SUPERSYMMETRY: COMPACTIFICATION, FLAVOR, AND DUALITIES

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SUPERSYMMETRY: COMPACTIFICATION, FLAVOR, AND DUALITIES

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We describe several new research directions in the area of supersymmetry.

In the context of low-energy supersymmetry, we show that the assumption of R-parity can be replaced with the minimal flavor violation hypothesis, solving the issue of nucleon decay and the new physics flavor problem in one stroke. The assumption of minimal flavor violation uniquely fixes the form of the baryon number violating vertex, leading to testable predictions. The NLSP is unstable, and decays promptly to jets, evading stringent bounds on vanilla supersymmetry from LHC searches, whereas the gravitino is long-lived, and can be a dark matter component. In the case of a sbottom LSP, neutral mesinos can form and undergo oscillations before decaying, leading to same sign tops, and allowing us to place constraints on the model in this case.

We show that this well-motivated phenomenology can be naturally explained by spontaneously breaking a gauged flavor symmetry at a high scale in the presence of additional vector-like quarks, leading to mass mixings which simultaneously generate the flavor structure of the baryon-number violating vertex and the Standard Model Yukawa couplings, explaining their minimal flavor violating structure. We construct a model which is robust against Planck suppressed corrections and which also solves the μ problem.

In the context of flux compactifications, we begin a study of the local geometry near a stack of D7 branes supporting a gaugino condensate, an integral component of the KKLT scenario for Kähler moduli stabilization. We obtain an exact

solution for the geometry in a certain limit using reasonable assumptions about symmetries, and argue that this solution exhibits BPS domain walls, as expected from field theory arguments. We also begin a larger program of understanding general supersymmetric compactifications of type IIB string theory, reformulating previous results in an $SL(2,\mathbb{R})$ covariant fashion.

Finally, we present extensive evidence for a new class of $\mathcal{N}=1$ gauge theory dualities relating different world-volume gauge theories of D3 branes probing an orientifold singularity. We argue that these dualities originate from the S-duality of type IIB string theory, much like electromagnetic dualities of $\mathcal{N}=4$ gauge theories.

BIOGRAPHICAL SKETCH

Ben Heidenreich attended Amherst College from 2001 to 2006, graduating *summa cum laude* with a double major in Physics and Music. He wrote an undergraduate honors thesis entitled "Searching for the Electron EDM using a Magnetically Polarized Solid" under the supervision of Larry Hunter. He has been a graduate student in Physics at Cornell University since August 2007, earning a Master's degree in 2011. His PhD research was supervised by Liam McAllister.

To my parents, who promise to read it.

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TABLE OF CONTENTS

LIST OF FIGURES

LIST OF TABLES

CHAPTER 1

INTRODUCTION

1.1 Two puzzles for particle physics

With the recent discovery of the Higgs boson [1, 2], the Standard Model (SM) of particle physics, table 1.1, is now complete. The model is simple and its experimental success has been tremendous: no significant deviation from its predictions have yet been found in the mountains of data produced by the Large Hadron Collider (LHC), nor in the precision measurements of the Large Electron-Positron Collider (LEP) near the electroweak scale (100 GeV).

Nonetheless, several deep mysteries remain. Most importantly, the nature of quantum gravity and how it fits into particle physics remains deeply unclear. Even beyond this, the need to account for neutrino masses and for dark matter necessitates the introduction of new particles somewhere below the Planck scale (10^{19} GeV). Consistency with cosmology requires a mechanism for inflation as well as baryogenesis (see e.g. [3, 4]), both of which likely require further modifications to the model.[1]

However, there are two problems in particular which most immediately indicate the need for new physics beyond the Standard Model.

Firstly, the long sought and recently discovered Higgs boson is expected to arise from a scalar field, the first fundamental scalar field ever found in nature. The absence of light scalar fields is easily explained: the mass of an interacting scalar field is typically subject to large quantum corrections. In the absence of a mechanism to cancel these corrections or some accidental cancellation between the quantum corrections and the bare mass, any interacting scalar field will generically

[1] See however [5] for an opposing viewpoint.

	$SU(3)_C$	$SU(2)_L$	$U(1)_Y$	#
q	$\square$	$\square$	$1/6$	3
$\bar{u}$	$\bar{\square}$	$\mathbf{1}$	$-2/3$	3
$\bar{d}$	$\bar{\square}$	$\mathbf{1}$	$1/3$	3
ℓ	$\mathbf{1}$	$\square$	$-1/2$	3
$\bar{e}$	$\mathbf{1}$	$\mathbf{1}$	1	3
h	$\mathbf{1}$	$\square$	$1/2$	1

$$\mathcal{L} = \mathcal{L}_{\text{kin}} + Y_u h q \bar{u} + Y_d h^\dagger q \bar{d} + Y_e h^\dagger \ell \bar{e} + c.c. - \lambda \left[h^\dagger h - \mu^2/2\lambda\right]^2$$

Table 1.1: The Standard Model of particle physics, in two-component notation. The fields $q, \bar{u}, \bar{d}, \ell, \bar{e}$ are left-handed Weyl fermions, and h is a complex scalar. The gauge bosons for the $SU(3) \times SU(2) \times U(1)$ gauge group are left implicit.

acquire a mass of order the cutoff scale of the theory. Since the cutoff of the Standard Model is potentially as high as the Planck scale, typical scalar masses should be of the same order, explaining the absence of light scalars. However, the Higgs field does not fit this pattern, since successful electroweak symmetry breaking requires its mass to be of order the electroweak scale, seventeen orders of magnitude below the Planck scale! This issue is known as the hierarchy problem, and has long motivated physicists to seek extensions to the Standard Model at or below the TeV scale.

Secondly, although the Standard Model excludes gravity, one might hope that including gravity and the (as yet unknown) details of its quantization will have little bearing on physics well below the Planck scale, where gravity is weak. Unfortunately, this reasonable expectation is at least partially violated by considering perhaps the simplest observable of a quantum field theory coupled to gravity: its vacuum energy. Unlike the situation in the absence of gravity, the vacuum energy now has observable consequences, since all forms of energy gravitate. In particular, a positive vacuum energy has an antigravitating effect, causing an accelerating expansion of the universe. This has catastrophic consequences, since vacuum energy (much like scalar masses) receives large quantum corrections, and might naively be expected to take a corrected value of order $(10^{19}\ \mathrm{GeV})^4$, which would rip apart the universe in an instant.

Clearly if such corrections are indeed present they must somehow cancel against the bare value to be consistent with observations. To make matters worse, the observed vacuum energy (the so-called "cosmological constant") is small but non-zero, roughly $(10^{-3}\ \mathrm{eV})^4$ in the experimentally-favored ΛCDM cosmology. Thus, any cancellation method must be imperfect yet very powerful to produce such a gigantic hierarchy between the corrections and the observed value. To further confuse the issue, the electroweak phase transition (wherein the Higgs field acquires a vev) produces an additional contribution of order $(100\ \mathrm{GeV})^4$, as does the quark condensate of Quantum Chromodynamics (QCD) at the scale $(100\ \mathrm{MeV})^4$. All of these contributions must somehow add up to give a nearly (but not exactly) vanishing result.

Both the hierarchy problem and the cosmological constant problem involve rampant quantum corrections which destroy a seemingly consistent semiclassical picture. To tame these corrections we could impose some additional symmetry which somehow contrives to cancel various contributions against each other. Indeed, there is a natural candidate capable of forcing such a cancellation, known as supersymmetry. We introduce supersymmetry in the next section, motivating its appearance by a study of local symmetries and quantum consistency of massless particles.

However, before proceeding it is prudent to mention another possible viewpoint on these mysteries, known as the anthropic principle. We review this viewpoint in appendix 1.A, arguing that the above considerations remain relevant even from this perspective.

1.2 What is supersymmetry?

Symmetry has long played a central role in physics. Newtonian dynamics reduces to the conservation of energy, momentum, and angular momentum, which in turn follow via Noether's theorem from the invariance of the laws of physics under translations in time and space as well as spatial rotations.[2] Special relativity is likewise founded on the principle of Poincaré invariance, which combines space and time translations with the Lorentz group of rotations and Lorentz boosts.

In particle physics, Poincare invariance restricts the allowed particle types to representations of the little group, i.e. the subgroup of the Lorentz group which leaves the particle's four-momentum p^μ invariant. For a massive particle, the little group consists of spatial rotations in the particle's rest frame, $SO(3)$, whose finite dimensional representations are the familiar $2j+1$ dimensional spin-j representations of quantum mechanics. Conversely, massless particles fall into $SO(2)$ representations[3] according to their helicity, where the spin-j representation is now only two-dimensional (consisting of helicities $\pm j$).

Hence, for $j \geq 1$, massless particles contain fewer degrees of freedom than massive ones, due to the absence of helicities h with $|h| < j$. Upon quantizing the theory, this discontinuity can lead to the appearance of negative norm states for massless particles of spin $j \geq 1$, resulting in inconsistencies. To resolve this dilemma, it is necessary to introduce a local symmetry which can remove the unwanted polarizations from the theory. For instance, Maxwell's equations admit an alternate formulation in terms of a four-vector potential A_μ, where $A'_\mu = A_\mu + \partial_\mu \lambda$ describes the same physics for any function λ. This *gauge symmetry* plays

[2]The symmetries of Newtonian physics also include Galilean boost invariance, whose corresponding conserved quantity is related to the center of mass position, see e.g. [6].

[3]The little group for massless particles is actually $ISO(2)$, leading to the possibility of so-called "continuous spin" representations, see for instance [7].

an essential role in quantizing the electromagnetic field, removing negative norm states from the theory via gauge-fixing ghosts.[4] The same considerations apply to any massless spin one field: an associated gauge symmetry is required to remove negative norm states from the quantum theory.

The general theory of relativity, which describes gravity, is built around the principle of *general covariance*, a local form of Lorentz invariance which implements the equivalence principle (the equivalence of acceleration and gravitational fields). Although quantizing gravity is famously subtle, a semiclassical theory of gravity can still be constructed as an effective field theory as in [9], much like a gauge theory which is not asymptotically free. In performing such a quantization, general covariance plays a similar role to gauge invariance, removing unwanted polarizations and negative norm states from the Hilbert space of the theory (see e.g. [7]).

Since local symmetries are essential to the quantization of both spin one fields (gauge bosons) and spin two fields (the graviton), and different spins have so far relied on different kinds of local symmetry, it is natural to ask whether higher spin fields can be quantized and what local symmetries will result. Surprisingly, this turns out to be impossible for interacting fields [10] unless *all* higher spins are included at once [11].[5] This suggests that gauge invariance and local Lorentz invariance are in some sense special among the universe of possible symmetries, and indeed these are the only local symmetries evident in nature.

However, we have so far considered only bosonic fields (integer spins). Spin-1/2 particles are abundant in nature, but (like scalars) their quantum description does

[4]More formally, the gauge symmetry defines a BRST cohomology on the naive Hilbert space of the theory, where the physical Hilbert space consists of cohomology classes. Since the negative norm states are not BRST closed, they are absent from the physical Hilbert space, see [8] for a concise review.

[5]An analogous (possibly equivalent) situation may occur in tensionless string theory.

not rely on a local symmetry. Conversely, much like higher-spin bosonic fields, massless fermions with $j > 2$ do not admit consistent interactions except in the presence of all higher spins.

Only one case remains to be considered, that of massless spin-3/2 fermions. Much like gauge bosons and the graviton, a local symmetry must be introduced to successfully quantize the theory. Such a symmetry is called a *local supersymmetry*. Local supersymmetry (SUSY) transformations turn out not to commute with local Lorentz transformations, hence the spin-3/2 particle can be thought of as fermionic partner of the graviton (called a gravitino), where together the two particles form a *supergravity multiplet*. More than one gravitino may be present, in which case there are several independent supersymmetries. The number of massless gravitinos — which we denote by $\mathcal{N}$ throughout this thesis — corresponds to the amount of unbroken supersymmetry.

In the energy ranges achievable in present-day colliders, gravity plays a negligible role, and it is often convenient to omit the graviton (hence also the gravitino) from theoretical models of particle physics. However, just as local Lorentz invariance has a corresponding global symmetry, local supersymmetry also has a global analog, which has profound consequences for the particle spectrum despite the absence of the corresponding gravitino. Much like Lorentz invariance, particles are now classified by representations of the supersymmetry group. The fermionic supercharges relate fermions and bosons, so these representations now contain multiple fields of different spin.

For instance, in the minimal $\mathcal{N} = 1$ case (one gravitino), the massless multiplets with $j \leq 2$ are the supergravity multiplet, consisting of the graviton and gravitino, the gauge multiplet, consisting of a gauge boson and a Weyl fermion (gaugino), and the chiral multiplet, consisting of a Weyl fermion and a complex scalar,[6] with analogous massive multiplets. Unbroken supersymmetry requires that both components of each multiplet have the same mass, charge, etc., and so predicts that for each observed particle there should be a *superpartner* with all the same properties but different spin.

This pairing between bosons and fermions produces the cancellation of quantum corrections we set out to find, since bosons and fermions contribute with opposite signs to these corrections. In the case of corrections to the scalar masses, this can be understood by the requirement that both components of the chiral multiplet have the same mass, whereas spin-1/2 fields are protected from large quantum corrections by a so-called "chiral symmetry." In the case of vacuum energy, the supersymmetry algebra which relates supercharges with energy and momentum (conserved charges of the Poincare algebra) ensures that the vacuum energy of a supersymmetric vacuum must vanish.[7]

Unlike gauge invariance and local Lorentz invariance, unbroken supersymmetry is not present in nature. For instance, there is no massless "photino" with couplings to charged matter similar to those of the photon, and superpartners for other light particles are likewise absent. Hence, the physics we observe is $\mathcal{N} = 0$.

However, this does not exclude supersymmetry from playing a role in fundamental physics, as it may be present in the UV theory yet spontaneously broken by the vacuum we live in. If supersymmetry is spontaneously broken near the weak scale,[8] this can explain the absence of large quantum corrections to the mass of

[6]There is also a representation consisting of spin-1 and spin-3/2 fields, but consistent interactions involving this representation will require $\mathcal{N} \geq 2$, since a massless interacting spin-3/2 field can only be a gravitino.

[7]In the case of local supersymmetry, supersymmetric vacua with negative vacuum energy are also possible.

[8]Technically, phenomenologically viable models of spontaneous (dynamical) supersymmetry breaking will break supersymmetry first in a hidden sector; while the effects of this breaking are

	$SU(3)_C$	$SU(2)_L$	$U(1)_Y$	#
q	$\square$	$\square$	$1/6$	3
$\bar{u}$	$\bar{\square}$	$\mathbf{1}$	$-2/3$	3
$\bar{d}$	$\bar{\square}$	$\mathbf{1}$	$1/3$	3
ℓ	$\mathbf{1}$	$\square$	$-1/2$	3
$\bar{e}$	$\mathbf{1}$	$\mathbf{1}$	1	3
h_u	$\mathbf{1}$	$\square$	$1/2$	1
h_d	$\mathbf{1}$	$\square$	$-1/2$	1

Table 1.2: The Minimal Supersymmetric Standard Model (MSSM). All of the fields shown in the table are chiral supermultiplets, consisting of a left-handed Weyl fermion and a complex scalar. The vector multiplets (consisting of gauge bosons and gauginos) for the $SU(3) \times SU(2) \times U(1)$ gauge group are left implicit.

the Higgs field, solving the hierarchy problem.

The minimal scenario (that with the fewest additional fields) which realizes this solution, known as the Minimal Supersymmetry Standard Model (MSSM), is shown in table 1.2 (see e.g. [12] for further details). At the renormalizable level, the supersymmetric part of the Lagrangian is determined by the *superpotential*:

$$W = \mu h_u h_d + Y_u q \bar{u} h_u + Y_d q \bar{d} h_d + Y_e \ell \bar{e} h_d \tag{1.1}$$

where R-parity (under which the SM superpartners are odd) has been imposed in addition to gauge invariance to forbid renormalizable lepton and baryon number violation (leading to proton decay). To give masses to the superpartners, a supersymmetry-breaking Lagrangian must be added:

$$\mathcal{L}_{\text{soft}} = m_{\tilde{q}} \tilde{q}^\dagger \tilde{q} + \ldots + A_u \tilde{q} \tilde{\bar{u}} h_u + \ldots + c.c. \tag{1.2}$$

where $\tilde{q}$ denotes the scalar component of the superfield q, etc. These "soft terms" consist of explicit mass terms for the squarks, sleptons and gauginos (superpartners of the quarks, leptons, and gauge bosons) and for the Higgs fields, as well as bi- and

felt in the visible (Standard Model) sector at the electroweak scale, they will occur at a higher scale in the hidden sector.

trilinear couplings between the scalar fields analogous to the allowed superpotential terms, and are understood to be generated through couplings of the MSSM fields to some hidden sector which breaks supersymmetry spontaneously (giving a mass to the gravitino).

Despite its complications, the MSSM is a relatively simple extension of the Standard Model, consisting mainly of the same fields as before plus their superpartners. Above the electroweak scale, the Higgs mass is protected from quantum corrections by the inclusion of the superpartners (especially the stop squark, superpartner of the top quark), and the theory is apparently UV complete up to the Planck scale, modulo questions about the origin of supersymmetry breaking and other issues which existed in the Standard Model itself.[9]

Thus, low-scale supersymmetry solves the hierarchy problem. What about the cosmological constant problem? Unfortunately, although in principle supersymmetry can also solve this problem, it would have to be broken at a much lower scale — 10^{-3} eV rather than between 100 GeV and 1 TeV (and yet higher in the hidden sector) — in order to do so. This is one reason why the cosmological constant problem remains one of the deepest mysteries in physics.

1.3 Why study supersymmetry?

The MSSM is over thirty years old, almost as old as the Standard Model itself. It has long been known that low-scale supersymmetry solves the hierarchy problem, whereas it clearly does not solve the cosmological constant problem. It might seem a purely experimental question (as yet unresolved) whether low-scale supersymmetry is realized in nature, with no substantial theoretical questions yet to be answered.

[9]The MSSM, like all extensions of the Standard Model, does introduce flavor problems, which are extensively discussed in chapters 5 and 7.

This view is incorrect for a number of reasons. In the most direct sense, it defies credulity to posit that all possible theoretical scenarios have been considered and explored. Experimental results could still prove to be surprisingly. Beyond this obvious caveat, we highlight three areas — by no means a comprehensive list — where important theoretical questions remain. All three areas will be discussed further in later chapters.

Firstly, the ability of supersymmetry to solve the hierarchy problem depends crucially on the Standard Model superpartners (especially the stop squark) having masses near the electroweak scale. Unfortunately, LHC searches to date appear to exclude squarks with masses less than about 1 TeV, inevitably reintroducing some degree of tuning and hence a "little" hierarchy problem. However, these exclusions rely on the additional assumption of "R-parity." While prior to the LHC era R-parity was widely (but not universally) accepted for reasons we review in section 5.1, more work is now needed to explore well-motivated scenarios for R-parity violation if low-energy supersymmetry is not to be wholly abandoned as a possible solution to the hierarchy problem. One possible approach is described in chapter 5 and discussed further in chapters 6 and 7.

Secondly, supersymmetry may play an important role in a consistent theory of quantum gravity. Indeed, string theory — by far the best understood candidate for such a theory — is inherently supersymmetric. In string theory, the need to accommodate a small positive cosmological constant once again favors low-scale supersymmetry, which also helps string theorists maintain control over their calculations. Yet supersymmetry comes with its own problems, most notably a proliferation of "moduli," flat directions in the scalar potential corresponding to the size and shape of the six compact dimensions (which together with the four visible dimensions fill out string theory's native ten dimensions) or to the positions of branes along the compact dimensions, etc. When supersymmetry is broken, these flat directions invariably receive quantum corrections, typically driving the moduli vevs to large or small values where theoretical control over the calculation is lost.

To address this issue, the moduli must either be stabilized supersymmetrically prior to supersymmetry breaking, or else supersymmetry breaking itself must be carefully engineered to ensure that the remaining moduli are stabilized in a controllable regime. The former approach, which requires a nontrivial embedding of the residual four-dimensional $\mathcal{N} = 1$ supersymmetry within the $\mathcal{N} = 8$ supersymmetry native to string theory, is described in more detail in chapters 2 and 3.

Finally, even if low-energy supersymmetry is excluded — indeed, even if string theory itself were shown not to describe nature — supersymmetric theories possess a certain simplicity which can further our understanding of nonsupersymmetric theories as well. For instance, when studying the strongly coupled dynamics of non-supersymmetric gauge theories, supersymmetric gauge theories can be thought of as analogous to the Ising model in condensed matter physics, a simple computable model with the right qualitative features to explain the general behavior of the physical system of interest.[10] Thus, the strongly coupled behavior of SUSY gauge theories is interesting in its own right, whereas it is usually more easily determined than that of a non-supersymmetric gauge theory such as QCD.

In particular, the study of supersymmetric dualities has proven to be fruitful in understanding these theories. These dualities relate a strong coupled SUSY gauge theory to another SUSY gauge theory (in some cases weakly coupled) at low

[10]String theory could be viewed in a similar light when considering questions of quantum gravity as a whole, though the analogy fails in part for two reasons. Firstly, string theory as a theory of quantum gravity is only computable in very special circumstances using present techniques (though it has already provided valuable insights in these areas). Secondly, it is not known in principle whether consistent theories of quantum gravity exist which are *not* string theory.

energies, and provide crucial insights into their infrared behavior. Such dualities are considered in more depth in chapter 4.

1.4 Outline of thesis

The outline of the remainder of this thesis is as follows:

In chapter 2, we discuss the effects of nonperturbative stabilization of type IIB Kähler moduli in the KKLT scenario on the geometry of the compact dimensions. We focus on the vicinity of the D7 branes supporting a gaugino condensate, choosing a simple topology for the rigid four cycle wrapped by the branes and adding an O7 plane to cancel tadpoles. Assuming unbroken supersymmetry and that quantum effects do not break the isometry group of the four cycle, we obtain an exact solution describing the stabilized geometry near the seven-branes. We find calibrated domain walls in our solution coming from D3 branes wrapping a torsion cycle. These domain walls reproduce the expected vacuum structure of the condensing gauge theory.

In chapter 3, we consider the general conditions for a supersymmetric compactification of type IIB string theory. Building on previous work, we express these conditions in an $SL(2,\mathbb{R})$ covariant fashion, making the redundancies of type IIB supergravity manifest. We find that solutions fall into two classes, "chargeless" and "charged" solutions, distinguished by the appearance of a globally defined constant $SL(2,\mathbb{R})$ doublet in the latter case. We focus our attention on chargeless solutions, which include all AdS vacua. The formalism developed in this chapter may prove useful in generalizing F-theory constructions beyond the well-studied conformally Calabi-Yau regime.

In chapter 4, we consider D3 branes probing an orientifold of the $\mathbb{C}^3/\mathbb{Z}_3$ geometry, a closely related setup to that considered in chapter 2. We show that the resulting $\mathcal{N}=1$ worldvolume gauge theories exhibit new dualities, which we interpret as arising from S-duality in string theory, much like electromagnetic duality for $\mathcal{N} = 4$ gauge theories. We present extensive checks of the duality, including matching the superconformal index between the two theories, and discuss the infrared behavior. We argue that this new class of dualities generalizes to other orientifold singularities, several of which we discuss in some detail. In particular, gauge theories arising from the dP_1 singularity exhibit a deformed moduli space for a certain value of N, and we show that the dual theories produce the same moduli space through apparently different mechanisms.

In chapter 5, we argue that the assumption of R-parity in the MSSM can be replaced by the principle of minimal flavor violation (MFV), which is sufficient to ensure stability of the proton while evading LHC searches which rely on large amounts of missing transverse energy. The model depends on only one new parameter in addition to those present in minimal flavor violating versions of the MSSM. We discuss the LHC phenomenology of the model, suggesting that a stop NLSP is particularly natural and difficult to rule out, whereas a gravitino LSP can be a component of dark matter. We also discuss what complications arise upon incorporating neutrino masses into this framework, and show that doing so imposes only weak constraints.

In chapter 6, we discuss a distinctive signature of the model introduced in chapter 5 which arises for certain parameter choices. In particular, we argue that a sbottom NLSP is sufficiently long lived to hadronize and to undergo mesino-antimesino oscillations when neutral mesinos are produced. This leads to the distinctive signature of same-sign tops for a substantial fraction of the resulting decays, allowing a range of candidate sbottom NLSP masses to be excluded using existing searches for same-sign dileptons. In an appendix, we clarify an older calculation regarding the mesino oscillation rate.

Finally, in chapter 7 we discuss a high-scale model which naturally reproduces the MFV SUSY structure postulated in chapter 5. This is accomplished by introducing three vector-like generations of quarks together with a gauged flavor symmetry. Having done so, the Yukawa couplings and the baryon-number violating vertex become flavor universal at high scales. Upon spontaneously breaking the gauged flavor symmetry, the extra quarks become massive, and are integrated out. The hierarchical Yukawa couplings arise from the mixings angles between the light and heavy quarks, which appear in the same form in the baryon number violating vertex, producing an R-parity violating superpotential of the MFV SUSY form. We show that the model can be made robust against Planck-suppressed corrections by introducing a gauged discrete $\mathbb{Z}_{11}$ R-symmetry, which simultaneously solves the μ problem. We argue that deviations from minimal flavor violation are strongly suppressed, and compute the soft terms in a gravity mediated scenario.

1.A On the anthropic principle

The need to avoid fine tuning, particular that of bare couplings against their quantum corrections, is known as "naturalness." We have applied this idea in motivating supersymmetry as a solution to the hierarchy problem (and a partial solution to the cosmological constant problem). Yet naturalness is not universally accepted as a necessary physical principle. Critics of the idea most often cite the anthropic principle, which we now review, as an alternative explanation for fine tunings.

The anthropic principle is perhaps best explained by analogy with the search for extraterrestrial life, in particular life on other planets. A study of our own solar system suggests that not every planet can support life. One is then led to wonder why the Earth has just the right characteristics (temperature, magnetic field, etc.) which appear to be necessary for the evolution of intelligent life forms. A natural explanation is that there are many terrestrial planets in the universe, and that we find ourselves on a hospitable one *because* the right conditions are present for the evolution of intelligent life forms to occur. This explanation, known as the *anthropic principle*, assumes a sufficiently large ensemble of candidate planets with sufficient variation such that the probability of intelligent life evolving somewhere in the universe is essentially unity.

So far, there is nothing particularly remarkable or controversial about this line of reasoning. Trouble can arise, however, when the anthropic principle is applied to justify particular features of our environment. One might, for instance, attempt to "predict" the magnitude of the Earth's magnetic field, its surface gravity, atmospheric composition, etc., using the anthropic principle. All of these quantities affect life as we know it, yet we are ill-equipped to understand exactly how they would affect alien life. Put differently, familiarity with our own circumstances and the parameters seemingly necessary for *human* life can prejudice *a posteriori* explanations based on the anthropic principle towards requiring these same parameters, even if they are not the right conditions for the existence of some alien life-form we have yet to encounter or understand.[11]

The anthropic principle can theoretically be applied to explain fine tunings in particle physics on two conditions: (i) there must be a multitude of different vacua with different vacuum energy, spectra, couplings, etc., and (ii) there must be a "multiverse" containing many universes populating the different vacua. These two conditions are thought to be satisfied in string theory, since (i) the string landscape is thought to contain more than 10^{500} vacua (see e.g. [13]), and (ii) eternal inflation can populate these different vacua (see e.g. [14]).[12]

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[11] The same objections are less applicable to *a priori* predictions about our environment based on the anthropic principle, yet the author is unaware of examples of such predictions, much less subsequently verified ones.

[12] While various objections can be lodged against both the string landscape and eternal inflation, we will not consider them here.

However, as explained above, concrete applications of the anthropic principle are fraught with difficulties (cf. [15]),[13] even laying aside the computability issues which arise in the string landscape. Furthermore, the multiverse is observationally inaccessible (unlike extrasolar planets), so *a posteriori* applications of the anthropic principle are not even in principle falsifiable. There remains some hope, however, that anthropic arguments can lead to true *a priori* predictions which are subsequently verified. Such an eventuality would be an experimental triumph for the multiverse picture.

Having provided a brief overview of the anthropic principle, we now consider how it affects naturalness. There are two relevant questions: (i) can the anthropic principle potentially "explain" (or at least justify) fine tunings, and (ii) does this render naturalness arguments moot? The answer to the first question is a qualified "yes," subject to the same subtleties we have emphasized above. The answer to the second question is almost certainly "no," as a simple application of the anthropic principle itself will show.

The anthropic principle suggests that, among the ensemble of possible universes, we live in the sub-ensemble of universes where intelligent life can exist. The universe we actually live in can be any member of this sub-ensemble. In particular, the laws of statistics suggest that our own universe should be a "generic" member of this sub-ensemble.[14] However, a generic member of this sub-ensemble should not exhibit any tunings *unless* these tunings are necessary for intelligent life to evolve within that universe.

[13]For instance, is the Standard Model gauge group, $SU(3) \times SU(2) \times U(1)$, necessary for intelligent life? Probably not, but in that case we are required to understand the dynamics of any gauge group which can be realized in the landscape in order to answer the question of whether intelligent life can exist in that particular vacuum. Such a question is hard enough when we confine our attention to simple questions like the low energy spectrum of a strongly coupled gauge theory. Understanding which spectra allow for intelligent life is unimaginably daunting.

[14]We ignore issues with the measure problem of eternal inflation in what follows, see for instance [16].

Thus, the anthropic principle suggests a new form of naturalness: fine tunings should be absent *except* those which are "necessary for the development of intelligent life." Unfortunately, this new principle is very vague and so not very useful in itself. However, it does demonstrate that naturalness arguments have a place in an anthropic picture; indeed the anthropic principle appears to *demand* naturalness in some modified form.

CHAPTER 2

DYNAMIC $SU(2)$ STRUCTURE FROM SEVEN-BRANES

We obtain a family of supersymmetric solutions of type IIB supergravity with dynamic $SU(2)$ structure,[1] which describe the local geometry near a stack of four D7-branes and one O7-plane wrapping a rigid four-cycle. The deformation to a generalized complex geometry is interpreted as a consequence of nonperturbative effects in the seven-brane gauge theory. We formulate the problem for seven-branes wrapping the base of an appropriate del Pezzo cone, and in the near-stack limit in which the four-cycle is flat, we obtain an exact solution in closed form. Our solutions serve to characterize the local geometry of nonperturbatively-stabilized flux compactifications.

2.1 Introduction

Two fundamental goals in string theory are characterizing the vacua of the theory and understanding strongly-coupled four-dimensional gauge theories from a ten-dimensional viewpoint. These problems intersect when the dynamics of a strongly-coupled four-dimensional gauge theory determines the potential for compactification moduli, as in flux compactifications of type IIB string theory, where gaugino condensation on seven-branes provides an important contribution to the potential for the Kähler moduli.

In this chapter we present a family of explicit local solutions that describe the region near a stack of seven-branes wrapping a rigid four-cycle. We argue that a subclass of our solutions encode seven-brane nonperturbative effects in ten-dimensional supergravity. We begin in section 2.1.1 by motivating the study of seven-brane gaugino condensation, then explain in section 2.1.2 why the corresponding solution will be a generalized complex geometry.

[1]This chapter is reprinted from Ben Heidenreich, Liam McAllister and Gonzalo Torroba, "Dynamic SU(2) Structure from Seven-branes," JHEP **1105**, 110 (2011), with permission.

2.1.1 Gaugino condensation in string compactifications

In a compactification of type IIB string theory on a Calabi-Yau threefold, classical vacua involving nonvanishing fluxes and localized D-brane and orientifold plane sources provide a rich array of four-dimensional theories with $\mathcal{N} = 1$ or $\mathcal{N} = 0$ supersymmetry. The Kähler moduli are typically unfixed in the classical vacuum and mediate gravitational-strength interactions that preclude the construction of realistic models of particle physics and cosmology. Perturbative and nonperturbative effects may be expected to give mass to the Kähler moduli, and in certain special cases the dominant effects can be computed.

The proposals for Kähler moduli stabilization of [17–19] incorporate nonperturbative effects arising from branes wrapping four-cycles in the compact space [20]: these can be either Euclidean D3-branes, or a stack of (p, q) seven-branes giving rise to a four-dimensional gauge theory that is strongly coupled in the infrared and generates a nonperturbative superpotential. Considerable efforts have been directed at understanding the four-dimensional effective theory incorporating the classical flux superpotential and the nonperturbative superpotential arising from wrapped seven-branes, but the corresponding ten-dimensional configuration that encodes the effects of nonperturbative dynamics on the seven-branes remains mysterious.

Why understand the imprint of four-dimensional nonperturbative physics in ten dimensions, given that the four-dimensional theory itself is well-understood? One powerful motivation comes from the utility of higher-dimensional locality in model building. For example, one can construct a supersymmetric visible sector on D-branes in one region of a compactification, and later incorporate soft terms induced by supersymmetry breaking in a distant region. A prerequisite for such an analysis is locality in the compact space, which manifestly requires a ten-dimensional solution. Thus, understanding locality in nonperturbatively-stabilized vacua requires a ten-dimensional solution encoding the effects that stabilize the Kähler moduli. A second motivation is that local geometries describing strong gauge dynamics on seven-branes could be glued into compact geometries as 'modules' effecting the stabilization of Kähler moduli. A third important motivation is that gravity solutions can shed new light on the dynamics of supersymmetric gauge theories on seven-branes via gauge-gravity duality, as we explain in section 2.1.2.

We are therefore led to search for explicit local solutions describing strong gauge dynamics on seven-branes wrapping rigid four-cycles. A natural class of local Calabi-Yau geometries containing rigid four-cycles are complex cones over del Pezzo surfaces. The possibilities for wrapping seven-branes on the del Pezzo base of such a cone are quite constrained: the total seven-brane charge must vanish (see section 2.2.1). A convenient choice obeying this constraint is a stack of four D7-branes that coincide with an O7-plane: the total seven-brane charge and tension vanish, but lower-dimensional brane charges may be induced on the stack, e.g. by α' corrections.[2] The resulting four-dimensional gauge theory is pure[3] super Yang-Mills, which exhibits gaugino condensation and chiral symmetry breaking at low energies.

We conclude that a promising setting for studying the backreaction of four-dimensional nonperturbative effects is a Calabi-Yau cone whose base, a del Pezzo surface, is the fixed-point locus of an orientifold action and in addition is wrapped by four D7-branes. In this chapter we formulate the problem in this general setting but provide a detailed solution for a simpler special case, in which one zooms in

[2]In fact, negative D3-brane charge and tension induced on the seven-branes will contribute to the formation of a singularity near the four-cycle.

[3]See section 2.7.1 for important comments on how the global anomaly of [21, 22] may constrain the resulting gauge group. For simplicity of presentation we will speak of gaugino condensation in pure super Yang-Mills throughout, while keeping in mind that the full gauge theory may be more complicated.

on the region near the seven-branes. Results for del Pezzo cones will be provided elsewhere.

2.1.2 Nonperturbative effects from seven-branes and generalized complex geometry

The above configuration of seven-branes will preserve four supercharges that are embedded in the type IIB ten-dimensional Majorana-Weyl supersymmetry generators ϵ^i as

$$\epsilon^i = \zeta_+ \otimes \eta^i_+ + \zeta_- \otimes \eta^i_- \quad , \quad i = 1, 2 \, , \tag{2.1}$$

where the conventions are those of [23]: ζ_+ is a positive-chirality four-dimensional spinor that generates four-dimensional $\mathcal{N} = 1$ supersymmetry transformations, and the η^i_+ are fixed positive-chirality six-dimensional spinors, with η^i_-, ζ_- the Majorana conjugates of η^i_+, ζ_+, respectively.

Taken as strictly classical sources, four D7-branes wrapping the del Pezzo fixed-point locus of an orientifold will preserve 'type B' supersymmetry [23], with $\eta^1_+ = \pm i \eta^2_+$. However, a supergravity solution encoding gaugino condensation requires a different supersymmetry, as we now argue. In a type B background (as characterized e.g. in [24]) a D3-brane experiences vanishing potential, while gaugino condensation on seven-branes is known to lift the D3-brane moduli space. Further evidence comes from [25], where it was established that in a type B solution, gaugino condensation from D7-branes on a rigid cycle sources imaginary anti-self-dual flux, of Hodge type (1,2), which is incompatible with the supersymmetry of the background. In fact, a supergravity solution encoding seven-brane gaugino condensation requires that the internal spinors η^1_+ and η^2_+ be distinct, as argued in section 2.4, and a direct analysis of the dilatino and gravitino variations becomes more involved. Mathematically, the two internal spinors define a *local* or *dynamic* $SU(2)$ structure that can be dealt with most efficiently in terms of generalized complex geometry [23], as originally proposed for D7-brane gaugino condensation in [26].[4]

Our goal in this chapter is to investigate supergravity solutions with dynamic $SU(2)$ structure that describe the gauge theory dynamics of compact seven-branes. The approach taken here is purely ten-dimensional, and we do not assume the existence of a four-dimensional nonperturbative superpotential. Rather, in the spirit of [27, 30], the supergravity solution describing seven-branes wrapped on an appropriate rigid cycle should already encode the effects of gaugino condensation.[5]

Let us comment on the relation between our approach and well-understood AdS/CFT descriptions of gaugino condensation in other systems. The gravity dual of pure super Yang-Mills is not known, and is expected to correspond to a regime of large curvature and strong corrections to classical supergravity. However, pure super Yang-Mills can be embedded into a branch of a large N quiver theory, or into a higher-dimensional gauge theory, leading to well-defined supergravity solutions, albeit with extra fields that are absent from the pure glue theory.

A celebrated example is the Klebanov-Strassler solution [30], in which an $SU(N) \times SU(N+M)$ quiver theory, which confines and breaks chiral symmetry in the infrared, is dual to a conformally-Calabi-Yau solution, the warped deformed conifold. In this case the boundary supersymmetry is of type B, facilitating embedding in well-understood flux compactifications [24]. In contrast, there is no known supergravity solution dual to a gauge theory with seven-branes wrapped on a rigid four-cycle. One obstacle is that taking a four-dimensional limit by shrinking the four-cycle generically gives chiral matter and an anomalous gauge theory.[6]

[4]An earlier example that displays gaugino condensation and $(1,2)$ three-form flux is the Polchinski-Strassler solution [27]; the ten-dimensional analysis of [28, 29] revealed the presence of an $SU(2)$ structure.

[5]See [31] for a ten-dimensional description of gaugino condensation in the heterotic string.

[6]Anomaly-free gauge theories from seven-branes on del Pezzo cones were constructed recently

The anomaly has to be canceled by adding extra ingredients, such as orientifolds, but these do not decouple from the low energy theory and complicate both the gauge theory analysis and the supergravity solution. Furthermore, the number of seven-branes cannot be taken to be large, so that it is difficult to obtain parametric control of the curvatures appearing on the gravity side.

One might expect that our noncompact supergravity solutions should capture the backreaction associated to the gauge theory dynamics. However, there is at present no fully-realized gauge-gravity duality for our system: our solutions do not include a large number of D3-branes, and the asymptotic geometry is very different from AdS_5. We will suggest that our solutions describe the behavior of the gauge theory in the deep infrared, after most of the degrees of freedom have renormalized away. Introducing a large number of color branes might yield a solution in which a more precise gauge/gravity dictionary can be constructed (see e.g. chapter 4), but we leave this question for the future.

2.1.3 Overview

Our main result is an explicit supersymmetric solution with dynamic (and in general, type-changing) $SU(2)$ structure, which describes the generalized complex geometry near a stack of four D7-branes and one O7-plane. This solution arises as a limiting case of configurations in which the seven-branes wrap a compact four-cycle: in the example we provide, this cycle is the base of the Calabi-Yau cone over $\mathbb{P}^2$, $\mathcal{O}(-3)_{\mathbb{P}^2}$. We construct our solution in closed form after solving the supersymmetry conditions for an AdS_4 vacuum in the limit of vanishing cosmological constant. The form of the supersymmetry conditions used here is $SL(2,\mathbb{R})$ covariant, which is very useful in classifying the resulting solutions.

The organization of this chapter is as follows. In section 2.2, we present some

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in [32].

essential geometric background for our analysis. In section 2.3.1, we describe a simple ansatz that will be our primary focus. In section 2.3.2, we extend our considerations to the $\mathbb{P}^2$ cone, and show that the ansatz of section 2.3.1 arises in a scaling limit that zooms in on the $\mathbb{P}^2$. In section 2.4, we briefly review dynamic $SU(2)$ structure, then present the full supersymmetry conditions for compactification to AdS_4 in an $SL(2,\mathbb{R})$ covariant form. In section 2.5 we solve these conditions, as well as the equations of motion, to obtain the most general 'AdS-like' supersymmetric solution to our ansatz. In section 2.6 we describe the regime of validity of supergravity and discuss a few key geometric properties of our solutions. In section 2.7 we present a preliminary analysis of the relation between the above solutions and nonperturbative effects on seven-branes, and indicate a few interesting applications and directions for future research. We conclude in section 2.8. The equations of motion for our ansatz are assembled in appendix 2.A.

2.2 Seven-branes and Orientifolds

We will study a stack of four D7-branes atop an O7-plane, wrapping a rigid four-cycle in a local Calabi-Yau geometry and preserving four supercharges. The specific local geometries that we will consider are resolved Calabi-Yau cones over del Pezzo surfaces. In each case, the del Pezzo base is a rigid shrinking divisor within the Calabi-Yau. These properties make Calabi-Yau cones over del Pezzos the natural choice for the purpose of obtaining local gauge theories that undergo gaugino condensation. In this chapter we will focus primarily on the simplest del Pezzo surface, $\mathbb{P}^2$, but we begin with topological considerations that are valid for any del Pezzo cone, and then proceed to describe the geometry of the resolved cone over $\mathbb{P}^2$, $\mathcal{O}(-3)_{\mathbb{P}^2}$.

2.2.1 Orientifolds of resolved del Pezzo cones

Wrapping seven-branes over the del Pezzo surface yields a supersymmetric field theory in four dimensions. In order to obtain a pure glue theory in which gaugino condensation can occur, we do not add seven-branes wrapping non-compact divisors. Then, owing to the rigidity of the del Pezzo within the Calabi-Yau, the four-dimensional field theory has no light matter, and develops a gaugino condensate in the infrared. One can imagine altering this situation in various ways to remove the gaugino condensate, for instance by adding D3-branes close to the seven-branes, giving light 3-7 strings. Thus, it makes sense to consider the supergravity background both with and without the effects of gaugino condensation. We first consider the situation without a condensate, in which case the background contains only D3-branes and seven-branes, and can be studied using F-theory.

The only compact divisor in these cones is the del Pezzo itself. This places a strong constraint on the allowed brane content consistent with tadpole cancellation, as we now show. Since stacks of seven-branes carry an $SL(2,\mathbb{Z})$ monodromy depending on their total charge, we consider the $SL(2,\mathbb{Z})$ monodromy structure of the solution. For a manifold M without branes, the allowed monodromies are given by homomorphisms $\Lambda : \pi_1(M) \to SL(2,\mathbb{Z})$. Seven-branes may be thought of as topological defects in the type IIB vacuum; thus, the seven-brane charges and monodromy structures associated with a given configuration of branes are classified by homomorphisms $\Lambda : \pi_1(M') \to SL(2,\mathbb{Z})$, where M' is given by M minus the worldvolumes of the seven-branes.

A del Pezzo cone may be viewed as a real cone M over a Sasaki-Einstein manifold which we refer to figuratively as the 'horizon.' Upon excising the tip, the resulting M' is homotopically equivalent to its horizon. Thus, the allowed seven-brane charges are constrained by the fundamental group of the horizon. It is known that these horizons are always simply connected up to torsion, and the only torsion groups that appear are $\mathbb{Z}_3$ (for the $\mathbb{P}^2$ cone) and $\mathbb{Z}_2$ (for the $\mathbb{P}^1 \times \mathbb{P}^1$ cone) [33]. Except in these special cases, we must cancel the D7-brane tadpole *locally*.[7] A well-understood way to do this is to wrap eight D7-branes on the four-cycle and then orientifold by a $\mathbb{Z}_2$ involution whose fixed point locus is the cycle itself.

It is important to distinguish between the 'upstairs' and 'downstairs' geometries of the resulting orientifold. From the perspective of perturbative string theory, it is natural to do computations in the upstairs geometry, in which we have eight D7-branes coinciding with an O7-plane wrapping the base of a resolved del Pezzo cone. At energies below the Kaluza-Klein scale the resulting gauge theory is pure glue $\mathcal{N} = 1$ super Yang-Mills, where all open string fields arise from 7-7 strings and live in the adjoint representation. Classically, the composite object carries *zero* seven-brane charge and tension, and generates no deficit angle.

From the perspective of F-theory, on the other hand, it is more natural to work in the downstairs geometry, where we identify under the involution to obtain a $\mathbb{Z}_2$ orbifold of the del Pezzo cone which carries a $\mathbb{Z}_2$ monodromy coming from the $-1 \in SL(2,\mathbb{Z})$. The eight D7-branes of the upstairs geometry are reduced to four by this identification. In addition, the orientifold plane appears in F-theory as a combination of *two* coincident (p,q) seven-branes that separate at strong coupling [34]. Thus, in the downstairs geometry, we obtain a stack of *six* (p,q) seven-branes. Each carries a deficit angle of $\pi/6$, for a total deficit angle of π, as required to match the deficit angle of $(\mathcal{O}(-3)_{\mathbb{P}^2})/\mathbb{Z}_2$.

Although seven-branes are often conveniently treated in F-theory, the more general supersymmetric backgrounds that will be relevant in our analysis are not

[7] This can be seen from the field theory perspective as well: the gauge theories corresponding to disallowed configurations of seven-branes will be rendered inconsistent by anomalies [32].

well-studied in F-theory, and we opt to work in ten-dimensional supergravity. (In section 2.6.3 we will explicitly demonstrate parametric control of the supergravity approximation.) We will generally work in the upstairs geometry, removing modes forbidden by the involution.

2.2.2 The Calabi-Yau geometry of the $\mathbb{P}^2$ cone

We now review the well-known geometry of the Calabi-Yau cone over $\mathbb{P}^2$, i.e. the resolution of the orbifold $\mathbb{C}^3/\mathbb{Z}_3$, where the $\mathbb{Z}_3$ acts by

$$z^i \to e^{2\pi i/3} z^i \tag{2.2}$$

on the $\mathbb{C}^3$ coordinates z^i. The Calabi-Yau metric has a $U(3) = SU(3) \times U(1)_\psi$ isometry that acts naturally on the z^i, where the $SU(3)$ subgroup acts on the $\mathbb{P}^2$ base in the natural way, and we normalize the $U(1)_\psi$ such that the z^i carry charge $+1/3$. Let us define the $U(3)$ invariant radial coordinate

$$\rho^2 \equiv \sum_{i=1}^{3} |z^i|^2 \,. \tag{2.3}$$

Each surface of constant ρ is diffeomorphic to the horizon, $S^5/\mathbb{Z}_3$. It is useful to think of this space as a Hopf fibration over $\mathbb{P}^2$, where $U(1)_\psi$ rotates the fibers. Locally, we can define a circle coordinate ψ with periodicity 2π, such that $e^{i\psi^{(i)}/3} = z^i/|z^i|$ for some arbitrarily chosen i, and $U(1)_\psi$ rotations are equivalent to shifts in ψ.

The $\mathbb{Z}_3$ orbifold singularity may be resolved into a $\mathbb{P}^2$. Viewed from the $\mathbb{C}^3/\mathbb{Z}^3 - \{0\}$ region, the size of the resolution is visible as a normalizable perturbation to the conical metric, as we now review. The conical Calabi-Yau metric for this space, either singular or resolved, can be obtained by taking the Kähler potential K to depend only on ρ, so that (cf. e.g. [35])

$$g_{mn}(y)dy^m dy^n = \partial_i\Big(z_{\bar{j}} K'(\rho^2)\Big)\, dz^i d\bar{z}^{\bar{j}} = K'(\rho^2) \sum_i dz^i d\bar{z}^{\bar{i}} + K''(\rho^2) \sum_{i,j} \bar{z}_i z_{\bar{j}} dz^i d\bar{z}^{\bar{j}} \,. \tag{2.4}$$

Here primes denote derivatives with respect to ρ^2 and $z_{\bar{i}} \equiv \delta_{\bar{i}j} z^j$. The Ricci-flatness condition $\det(\partial_i \bar{\partial}_j K) = 1$ admits the solution

$$\rho^2 K'(\rho^2) = (\rho^6 + \rho_0^6)^{1/3} \, . \tag{2.5}$$

Plugging (2.5) into (2.4) gives the metric on the resolved cone, where the origin $\rho = 0$ has been blown up into a finite $\mathbb{P}^2$ with size controlled by ρ_0.

We wrap eight D7-branes on the $\mathbb{P}^2$ base and orientifold under the involution

$$\sigma : z^i \to -z^i \, , \tag{2.6}$$

combined with $-1 \in SL(2, \mathbb{Z})$. This involution is holomorphic and reverses the holomorphic three-form

$$\Omega \propto \epsilon_{ijk} \, \mathrm{d}z^i \wedge \mathrm{d}z^j \wedge \mathrm{d}z^k \, , \tag{2.7}$$

as required to preserve $\mathcal{N} = 1$ supersymmetry. The fixed point locus is the $\mathbb{P}^2$ itself. Since the stack carries no D7-brane charge or tension in the upstairs geometry, the Calabi-Yau metric we have just derived remains valid. However, there is a net D3-brane charge and tension induced by α' corrections to the D7-brane action [36]. For the del Pezzo orientifold that we consider, the results of [37, 38] imply[8] that there is an induced negative D3-brane charge proportional to the Euler characteristic of the base. There is also a corresponding negative tension. These charges will backreact on the warp factor and C_4 in a manner which we derive in section 2.5.

Since the $\mathbb{P}^2$ is rigid, we expect gaugino condensation to ensue at low energies. The condensate will source imaginary anti-self-dual fluxes [25], and therefore the background will no longer be conformally Calabi-Yau. Our approach to this problem is to search for new supersymmetric supergravity solutions as candidates for the backreaction from nonperturbative effects, in some region away from the

[8]There are subtleties here involving cancellation of the global anomaly found in [21, 22]; cf. section 2.7.1.

branes where the supergravity approximation is valid. Let us reiterate that we do not directly incorporate gaugino condensation as a localized source (cf. [25, 26]), but instead obtain supergravity solutions that are consistent with the possibility of such backreaction.

The holomorphic three-form (2.7) is charged under $U(1)_\psi$, which will therefore be an R-symmetry in the gauge theory. The R-charge of Ω is the same as that of the four-dimensional superpotential, $R(\Omega) = +2$, so $e^{i\beta} \in U(1)_R$ corresponds to $e^{2i\beta} \in U(1)_\psi$. As in the gauge theory, we expect that $U(1)_R$ is anomalous, breaking to a discrete subgroup. Nonzero G_3 flux will further break the R-symmetry since G_3 must be odd under the subgroup generated by the spatial involution, $z^i \to -z^i$. We will see in section 2.7.1 that the R-symmetry breaks to $\mathbb{Z}_2$ for our $SU(2)$ structure solutions. We anticipate that, in regions of parameter space where our solution provides a gravity dual to gaugino condensation, this will be the gravity analogue of the expected spontaneous R-symmetry breaking in the gauge theory due to the expectation value of the gaugino bilinear. We assume that the backreacted solution does not break the remaining $SU(3)$ symmetry.

2.3 Ansatz for the Supergravity Solution

To simplify the problem, we will first focus on a small $\mathbb{R}^6$ patch near the stack. The supergravity ansatz for this limit is presented in §2.3.1. In section 2.3.2 we present the supergravity ansatz for the full Calabi-Yau cone over $\mathbb{P}^2$, and in section 2.3.3 we show that the two ansätze are related by a particular near-stack limit.

2.3.1 The near-stack region

We consider the $\mathbb{R}^6$ neighborhood of a small piece of the seven-brane stack, and approximate the stack as flat within this region. Imposing the $SU(3)$ symmetry group, the full geometry can be recovered from its local form in the $\rho \ll \rho_0$ limit (where ρ_0 determines the size of $\mathbb{P}^2$ as in (2.5)).

The first step is to define the correct 'near-stack limit'. We focus on the decomposition of the $U(3) = SU(3) \times U(1)_\psi$ isometry group in this limit, leaving a more detailed mapping of the supergravity fields to section 2.3.3. Consider the region $z^3 \neq 0$. We define coordinates

$$u^a \equiv z^a/z^3 \quad , \quad a = 1,2 \quad , \quad z \equiv \frac{1}{3}(z^3)^3 \, , \tag{2.8}$$

which are invariant under the $\mathbb{Z}_3$ orbifold action, and carry charges 0 and +1, respectively, under the $U(1)_\psi$. The $SU(3)$ decomposes into $SU(2) \times U(1)_T$, where the $SU(2)$ acts naturally on the u^a, and the $U(1)_T$ takes the form

$$u^a \to e^{i\theta} u^a \quad , \quad z \to e^{-2i\theta} z \, . \tag{2.9}$$

In addition, there are four generators of $SU(3)$ that mix the u^a and z. These take the infinitesimal form

$$z^a \to z^a + \theta^a z^3 + \mathcal{O}(\theta^2) \quad , \quad z^3 \to z^3 - \bar{\theta}_b z^b + \mathcal{O}(\theta^2) \, , \tag{2.10}$$

for complex θ^a, where $\theta_{\bar{b}} \equiv \delta_{\bar{b}a} \theta^a$. Thus,

$$u^a \to u^a \left(1 + u^b \bar{\theta}_b\right) + \theta^a + \mathcal{O}(\theta^2) \quad , \quad z \to z\left(1 - 3u^b \bar{\theta}_b\right) + \mathcal{O}(\theta^2) \, . \tag{2.11}$$

This is a nonlinear transformation on the coordinates, even at first order in θ^a. However, if we additionally approximate that $r_u \equiv \sum_a |u^a|^2 \ll 1$, then we obtain

$$u^a \to u^a + \theta^a + \mathcal{O}(r_u\theta, \theta^2) \quad , \quad z \to z + \mathcal{O}(r_u\theta, \theta^2) \, , \tag{2.12}$$

corresponding to $\mathbb{C}^2$ translations on the u^a. To accomplish this formally, we rescale

$$u^a \to \varepsilon u^a \quad , \quad z \to \varepsilon z \, , \tag{2.13}$$

and then truncate to leading order in ε.

We take (2.13) as the initial definition of the near-stack limit. (In section 2.6.3 we obtain a more precise definition (2.161) in terms of the effective codimension of the stack: in the near-stack limit, the seven-branes are real codimension two sources, while at longer distances they appear as real codimension six sources.) Geometrically, this limit corresponds to zooming in on a small neighborhood of a specified patch of the D7/O7 stack. As we have shown, 'small' $SU(3)$ transformations – those which map the small neighborhood to itself – decompose locally into $\mathbb{C}^2 \rtimes (SU(2) \times U(1)_T)$ transformations; different local patches are related by 'large' $SU(3)$ transformations.

The stack of seven-branes is located in the plane $z = 0$, and the involution takes $z \to -z$. We choose a circle coordinate ψ locally such that $z = r_z e^{i\psi}$ where r_z is real. In the near-stack limit, we find

$$r_z \approx \frac{1}{3}\rho^3 \equiv r\,, \tag{2.14}$$

where r is an alternate radial coordinate on the $\mathbb{P}^2$ cone that will be useful below. Thus, r and r_z match in the near-stack limit, and except where the distinction is important, we will denote both by r.

In our notation, the low energy effective action[9] for type IIB string theory written in Einstein frame is

$$S = \frac{1}{2\kappa_{10}^2}\int \mathrm{d}^{10}x\sqrt{-g^{(10)}}\left[\mathcal{R} - \frac{1}{2}\left((\nabla\phi)^2 + e^{-\phi}|H_3|^2 + e^{2\phi}|F_1|^2 \right.\right.$$
$$\left.\left. + e^{\phi}|\tilde{F}_3|^2 + \frac{1}{2}|\tilde{F}_5|^2\right)\right] - \frac{1}{4\kappa_{10}^2}\int C_4 \wedge H_3 \wedge F_3\,, \tag{2.15}$$

where $|F_p|^2 \equiv \frac{1}{p!}F^{M_1\ldots M_p}F^{\star}_{M_1\ldots M_p}$, $H_3 = \mathrm{d}B_2$, $F_p = \mathrm{d}C_{p-1}$, and

$$\tilde{F}_3 = F_3 - C_0 H_3 \;\;,\;\; \tilde{F}_5 = F_5 - \frac{1}{2}C_2 \wedge H_3 + \frac{1}{2}B_2 \wedge F_3\,. \tag{2.16}$$

[9] The equations of motion must be supplemented by the self-duality constraint $\tilde{F}_5 = \star_{10}\tilde{F}_5$.

We adopt a warped ansatz, with ten-dimensional metric

$$\mathrm{d}s_{10}^2 = e^{2A(y)}\, h_{\mu\nu}(x)\, \mathrm{d}x^\mu \mathrm{d}x^\nu + e^{-2A(y)}\, g_{mn}(y)\, \mathrm{d}y^m \mathrm{d}y^n\,, \tag{2.17}$$

and five-form field strength[10]

$$\tilde{F}_5 = (1+\star_{10})\, \Omega_4 \wedge \mathrm{d}\left[\alpha(y)\right]\,, \tag{2.18}$$

where α is a scalar related to C_4, $h_{\mu\nu}(x)$ is a maximally symmetric metric on $\mathbb{R}^{3,1}$ with cosmological constant $\Lambda = \mathcal{R}_{(4)}/4$ and volume form Ω_4, and $g_{mn}(y)$ (times the conformal factor e^{-2A}) gives the internal space metric. We do not assume that $g_{mn}(y)$ is Calabi-Yau, or indeed even complex. The axiodilaton $\tau(y) = C_0 + ie^{-\phi} = \tau_1 + i\,\tau_2$ varies over the compact space, and the three-form flux

$$G_3 \equiv \frac{1}{\sqrt{\tau_2}}\,(F_3 - \tau H_3) \tag{2.19}$$

points along the internal directions only.

Backreaction from the seven-branes changes the metric and sources various supergravity fields. We consider the most general ansatz compatible with the assumed symmetry group $\mathbb{C}^2 \rtimes (SU(2)\times U(1)_T)$. The internal metric g_{mn} must take the form

$$g_{mn}(y)\mathrm{d}y^m\mathrm{d}y^n = g_{rr}(r)\mathrm{d}r^2 + 2g_{r\psi}(r)\mathrm{d}r\mathrm{d}\psi + g_{\psi\psi}(r)\mathrm{d}\psi^2 + e^{2C(r)}\sum_a \mathrm{d}u^a\mathrm{d}\bar{u}^a\,. \tag{2.20}$$

This metric is not in general Hermitian with respect to the complex structure defined by (z, u^i), but can always be made Hermitian by a suitable coordinate redefinition that alters the complex structure,

$$z \to \lambda(r) z\,. \tag{2.21}$$

[10]With our sign conventions, the Hodge star associated with a D-dimensional metric g with volume form $\Omega_{(g)}$ is defined by $\star\Big(\mathrm{d}x^{m_1}\wedge\ldots\wedge\mathrm{d}x^{m_p}\Big) = \frac{1}{(D-p)!}\,\Omega_{(g)}^{m_1\ldots m_p}{}_{m_{p+1}\ldots m_D}\Big(\mathrm{d}x^{m_{p+1}}\wedge\ldots\wedge \mathrm{d}x^{m_D}\Big)$.

Using this, the metric may be brought to the form

$$\mathrm{d}s^2 = e^{-4B(r)}\left(\mathrm{d}r^2 + r^2\mathrm{d}\psi^2\right) + e^{2C(r)}\sum_a \mathrm{d}u^a\mathrm{d}\bar{u}^a\,. \tag{2.22}$$

The metric is now manifestly Hermitian, with Kähler form

$$J = \frac{i}{2}\left[e^{-4B(r)}\mathrm{d}z\wedge\mathrm{d}\bar{z} + e^{2C(r)}\sum_a \mathrm{d}u^a\wedge\mathrm{d}\bar{u}^a\right]. \tag{2.23}$$

The metric is Kähler if and only if $e^{2C(r)}$ is a constant.

We now consider the general form of G_3. Since there are no invariant one-forms pointing along the base (the $\mathrm{d}u^a$ and $\mathrm{d}\bar{u}^a$ directions), invariant three-forms must have two legs along the base and one along the fiber, and descend from invariant two-forms along the base. These are:

$$\omega_{1,1} \equiv \frac{i}{2}\sum_a \mathrm{d}u^a\wedge\mathrm{d}\bar{u}^a \quad , \quad \omega_{2,0} \equiv z\,\mathrm{d}u^1\wedge\mathrm{d}u^2\,. \tag{2.24}$$

Invariant three forms are constructed by wedging $\frac{1}{z}\mathrm{d}z$ and its conjugate into $\omega_{1,1}$, $\omega_{2,0}$, and $\omega_{0,2} \equiv \omega^\star_{2,0}$. Three-forms built out of $\omega_{1,1}$ are even under $z \to -z$, whereas those built from $\omega_{2,0}$ and $\omega_{0,2}$ are odd. G_3 then takes the general form:

$$\begin{aligned} G_3 = g_{3,0}\mathrm{d}z\wedge\mathrm{d}u^1\wedge\mathrm{d}u^2 + g_{2,1}e^{2i\psi}\mathrm{d}\bar{z}\wedge\mathrm{d}u^1\wedge\mathrm{d}u^2 \\ + g_{1,2}e^{-2i\psi}\mathrm{d}z\wedge\mathrm{d}\bar{u}^1\wedge\mathrm{d}\bar{u}^2 + g_{0,3}\mathrm{d}\bar{z}\wedge\mathrm{d}\bar{u}^1\wedge\mathrm{d}\bar{u}^2\,, \end{aligned} \tag{2.25}$$

where the $g_{p,q}$ are complex-valued functions of r only. Comparing with (2.23), we see that $J\wedge G_3 = 0$, so that G_3 is automatically primitive.

The scalars α, A, and τ can depend only on r. As previously remarked, the $U(1)_\psi$ symmetry is broken for solutions with non-vanishing G_3. In section 2.7.1 this breaking will be identified with the spontaneous breaking of the exact R-symmetry on the gauge theory side.

Although one can obtain an exact supersymmetric solution to the above system by writing down the equations of motion[11] and solving them directly, it is far

[11] For reference, we summarize the equations of motion in appendix 2.A.

easier to use the conditions for unbroken supersymmetry, which we will present in section 2.4 and solve in section 2.5. First, however, we generalize the preceding ansatz to a Calabi-Yau cone.

2.3.2 Seven-branes in the $\mathbb{P}^2$ cone

Having proposed an ansatz for the relatively simple geometry near a seven-brane stack, we now extend our analysis to seven-branes wrapping the $\mathbb{P}^2$ base of the resolved orbifold $\mathbb{C}^3/\mathbb{Z}^3$. We will verify that the complete supergravity ansatz proposed in section 2.3.1 emerges upon taking the near-stack limit of our result for the $\mathbb{P}^2$ cone. This connection provides valuable insight into the near-stack solution, in particular because knowledge of the solution for a compact four-cycle provides a regulator for divergences associated with the noncompact nature of the four-cycle in the near-stack ansatz.[12]

We develop the ansatz using the $\mathbb{C}^3/\mathbb{Z}_3$ coordinates, and later shift to a different chart appropriate for studying a small neighborhood of the resolved $\mathbb{P}^2$. As before, we wrap eight D7-branes on the resolved $\mathbb{P}^2$ and orientifold using the involution $z^i \to -z^i$ combined with $-1 \in SL(2,\mathbb{Z})$. The Calabi-Yau metric has an isometry group $U(3)$ which acts naturally on the z^i. However, $z^i \to -z^i$ lies within the $U(1)$ factor, so nonzero G_3 will spontaneously break the $U(1)$, leaving an $SU(3)$ symmetry group. We now study the ansatz that arises upon imposing this symmetry group. Since the orbifold action $\mathbb{Z}_3 \subset SU(3)$, $SU(3)$ singlets are never projected out by the orbifold, and we need not consider $\mathbb{Z}_3$ invariance separately.

[12]For example, α' corrections induce D3-brane charge on certain compact four-cycles, but the topological information determining this charge is lost in taking the near-stack limit.

Metric ansatz for the $\mathbb{P}^2$ cone

The backreacted ten-dimensional metric will be of the form (2.17), where the internal metric g_{mn} must be invariant under the $SU(3)$ symmetry. Since the symmetry group acts transitively on the horizon, we are free to choose a particular point or region on the horizon $S^5/\mathbb{Z}_3$, and we select the $z^1 = z^2 = 0$ plane. The symmetry group within this plane decomposes to $SU(2)\times U(1)$. The metric evaluated in this plane must take the $SU(2) \times U(1)$ invariant form:

$$g_{mn}\mathrm{d}y^m\mathrm{d}y^n = \mathfrak{f}_1(|z^3|)\left(\bar{z}^3/z^3\right)\mathrm{d}z^3\mathrm{d}z^3 + c.c. + \mathfrak{f}_2(|z^3|)\mathrm{d}z^3\mathrm{d}\bar{z}^3 \\ + \mathfrak{f}_3(|z^3|)(\mathrm{d}z^1\mathrm{d}\bar{z}^1 + \mathrm{d}z^2\mathrm{d}\bar{z}^2)\,. \quad (2.26)$$

Since $\mathfrak{f}_1$ is complex, the metric depends on four real functions. We can construct the global form of the metric by combining the invariant one-forms $\partial\rho^2$, $\bar{\partial}\rho^2$, and $\sum_i \mathrm{d}z^i\mathrm{d}\bar{z}^i$:

$$g_{mn}\mathrm{d}y^m\mathrm{d}y^n = \frac{1}{\rho^2}\mathfrak{f}_1(\rho)(\partial\rho^2)(\partial\rho^2) + c.c. + \frac{1}{\rho^2}\Big(\mathfrak{f}_2(\rho) - \mathfrak{f}_3(\rho)\Big)(\partial\rho^2)(\bar{\partial}\rho^2) \\ + \mathfrak{f}_3(\rho)\sum_i \mathrm{d}z^i\mathrm{d}\bar{z}^i\,. \quad (2.27)$$

This reduces to the local form given above at $z^1 = z^2 = 0$, as $\partial\rho^2 = \sum_i \bar{z}^i\mathrm{d}z^i$. Since ρ^2 is $SU(3)$ invariant, as is the complex structure, the ansatz (2.27) is manifestly invariant.

The metric (2.27) can always be made Hermitian by a suitable redefinition of the complex structure that preserves the $SU(3)$ symmetry,

$$z^i \to \lambda(\rho)z^i\,, \quad (2.28)$$

where $\lambda \in \mathbb{C}^\star$. One can show that $\mathfrak{f}_1$ can always be set to zero by an appropriate choice of λ. We define

$$\mathfrak{f}_2(\rho) \equiv \rho^4 e^{-4B(\rho)}\ \ ,\ \ \mathfrak{f}_3(\rho) \equiv \frac{1}{\rho^2}e^{2C(\rho)}\,, \quad (2.29)$$

to make the positivity of the metric explicit. Thus,

$$g_{mn}(y)\mathrm{d}y^m\mathrm{d}y^n = \left(\rho^2 e^{-4B(\rho)} - \frac{1}{\rho^4}\, e^{2C(\rho)}\right)\sum_{i,j}\bar{z}_i z_j dz^i d\bar{z}^j + e^{2C(\rho)}\,\frac{1}{\rho^2}\,\sum_i dz^i d\bar{z}^i\,. \tag{2.30}$$

The choice of notation is not accidental; we will see shortly that in the near-stack limit this form reduces to (2.22).

The corresponding Kähler form,

$$J = \frac{i}{2\rho^2}\Big(\rho^4 e^{-4B(\rho)} - \frac{1}{\rho^2}e^{2C(\rho)}\Big)\partial\rho^2\wedge\bar\partial\rho^2 + \frac{i}{2\rho^2}e^{2C(\rho)}\,\partial\bar\partial\rho^2\,, \tag{2.31}$$

can be rewritten as $J = e^{-4B}\chi_{1,1} + e^{2C}\omega_{1,1}$, where

$$\omega_{1,1} \equiv \frac{i}{2\rho^2}\left(\partial\bar\partial\rho^2 - \frac{1}{\rho^2}\partial\rho^2\wedge\bar\partial\rho^2\right) \quad , \quad \chi_{1,1}\equiv\frac{i\rho^2}{2}\partial\rho^2\wedge\bar\partial\rho^2\,. \tag{2.32}$$

Note that $\omega_{1,1}$ points along the base, and $\chi_{1,1}$ points along the fiber.

In components, the metric is

$$g_{i\bar{j}} = \frac{\rho^4}{2}e^{-4B}P_{i\bar{j}} + \frac{1}{2\rho^2}e^{2C}\left(\delta_{i\bar{j}} - P_{i\bar{j}}\right) \quad , \quad \text{for} \;\; P_{i\bar{j}} = \frac{1}{\rho^2}\bar{z}_i z_{\bar{j}} = \delta_{i\bar{k}}\delta_{\bar{j}l}P^{\bar{k}l} \quad , \quad z_{\bar{i}}\equiv\delta_{\bar{i}j}z^j\,, \tag{2.33}$$

with P^m_n a real rank one projector satisfying $P^i_j z^j = z^i$. The determinant of the metric is

$$8G \equiv 8\det g_{i\bar{j}} = \frac{8}{3!}\varepsilon^{ijk}\varepsilon^{\bar{l}\bar{m}\bar{n}}g_{i\bar{l}}g_{j\bar{m}}g_{k\bar{n}} = \sqrt{\det g_{mn}} = e^{-4B+4C}\,. \tag{2.34}$$

Kählerity and the Calabi-Yau metric

Let us rewrite the metric in terms of the alternate radial coordinate $r\equiv\frac{1}{3}\rho^3$. We find

$$\omega_{1,1} = \frac{i}{6r^2}\left(\partial\bar\partial r^2 - \frac{1}{r^2}\partial r^2\wedge\bar\partial r^2\right) \quad , \quad \chi_{1,1} = \frac{i}{2r^2}\partial r^2\wedge\bar\partial r^2\,. \tag{2.35}$$

Note that $\mathrm{d}\omega_{1,1}=0$ and $\mathrm{d}\chi_{1,1} = -3\,\mathrm{d}r^2\wedge\omega_{1,1}$. The Kähler condition now takes a particularly simple form:

$$\mathrm{d}J = \left[-3\,e^{-4B} + \frac{1}{2r}\left(e^{2C}\right)'\right]\mathrm{d}r^2\wedge\omega_{1,1} = 0\,. \tag{2.36}$$

Thus, Kählerity requires $e^{-4B} = \frac{1}{6r}\left(e^{2C}\right)'$, where $f' \equiv \frac{\mathrm{d}}{\mathrm{d}r}f$. For a Kähler metric, (2.34) implies

$$8G = \frac{1}{6r}\left(e^{2C}\right)' e^{4C} = \frac{1}{18r}\left(e^{6C}\right)' \,. \tag{2.37}$$

As a check, we consider the special case in which the metric is Calabi-Yau. The Ricci form is

$$\begin{aligned}\mathfrak{R} = -i\partial\bar{\partial}\log G = -\frac{i}{2r}\frac{\mathrm{d}}{\mathrm{d}r}\left[\frac{G'}{2rG}\right]\partial r^2 \wedge \bar{\partial} r^2 - \frac{iG'}{2rG}\partial\bar{\partial}r^2 \\ = -\frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}\left[\frac{rG'}{2G}\right]\chi_{1,1} - \frac{3rG'}{G}\omega_{1,1}\,.\end{aligned} \tag{2.38}$$

Thus, $G' = 0$. We find the solution $e^{6C} = 9\,\mathfrak{g}_0^3\,[r^2 + r_0^2]$, so that

$$e^{2C} = \mathfrak{g}_0\left[9\left(r^2 + r_0^2\right)\right]^{1/3} \quad , \quad e^{-4B} = \mathfrak{g}_0\left[9\left(r^2 + r_0^2\right)\right]^{-2/3} \,. \tag{2.39}$$

For $r_0 > 0$ this is the Calabi-Yau metric for the resolved $\mathbb{P}^2$ cone, and for $r_0 = 0$ it describes the singular $\mathbb{P}^2$ cone, for which the metric reduces to the canonical one on $\mathbb{C}^3$ with overall scale $\mathfrak{g}_0$.

G_3 ansatz for the $\mathbb{P}^2$ cone

We now enumerate the $SU(3)$-invariant p-forms of various ranks. The only invariant one-forms are $\partial\rho^2$ and its conjugate, which point along the fiber. In addition to the invariant two-forms $\chi_{1,1}$ and $\omega_{1,1}$ discussed above, pointing along the fiber and the base, respectively, there exists a complex invariant $(2,0)$ form pointing along the base:

$$\omega_{2,0} \equiv \frac{1}{6}\varepsilon_{ijk} z^i \mathrm{d}z^j \wedge \mathrm{d}z^k \,, \tag{2.40}$$

Including the conjugate $\omega_{0,2} \equiv \omega_{2,0}^\star$, this exhausts the list of invariant two-forms. All invariant p-forms for $p \geq 3$ can be written as wedge products of invariant one-forms and two-forms. Note that $\omega_{2,0} \wedge \omega_{1,1} = 0$, as both point along the base. These results can be checked by considering the $z^1 = z^2 = 0$ plane as in section 2.3.2.

We now consider the most general form of G_3 that preserves the $SU(3)$ and is odd under $z^i \to -z^i$. Since $\omega_{1,1}$ and $\chi_{1,1}$ are even under the involution, whereas $\omega_{2,0}$ and $\omega_{0,2}$ are odd, G_3 takes the general form

$$G_3 = g_{3,0}(r)\,\omega_{3,0} + g_{2,1}(r)\,\omega_{2,1} + g_{1,2}(r)\,\omega_{1,2} + g_{0,3}(r)\,\omega_{0,3}\,, \tag{2.41}$$

where

$$\omega_{3,0} \equiv \mathrm{d}z^1 \wedge \mathrm{d}z^2 \wedge \mathrm{d}z^3 = \frac{1}{r^2}\partial r^2 \wedge \omega_{2,0} = \mathrm{d}\omega_{2,0} \equiv \omega_{0,3}^{\star}\,, \tag{2.42}$$

$$\omega_{2,1} \equiv \frac{1}{r^2}\bar{\partial} r^2 \wedge \omega_{2,0} = \frac{1}{2\rho^2}\varepsilon_{klm} z_{\bar{j}} z^k \mathrm{d}\bar{z}^{\bar{j}} \wedge \mathrm{d}z^l \wedge \mathrm{d}z^m \equiv \omega_{1,2}^{\star}\,. \tag{2.43}$$

We immediately find $\omega_{1,1} \wedge G_3 = \chi_{1,1} \wedge G_3 = 0$; therefore G_3 is automatically primitive. In components, $G_{ijk} = \varepsilon_{ijk} g_{3,0}$ and $G_{\bar{i}jk} = \frac{1}{\rho^2} z_{\bar{i}} \varepsilon_{jkl} z^l g_{2,1}$.

D3-brane charge

The seven-brane stack wrapping the $\mathbb{P}^2$ can carry additional charges besides its seven-brane charge, which is fixed by the $-1 \in SL(2,\mathbb{Z})$ monodromy. However, since the horizon $S^5/\mathbb{Z}_3$ has vanishing third Betti number, F_3 and H_3 must be exact, and the stack cannot carry five-brane charge. We now show how to compute the D3-brane charge of the stack.

We define the D3-brane charge enclosed in a region R via the generalized Gauss's law:

$$Q_{\mathrm{D3}}(R) \equiv -\oint_{\partial R} \tilde{F}_5 = (2\pi)^4 \alpha'^2 N_{\mathrm{D3}}\,, \tag{2.44}$$

where D3-branes carry positive charge $2\kappa_{10}^2\,\mu_3 = (2\pi)^4\alpha'^2$, and there is a bulk contribution from the three-form fluxes, $Q_{\mathrm{D3}} = \int F_3 \wedge H_3 + Q_{\mathrm{loc}}$. Using (2.18), we obtain:

$$\tilde{F}_5 = \Omega_4 \wedge \mathrm{d}\alpha - e^{-8A} \star_6 \mathrm{d}\alpha\,. \tag{2.45}$$

Integrating over the $S^5/\mathbb{Z}_3$ at constant radius and accounting for the $\mathbb{Z}_2$ involution, we find

$$Q_{\rm D3}(r) = \oint r e^{4C-8A}\,\frac{{\rm d}\alpha}{{\rm d}r}\left(\frac{1}{2}\,\omega_{1,1}^2\wedge\omega\right) = \frac{\pi^3}{2}\,r e^{4C-8A}\,\frac{{\rm d}\alpha}{{\rm d}r}\,, \tag{2.46}$$

where $\omega \equiv \frac{1}{ir}(\partial-\bar\partial)r$, $\star_6\,\partial r = \frac{1}{2i}J^2\wedge\partial r$, and we use the periods

$$\int_{\mathbb{P}^2}\frac{1}{2}\,\omega_{1,1}^2 = \frac{1}{2}\,\pi^2 \quad , \quad \int_{S^5/\mathbb{Z}_3}\frac{1}{2}\,\omega_{1,1}^2\wedge\omega = \pi^3\,. \tag{2.47}$$

Since the integrands are closed, these integrals can be computed over any surface in the specified homology class. The periods (2.47) can also be used to compute the volume of the resolved $\mathbb{P}^2$:

$${\rm vol}(\mathbb{P}^2) = \frac{1}{2}\,\pi^2\,e^{4C(r)}\Big|_{r\to 0}\,. \tag{2.48}$$

The D3-brane defined by (2.44) is sourced in the bulk, and therefore not quantized. We define the Page charge [39] via the flux integral[13]

$$Q_{\rm D3}^{\rm Page}(R) \equiv -\oint_{\partial R}\left(\tilde F_5 + \frac{1}{2}\,C_2\wedge H_3 - \frac{1}{2}\,B_2\wedge F_3\right)\,. \tag{2.49}$$

The integrand is closed in the absence of sources, so the Page charge is not sourced in the bulk. In the absence of local sources coincident with ∂R, the Page charge is invariant under small gauge transformations of B_2 and C_2. It is not invariant under large gauge transformations unless F_3 and H_3 are exact when pulled back to ∂R. It has been argued [40] that the Page charge is quantized.

One can solve the Bianchi identity for G_3 to obtain

$$\mathcal{A}_2 \equiv \frac{1}{\sqrt{\tau_2}}\,(C_2-\tau B_2) = (g_{3,0}-g_{2,1})\,\omega_{2,0} + (g_{0,3}-g_{1,2})\,\omega_{0,2}\,. \tag{2.50}$$

It is then straightforward to compute the Page charge of the stack by the same method as above:

$$Q_{\rm D3}^{\rm Page} = \frac{\pi^3}{2}\left(r\,e^{4C-8A}\,\frac{{\rm d}\alpha}{{\rm d}r} + 2\,r^2\,|g_{3,0}-g_{2,1}|^2 - 2\,r^2\,|g_{1,2}-g_{0,3}|^2\right)\,, \tag{2.51}$$

[13] As there is more than one way to solve the $\tilde F_5$ Bianchi identity, there are inevitable ambiguities in defining the Page charge [40]. This difficulty does not arise in our setup due to the vanishing of the third Betti number for the horizon $\partial R = S^5/\mathbb{Z}_3$, so that all definitions are related by integration by parts.

where the right-hand side is independent of r as a consequence of the Bianchi identities. Since F_3 and H_3 are exact, the Page charge is gauge invariant. Henceforward, when we refer to the D3-brane charge of the solution, we mean the Page charge given by (2.51).

