate the first term in the right hand side of equation (4.24) in a series of steps. First, the contraction $\Delta\Omega^{(\mathcal{N})}_{AB}\ d\bar{X}^{(\mathcal{N}-1)}_s/ds\big|_{s=0}$ will have two pieces coming from the antisymetrization of the indices AB in equation (4.21c). The first one is

$$\left.\frac{d\bar{X}^{(\mathcal{N}-1)B}_s}{ds}\right|_{s=0} \times \left[\frac{\partial^2}{\partial Q^A_1 \partial Q^B_m} K^{(\mathcal{N})}_n(Q_1, \ldots, Q_n)\right]_{\{Q_j\}=Q}. \tag{4.25}$$

The derivative of $\bar{X}^{(\mathcal{N}-1)B}_s$ is evaluated at Q but we are allowed to move it inside the brackets and evaluate it at Q_m, since the bracket is evaluated at coincidence $\{Q_j\} = Q$. Using property (4.4), the contraction $d\bar{X}^B_s/ds\big|_{s=0}\ \partial/\partial Q^B$ acting on any function $f(\bar{X}_s(Q), s)$ will create a total derivative d/ds, minus a correction $\partial/\partial s$ due to the explicit time dependence of f

$$\begin{aligned} \frac{d\bar{X}^A}{ds}\frac{\partial}{\partial Q^A} f\big(\bar{X}_s(Q), s\big) &= \left.\frac{d}{d\Delta s}\right|_{\Delta s=0} f\big(\bar{X}_{s+\Delta s}(Q), s\big) \\ &= \left[\frac{d}{ds} - \frac{\partial}{\partial s}\right] f\big(\bar{X}_s(Q), s\big). \end{aligned} \tag{4.26}$$

Using the identity (4.26) in equation (4.25) we get

$$\Big\{\frac{\partial}{\partial Q_1^A}\int ds_1\dots ds_n\,\chi(s_1,\dots,s_n)\big[\frac{d}{ds_m}-\frac{\partial}{\partial s_m}\big]\times \\ \times\,\mathcal{G}_n\big(\bar{X}^{(\mathcal{N}-1)}_{s_1}(Q_1),\dots,\bar{X}^{(\mathcal{N}-1)}_{s_n}(Q_n);s_2-s_1,\dots,s_n-s_1\big)\Big\}_{\{Q_j\}=Q}. \tag{4.27}$$

We integrate by parts the total derivative d/ds_m, use property (4.15) to throw away boundary terms, relabel $s_m \leftrightarrow s_1$ and use the properties (4.9) and

$$\frac{d}{ds_j}\chi(s_1,\dots,s_n)=\begin{cases} +\delta(s_1) \ \text{ for}\, j=1 \\ -\delta(s_j) \ \text{ for}\, j=2,\dots,n \end{cases} \tag{4.28}$$

to get

$$\Big[\frac{\partial}{\partial Q_m^A}\int ds_2\dots ds_n\mathcal{G}_n(Q_1,\bar{X}^{(\mathcal{N}-1)}_{s_2}(Q_2)\dots,\bar{X}^{(\mathcal{N}-1)}_{s_n}(Q_n);s_2,\dots,s_n)\Big]_{\{Q_j\}=Q}. \tag{4.29}$$

Note that once we sum over n and m this term will give the $n-1$ last derivatives of the n-point function $\mathcal{G}_n$ in equation (4.10), which can be expressed as

$$\Big[\frac{\partial}{\partial Q^{A'}}\Phi(Q,Q',[\bar{X}^{(\mathcal{N}-1)}])\Big]_{Q'=Q}, \tag{4.30}$$

where the prime index in $Q^{A'}$ means that the derivative acts on Q' but not Q. Regarding the term proportional to the partial derivative $\partial/\partial s_m$in equation (4.27), note that we can pull

the sum $\sum_{m=2}^{n}$ from equation (4.21c) inside K to get

$$\begin{aligned}
&\Big[\frac{\partial}{\partial Q_1^A}\int ds_1\dots ds_n\,\chi(s_1,\dots,s_n)\Big[-\sum_{m=2}^{n}\frac{\partial}{\partial s_m}\Big]\times\\
&\times\,\mathcal{G}_n\big(\bar{X}_{s_1}^{(\mathcal{N}-1)}(Q_1),\dots,\bar{X}_{s_n}^{(\mathcal{N}-1)}(Q_n);s_2-s_1,\dots,s_n-s_1\big)\Big]_{\{Q_j\}=Q}\\
&=\Big[\frac{\partial}{\partial Q_1^A}\int ds_1\dots ds_n\,\chi(s_1,\dots,s_n)\frac{\partial}{\partial s_1}\times\\
&\times\,\mathcal{G}_n\big(\bar{X}_{s_1}^{(\mathcal{N}-1)}(Q_1),\dots,\bar{X}_{s_n}^{(\mathcal{N}-1)}(Q_n);s_2-s_1,\dots,s_n-s_1\big)\Big]_{\{Q_j\}=Q}
\end{aligned}\tag{4.31}$$

where we used a chain rule to replace the derivatives respect to all the s_m with a derivative respect to s_1.

Now, we move on to the other piece of the contraction $\Delta\Omega_{AB}^{(\mathcal{N})}\,d\bar{X}_s^{(\mathcal{N}-1)}/ds\big|_{s=0}$ coming from the antisymmetrization of indices AB in equation (4.21c)

$$-\left.\frac{d\bar{X}_s^{(\mathcal{N}-1)B}}{ds}\right|_{s=0}\times\left[\frac{\partial^2}{\partial Q_m^A\partial Q_1^B}K_n^{(\mathcal{N})}(Q_1,\dots,Q_n)\right]_{\{Q_j\}=Q}.\tag{4.32}$$

The contraction once again will give a total derivative d/ds_1 minus a correction $\partial/\partial s_1$ due to the explicit time dependence on K. Integrating by parts the total derivative recovers equation (4.29). The term proportional to the partial derivative is

$$\Big[\frac{\partial}{\partial Q_m^A}\int ds_1\dots ds_n\,\chi(s_1,\dots,s_n)\frac{\partial}{\partial s_1}\mathcal{G}_n\big(\bar{X}_{s_1}^{(\mathcal{N}-1)}(Q_1),\dots,\bar{X}_{s_n}^{(\mathcal{N}-1)}(Q_n);s_2-s_1,\dots,s_n-s_1\big)\Big]_{Q_n=\dots=Q}.\tag{4.33}$$

Now we apply the sum over m to this last term and add it to the term in equation (4.31) to create a derivative ∂_A acting on every argument of K. Putting equations (4.29), (4.31) and

(4.33) together we get that

$$\Delta\Omega^{(\mathcal{N})}_{AB}\left.\frac{d\bar{X}^{(\mathcal{N})}_s}{ds}\right|_{s=0} = \left[\frac{\partial}{\partial Q^{A'}}\Phi(Q,Q',[\bar{X}^{(\mathcal{N}-1)}])\right]_{Q'=Q} + \partial_A\Psi^{(\mathcal{N})}(Q) + O(\varepsilon^{\mathcal{N}+1}). \quad (4.34)$$

Plugging this into equation (4.23) we see that the first term on the right hand side of equation (4.34) cancels all the extra derivatives respect to the last (n-1) arguments of the n-point function in $\partial_A\Phi$ in equation (4.23). The last term in the right hand side of equation (4.34) cancels the term $\partial_A\Psi^{(\mathcal{N})}$ in equation (4.23) and we recover equation (4.13) up to corrections of order $O\left(\varepsilon^{\mathcal{N}+1}\right)$ as desired.

4.4 Alternative formulation of local Hamiltonian system

In this section we prove that, up to any order in $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n$, there exists a diffeomorphism in phase space that puts the symplectic form (4.21) in canonical form. We then apply this result up to second order and give explicit expressions for the diffeomorphism and the resulting Hamiltonian. We use boldface $\boldsymbol{V} = V^A\partial_A$ for vectors and tildes $\tilde{\omega} = \omega_A dQ^A$ for 1-forms. Indices will be raised and lowered by contraction with the first index on the zeroth order symplectic form Ω_{AB}.

We consider a one-parameter family of diffeomorphisms $\phi(\varepsilon) : \Gamma \to \Gamma$ that transform the $\mathcal{N}^{\text{th}}$ order Hamiltonian system $(\Omega_0 + \Delta\Omega^{(\mathcal{N})}, H^{(\mathcal{N})})$ to an equivalent Hamiltonian system

$$(\phi_*\Omega_0 + \phi_*\Delta\Omega^{(\mathcal{N})}, \phi_* H^{(\mathcal{N})}) \quad (4.35)$$

where ϕ_* is the pullback[†] defined by the diffeomorphism $\phi(\varepsilon)$.

We now specialize the diffeomorphism ϕ to make the new Hamiltonian system take the form $(\Omega_0, \phi_* H^{(\mathcal{N})})$, i.e. to make the transformed symplectic form coincide with the original zeroth order symplectic form

$$\phi_* \Omega^{(\mathcal{N})} = \Omega_0 + O(\varepsilon^{\mathcal{N}+1}). \tag{4.36}$$

First, note that we can express the perturbation (4.21) to the symplectic form $\Delta\Omega^{(\mathcal{N})}$ as an exact form. We define the 1-form

$$\xi_A^{(\mathcal{N})} = -\frac{1}{2} \sum_{n=2}^{N} \Big[\frac{\partial}{\partial Q_1^A} K_n^{(\mathcal{N})}(Q_1, \dots, Q_n) \Big]_{\{Q_j\}=Q} \tag{4.37}$$

such that the perturbation to the symplectic form is

$$\begin{aligned} \Delta\Omega_{AB}^{(\mathcal{N})}(Q) &= \big(d\tilde{\xi}^{(\mathcal{N})}\big)_{AB} \\ &= \partial_A \xi_B^{(\mathcal{N})} - \partial_B \xi_A^{(\mathcal{N})}. \end{aligned} \tag{4.38}$$

The 1-form $\tilde{\xi}^{(\mathcal{N})}$ is sourced by the $(\mathcal{N}-1)^{\text{th}}$ order flow and is accurate up to corrections of order $O(\varepsilon^{\mathcal{N}+1})$. Plugging equation (4.38) into the symplectic form (4.21) we get

$$\Omega^{(\mathcal{N})} = \Omega_0 + d\tilde{\xi}^{(\mathcal{N})} + O(\varepsilon^{\mathcal{N}+1}). \tag{4.39}$$

[†]As is well known, these transformations can be seen from a passive or an active viewpoint. The passive viewpoint considers the transformation to be a coordinate transformation, keeping all fields fixed. The active viewpoint considers the transformation as a field redefinition instead, with all coordinates unchanged. Both viewpoints are equivalent, but in this chapter we adopt the active viewpoint for clarity.

Now, consider a one-parameter family of diffemorphisms $\phi(\varepsilon) : \Gamma \to \Gamma$. We parametrize this diffeomorphism up to order $\mathcal{N}$ by $\mathcal{N}$ vector fields ζ_i with $i = 1, \ldots, \mathcal{N}$ as

$$\phi(\varepsilon) = \mathcal{D}_{\zeta_{\mathcal{N}}}(\varepsilon^{\mathcal{N}}) \circ \mathcal{D}_{\zeta_{\mathcal{N}-1}}(\varepsilon^{\mathcal{N}-1}) \circ \cdots \circ \mathcal{D}_{\zeta_1}(\varepsilon) \left[1 + O(\varepsilon^{\mathcal{N}+1})\right], \tag{4.40}$$

where the mapping $\mathcal{D}_\zeta(\varepsilon)$ moves any point ε units along the vector field ζ. The pullback ϕ_* can be expressed in terms of Lie derivatives as

$$\begin{aligned} \phi_* = 1 &+ \varepsilon \mathfrak{L}_{\zeta_1} + \frac{\varepsilon^2}{2} \mathfrak{L}_{\zeta_1} \mathfrak{L}_{\zeta_1} + \frac{\varepsilon^3}{6} \mathfrak{L}_{\zeta_1} \mathfrak{L}_{\zeta_1} \mathfrak{L}_{\zeta_1} \\ &+ \varepsilon^2 \mathfrak{L}_{\zeta_2} + \varepsilon^3 \mathfrak{L}_{\zeta_1} \mathfrak{L}_{\zeta_2} \\ &+ \varepsilon^3 \mathfrak{L}_{\zeta_3} + \ldots \end{aligned} \tag{4.41}$$

We want this diffeomorphism to make the symplectic form coincide with Ω_0 up to $\mathcal{N}^{\text{th}}$ order, as in equation (4.36). Combining equations (4.36) and (4.41) and inverting the pullback perturbatively, we can invert equation (4.36) to get

$$\begin{aligned} \Omega^{(\mathcal{N})} = \Omega_0 &- \varepsilon \mathfrak{L}_{\zeta_1} \Omega_0 + \frac{\varepsilon^2}{2} \mathfrak{L}_{\zeta_1} \mathfrak{L}_{\zeta_1} \Omega_0 - \frac{\varepsilon^3}{6} \mathfrak{L}_{\zeta_1} \mathfrak{L}_{\zeta_1} \mathfrak{L}_{\zeta_1} \Omega_0 \\ &- \varepsilon^3 \mathfrak{L}_{\zeta_2} \Omega_0 + \varepsilon^3 \mathfrak{L}_{\zeta_2} \mathfrak{L}_{\zeta_1} \Omega_0 - \varepsilon^3 \mathfrak{L}_{\zeta_3} \Omega_0 + \ldots \end{aligned} \tag{4.42}$$

We expand the 1-form (4.37) in powers of the formal parameter ε defined in equation (4.14)

$$\tilde{\xi}^{(\mathcal{N})} = \sum_{r=1}^{\mathcal{N}} \varepsilon^r \tilde{\xi}^{[r]} \tag{4.43}$$

where $\tilde{\xi}^{[r]}$ is the piece of $\tilde{\xi}^{(\mathcal{N})}$ of order $O(\varepsilon^r)$ and can be obtained by expanding the flow

$\bar{X}_s^{(\mathcal{N}-1)}(Q)$ in the definition (4.20) and plugging the expansion back into equation (4.37). We plug the expansion (4.43) into (4.39) and then into equation (4.42) and equate coefficients of powers of ε on both sides to obtain

$$\mathfrak{L}_{\zeta_1}\Omega_0 = -d\tilde{\xi}^{[1]}, \tag{4.44a}$$

$$\mathfrak{L}_{\zeta_2}\Omega_0 = -d\tilde{\xi}^{[2]} + \frac{1}{2}\mathfrak{L}_{\zeta_1}\mathfrak{L}_{\zeta_1}\Omega_0, \tag{4.44b}$$

$$\mathfrak{L}_{\zeta_3}\Omega_0 = -d\tilde{\xi}^{[3]} + \mathfrak{L}_{\zeta_2}\mathfrak{L}_{\zeta_1}\Omega_0 - \frac{1}{6}\mathfrak{L}_{\zeta_1}\mathfrak{L}_{\zeta_1}\mathfrak{L}_{\zeta_1}\Omega_0, \tag{4.44c}$$

$$\vdots$$

Using Cartan's magic formula and the fact that the symplectic form Ω_0 is closed, we can prove that the Lie derivative of the zeroth order symplectic form Ω_0 with respect to any vector field $\boldsymbol{V}$ is exact

$$\begin{aligned}\mathfrak{L}_{\boldsymbol{V}}\Omega_0 &= i_{\boldsymbol{V}}d\Omega_0 + d\big(i_{\boldsymbol{V}}\Omega_0\big)\\ &= d\big(i_{\boldsymbol{V}}\Omega_0\big)\\ &= d\tilde{V}.\end{aligned} \tag{4.45}$$

Here $i_{\boldsymbol{V}}\tilde{\omega}$ is the interior product, which contracts $\boldsymbol{V}$ with the first entry of any differential form it acts on. In the last line of equation (4.45) we used the zeroth order symplectic form to lower the index $V_B = V^A\Omega^0_{AB}$. Using identity (4.45), equation (4.44a) becomes

$$d\tilde{\zeta}_1 = -d\tilde{\xi}^{[1]}. \tag{4.46}$$

From the definition of $\tilde{\xi}$ in (4.37) we obtain the solution

$$\begin{aligned}\zeta_1^A =& \Omega_0^{AB}\xi_B^{[1]} \\ =& -\frac{1}{2}\Omega_0^{AB}\sum_{n=2}^{N}\Big[\frac{\partial}{\partial Q_1^B}K_n^{(1)}(Q_1,\ldots,Q_n)\Big]_{\{Q_j\}=Q}.\end{aligned} \tag{4.47}$$

Now, we use the identity (4.45) in equation (4.44b) to get

$$\begin{aligned}d\tilde{\zeta}_2 &= -d\tilde{\xi}^{[2]} + \frac{1}{2}\mathcal{L}_{\zeta_1}d\tilde{\zeta}_1 \\ &= -d\tilde{\xi}^{[2]} + d\big(\frac{1}{2}\mathcal{L}_{\zeta_1}\tilde{\zeta}_1\big).\end{aligned} \tag{4.48}$$

A solution of this equation for the second order vector field is

$$\zeta_2^A = \Omega_0^{AB}\xi_B^{(\mathcal{N},2)} - \frac{1}{2}\Omega_0^{AB}\big(\mathcal{L}_{\zeta_1}\tilde{\zeta}_1\big)_B. \tag{4.49}$$

It is easy to see that using equation (4.44) and the identity (4.45) and the fact that exterior derivatives and Lie derivatives commute, we can find solutions for the vector fields ζ_i that parametrize the diffeomorphism $\phi(\varepsilon)$ up to any order.