2.3.3 The near-stack limit of the $\mathbb{P}^2$ cone

We now apply the near-stack limit (2.13) to the above ansatz to recover the near-stack ansatz described in section 2.3.1 and to fix the relationship between the fields. As before, we apply the coordinate transformation (2.8), rescale as in (2.13), and truncate to leading order in ε. Finally, since ε is a formal expansion parameter, we set $\varepsilon = 1$. We find

$$\partial r^2 \to \partial r_z^2 \ \ , \ \ \chi_{1,1} \to \frac{i}{2}\,\mathrm{d}z \wedge \mathrm{d}\bar{z} \ \ , \ \ \omega_{1,1} \to \frac{i}{2}\left(\sum_a \mathrm{d}u^a \wedge \mathrm{d}\bar{u}^a\right) . \tag{2.52}$$

Thus, in particular

$$J \to \frac{i}{2}\Big(e^{-4B}\mathrm{d}z \wedge \mathrm{d}\bar{z} + e^{2C}\sum_a \mathrm{d}u^a \wedge \mathrm{d}\bar{u}^a\Big) . \tag{2.53}$$

We also find

$$G_3 \to g_{3,0}\mathrm{d}z\wedge\mathrm{d}u^1\wedge\mathrm{d}u^2 + g_{2,1}\frac{z}{\bar{z}}\mathrm{d}\bar{z}\wedge\mathrm{d}u^1\wedge\mathrm{d}u^2 + g_{1,2}\frac{\bar{z}}{z}\mathrm{d}z\wedge\mathrm{d}\bar{u}^1\wedge\mathrm{d}\bar{u}^2 + g_{0,3}\mathrm{d}\bar{z}\wedge\mathrm{d}\bar{u}^1\wedge\mathrm{d}\bar{u}^2 . \tag{2.54}$$

Therefore the quantities r, B, C, and $g_{p,q}$ defined in this section are appropriate generalizations far from the stack of the quantities r, B, C, and $g_{p,q}$ defined in section 2.3.1, and the two systems correspond in the near-stack limit. As an example, applying the near-stack limit to the Calabi-Yau metric for the resolved cone (2.39) gives a flat metric with B and C constant. The parameters r_0 and $\mathfrak{g}_0$ of the $\mathbb{P}^2$ ansatz can be recovered from

$$r_0 = \frac{1}{3}\,e^{2B_{\mathrm{ns}}+C_{\mathrm{ns}}} \ \ , \ \ \mathfrak{g}_0 = e^{4(C_{\mathrm{ns}}-B_{\mathrm{ns}})/3} , \tag{2.55}$$

where B_{ns} and C_{ns} are the constant values of B and C in the near-stack ansatz.

2.4 Supersymmetry Conditions

Our approach to learning about the gauge dynamics of compact seven-branes is to study the allowed supersymmetric backgrounds surrounding the seven-brane stack subject to some symmetry group, in the same spirit that early constructions of supersymmetric extremal black holes in supergravity (cf. e.g. [41]) presaged the appearance of D-branes as localized sources in supergravity.

In general, one expects AdS_4 vacua from compactifications that are stabilized by gaugino condensation on seven-branes [17]. This is because the superpotential generically develops a vev. If the compactification has finite warped volume, and therefore a finite four-dimensional Newton constant, the superpotential vev generates a negative cosmological constant, leading to an AdS_4 compactification. Noncompact solutions may be Minkowski, but we expect such solutions to arise in an appropriate decompactification limit of an AdS_4 solution.

We first consider general properties of supersymmetric AdS_4 compactifications, and return to the question of noncompactness below. As we will see, supersymmetric compactifications of type IIB supergravity to AdS_4 always have $SU(2)$ structure.[14] In our case, the $SU(2)$ structure is dynamic. As dynamic $SU(2)$ structure is less familiar than the more commonly studied strict $SU(3)$ structure, we now review dynamic $SU(2)$ structure in AdS_4 compactifications of type IIB supergravity, using tools from the more general field of generalized complex geometry.

2.4.1 Review of $SU(2)$ structure and generalized complex geometry

Having argued in section 2.1.2 that the supergravity solution should admit two spinors (η_+^1, η_+^2) that define a dynamic $SU(2)$ structure, the next step is to analyze

[14] While type-changing $SU(3)$ structure loci are possible, there is still a *local* $SU(2)$ structure away from these loci. Global, i.e. "strict," $SU(3)$ structure is impossible in AdS_4 compactifications [42].

the gravitino and dilatino variations for the supergravity ansatz introduced above. As shown in [23], the supersymmetry analysis simplifies considerably in terms of bispinors. Here we review the basic results from G-structures and generalized complex geometry needed for the rest of the chapter. For a recent review with further explanations and references, see e.g. [43].

We work in ten-dimensional Einstein frame, with metric ansatz (2.17),

$$\mathrm{d}s_{10}^2 = e^{2A(y)} h_{\mu\nu}(x) \mathrm{d}x^\mu \mathrm{d}x^\nu + e^{-2A(y)} g_{mn}(y) \mathrm{d}y^m \mathrm{d}y^n \,. \tag{2.56}$$

The warp and conformal factors are already made explicit and all the geometric quantities are constructed in terms of the internal metric g_{mn}. The four-dimensional metric $h_{\mu\nu}$ is that of AdS, with cosmological constant

$$\Lambda = -3|\mu|^2 \,. \tag{2.57}$$

The supersymmetry conditions were obtained in [23, 44], in string frame. The conversion from string frame to Einstein frame is done by modifying their warp factor $A_{(S)}$ and pure spinors $\Psi_{(S)}$ as follows:

$$A_{(S)} = A + \frac{\phi}{4} \;\; , \;\; \Psi_{(S)} = e^{(\phi/4 - A)\hat{p}} \Psi \,, \tag{2.58}$$

where the definition of the operator $\hat{p}$ is given by

$$\hat{p}\, C_p \equiv p\, C_p \tag{2.59}$$

for a p-form C_p. The rescaling in the pure spinor takes into account that the Mukai pairing $\langle \Psi, \overline{\Psi} \rangle$ is normalized by the volume of the internal metric $\mathrm{d}^6 y \sqrt{\det g_{mn}}$.

As in (2.1), we decompose the ten-dimensional Majorana-Weyl supersymmetry generators ϵ^i in Einstein frame,[15]

$$\epsilon^i = \zeta_+ \otimes \eta^i_+ + \zeta_- \otimes \eta^i_- \,. \tag{2.60}$$

[15]Using the conventions of [45], $\eta_- = C\eta_+^\star$, where C is the charge conjugation matrix.

The internal spinors η^i must have equal norms for an AdS_4 compactification [44]. Preservation of four-dimensional $\mathcal{N} = 1$ supersymmetry then imposes the normalization

$$|\eta^1_+|^2 = |\eta^2_+|^2 = e^A \,, \tag{2.61}$$

up to an arbitrary overall rescaling.

The two internal spinors may be combined into even and odd bispinors,

$$\Psi_+ = -8i\, e^{-A}\, \eta^1_+ \otimes \eta^{2\dagger}_+ \quad , \quad \Psi_- = -8i\, e^{-A}\, \eta^1_+ \otimes \eta^{2\dagger}_- \,, \tag{2.62}$$

where the extra warp factor dependence has been included for normalization purposes. Using the Clifford map, these bispinors are sums of forms of different degrees (polyforms). The supersymmetry conditions of [23] then become

$$\begin{aligned} \mathrm{d}_H \left(e^{(\phi/4-A)\hat{p}} e^{4A}\, \mathrm{Re}\,\Psi_+\right) &= -\,3e^{(\phi/4-A)\hat{p}} e^{3A-\phi/4}\, \mathrm{Re}(\bar{\mu}\Psi_-) \\ &\quad + e^{(2A-\phi/2)(3-\hat{p})} e^{4A+\phi} \star_6 \lambda(F)\,, \\ \mathrm{d}_H \left(e^{(\phi/4-A)\hat{p}} e^{2A-\phi/2}\, \mathrm{Im}\,\Psi_+\right) &= 0\,, \\ \mathrm{d}_H \left(e^{(\phi/4-A)\hat{p}} e^{3A-\phi/4}\, \Psi_-\right) &= -\,2i\mu\, e^{(\phi/4-A)\hat{p}} e^{2A-\phi/2}\, \mathrm{Im}\,\Psi_+\,. \end{aligned} \tag{2.63}$$

Here $\mathrm{d}_H \equiv \mathrm{d} - H\wedge$, and all fluxes are internal, with

$$F \equiv F_1 + \tilde{F}_3 + \tilde{F}_5^{(\mathrm{int})} \quad , \quad \lambda(A_p) = (-1)^{p(p-1)/2} A_p\,. \tag{2.64}$$

When supplemented by the p-form Bianchi identities, the above supersymmetry conditions imply all of the supergravity equations of motion (which we will verify explicitly in our examples).[16]

Now, let us consider the geometric properties of manifolds with $SU(2)$ structure [47]. It is convenient to introduce two globally defined orthonormal spinors η_+ and χ_+. They are related by a vector Θ_m,

$$\chi_+ = \frac{1}{2}\Theta^m \gamma_m \eta_- \,, \tag{2.65}$$

[16] In the presence of sources, one must also impose calibration conditions on the sources; cf. [46].

where $|\Theta|^2 = 2$. An $SU(2)$ structure is then characterized by the following invariant forms:

$$\Theta_m = \eta_-^\dagger \gamma_m \chi_+ \ , \ (J_2)_{mn} = -\frac{i}{2}\left(\eta_+^\dagger \gamma_{mn} \eta_+ - \chi_+^\dagger \gamma_{mn} \chi_+\right) \ , \ (\Omega_2)_{mn} = i\chi_+^\dagger \gamma_{mn} \eta_+ \,. \tag{2.66}$$

These satisfy

$$J_2 \wedge \Omega_2 = \Omega_2 \wedge \Omega_2 = 0 \ , \ J_2 \wedge J_2 = \frac{1}{2}\Omega_2 \wedge \overline{\Omega}_2 \ , \ \imath_\Theta \Omega_2 = \imath_\Theta J_2 = 0 \,. \tag{2.67}$$

Algebraically, the tangent bundle has a product structure, where Ω_2 and J_2 may be thought of as the holomorphic two-form and Kähler form for a complex-dimension two subspace of the tangent bundle, and Θ and $J_1 \equiv \frac{i}{2}\Theta \wedge \bar{\Theta}$ as the holomorphic one-form and Kähler form for the complex-dimension one complement. However, this product structure is typically not integrable, and the manifold need not be a direct product. Instead, we will think of the manifold as a line bundle with J_2 and Ω_2 pointing along the base and Θ and $\bar{\Theta}$ pointing along the fiber. This structure will turn out to be integrable in our examples, but this is not guaranteed in general.

The $SU(2)$ structure can be viewed locally as the intersection of two $SU(3)$ structures, each associated to one of the spinors. In particular, the $SU(3)$ structure from η_+ is defined by

$$J_{mn} = -i\eta_+^\dagger \gamma_{mn} \eta_+ \ , \ \Omega_{mnr} = i\eta_-^\dagger \gamma_{mnr} \eta_+ \,. \tag{2.68}$$

The forms (2.66) and (2.68) are related by

$$J = J_2 + \frac{i}{2}\Theta \wedge \bar{\Theta} \ , \ \Omega = \Theta \wedge \Omega_2 \,. \tag{2.69}$$

There are different ways of writing the spinors η_+^i in terms of (η_+, χ_+). In geometries with orientifold planes it is most convenient to average $\eta_+ \propto (\eta_+^1 + ie^{i\vartheta}\eta_+^2)$, where $ie^{i\vartheta}$ is the relative phase between the two spinors [48, 49]. Thus we take

$$\eta_+^1 = i\, e^{i\,\vartheta/2}\, e^{A/2}\left(\cos\frac{\varphi}{2}\,\eta_+ + \sin\frac{\varphi}{2}\,\chi_+\right) \ , \ \eta_+^2 = e^{-i\,\vartheta/2}\, e^{A/2}\left(\cos\frac{\varphi}{2}\,\eta_+ - \sin\frac{\varphi}{2}\,\chi_+\right) . \tag{2.70}$$

The warp factor is fixed by the normalization (2.61), and ϑ and φ parameterize the angle between the spinors,

$$e^{-A}\,\eta_+^{2\,\dagger}\eta_+^1 = i\,e^{i\vartheta}\cos\varphi\,. \tag{2.71}$$

Using this, the bispinors may be expressed in terms of the $SU(2)$ forms (2.66), yielding

$$\begin{aligned}\Psi_+ &= e^{i\vartheta}e^{\frac{1}{2}\Theta\wedge\bar{\Theta}}\left[\cos\varphi\left(1-\frac{1}{2}J_2^2\right)+\sin\varphi\,\mathrm{Im}\,\Omega_2 - iJ_2\right]\,,\\ \Psi_- &= \Theta\wedge\left[\sin\varphi\left(1-\frac{1}{2}J_2^2\right)-\cos\varphi\,\mathrm{Im}\,\Omega_2+i\,\mathrm{Re}\,\Omega_2\right]\,.\end{aligned} \tag{2.72}$$

It is straightforward to check by substituting this result into (2.63) that we must take $e^{i\,\vartheta} = \pm 1$ for AdS_4 solutions, where the extra sign can be absorbed by redefinitions. Thus, we take $\vartheta = 0$ without loss of generality.

The supersymmetry structure is characterized by the angle φ, which in turn determines the types of Ψ_+ and Ψ_-.[17] For static $SU(2)$ structure (type $(2,1)$), $\varphi = \pi/2$ and the two internal spinors are everywhere orthogonal. For strict $SU(3)$ structure (type $(0,3)$), $\varphi = 0$ and the spinors are everywhere parallel; the polyforms simplify to $\Psi_+ = e^{i\,\vartheta}e^{-iJ}$ and $\Psi_- = i\,\Omega_3$. For intermediate $SU(2)$ structure (type $(0,1)$), $0 < \varphi < \pi/2$, and the spinors are neither parallel nor orthogonal.

If φ varies along the internal manifold, the $SU(2)$ structure is said to be dynamic. Dynamic $SU(2)$ structure can be 'type-changing' if $\varphi = 0$ or $\varphi = \pi/2$ on some locus. Our solution will turn out to be dynamic; for certain values of the parameters, it is also type-changing with a $\varphi = \pi/2$ locus.

We now impose the orientifold projection $\mathcal{O} = \Omega_p(-1)^{F_L}\sigma$, where σ is the involution, F_L is the number of left-moving fermions, and Ω_p is the worldsheet parity. The involution on the pure spinors (Ψ_+, Ψ_-) should reduce to the known

[17] Here the 'type' of a polyform Ψ refers to the rank of the lowest-rank non-zero component of Ψ, as in [50]. In what follows, type (m,n) refers to Ψ_+ (Ψ_-) of type m (n).

result

$$\sigma(J) = J \ \ , \ \ \sigma(\Omega) = -\Omega\,, \tag{2.73}$$

for an O3/O7, so that we have [44],

$$\sigma(\Psi_+) = \lambda(\overline{\Psi}_+) \ \ , \ \ \sigma(\Psi_-) = \lambda(\Psi_-)\,, \tag{2.74}$$

with λ defined in (2.64). Applying (2.74) to (2.72) gives

$$\sigma(J_2) = J_2 \ \ , \ \ \sigma(\Omega_2) = -\Omega_2 \ \ , \ \ \sigma(\Theta) = \Theta \ \ , \ \ \sigma(\varphi) = \varphi\,. \tag{2.75}$$

In other words, in the basis (2.70) the orientifold action is realized as an explicit π rotation in the $(\mathrm{Re}\,\Omega_2, \mathrm{Im}\,\Omega_2)$ 'plane' of the space $(J_2, \mathrm{Re}\,\Omega_2, \mathrm{Im}\,\Omega_2)$ [49].

For strict $SU(3)$ structure compactifications, we must take the holomorphic three-form Ω to carry R-charge +2 [51]. Generalizing this, we see that Ψ_- should carry R charge +2. Thus, Θ and μ carry R-charge +2, while Ω_2 and J_2 are invariant.

Substituting (2.72) into (2.63), we obtain the supersymmetry conditions for compactifying type IIB supergravity to AdS_4. As remarked above, supersymmetric compactification to AdS_4 requires ${\eta^1_+}^\dagger\eta^1_+ = {\eta^2_+}^\dagger\eta^2_+$ and $\mathrm{Re}\,{\eta^2_+}^\dagger\eta^1_+ = 0$. We refer to Minkowski ($\mu = 0$) solutions that satisfy these conditions as 'AdS-like.' The set of 'AdS-like' solutions may be thought of as the closure of the set of AdS solutions, since a solution that arises upon taking a limit in parameter space of an AdS solution must still satisfy ${\eta^1_+}^\dagger\eta^1_+ = {\eta^2_+}^\dagger\eta^2_+$ and $\mathrm{Re}\,{\eta^2_+}^\dagger\eta^1_+ = 0$, provided that η^1 and η^2 vary continuously in this limit. Thus, we expect that noncompact solutions with gaugino condensation will be AdS-like. AdS-like solutions with strict $SU(3)$ structure are the well known type B supersymmetric solutions which arise from F-theory compactifications.

The supersymmetry conditions for AdS and AdS-like compactifications can be rewritten in an $SL(2,\mathbb{R})$ covariant form. In section 2.4.2, we briefly review $SL(2,\mathbb{R})$

covariant supergravity. In section 2.4.3, we present the covariant supersymmetry conditions (deferring a detailed derivation to chapter 3) and discuss their implications.

2.4.2 $SL(2,\mathbb{R})$ covariant supergravity

It is well known that the action (2.15) is invariant under $SL(2,\mathbb{R})$ transformations, where $\tau = C_0 + ie^{-\phi}$ transforms as $\tau \to \frac{a\tau+b}{c\tau+d}$, F_3 and H_3 mix as:

$$\begin{pmatrix} F_3 \\ H_3 \end{pmatrix} \to \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} F_3 \\ H_3 \end{pmatrix}, \tag{2.76}$$

with $ad-bc=1$, and the metric and C_4 are invariant. Accounting for brane sources as required by string theory, this $SL(2,\mathbb{R})$ breaks to the discrete subgroup $SL(2,\mathbb{Z})$, but this subgroup is gauged: monodromies are allowed, and indeed charged seven-branes carry $SL(2,\mathbb{Z})$ monodromies.

Since we are interested in studying nonperturbative effects on seven-branes, an approach that makes $SL(2,\mathbb{Z})$ (and indeed $SL(2,\mathbb{R})$) invariance manifest is indispensable (cf. e.g. [52]). It will be convenient to work with $\frac{1}{\tau_2}\,\mathrm{d}\tau$ and the complex field strength and potential

$$G_3 \equiv \frac{1}{\sqrt{\tau_2}}\left(F_3 - \tau H_3\right) \ , \ \mathcal{A}_2 \equiv \frac{1}{\sqrt{\tau_2}}\left(C_2 - \tau B_2\right). \tag{2.77}$$

These transform by a τ-dependent phase under $SL(2,\mathbb{R})$:

$$G_3 \to \frac{|c\tau+d|}{c\tau+d}\, G_3 \ , \ \frac{1}{\tau_2}\,\mathrm{d}\tau \to \left(\frac{|c\tau+d|}{c\tau+d}\right)^2 \frac{1}{\tau_2}\,\mathrm{d}\tau\,. \tag{2.78}$$

These are both examples of a more general transformation law,

$$\Omega \to \left(\frac{|c\tau+d|}{c\tau+d}\right)^{2Q} \Omega\,, \tag{2.79}$$

where Q is the 'S-charge' of Ω. In this language, G_3 and $\mathcal{A}_2$ carry charge $+1/2$ and $\frac{1}{\tau_2}\,\mathrm{d}\tau$ carries charge $+1$.

Since the τ-dependent phase is in general nonconstant, the derivative of an S-covariant quantity is not covariant. We introduce a covariant derivative $\partial_M \to D_M = \partial_M + iQK_M$ where K_M is a one-form connection. One can check that $K_M = \frac{1}{\tau_2}\,\partial_M \tau_1$ transforms appropriately, where $\tau_1 = \mathrm{Re}\,\tau$. Thus, we define the covariant exterior derivative:

$$\mathrm{D}\,\Omega \equiv \mathrm{d}\,\Omega + i\,Q\,\frac{1}{\tau_2}\,\mathrm{d}\tau_1 \wedge \Omega\,. \tag{2.80}$$

It is easy to check that D is not nilpotent: $\mathrm{D}^2 = \frac{iQ}{\tau_2^2}\,\mathrm{d}\tau_1 \wedge \mathrm{d}\tau_2$. We define the modified covariant exterior derivative for Ω of charge $+1/2$:

$$\mathrm{D}_\pm\,\Omega \equiv \mathrm{D}\,\Omega \pm \frac{i}{2\,\tau_2}\,\mathrm{d}\tau \wedge \Omega^\star\,. \tag{2.81}$$

Similarly, for Ω of charge $-1/2$, we define $\mathrm{D}_\pm\,\Omega \equiv \mathrm{D}\,\Omega \pm \frac{i}{2\,\tau_2}\,\mathrm{d}\bar\tau \wedge \Omega^\star = (\mathrm{D}_\mp\,\Omega^\star)^\star$. One can check that D_+ and D_- are both nilpotent.[18] The G_3 Bianchi identity now takes the simple form $\mathrm{D}_-\,G_3 = 0$, with the local solution $G_3 = \mathrm{D}_-\,\mathcal{A}_2$.

Expressed in terms of τ and G_3, the type IIB supergravity action (2.15) becomes

$$S = \frac{1}{2\kappa_{10}^2}\int \mathrm{d}^{10}x\sqrt{-g^{(10)}}\left[R - \frac{1}{2}\left(\frac{1}{\tau_2^2}\,|\mathrm{d}\tau|^2 + |G_3|^2 + \frac{1}{2}|\tilde F_5|^2\right)\right] - \frac{i}{8\kappa_{10}^2}\int C_4 \wedge G_3 \wedge G_3^\star\,, \tag{2.82}$$

where, in terms of G_3 and $\mathcal{A}_2$, $\tilde F_5$ takes the local form

$$\tilde F_5 = \mathrm{d}C_4 + \left(\frac{i}{4}\mathcal{A}_2 \wedge G_3^\star + c.c.\right)\,, \tag{2.83}$$

with the Bianchi identity $\mathrm{d}\,\tilde F_5 = \frac{i}{2}\,G_3 \wedge G_3^\star$. One can then rewrite the equations of motion as:[19]

$$\mathrm{D}_-\star_{10} G_3^\star = i\,G_3^\star \wedge \tilde F_5 \quad,\quad \mathrm{D}\star_{10}\left(\frac{1}{\tau_2}\,\mathrm{d}\bar\tau\right) = \frac{i}{2}\,G_3^\star \wedge \star_{10} G_3^\star\,, \tag{2.84}$$

[18]Note that $\mathrm{D}_\pm$ are not $\mathbb{C}$-linear ($i\mathrm{D}_+ = \mathrm{D}_- i$) and do not always obey the product rule.

[19]Here $|F_p|^2_{MN} \equiv \frac{1}{2(p-1)!}\,(F_p)^{M_1\ldots M_{p-1}}{}_M\,\left(F_p^\star\right)_{M_1\ldots M_{p-1}N} + c.c.$, so that $g^{MN}|F_p|^2_{MN} = p\,|F_p|^2$.

$$R_{MN} = \frac{1}{4\,\tau_2^2}\left(\nabla_M\tau\,\nabla_N\bar{\tau} + c.c.\right) + \frac{1}{2}\left[|G_3|^2_{MN} - \frac{1}{4}\,|G_3|^2\,g^{(10)}_{MN}\right] + \frac{1}{4}\,|\tilde{F}_5|^2_{MN}\,, \quad (2.85)$$

and in addition one must impose the self-duality constraint $\tilde{F}_5 = \star_{10}\tilde{F}_5$. The action and equations of motion are now manifestly covariant under $SL(2,\mathbb{R})$ transformations.[20]

Consider the additional $\mathbb{Z}_2$ symmetry of type IIB supergravity under which all of the RR fields are reversed (i.e. $C_p \to -C_p$ and $F_p \to -F_p$). This is an exact (gauged) symmetry of string theory which we denote $\mathbb{Z}_2^{(RR)}$, corresponding for instance to the involution associated with the O9 plane of the type I theory composed with the $-1 \in SL(2,\mathbb{Z})$. In the S-covariant language developed above, $\mathbb{Z}_2^{(RR)}$ takes the form:

$$G_3 \to -G_3^\star \quad , \quad \tau \to -\tau^\star \quad , \quad \tilde{F}_5 \to -\tilde{F}_5\,. \quad (2.86)$$

If Ω carries S-charge Q and transforms as $\Omega \to -\Omega^\star$ under $\mathbb{Z}_2^{(RR)}$, then $\mathrm{D}\,\Omega \to -(\mathrm{D}\,\Omega)^\star$. For $Q = \pm 1/2$, $\mathrm{D}_\pm\,\Omega \to -(\mathrm{D}_\pm\,\Omega)^\star$. Viewing $\mathbb{Z}_2^{(RR)}$ as $\mathrm{diag}(-1,1) \in SL_\pm(2,\mathbb{R})$, it combines with the $SL(2,\mathbb{R})$ invariance discussed above to generate the classical symmetry group $SL_\pm(2,\mathbb{R})$, of which the subgroup $SL_\pm(2,\mathbb{Z})$ is an exact (gauged) symmetry of string theory.

2.4.3 $SL(2,\mathbb{R})$ covariant supersymmetry conditions for AdS_4 compactifications

Henceforward, we work with the almost complex structure defined by the holomorphic three-form $\Omega \equiv \Omega_2 \wedge \Theta$ with associated Kähler form $J = J_1 + J_2$, where $J_1 \equiv (i/2)\Theta \wedge \bar{\Theta}$. Consistent with (2.75), we assign Ω_2 an S-charge of $-1/2$ and take J_2, Θ, and φ to be invariant. G_3 can be decomposed into pieces with zero, one, or two legs along the 'fiber' $(\Theta, \bar{\Theta})$ directions. The supersymmetry conditions

[20] The full supersymmetric action is also invariant [53].

directly imply $G_3 \wedge \Theta \wedge J_2 = G_3 \wedge \bar{\Theta} \wedge J_2 = 0$. Thus, a general decomposition takes the form [47]:

$$G_3 = J_1 \wedge \mathcal{G}_1 + J_2 \wedge \mathfrak{G}_1 + \mathfrak{g}_{3,0}\,\Omega_2 \wedge \Theta + \mathfrak{g}_{2,1}\,\Omega_2 \wedge \bar{\Theta} \\ + \mathfrak{g}_{1,2}\,\bar{\Omega}_2 \wedge \Theta + \mathfrak{g}_{0,3}\,\bar{\Omega}_2 \wedge \bar{\Theta} + \mathcal{G}_{1,1} \wedge \Theta + \mathfrak{G}_{1,1} \wedge \bar{\Theta}\,, \quad (2.87)$$

where $\mathcal{G}_1$ and $\mathfrak{G}_1$ are complex one-forms pointing along the base, the $\mathfrak{g}_{p,q}$ are complex scalars, and $\mathcal{G}_{1,1}$ and $\mathfrak{G}_{1,1}$ are complex primitive (1,1) forms pointing along the base. We also decompose the gradient into fiber and base directions:

$$\mathrm{d}f = (\partial_\Theta f)\,\Theta + \left(\bar{\partial}_\Theta f\right)\bar{\Theta} + \mathrm{d}_\Pi f\,. \quad (2.88)$$

Applying these decompositions to the supersymmetry conditions and simplifying (see chapter 3), we obtain:

$$\mathrm{d}\left[e^{4A}c_\varphi\right] = -3\,e^{2A}\,s_\varphi\,\mathrm{Re}\,(\bar{\mu}\,\Theta) + \mathrm{d}\alpha\,, \quad (2.89)$$

$$\mathrm{d}\left[e^{2A}s_\varphi\Theta\right] = 2i\mu\,(c_\varphi J_1 + J_2) \quad , \quad \mathrm{d}\left[c_\varphi J_1 + J_2\right] = 0\,, \quad (2.90)$$

$$\mathrm{D}_+\left[e^{2A}s_\varphi\Omega_2\right] = c_\varphi e^{4A}G_3^\star - ie^{4A}\star G_3^\star \\ + \frac{3\bar{\mu}}{2}(1+c_\varphi)\Omega_2 \wedge \Theta - \frac{3\mu}{2}(1-c_\varphi)\Omega_2 \wedge \bar{\Theta}\,, \quad (2.91)$$

$$\mathfrak{g}_{3,0} = -\frac{1-c_\varphi}{2s_\varphi}\,e^{-2A}\frac{i}{\tau_2}\partial_\Theta\tau\,, \qquad \mathfrak{g}_{2,1} = \frac{1+c_\varphi}{2s_\varphi}\,e^{-2A}\frac{i}{\tau_2}\bar{\partial}_\Theta\tau\,, \quad (2.92)$$

$$\mathfrak{g}_{1,2} = \bar{\mu}\,e^{-4A} + \frac{1+c_\varphi}{2s_\varphi}\,e^{-6A}\partial_\Theta\Phi_-\,, \quad \mathfrak{g}_{0,3} = \mu\,e^{-4A} - \frac{1-c_\varphi}{2s_\varphi}\,e^{-6A}\bar{\partial}_\Theta\Phi_+ \quad (2.93)$$

$$\mathcal{G}_{1,0} \wedge J_2 = -s_\varphi^{-1}e^{-2A}\left(\frac{i}{\tau_2}\,\mathrm{d}_\Pi\,\tau\right) \wedge \Omega_2 \quad , \quad \mathfrak{G}_{1,0} = -c_\varphi\mathcal{G}_{1,0}\,, \quad (2.94)$$

$$\mathcal{G}_{0,1} \wedge J_2 = -s_\varphi^{-1}e^{-6A}\left(\mathrm{d}_\Pi\left[e^{4A}\right] - c_\varphi\mathrm{d}_\Pi\alpha\right) \wedge \bar{\Omega}_2\,, \quad (2.95)$$

$$\mathfrak{G}_{0,1} \wedge J_2 = s_\varphi^{-1}e^{-6A}\left(c_\varphi\mathrm{d}_\Pi\left[e^{4A}\right] - \mathrm{d}_\Pi\alpha\right) \wedge \bar{\Omega}_2\,, \quad (2.96)$$

where c_φ and s_φ are shorthand for $\cos\varphi$ and $\sin\varphi$, and

$$\Phi_\pm \equiv e^{4A} \pm \alpha\,. \quad (2.97)$$

By referring to the charge assignments of Table 2.1, one can verify that the above equations are manifestly invariant under $SL(2,\mathbb{R})$, and also under a $U(1)_R$ symmetry.

	S-charge	R-charge		S-charge	R-charge
$G_3, \bar{\Omega}_2$	$+1/2$	0	$\mathfrak{g}_{1,2}$	0	-2
$\delta\tau/\tau_2$	$+1$	0	$\mathfrak{g}_{0,3}$	0	$+2$
$\mu, \Theta, \bar{\partial}_\Theta$	0	$+2$	$\mathcal{G}_1, \mathfrak{G}_1$	$+1/2$	0
$\mathfrak{g}_{3,0}$	$+1$	-2	$\mathcal{G}_{1,1}$	$+1/2$	-2
$\mathfrak{g}_{2,1}$	$+1$	$+2$	$\mathfrak{G}_{1,1}$	$+1/2$	$+2$

Table 2.1: The S-charge and R-charge of various fields; $\delta\tau$ represents any differential of τ.

In fact, the supersymmetry conditions (2.89 – 2.96) are covariant under the full $SL_\pm(2,\mathbb{R})$ classical symmetry group of type IIB supergravity, which is generated by $SL(2,\mathbb{R})$ transformations and by $\text{diag}(-1,1) \in SL_\pm(2,\mathbb{Z})$. The latter transformation, (2.86), takes $G_3 \to -G_3^\star$, $\tau \to -\tau^\star$, and $\alpha \to -\alpha$, along with $\Psi_\pm \to -\Psi_\pm$, so that

$$\Omega_2 \to -\Omega_2^\star \ , \ \ \varphi \to \varphi + \pi \ , \ \ J_2 \to -J_2 \, , \tag{2.98}$$

with the appropriate transformations on the components of G_3. In addition, the supersymmetry conditions possess the $\mathbb{Z}_2$ symmetry

$$\Theta \to -\Theta \ , \ \ \varphi \to -\varphi \ , \ \ \Omega_2 \to -\Omega_2 \, , \tag{2.99}$$

under which the polyforms Ψ_+ and Ψ_- are invariant.

The supersymmetry conditions for type B solutions are well known, and are easily derived from (2.63): τ must be holomorphic, J must be closed, Φ_- must vanish, G_3 must be primitive and of Hodge type $(2,1)$, and $\mathrm{d}(e^{\phi/2}\,\Omega) = 0$, which can be restated covariantly as $\mathrm{D}\Omega = 0$. Unlike type B solutions, AdS and AdS-like $SU(2)$ structure solutions need not be Kähler or even complex. Taking the $(1,2)$

component of (2.91) and applying (2.92, 2.94) we obtain

$$[\mathrm{d}\,\Omega_2]_{1,2} = \frac{1}{2}\, s_\varphi\, e^{2A} J_2 \wedge (\mathcal{G}_{1,0})^\star - \frac{1-c_\varphi}{s_\varphi}\, e^{2A}\, \bar\Theta \wedge (\mathcal{G}_{1,1})^\star\ . \tag{2.100}$$

Applying (2.90), we find $[\mathrm{d}\,\Omega]_{2,2} = [\mathrm{d}\,\Omega_2]_{1,2} \wedge \Theta$. Thus, the almost complex structure associated with $\Omega = \Omega_2 \wedge \Theta$ is integrable if and only if $\mathcal{G}_{1,0}$ and $\mathcal{G}_{1,1}$ both vanish. Similarly, the Kähler form J is not in general closed. Applying (2.89, 2.90), we find

$$\mathrm{d}J = -2\,\mu\, e^{-2A}\, \frac{1-c_\varphi}{s_\varphi} J_2 \wedge \mathrm{Re}\,\Theta - e^{-4A} \left(\frac{1-c_\varphi}{1+c_\varphi}\right) \mathrm{d}\,\Phi_+ \wedge J_1\,. \tag{2.101}$$

Adding (2.95) and (2.96), we see that $\mathrm{d}_\Pi\,\Phi_+$ vanishes if and only if $\mathcal{G}_{0,1} + \mathfrak{G}_{0,1} = 0$. Thus, J is closed if and only if the solution is Minkowski with $\mathcal{G}_{0,1} = -\mathfrak{G}_{0,1}$. Using (2.94), the above conditions on $\mathcal{G}_{0,1}$ and on the pair $\mathcal{G}_{0,1}$ and $\mathfrak{G}_{0,1}$ can be efficiently restated as the requirement that $G_{2,1}$ and $G_{1,2}$ be primitive, respectively.

In [26] it is argued that the appropriate generalization of the Gukov-Vafa-Witten on-shell flux superpotential [54] to generalized complex geometry solutions is the on-shell superpotential

$$W = \frac{1}{4\,\kappa_{10}^2} \int \left(e^{3A_{(S)}-\phi}\,\Psi_-^{(S)}\right) \wedge \lambda(F)\,, \tag{2.102}$$

where $A_{(S)}$ and $\Psi_-^{(S)}$ are related to the Einstein frame quantities A and Ψ_- by (2.58), λ is defined in (2.64), and the integral is over the compact manifold. Supersymmetric AdS solutions correspond to a nonvanishing superpotential vev. Using the supersymmetry conditions (2.89 – 2.96), we can evaluate (2.102) explicitly. We find

$$W = -\frac{\mu}{\kappa_{10}^2} \int \mathrm{d}^6 y\,\sqrt{g}\, e^{-4A} = -\frac{\mu}{\kappa_4^2}\,, \tag{2.103}$$

in exact agreement with the four-dimensional supergravity result $\Lambda = -3\,\kappa_4^4\,|W|^2$, where κ_4^2 is the four-dimensional Newton constant. The limit of rigid supersymmetry, $\kappa_4^2 \to 0$, is more subtle, as the integrand of (2.102) vanishes on-shell, but the integral is taken over an infinite volume.

2.5 Supergravity Solution in the Near-stack Region

In this section we search for supersymmetric solutions to the ansatz described in section 2.3.1. First, as a warmup we describe type B solutions to this ansatz. The axiodilaton must be holomorphic, but can only depend on the real coordinate r, so it is a constant, $\tau = C_0 + \frac{i}{g_s}$. Thus, the solution is conformally Calabi-Yau, with B and C constant, consistent with the near-stack limit of the metric (2.39). The only nonvanishing component of G_3 is $G_{2,1}$, which must be closed by the G_3 Bianchi identity; thus, $g_{2,1} = e^{2C}\, k_{2,1}/r^2$. Finally, we take $\alpha = e^{4A} = \Phi_+/2$, and solve the $\tilde{F}_5$ Bianchi identity, (2.194), which gives:

$$\frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}\left(r\,\left(\Phi_+^{-1}\right)'\right) = -2\,e^{-4C}\,|g_{2,1}|^2\,, \tag{2.104}$$

where primes denote derivatives with respect to r. We integrate to obtain $\Phi_+^{-1} = k_0 + k_1 \log r - \frac{1}{2r^2}|k_{2,1}|^2$. Comparing with (2.51), we see that k_1 is related to the D3-brane charge:

$$Q_{\mathrm{D3}} = -\pi^3\, r\, e^{4C}\,(\Phi_+^{-1})' + \pi^3\, r^2\,|g_{2,1}|^2 = -\pi^3\, e^{4C}\, k_1\,, \tag{2.105}$$

where $(\pi^2/2)e^{4C}$ is the volume of the resolved $\mathbb{P}^2$ and the near-stack approximation is valid for $r \ll r_0$, where $r_0 = \frac{1}{3}\,e^{2B+C}$, as in (2.55).

2.5.1 Supersymmetry conditions in the near-stack region

We now apply the supersymmetry conditions for AdS and AdS-like vacua (section 2.4.3) to the near-stack ansatz of section 2.3.1. We assume that the internal spinors η^1 and η^2 are singlets under the $SU(3)$ symmetry group, so that Ω_2, J_2, Θ and φ are also singlets under the $SU(3)$.

The only invariant one-forms are $\frac{1}{z}\,\mathrm{d}z$ and its conjugate, up to a radially-dependent factor. In order to satisfy the orthonormality conditions $g^{-1}(\Theta,\Theta) = 0$

and $g^{-1}(\Theta, \bar{\Theta}) = 2$, we must take

$$\Theta = e^{-2B+i\,\theta}\,\frac{r}{z}\,\mathrm{d}z\,, \tag{2.106}$$

for some real $\theta = \theta(r)$. Ω_2 must be a complex two-form that is odd under the involution $z \to -z$, and that satisfies $i_\Theta\,\Omega_2 = 0$, $\Omega_2 \wedge \Omega_2 = 0$, with the normalization $\mathrm{vol}_6 = \frac{1}{4} J_1 \wedge \Omega_2 \wedge \bar{\Omega}_2$. Therefore, we must have

$$\Omega_2 = e^{2C+i\,\xi}\,\frac{z}{r}\,\mathrm{d}u^1 \wedge \mathrm{d}u^2\,, \tag{2.107}$$

for some phase factor $\xi = \xi(r)$, and the complex structure defined by $\Omega = \Omega_2 \wedge \Theta$ is integrable by inspection. Then, (2.106) and (2.107) uniquely determine J_1 and J_2:

$$J_1 = \frac{i}{2}\,e^{-4B}\mathrm{d}z \wedge \mathrm{d}\bar{z} \quad , \quad J_2 = \frac{i}{2}\,e^{2C}\left(\mathrm{d}u^1 \wedge \mathrm{d}\bar{u}^1 + \mathrm{d}u^2 \wedge \mathrm{d}\bar{u}^2\right)\,. \tag{2.108}$$

Applying (2.90), we see that J_2 is closed, since $\varphi = \varphi(r)$ is a singlet scalar, and therefore $c_\varphi\,J_1$ is closed by (2.108). Thus, C is constant and the metric is Kähler. By (2.101), the solution must be Minkowski ($\mu = 0$).[21]

Applying $\mu = 0$ to (2.89) and integrating, we find

$$\alpha = c_\varphi\,e^{4A}\,, \tag{2.109}$$

where we fix the α shift symmetry. The first equation of (2.90) gives

$$e^{2A}\,s_\varphi\,\Theta = \frac{c_1}{z}\,\mathrm{d}z\,, \tag{2.110}$$

where c_1 is a complex constant. Thus,

$$r s_\varphi\,e^{2(A-B)+i\theta} = c_1\,, \tag{2.111}$$

and θ must be constant. Combining (2.109, 2.111) to eliminate φ, we find

$$\Phi_+\Phi_- = \frac{|c_1|^2}{r^2}\,e^{4(A+B)}\,. \tag{2.112}$$

[21] This is an artifact of the near-stack approximation: (2.90) requires $\mu \neq 0$ for $\varphi \neq 0$ solutions on the $\mathbb{P}^2$ cone. The AdS-like solutions obtained in this section may be viewed as 'approximately AdS,' in that they approximate solutions on the $\mathbb{P}^2$ cone with small cosmological constant.

Henceforth, we assume that $c_1 \neq 0$, since the special case $c_1 = 0$ is just a type B solution, as discussed above. For $c_1 \neq 0$, $\varphi \neq 0, \pi$, and so (2.109) implies that $|\alpha| < e^{4A}$, and therefore

$$\Phi_\pm > 0\,. \tag{2.113}$$

Now consider the decomposition of G_3, (2.87). $SU(3)$ invariance constrains $\mathcal{G}_1$, $\mathfrak{G}_1$, $\mathcal{G}_{1,1}$, and $\mathfrak{G}_{1,1}$ to vanish. Thus, we decompose:

$$G_3 = \mathfrak{g}_{3,0}\,\Omega_2 \wedge \Theta + \mathfrak{g}_{2,1}\,\Omega_2 \wedge \bar{\Theta} + \mathfrak{g}_{1,2}\,\bar{\Omega}_2 \wedge \Theta + \mathfrak{g}_{0,3}\,\bar{\Omega}_2 \wedge \bar{\Theta}\,, \tag{2.114}$$

where the $\mathfrak{g}_{p,q}$ are related to the $g_{p,q}$ of section 2.3.1 by

$$\begin{aligned} g_{3,0} &= e^{2(C-B)+i\,(\xi+\theta)}\mathfrak{g}_{3,0}\,, \qquad & g_{2,1} &= e^{2(C-B)+i\,(\xi-\theta)}\mathfrak{g}_{2,1}\,,\\ g_{1,2} &= e^{2(C-B)-i\,(\xi-\theta)}\mathfrak{g}_{1,2}\,, \qquad & g_{0,3} &= e^{2(C-B)-i\,(\xi+\theta)}\mathfrak{g}_{0,3}\,. \end{aligned} \tag{2.115}$$

The conditions (2.94 - 2.96) are now trivially satisfied. We use the above decomposition to write out the remaining conditions. Applying (2.109), (2.91) becomes

$$\mathrm{D}_+\left[e^{2A}\,s_\varphi\,\Omega_2\right] = \Phi_+\,(G_{3,0}+G_{1,2})^\star - \Phi_-\,(G_{2,1}+G_{0,3})^\star\,, \tag{2.116}$$

since G_3 is primitive. Next, (2.92, 2.93) become:

$$\mathfrak{g}_{3,0} = -\frac{r\,e^{-4A}}{4\,c_1}\,\Phi_-\,\frac{i\,\tau'}{\tau_2}\,, \qquad \mathfrak{g}_{2,1} = \frac{r\,e^{-4A}}{4\,\bar{c}_1}\,\Phi_+\,\frac{i\,\tau'}{\tau_2}\,, \tag{2.117}$$

$$\mathfrak{g}_{1,2} = \frac{r\,e^{-8A}}{4\,c_1}\,\Phi_+\Phi'_-\,, \qquad \mathfrak{g}_{0,3} = -\frac{r\,e^{-8A}}{4\,\bar{c}_1}\,\Phi_-\Phi'_+\,, \tag{2.118}$$

where we have

$$\mathrm{d}f(r) = f'(r)\,\mathrm{d}r = f'(r)\,e^{2A}\,s_\varphi\,\mathrm{Re}\left[\frac{r}{c_1}\,\Theta\right]\,, \tag{2.119}$$

so that

$$\partial_\Theta f = e^{2A}\,s_\varphi\,\frac{r}{2\,c_1}\,f'(r)\,. \tag{2.120}$$

It is convenient to fix an $SL(2,\mathbb{R})$ frame. To do so, we choose some radius r and perform an $SL(2,\mathbb{R})$ transformation to make τ'/τ_2 imaginary. Thus, at radius

r, $\mathfrak{g}_{3,0}\,\mathfrak{g}_{0,3}$ and $\mathfrak{g}_{2,1}\,\mathfrak{g}_{1,2}$ are both real. However, in this case the axion equation of motion (2.196) reads $C_0'' = 0$. Thus, in this frame the axion is constant, so that $i\tau'/\tau_2 = \phi'$. More general solutions can be recovered from this special case by an $SL(2,\mathbb{R})$ transformation.

We evaluate the left-hand side of (2.116), taking $\mathrm{d}\,C_0 = 0$:

$$\mathrm{D}_+ \left[e^{2A}\, s_\varphi\, \Omega_2\right] = \mathrm{d}\left[e^{2A}\, s_\varphi\, \Omega_2\right] - \frac{1}{2}\, e^{2A}\, s_\varphi\, \mathrm{d}\phi \wedge \bar{\Omega}_2\,. \tag{2.121}$$

Using (2.119), it is straightforward to check that the second term cancels against the $(1,2)+(0,3)$ component of the right-hand side of (2.116).[22] We are left with

$$\mathrm{d}\left[e^{2A}\, s_\varphi\, \Omega_2\right] = \Phi_+\, G^\star_{1,2} - \Phi_-\, G^\star_{0,3}\,. \tag{2.122}$$

We evaluate the exterior derivative using (2.107) and (2.119):

$$\begin{aligned}\mathrm{d}\left[e^{2A}\, s_\varphi\, \Omega_2\right] = \frac{1}{2\,c_1}\, e^{2A-i\,\xi} s_\varphi \frac{\mathrm{d}}{\mathrm{d}r}\Big(r e^{2A+i\,\xi} s_\varphi\Big)\, \Theta \wedge \Omega_2 \\ + \frac{r^2}{2\,\bar{c}_1}\, e^{2A-i\,\xi} s_\varphi \frac{\mathrm{d}}{\mathrm{d}r}\Big(\frac{1}{r} e^{2A+i\,\xi} s_\varphi\Big)\, \bar{\Theta} \wedge \Omega_2\,.\end{aligned} \tag{2.123}$$

Comparing with (2.118), we see that ξ must be a constant. After some manipulation, (2.122) becomes

$$\frac{1}{r^2}\,\frac{\mathrm{d}}{\mathrm{d}r}\Big(r^2 e^{-4A}\,\Phi_+\Phi_-\Big) = e^{-8A}\,\Phi_-^2\,\Phi_+' \ \ , \ \ r^2\,\frac{\mathrm{d}}{\mathrm{d}r}\Big(\frac{1}{r^2} e^{-4A}\,\Phi_+\Phi_-\Big) = e^{-8A}\,\Phi_+^2\,\Phi_-'\,. \tag{2.124}$$

Defining $\Xi_\pm \equiv \Phi_\pm^{-1}$, both halves of (2.124) reduce to

$$\frac{2}{r}\,(\Xi_+ + \Xi_-) = \Xi_-' - \Xi_+'\,. \tag{2.125}$$

Now (2.112) can be rewritten as

$$B = -\frac{1}{4}\log\left[\frac{|c_1|^2}{2\,r^2}\,(\Xi_+ + \Xi_-)\right]\,. \tag{2.126}$$

[22] This could have been anticipated from the integrability of the complex structure defined by $\Omega_2 \wedge \Theta$.

The supersymmetry conditions for the near-stack ansatz of section 2.3.1 are therefore equivalent to (2.113, 2.117, 2.118, 2.125, 2.126), where, in a general $SL(2,\mathbb{R})$ frame, $\mathrm{D}\, e^{i\,\xi} = 0$.[23]

2.5.2 Solutions in the near-stack limit

We define

$$\Delta(r) \equiv \frac{1}{2}\Big(\Xi_- - \Xi_+\Big) \quad , \quad f(r) \equiv \Xi_- + \Xi_+ = r\Delta' \,, \tag{2.127}$$

where the final equality follows from (2.125). To further constrain the solution, we must impose the equations of motion. In fact, we will only need two: the ϕ equation of motion and the B equation of motion.

First, from the dilaton equation of motion (2.196) we find

$$\frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}\left[r\phi'\right] = 8\, e^{4A-4C}\left(g^{3,0}g^{0,3} + g^{2,1}g^{1,2}\right) = -\frac{\Xi_+' + \Xi_-'}{\Xi_+ + \Xi_-}\,\phi' \,, \tag{2.128}$$

in the constant axion frame, so that

$$\phi' = \frac{2\,c_2}{r\,(\Xi_+ + \Xi_-)} \,, \tag{2.129}$$

where c_2 is another constant.

Now consider the B equation of motion (2.199), which can be written

$$\frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}\left[rB'\right] = \frac{\Xi_+'\Xi_-'}{2\,(\Xi_+ + \Xi_-)^2} + \frac{1}{8}\,(\phi')^2 \,. \tag{2.130}$$

Inserting (2.126), we find

$$\frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}\left[r\left(\Xi_+' + \Xi_-'\right)\right] = \frac{\left(\Xi_+'\right)^2 + \left(\Xi_-'\right)^2}{(\Xi_+ + \Xi_-)} - \frac{1}{2}\,(\Xi_+ + \Xi_-)\,(\phi')^2 \,. \tag{2.131}$$

It is straightforward to check that the $\Xi_\pm$ equations of motion (2.194, 2.195) are satisfied provided that (2.131) and the supersymmetry constraints are obeyed. Substituting (2.129) and (2.127), (2.131) takes the form

$$\frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}\left[rf'\right] = \frac{1}{f}\left[\frac{1}{2}\,(f')^2 + \frac{2}{r^2}\,(f^2 - c_2^2)\right] \,. \tag{2.132}$$

[23]Like Ω_2, $e^{i\xi}$ carries S-charge $-1/2$.

A general solution is of the form[24]

$$f(r) = c_3 r^2 + c_4 r^{-2} + c_5\,. \tag{2.133}$$

Substituting this result into (2.132), one obtains

$$c_2^2 = c_5^2 - 4\,c_3\,c_4\,. \tag{2.134}$$

We integrate once more to find

$$\Delta(r) = \frac{1}{2}\,c_3\,r^2 - \frac{1}{2}\,c_4\,r^{-2} + c_5\,\log\,r + c_6\,. \tag{2.135}$$

Thus,

$$\Xi_+ = (c_5/2 - c_6) + c_4\,r^{-2} - c_5\,\log\,r \quad , \quad \Xi_- = (c_5/2 + c_6) + c_3\,r^2 + c_5\,\log\,r\,, \tag{2.136}$$

and (2.129) becomes:

$$\phi' = \frac{2\,c_2}{r\,(c_3\,r^2 + c_4\,r^{-2} + c_5)}\,. \tag{2.137}$$

This can be integrated to give

$$\phi(r) = \phi_0 + \log\left[\frac{(c_5 + c_2)\,r^2 + 2\,c_4}{(c_5 - c_2)\,r^2 + 2\,c_4}\right]\,, \tag{2.138}$$

where we use (2.134).

The full solution is given by (2.136, 2.138, 2.117, 2.118, 2.126), where $\Xi_\pm \equiv \Phi_\pm^{-1}$ as above. It is straightforward to check that the remaining equations of motion (2.197, 2.200) are automatically satisfied. An interesting special case is where $c_2 = 0$, or $c_5^2 = 4\,c_3\,c_4$, so that $\phi' = 0$. This implies that $\mathfrak{g}^{3,0} = \mathfrak{g}^{2,1} = 0$, but $\mathfrak{g}^{1,2}$ and $\mathfrak{g}^{0,3}$ are nonvanishing. These are explicit examples of constant τ solutions that are not type B solutions.

[24]To see this, first differentiate (2.132) with respect to r, obtaining a third-order equation whose nonlinear terms can be eliminated using (2.132). The result gives $\frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}\left[r g'\right] = \frac{4}{r^2} g$ where $g(r) \equiv r f'(r)$. The solution is then easy to guess.

2.5.3 Beyond the near-stack limit

We now briefly consider the extension of our methods to the full $\mathbb{P}^2$ cone. We must have

$$\Theta = \frac{1}{r}\, e^{-2\,B + i\,\theta}\, \partial\, r^2\,, \tag{2.139}$$

which agrees with (2.106), since $r^2 \to |z|^2$ in the near-stack limit. We compute

$$\begin{aligned} \mathrm{d}\left[e^{2A}\, s_\varphi\, \Theta\right] &= \frac{i}{r}\frac{\mathrm{d}}{\mathrm{d}r}\left[r\, s_\varphi\, e^{2(A-B)+i\,\theta}\right]\chi_{1,1} + 6\, i\, r\, s_\varphi\, e^{2(A-B)+i\,\theta}\,\omega_{1,1} \\ &= 2\, i\, \mu \left(c_\varphi\, e^{-4B}\,\chi_{1,1} + e^{2C}\,\omega_{1,1}\right)\,, \end{aligned} \tag{2.140}$$

where we used (2.35, 2.90) and the decomposition $J_1 = \frac{i}{2}\,\Theta \wedge \bar{\Theta} = e^{-4B}\chi_{1,1}$ and therefore $J_2 = J - J_1 = e^{2C}\,\omega_{1,1}$. Comparing the terms proportional to $\omega_{1,1}$ in (2.140), we find

$$3\, r\, s_\varphi\, e^{2(A-B)+i\,\theta} = \mu\, e^{2C}\,. \tag{2.141}$$

Thus, for $s_\varphi \neq 0$, we must take $\mu \neq 0$, and the solution is always AdS. Using methods similar to those in section 2.5.1, one can check that supersymmetric AdS solutions with global $SU(3)$ symmetry of this type do exist. The resulting ODEs are nonlinear and difficult to solve except in certain special cases, and we defer further consideration of this problem to a later work.

Comparing (2.141) with (2.111), we see that c_1 of the near-stack ansatz is related to μ via

$$c_1 = \frac{\mu}{3}\, e^{2\,C_{\mathrm{ns}}}\,, \tag{2.142}$$

where C_{ns} is the constant near-stack value of C.

2.6 Geometry of the Near-stack Solution

2.6.1 Singularity structure

The solution found in section 2.5 depends on several parameters: the c_i ($i = 1 \ldots 6$), $g_s = e^{\phi_0}$, C, ξ, and C_0. The c_i must obey the relation (2.134), and in particular we must have

$$c_5^2 \geqslant 4 c_3 c_4 \,. \tag{2.143}$$

The explicit solution is given by

$$\Xi_+(r) = (c_5/2 - c_6) + c_4\, r^{-2} - c_5 \log r \,, \tag{2.144}$$

$$\Xi_-(r) = (c_5/2 + c_6) + c_3 r^2 + c_5 \log r \,, \tag{2.145}$$

$$\tau = C_0 + \frac{i}{g_s} \left(\frac{(c_5 - c_2) r^2 + 2c_4}{(c_5 + c_2) r^2 + 2c_4} \right) \,, \tag{2.146}$$

$$B = \frac{1}{4} \log \left[\frac{2\, r^4}{|c_1|^2 \,(c_4 + c_5 r^2 + c_3 r^4)} \right] \,, \tag{2.147}$$

with fluxes specified in (2.117, 2.118). The spinor angle φ varies with radius,

$$\cos\varphi = \alpha\, e^{-4A} = \frac{-c_4 + 2\, c_6\, r^2 + 2\, c_5\, r^2 \log r + c_3\, r^4}{c_4 + c_5\, r^2 + c_3\, r^4} \,, \tag{2.148}$$

so the solution has dynamic $SU(2)$ structure.

For the $\varphi \neq 0, \pi$ (i.e. non-type B) case that we are considering, the c_i must all be finite or vanishing, and the solution is regular and supersymmetric whenever the $\Xi_\pm$ are both positive. We now show that a singularity (i.e. a zero in either Ξ_+ or Ξ_-) always occurs at finite radius. Consider the sum (cf. (2.127)):

$$r^2 f(r) = r^2 (\Xi_+ + \Xi_-) = c_4 + r^2\, c_5 + r^4\, c_3 \,. \tag{2.149}$$

This must be positive as a necessary but insufficient condition for regularity and supersymmetry.

Assume that there exists a solution that is smooth at all finite radii. To maintain regularity in the large and small r regions, we must have $c_4 \geqslant 0$ and $c_3 \geqslant 0$, and moreover, since the discriminant of the quadratic polynomial $r^2 f(r)$ is non-negative by (2.143), c_5 must be nonnegative, or else $\Xi_+ + \Xi_-$ acquires a root at some finite radius. However, under these conditions Ξ_+ will become negative at small r unless $c_5 = 0$. If this is the case, then either c_3 or c_4 must vanish by (2.143), and c_6 must be respectively either positive or negative to obtain a regular solution anywhere. In either case, one of Ξ_+ or Ξ_- is constant, whereas the other crosses zero at some finite radius. Thus, a singularity will always occur at some radius. The constraint (2.143) plays a crucial role in this argument.

A singularity at finite radius is always of the $e^A \to \infty$ type. It is straightforward to check that curvature invariants diverge and the singularity is physical. Moreover, horizons, characterized by $e^A \to 0$, can only occur for $r \to 0$ and/or for $r \to \infty$, so the singularity is naked.

We now classify the available regions of parameter space for which a regular solution exists at some radius. Clearly we must have $f = \Xi_+ + \Xi_- > 0$. However, this is also sufficient at any given point for some choice of c_6, since we can always set $\Xi_+ = \Xi_-$ at any point of interest by adjusting c_6. If either c_3 or c_4 is positive, then f is positive at large or small r, respectively, and c_6 can be chosen such that a regular region exists. For solutions with neither c_3 nor c_4 positive, one can check that a region of positive f exists so long as c_5 is positive and the inequality $c_5^2 \geqslant 4c_3c_4$ is not saturated.

The space of available (c_3, c_4, c_5) can be imagined as $\mathbb{R}^3$ minus two cones, a 'positive' cone in the region $c_3, c_4 > 0$ given by $c_5^2 < 4c_3c_4$, and a 'negative' cone in the region $c_3, c_4 < 0$ given by $c_5 \leqslant \sqrt{4c_3c_4}$. The surface of the positive cone is available, and consists of the constant dilaton solutions ($c_2 = 0$), whereas the surface of the negative cone is unavailable. Not all points in this space are physically distinct, as radial rescalings $r \to \lambda r$ trace out hyperbolae $c_3\, c_4 = \text{const}$. The (c_3, c_4, c_5) parameter space is depicted in Figure 2.1.

2.6.2 Constant dilaton solutions

The space of solutions is large and complicated, and without an explicit mechanism for resolving the singularity and lacking an asymptotic AdS region for comparisons

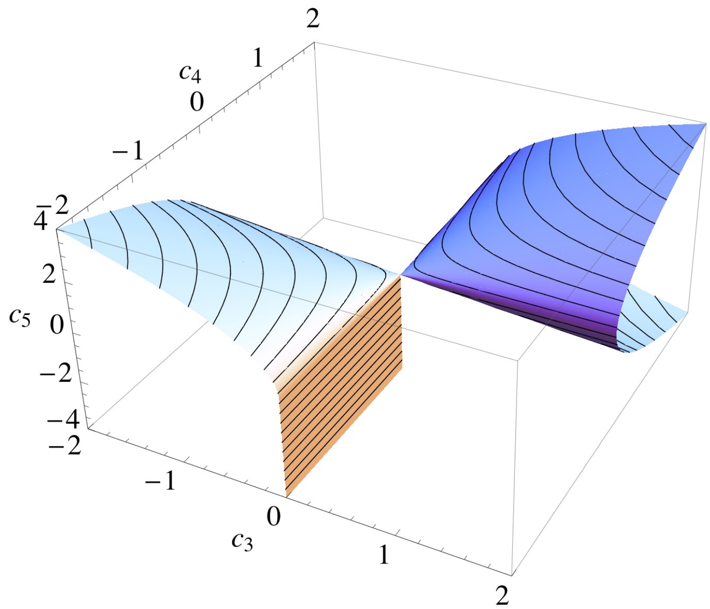

Figure 2.1: (c_3, c_4, c_5) parameter space. The region inside the cones is excluded. The surface of the negative cone is also excluded, whereas the surface of the positive cone consists of constant dilaton solutions. Hyperbolae of constant $c_3\, c_4$ are related by radial rescaling.

with a boundary field theory, it is hard to anticipate which of these solutions will be realized physically. We will focus on one of the simplest classes, in which the dilaton is constant ($c_2 = 0$).[25] These are the solutions that lie on the positive cone $c_5^2 = 4c_3c_4$ with $c_3, c_4 \geqslant 0$, as noted above. We will not consider the special cases where either c_3 or c_4 vanishes, so $c_5 \neq 0$ in general. It is convenient to reparameterize:

$$c_3 = \frac{|c_5|}{2\, r_\star^2} \quad , \quad c_4 = \frac{|c_5|\, r_\star^2}{2} \quad , \quad c_6 = \frac{1}{2}\,\delta\,|c_5| - c_5 \log r_\star \,, \tag{2.150}$$

where c_5 is related to the D3-brane charge. Using (2.51), we compute

$$Q = \pi^3\, e^{4C} c_5 \,, \tag{2.151}$$

[25]One motivation for considering this class of solutions is that string loop corrections can be controlled parametrically by taking $g_s \ll 1$. Although g_s is still available as a control parameter, the situation is more complicated when the dilaton runs.

where Q is the Page charge. We find that

$$\Xi_+(r) = \frac{|Q|}{2\,\pi^3}\,e^{-4\,C}\left[\frac{r_\star^2}{r^2} + \mathrm{sgn}(Q)\,(1 - 2\log r/r_\star) - \delta\right]\,, \tag{2.152}$$

$$\Xi_-(r) = \frac{|Q|}{2\,\pi^3}\,e^{-4\,C}\left[\frac{r^2}{r_\star^2} + \mathrm{sgn}(Q)\,(1 + 2\log r/r_\star) + \delta\right]\,, \tag{2.153}$$

where $\mathrm{sgn}(Q)$ is the sign of Q.

We consider the case of positive and negative Q separately. For positive Q, Ξ_- is a strictly increasing function of r, whereas Ξ_+ is a strictly decreasing function of r. Then, $f = \Xi_+ + \Xi_- \propto \frac{(r^2+r_\star^2)^2}{r^2 r_\star^2}$ is positive everywhere, so the solution can be made regular in any region by an appropriate choice of δ. However, $\Xi_+ \to -\infty$ as $r \to \infty$ and $\Xi_- \to -\infty$ as $r \to 0$, so the solution is only valid between two radii r_1 and $r_2 > r_1$ where Ξ_- and Ξ_+ cross zero, respectively. One can easily show that

$$r_2 = r_\star\left[W_0(e^{\delta-1})\right]^{-1/2}\quad,\quad r_1 = r_\star\left[W_0(e^{-\delta-1})\right]^{1/2}\,, \tag{2.154}$$

where W_0 is the main branch of the Lambert W-function. In particular, the ratio of the two scales is

$$(r_2/r_1)^2 = \left[W_0(e^{\delta-1})W_0(e^{-\delta-1})\right]^{-1}\,. \tag{2.155}$$

This ratio is minimized at $\delta = 0$, where it takes the value $r_2/r_1 = \left[W_0(1/e)\right]^{-1} \simeq 3.59$. For $|\delta| > 0$, the ratio increases, and asymptotically for large $|\delta|$, we find

$$(r_2/r_1)^2 \to |\delta|^{-1}e^{1+|\delta|}\,, \tag{2.156}$$

so the region of regularity can be made very large for modest values of δ.

For the case of negative Q, Ξ_+ and Ξ_- have a single minimum at $r_\star$, and $f \propto \frac{(r^2-r_\star^2)^2}{r^2 r_\star^2} = 0$ at $r = r_\star$, so the solution can be made regular anywhere but at $r_\star$. For any choice of δ, the solution is regular for $r > r_2$ and $r < r_1$, where for $\delta > 0$ both singularities are due to Ξ_+ crossing zero, and for $\delta < 0$ both are due to Ξ_- crossing zero. The radii of the singularities for $\delta > 0$ are

$$r_1 = r_\star\left[-W_{-1}(-e^{-1-\delta})\right]^{-1/2}\quad,\quad r_2 = r_\star\left[-W_0(-e^{-1-\delta})\right]^{-1/2}\,, \tag{2.157}$$

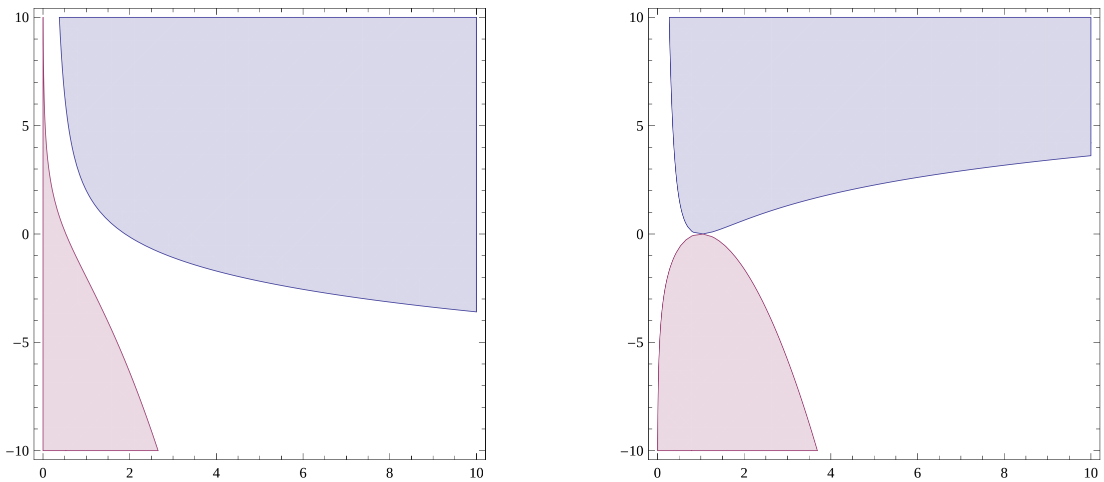

Figure 2.2: The singularity structure of the constant dilaton solutions. The horizontal axis is $r/r_\star$ and the vertical axis is δ. The nonsingular region is unshaded and Ξ_+ (Ξ_-) is negative in the blue (red) region. The first (second) plot corresponds to positive (negative) Q.

where W_{-1} is the lower branch of the Lambert W-function. For $\delta < 0$, we have instead

$$r_1 = r_\star \left[-W_0(-e^{\delta-1})\right]^{1/2} \quad , \quad r_2 = r_\star \left[-W_{-1}(-e^{\delta-1})\right]^{1/2} . \tag{2.158}$$

For the special case $\delta = 0$, $r_1 = r_2 = r_\star$, and the solution is regular everywhere else. This special case has interesting properties. For instance, the spinor angle is finite everywhere:

$$\cos\varphi = \frac{r^4 - r_\star^4 - 4\,r^2\log(r/r_\star)}{(r^2 - r_\star^2)^2} . \tag{2.159}$$

We see that φ interpolates between $\varphi = 0$ as $r \to \infty$ and $\varphi = \pi$ as $r \to 0$, passing through a type-changing locus ($\varphi = \pi/2$) coincident with the singularity at $r = r_\star$.

The singularity structure of constant dilaton solutions is illustrated in Figure 2.2 for both positive and negative Q. The special case discussed above corresponds to $\delta = 0$ in the second plot of the figure.

2.6.3 Domain of validity of the near-stack solutions

While we have obtained a family of closed-form solutions to the supergravity equations, our result is subject to corrections from a number of sources. Though string loop corrections can be controlled parametrically for constant dilaton solutions, the solutions we have obtained are always singular at some finite radius, and so α' corrections will invariably be large in some regions. Furthermore, the near-stack approximation employed in section 2.3.1 also has a finite region of validity. Generically we expect an interplay between these two sources of corrections, as curvatures should fall off at large distances whereas the near-stack approximation is valid at short distances. We now estimate the size of corrections from various sources, and show that in some regions of parameter space we can parametrically suppress all corrections in some finite interval $r_{\rm near} < r < r_{\rm far}$.

Corrections to the near-stack limit

Corrections to the near-stack limit of section 2.3.1 come in two forms. At large distances from the stack wrapping the resolved $\mathbb{P}^2$, the resolution appears as a small perturbation to the geometry, and the seven-branes appear to be codimension six sources, rather than codimension two sources as they do in the near-stack limit. This effect manifests itself in the Einstein equations for the $\mathbb{P}^2$ cone as extra terms of order $(r/r_0)^2$ and $(r/r_0)^4$, where r_0 is the radial scale of the blow-up, as in (2.55); that is, one can show that the Einstein equations (2.198, 2.199, 2.200), written in the form

$$r^2\, B'' = \dots \quad , \quad r^2\, C'' = \dots \quad , \quad 2\, r\, C' = \dots\, , \tag{2.160}$$

receive corrections of the form $r^2\, e^{-4B-2C}$ and $r^4\, e^{-8B-4C}$. (In the remainder of this section, we will omit numerical factors in expressions of the form $a \ll b$.)

Assuming that dimensionless terms of the form rB', etc., are of order unity, the extra terms are suppressed if

$$r^2\, e^{-4B-2C} \ll 1\,. \tag{2.161}$$

By (2.55), this is equivalent to the requirement $(r/r_0)^2 \ll 1$.

The other source of corrections to the near-stack limit is the finite cosmological constant $\Lambda = -3\,|\mu|^2$. This sources new terms in the Einstein equations and the warp factor equation of the form $r^2\, e^{-4(A+B)}\,\Lambda$, as in (2.195, 2.198, 2.199, 2.200). Applying the relation (2.142), we obtain the requirement

$$r^2\, e^{-4(A+B+C)}\,|c_1|^2 \ll 1\,. \tag{2.162}$$

Using (2.112, 2.126) the conditions (2.161, 2.162) can be rewritten in the form:

$$|c_1|^2\, e^{-2C}\,(\Xi_+ + \Xi_-) \ll 1 \;\;,\;\; |c_1|^4\, e^{-4C}\,\Xi_+\,\Xi_- \ll 1\,. \tag{2.163}$$

However, for $\Xi_\pm > 0$,

$$|\Xi_+\,\Xi_-| \le \frac{1}{4}\,(\Xi_+ + \Xi_-)^2\,. \tag{2.164}$$

Therefore, for supersymmetric solutions, the second bound in (2.163) is implied by the first.

For solutions with nonvanishing Q, it is convenient to extract an overall scale from $\Xi_\pm$:

$$\Xi_\pm = \frac{|Q|}{2\,\pi^3}\, e^{-4\,C}\, F_\pm\,, \tag{2.165}$$

where for constant dilaton solutions, the $F_\pm$ are given by the bracketed terms in (2.152, 2.153). The bound (2.163) becomes

$$\zeta\,|Q|\,(F_+ + F_-) \ll 1\,, \tag{2.166}$$

where $\zeta \equiv |c_1|^2\, e^{-6\,C}$. The constants c_1 and e^C naturally appear in physical quantities in this combination, since both can be rescaled by redefining the four-dimensional coordinates $x^\mu \to \lambda\, x^\mu$ and absorbing the rescaling into the warp factor and the six-dimensional metric, leaving the ten-dimensional metric invariant; ζ is invariant under this rescaling.

Curvature corrections

To estimate the size of α' corrections, we compute the ten-dimensional Riemann tensor components in Einstein frame. We assume that corrections can be suppressed if these components (expressed in an orthonormal basis) can be made parametrically small in units of α'. The size of α' corrections in Einstein frame will actually depend on g_s, since the F-string tension depends on g_s in this frame. However, if we restrict our attention to constant dilaton solutions, and fix some small value of g_s to suppress string loop corrections, this will only modify the curvature scale at which corrections set in by a fixed factor, and parametric control of the Einstein-frame curvature is still sufficient to suppress corrections. We will not consider corrections involving the other supergravity fields.[26]

The ten-dimensional Riemann tensor for the metric (2.17) can be computed using standard methods. Expressing everything in an orthonormal basis, one finds terms of the form $e^{-2A}\,[R_{(4)}]_{\mu\nu\rho\sigma}$, $e^{2A}\,[R_{(6)}]_{mnpq}$, $e^{2A}\,(\nabla_m A)(\nabla_n A)$, and $e^{2A}\,\nabla_m\nabla_n A$, where $[R_{(4)}]_{\mu\nu\rho\sigma}$ are the components of the four-dimensional Riemann tensor computed from $h_{\mu\nu}$, $[R_{(6)}]_{mnpq}$ are the components of the (unwarped) six-dimensional Riemann tensor computed from g_{mn}, and ∇ is the connection computed from g_{mn}. We assume that the near-stack approximation holds in the region of interest, and use the ansatz of section 2.3.1 to compute

$$
\begin{aligned}
e^{2A}[R_{(6)}]_{\hat{r}\hat{\psi}\hat{r}\hat{\psi}} &= 2\,e^{2A+4B}\left(B'' + \frac{1}{r}B'\right)\ , \quad e^{2A}\,(\nabla_{\hat{r}}A)(\nabla_{\hat{r}}A) = e^{2A+4B}(A')^2\,,\\
e^{2A}\,\nabla_{\hat{r}}\nabla_{\hat{r}}A &= e^{2A+4B}(A'' + 2A'B')\ , \quad e^{2A}\,\nabla_{\hat{\psi}}\nabla_{\hat{\psi}}A = e^{2A+4B}\left(\frac{1}{r}A' - 2A'B'\right)\,,
\end{aligned}
\tag{2.167}
$$

where all other terms vanish. If we repeat this computation for the full $\mathbb{P}^2$ geometry,

[26] One might hope that these corrections are also suppressed when the Riemann components are small, but not all of these corrections are known, even at leading order, so more detailed estimates may be misguided.

we obtain extra terms which are suppressed in the near-stack limit, but which scale similarly to those above. However, we also obtain a contribution from the cosmological constant which scales distinctly:

$$e^{-2A}[R_{(4)}]^{\mu}{}_{\nu\rho\sigma} = -|\mu|^2 \, e^{-2A} \left(\delta^{\mu}_{\rho} \, \eta_{\nu\sigma} - \delta^{\mu}_{\sigma} \, \eta_{\nu\rho}\right) \, . \tag{2.168}$$

Taking rA' and other dimensionless derivatives of A and B to be of order unity, all the components in (2.167) are of the same order, as determined by the prefactor, whose square is:

$$\frac{1}{r^4} \, e^{4A+8B} = \left(\frac{|c_1|^4}{2} \, \Xi_+ \, \Xi_- \, (\Xi_+ + \Xi_-)\right)^{-1} = \left(\frac{|Q|}{2\pi^3}\right)^{-3} \zeta^{-2} \, \frac{2}{F_+ \, F_- \, (F_+ + F_-)} \, , \tag{2.169}$$

where we have used (2.165). By contrast, the square of the curvature induced by the cosmological constant, (2.168), is:

$$|\mu|^4 \, e^{-4A} = \frac{81 \, \zeta^2 \, |Q|}{2\pi^3} \, \frac{2 \, F_+ \, F_-}{F_+ + F_-} \, , \tag{2.170}$$

using (2.142).

Regions of parametric control

To suppress α' corrections, we should parametrically suppress (2.169, 2.170) while also satisfying (2.166). Rewriting $\zeta|Q| = \epsilon_1$ and $|Q|^{-3}\zeta^{-2} = \epsilon_2$, we see that near-stack corrections can be controlled by taking $\epsilon_1 \ll 1$, while curvature corrections can be controlled by taking ϵ_2 to be well below some fiducial curvature scale. The curvature (2.170) is then suppressed by $\epsilon_1^4 \, \epsilon_2$. However, in this limit $|Q| = 1/(\epsilon_1^2 \epsilon_2)$, so the D3-brane charge must be taken to be large.

In a physical solution, we expect that the D3-brane charge is determined by the worldvolume dynamics on the seven-brane stack, and is not a free parameter. If we add D3-branes near the tip, the 3-7 strings become light and introduce light matter into the worldvolume theory, precluding gaugino condensation. Thus, $|Q|$

is not a free parameter, and we must look elsewhere for parametric control of the curvature. Comparing (2.166) and (2.169), we see that control may be possible for $F_\pm \gg 1$ with ζ scaled appropriately (e.g. $\zeta \propto F^{-4/3}$). For constant dilaton solutions, (2.152) gives

$$F_+ + F_- = \frac{r_\star^2}{r^2} + \frac{r^2}{r_\star^2} + 2\,\mathrm{sgn}(Q)\,. \tag{2.171}$$

Thus, $F_\pm \gg 1$ requires $r \gg r_\star$ or $r \ll r_\star$.

Solutions in the region $r \ll r_\star$ are unlikely to be physical for the following reason. Referring to Figure 2.2, we see that such a solution can only be regular for $r < r_s$ where $r_s \leq r_\star$. The large-distance singularity at $r = r_s$ is not obviously due to localized sources, and should be removed by compactification, rather than by curvature corrections. However, corrections to the near-stack limit (the first step towards compactification) *decrease* with increasing r in the region $r < r_\star$, since $\Xi_+ + \Xi_-$ has a single minimum at $r = r_\star$.

Thus, we restrict our attention to the $r \gg r_\star$ case. For simplicity, we consider the case where Ξ_+ and Ξ_- cross at some large radius $r_{\rm eq} \gg r_\star$. Thus, at $r_{\rm eq}$, $F_+ = F_- \approx r_{\rm eq}^2/(2\,r_\star^2)$. In this limit, one can show that the curvature terms are indeed suppressed. At leading order in $r_\star \ll r_{\rm eq}, r$, we find

$$r^2 A''(r) \to \frac{6\,r^2\,r_{\rm eq}^2 - r_{\rm eq}^4}{2\,(2\,r^2 - r_{\rm eq}^2)^2}\,, \qquad r A'(r) \to \frac{-r_{\rm eq}^2}{2\,(2\,r^2 - r_{\rm eq}^2)}\,, \tag{2.172}$$

$$r^2 B''(r) \to -\frac{3\,r_\star^2}{r^2}\,, \qquad r B'(r) \to \frac{r_\star^2}{r^2}\,, \tag{2.173}$$

so the derivatives are of order one or smaller for $r \gtrsim r_{\rm eq}$, and the size of the Riemann components is determined by the prefactor (2.169). Thus, our previous arguments apply, and we conclude that both curvature and near-stack corrections can be suppressed at $r \sim r_{\rm eq}$.[27] Depending on the parameters, near-stack corrections will

[27]For $r_{eq} \gg r_\star$ the singularity occurs at $r_{eq}/\sqrt{2}$, but the ten-dimensional distance between $r_{\rm eq}$ and the singularity is proportional to re^{-2B-A}, and will be large in string units when the curvatures are small.

become important at some $r_{\text{far}} > r_{\text{eq}}$ and curvature corrections at some $r_{\text{near}} < r_{\text{eq}}$. Near $r = r_{\text{eq}}$ both types of corrections are small, but $\varphi \sim \pi/2$ and the solution cannot be described by perturbations about a type B background.

As illustrated by this example, the solutions we have obtained describe physics inaccessible to previous approaches. We stress that the above is not a complete classification of regions of parametric control. We leave further exploration of the large parameter space of solutions, including solutions where the dilaton runs, to a future work.

2.7 Towards the Physics of the Solutions

Our choice to study four D7-branes atop an O7-plane on a rigid four-cycle was strongly motivated by the fact that the corresponding seven-brane gauge theory confines in the infrared, but our analysis so far has exclusively involved ten-dimensional supergravity, without any input of gauge theory physics. We have obtained a family of exact noncompact solutions parameterized by a number of integration constants, but we expect that some of these constants are actually *fixed* by a proper inclusion of nonperturbative source terms localized on the seven-branes (cf. [26], [25]), or by matching to the seven-brane gauge theory.

In this section we present a preliminary analysis of the relation between the solutions obtained in section 2.5 and the dynamics of the four-dimensional gauge theory on seven-branes in the $\mathbb{P}^2$ cone. Potential applications of our results to the ten-dimensional description of KKLT vacua and geometric transitions for seven-branes are also discussed.

2.7.1 Gaugino condensation in supergravity

Let us start by briefly discussing the gauge theory supported on the seven-brane stack wrapping the rigid holomorphic $\mathbb{P}^2$. An important subtlety in obtaining the worldvolume gauge theory is the following. Since $\mathbb{P}^2$ is not spin, wrapping D7-branes on it gives rise to global anomaly [21, 22], whose cancellation requires a nontrivial gauge bundle that will break $SO(8)$ down to (at most) $U(4)$. While it is not clear to us how the anomaly constraint is modified in the presence of the O7-plane, if we assume that the anomaly does persist, the resulting gauge group need not be asymptotically free.

In this chapter we will assume that the seven-brane gauge group generates a gaugino condensate in the infrared, postponing a proper treatment of the Freed-Witten anomaly to future work. Fortunately, the methods we have developed apply equally well to the Calabi-Yau cone over $\mathbb{P}^1 \times \mathbb{P}^1$. The gauge theory analysis there is simpler, because $\mathbb{P}^1 \times \mathbb{P}^1$ is spin, but the supergravity analysis becomes slightly more involved than in the $\mathbb{P}^2$ cone.

Let us now turn to gaugino condensation. Denoting the dynamical scale by Λ and the dual Coxeter number of the nonabelian group by $\mathfrak{c}_2$ (not to be confused with the integration constant c_2), we have

$$\langle \lambda\lambda \rangle \sim \alpha_{\mathfrak{c}_2} \Lambda^3 \,, \tag{2.174}$$

where $\alpha_{\mathfrak{c}_2}$ is a $\mathfrak{c}_2$-th root of unity. The nonperturbative superpotential is also proportional to the gaugino bilinear $\langle \lambda\lambda \rangle$. Recall that the $U(1)_R$ symmetry that acts on the gauginos is anomalous at the quantum level:

$$\lambda \to e^{i\theta}\lambda \;\;,\;\; \tau_{YM} \to \tau_{YM} + \frac{\mathfrak{c}_2}{\pi}\,\theta \,. \tag{2.175}$$

The exact symmetry is thus reduced to a discrete $\mathbb{Z}_{2\mathfrak{c}_2}$. Moreover, this is spontaneously broken to $\mathbb{Z}_2$ by (2.174), leading to $\mathfrak{c}_2$ inequivalent vacua.

Gaugino condensation and IASD flux

To connect the gauge theory dynamics to our supergravity solution,[28] we use the results of [25], which showed that gaugino condensation on D7-branes sources imaginary anti-self-dual (IASD) flux in the space surrounding the branes. Using the classical DBI coupling between D7-brane gauginos λ and bulk fluxes [55],

$$\mathcal{L} \supset \frac{a}{\alpha'^2} \int_{\mathbb{P}^2} \sqrt{g}\, G_3 \cdot \Omega\, \bar{\lambda}\bar{\lambda} + c.c.\,, \tag{2.176}$$

the flux equation of motion, expanded around a background containing exclusively imaginary self-dual fluxes, was found to be [25]

$$\mathrm{d}\Big[e^{4A}\,(\star G_3 - i\, G_3)\Big] = \frac{4ia\kappa_{10}^2}{g_s\alpha'^2}\, \mathrm{d}\Big[\lambda\lambda\,\bar{\Omega}\,\delta(\mathbb{P}^2)\Big]\,. \tag{2.177}$$

Here $\star$ is the six-dimensional Hodge star, $\delta(\mathbb{P}^2)$ is a delta-function localizing on $\mathbb{P}^2$, and a is a dimensionless constant.

The nonzero expectation value (2.174) provides a localized source for flux in ten dimensions via (2.177), and the resulting flux is IASD with Hodge type $(1,2)$. Compelling evidence for this proposal comes from the fact that for D7-branes wrapping a given four-cycle in a local geometry, the $G_{1,2}$ flux background induced by the coupling (2.177) precisely encodes the superpotential of the four-dimensional gauge theory: a D3-brane probing the flux background sourced by (2.177) experiences the superpotential derived upon reduction to four dimensions [25].

The analysis of [25] was performed in an expansion around type B backgrounds [24], but focused on the theory of probe D3-branes[29] and consistently neglected perturbations to the metric and dilaton: such perturbations are clearly

[28] A precise matching between supergravity and gauge theory requires having a smooth solution, which we lack at present. Our discussion here will be qualitative, and limited to showing that the family of supersymmetric solutions found above has the required ingredients to encode gaugino condensation.

[29] Consideration of more general D-brane probes [56, 57] of our solutions is likely to lead to substantial physical insight, but is beyond the scope of the present work.

present as a consequence of the IASD flux sourced by the gaugino condensate, but do not contribute to the D3-brane scalar potential until third order. However, it was natural to expect that the full solution of all the equations of motion would unite the proposals of [26], in which the background is a generalized complex geometry, and of [25], in which $G_{1,2}$ flux plays the central role. In this chapter we have exhibited a solution with dynamic $SU(2)$ structure that crucially involves $G_{1,2}$ flux, thereby illuminating the relationship between [26] and [25].

R-symmetry breaking and domain walls

Having explained the relation between three-form fluxes and gaugino condensation, let us discuss how the pattern of R-symmetry breaking described above may be encoded in the supergravity solution. Referring to Table 2.1, we see that $R(\Theta) = +2$ and $R(\Omega_2) = 0$. Comparing (2.110, 2.107) with the geometric action of $U(1)_\psi$ (appropriately normalized as in section 2.2.2), we see that c_1 and $e^{i\xi}$ must carry R-charges $+2$ and -2 respectively. This suggests that, in a four-dimensional off-shell formulation (where the equations of motion of the effective action reproduce the ten-dimensional equations), some combination of c_1 and $e^{i\xi}$ becomes a fluctuating space-time field with nonvanishing R-charge, whose expectation value spontaneously breaks the exact R-symmetry to $\mathbb{Z}_2$.

Gaugino condensation has a similar off-shell description in terms of the glueball superfield $S = -\frac{1}{32\pi^2}\mathrm{tr}\, W_\alpha W^\alpha$ and the Veneziano-Yankielowicz superpotential [58]. On-shell this field acquires a nonzero expectation value proportional to (2.174) and reproduces the nonperturbative superpotential. It is natural to conjecture that the combination of c_1 and $e^{i\xi}$ mentioned above is dual to S; then the on-shell superpotential (2.102, 2.103) would have to agree with the gaugino-condensate superpotential.[30]

We expect the appearance of domain walls due to the spontaneous breaking of the exact R-symmetry $\mathbb{Z}_{2\mathfrak{c}_2} \to \mathbb{Z}_2$; these should correspond to wrapped branes in the gravity solution. We now argue that D3-branes wrapping a loop inside the $S^5/\mathbb{Z}_3$ have the right properties to be domain walls in our solution.

The $S^5/\mathbb{Z}_3$ has fundamental group $\mathbb{Z}_3$ generated by a loop where ψ runs from $0 \to 2\pi$. However, due to the involution, a D3-brane wrapping from $0 \to \pi$ is permitted, where the two ends are identified under the involution $\psi \to \psi + \pi$. This corresponds to the generator of the $\mathbb{Z}_6$ fundamental group of the horizon in the downstairs geometry. We compute the tension of the domain wall by evaluating the DBI action for the brane in the probe approximation:

$$T_{(\text{wall})} = \mu_3 \oint \mathrm{d}s_{(6)}\, e^{2A} = \mu_3\, \pi\, r\, e^{2(A-B)} = \mu_3\, \pi\, |c_1|\, \frac{|\Xi_+ + \Xi_-|}{2\, |\Xi_+\, \Xi_-|^{1/2}} \geq \mu_3\, \pi\, |c_1|\,, \tag{2.178}$$

where the bound is saturated if and only if $\Xi_+ = \Xi_-$ at the location of the domain wall. Thus, the tension is minimized at a $\varphi = \pi/2$ locus, in which case the wall is a half-BPS state [57]. The analysis of section 2.6.3 describes one example where the supergravity approximation is valid at such a locus.

The $\mathbb{Z}_6$ fundamental group in the downstairs geometry implies that the number of domain walls of this type is conserved modulo six. Thus, the gauge theory has six vacua, consistent with gaugino condensation in $SO(8)$ pure super Yang-Mills. We anticipate that these vacua are related by exact R-symmetry transformations. The BPS domain wall tension (2.178) can then be used to infer the precise value of the superpotential vev. We postpone a more detailed study of these domain walls and other wrapped branes to a future work (see however [60]).

Finally, we recall that the extension of the gravity solution to the full $\mathbb{P}^2$ cone us-

[30]For D5-branes or D6-branes in the conifold this matching was obtained in the large N duality of [59].

ing the ansatz of section 2.3 reveals that the space-time becomes AdS, as explained in §2.5.3. We have not obtained a satisfactory interpretation of this restriction from the viewpoint of the gauge theory.

2.7.2 Applications

Our solution has a range of interesting applications, two of which we now describe.

Ten-dimensional consistency of KKLT vacua

One application would be to study the ten-dimensional consistency conditions for KKLT vacua. As vacua of the four-dimensional effective theory, these solutions are reasonably well understood, but are known to violate constraints that emerge from the ten-dimensional type IIB supergravity equations of motion with classical sources. Specifically, from the external Einstein equations and the five-form Bianchi identity, one obtains

$$\nabla^2\Phi_- = \frac{1}{4}\, e^{8A} \left|\star G_3 - i\, G_3\right|^2 + \mathcal{R}_4 + e^{-4A}\left|\nabla\Phi_-\right|^2 + \mathcal{S}_{\mathsf{local}}\,, \tag{2.179}$$

where $\mathcal{R}_4$ is the four-dimensional Ricci scalar, and [24]

$$\mathcal{S}_{\mathsf{local}} = 2\kappa_{10}^2\, e^{2A}\left(\frac{e^{2A}}{4} T^m_m - \frac{e^{-2A}}{4} T^\mu_\mu - \mu_3\rho_3\right)_{\mathsf{local}}, \tag{2.180}$$

where T^m_n and T^μ_ν are the internal and external components of the ten-dimensional stress-energy tensor T^M_N (with indices raised by the unwarped metrics g^{mn} and $h^{\mu\nu}$, respectively), and ρ_3 is the D3-brane charge density. An anti-D3-brane at position y_0 provides a *positive* localized source,

$$\mathcal{S}^{\overline{D3}}_{\mathsf{local}} = 4\,\kappa_{10}^2\,\mu_3\, e^{8A}\,\delta^6(y-y_0)\,, \tag{2.181}$$

whereas D3-branes, O3-planes, and O7-planes, like all other local sources allowed in the solutions of [24], provide a vanishing contribution to $\mathcal{S}_{\mathsf{local}}$. Then, noting that the integral of the left-hand side of (2.179) over a compact space vanishes, one learns that a de Sitter solution is possible only if a suitable localized negative contribution to the right-hand side is present. We will denote such a contribution as $\rho_-(y)$.

We emphasize that negative tension alone does not suffice to produce a contribution to ρ_-, as is evident from the fact that O3-planes and O7-planes do not contribute. Localized classical sources that do contribute to ρ_- include anti-O3-planes and O5-planes [24], but to our knowledge such objects have not played a role in the construction of consistent de Sitter vacua in the framework of [17].

A natural guess is that the four-dimensional nonperturbative effects that led to stability in the effective theory – i.e., gaugino condensation on D7-branes – will provide new sources in the ten-dimensional equations of motion. In fact, the stress tensor contribution given in (2.179) arises from varying the classical action for *bosonic* fields with respect to the metric. The stress tensor by definition involves the variation of the complete action, but fermionic expectation values vanish in classical vacua, and the corresponding contributions may then be omitted. However, in the solutions we have considered, the fermion bilinear $\lambda\lambda$ plays an essential role, and contributions proportional to $\langle\lambda\lambda\rangle$ should be retained.

Specifically, to incorporate the effects of gaugino condensation one should also vary the coupling (2.176) with respect to the metric. The result is a new contribution to (2.180) that is *negative* and proportional to $|\langle\lambda\lambda\rangle|^2$. We speculate that this negative contribution could suffice to establish that KKLT vacua can be lifted to consistent ten-dimensional solutions, but we leave a thorough investigation of this point for the future.

On del Pezzo transitions with seven-branes

A more speculative application of our result is to a geometric transition for seven-branes. It would be interesting to understand when a divisor wrapped by seven-branes can be contracted in such a way that the resulting singularity can be deformed. The only divisors in a Calabi-Yau threefold that admit birational contraction followed by deformation to a new Calabi-Yau are the del Pezzo surfaces $\mathbb{P}^1 \times \mathbb{P}^1$ and dP_k, $k \geq 2$ [61, 62].[31] (The del Pezzo surfaces $dP_0 = \mathbb{P}^2$ and dP_1 can be contracted, but the resulting singular varieties cannot be deformed to smooth Calabi-Yau threefolds.)

We would like to understand when del Pezzo transitions can occur for divisors wrapped by seven-branes, motivated by the rich physics of geometric transitions involving D5-branes. The role of gaugino condensation in the conventional D5-brane geometric transition [59] is well understood, and strongly suggests that for seven-branes it is also important to characterize the effect of gaugino condensation in the geometry. Our result prepares the tools for such an analysis, but exploring a seven-brane geometric transition in detail is beyond the scope of this work. We also observe that in any case where a del Pezzo transition with seven-branes *is* possible, so that a smooth geometry is obtained after the deformation, the resulting absence of a local source for ρ_- as described in section 2.7.2 presents an obstacle to obtaining consistent de Sitter vacua.

2.8 Conclusions

We have obtained a family of exact, noncompact, supersymmetric solutions of type IIB supergravity with dynamic $SU(2)$ structure. The core of each solution is a stack of four D7-branes atop an O7-plane on a flat non-compact four-cycle. We

[31] Strictly speaking, del Pezzo surfaces are the only possible exceptional divisors for 'primitive' contractions, from which more general contractions can be constructed [62].

argued that this solution describes the region near a small patch of a compact, rigid four-cycle, within which we have assumed rotational symmetry. We gave strong evidence for this claim by presenting an ansatz for a corresponding configuration of four D7-branes and an O7-plane wrapping the $\mathbb{P}^2$ base of the simplest del Pezzo cone, and showing that in the near-stack limit our flat ansatz is recovered. Decompactifying the four-cycle destroys information about induced charges and sends the seven-brane gauge coupling to zero, while drastically simplifying the equations of motion. By comparing the flat-space solution to the $\mathbb{P}^2$ cone configuration, we were able to interpret certain aspects of our solution as arising from seven-brane gauge dynamics or from induced D3-brane charge and tension.

For compact, rigid four-cycles, the seven-brane gauge theory undergoes gaugino condensation at low energies. In this chapter we have identified a class of exact solutions as candidates for the ten-dimensional backreaction of seven-brane gaugino condensation. Our approach was strictly ten-dimensional and did not incorporate nonperturbative source terms, in contrast to, but not in contradiction with, [26],[32] which proposed that gaugino condensation provides a localized source term that induces a deformation to a generalized complex geometry. It would be valuable to understand the relationship between these approaches.

Although the seven-brane charge necessarily vanishes in our solutions, the three-brane charge and tension induced on seven-branes wrapping $\mathbb{P}^2$ can be negative, so that at short distances one expects singular behavior typical of orientifolds. We indeed find a singularity with divergent warp factor near the seven-branes. It would be very interesting to understand if this singularity is ultimately removed by strong gauge dynamics on the seven-branes.

[32]See e.g. [27–29] for earlier connections between gaugino condensation and generalized complex geometry, and [25] for evidence that gaugino couplings to flux source ten-dimensional deformations.

Stacks of seven-branes wrapping rigid four-cycles are ubiquitous in type IIB compactifications, and understanding their effects in ten-dimensional supergravity is an important step toward characterizing the resulting four-dimensional effective theories. In particular, decoupling arguments analogous to [63] that invoke extradimensional locality require a ten-dimensional description, and the effective theory of D3-branes is most efficiently described by geometrizing seven-brane gauge dynamics [25], as we have done here. The configuration we have presented is arguably the simplest nontrivial example of the backreaction of seven-brane nonperturbative effects, because the seven-brane charges vanish and the four-cycle is highly symmetric. Our approach can be extended to configurations with less symmetry, such as seven-branes wrapping the base of a suitable del Pezzo cone; we argued that analogous solutions exist for the $\mathbb{P}^2$ cone. It would be very interesting to construct additional examples and explore their implications, both as windows into the dynamics of the seven-brane gauge theory, and as descriptions of local regions of stabilized type IIB compactifications.

2.A Equations of Motion for the Near-stack Ansatz

We now present the general equations of motion for the warped ansatz (2.17, 2.18), and then specialize to the near-stack limit. Rewriting the covariant action (2.82) in terms of a four-dimensional Lagrangian density,

$$S = \int \mathrm{d}^4 x \sqrt{-h}\, \mathcal{L}\,, \tag{2.182}$$

where $\mathcal{L}$ is given by an integral over the internal space, and applying the warped ansatz (2.17, 2.18),[33] we find the vacuum Lagrangian:

$$\mathcal{L} = \frac{1}{2\kappa_{10}^2} \int \mathrm{d}^6 y \sqrt{g} \left[\mathcal{R} + \frac{1}{2} e^{-8A} (\nabla \alpha)^2 - 8 (\nabla A)^2 \right.$$

[33] As usual, there are some surmountable subtleties relating to the self-duality of $\tilde{F}_5$.

$$+4\Lambda e^{-4A} - \frac{1}{2}\left(\frac{1}{\tau_2^2}|\mathrm{d}\tau|^2 + e^{4A}|G_3|^2\right)\Bigg] - \frac{i}{4\kappa_{10}^2}\int \alpha\, G_3 \wedge G_3^\star\,, \quad (2.183)$$

where $\Lambda = \mathcal{R}_{(4)}/4$ is the four-dimensional cosmological constant, $\tau_2 = \mathrm{Im}\,\tau$, contractions are made using the unwarped metric g_{mn}, and $G_3 = \mathrm{D}_-\mathcal{A}_2$, so that $\mathrm{D}_-\,G_3 = 0$. The corresponding equations of motion are:

$$\mathrm{d}\left(e^{-8A}\star_6 \mathrm{d}\alpha\right) = -\frac{i}{2}\,G_3 \wedge G_3^\star\,, \quad (2.184)$$

$$\nabla^2 A = \frac{1}{8}e^{4A}|G_3|^2 + \frac{1}{4}e^{-8A}(\nabla\alpha)^2 + \Lambda e^{-4A}\,, \quad (2.185)$$

$$\mathrm{D}_-\left(e^{4A}\star_6 G_3^\star\right) = -i\,\mathrm{d}\alpha \wedge G_3^\star\,, \quad (2.186)$$

$$\mathrm{D}\star_6\left(\frac{1}{\tau_2}\,\mathrm{d}\bar\tau\right) = \frac{i}{2}e^{4A}G_3^\star \wedge \star_6 G_3^\star\,, \quad (2.187)$$

$$\begin{aligned} R_{mn} &= 8\,\nabla_m A\,\nabla_n A - \frac{1}{2}e^{-8A}\,\nabla_m\alpha\,\nabla_n\alpha \\ &\quad + \frac{1}{4\tau_2^2}\left[\nabla_m\tau\nabla_n\bar\tau + c.c.\right] + \frac{1}{2}e^{4A}\,\hat{T}_{mn} - \Lambda e^{-4A}g_{mn}\,, \end{aligned} \quad (2.188)$$

where

$$\hat{T}^m_n = \frac{1}{4}(G^{mpq}\bar{G}_{npq} + \bar{G}^{mpq}G_{npq}) - \frac{1}{12}\bar{G}^{pqr}G_{pqr}\delta^m_n\,. \quad (2.189)$$

To work out the Einstein equations, we compute $\hat{T}$ in a complex basis. It is straightforward to check that $\hat{T}^\mu_\nu = 0$ if μ and ν are both holomorphic indices; this is a consequence of the primitivity of G_3. Of the mixed, $\hat{T}^{\bar\mu}_\nu$ components, all except $\hat{T}^{\bar z}_z = (\hat{T}^z_{\bar z})^\star$ must vanish by symmetry, and we find

$$\hat{T}_{zz} = 4\,e^{-4C}\frac{\bar z}{z}\left[g^{3,0}\bar{g}^{2,1} + g^{1,2}\bar{g}^{0,3}\right]\,. \quad (2.190)$$

After a straightforward computation, we find the Ricci components

$$R_{zz} = -\frac{\bar z}{z}\left[C'' + 4B'C' + (C')^2 - \frac{1}{r}C'\right]\,, \quad (2.191)$$

$$R_{z\bar z} = \left[B'' - C'' + \frac{1}{r}(B' - C') - (C')^2\right]\,, \quad (2.192)$$

$$R_{u^i\bar{u}^{\bar j}} = -\frac{1}{2}e^{4B+2C}\left[C'' + \frac{1}{r}C' + 4(C')^2\right]\delta_{i\bar j}\,. \quad (2.193)$$

Using these formulae, one can write down the Einstein equations in terms of B and C. Applying the remaining equations of motion to the ansatz of section 2.3.1 in like fashion, we find four real second order equations of motion for A, α, B, and C, along with one complex second order equation of motion for τ, four real first order equations of motion for the $g_{p,q}$, and one complex constraint coming from the r, ψ component of the Einstein equations.

The α and A equations of motion are

$$\frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}\left(re^{4(C-2A)}\alpha'\right) = 4\sum_{p=0}^{3}(-1)^p|g^{p,3-p}|^2\,, \tag{2.194}$$

$$\frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}\left(re^{4C}A'\right) = e^{4A}\sum_{p=0}^{3}|g^{p,3-p}|^2 + \frac{1}{4}e^{4(C-2A)}(\alpha')^2 + \Lambda e^{4(C-B-A)}\,, \tag{2.195}$$

where primes denote derivatives with respect to r. The τ equation of motion is

$$\frac{1}{r\tau_2}\frac{\mathrm{d}}{\mathrm{d}r}\left(re^{4C}\tau'\right) + \frac{i}{\tau_2^2}e^{4C}(\tau')^2 = -8ie^{4A}\left(g^{3,0}g^{0,3} + g^{2,1}g^{1,2}\right)\,. \tag{2.196}$$

The G_3 equations of motion and Bianchi identities are

$$\begin{aligned}
\frac{1}{r\sqrt{\tau_2}}\frac{\mathrm{d}}{\mathrm{d}r}\left[r\sqrt{\tau_2}e^{4A}g^{3,0}\right] + \frac{i\tau'}{2\tau_2}e^{4A}\left[g^{3,0}+\bar{g}^{1,2}\right] &= \frac{e^{4A}}{r}g^{3,0} + \frac{1}{2}(e^{4A}-\alpha)'\left[g^{3,0}-g^{2,1}\right]\,,\\
\frac{1}{r\sqrt{\tau_2}}\frac{\mathrm{d}}{\mathrm{d}r}\left[r\sqrt{\tau_2}e^{4A}g^{2,1}\right] + \frac{i\tau'}{2\tau_2}e^{4A}\left[g^{2,1}+\bar{g}^{0,3}\right] &= -\frac{e^{4A}}{r}g^{2,1} - \frac{1}{2}(e^{4A}+\alpha)'\left[g^{3,0}-g^{2,1}\right]\,,\\
\frac{1}{r\sqrt{\tau_2}}\frac{\mathrm{d}}{\mathrm{d}r}\left[r\sqrt{\tau_2}e^{4A}g^{1,2}\right] + \frac{i\tau'}{2\tau_2}e^{4A}\left[\bar{g}^{3,0}+g^{1,2}\right] &= -\frac{e^{4A}}{r}g^{1,2} - \frac{1}{2}(e^{4A}-\alpha)'\left[g^{0,3}-g^{1,2}\right]\,,\\
\frac{1}{r\sqrt{\tau_2}}\frac{\mathrm{d}}{\mathrm{d}r}\left[r\sqrt{\tau_2}e^{4A}g^{0,3}\right] + \frac{i\tau'}{2\tau_2}e^{4A}\left[\bar{g}^{2,1}+g^{0,3}\right] &= \frac{e^{4A}}{r}g^{0,3} + \frac{1}{2}(e^{4A}+\alpha)'\left[g^{0,3}-g^{1,2}\right]\,.
\end{aligned} \tag{2.197}$$

The B and C equations of motion are

$$C'' + \frac{1}{r}C' + 4(C')^2 = \Lambda e^{-4(A+B)}\,, \tag{2.198}$$

$$B'' + \frac{1}{r}B' + 3(C')^2 = 2(A')^2 - \frac{1}{8}e^{-8A}(\alpha')^2 + \frac{1}{8\tau_2^2}|\tau'|^2 + \frac{1}{2}e^{-4(A+B)}\Lambda\,, \tag{2.199}$$

and the constraint takes the form

$$C' \left[\frac{2}{r} + 3C' - 4B' \right] = 2(A')^2 - \frac{1}{8} e^{-8A} (\alpha')^2 + \frac{1}{8\tau_2^2} |\tau'|^2 + 2e^{4(A-C)} \left(g_{3,0}\, \bar{g}_{2,1} + g_{1,2}\, \bar{g}_{0,3} \right) + e^{-4(A+B)} \Lambda \,. \quad (2.200)$$

CHAPTER 3

$\mathrm{SL}(2,\mathbb{R})$ COVARIANT CONDITIONS FOR $\mathcal{N}=1$ FLUX VACUA

Four-dimensional supersymmetric $\mathcal{N}=1$ vacua of type IIB supergravity are elegantly described by generalized complex geometry.[1] However, this approach typically obscures the $\mathrm{SL}(2,\mathbb{R})$ covariance of the underlying theory. We show how to rewrite the pure spinor equations of Graña, Minasian, Petrini and Tomasiello (hep-th/0505212) in a manifestly $\mathrm{SL}(2,\mathbb{R})$ covariant fashion. Solutions to these equations fall into two classes: "charged" solutions, such as those containing D5-branes, and "chargeless" solutions, such as F-theory solutions in the Sen limit and AdS_4 solutions. We derive covariant supersymmetry conditions for the chargeless case, allowing general $\mathrm{SU}(3)\times\mathrm{SU}(3)$ structure. The formalism presented here greatly simplifies the study of the ten-dimensional geometry of general supersymmetric compactifications of F-theory.

3.1 Introduction

The first step towards understanding the phenomenological implications of string theory is to understand the four-dimensional vacua of the theory. These vacua are thought to be numerous [13], though many subtleties remain to gain a complete understanding of even one example. Progress has been hindered by the appearance of apparent flat directions (moduli) in the scalar-field effective potential of supersymmetric vacua. Known solutions to this problem involve nonperturbative effects [17–19].

Supergravity, the low energy limit of string theory, has so far proved to be an invaluable tool in the pursuit of vacua. In particular, the study of Calabi-Yau geometry has yielded a detailed understanding of the massless spectra of unstabilized $\mathcal{N}=1$ orientifolds (see e.g. [64]). Naturally, one would like to classify all string vacua if possible. While this is a very difficult problem, significant progress

[1] This chapter is reprinted from Ben Heidenreich, "$\mathrm{SL}(2,\mathbb{R})$ covariant conditions for $\mathcal{N}=1$ flux vacua," JHEP **1110**, 057 (2011), with permission.

can be made if we restrict our attention to vacua with a geometric supergravity description, with the necessary[2] addition of localized brane sources. The geometry of such vacua need not be Calabi-Yau in any sense, and more general tools are needed.

In the case of unbroken $\mathcal{N} = 1$ supersymmetry, significant progress in this direction has been made already using generalized complex geometry [50, 65]. Graña, Minasian, Petrini, and Tomasiello [23, 44] have shown that the conditions for unbroken $\mathcal{N} = 1$ supersymmetry in type IIA/B supergravity can be rewritten as a set of relatively simple algebraic and differential conditions on a pair of compatible pure spinors. Supersymmetric brane embeddings may also be described elegantly using generalized calibrations [56, 57, 66]. Moreover, a few explicit examples of solutions of this type are now known in type IIA [44, 49, 67–75] and IIB [44, 49, 74, 76], see also chapter 2.

One serious drawback of the pure-spinor equations of [23] is that they obscure the $\mathrm{SL}(2, \mathbb{R})$ invariance of type IIB supergravity. Though this may seem to be merely an aesthetic problem at first glance, it becomes a serious obstacle in the presence of nontrivial $\mathrm{SL}(2, \mathbb{Z})$ monodromies, such as arise in F-theory compactifications (see e.g. [77, 78]). Though the pure spinors are invariant under axion shifts, and transform in a known fashion under orientifold involutions [44], their behavior under more general $\mathrm{SL}(2, \mathbb{R})$ transformations is significantly more complicated, and, we believe, not previously described in the literature.[3]

It is therefore advantageous to restate the $\mathcal{N} = 1$ supersymmetry conditions in an $\mathrm{SL}(2, \mathbb{R})$ covariant fashion, both as a consistency check, and to better describe the geometry of stabilized F-theory compactifications. In this chapter, we carry

[2]In particular, local sources of negative tension (e.g. O-planes) are required for a geometric vacuum solution with $\mathcal{N} \leq 1$ supersymmetry and a non-negative cosmological constant.

[3]In principle, some of these results may be encoded in the far more ambitious approach of exceptional generalized geometry [79–81], though decoding them is far from trivial.

out this computation for a special, yet important, class of "chargeless" solutions. In particular, all AdS_4 solutions are chargeless, and, we will argue, such solutions encompass all possible supersymmetric deformations of SU(3) structure F-theory compactifications. We find that the covariant conditions obtained here are easier to work with than the original pure spinor equations, though they are less compactly stated.

While the general conditions for $\mathcal{N} = 1$ supersymmetry are interesting in their own right, the problem of moduli stabilization provides additional impetus for studying them. It has been suggested [26] that gaugino condensation on seven-branes, a key ingredient of moduli stabilization in several scenarios, sources a generalized complex geometry. An improved understanding of such geometries (see [25, 82], and chapter 2) could lead to a better understanding of known scenarios for moduli stabilization, or to additional scenarios for stabilizing moduli.

In section 3.2 we review the algebraic and differential conditions for unbroken $\mathcal{N} = 1$ supersymmetry, as laid out by [23]. In section 3.3, we review the $SL(2, \mathbb{R})$ covariance of type IIB supergravity and rewrite the pure spinor equations in Einstein frame. We then show that $\mathcal{N} = 1$ solutions fall into two classes, which we call "charged" and "chargeless," and review chargeless SU(3) structure, the well-known "F-theory" solutions. In section 3.4, we show how to characterize the $SU(3)\times SU(3)$ structure of a general chargeless solution in terms of $SL(2, \mathbb{R})$ covariant forms, and, writing the pure spinors in terms of these forms, we derive $SL(2, \mathbb{R})$ covariant supersymmetry conditions for these solutions, (3.98 – 3.103). (The technical details of this computation are presented separately in appendix 3.A.) In section 3.5 review the flux equations of motion and show that they follow from the supersymmetry conditions and Bianchi identities. We also show that the flux superpotential proposed in [26] is $SL(2, \mathbb{R})$ invariant. In §3.6 we discuss future directions and conclude.

3.2 The pure-spinor equations for general $\mathcal{N}=1$ vacua

The supersymmetry transformations of type IIB supergravity are generated by two ten-dimensional Majorana-Weyl spinors of the same chirality, which we denote by ϵ^i, for $i=1,2$. For $\mathcal{N}=1$ solutions, these should be determined by a single four-dimensional Weyl spinor ζ_+ (the generator of $\mathcal{N}=1$ supersymmetry transformations.) The most general relation compatible with unbroken $\mathcal{N}=1$ supersymmetry is [23]

$$\epsilon^i = \zeta_+ \otimes \eta^i_+ + \zeta_- \otimes \eta^i_- \ , \tag{3.1}$$

where η^i_+ are a pair of fixed positive-chirality internal spinors and ζ_- and η^i_- are the Majorana conjugates of ζ_+ and η^i_+ respectively.

We construct bispinors $\slashed{\Upsilon}_+$ and $\slashed{\Upsilon}_-$ from the spinors η^i_+:

$$\slashed{\Upsilon}_\pm = -8i\eta^1_+ \otimes \eta^{2\,\dagger}_\pm \ . \tag{3.2}$$

Using the Clifford map, $\slashed{\Upsilon}_\pm$ can be rexpressed as polyforms $\Upsilon_\pm$ [44]

$$\Upsilon_\pm = \sum_p \frac{1}{p!}(\Upsilon_\pm)^{(p)}_{m_1\ldots m_p}\mathrm{d}y^{m_1}\wedge\ldots\mathrm{d}y^{m_p} \longleftrightarrow \slashed{\Upsilon}_\pm = \sum_p \frac{1}{p!}(\Upsilon_\pm)^{(p)}_{m_1\ldots m_p}\gamma^{m_1\ldots m_p} \tag{3.3}$$

where $\gamma^{m_1\ldots m_p} = \gamma^{[m_1}\ldots\gamma^{m_p]}$ denotes the antisymmetrized product of gamma matrices.

As direct products of spinors, the bispinors $\slashed{\Upsilon}_\pm$ are not generic, and satisfy certain algebraic constraints. We review these constraints in section 3.2.1, rewriting them as constraints on the polyforms $\Upsilon_\pm$ using the language of G-structures. Unbroken supersymmetry imposes additional differential conditions on the polyforms $\Upsilon_\pm$, which we review in section 3.2.2. Taken together, the conditions in section 3.2.1 and section 3.2.2 are sufficient for unbroken $\mathcal{N}=1$ supersymmetry, and it is no longer necessary to construct the original spinors η^i_+ explicitly.

3.2.1 Algebraic constraints – SU(3) × SU(3) structure

The algebraic constraints on $\Upsilon_\pm$ can be expressed in terms of pure spinors. A pure spinor is a spinor which is annihilated by one-half of the gamma matrices in the associated Clifford algebra,[4] the maximum possible number. Spinors of SO(d) for $d \leq 6$ are always pure, whereas for $d > 6$, purity imposes a non-trivial constraint.

Bispinors of SO(d) can be viewed as spinors of SO(d,d), whose associated Clifford algebra consists of the d gamma matrices of SO(d) γ^m acting on the left $\Gamma^m_L = \gamma^m \otimes 1$ and the same gamma matrices acting on the right combined with the bispinor chirality operator $\Gamma^m_R = (1 \otimes \gamma^m)\Gamma$, where $\Gamma = \gamma \otimes \gamma = (-1)^{d/2}(\gamma^1 \dots \gamma^d) \otimes (\gamma^1 \dots \gamma^d)$. Left-acting and right-acting gamma matrices commute, whereas all gamma matrices anticommute with the chirality operator Γ. Thus, Γ^m_L and Γ^m_R satisfy an SO(d,d) Clifford algebra:

$$\{\Gamma^m_L, \Gamma^n_L\} = 2\,\delta^{mn} \quad , \quad \{\Gamma^m_R, \Gamma^n_R\} = -2\,\delta^{mn} \quad , \quad \{\Gamma^m_L, \Gamma^n_R\} = 0 \tag{3.4}$$

Viewed as spinors of SO(6, 6), the bispinors $\Upsilon_\pm$ are pure; they are annihilated by three left-acting gamma matrices and three right-acting gamma matrices due to the (automatic) purity of the $\eta^i_\pm$ in $d = 6$. Moreover, the three left-acting gamma matrices Γ^i_+ are common annihilators of $\Upsilon_\pm$, and the three right-acting gamma matrices Γ^i_- which annihilate Υ_+ also annihilate $\Upsilon^c_- = 8i\eta^1_- \otimes \eta^{2\,\dagger}_+$, where $\Upsilon^c = C\Upsilon^\star C^\dagger$ denotes the charge conjugate of a bispinor and C is the (unitary) SO(6) charge conjugation matrix.[5]

Thus, the bispinors $\Upsilon_\pm$ divide the twelve-dimensional space of bispinor gamma matrices into four (complex) subspaces $\mathcal{V}_+ = \{\Gamma^i_+\}$, $\mathcal{V}_- = \{\Gamma^i_-\}$, and $\bar{\mathcal{V}}_\pm = \{\Gamma^{i\,\dagger}_\pm\}$. The subspaces $\mathcal{V}_+$ and $\mathcal{V}_-$ are positive and negative respectively, in the sense that $\{\Gamma^i_+, \Gamma^{j\,\dagger}_+\}$ is a positive-definite Hermitean matrix, defining an SU(3) Clifford

[4]We take the Clifford algebra to be even-dimensional.

[5]We use the spinor conventions of [45].

algebra, whereas $\{\Gamma^i_-, \Gamma^{j\,\dagger}_-\}$ is negative-definite.

It turns out that the properties enumerated in the last few paragraphs are sufficient to establish that $\Upsilon_\pm$ take the form (3.2) for some spinors η^i_+ [23]. We say that SO(6) bispinors $\Upsilon_\pm$ are compatible pure spinors if the spaces $\mathcal{V}_\pm$ defined above are of dimension three and respectively positive and negative in the sense defined above.[6] Thus, $\Upsilon_\pm$ are compatible pure spinors if and only if they take the form (3.2). Moreover the η^i_+ can be reconstructed from $\Upsilon_\pm$, up to the obvious rescaling $\eta^1_+ \to \lambda\eta^1_+$ and $\eta^2_+ \to \lambda^{-1}\eta^2_+$ for real λ [23].

It is convenient to re-express the requirement of compatible pure spinors in $d = 6$ in terms of the polyforms $\Upsilon_\pm$ using the language of G-structures. In special cases, $\Upsilon_\pm$ define SU(3) or SU(2) structures on the tangent bundle. We review the properties of SU(3) and SU(2) structures in section 3.2.1 before applying them to the general case in section 3.2.1, where $\Upsilon_\pm$ define an SU(3) $\times$ SU(3) structure on the tangent plus cotangent bundle.

SU(3) **and** SU(2) **structures**

A G-structure on a real (complex) d-dimensional vector bundle is a subbundle of the associated vector bundle with reduced structure group $G \subset \mathrm{GL}(d,\mathbb{R})$ ($G \subset \mathrm{GL}(d,\mathbb{C})$). An almost complex structure on a d-dimensional manifold defines a $\mathrm{GL}(d/2,\mathbb{C})$ structure on the tangent bundle by restricting to holomorphic bases. With the addition of a hermitean metric, the structure group is reduced to $\mathrm{U}(d/2)$ by restricting to orthonormal frames. The structure group may be further reduced to $\mathrm{SU}(d/2)$ using a complex non-degenerate "volume element", that is a globally defined decomposable $d/2$ form Ω satisfying $\Omega\wedge\bar{\Omega} \neq 0$. In fact, the entire $\mathrm{SU}(d/2)$ structure may be specified using Ω (which specifies the almost complex structure)

[6]These conditions are sufficient to establish the purity of $\Upsilon_\pm$, and generalize readily to any even dimension.

and a nondegenerate two-form, J, the Kähler form, where the two must satisfy the compatibility condition, $J \wedge \Omega = 0$, and the associated metric must be positive definite. It is conventional to normalize such that

$$\frac{1}{n!} J^n = \frac{i^n}{2^n} (-1)^{n(n-1)/2} \, \Omega \wedge \bar{\Omega} \neq 0 \,, \tag{3.5}$$

which combines the nondegeneracy conditions for J and Ω, where $n = d/2$. The positive-definiteness of the associated metric is equivalent to the condition:

$$- iJ (v, \bar{v}) > 0 \,, \tag{3.6}$$

for any nonvanishing holomorphic vector v (i.e. satisfying $v \neq 0, \iota_{\bar{v}} \Omega = 0$). This last condition is "topological," since the nondegeneracy of J implies that the associated metric is positive definite everywhere so long as this is true at any one point.

Given an SU($d/2$) structure, the defining forms J and Ω can be uniquely reconstructed.[7] Moreover, associated to any SU($d/2$) structure, we have a conjugate SU($d/2$) structure, defined by $-J$ and $\Omega^\star$.

By providing a global decomposition of Ω into lower-rank forms, we obtain a reduced structure group. For $d = 6$, the only nontrivial example of this is an SU(2) structure, where we decompose

$$\Omega = \Omega_2 \wedge \Theta \,. \tag{3.7}$$

Taking Θ to be holomorphic and orthonormal with respect to the metric defined by Ω, J, and $\Omega_2 = \frac{1}{2} \iota_{\bar{\Theta}} \Omega$, we see that $J_2 \equiv J - J_1$ has rank two, where $J_1 \equiv \frac{i}{2} \Theta \wedge \bar{\Theta}$. The SU(2) structure, defined by Θ, J_2, and Ω_2, must satisfy[8]

$$J_2 \wedge \Omega_2 = \Omega_2 \wedge \Omega_2 = 0 \;\;,\;\; \Omega_2 \wedge \bar{\Omega}_2 = 2J_2^2 \;\;,\;\; J_1 \wedge J_2^2 \neq 0 \,, \tag{3.8}$$

[7] Given a local holomorphic orthonormal frame $\{\theta^1, \ldots, \theta^n\}$, $\Omega \equiv \theta^1 \wedge \ldots \wedge \theta^n$ and $J = \frac{i}{2} \delta_{i\bar{j}} \theta^i \wedge \bar{\theta}^{\bar{j}}$.

[8] The condition $\Omega_2 \wedge \Omega_2 = 0$ implies that Ω_2 is rank one (viewed as an antisymmetry matrix,) and therefore decomposable, whereas the conditions $2J_2^2 = \Omega_2 \wedge \bar{\Omega}_2 \neq 0$ and $J_2 \wedge \Omega_2 = 0$ imply that J_2 is rank 2.

and the associated metric must be positive definite as necessary and sufficient conditions for J_2, Ω_2, Θ to define an SU(2) structure. This last condition can be restated as the requirement

$$-\, iJ_2\,(v,\bar{v}) > 0\;, \tag{3.9}$$

for any nonvanishing vector v satisfying $\iota_{\bar{v}}\Omega_2 = \iota_v\Theta = \iota_{\bar{v}}\Theta = 0$. As before, this requirement is topological, given the nondegeneracy of J_2.

An SU(2) structure induces a natural decomposition into "base" (J_2, Ω_2) and "fiber" $(\Theta, \bar{\Theta})$ directions; a nonvanishing vector v is said to "point along the base" if $\iota_v J_1 = 0$, and along the fiber if $\iota_v J_2 = 0$, and a nonvanishing one-form ω is said to point along the base if $\omega \wedge J_2^2 = 0$, and along the fiber if $\omega \wedge J_1 = 0$.

The conditions on an SU(2) structure possess a remarkable symmetry. To make this manifest, we define a vector of real two-forms $\Omega^i = \{\mathrm{Re}\,\Omega_2, \mathrm{Im}\,\Omega_2, J_2\}$ and the real four-form $\omega_4 = J_2^2$. The SU(2) structure conditions may now be rewritten as[9]

$$\Omega^i \wedge \Omega^j = \delta^{ij}\omega_4 \quad , \quad J_1 \wedge \omega_4 \neq 0\;, \tag{3.10}$$

together with the condition that the associated metric is positive definite, which can be rewritten as:

$$-\, \varepsilon_{ijk}\Omega^i\left(v^j, v^k\right) > 0\;, \tag{3.11}$$

where v^i is any nonvanishing triplet of real vectors satisfying $\delta_{ij}\iota_{v^i}\Omega^j = 0$. Thus, given an SU(2) structure (Ω^i, Θ), we can obtain a new SU(2) structure by performing an (in principle spatially dependent) SO(3) rotation on the Ω^i. Moreover, the induced metric is invariant under these rotations. The induced SU(3) structure $J = J_1 + J_2$, $\Omega = \Omega_2 \wedge \Theta$ is *not* invariant. Thus, an SU(2) structure defines many *different* SU(3) structures [47], depending on the choice of rotation.

[9]The algebraic structure is similar to that of hyper-Kähler manifolds.

Compatibility and $\mathrm{SU}(3) \times \mathrm{SU}(3)$ structure

We return to the compatibility conditions on $\Upsilon_\pm$ using the language of G-structures laid out in the previous section.

A bispinor Υ is pure if and only if the corresponding polyform takes the form

$$\Upsilon = \Omega_k \wedge e^{B-iJ} \ , \tag{3.12}$$

for real two-forms B and J, where Ω_k is a decomposable k-form and k is the *type* of the pure spinor. Even (odd) chirality pure spinors have even (odd) type; thus, by (3.2), Υ_+ is even rank and Υ_- is odd rank. It turns out that the only (Υ_+, Υ_-) types consistent with the compatibility conditions are $(0,3)$, $(2,1)$, and $(0,1)$. The types of the pure spinors can change over the compactification manifold, a phenomenon known as "type-changing."

The compatibility conditions on $\Upsilon_\pm$ can be restated as the requirement that $\Upsilon_\pm$ define a local SU(2) structure (types $(0,1)$ or $(2,1)$) or SU(3) structure (type $(0,3)$). If $(0,3) \leftrightarrow (0,1)$ type-changing occurs, then neither G-structure is globally defined.

To restate the compatibility conditions in this way, it is helpful to work with normalized pure-spinors. The bispinor norm corresponds to the Mukai pairing:

$$\langle \Psi, \Phi \rangle \equiv [\Psi \wedge (-1)^{\hat{p}(\hat{p}-1)/2} \Phi]_{\mathrm{top}} \ , \tag{3.13}$$

where $\hat{p}$ measures the rank of a form, $\hat{p} F_p = p F_p$. We have

$$\langle \Upsilon_+, \bar{\Upsilon}_+ \rangle = \langle \Upsilon_-, \bar{\Upsilon}_- \rangle = (-2i)^3 f_a f_b \Omega_6 \ , \tag{3.14}$$

where Ω_6 is the volume element of the metric used to define the spinors, and $f_a = \eta_+^{1\dagger} \eta_+^1$, $f_b = \eta_+^{2\dagger} \eta_+^2$. We define normalized polyforms[10]

$$\Psi_\pm = \frac{1}{\sqrt{f_a f_b}} \Upsilon_\pm \tag{3.15}$$

[10] The case of vanishing norm, $f_a = 0$ or $f_b = 0$, is not encompassed by the approach of [23].

so that $\langle\Psi_+,\bar{\Psi}_+\rangle = \langle\Psi_-,\bar{\Psi}_-\rangle = (-2i)^3\Omega_6$.

One can show that, wherever $\Psi_\pm$ have types $(0,3)$, they must take the form

$$\Psi_+ = e^{i\vartheta}e^{-iJ} \quad , \quad \Psi_- = \Omega \; , \tag{3.16}$$

where J and Ω define an SU(3) structure and ϑ is an additional phase factor. By contrast, wherever $\Psi_\pm$ have types $(0,1)$ or $(2,1)$, they must take the form (see e.g. [43]):

$$\Psi_+ = e^{i\vartheta}e^{-iJ_1}\wedge[c_\varphi e^{-iJ_2} - s_\varphi\Omega_2] \quad , \quad \Psi_- = \Theta\wedge[c_\varphi\Omega_2 + s_\varphi e^{-iJ_2}] \; , \tag{3.17}$$

where Θ, J_2, Ω_2 define an SU(2) structure, φ is the "spinor angle," and we use the shorthands $c_\varphi = \cos\varphi$ and $s_\varphi = \sin\varphi$. Though the types of the pure spinors $\Psi_\pm$ may vary across the compact space, either (3.16) or (3.17) must apply at any point; as such, these two equations are equivalent to the compatibility conditions.

The pure spinors $\Psi_\pm$ have types $(2,1)$ where $\varphi = \pi/2$ and types $(0,3)$ where $\varphi = 0$, though in the latter case the SU(2) structure need not be well defined, and only $\Omega = \Theta\wedge\Omega_2$ and $J = J_1 + J_2$ need be single-valued, defining a (local) SU(3) structure. Otherwise, for generic spinor angles such that $s_\varphi, c_\varphi \neq 0$, the pure spinors have types $(0,1)$.

From the perspective of generalized complex geometry, $\Psi_\pm$ define an SU(3) $\times$ SU(3) on the tangent plus cotangent bundle [43, 50]. This is not essential to our discussion, though we refer to the compatible pure spinors as an SU(3) $\times$ SU(3) structure for want of a better label.

3.2.2 Differential constraints

Having stated the algebraic conditions on $\Upsilon_\pm$ in the form (3.16, 3.17), we now review the differential conditions on $\Upsilon_\pm$ and the spinor norms f_a and f_b for unbroken $\mathcal{N} = 1$ supersymmetry.

We adopt the supergravity conventions of [23, 44] in string-frame. Subsequently, we employ compatible Einstein-frame conventions, which are outlined in section 3.3.1. The string-frame compactification metric takes the form:

$$\mathrm{d}s^2_{(S)} = e^{2A^{(S)}(y)}\mathrm{d}s^2_{(4)} + g^{(S)}_{mn}(y)\,\mathrm{d}y^m\mathrm{d}y^n \ , \tag{3.18}$$

where $A^{(S)}$ is the string-frame warp factor, $g^{(S)}_{mn}$ is the string-frame warped metric, and $\mathrm{d}s^2_{(4)}$ is a maximally isotropic four-dimensional metric (either Minkowski or anti-de-Sitter.) We define the field-strength polyform

$$F = F_1 + \tilde{F}_3 + \tilde{F}_5^{(\mathrm{int})} \ , \tag{3.19}$$

where $\tilde{F}_5^{(\mathrm{int})}$ denotes the internal components of $\tilde{F}_5 = (1+\star_{10})\,\tilde{F}_5^{(\mathrm{int})} = \tilde{F}_5^{(\mathrm{int})} + e^{4A^{(S)}}\Omega_4 \wedge \star_6^{(S)}\tilde{F}_5^{(\mathrm{int})}$,[11] so that F has internal components only. In terms of F, the source-free RR equations of motion and Bianchi identities take the form

$$\mathrm{d}_H F = 0 \ \ , \ \ \mathrm{d}_H\left[e^{4A^{(S)}}\star_6^{(S)} F\right] = 0 \ , \tag{3.20}$$

where $\mathrm{d}_H F \equiv \mathrm{d}F - H\wedge F$.

The Clifford map is frame-dependent; we denote polyforms constructed using the string-frame metric (3.18) with a superscript, as in $\Upsilon^{(S)}_\pm$, and those constructed using the Einstein-frame metric (3.45) without. Demanding that the supersymmetry variations vanish, one obtains the differential conditions [23]:

$$\begin{aligned}\mathrm{d}_H\left[e^{2A^{(S)}-\phi}\Upsilon^{(S)}_+\right] &= -3e^{A^{(S)}-\phi}\,\mathrm{Re}\left[\bar{\mu}\Upsilon^{(S)}_-\right] - e^{2A^{(S)}-\phi}\mathrm{d}A^{(S)}\wedge\bar{\Upsilon}^{(S)}_+ \\ &\quad + \frac{1}{2}e^{2A^{(S)}}\left[(f_a+f_b)\hat{\star}_6^{(S)}F - i(f_a-f_b)F\right] \ , \end{aligned} \tag{3.21}$$

$$\mathrm{d}_H\left[e^{2A^{(S)}-\phi}\Upsilon^{(S)}_-\right] = -2i\mu e^{A^{(S)}-\phi}\,\mathrm{Im}\left(\Upsilon^{(S)}_+\right) \ , \tag{3.22}$$

$$\mathrm{d}f_a = f_b\mathrm{d}A^{(S)} \ \ , \ \ \mathrm{d}f_b \ = \ f_a\mathrm{d}A^{(S)} \ , \tag{3.23}$$

---

[11] To establish sign conventions, the Hodge star associated with a D-dimensional metric g with volume form $\Omega_{(g)}$ is defined by $\star[\mathrm{d}x^{m_1}\wedge\ldots\wedge\mathrm{d}x^{m_p}] = \frac{1}{(D-p)!}\,\Omega_{(g)}{}^{m_1\ldots m_p}{}_{m_{p+1}\ldots m_D}[\mathrm{d}x^{m_{p+1}}\wedge\ldots\wedge\mathrm{d}x^{m_D}]$.

where μ is related to the vacuum expectation value of the superpotential, $\langle W\rangle = \mu/\kappa_4^2$ (see (3.123)), so that the cosomological constant is $\Lambda = -3|\mu|^2$,[12] and $\hat{\star}_6 F \equiv (-1)^{\hat{p}(\hat{p}-1)/2} \star_6 F$.

The differential conditions on the spinor norms (3.23) can be immediately integrated, giving

$$f_a = k_0 e^{A^{(S)}} + k_1 e^{-A^{(S)}} \quad , \quad f_b = k_0 e^{A^{(S)}} - k_1 e^{-A^{(S)}} \ . \tag{3.24}$$

Since $f_a, f_b \geqslant 0$, $k_0 > 0$, and we can set $k_0 = 1$ by rescaling the spinors η_+^i. From (3.15), we obtain $\Psi_\pm^{(S)} = \kappa^{-1} e^{-A^{(S)}} \Upsilon_\pm^{(S)}$, where $\kappa = \sqrt{1 - k_1^2 e^{-4A^{(S)}}}$. Expressed in terms of $\Psi_\pm^{(S)}$, the pure spinor equations (3.21, 3.22) become

$$\mathrm{d}_H \left[\kappa e^{4A^{(S)}-\phi}\, \mathrm{Re}\, \Psi_+^{(S)}\right] = -3\kappa e^{3A^{(S)}-\phi}\, \mathrm{Re}\left[\bar{\mu}\Psi_-^{(S)}\right] + e^{4A^{(S)}} \hat{\star}_6^{(S)} F \ , \tag{3.25}$$

$$\mathrm{d}_H \left[\kappa e^{2A^{(S)}-\phi}\, \mathrm{Im}\, \Psi_+^{(S)}\right] = -k_1 F \ , \tag{3.26}$$

$$\mathrm{d}_H \left[\kappa e^{3A^{(S)}-\phi} \Psi_-^{(S)}\right] = -2i\mu\kappa e^{2A^{(S)}-\phi}\, \mathrm{Im}\, \Psi_+^{(S)} \ , \tag{3.27}$$

The conditions (3.25, 3.26, 3.27), together with the algebraic conditions on $\Psi_\pm^{(S)}$ (3.16, 3.17) are necessary and sufficient conditions for unbroken four-dimensional $\mathcal{N} = 1$ supersymmetry, except in the degenerate case $\kappa = 0$ [23].[13] The $\mathrm{U}(1)_R$ symmetry associated with four-dimensional $\mathcal{N} = 1$ supersymmetry takes the form

$$\Psi_-^{(S)} \to e^{i\theta} \Psi_-^{(S)} \quad , \quad \mu \to e^{i\theta} \mu \ , \tag{3.28}$$

where the superpotential rotates $W \to e^{i\theta} W$.

The one-form component of (3.26) implies that

$$k_2 \equiv \kappa e^{2A^{(S)}-\phi}\, \mathrm{Im}\, \Psi_{+(0)}^{(S)} + k_1 C_0 \ , \tag{3.29}$$

is a constant, where the subscript denotes the zero-form component. AdS ($\mu \neq 0$) solutions to these conditions are more restricted than Minkowski ($\mu = 0$) solutions.

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[12] In our conventions, the cosmological constant is one-quarter of the Ricci scalar: $\Lambda = R_{(4)}/4$.

[13] One can incorporate the degenerate case using $\mathrm{SL}(2,\mathbb{R})$ covariance [83].

In particular, (3.27) implies $\mathrm{Im}\,\Psi^{(S)}_{+(0)} = 0$ for $\mu \neq 0$. Moreover, applying d_H to (3.27) and imposing the source-free Bianchi identity $\mathrm{d}H = 0$ as well as (3.26), we find $\mu k_1 F = 0$. Thus, for $\mu \neq 0$, either k_1 or F must vanish. However, $F = 0$ implies that $A^{(S)}$ is constant [23], so that κ is constant, and we may take $k_1 = 0$ without altering the supersymmetry conditions. With this caveat, we conclude that AdS solutions require $k_1 = k_2 = 0$.

3.3 The covariant conditions – setup

We wish to restate the $\mathcal{N} = 1$ supersymmetry conditions in a way which makes the $\mathrm{SL}(2,\mathbb{R})$ covariance of type IIB supergravity manifest. In principle, one could do this by repeating the steps taken by [23, 44] in deriving the pure spinor equations starting with a manifestly $\mathrm{SL}(2,\mathbb{R})$ covariant formulation of type IIB supergravity and maintaining covariance at each step. However, we find it more convenient to work with the pure spinors equations (3.25, 3.26, 3.27). It is then necessary to guess how the pure spinors $\Psi_\pm$ transform under $\mathrm{SL}(2,\mathbb{R})$. This guess can then be validated by showing that the supersymmetry conditions are covariant.

We examine this last inference in detail. Suppose that we misidentify the transformation of the pure spinors under $\mathrm{SL}(2,\mathbb{R})$, yet find that the supersymmetry conditions are covariant. Cancelling the $\mathrm{SL}(2,\mathbb{R})$ transformation of the supergravity fields using a genuine $\mathrm{SL}(2,\mathbb{R})$ transformation, we find an $\mathrm{SL}(2,\mathbb{R})$ symmetry of the pure spinor equations under which all supergravity fields are invariant but the pure spinors transform nontrivially. A symmetry of this type can only be an R-symmetry. Thus, we conclude that there exists a homomorphism from $\mathrm{SL}(2,\mathbb{R})$ to the R-symmetry group G_R, so that G_R contains a subgroup $\mathrm{SL}(2,\mathbb{R})/H$, where H is a proper normal subgroup of $\mathrm{SL}(2,\mathbb{R})$.[14] The only possibilites are $H = \{1\}$,

[14] H must be a proper subgroup because the pure spinors transform nontrivially by assumption.

$H = \{1, -1\}$, so that G_R must contain an $\mathrm{SL}(2,\mathbb{R})$ or $\mathrm{PSL}(2,\mathbb{R})$ subgroup. This is obviously impossible for $\mathcal{N} = 1$ vacua, since then $G_R \cong \mathrm{U}(1)$. In fact, it is still impossible for extended supersymmetry ($\mathcal{N} \geqslant 2$), since G_R is in general a compact Lie group, whose Lie algebra does not have subalgebras isomorphic to the split Lie algebra $\mathfrak{sl}_2(\mathbb{R})$. Thus, we conclude that the $\mathrm{SL}(2,\mathbb{R})$ transformation properties of the pure spinors are uniquely determined by the covariance of the pure spinor equations.

This argument relies on the full $\mathrm{SL}(2,\mathbb{R})$ invariance of type IIB supergravity. While only an $\mathrm{SL}(2,\mathbb{Z})$ subgroup is nonanomalous in the quantum theory, the conditions derived in [23] follow from *classical* type IIB supergravity, and therefore necessarily possess the full $\mathrm{SL}(2,\mathbb{R})$ invariance. Thus, our approach is not only valid, but additionally presents a highly nontrivial consistency check on the pure spinor equations (3.25, 3.26, 3.27), which were derived without reference to $\mathrm{SL}(2,\mathbb{R})$ invariance.

Type IIB supergravity has an even slightly larger, $\mathrm{SL}_\pm(2,\mathbb{R})$ invariance, where negative and positive determinant transformations are connected by "charge conjugation," a $\mathbb{Z}_2$ symmetry which reverses all RR fields and leaves the NSNS fields invariant. Charge conjugation acts simply on the pure spinors, taking $\Psi_\pm^{(S)} \to -\Psi_\pm^{(S)}$.[15]

In section 3.3.1, we review how the $\mathrm{SL}_\pm(2,\mathbb{R})$ invariance of type IIB string theory can be made explicit to develop the notation necessary to write down the covariant supersymmetry conditions. In section 3.3.2, we rewrite the pure spinor equations in Einstein frame, and in section 3.3.3, we show that solutions fall into two classes, "charged" and "chargeless" solutions. After reviewing chargeless solutions with strict SU(3)-structure in section 3.3.4, the often-studied "F-theory"

[15] This is similar to the O5/O9 involution, which combines charge conjugation with $-1 \in \mathrm{SL}(2,\mathbb{Z})$, but has a more complicated action on the pure spinors.

solutions with imaginary self-dual G_3 flux, we consider general chargeless solutions in section 3.4.

3.3.1 The $\mathrm{SL}_{\pm}(2,\mathbb{R})$ covariance of type IIB supergravity

The bosonic low energy effective action for type IIB string theory written in Einstein frame is[16]

$$S = \frac{1}{2\kappa_{10}^2}\int \mathrm{d}^{10}x\sqrt{-g^{(10)}}\left[R_{(10)} - \frac{1}{2}\left((\nabla\phi)^2 + e^{-\phi}|H_3|^2 \right.\right.$$
$$\left.\left. + e^{2\phi}|F_1|^2 + e^{\phi}|\tilde{F}_3|^2 + \frac{1}{2}|\tilde{F}_5|^2\right)\right] - \frac{1}{4\kappa_{10}^2}\int C_4\wedge H_3\wedge F_3\ , \quad (3.30)$$

where $|F_p|^2 \equiv \frac{1}{p!}F^{M_1\ldots M_p}F^{\star}_{M_1\ldots M_p} = F_p\cdot F_p^{\star}$, $H_3 = \mathrm{d}B_2$, $F_p = \mathrm{d}C_{p-1}$,

$$\tilde{F}_3 = F_3 - C_0 H_3\ \ ,\ \ \tilde{F}_5 = F_5 - \frac{1}{2}C_2\wedge H_3 + \frac{1}{2}B_2\wedge F_3\ , \quad (3.31)$$

and the equations of motion must be suplemented by the self-duality constraint, $\tilde{F}_5 = \star_{10}\tilde{F}_5$. The bosonic fields may be arranged into singlets, doublets, and triplets as follows:

$$C_2^i = \begin{pmatrix} C_2 \\ B_2 \end{pmatrix}\ ,\ \ F_3^i = \begin{pmatrix} F_3 \\ H_3 \end{pmatrix}\ ,\ \ \phi^{ij} = \begin{pmatrix} C_0^2 e^{\phi} + e^{-\phi} & C_0 e^{\phi} \\ C_0 e^{\phi} & e^{\phi} \end{pmatrix}\ , \quad (3.32)$$

where ϕ^{ij} only carries two degrees of freedom due to the constraint $\det\phi^{ij} = 1$, and $g^{(10)}$, C_4, and $\tilde{F}_5$ are singlets. The action can then be rewritten as

$$S = \frac{1}{2\kappa_{10}^2}\int \mathrm{d}^{10}x\sqrt{-g^{(10)}}\left[R_{(10)} + \frac{1}{4}F_1^{ij}\cdot(F_1)_{ij} - \frac{1}{2}\phi_{ij}F_3^i\cdot F_3^j - \frac{1}{4}|\tilde{F}_5|^2\right]$$
$$+ \frac{\varepsilon_{ij}}{8\kappa_{10}^2}\int C_4\wedge F_3^i\wedge F_3^j\ . \quad (3.33)$$

where

$$F_1^{ij} = \mathrm{d}\phi^{ij}\ \ ,\ \ \tilde{F}_5 = \mathrm{d}C_4 - \frac{1}{2}\varepsilon_{ij}C_2^i\wedge F_3^j\ \ ,\ \ \varepsilon_{12} = -\varepsilon_{21} = -\varepsilon^{12} = \varepsilon^{21} = +1\ , \quad (3.34)$$

[16] We employ the supergravity conventions of chapter 2 in Einstein frame.

and indices are raised and lowered by left multiplication by ε_{ij} or ε^{ij}, so that $(F_3)_i = \varepsilon_{ij} F_3^j$. Invariance of the action under global $\Lambda^i{}_j \in \mathrm{SL}_\pm(2,\mathbb{R})$ transformations is now manifest, where

$$F_3^i \to \Lambda^i{}_j F_3^j \quad , \quad \phi^{ij} \to \Lambda^i{}_k \Lambda^j{}_l \phi^{kl} \quad , \quad \tilde{F}_5 \to (\det \Lambda)\, \tilde{F}_5 \; . \tag{3.35}$$

An $\mathrm{SL}_\pm(2,\mathbb{Z})$ subgroup of this classical $\mathrm{SL}_\pm(2,\mathbb{R})$ symmetry of type IIB supergravity is an exact gauged symmetry of type IIB string theory, as manifested by the presence of branes (seven-branes and orientifold planes) in the spectrum carrying monodromies of this type. This symmetry group has a geometric interpretation in F-theory as modular transformations on an elliptic fibration.

In some contexts, it is convenient to re-express the bosonic fields in complex combinations. We define $\tau \equiv C_0 + i e^{-\phi} = \tau_1 + i\tau_2$, so that under $\mathrm{SL}(2,\mathbb{R})$ transformations

$$\tau \to \frac{a\tau + b}{c\tau + d} \quad , \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2,\mathbb{R}) \; . \tag{3.36}$$

One can then check that the complex doublet $t^i \equiv \frac{1}{\sqrt{\tau_2}} \binom{\tau}{1}$ transforms by an additional phase under $\mathrm{SL}(2,\mathbb{R})$:

$$t^i \to \left(\frac{|c\tau + d|}{c\tau + d} \right) \Lambda^i{}_j t^j \; , \tag{3.37}$$

which motivates the definitions of the following complex combinations:

$$G_1 \equiv \frac{i}{2} t_i t_j F_1^{ij} = \frac{1}{\tau_2} \mathrm{d}\tau \quad ,$$
$$\mathcal{A}_2 \equiv t_i C_2^i = \frac{1}{\sqrt{\tau_2}} (C_2 - \tau B_2) \quad , \quad G_3 \equiv t_i F_3^i = \frac{1}{\sqrt{\tau_2}} (F_3 - \tau H_3) \quad , \tag{3.38}$$

all of which transform by a phase under $\mathrm{SL}(2,\mathbb{R})$ transformations, and which we label as charge $Q = +1$, $Q = +1/2$, and $Q = +1/2$ respectively, according to the power $2Q$ of the phase factor that they transform by. Notably, half-integer charged quantities change sign under $-1 \in \mathrm{SL}(2,\mathbb{Z})$ transformations, whereas τ and integer-charged quantities are invariant.

Since the phase factor is spatially dependent in general, it is necessary to introduce a covariant derivative:

$$\mathrm{D}\Omega \equiv \mathrm{d}\Omega + \frac{iQ}{\tau_2}\mathrm{d}\tau_1 \wedge \Omega \quad , \quad \mathrm{D}^2\Omega = -\frac{Q}{2} G_1 \wedge G_1^\star \wedge \Omega \ , \tag{3.39}$$

so that $\mathrm{D}\Omega$ carries the same charge as Ω, though the operator is no longer nilpotent. We also define the nilpotent covariant derivatives:

$$\mathrm{D}_\pm\Omega = \mathrm{D}\Omega \pm \frac{i}{2} G_1 \wedge \Omega^\star \quad , \quad \mathrm{D}_\pm\tilde{\Omega} = \mathrm{D}\tilde{\Omega} \pm \frac{i}{2} G_1^\star \wedge \tilde{\Omega}^\star \ , \tag{3.40}$$

for $\Omega, \tilde{\Omega}$ of charge $+1/2$ and $-1/2$ respectively. However, these operators are not $\mathbb{C}$-linear (e.g. $i\mathrm{D}_\pm = \mathrm{D}_\mp i$), and so the usual Leibniz rule is not obeyed in general, though the following identities may be used:

$$\mathrm{D}_\pm\left(\Omega_p \wedge F_q\right) = \mathrm{D}_\pm\Omega_p \wedge F_q + (-1)^p\, \Omega_p \wedge \mathrm{d}F_q \ , \tag{3.41}$$

$$\mathrm{d}\left(\Omega_p \wedge \tilde{\Omega}_q + c.c.\right) = \mathrm{D}_\pm\Omega_p \wedge \tilde{\Omega}_q + (-1)^p\, \Omega_p \wedge \mathrm{D}_\pm\tilde{\Omega}_q + c.c. \ , \tag{3.42}$$

for F_q real and neutral, and Ω_p, $\tilde{\Omega}_q$ of charge $+1/2$ and $-1/2$ respectively (or vice versa).

Using this notation, the G_3 Bianchi identity becomes $\mathrm{D}_- G_3 = 0$, which is solved locally by $G_3 = \mathrm{D}_-\mathcal{A}_2$. The G_1 Bianchi identity becomes $\mathrm{D}G_1 = 0$, and the $\tilde{F}_5$ Bianchi identity and local solution become

$$\mathrm{d}\tilde{F}_5 = \frac{i}{2} G_3 \wedge G_3^\star \quad , \quad \tilde{F}_5 = \mathrm{d}C_4 + \frac{i}{4}\mathcal{A}_2 \wedge G_3^\star + c.c. \ . \tag{3.43}$$

Extended $\mathrm{SL}_\pm(2,\mathbb{R})$ transformations may be generated by combining $\mathrm{SL}(2,\mathbb{R})$ transformations with charge conjugation, i.e. $\left(\begin{smallmatrix} -1 & 0 \\ 0 & 1 \end{smallmatrix}\right) \in \mathrm{SL}_\pm(2,\mathbb{R})$, which acts by negative complex conjugation:

$$\tau \to -\tau^\star \quad , \quad G_3 \to -G_3^\star \quad , \quad \tilde{F}_5 \to -\tilde{F}_5 \ , \tag{3.44}$$

where $\mathrm{D}\Omega$ and $\mathrm{D}_\pm\Omega$ once again have the same transformations properties as Ω.

3.3.2 The pure-spinor equations in Einstein frame

As a first step towards covariantization, we now show how to rewrite the pure spinor equations (3.25, 3.26, 3.27) in terms of the Einstein-frame quantities. We take the following ansatz for the Einstein-frame metric:

$$\mathrm{d}s_{10}^2 = e^{2A(y)}\,\mathrm{d}s_{(4)}^2 + e^{-2A(y)}\,g_{mn}(y)\,\mathrm{d}y^m\mathrm{d}y^n\ . \tag{3.45}$$

Thus, the warp-factor A and unwarped metric g are related to their string-frame counterparts by

$$A^{(S)} = A + \phi/4\ \ ,\ \ g_{mn}^{(S)} = e^{\phi/2-2A}g_{mn}\ , \tag{3.46}$$

where $g^{(S)}$ is the warped metric which appears in (3.18).

It is convenient to work with compatible pure spinors whose associated metric is g rather than $g^{(S)}$. From (3.16, 3.17), we see that this can be accomplished by the rescaling $\Psi_\pm^{(S)} \equiv e^{(\phi/4-A)\hat{p}}\Psi_\pm$, We also rewrite the Hodge star as $\hat{\star}_6^{(S)}F = e^{(2A-\phi/2)(3-\hat{p})}\hat{\star}_6 F$, where $\star_6$ is the Hodge star associated with g.

Applying these replacements, the pure spinor equations become:

$$\mathrm{d}_H\left[\kappa e^{(\phi/4-A)\hat{p}}e^{4A}\,\mathrm{Re}\,\Psi_+\right] = -3\kappa e^{(\phi/4-A)(\hat{p}-3)}e^{\phi/2}\,\mathrm{Re}\left[\bar{\mu}\Psi_-\right] + e^{(2A-\phi/2)(5-\hat{p})}e^{2\phi}\hat{\star}_6 F\ , \tag{3.47}$$

$$\mathrm{d}_H\left[\kappa e^{(\phi/4-A)(\hat{p}-2)}\,\mathrm{Im}\,\Psi_+\right] = -k_1 F\ , \tag{3.48}$$

$$\mathrm{d}_H\left[\kappa e^{(\phi/4-A)(\hat{p}-3)}e^{\phi/2}\Psi_-\right] = -2i\mu\kappa e^{(\phi/4-A)(\hat{p}-2)}\,\mathrm{Im}\,\Psi_+\ , \tag{3.49}$$

where

$$\kappa = \sqrt{1-k_1^2e^{-4A-\phi}}\ . \tag{3.50}$$

Depending on the spinor angle, the pure spinors $\Psi_\pm$ must either take the form (3.16) or (3.17).

The supersymmetry conditions imply that $e^{8A}\star_6\tilde{F}_5^{(\mathrm{int})}$ is closed. Thus, we may take the local ansatz $e^{8A}\star_6\tilde{F}_5^{(\mathrm{int})} = \mathrm{d}\alpha$. Due to the ten-dimensional self-duality of

$\tilde{F}_5$, this implies

$$\tilde{F}_5 = (1 + \star_{10})\, \Omega_4 \wedge \mathrm{d}\alpha \ , \tag{3.51}$$

so that α is related to the external components of C_4 via $C_4^{(\mathrm{ext})} = \alpha\Omega_4$, where Ω_4 is the volume-form for the external directions. The Minkowski supersymmetry conditions actually imply that $e^{8A} \star_6 \tilde{F}_5^{(\mathrm{int})}$ is exact, so that α is globally defined, though this need not be the case for $\mu \neq 0$.

3.3.3 Charged and Chargeless solutions

Solutions to the pure spinor equations fall into two categories, charged solutions and chargeless solutions, as we now demonstrate.

Consider the one-form component of (3.47), the two-form component of (3.49), and the three-form component of (3.48):

$$\mathrm{d}\left[\kappa e^{4A}\,\mathrm{Re}\,\Psi_+^{(0)}\right] = -3\kappa e^{2A}\,\mathrm{Re}[\bar{\mu}\Psi_-^{(1)}] + \mathrm{d}\alpha \ , \tag{3.52}$$

$$\mathrm{d}\left[\kappa e^{2A}\Psi_-^{(1)}\right] = -2i\mu\kappa\,\mathrm{Im}\,\Psi_+^{(2)} \ , \tag{3.53}$$

$$\mathrm{d}\left[\kappa\,\mathrm{Im}\,\Psi_+^{(2)}\right] = k_2 H_3 - k_1 F_3 \ , \tag{3.54}$$

where we rewrite the last equation using (3.29) in the form

$$k_2 \equiv \kappa e^{2A-\phi/2}\,\mathrm{Im}\,\Psi_+^{(0)} + k_1 C_0 \ , \tag{3.55}$$

in order to eliminate $\mathrm{Im}\,\Psi_+^{(0)}$, where k_2 is constant as a result of the one-form component of (3.48). These equations are consistent with the $\mathrm{SL}(2,\mathbb{R})$ invariance of $\kappa\,\mathrm{Re}\,\Psi_+^{(0)}$, $\kappa\Psi_-^{(1)}$, and $\kappa\,\mathrm{Im}\,\Psi_+^{(2)}$, provided that we identify

$$Q^i = \begin{pmatrix} k_2 \\ k_1 \end{pmatrix} \ , \tag{3.56}$$

as an $\mathrm{SL}(2,\mathbb{R})$ doublet. Indeed, these same forms define calibrations for space-filling, domain-wall, and cosmic-string D3 branes respectively [56, 57], and therefore must be $\mathrm{SL}(2,\mathbb{R})$ invariant due to the $\mathrm{SL}(2,\mathbb{Z})$ invariance of the D3 brane, as classical supergravity does not distinguish between $\mathrm{SL}(2,\mathbb{R})$ and $\mathrm{SL}(2,\mathbb{Z})$.

As a further consistency check on this proposal, note that the pure spinors always satisfy $|\Psi_-^{(1)}|^2/2 + |\Psi_+^{(0)}|^2 = 1$.[17] Defining $\eta \equiv \kappa \operatorname{Re} \Psi_+^{(0)}$ and $\theta \equiv \kappa \Psi_-^{(1)}$, and using (3.55) to eliminate $\operatorname{Im} \Psi_+^{(0)}$, we obtain:

$$|\theta|^2/2 + \eta^2 = 1 - k_1^2 e^{-4A-\phi} - e^{-4A+\phi}\left(k_2 - k_1 C_0\right)^2 = 1 - |\chi|^2 \ , \tag{3.57}$$

where

$$\chi \equiv \frac{e^{-2A}}{\sqrt{\tau_2}}(k_2 - \tau k_1) = e^{-2A} t_i Q^i \ , \tag{3.58}$$

carries charge $+1/2$. Under these assumptions, (3.57) is manifestly covariant.

As a final consistency check, note that the $-1 \in \mathrm{SL}(2,\mathbb{Z})$ involution of an O3/O7 plane takes the form [44]:

$$\kappa\Psi_+ \to (-1)^{\hat{p}(\hat{p}-1)/2}\, \kappa\bar{\Psi}_+ \ \ , \ \ \kappa\Psi_- \to (-1)^{\hat{p}(\hat{p}-1)}\, \kappa\Psi_- \ . \tag{3.59}$$

Thus, $\kappa\Psi_-^{(3)}$, $\kappa \operatorname{Im}\Psi_+^{(0)}$, $\kappa \operatorname{Re}\Psi_+^{(2)}$, $\kappa \operatorname{Im}\Psi_+^{(4)}$, and $\kappa \operatorname{Re}\Psi_+^{(6)}$ change sign under $-1 \in \mathrm{SL}(2,\mathbb{Z})$, whereas the other components are invariant. Therefore, it is consistent to assume that these components consist of sums of half-integer charged terms, whereas the other components consist of sums of neutral and/or integer charged terms.[18] This is consistent with the $\mathrm{SL}(2,\mathbb{R})$ invariance of $\kappa \operatorname{Re}\Psi_+^{(0)}$, $\kappa\Psi_-^{(1)}$, and $\kappa \operatorname{Im}\Psi_+^{(2)}$.

We refer to solutions with $Q^i = 0$, whether Minkowski or AdS, as "chargeless,"[19] and those with $Q^i \neq 0$ as "charged," due to the presence of a globaly defined $\mathrm{SL}(2,\mathbb{R})$ doublet of constants. Strict SU(3)-structure solutions with supersymmetric five-branes (e.g. [84]) form a well known class of *charged* solutions,

[17] The one-form Θ in (3.17) is normalized so that $|\Theta|^2 = g^{-1}\left(\Theta, \bar{\Theta}\right) = 2$.

[18] One can verify that this is correct by using the known $\mathrm{SL}(2,\mathbb{R})$ transformation law for the supersymmetry generators ϵ^i [83]. I would like to thank P. Koerber for helpful discussions and correspondence on this point.

[19] Solutions of this type were termed "AdS-like" in chapter 2.

whereas strict SU(3)-structure solutions with supersymmetric three- and/or seven-branes (e.g. [51]) form another well-known class of *chargeless* solutions.

As we saw in section 3.2.2, AdS solutions are always chargeless. Moreover, calibrated space-filling D3 branes can only occur in chargeless solutions, as space-filling (anti-) D3 branes are calibrated where $\eta = +1$ ($\eta = -1$), which, by (3.57), implies $\chi = 0$. Additionally, the presence of a globally defined charge doublet restricts the allowable monodromies to D7 brane and O5 plane monodromies (or $SL(2,\mathbb{Z})$ conjugates of these, depending on the frame); this is inconsistent with the seven-brane configurations found in F-theory setups. For these reasons we focus on chargeless solutions in this chapter, for which, moreover, the supersymmetry conditions take a somewhat simpler form.

Solutions with $\eta = 1$, sometimes referred to as "F-theory solutions" since they arise from compactifications of F-theory on Calabi-Yau four-folds in the Sen limit, form a special well-studied class of examples. We review the supersymmetry conditions for this case in the next section, before moving on to consider general chargeless solutions.

3.3.4 F-theory solutions

By (3.57), $\eta^2 = 1$ implies $\chi = 0$ and $\theta = 0$. Thus, the pure spinors have types $(0, 3)$, and define an SU(3) structure

$$\Psi_+ = \eta e^{-iJ} \quad , \quad \Psi_- = i\Omega \; , \tag{3.60}$$

where Ω is a decomposable three-form[20] and J a nondegenerate real two-form such that

$$\frac{i}{8}\Omega \wedge \bar{\Omega} = \frac{1}{6}J^3 \neq 0 \quad , \quad J \wedge \Omega = 0 \; . \tag{3.61}$$

[20]The i in (3.60) is purely coventional.