4.4.1 Transformed second order Hamiltonian

We now compute the transformed Hamiltonian function (4.35) starting with the expression (4.18) for the $\mathcal{N}^{\text{th}}$ order Hamiltonian $H^{(\mathcal{N})}$ and specializing to second order for simplicity.

The second order Hamiltonian will be expressed in terms of following functions

$$\Phi^{(2)}(Q) = \Phi(Q, Q, [\bar{X}^{(1)}]), \tag{4.50a}$$

$$\Psi^{(2)}(Q) = \Psi(Q, [\bar{X}^{(1)}]) \tag{4.50b}$$

where the right hand side terms were defined in equations (4.10), (4.16) and (4.19). Both $\Phi^{(2)}$ and $\Psi^{(2)}$ in equations (4.50a) and (4.50b) have contributions of order $O(\varepsilon)$ and $O(\varepsilon^2)$.

We now specialize the order of the expansion of the diffeomorphism (4.40) to second order. Its action on the Hamiltonian will produce a new Hamiltonian $\hat{H}^{(2)} = \phi_* H^{(2)}$ given by

$$\hat{H}^{(2)} = \left(1 + \varepsilon \mathcal{L}_{\zeta_1} + \varepsilon^2 \mathcal{L}_{\zeta_2} + \frac{1}{2}\varepsilon^2 \mathcal{L}_{\zeta_1}\mathcal{L}_{\zeta_1}\right) H^{(2)} + O(\varepsilon^3). \tag{4.51}$$

We can simplify this expression using the results (4.47) and (4.49) for ζ_1 and ζ_2. We can also use equation (4.43) to regroup $\varepsilon\xi^A_{[1]} + \varepsilon^2 \xi^A_{[2]} = \xi^A_{(2)} + O(\varepsilon^3)$. The result is

$$\begin{aligned}\hat{H}^{(2)} = H^{(2)} &- \xi^A_{(2)}\partial_A H^{(2)} - \frac{1}{2}\Omega_0^{AB}\left(\mathcal{L}_{\xi_{(2)}}\tilde{\xi}_{(2)}\right)_B \partial_A H^{(2)} \\ &+ \frac{1}{2}\xi^A_{(2)}\partial_A\left(\xi^B_{(2)}\partial_B H^{(2)}\right) + O(\varepsilon^3).\end{aligned} \tag{4.52}$$

In order to calculate $\hat{H}$ we'll make frequent use of the equation of motion (4.34), specialized to $\mathcal{N} = 2$, which becomes

$$\Delta\Omega^{(2)}_{AB}\left.\frac{d\bar{X}^{(1)B}_s}{ds}\right|_{s=0} = \left[\frac{\partial}{\partial Q^{A'}}\Phi(Q, Q', [\bar{X}^{(1)}])\right]_{Q'=Q} + \partial_A \Psi^{(2)}(Q) + O(\varepsilon^3). \tag{4.53}$$

We'll also use

$$\xi^{(2)}_A \frac{d\bar{X}^{(1)A}}{ds} = \frac{1}{2}\Phi^{(2)} + \Psi^{(2)} \tag{4.54}$$

which can be derived from equation (4.47) using techniques similar to the ones in subsection 4.3.1 (See, for example, equation (4.25)).

The first correction in equation (4.52) is $\xi^A_{(2)}\partial_A H^{(2)}$. We use the equations of motion (4.23) to replace $\partial_A H^{(2)}$ by $\Omega^{(2)}_{AB} d\bar{X}^{(1)B}/ds$

$$\xi^A_{(2)}\partial_A H^{(2)} = \xi^A_{(2)}\left(\Omega^0_{AB} + \Delta\Omega^{(2)}_{AB}\right)\frac{d\bar{X}^{(1)B}}{ds} + O(\varepsilon^3) \tag{4.55}$$

Now, we use identity (4.54) for the first term and identity (4.53) for the second term to get

$$\begin{aligned} \xi^A_{(2)}H^{(2)} = &\frac{1}{2}\Phi^{(2)} + \Psi^{(2)} \\ &+ \xi^A_{(2)}\left[\frac{\partial}{\partial Q^{A'}}\Phi(Q, Q', [\bar{X}^{(1)}])\right]_{Q'=Q} + \xi^A_{(2)}\partial_A\Psi^{(2)} + O(\varepsilon^3). \end{aligned} \tag{4.56}$$

The second correction term in equation (4.52) is more involved, let's simplify it first. Using Cartan's magic formula we can write

$$\begin{aligned} \mathcal{L}_{\xi_{(2)}}\tilde{\xi}_{(2)} &= i_{\xi_{(2)}} d\tilde{\xi}_{(2)} \\ &= i_{\xi_{(2)}}\Delta\Omega^{(2)} \end{aligned} \tag{4.57}$$

where we used equation (4.38) to replace $d\tilde{\xi}_{(2)}$ by the correction to the symplectic form $\Delta\Omega^{(2)}$. Next, we use the equations of motion to replace $\partial_A H^{(2)}$ by $\Omega^0_{AB}\frac{d\bar{X}^{(1)B}}{ds} + O(\varepsilon^2)$. Com-

bining this with equation (4.57), the second term in equation (4.52) becomes

$$-\frac{1}{2}\Omega_0^{AB}\xi_{(2)}^{C}\Delta\Omega_{CB}\Omega_{AD}^{0}\frac{d\bar{X}^{(1)D}}{ds}. \tag{4.58}$$

Now, we use identity (4.53) to rewrite this as

$$\frac{1}{2}\xi_{(2)}^{A}\left[\frac{\partial}{\partial Q^{A'}}\Phi(Q,Q',[\bar{X}^{(1)}])\right]_{Q'=Q}+\frac{1}{2}\xi_{(2)}^{A}\partial_A\Upsilon^{(2)}(Q)+O(\varepsilon_n^3). \tag{4.59}$$

The last term in equation (4.52) is

$$\frac{1}{2}\xi_{(2)}^{A}\partial_A\big(\xi_{(2)}^{B}\partial_B H^{(2)}\big). \tag{4.60}$$

Again, we use Hamilton's equations to replace $\partial_B H^{(2)} = \Omega_{BC}^{0}\frac{\bar{X}^{(1)C}}{ds} + O(\varepsilon^2)$. We then use identity (4.54) to get

$$\frac{1}{2}\xi_{(2)}^{A}\partial_A\big(\frac{1}{2}\Phi^{(2)}+\Upsilon^{(2)}\big). \tag{4.61}$$

Combining equations (4.56), (4.59) and (4.61) and plugging them into equation (4.52), the final expression for the new Hamiltonian is

$$\begin{aligned}\hat{H}^{(2)} = H_0 &+ \frac{1}{2}\Phi^{(2)} - \frac{1}{4}\xi_{(2)}^{A}\partial_A\Phi^{(2)} \\ &+ \frac{1}{2}\xi_{(2)}^{A}\left[\frac{\partial}{\partial Q^{A}}\Phi(Q,Q',[\bar{X}^{(1)}])\right]_{Q'=Q}+O(\varepsilon^3).\end{aligned} \tag{4.62}$$

Note that the third and fourth terms include contributions of order $O(\varepsilon^3)$ which could be discarded without affecting the accuracy of the result.

4.5 Application: Binary systems in the post-Newtonian approximation

The motion of binary systems in general relativity can be studied in the post-Newtonian approximation, where their dynamics is expanded in powers of $1/c^2$. A term of order $1/c^{2n}$ is called nPN in the literature. In [55], Damour, Jaranowski and Schäfer give an explicit expression for the 4PN non-local Hamiltonian[‡] of two non-spinning point particles with phase space coordinates $Q^A = (\mathbf{x}_a, \mathbf{p}_a)$ and masses m_a with $a = 1, 2$ and boldface representing 3-vectors. Following the notation of this chapter, we use X_s for a trajectory in phase space parametrized by s. Their result is

$$H_{\leq 4\text{PN}}(Q, [X]) = H_{<4\text{PN}}(Q) + H^{\text{local}}_{4\text{PN}}(Q) + H^{\text{non-local}}_{4\text{PN}}(Q, [X]) \tag{4.63}$$

where $H_{<4\text{PN}}$ gathers all the contributions of order 3PN or less and $H^{\text{local}}_{4\text{PN}}$ gives the local piece of the 4PN Hamiltonian. We'll focus on the last term, which is written in terms of the quadrupole moment

$$I^{ij}(\mathbf{x}_a) = \sum_{a=1}^{2} m_a \left(x_a^i x_a^j - \frac{1}{3}\delta^{ij}|\mathbf{x}_a|^2 \right) \tag{4.64}$$

as

$$H^{\text{non-local}}_{4\text{PN}}(Q, [X]) = \frac{1}{c^8} \mathcal{C}\, \ddot{\ddot{I}}^{ij}(Q) \int_{-\infty}^{\infty} d\tau \frac{\ddot{\ddot{I}}_{ij}\left(X_\tau\right)}{|\tau|} \tag{4.65}$$

[‡] In a follow-up paper [56], the same authors utilize an (infinite-)order-reduction of the nonlocal dynamics to a local dynamical system. This procedure is similar to the one carried in section 4.2.1 and, similarly, doesn't result in a Hamiltonian system. Instead, the procedure determines a pseudo-Hamiltonian dynamical system (see appendix C for details).

where $\mathcal{C}$ is a normalization factor whose value is not important here. $H_{4\text{PN}}^{\text{non-local}}$ is the non-local or "tail" piece of the 4PN Hamiltonian. The non-locality arises from the integral over the full trajectory X_τ. In equation (5.1) of [55], they also derive a non-local contribution to the action principle from which $H_{4\text{PN}}^{\text{non-local}}$ can be derived, which is

$$S_{\text{nl}}[X] = -\frac{1}{c^8}\mathcal{C}\int d\tau d\tau' \frac{\dddot{I}^{ij}(X_\tau)\,\ddot{I}_{ij}(X_{\tau'})}{|\tau-\tau'|}. \tag{4.66}$$

Note that in equations (4.65) and (4.66) we are dropping the regularization prescription used in [57] to take care of the ultraviolet divergences of $H_{4\text{PN}}^{\text{non-local}}$ that occur at the coincidence limit $\tau \to \tau'$. The regularization can be reapplied after a local Hamiltonian is obtained.

We now show that the dynamical system (4.63) can be cast as a local Hamiltonian system by using the results of Section 4.3. We define a two-point function

$$\mathcal{G}_2(Q_1, Q_2, \sigma) = \mathcal{C}\frac{\dddot{I}^{ij}(Q_1)\,\ddot{I}_{ij}(Q_2)}{|\sigma|} \tag{4.67}$$

such that the non-local action in (4.66) takes the form of equation (4.8). Following the steps of section 4.3, we can evaluate the functional dependence of the non-local Hamiltonian (4.65) on the 0PN flow $\bar{X}^{(0)}$, which is the Newtonian solution to the equations of motion. It is not necessary to include corrections of order $1/c^2$ or higher in the flow, since that would give corrections to the Hamiltonian at 5PN and higher.

It follows that the non-local Hamiltonian $H_{\leq 4\text{PN}}$ admits a local Hamiltonian description up to $O(1/c^8)$, with Hamiltonian function and symplectic form given by the results in section (4.3).

If some new truth, O Friend! thy toil discover,

If thine eyes first by some fair form be blest,

Love it for what it is, and as a lover

Gaze, or with joy receive thine honoured guest:

The new-found Thought, set free, awhile may hover

Gratefully near thee, but it cannot rest.

Sir William Rowan Hamilton

5

The conservative sector of the dynamics of binary systems (Adapted from [3, 4])

5.1 Context

This chapter applies the results of chapter 4 to obtain the conservative sector of the dynamics of binary systems in the small mass-ratio regime. As we discussed in chapter 2, we would like

to define a splitting of the self-force that acts on a small body into conservative and dissipative pieces. The former modifies the orbital parameters, while the latter drives the inspiral through the emission of radiation. To linear order in the mass ratio, there is a well-known standard definition of a splitting determined by selecting the symmetric or antisymmetric piece of the Green function (cf. Equation (2.14)). Prior to the results presented later in this chapter and published in References [3, 4], it was not known whether the conservative dynamics admits a Hamiltonian description, although partial results were obtained in Reference [58]. The main obstacle to deriving a Hamiltonian description is the functional dependence of the self-force on the past worldline of the particle, since a Hamiltonian function should be a local function on phase space. Furthermore, there are several natural candidate definitions for a splitting of the self-force into conservative and dissipative pieces. In this chapter, we argue for one of these definitions and show that the corresponding conservative dynamics is Hamiltonian [3, 4].

The two ingredients necessary to derive this result are: 1) the second order equations of motion (3.95) derived in chapter 3 in terms of the regular field $\hat{\varphi}$ and 2) the perturbative local Hamiltonian description of non-local dynamics derived in Chapter 4.

The chapter is organized as follows. In Section 5.2, we give the definition of pseudo-Hamiltonian descriptions of dynamical systems, and explain their relation to the non-local action principles used in Chapter 4. In section 5.3, we derive the Hamiltonian description of the conservative gravitational self-force in static spacetimes, to linear order in the mass ratio. We analyze resonant orbits, the possible breakdown of integrability, and the first law of binary black hole mechanics. In Section 5.4, we extend those results to include linear-in-spin effects. In section 5.5, we consider a body coupled to a long-ranged, self-interacting scalar field, as a toy model for general relativity. First, we specify a unique prescription prescription

for defining conservative and dissipative sectors of the dynamics to second order in the body's charge, using the formalism of Chapter **??**. Second, we use the results of Chapter 4 to show that the conservative sector is Hamiltonian.

5.2 Pseudo-Hamiltonian description of dynamical systems

We define a *pseudo-Hamiltonian* dynamical system to consist of a phase space Γ, a closed, non-degenerate two form Ω_{AB} and a smooth pseudo-Hamiltonian function $\mathcal{H} : \Gamma \times \Gamma \to \mathbb{R}$, for which the dynamics are given by integral curves of the vector field

$$v^A = \Omega^{AB} \frac{\partial}{\partial Q^B} \mathcal{H}(Q, Q')|_{Q'=Q}\,, \tag{5.1}$$

where $\Omega^{AB}\Omega_{BC} = \delta^A_C$ and Q^A are coordinates on Γ.