The choices $\eta = \pm 1$ are related by charge conjugation, under which $\Psi_\pm \to -\Psi_\pm$. We consider the case $\eta = +1$. The pure spinor equations (3.47, 3.48, 3.49) reduce to

$$e^{4A} = \alpha \quad , \quad 0 = H_3 - e^{\phi} \star_6 \tilde{F}_3 \;\; , \;\; \mathrm{d}\left[-\frac{1}{2}e^{\phi}J^2\right] = e^{2\phi} \star F_1 \; , \tag{3.62}$$

$$\mathrm{d}J = 0 \quad , \quad 0 = J \wedge H_3 \;\; , \;\; \mathrm{d}[e^{\phi/2}\Omega] = 0 \;\; , \;\; 0 = \Omega \wedge H_3 \; . \tag{3.63}$$

The conditions involving three-form flux collectively imply that G_3 is primitive with Hodge type $(2,1)$. Imposing $\mathrm{d}J = 0$, the last equation of (3.62) implies that τ is holomorphic. These conditions, together with $e^{4A} = \alpha$, are manifestly $\mathrm{SL}(2,\mathbb{R})$ covariant if the complex structure associated to Ω is taken to be $\mathrm{SL}(2,\mathbb{R})$ invariant, which implies that J is also invariant. The third equation of (3.63) may be rewritten in the covariant form $\mathrm{D}\Omega = 0$, provided that we take Ω to carry charge $-1/2$ under $\mathrm{SL}(2,\mathbb{R})$, where the equivalence of this expression with (3.63) follows from the holomorphicity of τ. Thus, the conditions on chargeless SU(3) structure vacua, commonly known as "F-theory" solutions, may be written in the simple covariant form:

$$e^{4A} = \alpha \; , \; G_3 \wedge J = G_3 \wedge \Omega = 0 \; , \; \star G_3 = iG_3 \; , \; \mathrm{d}J = 0 \; , \; \frac{1}{\tau_2}\mathrm{d}\tau \wedge \Omega = 0 \; , \; \mathrm{D}\Omega = 0 \, , \tag{3.64}$$

where we take Ω to carry charge $-1/2$ and J to be neutral.

The supersymmetry conditions for $\eta = -1$ are similar:

$$e^{4A} = -\alpha \; , \; G_3 \wedge J = G_3 \wedge \Omega = 0 \; , \; \star G_3 = -iG_3 \; , \; \mathrm{d}J = 0 \; , \; \frac{1}{\tau_2}\mathrm{d}\tau \wedge \bar{\Omega} = 0 \; , \; \mathrm{D}\Omega = 0 \, , \tag{3.65}$$

except that we must now take Ω to carry charge $+1/2$, due to the fact that τ is now anti-holomorphic. We address this apparent discrepancy between the cases $\eta = \pm 1$ in the next section.

3.4 The chargeless supersymmetry conditions

Having classified $\mathcal{N} = 1$ flux vacua of type IIB supergravity into charged and chargeless backgrounds, we now consider general chargeless solutions. We show how to describe a general chargeless $\mathrm{SU}(3) \times \mathrm{SU}(3)$ structure in an $\mathrm{SL}(2,\mathbb{R})$ covariant fashion in section 3.4.1, and then derive covariant versions of the pure spinor equations for these solutions in section 3.4.2.

3.4.1 Chargeless $\mathrm{SU}(3) \times \mathrm{SU}(3)$ structure

We have shown that for $\eta = 1$, the Kähler form J associated to the SU(3) structure is $\mathrm{SL}(2,\mathbb{R})$ invariant, whereas the holomorphic three-form Ω carries charge $-1/2$. Deforming away from $\eta = 1$ slightly, the SU(3) structure decomposes into a local SU(2) structure as follows:

$$J = J_1 + J_2 \ \ , \ \ \Omega = \Omega_2 \wedge \Theta \, . \tag{3.66}$$

Since $\theta = \sqrt{1-\eta^2}\Theta$ by (3.17, 3.57), we conclude that Θ is $\mathrm{SL}(2,\mathbb{R})$ invariant, and therefore that J_2 is invariant and that Ω_2 carries charge $-1/2$. In order to preserve these charge assignments for arbitary η, we rewrite the ansatz (3.17) by performing an SO(3) rotation on the Ω^i:[21] [22]

$$\Psi_+ = e^{-iJ_1} \wedge \left[c_\varphi \left(1 - \frac{1}{2} J_2^2 \right) + s_\varphi \operatorname{Im} \Omega_2 - iJ_2 \right] \, , \tag{3.67}$$

$$\Psi_- = \Theta \wedge \left[s_\varphi \left(1 - \frac{1}{2} J_2^2 \right) - c_\varphi \operatorname{Im} \Omega_2 + i \operatorname{Re} \Omega_2 \right] \, , \tag{3.68}$$

so that $j \equiv -\operatorname{Im} \Psi_+^{(2)} = c_\varphi J_1 + J_2$ is manifestly invariant.

While (3.67, 3.68) completely specify how the pure spinors transform under $\mathrm{SL}(2,\mathbb{R})$ in the chargeless case, the SU(2) structure forms Θ, J_1, J_2 and Ω_2 need

[21] Recall that $e^{i\vartheta} = \pm 1$ for a chargeless solution, where the extra sign can be absorbed by redefinitions.

[22] A similar basis was used in [49].

not be globally defined if SU(3)-structure loci ($s_\varphi = 0$) are present. Instead, we consider the charge $-1/2$ forms

$$\omega \equiv s_\varphi \Omega_2 \ , \ \ \beta = \frac{1}{2}(1+c_\varphi)\Omega_2 \wedge \Theta \ , \ \ \gamma = \frac{1}{2}(1-c_\varphi)\Omega_2 \wedge \bar{\Theta} \ , \tag{3.69}$$

in addition to the invariant forms $\eta = c_\varphi$, $\theta = s_\varphi \Theta$ and $j = c_\varphi J_1 + J_2$ defined previously. All of these forms are globally defined up to $\mathrm{SL}_\pm(2,\mathbb{Z})$ monodromies, as they can be extracted from the pure spinors directly. In particular,

$$\begin{gathered}\eta = \operatorname{Re}\Psi_+^{(0)} \ , \ \theta = \Psi_-^{(1)} \ , \ j = -\operatorname{Im}\Psi_+^{(2)} \ , \ \operatorname{Im}\omega = \operatorname{Re}\Psi_+^{(2)} \ , \ \operatorname{Re}\omega = \iota_{\operatorname{Re}\theta}\operatorname{Im}\Psi_-^{(3)} \ , \\ \beta = \frac{1}{2i}(1+\eta)\Psi_-^{(3)} - \frac{1}{2i}\operatorname{Im}\omega \wedge \theta \ , \ \gamma^\star = \frac{1}{2i}(1-\eta)\Psi_-^{(3)} + \frac{1}{2i}\operatorname{Im}\omega \wedge \theta \ ,\end{gathered} \tag{3.70}$$

where the inner product is computed using the associated metric. The original pure spinors can be reconstructed using only these forms:

$$\Psi_+ = \eta + \operatorname{Im}\omega - ij - \frac{1}{2}\eta j^2 - j\wedge j_1 - i\operatorname{Im}\omega \wedge \tilde{j} + \frac{i}{6}j^3 \ , \ \ \Psi_- = \theta + i(\beta+\gamma^\star) - \frac{1}{2}j^2 \wedge \theta \ , \tag{3.71}$$

where $j_1 \equiv \frac{i}{2}\theta \wedge \bar{\theta} = s_\varphi^2 J_1$ and $\tilde{j} \equiv \eta j + j_1 = J_1 + c_\varphi J_2$.

We refer to the forms $\eta, \theta, j, \omega, \beta$ and γ as the (chargeless) SU(3) $\times$ SU(3) structure henceforward, since they are collectively equivalent to the chargeless pure spinors by (3.70) and (3.71). The compatibility and purity of $\Psi_\pm$ impose certain conditions on the SU(3) $\times$ SU(3) structure forms. These are readily derived by requiring that

$$\Theta = \frac{1}{\sqrt{1-\eta^2}}\theta \ , \ \ J_2 = \frac{1}{1-\eta^2} j_2 \ , \ \ \Omega_2 = \frac{1}{\sqrt{1-\eta^2}}\omega \ , \tag{3.72}$$

define an SU(2) structure for $\eta^2 < 1$, where $j_2 \equiv j - \eta \tilde{j} = s_\varphi^2 J_2$, and that

$$\Omega = \beta \ , \ \ J = j \ , \tag{3.73}$$

or

$$\Omega = \gamma^\star \ , \ \ J = -j \ , \tag{3.74}$$

define an SU(3) structure for $\eta = \pm 1$ respectively, where ω, θ vanish in the latter two cases, γ vanishes for $\eta = 1$, and β vanishes for $\eta = -1$. Writing out these conditions using (3.8, 3.61) and simplifying, we find that the SU(3) $\times$ SU(3) structure must satisfy:

$$\eta^2 \leqslant 1 \;\;,\;\; \frac{1}{2} j_1 \wedge j^2 + \frac{1}{6}\eta j^3 = \frac{i}{4(1+\eta^2)}\left[\beta \wedge \bar{\beta} - \gamma \wedge \bar{\gamma}\right] \neq 0 \;, \tag{3.75}$$

$$(1-\eta)\,\beta = \frac{1}{2}\omega \wedge \theta \;\;,\;\; (1+\eta)\,\gamma = \frac{1}{2}\omega \wedge \bar{\theta} \;, \tag{3.76}$$

$$\omega \wedge \omega = 0 \;\;,\;\; j \wedge \omega = \frac{i}{2}\eta\left[\theta \wedge \gamma - \bar{\theta} \wedge \beta\right] \;, \tag{3.77}$$

$$\frac{1}{2}\omega \wedge \bar{\omega} = j^2 - \tilde{j}^2 \;\;,\;\; j \wedge \beta = j \wedge \gamma = 0 \;, \tag{3.78}$$

as well as the requirement that both β and γ are decomposable and the topological condition that the associated metric is positive definite. These conditions are necessary and sufficient to define an SU(3) $\times$ SU(3) structure, and ensure in particular that $\omega = \theta = 0$ for $\eta = \pm 1$.

Referring to (3.70), we see that charge conjugation $\Psi_\pm \to -\Psi_\pm$ has the following action on the SU(3) $\times$ SU(3) structure:

$$\eta \to -\eta \;\;,\;\; \theta \to -\theta \;\;,\;\; j \to -j \;\;,\;\; \omega \to \omega^\star \;\;,\;\; \beta \to -\gamma^\star \;\;,\;\; \gamma \to -\beta^\star \;. \tag{3.79}$$

This explains the apparent discrepancy in the previous section where the holomorphic three-form carried opposite charge for $\eta = +1$ F-theory solutions and their charge conjugate $\eta = -1$ counterparts, as the holomorphic three-form is given by β (with $\gamma = 0$) in the first case, and $\gamma^\star$ (with $\beta = 0$) in the second, so that charge conjugation takes $\Psi_-^{(3)} \to -\Psi_-^{(3)}$ as expected.

3.4.2 The covariant "pure-spinor" equations

We now show how to rewrite the pure spinor equations (3.47, 3.48, 3.49) as covariant differential conditions on the chargeless SU(3) $\times$ SU(3) structure. Writing them out rank by rank using (3.71), we obtain:

$$\mathrm{d}\left[e^{4A}\eta\right] = \mathrm{d}\alpha - 3e^{2A}\,\mathrm{Re}\left[\bar{\mu}\theta\right] \quad , \quad \mathrm{d}\left[e^{2A}\theta\right] \;=\; 2i\mu j \quad , \quad \mathrm{d}j \;=\; 0 \; , \tag{3.80}$$

$$\mathrm{d}\left[e^{2A+\phi/2}\,\mathrm{Im}\,\omega\right] = \eta e^{4A}H_3 - e^{4A+\phi}\star\tilde{F}_3 + 3e^{\phi/2}\,\mathrm{Im}\left[\bar{\mu}(\beta+\bar{\gamma})\right] \; , \tag{3.81}$$

$$\mathrm{d}\left[ie^{\phi/2}\left(\beta+\bar{\gamma}\right)\right] = e^{2A}H_3\wedge\theta + 2i\mu e^{\phi/2-2A}\,\mathrm{Im}\,\omega\wedge\tilde{j} \; , \tag{3.82}$$

$$\mathrm{d}\left[e^{\phi}\left(\frac{1}{2}\eta j^2 + j\wedge j_1\right)\right] = -e^{2A+\phi/2}\,\mathrm{Im}\,\omega\wedge H_3 - e^{2\phi}\star F_1 - \frac{3}{2}e^{\phi-2A}j^2\wedge\mathrm{Re}\left[\bar{\mu}\theta\right] \; , \tag{3.83}$$

$$\mathrm{d}\left[e^{\phi/2-2A}\,\mathrm{Im}\,\omega\wedge\tilde{j}\right] = j\wedge H_3 \; , \tag{3.84}$$

$$\mathrm{d}\left[\frac{1}{2}e^{\phi-2A}j^2\wedge\theta\right] = ie^{\phi/2}(\beta+\gamma^{\star})\wedge H_3 + 2i\mu e^{\phi-4A}\frac{1}{6}j^3 \; . \tag{3.85}$$

The conditions (3.80) are already covariant. We decompose the conditions (3.81 – 3.85) into covariant pieces, introducing noncovariant undetermined currents, which we label as "separation forms." To accomplish this decomposition, we use the following replacements

$$e^{\phi/2}\tilde{F}_3 = \frac{1}{2}\left(G_3+G_3^{\star}\right) \quad , \quad e^{-\phi/2}H_3 = \frac{i}{2}\left(G_3-G_3^{\star}\right) \; , \tag{3.86}$$

$$e^{\phi}F_1 = \frac{1}{2\tau_2}\left(\mathrm{d}\tau+\mathrm{d}\tau^{\star}\right) \quad , \quad \mathrm{d}\phi = \frac{i}{2\tau_2}\left(\mathrm{d}\tau-\mathrm{d}\tau^{\star}\right) \; , \tag{3.87}$$

as well as the useful identities

$$\mathrm{Im}\left[\mathrm{D}_+\xi\right] = e^{-\phi/2}\mathrm{d}\left[e^{\phi/2}\,\mathrm{Im}\,\xi\right] \quad , \quad e^{-\phi/2}\mathrm{d}\left[e^{\phi/2}\xi\right] = \mathrm{D}\xi + \frac{i}{2\tau_2}\mathrm{d}\tau\wedge\xi \; , \tag{3.88}$$

for ξ of charge $-1/2$.

The three-form equation (3.81) decomposes into

$$\mathrm{D}_+\left[e^{2A}\omega\right] = \eta e^{4A}G_3^{\star} - ie^{4A}\star G_3^{\star} + 3\bar{\mu}\beta - 3\mu\gamma + \mathcal{I}_3 \; , \tag{3.89}$$

where $\mathcal{I}_3$ is a real separation form. The four-form equation (3.82) decomposes into

$$\mathrm{D}\beta - \frac{i}{2\tau_2}\mathrm{d}\tau^{\star}\wedge\gamma^{\star} = -\frac{1}{2}e^{2A}G_3^{\star}\wedge\theta - i\mu e^{-2A}\omega\wedge\tilde{j} + \mathcal{J}_4 \; , \tag{3.90}$$

$$\mathrm{D}\gamma - \frac{i}{2\tau_2}\mathrm{d}\tau^\star \wedge \beta^\star = \frac{1}{2}e^{2A}G_3^\star \wedge \bar{\theta} - i\bar{\mu}e^{-2A}\omega \wedge \tilde{j} - \mathcal{J}_4^\star \ , \tag{3.91}$$

where $\mathcal{J}_4$ is a complex separation form. The first five-form equation (3.83) decomposes into:

$$\star\frac{1}{\tau_2}\mathrm{d}\tau = -\frac{i}{\tau_2}\mathrm{d}\tau \wedge \left(\frac{1}{2}\eta j^2 + j \wedge j_1\right) + \frac{1}{2}e^{2A}\omega^\star \wedge G_3 + \mathcal{J}_5 \ , \tag{3.92}$$

$$\mathrm{d}\left[\frac{1}{2}\eta j^2 + j \wedge j_1\right] = -\frac{1}{2}e^{2A}\,\mathrm{Re}\left[\omega \wedge G_3\right] - \frac{3}{2}e^{-2A}j^2 \wedge \mathrm{Re}\left[\bar{\mu}\theta\right] - \mathrm{Re}\,\mathcal{J}_5 \ , \tag{3.93}$$

where $\mathcal{J}_5$ is a complex separation form. The second five-form equation (3.84) decomposes into:

$$\mathrm{D}_+\left[e^{-2A}\omega \wedge \tilde{j}\right] = j \wedge G_3^\star + \mathcal{I}_5 \ , \tag{3.94}$$

where $\mathcal{I}_5$ is a real separation form. Finally, the six-form equation (3.85) decomposes into:

$$\mathrm{d}\left[\frac{1}{2}e^{-2A}j^2 \wedge \theta\right] = \frac{1}{3}i\mu e^{-4A}j^3 - \frac{1}{2}\left(\beta \wedge G_3 - \gamma^\star \wedge G_3^\star\right) + \frac{1}{2}(\mathcal{J}_6 - \mathcal{K}_6) \ , \tag{3.95}$$

$$0 = \gamma^\star \wedge G_3 + \frac{i}{2\tau_2}e^{-2A}\mathrm{d}\tau \wedge j^2 \wedge \theta + \mathcal{J}_6 \ , \tag{3.96}$$

$$0 = \beta \wedge G_3^\star + \frac{i}{2\tau_2}e^{-2A}\mathrm{d}\tau^\star \wedge j^2 \wedge \theta + \mathcal{K}_6 \ , \tag{3.97}$$

where $\mathcal{J}_6$ and $\mathcal{K}_6$ are complex separation forms.

To show that the conditions (3.89 – 3.97) are covariant, it is sufficient to prove that all the separation forms $\mathcal{I}_3$, $\mathcal{J}_4$, $\mathcal{I}_5$, $\mathcal{J}_5$, $\mathcal{J}_6$, and $\mathcal{K}_6$ must vanish. The derivation is rather technical. We consider the cases $\eta^2 = 1$ and $\eta^2 < 1$ separately, either one of which must hold at any point of interest, regardless of whether either is true globally. In the former case we apply the Hodge and primitivity decompositions with respect to the local SU(3) structure (3.73) or (3.74), and in the latter we decompose with respect to the local SU(2) structure (3.72). In either case, applying the SU(3) × SU(3) stucture constraints (3.75 – 3.78) and the covariant conditions (3.80), one can show that all separation forms must vanish. This derivation is summarized in appendix 3.A.

We then obtain the explicitly covariant supersymmetry conditions:[23]

$$\mathrm{d}\left[e^{4A}\eta\right] = \mathrm{d}\alpha - 3e^{2A}\,\mathrm{Re}\left[\bar{\mu}\theta\right] \;\;,\;\; \mathrm{d}\left[e^{2A}\theta\right] = 2i\mu j \;\;,\;\; \mathrm{d}j = 0\;, \tag{3.98}$$

$$\mathrm{D}_+\left[e^{2A}\omega\right] = \eta e^{4A}G_3^\star - ie^{4A}\star G_3^\star + 3\bar{\mu}\beta - 3\mu\gamma\;, \tag{3.99}$$

$$\mathrm{D}\beta - \frac{i}{2\tau_2}\mathrm{d}\tau^\star\wedge\gamma^\star = -\frac{1}{2}e^{2A}G_3^\star\wedge\theta - i\mu e^{-2A}\omega\wedge\tilde{j}\;, \tag{3.100}$$

$$\mathrm{D}\gamma - \frac{i}{2\tau_2}\mathrm{d}\tau^\star\wedge\beta^\star = \frac{1}{2}e^{2A}G_3^\star\wedge\bar{\theta} - i\bar{\mu}e^{-2A}\omega\wedge\tilde{j}\;, \tag{3.101}$$

$$\mathrm{D}_+\left[e^{-2A}\omega\wedge\tilde{j}\right] = j\wedge G_3^\star\;, \tag{3.102}$$

$$\mathrm{d}\left[\frac{1}{2}e^{-2A}j^2\wedge\theta\right] = \frac{1}{3}i\mu e^{-4A}j^3 - \frac{1}{2}\left(\beta\wedge G_3 - \gamma^\star\wedge G_3^\star\right)\;, \tag{3.103}$$

as well as

$$\star\frac{1}{\tau_2}\mathrm{d}\tau = -\frac{i}{\tau_2}\mathrm{d}\tau\wedge\left(\frac{1}{2}\eta j^2 + j\wedge j_1\right) + \frac{1}{2}e^{2A}\omega^\star\wedge G_3\;, \tag{3.104}$$

$$\mathrm{d}\left[\frac{1}{2}\eta j^2 + j\wedge j_1\right] = -\frac{1}{2}e^{2A}\,\mathrm{Re}\left[\omega\wedge G_3\right] - \frac{3}{2}e^{-2A}j^2\wedge\mathrm{Re}\left[\bar{\mu}\theta\right]\;, \tag{3.105}$$

$$0 = \gamma^\star\wedge G_3 + \frac{i}{2\tau_2}e^{-2A}\mathrm{d}\tau\wedge j^2\wedge\theta\;, \tag{3.106}$$

$$0 = \beta\wedge G_3^\star + \frac{i}{2\tau_2}e^{-2A}\mathrm{d}\tau^\star\wedge j^2\wedge\theta\;. \tag{3.107}$$

In fact, (3.98 – 3.103), together with the algebraic constraints (3.75 – 3.78), imply the remaining equations, (3.104 – 3.107). This redundancy is not surprising, given that five noncovariant equations (3.81 – 3.85) have been shown to imply, taken together with (3.75 – 3.78, 3.80), nine covariant equations (3.99 – 3.107).

To establish this last result, we follow similar steps to those taken to derive the covariant conditions. We add arbitrary forms $\mathcal{J}_5$, $\mathcal{M}_5$, $\mathcal{J}_6$, and $\mathcal{K}_6$ respectively to (3.104 – 3.107) and then show that these forms must vanish upon imposing the other supersymmetry conditions. The math is now very similar to that used to derive the covariant conditions. In particular, for $\eta^2 < 1$, the same steps that led

[23]Recall that the forms η, θ, j, ω, β, and γ are equivalent to the chargeless pure spinors by (3.70, 3.71); we prefer them to the pure spinors due to their simpler transformation properties under $\mathrm{SL}(2,\mathbb{R})$.

to (3.137, 3.140) show that $\mathcal{J}_6 = \mathcal{K}_6 = 0$ as a consequence of (3.99, 3.100, 3.101). Similarly, the steps which led to (3.141) and (3.176, 3.180) show that $\mathcal{J}_5 = 0$, and those which led to (3.146) and (3.179) show that $\mathcal{M}_5 = 0$. The special case $\eta = \pm 1$ is readily verified.

Even so, the redundant conditions (3.104 – 3.107) are sometimes useful in computations.

3.5 Consistency Checks

Having established our main results, the distinction between charged and chargeless solutions, the $\mathrm{SL}(2,\mathbb{R})$ transformation properties of the chargeless pure spinors (3.71), and the $\mathrm{SL}(2,\mathbb{R})$-covariant supersymmetry conditions (3.98 – 3.103), we now perform a few additional computations as consistency checks on these results. In section 3.5.1, we review the supergravity Bianchi identities, and show that they imply the flux equations of motion upon imposition of the supersymmetry conditions, a known result which we are able to rederive relatively easily. In section 3.5.2, we show that the flux superpotential proposed in [26] is $\mathrm{SL}(2,\mathbb{R})$ invariant, a new result which presents a further consistency check on our calculation and on the proposed superpotential.

3.5.1 Equations of motion

In addition to the supersymmetry conditions, four-dimensional $\mathcal{N} = 1$ vacua must satisfy the supergravity Bianchi identities[24]

$$\mathrm{d}\left[e^{-8A} \star \mathrm{d}\alpha\right] = -\frac{i}{2} G_3 \wedge G_3^\star \quad , \quad \mathrm{D}_- G_3 = 0 \; , \tag{3.108}$$

[24]Our distinction between "Bianchi identities" and "equations of motion" is consistent with that made in [23]; the RR Bianchi identity $\mathrm{d}F_1 = 0$ is implicit in this formalism, since τ is specified directly.

and equations of motion:

$$\mathrm{d}\star\mathrm{d}A = \frac{1}{8}e^{4A}G_3\wedge\star G_3^\star + \frac{1}{4}e^{-8A}\mathrm{d}\alpha\wedge\star\mathrm{d}\alpha + \Lambda e^{-4A}\Omega_6\ , \tag{3.109}$$

$$\mathrm{D}_+\left[e^{4A}\star G_3\right] = i\mathrm{d}\alpha\wedge G_3\ \ ,\ \ \mathrm{D}\left[\star\frac{1}{\tau_2}\mathrm{d}\tau\right] = -\frac{i}{2}e^{4A}G_3\wedge\star G_3\ , \tag{3.110}$$

$$\begin{aligned} R_{mn} = 8\nabla_m A\nabla_n A - \frac{1}{2}e^{-8A}\nabla_m\alpha\nabla_n\alpha + \frac{1}{4\tau_2^2}\left[\nabla_m\tau\nabla_n\bar{\tau} + c.c.\right] \\ + \frac{1}{2}e^{4A}\hat{T}_{mn} - \Lambda e^{-4A}g_{mn}\ , \end{aligned} \tag{3.111}$$

where $\Lambda = \mathcal{R}_{(4)}/4$ is the four-dimensional cosmological constant, R_{mn} is the Ricci tensor formed from the unwarped metric g_{mn}, contractions and Hodge duals are formed using g_{mn}, and

$$\hat{T}^m_n = \frac{1}{4}\left(G^{mpq}\bar{G}_{npq} + \bar{G}^{mpq}G_{npq}\right) - \frac{1}{12}\bar{G}^{pqr}G_{pqr}\delta^m_n\ . \tag{3.112}$$

Fortunately, one can show that the supersymmetry conditions, combined with the Bianchi identities (3.108), imply the remaining equations of motion (3.109, 3.110, 3.111) [85]. This result can be extended to include calibrated D-branes, wherein the bulk supersymmetry conditions and Bianchi identities, together with the calibration equations, imply all remaining bulk and brane equations of motion [46].

For completeness, we partially reproduce this result for the chargeless solutions considered here. While we do not impose calibration conditions, we do not exclude sources explicitly, and do not impose the source-free Bianchi identities in the following derivation, apart from the Bianchi identity $\mathrm{D}\left[\frac{1}{\tau_2}\mathrm{d}\tau\right] = 0$.[25]

Applying D_+ to (3.99) and simplifying using (3.100, 3.101, 3.98), we find:

$$\mathrm{D}_+\left[e^{4A}\star G_3\right] = i\mathrm{d}\alpha\wedge G_3 + i\eta e^{4A}\mathrm{D}_-G_3 \tag{3.113}$$

[25] Violating the Bianchi identity $\mathrm{D}\left[\frac{1}{\tau_2}\mathrm{d}\tau\right] = 0$ in an $\mathrm{SL}(2,\mathbb{Z})$ covariant formalism requires $\mathrm{SL}(2,\mathbb{R})$ to be gauged and spontaneously broken to $\mathrm{SL}(2,\mathbb{Z})$, which is beyond the scope of this work. This gauging is necessary in the vicinity of a seven-brane, due to the topological defect caused by the monodromy.

Thus, in the absence of sources, the G_3 Bianchi identity implies the G_3 equation of motion. Applying D to (3.104) and simplifying using (3.105, 3.99, 3.106, 3.107), we find:

$$\mathrm{D}\left[\star\frac{1}{\tau_2}\mathrm{d}\tau\right] = -\frac{i}{2}e^{4A}G_3\wedge\star G_3 + \frac{1}{2}e^{2A}\omega^\star\wedge\mathrm{D}_-G_3 \tag{3.114}$$

Thus, in the source-free case, the axodilaton equation of motion also follows from the G_3 Bianchi identity.

To obtain the warp-factor equation of motion from the Bianchi identities, we use the identity

$$0 = \frac{1}{2}e^{-6A}\star\mathrm{d}\left[(1-\eta^2)e^{8A}\right] - \frac{1}{2}e^{4A}\,\mathrm{Im}\left[\omega\wedge G_3\right] - \frac{3}{2}\eta j^2\wedge\mathrm{Im}\left[\bar{\mu}\theta\right] \tag{3.115}$$

which can be shown to follow from the supersymmetry conditions. Combining this with ηe^{-4A} times the Hodge star of (3.98), we obtain:

$$0 = e^{-4A}\star\mathrm{d}\left[e^{4A}\right] - \frac{1}{2}e^{2A}\,\mathrm{Im}\left[\omega\wedge G_3\right] - \eta e^{-4A}\star\mathrm{d}\alpha \tag{3.116}$$

Taking the exterior derivative of this equation, and applying (3.99, 3.105, 3.103), we obtain:

$$\begin{aligned} 0 = 4\mathrm{d}\star\mathrm{d}A &- \frac{1}{2}e^{4A}G_3\wedge\star G_3^\star - e^{-8A}\mathrm{d}\alpha\wedge\star\mathrm{d}\alpha + 12|\mu|^2e^{-4A}\left[\frac{1}{6}\eta j^3 + \frac{1}{2}j^2\wedge j_1\right] \\ &- \eta e^{4A}\left[\mathrm{d}\left[e^{-8A}\star\mathrm{d}\alpha\right] + \frac{i}{2}G_3\wedge G_3^\star\right] - \frac{1}{2}e^{2A}\,\mathrm{Im}\left[\omega\wedge\mathrm{D}_-G_3\right] \end{aligned} \tag{3.117}$$

The first line is the source-free A equation of motion with cosmological constant $\Lambda = -3|\mu|^2$. Thus, this too follows from the Bianchi identities in the absence of sources.

The chargeless supersymmetry conditions also impose constraints upon D_-G_3 itself. Applying D to (3.100, 3.101) and simplifying using (3.102), and applying D_+ to (3.102) itself, we find:

$$\mathrm{D}_-G_3\wedge\theta = \mathrm{D}_-G_3\wedge\bar{\theta} = \mathrm{D}_-G_3\wedge j = 0 \tag{3.118}$$

Moreover, taking the exterior derivative of (3.105) and simplifying using (3.99, 3.103, 3.104), we obtain:

$$\mathrm{Re}\,[\omega \wedge \mathrm{D}_{-} G_3] = 0 \tag{3.119}$$

These equations constrain the form of possible source terms consistent with $\mathcal{N} = 1$ supersymmetry, though such questions are more thoroughly addressed by the study of D-brane calibrations [56, 57, 66].

While it is possible to derive the Einstein equations (3.111) from the supersymmetry conditions and Bianchi identities [85], we do not attempt to reproduce such a computation in this context.

3.5.2 The flux superpotential

The off-shell flux superpotential[26]

$$W = \frac{1}{4\kappa_{10}^2} \int \left\langle e^{3A^{(S)}-\phi} \Psi_{-}^{(S)}, F + i\,\mathrm{d}_H \left[e^{-\phi}\, \mathrm{Re}\, \Psi_{+}^{(S)} \right] \right\rangle \ , \tag{3.120}$$

has been proposed [26, 86] as an appropriate generalization of the well-known Gukov-Vafa-Witten superpotential [54, 87] to general $\mathrm{SU}(3) \times \mathrm{SU}(3)$ structure compactifications with equal spinor norms ($k_1 = 0$). Since $\mathrm{SL}_{\pm}(2,\mathbb{Z})$ is an exact gauged symmetry of string theory, W must be $\mathrm{SL}_{\pm}(2,\mathbb{Z})$ invariant. As usual, we expect that this is enhanced to $\mathrm{SL}_{\pm}(2,\mathbb{R})$ invariance in tree-level supergravity, so that the integrand of (3.120) must be neutral under $\mathrm{SL}(2,\mathbb{R})$.

We now verify that this is the case for chargeless solutions. Applying the chargeless ansatz (3.71) to (3.120) and simplifying the integrand using (3.75 – 3.78), we obtain:

$$W = \frac{i}{4\kappa_{10}^2} \int \left(G_3 \wedge \beta + G_3^\star \wedge \gamma^\star + \frac{1}{2} \beta \wedge \mathrm{D}[e^{-2A} \omega^\star] - \frac{1}{2} \gamma^\star \wedge \mathrm{D}[e^{-2A} \omega] \right)$$

[26] Specifically, this is the superpotential of four-dimensional "Weyl-invariant supergravity," as explained in detail in [26]. The more-standard Einstein supergravity superpotential is then $W_E = e^{-K/2} W$, where K is the Kähler potential. I thank P. Cámara for discussions on this point.

$$+\frac{1}{4\kappa_{10}^2}\int\Big(e^{2A}\,\theta\wedge\tilde{F}_5 - i\,\theta\wedge j\wedge \mathrm{d}[\eta e^{-2A}j] - i\,e^{-2A}j\wedge j_1\wedge \mathrm{d}\theta\Big)\ . \tag{3.121}$$

Comparing with (3.79), we see that the superpotential is $\mathrm{SL}_{\pm}(2,\mathbb{R})$ invariant. Moreover, for $\eta = 1$, it truncates to the familiar Gukov-Vafa-Witten result:

$$W = \frac{i}{4\kappa_{10}^2}\int G_3\wedge\Omega\ , \tag{3.122}$$

where $\Omega = \beta$ is the holomorphic three-form.

Applying the supersymmetry conditions (3.98 – 3.107) to (3.121) and simplifying the integrand, we find

$$\langle W\rangle = \frac{\mu}{\kappa_{10}^2}\int \mathrm{d}^6y\sqrt{g}e^{-4A} = \frac{\mu}{\kappa_4^2}\ , \tag{3.123}$$

for a supersymmetric vacuum, where κ_4^2 is the four-dimensional Newton constant. This is consistent with the supergravity result $\Lambda = -3\kappa_4^4|W|^2 = -3|\mu|^2$.

The $\mathrm{SL}(2,\mathbb{R})$ covariance of (3.121) is a highly non-trivial consistency check on the proposed superpotential (3.120), as the latter was developed using D-brane and Euclidean D-brane physics [26] without imposing $\mathrm{SL}(2,\mathbb{Z})$ invariance.

3.6 Conclusions

We have shown that geometric $\mathcal{N} = 1$ vacua of type IIB string theory fall into two classes, which we label chargeless and charged solutions. Chargeless solutions are particularly interesting from the perspective of F-theory, as they allow in-principle arbitrary combinations of $\mathrm{SL}(2,\mathbb{Z})$ monodromies. We have derived simple algebraic (3.75 – 3.78) and differential (3.98 – 3.103) conditions for chargeless supersymmetric solutions which are manifestly $\mathrm{SL}(2,\mathbb{R})$ covariant. Together with the Bianchi identities (3.108), these are necessary and sufficient conditions for chargeless supersymmetry. The success of this endeavor is a non-trivial consistency check on the pure-spinor equations of [23], which do not make the $\mathrm{SL}(2,\mathbb{R})$ invariance of the theory manifest.

We have also demonstrated that the flux superpotential proposed in [26] is $\mathrm{SL}_{\pm}(2,\mathbb{R})$ invariant for chargeless $\mathrm{SU}(3)\times\mathrm{SU}(3)$ structure, obtaining the covariant expression (3.121).

The formalism presented here should prove useful to the study of generalized F-theory solutions, where $\mathrm{SL}(2,\mathbb{Z})$ covariance plays an essential role. It also provides a useful alternative perspective on previous approaches to the classification of $\mathcal{N}=1$ vacua using generalized complex geometry.

One might hope to extend these methods to charged solutions. Indeed, in the case of strict SU(3) structure, the calculation is relatively straightforward, and results in a clean restatement of the supersymmetry conditions on an already well-studied class of vacua. There are indications that the $\mathrm{SL}(2,\mathbb{R})$-covariant supersymmetry conditions on general charged vacua should be relatively simple, but an explicit derivation of these conditions is hampered by the difficulty in determining the $\mathrm{SL}(2,\mathbb{R})$ transformation properties of the pure spinors, since the considerations of §3.4.1 no longer apply. A more direct approach using the known $\mathrm{SL}(2,\mathbb{R})$ transformation law for the supersymmetry generators ϵ^i may be indicated. We return to these questions in a future work [83].

3.A Derivation of the chargeless SUSY conditions

In section 3.4.2, we showed that the pure spinor equations (3.47, 3.48, 3.49) can be rewritten in the form (3.80, 3.89 – 3.97) for arbitrary real separation forms $\mathcal{I}_3$ and $\mathcal{I}_5$ and complex separation forms $\mathcal{J}_4$, $\mathcal{J}_5$, $\mathcal{J}_6$ and $\mathcal{K}_6$. We now show that these separation forms all vanish, proving that the supersymmetry conditions are covariant.

We consider the cases $\eta^2 = 1$ and $\eta^2 < 1$ separately, in §3.A.1 and §3.A.2

respectively.

3.A.1 SU(3) structure loci ($\eta = \pm 1$)

We first prove that the separation forms vanish at a locus where $\eta^2 = 1$. We consider the case $\eta = +1$ ($\eta = -1$ is related to this by charge conjugation). Note that the SU(3) × SU(3) structure constraints (3.75 – 3.78) and (3.80) imply the conditions:

$$\mathrm{d}\theta = 2i\mu e^{-2A} j \ , \quad \mathrm{d}\eta = 0 \quad , \ \mathrm{d}\gamma = 0 \ , \tag{3.124}$$

$$[\mathrm{d}\omega]_{(0,3)} = [\mathrm{d}\omega]_{(1,2)} = 0 \ , \ j \wedge \mathrm{d}\omega = 0 \ , \ \mathrm{d}\omega \wedge \beta^\star = -4i\bar{\mu}e^{-2A} j^3 \ , \tag{3.125}$$

where $\mathrm{D}\omega = \mathrm{D}_+\omega = \mathrm{d}\omega$, since $\omega = 0$, so that the connection terms vanish, and the Hodge decomposition is taken with respect to the locally defined almost complex structure β. This complex structure need not be integrable. However, the $(1,2)$ component of $\mathrm{d}\omega$ must still vanish, since ω is a $(2,0)$ form which vanishes where $\eta = 1$, so that $\omega \wedge f_{(2,1)} = 0$ for any $(2,1)$ form f. Taking the exterior derivative and imposing $\omega = 0$, we recover $[\mathrm{d}\omega]_{(1,2)} = 0$ since f is arbitrary.

Written out, (3.89 – 3.97) reduce to:

$$e^{2A}\mathrm{d}\omega = e^{4A}[G_3^\star - i \star G_3^\star] + 3\bar{\mu}\beta + \mathcal{I}_3 \ , \ \mathrm{D}\beta = \mathcal{J}_4 \ , \ \frac{i}{2\tau_2}\mathrm{d}\tau^\star \wedge \beta^\star = \mathcal{J}_4^\star \ , \tag{3.126}$$

$$\star\frac{1}{\tau_2}\mathrm{d}\tau = -\frac{i}{2\tau_2}\mathrm{d}\tau \wedge j^2 + \mathcal{J}_5 \ , \ \mathrm{Re}\,\mathcal{J}_5 = 0 \ , \ j \wedge G_3^\star + \mathcal{I}_5 = 0 \ , \tag{3.127}$$

$$0 = \frac{2}{3}i\mu e^{-4A} j^3 + \frac{1}{2}\beta \wedge G_3 - \frac{1}{2}(\mathcal{J}_6 - \mathcal{K}_6) \ , \ \mathcal{J}_6 = 0 \ , \ \beta \wedge G_3^\star + \mathcal{K}_6 = 0 \ . \tag{3.128}$$

Wedging $\beta^\star$ into the first equation of (3.126) and simplifying, we find $\mathcal{I}_3 \wedge \beta^\star = 0$. Therefore, since $\mathcal{I}_3$ is real, $\mathcal{I}_3 = (2,1) \oplus (1,2)$. However, the rest of equation only has $(3,0) \oplus (2,1)^{\mathrm{P}} \oplus (1,2)^{\mathrm{NP}} \oplus (0,3)$ components.[27] Since $\mathcal{I}_3$ is real, this implies that it must vanish, and therefore in particular $G^{\mathrm{NP}}_{(2,1)} = G_{(3,0)} = 0$. Thus, $\mathcal{K}_6 = 0$ and $\mathcal{I}_5 = j \wedge (2,1)^{\mathrm{NP}}$ and is therefore vanishing, since it is real.

[27] The superscripts P and NP denote primitive and non-primitive components.

Writing out the Hodge star in the first equation of (3.127), we find:

$$\frac{i}{2\tau_2} j^2 \wedge \bar{\partial}\tau = \mathcal{J}_5 \tag{3.129}$$

However, since $\mathcal{J}_5$ is imaginary, we conclude that it must vanish, and therefore $\bar{\partial}\tau = 0$. Applying this to (3.126), we find $\mathcal{J}_4 = 0$. Thus, all the separation forms must vanish at a locus where $\eta = 1$. A similar argument applies to the case $\eta = -1$.

3.A.2 Local SU(2) structure ($\eta^2 < 1$)

Now consider a point where $\eta^2 < 1$; we can define a local SU(2) structure J_2, Ω_2, and Θ via (3.72). Using this SU(2) structure, we can decompose an arbitrary forms according to their θ and $\bar{\theta}$ fiber components, as well as their Hodge type and (for $(1,1)$ forms) their primitivity along the base. Thus, for instance, an arbitrary three-form decomposes as

$$\begin{aligned} M_3 = {} & M_{\theta\bar{\theta};1,0} + M_{\theta\bar{\theta};0,1} + M_{;2,1} + M_{;1,2} + M_{\theta;2,0} + M_{\theta;0,2} + M_{\bar{\theta};2,0} + M_{\bar{\theta};0,2} \\ & + M_{\theta;(1,1)^{\mathrm{P}}} + M_{\bar{\theta};(1,1)^{\mathrm{P}}} + M_{\theta;(1,1)^{\mathrm{NP}}} + M_{\bar{\theta};(1,1)^{\mathrm{NP}}} \,, \end{aligned} \tag{3.130}$$

and an arbitrary four-form as

$$N_4 = N_{;2,2} + N_{\theta\bar{\theta};(1,1)^{\mathrm{NP}}} + N_{\theta\bar{\theta};(1,1)^{\mathrm{P}}} + N_{\theta\bar{\theta};2,0} + N_{\theta\bar{\theta};0,2} + N_{\theta;2,1} + N_{\theta;1,2} + N_{\bar{\theta};2,1} + N_{\bar{\theta};1,2} \,. \tag{3.131}$$

Note the use of the semicolon to distinguish this from an ordinary Hodge decomposition; e.g. $M_{;1,2} \neq M_{(1,2)}$, since the former has legs along the base only. Many of these components can be written as scalars times the SU(2) structure forms, for instance $M_{\theta;(1,1)^{\mathrm{NP}}} \propto \Theta \wedge J_2$ and $N_{;2,2} \propto J_2^2$, etc.

We use these decompositions to show that the separation forms vanish. To begin with, we consider the $\theta \wedge (1,1)^{\mathrm{P}}$ and $\bar{\theta} \wedge (1,1)^{\mathrm{P}}$ components of (3.89), along with the $J_1 \wedge (1,1)^{\mathrm{P}}$ components of (3.90, 3.91):

$$e^{2A} \left[\mathrm{D}\omega\right]_{\theta;(1,1)^{\mathrm{P}}} = (1+\eta) e^{4A} \left[G_{\bar{\theta};(1,1)^{\mathrm{P}}}\right]^{\star} + \mathcal{I}_{\theta;(1,1)^{\mathrm{P}}} \,, \tag{3.132}$$

$$e^{2A}\,[\mathrm{D}\omega]_{\bar\theta;(1,1)^{\mathrm{P}}} = -(1-\eta)e^{4A}\left[G_{\theta;(1,1)^{\mathrm{P}}}\right]^{\star} + \mathcal{I}_{\bar\theta;(1,1)^{\mathrm{P}}}\ , \tag{3.133}$$

$$[\mathrm{D}\omega]_{\bar\theta;(1,1)^{\mathrm{P}}}\wedge\theta = -e^{2A}(1-\eta)\left[G_{\theta;(1,1)^{\mathrm{P}}}\right]^{\star}\wedge\theta + 2(1-\eta)\mathcal{J}_{\theta\bar\theta;(1,1)^{\mathrm{P}}}\ , \tag{3.134}$$

$$[\mathrm{D}\omega]_{\theta;(1,1)^{\mathrm{P}}}\wedge\bar\theta = e^{2A}(1+\eta)\left[G_{\bar\theta;(1,1)^{\mathrm{P}}}\right]^{\star}\wedge\bar\theta - 2(1+\eta)\left[\mathcal{J}_{\theta\bar\theta;(1,1)^{\mathrm{P}}}\right]^{\star}\ . \tag{3.135}$$

Wedging $\bar\theta$ and θ into (3.132) and (3.133) respectively, and combining them with (3.135) and (3.134) to eliminate G_3, we find

$$0 = e^{-2A}\mathcal{I}_{\theta;(1,1)^{\mathrm{P}}}\wedge\bar\theta + 2(1+\eta)\left[\mathcal{J}_{\theta\bar\theta;(1,1)^{\mathrm{P}}}\right]^{\star}\ \ ,\ \ 0 = e^{-2A}\mathcal{I}_{\bar\theta;(1,1)^{\mathrm{P}}}\wedge\theta - 2(1-\eta)\mathcal{J}_{\theta\bar\theta;(1,1)^{\mathrm{P}}}\ . \tag{3.136}$$

Using the reality of $\mathcal{I}_3$, we deduce that $\mathcal{I}_{\theta;(1,1)^{\mathrm{P}}} = \mathcal{I}_{\bar\theta;(1,1)^{\mathrm{P}}} = \mathcal{J}_{\theta\bar\theta;(1,1)^{\mathrm{P}}} = 0$.

We extract further components of the separation forms by wedging them into various of the SU(2) structure forms. Wedging $j\wedge\theta$ and $j\wedge\bar\theta$ into (3.89) and combining with θ and $\bar\theta$ wedged into (3.94), we find that $\mathcal{I}_5\wedge\theta = \mathcal{I}_3\wedge j\wedge\theta = 0$, as well as $G_3\wedge j\wedge\theta = G_3\wedge j\wedge\bar\theta = 0$. Wedging j and θ into (3.90), we obtain $\mathcal{J}_4\wedge\theta = \mathcal{J}_4\wedge j = 0$.

Now consider β and γ wedged into (3.89). Integrating by parts, applying (3.90, 3.91), and using (3.96, 3.97) to eliminate G_3, we obtain:

$$\mathcal{I}_3\wedge\beta + e^{2A}\mathcal{J}_4\wedge\omega + 2\eta e^{4A}\mathcal{K}_6 = 0\ \ ,\ \ \mathcal{I}_3\wedge\gamma - e^{2A}\mathcal{J}_4^{\star}\wedge\omega + 2\eta e^{4A}\mathcal{J}_6^{\star} = 0\ , \tag{3.137}$$

where we make use of the identities

$$\omega^{\star}\wedge\beta = (1+\eta)j^2\wedge\theta\ \ ,\ \ \omega\wedge\gamma^{\star} = (1-\eta)j^2\wedge\theta\ , \tag{3.138}$$

and

$$\star\,\theta = -\frac{i}{2}j^2\wedge\theta\ \ ,\ \ \star\beta = -i\beta\ \ ,\ \ \star\gamma = i\gamma\ \ ,\ \ \star(j\wedge\theta) = -ij\wedge\theta \tag{3.139}$$

We also consider $\frac{1}{2}\omega\wedge\theta$ and $\frac{1}{2}\omega\wedge\bar\theta$ wedged into (3.89). Integrating by parts and using (3.96, 3.97) to eliminate G_3, we obtain:

$$\mathcal{I}_3\wedge\beta + (1+\eta)e^{4A}\mathcal{K}_6 = 0\ \ ,\ \ \mathcal{I}_3\wedge\gamma - (1-\eta)e^{4A}\mathcal{J}_6^{\star} = 0 \tag{3.140}$$

where we cancel an overall factor of $(1-\eta)$ from the first equation and $(1+\eta)$ from the second; these equations still hold in the special case $\eta = \pm 1$, since they then follow from (3.137).

Wedging θ and $\bar{\theta}$ into (3.92) and using (3.96, 3.97) to eliminate G_3 as before, we obtain:

$$\mathcal{J}_5 \wedge \theta + (1+\eta) e^{2A} \mathcal{J}_6 = 0 \ , \quad \mathcal{J}_5 \wedge \bar{\theta} + (1-\eta) e^{2A} \mathcal{K}_6^\star = 0 \ . \tag{3.141}$$

Next, consider (3.93) wedged into θ:

$$0 = -i\mu j^2 \wedge j_1 + e^{2A} \mathrm{d}\eta \wedge j^2 \wedge \theta + e^{4A}(1-\eta) G_3 \wedge \beta + e^{4A}(1+\eta) G_3^\star \wedge \gamma^\star + 2e^{2A} \operatorname{Re} \mathcal{J}_5 \wedge \theta \ . \tag{3.142}$$

We compare this with the wedge product of $\beta^\star$ and $\gamma^\star$ with (3.89). Integrating by parts, applying (3.90, 3.91), and simplifying, we obtain:

$$\begin{aligned} &-i\bar{\mu} j^2 \wedge j_1 - e^{2A} \mathrm{d}\eta \wedge j^2 \wedge \bar{\theta} - (1+\eta) e^{4A} G_3 \wedge \gamma \\ &\qquad - (1-\eta) e^{4A} G_3^\star \wedge \beta^\star + e^{2A} \omega \wedge \mathcal{J}_4^\star + \mathcal{I}_3 \wedge \beta^\star = 0 \,, \end{aligned} \tag{3.143}$$

$$\begin{aligned} &-i\mu j^2 \wedge j_1 + e^{2A} \mathrm{d}\eta \wedge j^2 \wedge \theta + (1-\eta) e^{4A} G_3 \wedge \beta \\ &\qquad + (1+\eta) e^{4A} G_3^\star \wedge \gamma^\star - e^{2A} \omega \wedge \mathcal{J}_4 + \mathcal{I}_3 \wedge \gamma^\star = 0 \ , \end{aligned} \tag{3.144}$$

where we use

$$2(1+\eta) j^3 + \omega \wedge \omega^\star \wedge \tilde{j} - 3i\beta \wedge \beta^\star = -2(1-\eta) j^3 + \omega \wedge \omega^\star \wedge \tilde{j} + 3i\gamma \wedge \gamma^\star = -j^2 \wedge j_1 \ , \tag{3.145}$$

which can be verified a number of different ways. Thus, we find:

$$\mathcal{I}_3 \wedge \beta^\star + e^{2A} \omega \wedge \mathcal{J}_4^\star + 2e^{2A} \operatorname{Re} \mathcal{J}_5 \wedge \bar{\theta} = 0 \ , \quad \mathcal{I}_3 \wedge \gamma^\star - e^{2A} \omega \wedge \mathcal{J}_4 - 2e^{2A} \operatorname{Re} \mathcal{J}_5 \wedge \theta = 0 \ . \tag{3.146}$$

Finally, consider $\frac{1}{2}\omega^\star \wedge \theta$ wedged into (3.89). Integrating by parts and simplifying, we obtain:

$$0 = 2i\mu\eta j^2 \wedge j_1 - e^{-2A}\mathrm{d}\left[e^{4A}(1-\eta^2)\right] \wedge j^2 \wedge \theta + (1+\eta)^2 e^{4A} G_3^\star \wedge \gamma^\star$$
$$- (1-\eta)^2 e^{4A} G_3 \wedge \beta + (1+\eta)\mathcal{I}_3 \wedge \gamma^\star + (1-\eta)\mathcal{I}_3 \wedge \beta \ , \quad (3.147)$$

where we use

$$2(1-\eta^2)j^3 = 3(1+\eta)i\gamma \wedge \gamma^\star + 3(1-\eta)i\beta \wedge \beta^\star \ , \quad \omega \wedge \omega^\star \wedge j = 2\eta j^2 \wedge j_1 \ . \quad (3.148)$$

To simplify the above expression, we employ (3.95), written in the form:

$$0 = -\frac{2}{3}i\mu j^3 + \frac{1}{2}e^{-2A}\mathrm{d}e^{4A} \wedge j^2 \wedge \theta - \frac{1}{2}e^{4A}\left(\beta \wedge G_3 - \gamma^\star \wedge G_3^\star\right) + \frac{1}{2}e^{4A}(\mathcal{J}_6 - \mathcal{K}_6) \ , \quad (3.149)$$

as well as (3.142). We find

$$0 = (1+\eta)\mathcal{I}_3 \wedge \gamma^\star + (1-\eta)\mathcal{I}_3 \wedge \beta - 4\eta e^{2A}\,\mathrm{Re}\,\mathcal{J}_5 \wedge \theta + (1-\eta^2)e^{4A}(\mathcal{J}_6 - \mathcal{K}_6) \ . \quad (3.150)$$

Equations (3.137, 3.140, 3.141, 3.146, 3.150) constitute nine conditions on the eight variables $\mathcal{I}_3 \wedge \beta$, $\mathcal{I}_3 \wedge \gamma$, $\mathcal{J}_4 \wedge \omega$, $\mathcal{J}_4 \wedge \omega^\star$, $\mathcal{J}_6$, $\mathcal{K}_6$, $\mathcal{J}_5 \wedge \theta$, and $\mathcal{J}_5 \wedge \bar{\theta}$. Thus, one might expect that we can solve for all eight variables. Indeed this can be done, even without (3.150); it is straightforward to check that all of them must vanish:

$$\mathcal{I}_3 \wedge \beta \ = \ \mathcal{I}_3 \wedge \gamma \ = \ 0 \ , \quad \mathcal{J}_4 \wedge \omega \ = \ \mathcal{J}_4 \wedge \omega^\star \ = \ 0 \ ,$$
$$\mathcal{J}_6 \ = \ \mathcal{K}_6 \ = \ 0 \ , \quad \mathcal{J}_5 \wedge \theta \ = \ \mathcal{J}_5 \wedge \bar{\theta} \ = \ 0 \ . \quad (3.151)$$

Together with the constraints derived previously, this implies that $\mathcal{I}_3$, $\mathcal{J}_4$, $\mathcal{I}_5$ and $\mathcal{J}_5$ take the form:

$$\mathcal{I}_3 = \mathcal{I}_{\theta\bar{\theta};1,0} + \mathcal{I}_{;2,1} + c.c. \ , \quad \mathcal{J}_4 = \mathcal{J}_{\theta;2,1} + \mathcal{J}_{\theta;1,2} \ ,$$
$$\mathcal{I}_5 = \mathcal{I}_{\theta\bar{\theta};2,1} + c.c. \ , \quad \mathcal{J}_5 = \mathcal{J}_{\theta\bar{\theta};2,1} + \mathcal{J}_{\theta\bar{\theta};1,2} \ . \quad (3.152)$$

To extract the relevant components of (3.90, 3.91), we wedge them into $\bar{\theta}$ and θ respectively and apply (3.94) to obtain:

$$0 = -e^{-4A}\mathrm{d}\left[e^{4A}(1+\eta)\right] \wedge \omega \wedge \tilde{j} - e^{2A}\left[j + \tilde{j}\right] \wedge G_3^\star + \frac{i}{\tau_2}\mathrm{d}\tau^\star \wedge \omega^\star \wedge \tilde{j}$$

$$+ i\mathcal{J}_4 \wedge \bar{\theta} - e^{2A}(1+\eta)\mathcal{I}_5 \ , \tag{3.153}$$

$$0 = -e^{-4A}\mathrm{d}\left[e^{4A}(1-\eta)\right] \wedge \omega \wedge \tilde{j} - e^{2A}\left[j - \tilde{j}\right] \wedge G_3^\star + \frac{i}{\tau_2}\mathrm{d}\tau^\star \wedge \omega^\star \wedge \tilde{j}$$

$$+ i\mathcal{J}_4^\star \wedge \theta - e^{2A}(1-\eta)\mathcal{I}_5 \ . \tag{3.154}$$

Similarly, to extract the relevant components of (3.89), we wedge it into j and j_1 and simplify using (3.94) to obtain:

$$0 = -e^{-2A}\mathrm{d}\left[e^{4A}\eta\right] \wedge \omega \wedge \tilde{j} - ie^{4A} \star G_3^\star \wedge j + \mathcal{I}_3 \wedge j - \eta e^{4A}\mathcal{I}_5 \ , \tag{3.155}$$

$$0 = -e^{-6A}\mathrm{d}\left[(1-\eta^2)e^{8A}\right] \wedge \omega \wedge \tilde{j} - e^{4A}G_3^\star \wedge \left[j - \eta\tilde{j}\right] - ie^{4A} \star G_3^\star \wedge j_1$$

$$+ \mathcal{I}_3 \wedge j_1 - e^{4A}(1-\eta^2)\mathcal{I}_5 \ . \tag{3.156}$$

To simplify these expressions further, we use the identities:

$$j \wedge \star\hat{\Omega} = -i\tilde{j} \wedge \left(\hat{\Omega}_{2,1} - \hat{\Omega}_{1,2}\right) \quad , \quad \tilde{j} \wedge \star\hat{\Omega} = -ij \wedge \left(\hat{\Omega}_{2,1} - \hat{\Omega}_{1,2}\right) \ , \tag{3.157}$$

where the Hodge decomposition is with respect to β (or, equivalently, γ) and $\hat{\Omega}$ is any three-form satisfying $\hat{\Omega}_{\theta;(1,1)^{\mathrm{NP}}} = \hat{\Omega}_{\bar{\theta};(1,1)^{\mathrm{NP}}} = 0$. To prove these identities, note that we can decompose $\hat{\Omega} = j \wedge v + \tilde{j} \wedge w + \ldots$, where the omitted terms vanish when wedged into j and $\tilde{j}$. One can then show using the primitivity decomposition that

$$\star\,\hat{\Omega} = -i\left(v_{1,0} - v_{0,1}\right) \wedge \tilde{j} - i(w_{1,0} - w_{0,1}) \wedge j + \ldots \ . \tag{3.158}$$

The identities (3.157) are now easily verified.

Thus, (3.155, 3.156) become:[28]

$$0 = -e^{-2A}\mathrm{d}[e^{4A}\eta] \wedge \omega \wedge \tilde{j} - e^{4A}G^\star_{(1,2)} \wedge \tilde{j} + \mathcal{I}_{(2,1)} \wedge j - \eta e^{4A}\mathcal{I}_{(3,2)} \ , \tag{3.159}$$

$$0 = e^{4A}G^\star_{(2,1)} \wedge \tilde{j} + \mathcal{I}_{(1,2)} \wedge j - \eta e^{4A}\mathcal{I}_{(2,3)} \ , \tag{3.160}$$

$$0 = -e^{-6A}\mathrm{d}[(1-\eta^2)e^{8A}] \wedge \omega \wedge \tilde{j}$$

[28]To clarify notation, $G^\star_{(p,q)} \equiv \left[G_{(p,q)}\right]^\star = [G^\star]_{(q,p)} \neq [G^\star]_{(p,q)}$.

$$- 2e^{4A}G^\star_{(1,2)} \wedge j_2 + \mathcal{I}_{(2,1)} \wedge j_1 - e^{4A}(1-\eta^2)\mathcal{I}_{(3,2)} \ , \tag{3.161}$$

$$0 = \mathcal{I}_{(1,2)} \wedge j_1 - e^{4A}(1-\eta^2)\mathcal{I}_{(2,3)} \ , \tag{3.162}$$

where $j_1 = \tilde{j} - \eta j$ and $j_2 = j - \eta\tilde{j}$. Similarly, (3.153, 3.154) become:

$$0 = -e^{-4A}\mathrm{d}[e^{4A}(1+\eta)] \wedge \omega \wedge \tilde{j}$$

$$- e^{2A}[j+\tilde{j}] \wedge G^\star_{(1,2)} + i\mathcal{J}_{(3,1)} \wedge \bar{\theta} - e^{2A}(1+\eta)\mathcal{I}_{(3,2)} \ , \tag{3.163}$$

$$0 = -e^{2A}[j+\tilde{j}] \wedge G^\star_{(2,1)} + \frac{i}{\tau_2}\mathrm{d}\tau^\star \wedge \omega^\star \wedge \tilde{j} + i\mathcal{J}_{(2,2)} \wedge \bar{\theta} - e^{2A}(1+\eta)\mathcal{I}_{(2,3)} \ , \tag{3.164}$$

$$0 = -e^{-4A}\mathrm{d}[e^{4A}(1-\eta)] \wedge \omega \wedge \tilde{j}$$

$$- e^{2A}[j-\tilde{j}] \wedge G^\star_{(1,2)} + i\mathcal{J}^\star_{(2,2)} \wedge \theta - e^{2A}(1-\eta)\mathcal{I}_{(3,2)} \ , \tag{3.165}$$

$$0 = -e^{2A}[j-\tilde{j}] \wedge G^\star_{(2,1)} + \frac{i}{\tau_2}\mathrm{d}\tau^\star \wedge \omega^\star \wedge \tilde{j} + i\mathcal{J}^\star_{(3,1)} \wedge \theta - e^{2A}(1-\eta)\mathcal{I}_{(2,3)} \ . \tag{3.166}$$

We combine these equations to eliminate G_3 and $\mathrm{d}\tau$, leaving:

$$0 = 2\mathcal{I}_{(1,2)} \wedge j + ie^{2A}\mathcal{J}_{(2,2)} \wedge \bar{\theta} - ie^{2A}\mathcal{J}^\star_{(3,1)} \wedge \theta - 4\eta e^{4A}\mathcal{I}_{(2,3)} \ , \tag{3.167}$$

$$0 = \mathcal{I}_{(1,2)} \wedge j_1 - e^{4A}(1-\eta^2)\mathcal{I}_{(2,3)} \ , \tag{3.168}$$

$$0 = -2\mathcal{I}_{(2,1)} \wedge j + ie^{2A}\mathcal{J}_{(3,1)} \wedge \bar{\theta} - ie^{2A}\mathcal{J}^\star_{(2,2)} \wedge \theta \ , \tag{3.169}$$

$$0 = -\mathcal{I}_{(2,1)} \wedge [j_1 + 2\eta j] + ie^{2A}\mathcal{J}_{(3,1)} \wedge \bar{\theta} + ie^{2A}\mathcal{J}^\star_{(2,2)} \wedge \theta - e^{4A}(1-\eta^2)\mathcal{I}_{(3,2)} \ . \tag{3.170}$$

Applying the Hodge decomposition to (3.92, 3.93) and extracting the relevant components, we obtain:

$$0 = \mathcal{J}_{(3,2)} \ , \tag{3.171}$$

$$0 = -\frac{i}{\tau_2}\bar{\partial}_\Pi\tau \wedge \tilde{j} \wedge j_2 + \frac{1}{4}e^{2A}(1-\eta^2)\omega^\star \wedge \hat{G}_{(2,1)} + \frac{1}{2}(1-\eta^2)\mathcal{J}_{(2,3)} \ , \tag{3.172}$$

$$0 = \frac{1}{2}e^{-8A}\bar{\partial}_\Pi[(1-\eta^2)e^{8A}] \wedge \tilde{j} \wedge j_2 - \frac{1}{4}e^{2A}(1-\eta^2)\omega^\star \wedge \hat{G}^\star_{(1,2)} - (1-\eta^2)[\mathrm{Re}\,\mathcal{J}_5]_{(2,3)} \ , \tag{3.173}$$

where $\bar{\partial}_\Pi$ is the projection of the scalar gradient onto antiholomorphic directions along the base, $j_2 = j - \eta\tilde{j}$, and $\hat{G}_3 = G_{\theta\bar{\theta};1,0} + G_{\theta\bar{\theta};0,1} + G_{;2,1} + G_{;1,2}$ consists of the components of G_3 with an even number of legs along the fiber. The latter two equations can be usefully restated using the identity:

$$v_{1,0} \wedge J_2 = \frac{1}{2} w_{0,1} \wedge \Omega_2 \quad \leftrightarrow \quad w_{0,1} \wedge J_2 = -\frac{1}{2} v_{1,0} \wedge \bar{\Omega}_2 \ , \tag{3.174}$$

for any SU(2) structure, where v and w point along the base. In particular,

$$\hat{v}_{2,1} \wedge j_2 = \frac{1}{4}(1-\eta^2)\hat{w}_{1,2} \wedge \omega \quad \leftrightarrow \quad \hat{w}_{1,2} \wedge j_2 = -\hat{v}_{2,1} \wedge \omega^\star \ , \tag{3.175}$$

for any v, w with an even number of legs along the fiber, since $j_2 = s_\varphi^2 J_2$. Thus, taking $\mathcal{J}_5 = \left[\frac{1}{2}\eta j^2 + j \wedge j_1\right] \wedge \mathcal{J}_1$, where $\mathcal{J}_1$ points along the base, we obtain:

$$\mathcal{J}_{(1,0)} = 0 \ , \tag{3.176}$$

$$0 = \frac{1}{2}\mathcal{J}_{(0,1)} \wedge \omega \wedge \tilde{j} - \frac{i}{\tau_2}\mathrm{d}\tau \wedge \omega \wedge \tilde{j} - e^{2A} j_2 \wedge \hat{G}_{(2,1)} \ , \tag{3.177}$$

$$0 = \frac{1}{2}e^{-8A}\mathrm{d}\left[(1-\eta^2)e^{8A}\right] \wedge \omega \wedge \tilde{j} - [\mathrm{Re}\,\mathcal{J}_1]_{(0,1)} \wedge \omega \wedge \tilde{j} + e^{2A} j_2 \wedge \hat{G}^\star_{(1,2)} \ . \tag{3.178}$$

Combining with (3.160, 3.161, 3.164) to eliminate G_3, we find:

$$0 = \mathcal{I}_{(2,1)} \wedge j_1 - e^{4A}(1-\eta^2)\mathcal{I}_{(3,2)} - 2e^{2A}\left[\mathrm{Re}\,\mathcal{J}_1\right]_{(0,1)} \wedge \omega \wedge \tilde{j} \ , \tag{3.179}$$

$$0 = e^{2A}\mathcal{J}^\star_{(0,1)} \wedge \omega^\star \wedge \tilde{j} - 2ie^{2A}\mathcal{J}_{(2,2)} \wedge \bar{\theta} + 2e^{4A}(1+\eta)^2\mathcal{I}_{(2,3)} - 2(1+\eta)\mathcal{I}_{(1,2)} \wedge j \ . \tag{3.180}$$

The conditions (3.167 – 3.170, 3.176, 3.179, 3.180) constitute seven equations in seven unknowns: $\mathcal{I}_{(2,1)} \wedge j$, $\mathcal{I}_{(2,1)} \wedge \tilde{j}$, $\mathcal{I}_{(3,2)}$, $\mathcal{J}_{(3,1)} \wedge \bar{\theta}$, $\mathcal{J}_{(2,2)} \wedge \bar{\theta}$, $\mathcal{J}_{(1,0)} \wedge \omega^\star \wedge \tilde{j}$ and $\mathcal{J}_{(0,1)} \wedge \omega \wedge \tilde{j}$. One can check that the only solution is

$$\mathcal{I}_{(2,1)} \wedge j = \mathcal{I}_{(2,1)} \wedge \tilde{j} = 0 \ , \ \mathcal{I}_{(3,2)} = 0 \ ,$$
$$\mathcal{J}_{(3,1)} \wedge \bar{\theta} = \mathcal{J}_{(2,2)} \wedge \bar{\theta} = 0 \ , \ \mathcal{J}_{(1,0)} = \mathcal{J}_{(0,1)} = 0 \ . \tag{3.181}$$

Taken together with the constraints derived previously, we see that all separation forms must vanish, so that the supersymmetry conditions are manifestly $SL(2,\mathbb{R})$ covariant.

CHAPTER 4

NEW $\mathcal{N} = 1$ DUALITIES FROM ORIENTIFOLD TRANSITIONS

We report on a broad new class of $\mathcal{N} = 1$ gauge theory dualities[1] which relate the worldvolume gauge theories of D3 branes probing different orientifolds of the same Calabi-Yau singularity. In this chapter, we focus on the simplest example of these new dualities, arising from the orbifold singularity $\mathbb{C}^3/\mathbb{Z}_3$. We present extensive checks of the duality, including anomaly matching, partial moduli space matching, matching of discrete symmetries, and matching of the superconformal indices between the proposed duals. We then present a related duality for the dP_1 singularity, as well as dualities for the $\mathbb{F}_0$ and $Y^{4,0}$ singularities, illustrating the breadth of this new class of dualities. In a companion paper, we show that certain infinite classes of geometries which include $\mathbb{C}^3/\mathbb{Z}_3$ and dP_1 all exhibit such dualities, and argue that their ten-dimensional origin is the $SL(2, \mathbb{Z})$ self-duality of type IIB string theory.

4.1 Introduction

One of the most remarkable achievements of the study of supersymmetric gauge theories has been the discovery of strong/weak gauge theory dualities, and the correspondent increase in our understanding of (supersymmetric) strongly coupled gauge theories. A prototypical example of such dualities — and indeed the most important of the $\mathcal{N} = 1$ dualities — is the duality, due to Seiberg [88, 89], between supersymmetric QCD with N_C colors and N_F flavors and supersymmetric QCD with $N_F - N_C$ colors, N_F flavors, and additional gauge singlets interacting with the dual quarks via the superpotential. The duality, an infrared correspondence between two gauge theories which differ in the ultraviolet, allows the infrared behavior of supersymmetric QCD to be understood for all values of N_F and N_C.

The success of Seiberg duality has motivated a thorough study of further dualities of this type, ranging from natural generalizations to SO and USp gauge

[1] This chapter previously appeared as Iñaki García-Etxebarria, Ben Heidenreich and Timm Wrase, "New $\mathcal{N} = 1$ dualities from orientifold transitions. Part I: Field Theory," arXiv:1210.7799 [hep-th].

groups [88, 90, 91], generalizations with adjoint matter and a superpotential [92–94], models with antisymmetric tensor matter [95–98], "self-dual" theories [99, 100], to yet more complicated examples (see e.g. [101, 102]), in addition to the classifications of various types of confining gauge theories [103–106] where the confined phase has a weakly coupled dual description without a dual gauge group.

Seiberg duality often admits a very natural and enlightening embedding in string theory, where it appears in the context of brane systems [107–110], the duality cascade [30, 111, 112], toric duality [113, 114], and geometric transitions [115, 116]. (Many of these are related manifestations of the same phenomenon, where Seiberg duality is realized as the effect of passing NS5 branes through each other in a particular T-dual picture [114].) String theory also supplies some contexts where Seiberg duality can be exact [112]. As such, the two fields have enjoyed a largely symbiotic relationship.

Another gauge theory duality of a different nature also enjoys a close relationship to string theory. Montonen-Olive duality [117–119], which relates $\mathcal{N} = 4$ super-Yang Mills to itself at different couplings, is directly related to the $SL(2,\mathbb{Z})$ self-duality of type IIB string theory.[2] In particular, the appearance of an $SL(2,\mathbb{Z})$ Montonen-Olive duality in the world-volume gauge theory of D3 branes in a flat background follows from the invariance of the D3 under $SL(2,\mathbb{Z})$, which nonetheless acts nontrivially on the world-volume gauge field (as an electromagnetic duality) and gauge coupling (as a strong/weak duality), reproducing the action of Montonen-Olive duality on the gauge theory.

Montonen-Olive duality is different from Seiberg duality in some important ways. Unlike Seiberg duality, Montonen-Olive duality is an exact duality, in the

[2]The term "S-duality" is sometimes used to refer to the entire $SL(2,\mathbb{Z})$ self-duality. In this chapter we will use it to refer specifically to the $\tau \to -1/\tau$ element of the $SL(2,\mathbb{Z})$ duality of type IIB string theory.

sense that it gives various superficially distinct but quantum equivalent formulations of a single physical theory, with each of the formulations most suitable for certain values of the Yang-Mills coupling constant. There is no flow wherein distinct gauge theories converge on the same infrared fixed point. Indeed, due to maximal supersymmetry, there is no flow whatsoever, and when one description is weakly coupled S-dual descriptions are necessarily strongly coupled (at all energy scales).

In this chapter, we construct $\mathcal{N} = 1$ analogs of Montonen-Olive duality.[3] $\mathcal{N} = 1$ gauge theories are interesting for many reasons: unlike $\mathcal{N} = 4$ gauge theories, they can exhibit chirality, confinement, and dynamical supersymmetry breaking, among other things. Our new class of $\mathcal{N} = 1$ variants of Montonen-Olive duality provide an interesting counterpoint to known examples of Seiberg duality, while illuminating the dynamics of interesting gauge theories via the duality. Moreover, our examples also serve to illustrate which of the aforementioned features of Montonen-Olive duality are due to extended supersymmetry, and which persist with less supersymmetry.

Since Montonen-Olive duality arises from $SL(2,\mathbb{Z})$ acting on the world-volume gauge theory of D3 branes in a flat background, a natural place to look for analogous dualities with less supersymmetry is in the world-volume gauge theory of D3 branes probing a Calabi-Yau singularity. Since the geometry is $SL(2,\mathbb{Z})$ invariant, these gauge theories are expected to exhibit an $SL(2,\mathbb{Z})$ self-duality as well. Unfortunately, there are virtually no available checks of this conjecture. The usual consistency checks, such as anomaly matching and moduli space matching, are trivial and hence meaningless in the case of a self-duality.

[3] To our knowledge, the only $\mathcal{N} = 1$ examples of Montonen-Olive duality discussed in the literature are mass deformations of $\mathcal{N} = 4$ theories (see e.g. [120, 121]) or of certain $\mathcal{N} = 2$ theories with a similar $SL(2,\mathbb{Z})$ duality [122]. Our examples are not directly related to either $\mathcal{N} = 4$ or $\mathcal{N} = 2$ theories.

Fortunately, other types of Montonen-Olive duality are possible. By placing k D3 branes atop an O3 plane in flat space, one obtains an $\mathcal{N} = 4$ $SO(2k)$, $SO(2k+1)$, or $USp(2k)$ gauge theory (depending on the type of O3 plane). Whereas $SO(2k)$ is again self-dual under Montonen-Olive duality, $SO(2k+1)$ and $USp(2k)$ are exchanged under the duality due to the S-duality transformation properties of the respective O3 planes [123]. Thus, in order to construct an $\mathcal{N} = 1$ analog, we will consider the world-volume gauge theory of D3 branes probing an orientifolded Calabi-Yau singularity, where $SL(2, \mathbb{Z})$ can act nontrivially on the orientifold plane, leading to dual theories with distinct gauge groups.

While such a construction generally involves collapsed O7 planes, rather than O3 planes, and the appearance of fractional branes at small volume further complicates the situation, we argue in a companion paper [60] that S-duality nonetheless acts simply on the entire system. Analogously to the $\mathcal{N} = 4$ case discussed above, we argue that the fractional O7 planes undergo an "orientifold transition" at strong coupling, exchanging $O7^-$ and $O7^+$ planes while emitting / absorbing a number of fractional branes during the process. Understanding such a transition is one important motivation for our work, but we defer further details to [60].

There are numerous additional motivations for studying duality in this context. While world-volume gauge theories on D3 branes probing (toric) singularities have been exhaustively studied, orientifolded singularities have received comparatively little attention. Systematic tools for the construction of many examples are available [124], whereas very few examples have been studied in any detail (see for example [32]). Furthermore, the gauge theories we study are highly nontrivial chiral gauge theories with tree-level superpotentials, tensor matter, and a nontrivial flow. Depending on the singularity and the number of D3 branes, a range of interesting IR behavior arises. In particular in the limited sample of models we analyze, we find a runaway superpotential, confinement with chiral symmetry breaking, a free magnetic phase, or a nontrivial superconformal fixed point.

These gauge theories are also interesting from the point of view of moduli stabilization, as the nonperturbative dynamics of these gauge theories for sufficiently low N can lift D3 brane moduli and potentially Kähler moduli as well. Indeed, a number of interesting Calabi-Yau singularities correspond to rigid shrinking divisors, whereas blown-up versions of these have played an important role in stabilizing Kähler moduli in geometric compactifications of type IIB string theory [17, 18]. The results of chapter 2 hint that an AdS/CFT description of the dynamics may be possible, though pitfalls abound due to the necessity of low N in this context to obtain gauge theories which are not approximately superconformal.

The outline of this chapter is as follows. In section 4.2, we review some basic facts about Montonen-Olive duality which illustrate how it is distinct from Seiberg duality. In section 4.3, we present the simplest example of a new duality, relating two possible gauge theories for D3 branes probing the orientifolded $\mathbb{C}^3/\mathbb{Z}_3$ singularity. We present several nontrivial consistency checks and discuss an example of the duality. In section 4.4, we compute the superconformal indices for the proposed dual gauge theories and show that they match (up to the limits of our computational resources), a very nontrivial check of the proposed duality. In section 4.5, we discuss the infrared physics of these gauge theories using Seiberg duality. In section 4.6, we discuss further examples of the duality coming from different ten-dimensional geometries, with particular attention to the dP_1 singularity. The corresponding gauge theories can be blown down to recover the $\mathbb{C}^3/\mathbb{Z}_3$ gauge theories, and exhibit interesting behavior at low N. We also briefly discuss dualities which arise in the $\mathbb{F}_0$ and $Y^{4,0}$ geometries, some of which appear to have a different origin in string theory, unrelated to $SL(2,\mathbb{Z})$. We leave a detailed treatment of these dualities to a future work. We present our conclusions in section 4.7.

We provide several appendices for the reader's benefit. In appendix 4.A we review the language of quiverfolds, a generalization of quiver gauge theories which arise naturally in the presence of orientifold planes. In appendix 4.B we review a useful mathematical tool for anomaly matching. In appendix 4.C, we show that holomorphic combinations of couplings which are invariant under all possible spurious and/or anomalous global symmetries are RGE invariant. In appendix 4.D, we show how the string coupling can be related to the gauge and superpotential couplings of a D-brane gauge theory by moving out along the Coulomb branch. In appendix 4.E we discuss some technical details of the computation of the superconformal index. In appendix 4.F, we relate the matching of certain baryonic directions in the moduli space of the prospectively dual $\mathbb{C}^3/\mathbb{Z}_3$ theories to a group theoretic conjecture and provide evidence for this conjecture. Finally, in appendix 4.G, we relate the matching of the superconformal indices to a conjectural identity for elliptic hypergeometric integrals.

In companion papers [60, 125], we discuss the construction of these orientifold gauge theories using exceptional collections as well as details of their gravity duals, focusing on string theoretic arguments that the dual gauge theories are connected by ten-dimensional S-duality. We also discuss the nature of the orientifold transition which seems to govern the duality, and construct infinite families of geometries which exhibit similar dualities.

4.2 Review of Montonen-Olive duality

In this section, we review certain aspects of Montonen-Olive duality which will be important for the present work.

Rigid $\mathcal{N} = 4$ gauge theories are characterized by their gauge group and by their holomorphic gauge coupling, which takes the form

$$\tau_{\text{YM}} = \frac{\theta_{\text{YM}}}{2\pi} + \frac{4\pi i}{g_{\text{YM}}^2}\,, \tag{4.1}$$

for an $SU(N)$ gauge theory. Such gauge theories are easily realized in string theory; for instance, the world-volume gauge theory on N D3 branes probing a smooth background is an $\mathcal{N}=4$ $U(N)$ gauge theory with holomorphic gauge coupling equal to the type IIB axio-dilaton, $\tau_{\text{10d}} = C_0 + ie^{-\phi}$, though the extended supersymmetry may be broken by irrelevant operators in a smoothly curved background or by relevant operators in the presence of flux (see for example [27, 126]).

Montonen-Olive duality relates such a gauge theory to a dual theory at a different coupling under the action of the modular group, $SL(2,\mathbb{Z})$:

$$\tau' = \frac{a\tau + b}{c\tau + d}\,. \tag{4.2}$$

In particular, unless the modular transformation is of the form $\tau \to \tau + n$ (enforcing the periodicity of the theta angle), it is straightforward to check that the original and dual descriptions cannot both be weakly coupled. The string realization of this duality, in the case where the gauge theory arises as the world-volume gauge theory on a stack of D3 branes, is the $SL(2,\mathbb{Z})$ self-duality of type IIB string theory.

It is important to bear in mind that Montonen-Olive duality is not literally a "duality" (a word whose root is "two"): a weakly coupled $\mathcal{N}=4$ theory has not just one but an *infinite number* of strongly-coupled dual descriptions. Alternately phrased, by deforming a weakly-coupled $\mathcal{N}=4$ gauge theory to strong coupling, we encounter an infinite number of phases with a weakly-coupled dual description.

To illustrate this intricate and fascinating behavior, we observe that the modular invariant $j(\tau)$ is approximately $e^{-2\pi i\tau}$ at weak coupling, so that $|j(\tau)| \to \infty$ is a modular-invariant definition of weak coupling. A plot of $|j(\tau)|$ on the upper half plane (conformally mapped to a disk) is shown in figure 4.1. The infinite order

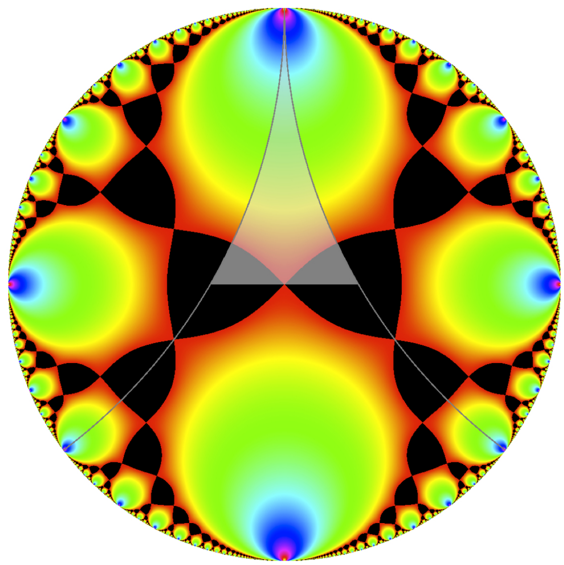


Figure 4.1: The modular invariant $|j(\tau)|$ plotted across the upper half plane $\mathbb{H}$, where $\mathbb{H}$ is conformally mapped to the unit disk via $w = \frac{1+i\tau}{i+\tau}$, so that $\tau = +i\infty$ (weak coupling) lies at the top edge of the disk, $\tau = i$ (intermediate coupling) in the center and $\tau = 0$ (strong coupling) at the bottom, whereas the left and right edges correspond to $\tau = -1$ and $\tau = +1$ respectively, and the Im $\tau = 0$ axis spans the perimeter. The black regions, corresponding to $|j(\tau)| < 12^3$, serve to divide the plane into an infinite number of disjoint colored regions, each containing a subregion with a weakly coupled dual description ($|j(\tau)| \to \infty$), colored blue/purple. The superimposed curved grey lines illustrate the boundary of the region $|\mathrm{Re}\ \tau| < 1/2$, a fundamental domain under the identification $\tau \to \tau + 1$. The transformation $\tau \to -1/\tau$ is equivalent to inversion through the center of the disk, and the shaded triangular region is the canonical fundamental domain for $SL(2,\mathbb{Z})$.

of $SL(2,\mathbb{Z})$ leads to a fractal structure, as seen in the figure. Thus, the behavior of these theories at strong coupling is very rich, with many dual weakly coupled descriptions in the strong coupling limit, depending on the exact value of the theta angle. The weakly coupled dual descriptions become free as Im $\tau \to 0$ for rational values of $\theta/2\pi$, and are therefore dense along the Re τ axis.

For an $SU(N)$ gauge group, all the dual descriptions have the same perturbative gauge group.[4] This is a consequence of the invariance of the D3 brane under

[4]Globally $SU(N)$ differs from its dual by a $\mathbb{Z}_N$ factor coming from its center [117]. This is

$SL(2,\mathbb{Z})$. We now consider Montonen-Olive duality for SO and USp gauge groups. For an $SO(2k)$ gauge group, the duals likewise have the same gauge group; equivalently, the $\mathrm{O3}^-$ plane is $SL(2,\mathbb{Z})$ invariant [123]. For an $SO(2k+1)$ gauge group, however, the dual descriptions have gauge group $USp(2k)$. In string theory, this corresponds to the fact that the $(\tau \to -1/\tau)$ S-dual of the $\widetilde{\mathrm{O3}}^-$ — another name for an $\mathrm{O3}^-$ plus a single (pinned) D3 brane[5] — is an $\mathrm{O3}^+$. Thus, S-duality maps

$$\mathrm{O3}^- + (2k+1)\ \mathrm{D3's} \longrightarrow \mathrm{O3}^+ + 2k\ \mathrm{D3's}\,. \tag{4.3}$$

This is a well known example of what we will call an "orientifold transition",[6] wherein strongly coupled orientifold planes recombine with branes to form a different, weakly coupled orientifold plane. Examples of this phenomenon with fractional O7 planes and $\mathcal{N}=1$ supersymmetry are considered in [60], and play an important role in the new dualities discussed in this chapter.

The upshot of the previous paragraph is that Montonen-Olive duality relates strongly coupled $\mathcal{N}=4$ gauge theories with $SO(2k+1)$ and $USp(2k)$ gauge groups to each other. This is not the whole story, however. Because the $\mathrm{O3}^+$ and $\widetilde{\mathrm{O3}}^-$ are related by S-duality, they must form some $SL(2,\mathbb{Z})$ multiplet. However, the multiplet is as yet incomplete. To see this, consider the $SL(2,\mathbb{Z})$ generators T : $\tau \to \tau+1$ and $S : \tau \to -1/\tau$. The $\widetilde{\mathrm{O3}}^-$ is T-invariant; therefore ST maps $\widetilde{\mathrm{O3}}^-$ to $\mathrm{O3}^+$. However, since $(ST)^3 = 1$, it cannot be true that ST maps $\mathrm{O3}^+$ back to $\widetilde{\mathrm{O3}}^-$. We denote the ST image of $\mathrm{O3}^+$ as $\widetilde{\mathrm{O3}}^+$, where the three O3 planes form a

compatible with the D3 brane being invariant under $SL(2,\mathbb{Z})$ since the gauge group on N D3 branes is actually $U(N)$, which is self-dual (see for example [127]). We will ignore these global subtleties in the remainder of the chapter.

[5]The single additional D3 brane is "pinned" to the orientifold plane because it is its own orientifold image. A pinned brane arises whenever an odd number of (upstairs) branes is placed atop an orientifold plane, giving rise to an $SO(2k+1)$ gauge group.

[6]The term "orientifold transition" was used in a different context in [128]. We do not mean to imply that our physical mechanism is the same, just that we also have a change in the orientifold type.

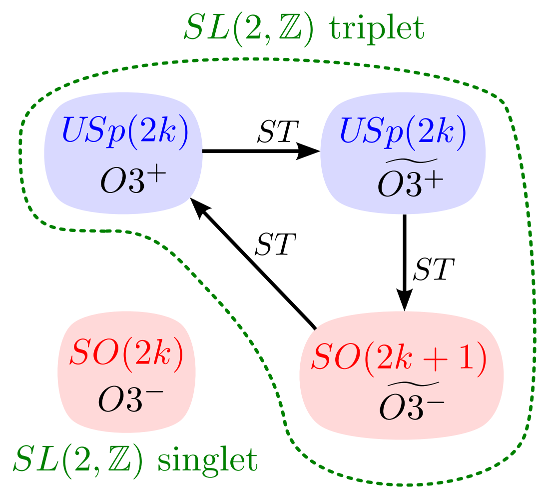


Figure 4.2: Summary of the four gauge theories that arise from placing D3 branes on top of the four different O3 planes.

triplet under $SL(2, \mathbb{Z})$ [123]. We summarize the resulting structure in figure 4.2.

The complete action of $SL(2, \mathbb{Z})$ on the triplet is as follows: S exchanges the $\mathrm{O3}^+$ and $\widetilde{\mathrm{O3}}^-$, leaving the $\widetilde{\mathrm{O3}}^+$ invariant, whereas T exchanges the $\mathrm{O3}^+$ and $\widetilde{\mathrm{O3}}^+$, leaving the $\widetilde{\mathrm{O3}}^-$ invariant, so that ST cyclically permutes the three O3 planes. Since T is a perturbative duality, the $\widetilde{\mathrm{O3}}^+$ also gives rise to an $USp(2k)$ gauge group, and is perturbatively equivalent to the $\mathrm{O3}^+$, the two configurations being distinguished non-perturbatively by their spectrum of BPS states [129]. It is possible to rephrase this by saying that the two different $\mathrm{O3}^+$ planes give rise to the same gauge theory at different theta angles. In particular, $\tau \to \tau + 2$ leaves the O3 plane type invariant, and defines the periodicity of the theta angle in the corresponding gauge theory, whereas $\tau \to \tau + 1$ exchanges the two O3 plane types. Thus, the gauge theories corresponding to $2k$ D3 branes atop an $\mathrm{O3}^+$ and $\widetilde{\mathrm{O3}}^+$ can be identified with each other upon shifting the theta angle by a half period.

To illustrate the nature of these dualities, we show how the weakly coupled description changes as a function of the holomorphic gauge coupling in figure 4.3.

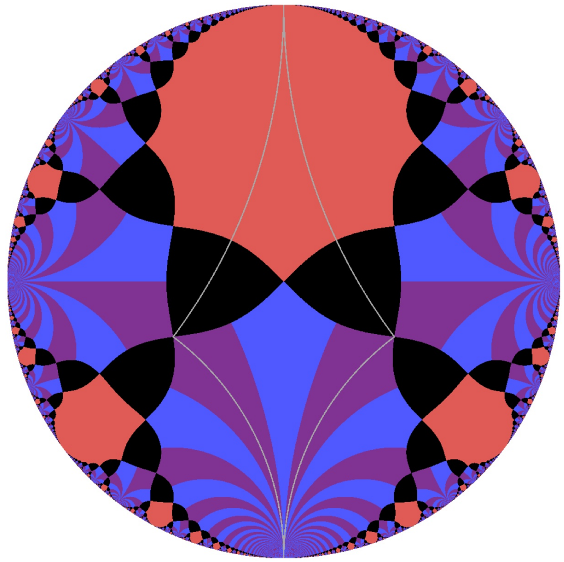


Figure 4.3: A schematic illustration of the phase structure of $\mathcal{N}=4$ $SO(2k+1)$ and $USp(2k)$ gauge theories as a function of τ, patterned after figure 4.1. The different colors indicate the type of the O3 plane (and hence the gauge group) in the dual weakly coupled phase for each value of τ, where red, blue and purple correspond to an $\widetilde{\mathrm{O3}}^{-}$, $\mathrm{O3}^{+}$ or $\widetilde{\mathrm{O3}}^{+}$, respectively, and the latter two possibilities are distinguished by requiring $-1/2 < \mathrm{Re}\,\tau \leq 1/2$ in the dual theory. Thus, the red regions have a dual weakly coupled $SO(2k+1)$ description, whereas the blue/purple striped regions have a dual weakly coupled $USp(2k)$ description. The thin grey lines outline a fundamental region for $\Gamma_0(2)$, the self-duality group for the $SO(2k+1)$ theory. Note that the region where each dual theory is perturbative is most likely smaller than the colored region indicated here (see figure 4.1).

As can be seen in the figure, each gauge theory has additional self-dualities as well as the dualities which relate the different theories. For example, the self-duality group for $SO(2k+1)$ is the subgroup $\Gamma_0(2) \subset SL(2,\mathbb{Z})$ of elements $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ for which c is even (hence a and d are odd, since $ad - bc = 1$), whereas for $USp(2k)$ it is the conjugate subgroup consisting of elements for which b is even.

4.3 Duality for $\mathbb{C}^3/\mathbb{Z}_3$

In this section, we examine the simplest of our new $\mathcal{N}=1$ dualities. In the $\mathcal{N}=4$ examples discussed above, the six directions transverse to the D3 branes form a

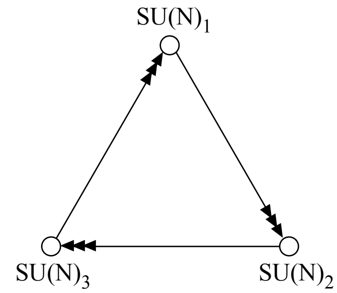


Figure 4.4: The quiver gauge theory for $\mathbb{C}^3/\mathbb{Z}_3$.

flat $\mathbb{R}^6$ or equivalently $\mathbb{C}^3$ transverse space, leading to gauge theories with maximal supersymmetry, where the $SU(4) \cong SO(6)$ R-symmetry is just the rotational isometry group of $\mathbb{R}^6$. To obtain an $\mathcal{N} = 1$ theory at low energies, we must either switch on flux or introduce singularities. We choose to do the latter.

A simple and well-known example of such a transverse space is the $\mathbb{C}^3/\mathbb{Z}_3$ orbifold, where the orbifold action on the transverse complex coordinates is

$$z^i \to e^{2\pi i/3} z^i \,. \tag{4.4}$$

The singularity can be resolved by blowing up a $\mathbb{P}^2$ exceptional divisor. Placing D3 branes at the singularity leads to the $\mathcal{N} = 1$ $SU(N)^3$ quiver gauge theory shown in figure 4.4. The orbifold reduces the isometry of the transverse space to $SU(3) \times U(1)$, with the $U(1)$ appearing as an R-symmetry in the $\mathcal{N} = 1$ gauge theory.

We consider an orientifold of this configuration, since, as we argued in the previous section, the $SL(2,\mathbb{Z})$ dual descriptions of gauge theories arising from D3 branes at singularities all have the same gauge group and matter content. We choose the orientifold involution $z^i \to -z^i$ which corresponds to an O7 plane wrapping the shrunken $\mathbb{P}^2$. As the resulting configuration is essentially a $\mathbb{Z}_3$ orbifold of the $\mathcal{N} = 4$ orientifolds considered in the previous section, we will argue that the strong coupling behavior is closely analogous. In this chapter we focus on the characteristics of the resulting gauge theories, deferring a detailed discussion of the analogy between the gravity duals to [60].

In appendix 4.A we discuss in general how to "orientifold" a quiver gauge theory and apply this procedure to two explicit examples. In particular in section 4.A.2 we study the orientifolds of the $\mathbb{C}^3/\mathbb{Z}_3$ orbifold theory for the orientifold involution $z^i \to -z^i$. Counting $SO(2N)$ and $SO(2N+1)$ as two separate cases, we find that there are three possible gauge theories arising on D3 branes probing this orientifolded singularity. They correspond to a shrunken $\mathbb{P}^2$ that is wrapped by an $O7^+$ plane or to an $O7^-$ plane with and without a pinned D3 brane respectively.

We will argue that two of these gauge theories are dual, whereas the third is self-dual, analogous to the $\mathcal{N}=4$ $SO(2k+1)$, $USp(2k)$ and $SO(2k)$ gauge theories discussed above. The dualities studied here are merely the simplest examples of a large class of $\mathcal{N}=1$ dualities between orientifold gauge theories, some of which we will study in detail in this chapter as well as in [60, 125], where we will discuss many more examples.

The $\mathbb{C}^3/\mathbb{Z}_3$ orientifold we discuss here was, to the best of our knowledge, first studied in [130–133] and recently revisited from the dimer point of view in [32, 124] and applied to the problem of moduli stabilization in chapter 2 and also in [134]. Orientifolds of $\mathbb{C}^3/\mathbb{Z}_3$ and related abelian orbifolds have also proven to be an interesting testing ground for studying non-perturbative dynamics in string theory [135–139]. For the involution discussed above, one finds $SO(N-4)\times SU(N)$ and $USp(\tilde{N}+4)\times SU(\tilde{N})$ gauge theories for fractional $O7^-$ and $O7^+$ planes, respectively. Both theories have a non-anomalous R-symmetry in addition to a global $SU(3)$ symmetry, corresponding to the $SU(3)\times U(1)$ isometry of the transverse space. A careful analysis reveals that both models also have a discrete "baryonic" $\mathbb{Z}_3$

symmetry. The two models are[7]

	$SO(N-4)$	$SU(N)$	$SU(3)$	$U(1)_R$	$\mathbb{Z}_3$
A^i	$\square$	$\overline{\square}$	$\square$	$\frac{2}{3}+\frac{2}{N}$	ω_{3N}
B^i	$\mathbf{1}$	$\begin{smallmatrix}\square\\\square\end{smallmatrix}$	$\square$	$\frac{2}{3}-\frac{4}{N}$	ω_{3N}^{-2}

(4.5)

	$USp(\tilde{N}+4)$	$SU(\tilde{N})$	$SU(3)$	$U(1)_R$	$\mathbb{Z}_3$
$\tilde{A}^i$	$\square$	$\overline{\square}$	$\square$	$\frac{2}{3}-\frac{2}{\tilde{N}}$	$\omega_{3\tilde{N}}$
$\tilde{B}^i$	$\mathbf{1}$	$\square\square$	$\square$	$\frac{2}{3}+\frac{4}{\tilde{N}}$	$\omega_{3\tilde{N}}^{-2}$

(4.6)

where $\omega_n \equiv e^{2\pi i/n}$, $\tilde{N}$ is even, and the tree-level superpotentials are

$$W = \frac{\lambda}{2}\epsilon_{ijk}\mathrm{Tr}\, A^i A^j B^k\,, \quad \tilde{W} = \frac{\tilde{\lambda}}{2}\epsilon_{ijk}\mathrm{Tr}\,\tilde{A}^i\tilde{A}^j\tilde{B}^k\,, \tag{4.7}$$

respectively, where λ and $\tilde{\lambda}$ are superpotential couplings.

Note that we label the discrete symmetry as a $\mathbb{Z}_3$ even though the cube of the generator is not the identity. This is because the cube of the generator lies within the $\mathbb{Z}_N$ or $\mathbb{Z}_{\tilde{N}}$ center of $SU(N)$ or $SU(\tilde{N})$, so we obtain a $\mathbb{Z}_3$ symmetry upon composing the generator with an element of $SU(N)$ or $SU(\tilde{N})$ whose cube is the inverse of the cube of the generator. This latter $\mathbb{Z}_3$ symmetry is equivalent to the discrete symmetry indicated in the charge table up to a constant gauge transformation.[8]

The $SO(N-4)\times SU(N)$ gauge theories have a classical moduli space which includes directions corresponding to moving D3 branes away from the singularity. The gauge group is then Higgsed down to $SO(N-4-2k)\times SU(N-2k)\times U(k)$ where k is the number of (downstairs) D3 branes removed from the O-plane, corresponding to the $U(k)$ factor in the Higgsed gauge group. After integrating

[7]These two gauge theories are related by a negative rank duality as explained in appendix 4.B.

[8]In general, a discrete symmetry can be rewritten as a $\mathbb{Z}_k$ discrete symmetry times a constant gauge transformation whenever the kth power of the generator lies within the gauge group.

out massive matter, the $U(k)$ decouples from the rest of the gauge group in the IR, giving a separate $\mathcal{N} = 4$ gauge theory corresponding to k D3 branes probing a smooth region of the Calabi-Yau cone. Meanwhile, the remaining $SO(N-4-2k)\times SU(N-2k)$ reproduces the original gauge theory at a different rank $N' = N-2k$. Thus, we see that the moduli space of the $SO(N-4) \times SU(N)$ family of gauge theories falls into two disconnected components for even and odd N respectively, where all even N theories are connected by the above process, as are all odd N theories. Similarly, all $USp(\tilde{N}+4) \times SU(\tilde{N})$ theories are interconnected by an analogous motion in moduli space, where $\tilde{N}$ must be even for $USp(\tilde{N}+4)$ to exist.

In all, we have obtained three distinct families of gauge theories corresponding to D3 branes probing the orientifolded $\mathbb{C}^3/\mathbb{Z}_3$ singularity, all corresponding to the same geometric orientifold involution. Two of these theories, the $SO(N-4) \times SU(N)$ theories for even and odd N, are distinguished from each other by the presence or absence of a pinned D3 brane and its corresponding half-integral D3 brane charge, while the $USp(\tilde{N}+4)\times SU(\tilde{N})$ theory corresponds to a compact $O7^+$ plane rather than a compact $O7^-$ plane. Regardless of the O-plane type, the seven-brane tadpole is cancelled locally by (anti-)D7 branes, and the two configurations have the same $SL(2,\mathbb{Z})$ monodromy.[9]

The situation is closely analogous to the three gauge theories $SO(2k)$, $SO(2k+1)$, $USp(2k)$ appearing in the $\mathcal{N} = 4$ case, and we therefore hypothesize that one of the SO families enjoys an $SL(2,\mathbb{Z})$ self-duality, whereas the other SO family and the USp family are related by an $SL(2,\mathbb{Z})$ duality. In the remainder of this section and in section 4.4, we present strong field theoretic evidence for the latter duality, based on the matching of various computable infrared observables, and explore some of its properties.

[9]We argue in [60] that the geometric duals are distinguished from each other by different (S-dual) choices of discrete torsion.

We begin by discussing 't Hooft anomaly matching in section 4.3.1, leading to a more precise statement of the proposed duality. We then provide further evidence for the duality by a partial matching between the moduli spaces of the two theories. In section 4.3.2, we highlight an important limitation of our methods which is nonetheless linked to the nature of the duality, and in section 4.3.3 we discuss a specific, finite N example of the proposed duality.

We continue our discussion of these gauge theories in the following sections. In section 4.4, we provide further strong evidence for the proposed duality by comparing the superconformal indices between the prospectively dual theories, and in section 4.5, we discuss their infrared physics using Seiberg duality.

4.3.1 Classic checks of the duality

As a precursor to anomaly matching, we note that the dual theories should have the same global symmetry groups. In particular, for N or $\tilde{N}$ not divisible by three the baryonic $\mathbb{Z}_3$ is equivalent to the $\mathbb{Z}_3$ center of $SU(3)$ composed with a constant gauge transformation, and therefore lies within the continuous symmetry group, whereas for $N = 0 \mod 3$ the $\mathbb{Z}_3$ is distinct.[10] Moreover, there is an additional $\mathbb{Z}_2$ "color conjugation" symmetry (see e.g. [140]) for the SO theory with even N, which comes from the outer automorphic group of $SO(2n)$. In net, the global symmetry group for the SO theories is $SU(3) \times U(1)_R \times \mathbb{Z}_{\mathrm{gcf}(6,N)}$, whereas for the USp theories it is $SU(3) \times U(1)_R \times \mathbb{Z}_{\mathrm{gcf}(3,\tilde{N})}$. Since the global symmetry groups must match between dual descriptions, this suggests that the ranks of the dual pair must be related as follows:

$$N = \tilde{N} + 3k\,, \tag{4.8}$$

for some odd integer k to be determined.

[10] In this case the center of $SU(3)$ lies within center of the $SU(N)$ or $SU(\tilde{N})$ gauge group.

$SO(N-4)\times SU(N)$ theory:

$SU(3)^3$	$\frac{3}{2}(N-3)N$
$SU(3)^2\,U(1)_R$	$-\frac{1}{2}(N-3)N-6$
$U(1)_R^3$	$\frac{4}{3}(N-3)N-33$
$U(1)_R$	-9
$SU(3)^2\,\mathbb{Z}_3$	1
$\mathbb{Z}_3$	1

$USp(\tilde{N}+4)\times SU(\tilde{N})$ theory:

$SU(3)^3$	$\frac{3}{2}\tilde{N}(\tilde{N}+3)$
$SU(3)^2\,U(1)_R$	$-\frac{1}{2}\tilde{N}(\tilde{N}+3)-6$
$U(1)_R^3$	$\frac{4}{3}\tilde{N}(\tilde{N}+3)-33$
$U(1)_R$	-9
$SU(3)^2\,\mathbb{Z}_3$	1
$\mathbb{Z}_3$	1

Table 4.1: The anomalies for the $\mathbb{C}^3/\mathbb{Z}_3$ orientifold gauge theories. In our notation, $G^2\mathbb{Z}_k=\prod_i \eta_i^{2T(r_i)}$ and $(\text{grav})^2\mathbb{Z}_k=\prod_i \eta_i^2$, where η_i is the multiplicative charge of ith Weyl fermion under the generator of the $\mathbb{Z}_k$ discrete symmetry. For a discrete anomaly η, the Jacobian for the symmetry transformation in the path integral is η^n, where n is the instanton number for the background in question; therefore, the anomaly vanishes iff $\eta=1$.

The global anomalies for the two models are shown in table 4.1.[11] We see that the anomalies match between the two theories for $N=\tilde{N}+3$, in agreement with the restriction from matching the global symmetry groups discussed above. In [60], it is shown that this rank relation agrees with D3 charge conservation, as it must. Since $\tilde{N}$ is necessarily even, this is evidence for a possible duality between the $SO(N-4)\times SU(N)$ theory for odd N and the $USp(N+1)\times SU(N-3)$ theory. It will also follow from the arguments of [60] that the $SO(N-4)\times SU(N)$ theory for even N is self-dual.

We now consider the moduli spaces of the prospectively dual theories, which is classically equivalent to the affine variety parameterized by the holomorphic gauge invariant operators identified under the F-term conditions and classical constraints [142]. In general, a holomorphic gauge invariant of the $SO(N-4)\times SU(N)$

[11] Here and in future we only write the $G^2\mathbb{Z}_k$ and $(\text{grav})^2\mathbb{Z}_k$ discrete anomalies for G nonabelian, as the remaining discrete "anomalies" need not match between two dual theories [140], and do not appear as anomalies in the path integral measure [140, 141].

theory takes the form

$$\mathcal{O}^{N_A,N_B} = A^{N_A} B^{N_B}\,, \tag{4.9}$$

for some particular choice of contraction of the gauge indices. Such operators may be classified as "mesons" or "baryons", depending on whether the $SU(N)$ Levi-Civita symbol is irreducibly involved in the contraction of gauge indices or not, i.e. on whether the baryonic charge

$$Q_A \equiv (N_A - 2N_B)/N \tag{4.10}$$

is vanishing or not. The corresponding $U(1)_A$ is anomalous, with an anomaly-free $\mathbb{Z}_3$ subgroup that was identified above:

$$Q_3 = \omega_3^{Q_A}\,, \tag{4.11}$$

where the Q_A charge of a gauge invariant operator is necessary integral, since $\mathbb{Z}_N \subset SU(N)$ lies within the gauge group.

No $SO(N-4)$ gauge invariants exist for the case $N_A = 1$ with $N > 5$. Thus, baryonic operators can be further subdivided into those with $Q_A > 0$, which can be "factored"[12] as

$$\mathcal{O} = (A^N)^{n_1}(AAB)^{n_2}\,, \tag{4.12}$$

and those with $Q_A < 0$, which can be "factored" as

$$\mathcal{O} = (B^N)^{n_1}(AAB)^{n_2}\,, \tag{4.13}$$

for integral powers n_1 and n_2.

We will focus on the "irreducible" baryons, of the form $\mathcal{O}_k^{(A)} \equiv A^{kN}$ and $\mathcal{O}_k^{(B)} = B^{kN}$. These have R-charges

$$Q_R(A^{kN}) = \frac{2(N+3)}{3}k\,, \quad Q_R(B^{kN}) = \frac{2(N-6)}{3}k\,, \tag{4.14}$$

[12] We do not mean to imply that the gauge index contractions factorize in the indicated manner.

and in both cases the $\mathbb{Z}_3$ charge ω_3^k. "Reducible" baryons are similar, but with an additional R-charge of $+2$ for every factor of (AAB) which appears.

The holomorphic gauge invariants of the $USp(\tilde{N}+4) \times SU(\tilde{N})$ theory can be similarly classified, where now the irreducible baryons have R-charges

$$Q_R(\tilde{A}^{k\tilde{N}}) = \frac{2(\tilde{N}-3)}{3}k\,, \quad Q_R(\tilde{B}^{k\tilde{N}}) = \frac{2(\tilde{N}+6)}{3}k\,, \tag{4.15}$$

with the $\mathbb{Z}_3$ charge ω_3^k, as before.

In general, mesons and reducible and irreducible baryons can all be intermixed in the duality relations between the two theories. However, in certain cases only one type of operator with the correct R-charge exists. In particular, this occurs in the following cases for the $SO(N-4) \times SU(N)$ theory:

1. For $Q_R < 2(N-6)$ and $Q_3 = \omega_3^0$, only mesonic operators are possible.
2. For $Q_3 = \omega_3$ and $Q_3 = \omega_3^{-1}$, the minimum possible R-charges are $\frac{2(N-6)}{3}$ and $\frac{4(N-6)}{3}$, respectively, corresponding to the irreducible baryons B^N and B^{2N}.

Similarly, for the $USp(\tilde{N}+4) \times SU(\tilde{N})$ theory:

1. For $Q_R < 2(\tilde{N}-3)$ and $Q_3 = \omega_3^0$, only mesonic operators are possible.
2. For $Q_3 = \omega_3$ and $Q_3 = \omega_3^{-1}$, the minimum possible R-charges are $\frac{2(\tilde{N}-3)}{3}$ and $\frac{4(\tilde{N}-3)}{3}$, respectively, corresponding to the irreducible baryons $\tilde{A}^{\tilde{N}}$ and $\tilde{A}^{2\tilde{N}}$.

This suggests the matching:

$$B^N \longleftrightarrow \tilde{A}^{\tilde{N}}\,, \quad B^{2N} \longleftrightarrow \tilde{A}^{2\tilde{N}}\,, \tag{4.16}$$

between the $Q_3 = \omega_3$ and $Q_3 = \omega_3^{-1}$ operators of minimum possible R-charge in both theories. In particular, these operators must have the same R-charge, i.e.

$$\frac{2(N-6)}{3} = \frac{2(\tilde{N}-3)}{3}\,, \tag{4.17}$$

which reproduces the rank relation $N = \tilde{N} + 3$ that we saw from the anomaly matching conditions.

The $SU(3)$ representations of these operators should also match. For $\tilde{A}^{\tilde{N}}$, the $SU(3)$ representation can be determined as follows: the symplectic invariant contracts the $\tilde{A}$'s in pairs, and the operator therefore factors as $(\tilde{A}^2)^{\tilde{N}/2}$. The $\tilde{B}$ F-term condition implies that the non-vanishing component of the $USp(\tilde{N}+4)$ invariant $\tilde{A}^2$ transforms as $(\overline{\begin{smallmatrix}\square\\ \square\end{smallmatrix}}, \square\!\square)_{4/3-4/\tilde{N}}$ under $SU(\tilde{N}) \times SU(3) \times U(1)_R$. Thus, $\tilde{A}^{\tilde{N}} = (\tilde{A}^2)^{\tilde{N}/2}$ takes the form of a "Pfaffian" of $\tilde{A}^2$, which is symmetric in its factors. The non-vanishing gauge-invariant component of $\tilde{A}^{\tilde{N}}$ therefore transforms in the $SU(3)$ representation

$$\mathrm{Sym}^{\tilde{N}/2}(\square\!\square) \equiv \underbrace{\square\!\square \otimes_S \square\!\square \otimes_S \ldots \otimes_S \square\!\square}_{\tilde{N}/2}\,, \tag{4.18}$$

where $\otimes_S$ denotes the symmetric tensor product.

For B^N, the F-term conditions impose no additional constraints. Using the computer algebra package `LiE` [143], one can show that the gauge invariant component of B^N also transforms in $\mathrm{Sym}^{\tilde{N}/2}(\square\!\square)$ for $N = 5, 7, 9, 11$ and 13 and $\tilde{N} = N-3$, whereas checking that this holds for larger N is too computationally expensive using `LiE` directly. Using the more efficient approach explained in appendix 4.E.1 we have verified agreement up to $N = 21$. It would be desirable to have an argument for all N, and while we do not have a general proof, in appendix 4.F we show how agreement between the $SU(3)$ representations of $\tilde{A}^{\tilde{N}}$ and B^N follows from a certain conjectural mathematical identity involving representations of the symmetric group, and we provide additional evidence for this identity.

As a further check, we should be able to match the mesonic operators with $Q_R < 2(N-6) = 2(\tilde{N}-3)$ between the two theories. Such operators can be factored into products of single-trace operators of the form:

$$\mathcal{O}_n^{i_1 j_1 k_1 \ldots i_n j_n k_n} \equiv \mathrm{Tr}\,(A^{i_1})^T B^{j_1} A^{k_1} \ldots (A^{i_n})^T B^{j_n} A^{k_n}\,, \tag{4.19}$$

where the F-term conditions imply that $\mathcal{O}_n$, with $Q_R = 2n$, is totally symmetric in its $3n$ $SU(3)$ indices. A similar argument goes through for the $USp(\tilde{N}+4)\times SU(\tilde{N})$ theory, resulting in the same spectrum of single-trace operators.

4.3.2 Limitations from the perturbativity of the string coupling

Before turning to specific examples of the duality, we briefly review some general obstructions to having a perturbative description of the D-brane gauge theories obtained from quantization of open strings. For a "perturbative description", we require that there exists some energy scale at which the gauge theory is weakly coupled, rather than weak coupling in the infrared.[13] The nature of these obstructions will also serve to illustrate how our proposed duality differs from Seiberg duality.

The one-loop beta function for a supersymmetric gauge theory is given by [12]:

$$\beta(g) \equiv \frac{dg}{d\ln\mu} = -\frac{g^3}{16\pi^2}\left(3\,T(\mathrm{Adj}) - \sum_i T(r_i)\right), \tag{4.20}$$

where $T(r)$ denotes the Dynkin index for the representation r,[14] Adj denotes the adjoint representation, and the sum is taken over all chiral superfields. If g is taken to be the holomorphic gauge coupling, then this result is exact, whereas the corresponding exact result for the physical gauge coupling depends on the anomalous dimensions of the chiral superfields.

The one-loop beta function coefficients (the term within parentheses in (4.20)) for the gauge group factors of the SO theory are:

$$b_{SO} = -18\,, \quad b_{SU} = 9\,, \tag{4.21}$$

[13] As we shall see, all of the D-brane gauge theories considered above are strongly coupled in the infrared, so the latter requirement is too strong.

[14] We employ the conventions $T(\square) = \frac{1}{2}$ for SU and USp gauge groups, and $T(\square) = 1$ for SO gauge groups.

whereas for those of the USp theory they are

$$\tilde{b}_{USp} = 9\,, \quad \tilde{b}_{SU} = -9\,. \tag{4.22}$$

Since the beta functions for the two gauge group factors have opposite signs, neither gauge theory is either IR free or asymptotically free, and the perturbative description will be valid at most in a finite range of energy scales. More precisely, a perturbative description at any scale is only possible if there is a separation between the dynamical scales, $\Lambda_{SO} \gg \Lambda_{SU}$ or $\tilde{\Lambda}_{SU} \gg \tilde{\Lambda}_{USp}$, along with a small superpotential coupling ($\lambda \ll 1$ or $\tilde{\lambda} \ll 1$) somewhere between these two scales. We will work in this limit. While it is possible in principle to incorporate corrections which are subleading in an expansion in small $\Lambda_{SU}/\Lambda_{SO}$ or $\tilde{\Lambda}_{USp}/\tilde{\Lambda}_{SU}$, this can be very difficult in practice, and we will not attempt to do so.

Conversely, the duality we propose partly addresses the question of what happens to the gauge theory in the limit where the dynamical scales have an inverted hierarchy. To see why, note that the string coupling is given by

$$\tau_{10\mathrm{d}} = \frac{1}{2\pi i} \ln\left[\lambda^{6(N-2)} \Lambda_{SO(N-4)}^{-18} \Lambda_{SU(N)}^{18}\right]\,, \quad \tau_{10\mathrm{d}} = \frac{1}{\pi i} \ln\left[\tilde{\lambda}^{3(\tilde{N}+2)} \Lambda_{USp(\tilde{N}+4)}^{9} \Lambda_{SU(\tilde{N})}^{-9}\right]\,, \tag{4.23}$$

for the prospective dual theories, up to a multiplicative numerical factor within the square brackets. This result can be established in a variety of ways; for completeness, we present a Coulomb branch computation of it in appendix 4.D. The result is also intuitive: a perturbative gauge theory necessarily corresponds to a weak string coupling.

The duality we propose acts as a modular transformation on $\tau_{10\mathrm{d}}$, mapping any perturbative string coupling to a nonperturbative one. Conversely, deforming to strong string coupling and applying the duality, we obtain a dual description with a weak string coupling. Thus, since the string coupling is linked to the hierarchy $\Lambda_{SU}/\Lambda_{SO}$ or $\tilde{\Lambda}_{USp}/\tilde{\Lambda}_{SU}$, the duality provides at least partial information about the behavior of these gauge theories with an inverted hierarchy $\Lambda_{SU} \gg \Lambda_{SO}$ or $\tilde{\Lambda}_{USp} \gg \tilde{\Lambda}_{SU}$.[15]

By contrast, Seiberg duality is generally used to understand the infrared behavior of a gauge theory which is perturbative at some scale, an illustration of the different natures of these two types of duality. While we can repeatedly apply Seiberg duality (together with deconfinement) to the individual gauge group factors, in our experience such an exercise never reproduces the prospective dual gauge theory,[16] providing further circumstantial evidence that the duality is not a Seiberg duality in the usual sense. Indeed, if we were able to do so, we would have to somehow reconcile the complicated gauge coupling relations which result from applying modular transformations to (4.23) with the algebraic relationships between dynamical scales predicted by Seiberg duality.

With these considerations in mind, we turn to a specific example of the proposed duality.

4.3.3 Case study: the $SU(5) \longleftrightarrow USp(6) \times SU(2)$ duality

Since we are constrained to $N \geq 4$ and $\tilde{N} \geq 0$ to have gauge groups of non-negative rank, the lowest rank duality we expect to find is between the $SU(5)$ and $USp(6) \times SU(2)$ gauge theories:

$$
\begin{array}{c|c|cc}
 & SU(5) & SU(3) & U(1)_R \\
\hline
A^i & \overline{\square} & \square & 16/15 \\
B^i & \begin{smallmatrix}\square\\ \square\end{smallmatrix} & \square & -2/15
\end{array}
\quad \longleftrightarrow \quad
\begin{array}{c|cc|cc}
 & USp(6) & SU(2) & SU(3) & U(1)_R \\
\hline
\tilde{A}^i & \square & \square & \square & -1/3 \\
\tilde{B}^i & \mathbf{1} & \square\square & \square & 8/3
\end{array}
\tag{4.24}
$$

$$
W = \tfrac{1}{2}\lambda\, \epsilon_{ijk} A^i_m A^j_n B^{mn;\,k}\ , \qquad W = \tfrac{1}{2}\tilde{\lambda}\, \Omega^{ab} \epsilon_{ijk} \tilde{A}^i_{a;\,m} \tilde{A}^j_{b;\,n} \tilde{B}^{mn;\,k}\ ,
$$

[15]While the Lagrangian definition of the gauge theory may be insufficient in this case, in principle string theory provides a complete definition for any N via the AdS/CFT correspondence, although this definition is impractical for computations except in the large N limit.

[16]Repeated application of Seiberg duality to these gauge theories requires a seemingly never-ending chain of deconfinements, leading to more and more gauge group factors. While one can imagine some of these factors eventually reconfining after several steps, we have not found this to be the case in our limited explorations of the matter.

where Ω denotes the symplectic invariant. We characterize the classical moduli space of both theories, and show that both generate a runaway superpotential.

We discuss higher rank examples in section 4.5.

The $USp(6) \times SU(2)$ theory

The F-term conditions are

$$\Omega^{ab} \tilde{A}^{[i}_{a;\,m} \tilde{A}^{j]}_{b;\,n} = 0\,, \quad \tilde{A}^{[j}_{b;\,n} \tilde{B}^{k];\,mn} = 0\,. \tag{4.25}$$

The first condition implies that all non-vanishing $USp(6)$ holomorphic gauge invariants are built from

$$\mathcal{A}^{ij} = \Omega^{ab} \epsilon^{mn} \tilde{A}^{i}_{a;\,m} \tilde{A}^{j}_{b;\,n}\,, \tag{4.26}$$

which transforms as a $\Box\!\Box_{-2/3}$ under $SU(3) \times U(1)_R$. The remaining holomorphic gauge invariants are easily cataloged:

$$\mathcal{B}^{ij} = \epsilon_{mp}\epsilon_{nq} \tilde{B}^{i;\,mn} \tilde{B}^{j;\,pq}\,, \quad \mathcal{B} = \frac{1}{6} \epsilon_{ijk}\epsilon_{np}\epsilon_{qr}\epsilon_{sm} \tilde{B}^{i;\,mn} \tilde{B}^{j;\,pq} \tilde{B}^{k;\,rs}\,, \tag{4.27}$$

which transform as $\Box\!\Box_{16/3}$ and $\mathbf{1}_8$, respectively, and obey the constraint $\mathcal{B}^2 = \frac{1}{2} \det \mathcal{B}^{ij}$.

The second F-term condition implies a constraint relating $\mathcal{A}^{ij}$ and $\mathcal{B}^{ij}$. In particular,

$$\mathcal{A}^{ij}\mathcal{B}^{kl} = \Omega^{ab}\epsilon^{mn}\epsilon_{pr}\epsilon_{qs} \tilde{A}^{i}_{a;\,m} \tilde{A}^{j}_{b;\,n} \tilde{B}^{k;\,pq} \tilde{B}^{l;\,rs} = 2\Omega^{ab}\epsilon_{qs} \tilde{A}^{i}_{a;\,m} \tilde{A}^{j}_{b;\,n} \tilde{B}^{k;\,mq} \tilde{B}^{l;\,ns}\,, \tag{4.28}$$

where we apply the first F-term condition to simplify the right-hand side. The second F-term condition then implies the constraint

$$\mathcal{A}^{i[j}\mathcal{B}^{k]l} = 0\,. \tag{4.29}$$

Thus, the classical moduli space has three distinct branches:

1. A branch with $\mathcal{B}^{ij} = 0$, parameterized by $\mathcal{A}^{ij} \neq 0$. For generic (full-rank) $\mathcal{A}^{ij}$, $USp(6) \times SU(2)$ breaks to a diagonal $SU(2)$, whereas for rank-deficient $\mathcal{A}^{ij}$, a larger gauge symmetry remains: $USp(4) \times SU(2)$ for $\mathcal{A}^{ij}$ rank one and $SU(2) \times SU(2)$ for $\mathcal{A}^{ij}$ rank two.

2. A branch with $\mathcal{A}^{ij} = 0$, parameterized by $\mathcal{B}^{ij} \neq 0$ (and $\mathcal{B}$). For $\mathcal{B}^{ij}$ rank two or three, the $SU(2)$ gauge factor is completely broken, whereas $SU(2) \to U(1)$ for $\mathcal{B}^{ij}$ rank one.

3. A branch with $\mathcal{A}^{ij} = e^{i\phi} \cos\theta \, v^i v^j$ and $\mathcal{B}^{ij} = e^{-i\phi} \sin\theta \, v^i v^j$. $USp(6) \times SU(2)$ breaks to $USp(4) \times U(1)$, except for when $\theta = 0$ or $\theta = \pi/2$, where this branch intersects branches 1 and 2, respectively.

We now discuss quantum corrections to this picture. The $USp(6)$ gauge factor is asymptotically free, whereas the $SU(2)$ gauge factor is infrared free. Thus, the infrared dynamics are primary governed by $USp(6)$ (to leading order in $\Lambda_{USp(6)} \ll \Lambda_{SU(2)}$), which generates an ADS superpotential:[17]

$$W_{\text{ADS}} = \frac{\Lambda_{\text{Sp}}^9}{\det \tilde{A}} \,, \tag{4.30}$$

where $\tilde{A}$ is viewed as a 6×6 matrix over $USp(6) \times (SU(2) \times SU(3))$.

We now consider the effect of the ADS superpotential on the moduli space. It is helpful to rewrite $\tilde{B}$ in the form:

$$\tilde{B}^i_{mn} = \frac{1}{2} \epsilon^{ijk} \hat{B}_{j_m k_n} \,, \tag{4.31}$$

where $\hat{B}$ transforms as $\overline{\begin{smallmatrix}\Box\\\Box\end{smallmatrix}}$ under a (fictitious) $SU(6) \supset SU(2) \times SU(3)$ flavor symmetry, which is broken by the constraint:

$$\epsilon^{mn} \hat{B}_{j_m k_n} = 0 \,, \tag{4.32}$$

[17]The same superpotential was obtained via a direct string computation in [136].

as well as the (weak) gauging of $SU(2)$. We impose the constraint via a Lagrange multiplier field M^{ij}:

$$W = \frac{1}{2}\tilde{\lambda}\,\Omega^{ab}\tilde{A}_a^M\tilde{A}_b^N\hat{B}_{MN} + \frac{1}{2}\epsilon^{mn}\hat{B}_{i_m j_n}M^{ij} + \frac{\Lambda_{\mathrm{Sp}}^9}{\det\tilde{A}}\,, \tag{4.33}$$

where M, N index the (fictitious) $SU(6)$. The $\hat{B}$ F-term condition is then

$$\tilde{\lambda}\,\Omega^{ab}\tilde{A}_a^{i_m}\tilde{A}_b^{j_n} = -\epsilon^{mn}M^{ij}\,, \tag{4.34}$$

so M^{ij} is related to the holomorphic gauge invariant $\mathcal{A}^{ij}$. Finally, the $\tilde{A}$ F-term condition reads:

$$\tilde{\lambda}\,\Omega^{ab}\tilde{A}_b^N\hat{B}_{MN} = \frac{\Lambda_{\mathrm{Sp}}^9}{\det\tilde{A}}\left(\tilde{A}^{-1}\right)^a_M\,, \tag{4.35}$$

or

$$\hat{B}_{MN} = -\frac{\Lambda_{\mathrm{Sp}}^9}{\det\tilde{A}}\left[\left(\tilde{\lambda}\,\tilde{A}^T\Omega\tilde{A}\right)^{-1}\right]_{MN}\,. \tag{4.36}$$

Applying (4.34), we obtain

$$\hat{B}_{i_m j_n} = -\frac{\Lambda_{\mathrm{Sp}}^9}{\det\tilde{A}}\,\epsilon_{mn}\,M_{ij}^{-1}\,. \tag{4.37}$$

However, this is incompatible with the constraint (4.32). Therefore, supersymmetry is broken.

In particular, for generic $|\tilde{A}| \gg |\tilde{\lambda}^{-1/9}\Lambda_{\mathrm{Sp}}|$, the classical superpotential dominates, and the classical F-terms set $\tilde{B} = 0$. We obtain a semiclassical "moduli-space" parameterized by $\mathcal{A}^{ij}$, subject to a runaway scalar potential generated by the ADS superpotential:[18]

$$W_{\mathrm{eff}} \sim \frac{\Lambda_{\mathrm{Sp}}^9}{\det\mathcal{A}}\,. \tag{4.38}$$

[18] In particular, branch 1 of the classical moduli space is approximately flat for large $\det\mathcal{A}^{ij}$. While other approximately flat regions corresponding to the other branches of moduli space may exist, they are not semiclassical, in that the classical superpotential must be made to cancel the large vacuum energy arising from the ADS superpotential.

The $SU(5)$ theory

The F-term conditions are:

$$A_a^{[i}A_b^{j]} = 0\,, \quad A_a^{[j}B^{k];\,ab} = 0\,. \tag{4.39}$$

We now characterize the classical moduli space. The first F-term constraint implies that

$$\langle A_a^i \rangle = v^i u_a\,, \tag{4.40}$$

where we may choose $u_a u^{*a} = 1$ without loss of generality. Suppose that $\langle A_a^i \rangle \neq 0$. We gauge fix such that $u_a = (0,0,0,0,1)$. Thus, the second F-term constraint implies

$$\langle B^{i;\hat{a}5} \rangle = b^{\hat{a}} v^i\,, \quad B^{i;\hat{a}\hat{b}} = b^{i;\hat{a}\hat{b}}\,, \tag{4.41}$$

where $\hat{a}, \hat{b} = 1 \ldots 4$.

Due to the first F-term constraint, the only possible non-vanishing holomorphic gauge invariant involving A is the following:

$$\mathcal{O}^{ijkl} = A_a^i B^{j;\,ab} B^{k;\,cd} B^{l;\,ef} \epsilon_{bcdef}\,. \tag{4.42}$$

However, applying the above gauge-fixed forms for $\langle A \rangle$ and $\langle B \rangle$, we find that this also vanishes. This suggests that $\langle A \rangle = 0$ once the D-term conditions are imposed, which can be verified by an explicit computation.[19]

Since the F-term conditions are then identically satisfied, the classical moduli space is the subset of that of the $\lambda = 0$ theory (without a superpotential) where

[19]In fact, to show that $\langle \Phi \rangle = 0$ in all supersymmetric vacua for some field Φ, it is sufficient to show that for every solution to the F-term conditions with $\langle \Phi \rangle \neq 0$, another solution with $\langle \Phi \rangle = 0$ exists with all holomorphic gauge invariants taking the same vevs. This is because the latter solution must be equivalent to the unique D-flat solution with the same holomorphic-gauge-invariant vevs under an extended complexified gauge transformation [142], but such a gauge transformation will never regenerate a vev for Φ.

$\langle A \rangle = 0$. This theory is s-confining, with the confined description [103, 104]:

$$(4.43)$$

	$SU(5)$	$SU(3)$	$SU(3)$	$U(1)$	$U(1)_R$
A^i	$\overline{\square}$	$\square$	$\mathbf{1}$	-3	$16/15$
B^i	$\begin{smallmatrix}\square\\\square\end{smallmatrix}$	$\mathbf{1}$	$\square$	1	$-2/15$
$T_i^m = A^2 B$		$\overline{\square}$	$\square$	-5	2
$U^{i;m}_{\ \ n} = AB^3$		$\square$	Adj	0	$2/3$
$V^{mn} = B^5$		$\mathbf{1}$	$\square\square$	5	$-2/3$

with the dynamical superpotential:

$$W = \frac{1}{\Lambda^9}\left(\epsilon_{mnp}\, T_i^m U^{i;n}_{\ \ q} V^{pq} - \frac{1}{3}\epsilon_{ijk}\, U^{i;m}_{\ \ p} U^{j;n}_{\ \ m} U^{k;p}_{\ \ n}\right) , \tag{4.44}$$

up to an overall multiplicative factor, where

$$\begin{aligned} T_i^m &\equiv \frac{1}{2}\epsilon_{ijk} A_a^j A_b^k B^{ab;m} , \qquad U^{i;m}_{\ \ n} \equiv \frac{1}{12}\epsilon_{npq}\epsilon_{bcdef} A_a^i B^{ab;p} B^{cd;q} B^{ef;m} , \\ V^{mn} &\equiv \frac{1}{160}\epsilon_{pqr}\epsilon_{a_1 b_1 c_1 d_1 e_1}\epsilon_{a_2 b_2 c_2 d_2 e_2} B^{a_1 a_2;p} B^{b_1 c_1;q} B^{b_2 c_2;r} B^{d_1 e_1;m} B^{d_2 e_2;n} . \end{aligned} \tag{4.45}$$

One feature of s-confining theories is that their classical and quantum moduli spaces match. Thus, the above spectrum of gauge invariants describes the classical moduli space of the $\lambda = 0$ theory, subject to the classical constraints, which are equivalent to the F-term conditions arising from the dynamical superpotential. Setting $\langle A \rangle = 0$, we find $\langle T \rangle = \langle U \rangle = 0$, with no remaining F-term constraints on V. Thus, the classical moduli space of the $\lambda \neq 0$ theory is parameterized by V^{ij}, which transforms as a $\square\square_{-2/3}$ under the $SU(3) \times U(1)_R$ preserved by the superpotential. This matches branch 1 of the classical moduli space of the $USp(6) \times SU(2)$ theory described above, and both are parameterized by the baryon discussed in section 4.3.1.

To obtain a quantum description of this theory, we perturb the s-confining theory (without superpotential) by the classical superpotential AAB,[20] which can

[20]There may also be instanton corrections to the superpotential, due to the completely broken

now be written in the form:

$$W_{\text{class}} = \lambda T^i_i \,,$$

which breaks $SU(3) \times SU(3) \times U(1) \to SU(3)_{\text{diag}}$, but preserves $U(1)_R$. One can show that the resulting F-term conditions cannot be solved, and therefore supersymmetry is broken [144].

For simplicity, we restrict our attention to the case where V^{ij} is full rank. We are then entitled to make the field redefinitions:

$$T^i_j = \hat{T}^i_j + \frac{\lambda^2 \Lambda^{18}}{4 \det V} \delta^i_j \,, \quad U^{ij}_l = \hat{U}^{ij}_k - \frac{1}{2} \lambda \, \Lambda^9 \, \epsilon^{ijk} V^{-1}_{kl} \,. \tag{4.46}$$

The resulting superpotential is

$$W = \frac{1}{\Lambda^9} \, \epsilon_{ijk} \left(\hat{T}^i_l \hat{U}^{lj}_m V^{mk} + \frac{1}{3} \hat{U}^{il}_n \hat{U}^{jm}_l \hat{U}^{kn}_m \right) + \frac{\lambda}{2} \left(\hat{U}^{ik}_l \hat{U}^{lj}_k - \hat{U}^{ij}_l \hat{U}^{kl}_k \right) V^{-1}_{ij} + \frac{\lambda^3 \Lambda^{18}}{4 \det V} \,. \tag{4.47}$$

To show that there are no F-flat solutions, we first show that $\hat{T} = \hat{U} = 0$ is the only solution to $\hat{T}$ and $\hat{U}$ F-term conditions for full-rank V. Note that in this case, V^{ij} can always be brought to the form $V^{ij} \propto \delta^{ij}$ after a complexified $SU(3)$ transformation. As the F-term conditions are appropriately covariant under this complexified symmetry transformation, it is sufficient to show that $\hat{T} = \hat{U} = 0$ is the only solution for $V^{ij} = z \, \delta^{ij}$.

In this case, the $\hat{T}$ F-term conditions reduce to

$$\epsilon_{ijk} \hat{U}^{lj}_k = 0 \,, \tag{4.48}$$

so that $\hat{U}^{ij}_k = \hat{U}^{ik}_j$. The $\hat{U}$ F-term conditions are:

$$0 = \frac{1}{\Lambda^9} \left(\epsilon_{inm} \hat{T}^i_k \, z + \epsilon_{ijk} \hat{U}^{il}_n \hat{U}^{jm}_l \right) + \frac{\lambda}{2z} \left(\hat{U}^{mk}_n + \hat{U}^{nm}_k - \delta_{kn} \hat{U}^{im}_i - \delta_{km} \hat{U}^{ii}_n \right) \,. \tag{4.49}$$

$SO(N-4)$ gauge group. These are subleading for $g_s \ll 1$ and vevs $\sim \Lambda_{SU}$, but could play a role for very large vevs.

Extracting the component which is symmetric in $n \leftrightarrow m$, we obtain

$$\hat{U}_k^{(nm)} - \delta_{k(n} \hat{U}^{ii}_{m)} = 0 \tag{4.50}$$

after applying the T F-term condition. Contracting with δ_{km} we find $\hat{U}^{ii}_j = 0$, so the above condition reduces to

$$\hat{U}_k^{mn} + \hat{U}_k^{nm} = 0 \,. \tag{4.51}$$

Together with the $\hat{T}$ F-term condition, this is sufficient to show that $\hat{U} = 0$. The remaining components of the $\hat{U}$ F-term condition then imply that $\hat{T} = 0$. By the above argument, these results apply for arbitrary (full-rank) V.

Having solved the $\hat{T}$ and $\hat{U}$ F-term conditions for $\hat{T}$ and $\hat{U}$, we may "integrate out" these fields, leaving the effective superpotential:

$$W_{\text{eff}} = \frac{\lambda^3 \Lambda^{18}}{4 \det V} \,, \tag{4.52}$$

for V^{ij}, which has no F-flat solutions and generates a runaway scalar potential, much like the $USp(6) \times SU(2)$ theory.

4.4 Matching of superconformal indices

In this section we discuss another very nontrivial test of the proposed duality: the matching between the superconformal indices of the two gauge theories. The discussion is inherently somewhat technical in nature, and readers primarily interested in the gauge theoretic consequences of the proposed duality may wish to skip ahead to section 4.5.

Superconformal indices for $\mathcal{N} = 1$ theories compactified on $S^3 \times \mathbb{R}$ [145, 146], while being a relatively recent development, have already provided important insights into the topic of dualities. In particular, equality of the superconformal index provides very strong support for a number of known and conjectured Seiberg dualities between $\mathcal{N}=1$ theories [147–156] and S-dualities in $\mathcal{N}=2$ [157–160] and $\mathcal{N}=4$ theories [161]. It also proves to be a very useful tool in the study of holography [146]. In this section we will present evidence for the agreement of the superconformal indices of the dual pair of theories presented in section 4.3. As we will see momentarily, the agreement relies on extremely non-trivial group-theoretical identities, providing very strong support for our conjectured duality.

It is not our intention to give a detailed discussion of the superconformal index here (we refer the interested reader to [145, 146, 148, 153] for very readable expositions of the topic), but we will briefly review in this section the basic elements that enter into its computation in order to settle notation. Consider a four dimensional theory compactified on $S^3\times\mathbb{R}$. The superconformal algebra has generators $J_\pm, J_3, \overline{J}_\pm, \overline{J}_3$ (associated to rotations on the S^3), supersymmetry generators $Q_\alpha, \overline{Q}_{\dot\alpha}$, translations P_μ, special superconformal generators $K_\mu, S_\alpha, \overline{S}_{\dot\alpha}$, superconformal dilatations H and the $U(1)$ R-symmetry generator R. Define $Q=Q_1$, which implies [146] $Q^\dagger = S_1$. The superconformal algebra then gives:

$$2\{Q^\dagger, Q\} = H - 2J_3 - \frac{3}{2}R \equiv \mathcal{H}\,. \tag{4.53}$$

The superconformal index is then defined by:

$$\mathrm{Tr}\,(-1)^F e^{-\beta\mathcal{H}} M\,, \tag{4.54}$$

with F the fermion number operator, and M any symmetry commuting with Q and $Q^\dagger$. Let us choose M to be generated by $\mathcal{R}\equiv R+2J_3$, $\overline{J}_3$, and gauge and flavor group elements g, f. Introducing appropriate chemical potentials, the refined index is thus given by:

$$\mathcal{I}(t,x,f) = \int dg\,\mathrm{Tr}\,(-1)^F e^{-\beta\mathcal{H}} t^{\mathcal{R}} x^{2\overline{J}_3} f g\,, \tag{4.55}$$

where we have integrated over the gauge group in order to count singlets only (we will have more to say about this integration below). It was argued in [146] that, exactly as in the case of the Witten index [162], the index (4.55) receives contributions only from states annihilated by Q and $Q^\dagger$, and thus the index does not actually depend on β, which plays the role of a regulator only.

In order to actually compute (4.55) we follow the prescription in [147], which gives a systematic way of computing the superconformal index in terms of the fields in the weakly coupled Lagrangian description of the theory, when one is available. For a general weakly coupled theory $\mathcal{T}$ with gauge group G and flavor group F, neither necessarily simple, and matter fields X_i with superconformal R-charge r_i in the representation $R_G^i \otimes R_F^i$ of $G \times F$, not necessarily irreducible, one constructs the letter

$$\begin{aligned} i_{\mathcal{T}}(t,x,g,f) = & \frac{(2t^2 - t(x+x^{-1}))\chi_{\mathrm{Adj}}(g)}{(1-tx)(1-tx^{-1})} \\ & + \frac{\sum_i \left(t^{r_i} \chi_{R_G^i}(g)\, \chi_{R_F^i}(f) - t^{2-r_i} \chi_{\overline{R_G^i}}(g)\, \chi_{\overline{R_F^i}}(f) \right)}{(1-tx)(1-tx^{-1})} \, . \end{aligned} \tag{4.56}$$

Here t,x,g,f are the same as in (4.55). $\chi_R(g)$ denotes the character of g in the representation R, and we denote with bars the complex conjugate representations. Once we have the letter (4.56) for $\mathcal{T}$, the superconformal index $\mathcal{I}_{\mathcal{T}}$ is obtained by taking the plethystic exponential, and integrating over the gauge group:

$$\mathcal{I}_{\mathcal{T}}(t,x,f) = \int dg \, \exp\left[\sum_{k=1}^{\infty} \frac{1}{k} i_{\mathcal{T}}(t^k, x^k, g^k, f^k) \right] . \tag{4.57}$$

Here dg denotes the Haar measure on the group G.[21]

We will thus compute the index in a weakly coupled, non-conformal description of the theory, and will assume that this gives the right index for the theory at its (presumed) superconformal fixed point in the IR. In the case of the theory compactified on S^3, one can argue [165] that since the index is independent of the

[21]We refer the reader to [163, 164] for nice references to the Lie group representation theory that we will need. We will give explicit expressions for the Haar measure of the groups of interest to us in section 4.4.2.

radius r of the S^3 it is independent of the dimensionless $r\Lambda$ coupling, and thus it is invariant under the RG flow and changes in the coupling constant. In order for this to be the case, we need to choose M in (4.54) constant along the flow, and in particular it should agree with the value of M at the IR superconformal fixed point. In particular, we need to choose the right value of the superconformal R-charge — determined using a-maximization [166], for instance — in constructing M.

Ideally, one would compute a closed form expression for (4.57) in the two dual phases, and then show that the two expressions agree for all N. Unfortunately we have not been able to prove the equality of the resulting indices, but in sections 4.4.1 and 4.4.2 we will provide very non-trivial evidence for the matching of the functions in two particularly tractable limits. The exact matching will thus remain a well motivated conjecture about elliptic hypergeometric integrals, which we formulate precisely in appendix 4.G.

4.4.1 Expansion in t

The first limit corresponds to an expansion around $t = 0$. Expanding (4.57) is elementary, but the integration over the gauge group requires some more advanced technology. In particular, one needs to use orthogonality of the characters under integration:

$$\int dg\, \chi_{R_i}(g)\chi_{\overline{R_j}}(g) = \delta_{ij}\,. \tag{4.58}$$

where R_i and R_j are irreps of G. When expanding the plethystic exponential (4.57) one encounters expressions of the form (we will deal with higher powers of g momentarily):

$$\int dg\, \chi_{R_1}(g)\cdots\chi_{R_n}(g)\,. \tag{4.59}$$

By using the well know property of the characters $\chi_{R_1}(g)\chi_{R_2}(g) = \chi_{R_1 \otimes R_2}(g)$, and then plugging the resulting expression into (4.58) with the second term being the character of the trivial representation (i.e. just 1), we obtain that (4.59) just counts the number of singlets in $R_1 \otimes \ldots \otimes R_n$.

When expanding (4.57) we will also encounter terms of the form $\chi_R(g^n)$. The act of decomposing such terms into characters of irreducible representations with group element g is known as applying the n-th Adams operator A_n to R. As an example, consider the fundamental representation $\square$ for $SU(N)$, which has character:[22]

$$\chi_{\square}(g) = \sum_{i=1}^{N} t_i \, , \tag{4.60}$$

where t_i are the elements of g on the maximal torus of $SU(N)$. Similarly, for the symmetric $\square\square$ and antisymmetric $\begin{smallmatrix}\square\\ \square\end{smallmatrix}$ representations we have:

$$\chi_{\square\square}(g) = \sum_{1 \le i \le j \le N} t_i t_j + \sum_{i=1}^{N} t_i^2 \, , \tag{4.61}$$

$$\chi_{\begin{smallmatrix}\square\\ \square\end{smallmatrix}}(g) = \sum_{1 \le i \le j \le N} t_i t_j \, . \tag{4.62}$$

It is thus clear that $\mathsf{A}_2(\square) = \square\square - \begin{smallmatrix}\square\\ \square\end{smallmatrix}$, or in terms of characters:

$$\chi_{\square}(g^2) = \chi_{\square\square}(g) - \chi_{\begin{smallmatrix}\square\\ \square\end{smallmatrix}}(g) \, . \tag{4.63}$$

Proceeding systematically in this way, one can decompose the plethystic exponential, up to any order, into sums of products of characters of irreps, which can then be easily integrated over the gauge group. The flavor characters can be dealt with similarly, and we will give the final results in terms of irreps. In the flavor case it is particularly important to do the decomposition into irreducible representations

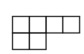
[22] Characters for representations of Lie groups can be worked out systematically using the Weyl character formula, see for example [163].

since there are important cancellations between terms, we will give an example below.

When the problem is formulated in this way the rest of the computation is conceptually straightforward, but doing this by hand quickly turns impossible, and the aid of computer systems is required for doing any non-trivial computations. We took advantage of the computer algebra package `LiE` [143] for doing the relevant group decompositions and Adams operations, and the mathematics software system `Sage` [167] for the polynomial manipulations.[23]

With this technology in place, the actual computation of the indices is straightforward, if lengthy. We obtain perfect agreement of the indices between the two dual theories in section 4.3 up to the degrees that we checked. In particular, for $SO(3) \times SU(7) \leftrightarrow USp(8) \times SU(4)$, we obtain the index:

$$\begin{aligned}\mathcal{I}_{SO/USp}(t,x,f) = 1 &+ t^{\frac{2}{3}}\left[\chi_{0,2}(f)+\chi_{4,0}(f)\right] \\ &+ t^{\frac{4}{3}}\left[2\chi_{0,4}(f)+2\chi_{2,0}(f)+\chi_{3,1}(f)+2\chi_{4,2}(f)+\chi_{8,0}(f)\right] \\ &+ t^{\frac{5}{3}}(x+x^{-1})\left[\chi_{0,2}(f)+\chi_{4,0}(f)\right] \\ &+ t^2\big[4+3\chi_{0,6}(f)+\chi_{1,4}(f)+5\chi_{2,2}(f)+3\chi_{3,3}(f) \\ &\quad + 2\chi_{4,1}(f)+3\chi_{4,4}(f)+\chi_{5,2}(f)+4\chi_{6,0}(f)+\chi_{6,3}(f) \\ &\quad + \chi_{7,1}(f)+2\chi_{8,2}(f)+\chi_{12,0}(f)\big]+\ldots \end{aligned} \tag{4.64}$$

where we have omitted terms of higher order in t. We have denoted the $SU(3)$ representation by its Dynkin labels, so for example the representation with $(2,2)$ Dynkin labels can be described as  in terms of ordinary Young tableaux. Notice that, as we were indicating before, even at this relatively low order the matching of the indices is very non-trivial, with rather complicated character poly-

[23]The actual code we used for the calculation can be downloaded from

`http://cern.ch/inaki/SCI.tar.gz`

nomials appearing. Furthermore, the agreement is only obtained after some rather involved group theory cancellations. As a particularly simple example, consider the $t^{\frac{2}{3}}$ term. In the $USp(8) \times SU(4)$ theory the relevant contribution after doing the gauge integration is of the form:

$$\mathcal{I}_{USp}(x,t,f) = t^{\frac{2}{3}} \left[\frac{1}{8}\chi_{\Box}^4(f) + \frac{1}{4}\chi_{\Box}^2(f)\chi_{\Box}(f^2) + \frac{3}{8}\chi_{\Box}^2(f^2) + \frac{1}{4}\chi_{\Box}(f^4) \right] + \dots \qquad (4.65)$$

where we have ignored terms of other orders in t. On the other hand, the corresponding expression for the $SO(3) \times SU(7)$ theory is given by:

$$\begin{aligned} \mathcal{I}_{SO}(x,t,f) = t^{\frac{2}{3}} \Bigg[& \frac{1}{140}\chi_{\Box}^7(f) + \frac{1}{40}\chi_{\Box}^5(f)\chi_{\Box}(f^2) - \frac{1}{8}\chi_{\Box}(f)\chi_{\Box}^3(f^2) \\ & - \frac{1}{4}\chi_{\Box}(f)\chi_{\Box}(f^2)\chi_{\Box}(f^4) + \frac{1}{10}\chi_{\Box}^2(f)\chi_{\Box}(f^5) \\ & + \frac{1}{10}\chi_{\Box}(f^2)\chi_{\Box}(f^5) + \frac{1}{7}\chi_{\Box}(f^7) \Bigg] + \dots \end{aligned} \qquad (4.66)$$

again ignoring terms of different degree in t. We clearly see that both expressions look rather different, and only agree after using some non-trivial group theoretical identities involving the Adams operator.

One can proceed similarly for other ranks. As we have seen in section 4.3.3 reducing the rank leads to a runaway theory, so we will restrict ourselves to larger ranks. In particular we have calculated the superconformal index for $SO(5) \times SU(9) \leftrightarrow USp(10) \times SU(6)$ and $SO(7) \times SU(11) \leftrightarrow USp(12) \times SU(8)$ up to order t^4 and found in both cases perfect agreement.

It is interesting to construct explicitly the states that are annihilated by $\mathcal{H}$ and therefore contribute to the superconformal index (4.55). They are the scalar components of the chiral multiplets, the right-handed chiral fermions in the complex conjugate representation as well as the gauginos and field strengths of the gauge groups. The fields that contribute for $SO(N-4) \times SU(N)$ are shown in table 4.2.

Field	$SO(N-4)\times SU(N)$	$SU(3)$	t exponent	$SU(2)_r$
$A^i_{(l)}$	$(\square,\overline{\square})$	$\square$	$\frac{2}{3}+\frac{2}{N}+l$	$l+1$
$B^i_{(l)}$	$(\mathbf{1},\begin{smallmatrix}\square\\\square\end{smallmatrix})$	$\square$	$\frac{2}{3}-\frac{4}{N}+l$	$l+1$
$\bar{\psi}^A_{(l)}$	$(\square,\square)$	$\overline{\square}$	$\frac{4}{3}-\frac{2}{N}+l$	$l+1$
$\bar{\psi}^B_{(l)}$	$\left(\mathbf{1},\overline{\begin{smallmatrix}\square\\\square\end{smallmatrix}}\right)$	$\overline{\square}$	$\frac{4}{3}+\frac{4}{N}+l$	$l+1$
$\lambda^{SO}_{(l)}$	$(\begin{smallmatrix}\square\\\square\end{smallmatrix},\mathbf{1})$	$\mathbf{1}$	$1+l$	$l\oplus(l+2)$
$F^{SO}_{(l)}$	$(\begin{smallmatrix}\square\\\square\end{smallmatrix},\mathbf{1})$	$\mathbf{1}$	$2+l$	$(l+1)\oplus(l+1)$
$\lambda^{SU}_{(l)}$	$(\mathbf{1},\mathrm{Adj})$	$\mathbf{1}$	$1+l$	$l\oplus(l+2)$
$F^{SU}_{(l)}$	$(\mathbf{1},\mathrm{Adj})$	$\mathbf{1}$	$2+l$	$(l+1)\oplus(l+1)$

Table 4.2: The fields which contribute to the superconformal index for $SO(N-4)\times SU(N)$, where the $SU(2)_r$ column denotes the representation under the $SU(2)$ group generated by $\overline{J}_\pm, \overline{J}_3$.

The superconformal index counts gauge invariant combinations of these fields and we can explicitly construct these combinations to check our result (4.64). For the $SO(3)\times SU(7)$ theory, the gauge invariants which contribute at the lowest orders in the Taylor expansion about $t=0$ are shown in table 4.3.[24] Taking into account the factor $(-1)^F$ we find perfect agreement with (4.64).

Likewise we can check the gauge invariant contributions for the $USp(8)\times SU(4)$ theory. We again find perfect agreement with (4.64) as is shown in detail in appendix 4.E.

4.4.2 Large N

Using the tools given above, one can go as high in N and t as desired, limited only by computing resources and patience. In this section we will approach the computation of the index from a complementary perspective, namely we will compute

[24]For the operator $\left(B^i_{(0)}\right)^{21}$ a direct computation of the representation under the flavor group takes very long so that we devised a refined method that is explained in appendix 4.E.1.

operator	t exp.	$2\overline{J}_3$	$SU(3)$ character
$\left(B^i_{(0)}\right)^7$	$\frac{2}{3}$	0	$\chi_{0,2}(f)+\chi_{4,0}(f)$
$\left(B^i_{(0)}\right)^{14}$	$\frac{4}{3}$	0	$2\chi_{0,4}(f)+2\chi_{2,0}(f)+\chi_{3,1}(f)+2\chi_{4,2}(f)+\chi_{8,0}(f)$
$\left(B^i_{(0)}\right)^6 B^i_{(1)}$	$\frac{5}{3}$	± 1	$\chi_{0,2}(f)+\chi_{4,0}(f)$
$[\lambda^{SO}_{(0)}]^2$	2	0	1
$[\lambda^{SU}_{(0)}]^2$	2	0	1
$A_{(0)}\psi^A_{(0)}$	2	0	$1+\chi_{1,1}(f)$
$B_{(0)}\psi^B_{(0)}$	2	0	$1+\chi_{1,1}(f)$
$(A_{(0)})^2 B_{(0)}$	2	0	$1+\chi_{1,1}(f)$
$\left(B^i_{(0)}\right)^{21}$	2	0	$3+3\chi_{0,6}(f)+\chi_{1,1}(f)+\chi_{1,4}(f)+5\chi_{2,2}(f)$ $+3\chi_{3,3}(f)+2\chi_{4,1}(f)+3\chi_{4,4}(f)+\chi_{5,2}(f)$ $+4\chi_{6,0}(f)+\chi_{6,3}(f)+\chi_{7,1}(f)+2\chi_{8,2}(f)+\chi_{12,0}(f)$

Table 4.3: The gauge-invariants contributing to the superconformal index of the $SO(3)\times SU(7)$ theory at the lowest orders in the Taylor expansion about $t=0$, where $()^m$ denotes taking the m-th symmetrized tensor product and $[]^m$ taking the m-th antisymmetrized tensor product.

the index in the dual pair of $SO(N-4)\times SU(N)$ and $USp(N+1)\times SU(N-3)$ theories when $N\to\infty$, following the discussion in [146, 148, 165].

Let us start by taking the large N limit of the Haar measures for group integration over the ABC Lie groups appearing in our construction. Starting with $SU(N)$, the explicit form of the integral of a gauge invariant function $f(g)$ (such as a function of group characters) over the group is given by [148]:

$$\int dg f(g) = \frac{1}{N!}\oint \prod_{j=1}^{N-1} \frac{dx_i}{2\pi i x_i}\Delta(x)\Delta(x^{-1})f(x)\,, \tag{4.67}$$

with $\Delta(t)=\prod_{i<j}(t_i-t_j)$, and the integration can be taken to be on the unit circle around $x_i=0$. Parameterizing $x=e^{i\theta}$, the integral (4.67) can be equivalently

rewritten as:

$$\int dg f(g) = \frac{1}{N!}\frac{1}{(2\pi)^{N-1}} \int \prod_{j=1}^{N-1} d\theta_j \; \Delta(e^{i\theta})\Delta(e^{-i\theta}) f(\theta) \,. \tag{4.68}$$

Using now that $\sum_{n=1}^{\infty} x^n/n = -\log(1-x)$, this can be conveniently rewritten as:

$$\int dg f(g) = \frac{1}{N!}\frac{1}{(2\pi)^{N-1}} \int \prod_{k=1}^{N-1} d\theta_k \exp\left(-\sum_{n=1}^{\infty} \frac{1}{n} \sum_{i\neq j} e^{in(\theta_i-\theta_j)}\right) f(\theta) \,. \tag{4.69}$$

Let us point out in passing that this expression, modulo some constant factors, can be also rewritten as

$$\int dg f(g) \propto \int \prod_{k=1}^{N-1} d\theta_k \exp\left(-\sum_{n=1}^{\infty} \frac{1}{n} \chi_{\rm Adj}(e^{in\theta})\right) f(\theta) \,, \tag{4.70}$$

where $\chi_{\rm Adj}(e^{in\theta})$ denotes the character of x^n in the adjoint. This structure also applies to the USp and SO cases we analyze below.

In the large N limit, we replace the sum over eigenvalues $\sum_i$ with a continuous integral $N\int d\alpha$. We also have that θ becomes a continuous function $\theta(\alpha)$. It is convenient to change the variable of integration to θ itself: $\int d\alpha \to \int d\theta \rho(\theta)$, where we have denoted the Jacobian $\rho(\theta) = d\alpha/d\theta$. Doing these changes, we have that at large N:

$$\begin{aligned} \sum_{i\neq j} e^{in(\theta_i-\theta_j)} &= \left(\sum_i e^{in\theta_i}\right)\left(\sum_j e^{-in\theta_j}\right) - N \\ &\to N^2 \left(\int d\theta\rho\, e^{in\theta}\right)\left(\int d\theta\rho\, e^{-in\theta}\right) - N \,. \end{aligned} \tag{4.71}$$

In what follows we will drop constant terms (those independent of ρ) for simplicity, we will account for their effect by fixing the normalization of the final result. It is also convenient to introduce, as in [148], $\rho_n = N\int d\theta\, \rho\, e^{in\theta}$. With these changes, we have that:

$$\sum_{i\neq j} e^{in(\theta_i-\theta_j)} \to |\rho_n|^2 \,, \tag{4.72}$$

and the integration becomes simply a product of complex gaussian integrals:

$$
\begin{aligned}
\int dg f(g) &\to \int \prod_{n=1}^{\infty} \left(\frac{i\, d^2\rho_n}{2\pi n} e^{-\frac{1}{n}|\rho_n|^2} \right) f(\rho) \\
&\equiv \int \prod_{n=1}^{\infty} [d^2\rho_n] f(\rho) \equiv \int [d^2\rho] f(\rho) \,,
\end{aligned}
\tag{4.73}
$$

where we have introduce some convenient notation, and imposed unit normalization for the large N measure: $\int [d^2\rho_n]\, 1 = 1$.

We can proceed similarly for the other cases of interest to us. For the $USp(2N)$ group, the Haar measure is given by:

$$
\int dg f(g) = \frac{(-1)^N}{2^N\, N!} \oint \left(\prod_{j=1}^{N} \frac{dx_j}{2\pi i x_j} \big(x_j - x_j^{-1}\big)^2 \right) \Delta(x + x^{-1})^2 f(x) \,. \tag{4.74}
$$

By an argument very similar to the above, we can rewrite this as (ignoring constant prefactors):

$$
\begin{aligned}
\int dg f(g) \propto \int \prod_{j=1}^{N} d\theta_j \exp\Big[- \sum_{n=1}^{\infty} \frac{1}{2n} \Big\{ \Big(\sum_k \big(e^{in\theta_k} + e^{-in\theta_k}\big) \Big)^2 \\
+ \sum_k \big(e^{2in\theta_k} + e^{-2in\theta_k}\big) \Big\} \Big] f(\theta) \,.
\end{aligned}
\tag{4.75}
$$

It is thus natural to introduce $\gamma = d\alpha/d\theta$ as before, and to define the real variable $\gamma_n \equiv N \int d\theta\, \gamma(\theta)\, (e^{in\theta} + e^{-in\theta})$. The resulting measure is again an infinite product of (real, in this case) Gaussian integrals, which when properly normalized can be written as:

$$
\begin{aligned}
\int dg f(g) &\to \int \left[\prod_{n=1}^{\infty} \frac{d\gamma_n}{\sqrt{2\pi n}} \exp\Big(-\frac{1}{2n} (\gamma_n + 1)^2 \Big) \right] f(\gamma) \\
&\equiv \int \prod [d\gamma_n] f(\gamma) \equiv \int [d\gamma] f(\gamma) \,.
\end{aligned}
\tag{4.76}
$$

Finally, for $SO(2N+1)$, the process works very similarly to $USp(2N)$. The integration over the gauge group is given by

$$
\int dg f(g) = \frac{(-1)^N}{2^N\, N!} \oint \left(\prod_{j=1}^{N} \frac{dx_j}{2\pi i x_j} \big(\sqrt{x_j} - \sqrt{x_j}^{-1}\big)^2 \right) \Delta(x + x^{-1})^2 f(x) \,, \tag{4.77}
$$

which at large N becomes

$$\int dg f(g) \to \int \left[\prod_{n=1}^{\infty} \frac{d\gamma_n}{\sqrt{2\pi n}} \exp\left(-\frac{1}{2n}(\gamma_n - 1)^2\right)\right] f(\gamma) \\ \equiv \int \prod [d\gamma_n] f(\gamma) \equiv \int [d\gamma] f(\gamma) \,, \tag{4.78}$$

where we have introduced $\gamma_n \equiv 1 + N \int d\theta\, \gamma(\theta)\,(e^{in\theta} + e^{-in\theta})$. We have chosen to shift the definition of γ_n by 1 in order to make the argument for the equality of the indices below more straightforward.