Systems described by equations like (4.13) can be described in terms of a pseudo-Hamiltonian function, first defined in Reference [3]. We now detail the relation between pseudo-Hamiltonians and the formalism developed in chapter 4. Although pseudo-Hamiltonians are not necessary to derive the results in the following sections, they provide a simple language to understand the self-force corrections to the Hamiltonian.

The perturbative local dynamical systems derived in Subsection 4.2.1 are examples of pseudo-Hamiltonian systems which are perturbations of a Hamiltonian system. The symplectic form and pseudo-Hamiltonian up to $\mathcal{N}^{\text{th}}$ order are

$$\Omega_{AB} = \Omega_{0\,AB}, \tag{5.2a}$$

$$\mathcal{H}^{(\mathcal{N})}(Q, Q') = H_0(Q) + \Phi(Q, Q', [\bar{X}^{(\mathcal{N}-1)}]), \tag{5.2b}$$

where $\Phi(Q, Q', [X])$ is defined in equation (4.10). We repeat its definition here

$$\Phi(Q, Q', [X]) = \sum_{n=2}^{N} \varepsilon_n \int ds_2 \dots ds_n \mathcal{G}_n\big(Q, X_{s_2}(Q'), \dots, X_{s_n}(Q'); s_2, \dots, s_n\big). \tag{5.3}$$

The local equations of motion (4.23) are obtained by plugging the pseudo-Hamiltonian system (5.2) into equation (5.1).

In chapter 4, we derived pseudo-Hamiltonian equations of motion from a non-local action principle. However, pseudo-Hamiltonians can be used in a broader context and need not be derived from a variational principle. In such cases, the n-point functions $\mathcal{G}_n$ that appear in the definition (5.3) of $\Phi(Q, Q', [X])$ need not be fully symmetric (cf. property (4.9)). A pseudo-Hamiltonian system obtained by starting from equations (5.2b) and (5.3), without imposing that the n-point functions obey the symmetry property (4.9), can include dissipative effects [59]. In the context of the first-order gravitational self-force, for example, we can construct a pseudo-Hamiltonian using the retarded Green function, which encodes both dissipative and conservative effects, as opposed to the time-symmetric Green function, which only describes the conservative piece of the dynamics.

5.3 Gravitational case: Linear order in the mass ratio

We start by explaining how the motion of a particle under the action of its conservative first-order gravitational self-force can be cast as a pseudo-Hamiltonian, as defined in Section 5.2.

For the zeroth order geodesic motion we use phase space coordinates (x^μ, p_μ) with sym-

plectic form $\Omega_0 = dp_\mu \wedge dx^\mu$ and Hamiltonian*

$$H_0 = -\sqrt{-g^{\mu\nu}(x)p_\mu p_\nu}. \tag{5.4}$$

The time parameter τ associated with this Hamiltonian is then proper time normalized with respect to g_{ab}, while the conserved value of $-H_0$ is the mass of the particle.

For first-order motion, consider a particle at location $x^{\mu'}$ with an initial 4-momentum $p_{\mu'}$. Writing $Q' = (x', p')$, we denote by† $\phi_{\tau'}(Q') = [x^{\bar{\mu}}(\tau'), p_{\bar{\mu}}(\tau')]$ the geodesic with initial data Q'. From this geodesic we can compute the Lorenz gauge metric perturbation

$$h^{\mu\nu}(x; Q') = \frac{1}{\sqrt{-g^{\mu'\nu'}p_{\mu'}p_{\nu'}}} \int d\tau' G^{\mu\nu\bar{\mu}\bar{\nu}}_{C,R}[x, x'(\tau')]p_{\bar{\mu}}(\tau')p_{\bar{\nu}}(\tau'). \tag{5.5}$$

Here, the conservative regularized Green's function $G^{\mu\nu\bar{\mu}\bar{\nu}}_{C,R}$ is the average of the retarded and advanced Green's functions, regularized according to the Detweiler-Whiting prescription [30, 38]. The conservative forced motion of the particle is then equivalent at linear order to geodesic motion in the metric $g_{\mu\nu} + h_{\mu\nu}$, where Q' is held fixed when evaluating the geodesic equation and then evaluated at $Q' = Q$ [37, 38].

We can therefore obtain a pseudo-Hamiltonian description of the dynamics by replacing the metric $g_{\mu\nu}(x)$ in Eq. (5.4) with $g_{\mu\nu}(x) + h_{\mu\nu}(x, Q')$ (See Appendix B.3 for a derivation of the first-order equations of motion from this pseudo-Hamiltonian). Expanding to linear

*This differs from the Hamiltonian of [58] in that it includes a square root, which is necessary to make $G(Q, Q')$ symmetric in Eq. (5.7) below.

†Our index conventions are unadorned indices for the point $Q = (x, p)$, primed indices for the point $Q' = (x', p')$, and barred indices for $\phi_{\tau'}(Q')$.

order in $h_{\mu\nu}$, we get[‡]

$$\mathcal{H}(Q,Q') = H_0(Q) + \varepsilon \int G\big[Q, \phi_{\tau'}(Q')\big] d\tau', \tag{5.6}$$

where the 2-point function is

$$G(Q,Q') = -\frac{G^{\mu\nu\mu'\nu'}_{C,R}(x,x') p_\mu p_\nu p_{\mu'} p_{\nu'}}{2\sqrt{-g^{\lambda\sigma}p_\lambda p_\sigma}\sqrt{-g^{\lambda'\sigma'}p_{\lambda'}p_{\sigma'}}}. \tag{5.7}$$

The proof of chapter 4 requires that the 2-point function (5.7) is fully symmetric and decays to zero as we take its arguments to infinity. The symmetry condition is immediately satisfied, since we constructed $G(Q,Q')$ from the conservative piece of the retarded regularized 2-point function. Furthermore, recall that chapter 4 derives Hamiltonian dynamics from an action principle. Therefore, the symmetry condition will be automatically satisfied for any $G(Q,Q')$ we plug into the non-local action principle (4.8), since the action principle picks its symmetric piece due to double integration in the affine parameter s. In other words, deriving the equations of motion from a non-local action principle only gives conservative dynamics. The second condition will be satisfied if the retarded Green function falls off at late times. This is known to be true for scalar fields in a class of stationary space times [60],

[‡]Similar constructions work for scalar and electromagnetic self-forces. For a particle endowed with a scalar charge q and electromagnetic charge e we replace the initial Hamiltonian expression (5.4) with $-\sqrt{-g^{\mu\nu}(p_\mu - eA_\mu)(p_\nu - eA_\nu)} - q\Phi$. The expression (5.7) gets replaced by $-q^2 G_{\rm sc}(x,x')$ in the scalar case, where $G_{\rm sc}$ is the scalar Green's function, and with

$$-\frac{e^2 G^{\mu\mu'}(x,x') p_\mu p_{\mu'}}{\sqrt{-g^{\lambda\sigma}p_\lambda p_\sigma}\sqrt{-g^{\lambda'\sigma'}p_{\lambda'}p_{\sigma'}}}$$

in the electromagnetic case, where $G^{\mu\mu'}$ is the Lorenz gauge electromagnetic Green's function.

while for black holes it is a lore of the field that perturbations decay at late times as a power law [61]. This decay was shown for the Weyl scalars in black hole spacetimes by Barack [62], and it is also generally believed to be true for tensor perturbations, although it has not yet been established rigorously; see Refs. [63, 64] for recent developments.

Then, it follows from the results of Chapter 4 (cf. Section 4.3) that the linear gravitational self-force admits a local Hamiltonian description. Furthermore, there is a coordinate change, defined in Section 4.4 of chapter 4, such that the symplectic form takes canonical form and the Hamiltonian, given by Equation (4.62), is

$$H(Q) = -\sqrt{-g^{\alpha\beta}_{(0)} p_\alpha p_\beta} + \varepsilon H_1(Q) \tag{5.8}$$

where the first order Hamiltonian is

$$H_1(Q) = \frac{1}{2}\int G\big[Q, \phi_{\tau'}(Q)\big] d\tau'. \tag{5.9}$$

5.3.1 Specialization to motion near a black hole

We now specialize to the motion of a particle orbiting a Kerr black hole. In this context, it is useful to derive an explicit form for the Hamiltonian in action angle variables.

We use the variables $(q^\alpha, j_\alpha) = (q^t, q^r, q^\theta, q^\varphi, j_t, j_r, j_\theta, j_\varphi)$ defined in Refs. [40, 65], deformed via the coordinate transformation defined in Section 4.4 of chapter 4. In these variables the symplectic form is $\Omega = dj_\alpha \wedge dq^\alpha$ and the full Hamiltonian from equation (5.8) is

$$H = H_0(j_\alpha) + H_1(q^\alpha, j_\alpha). \tag{5.10}$$

The zeroth order geodesic motion is given by $q^\alpha(\tau) = q_0^\alpha + \Omega_0^\alpha(j)\tau, j_\alpha = \text{const}$, where $\Omega_0^\alpha = \partial H_0/\partial j_\alpha$ are the zeroth order frequencies.

We now fix a value m of the conserved quantity $-H$, which is the mass of the particle to leading order. For describing motion on the mass shell $H = -m$ it will be convenient to define rescaled versions of the symplectic form and Hamiltonian,

$$\hat{\Omega}_{AB} = \Omega_{AB}/m, \quad \hat{H} = H/m. \tag{5.11}$$

This rescaling preserves Hamilton's equations. Using the fact that under the transformation $(x^\mu, p_\mu) \to (x^\mu, sp_\mu)$ with $s > 0$ we have $(q^\alpha, j_\alpha) \to (q^\alpha, sj_\alpha)$ [40], $H_0 \to sH_0$ and $H_1 \to s^2 H_1$ [cf. Eq. (5.7)], the dynamical system can be written as

$$\hat{\Omega} = dJ_\alpha \wedge dq^\alpha, \quad \hat{H} = \hat{H}_0(J) + m\hat{H}_1(q,J), \tag{5.12}$$

where $J_\alpha = j_\alpha/m$.

Motion on this mass shell can be described in terms a 6 dimensional Hamiltonian system, which can be derived from the 8 dimensional system (5.12) as follows [53]. Because of the symmetries of the Kerr background the Hamiltonian is independent of q^t, $\hat{H} = \hat{H}(q^i, J)$ where $q^i = (q^r, q^\theta, q^\varphi)$. Consider paths in the 9-dimensional extended phase space (q, J, τ) that join an initial point (q_1, J_1, τ_1) to a final point (q_2, J_2, τ_2). Paths that extremize the line integral of the Poincaré-Cartan one form $\int [J_\alpha dq^\alpha - \hat{H}(q^i, J)d\tau]$, with $\delta q^\alpha = \delta\tau = 0$ at the endpoints, satisfy the 8-dimensional Hamilton equations of motion [53]. We now restrict to paths lying within the surface $\hat{H} = -1$. Within this surface we can solve for $J_t = -h(q^i, J_i)$

in terms of the other parameters from the equation

$$\hat{H}[q^i, -h(q^i,J_i),J_i] = -1, \tag{5.13}$$

where $J_i = (J_r,J_\theta,J_\varphi)$. The line integral now reduces to

$$\int \left[J_i dq^i - h dq^t\right] + (\tau_2 - \tau_1). \tag{5.14}$$

The second term is a constant and the first term is an extremum under the variation of paths $[q^i(q^t),J_i(q^t)]$ that connect the two endpoints for which $\delta q^i = 0$. Hence, we obtain a 6-dimensional Hamiltonian system with Hamiltonian $h(q^i,J_i)$, time parameter q^t and symplectic form $dJ_i \wedge dq^i$. By combining Eqs. (5.11) and (5.13) it follows that the Hamiltonian can be expanded as

$$h(q^i,J_i) = h_0(J_i) + m h_1(q^i,J_i) + O(m^2). \tag{5.15}$$

where h_0 and h_1 are given by $\hat{H}_0(-h_0,J_i) = -1$ and $h_1 = \hat{H}_1(q^i,-h_0,J_i)/\Omega_0^t$. The zeroth order frequencies are now $\omega_0^i = \partial h_0/\partial J_i = \Omega_0^i/\Omega_0^t$.

The Hamiltonian perturbation h_1 is independent of q^φ due to the symmetry of the Kerr background, and can be expanded in Fourier modes[§] on the torus parameterized by $\mathbf{q} =$

[§]It is possible to obtain an explicit formula for the coefficients $h_{1\,\mathbf{k}}$ starting from a Fourier expansion of the function (2.28) in action angle variables

$$G(q,J,q',J') = \int d\omega \sum_{m,\mathbf{k},\mathbf{k}'} e^{-i\omega(q^t-q'^t)-im(q^\varphi-q'^\varphi)} e^{i\mathbf{k}\cdot\mathbf{q}+i\mathbf{k}'\cdot\mathbf{q}'} G_{\omega m\mathbf{k}\mathbf{k}'}(J,J').$$

Combining this with Eqs. (5.8), (5.13), (5.15) and (5.16) gives

$$h_{1\,\mathbf{k}} = \frac{\pi}{(\Omega_0^t)^2} \sum_{m,\mathbf{l}} G_{\omega m(\mathbf{k}/2+\mathbf{l}/2)(\mathbf{k}/2-\mathbf{l}/2)}(J_t,J_i,J_t,J_i),$$

(q^r, q^θ):

$$h_1(\mathbf{q}, J_i) = \sum_{k_r=-\infty}^{\infty} \sum_{k_\theta=-\infty}^{\infty} e^{i\mathbf{k}\cdot\mathbf{q}} h_{1\mathbf{k}}(J_i). \tag{5.16}$$

5.3.2 Application: Integrability of dynamics

We now turn to discussing some applications. Since the motion is Hamiltonian, one can ask whether it is also integrable. It will be integrable to linear order if and only if all the resonant mode amplitudes vanish, that is,

$$h_{1\mathbf{k}}(J_i) = 0 \quad \text{whenever} \quad \mathbf{k}\cdot\omega_0(J_i) = 0,\ \mathbf{k} \neq 0. \tag{5.17}$$

This is easy to see, since under a linearized canonical transformation with generating function $G(\mathbf{q}, J_i) = \sum_{\mathbf{k}} \exp[i\mathbf{k}\cdot\mathbf{q}] G_{\mathbf{k}}(J_i)$ we have $h_{1\mathbf{k}} \to h_{1\mathbf{k}} + i(\mathbf{k}\cdot\omega_0) G_{\mathbf{k}}$. Thus choosing $G_{\mathbf{k}}(J_i) = -ih_{1\mathbf{k}}(J_i)/\mathbf{k}\cdot\omega_0$ yields $h_{1\mathbf{k}} = 0$ for all nonzero $\mathbf{k}$ and thus an integrable system[¶], and this choice is possible without divergences only when the condition (5.17) is satisfied. Conversely, if the system is integrable there must exist perturbed versions $J_i + m\delta J_i$ of the action variables which have vanishing Poisson brackets with the Hamiltonian $h_0 + mh_1$, which yields at linear order the relation

$$k_i h_{1\mathbf{k}} = (\mathbf{k}\cdot\omega_0)\delta J_{i\mathbf{k}} \tag{5.18}$$

between Fourier components, enforcing the condition (5.17).

An alternative version of the integrability condition (5.17) is that the average of the conservative time derivative of the Carter constant $Q(J_i)$ over any orbit on any resonant torus

where we sum over all pairs of integers $\mathbf{l} = (l_r, l_\theta)$ for which $k_r + l_r$ and $k_\theta + l_\theta$ are even, and we evaluate at $J_t = -h_0(J_i)$ and at $\omega = m\omega_0^\varphi + (\mathbf{k} - \mathbf{l})\cdot\omega_0/2$.