In order to rewrite the superconformal index (4.56), (4.57) at large N, we need to find out the large N limit of the characters of the representations appearing in our theory. Consider for instance the symmetric representation of $SU(N)$. Its character is given by:

$$\chi_{\square\square}(x) = \sum_{i<j} x_i x_j + \sum_{i=1}^{N} x_i^2 = \frac{1}{2}\left(\sum_{i\neq j} x_i x_j\right) + \sum_{i=1}^{N} x_i^2 \\ = \frac{1}{2}\left(\left(\sum_i x_i\right)^2 + \sum_{i=1}^{N} x_i^2\right) . \tag{4.79}$$

Introducing ρ_n as before, this can be rewritten as:

$$\chi_{\square\square}(x) \to \frac{1}{2}(\rho_1^2 + \rho_2) \,. \tag{4.80}$$

Other representations can be treated similarly, let us just quote the results that we will need. For $SU(N)$ we have:

$$\chi_{SU,\square}(x^n) = \sum_i x_i^n \quad \to \ \rho_n \,, \tag{4.81}$$

$$\chi_{SU,\boxminus}(x^n) = \sum_{i<j} x_i^n x_j^n - \sum_{i=1}^{N} x_i^{2n} \quad \to \ \frac{1}{2}(\rho_n^2 - \rho_{2n}) \,, \tag{4.82}$$

$$\chi_{SU,\mathrm{Adj}}(x^n) = \sum_{i,j} x_i^n x_j^{-n} - 1 \quad \to \ |\rho_n|^2 - 1 \,. \tag{4.83}$$

For $USp(2N)$ we have:

$$\chi_{USp,\square}(x^n) = \sum_i (x_i^n + x_i^{-n}) \quad \to \ \gamma_n \,, \tag{4.84}$$

$$\begin{aligned}\chi_{USp,\text{Adj}}(x^n) = &\sum_{i<j}(x_i^n x_j^n + x_i^n x_j^{-n} + x_i^{-n}x_j^n + x_i^{-n}x_j^{-n}) \\ &+\sum_i (x_i^{2n}+x_i^{-2n}) + N \quad \to \ \frac{1}{2}(\gamma_n^2+\gamma_{2n})\,,\end{aligned} \tag{4.85}$$

and similarly for $SO(2N+1)$:

$$\chi_{SO,\square}(x^n) = \sum_i (x_i^n + x_i^{-n}) + 1 \quad \to \ \gamma_n\,, \tag{4.86}$$

$$\begin{aligned}\chi_{SO,\text{Adj}}(x^n) = &\sum_{i<j}(x_i^n x_j^n + x_i^n x_j^{-n} + x_i^{-n}x_j^n + x_i^{-n}x_j^{-n}) \\ &+\sum_i (x_i^{n}+x_i^{-n}) + N \quad \to \ \frac{1}{2}(\gamma_n^2-\gamma_{2n})\,.\end{aligned} \tag{4.87}$$

The equality of the indices at large N between the $SO\times SU$ and $USp\times SU$ cases now follows from a simple redefinition of the integration variables: $\rho_n \leftrightarrow -\rho_n$, $\gamma_n \leftrightarrow -\gamma_n$. Indeed, under this change of variables, for the measures of integration we have that $[d^2\rho]$ stays invariant, while $[d\gamma]_{SO}$ gets exchanged with $[d\gamma]_{USp}$. Similarly, we have that $\chi_{SO,\text{Adj}} \leftrightarrow \chi_{USp,\text{Adj}}$, the symmetric and antisymmetric characters of SU get exchanged, and the character of the bifundamental, given by $\rho_n\gamma_n$, stays invariant. This is precisely the map between the two dual theories.

It is also instructive to compare the result of the large N computation in this section with the low N computations in the previous section. From the discussion in subsection 4.3.1, baryons start contributing at order $t^{\frac{2N}{3}-4}$, and thus disappear in the large N limit of the expressions above. The mesonic contributions have N independent t exponent, and survive the limit. This means in particular that the t expansion becomes N independent for large N. As an illustration, for $N = 15$ we find that the direct low N computation and the large N computation agree up to order 5 in the t expansion, with the result:

$$\begin{aligned}
\mathcal{I}_{SO/USp}(t,x,f) =& 1+t^2\big[-\chi_{1,1}(f)+1\big] \\
& -t^3(x+x^{-1})\big[\chi_{1,1}(f)+\chi_{3,0}(f)\big] \\
& -t^4(x^2+x^{-2})\big[\chi_{1,1}(f)+\chi_{3,0}(f)\big] \\
& +t^4\big[\chi_{0,3}(f)-2\chi_{1,1}(f)+\chi_{6,0}(f)\big] \\
& -t^5(x^3+x^{-3})\big[\chi_{1,1}(f)+\chi_{3,0}(f)\big] \\
& +t^5(x^1+x^{-1})\big[\chi_{0,3}(f)+2\chi_{2,2}(f)+2\chi_{4,1}(f)+\chi_{6,0}(f)\big]+\ldots
\end{aligned} \tag{4.88}$$

In addition to the physical arguments for the duality presented in the rest of this chapter, we find the "experimental" evidence for the agreement of the indices presented in this and the previous subsection compelling enough to conjecture the equality of the indices for all values of N:

$$\mathcal{I}_{USp} = \mathcal{I}_{SO}\,. \tag{4.89}$$

In appendix 4.G we reformulate this equality in terms of elliptic hypergeometric integrals. This leads us to a conjecture about elliptic hypergeometric functions that could potentially be proven along the lines of [168].

4.5 Infrared behavior

We now discuss the infrared behavior of these gauge theories, and what it implies about our proposed duality.

Before turning to specific examples where the infrared behavior can be determined using Seiberg duality, we first note that the string coupling (4.23) is constant along the RG flow, i.e. it is "exactly dimensionless" (its exact quantum-corrected scaling-dimension vanishes). In the large N limit, this result follows from the no-scale structure of the supergravity dual. In appendix 4.C, we argue that this persists at finite N, and that the string coupling is neither perturbatively nor nonperturbatively renormalized (at the origin of moduli space).

The fact that τ_{10d} is exactly dimensionless can have important consequences for the infrared behavior. Generically, this implies that the infrared fixed point is actually a fixed line parameterized by τ_{10d}. The string coupling therefore maps to an exactly marginal operator at the superconformal fixed point. This is to be expected: as we saw in section 4.2, an $SL(2,\mathbb{Z})$ duality generally incorporates self-dualities relating each gauge theory to itself at different values of the couplings, whereas it has been suggested that the occurrence of self-dualities is closely tied to that of exactly marginal operators [100, 122], with the corresponding deformation interpolating between the dual descriptions in the infrared.

Thus, in general the two fixed points reached by the dual theories in their respective perturbative regimes will occur at different locations along a line of fixed points parameterized by the string coupling. Since the theories are connected by a continuous deformation, the global anomalies, the superconformal index, and the topology of the moduli space should match between the two fixed points, provided that a discontinuous "phase transition" does not occur in between; we have argued that these data do indeed match in section 4.3 and section 4.4.

In some cases, the infrared behavior may be different. In particular, the string coupling, despite being exactly dimensionless along the flow, does not always correspond to a deformation of the fixed point. Instead, the flows may converge to a single fixed point; this can happen when the string coupling becomes ill-defined at that point, for instance when its constituent couplings approach some limit. As a toy example, consider an $SU(N)^2$ gauge group with N_F $(\square,\overline{\square}) \oplus (\overline{\square},\square)$ bifundamental "flavors" and no superpotential. If $N_F \geq 3$, then the two gauge theories are infrared free, whereas

$$\frac{1}{g_1^2} - \frac{1}{g_2^2}\,, \tag{4.90}$$

is an exactly dimensionless coupling. However, while (4.90) is constant along the flow, as $g_1 \to 0$ the difference between the gauge couplings g_1 and g_2 also flows to zero, and in the deep infrared the theory is free, independent of the initial values of the couplings. In these cases, since the string coupling is irrelevant at the fixed point, the infrared physics should not depend on $\tau_{10\mathrm{d}}$, and the two fixed points should be the same, as in Seiberg duality.

We now consider specific examples. In section 4.3.3, we saw the both the SO and USp theories have a dynamically generated runaway superpotential for $N = 5$ ($\tilde{N} = 2$). We now attempt to determine the infrared behavior of these gauge theories for larger values of N.

It turns out that the USp theories are in general somewhat more tractable than the SO theories, so we focus on the former, extracting predictions for the IR behavior of the latter. We begin by discussing the cases $N = 7$ and $N = 9$ in section 4.5.1 and section 4.5.2, respectively, where the infrared behavior can be determined using known dualities. In section 4.5.3, we speculate about the infrared behavior for $N > 9$.

4.5.1 The $USp(8) \times SU(4)$ theory

The prospective dual theories for $N = 7$ are:

	$SO(3)$	$SU(7)$	$SU(3)$	$U(1)_R$
A^i	$\square$	$\overline{\square}$	$\square$	$\frac{20}{21}$
B^i	1	$\begin{smallmatrix}\square\\\square\end{smallmatrix}$	$\square$	$\frac{2}{21}$

$$W = \tfrac{1}{2}\lambda\,\delta^{ab}\epsilon_{ijk}A^i_{a;m}A^j_{b;n}B^{mn;k}\,,$$

$\longleftrightarrow$

	$USp(8)$	$SU(4)$	$SU(3)$	$U(1)_R$
$\tilde{A}^i$	$\square$	$\overline{\square}$	$\square$	$\frac{1}{6}$
$\tilde{B}^i$	1	$\square\square$	$\square$	$\frac{5}{3}$

$$W = \tfrac{1}{2}\tilde{\lambda}\,\Omega^{ab}\epsilon_{ijk}\tilde{A}^i_{a;\,m}\tilde{A}^j_{b;\,n}\tilde{B}^{mn;\,k}\,. \tag{4.91}$$

We focus on the $USp(8) \times SU(4)$ theory, showing that it has an infrared-free dual description with a quantum moduli space.

The IR dynamics of this theory are particularly easy to describe, as the $USp(8)$ factor is s-confining, leaving an $SU(4) \cong SO(6)$ gauge theory in the confined description which can be Seiberg dualized to obtain an IR free description.

The dynamics of the s-confined $USp(8)$ can be described in terms of the meson

$$M^{IJ} = \Omega^{ab} \tilde{A}_a^I \tilde{A}_b^J \,, \tag{4.92}$$

with the superpotential

$$W = \frac{1}{\Lambda_{\text{Sp}}^9} \, \text{Pf}\, M \,, \tag{4.93}$$

where the indices I, J parameterize a fictitious $SU(12) \subset SU(4) \times SU(3)$. M decomposes into irreps Ψ and Φ transforming as $(\overline{\square\square}, \overline{\square})$ and $\left(\overline{\begin{smallmatrix}\square\\\square\end{smallmatrix}}, \square\square\right)$ under $SU(4) \times SU(3)$, respectively, where the superpotential now takes the form

$$W \sim \frac{1}{\Lambda_{\text{Sp}}^3} \left(\Phi^6 + \Phi^5 \Psi + \ldots\right) + \tilde{\lambda} \Lambda_{\text{Sp}} \Psi \tilde{B} \,, \tag{4.94}$$

where we suppress the index structure for simplicity, and we absorb a factor of Λ_{Sp}^{-1} into the definition of Φ and Ψ to make them dimension-one fields. Thus, Ψ and $\tilde{B}$ acquire a mass and can be integrated out, leaving the superpotential

$$W \sim \frac{1}{\Lambda_{\text{Sp}}^3} \Phi^6 \,, \tag{4.95}$$

where the remaining terms are exactly those generated by the Pfaffian for $\Psi = 0$.

It is now instructive to rewrite the gauge group $SU(4)$ as $SO(6)$, under which the Φ transform as a vector. The gauge-invariant meson Φ^2 transforms as $\square\square\square\square_{+2/3} \oplus \overline{\square\square}_{+2/3}$ under $SU(3) \times U(1)_R$, corresponding to the baryon $\tilde{A}^4$ in the original theory. In terms of this meson, the superpotential takes the form:

$$W \sim \frac{1}{\Lambda_{\text{Sp}}^3} \left((\Phi^2)^3 + \det \Phi\right) \,, \tag{4.96}$$

where we suppress the index structure and numerical prefactors for simplicity, and $\det\Phi$ denotes the lone $SO(6)$ baryon, which is automatically $SU(3)$ invariant.

Applying Seiberg duality, we obtain the $SO(4)$ gauge theory:

	$SO(4)$	$SU(3)$	$U(1)_R$
Q	$\square$	$\overline{\square\square}$	$\frac{2}{3}$
$\mathcal{A}$	$\mathbf{1}$	$\square\square\square \oplus \overline{\square\square}$	$\frac{2}{3}$

(4.97)

with the superpotential

$$W \sim \lambda_1 \mathcal{A}^3 + \lambda_2 \mathcal{A}\, Q^2 \,, \tag{4.98}$$

where $\mathcal{A} = \frac{1}{\lambda_1^{1/3}\Lambda_{\mathrm{Sp}}}\Phi^2$. The baryon $\det\Phi$ in the superpotential (4.96) maps to a glueball $\epsilon^{ijkl}W_{ij}W_{kl}$ in the dual theory [88], which causes a splitting between the two gauge couplings, τ_1 and τ_2, of the $SU(2)\times SU(2) \cong SO(4)$ gauge group. Performing scale matching at each step in this chain of dualities, we find that

$$\tau_1 - \tau_2 \sim e^{\pi i \tau_{10\mathrm{d}}/2} \,, \tag{4.99}$$

where $\tau_{10\mathrm{d}}$ is the ten-dimensional axio-dilaton, which is related to the other couplings by

$$e^{\pi i \tau_{10\mathrm{d}}} = \lambda_1^2 \lambda_2^{-6} e^{-\pi i(\tau_1+\tau_2)} \,. \tag{4.100}$$

Thus, the splitting between the gauge couplings, (4.99), is nonperturbatively suppressed at weak string coupling.

We now consider the infrared behavior of this theory. Since the beta function coefficient of $SO(4)$ vanishes, the theory has a free fixed point. We argue that this fixed point is attractive. The exact beta functions are:[25]

$$\beta(g_i) = -\frac{3g_i^3}{8\pi^2}\frac{\gamma_Q}{1-g_i^2/4\pi^2} \,, \quad \beta(\lambda_1) = \frac{3}{2}\lambda_1\gamma_{\mathcal{A}} \,, \quad \beta(\lambda_2) = \frac{1}{2}\lambda_2(\gamma_{\mathcal{A}} + 2\gamma_Q) \,, \tag{4.101}$$

[25] The argument given here is somewhat of an oversimplification since $\mathcal{A}$ is not an irrep, and therefore λ_1 and λ_2 correspond to more than one physical coupling. However, it is straightforward to account for the additional complications which arise in a more careful treatment.

where γ_Q and $\gamma_{\mathcal{A}}$ are the anomalous dimensions of Q and $\mathcal{A}$, which take the form

$$\gamma_Q = \frac{k_2|\lambda_2|^2}{192\pi^2} - \frac{3}{16\pi^2}(g_1^2 + g_2^2)\,, \quad \gamma_{\mathcal{A}} = \frac{k_1|\lambda_1|^2}{112\pi^2} + \frac{k_2|\lambda_2|^2}{336\pi^2}\,, \tag{4.102}$$

at the one-loop level, where we use the one-loop result (see e.g. [169])

$$\gamma_i \simeq \frac{n_i k_\lambda |\lambda|^2}{16\pi^2|r_i|} - \frac{g^2}{4\pi^2}C(r_i)\,, \tag{4.103}$$

for a chiral superfield ϕ_i in the representation r_i of the gauge group G, where $C(r) = |G|T(r)/|r|$ is the quadratic Casimir operator, $W = \lambda \prod_i \phi_i^{n_i}$ with $\sum_i n_i = 3$, and k_λ is a positive real constant which depends on the index structure and normalization of the superpotential, which we will not need to compute.

A weakly-coupled flow can be approximated as follows. The gauge coupling does not run at one loop, so we initially treat it as a constant, whereas the superpotential couplings run to the "fixed point" $k_1|\lambda_1|^2 \sim 0$ and $k_2|\lambda_2|^2 \sim 28(g_1^2 + g_2^2)$. Thus,

$$\gamma_Q \sim -\frac{1}{24\pi^2}(g_1^2 + g_2^2)\,, \tag{4.104}$$

at the end of the one-loop flow. The remainder of the flow occurs more slowly, at the two-loop level, and can be approximated by substituting (4.104) into the beta function (4.101), giving

$$\beta(g_i) \simeq \frac{1}{(8\pi^2)^2}\, g_i^3 (g_1^2 + g_2^2)\,, \tag{4.105}$$

in the weak-coupling limit, where two-loop running can be treated adiabatically with respect to one-loop running. Thus, the gauge couplings (and hence λ_2) run to zero in the infrared, and the theory becomes free.

This is one example where the string coupling (4.99, 4.100) is an irrelevant deformation at the infrared fixed point, as discussed previously. This is consistent because the string coupling corresponds to a ratio of couplings which remains constant as the flow approaches the infrared fixed point, and therefore the exactly dimensionless coupling parameterizes a family of flows, all of which converge to the same free fixed point.

While the chain of dualities we have employed to arrive at this infrared-free description is valid at weak string coupling, the above discussion suggests that the infrared fixed point is perturbatively independent of the string coupling. If this persists nonperturbatively, then the same $SO(4)$ gauge theory should also describe the infrared behavior of the $SO(3) \times SU(7)$ gauge theory. It would be interesting to pursue this point further.

We now consider the moduli space of this theory. The F-term conditions take the form:

$$\mathcal{I}_{IJKLMN}\mathcal{A}^{KL}\mathcal{A}^{MN} + \delta_{ab} Q_I^a Q_J^b = 0\,, \quad \mathcal{A}^{IJ} Q_J^b = 0\,, \tag{4.106}$$

where I and J index the six components of a $\square\!\square$ of $SU(3)$, so that $\mathcal{A}^{IJ} = \mathcal{A}^{JI}$, and $\mathcal{I}_{IJKLMN}$ is an appropriate $SU(3)$ invariant. The first equation fixes the $SO(4)$ meson Q^2 in terms of $\mathcal{A}^2$. Since the $SO(4)$ baryon Q^4 obeys a classical constraint of the schematic form $(Q^4)^2 = (Q^2)^4$, its vev is fixed in terms of that of the meson Q^2 up to a sign, and therefore the classical moduli space is locally parameterized by the gauge invariant $\mathcal{A}$, corresponding to the baryon discussed in section 4.3.1.

However, not all $\mathcal{A}$ vevs can be extended to solutions to (4.106). In particular, the complete F-term conditions imply the following constraints on $\mathcal{A}$:

$$\mathcal{I}_{IKLMNP}\mathcal{A}^{JK}\mathcal{A}^{LM}\mathcal{A}^{NP} = 0\,, \quad \mathrm{cof}\,\left(\mathcal{I}_{IJKLMN}\mathcal{A}^{KL}\mathcal{A}^{MN}\right) = 0\,, \tag{4.107}$$

where cof denotes the matrix of cofactors, the first constraint arises upon contracting the first condition from (4.106) with $\mathcal{A}^{JP}$ and applying the second condition, and the second constraint follows from the classical constraint that $(Q^2)_{IJ}$ has rank at most four.

One can show that the constraints (4.107) are necessary and sufficient for a choice of Q_I^a to exist which satisfies (4.106), and therefore characterize the classical moduli space of the theory.[26] However, we have not yet demonstrated that any nontrivial solutions to these equations exist. Moreover, the quantum moduli space may differ from the classical moduli space if, for instance, an F-flat $\mathcal{A}$ vev with $\langle Q \rangle = 0$ gives a mass to too many flavors, generating a dynamical superpotential.

To address these issues, it is more convenient to write the superpotential as

$$W = \det \mathcal{A}^{ijkl} + c_1 \mathcal{A}_{ij} \mathcal{A}_{kl} \mathcal{A}^{ijkl} + c_2 \det \mathcal{A}_{ij} + \left(\mathcal{A}^{ijkl} + \epsilon^{ikm} \epsilon^{jln} \mathcal{A}_{mn} \right) \delta_{ab} Q^a_{ij} Q^b_{kl} \,, \tag{4.108}$$

in a non-canonically-normalized basis, where $\mathcal{A}^{ijkl}$ and $\mathcal{A}_{ij}$ denote the irreducible $\square\square\square\square$ and $\overline{\square\square}$ components of $\mathcal{A}$, respectively, and

$$\det M^{i_1 \dots i_{2p}} \equiv \frac{1}{d!} \epsilon_{i_{11} \dots i_{1d}} \dots \epsilon_{i_{(2p)1} \dots i_{(2p)d}} M^{i_{11} \dots i_{(2p)1}} \dots M^{i_{1d} \dots i_{(2p)d}} \,, \tag{4.109}$$

denotes an $SU(d)$ invariant formed from d copies of a $2p$-index tensor M which generalizes the determinant of a matrix. c_1 and c_2 are numerical prefactors corresponding to exactly marginal couplings, whose explicit values can be determined by relating $\mathcal{A}^3$ to the Pfaffian superpotential generated by the s-confinement of $USp(8)$. An explicit computation gives $c_1 = -\frac{3}{4}$ and $c_2 = \frac{3}{2}$.

It is now straightforward to find directions in the classical moduli space. For instance $\langle \mathcal{A}^{1111} \rangle \neq 0$ with all other vevs vanishing satisfies the F-term conditions. As this gives a mass to only one $SO(4)$ flavor, this suggests that this direction is part of the quantum moduli space, which is therefore nonempty. It would be interesting to better understand which parts of the moduli space defined by (4.107), if any, are lifted by quantum effects.

In summary, we find that the $USp(8) \times SU(4)$ theory has a dual description with a free infrared fixed point and a quantum moduli space. Our proposed duality

[26] If $\mathcal{I}_{IJKLMN} \mathcal{A}^{KL} \mathcal{A}^{MN}$ has (maximal) rank four, then the Q^4 baryon is non-vanishing, and the moduli space has two branches corresponding to the sign of Q^4 which are related by the spontaneously broken $\mathbb{Z}_2$ outer automorphism of $SO(4)$.

would seem to imply that the $SO(3)\times SU(7)$ theory has these features as well, and it would be interesting to check this in more detail to gain a better understanding of the proposed duality.

4.5.2 The $USp(10)\times SU(6)$ theory

The prospective dual theories for $N=9$ are:

	$SO(5)$	$SU(9)$	$SU(3)$	$U(1)_R$
A^i	$\square$	$\overline{\square}$	$\square$	$\frac{8}{9}$
B^i	$\mathbf{1}$	$\begin{smallmatrix}\square\\\square\end{smallmatrix}$	$\square$	$\frac{2}{9}$

$\longleftrightarrow$

	$USp(10)$	$SU(6)$	$SU(3)$	$U(1)_R$
$\tilde{A}^i$	$\square$	$\overline{\square}$	$\square$	$\frac{1}{3}$
$\tilde{B}^i$	$\mathbf{1}$	$\square\square$	$\square$	$\frac{4}{3}$

$$W=\tfrac{1}{2}\lambda\,\delta^{ab}\epsilon_{ijk}A^i_{a;m}A^j_{b;n}B^{mn;\,k}\,,\qquad W=\tfrac{1}{2}\tilde{\lambda}\,\Omega^{ab}\epsilon_{ijk}\tilde{A}^i_{a;\,m}\tilde{A}^j_{b;\,n}\tilde{B}^{mn;\,k}\,. \tag{4.110}$$

We focus on the $USp(10)\times SU(6)$ theory, showing that it has a line of infrared fixed points including a free fixed point.

We Seiberg-dualize the $USp(10)$ gauge group to obtain the theory

	$USp(4)$	$SU(6)$	$SU(3)$	$U(1)_R$
ϕ_i	$\square$	$\square$	$\overline{\square}$	$\frac{2}{3}$
ψ^{ij}	$\mathbf{1}$	$\overline{\begin{smallmatrix}\square\\\square\end{smallmatrix}}$	$\square\square$	$\frac{2}{3}$

(4.111)

with the superpotential

$$W=\frac{1}{2}\hat{\lambda}\,\Omega_{ab}\,\phi_i^{a;\,m}\phi_j^{b;\,n}\psi^{ij}_{mn}\,, \tag{4.112}$$

after integrating out massive matter. The beta function coefficients for both gauge groups vanish, and the (exactly marginal) string coupling takes the form

$$\tau_{\rm 10d}=\frac{1}{\pi i}\ln\left[\hat{\lambda}^{24}e^{4\pi i\tau_{\rm Sp}}e^{2\pi i\tau_{\rm SU}}\right]\,. \tag{4.113}$$

We find the exact beta functions

$$\beta(g_{\rm Sp})=-\frac{9g^3_{\rm Sp}}{16\pi^2}\frac{\gamma_\phi}{1-3g^2_{\rm Sp}/8\pi^2}\,,\quad \beta(g_{\rm SU})=-\frac{3g^3_{\rm SU}}{8\pi^2}\frac{\gamma_\phi+2\gamma_\psi}{1-3g^2_{\rm SU}/4\pi^2}\,,$$
$$\beta(\hat{\lambda})=\frac{1}{2}\hat{\lambda}(2\gamma_\phi+\gamma_\psi)\,, \tag{4.114}$$

where the anomalous dimensions γ_ϕ and γ_ψ take the form

$$\gamma_\phi \simeq \frac{k|\hat{\lambda}|^2}{576\pi^2} - \frac{35g_{\rm SU}^2}{48\pi^2} - \frac{5g_{\rm Sp}^2}{16\pi^2}\,, \quad \gamma_\psi \simeq \frac{k|\hat{\lambda}|^2}{1440\pi^2} - \frac{7g_{\rm SU}^2}{6\pi^2}\,, \tag{4.115}$$

at one loop, applying (4.103). As in section 4.5.1, we separate the flow into one-loop and higher-loop portions. At one loop, the gauge couplings do not run, and the superpotential coupling runs to the "fixed point"

$$k|\hat{\lambda}|^2 \sim 30\,(21g_{\rm SU}^2 + 5g_{\rm Sp}^2)\,. \tag{4.116}$$

Thus, after the one-loop running, we have

$$\gamma_\phi \simeq \frac{5}{96\pi^2}\left(7g_{\rm SU}^2 - g_{\rm Sp}^2\right)\,, \quad \gamma_\psi \simeq \frac{5}{48\pi^2}\left(g_{\rm Sp}^2 - 7g_{\rm SU}^2\right)\,, \tag{4.117}$$

Putting these into the beta functions for the gauge couplings, we obtain

$$\beta(g_{\rm Sp}) \simeq \frac{15g_{\rm Sp}^3}{2(16\pi^2)^2}\left(g_{\rm Sp}^2 - 7g_{\rm SU}^2\right)\,, \quad \beta(g_{\rm SU}) \simeq \frac{15g_{\rm SU}^3}{(16\pi^2)^2}\left(7g_{\rm SU}^2 - g_{\rm Sp}^2\right)\,, \tag{4.118}$$

under the same assumption of adiabaticity as before.

By inspection, we see that $\frac{2}{g_{\rm Sp}^2} + \frac{1}{g_{\rm SU}^2}$ is constant along the two-loop flow under the stated assumptions. Indeed, this combination corresponds approximately to the exactly marginal coupling (4.113) along this flow,

$$\frac{1}{8\pi g_s} \sim \frac{2}{g_{\rm Sp}^2} + \frac{1}{g_{\rm SU}^2}\,, \tag{4.119}$$

since the logarithm of the superpotential coupling, fixed by (4.116) in the adiabatic approximation, is small compared to $1/g^2$. Thus, the two-loop flow lines lie along contours of constant $\frac{2}{g_{\rm Sp}^2} + \frac{1}{g_{\rm SU}^2}$, and converge on the fixed line $g_{\rm Sp}^2 \simeq 7g_{\rm SU}^2$ and $k|\hat{\lambda}|^2 \simeq 240g_{\rm SU}^2$, with the final position along the fixed line dictated by the string coupling, as in (4.119).

Since the superpotential coupling and theta angles define one physical phase among them, there is a complex line of infrared fixed points parameterized by $\tau_{\rm 10d}$, where weak string coupling corresponds to a weakly coupled gauge theory and vice versa. Thus, unlike the previous example, the string coupling corresponds to a marginal deformation at the infrared fixed point, and affects the physics there. As such, we cannot readily infer the complete infrared behavior of the prospectively dual $SO(5) \times SU(9)$ theory from the above treatment, as this corresponds to a portion of the infrared fixed line which is strongly coupled in the $USp(4) \times SU(6)$ description.

4.5.3 The infrared behavior for $N > 9$

While the $N = 7$ and $N = 9$ examples treated in section 4.5.1 and section 4.5.2 are distinct in a number of ways, they both share the feature that the infrared physics is perturbatively accessible in some dual description, i.e. that there is a weakly coupled dual description, at least for certain values of the string coupling. We now ask whether this can hold more generally, for $N > 9$.

At any free fixed point, all the fundamental chiral superfields will have dimension one and the corresponding superconformal R-charge $+2/3$. If we assume that no accidental $U(1)$ symmetries appear along the flow, then the superconformal R-charge of gauge invariant operators can be determined via a-maximization [166], whereas the assumption of a free fixed point requires that the R-charge of such an operator be an integer multiple of $2/3$.

Indeed, since an arbitrary gauge invariant of the SO theory takes the form (4.12) or (4.13), it is easy to check that all such operators have R-charge $Q_R = \frac{2}{3}n$ for $n > 0$ and $N \geq 7$, whereas a similar argument applies to the USp theory for $\tilde{N} \geq 4$. This is suggestive and nontrivial evidence for a free fixed point, which we have already shown to occur for the cases $\tilde{N} = 4, 6$.

If such a fixed point exists, the $U(1)_R^3$ and $U(1)_R$ anomalies further constrain its form. In particular, a collection of N_χ chiral superfields with $Q_R = +2/3$

interacting via a gauge group G have the following anomalies

$$U(1)_R^3 = |G| - \frac{1}{27} N_\chi \,, \quad U(1)_R = |G| - \frac{1}{3} N_\chi \,. \tag{4.120}$$

Therefore,

$$|G| = \frac{1}{8}\left(9U(1)_R^3 - U(1)_R\right) \,, \quad N_\chi = \frac{27}{8}\left(U(1)_R^3 - U(1)_R\right) \,. \tag{4.121}$$

Thus,

$$|G| = \frac{3}{2}N(N-3) - 36 \,, \quad N_\chi = \frac{9}{2}N(N-3) - 81 \,, \tag{4.122}$$

for the case at hand. Conservation of the superconformal R-charge implies that the semi-simple component of G must have vanishing beta function coefficient, whereas any $U(1)$ factors must decouple.

Even if we assume that G is semisimple, for large N there are many possible product gauge groups which can reproduce the dimension formula (4.122). One possibility, which explains the pattern for all $N \geq 7$, is

$$G = [USp(4) \times SU(6)]^{\frac{N-7}{2}} \times SO(4)^{\left(\frac{N-9}{2}\right)^2} \,. \tag{4.123}$$

However, for any fixed N, there remain many possible spectra for this gauge group with vanishing beta function coefficients. While there are many further consistency checks one can apply to any specific candidate description, no obvious candidate presents itself. Moreover, the possibilities are yet broader if we allow for accidental $U(1)$ symmetries.

Should such an infrared description be found, it would be interesting to understand if it has a direct string theory interpretation, e.g. in terms of branes. We leave further study of the infrared behavior of these theories to a future work.

4.6 Further examples

So far we have focused on a single example of a new $\mathcal{N} = 1$ $SL(2,\mathbb{Z})$ duality which arises on the world-volume of D3 branes probing the orientifolded $\mathbb{C}^3/\mathbb{Z}_3$ singularity. While this example is closely analogous to the known $\mathcal{N}=4$ examples, making the parallels easier to grasp, it is but one example of a previously unexplored class of dualities of this type. In this section, we aim to briefly illustrate the breadth of this class, and also to point out other new dualities which arise from D3 branes probing orientifolded singularities but which appear to be of a different origin. We focus on a few simple examples, and defer further examples to [60, 125].

We begin by discussing the Calabi-Yau cone over dP_1 (a real cone over $Y^{2,1}$),[27] which provides a simple, non-orbifold example of the $SL(2,\mathbb{Z})$ dualities we have focused on. The resulting gauge theories are related to the $\mathbb{C}^3/\mathbb{Z}_3$ theories by Higgsing, and exhibit interesting infrared physics. We discuss anomaly matching, moduli space matching, and Higgsing for all N, before treating a specific example where the quantum moduli spaces can be shown to match exactly.

We then briefly discuss two other non-orbifold examples given by the Calabi-Yau cones over $Y^{2,0}$ and $Y^{4,0}$,[28] both of which exhibit different, more complicated patterns of dualities.

4.6.1 Complex cone over dP_1

We begin by considering the complex cone over the first del Pezzo surface dP_1, which can be obtained by blowing up $\mathbb{P}^2$ at a point. We are interested in orientifolds of this configuration corresponding to a compact O7 plane wrapping the del Pezzo base. As shown in figure 4.5, only one involution of the parent quiver exists which satisfies rule I of appendix 4.A, up to the choice of fixed element signs. Moreover, only two choices for these signs lead to theories which can be anomaly free without the addition of non-compact "flavor" D7 branes. As we show in [125] using brane

[27] See [170, 171] for more on the infinite class of Sasaki-Einstein manifolds known as the $Y^{p,q}$.

[28] The real cone over $Y^{2,0}$ is the same as the Calabi-Yau cone over the zeroth Hirzebruch surface $\mathbb{F}_0 \equiv \mathbb{P}^1 \times \mathbb{P}^1$.

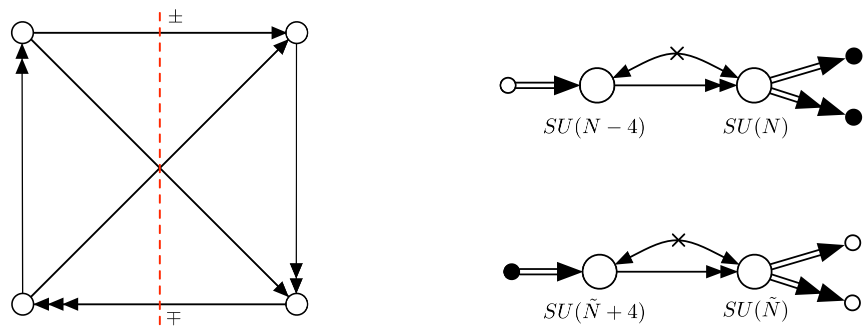


Figure 4.5: The left side shows the quiver gauge theory for dP_1, with the involution of interest indicated by the dashed line. The resulting quiverfold (see appendix 4.A) theories for the two sign choices are shown on the right.

tiling methods, these involutions also satisfy rule II and lead to superpotentials which inherit the $SU(2)\times U(1)_X\times U(1)_R$ geometric flavor symmetries of the parent theory, as expected for a compact O7 plane.

The two possible sign choices lead to the orientifold gauge theory

	$SU(N-4)$	$SU(N)$	$SU(2)$	$U(1)_X$	$U(1)_B$	$U(1)_R$
A^i	$\square$	$\overline{\square}$	$\square$	$\frac{N-2}{N-4}$	$-\frac{2(N-1)}{N(N-4)}$	$-\frac{8}{N(N-4)}$
Y	$\overline{\square}$	$\overline{\square}$	$\mathbf{1}$	$-\frac{N-2}{N-4}$	$\frac{(N+2)}{N(N-4)}$	$\frac{N^2-8}{N(N-4)}$
Z	$\overline{\square\square}$	$\mathbf{1}$	$\mathbf{1}$	$-\frac{N}{N-4}$	$\frac{3}{N-4}$	$\frac{N}{N-4}$
B^i	$\mathbf{1}$	$\begin{smallmatrix}\square\\\square\end{smallmatrix}$	$\square$	0	$\frac{1}{N}$	$\frac{N-4}{N}$
X	$\mathbf{1}$	$\begin{smallmatrix}\square\\\square\end{smallmatrix}$	$\mathbf{1}$	-1	$\frac{1}{N}$	$\frac{N-4}{N}$

(4.124)

with superpotential

$$W=\epsilon_{ij}\mathrm{Tr}\left[B^iA^jY+XA^iZA^j\right], \tag{4.125}$$

as well as the theory[29]

	$SU(\tilde{N}+4)$	$SU(\tilde{N})$	$SU(2)$	$U(1)_X$	$U(1)_B$	$U(1)_R$
$\tilde{A}^i$	$\square$	$\overline{\square}$	$\square$	$\frac{\tilde{N}+2}{\tilde{N}+4}$	$-\frac{2(\tilde{N}+1)}{\tilde{N}(\tilde{N}+4)}$	$-\frac{8}{\tilde{N}(\tilde{N}+4)}$
$\tilde{Y}$	$\overline{\square}$	$\overline{\square}$	$\mathbf{1}$	$-\frac{\tilde{N}+2}{\tilde{N}+4}$	$\frac{(\tilde{N}-2)}{\tilde{N}(\tilde{N}+4)}$	$\frac{\tilde{N}^2-8}{\tilde{N}(\tilde{N}+4)}$
$\tilde{Z}$	$\overline{\begin{smallmatrix}\square\\\square\end{smallmatrix}}$	$\mathbf{1}$	$\mathbf{1}$	$-\frac{\tilde{N}}{\tilde{N}+4}$	$\frac{3}{\tilde{N}+4}$	$\frac{\tilde{N}}{\tilde{N}+4}$
$\tilde{B}^i$	$\mathbf{1}$	$\square\square$	$\square$	0	$\frac{1}{\tilde{N}}$	$\frac{\tilde{N}+4}{\tilde{N}}$
$\tilde{X}$	$\mathbf{1}$	$\square\square$	$\mathbf{1}$	-1	$\frac{1}{\tilde{N}}$	$\frac{\tilde{N}+4}{\tilde{N}}$

(4.126)

with superpotential

$$W = \epsilon_{ij} \mathrm{Tr}\,[\tilde{B}^i \tilde{A}^j \tilde{Y} + \tilde{X} \tilde{A}^i \tilde{Z} \tilde{A}^j]\,, \tag{4.127}$$

where in either case the gauge indices are cyclically contracted. Henceforward, for want of a better label we refer to these theories as "Theory A" and "Theory B", respectively. For ease of presentation we have chosen a basis for the R-symmetry which does not satisfy a-maximization, as the superconformal R-charges are irrational.

The global anomalies for both theories are shown in table 4.4. We see that the anomalies match between the two theories for $\tilde{N} = N - 2$ provided that N is odd. For even N the $SU(2)^3$ Witten anomalies do not match, and the theories are not dual.[30]

For completeness, we also present the a-maximizing superconformal R-charge, which is a linear combination:[31]

$$U(1)_R^{(\mathrm{sc})} = U(1)_R + a_X U(1)_X + a_B U(1)_B\,. \tag{4.128}$$

[29]Up to charge conjugation of the global $U(1)_B$ this theory is the negative rank dual of the first theory (see appendix 4.B for details on negative rank duality).

[30]One can also show that holomorphic gauge invariants do not match between the two theories. For instance, the B-theory operator $\tilde{Z}^{(\tilde{N}+4)/2}$, defined for even $\tilde{N}$, has no dual in the A-theory.

[31]As remarked in [172], it is important to do a-maximization over the symmetries preserved by the superpotential, as we do here. It is easy to verify that in the large N regime our results agree with those in [172], as they should since the orientifold corrections are subleading in this limit [173].

	Theory A	Theory B
$SU(2)^3$	$(-1)^{\frac{3}{2}N(N-3)}$	$(-1)^{\frac{3}{2}\tilde{N}(\tilde{N}+3)}$
$SU(2)^2\,U(1)_X$	$N(N-2)$	$\tilde{N}(\tilde{N}+2)$
$SU(2)^2\,U(1)_B$	$-\frac{3}{2}(N-1)$	$-\frac{3}{2}(\tilde{N}+1)$
$U(1)_X^2\,U(1)_B$	$-(N-1)$	$-(\tilde{N}+1)$
$U(1)_X\,U(1)_B^2$	2	2
$SU(2)^2\,U(1)_R$	$-N(N-2)-6$	$-\tilde{N}(\tilde{N}+2)-6$
$U(1)_X^2\,U(1)_R$	$-2N(N-2)-4$	$-2\tilde{N}(\tilde{N}+2)-4$
$U(1)_B^2\,U(1)_R$	-8	-8
$U(1)_X\,U(1)_B\,U(1)_R$	$4(N-1)$	$4(\tilde{N}+1)$
$U(1)_X\,U(1)_R^2$	$2N(N-2)$	$2\tilde{N}(\tilde{N}+2)$
$U(1)_B\,U(1)_R^2$	$-4(N-1)$	$-4(\tilde{N}+1)$
$U(1)_R^3$	-34	-34
$U(1)_R$	-10	-10

Table 4.4: The non-vanishing anomalies for theory A and theory B, where we use a multiplicative notation for the $SU(2)^3$ Witten anomaly analogous to that used for discrete symmetries in section 4.3.1. The $U(1)_B^3$, $U(1)_X^3$, $U(1)_B$, and $U(1)_X$ anomalies vanish in both theories.

A-maximization in theory A gives

$$a_B = \frac{a_X^2 - 8a_X + 4}{4(a_X - 4)}(N-1)\,, \tag{4.129}$$

where a_X is a solution to the quartic equation

$$0 = (N-1)^2(a_X^2 - 4)[3a_X^2 - 16a_X + 4] + 16(2a_X + 1)(a_X - 4)^2\,, \tag{4.130}$$

in the range $a_X \in \left(-\frac{1}{2}, \frac{2}{3}(4-\sqrt{13})\right)$, whereas exactly one solution lies in this range for any $N > 1$. For example, we obtain approximately

N	2	3	4	5	6	7	8	9	10	11	12
a_X	-0.431	-0.270	-0.113	0	0.074	0.122	0.155	0.178	0.194	0.207	0.216

(4.131)

The result for $N = 5$ is exact, giving $a_X = 0$ and $a_B = -1$. For large N, a_X asymptotically approaches $\frac{2}{3}(4 - \sqrt{13}) \approx 0.263$. The same considerations apply in theory B upon replacing $N - 1 \to \tilde{N} + 1$.

In the following sections, we explore the prospective duality for odd N. We will also have more to say about the case of even N in the next section.

Moduli space and Higgsing

We begin by considering the mapping between the moduli spaces of the two theories, which is equivalently expressed as a map between the holomorphic gauge invariants subject to the F-term conditions.

To obtain this map, we consider certain "minimal" operators, i.e. operators whose $U(1)$ charges cannot be obtained as the sum of the $U(1)$ charges of two or more non-vanishing operators. Operators of this type can only mix with other minimal operators under the duality, whereas generically no two minimal operators share the same $U(1)$ charges, leading to a unique matching between the minimal operators of the dual theories.

To find minimal operators, we begin by classifying irreducible "gauge-invariant" monomials in the fundamental fields, i.e. formal products of the fields (disregarding gauge-indices) which are neutral under the $\mathbb{Z}_{N-4} \times \mathbb{Z}_N$ or $\mathbb{Z}_{\tilde{N}+4} \times \mathbb{Z}_{\tilde{N}}$ gauge-group center, and which cannot be factored into two or more gauge-invariant pieces. The resulting finite list generates all gauge-invariant monomials, a subset of which will correspond to actual gauge-invariant operators. Using this classification, it is possible to show that certain candidate operators are minimal.

Using these methods, we obtain the following minimal operators in theory A for odd N:

	$U(1)_B$	$U(1)'_X$	$U(1)'_R$
$A^2(Y\|AZ)^2(B\|X)^2$	0	$[-2,2]$	4
$B^2X(B\|X)^{N-3}$	1	$[-\frac{N-3}{2},\frac{N-3}{2}]$	$N-5$
$Z^{N-4-2k}(Y^2X)^{2k}$	3	$\frac{N-3}{2}-4k$	$N-3+4k$
$A^{N-4}B(B\|X)^{N-3}$	-1	$[-\frac{N-3}{2},\frac{N-3}{2}]$	$N-5$
$A^N(Y\|AZ)^4(B\|X)^2$	-2	$[-3,3]$	8
$A^{p(N-4)}(B\|X)^{N-2p}$	$1-2p$	$[-\frac{N+1-2p}{2},\frac{N-1-2p}{2}]$	$N-5$

(4.132)

where $2\le p\le\frac{N-1}{2}$, $(x|y)^n$ denotes a monomial of degree n in x and y, and we employ a slightly different basis for the $U(1)$ charges:

$$U(1)'_X=U(1)_X+\frac{N-1}{2}U(1)_B\,,\quad U(1)'_R=U(1)_R-U(1)_B\,. \tag{4.133}$$

A similar analysis in theory B for odd $\tilde{N}$ gives

	$U(1)_B$	$U(1)'_X$	$U(1)'_R$
$\tilde{A}^2(\tilde{Y}\|\tilde{A}\tilde{Z})^2(\tilde{B}\|\tilde{X})^2$	0	$[-2,2]$	4
$\tilde{Y}\tilde{Z}^2(\tilde{Y}\|\tilde{A}\tilde{Z})^{\tilde{N}-1}$	1	$[-\frac{\tilde{N}-1}{2},\frac{\tilde{N}-1}{2}]$	$\tilde{N}-3$
$\tilde{Z}^{\tilde{N}+3-2k}(\tilde{Y}^2\tilde{X})^{2k+1}$	3	$\frac{\tilde{N}-1}{2}-4k$	$\tilde{N}-1+4k$
$\tilde{A}^{\tilde{N}+1}\tilde{Z}(\tilde{Y}\|\tilde{A}\tilde{Z})^{\tilde{N}-1}$	-1	$[-\frac{\tilde{N}-1}{2},\frac{\tilde{N}-1}{2}]$	$\tilde{N}-3$
$\tilde{A}^{\tilde{N}+6}(\tilde{Y}\|\tilde{A}\tilde{Z})^2(\tilde{B}\|\tilde{X})^4$	-2	$[-3,3]$	8
$\tilde{A}^{p(\tilde{N}+2)-2}(\tilde{Y}\|\tilde{A}\tilde{Z})^{\tilde{N}+2-2p}$	$1-2p$	$[-\frac{\tilde{N}+3-2p}{2},\frac{\tilde{N}+1-2p}{2}]$	$\tilde{N}-3$

(4.134)

Thus, the spectrum of minimal operators appears to match between the two theories for $\tilde{N}=N-2$, a highly nontrivial check of the proposed duality.[32]

Several comments are in order. Firstly, while this may not comprise a complete list of minimal operators, one can show that all of the listed operators are minimal, and that no other minimal operators share the same $U(1)$ charges, so the matching

[32]It would be instructive to also compute the $SU(2)$ representations of these operators.

is reliable. Secondly, to obtain this matching, it is necessary to carefully account for the structure of the gauge-index contraction as well as the F-term conditions. For example, consider the operator $Z^{N-4-p}(Y^2X)^p$. Each X factor must appear in the combination $X^{mn}Y^a_mY^b_n$, which is therefore antisymmetric in the $SU(N-4)$ indices. Since Z^{ab} is symmetric, a gauge invariant index contraction exists if and only if an even number of XY^2 factors appear, i.e. if and only if p is even. By contrast, in the operator $\tilde{Z}^{\tilde{N}+4-p}(\tilde{Y}^2\tilde{X})^p$ the symmetry properties are reversed, and an even number $\tilde{Z}$ factors must appear, i.e. p must be *odd* (since $\tilde{N}$ is odd).

These particular operators are also interesting in that they correspond to Higgsing to the dP_0 theories studied in section 4.3. In particular, the operator $Z^{N-4-2k}(Y^2X)^{2k}$ corresponds to Higgsing the A theory $SU(N-4)\times SU(N)$ to $SO(N-4-2k)\times SU(N-2k)$, whereas the operator $\tilde{Z}^{\tilde{N}+3-2k}(\tilde{Y}^2\tilde{X})^{2k+1}$ corresponds to Higgsing the B theory $SU(\tilde{N}+4)\times SU(\tilde{N})$ to $USp(\tilde{N}+3-2k)\times SU(\tilde{N}-2k-1)$.[33] Consistent with the proposed operator mapping, we observe that the resulting theories are related by the duality proposed in section 4.3. This is another nontrivial consistency check.

At this point, it is also instructive to consider the behavior of the even-N theories under Higgsing. Turning on a vev for $Z^{N-4-2k}(Y^2X)^{2k}$ once again corresponds to Higgsing theory A to $SO(N-4-2k)\times SU(N-2k)$, where now the resulting theory is conjectured to be self-dual under S-duality, suggesting that the A theory for even N is also self-dual. However, things are quite different in the even-$\tilde{N}$ B theory. Here, the simplest Higgsing, corresponding to the operator $\tilde{Z}^{(\tilde{N}+4)/2}$, breaks $SU(\tilde{N}+4)\times SU(\tilde{N})$ to $USp(\tilde{N}+4)\times SU(\tilde{N})$, where now the resulting theory is *not* a singlet under S-duality, inconsistent with self-duality for the parent theory, while on the other hand there is no candidate dual for the parent theory.

[33] Note that from this viewpoint, the dP_0 theories enjoy an unbroken $\mathbb{Z}_3$ baryonic symmetry precisely because they are obtained by turning on a vev for an operator with $Q_B = 3$.

We hypothesize that the even-$\tilde{N}$ B theory is inconsistent in string theory, potentially due to an uncanceled K-theory (discrete) tadpole. We hope to verify this through explicit computation of the K-theory tadpoles in future work.

Having discussed some generic features of the proposed odd-N duality, we next discuss a particularly tractable example with a deformed quantum moduli space.

Case study: the $SU(5) \longleftrightarrow SU(7) \times SU(3)$ duality

The lowest rank example of the proposed duality between the A and B theories is for $N = 5$. This example turns out to be particularly tractable, and we now show that the dual theories have biholomorphic quantum-deformed moduli spaces. The $SU(7) \times SU(3)$ theory turns out to be somewhat more intuitive, so we begin by discussing this theory, after which we briefly explain how to show that the $SU(5)$ theory has the same moduli space.

Theory B We consider the B-theory $SU(7) \times SU(3)$:

	$SU(7)$	$SU(3)$	$SU(2)$	$U(1)_B$	$U(1)'_X$	$U(1)'_R$
$\tilde{A}^i$	$\overline{\square}$	$\square$	$\square$	$-\frac{8}{21}$	$-\frac{1}{21}$	0
$\tilde{Y}$	$\square$	$\square$	$\mathbf{1}$	$\frac{1}{21}$	$-\frac{13}{21}$	0
$\tilde{Z}$	$\begin{smallmatrix}\square\\ \square\end{smallmatrix}$	$\mathbf{1}$	$\mathbf{1}$	$\frac{3}{7}$	$\frac{3}{7}$	0
$\tilde{B}^i$	$\mathbf{1}$	$\overline{\square\square}$	$\square$	$\frac{1}{3}$	$\frac{2}{3}$	2
$\tilde{X}$	$\mathbf{1}$	$\overline{\square\square}$	$\mathbf{1}$	$\frac{1}{3}$	$-\frac{1}{3}$	2

(4.135)

with the superpotential:

$$W = \tilde{\lambda}\, \epsilon_{ij} \mathrm{Tr}\, [\tilde{B}^i \tilde{A}^j \tilde{Y}] + \frac{1}{2\tilde{\mu}} \epsilon_{ij} \mathrm{Tr}\, [\tilde{X} \tilde{A}^i \tilde{Z} \tilde{A}^j] \,. \tag{4.136}$$

All possible $SU(7)$ gauge invariants are products of the following:

$$\mathcal{Y}^{Im} \equiv \tilde{A}^I_a \tilde{Y}^{a;m} \,, \quad \mathcal{Q}^m \equiv \frac{1}{48} \epsilon_{abcdefg} \tilde{Y}^{a;m} \tilde{Z}^{bc} \tilde{Z}^{de} \tilde{Z}^{fg} \,, \tag{4.137}$$

$$\mathcal{Z}^{IJ} \equiv \tilde{A}^I_a \tilde{A}^J_b \tilde{Z}^{ab}\,, \quad \Phi \equiv \frac{1}{48}\epsilon_{mnp}\epsilon_{abcdefg}\tilde{Y}^{a;m}\tilde{Y}^{b;n}\tilde{Y}^{c;p}\tilde{Z}^{de}\tilde{Z}^{fg}\,,$$

where $a, b, \ldots$ index $SU(7)$, $m, n, \ldots$ index $SU(3)$, and $I, J, \ldots$ index a fictitious $SU(6) \supset SU(3) \times SU(2)$. There is a classical constraint:

$$\frac{1}{2}\epsilon_{mnp}(\mathrm{Pcf}\mathcal{Z})_{IJ}\mathcal{Y}^{Im}\mathcal{Y}^{Jn}\mathcal{Q}^p = (\mathrm{Pf}\mathcal{Z})\Phi\,, \tag{4.138}$$

where we define

$$\begin{aligned} \mathrm{Pf}M &\equiv \frac{1}{2^n n!}\epsilon_{i_1 j_1 \ldots i_n j_n} M^{i_1 j_1} \ldots M^{i_n j_n}\,, \\ (\mathrm{Pcf}M)_{ij} &\equiv \frac{1}{2^{n-1}(n-1)!}\epsilon_{ij i_2 j_2 \ldots i_n j_n} M^{i_2 j_2} \ldots M^{i_n j_n}\,, \end{aligned} \tag{4.139}$$

for a $2n \times 2n$ antisymmetric matrix M^{ij} and “Pcf” stands for “Pfaffian cofactor”, since for M invertible it takes the form $\mathrm{Pcf}M = (\mathrm{Pf}M)[M^{-1}]^T$, much like $\mathrm{cof}M = (\det M)[M^{-1}]^T$ for an arbitrary invertible matrix M.

The classical constraint is quantum modified to [174]

$$\frac{1}{2}\epsilon_{mnp}(\mathrm{Pcf}\mathcal{Z})_{IJ}\mathcal{Y}^{Im}\mathcal{Y}^{Jn}\mathcal{Q}^p - (\mathrm{Pf}\mathcal{Z})\Phi = \Lambda^{14}_{SU(7)}\,. \tag{4.140}$$

This equation describes the quantum moduli space of the $SU(7)$ gauge theory when we take the $SU(3)$ gauge coupling and superpotential couplings to zero.

We now account for the finiteness of these couplings. In particular, the superpotential couplings give a mass to certain components of $\mathcal{Y}$ and $\mathcal{Z}$, so that on the moduli space we must have

$$\mathcal{Y}^{m_i n} = \epsilon^{mnp}\mathcal{Y}^i_p\,, \quad \mathcal{Z}^{m_i n_j} = \epsilon^{mnp}\mathcal{Z}^{ij}_p\,, \tag{4.141}$$

where $m_i = m + 3(i-1)$ indexes the fictitious $SU(6)$. Since the LHS of (4.140) contains only $SU(3)$ baryons built from $SU(3)$ fundamentals, $SU(3)$ is completely broken everywhere in the moduli space, leading to a confined description where the effect of gauging $SU(3)$ is to remove 8 Higgsed degrees of freedom and their superpartners.

Thus, the moduli space is parameterized by the operators:

	$SU(3)$	$SU(2)$	$U(1)_B$	$U(1)'_X$	$U(1)'_R$
$\mathcal{Y}^i_m = (\tilde{Y}\tilde{A}^i)$	$\bar{\square}$	$\mathbf{2}$	$-\frac{1}{3}$	$-\frac{2}{3}$	0
$\mathcal{Z}^{ij}_m = (\tilde{Z}\tilde{A}^i\tilde{A}^j)$	$\bar{\square}$	$\mathbf{3}$	$-\frac{1}{3}$	$\frac{1}{3}$	0
$\mathcal{Q}^m = (\tilde{Z}^3\tilde{Y})$	$\square$	$\mathbf{1}$	$\frac{4}{3}$	$\frac{2}{3}$	0
$\Phi = (\tilde{Z}^2\tilde{Y}^3)$	$\mathbf{1}$	$\mathbf{1}$	1	-1	0

(4.142)

subject to the gauging of $SU(3)$ and the quantum-modified constraint.[34] Therefore, the dimension of the moduli space is:

$$\dim \mathcal{M} = 19 - 8 - 1 = 10\,. \tag{4.143}$$

Since all operators are neutral under $U(1)'_R$, there is an unbroken $U(1)'_R$ everywhere in the moduli space.

A complete list of $SU(3)$ gauge invariants formed from these four fields is:

	$SU(2)$	$U(1)_B$	$U(1)'_X$	$U(1)'_R$
$\mathcal{Y}^2\mathcal{Z}$	$\mathbf{3}$	-1	-1	0
$\mathcal{Y}\mathcal{Z}^2$	$\mathbf{4}\oplus\mathbf{2}$	-1	0	0
$\mathcal{Z}^3$	$\mathbf{1}$	-1	1	0
$\mathcal{Z}\mathcal{Q}$	$\mathbf{3}$	1	1	0
$\mathcal{Y}\mathcal{Q}$	$\mathbf{2}$	1	0	0
Φ	$\mathbf{1}$	1	-1	0

(4.144)

Since there are a total of 16 invariants, still subject to one modified constraint, we conclude that there are five further "classical" constraints relating these $SU(3)$ composites. To make these constraints explicit, we define:

$$\bar{\mathcal{Q}}^A_m \equiv \{\mathcal{Y}^i_m, \mathcal{Z}^\alpha_m\}\,, \tag{4.145}$$

[34]Note that this spectrum has an $SU(3)^3$ gauge anomaly, but this is fine because $SU(3)$ is completely broken on the moduli space. Adding an $SU(7)$ flavor to the original theory and s-confining leads to an anomaly free spectrum for $SU(3)$. Upon adding a mass for the additional flavor, one obtains a tadpole in the s-confined description, whereupon the additional fields are set to zero by the F-term conditions, leaving the moduli given here.

where A indexes a fictitious $SU(5) \supset SU(2) \times SO(3)$, and $\mathcal{Z}^\alpha_m \equiv \frac{1}{2}\sigma^\alpha_{ij}\mathcal{Z}^{ij}_m$ with the $SU(2) \cong SO(3)$ conventions:

$$\sigma_i^{\alpha\, j} = \left\{ \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \right\},$$

$$\sigma^{ij}_\alpha = \epsilon^{ik}\sigma_k^{\alpha\, j}\,, \quad \sigma^\alpha_{ij} = \sigma_i^{\alpha\, k}\epsilon_{kj}\,, \quad \epsilon^{12} = \epsilon_{12} = +1\,. \tag{4.146}$$

The classical constraints then take the form:

$$[\bar{\mathcal{Q}}^3]_{AB}[\mathcal{Q}\bar{\mathcal{Q}}]^B = 0\,, \quad \epsilon^{ABCDE}[\bar{\mathcal{Q}}^3]_{AB}[\bar{\mathcal{Q}}^3]_{CD} = 0\,. \tag{4.147}$$

Although both equations appear to have five components, examining small fluctuations about a background with $\bar{\mathcal{Q}} \neq 0$ satisfying these constraints gives only three independent constraints from the second equation, and a further two from the first, for a total of five constraints, as expected.

We define:

$$\Psi \equiv \det \mathcal{Z}^\alpha_m\,, \quad \Psi^i_\alpha \equiv \mathcal{Y}^i_m[\mathrm{cof}\,\mathcal{Z}]^m_\alpha\,, \quad \Psi^\alpha \equiv \frac{1}{2}\epsilon_{ij}\epsilon^{mnp}\mathcal{Y}^i_m\mathcal{Y}^j_n\mathcal{Z}^\alpha_p\,, \tag{4.148}$$

$$\Phi^i \equiv \mathcal{Q}^m\mathcal{Y}^i_m\,, \quad \Phi^\alpha \equiv \mathcal{Q}^m\mathcal{Z}^\alpha_m\,. \tag{4.149}$$

In terms of these gauge invariants, the classical constraints becomes:

$$\Psi\Phi^j - \Psi^j_\alpha\Phi^\alpha = 0\,, \quad \epsilon_{ij}\Psi^i_\alpha\Phi^j + \epsilon_{\alpha\beta\gamma}\Psi^\beta\Phi^\gamma = 0\,, \tag{4.150}$$

and

$$\Psi^i_\alpha\Psi^\alpha = 0\,, \quad \Psi\Psi^\alpha - \frac{1}{2}\epsilon^{\alpha\beta\gamma}\epsilon_{ij}\Psi^i_\beta\Psi^j_\gamma = 0\,. \tag{4.151}$$

After a somewhat longer computation, we find that the quantum modified constraint takes the form:

$$i\Phi^i\Psi^j_\alpha\sigma^\alpha_{ij} - \Phi_\alpha\Psi^\alpha - 2i\Phi\Psi = \Lambda^{14}_{SU(7)}\,. \tag{4.152}$$

Together, (4.150, 4.151, 4.152) completely describe the deformed moduli space of the quantum theory.

The maximal unbroken flavor symmetry is $SU(2)\times U(1)'_{B+X}\times U(1)'_R$, which is attained when we take Φ and Ψ to be non-vanishing with all other fields vanishing. We can then solve the constraints to eliminate Φ^i, Ψ^α, and Φ:

$$\Phi^i = \frac{1}{\Psi}\Psi^i_\alpha\Phi^\alpha\,, \quad \Psi^\alpha = \frac{1}{2\Psi}\epsilon^{\alpha\beta\gamma}\epsilon_{ij}\Psi^i_\beta\Psi^j_\gamma\,, \quad \Phi = \frac{i}{2\Psi}\Lambda^{14}_{SU(7)}\,, \tag{4.153}$$

whereupon the remaining constraints are trivially satisfied. The light modes along this line are therefore:

	$SU(2)$	$U(1)'_{B+X}$	$U(1)'_R$
Ψ^i_α	$\mathbf{4}\oplus\mathbf{2}$	-1	0
Ψ	$\mathbf{1}$	0	0
Φ^α	$\mathbf{3}$	2	0

(4.154)

One can check that the global $SU(2)\times U(1)'_{B+X}\times U(1)'_R$ anomalies match those of the original description, as expected.

Theory A We now consider the dual theory:[35]

	$SU(5)$	$SU(2)$	$U(1)_B$	$U(1)'_X$	$U(1)'_R$
A^i	$\overline{\square}$	$\square$	$-\frac{8}{5}$	$-\frac{1}{5}$	0
Y	$\overline{\square}$	$\mathbf{1}$	$\frac{7}{5}$	$-\frac{1}{5}$	2
Z	$\mathbf{1}$	$\mathbf{1}$	3	1	2
B^i	$\begin{smallmatrix}\square\\\square\end{smallmatrix}$	$\square$	$\frac{1}{5}$	$\frac{2}{5}$	0
X	$\begin{smallmatrix}\square\\\square\end{smallmatrix}$	$\mathbf{1}$	$\frac{1}{5}$	$-\frac{3}{5}$	0

(4.155)

with the superpotential:

$$W = \lambda\,\epsilon_{ij}Y_m A^i_n B^{jmn} + \frac{1}{2\mu}\epsilon_{ij}ZA^i_m A^j_n X^{mn}\,. \tag{4.156}$$

[35] Reference [175] discusses a similar theory in the context of dynamical supersymmetry breaking.

As reviewed in section 4.3.3, taking $W \to 0$, the $SU(5)$ gauge theory has an s-confined description:

	$SU(5)$	$SU(3)_a$	$SU(3)_b$	$U(1)_B^{(s)}$	$U(1)_R^{(s)}$
$\mathcal{A}^I$	$\overline{\square}$	$\square$	$\mathbf{1}$	$-\frac{3}{5}$	$2/3$
$\mathcal{B}^I$	$\begin{smallmatrix}\square\\\square\end{smallmatrix}$	$\mathbf{1}$	$\square$	$\frac{1}{5}$	0
Z	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$	3	2
$T^I_J = \mathcal{A}^2\mathcal{B}$		$\overline{\square}$	$\square$	-1	$4/3$
$U^{I;J}_{\ K} = \mathcal{A}\mathcal{B}^3$		$\square$	Adj	0	$2/3$
$V^{IJ} = \mathcal{B}^5$		$\mathbf{1}$	$\square\square$	1	0

(4.157)

with the dynamical superpotential:

$$W = \frac{1}{\Lambda^9}\left(\epsilon_{IJK} T^I_L U^{L;J}_{\ M} V^{MK} - \frac{1}{3}\epsilon_{IJK} U^{I;L}_{\ N} U^{J;M}_{\ L} U^{K;N}_{\ M}\right) , \tag{4.158}$$

where $I, J, \ldots = 1, 2, 3$, and we omit an unimportant $U(1)$ global symmetry under which only the additional singlet Z is charged.

Thus, deforming the resulting theory by the tree-level superpotential, we obtain:

$$W = \frac{1}{\Lambda^9}\left(\epsilon_{IJK} T^I_L U^{L;J}_{\ M} V^{MK} - \frac{1}{3}\epsilon_{IJK} U^{I;L}_{\ N} U^{J;M}_{\ L} U^{K;N}_{\ M}\right) + \lambda T^i_i + \frac{1}{\mu} Z\, T^3_3 \, , \tag{4.159}$$

where $i, j, \ldots = 1, 2$. The superpotential partially breaks the flavor symmetries of the pure s-confining theory. In particular, $SU(3)_a \times SU(3)_b \to SU(2) \times U(1)_a \times U(1)_b$ where $U(1)_a$ and $U(1)_b$ denote the diag$(1/3, 1/3, -2/3)$ elements of each $SU(3)$, and the unbroken $U(1)$ linear combinations are:

$$U(1)'_R = U(1)_R^{(s)} - 2U(1)_a \, , \quad U(1)_B = U(1)_B^{(s)} - 3U(1)_a \, , \quad U(1)'_X = U(1)_a + U(1)_b \, . \tag{4.160}$$

The F-term conditions now read:

$$\frac{1}{\Lambda^9}\epsilon_{IJK} U^{L;J}_{\ M} V^{MK} + \lambda \delta^i_I \delta^L_i + \frac{1}{\mu} Z \delta^3_I \delta^L_3 = 0 \, , \qquad T^3_3 = 0 \, ,$$

$$\epsilon_{IJK} T^I_L V^{MK} - \epsilon_{ILK} U^{I;M}_{\ \ N} U^{K;N}_{\ \ J} = 0\,, \qquad \epsilon_{IJK} T^I_L U^{L;J}_{\ \ M} = 0\,. \tag{4.161}$$

It is straightforward to show, using the Gröbner basis algorithm,[36] that solutions to these equations must satisfy:

$$T^I_J = 0\,, \quad Z = 0\,, \quad U^{3;I}_{\ \ J} = 0\,, \quad U^{(i;j)}_{\ \ 3} = 0\,, \quad U^{(i;j)}_{\ \ i} = 0\,, \quad U^{i;3}_{\ \ i} = 0\,. \tag{4.162}$$

We decompose the non-vanishing fields as follows:

$$\begin{aligned} U^{i;j}_{\ \ 3} &= \psi \epsilon^{ij}\,, \quad U^{i;3}_{\ \ j} = \sigma^{\alpha i}_j \psi^\alpha\,, \quad V^{ij} = \sigma^{ij}_\alpha \phi^\alpha\,, \quad V^{i3} = \phi^i\,, \\ V^{33} &= \phi\,, \qquad U^{i;j}_{\ \ k} = \frac{1}{3}\psi^j_\alpha \sigma^{\alpha i}_k - \frac{1}{3}\sigma^{ij}_\alpha \psi^\alpha_k - \psi^i_\alpha \sigma^{\alpha j}_k\,, \end{aligned} \tag{4.163}$$

where $\psi^\alpha_k \equiv \psi^j_\alpha \epsilon_{jk}$, our remaining conventions are given in (4.146), and the fields transform as

	$SU(2)$	$U(1)_B$	$U(1)'_X$	$U(1)'_R$
$\psi = AB^3$	$\mathbf{1}$	-1	1	0
$\psi^i_\alpha = AB^2X$	$\mathbf{4} \oplus \mathbf{2}$	-1	0	0
$\psi^\alpha = ABX^2$	$\mathbf{3}$	-1	-1	0
$\phi^\alpha = B^4X$	$\mathbf{3}$	1	1	0
$\phi^i = B^3X^2$	$\mathbf{2}$	1	0	0
$\phi = B^2X^3$	$\mathbf{1}$	1	-1	0

(4.164)

under the global symmetries.

The F-term conditions involving non-vanishing fields are:

$$\epsilon_{iJK} U^{l;J}_{\ \ M} V^{MK} + \lambda \Lambda^9 \delta^l_i = 0\,, \quad \epsilon_{jk} U^{i;j}_{\ \ M} V^{Mk} = 0\,, \quad \epsilon_{ik} U^{i;M}_{\ \ N} U^{k;N}_{\ \ J} = 0\,. \tag{4.165}$$

Applying the above decomposition and simplifying, we eventually obtain:

$$\psi \phi^i - 2\psi^i_\alpha \phi^\alpha \ = \ 0\,, \quad \epsilon_{ij} \psi^i_\alpha \phi^j + i\epsilon_{\alpha\beta\gamma} \phi^\beta \psi^\gamma = 0\,,$$

[36] We use the `Elimination[]` function of the `Stringvacua` package [176], which uses `SINGULAR` [177] for computations.

$$
\begin{aligned}
\psi^i_\alpha \psi^\alpha &= 0\,, \quad \psi\psi^\alpha - i\epsilon^{\alpha\beta\gamma}\epsilon_{ij}\psi^i_\beta\psi^j_\gamma = 0\,, \\
\sigma^\alpha_{ij}\psi^i_\alpha\phi^j + \psi\phi + \psi_\alpha\phi^\alpha &= -\lambda\Lambda^9\,.
\end{aligned}
\tag{4.166}
$$

Upon replacing:

$$
\psi \to \frac{2}{m^5}\Psi\,, \quad \psi^\alpha \to \frac{i}{m^3}\Psi^\alpha\,, \quad \psi^i_\alpha \to \frac{1}{m^4}\Psi^i_\alpha\,,
$$
$$
\phi \to -\Phi\,,\ \phi^i \to \frac{1}{m}\Phi^i\,, \quad \phi^\alpha \to \frac{1}{m^2}\Phi^\alpha\,, \tag{4.167}
$$

for some mass scale m, we recover the exact constraint equations for the moduli space of theory B for $\Lambda^{14}_{SU(7)} = -i\lambda m^5 \Lambda^9_{SU(5)}$. Thus, the moduli spaces are biholomorphic.

4.6.2 Complex cone over $\mathbb{F}^0$

We now consider the Calabi-Yau cone over $\mathbb{F}_0 = \mathbb{P}^1 \times \mathbb{P}^1$, a $\mathbb{Z}_2$ orbifold of the conifold which is the same as the real cone over $Y^{2,0}$. As shown in figure 4.6(a)–(b), there are two different toric[37] quiver gauge theories ("phases") which describe D3 branes probing this singularity. These theories, which we denote by phase I and phase II, are related by Seiberg duality on one of the nodes.

In the $\mathbb{C}^3/\mathbb{Z}_3$ and dP_1 examples studied previously, there was only one toric phase, and we found a duality relating two different orientifolds of that phase which differed by exchanging SO and USp groups and symmetric and antisymmetric tensor matter, a "negative rank duality" as explained in appendix 4.B. We argue in [60] that these dualities relate to the $SL(2,\mathbb{Z})$ self-duality of type IIB string theory. Interestingly, negative rank duality also partly "explains" the pattern of $\mathcal{N} = 4$ dualities between SO and USp theories, suggesting that it may have some physical interpretation relating to Montonen-Olive duality and its $\mathcal{N} = 1$ analogues.

[37] In this context, a toric quiver gauge theory is one for which the number of arrows entering and exiting each node is the same.

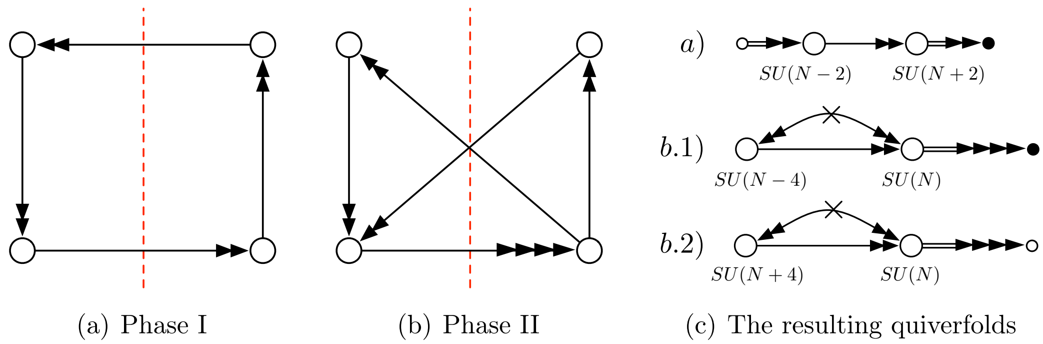


Figure 4.6: (a)–(b) The two Seiberg-dual quiver gauge theories for $\mathbb{F}_0$. The red dashed lines indicate orientifold involutions compatible with the $SU(2) \times SU(2)$ isometry of the base. (c) The $SU(2) \times SU(2)$-preserving anomaly-free quiverfolds that result from orientifolding these theories.

By contrast, for the $\mathbb{F}_0$ orientifolds we now study, the negative rank duals are either trivially equivalent or related by Seiberg duality. Instead, we find a nontrivial duality between orientifolds of the two different phases. Although the two phases are related by Seiberg duality in the parent theory, the resulting orientifolds are not obviously related in this way, giving yet another new field theory duality.[38]

As in our previous examples, we wish to consider orientifolds corresponding to compact O7 planes wrapping the base $\mathbb{F}_0$. This is equivalent to the requirement that the orientifold preserves the $SU(2) \times SU(2)$ isometry of the base. Only the involutions pictured in figure 4.6(a)–(b) do so, and of the fixed-element sign choices compatible with $SU(2) \times SU(2)$ invariance, only one choice for phase I and two for phase II lead to anomaly-free theories, giving the theories pictured in figure 4.6(c).[39]

Notice that the sole orientifold of phase I is its own negative rank dual, whereas the two orientifolds of phase II are related by negative rank duality. As we shall

[38]We cannot eliminate the possibility that a chain of deconfinements, dualizations, and reconfinements might relate the two theories via known dualities, but we have not been able to find such a chain.

[39]See [125] for a more detailed, brane tiling-based derivation of these orientifolds.

see, the latter two theories are Seiberg dual upon dualizing the left-node. We now discuss the orientifolds of each phase in turn, providing evidence for a duality between the orientifolds of the different phases.

Phase I

We obtain the orientifold theory:

	$SU(N-2)$	$SU(N+2)$	$SU(2)_1$	$SU(2)_2$	$U(1)_B$	$U(1)_R$	$\mathbb{Z}_2$
A^i	$\square$	$\overline{\square}$	$\square$	$\mathbf{1}$	$\frac{N}{N^2-4}$	$\frac{1}{2}-\frac{6}{N^2-4}$	$\omega_{2(N-2)}$
B^i	$\overline{\square\square}$	$\mathbf{1}$	$\mathbf{1}$	$\square$	$-\frac{1}{N-2}$	$\frac{1}{2}+\frac{3}{N-2}$	$\omega_{2(N-2)}^{-2}$
C^i	$\mathbf{1}$	$\begin{smallmatrix}\square\\ \square\end{smallmatrix}$	$\mathbf{1}$	$\square$	$-\frac{1}{N+2}$	$\frac{1}{2}-\frac{3}{N+2}$	1

(4.168)

with superpotential given by

$$W = \epsilon_{ij}\epsilon_{kl}\mathrm{Tr}\left(A^i B^k A^j C^l\right) . \tag{4.169}$$

For odd N, the $\mathbb{Z}_2$ discrete symmetry is gauge equivalent[40] to the $\mathbb{Z}_2$ center of $SU(2)_1$, whereas for even N it is a distinct global symmetry. Thus, the global symmetry group is actually $SU(2)_1 \times SU(2)_2 \times U(1)_B \times U(1)_R \times \mathbb{Z}_{\mathrm{gcf}(2,N)}$.

The global anomalies for this theory are shown in table 4.4(a). Note that the anomalies are invariant under $N \to -N$ combined with a charge conjugation of $U(1)_B$. This invariance corresponds to taking the negative rank dual as explained in appendix 4.B. While it led to two different gauge theories in the previous examples, one can check that in this case it maps the above theory to itself.

[40]Here and in future by "gauge equivalent" we mean that the two generators are related by composition with a (constant) gauge transformation, see the discussion at the beginning of section 4.3.

(a) Phase I

$SU(2)^3_{1/2}$	$(-1)^N$
$SU(2)^2_{1/2}\,U(1)_B$	$\pm N$
$SU(2)^2_{1/2}\,U(1)_R$	$-\frac{1}{2}(N^2+8)$
$U(1)^2_B\,U(1)_R$	-2
$U(1)^3_R$	$\frac{3}{2}N^2-34$
$U(1)_R$	-10
$SU(2)^2_1\,\mathbb{Z}_2$	$(-1)^N$
$SU(2)^2_2\,\mathbb{Z}_2$	$-(-1)^N$
$\mathbb{Z}_2$	1

(b) Phase II

$SU(2)^3_{1/2}$	$(-1)^N$
$SU(2)^2_{1/2}\,U(1)_B$	$\pm N$
$SU(2)^2_{1/2}\,U(1)_R$	$-\frac{1}{2}(N^2+8)$
$U(1)^2_B\,U(1)_R$	-2
$U(1)^3_R$	$\frac{3}{2}N^2-34$
$U(1)_R$	-10
$SU(2)^2_1\,\mathbb{Z}_4$	-1
$SU(2)^2_2\,\mathbb{Z}_4$	$-(-1)^N$
$\mathbb{Z}_4$	1

Table 4.5: The non-vanishing anomalies for the orientifolds of the different phases of $\mathbb{F}_0$, where we use a multiplicative notation for discrete and Witten anomalies as before (see section 4.3.1). The $U(1)_B$, $U(1)^3_B$, and $U(1)_B\,U(1)^2_R$ anomalies all vanish.

Phase II

We obtain the orientifold theory:

	$SU(N-4)$	$SU(N)$	$SU(2)_1$	$SU(2)_2$	$U(1)_B$	$U(1)_R$	$\mathbb{Z}_4$
A^i	$\square$	$\overline{\square}$	$\square$	$\mathbf{1}$	$\frac{1}{N-4}$	$\frac{1}{2}+\frac{2}{N}$	$\omega_{4N}^{-1}\,\omega_{4(N-4)}^{-1}$
B^i	$\overline{\square}$	$\overline{\square}$	$\mathbf{1}$	$\square$	$-\frac{1}{N-4}$	$\frac{1}{2}+\frac{2}{N}$	$\omega_{4N}^{-1}\,\omega_{4(N-4)}$
$C^{i;j}$	$\mathbf{1}$	$\begin{smallmatrix}\square\\\square\end{smallmatrix}$	$\square$	$\square$	0	$1-\frac{4}{N}$	ω_{4N}^2

(4.170)

with superpotential

$$W = \epsilon_{ij}\epsilon_{kj}\mathrm{Tr}\left(A^i C^{j;k} B^l\right), \qquad (4.171)$$

as well as the theory

	$SU(N+4)$	$SU(N)$	$SU(2)_1$	$SU(2)_2$	$U(1)_B$	$U(1)_R$	$\mathbb{Z}_4$
$\tilde{A}^i$	$\square$	$\overline{\square}$	$\square$	$\mathbf{1}$	$\frac{1}{N+4}$	$\frac{1}{2}-\frac{2}{N}$	$\omega_{4N}\,\omega_{4(N+4)}$
$\tilde{B}^i$	$\overline{\square}$	$\overline{\square}$	$\mathbf{1}$	$\square$	$-\frac{1}{N+4}$	$\frac{1}{2}-\frac{2}{N}$	$\omega_{4N}\,\omega_{4(N+4)}^{-1}$
$\tilde{C}^{i;j}$	$\mathbf{1}$	$\square\square$	$\square$	$\square$	0	$1+\frac{4}{N}$	ω_{4N}^{-2}

(4.172)

with superpotential

$$\tilde{W} = \epsilon_{ij}\epsilon_{kj}\mathrm{Tr}\left(\tilde{A}^i\tilde{C}^{j;k}\tilde{B}^l\right). \tag{4.173}$$

For odd N, the $\mathbb{Z}_4$ discrete symmetry is gauge-equivalent to the $\mathbb{Z}_2$ center of $SU(2)_1$, whereas for $N = 4k+2$ the $\mathbb{Z}_2 \subset \mathbb{Z}_4$ subgroup is gauge-equivalent to the $\mathbb{Z}_2$ center of $SU(2)_1$, and for $N = 4k$ the $\mathbb{Z}_4$ is a distinct global symmetry. Thus, the global symmetry group is $SU(2)_1 \times SU(2)_2 \times U(1)_B \times U(1)_R \times \mathbb{Z}_{\mathrm{gcf}(4,N)}$.

It is straightforward to check that these two theories, which are related by negative rank duality, are also related by Seiberg dualizing the $SU(N \pm 4)$ gauge group factor and integrating out massive matter.

Relationship between the two phases

The global anomalies for these theories are shown in table 4.4(b), where for simplicity we do not display the anomalies of the Seiberg dual theories separately; one can verify that they match as expected. More importantly, we see that the phase I and phase II orientifolds have matching anomalies for odd $N_{\text{phase I}} = N_{\text{phase II}}$. For even N the global symmetry groups do not match, and the theories are not dual.[41]

It is interesting to understand better the nature of this prospective duality between orientifolds of the two phases. We will present evidence in [125] that the

[41] Although the global symmetry groups match for $N = 2k+2$, by removing a single D3 brane we reduce $N \to N-2$, after which the global symmetry groups no longer match, so there is no duality for even N. One can also show this by constructing holomorphic gauge invariants in one theory with no dual in the other theory, for example the phase I operator $C^{\frac{N+2}{2}}$.

embeddings in string theory for the two phases are related as in realizations of ordinary Seiberg duality, so we can expect the nature of the duality relating the two phases phases to be an infrared duality closely analogous to it. However, we emphasize that this duality is not obviously derivable from known examples of Seiberg duality. In [125], we also argue that the action of IIB S-duality on the D-brane configuration describing each phase reproduces the field theory dualities inside each phase that we just studied: it is a self-duality in phase I and it exchanges the two theories in phase II.

4.6.3 The real cone over $Y^{4,0}$

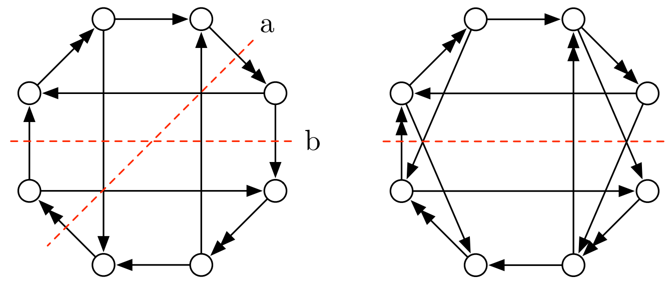
Before concluding, we present one final example of new dualities relating the world-volume gauge theories of D3 branes probing a Calabi-Yau singularity. Much like the $\mathbb{F}_0$ example studied above, this example exhibits interesting new patterns of dualities which appear distinct from the $\mathbb{C}_3/\mathbb{Z}_3$ and dP_1 examples discussed previously.

We consider the real cone over $Y^{4,0}$ which, like the cone over $Y^{2,0}$ considered above, is an orbifold of the conifold. There are five toric quiver gauge theories which describe D3 branes probing this singularity,[42] all of which are Seiberg dual. We focus on the two phases pictured in figure 4.7 and on the involutions also pictured there.[43]

For simplicity, we restrict our attention to the anomaly-free orientifolds pictured in figure 4.8. One can show that these orientifolds correspond to compact O7 planes, and preserve the full $SU(2) \times U(1)_X \times U(1)_R$ isometry group of $Y^{4,0}$. We now briefly discuss each of the three theories in turn, after which we illustrate a

[42]See [178, 179] for a classification of the toric phases of D3 branes probing a $Y^{p,q}$ singularity.

[43]Two of the remaining three phases also admit involutions, and several of the resulting orientifold theories are manifestly Seiberg dual to those considered here.



(a) Phase I (b) Phase II

Figure 4.7: Two of the five Seiberg-dual toric quiver gauge theories for $Y^{4,0}$. The red dashed lines indicate the orientifold involutions we will consider.

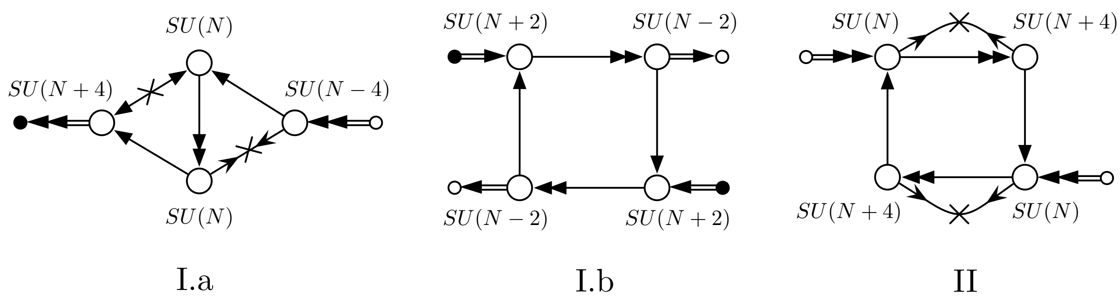


Figure 4.8: Quiverfolds for the anomaly-free orientifold gauge theories we will consider, arranged by the parent quiver and involution used to generate them (see figure 4.7). The phase II quiverfold has a negative rank dual which is not pictured, as it is manifestly Seiberg dual to the quiverfold which is shown. The phase I quiverfolds are "self-dual" under negative rank duality.

potential duality between them using anomaly matching.

Phase I, Involution a

We obtain the orientifold theory:

	$SU(N+4)$	$SU(N)$	$SU(N)$	$SU(N-4)$	$SU(2)$	$U(1)_B$	$U(1)_X$	$U(1)_R$
A^i	$\begin{smallmatrix}\square\\\square\end{smallmatrix}$	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$	$\square$	$\frac{1}{N+4}$	0	$\frac{1}{2}-\frac{6}{N+4}$
S^i	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$	$\overline{\square\square}$	$\square$	$\frac{1}{N-4}$	0	$\frac{1}{2}+\frac{6}{N-4}$
P_{12}	$\overline{\square}$	$\overline{\square}$	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$	$-\frac{N+2}{N(N+4)}$	$-\frac{N-4}{N}$	$\frac{1}{2}+\frac{3}{N+4}$
P_{13}	$\overline{\square}$	$\mathbf{1}$	$\square$	$\mathbf{1}$	$\mathbf{1}$	$-\frac{N+2}{N(N+4)}$	$\frac{N-4}{N}$	$\frac{1}{2}+\frac{3}{N+4}$
P_{23}^i	$\mathbf{1}$	$\square$	$\overline{\square}$	$\mathbf{1}$	$\square$	$\frac{1}{N}$	0	$\frac{1}{2}$
P_{24}	$\mathbf{1}$	$\overline{\square}$	$\mathbf{1}$	$\square$	$\mathbf{1}$	$-\frac{N-2}{N(N-4)}$	$\frac{N+4}{N}$	$\frac{1}{2}-\frac{3}{N-4}$
P_{34}	$\mathbf{1}$	$\mathbf{1}$	$\square$	$\square$	$\mathbf{1}$	$-\frac{N-2}{N(N-4)}$	$-\frac{N+4}{N}$	$\frac{1}{2}-\frac{3}{N-4}$

(4.174)

with superpotential

$$W=\epsilon_{ij}A^iP_{12}P_{23}^jP_{13}+\epsilon_{ij}S^iP_{24}P_{23}^jP_{34}\,, \tag{4.175}$$

where there is an additional discrete $\mathbb{Z}_{\mathrm{gcf}(4,N)}$ symmetry for even N, which we omit from the charge table for simplicity, as it will not play a large role in our analysis.

Phase I, Involution b

We obtain the orientifold theory:

	$SU(N+2)$	$SU(N-2)$	$SU(N+2)$	$SU(N-2)$	$SU(2)$	$U(1)_B$	$U(1)_X$	$U(1)_R$
T_1	$\overline{\begin{smallmatrix}\square\\\square\end{smallmatrix}}$	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$	$-\frac{1}{N+2}$	$\frac{N+4}{N+2}$	$\frac{1}{2}-\frac{7}{N+2}$
T_2	$\mathbf{1}$	$\square\!\square$	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$	$-\frac{1}{N-2}$	$-\frac{N-4}{N-2}$	$\frac{1}{2}+\frac{7}{N-2}$
T_3	$\mathbf{1}$	$\mathbf{1}$	$\overline{\begin{smallmatrix}\square\\\square\end{smallmatrix}}$	$\mathbf{1}$	$\mathbf{1}$	$-\frac{1}{N+2}$	$-\frac{N+4}{N+2}$	$\frac{1}{2}-\frac{7}{N+2}$
T_4	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$	$\square\!\square$	$\mathbf{1}$	$-\frac{1}{N-2}$	$\frac{N-4}{N-2}$	$\frac{1}{2}+\frac{7}{N-2}$
P_{12}^i	$\square$	$\overline{\square}$	$\mathbf{1}$	$\mathbf{1}$	$\square$	$\frac{N}{N^2-4}$	$-\frac{2N}{N^2-4}$	$\frac{1}{2}-\frac{14}{N^2-4}$
P_{23}	$\mathbf{1}$	$\square$	$\overline{\square}$	$\mathbf{1}$	$\mathbf{1}$	$-\frac{N}{N^2-4}$	$\frac{N^2}{N^2-4}$	$\frac{1}{2}+\frac{14}{N^2-4}$
P_{34}^i	$\mathbf{1}$	$\mathbf{1}$	$\square$	$\overline{\square}$	$\square$	$\frac{N}{N^2-4}$	$\frac{2N}{N^2-4}$	$\frac{1}{2}-\frac{14}{N^2-4}$
P_{41}	$\overline{\square}$	$\mathbf{1}$	$\mathbf{1}$	$\square$	$\mathbf{1}$	$-\frac{N}{N^2-4}$	$-\frac{N^2}{N^2-4}$	$\frac{1}{2}+\frac{14}{N^2-4}$

(4.176)

with superpotential

$$W = \frac{1}{2}\epsilon_{ij}T_1P_{12}^iP_{12}^jT_2 + \frac{1}{2}\epsilon_{ij}T_3P_{34}^iP_{34}^jT_4 + \epsilon_{ij}P_{12}^iP_{23}P_{34}^jP_{41}\,. \tag{4.177}$$

As before, there is an additional discrete $\mathbb{Z}_{\mathrm{gcf}(4,N)}$ symmetry for even N.

Phase II

We obtain the orientifold theory:

	$SU(N+4)$	$SU(N)$	$SU(N+4)$	$SU(N)$	$SU(2)$	$U(1)_B$	$U(1)_X$	$U(1)_R$
P_{12}	$\square$	$\overline{\square}$	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$	$-\frac{1}{N+4}$	$-\frac{N+2}{N+4}$	$\frac{1}{2}+\frac{2(N+8)}{N(N+4)}$
P_{23}	$\mathbf{1}$	$\square$	$\square$	$\mathbf{1}$	$\mathbf{1}$	$-\frac{1}{N+4}$	$-\frac{N+6}{N+4}$	$\frac{1}{2}-\frac{2(3N+8)}{N(N+4)}$
P_{34}	$\mathbf{1}$	$\mathbf{1}$	$\square$	$\overline{\square}$	$\mathbf{1}$	$-\frac{1}{N+4}$	$\frac{N+2}{N+4}$	$\frac{1}{2}+\frac{2(N+8)}{N(N+4)}$
P_{41}	$\square$	$\mathbf{1}$	$\mathbf{1}$	$\square$	$\mathbf{1}$	$-\frac{1}{N+4}$	$\frac{N+6}{N+4}$	$\frac{1}{2}-\frac{2(3N+8)}{N(N+4)}$
X_2^i	$\mathbf{1}$	$\overline{\square\!\square}$	$\mathbf{1}$	$\mathbf{1}$	$\square$	0	1	$1+\frac{8}{N}$
X_4^i	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$	$\overline{\square\!\square}$	$\square$	0	-1	$1+\frac{8}{N}$
T_{41}^i	$\overline{\square}$	$\mathbf{1}$	$\mathbf{1}$	$\square$	$\square$	$\frac{1}{N+4}$	$-\frac{2}{N+4}$	$\frac{1}{2}-\frac{2(N+8)}{N(N+4)}$
T_{23}^i	$\mathbf{1}$	$\square$	$\overline{\square}$	$\mathbf{1}$	$\square$	$\frac{1}{N+4}$	$\frac{2}{N+4}$	$\frac{1}{2}-\frac{2(N+8)}{N(N+4)}$

(4.178)

$SU(2)^2\, U(1)_B$	$2N$
$U(1)_X^2\, U(1)_B$	$-4N$
$SU(2)^2\, U(1)_R$	$-N^2-24$
$U(1)_B^2\, U(1)_R$	-4
$U(1)_X^2\, U(1)_R$	$-2(N^2+32)$
$U(1)_R^3$	$3N^2-164$
$U(1)_R$	-20

Table 4.6: The non-vanishing anomalies of the three different $Y^{4,0}$ theories. We omit discrete anomalies for simplicity, as there are no discrete symmetries for odd N, whereas we argue that no dualities are possible for even N.

with superpotential

$$W = \epsilon_{ij} X_2^i P_{23} T_{23}^j + \epsilon_{ij} X_4^i P_{41} T_{41}^j + \epsilon_{ij} P_{12} T_{23}^i P_{34} T_{41}^j \,. \tag{4.179}$$

In this case, there is an additional discrete $\mathbb{Z}_{\mathrm{gcf}(8,N)}$ symmetry for even N.

Relationship between the different orientifolds

One can show that the anomalies involving the continuous global symmetries match between all three theories considered above, where the non-vanishing anomalies (excluding discrete anomalies) are shown in table 4.6.

For odd N there are no discrete symmetries and therefore all three theories have matching global symmetry groups and anomalies. For even N not divisible by eight, the global symmetry groups again match between all three theories. However, by removing k D3 branes we can reduce $N \to N-2k$. Thus, consistency along the Coulomb branch rules out a duality between phase I and phase II for even N, since $\mathbb{Z}_{\mathrm{gcf}(8,N)} \neq \mathbb{Z}_{\mathrm{gcf}(4,N)}$ for $N = 8p$.

This leaves open the possibility of a duality between the two orientifolds of phase I for even N. However, one can show that the operator spectra do not match in this case. Specifically, we can compare the baryons of minimum R-charge in both theories:

$$
\begin{array}{c|cc}
 & \text{I.a} & \text{I.b} \\
\hline
\text{Baryon} & A^{\frac{N+4}{2}} & T_1^{\frac{N+2}{2}} \text{ or } T_3^{\frac{N+2}{2}} \\
Q_B & 1/2 & -1/2 \\
Q_X & 0 & \frac{N+4}{2} \text{ or } -\frac{N+4}{2} \\
Q_R & \frac{N}{4}-2 & \frac{N}{4}-3
\end{array}
\tag{4.180}
$$

Clearly the operators do not match each other, which is inconsistent with a duality between these two theories for even N. Moreover, in the phase II theory only integral Q_B is possible, so these operators have no dual there either, consistent with the mismatch in discrete symmetries explained above. Thus, we conclude that there are no dualities between the different theories for even N.

For odd N, these issues do not arise, as the above operators are no longer well-defined, and only integral Q_B is possible in all three theories. As an additional check that a duality can occur for this case, we again consider the baryons of minimal R-charge. For theory I.a, we find the baryons $P_{24}^{N-4}P_{12}^4A^2$ and $P_{34}^{N-4}P_{13}^4A^2$, which transform as $\left(\square\square, -1, \pm(N-4), \frac{N-4}{2}\right)$ under $SU(2)\times U(1)_B\times U(1)_X\times U(1)_R$. Consistent with the proposed duality, we find that the theory I.b baryons $T_1^N(T_3P_{34}^2P_{41}^2)^2$ and $T_3^N(T_1P_{12}^2P_{23}^2)^2$ have the same charges under the global symmetries, as do the theory II baryons $P_{41}^N(P_{12}^2P_{23}T_{23})^2$ and $P_{23}^N(P_{34}^2P_{41}T_{41})^2$, where the F-term conditions play a nontrivial role in the latter two cases.

The duality between the two orientifolds of phase I is an intriguing new feature of this geometry which does not appear in the simpler examples we considered previously. We leave further discussion of it to a future work.

4.7 Conclusions

In this chapter, we showed that the $\mathcal{N}=1$ gauge theories arising on D3 branes probing orientifolded Calabi-Yau singularities exhibit a rich class of gauge theory dualities not previously explored in the literature. We focused on a particular example of these dualities, corresponding to the well-known $\mathbb{C}_3/\mathbb{Z}_3$ singularity, providing extensive checks for the proposed duality, including anomaly matching, matching of discrete symmetries, moduli space matching, and matching of the superconformal indices. In some instances the matching of various quantities between the two theories follows from, or would imply, some remarkable mathematical identities, see for example appendices 4.F and 4.G. Together, the success of these checks presents a compelling argument for the existence of a duality.

In [125], we argue that this duality originates from the $SL(2,\mathbb{Z})$ self-duality of type IIB string theory, and is therefore a close cousin of the more familiar Montonen-Olive duality of $\mathcal{N}=4$ theories. In section 4.3.2 we show that $SL(2,\mathbb{Z})$ then acts in the usual way on a particular combination of holomorphic couplings which is constant along the RG flow and which corresponds to the axio-dilaton of type IIB string theory. We conclude that the dual descriptions we find are different weakly coupled limits of a single theory — the theory of branes at the orientifolded singularity — valid for complementary ranges of axio-dilaton vevs. These features make it clear that this $\mathcal{N}=1$ duality is of a different type than the more usually considered Seiberg (infrared) duality. Rather, it is more closely analogous to Montonen-Olive duality, differing only by the reduced supersymmetry and consequently richer dynamics.

As the axio-dilaton corresponds to a holomorphic combination of couplings which is RGE invariant, in general these theories will flow to a complex fixed line parameterized by the axio-dilaton. (We have demonstrated that this occurs in a specific example where part of the fixed line is perturbatively accessible.) The $SL(2,\mathbb{Z})$ duality group therefore acts nontrivially on the fixed line, much as in the $\mathcal{N}=1^*$ theories already understood in the literature [120, 122], which are mass deformations of $\mathcal{N} = 2$ or $\mathcal{N} = 4$ theories and inherit their $SL(2, \mathbb{Z})$ duality directly from that of the parent theory. In certain special cases, however, the flows corresponding to different values of the string coupling converge to a single fixed point. In these cases, one of which we discuss in the text, the $SL(2, \mathbb{Z})$ duality gives rise to an infrared duality relating the two dual theories, both taken at weak string coupling as in ordinary Seiberg duality.

The orientifolded $\mathbb{C}_3/\mathbb{Z}_3$ singularity is but one example among many geometries that exhibit these dualities. We expect that D3 branes probing any orientifolded Calabi-Yau singularity will exhibit an $SL(2, \mathbb{Z})$ duality so long as the O7 planes are compact, though in some cases it is only a self-duality. For example, the dP_1 singularity is a closely related geometry giving rise to a dual pair of gauge theories related to the $\mathbb{C}^3/\mathbb{Z}_3$ theories by Higgsing (corresponding to blowing down a two-cycle in the dP_1 base). These theories exhibit interesting dynamics, such as a quantum-deformed moduli space for the lowest rank example which we were able to completely match between the dual theories. It would be interesting to understand the dynamics of these theories for larger N. In [60, 125] we provide infinite classes of geometries which generalize $\mathbb{C}_3/\mathbb{Z}_3$ and dP_1, all of which exhibit similar dualities.

In addition to $SL(2, \mathbb{Z})$ dualities, more complicated geometries (such as the $\mathbb{F}_0$ singularity) also exhibit other interesting dualities. The simplest of these appear to be closely related to Seiberg duality, at least from the perspective of string theory, as we argue in [125]. However, from the field theory perspective, they are new (presumably infrared) dualities not readily derivable from the Seiberg duals known in the literature. We also find indications of further dualities whose string theoretic origin is unclear, such as the duality relating the two orientifolds of phase I of $Y^{4,0}$. It would be interesting to better understand the nature and origin of these dualities.

We anticipate that further study of these dualities will lead to new insights concerning both string theory and gauge theories. In particular, on the gauge theory side, our work helps to substantially expand the universe of known dualities to cases where product gauge groups play a pivotal role, and radically broadens the contexts in which $SL(2,\mathbb{Z})$ dualities are seen to arise. The infinite variety of Calabi-Yau singularities provides plenty of room for further study, which could reveal further types of duality or further illuminate the dualities we have considered here.

Finally, given a clear understanding of when dualities are expected to occur in string theory, it might be possible to construct examples of $\mathcal{N} = 0$ dualities. Indeed, this has recently been attempted for the case where supersymmetry is broken by antibranes [180].[44] Our work suggests a related program of dualities from anti-branes at Calabi-Yau singularities, or from branes probing SUSY-breaking singularities, such as non-supersymmetric orbifolds. While string theory seems to suggest that both cases should lead to $SL(2,\mathbb{Z})$ dualities, this seems extraordinary from the field theory perspective, making it a natural topic for further research.

4.A Quiverfolds

As the main focus of this chapter is dualities relating gauge theories arising on the world-volumes of D3 branes probing orientifolded Calabi-Yau singularities, it is useful to establish some general facts about these gauge theories.

While D-brane gauge theories are quiver gauge theories, the introduction of O-planes leads to a slightly more general class of theories which we refer to as "quiverfold" gauge theories. Quiverfold gauge theories admit more general gauge

[44]Other interesting work somewhat related in spirit, although focusing on non-supersymmetric analogues of Seiberg duality, can be found in [181], based on ideas reviewed in [182].

groups and matter content than quiver gauge theories. While quiver gauge theories allow only SU gauge group factors as well as adjoint and bifundamental $(\square,\overline{\square})$ or $(\overline{\square},\square)$ matter, quiverfold gauge theories also allow SO and USp groups, as well as two-index tensor matter and bifundamental matter in the $(\square,\square)$ or $(\overline{\square},\overline{\square})$ representations.

Such gauge theories cannot be described by standard quiver diagrams (directed graphs), and we develop a more general diagrammatic notation in section 4.A.3, which we call a "quiverfold diagram". One can show that any connected quiverfold diagram which is not a strict quiver can be thought of (in a precise way which we later make clear) as the result of "folding" a quiver in half along a line of $\mathbb{Z}_2$ symmetry, which is the inspiration for the term "quiverfold".

Before discussing quiverfolds in section 4.A.3, we first motivate their introduction by "deriving" a set of rules for obtaining orientifold gauge theories from their parent (orientifold-free) quiver gauge theory in section 4.A.1 and applying these rules to a few simple examples in section 4.A.2. As shown in [125], these rules are equivalent to well-established results in the literature on orientifolding toric Calabi-Yau singularities using brane tilings [124]. While the brane tiling method has some computational advantages relating to the superpotential, our approach (following [183]) is somewhat more intuitive, and we focus on it here for that reason, deferring further discussion of brane tiling methods to [125].

4.A.1 Orientifolding a quiver gauge theory

We consider a quiver gauge theory describing a collection of D-branes probing some background. Each node in the quiver corresponds to a stack of identical D-branes, with an associated $U(N)$ gauge group. Arrows in the quiver, bifundamental matter in the quiver gauge theory, correspond to open strings stretched between the stacks of branes at their intersections.

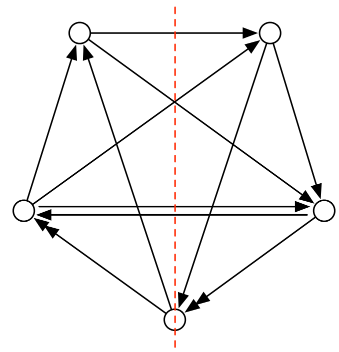


Figure 4.9: An example of an involution of a quiver. The quiver theory pictured here describes the toric PdP_2 singularity [184].

To this picture, we now add orientifold planes (O-planes). The associated involution, σ, must map the background and the collection of branes onto itself (up to certain signs and orientations), and squares to the identity. Thus, the involution defines an order-two permutation on the nodes of the quiver. Moreover, the involution maps open strings to *oppositely oriented* open strings. Thus, the involution also defines an order-two permutation on the arrows of the quiver, such that for any arrow $X : A \to B$ connecting node A to node B, the arrow's orientifold image $X' : B' \to A'$ connects B' to A', where A' and B' are the orientifold images of the nodes A and B, respectively.

The observations of the last paragraph may be summarized as follows:

> **Rule I:** The O-plane involution defines a $\mathbb{Z}_2$ automorphism of the quiver which reverses the directions of arrows.

An example of the resulting involution is shown in figure 4.9. Note that not every quiver has an involution, in the sense defined above. A necessary condition is that the quiver be isomorphic to its charge conjugate (the same quiver with the arrows reversed). This corresponds to the fact that not all brane configurations can be orientifolded, since the branes must then come in image pairs under the involution.

D-brane gauge theories generically come with a classical (tree-level) superpotential,[45] which is determined by the geometry and brane configuration. Since these objects are appropriately covariant under the involution, we conclude that the superpotential must also be appropriately covariant: that is, $W \to W'$, where W' is equivalent to W up to some symmetry transformation. In the examples which follow, we shall see that the appropriate restriction is in fact:

> **Rule II:** The superpotential of the parent theory is invariant under the involution.

Notice that if we impose the same requirement on the (generally unknown) Kähler potential, this implies that the corresponding gauge theory has a *color-conjugation* symmetry.[46] The orientifold theory results from identifying the chiral and vector superfields related by the involution. This can be restated as:

> **Rule III:** The orientifold gauge theory is derived from the parent theory by gauging the involution.

Note that the above rules should only be interpreted at the classical level. For instance, the gauge group ranks compatible with anomaly cancellation are generally different in the parent and orientifold theories, corresponding to the tadpoles (RR charge) carried by the O-planes.

We have presented a heuristic argument (following [183]) for a set of rules relating the world-volume theories of stacks of D-branes to the world-volume gauge theories of their orientifolds. To the extent that these arguments hold, the above rules should be viewed as *necessary* (but potentially insufficient) conditions which must be satisfied by consistent orientifold involutions. We now illustrate these arguments with a pair of examples.

---

[45] We restrict our attention to supersymmetric brane configurations and orientifolds.

[46] Since in general there are multiple gauge groups, the theory can still be chiral (cf. [140]).

4.A.2 Examples

$\mathcal{N} = 4$ orientifolds

We consider the world-volume gauge theory of N parallel D3 branes in flat space, which is $\mathcal{N} = 4$ $SU(N)$ super-Yang-Mills. This theory has an $\mathcal{N} = 1$ description with three adjoint chiral superfields Φ^i, $i \in \{1, 2, 3\}$, and the superpotential:

$$W = \frac{1}{3}\,\epsilon_{ijk} \mathrm{Tr}\,\Phi^i \Phi^j \Phi^k\,, \tag{4.181}$$

up to a superpotential coupling which can be removed by rescaling the fields. However, in this language only an $SU(3) \times U(1)_R$ subgroup of the $SU(4)_R$ symmetry is manifest, where Φ^i transforms as $\square_{+2/3}$.

We consider orientifolds of this theory, imposing the rules from the previous section. We first consider the action of the involution on the gauge bosons. It is well known that only (products of) $U(N)$, $SO(N)$ and $USp(N)$ gauge groups are possible in perturbative string theory. In particular, the involution must act on the gauge bosons as follows:

$$A \to \pm M A^T M^\dagger\,, \tag{4.182}$$

where M must be unitary to leave the gauge kinetic term invariant. Since the involution squares to the identity, we find $MM^* = \pm 1$ so that $M^T = \pm M$. In the case where M is symmetric, it can be diagonalized by a gauge transformation, giving $M = \mathbb{1}$. The remaining unbroken gauge symmetry is $SO(N)$, and we choose

$$A \to -A^T \tag{4.183}$$

to ensure that the invariant gauge bosons correspond to the generators of $SO(N)$.

Conversely, if M is antisymmetric, it can be put into the form

$$M = \Omega = \begin{pmatrix} 0 & 1 & 0 & 0 & \cdots \\ -1 & 0 & 0 & 0 & \\ 0 & 0 & 0 & 1 & \\ 0 & 0 & -1 & 0 & \\ \vdots & & & & \ddots \end{pmatrix} , \qquad (4.184)$$

and the remaining unbroken gauge symmetry is $USp(N)$. We then choose the involution

$$A \to \Omega A^T \Omega \qquad (4.185)$$

once again to ensure that the invariant gauge bosons correspond to the generators of $USp(N)$.

We now consider the action of the involution on the (adjoint) Weyl fermions ψ^i, $i \in 1 \ldots 4$:

$$\psi^i \to \Lambda^i_j \mathcal{I} (\psi^j)^T \mathcal{I}^* , \qquad (4.186)$$

where $\mathcal{I}$ acts on the gauge indices. Invariance under the remaining $SO(N)$ or $USp(N)$ gauge symmetry requires that $\mathcal{I} = \mathbb{1}$ or $\mathcal{I} = \Omega$, respectively, up to an overall factor which can be absorbed into Λ^i_j. Invariance of the kinetic term requires Λ^i_j to be unitary, which can be diagonalized after an $SU(4)_R$ transformation, taking the form $\Lambda^i_j = \mathrm{diag}(\pm_1 1, \pm_2 1, \pm_3 1, \pm_4 1)$. For each positive (negative) eigenvalue of Λ^i_j, the corresponding Weyl fermion projects down to its invariant symmetric (antisymmetric) component. To preserve at least $\mathcal{N} = 1$ supersymmetry, at least one sign must be -1 ($+1$) to form a vector multiplet with the $SO(N)$ ($USp(N)$) gauge bosons, which we take to be ($\pm_4$) WLOG. In $\mathcal{N} = 1$ language, the remaining signs specify the action of the involution on the adjoint chiral superfields:

$$\Phi^i \to \hat{\Lambda}^i_j \mathcal{I} (\Phi^j)^T \mathcal{I}^* , \qquad (4.187)$$

where $\hat{\Lambda}^i_j = \text{diag}(\pm_1, \pm_2, \pm_3)$. The superpotential transforms as:

$$\begin{aligned}
W \to W' &= \frac{1}{3}\epsilon_{ijk}\hat{\Lambda}^i_{i'}\hat{\Lambda}^j_{j'}\hat{\Lambda}^k_{k'}\text{Tr}\left[\mathcal{I}(\Phi^{i'})^T\mathcal{I}^*\mathcal{I}(\Phi^{j'})^T\mathcal{I}^*\mathcal{I}(\Phi^{k'})^T\mathcal{I}^*\right] \\
&= \frac{1}{3}\det(\hat{\Lambda})(\pm_{\text{Sp}})^3\,\epsilon_{ijk}\text{Tr}\,\Phi^k\Phi^j\Phi^i \\
&= -(\pm_{\text{Sp}})\det(\hat{\Lambda})\,W\,, \qquad (4.188)
\end{aligned}$$

where $(\pm_{\text{Sp}})$ is $+1$ (-1) for an *SO* (*USp*) projection, so that $\mathcal{I}\mathcal{I}^* = \pm_{\text{Sp}}\mathbb{1}$. Thus, invariance of the superpotential requires that

$$(\pm_{\text{Sp}})(\pm_1)(\pm_2)(\pm_3) = -1\,. \qquad (4.189)$$

This is our first example of a "sign rule" [124]: a restriction on the form of the involution, and thus the spectrum of the orientifold theory, due to the requirement that W is invariant.

In $\mathcal{N} = 4$ language, the above sign rule amounts to the requirement $\det\Lambda = 1$, since $\pm_4 = -(\pm_{\text{Sp}})$. For an *SO* projection, the possibilities are $\Lambda = \text{diag}(-,-,-,-)$ and $\Lambda = \text{diag}(+,+,-,-)$, corresponding to the spectrum of an $\mathcal{N} = 4$ $SO(N)$ gauge theory and an $\mathcal{N} = 2$ $SO(N)$ gauge theory with a hypermultiplet in the symmetric representation, respectively. Similarly, for the *USp* projection, the possibilities are $\Lambda = \text{diag}(+,+,+,+)$ and $\Lambda = \text{diag}(-,-,+,+)$, corresponding to the spectrum of an $\mathcal{N} = 4$ $USp(N)$ gauge theory and an $\mathcal{N} = 2$ $USp(N)$ gauge theory with a hypermultiplet in the antisymmetric representation, respectively.

By comparison, D3 branes are mutually supersymmetric with coincident O3 and O7 planes: N D3 branes atop an O3$^-$ (O3$^+$) gives rise to an $\mathcal{N} = 4$ $SO(N)$ ($USp(N)$) world-volume gauge theory, whereas N D3 branes atop an O7$^-$ (O7$^+$) gives rise to an $\mathcal{N} = 2$ $USp(N)$ ($SO(N)$) world-volume gauge theory, in agreement with the sign rule (4.189). This agreement relies on our choice of rule II as the correct restriction on the transformation of W under the involution. Had we imposed $W \to -W'$ for instance, we would have obtained spectra with only $\mathcal{N}=1$ supersymmetry, which are not realized in string theory as the world-volume gauge theory of a stack of D3 branes coincident with an O-plane in a flat background.

In fact, the geometric involution of the Op brane can be computed directly from the action of the involution on the open string fields, (4.187). We form gauge invariant single trace mesons:

$$Z^{i_1 i_2 \ldots i_n} \equiv \operatorname{Tr} \Phi^{i_1} \Phi^{i_2} \ldots \Phi^{i_n} \,. \tag{4.190}$$

Upon imposing the F-term conditions, we obtain $[\Phi^i, \Phi^j] = 0$, so that $Z^{i_1 i_2 \ldots i_n}$ is totally symmetric in its indices. Acting with the involution (4.187), we obtain:

$$Z^{i_1 i_2 \ldots i_n} \to \left[(\pm_{\mathrm{Sp}})\hat{\Lambda}^{i_1}_{i'_1}\right] \left[(\pm_{\mathrm{Sp}})\hat{\Lambda}^{i_2}_{i'_2}\right] \ldots \left[(\pm_{\mathrm{Sp}})\hat{\Lambda}^{i_n}_{i'_n}\right] Z^{i'_1 i'_2 \ldots i'_n} \,, \tag{4.191}$$

modulo F-terms, where the extra signs $\pm_{\mathrm{Sp}}$ come from factors of $\Omega^2 = -1$ which appear in the trace for symplectic projections. Geometrically, Z^i corresponds to the coordinates z^i of the $\mathbb{C}^3$ in which the D3 branes are embedded. Thus, the geometric involution is simply:

$$z^i \to (\pm_{\mathrm{Sp}})\hat{\Lambda}^i_j z^j \,. \tag{4.192}$$

It is straightforward to check that this reproduces the O3 and O7 involutions for the $\mathcal{N}=4$ and $\mathcal{N}=2$ cases considered above. For example, choosing $\hat{\Lambda} = (-,+,+)$ with an SO projection, we obtain $z^1 \to -z^1$, $z^2 \to z^2$, $z^3 \to z^3$, corresponding to an O7 plane at $z^1 = 0$, whereas choosing $\hat{\Lambda} = (+,+,+)$ with an USp projection, we obtain $z^i \to -z^i$, corresponding to an O3 plane at the origin.

To obtain the superpotential of the orientifold theory, we replace the fields with their projections:

$$\Phi^i \to \phi^i \mathcal{I}^* \,, \tag{4.193}$$

where invariance under the involution requires that

$$\phi^i = \hat{\Lambda}^i_j \, (\phi^j)^T \,, \tag{4.194}$$

so that for $\hat{\Lambda} = \mathrm{diag}(\pm_1, \pm_2, \pm_3)$, ϕ^i is symmetric (antisymmetric) when $\pm_i$ is positive (negative), as previously noted. Applying this replacement to the superpotential, we obtain

$$W = \frac{1}{6}\epsilon_{ijk} \mathrm{Tr}\, \phi^i \phi^j \phi^k \,, \tag{4.195}$$

where for USp projections the trace implicitly includes factors of Ω between each pair of fields, and we include an extra factor of 1/2 by convention, the overall normalization being arbitrary up to field redefinitions. Written out explicitly, we obtain:

$$W = \frac{1}{2}\mathrm{Tr}\, \phi^1\phi^2\phi^3 - \frac{1}{2}\mathrm{Tr}\, \phi^3\phi^2\phi^1 \,, \tag{4.196}$$

while $\mathrm{Tr}\, M = \mathrm{Tr}\, M^T$ implies that $\mathrm{Tr}\, \phi^3\phi^2\phi^1 = (\pm_{\mathrm{Sp}})(\pm_1)(\pm_2)(\pm_3)\mathrm{Tr}\, \phi^1\phi^2\phi^3$, where the first sign $(\pm_{\mathrm{Sp}})^3 = (\pm_{\mathrm{Sp}})$ comes from $\Omega^T = -\Omega$. Thus, imposing (4.189), the superpotential reduces to

$$W = \mathrm{Tr}\, \phi^1\phi^2\phi^3 \,, \tag{4.197}$$

whereas imposing $W' = -W$ and following the same procedure, we would obtain a vanishing superpotential. Moreover, the superpotential (4.197) is exactly that required by the extended supersymmetry of the corresponding brane configurations.

Orientifolds of $\mathbb{C}^3/\mathbb{Z}_3$

Next, we consider N D3 branes probing the orbifold singularity $\mathbb{C}^3/\mathbb{Z}_3$, with the orbifold action $z^i \to e^{2\pi i/3} z^i$. The resulting $\mathcal{N} = 1$ quiver gauge theory, shown in

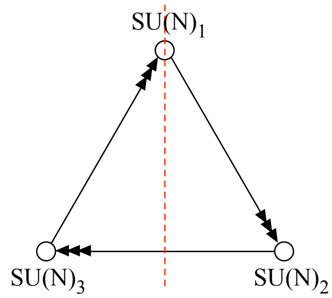


Figure 4.10: The quiver for $\mathbb{C}^3/\mathbb{Z}_3$, with the involution of interest indicated by the dashed line.

figure 4.4 (which we reproduce in figure 4.10 for convenience), is well known. The corresponding superpotential is:

$$W = \epsilon_{ijk} \text{Tr}\, X^i_{12} X^j_{23} X^k_{31}\,, \tag{4.198}$$

up to a superpotential coupling which can be removed by rescaling the fields. An $SU(3) \times U(1)_R$ symmetry is manifest under which the X^i_{AB} transform as $\square_{+2/3}$.

Applying the rules of section 4.A.1, we search for orientifolds of this configuration. Inspecting the quiver, one can easily check that rule I implies that the involution must fix one node and exchange the other two. As the quiver has a $\mathbb{Z}_3$ symmetry, we take the fixed node to be node 1 WLOG. The action of the involution on the chiral superfields is then:

$$X^i_{12} \to \Lambda^i_{\ j}\, \mathcal{I}_1 (X^j_{31})^T \delta^*_{32}\,, \quad X^i_{23} \to \Sigma^i_{\ j}\, \delta_{23} (X^j_{23})^T \delta^*_{23}\,, \quad X^i_{31} \to (\Lambda^i_{\ j})^\dagger\, \delta_{32} (X^j_{12})^T \mathcal{I}^*_1\,, \tag{4.199}$$

where Λ and Σ are unitary matrices, $\delta_{23} = \delta^T_{32} = \mathbb{1}$ breaks $SU(N)_2 \times SU(N)_3 \to SU(N)$, and $\mathcal{I}_1 = \mathbb{1}$ or Ω, depending on whether we choose an SO or USp projection for the fixed node, respectively. Moreover, since the involution squares to the identity, $\Sigma^2 = \mathbb{1}$, so that Σ is both unitary and Hermitian.

We compute the orientifold image of the superpotential:

$$W \to W' \;=\; \epsilon_{ijk}\, \Lambda^i_{\ i'} \Sigma^j_{\ j'} (\Lambda^k_{\ k'})^\dagger\, \text{Tr}\, \mathcal{I}_1 (X^{i'}_{31})^T \delta^*_{32}\, \delta_{23} (X^{j'}_{23})^T \delta^*_{23}\, \delta_{32} (X^{k'}_{12})^T \mathcal{I}^*_1$$

$$= \quad (\pm_{\rm Sp})\, \epsilon_{ijk}\, \Lambda^i_{\ i'} \Sigma^j_{\ j'} (\Lambda^k_{\ k'})^\dagger \, {\rm Tr}\, X_{12}^{k'} X_{23}^{j'} X_{31}^{i'} \,. \tag{4.200}$$

Therefore, invariance of the superpotential requires:

$$\epsilon_{ijk}\, \Lambda^i_{\ i'} \Sigma^j_{\ j'} (\Lambda^k_{\ k'})^\dagger = -(\pm_{\rm Sp})\, \epsilon_{i'j'k'} \,. \tag{4.201}$$

In fact, this is only possible if $\Lambda = e^{i\theta}\Sigma$,[47] where the phase factor can be removed by rotating $X_{12}^i \to e^{i\theta/2} X_{12}^i$ and $X_{31}^i \to e^{-i\theta/2} X_{31}^i$ (leaving the superpotential invariant). Thus, we take $\Lambda = \Sigma$, where the invariance of the superpotential requires

$$\det \Sigma = -(\pm_{\rm Sp}) \,. \tag{4.202}$$

After an $SU(3)$ transformation, we obtain $\Sigma = {\rm diag}(\pm_1, \pm_2, \pm_3)$, and the requirement that the superpotential be invariant takes the form of a sign rule:

$$(\pm_1)(\pm_2)(\pm_3)(\pm_{\rm Sp}) = -1 \,. \tag{4.203}$$

Thus, for an SO projection, there are two possible involutions $\Sigma = {\rm diag}(-,-,-)$ and $\Sigma = {\rm diag}(-,+,+)$ up to an $SU(3)$ transformation. The spectrum of the latter theory turns out to be anomalous for any choice of gauge group ranks,[48] so we will focus on the first possibility. Similarly, for an USp projection, $\Sigma = {\rm diag}(+,+,+)$ and $\Sigma = {\rm diag}(-,-,+)$ are possible, again up to an $SU(3)$ transformation, with the latter being anomalous for any choice of ranks.

The anomalous orientifolds correspond to non-compact O7 planes, whereas the remaining possibilities correspond to compact O7 planes [32]. We verify this by computing the geometric involution. Consider mesons of the form:

$$Z^{ijk} \equiv {\rm Tr}\, X_{12}^i X_{23}^j X_{31}^k \,. \tag{4.204}$$

[47] In general whenever $0 \neq \epsilon_{ijk} A^i_{i'} B^j_{j'} C^k_{k'} \propto \epsilon_{i'j'k'}$, then $A \propto B, C$.

[48] The anomaly can be cancelled by introducing non-compact "flavor" D7 branes into the geometry [32].

Upon imposing the F-term conditions, we find that Z^{ijk} is totally symmetric in its indices. Applying the involution (4.199), we obtain:

$$Z^{ijk} \to (\pm_{\rm Sp})\, \Sigma^i_{i'} \Sigma^j_{j'} \Sigma^j_{j'} Z^{i'j'k'} = [(\pm_{\rm Sp})\Sigma^i_{i'}]\,[(\pm_{\rm Sp})\Sigma^j_{j'}]\,[(\pm_{\rm Sp})\Sigma^k_{k'}]\, Z^{i'j'k'} \,, \quad (4.205)$$

where the sign $\pm_{\rm Sp}$ comes from the $\Omega^2 = -1$ which appears in the trace for USp projections. The mesons Z^{ijk} correspond to the coordinates $z^i z^j z^k$ of $\mathbb{C}^3/\mathbb{Z}_3$; thus, we read off the geometric involution

$$z^i \to (\pm_{\rm Sp})\, \Sigma^i_j z^j \,. \quad (4.206)$$

From this, it is easy to check that the anomaly-free orientifolds, $(-,-,-)$ and $(+,+,+)$ for SO and USp respectively, correspond to compact O7 planes, with the involution $z^i \to -z^i$, whereas the anomalous orientifolds, $(-,+,+)$ and $(+,-,-)$ for SO and USp respectively, correspond to non-compact O7 planes, with the involution $z^1 \to -z^1$, $z^2 \to z^2$, $z^3 \to z^3$.

To derive the superpotential of the orientifold theory, we replace:

$$\begin{aligned} X^i_{12} &\to \Sigma^i_j\, A^j \,, \\ X^i_{23} &\to \Sigma^i_k\, B^j\, \delta^*_{23} \,, \\ X^i_{31} &\to \delta_{32}\, (A^j)^T \mathcal{I}^*_1 \,, \end{aligned} \quad (4.207)$$

where invariance under the involution requires that:

$$B^i = \Sigma^i_j\, (B^j)^T \,, \quad (4.208)$$

so that for $\Sigma = {\rm diag}(\pm_1, \pm_2, \pm_2)$, B^i is symmetric (antisymmetric) when $\pm_i$ is positive (negative). Applying these replacements to the superpotential, we obtain:

$$W = \frac{1}{2} (\det \Sigma)\, \epsilon_{ijk} \Sigma^k_l \,{\rm Tr}\, A^i B^j (A^l)^T \,, \quad (4.209)$$

where for USp projections the use of Ω in the trace is implicit. Since $\mathrm{Tr}\, M = \mathrm{Tr}\, M^T$, this can also be written as:

$$W = (\pm_{\mathrm{Sp}})\frac{1}{2}(\det\Sigma)\,\epsilon_{ijk}\Sigma^k_l\,\Sigma^j_m\,\mathrm{Tr}\,A^l B^m (A^i)^T = -(\pm_{\mathrm{Sp}})(\det\Sigma)\,W\,. \qquad (4.210)$$

Thus, as before, the sign rule (4.202) is necessary to ensure that the superpotential of the orientifold theory does not vanish.

For the cases $\Sigma = \mathrm{diag}(-,-,-)$ and $\Sigma = \mathrm{diag}(+,+,+)$ for SO and USp projections, respectively, the superpotential simplifies:

$$W = \frac{1}{2}\,\epsilon_{ijk}A^i A^j B^k\,, \qquad (4.211)$$

where we leave the contractions of gauge indices implicit. The resulting theories have the same $SU(3)\times U(1)_R$ flavor symmetries as the parent quiver theory, and are discussed more thoroughly in section 4.3.

4.A.3 General Quiverfolds

In the simple examples discussed above, we applied the rules of section 4.A.1 in a straightforward (if tedious) fashion to rederive known results. We now discuss some general features of this program applied to arbitrary quiver gauge theories. Specifically, we show how to derive the gauge group and spectrum of the orientifold theory graphically using the quiver diagram, and define a suitable generalization of the quiver to represent these data.

For the purposes of this discussion, we mainly ignore the superpotential, though we emphasize that rule II is generally very restrictive, and not all involutions of the quiver will leave the superpotential invariant. An explicit computation to check that W is invariant under the involution can be tedious, and for toric singularities the problem is well suited to brane tiling methods, as originally formulated in [124] and reviewed in [125].

Rule I implies that the quiver of the parent theory possesses a $\mathbb{Z}_2$ charge conjugation (arrow reversing) symmetry representing the involution in question. We embed the quiver in $\mathbb{R}^2$ such that this symmetry is manifest as a reflection through a fixed line, as in figure 4.9.[49] In the resulting figure, fixed nodes must lie along the fixed line, and fixed edges will intersect it perpendicularly, whereas any unfixed edge which crosses the fixed line must intersect another edge (its image) at the point of crossing.

To obtain the gauge group and spectrum of the orientifold theory we cut the plane in two along the fixed line, discarding one half of it and labeling each node and perpendicular (fixed) edge along the boundary with a sign. The resulting diagram on the half-plane, which we call a "quiverfold", specifies the gauge group and spectrum of the orientifold theory as follows: each node away from the boundary ("whole" node) corresponds to an SU gauge group, whereas each $+$ ($-$) node along the boundary ("half" node) corresponds to an SO (USp) gauge group. Each arrow away from the boundary ("uncrossed" (whole) edge) corresponds to bifundamental ($\square, \bar{\square}$) matter in the usual way, while each arrow intersecting the boundary obliquely is joined to its image arrow to form an edge ("crossed" (whole) edge) with opposite orientations associated to each end, and corresponding to ($\square, \square$) or ($\bar{\square}, \bar{\square}$), depending on the orientation of the arrows. Finally, each $+$ ($-$) edge ending perpendicularly on the boundary ("half" edge) corresponds to symmetric (antisymmetric) matter.

An example quiverfold is shown in figure 4.11(a). As shown in figure 4.11(b – c), the quiverfold can be drawn without the boundary line by using appropriate symbols to denote the fixed elements and crossed edges. From this perspective, a quiverfold is just an "enhanced" quiver, with a few additional representations and gauge groups allowed. Just as the world-volume gauge theory on intersecting

[49]While this is always possible to do, in general there are many possible embeddings. For a fixed involution, all embeddings will give the same quiverfold, as discussed below.

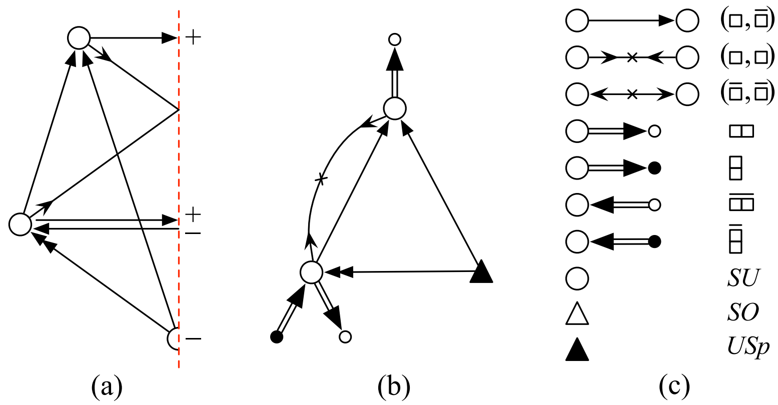


Figure 4.11: (a) An example of a quiverfold. The parent quiver is shown in figure 4.9. (b) The quiverfold can be redrawn without the fixed line using appropriate symbols, defined in (c).

D-branes can always be represented by a quiver gauge theory, orientifolds of these configurations can always be represented by a quiverfold (to the extent that rule I holds), which is then a very useful tool for concisely stating the gauge group and spectrum.

Note that some apparently different quiverfolds are isomorphic. In particular, any whole node of the quiverfold can be charge conjugated, yielding a new, equivalent quiverfold with different crossed and uncrossed edges; this corresponds to swapping the positions of a node and its image in the original $\mathbb{Z}_2$ symmetric embedding of the quiver. In a strict quiver, there is no analogous operation: since crossed edges are not allowed, charge conjugation can only be applied to the quiver as a whole. Furthermore, not every edge of a quiverfold is directed at both ends, since arrows entering and exiting half nodes are equivalent. Thus, edges connecting a half node to a whole node have a single direction (they cannot be crossed), whereas edges connecting two half-nodes are undirected. Taking into account these isomorphisms,[50] it is straightforward to show that different $\mathbb{R}^2$ embeddings of the

[50]There is moreover an isomorphism between a crossed edge connecting a whole node to itself (or a whole edge connecting a half-node to itself) and two half edges of opposite sign and

same involution (with the same choice of fixed-element signs) lead to the same quiverfold. Moreover, given a quiverfold, it is possible to uniquely reconstruct the parent quiver and involution by embedding the quiverfold on the half-plane with fixed elements on the boundary, as above.

It should be emphasized that just as a quiver is a direct pictorial representation of a certain class of gauge theories (quiver gauge theories), a quiverfold is also a direct pictorial representation of a certain (somewhat broader) class of gauge theories, which we call quiverfold gauge theories. Just as gauge invariant (mesonic) operators are directed loops in the quiver diagram, gauge invariant (mesonic) operators are loops in the quiverfold,[51] subject to the requirement that the loop enter and exit each whole node on oppositely directed edges. However, in some cases the mesonic operator corresponding to such a loop vanishes due to symmetry (e.g. it takes the form $\mathrm{Tr}\, M$ where M is antisymmetric).

While quiverfolds are useful for computing and representing the gauge group and spectrum of a given orientifold, the set of involutions consistent with rule I is usually a superset of those involutions consistent with both rules I and II: as we saw in section 4.A.2, the invariance of the superpotential imposes important constraints, such as the sign rules (4.189), (4.203) and (in the latter case) the alignment of the flavor rotations Λ^i_j and Σ^i_j.

It is possible to reformulate rule II graphically by describing the parent gauge theory and the involution in terms of a brane tiling, rather than a quiver diagram. We refer the interested reader to [124] for further details and references. As shown in [125], this approach is equivalent to the one outlined here. Regardless of the

like orientation connected to the node in question. While the involutions which give rise these configurations appear different, they are related by a nonabelian flavor symmetry of the parent theory, and the resulting spectra are the same.

[51] If the loop includes a half-edge, it doubles back on itself at this point, reentering the same node it just exited.

method used to apply these rules, quiverfold diagrams provide an intuitive and precise representation of the gauge group and spectrum of the orientifold gauge theory, much like quiver diagrams for D-brane gauge theories.

4.B Negative rank duality

In this appendix we review a fact about continuing $SU(N)$, $SO(N)$ and $USp(N)$ groups to negative rank that turns out to be very useful in the anomaly matching discussion in the main text. We refer the reader to chapter 13 in [185] for more details and further references. As we explain below, this continuation relates for example an $SU(-N)$ gauge theory to an $\widetilde{SU(N)}$ gauge theory and is often referred to as negative rank duality although the two related theories are generically *not* dual in the physical sense. In particular the two related gauge theories have generically different anomalies.

For an $SU(N)$ gauge theory with matter in certain representation we can exchange symmetrization and antisymmetrization (i.e. reflect the Young tableau across the diagonal) and at the same time replace N with $-N$. This leads to a new gauge theory we denote $\widetilde{SU(N)}$. As was first noticed by [186], for $SO(N)$ and $USp(N)$ theories we can likewise obtain a negative rank dual theory by exchanging symmetrization and antisymmetrization and replacing the $SO(N)$ symmetric bilinear invariant δ_{ab} by the $USp(N)$ antisymmetric bilinear invariant Ω_{ab} and replacing N by $-N$: $SO(-N) \cong \widetilde{USp(N)}$, $USp(-N) \cong \widetilde{SO(N)}$.

In [185] it is proven that under these dualities any scalar quantity becomes the dual scalar quantity up to potentially an overall sign. In particular, if we have a matter field transforming in the representation r which has a Young tableau with p boxes and $\tilde{r}$ denotes the transposed tableau obtained by a flip across the diagonal, then the dimensions of the corresponding representations are related by[52]

$$d_N(r) = (-1)^p d_{-N}(\tilde{r})\,. \tag{4.212}$$

Thanks to the theorems of [185] that we mentioned above, the proof is simple since we only need to determine the overall sign $(-1)^p$: Any representation with p boxes in the Young tableau has a leading N scaling that is given by N^p so that the overall sign under changing $N \to -N$ is $(-1)^p$, which gives the stated result.

Below we study the anomalies of negative rank dual theories of a generic gauge theory (see [187] for related results). For that we need the transformation properties of the Dynkin index $T(r)$ and anomaly coefficient $A(r)$ under the negative rank duality. These are again determined by the leading N scaling. Contrary to the dimension the Dynkin index and anomaly coefficient of the fundamental representation are independent of N. However, similarly to the dimension any extra box in the Young tableau leads to an extra factor of N so that one finds

$$T_N(r) = (-1)^{p-1} T_{-N}(\tilde{r})\,, \qquad A_N(r) = (-1)^{p-1} A_{-N}(\tilde{r})\,. \tag{4.213}$$

To prove this, we can again derive the leading N scaling by calculating the Dynkin index and anomaly coefficient for the tensor product of p fundamental representations. The Dynkin index $T(r)$ is defined by $(T_r^a)_n^m (T_r^b)_m^n = T(r)\delta^{ab}$, where the T_r^a are the generators for the representation r. Taking the tensor product with another fundamental representation introduces a factor of N in $T(r)$ and taking the tensor product of a fundamental with $(p-1)$ fundamental representation leads to the above result. Explicitly, for $SU(N)$ we can choose one of the generators in the fundamental representation to be $T_\square^1 = \frac{1}{\sqrt{2((N-1)^2+N-1)}}\mathrm{diag}(1,1,\ldots,1,-(N-1))$, which leads to $T(\square) = \frac{1}{2}$. The leading N scaling for any representation with p

[52]This statement only holds for representations with fixed, N independent p. In particular we should think of the anti-fundamental representation of $SU(N)$ as having $p=1$ and not $p=N-1$ and similarly for the adjoint representation we take $p=2$.

boxes is the same as the leading N scaling for the tensor product of p fundamentals. Taking $p-1$-times the tensor product of the above generator with $\mathbb{1}_N$ we obtain a generator for the representation that is given by the tensor product of p fundamentals and we find the leading N term $T(r) \propto N^{p-1}$, which is true for all irreducible representation of $SU(N)$ with p boxes. Similarly one can explicitly work out the scaling for $SO(N)$ and $USp(N)$. For the anomaly coefficient of $SU(N)$, we us the following result from [188]: $A(r_1 \otimes r_2) = d(r_1)A(r_2) + d(r_2)A(r_1)$. Together with the fact that $A(\Box)$ is N independent this leads to the leading N scaling $A(r) \propto N^{p-1}$ which completes the proof.

We now show that for any gauge theory the negative rank dual is free of gauge anomalies if all chiral matter representations have dimensions that are even under the negative rank dual i.e. whenever for every chiral field the number of all boxes in the Young tableaux of all gauge theories we are dualizing is even. Furthermore, the global anomalies of the two theories are related by replacing the rank of each gauge group factor we are dualizing with its negative and adding an overall minus sign whenever the global anomaly involves a non-abelian gauge group that is also being dualized.[53]

We take the combined gauge and global symmetry group to be $G = U(1)_1 \times \ldots \times U(1)_m \times G_1 \times \ldots \times G_n$, where G_a are SU, SO or USp groups. We denote the chiral matters fields by χ, the corresponding $U(1)_i$ charges by q_i^χ [54] and the matter dimension by $d(\chi) = \prod_{a=1}^n d(r_\chi^{G_a})$ where $r_\chi^{G_a}$ denotes the representation of

[53]In the absence of an $U(1)_R$ symmetry a negative rank dual theory is also anomaly free if all matter representation have dimensions that are odd under the negative rank dual. In that case all global symmetries pick up an extra overall minus sign.

[54]We assume for simplicity in the discussion below that the q_i^χ do not change sign under the negative rank duality. This condition can be relaxed so that for fixed i the q_i^χ, $\forall\chi$ change sign. This can lead to an extra overall minus sign in global anomalies involving $U(1)_i$.

χ under the group G_a. The $U(1)^3$ and $U(1)$ anomalies are given by

$$U(1)_i \, U(1)_j \, U(1)_k = \sum_\chi d(\chi) q_i^\chi q_j^\chi q_k^\chi \,, \tag{4.214}$$

$$U(1)_i = \sum_\chi d(\chi) q_i^\chi \,, \tag{4.215}$$

where the sums are over all chiral superfields χ. Whenever all chiral matter fields satisfy $d(\chi) = d(\tilde{\chi})$, then the above anomalies are unchanged after dualizing any of the G_a. The $G^2 \, U(1)$ and G^3 anomalies are

$$G_a^2 \, U(1)_i = \sum_\chi \frac{d(\chi)}{d(r_\chi^{G_a})} T(r_\chi^{G_a})) q_i^\chi \,, \tag{4.216}$$

$$G_a^3 = \sum_\chi \frac{d(\chi)}{d(r_\chi^{G_a}) A(r_\chi(G_a))} \,. \tag{4.217}$$

If G_a does not undergo a negative rank transition then the above anomalies are unchanged. In the case that G_a undergoes a negative rank duality we use the fact that $T(r)/d(r) = -T(\tilde{r})/d(\tilde{r})$ and $A(r)/d(r) = -A(\tilde{r})/d(\tilde{r})$ to find that both of the anomalies above pick up an extra minus sign. In particular this means that all the gauge and mixed anomalies that do not involve the R-symmetry still vanish after the negative rank transition. In our examples the global non-abelian gauge groups will not undergo a negative rank transition so that none of the global anomalies pick up an extra minus sign. They are simply given by replacing the ranks of all the gauge groups that undergo the negative rank duality with their negative ranks.

Next we calculate the anomalies that involve the R-symmetry

$$U(1)_R^3 = \sum_\chi d(\chi)(q_R^\chi - 1)^3 + d(G_{gauge}) \,, \tag{4.218}$$

$$U(1)_i \, U(1)_R^2 = \sum_\chi d(\chi) q_i (q_R^\chi - 1)^2 \,, \tag{4.219}$$

$$U(1)_i \, U(1)_j \, U(1)_R = \sum_\chi d(\chi) q_i q_j (q_R^\chi - 1) \,, \tag{4.220}$$

$$U(1)_R = \sum_\chi d(\chi)(q_R^\chi - 1) + d(G_{gauge}) \,, \tag{4.221}$$

$$G_a^2 U(1)_R \;=\; \sum_\chi \frac{d(\chi)}{d(r_\chi^{G_a})} T(r_\chi^{G_a})(q_R^\chi - 1) + T(\mathrm{Adj}_{G_a})\,. \quad (4.222)$$

Above $d(G_{gauge})$ denotes the dimension of the entire gauge group (excluding the global symmetry group) and $T(\mathrm{Adj}_{G_a})$ denotes the Dynkin index of the adjoint of G_a, if G_a is part of the gauge group. If G_a is part of the global symmetry group, then there are no gauginos that contribute and we have to set $T(\mathrm{Adj}_{G_a}) = 0$. For the SU, SO and USp groups the group dimension has always even parity under the negative rank transition. Thus $d(G_{gauge})$ is even and as mentioned above $T(\mathrm{Adj}_{G_a})$ is odd, if G_a undergoes the negative rank transition since $p = 2$. This means that only the last of the anomalies above picks up an overall minus sign if G_a undergoes the negative rank transition. We thus conclude that all gauge and mixed anomalies vanish after the transition. In the case where none of the global non-abelian symmetry groups undergo a negative rank transition we can furthermore conclude that all anomalies of the negative rank dual theory are obtained by replacing the ranks of all gauge group factors that undergo the transition with their negative.

A simple example of two negative rank dual theories has already appeared above in section 4.3. Both theories are related by taking the negative rank dual of both gauge group factors. The $SO(N-4) \times SU(N)$ extrapolated to negative N is $SO(-(N+4)) \times SU(-N)$ which dualizes to $USp(N+4) \times SU(N)$. We also have to flip the Young tableaux so that the antisymmetric representation of $SU(-N)$ becomes the symmetric representation of $SU(N)$. The usefulness of this duality is that we did not have to calculate the anomalies in subsection 4.3.1 for both theories, since they are related by changing the sign of N. Since the anomalies depend on $N(N-3)$ which becomes $N(N+3)$ we see that the two negative rank dual theories are *not* dual in the physical sense since they have different anomalies. In this particular case the negative rank dual is however dual to the original theory after shifting the ranks of the gauge groups.

4.C Exactly dimensionless couplings

Under certain assumptions, a sufficient condition for a holomorphic coupling to be constant along the RG flow is that it be neutral under all possible flavor symmetries, in particular those which are spurious and/or anomalous.

We focus first on the case where there are no nonabelian flavor symmetries. Due to various nonrenormalization theorems (see e.g. [12]), holomorphic couplings are not perturbatively renormalized apart from the one-loop running of holomorphic gauge couplings. Thus, in the absence of nonperturbative renormalization of these couplings, holomorphic couplings are independent of scale, provided we replace the scale-dependent holomorphic gauge couplings $\tau(\mu)$ with the holomorphic dynamical scale $\Lambda \equiv \mu e^{2\pi i \tau(\mu)/b}$, where $b = 3T(\mathrm{Adj}) - T(\mathrm{mat})$ is the one-loop beta function coefficient (if $b = 0$ then Λ is ill-defined but τ itself is independent of scale).

However, non-holomorphic couplings are not likewise protected against renormalization, and in particular chiral superfields are subject to wave-function renormalization through corrections to the Kähler potential. Rescaling the chiral superfields to restore canonical normalization leads to rescaling anomalies which alter the values of the holomorphic couplings, leading to a nontrivial running for their physical (canonically normalized) counterparts. In particular, the rescaling may be realized as a complexification of a $U(1)$ symmetry under which the chiral superfield in question is charged, whereas the corresponding $U(1)$ may be spurious and/or anomalous, leading to a rescaling of the corresponding spurions (superpotential couplings) and/or the holomorphic dynamical scale(s) of the gauge theory [12]. However, if a certain holomorphic combination of couplings is neutral under all of these $U(1)$'s, then it is unaffected by the rescaling, and therefore the corresponding physical coupling is scale independent (has vanishing anomalous dimension). Such a coupling is exactly dimensionless if and only if it is classically dimensionless. This is readily shown to be equivalent to the requirement that the coupling is neutral under the $U(1)_R$ under which all chiral superfields carry charge $+2/3$.

Thus, a holomorphic coupling corresponds to an exactly dimensionless physical (canonically normalized) coupling if it is neutral under all possible $U(1)$ and $U(1)_R$ symmetries[55] (since an arbitrary $U(1)_R$ is a linear combination of an arbitrary $U(1)$ with the "canonical" $U(1)_R$ considered above), assuming that none of the constituent couplings are nonperturbatively renormalized. While the converse need not be true, the existence of an exactly marginal holomorphic coupling which violates these conditions imposes a nontrivial relation on the anomalous dimensions along the flow. Since anomalous dimensions typically cannot be computed exactly away from an infrared fixed point, computable examples without extended supersymmetry must satisfy these conditions.

If the gauge group is semi-simple,[56] a straightforward counting argument gives the number N_0 of exactly dimensionless couplings of this type for a model with N_G (simple) gauge groups, N_W superpotential terms (each with a corresponding coupling), N_χ chiral superfields, and $N_{U(1)}$ linearly independent "good" $U(1)$ or $U(1)_R$ symmetries (not broken by gauge anomalies or by the superpotential):

$$N_0 = N_{U(1)} + N_G + N_W - (N_\chi + 1)\,. \tag{4.223}$$

The argument is as follows: there are $N_\chi + 1$ linearly independent spurious and/or anomalous $U(1)$ or $U(1)_R$ symmetries in general, whereas the "good" $U(1)$'s are those under which the $N_G + N_W$ holomorphic couplings are neutral, and can be represented by vectors of length $N_\chi + 1$ which are annihilated by the $(N_G + N_W) \times$

[55]This is closely related to the criteria for an exactly marginal operator at the superconformal fixed point [189].

[56]If the gauge group contains a $U(1)$ factor, then this argument still applies so long as we consider a global $U(1)$ with a non-vanishing $U(1)_{\text{gauge}}U(1)^2_{\text{global}}$ anomaly to be a "good" $U(1)$.

$(N_\chi + 1)$ matrix of $U(1)$ charges of the holomorphic couplings acting on the left. The rank of this matrix is therefore $N_\chi + 1 - N_{U(1)}$. By contrast, an exactly dimensionless coupling is a product of holomorphic couplings which is neutral under all the $U(1)$'s, and can be represented by a vector of length $N_G + N_W$ which is annihilated by the same matrix acting on the right. Since row rank and column rank are equal, the number of linearly independent vectors of this type is $N_G + N_W - (N_\chi + 1 - N_{U(1)})$, reproducing the above formula.

These arguments must be modified to include any (potentially spurious) non-abelian flavor symmetries, since chiral multiplets with the same gauge quantum numbers are subject to kinetic mixing (unless forbidden by the global symmetries). In particular, the candidate combination of couplings must also be neutral under these non-abelian symmetries in addition to the $U(1)$ and $U(1)_R$ symmetries as a sufficient condition for exact marginality.

Let G_F denote the semisimple component of the spurious flavor symmetries. Since only G_F-singlet combinations of couplings can appear in our candidate exactly dimensionless couplings and the holomorphic gauge couplings are all neutral under G_F, we need only consider G_F-invariant combinations of superpotential couplings. We can then treat G_F as if it were gauged (without the corresponding gauge coupling). Thus, the above counting argument still holds, where now N_χ counts the number of irreducible G_F multiplets, N_W the number of independent G_F invariant combinations of superpotential couplings, and $N_{U(1)}$ counts the number of "good" $U(1)$ or $U(1)_R$ symmetries which commute with G_F.

4.C.1 On nonperturbative effects

So far we have ignored the possibility that the holomorphic couplings run due to nonperturbative effects. While it is not possible to exclude this in general, such effects are also constrained by nonrenormalization theorems, and are known to be absent in some simple cases, such as pure $\mathcal{N}=1$ super-Yang-Mills [190].

In particular, for the gauge theories studied in this paper, we are interested in whether the string coupling $\tau_{10\text{d}}$ (4.23) — which is not perturbatively renormalized by the above criteria — can run nonperturbatively. A spurion analysis reveals that the exact (Wilsonian) beta function must take the form:

$$\mu\frac{d}{d\mu}\tau_{10\text{d}} = f(\tau_{10\text{d}}) \tag{4.224}$$

where $f(\tau_{10\text{d}})$ is a holomorphic function satisfying $f(+i\infty)=0$ due to the lack of perturbative running, and f cannot depend on any other holomorphic couplings due to constraints imposed by the spurious and/or anomalous $U(1)$ symmetries.

We first consider the SO theory for even N, where $SL(2,\mathbb{Z})$ covariance requires that

$$f\left(\frac{a\tau+b}{c\tau+d}\right) = (c\tau+d)^{-2}f(\tau) \tag{4.225}$$

Hence $f(\tau)$ is a modular form[57] of weight -2. However, no such holomorphic modular form exists. Instead, such a modular form is necessarily *meromorphic*, with poles in the upper half plane $\mathbb{H}$ where the beta function blows up at finite coupling. Such poles signal a breakdown of the Wilsonian description, and are likely inconsistent. Analogous statements hold for odd N (and for the USp theory) where $SL(2,\mathbb{Z})$ becomes $\Gamma_0(2)$ and f is a level-two modular form. Hence, we conclude that $f(\tau)=0$, and $\tau_{10\text{d}}$ is not renormalized in either theory.

4.D Coulomb branch computation of the string coupling

In this appendix, we provide a derivation of (4.23) for completeness. A similar computation can be done for gauge theories arising from more complicated geometries.

[57] In fact it is a cusp form, since $f(+i\infty)=0$.

To establish this result, we consider the $SO(N-4+2k)\times SU(N+2k)$ theory and switch on a mesonic vev, removing k D3 branes from the orientifold plane and breaking the gauge group down to $SO(N-4)\times SU(N)\times U(k)$, where the last factor corresponds to the $\mathcal{N}=4$ gauge theory on the k D3 branes. The holomorphic gauge coupling of $U(k)$ is therefore equal to the ten-dimensional axio-dilaton, and by performing scale matching at each step of the computation, we can relate it to the couplings of the $SO(N-4)\times SU(N)$ theory, giving (4.23).

We now sketch the details of this argument. For simplicity, we routinely drop numerical factors throughout the computation, only keeping track of the dependence on the couplings. We aim to turn on a vev which breaks

$$SO(N-4+2k)\times SU(N+2k)\longrightarrow SO(N-4)\times SU(N)\times U(k)\,, \tag{4.226}$$

corresponding to removing D3 branes from the orientifold plane. In particular, a suitable B vev will break $SU(N+2k)\to SU(N)\times USp(2k)$, whereas an A vev will then break $SO(N-4+2k)\times USp(2k)\to SO(N-4)\times U(k)$, since Higgsing a bifundamental breaks $SO(2k)\times USp(2k)\to U(k)$. For a suitable normalization of the $U(1)$ component, we have the decomposition

$$\square\to\square_{+1}\oplus\overline{\square}_{-1}\,, \tag{4.227}$$

for both $SO(2k)\to U(k)$ and $USp(2k)\to U(k)$. Thus, decomposing A^i and B^i into irreps of $SO(N-4)\times SU(N)\times U(k)$, we find

$$\begin{aligned} A &\to (\square,\overline{\square},1)\oplus(\square,1,\square_{+1}\oplus\overline{\square}_{-1})\oplus(1,\overline{\square},\square_{+1}\oplus\overline{\square}_{-1}) \\ &\quad\oplus\Big(1,1,\square\!\square_{+2}\oplus\begin{smallmatrix}\square\\\square\end{smallmatrix}_{+2}\oplus\mathrm{Adj}_0\oplus\mathrm{Adj}_0\oplus\overline{\square\!\square}_{-2}\oplus\overline{\begin{smallmatrix}\square\\\square\end{smallmatrix}}_{-2}\Big)\,, \end{aligned} \tag{4.228}$$

$$B \to \Big(1,\begin{smallmatrix}\square\\\square\end{smallmatrix},1\Big)\oplus(1,\square,\square_{+1}\oplus\overline{\square}_{-1})\oplus\Big(1,1,\begin{smallmatrix}\square\\\square\end{smallmatrix}_{+2}\oplus\overline{\begin{smallmatrix}\square\\\square\end{smallmatrix}}_{-2}\oplus\mathrm{Adj}_0\Big)\,, \tag{4.229}$$

for each of the three $SU(3)$ "flavors" of each field, where Adj_0 denotes the reducible $U(k)$ adjoint representation of dimension k^2, containing both singlet and trace-free irreps.

We choose to turn on a vev for the singlet components of A^3 and B^3 only, which implies that only components of these fields can be Higgsed.[58] We have broken $3(2N-3)k+5k^2$ generators, therefore

$$\frac{3(N+2k)(N+2k-3)}{2} - 3(2N-3)k - 5k^2 = \frac{3N(N-3)}{2} + k^2 \qquad (4.230)$$

chiral superfields remain unHiggsed. The only way to get the correct scaling with N and k is if the unHiggsed fields are

$$(\Box, \overline{\Box}, 1) \oplus (1, \begin{smallmatrix}\Box\\\Box\end{smallmatrix}, 1) \oplus (1, 1, \mathrm{Adj}_0)\,, \qquad (4.231)$$

coming from A^3, B^3, and a linear combination of the two, respectively. Thus, the matter content just below the Higgsing scale v is precisely:

origin	$SO(N-4)$	$SU(N)$	$U(k)$	$SU(2)$	$U(1)'_R$	#
A	$\Box$	$\overline{\Box}$	1	$\Box$	$1+\frac{2}{N}$	1
A	$\Box$	$\overline{\Box}$	1	1	$\frac{2}{N}$	1
B	1	$\begin{smallmatrix}\Box\\\Box\end{smallmatrix}$	1	$\Box$	$1-\frac{4}{N}$	1
B	1	$\begin{smallmatrix}\Box\\\Box\end{smallmatrix}$	1	1	$-\frac{4}{N}$	1
A	$\Box$	1	$\Box_{+1} \oplus \overline{\Box}_{-1}$	$\Box$	1	1
A	1	$\overline{\Box}$	$\Box_{+1} \oplus \overline{\Box}_{-1}$	$\Box$	1	1
B	1	$\Box$	$\Box_{+1} \oplus \overline{\Box}_{-1}$	$\Box$	1	1
A	1	1	$\Box\Box_{+2} \oplus \overline{\Box\Box}_{-2}$	$\Box$	1	1
A, B	1	1	$\begin{smallmatrix}\Box\\\Box\end{smallmatrix}_{+2} \oplus \overline{\begin{smallmatrix}\Box\\\Box\end{smallmatrix}}_{-2}$	$\Box$	1	2
$A^{\times 2}, B$	1	1	Adj_0	$\Box$	1	3
A/B	1	1	Adj_0	1	0	1

(4.232)

where the unbroken flavor symmetry is $SU(2) \times U(1)'_R$, with

$$U(1)'_R = U(1)_R + \mathrm{diag}_{SU(3)}\left(\frac{1}{3}, \frac{1}{3}, -\frac{2}{3}\right) + \mathrm{diag}_{SU(N+2k)}\left(\frac{2}{N+2k}, \ldots, -\frac{4k}{N(N+2k)}, \ldots\right). \qquad (4.233)$$

[58] One can show by explicit computation that a vev of this type satisfies the D-term conditions.

Note that, due to the unbroken global symmetries, the chiral superfields above and below the line cannot couple to each other at the renormalizable level.

One can check that the $U(1) \subset U(k)$ charged fields all receive masses at the scale λv from the superpotential which descends from λAAB, as do two of the three Adj_0 $SU(2)$ doublets, leaving

	$SO(N-4)$	$SU(N)$	$U(k)$	$SU(3)$	$U(1)_R$
A^i	$\square$	$\overline{\square}$	1	$\square$	$\frac{2}{3}+\frac{2}{N}$
B^i	1	$\begin{smallmatrix}\square\\ \square\end{smallmatrix}$	1	$\square$	$\frac{2}{3}-\frac{4}{N}$
Φ^i	1	1	Adj_0	$\square$	$\frac{2}{3}$

(4.234)

where we can now formally restore $SU(3) \times U(1)_R$ invariance, and the superpotential now takes the form:

$$W \sim \lambda\, \varepsilon_{ijk} \delta^{ab} A^i_{a;m} A^j_{b;n} B^{k;mn} + \lambda\, \varepsilon_{ijk} \mathrm{Tr}[\Phi^i \Phi^j \Phi^k]\,, \tag{4.235}$$

where the vev $\langle \Phi^i \rangle = v^i \mathbb{1}$ breaks $SU(3) \times U(1)_R \to SU(2) \times U(1)'_R$, but decouples from the other fields. The $U(k)$ gauge group factor decouples from the rest of the theory and flows to an $\mathcal{N}=4$ superconformal fixed point in the infrared. To make the enhanced supersymmetry manifest, we rescale $\Phi \to \lambda^{-1/3}\Phi$, setting the superpotential coupling to 1 (up to a numerical factor) in the holomorphic basis.

To determine the gauge couplings of the low energy theory, we compute the beta function coefficients $b = 3T_{\mathrm{Adj}} - T_{\mathrm{mat}}$ above, between, and below the scales v and λv and apply the scale matching relations. Above or below both scales, we have:

$$b_{SO} = -18\,, \quad b_{SU} = 9\,, \tag{4.236}$$

whereas between the two scales we find:

$$b'_{SO} = -18 - 4k\,, \quad b'_{SU} = 9 - 4k\,. \tag{4.237}$$

In either case, we have the scale matching relations

$$\left(\frac{\Lambda}{v}\right)^{b} = \left(\frac{\Lambda'}{v}\right)^{b'}, \quad \left(\frac{\Lambda'}{\lambda v}\right)^{b'} = \left(\frac{\Lambda''}{\lambda v}\right)^{b}, \tag{4.238}$$

so that

$$(\Lambda'')^{b} = \lambda^{b-b'}\Lambda^{b}. \tag{4.239}$$

Thus, in net

$$\Lambda^{9}_{SU(N)} = \lambda^{4k}\Lambda^{9}_{SU(N+2k)}, \quad \Lambda^{-18}_{SO(N-4)} = \lambda^{4k}\Lambda^{-18}_{SO(N-4+2k)}. \tag{4.240}$$

Now consider the $SU(k) \subset U(k)$ factor.[59] We have

$$b_{SU(k)} = -(6(N-2) + 10k), \tag{4.241}$$

between the scales v and λv, whereas scale matching at the scale v gives:

$$\left(\frac{\Lambda_{SU(k)}}{v}\right)^{-6(N-2)-10k} = \left(\frac{\Lambda_{SO(N-4+2k)}}{v}\right)^{-18}\left(\frac{\Lambda_{SU(N+2k)}}{v}\right)^{18}, \tag{4.242}$$

since the index of embedding [191] for $SU(k) \subset SO(2k)$ is 1 whereas it is 2 for $SU(k) \subset USp(2k) \subset SU(2k)$. Evaluating the holomorphic gauge coupling at the scale λv, we obtain

$$\begin{aligned} \tau_{k,N} &= \frac{1}{2\pi i}\ln\left(\frac{\Lambda_{SU(k)}}{\lambda v}\right)^{-(6(N-2)+10k)} \\ &= \frac{1}{2\pi i}\ln\left[\lambda^{6(N-2)+10k}\Lambda^{-18}_{SO(N-4+2k)}\Lambda^{18}_{SU(N+2k)}\right], \end{aligned} \tag{4.243}$$

which can be rewritten as

$$\tau_{k,N} = \frac{1}{2\pi i}\ln\left[\lambda^{6(N-2)-2k}\Lambda^{-18}_{SO(N-4)}\Lambda^{18}_{SU(N)}\right], \tag{4.244}$$

using (4.240). Due to the vanishing of the beta function coefficient, the holomorphic gauge coupling does not run below the scale λv. However, rescaling Φ^{i} to

[59]We ignore the $U(1) \subset U(k)$ henceforward for simplicity.

make $\mathcal{N} = 4$ supersymmetry manifest alters τ due to a rescaling anomaly. We find:

$$\hat{\tau} = \tau_{k,N} + \frac{1}{2\pi i} \ln \lambda^{2k} = \frac{1}{2\pi i} \ln \left[\lambda^{6(N-2)} \Lambda_{SO(N-4)}^{-18} \Lambda_{SU(N)}^{18} \right] . \tag{4.245}$$

Note that the dependence on k disappears. Moreover, (4.245) is also independent of N, which can be verified by applying (4.240).