[¶] The resulting Hamiltonian coincides with that found by Ref. [58], who excluded resonances.

should vanish. Computing a time derivative using Eqs. (5.15) and (5.16) gives $dQ/d\tau = \Omega_0^t(\partial Q/\partial J_i)dJ_i/dq^t = -i\Omega_0^t(\partial Q/\partial J_i)\sum_{\mathbf{k}} k_i h_{1\mathbf{k}} e^{i\mathbf{k}\cdot\mathbf{q}}$. Now using $\mathbf{q}(\tau) = \mathbf{q}_0 + \Omega_0\tau$, writing the resonant vectors as $\mathbf{k} = N\mathbf{k}_0 = N(n, -p, 0)$ for integers N and taking an orbit average gives[‖]

$$\left\langle \frac{dQ}{d\tau} \right\rangle = -i\Omega_0^t \left(n\frac{\partial Q}{\partial J_r} - p\frac{\partial Q}{\partial J_\theta} \right) \sum_{N=-\infty}^{\infty} N h_{1N\mathbf{k}_0} e^{iNq_{\rm res}}, \tag{5.19}$$

where $q_{\rm res} = \mathbf{k}_0 \cdot \mathbf{q}_0 = nq_0^r - pq_0^\theta$ is the resonant combination of the phases. The left hand side vanishing for all $q_{\rm res}$ is equivalent to all the resonant amplitudes $h_{1N\mathbf{k}_0}$ vanishing.

Flanagan conjectured in Ref. [66] that the linear integrability condition (5.17) is satisfied in Kerr, based on the fact that enhanced symmetries present in the post-Newtonian limit enforce this condition. However, this was a weak argument since it is possible for symmetries to be present only near the boundary of phase space that corresponds to the post-Newtonian limit and not in the interior. Indeed, in past years Nasipak and Evans have shown numerically that $\langle dQ/d\tau\rangle = 0$ fails for conservative scalar self-forces in Kerr on resonances [67, 68]. The gravitational self-force case is presumably similar, although this will need to be confirmed numerically (see Ref. [69]).

If the gravitational case is indeed non-integrable, the qualitative consequences for the conservative dynamics are well understood in general contexts from the theory of weakly perturbed Hamiltonian systems [53, 70]. They have been explored in the context of tidal and other perturbations to extreme mass ratio inspirals in Refs. [71–74]. Suppose we focus at-

[‖] We neglect in this calculation the coordinate transformation from Section 4.4, used to put the symplectic form into canonical form, because under $J_i \to J_i + \delta J_i$ we have $\dot{J}_i \to \dot{J}_i + \omega_0^j \partial \delta J_i/\partial q^j$ and the resonant Fourier components of the correction evaluated on a resonant torus vanish.

tention on one resonant torus $J_i = J_i^*$ and neglect the effect of other resonances. First, away from this torus the invariant tori J_i = constant are deformed [cf. Eq. (5.18)] but preserved (as predicted by the KAM theorem [53]). Second, within a shell of width $J_i - J_i^* \sim \sqrt{m}$ the dynamics is altered: In the $m \to 0$ limit the resonant torus is destroyed and replaced by a number of islands of size $\sim \sqrt{m}$ in phase space within which the motion is integrable** [74]. One can define action angle variables within each island, but they do not join continuously onto the global action angle variables. At finite m chaotic regions develop within the shell. Third, motion that starts within the shell is confined to remain within it by the surrounding surviving invariant tori, since the system is effectively two dimensional (J_φ is conserved) [70]. There are no large excursions to $J_i - J_i^* \sim O(1)$, unlike in higher dimensions.

When one considers the full $O(m)$ dynamics with the dissipative component of the self force included, the non-integrable mode coefficients $h_{1\mathbf{k}}$ can drive transient resonances which give $O(\sqrt{m})$ kicks to the action variables J_i [66], and also sustained resonances in which the orbit evolves along a non-adiabatic path in the space of parameters J_i maintaining the condition $\mathbf{k} \cdot \omega_0(J_i) = 0$ [75]. However neither of these are smoking gun signatures of the breakdown of integrability, since both can be produced when $h_{1\mathbf{k}} = 0$ by the oscillatory dissipative components of the self force [75].

Non-integrability would also complicate the dynamics away from the resonant islands in phase space. If one computes the dynamics using the linear prescription described after Eq. (5.17) for eliminating the oscillatory terms in the Hamiltonian (5.15), ignoring the divergences, the resulting fractional errors caused by the nearest strong resonance scale as

**This can be seen explicitly in the description of the near-resonance dynamics derived by van de Meent, Eq. (18) of Ref. [75], dropping the dissipative terms (the first term on the right hand side and half of the oscillatory terms); the solutions consist of rotational or librational (islands) motions, depending on the energy.

$\sim m^2|h_{1\mathrm{k}}|^2(J-J^*)^{-4}$. It is possible to achieve smaller errors $\sim m^3|h_{1\mathrm{k}}|^3(J-J^*)^{-6}$ by using a second order canonical transformation to eliminate the oscillatory terms in (5.15) from h_1 through $O(m^2)$, at the price of a more complicated description of the dynamics. In either case the errors become of order unity in the vicinity of the resonant islands.

5.3.3 Application: First law of binary black hole mechanics

In the absence of resonances, our Hamiltonian (5.10) directly yields a version of the first law, as in Ref. [58]. We eliminate all q dependent terms in (5.10) using a canonical transformation as described after Eq. (5.17). We regard H as a function $H = H(j_\alpha, \mathcal{M}_{\mathrm{irr}}, S_{\mathrm{bh}})$ of the action variables j_α and of the irreducible mass $\mathcal{M}_{\mathrm{irr}}$ and spin S_{bh} of the large black hole. Taking a variation and using $H = -m$ gives

$$-\delta m = \Omega^t \delta j_t + \Omega^i \delta j_i + \frac{\partial H}{\partial \mathcal{M}_{\mathrm{irr}}}\delta \mathcal{M}_{\mathrm{irr}} + \frac{\partial H}{\partial S_{\mathrm{bh}}}\delta S_{\mathrm{bh}}, \tag{5.20}$$

where $\Omega^\alpha = \partial H/\partial j_\alpha$ are the frequencies accurate to subleading order in m. Identifying $-j_t$ as the orbital energy E, dividing by Ω^t, and adding the variation of the background black hole mass $\mathcal{M}_{\mathrm{bh}}(\mathcal{M}_{\mathrm{irr}}, S_{\mathrm{bh}})$ gives

$$\delta(\mathcal{M}_{\mathrm{bh}} + E) = z\delta m + \omega^i \delta j_i + z_{\mathrm{bh}}\delta \mathcal{M}_{\mathrm{irr}} + \Omega_{\mathrm{bh}}\delta S_{\mathrm{bh}}. \tag{5.21}$$

Here $z = 1/\Omega^t$ is the redshift invariant, $\omega^i = \Omega^i/\Omega^t$, $z_{\mathrm{bh}} = \partial \mathcal{M}_{\mathrm{bh}}/\partial \mathcal{M}_{\mathrm{irr}} + z\partial H/\partial \mathcal{M}_{\mathrm{irr}}$ and $\Omega_{\mathrm{bh}} = \partial \mathcal{M}_{\mathrm{bh}}/\partial S_{\mathrm{bh}} + z\partial H/\partial S_{\mathrm{bh}}$. Equation (5.21) yields a form of the first law for binaries[††]

[††]Equation (5.21) is not quite the conventional form of the first law beyond the leading order in m. The conventional form would require the quantity $\mathcal{M}_{\mathrm{bh}} + E$ on the left hand side to coincide with the Bondi mass to $O(m^2)$, whereas it is known to coincide only to $O(m)$ [76]. Additionally we have identified the on-shell

to subleading order in m.

This derivation of the first law required the integrability assumption (5.17). We now explain how the first law would break down if that assumption is violated as discussed above. The first law requires a labeling of time-averaged orbits by some smooth set of parameters. However, when (5.17) is violated integrable motions near a resonance fall into different types that are disconnected from one another. Within an island one can define new action angle variables $\tilde{q}^i, \tilde{j}_i$, but these cannot join smoothly onto the deformed action-angle variables outside the islands. Thus, the best one can hope for is set of distinct first laws, one for each disconnected component of integrable motion. Also the number of such components is formally infinite, since the resonances are dense in phase space. In practice only the few resonances for which the order $|k_r| + |k_\theta|$ is not large will be significant: the width of an island scales as $\sim \sqrt{m}|h_{1\mathrm{k}}|$ [75] which will go exponentially to zero as the order increases, assuming the Hamiltonian is a smooth function on the torus.

5.3.4 Application: Gauge invariant observables

Gauge invariant observables such as invariant redshifts, frequencies of innermost stable circular orbits, etc. have proven enormously useful for cross checks between different computational methods [37]. The simple form (5.8) of our Hamiltonian may be helpful for computing such observables, since one expects the complicated phase space coordinate transformation that makes the symplectic form canonical not to be relevant for gauge invariant observables.

value of the Hamiltonian with minus the particle mass m, and this relation can have a correction to subleading order in m. Nevertheless, our form is sufficient to illustrate the difficulties caused by non-integrability, which are generic for all forms of the first law.

For generic orbits, non-integrability of the dynamics would impede the definition of such observables. For example one can no longer label orbits by their three fundamental frequencies of motion. However new gauge invariant observables do arise in this context, the resonant amplitudes h_{1k} themselves, for which the action-angle variables are defined geometrically at zeroth order [40] and which at first order are invariant under linearized phase space coordinate transformations. These observables are not accessible from within post-Newtonian or post-Minkowski theory, but could be useful for comparisons between self-force theory and numerical relativity.

5.4 Gravitational case: Linear-in-spin conservative effects

We now discuss how to include linear-in-spin effects in the Hamiltonian description of the conservative self-force. We will follow the same steps of the previous section: 1) Express the dynamics of a test spinning point-particle in terms of a Hamiltonian function, 2) Replace the background metric on the zeroth order Hamiltonian by the effective metric, 3) Expand the Hamiltonian to linear order in the mass ratio to obtain the first order correction to the Hamiltonian, which will be nonlocal and 4) Use the results from chapter 4 to obtain a local Hamiltonian description.

5.4.1 Hamiltonian description of the motion of a spinning test particle

The motion of an extended body in general relativity, neglecting self gravity, can be reduced to the motion of a point particle of mass μ endowed with a series of mass and current multipole

moments [39, 44, 77, 78]. If we restrict ourselves to the pole-dipole approximation, where only the mass and spin are included, the dynamics are given by the well-known Mathisson-Papapetrou-Dixon (MPD) equations [47, 79, 80]

$$\nabla_{\boldsymbol{u}} p_\mu = -\frac{1}{2} R_{\mu\nu\alpha\beta} u^\nu S^{\alpha\beta}, \tag{5.22a}$$

$$\nabla_{\boldsymbol{u}} S^{\alpha\beta} = 2 p^{[\alpha} u^{\beta]}. \tag{5.22b}$$

Here

$$\frac{dx^\mu}{d\tau} = u^\mu \tag{5.23}$$

is the 4-velocity of the particle, $S^{\alpha\beta}$ is its spin tensor, p_μ is its 4-momentum, $\nabla_{\boldsymbol{u}} = u^\alpha \nabla_\alpha$ is the covariant derivative respect to proper time τ, and $R_{\mu\nu\alpha\beta}$ is the Riemann tensor. The set of equations (5.22) and (5.23) comprises 14 equations for 17 independent unknowns $x^\mu(\tau)$, $u^\mu(\tau)$, $p_\mu(\tau)$ and $S^{\alpha\beta}(\tau)$. Hence the dynamical system is not yet completely specified. This incompleteness arises because of the freedom to choose different definitions of the center-of-mass worldline $x^\mu(\tau)$ of the extended body [81, 82]. A definition can be chosen by imposing a so-called spin supplementary condition of the form

$$S^{\alpha\beta} V_\beta = 0, \tag{5.24}$$

for some timelike vector V_β.

In this section, we use the Tulczyjew-Dixon spin supplementary condition [47, 77]

$$S^{\alpha\beta}p_\beta = 0, \tag{5.25}$$

which reduces the MPD equations to

$$\begin{aligned}
\frac{dx^\mu}{d\tau} &= \frac{1}{\mu}g^{\mu\nu}p_\nu - \frac{1}{2\mu^3}R_{\sigma\rho\alpha\beta}S^{\sigma\mu}p^\rho S^{\alpha\beta} + O(S^4) && \text{(5.26a)}\\
\nabla_{\boldsymbol{u}}p_\mu &= -\frac{1}{2\mu}R_{\mu\nu\alpha\beta}S^{\alpha\beta}p^\nu + \\
&\quad \frac{1}{4\mu^3}R_{\mu\nu\alpha\beta}S^{\alpha\beta}R_{\sigma\rho\delta\gamma}S^{\sigma\nu}p^\rho S^{\delta\gamma} + O(S^5) && \text{(5.26b)}\\
\nabla_{\boldsymbol{u}}S^{\alpha\beta} &= -\frac{1}{\mu}u^{[\alpha}S^{\beta]\sigma}u^\rho S^{\mu\nu}R_{\sigma\rho\mu\nu} + O(S^4). && \text{(5.26c)}
\end{aligned}$$

Here we have defined the particle mass

$$\mu = \sqrt{-g^{\alpha\beta}p_\alpha p_\beta}. \tag{5.27}$$

We now consider the effect of keeping only terms linear in spin in the context of extreme mass ratio inspirals, where the spin scales as $S \propto \mu^2$. The non-linear terms in spin would induce changes $\Delta x = \mu \times \varepsilon$, $\Delta p = \mu \times \varepsilon^3$ and $\Delta S = \mu^2 \times \varepsilon$ over the inspiral time $\Delta\tau \propto \frac{M^2}{\mu}$. These are all smaller than the effect of the conservative piece of the gravitational self-force and thus can be ignored. Keeping only linear terms in spin, we get

$$\frac{dx^\mu}{d\tau} = \frac{1}{\mu} g^{\mu\nu} p_\nu \tag{5.28a}$$

$$\nabla_{\boldsymbol{u}} p_\mu = -\frac{1}{2\mu} R_{\mu\nu\alpha\beta} S^{\alpha\beta} p^\nu \tag{5.28b}$$

$$\nabla_{\boldsymbol{u}} S^{\alpha\beta} = 0. \tag{5.28c}$$

We note that the spin supplementary condition (5.25) is not preserved by the dynamics (5.28). This arises because we are working to linear order in spin. In this paper we shall adopt the equations (5.28) as the definition of the dynamical system we are working with, even though this definition is formally inconsistent with the spin supplementary condition from which it was derived. The inconsistency is higher order in spin and so can be safely ignored for our purposes.

Let Γ_s denote the phase space consisting of the bundle over spacetime with coordinates $(x^\mu, p_\nu, S^{\alpha\beta})$. As is well known, there exists a Hamiltonian function and a Poisson bracket structure on Γ_s that give rise to the dynamical system (5.28) [83–88]. The Poisson brackets are

$$\{x^\mu, x^\nu\} = 0, \tag{5.29a}$$

$$\{x^\mu, p_\nu\} = \delta^\mu_\nu, \tag{5.29b}$$

$$\{p_\mu, p_\nu\} = -\frac{1}{2} R_{\mu\nu\alpha\beta} S^{\alpha\beta}, \tag{5.29c}$$

$$\{x^\mu, S^{\alpha\beta}\} = 0, \tag{5.29d}$$

$$\{S^{\alpha\beta}, p_\mu\} = -\Gamma^\alpha_{\mu\rho} S^{\rho\beta} - \Gamma^\beta_{\mu\rho} S^{\alpha\rho}, \tag{5.29e}$$

$$\{S^{\mu\nu}, S^{\alpha\beta}\} = 2g^{\mu[\beta} S^{\alpha]\nu} - 2g^{\nu[\beta} S^{\alpha]\mu}, \tag{5.29f}$$

and the Hamiltonian H_0 is

$$H_0(x,p,S) = -\sqrt{-g^{\mu\nu}p_\mu p_\nu}. \tag{5.30}$$

It will be convenient to make a change of coordinates on phase space to simplify the form (5.29) of the Poisson brackets [89]. We choose an arbitrary orthonormal basis $\boldsymbol{e}_\Lambda = e_\Lambda^\alpha \partial_\alpha$ for $0 \le \Lambda \le 3$, with $\boldsymbol{e}_\Lambda \cdot \boldsymbol{e}_\Sigma = \eta_{\Lambda\Sigma}$, the Minkowski metric with signature $(-1,1,1,1)$. We use upper case Greek indices for orthonormal basis indices and lower case Greek indices for spacetime indices. We define the dual basis $\mathbf{e}^\Lambda = e^\Lambda_{\ \mu} dx^\mu$ by $e^\Lambda_{\ \mu} e^{\ \mu}_\Sigma = \delta^\Lambda_\Sigma$, and the components of the spin connection by

$$\omega_{\alpha\Lambda\Sigma} = e_{\Lambda\rho} \nabla_\alpha e^\rho_\Sigma. \tag{5.31}$$

We define new phase space coordinates $(x^\alpha, \pi_\alpha, S^{\Lambda\Pi})$ by

$$\pi_\alpha = p_\alpha - \frac{1}{2}\omega_{\alpha\Lambda\Sigma}\, e^\Lambda_{\ \mu}\, e^\Sigma_{\ \nu}\, S^{\mu\nu}, \tag{5.32a}$$

$$S^{\Lambda\Sigma} = e^\Lambda_{\ \mu}\, e^\Sigma_{\ \nu}\, S^{\mu\nu}. \tag{5.32b}$$

In these new coordinates the only non-vanishing Poisson brackets are

$$\{x^\mu, \pi_\nu\} = \delta^\mu_\nu, \tag{5.33a}$$

$$\{S^{\Theta\Pi}, S^{\Gamma\Lambda}\} = 2\eta^{\Theta[\Lambda} S^{\Gamma]\Pi} - 2\eta^{\Pi[\Lambda} S^{\Gamma]\Theta}. \tag{5.33b}$$

Substituting the coordinate change (5.32) into the Hamiltonian (5.30) and linearizing in spin

gives the form of the Hamiltonian in these coordinates

$$H_0(x,\pi,S) = -\sqrt{-g^{\mu\nu}\pi_\mu\pi_\nu} + \frac{g^{\mu\nu}\pi_\mu\omega_{\nu\Theta\Pi}S^{\Theta\Pi}}{2\sqrt{-g^{\mu\nu}\pi_\mu\pi_\nu}}. \tag{5.34}$$

It will also be convenient to define a new mass parameter m related to the norm of the new momentum 4-vector

$$m = \sqrt{-g^{\alpha\beta}\pi_\alpha\pi_\beta}, \tag{5.35}$$

which is related to our previously defined mass (5.27) by $m = \mu + O(S)$. In the following sections we will expand the Hamiltonian of the system in powers of m and S, by counting factors of π_μ and $S^{\Lambda\Pi}$. Using this counting the first term in the Hamiltonian (5.34) is $O(m)$ while the second one is $O(S)$.

Although the Hamiltonian function (5.34) and Poisson structure (5.33) give rise to the dynamical system (5.28) on Γ_s, the dynamical system is not Hamiltonian since the Poisson structure (5.33) is degenerate. The degeneracy is due to the existence of two Casimir invariants [85, 86]

$$S_*^2 = \frac{1}{8}\varepsilon_{\Gamma\Sigma\Xi\Pi}S^{\Gamma\Sigma}S^{\Xi\Pi}, \tag{5.36a}$$

$$S_\circ^2 = \frac{1}{2}\eta_{\Gamma\Sigma}\eta_{\Xi\Pi}S^{\Gamma\Xi}S^{\Sigma\Pi}, \tag{5.36b}$$

which satisfy $\{S_*, F\} = \{S_\circ, F\} = 0$ for any function F on phase space. Denote by $y^{\mathcal{A}}$ abstract coordinates on Γ_s, the Poisson structure can be written as a tensor $\Omega^{\mathcal{AB}}$, and its degeneracy implies that a symplectic form $\Omega_{\mathcal{AB}}$ satisfying $\Omega_{\mathcal{AB}}\Omega^{\mathcal{BC}} = \delta^{\mathcal{A}}_{\mathcal{C}}$ does not exist. Thus, Γ_s is a Poisson manifold but not a symplectic manifold.

We can overcome this difficulty and obtain a true Hamiltonian description of the dynamics as follows, following [85, 86]. Fix values $S_\circ$ and S_* of the Casimirs, and consider the corresponding submanifold Γ of Γ_s. Denoting by Q^A abstract coordinates on Γ, and by $y^\mathcal{A} = y^\mathcal{A}(Q^B)$ the embedding map. There exists an invertible Poisson structure Ω^{AB} on Γ whose pushforward

$$\Omega^{\mathcal{A}\mathcal{B}} = \frac{\partial y^\mathcal{A}}{\partial Q^A}\frac{\partial y^\mathcal{B}}{\partial Q^B}\Omega^{AB} \tag{5.37}$$

to Γ_s coincides with the Poisson structure (5.33). It follows that the dynamical vector field $v^\mathcal{A} = \Omega^{\mathcal{A}\mathcal{B}}\partial_\mathcal{B} H_0$ on Γ_s is the pushforward $v^A \partial y^\mathcal{A}/\partial Q^A$ of the Hamiltonian vector field $v^A = \Omega^{AB}\partial_B \bar{H}_0$ on Γ, where $\bar{H}_0$ is the pullback of H_0 to Γ (below we will drop the bar). Thus, the dynamics restricted to Γ is Hamiltonian and Γ is a symplectic manifold.

We now review the construction of the nondegenerate Poisson structure on Γ [86]. We specialize to the region of Γ where $S_\circ^2 \geq 0$ and $S_*^2 = 0$. This is true for any spin supplementary condition $S^{\alpha\beta}f_\beta = 0$ where the vector f_β is time-like [‡‡][86]. We define coordinates

[‡‡]Note that in this case we know that the spin supplementary condition is not preserved by the equations of motion given by equation (5.28). However the conditions on $S_\circ^2$ and S_*^2 still are, since they are Casimir invariants and thus commute with all functions in phase space, including the Hamiltonian.

$\{x^\mu, \pi_\mu, \sigma, \rho_\sigma, \varsigma, \rho_\varsigma\}$ on this patch of Γ by the relations

$$S^{23} = X\cos\sigma, \tag{5.38a}$$

$$S^{31} = X\sin\sigma, \tag{5.38b}$$

$$S^{12} = \rho_\sigma, \tag{5.38c}$$

$$S^{01} = Y\rho_\sigma \sin\varsigma\cos\sigma + Y\rho_\varsigma \cos\varsigma\sin\sigma + XZ\cos\sigma, \tag{5.38d}$$

$$S^{02} = Y\rho_\sigma \sin\varsigma\sin\sigma - Y\rho_\varsigma \cos\varsigma\cos\sigma + XZ\sin\sigma, \tag{5.38e}$$

$$S^{03} = Z\rho_\sigma - XY\sin\varsigma, \tag{5.38f}$$

with

$$X = \sqrt{\rho_\varsigma^2 - \rho_\sigma^2}, \quad Y = \sqrt{1 - \frac{S_\circ^2}{\rho_\varsigma^2} - \frac{S_*^2}{\rho_\varsigma^4}}, \quad Z = \frac{S_*^2}{\rho_\varsigma^2}. \tag{5.39}$$

We define a Poisson structure on this patch of Γ by

$$\{\sigma, \rho_\sigma\} = 1, \tag{5.40a}$$

$$\{\varsigma, \rho_\varsigma\} = 1, \tag{5.40b}$$

$$\{x^\mu, \pi_\nu\} = \delta^\mu_\nu, \tag{5.40c}$$

with all other brackets vanishing. This is equivalent to the symplectic form $= d\rho_\sigma \wedge d\sigma + d\rho_\varsigma \wedge d\varsigma + d\pi_\mu \wedge dx^\mu$. One can check that the pushforward of the Poisson structure (5.40) using the embedding (5.38) and (5.39) gives the Poisson structure (5.33).

To summarize, the Hamiltonian system on the twelve dimensional phase space Γ is given by the Poisson brackets (5.40), and by the Hamiltonian (5.34) expressed in terms of the coordinates $\{x^\mu, \pi_\mu, \sigma, \rho_\sigma, \varsigma, \rho_\varsigma\}$ using the map (5.38) and (5.39).

5.4.2 Pseudo-Hamiltonian description of the motion of a self-gravitating spinning particle

In the previous subsection, we defined phase-space coordinates $Q^A = (x^\mu, \pi_\mu, \sigma, \rho_\sigma, \varsigma, \rho_\varsigma)$ on Γ, with symplectic form (5.40) and Hamiltonian (5.34). The equations of motion (5.28) give the dynamics to zeroth order in self-gravity, but contain effects first order in spin.

For the first-order motion, consider a particle at location $x^{\mu'}$ with initial 4-momentum $p_{\mu'}$ and initial spin $S^{\Lambda'\Sigma'}$ [here the spin variable $S^{\Lambda'\Sigma'}$ should be understood to be a shorthand for the four variables $\sigma, \rho_\sigma, \varsigma, \rho_\varsigma$ defined in Eq. (5.38)]. Writing $Q' = (x^{\mu'}, \pi_{\mu'}, S^{\Lambda'\Sigma'})$, we denote by $\phi_{\tau'}(Q') = [x^{\bar\mu}(\tau'), \pi_{\bar\mu}(\tau'), S^{\bar\Lambda\bar\Sigma}(\tau')]$ the solution to the zeroth order motion and spin evolution (5.28), where τ' is proper time. We can compute from this zeroth order motion a metric perturbation as follows. Inserting the stress energy tensor of a spinning point particle given by Eq. (9) of Ref. [90] into the linearized Einstein equation gives the Lorenz gauge retarded regularized metric perturbation

$$\begin{aligned} h^{\mu\nu}(x, Q') &= \int d\tau' G^{\mu\nu\bar\mu\bar\nu}_{+,R}[x, \bar x(\tau')] \frac{p_{\bar\mu}(\tau') p_{\bar\nu}(\tau')}{\sqrt{-g^{\bar\lambda\bar\sigma} p_{\bar\lambda} p_{\bar\sigma}}} \\ &\quad - \int d\tau' \nabla_{\bar\rho} G^{\mu\nu\bar\mu\bar\nu}_{+,R}[x, \bar x(\tau')] \frac{p_{\bar\mu}(\tau') e_{\bar\Lambda\bar\tau} e^{\bar\rho}_{\bar\Sigma} S^{\bar\Lambda\bar\Sigma}}{\sqrt{-g^{\bar\lambda\bar\sigma} p_{\bar\lambda} p_{\bar\sigma}}}. \end{aligned} \tag{5.41}$$

Here barred indices indicate quantities that are evaluated at $x^{\bar\mu}(\tau')$, and $\nabla_{\bar\rho}$ acts only on the second argument of the Green's function. The factor of $\sqrt{-\boldsymbol{p}^2}$ could be evaluated either at x^μ or at $x^{\bar\mu}$, since it is conserved by the dynamics (5.28); we choose the latter for later convenience. Note that we are using the retarded regularized 2-point function so that the metric perturbation h is really the *effective* metric perturbation. Now it is known that the self-forced

and self-torqued motion of the spinning particle is given at linear order by evaluating the equations of motion (5.28) in the metric $g_{\mu\nu} + h_{\mu\nu}$, where Q' is held fixed when evaluating the equations and then evaluated at $Q' = Q$ [90]. It follows that we can obtain a pseudo-Hamiltonian description of the dynamics by making the replacements

$$g_{\mu\nu}(x) \quad \to \quad g_{\mu\nu}(x) + h_{R\,\mu\nu}(x, Q'), \tag{5.42a}$$

$$e_\Lambda^{\ \mu} \quad \to \quad e_\Lambda^{\ \mu} - \frac{1}{2} e_\Lambda^{\ \nu}\, g^{\sigma\mu}\, h_{\nu\sigma}(x, Q') \tag{5.42b}$$

in the Hamiltonian (5.34) and expanding to linear order. Here the perturbation to the orthonormal basis is chosen to maintain orthonormality. Note that in order to apply the results of chapter 4, we must use the form (5.34), (5.40) of the dynamical system for which the symplectic form is constant and so not modified by the substitutions (5.42), rather than the original form (5.29), (5.30). This is because the formalism of chapter 4 is tailored for non-local corrections to the Hamiltonian function, while keeping the symplectic form Ω_0 unperturbed.

To complete the pseudo-Hamiltonian formulation of the dynamics, we need to write the pseudo-Hamiltonian in terms of the phase space variables $(x^\mu, \pi_\nu, S_{\Lambda\Sigma})$. We start by writing the metric perturbation (5.41) in terms of the new momentum variable (5.32a) and expanding to linear order in spin, which gives

$$h^{\alpha\beta}(x, Q') = h^{\alpha\beta}_{(m)}(x, Q') + h^{\alpha\beta}_{(S)}(x, Q'), \tag{5.43}$$

where

$$h^{\alpha\beta}_{(m)}(x,Q') = \int d\tau' G^{\alpha\beta\bar{\mu}\bar{\nu}}_{+,R}\big[x,\bar{x}(\tau')\big]\frac{\pi_{\bar{\mu}}(\tau')\pi_{\bar{\nu}}(\tau')}{\sqrt{-g^{\bar{\rho}\bar{\sigma}}\pi_{\bar{\rho}}\pi_{\bar{\sigma}}}}, \tag{5.44a}$$

$$\begin{aligned} h^{\alpha\beta}_{(S)}(x,Q') &= \int d\tau' G^{\alpha\beta\bar{\mu}\bar{\nu}}_{+,R}\big[x,\bar{x}(\tau')\big]\frac{\pi_{\bar{\mu}}\omega_{\bar{\nu}\bar{\Theta}\bar{\Pi}}S^{\bar{\Theta}\bar{\Pi}}}{\sqrt{-g^{\bar{\rho}\bar{\sigma}}\pi_{\bar{\rho}}\pi_{\bar{\sigma}}}} \\ &\quad - \int d\tau' \nabla_{\bar{\rho}} G^{\alpha\beta\bar{\mu}\bar{\nu}}_{+,R}\big[x,\bar{x}(\tau')\big]\frac{\pi_{\bar{\mu}}(\tau')e_{\bar{\Theta}\bar{\nu}}e^{\bar{\rho}}_{\bar{\Pi}}S^{\bar{\Theta}\bar{\Pi}}(\tau')}{\sqrt{-g^{\bar{\rho}\bar{\sigma}}\pi_{\bar{\rho}}\pi_{\bar{\sigma}}}} \\ &\quad + \tfrac{1}{2}\int d\tau' G^{\alpha\beta\bar{\mu}\bar{\nu}}_{+,R}\big[x,\bar{x}(\tau')\big]\frac{\pi_{\bar{\mu}}\pi_{\bar{\nu}}\pi^{\bar{\rho}}\omega_{\bar{\rho}\bar{\Lambda}\bar{\Theta}}S^{\bar{\Lambda}\bar{\Theta}}}{\big[-g^{\bar{\sigma}\bar{\lambda}}\pi_{\bar{\sigma}}\pi_{\bar{\lambda}}\big]^{3/2}}. \end{aligned} \tag{5.44b}$$

Below we will need the metric perturbation $h^{\mu\nu}$ accurate to $O(m)$ and $O(S)$, and we can neglect $O(m^2)$, $O(mS)$ and $O(S^2)$ contributions. Hence in the expressions (5.44) it is sufficient to use the geodesic worldline rather than the solution to Eqs. (5.28) which incorporates $O(S)$ corrections to the worldline. We next make the replacements (5.42) in the Hamiltonian (5.34). This yields the pseudo-Hamiltonian

$$\begin{aligned} \mathcal{H}^{+,R}(Q,Q') &= -\sqrt{-g^{\mu\nu}\pi_\mu\pi_\nu} + \frac{g^{\mu\nu}\pi_\mu\omega_{\nu\Theta\Pi}S^{\Theta\Pi}}{2\sqrt{-g^{\mu\nu}\pi_\mu\pi_\nu}} - \frac{h^{\mu\nu}(x,Q')\pi_\mu\pi_\nu}{2\sqrt{-g^{\mu\nu}\pi_\mu\pi_\nu}} \\ &\quad - \frac{h^{\mu\nu}(x,Q')\pi_\mu\omega_{\nu\Theta\Pi}S^{\Theta\Pi}}{2\sqrt{-g^{\mu\nu}\pi_\mu\pi_\nu}} + \frac{g^{\mu\nu}\pi_\mu e^\alpha_\Theta e^\beta_\Pi h_{\nu[\alpha;\beta]}S^{\Theta\Pi}}{2\sqrt{-g^{\mu\nu}\pi_\mu\pi_\nu}} \\ &\quad - \frac{1}{4}\frac{h^{\mu\nu}(x,Q')\pi_\mu\pi_\nu\pi^\rho\omega_{\rho\Lambda\Theta}S^{\Lambda\Theta}}{\big[-g^{\sigma\lambda}\pi_\sigma\pi_\lambda\big]^{3/2}}, \end{aligned} \tag{5.45}$$

where we used that the perturbation to the spin connection is $\delta\omega_{\mu\Lambda\Pi} = e^\alpha_\Lambda e^\beta_\Pi h_{\mu[\alpha;\beta]}$.

As an aside, we can verify as follows that the the pseudo-Hamiltonian (5.45) with symplectic form (5.40) gives the correct dynamics for a spinning particle under the effect of the first order gravitational self-force. Using Eq. (5.1) we obtain for the equations of motion $\nabla_{\boldsymbol{u}}u^\mu = a^\mu$ and $\nabla_{\boldsymbol{u}}S^{\mu\nu} = N^{\mu\nu}$, where the self-acceleration a^μ and self-torque $N^{\mu\nu}$ are given

by

$$
\begin{aligned}
a^{\mu} &= -\frac{1}{2}\left[g^{\mu\lambda}+u^{\mu}u^{\lambda}\right]\left[2h_{\lambda\rho;\sigma}-h_{\rho\sigma;\lambda}\right]u^{\rho}u^{\sigma} \\
&- \frac{1}{2m}R^{\mu}{}_{\alpha\beta\gamma}\left[1-\frac{1}{2}h^{(m)}_{\rho\gamma}u^{\rho}u^{\gamma}\right]u^{\alpha}S^{\beta\gamma} \\
&+ \frac{1}{2m}\left[g^{\mu\nu}+u^{\mu}u^{\nu}\right]\left[2h^{(m)}_{\nu(\alpha;\beta)\gamma}-h^{(m)}_{\alpha\beta;\nu\gamma}\right]u^{\alpha}S^{\beta\gamma}, \qquad (5.46a) \\
N^{\mu\nu} &= u^{(\rho}S^{\sigma)[\mu}g^{\mu]\lambda}\left[2h^{R(m)}_{\lambda\rho;\sigma}-h^{(m)}_{\rho\sigma;\lambda}\right], \qquad (5.46b)
\end{aligned}
$$

and where the metric perturbation $h_{\mu\nu}$ has been evaluated at $Q'=Q$ after the derivatives have been taken. These equations agree with those of Ref. [90]. They can also be obtained by making the substitutions (5.42) in the equations of motion (5.28). As discussed in the introduction, we keep only terms of order $O(m^2)$, $O(S)$ and $O(mS)$ in the self force, and $O(mS)$ in the self-torque, which explains why we have replaced $h^{\mu\nu}_{R}$ with $h^{\mu\nu}_{R\,(m)}$ [cf. Eq. (5.44a)] in some of the terms in (5.46).

5.4.3 Hamiltonian formulation of the conservative motion of a self-gravitating spinning particle

In this subsection we show that the motion of a spinning point particle under the action of the first order conservative self force is Hamiltonian, by combining the pseudo-Hamiltonian formulation of the motion derived in Sec. 5.4.2 with the general result of Chapter 4. To do this, we need to read off the function $G(Q,Q')$ in phase space defined by Eqs. (5.3), and verify that it satisfies the right symmetry and fall-off conditions.

We start by specializing to the conservative sector of the dynamics. As described in section 5.3 in the nonspinning case, this is achieved by replacing in the pseudo-Hamiltonian (5.45)

the regularized retarded Green's function $G^{\mu\nu\,\bar{\mu}\bar{\nu}}_{+,R}$ with the average $G^{\mu\nu\,\bar{\mu}\bar{\nu}}_{C,R}$ of the retarded and advanced Green's functions, regularized in the same way, and replacing the metric perturbation $h_{\mu\nu}$ with its conservative piece $h^{C,R}_{\mu\nu}$. Note that this Green's function obeys the symmetry property

$$G^{\mu\nu\alpha'\beta'}_{C,R}(x,x') = G^{\alpha'\beta'\mu\nu}_{C,R}(x',x). \tag{5.47a}$$

Next, by comparing the pseudo-Hamiltonian given by Eqs. (5.43) and (5.45) with the general form given by Eqs. (5.2b), and (5.3), we obtain for the function $G(Q,Q')$ on phase space

$$\begin{aligned} G(Q,Q') \;=\; & \frac{1}{4}NN'\Big[-2\pi_\mu\pi_\nu\pi_{\rho'}\pi_{\sigma'}G^{\mu\nu\rho'\sigma'}_{C,R}(x,x') \\ & -2\pi_\mu\pi_\nu S^{\Theta'\Pi'}\pi_{\rho'}\omega_{\sigma'\Theta'\Pi'}G^{\mu\nu\rho'\sigma'}_{C,R}(x,x') \\ & -2\pi_{\rho'}\pi_{\sigma'}S^{\Theta\Pi}\pi_\mu\omega_{\nu\Theta\Pi}G^{\mu\nu\rho'\sigma'}_{C,R}(x,x') \\ & +2\pi_\mu\pi_\nu\pi_{\rho'}e_{\Theta'\sigma'}e^{\lambda'}_{\Pi'}S^{\Theta'\Pi'}\nabla_{\lambda'}G^{\mu\nu\rho'\sigma'}_{C,R}(x,x') \\ & +2\pi_{\rho'}\pi_{\sigma'}\pi_\mu e_{\Theta\nu}e^{\lambda}_{\Pi}S^{\Theta\Pi}\nabla_{\lambda}G^{\mu\nu\rho'\sigma'}_{C,R}(x,x') \\ & -N'^2\pi_\mu\pi_\nu\pi_{\alpha'}\pi_{\beta'}\pi_{\rho'}\omega^{\rho'}{}_{\Theta'\Pi'}S^{\Theta'\Pi'}G^{\mu\nu\alpha'\beta'}_{C,R}(x,x') \\ & -N^2\pi_{\alpha'}\pi_{\beta'}\pi_\mu\pi_\nu\pi_\rho\omega^{\rho}{}_{\Theta\Pi}S^{\Theta\Pi}G^{\mu\nu\alpha'\beta'}_{C,R}(x,x')\Big], \end{aligned} \tag{5.48}$$

where

$$N = \frac{1}{\sqrt{-g^{\alpha\beta}\pi_\alpha\pi_\beta}}, \quad N' = \frac{1}{\sqrt{-g^{\rho'\sigma'}\pi_{\rho'}\pi_{\sigma'}}}. \tag{5.49}$$

Because of the symmetry property (5.47a) of the Green's function, the function (5.48) satisfies the required symmetry property. It also satisfies the required fall-off conditions, for the

same reasons discussed in section 5.3.

It now follows from the results of chapter 4 that the dynamical system (5.46) admits a Hamiltonian description. The Hamiltonian function is

$$H(Q) = -\sqrt{-g^{\mu\nu}\pi_\mu\pi_\nu} + \frac{g^{\mu\nu}\pi_\mu\omega_{\nu\Theta\Pi}\mathcal{S}^{\Theta\Pi}}{2\sqrt{-g^{\mu\nu}\pi_\mu\pi_\nu}} + H_1(Q) \tag{5.50}$$

where the self-force correction is

$$H_1(Q) = \frac{1}{2}\int G\big[Q, \phi_{\tau'}(Q)\big]d\tau' \tag{5.51}$$

and the 2-point function $G(Q, Q')$ is given by equation (5.48). The symplectic form is given by (5.40), in the phase-space coordinates discussed in section 4.4.

5.5 Scalar case: Second order in the scalar charge

Recall that in Section 3.5, we derived the second-order point-particle effective scalar self-force, given by equation (3.95). To second order in the scalar charge $\hat{q}$, the equations of motion for a point-particle with mass m and worldline γ_s is

$$m\frac{D\dot{\gamma}^\mu}{ds} = -\hat{q}\big(g^{\mu\nu} + \gamma^\mu\gamma^\nu\big)\nabla_\nu\hat{\varphi}(x) \tag{5.52}$$

where the effective field is

$$\hat{\varphi}(x) = \bar{\varphi} + \hat{q}\int G_2^{+,R}(x,\gamma_{s'})ds' + \hat{q}^2\int G_3^{;R}(x,\gamma_s,\gamma_{s'})ds'ds'' + O(\hat{q}^3). \tag{5.53}$$

We specialize to a flat spacetime and a nonlinear potential model, as described in section 3.6. Then, the effective two and three point functions are

$$\begin{aligned}
G_2^{+,R}(x,x') &= G_2^{+}(x,x') - G_{2,0}(x,x') && \text{(5.54a)}\\
G_3^{+,R}(x,x',x'') &= \frac{1}{2}\int V^{(3)}\left[\bar{\varphi}(y)\right]G_2^{+}(x,y)G_2(y,x')G_2^{+}(y,x'')dV_y && \text{(5.54b)}\\
&- G_{3,0}(x,x',x'')\\
&- \int G_2^{+,R}(x,y)G_{2,1}(x',x'',y)dV_y\\
&- \int G_2^{+,R}(y,x')G_{2,1}(x,x'',y)dV_y\\
&- \int G_2^{+,R}(y,x'')G_{2,1}(x,x',y)dV_y.
\end{aligned}$$

Here, we are combining equations (3.82) and (3.99) and repeating them in this section for simplicity. Just like in the previous two sections, we will analyze the motion perturbatively, and replace the functional dependence on an unspecified worldline for the solution to the previous order. Equation (5.52) can then be derived from the following pseudo-Hamiltonian system. We choose phase-space variables $Q^A = (x^\mu, p_\nu)$, a symplectic form $\Omega = dp_\mu \wedge dx^\mu$ and define the solution to equation (5.52) to nth order as $\gamma_s^{(n)}(Q)$, where Q is the initial conditions

of the solution. Then, we can define a pseudo-Hamiltonian function

$$\begin{aligned}\mathcal{H}(Q,Q') = &- \sqrt{-g^{\mu\nu}p_\mu p_\nu} + \hat{q}\bar{\varphi}(x) + \\ &+\hat{q}^2\int G_2^{+,R}\left[x,\gamma_{s'}^{(1)}(Q')\right]ds' + \hat{q}^3\int G_3^{+,R}\left[x,\gamma_{s'}^{(0)}(Q'),\gamma_{s''}^{(0)}(Q'')\right]ds'ds''.\end{aligned} \tag{5.55}$$

Note that this pseudo-Hamiltonian, written in terms of the n-point functions (5.54) produces the full, retarded dynamics (See Appendix B.4 for a derivation of the equations of motion from this pseudo-Hamiltonian). As discussed in chapter 4, a pseudo-Hamiltonian of this kind is equivalent to a Hamiltonian description if the n-point functions are fully symmetric. This property gives a prescription for selecting the conservative sector of the self-force. We expand the retarded Green function in terms of the conservative and dissipative 2-point functions

$$G^C = \frac{G^+ + G^-}{2} \tag{5.56a}$$

$$G^D = \frac{G^+ - G^-}{2} \tag{5.56b}$$

Then, the symmetric piece of the effective 2-point function is

$$G^{C,R}(x,x') = G^C(x,x') - G_{2,0}(x,x'). \tag{5.57}$$

This is equivalent to the rule that conservative dynamics can be derived by replacing the retarded Green function by its symmetric piece.

At second order, this is no longer true. The symmetric piece of the effective 3-point func-

tion is

$$
\begin{aligned}
G_3^{\mathrm{sym},R}(x,x',x'') = &\frac{1}{2}\int V^{(3)}\left[\bar{\varphi}(y)\right] G^C(y,x) G^C(y,x') G^C(y,x'') dV_y \\
&+ \frac{1}{2}\int V^{(3)}\left[\bar{\varphi}(y)\right] G_2^D(x,y) G_2^D(x',y) G_2^D(x'',y) dV_y \\
&- G_{3,0}(x,x',x'') \\
&- \int G^{C,R}(y,x) G_{2,1}(x',x'',y) dV_y \\
&- \int G^{C,R}(y,x') G_{2,1}(x,x'',y) dV_y \\
&- \int G^{C,R}(y,x'') G_{2,1}(x,x',y) dV_y .
\end{aligned}
\tag{5.58}
$$

Notice that the second term is the only term that is fully symmetric under the exchange of arguments that involves the dissipative 2-point function.

At this point, we invoke another property of conservative systems, which is not related to the proof of Chapter 4: Conservative systems must be time-reversal invariant. A time-reversal transformation can be applied by exchanging retarded and advanced Green functions. This transformation preserves the conservative 2-point function G^C but it flips the sign of the dissipative 2-point function G^D. Since there are three powers of the dissipative 2-point function on the second term in equation (5.58), we conclude that this term violates time-reversal in-

variance. Therefore, the conservative regularized 3-point function is given by

$$\begin{aligned} G_3^{C,R}(x,x',x'') = \frac{1}{2}\int V^{(3)}\left[\bar{\varphi}(y)\right] G^C(y,x)G^C(y,x')G^C(y,x'')dV_y \\ - G_{3,0}(x,x',x'') \\ - \int G^{C,R}(y,x)G_{2,1}(x',x'',y)dV_y \\ - \int G^{C,R}(y,x')G_{2,1}(x,x'',y)dV_y \\ - \int G^{C,R}(y,x'')G_{2,1}(x,x',y)dV_y. \end{aligned} \tag{5.59}$$

We see that conservative dynamics can be obtained by the prescription to replace every retarded Green function with its symmetric piece, as was the case with the linear-order self-force.

We conclude that the unique prescription to select the conservative piece of the self-force is to pick the n-point functions which are symmetric under exchange of arguments and invariant under time-reversal transformations. The conservative sector defined by this prescription is then Hamiltonian.

6

Conclusion

This thesis was guided by two questions. First, how do we study the dynamics of binary systems when their self-interactions are strong? Second, is it possible to split the dynamics into conservative and dissipative sectors and, if so, is there a Hamiltonian formulation of the former? These questions play a fundamental role in our understanding of binary systems and are important if we want to model gravitational-wave production accurately.

In Chapters 1 and 2, we reviewed the history of binary system dynamics and derived a series of basic results in the small mass-ratio approximation. We studied the gravitational self-force and discussed the issues that arise at second order when a naive perturbative expansion is carried to second order. The main obstacles to study extreme mass-ratio inspirals were connected to the point-particle limit of systems with strong self-fields.

In Chapter 3, we developed a nonperturbative method to study self-interacting extended objects in scalar theories. Our approach produced renormalized equations of motion, where the object's self-field is reabsorbed into an effective charge density and stress-energy tensor such that the object behaves as a test body moving in an effective external field. This approach provides a clean split between self and external fields and presents us with a physically motivated picture to understand the regularization schemes usually used in the self-force community. The description of extended objects as effective test objects in an external field could be useful to study tidal effects in the strong-gravity regime, where non-linearities usually blur the distinction between external and self-fields. This is an interesting avenue for future work.

The approach developed in Chapter 3 produced nonlocal equations of motion, in the sense that the self-force is a functional of the complete history of the object. This was the starting point of Chapter 4, where we proved that a certain class of non-local dynamical systems admits a local Hamiltonian description. The proof can be applied iteratively to any order in perturbation theory and is useful in the context of the self-force formulation, as well as the post-Newtonian expansion. We applied the proof to 4PN dynamics and derived a local Hamiltonian description, something that was not clearly understood in the literature before.

Finally, in Chapter 5, we applied the results of Chapter 4 to the self-force problem. We derived the local Hamiltonian description of the gravitational self-force to linear order in

mass and spin. We used its Hamiltonian formulation to study integrability, resonant orbits, and the first law of binary black hole dynamics. We found that if the Fourier components of the self-force are nonvanishing over the resonant frequencies of the zeroth-order phase space, then integrability is broken.

In future work, it would be interesting to extend the non-perturbative approach of Chapter 3 to self-interacting systems to the gravitational case and to derive the gravitational second-order self-force this way. After this, it should be straightforward to apply the results of Chapter 4 to obtain its Hamiltonian formulation. Furthermore, our study of self-interacting extended objects is well suited for an application to tidal effects and their imprints on the gravitational-wave production of binary systems.

A

General Theory of bitensors

In this appendix, we follow the Living Reviews article [30] for some useful aspects on the theory of bitensors in curved spacetimes. We finish the appendix with an explicit construction of the 2-point parametrices introduced in Chapter 3, which follows the work in Ref. [43].

A.1 Synge's Worldfunction

One of the most useful bitensors is Synge's worldfunction $\sigma(x_1, x_2)$. Given a segment of a geodesic z_λ parametrized by λ, with initial and final points $x_1 = z_{\lambda_1}$ and $x_2 = z_{\lambda_2}$, Synge's worldfunction is

$$\sigma(x_1, x_2) = \frac{1}{2}(\lambda_2 - \lambda_1) \int_{\lambda_1}^{\lambda_2} g_{\mu\nu}(z_\lambda) \frac{dz^\mu}{d\lambda} \frac{dz^\nu}{d\lambda} d\lambda. \tag{A.1}$$

Synge's worldfunction is therefore half the squared geodesic distance between its arguments. It is defined only when x_2 is in the normal convex neighborhood of x_1, as this guarantees the uniqueness of the geodesic that connects them.

The world function $\sigma(x_1, x_2)$ can be differentiated with respect to either argument. We use numbered indices to represent derivatives respect to the corresponding argument, such that $\sigma_{\alpha_1} \equiv \partial\sigma/\partial x_1^{\alpha_1}$ and $\sigma_{\alpha_2} \equiv \partial\sigma/\partial x_2^{\alpha_2}$. This notation extends to any number of derivatives, for example,

$$\sigma_{\alpha_1\beta_1\alpha_2} \equiv= \nabla_{\alpha_2} \nabla_{\beta_1} \nabla_{\alpha_1} \sigma(x_1, x_2) \tag{A.2}$$

is a rank $(0, 2)$ tensor on its first argument and a 1-form on its second argument.

The first derivative of Synge's worldfunction is proportional to the tangent vector along the geodesic connecting its two arguments

$$\sigma_{\alpha_2}(x_1, x_2) = (\lambda_2 - \lambda_1) g_{\alpha_2\beta_2} \frac{dz^{\beta_2}}{d\lambda}. \tag{A.3}$$

The derivative with respect to the other argument satisfies $\sigma_{\alpha_1}(x_1, x_2) = -\sigma_{\alpha_2}(x_1, x_2)$. From

this identity it follows that

$$g^{\alpha_1\beta_1}\sigma_{\alpha_1}\sigma_{\beta_1} = 2\sigma, \tag{A.4}$$

and

$$\sigma^{\alpha_1}_{;\beta_1}\sigma^{\beta_1} = \sigma^{\alpha_1} \tag{A.5}$$

and similarly for the other argument.

The coincidence limit of Synge's worldfunction and its derivatives can also be calculated. Given a bitensor $F(x,x')$, we denote the coincidence limit by $[F] = F(x,x)$. Then, the coincidence limit of the world function and its derivatives is

$$[\sigma] = 0, \tag{A.6a}$$

$$[\sigma_{\alpha_1}] = [\sigma_{\alpha_2}] = 0, \tag{A.6b}$$

$$[\sigma_{\alpha_1\beta_1}] = -g_{\alpha_1\beta_1}, \tag{A.6c}$$

$$[\sigma_{\alpha_2\beta_2}] = g_{\alpha_2\beta_2}. \tag{A.6d}$$

A.2 Hadamard 2-point functions

Consider the singular two-point functions discussed in Chapter 3. A singular two-point function $G(x,x')$ is symmetric in its arguments, vanishes for timelike separations and satisfies some differential equation. We discussed Green functions, whose equations are exactly sourced by a delta function; and parametrices, whose equations are only approximately sourced by a delta function. In this section, we review the construction of Green functions

and parametrices. The Hadamard ansatz for such a 2-point function is

$$G(x,x') = U(x,x')\delta(\sigma) + V(x,x')\theta(\sigma) \tag{A.7}$$

where $U(x,x')$ and $V(x,x')$ are two smooth biscalars and σ is Synge's worldfunction connecting x and x'. Fixing x as the base point, $U(x,x')$ is the direct piece of the 2-point function, which has support on the lightcone emanating from x, while $V(x,x')$ is the tail piece, which has support only outside the lightcone.

We can derive equations for $U(x,x')$ and $V(x,x')$ for a nonlinear potential model in curved spacetime using the equation for the singular 2-point function

$$\Big[\Box + k(\bar{\varphi})\Big] G(x,x') = \delta(x,x'). \tag{A.8}$$

Here,$\bar{D} = \Box + k(\bar{\varphi})$. We use $k(\varphi)$ as the nonlinear potential as opposed to $V(\varphi)$ to distinguish the potential from the tail term $V(x,x')$. $\bar{\varphi}$ is the background scalar field. We now plug the ansatz A.7 into Equation 3.6. Using known properties of delta and Heaviside functions (See [30] for a detailed derivation), we obtain a series of differential equations and boundary conditions for $U(x,x')$ and $V(x,x')$. The direct term satisfies

$$2U(x_1,x_2)_{,\alpha_2}\sigma^{\alpha_2} + \left(\sigma^{\alpha_2}_{\ \alpha_2} - 4\right)U(x_1,x_2) = 0, \tag{A.9a}$$

$$[U] = 1. \tag{A.9b}$$

These two equations uniquely determine $U(x_1, x_2)$. The solution is

$$U(x_1, x_2) = \Delta^{1/2}(x_1, x_2) \tag{A.10}$$

where $\Delta(x_1, x_2)$ is the van Vleck determinant [30]. The tail piece satisfies

$$\bar{D}V(x_1, x_2) = 0, \tag{A.11a}$$

$$\left\{\left(\sigma^{\alpha_2}_{\ \alpha_2} - 2\right)V + 2\sigma^{\alpha_2}V_{;\alpha_2}\right\}_{\sigma=0} = -\bar{D}U|_{\sigma=0}\,. \tag{A.11b}$$

The first equation tells us that $V(x, x')$ is a homogeneous solution. The second equation provides boundary conditions along the lightcone, determined by $\sigma = 0$.

Note that the direct term is independent of the nonlinear potential $k(\varphi)$. The explanation is that the direct piece $U(x, x')$ describes radiation traveling from x_1 to x_2 without dispersion. In contrast, the tail term $V(x, x')$ describes the effect of radiation that is scattered off the nonlinear potential so that x and x' need not be light-like.

A.3 Hadamard Expansion and Parametrices

In this section, we solve the equations for $V(x, x')$ using a Hadamard expansion [43]. It is not guaranteed that the expansion will always converge. However, a truncated solution which approximately solves Equation (A.11) can always be found, in which case $G(x, x')$ is not a Green function, but a parametrix instead. We also specialize to flat spacetimes, such that $\sigma^{\alpha}_{\ \alpha} = 4$ and $U = 1$.

A Hadamard expansion is a series in powers of $\sigma(x_1, x_2)$

$$V(x_1, x_2) = \sum_{p=0}^{\infty} V_p(x_1, x_2)\sigma^p(x_1, x_2). \tag{A.12}$$

In the following steps, we will truncate this sum to a finite order P. Plugging Equation (A.12) into Equation (A.11a), we get

$$\bar{D}V(x_1, x_2) = \sum_{p=0}^{\infty} \sigma^p \left\{ 2(p+1)\sigma^\alpha \partial_\alpha V_{p+1} + \bar{D}V_p + 2(p+1)pV_{p+1} + 4(p+1)V_{p+1} \right\} \tag{A.13}$$

where all indices represent tensors on x_2 and we omit the notation α_2 for short. Setting $\bar{D}V = 0$ we get a series of equations for each V_p

$$\left[\sigma^\alpha \partial_\alpha + p + 2\right] V_{p+1} = -\frac{\bar{D}V_p}{2(p+1)}. \tag{A.14}$$

This equation is of the form

$$\Big[\sigma^\alpha \partial_\alpha + \kappa\Big] f(x_1, x_2) = F(x_1, x_2). \tag{A.15}$$

Given a geodesic $\gamma(s; x_1, x_2)$ with endpoints

$$\gamma(0; x_1, x_2) = x_1, \tag{A.16a}$$

$$\gamma(1; x_1, x_2) = x_2, \tag{A.16b}$$

we can express σ^α as

$$\sigma^\alpha\big[x_1, \gamma(s)\big] = s\frac{d\gamma^\alpha}{ds}. \tag{A.17}$$

Then, it holds that

$$\frac{d}{ds}\left\{f\big[x, \gamma(s)\big]s^\kappa\right\} = s^{\kappa-1}F\big[x_1, \gamma(s)\big]. \tag{A.18}$$

Therefore, we obtain a solution

$$V_{p+1}(x_1, x_2) = -\int_0^1 s^{p+1}\frac{\bar{D}V_p\big[x_1, \gamma(s; x_1, x_2)\big]}{2(p+1)}. \tag{A.19}$$

If we can find an expression for V_0, then Equation (A.19) gives us all V_p for $p > 0$. We now use the boundary condition $A.11b$. Since it is evaluated on the lightcone, all V_p with $p > 0$ vanish and the boundary condition becomes

$$\sigma^\alpha\partial_\alpha V_0 + V_0 = -\frac{1}{2}k''(\bar{\varphi}). \tag{A.20}$$

This is of the same form as Equation A.15, so we get

$$V_0(x_1, x_2) = -\frac{1}{2}\int_0^1 k''\left[\bar{\varphi}\big[\gamma(s; x_1, x_2)\big]\right] ds. \tag{A.21}$$

Equations (A.19) and (A.21) fully determine the tail term $V(x_1, x_2)$. However, it is not clear that the Hadamard expansion (A.12) converges. If we truncate the sum at a finite order P, then the tail term satisfies a "modified" homogeneous equation

$$\bar{D}V(x_1, x_2) = \mathfrak{s}(x_1, x_2) \tag{A.22}$$

where the extra source term is determined by the last term on the sum

$$\mathfrak{z}(x_1, x_2) = \sigma^P \bar{D} V_P. \tag{A.23}$$

Furthermore, as long as we truncate the sum at a high enough order P, this source term will correspond to a high multipolar moment and will drop off any finite, low-order Multipolar expansion.

B

Derivation of equations of motion

B.1 Covariant techniques for phase-space variations

In this section, we review a set of useful techniques for phase-space variations. We follow conventions from Justin Vines' thesis [91].

Consider a biscalar function of position and 4-velocity, such as a Lagrangian $L(x, \dot{x})$. The

variation of L is

$$\delta L = \frac{\partial L}{\partial x^\mu}\delta x^\mu + \frac{\partial L}{\partial \dot{x}^\mu}\delta \dot{x}^\mu. \tag{B.1}$$

There is a way to explicitly make this variation covariant. We add and subtract an extra term to get

$$\begin{aligned}\delta L &= \frac{\partial L}{\partial x^\mu}\delta x^\mu + \frac{\partial L}{\partial \dot{x}^\mu}\delta \dot{x}^\mu \\ &= \left[\frac{\partial L}{\partial x^\mu} - \dot{x}^\nu \Gamma^\rho_{\nu\mu}\frac{\partial L}{\partial \dot{x}^\rho}\right]\delta x^\mu + \frac{\partial L}{\partial \dot{x}^\mu}\left[\delta \dot{x}^\mu + \dot{x}^\nu \Gamma^\mu_{\nu\rho}\delta x^\mu\right] \\ &= \tilde{\nabla}_\mu L \delta x^\mu + \frac{\partial L}{\partial \dot{x}^\mu} D_\lambda \delta x^\mu\end{aligned} \tag{B.2}$$

where we defined the horizontal covariant derivative acting on a biscalar on configuration space

$$\tilde{\nabla}_\mu L(x,\dot{x}) = \frac{\partial L}{\partial x^\mu} - \dot{x}^\nu \Gamma^\rho_{\nu\mu}\frac{\partial L}{\partial \dot{x}^\rho} \tag{B.3}$$

and the usual covariant derivative

$$D_\lambda = \dot{z}^\nu \nabla_\nu \tag{B.4}$$

The horizontal derivative satisfies a series of useful properties. When acting on a tensor it can be generalized by replacing the first derivative by a covariant one

$$\tilde{\nabla}_\mu T^{\alpha_1\ldots\alpha_n} = \nabla_\mu T^{\alpha_1\ldots\alpha_n} - \dot{x}^\nu \Gamma^\rho_{\nu\mu}\frac{\partial T^{\alpha_1\ldots\alpha_n}}{\partial \dot{x}^\rho}. \tag{B.5}$$

Also, when acting on a tensor $F_\alpha(x)\dot{x}^\alpha$ contracted with a 4-velocity it goes through becoming

a covariant derivative

$$\begin{aligned}\tilde{\nabla}_\mu\Big[F_\alpha(x)\dot{x}^\alpha\Big] &= \frac{\partial F_\alpha}{\partial x^\mu}\dot{x}^\alpha - \dot{x}^\nu\Gamma^\rho_{\nu\mu}F_\rho \\ &= \Big[\frac{\partial F_\alpha}{\partial x^\mu} - \Gamma^\rho_{\alpha\mu}F_\rho\Big]\dot{x}^\alpha \\ &= \nabla_\mu F_\alpha \dot{x}^\alpha\end{aligned} \tag{B.6}$$

We can also apply D_λ to a scalar function in configuration space. We get

$$\begin{aligned}D_\lambda L(x,\dot{x}) &= \dot{x}^\mu\frac{\partial L}{\partial x^\mu} + \ddot{x}^\mu\frac{\partial L}{\partial \dot{x}^\mu} \\ &= \dot{x}^\mu\Big[\frac{\partial L}{\partial x^\mu} - \dot{x}^\nu\Gamma^\rho_{\nu\mu}\frac{\partial L}{\partial \dot{x}^\rho}\Big] + \Big[\ddot{x}^\rho + \Gamma^\rho_{\nu\mu}\dot{x}^\nu\dot{x}^\mu\Big]\frac{\partial L}{\partial \dot{x}^\rho} \\ &= \dot{x}^\mu\tilde{\nabla}_\mu L + a^\mu\frac{\partial L}{\partial \dot{x}^\mu}\end{aligned} \tag{B.7}$$

where the aceleration is defined in the usual way as

$$a^\mu = D_\lambda\dot{x}^\mu. \tag{B.8}$$

A similar construction can be built for biscalars in phase space. Consider a Hamiltonian function $H(x,p)$. A variation on phase space is

$$\begin{aligned}\delta H &= \frac{\partial H}{\partial x^\mu}\delta x^\mu + \frac{\partial H}{\partial p_\mu}\delta p_\mu \\ &= \Big[\frac{\partial H}{\partial x^\mu} + p_\alpha\Gamma^\alpha_{\mu\nu}\frac{\partial H}{\partial p_\nu}\Big]\delta x^\mu + \frac{\partial H}{\partial p_\mu}\Big[\delta p_\mu - p_\alpha\Gamma^\alpha_{\mu\nu}\delta x^\mu\Big] \\ &= \tilde{\nabla}_\mu H\delta x^\mu + \frac{\partial H}{\partial p_\mu}\Delta p_\mu\end{aligned} \tag{B.9}$$

where we defined the covariant variation

$$\Delta p_\rho = \delta p_\rho - p_\alpha\Gamma^\alpha_{\mu\rho}\delta x^\mu \tag{B.10}$$

which is the variation that the 1-form $\tilde{p}$ experiences due to the coordinate change: the first term is the explicit change in its coordinates p_μ and the second term is the change in the basis $\tilde{e}^\mu$ due to the change in position Δx^μ.

We use the symbol $\tilde{\nabla}_\mu$ for horizontal derivatives in configuration and phase space. The distinction is clear from context.

B.2 Geodesic Motion

In this section, we derive the equations of motion of a free-falling point particle using the Lagrangian and Hamiltonian formulations. We then discuss the differences between using the "square-root" Hamiltonian and the "quadratic" Hamiltonian.

B.2.1 Lagrangian formulation

Consider a point particle with mass m following a geodesic $x(\lambda)$ on a manifold with metric tensor g_{ab}. The action principle is

$$\begin{aligned} S &= -m \int d\tau \\ &= -m \int \sqrt{-g_{\mu\nu} \frac{dx^\mu}{d\lambda} \frac{dx^\nu}{d\lambda}} d\lambda. \end{aligned} \tag{B.11}$$

Here, τ is proper time respect to the metric tensor g_{ab} and λ is an arbitrary time parameter. Using the techniques from section B.1, the Euler-Lagrange equations are

$$\frac{D}{d\lambda} \frac{dx^\mu}{d\lambda} - \kappa \frac{dx^\mu}{d\lambda} = 0, \tag{B.12}$$

where κ measures the failure of λ to be an affine parameter on the geodesic x

$$\kappa \equiv \frac{1}{\sqrt{-g_{\mu\nu}\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda}}}\frac{d}{d\lambda}\sqrt{-g_{\mu\nu}\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda}}. \tag{B.13}$$

The key feature of the "square-root" Lagrangian is that it is explicitly reparametrization invariant. Indeed, choosing proper time τ as the parametrization, we get $\sqrt{-g_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau}} = 1$ and the Euler-Lagrange equations become

$$\frac{D}{d\tau}\frac{dx^\mu}{d\tau} = 0, \tag{B.14}$$

which is the usual geodesic equation written in terms of proper time.

An alternative Lagrangian formulation can be obtained by giving up reparametrization invariance. We use instead the "quadratic" Lagrangian

$$L = \frac{m}{2}g_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau}. \tag{B.15}$$

The equations of motion are

$$\frac{D}{d\tau}\frac{dx^\mu}{d\tau} = 0. \tag{B.16}$$

In other words, this Lagrangian formulation is not reparametrization invariant and is written directly in terms of proper time.

B.2.2 Hamiltonian formulation

It is not possible to derive a Hamiltonian formulation from the "square-root" Lagrangian, using a Legendre transformation. This is because reparametrization invariance fixes $H = 0$

identically. We are left with two options, the first one is to derive a Hamiltonian formulation from the "quadratic" Lagrangian. The Legendre transformation yields the Hamiltonian function

$$H = \frac{g^{\mu\nu} p_\mu p_\nu}{2m} \tag{B.17}$$

where m is a constant parameter. Hamilton equations are

$$\frac{dx^\mu}{d\tau} = \frac{p^\mu}{m}, \tag{B.18a}$$

$$\frac{Dp_\mu}{d\tau} = 0. \tag{B.18b}$$

The main problem with this formulation is that the 4-moment is constraint by the mass-shell equation $p^2 = -m^2$.

An alternative is to give up the Legendre transformation altogether and start directly from the Hamiltonian function

$$H = -\sqrt{-g^\mu p_\mu p_\nu}. \tag{B.19}$$

Hamilton equations are then

$$\frac{dx^\mu}{d\tau} = \frac{p^\mu}{\sqrt{-g^{\mu\nu} p_\mu p_\nu}}, \tag{B.20a}$$

$$\frac{Dp_\mu}{d\tau} = 0. \tag{B.20b}$$

Now, the 4-momentum is not constraint by the mass parameter m. Its norm p^2 is preserved by the dynamics, but we are free to pick any mass-shell.

In more technical terms, the generated by the "quadratic" Hamiltonian (B.17) are con-

straint to a submanifold of phase space, determined by the mass-shell condition $p^2 = -m^2$. The "square-root" Hamiltonian gives dynamics on the full phase space, and the dynamics produces phase-space trajectories that preserve the norm of the 4-momentum, determined by the initial conditions. In any case, both formulations give up reparametrization invariance and give the equations of motion directly written in proper time. This is a feature of Hamiltonian formulations and the role of the Hamiltonian function as the generator of time translations.

B.3 First-order gravitational self-force in Hamiltonian formulation

Consider a point-particle moving along a geodesic z_σ on an effective spacetime given by the metric tensor

$$g^{\text{eff}}_{\mu\nu} = g_{\mu\nu} + \varepsilon h_{\mu\nu} + O(\varepsilon^2). \tag{B.21}$$

The metric perturbation sourced by a point particle is [92]

$$h_{\mu\nu}(x;[z,p]) = \int G_{\mu\nu\alpha'\beta'}\left(x, z_{\sigma'}\right) p^{\alpha'} \frac{dz^{\beta'}}{d\sigma} d\sigma'. \tag{B.22}$$

We use the "square-root" Hamiltonian formulation for the effective metric and expand to linear order in ε to obtain a nonlocal Hamiltonian

$$H = -\sqrt{-g^{\mu\nu}p_\mu p_\nu} - \varepsilon \frac{1}{2\sqrt{-g^{\mu\nu}p_\mu p_\nu}} h^{\mu\nu} p_\mu p_\nu. \tag{B.23}$$

Next, define phase-space variables $Q^A = (x^\mu, p_\mu)$ and use the zeroth order solutions to express the metric perturbation in terms of phase-space variables only. We define the zeroth-order Hamiltonian flow by $\phi_\sigma(Q) = \left(x^{\bar{\mu}}, p_{\bar{\mu}}\right)$ where barred indices mean that the variable is a zeroth order solution which depends on some initial condition Q'. The metric perturbation is

$$h_{\mu\nu}(x, Q') = \int G_{\mu\nu\alpha'\beta'}\left(x, \bar{x}_{\sigma'}\right) \frac{p^{\bar{\alpha}'} p^{\bar{\beta}'}}{\sqrt{-g^{\alpha'\beta'} p_{\bar{\alpha}'} p_{\bar{\beta}'}}} d\sigma'. \tag{B.24}$$

Plugging the metric perturbation (B.24) back into the nonlocal Hamiltonian function, we get a pseudo-Hamiltonian function

$$H(Q, Q') = -\sqrt{-g^{\mu\nu} p_\mu p_\nu} - \varepsilon \int G_{\mu\nu\alpha'\beta'}\left(x, \bar{x}_{\sigma'}\right) \frac{p^\mu p^\nu p^{\bar{\alpha}'} p^{\bar{\beta}'}}{2\sqrt{-g^{\mu\nu} p_\mu p_\nu}\sqrt{-g^{\alpha'\beta'} p_{\bar{\alpha}'} p_{\bar{\beta}'}}} d\sigma' \tag{B.25}$$

From the last term on the right hand side, we can read the 2-point function $G(Q, Q')$ from equation (5.7) in section 5.3.

Using the covariant techniques from section B.1 and defining $m = \sqrt{-g^{\mu\nu} p_\mu p_\nu}$ for short, Hamilton equations are

$$\frac{dx^\mu}{d\sigma} = \frac{p^\mu}{m} - \varepsilon \frac{p^\nu}{m} h^\mu{}_\nu - \frac{\varepsilon}{2} \frac{p^\mu}{m} \frac{p^\alpha}{m} \frac{p^\beta}{m} h_{\alpha\beta} \tag{B.26a}$$

$$\frac{Dp_\mu}{d\sigma} = \frac{\varepsilon}{2} h_{\alpha\beta;\mu} \frac{p^\alpha p^\beta}{m} \tag{B.26b}$$

Plugging Equation (B.26a) into Equation (B.26b) and keeping only terms first order in ε, we

get

$$\frac{D}{d\sigma}\left(m\frac{dx^\mu}{d\sigma}\right) = \varepsilon\frac{m}{2}h_{\alpha\beta}^{\ \ ;\mu}\frac{dx^\alpha}{d\sigma}\frac{dx^\beta}{d\sigma} - \varepsilon m h^\mu_{\ \alpha;\beta}\frac{dx^\alpha}{d\sigma}\frac{dx^\beta}{d\sigma} - \varepsilon\frac{m}{2}h_{\alpha\beta;\gamma}\frac{dx^\alpha}{d\sigma}\frac{dx^\beta}{d\sigma}\frac{dx^\gamma}{d\sigma}\frac{dx^\mu}{d\sigma}. \tag{B.27}$$

The "square-root" Hamiltonian gives equations of motion written in terms of proper time, which means that σ is proper time respect to the effective metric g^{eff}_{ab}. We want to obtain equations of motion in terms of proper time respect to the background metric τ. These parameters relate by

$$\frac{d\sigma}{d\tau} = 1 - \frac{\varepsilon}{2}h_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau}. \tag{B.28}$$

This means that we are allowed to replace σ by τ on any term which is already order $O(\varepsilon)$, since the error will be $O(\varepsilon^2)$. Furthermore, comparing Equation (B.28) with the geodesic equation in an arbitrary time parameter (B.12), we see that the derivative of m on the left hand side of equation (B.27) will provide exactly the factor of κ that relates proper time respect to the background metric and the effective metric. We can therefore rewrite all terms with respect to the background proper time τ, such that all factors of m cancel out and we get

$$\frac{Du^\mu}{d\tau} = \varepsilon\frac{1}{2}h_{\alpha\beta}^{\ \ ;\mu}u^\alpha u^\beta - \varepsilon h^\mu_{\ \alpha;\beta}u^\alpha u^\beta - \varepsilon\frac{1}{2}h_{\alpha\beta;\gamma}u^\alpha u^\beta u^\gamma u^\mu, \tag{B.29}$$

where we defined the 4-velocity with respect to background proper time

$$u^\mu \equiv \frac{dx^\mu}{d\tau}.$$

Equation (B.29) is the correct first-order self-force, first introduced in Equation (2.30).

B.4 Second-order scalar self-force in Hamiltonian formulation

In section 3.5, we derived the second-order equations of motion for a scalar point particle with mass m and scalar charge $\hat{q}$

$$\frac{D}{d\tau}\left(mu^{\mu}\right) = -g^{\mu\nu}\nabla_{\nu}\hat{\varphi} \tag{B.30}$$

where the effective field is

$$\hat{\varphi}(x;z] = \bar{\varphi}(x) + \hat{q}\int G_2^{+,R}(x,z_{\tau'})d\tau' + \hat{q}^2\int G_3^{+,R}\left(x,z_{\tau'},z_{\tau''}\right)d\tau' d\tau'' \tag{B.31}$$

and the mass satisfies

$$\frac{dm}{d\tau} = u^{\mu}\nabla_{\mu}\hat{\varphi}. \tag{B.32}$$

Plugging Equation (B.32) into (B.30), we get

$$m\frac{Du^{\mu}}{d\tau} = -\left(g^{\mu\nu} + u^{\mu}u^{\nu}\right)\nabla_{\nu}\hat{\varphi}. \tag{B.33}$$

We are interested in the dynamics up to cubic order in $\hat{q}$, which corresponds to the second-order self-force. We define $z_{\tau}(Q)$ to be the first-order solution, with initial conditions $Q^A =$

(x^a, p_a). In this section, we prove that the pseudo-Hamiltonian

$$\begin{aligned}\mathcal{H}(Q,Q') = &-\sqrt{-g^{ab}p_a p_b} + \hat{q}\bar{\varphi} + \hat{q}^2 \int G_2^{+,R}\left[x, z_{\tau'}(Q')\right] d\tau' \\ &+ \hat{q}^3 \int G_3^{+,R}\left[x, z_{\tau'}(Q'), z_{\tau''}(Q')\right] d\tau' d\tau''\end{aligned} \tag{B.34}$$

returns the equations of motion (B.30). Hamilton equations are

$$\frac{dx^\mu}{d\tau} = \frac{p^m u}{\sqrt{-g^{\alpha\beta}p_\alpha p_\beta}} \tag{B.35a}$$

$$\begin{aligned}\frac{Dp_\mu}{d\tau} = &-\hat{q}\nabla_\mu\bar{\varphi} - \hat{q}^2\nabla_\mu \int G_2^{+,R}\left[x, z_{\tau'}(Q')\right] d\tau' \\ &- \hat{q}^3\nabla_\mu \int G_3^{+,R}\left[x, z_{\tau'}(Q'), z_{\tau''}(Q')\right] d\tau' d\tau''.\end{aligned} \tag{B.35b}$$

Defining $m = \sqrt{-g^{ab}p_a p_b}$ and plugging equation (B.35a) into (B.35b), we recover the Equations of motion (B.30).

C

Alternative description of non-local dynamical systems

C.1 Relation to the work of Llosa and Vives

Llosa and Vives [54] consider non-local-in-time action principles in configuration space $(x^\mu, \dot{x}^\mu)$, which they describe as a non-local Lagrangian $L[x]$ which is a functional of x. In Chapter 4

we instead consider an action functional of a phase-space trajectory $Q^A = (x^\mu, p_\mu)$. Furthermore, they do not carry the perturbative expansion of the non-localities explicitly but rather leave the non-local piece of the action principle unspecified. This affects their final results in two ways. First, their expressions for the local Hamiltonian and symplectic forms depend on functional derivatives of the action functional. Second, without using a perturbative expansion of the nonlocalities, the space of initial data for the Hamiltonian flow is not defined. They assume that an order reduction procedure to make the dynamics local exists and instead work with this unspecified space of initial data. In this paper, we expand the nonlocal-in-time piece of the action functional as a series of integrals of N-point functions, which allows us to derive a simpler local Hamiltonian and symplectic form, expressed explicitly in terms of integrals of said N-point functions, evaluated on points in the unperturbed phase space, which constitutes our space of initial data. Although it is possible that the results of Section 4.3 could be obtained from results in their work, our results are derived using a different method and provide a simpler and more streamlined framework for studying nonlocal-in-time perturbations to all orders. Sections 4.4 and 4.5 are entirely original results.

D

Strong causality condition

The causality condition (**??**) is not entirely satisfactory because it is intrinsically perturbative, as opposed to the locality condition (3.35), which doesn't require a perturbative expansion of the generating functional. One might be tempted to impose a stronger condition

$$\frac{\delta\hat{\varphi}(x;\hat{\rho},\varphi]}{\delta\hat{\rho}(y)} = 0 \text{ for timelike } (x,y) \tag{D.1}$$

which we call *strong causality condition*. This would have two advantages: First, it is nonperturbative. Second, the nonlocality in time of the dependence of the effective field $\hat{\varphi}$ on the physical charge density ρ is limited to one light-crossing time into the causal past/future. This contrasts with the n light-crossing times obtained for the weaker causality condition (3.37), used in the body of the chapter.

However, condition (D.1) also imposes a constraint

$$\hat{\mathcal{G}}_n(x_1, \ldots, x_n) = 0 \text{ if there's at least one timelike pair,} \tag{D.2}$$

which is much stronger than the constraint (3.57). The constraint (D.2) is probably too strong, in the sense that it may not be possible to find any n-point functions which satisfy it, together with the rest of our requirements. In particular, the constraint (D.2) implies some non-trivial identities which must be satisfied by the lower order n-point functions $\mathcal{G}_n$. From Equation (3.52b), we find that when (x_1, x_2) is timelike, we must have

$$\mathcal{G}_3(\underbrace{x_1, x_2}_{\text{timelike}}, x_3) = \frac{1}{2}\int \mathcal{G}_{2,1}(x_1, x_3; y)\mathcal{G}_2(y, x_2)dV_y + \frac{1}{2}\int \mathcal{G}_{2,1}(x_2, x_3; y)\mathcal{G}_2(y, x_1)dV_y \tag{D.3}$$

It seems unlikely that 3-point functions exist which satisfy this requirement as well as our other assumptions. For this reason we use the weaker causality condition (**??**) in the body of the thesis.

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