Since the holomorphic gauge coupling on D3 branes probing a smooth background is just $\tau_{10\text{d}}$ evaluated in that background, we interpret (4.245) as the ten-dimensional axio-dilaton. Note that the result is independent of v, as expected from the constant axio-dilaton profile of the dual geometry at large N.

The computation for the $USp(\tilde{N}+4) \times SU(\tilde{N})$ theory is closely analogous, and we obtain the result

$$\tau_{10\text{d}} = \frac{1}{2\pi i} \ln \left[\tilde{\lambda}^{6(\tilde{N}+2)} \tilde{\Lambda}_{USp(\tilde{N}+4)}^{18} \tilde{\Lambda}_{SU(\tilde{N})}^{-18} \right] \tag{4.246}$$

in place of (4.245). However, at this point an important subtlety arises, since the factor inside the log is a perfect square. This can be rewritten as

$$\tau_{10\text{d}} = \frac{1}{\pi i} \ln \left[\tilde{\lambda}^{3(\tilde{N}+2)} \tilde{\Lambda}_{USp(\tilde{N}+4)}^{9} \tilde{\Lambda}_{SU(\tilde{N})}^{-9} \right] , \tag{4.247}$$

but there is an ambiguity, since

$$\tau_{10\text{d}} = \frac{1}{\pi i} \ln \left[\tilde{\lambda}^{3(\tilde{N}+2)} \tilde{\Lambda}_{USp(\tilde{N}+4)}^{9} \tilde{\Lambda}_{SU(\tilde{N})}^{-9} \right] + 1 \tag{4.248}$$

is also consistent with (4.246), depending on which sign we take for the square root. The resolution to this puzzle is that the two answers correspond to different types of O-planes, much like the distinction between O3^+ and $\widetilde{\text{O3}}^+$ planes in the $\mathcal{N} = 4$ examples discussed in section 4.2.

4.E Details of the superconformal index for $N = 7$

In this appendix we discuss some technical details of the computation of the superconformal index for the $SO(3) \times SU(7) \leftrightarrow USp(8) \times SU(4)$ dual pair. In section 4.E.1 we present some technical details related to the calculation of the $SU(3)$ representation of $\left(B^i_{(0)}\right)^{21}$ for the $SO(3) \times SU(7)$ theory whereas in section 4.E.2 we present the rather lengthy results related to the calculation of the superconformal index for the $USp(8) \times SU(4)$ theory (cf. subsection 4.4.1).

4.E.1 A note on computing $\left(B^i_{(0)}\right)^{21}$ efficiently

In the simplest cases, the representation under the flavor group of the gauge singlets contributing to the superconformal index can be computed straightforwardly using a computer algebra program such as `LiE` [143]. However, the computation becomes more and more expensive as one studies larger and larger baryons, and already for $\left(B^i_{(0)}\right)^{21}$ direct computation becomes intractable. One can then use a different and more efficient method, which we now explain.

The first observation is that B lives in a tensor product representation $E \otimes F$ of $SU(7) \times SU(3)$. The m-th symmetric tensor product (in our case $m = 21$) representation of a tensor product decomposes as [192, 193]:

$$\mathrm{Sym}^m(E \otimes F) = \sum_{|\lambda|=m} L_\lambda E \otimes L_\lambda F\,. \tag{4.249}$$

Here we are summing over all partitions λ of m (i.e. all standard Young tableaux with m boxes), and L_λ is the Schur functor for λ. This expression already provides an important simplification of the calculation, since F is the fundamental of $SU(3)$, and thus $L_\lambda F$ is just the $SU(3)$ representation described by the Young tableau λ. If λ has more than 3 rows this terms vanishes, and we can ignore it in the sum.

We are left with computing the number of singlets in $L_\lambda E = L_\lambda(\bigwedge^2 f)$, with f the fundamental representation of $SU(7)$. This can be done from general properties of plethysms. In particular, denoting by μ the $1+1$ partition of 2 corresponding to the antisymmetric $\bigwedge^2$, we can apply the formula [194]:[60]

$$L_\lambda L_\mu = \frac{1}{m!} \sum_{|\kappa|=m} C(\kappa)\, \chi^\lambda_\kappa \bigotimes_{i=1}^{\ell(\kappa)} \mathsf{A}_{\kappa_i}(\mu)\,. \tag{4.250}$$

Here $C(\kappa)$ denotes the order of the elements of cycle class κ in the symmetric group $S_{|\lambda|}$, χ^λ_κ is the character χ^λ of elements of cycle type κ evaluated in the representation of $S_{|\lambda|}$ associated to λ, and $\ell(\kappa)$ is the number of parts (rows) of the partition κ. This formula follows from well known facts, let us give a quick proof. It is convenient to switch to the representation in terms of symmetric polynomials [192], in which the left hand side of (4.250) is given by $s_\lambda \circ s_\mu$, with "$\circ$" is the plethysm operator, and s_λ and s_μ are the symmetric Schur functions indexed by the partitions λ and μ respectively. Decomposing s_λ in terms of power symmetric functions p_κ indexed by the partition κ we have [192]:

$$s_\lambda = \frac{1}{m!} \sum_{|\kappa|=m} C(\kappa)\, \chi^\lambda_\kappa\, p_\kappa\,. \tag{4.251}$$

Formula (4.250) now follows using $p_\kappa = \prod_{i=1}^{\ell(\kappa)} p_{\kappa_i}$, the fact that $(ab)\circ c = (a\circ c)(b\circ c)$, and the definition of plethysm with a fundamental power symmetric polynomial: $p_n \circ \mu(x) = \mu(x^n)$.

The second simplification in the calculation now comes from observing that the tensor product of Adams operators appearing in this formula is actually independent of λ. It also happens to be the most expensive part of the computation, so it just needs to be calculated once. Making this manifest, the final formula we computed is effectively:

$$\mathrm{Sym}^m(E \otimes F) = \frac{1}{m!} \sum_{|\kappa|=m} C(\kappa) \left(\sum_{|\lambda|=m} \chi^\lambda_\kappa\, L_\lambda F \right) \left\langle \bigotimes_{i}^{\ell(\kappa)} \mathsf{A}_{\kappa_i}(\mu) \right\rangle\,, \tag{4.252}$$

where the brackets indicate taking the singlet part only.

[60]One could alternatively use the formula in example I.8.9 of [192], in terms of generalized Kostka numbers. See also [195, 196] for similar formulas, and appendix 4.F for a more analytic approach to the problem based on the discussion in [195–197].

Field	$USp(\tilde{N}+4)\times SU(\tilde{N})$	$SU(3)$	t exponent	$SU(2)_r$
$\tilde{A}^i_{(l)}$	$(\square,\overline{\square})$	$\square$	$\frac{2}{3}-\frac{2}{\tilde{N}}+l$	$l+1$
$\tilde{B}^i_{(l)}$	$(1,\square\square)$	$\square$	$\frac{2}{3}+\frac{4}{\tilde{N}}+l$	$l+1$
$\bar{\psi}^{\tilde{A}}_{(l)}$	$(\square,\square)$	$\overline{\square}$	$\frac{4}{3}+\frac{2}{\tilde{N}}+l$	$l+1$
$\bar{\psi}^{\tilde{B}}_{(l)}$	$(1,\overline{\square\square})$	$\overline{\square}$	$\frac{4}{3}-\frac{4}{\tilde{N}}+l$	$l+1$
$\lambda^{USp}_{(l)}$	$(\square\square,1)$	1	$1+l$	$l\oplus(l+2)$
$F^{USp}_{(l)}$	$(\square\square,1)$	1	$2+l$	$(l+1)\oplus(l+1)$
$\lambda^{SU}_{(l)}$	$(1,\mathrm{Adj})$	1	$1+l$	$l\oplus(l+2)$
$F^{SU}_{(l)}$	$(1,\mathrm{Adj})$	1	$2+l$	$(l+1)\oplus(l+1)$

Table 4.7: The fields which contribute to the superconformal index for $USp(\tilde{N}+4)\times SU(\tilde{N})$, where the $SU(2)_r$ column denotes the representation under the $SU(2)$ group generated by $\overline{J}_\pm, \overline{J}_3$.

4.E.2 Check of the superconformal index calculation for $USp(8)\times SU(4)$

In this section we present some rather lengthy results related to the calculation of the superconformal index for the $USp(8)\times SU(4)$ theory (cf. section 4.4.1).

The fields that contribute to the superconformal index for the $USp(\tilde{N}+4)\times SU(\tilde{N})$ theory are shown in table 4.7. The gauge invariant contributions for the $USp(8)\times SU(4)$ theory up to order t^2 are shown in tables 4.8 and 4.9. Taking into account the factor $(-1)^F$ we find perfect agreement with (4.64).

4.F On the decomposition of certain generalized Specht modules

One of the arguments presented in section 4.3 for the agreement between the two dual theories relied on the matching of the flavor representation of baryons with minimal R-charge between the two descriptions of the theory. In particular, we could argue that for all values of N the baryon $\tilde{A}^{N-3}$ in the $USp(N+1)\times SU(N-3)$

operator	t ex.	$2\bar{J}_3$	$SU(3)$ character
$(\tilde{A}_{(0)})^4$	$\frac{2}{3}$	0	$\chi_{0,2}+\chi_{4,0}$
$(\tilde{A}_{(0)})^8$	$\frac{4}{3}$	0	$5\chi_{0,4}+2\chi_{1,2}+5\chi_{2,0}+3\chi_{3,1}+3\chi_{4,2}+\chi_{8,0}$
$(\tilde{A}_{(0)})^6\psi^{\tilde{B}}_{(0)}$	$\frac{4}{3}$	0	$\chi_{0,1}+3\chi_{0,4}+4\chi_{1,2}+3\chi_{2,0}+3\chi_{3,1}+\chi_{4,2}+\chi_{5,0}$
$(\tilde{A}_{(0)})^4[\psi^{\tilde{B}}_{(0)}]^2$	$\frac{4}{3}$	0	$2\chi_{0,1}+2\chi_{1,2}+\chi_{3,1}+\chi_{5,0}$
$(\tilde{A}_{(0)})^2[\psi^{\tilde{B}}_{(0)}]^3$	$\frac{4}{3}$	0	$\chi_{0,1}$
$(\tilde{A}_{(0)})^3\tilde{A}_{(1)}$	$\frac{5}{3}$	±1	$\chi_{0,2}+\chi_{1,0}+2\chi_{2,1}+\chi_{4,0}$
$\lambda^{USp}_{(0)}(\tilde{A}_{(0)})^4$	$\frac{5}{3}$	±1	$\chi_{1,0}+\chi_{2,1}$
$\lambda^{SU}_{(0)}(\tilde{A}_{(0)})^4$	$\frac{5}{3}$	±1	$\chi_{1,0}+2\chi_{2,1}$
$\lambda^{SU}_{(0)}\psi^{\tilde{B}}_{(0)}(\tilde{A}_{(0)})^2$	$\frac{5}{3}$	±1	$\chi_{1,0}+\chi_{2,1}$

Table 4.8: Gauge invariant contributions to the superconformal index for $USp(8)\times SU(4)$ of order less than t^2, where $()^*$ denotes the symmetric tensor product and $[]^*$ the antisymmetric tensor product.

theory transforms in the $\mathrm{Sym}^{(N-3)/2}(\square\square)$ representation of the $SU(3)$ flavor group, where $\mathrm{Sym}^k(R)$ denotes the k-th symmetric power of the representation R. We also argued, and checked in a number of examples, that there is a corresponding minimal R-charge baryon of the form B^N on the $SO(N-4)\times SU(N)$ side, transforming in the same representation of the flavor group. The duality conjectured in this chapter then requires the group theoretical identity[61]

$$\left\langle \mathrm{Sym}^N\left(\begin{smallmatrix}\square\\\square\end{smallmatrix}_{SU(N)}\otimes\square_{SU(3)}\right)\right\rangle \cong \mathrm{Sym}^{(N-3)/2}(\square\square) \tag{4.253}$$

to hold (as representations of the flavor $SU(3)$), where the angle brackets denote taking the singlet part under $SU(N)$. In this appendix we would like to demystify this expression somewhat by reformulating it as an statement about representations of the symmetric group S_N, and give some additional evidence for its validity based on this new viewpoint. The interested reader can find nice reviews of the required

[61]In this appendix, as in the rest of the chapter, we will be assuming that N is odd.

operator	t ex.	$2\bar{J}_3$	$SU(3)$ character
$(\tilde{A}_{(0)})^{12}$	2	0	$16+8\chi_{0,3}+8\chi_{0,6}+22\chi_{1,1}+13\chi_{1,4}+42\chi_{2,2}$ $+\chi_{2,5}+12\chi_{3,0}+19\chi_{3,3}+20\chi_{4,1}+8\chi_{4,4}$ $+8\chi_{5,2}+15\chi_{6,0}+2\chi_{6,3}+4\chi_{7,1}+3\chi_{8,2}+\chi_{12,0}$
$(\tilde{A}_{(0)})^{10}\psi^{\tilde{B}}_{(0)}$	2	0	$16+28\chi_{0,3}+5\chi_{0,6}+54\chi_{1,1}+27\chi_{1,4}+68\chi_{2,2}$ $+4\chi_{2,5}+37\chi_{3,0}+32\chi_{3,3}+41\chi_{4,1}+6\chi_{4,4}$ $+17\chi_{5,2}+14\chi_{6,0}+\chi_{6,3}+6\chi_{7,1}+\chi_{8,2}+\chi_{9,0}$
$(\tilde{A}_{(0)})^{8}[\psi^{\tilde{B}}_{(0)}]^2$	2	0	$6+35\chi_{0,3}+52\chi_{1,1}+21\chi_{1,4}+46\chi_{2,2}+5\chi_{2,5}$ $+43\chi_{3,0}+22\chi_{3,3}+35\chi_{4,1}+\chi_{4,4}+13\chi_{5,2}+4\chi_{6,0}$ $+\chi_{6,3}+3\chi_{7,1}+\chi_{9,0}$
$(\tilde{A}_{(0)})^{6}[\psi^{\tilde{B}}_{(0)}]^3$	2	0	$6+18\chi_{0,3}+26\chi_{1,1}+7\chi_{1,4}+22\chi_{2,2}+2\chi_{2,5}$ $+21\chi_{3,0}+7\chi_{3,3}+14\chi_{4,1}+3\chi_{5,2}+2\chi_{6,0}+\chi_{6,3}$
$(\tilde{A}_{(0)})^{4}[\psi^{\tilde{B}}_{(0)}]^4$	2	0	$4+3\chi_{0,3}+9\chi_{1,1}+\chi_{1,4}+9\chi_{2,2}$ $+3\chi_{3,0}+\chi_{3,3}+2\chi_{4,1}+\chi_{6,0}$
$(\tilde{A}_{(0)})^{2}[\psi^{\tilde{B}}_{(0)}]^5$	2	0	$1+2\chi_{1,1}+2\chi_{2,2}+\chi_{3,0}$
$[\psi^{\tilde{B}}_{(0)}]^6$	2	0	$\chi_{3,0}$
$[\tilde{\lambda}^{USp}_{(0)}]^2$	2	0	1
$[\tilde{\lambda}^{SU}_{(0)}]^2$	2	0	1
$A_{(0)}\psi^{A}_{(0)}$	2	0	$1+\chi_{1,1}$
$B_{(0)}\psi^{B}_{(0)}$	2	0	$1+\chi_{1,1}$
$(A_{(0)})^2B_{(0)}$	2	0	$1+\chi_{1,1}$

Table 4.9: Gauge invariant contributions to the superconformal index for $USp(8)\times SU(4)$ of order t^2.

introductory material in [192, 194, 198]. We will also make use of the generalized Specht modules introduced by Doran in [197] (see also [195, 196]). We will show that in this context (4.253) follows from a conjectured decomposition of certain generalized Specht module into ordinary Specht modules.

We start by using the decomposition of the symmetric power of a tensor product into a sum of ordinary tensor products [192, 193]:

$$\mathrm{Sym}^N(E \otimes F) = \sum_{|\lambda|=N} L_\lambda E \otimes L_\lambda F\,, \tag{4.254}$$

where L_λ is the Schur functor for the partition λ, and the sum is over partitions of N. In our particular case we have $E = \begin{smallmatrix}\square\\\square\end{smallmatrix}_{SU(N)}$ and $F = \square_{SU(3)}$. Taking the $SU(N)$ singlet part:

$$\left\langle \mathrm{Sym}^N(E \otimes F) \right\rangle = \sum_{|\lambda|=N} \langle L_\lambda E\rangle \; L_\lambda F\,. \tag{4.255}$$

We thus see that we are left to enumerate the λ for which $L_\lambda \begin{smallmatrix}\square\\\square\end{smallmatrix}$ contains $SU(N)$ singlets. As in appendix 4.E, in order to do this it is convenient to work with symmetric polynomials [192] instead of directly representations, so we rewrite (4.255) as:

$$\left\langle \mathrm{Sym}^N(E \otimes F) \right\rangle \cong \sum_{|\lambda|=N} \langle s_\lambda \circ e_2\rangle \; s_\lambda\,, \tag{4.256}$$

where "$\circ$" denotes plethysm, s_λ is the Schur symmetric function indexed by the partition λ, and e_2 is the elementary symmetric function or order 2 associated with the antisymmetric. Expanding s_λ into power symmetric polynomials p_ρ:

$$\left\langle \mathrm{Sym}^N(E \otimes F) \right\rangle \cong \frac{1}{N!} \sum_{|\lambda|=N} \sum_{|\rho|=N} C(\rho) \chi^\lambda_\rho \chi \, \langle p_\rho \circ e_2\rangle \; s_\lambda\,, \tag{4.257}$$

where ρ is a partition of N, and as in (4.250) we have introduced the order $C(\rho)$ of the cycle class ρ in $S_{|\rho|}$, and the character χ^λ_ρ of elements of cycle type ρ in the representation indexed by λ. We can now use [195, 196, 199]:

$$p_\rho \circ e_2 = \sum_{|\kappa|=2N} \chi_\rho^{\kappa',N} s_\kappa \,, \tag{4.258}$$

where κ runs over partitions of $2N$, and κ' denotes the transpose of κ. $\chi_\rho^{\kappa',N}$ is the character of cycles of type ρ in the generalized Specht module $S^{\kappa',N}$ [197], which we will describe further momentarily. (Notice that the formula given in [195, 196] acts on h_2 rather than e_2, but we can easily obtain (4.258) by acting with the involution ω exchanging e_2 and h_2 [192], which gives the transpose of κ.) There is a single partition of $2N$ giving rise to a gauge singlet of $SU(N)$, it is the partition $2N = 2+2+\ldots \equiv 2^N$ (in standard notation for partitions). Taking into account that the transpose partition of 2^N is just N^2, we finally get:

$$\langle p_\rho \circ e_2 \rangle = \chi_\rho^{N^2,N} \,. \tag{4.259}$$

Now, the generalized Specht module $S^{N^2,N}$ is a (in general reducible) representation of the permutation group S_N, so let us write $S^{N^2,N} \cong \bigoplus_\mu c_\mu S^\mu$ for its decomposition into irreducible representations of S_n, the Specht modules S^μ, indexed by the partitions μ of N. Using linearity of characters, we find that

$$\langle p_\rho \circ e_2 \rangle = \sum_\mu c_\mu \chi_\rho^\mu \,. \tag{4.260}$$

Plugging this back in (4.257), we obtain

$$\begin{aligned} \left\langle \mathrm{Sym}^N(E \otimes F) \right\rangle &\cong \sum_{|\mu|=N} c_\mu \sum_{|\lambda|=N} \left(\frac{1}{N!} \sum_{|\rho|=N} C(\rho) \chi_\rho^\lambda \chi_\rho^\mu \right) s_\lambda \\ &= \sum_{|\mu|=N} c_\mu s_\mu \,, \end{aligned} \tag{4.261}$$

where we have used orthogonality of characters to set the term in parenthesis to $\delta_{\lambda\mu}$. We thus find the remarkably simple result that the flavor representation of our baryon is just the $SU(3)$ representation associated to the generalized Specht module $S^{N^2,N}$.[62] Furthermore we conjecture that the following decomposition holds for all N:

$$S^{N^2,N} \cong \bigoplus_{k} S^{\kappa} \,, \tag{4.262}$$

where κ runs over the three element partitions $\kappa_1 + \kappa_2 + \kappa_3$ of N such that all κ_i are odd numbers. Before presenting the evidence that we have found for this conjecture, let us show that this implies (4.253). The right hand side is given by [192, 193]:

$$\mathrm{Sym}^{(N-3)/2}(\square\square) \cong \bigoplus_{|\lambda|=N-3} S^{\lambda} \,, \tag{4.263}$$

where the parts of the partition λ are all even numbers. Using the fact that we are interested in representations of $SU(3)$, we can restrict the sum to partitions with 3 parts at most (the rest vanish as $SU(3)$ representations). We also notice that since $\kappa_i \in 2\mathbb{Z}+1$ and $\kappa_i \geq 0$, we have $\kappa_i \geq 1$. Removing a column of three boxes on the leftmost column of a Young tableau gives rise to $SU(3)$ isomorphic representations, so we may just as well send $\kappa_i \to \tilde{\kappa}_i = \kappa_i - 1$, where now $\tilde{\kappa}$ is a partition of $N-3$ with all parts even. We have thus just obtained a natural isomorphism between (4.262) and (4.263) as $SU(3)$ representations, as we wanted.

Coming back to our conjecture (4.262), we have found various pieces of evidence for its validity. First of all, direct computation (using `LiE` [143]) shows that the identity holds for all odd N between 3 and 21. More conceptually, it is possible to show (by using a straightforward modification of the straightening procedure based on Garnir elements, for example) that $S^{N^2,N}$ has a basis indexed by the semistandard tableaux of shape N^2 and weight 2^N. On the other hand it is well known that any ordinary Specht module S^{μ} has a basis indexed by standard

[62] To each ordinary Specht module S^{λ} we can associate in the usual way the $SU(3)$ representation with Young tableau λ. Since $S^{N^2,N}$ is a sum of ordinary Specht modules we associate to it the corresponding sum of $SU(3)$ representations.

tableaux of shape λ. So in order for the dimensions of the corresponding modules to match it should hold that the number of semistandard tableaux of shape N^2 and weight 2^N should be the sum of the number of standard tableaux with shapes as in formula (4.262). This enumeration task is well suited to a computer (we used SAGE [167]), and by direct computation it is easy to see that the dimensions match up to $N = 45$.

4.G A conjectured identity for elliptic hypergeometric integrals

In this appendix we will reformulate the conjecture (4.89), $\mathcal{I}_{USp} = \mathcal{I}_{SO}$, in terms of elliptic hypergeometric integrals,[63] giving rise to a conjecture about elliptic hypergeometric functions that could perhaps be proven along the lines of [168] (we will not attempt to prove it in this chapter). One point of mathematical interest is that, since the physical process behind our conjectured duality seems to be qualitatively different from ordinary Seiberg duality (this is particularly clear when formulated in string theory [60]), one may expect that (4.89) is a new fundamental identity between elliptic hypergeometric functions, independent from the one proven by Rains [168].

It is by now an standard exercise to reformulate the superconformal index in terms of elliptic hypergeometric functions (following [148]) so we will be somewhat brief. Let us start on the $USp \times SU$ side, which we will parametrize as $USp(2M) \times SU(L)$ (so one has $2M \equiv N+1$, $L \equiv N-3$, assuming that the dual theory was $SO(N-4) \times SU(N)$). The index (4.57) factorizes into:

$$\mathcal{I}_{USp}(t,x,f) = \int_{USp} [dz_1] \int_{SU} [dz_2] \; \mathcal{I}_{\overline{\square}}(t,x,z_1,z_2,f)\, \mathcal{I}_{\square\square}(t,x,z_2,f)\,. \tag{4.264}$$

[63] We refer the reader to [200, 201] for the original works on hypergeometric integrals, and to [202] for a nice review of the field.

As in [153], we have absorbed the contribution to the index coming from vector bosons into the integration measure. Explicit expressions can be found in [153], and we reproduce them here for the convenience of the reader, adapted to our notation:

$$\int_{SU(N)} [dz] \equiv \frac{1}{N!} \oint \left(\prod_{a=1}^{N-1} \frac{dz_a}{2\pi i z_a} (tx; tx) \left(\frac{t}{x}; \frac{t}{x} \right) \right) \frac{1}{\prod_{1 \le b < c \le N} \Gamma(z_b z_c^{-1}, z_b^{-1} z_c)} \Bigg|_{\prod z_a = 1} , \tag{4.265}$$

$$\int_{USp(2N)} [dz] \equiv \frac{1}{N!} \oint \left(\prod_{a=1}^{N} \frac{dz_a}{4\pi i z_a} \frac{(tx; tx) \left(\frac{t}{x}; \frac{t}{x} \right)}{\Gamma(z_a^2, z_a^{-2})} \right) \frac{1}{\prod_{1 \le b < c \le N} \Gamma(z_b z_c, z_b z_c^{-1}, z_b^{-1} z_c, z_b^{-1} z_c^{-1})} , \tag{4.266}$$

$$\int_{SO(2N+1)} [dz] \equiv \oint \frac{1}{N!} \left(\prod_{a=1}^{N} \frac{dz_a}{4\pi i z_a} \frac{(tx; tx) \left(\frac{t}{x}; \frac{t}{x} \right)}{\Gamma(z_a, z_a^{-1})} \right) \frac{1}{\prod_{1 \le b < c \le N} \Gamma(z_b z_c, z_b z_c^{-1}, z_b^{-1} z_c, z_b^{-1} z_c^{-1})} . \tag{4.267}$$

where we have introduced the following standard special functions:[64]

$$\Gamma(u; t, x) = \prod_{a,b \ge 0} \frac{1 - u^{-1} t^{a+b+2} x^{a-b}}{1 - u t^{a+b} x^{a-b}} , \tag{4.268}$$

$$\theta(u; y) = \prod_{a \ge 0} (1 - u y^a)(1 - u^{-1} y^{a+1}) , \tag{4.269}$$

$$(u; y) = \prod_{a \ge 0} (1 - u y^a) . \tag{4.270}$$

Finally, we have also introduced the short-hand notation

$$\Gamma(u) \equiv \Gamma(u; t, x) , \tag{4.271}$$

$$\Gamma(u_1, \ldots, u_k) \equiv \prod_{i=1}^{k} \Gamma(u_i) . \tag{4.272}$$

The ordinary Γ function (i.e. the generalization of the factorial) will play no role in our discussion, so by $\Gamma(u)$ we will always mean (4.271).

[64]It is common in the literature to introduce the new variables $p = tx$, $q = tx^{-1}$, and express the integrals in terms of these, but we will keep using the t and x variables we have been using so far.

We are left to evaluate the contribution $\mathcal{I}_{\overline{\Box}}$ from the bifundamental, and the contribution $\mathcal{I}_{\Box\Box}$ from the symmetric. Let us start by $\mathcal{I}_{\overline{\Box}}$. This fields transforms in the bifundamental of $USp(2M) \times SU(L)$, and accordingly its one-letter index is given by:

$$i_{\overline{\Box}}(t, x, f, z_1, z_2) = \frac{1}{(1-tx)(1-tx^{-1})} \left[t^r \chi_{\overline{\Box}} - t^{(2-r)} \chi_{\Box} \right] . \tag{4.273}$$

Here we have introduced the total character $\chi_{\overline{\Box}} = \chi_{\Box}(z_1)\chi_{\overline{\Box}}(z_2)\chi_{\Box}(f)$, and its conjugate $\chi_{\Box}$. It is convenient to expand these characters into elementary monomials:

$$\begin{aligned} \chi_{\overline{\Box}} &= \chi_{\Box}(z_1)\chi_{\overline{\Box}}(z_2)\chi_{\Box}(f) \\ &= \left(\sum_{a=1}^{M} (z_{1,a} + z_{1,a}^{-1}) \right) \left(\sum_{b=1}^{L} \frac{1}{z_{2,b}} \right) \left(\sum_{c=1}^{3} f_c \right) \\ &= \sum_{a,b,c} \frac{z_{1,a} f_c}{z_{2,b}} + \sum_{a,b,c} \frac{z_{1,a}^{-1} f_c}{z_{2,b}} \\ &\equiv \sum_q \eta_q . \end{aligned} \tag{4.274}$$

Where η_q is a monomial in the expansion, and q an unified index. The subindices denote projection of the group elements into the maximal torus, and for SU characters there are constraints of the form $\prod z_{i,a} = 1$, which we will not indicate explicitly in what follows. Expanding the denominator in (4.273), we have:

$$i_{\overline{\Box}}(t, x, f, z_1, z_2) = \sum_{a,b \geq 0} \sum_q t^{a+b} x^{a-b} (t^r \eta_q - t^{(2-r)} \eta_q^{-1}) . \tag{4.275}$$

The plethystic exponent $\mathbb{E}_{\overline{\Box}}$ in (4.57) then becomes:

$$\begin{aligned} \mathbb{E}_{\overline{\Box}} &= \sum_{k=1}^{\infty} \frac{1}{k} i_{\overline{\Box}}(t^k, x^k, z_1^k, z_2^k, f^k) \\ &= \sum_{k=1}^{\infty} \frac{1}{k} \sum_{a,b \geq 0} \sum_q t^{k(a+b)} x^{k(a-b)} (t^{kr} \eta_q^k - t^{k(2-r)} \eta_q^{-k}) \\ &= \sum_{a,b \geq 0} \sum_q \log \left(\frac{1 - t^{2+a+b-r} x^{a+b} \eta_q^{-1}}{1 - t^{a+b+r} x^{a-b} \eta_q} \right) , \end{aligned} \tag{4.276}$$

and the superconformal index (4.57)

$$
\begin{aligned}
\mathcal{I}_{\overline{\square}}(t,x,z_1,z_2,f) &= \exp(\mathbb{E}_{\overline{\square}}) \\
&= \prod_q \prod_{a,b\geq 0} \left(\frac{1-t^{2+a+b-r}x^{a+b}\eta_q^{-1}}{1-t^{a+b+r}x^{a-b}\eta_q} \right) \\
&= \prod_q \Gamma(t^r \eta_q)\,,
\end{aligned}
\tag{4.277}
$$

where we have used the definition (4.268). Expanding η_q back from its definition (4.274), we finally obtain:

$$
\mathcal{I}_{\overline{\square}}(t,x,z_1,z_2,f) = \prod_{a=1}^{M}\prod_{b=1}^{L}\prod_{c=1}^{3} \Gamma\left(\frac{z_{1,a}f_c}{z_{2,b}}, \frac{f_c}{z_{1,a}z_{2,b}} \right) . \tag{4.278}
$$

The $\mathcal{I}_{\square\square}$ contribution can be computed similarly, one just needs the group character for the symmetric of $SU(N)$:

$$
\chi_{\square\square}(z_2) = \sum_{1\leq i<j\leq N} z_{2,i}z_{2,j} + \sum_{i=1}^{N} z_{z,i}^2\,. \tag{4.279}
$$

Proceeding as above, we get:

$$
\mathcal{I}_{\square\square}(t,x,z_2,f) = \prod_{c=1}^{3} \left(\prod_{1\leq i<j\leq L} \Gamma\big(t^r z_{2,i}z_{2,j}f_c\big) \right) \cdot \left(\prod_{i=1}^{L} \Gamma\big(t^r z_{2,i}^2 f_c\big) \right) . \tag{4.280}
$$

Plugging (4.278) and (4.280) into (4.264) one gets an explicit integral expression for the index in this phase.

Going to the dual theory, let us parametrize the gauge groups by $SO(2M+1)\times SU(N)$ (we have $M=\frac{N-5}{2}$). The superconformal index is now given by:

$$
\mathcal{I}_{SO}(t,x,f) = \int_{SO}[dz_1]\int_{SU}[dz_2]\ \mathcal{I}_{\overline{\square}}(t,x,z_1,z_2,f)\,\mathcal{I}_{\genfrac{}{}{0pt}{}{\square}{\square}}(t,x,z_1,z_2,f)\,. \tag{4.281}
$$

In order to compute the contributions to the index from the bifundamental and the antisymmetric we need the following group characters:

$$
\chi_{\square}^{SO(2M+1)}(z_1) = 1 + \sum_{a=1}^{M}(z_{1,a}+z_{1,a}^{-1})\,, \tag{4.282}
$$

$$\chi^{SU(N)}_{\substack{\square\\\square}}(z_2) = \sum_{1\leq i<j\leq N} z_{2,i}z_{2,j}\,. \tag{4.283}$$

Proceeding as above, one thus gets:

$$\mathcal{I}_{\overline{\square}}(t,x,z_1,z_2,f) = \prod_{b=1}^{N}\prod_{c=1}^{3}\Gamma\big(t^r z_{2,b}^{-1} f_c\big)\prod_{a=1}^{M}\Gamma\left(\frac{t^r z_{1,a} f_c}{z_{2,b}}, \frac{t^r f_c}{z_{1,a}z_{2,b}}\right)\,, \tag{4.284}$$

$$\mathcal{I}_{\substack{\square\\\square}}(t,x,z_2,f) = \prod_{c=1}^{3}\left(\prod_{1\leq i<j\leq L}\Gamma\big(t^r z_{2,i}z_{2,j} f_c\big)\right)\,, \tag{4.285}$$

and plugging these expressions into (4.281) gives the expression for the superconformal index in terms of elliptic hypergeometric functions, as desired.

Now that we have the explicit expression in terms of elliptic hypergeometric functions, one is left to prove the identity (4.89). Given the physical interpretation of our duality as a strong/weak duality, a preliminary first step would be to prove the analogous index identity between $SO(2N+1)$ and $USp(2N)$ gauge groups in $\mathcal{N}=4$. To our knowledge a complete proof has not been found yet, although in [161] it has been shown that the relevant superconformal indices agree in a number of simplifying limits.

CHAPTER 5

MFV SUSY: A NATURAL THEORY FOR R-PARITY VIOLATION

We present an alternative approach to low-energy supersymmetry.[1] Instead of imposing R-parity we apply the minimal flavor violation (MFV) hypothesis to the R-parity violating MSSM. In this framework, which we call MFV SUSY, squarks can be light and the proton long lived without producing missing energy signals at the LHC. Our approach differs from that of Nikolidakis and Smith in that we impose holomorphy on the MFV spurions. The resulting model is highly constrained and R-parity emerges as an accidental approximate symmetry of the low-energy Lagrangian. The size of the small R-parity violating terms is determined by the flavor parameters, and in the absence of neutrino masses there is only one renormalizable R-parity violating interaction: the baryon-number violating $\bar{u}\bar{d}\bar{d}$ superpotential term. Low energy observables (proton decay, dinucleon decay and $n-\bar{n}$ oscillation) pose only mild constraints on the parameter space. LHC phenomenology will depend on whether the LSP is a squark, neutralino, chargino or slepton. If the LSP is a squark it will have prompt decays, explaining the non-observation of events with missing transverse energy at the LHC.

5.1 Introduction

Supersymmetric extensions of the standard model do not automatically posses the requisite global symmetries of the standard model: baryon and lepton number violation can be mediated by squark and gaugino exchange, and flavor-non-universal soft breaking terms can mediate flavor-changing neutral currents (FCNCs). In order to remove baryon and lepton number violating processes one usually assumes the additional presence of R-parity, while to remove FCNCs one usually assumes flavor universality (possibly at a high scale). R-parity has very important consequences for the phenomenology of the MSSM: it renders the lightest superpartner

[1]This chapter is reprinted from Csaba Csáki, Yuval Grossman and Ben Heidenreich, "Minimal Flavor Violation Supersymmetry: A Natural Theory for R-Parity Violation," Phys. Rev. D **85**, 095009 (2012), with permission.

stable, forces superpartners to be pair-produced, and implies that (when produced) superparticles will always decay to the LSP, which will escape the detector, resulting in events with large missing energy.

R-parity is clearly not necessary [203–208]: very small R-parity violating terms can be added to the supersymmetric Lagrangian, fundamentally changing the phenomenology of the model without conflicting with any current experimental bound (for an excellent review see [209]). The introduction of R-parity is therefore linked to the idea of naturalness: if R-parity were not imposed, many dimensionless couplings in the superpotential would have to be extremely small in order to ensure a sufficiently long-lived proton.

LHC data, however, is beginning to place severe constraints on the R-parity conserving MSSM, ruling out squark masses up to about 1 TeV in some scenarios, due to the absence of the expected missing transverse energy events [210, 211]. Increasing the scale of supersymmetry breaking leads to increasingly large radiative corrections to the Higgs mass, suggesting that low-scale supersymmetry with R-parity may not be the correct solution to the hierarchy problem. In light of this it is natural to consider R-parity violation, which allows the LSP to decay promptly, and thus evades searches based on missing transverse energy or displaced vertices. However, besides naturalness, such an undertaking suffers from a proliferation of undetermined couplings, making it very difficult to constrain the theory from experimental data.

Here, we consider an alternate approach to low-energy supersymmetry. Instead of assuming R-parity, we only impose the minimal flavor violation hypothesis on the theory [212–215], positing that the non-abelian flavor symmetries are only broken by the holomorphic spurions corresponding to the Yukawa couplings.[2] As a con-

[2]While the most general flavor symmetry, $U(3)^5$, is not semi-simple, the abelian $U(1)^5$ component contains R-parity, and would imply the complete absence of lepton- and baryon-number

sequence, all R-parity violating operators will be suppressed by Yukawa couplings and CKM factors, and the smallness of the R-parity violating terms is explained in terms of the smallness of the flavor parameters. We find that this assumption is sufficient to naturally avoid present bounds on baryon- and lepton-number violation, while automatically suppressing FCNCs as in any MFV model. Thus, we are able to replace two independent ad-hoc assumptions, those of R-parity and flavor universality, with the single assumption of minimal flavor violation. R-parity then emerges as an approximate accidental symmetry of the low-energy Lagrangian, where the R-parity breaking terms are determined by the flavor sector.

We will argue that the simplest form of this model is viable with natural $\mathcal{O}(1)$ coefficients for all operators and low, $\sim 100-300$ GeV, superpartner masses. This provides a natural alternative framework for studying supersymmetric extensions of the standard model. While the R-parity violating couplings are sufficiently small to prevent proton decay, they are sufficiently large to make the LSP decay promptly. The phenomenology is distinctive, and depends on only a relatively small number of unknown $\mathcal{O}(1)$ parameters, in contrast to the generic R-parity violating MSSM.

The idea that minimal flavor violation can replace R-parity was originally explored in an important paper by Nikolidakis and Smith a few years ago [216] (see also [217]).[3] Our approach differs from theirs in that we take the spurions to be holomorphic, which is necessary since they appear in the superpotential as Yukawa couplings, and should be thought of as VEVs of chiral superfields. Thus, $Y^\dagger$ cannot appear in the superpotential, nor in soft-breaking A-terms,[4] which, combined with

---

violating operators. In our spurion analysis, we only impose the nonabelian $SU(3)^5$ component.

[3]We thank Yossi Nir for pointing out these references to us.

[4]Nonholomorphic corrections to the A-terms are possible. However, these corrections are subleading, as explored in appendix 5.B. In addition, bilinear corrections to the superpotential can be generated nonholomorphically at the scale m_{soft}.

the MFV hypothesis, severely constrains the form of these terms.

We will show that in the absence of neutrino masses there is no holomorphic invariant violating lepton number, and there is only a single renormalizable term violating baryon number, the $\bar{u}\bar{d}\bar{d}$ term in the superpotential. Furthermore, an unbroken $\mathbb{Z}_3^L$ subgroup of $\mathrm{U}(1)_L$ — a necessary consequence of MFV — ensures that the first non-holomorphic (Kähler) corrections violating lepton number appear at dimension eight, and are very strongly suppressed for even a moderately high cutoff scale. Thus, in the limit of vanishing neutrino masses the proton will be effectively stable. The constraints from $n-\bar{n}$ oscillations are easily satisfied, while those from dinucleon decay place a mild upper bound on $\tan\beta$ for light squark masses.

Majorana neutrino masses require additional holomorphic spurions charged under $\mathbb{Z}_3^L$, and we find that once they are incorporated into the model through the seesaw mechanism, current bounds on proton decay will impose interesting, though not too onerous, constraints on the right-handed neutrino sector. Other methods of neutrino mass generation should also be constrained by proton stability.

The phenomenology of such models is largely determined by the choice of the LSP. If it is a squark, it can decay directly via the baryon number violating $\bar{u}\bar{d}\bar{d}$ vertex, which yields a lifetime short enough for these decays to be prompt. If a sparticle other than a squark is the LSP (such as a neutralino, chargino or slepton) then the decays will involve more particles in the final state and the lifetime will increase, potentially leading to displaced vertices, and in some cases also to missing energy via neutrinos and tops in the final state.

This chapter is organized as follows. In section 5.2 we introduce the MFV SUSY framework and list possible superpotential terms, neglecting neutrino masses. In section 5.3, we focus on the most interesting of these terms, a baryon number vio-

	$\mathrm{SU}(3)_C$	$\mathrm{SU}(2)_L$	$\mathrm{U}(1)_Y$
Q	$\square$	$\square$	$1/6$
$\bar{u}$	$\bar{\square}$	$\mathbf{1}$	$-2/3$
$\bar{d}$	$\bar{\square}$	$\mathbf{1}$	$1/3$
L	$\mathbf{1}$	$\square$	$-1/2$
$\bar{e}$	$\mathbf{1}$	$\mathbf{1}$	1
H_u	$\mathbf{1}$	$\square$	$1/2$
H_d	$\mathbf{1}$	$\square$	$-1/2$

Table 5.1: The MSSM fields and their representations under the SM gauge group.

lating vertex. In section 5.4, we discuss constraints arising from $n - \bar{n}$ oscillations and dinucleon decay induced by this vertex. In section 5.5, we modify the model to incorporate neutrino masses, focusing on the seesaw mechanism, and list the relevant operators, VEVs, and mixings. In section 5.6, we discuss constraints on the right-handed neutrino sector arising from bounds on proton decay. In section 5.7 we estimate the LSP lifetime and comment on LHC signals/constraints. We conclude in section 5.8. In a collection of appendices, we classify all possible holomorphic superpotential terms, discuss nonholomorphic corrections from supersymmetry breaking, argue that diagrams other than those considered in the main text will be subdominant for the processes of interest, and show that higher-dimensional operators will not affect our conclusions for a sufficiently high cutoff.

5.2 MFV SUSY without neutrino masses

We first consider the limit of vanishing neutrino masses (we introduce them in section 5.5). The MSSM consists of the standard model (SM) gauge group $\mathrm{SU}(3)_C \times \mathrm{SU}(2)_L \times \mathrm{U}(1)_Y$, together with the usual chiral superfields as shown in Table 5.1. The matter fields $Q, \bar{u}, \bar{d}, L$, and $\bar{e}$ are flavored, and come in three generations. The superpotential

$$W = \mu H_u H_d + Y_e L H_d \bar{e} + Y_u Q H_u \bar{u} + Y_d Q H_d \bar{d}\,, \tag{5.1}$$

is necessary to generate the SM fermion masses and charged higgsino masses. The additional (renormalizable) superpotential terms allowed by gauge invariance are

$$W' = \lambda LL\bar{e} + \lambda' QL\bar{d} + \lambda'' \bar{u}\bar{d}\bar{d} + \bar{\mu} LH_u \,. \tag{5.2}$$

These superpotential terms violate lepton and baryon number, and therefore should be absent or very small. The traditional approach is to impose a $\mathbb{Z}_2$ symmetry, called matter parity, under which the matter fields $Q, \bar{u}, \bar{d}, L$, and $\bar{e}$ are odd and the Higgs fields H_u and H_d are even. This $\mathbb{Z}_2$ symmetry forbids all unwanted superpotential terms in W', leaving only those in (5.1). A combination of matter parity with a discrete subgroup of the Lorentz group gives R-parity, under which all SM fields are even and superpartners odd.

The imposition of R-parity is not the only ad-hoc assumption needed to make the MSSM phenomenologically acceptable. Soft terms needed to break supersymmetry and mass-up the superpartners generically induce large flavor-changing neutral currents. In order to reduce FCNCs, one usually imposes flavor universality: i.e. the assumption that at some scale all soft breaking masses are flavor universal and the A-terms are proportional to the corresponding Yukawa couplings.

Our approach will be to replace these two ad-hoc assumptions with the single assumption of Minimal Flavor Violation (MFV). MFV is based on the observation that apart from the μ term, most of the terms in the superpotential (5.1) are small due to the smallness of the Yukawa couplings. It is then natural to analyze the spurious symmetries preserved by the μ-term but broken by the Yukawa couplings, which are given in Table 5.2. Excepting $U(1)_{B-L}$ and a $U(1)^2$ subgroup of $\mathrm{SU}(3)_L \times \mathrm{SU}(3)_e$ representing intergenerational lepton number differences, the Yukawa couplings are charged under all of these symmetries, which are therefore broken by the superpotential.

The basic assumption of minimal flavor violation [212–215] is that the Yukawa

	$\mathrm{SU}(3)_Q$	$\mathrm{SU}(3)_u$	$\mathrm{SU}(3)_d$	$\mathrm{SU}(3)_L$	$\mathrm{SU}(3)_e$	$\mathrm{U}(1)_{B-L}$	$\mathrm{U}(1)_H$
Q	$\bar{\square}$	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$	$1/3$	0
$\bar{u}$	$\mathbf{1}$	$\square$	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$	$-1/3$	0
$\bar{d}$	$\mathbf{1}$	$\mathbf{1}$	$\square$	$\mathbf{1}$	$\mathbf{1}$	$-1/3$	0
L	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$	$\bar{\square}$	$\mathbf{1}$	-1	0
$\bar{e}$	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$	$\square$	1	0
H_u	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$	0	1
H_d	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$	0	-1
Y_u	$\square$	$\bar{\square}$	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$	0	-1
Y_d	$\square$	$\mathbf{1}$	$\bar{\square}$	$\mathbf{1}$	$\mathbf{1}$	0	1
Y_e	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$	$\square$	$\bar{\square}$	0	1

Table 5.2: The transformation properties of the chiral superfields and the spurions under the non-anomalous flavor symmetries preserved by the μ term. We omit discrete symmetries and a non-anomalous $\mathrm{U}(1)_R$ which is broken by the soft terms, including the B_μ term.

couplings Y_u, Y_d, and Y_e are the *only* spurions which break the nonabelian $\mathrm{SU}(3)^5$ flavor symmetry. No assumption on baryon or lepton number is made. Thus, while flavor non-singlet terms may be written in the superpotential, or as soft breaking terms, their coefficients must be built out of combinations of Yukawa couplings and their complex conjugates in a way which respects the underlying spurious flavor symmetry. The main new ingredient in applying MFV to SUSY theories is that the spurions also have to be assigned to representations of supersymmetry. Since the spurions $Y_{u,d,e}$ appear in the superpotential in the Yukawa terms, the most natural assumption is to assign these spurions to chiral superfields, with the expectation that in a UV completion these spurions would emerge as VEVs of some heavy chiral superfields. This assignment for the spurions ensures that the conjugate Yukawa couplings $Y^\dagger$ *cannot* appear in the superpotential, which will lead to a very restrictive ansatz, both for R-parity violating terms and for higher dimensional operators.

The MFV hypothesis can be shown to naturally suppress FCNCs [214, 215], thereby solving the new physics flavor problem. It is also RGE stable, due to the spurious flavor symmetries, which prevent flavor violating terms from being generated radiatively except those proportional to the original spurions themselves. As explored in [216], it is possible to impose the MFV hypothesis on spurious (and even anomalous) U(1) symmetries as well. However, we will not do so, since the abelian symmetries are not needed to suppress FCNCs, and furthermore, imposing such a hypothesis will generally lead to phenomenology which is closer to the R-parity conserving MSSM, while our primary goal is to demonstrate a viable supersymmetric model with vastly different phenomenology.

In addition to FCNCs, low-energy CP violation (CPV) searches and measurements also impose strong constraints on new physics. Experimentally, CPV has been discovered only in flavor changing processes in K and B decays. In the SM, this is explained by the fact that the only source of CPV is the one physical phase of the CKM matrix. When extending the SM, however, many new sources of CPV can arise, both in flavor changing as well as flavor conserving couplings. The MFV framework suppresses all new flavor-changing CPV effects, but does not address the problem of flavor diagonal sources of CPV. In SUSY, in particular, new flavor diagonal couplings can give rise to large EDMs, and thus the new phases cannot be order one, and must be tuned to satisfy experimental constraints [218]. Within MFV, one solution is to assume that all CP violating spurions come from the Yukawa matrices. We will not consider the problem of CP violation any further in this chapter, as we do not expect that the problem will be qualitatively different for MFV SUSY than for other MFV models [219].

Thus, we will make the "minimal" assumption that the holomorphic spurions Y_u, Y_d, Y_e are the only sources of $\mathrm{SU}(3)^5$ breaking, discarding R-parity as a means of stabilizing the proton. This assumption, together with the holomorphy of the Yukawa couplings, turns out to be very restrictive. It is straightforward to find

	$\mathrm{SU}(3)_C$	$\mathrm{SU}(2)_L$	$\mathrm{U}(1)_Y$	$\mathrm{U}(1)_B$	$\mathrm{U}(1)_L$	$\mathbb{Z}_2^R$
(QQQ)	$\mathbf{1}$	$\square\square\square$	$1/2$	1	0	$-$
$(QQ)Q$	$\mathbf{8}$	$\square$	$1/2$	1	0	$-$
$(Y_u\bar{u})(Y_u\bar{u})(Y_d\bar{d})$	$\mathbf{8}\oplus\mathbf{1}$	$\mathbf{1}$	-1	-1	0	$-$
$(Y_u\bar{u})(Y_d\bar{d})(Y_d\bar{d})$	$\mathbf{8}\oplus\mathbf{1}$	$\mathbf{1}$	0	-1	0	$-$
$\det\bar{u}$	$\mathbf{1}$	$\mathbf{1}$	-2	-1	0	$-$
$\det\bar{d}$	$\mathbf{1}$	$\mathbf{1}$	1	-1	0	$-$
$QY_u\bar{u}$	$\mathbf{8}\oplus\mathbf{1}$	$\square$	$-1/2$	0	0	$+$
$QY_d\bar{d}$	$\mathbf{8}\oplus\mathbf{1}$	$\square$	$1/2$	0	0	$+$
$LY_e\bar{e}$	$\mathbf{1}$	$\square$	$1/2$	0	0	$+$
H_u	$\mathbf{1}$	$\square$	$1/2$	0	0	$+$
H_d	$\mathbf{1}$	$\square$	$-1/2$	0	0	$+$

Table 5.3: The irreducible holomorphic flavor singlets. We omit flavor-singlet spurions (irrelevant to our analysis) as well as flavor singlets formed from $\mathrm{SU}(3)_C \times \mathrm{SU}(2)_L$ contractions of products of the operators listed here.

the complete list of irreducible holomorphic flavor singlets, shown in Table 5.3. The superpotential is therefore built from gauge invariant combinations of these operators. In particular, since none of these operators carry lepton number, $\mathrm{U}(1)_L$ is an exact symmetry of the superpotential.

While holomorphy also forbids lepton number violation in the soft breaking A terms, lepton number violation can still occur in the Kähler potential, and in bilinear superpotential terms,[5] B terms, and the soft mass mixing term $\tilde{L}\tilde{H}_d^{\star}+c.c.$. However, while such terms will play an important role when we introduce neutrino masses in section 5.5, in the case of massless neutrinos they are absent for the following symmetry reason. There is a $\mathbb{Z}_3^L \in \mathrm{SU}(3)_L \times \mathrm{SU}(3)_e$ symmetry of the form:

$$L \to \omega L \ , \ \ \bar{e} \to \omega^{-1}\bar{e} \ , \ \ Y_e \to Y_e \ , \tag{5.3}$$

where $\omega \equiv e^{2\pi i/3}$ and the other fields and spurions are not charged under $\mathbb{Z}_3^L$. In particular, $\mathbb{Z}_3^L$ lies within the $\mathbb{Z}_3 \times \mathbb{Z}_3$ center of $\mathrm{SU}(3)_L \times \mathrm{SU}(3)_e$, and is also a $\mathbb{Z}_3$

[5]These can be generated nonholomorphically after SUSY breaking, as shown in appendix 5.B.

subgroup of $U(1)_L$. As all spurions are neutral under $\mathbb{Z}_3^L$, we conclude that lepton number can only be violated in multiples of three. Soft terms of this type are not possible, whereas the lowest-dimension $\Delta L = \pm 3$ Kähler potential corrections are dimension eight, and are strongly suppressed for a sufficiently high cutoff.

Since, in the absence of light unflavored fermions, proton decay requires lepton number violation, we conclude that the proton is effectively stable for massless neutrinos. Thus, proton stability will only constrain the neutrino sector, as discussed in section 5.6.[6]

In addition to the R-parity conserving terms (5.1), MFV allows only one additional renormalizable correction to the superpotential:

$$W_{\rm BNV} = \frac{1}{2}\, w''(Y_u\,\bar{u})(Y_d\,\bar{d})(Y_d\,\bar{d})\,, \tag{5.4}$$

where w'' is an unknown $\mathcal{O}(1)$ coefficient. In combination with the MFV structure of the soft terms, most of the interesting phenomenology of our model arises from this baryon-number and R-parity violating term.

The Kähler potential need not be canonical, and is subject to non-universal corrections. At the renormalizable level, these take the form:

$$\begin{aligned} K = Q^\dagger &\left[1 + f_Q(Y_u Y_u^\dagger, Y_d Y_d^\dagger)^T + h.c.\right] Q + \bar{u}^\dagger \left[1 + Y_u^\dagger\, f_u(Y_u Y_u^\dagger, Y_d Y_d^\dagger) Y_u + h.c.\right] \bar{u} \\ &+ \bar{d}^\dagger \left[1 + Y_d^\dagger\, f_u(Y_u Y_u^\dagger, Y_d Y_d^\dagger) Y_d + h.c.\right] \bar{d} \\ &+ L^\dagger \left[1 + f_L(Y_e Y_e^\dagger)^T + h.c.\right] L + \bar{e}^\dagger \left[1 + f_e(Y_e^\dagger Y_e) + h.c.\right] \bar{e}\,, \end{aligned} \tag{5.5}$$

where the f_i are polynomials in the indicated (Hermitean) matrices. While the renormalizable Kähler potential can be made canonical by an appropriate change of basis, such a change of basis is not compatible with the holomorphy of the spurions. The situation is analogous to that of the supersymmetric beta function,

[6] The situation changes if the gravitino (or another unflavored fermion, such as an axino) is lighter than m_p. We discuss the resulting constraints on $m_{3/2}$ in section 5.6.

where the one-loop NSVZ result can be shown to be exact in an appropriate holomorphic basis, but the "physical" all-loop beta function is still subject to wave function renormalization, since the gauge boson kinetic term is non-canonical in the holomorphic basis. Similarly, in MFV SUSY the form of the superpotential is highly constrained, but the Kähler potential is still subject to a large number of unknown corrections. Fortunately, these unknown corrections are suppressed by the smallness of the Yukawa couplings.

The allowed A and B terms are in direct correspondence with the allowed superpotential terms, and carry the same flavor structure, except that the A-terms are subject to certain subleading non-holomorphic corrections:

$$\mathcal{L}_{\mathrm{soft}} \supset Y_u(1 + Y_u^\dagger Y_u + \ldots)\tilde{\bar{u}}(Y_d\tilde{\bar{d}})(Y_d\tilde{\bar{d}}) + (Y_u\tilde{\bar{u}})(Y_d\tilde{\bar{d}})Y_d(Y_d^\dagger Y_d + \ldots)\tilde{\bar{d}}\,, \tag{5.6}$$

and similar corrections to the other A terms, as explained in appendix 5.B. However, as with corrections to the Kähler potential, these corrections are suppressed by the smallness of the Yukawa couplings.

The soft breaking scalar masses have the same basic flavor structure as the Kähler terms listed above. This implies in particular that, while FCNCs can occur via squark exchange, they are suppressed by the GIM mechanism [220], just as in the standard model. This automatic suppression of FCNCs is a universal feature of MFV scenarios. We will quantify the flavor-changing squark mass-mixings in section 5.4.1.

We defer consideration of higher-dimensional operators to appendix 5.E, where we show that such operators will give subdominant contributions to baryon-number violating processes.

5.3 The baryon-number violating vertex

Most of the interesting phenomenology of our model arises from the interaction (5.4), which we now discuss in more detail. Performing an SU(3)^5 transformation, we choose a basis where

$$Y_u = \frac{1}{v_u} V_{CKM}^{\dagger} \begin{bmatrix} m_u & 0 & 0 \\ 0 & m_c & 0 \\ 0 & 0 & m_t \end{bmatrix} \quad , \quad Y_d = \frac{1}{v_d} \begin{bmatrix} m_d & 0 & 0 \\ 0 & m_s & 0 \\ 0 & 0 & m_b \end{bmatrix} \quad ,$$

$$Y_e = \frac{1}{v_d} \begin{bmatrix} m_e & 0 & 0 \\ 0 & m_\mu & 0 \\ 0 & 0 & m_\tau \end{bmatrix} \quad , \tag{5.7}$$

where V_{CKM} is the CKM matrix and $v_{u,d} = \langle H_{u,d} \rangle$ are the Higgs VEVs, with $v^2 = v_u^2 + v_d^2 \approx (174 \text{ GeV})^2$ the standard model Higgs VEV. Since the Yukawa couplings are RG dependent quantities, we should in principle evaluate them at the squark-mass scale to estimate (5.4), integrate out the superpartners, and then run the resulting couplings down to the QCD scale. However, to obtain a rough estimate, it is sufficient to estimate them using the following low-energy quark masses [221]:

$$\begin{aligned} &m_u \sim 3 \text{ MeV} \quad , \quad m_c \sim 1.3 \text{ GeV} \ , \ m_t \sim 173 \text{ GeV} \sim v \ , \\ &m_d \sim 6 \text{ MeV} \quad , \quad m_s \sim 100 \text{ MeV} \ , \ m_b \sim 4 \text{ GeV} \ , \end{aligned} \tag{5.8}$$

together with the lepton masses:

$$m_e \simeq 0.511 \text{ MeV} \ , \ m_\mu \simeq 106 \text{ MeV} \ , \ m_\tau \simeq 1.78 \text{ GeV} \ . \tag{5.9}$$

For the magnitudes of the CKM elements, we take

$$V_{CKM} \sim \begin{pmatrix} 1 & \lambda & \lambda^3/2 \\ \lambda & 1 & \lambda^2 \\ \lambda^3 & \lambda^2 & 1 \end{pmatrix} , \tag{5.10}$$

where $\lambda \sim 1/5$ approximates all elements to better than 20% accuracy.

The lepton and down-type Yukawa couplings depend strongly on $\tan\beta \equiv v_u/v_d$. We consider a broad range, $3 \lesssim \tan\beta \lesssim 45$, where the lower bound is motivated by electroweak symmetry breaking, and the upper bound by perturbativity of the bottom Yukawa coupling, $y_b \lesssim 1$. Consistent with the lower bound $\tan\beta \gtrsim 3$, we will usually assume $\tan\beta \gg 1$, which simplifies many formulae.

Using the assumptions outlined above, we now estimate the size of the baryon-number violating term (5.4), which is conventionally written in the form:

$$W_{\rm BNV} = \frac{1}{2}\lambda''_{ijk}\epsilon^{abc}\bar{u}^i_a\bar{d}^j_b\bar{d}^k_c \,, \tag{5.11}$$

where a, b, c are color indices and i, j, k are the flavor indices, with summation over repeated indices understood. The factor of one-half is due to the anti-symmetry of the operator in the down-type flavor indices (which is a consequence of the color contraction). Using the basis (5.7), we find

$$\lambda''_{ijk} = w'' y^{(u)}_i y^{(d)}_j y^{(d)}_k \epsilon_{jkl} V^\star_{il} \,, \tag{5.12}$$

where $y^{(u)}_i$ and $y^{(d)}_i$ are the up and down-type Yukawa couplings, and the coupling scales like $(\tan\beta)^2$ for large $\tan\beta$. Using the CKM estimate (5.10), we find

$$\begin{aligned}
&\lambda''_{usb} \sim t^2_\beta \frac{m_b m_s m_u}{m_t^3}\,, && \lambda''_{ubd} \sim \lambda t^2_\beta \frac{m_b m_d m_u}{m_t^3}\,, && \lambda''_{uds} \sim \lambda^3 t^2_\beta \frac{m_d m_s m_u}{2\,m_t^3}\,, \\
&\lambda''_{csb} \sim \lambda t^2_\beta \frac{m_b m_c m_s}{m_t^3}\,, && \lambda''_{cbd} \sim t^2_\beta \frac{m_b m_c m_d}{m_t^3}\,, && \lambda''_{cds} \sim \lambda^2 t^2_\beta \frac{m_c m_d m_s}{m_t^3}\,, \\
&\lambda''_{tsb} \sim \lambda^3 t^2_\beta \frac{m_b m_s}{m_t^2}\,, && \lambda''_{tbd} \sim \lambda^2 t^2_\beta \frac{m_b m_d}{m_t^2}\,, && \lambda''_{tds} \sim t^2_\beta \frac{m_d m_s}{m_t^2}\,.
\end{aligned} \tag{5.13}$$

where we t_β as a shorthand for $\tan\beta$. Taking the extreme value $\tan\beta = 45$, and using the quark masses (5.8) and $\lambda \sim 1/5$, we obtain the numerical estimates shown in Table 5.4.

Due to the Yukawa suppression, the largest coupling, λ''_{tsb}, involves as many third-generation quarks as possible, without any first generation quarks. This

	$s\,b$	$b\,d$	$d\,s$
u	5×10^{-7}	6×10^{-9}	3×10^{-12}
c	4×10^{-5}	1.2×10^{-5}	1.2×10^{-8}
t	2×10^{-4}	6×10^{-5}	4×10^{-5}

Table 5.4: Numerical estimates of λ''_{ijk} for $\tan\beta = 45$ and $w'' \sim 1$.

coupling, however, will contribute subdominantly to low energy baryon number violation, due to the CKM suppression required for the third generation quarks to flavor change into first generation external state quarks.

There are many bounds on specific combinations of RPV couplings [209]. These bounds typically assume a generic form for the soft-masses, and thus do not necessarily apply to MFV SUSY. However, due to the flavor suppression, the predicted values of the RPV couplings in our case are small, and all of these bounds are satisfied.

5.4 Constraints from $\Delta B = 2$ processes

The baryon number violating interaction (5.4) will lead to baryon number violating processes which are, in theory, observable at low energy [222]. In particular, the most stringent limits on baryon number violation without lepton number violation come from the lower bound on the neutron-anti-neutron oscillation time [223]

$$\tau_{n-\bar{n}} \geq 2.44\times10^{8}\ \mathrm{s}\,, \tag{5.14}$$

and from the lower bound on the partial lifetime for $pp \to K^+K^+$ dinucleon decay [224]

$$\tau_{pp\to K^+K^+} \geq 1.7\times10^{32}\ \mathrm{yrs}\,. \tag{5.15}$$

Both limits come from null observation of ^{16}O decay to various final states in the Super-Kamiokande water Cherenkov detector. Present limits on other dinucleon partial lifetimes are somewhat weaker, at $\sim 10^{30}$ yrs [221].

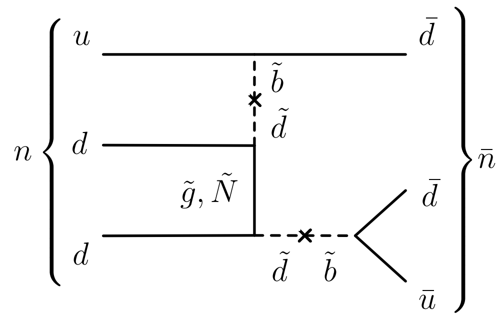


Figure 5.1: The leading contribution to $n - \bar{n}$ oscillation.

In this section, we will only consider the simplest, tree-level diagrams for the processes of interest. While these will turn out the be the dominant diagrams, it is necessary to check that other contributions are subdominant. We outline a systematic scheme for doing so in Appendices 5.C and 5.D.

5.4.1 $n - \bar{n}$ oscillations

There is a unique tree-level diagram for $n-\bar{n}$ oscillations, up to crossing symmetry, the choice of the exchanged fermionic sparticle, and the squark flavors (see Fig. 5.1). The down-type squarks cannot be first generation, due to the antisymmetry of λ''_{ijk} in the last two indices. Thus, to achieve the required flavor-changing, the squarks must change flavor via mass insertions, arising from soft-terms of the form:

$$\mathcal{L}_{\text{soft}} \supset m^2_{\text{soft}}\, \tilde{Q}^{\star} \left(Y_u Y_u^{\dagger} + Y_d Y_d^{\dagger} \right) \tilde{Q} + \dots \;, \tag{5.16}$$

where the omitted terms are higher order in the Yukawa couplings or are diagonal in the quark mass basis.

Thus, off-diagonal mass-mixing between left-handed down-type squarks of flavors i and j is suppressed by

$$V_{ij}^{(\text{neutral})} \equiv \frac{\delta m^2_{ij}}{m^2_{\text{soft}}} \sim \sum_k V^{\dagger}_{ik} \left[y_k^{(u)} \right]^2 V_{kj} \,, \tag{5.17}$$

with a similar expression for up-type squarks. The sum in (5.17) is dominated by the third generation except in the case of $V_{uc}^{(\text{neutral})}$, where there is a competitive (though not dominant) contribution from the second generation. We find:

$$V_{ds}^{(\text{neutral})} \sim \lambda^5 \quad , \quad V_{db}^{(\text{neutral})} \sim \lambda^3 \;\; , \;\; V_{sb}^{(\text{neutral})} \sim \lambda^2 \, ,$$
$$V_{uc}^{(\text{neutral})} \sim y_b^2 \, \lambda^5/2 \quad , \quad V_{ut}^{(\text{neutral})} \sim y_b^2 \, \lambda^3/2 \;\; , \;\; V_{ct}^{(\text{neutral})} \sim y_b^2 \, \lambda^2 \, . \qquad (5.18)$$

Since the squarks in Fig. 5.1 are initially right-handed, the required flavor changing is suppressed by an additional Yukawa coupling. Depending on the initial flavor of the squark, we obtain

$$\tilde{b}_R \to \tilde{d}_L \;\sim\; y_b \lambda^3 \quad , \quad \tilde{s}_R \to \tilde{d}_L \;\sim\; y_s \lambda^5 \, . \qquad (5.19)$$

As the vertex factor is also larger for a $\tilde{b}$ squark, $\tilde{b}_R \to \tilde{d}_L$ is clearly dominant.

Gathering all factors, we obtain the amplitude

$$\mathcal{M}_{n-\bar{n}} \sim \tilde{\Lambda} \, t_\beta^6 \, \lambda^8 \, \frac{m_u^2 m_d^2 m_b^4}{m_t^8} \left(\frac{\tilde{\Lambda}}{m_{\tilde{q}}} \right)^4 \left[g_s^2 \left(\frac{\tilde{\Lambda}}{m_{\tilde{g}}} \right) + \ldots \right] , \qquad (5.20)$$

where we write the hadronic matrix element as $\tilde{\Lambda}^6$, with $\tilde{\Lambda} \sim \Lambda_{QCD}$ in rough agreement with the estimates of [209, 225]. The omitted terms come from neutralino, rather than gluino, exchange and can be important if the gluino is very heavy.

The $n - \bar{n}$ oscillation time is approximately $t_{\text{osc}} \sim \mathcal{M}^{-1}$. Therefore, assuming that the tree-level amplitude (5.20) gives the dominant contribution, we find

$$t_{\text{osc}} \sim (9 \times 10^9 \text{ s}) \left(\frac{250 \text{ MeV}}{\tilde{\Lambda}} \right)^6 \left(\frac{m_{\tilde{q}}}{100 \text{ GeV}} \right)^4 \left(\frac{m_{\tilde{g}}}{100 \text{ GeV}} \right) \left(\frac{45}{\tan \beta} \right)^6 , \qquad (5.21)$$

where we take $\alpha_s \equiv g_s^2/4\pi \sim 0.12$. This must be compared to the experimental bound (5.14), $\tau \geq 2.44 \times 10^8$ s. Thus, unless we have substantially underestimated the hadronic matrix element, $n - \bar{n}$ oscillations place no constraint on our model.

5.4.2 Dinucleon decay

The simplest diagrams for dinucleon decay take the same form as the tree-level $n - \bar{n}$ diagram (see Fig. 5.1), with the addition of two spectator quarks, as shown in Fig. 5.2. There are two possibilities, depending on whether the exchanged sparticle is a chargino or a gluino/neutralino. In the former case, the squarks undergo charged flavor changing while converting to quarks, much like quarks exchanging a W boson; charge conservation then requires that one squark is up-type and the other down-type. In the latter case, the squark/quark/neutralino vertex is flavor diagonal, but neutral flavor changing via squark mass mixing is still possible.

For simplicity, we only consider diagrams of this type.[7] The external quarks must be light quarks, no more than two of which may be strange quarks. Since the quark legs do not change flavor, only ubs, ubd, uds, cds, and tds vertices may be used. By enumerating all possibilities, one can check that the dominant diagram involving chargino exchange combines a tds vertex with a ubs vertex, whereas the dominant diagram involving gluino/neutralino exchange combines two tds vertices with $\tilde{t} \to \tilde{u}$ flavor-changing mass mixing along the squark lines. The two diagrams are shown in Fig. 5.2, with flavor suppressions $y_u y_d y_s^2 y_b^2 \lambda^6/2$ for the chargino exchange diagram, and $y_d^2 y_s^2 y_b^4 \lambda^6/4$ for the gluino/neutralino exchange diagram. Ignoring order-one factors (including gauge couplings), the gluino/neutralino diagram is dominant if

$$\frac{y_d\, y_b^2}{2\, y_u} \simeq \frac{m_d}{2\, m_u} \left(\frac{m_b}{m_t}\right)^2 \tan^3\beta \gtrsim 1\,. \tag{5.22}$$

Thus, for $\tan\beta \gtrsim 12$ the gluino/neutralino diagram dominates; we focus on this possibility for the time being.

Following Goity and Sher [225], we obtain the dinucleon $NN \to KK$ width:

$$\Gamma \sim \rho_N \frac{128\pi\alpha_s^2\tilde{\Lambda}^{10}}{m_N^2 m_{\tilde{g}}^2 m_{\tilde{q}}^8} \left(\frac{\lambda^3 m_d m_s m_b^2}{2m_t^4} \tan^4\beta\right)^4 , \tag{5.23}$$

where $m_N \simeq m_p$ is the nucleon mass, $\rho_N \sim 0.25$ fm^{-3} is the nucleon density, and $\tilde{\Lambda}$ is the "hadronic scale," arising from the hadronic matrix element and phase-space

[7] For a more systematic treatment, see Appendices 5.C and 5.D.

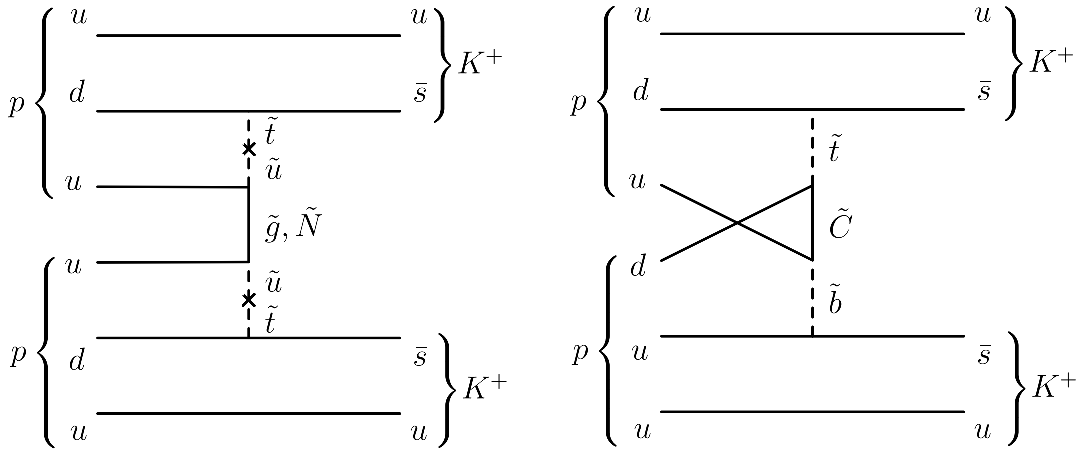


Figure 5.2: Dinucleon decay via neutral gaugino exchange (left) and chargino exchange (right).

integrals. Thus,

$$\tau_{NN\to KK} \sim \left(1.9\times 10^{32}\ \text{yrs}\right)\left(\frac{150\ \text{MeV}}{\tilde{\Lambda}}\right)^{10}\left(\frac{m_{\tilde{q},\tilde{g}}}{100\ \text{GeV}}\right)^{10}\left(\frac{17}{\tan\beta}\right)^{16}, \tag{5.24}$$

where, as before, we take $\alpha_s \sim 0.12$. Comparing with the experimental bound (5.15), $\tau \geq 1.7\times 10^{32}$ yrs, we obtain an upper bound

$$\tan\beta \lesssim 17\left(\frac{150\ \text{MeV}}{\tilde{\Lambda}}\right)^{5/8}\left(\frac{m_{\tilde{q},\tilde{g}}}{100\ \text{GeV}}\right)^{5/8}. \tag{5.25}$$

This bound is illustrated in Fig. 5.3.

There remains considerable uncertainty in the hadronic matrix element. Goity and Sher consider values for $\tilde{\Lambda}/m_{\tilde{q},\tilde{g}}$ between 10^{-3} and 10^{-6} [225]. An earlier paper by Barbieri and Masiero, while taking a substantially different approach, obtains a result consistent with $\tilde{\Lambda} \sim 150$ MeV [226]. We will take $\tilde{\Lambda} = 150$ MeV as a representative value. While this is somewhat smaller than the "natural" $\sim \Lambda_{\text{QCD}}$ scale that one might expect, the matrix element is expected to be suppressed by hard-core repulsion between the nucleons, motivating the yet-smaller scales considered by [225]. Due to the uncertainty in $\tilde{\Lambda}$, we leave the dependence on it explicit in (5.25); Fig. 5.3 illustrates the effect of varying $\tilde{\Lambda}$.

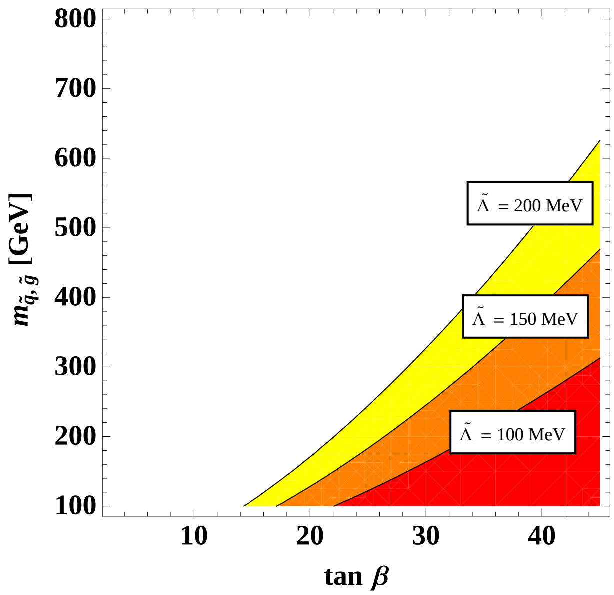


Figure 5.3: Constraints on $\tan\beta$ and superparter masses due to the nonobservation of dinucleon decay. The red region is excluded assuming that $\tilde{\Lambda} \geq 100$ MeV, whereas the orange region is also excluded when $\tilde{\Lambda} \geq 150$ MeV, and the yellow for $\tilde{\Lambda} \geq 200$ MeV.

Assuming $m_{\tilde{q},\tilde{g}} \gtrsim 100$ GeV, the charged flavor-changing diagram does not alter the above bounds, since both amplitudes increase with $\tan\beta$, whereas the neutral flavor-changing diagram is already sufficiently suppressed at $\tan\beta \sim 12$, below which charged flavor-changing becomes dominant.

5.5 Incorporating neutrino masses

We have seen that in the absence of neutrino masses the MFV SUSY approach approximately conserves lepton number, leaving an exact $\mathbb{Z}_3^L$ lepton number symmetry unbroken. To introduce neutrino masses, we therefore require additional spurions, which will lead to additional allowed operators in the Lagrangian [227, 228]. It is important to fully characterize such operators as, in combination with the baryon number violating vertex (5.4), they can induce proton decay.

We focus on the see-saw mechanism to generate Majorana masses for the

	$\mathrm{SU}(3)_L$	$\mathrm{SU}(3)_e$	$\mathrm{SU}(3)_N$	$\mathrm{U}(1)_{B-L}$	$\mathrm{U}(1)_H$	$\mathrm{U}(1)_N$
L	$\bar{\square}$	$\mathbf{1}$	$\mathbf{1}$	-1	0	0
$\bar{e}$	$\mathbf{1}$	$\square$	$\mathbf{1}$	1	0	0
$\bar{N}$	$\mathbf{1}$	$\mathbf{1}$	$\square$	1	0	1
Y_e	$\square$	$\bar{\square}$	$\mathbf{1}$	0	1	0
Y_N	$\square$	$\mathbf{1}$	$\bar{\square}$	0	-1	-1
M_N	$\mathbf{1}$	$\mathbf{1}$	$\overline{\square\square}$	-2	0	-2

Table 5.5: The spurious leptonic flavor symmetries of the MSSM with right-handed neutrinos. We omit discrete and anomalous symmetries.

neutrinos. We add three right-handed sterile neutrinos, $\bar{N}$, which obtain Majorana masses at a heavy scale M_R. Through a Yukawa coupling Y_N to the left-handed neutrinos, this gives the left-handed neutrinos a small Majorana mass of order $Y_N^2 v^2/M_R$ upon electroweak symmetry breaking. Due to the additional flavored field, the nonabelian spurious symmetry of the lepton sector is extended to $\mathrm{SU}(3)_L \times \mathrm{SU}(3)_e \times \mathrm{SU}(3)_N$. The superpotential required to generate neutrino masses is

$$W_{\text{lept}} = Y_e L H_d \, \bar{e} + Y_N L H_u \bar{N} + \frac{1}{2} M_N \bar{N} \bar{N} \,, \tag{5.26}$$

where the elements of M_N are assumed to be of order M_R. Thus, there are now three spurions in the lepton sector: Y_e, Y_N and M_N. The transformation properties of the leptonic sector under the spurious symmetries are shown in Table 5.5. As before, we do not impose the MFV hypothesis on the (spurious) U(1) symmetries.

A subtlety arises when applying the MFV hypothesis to M_N, since it is dimensionful. Instead, we will expand in the dimensionless spurion:

$$\mu_N \equiv \frac{1}{\Lambda_R} M_N \,, \tag{5.27}$$

where Λ_R is an unknown heavy scale. Perturbativity of the spurion expansion requires $M_R \lesssim \Lambda_R$. In addition $\Lambda_R \gg m_{\text{soft}}$ is required for a valid low-energy description. Otherwise, Λ_R is an unknown scale, which may or may not be related to other cutoff scales in the theory.

	$SU(2)_L$	$U(1)_Y$	$U(1)_L$	$\mathbb{Z}_2^R$
$(LL)(\tilde{Y}_N M_N \tilde{Y}_N)(LL)$	$\mathbf{1}$	-2	4	$+$
$(LL)(\tilde{Y}_N M_N \tilde{Y}_N)(Y_e\bar{e})$	$\mathbf{1}$	0	1	$-$
$(LL)\tilde{Y}_N M_N \bar{N}$	$\mathbf{1}$	-1	1	$-$
$L(Y_N \tilde{M}_N Y_N)(Y_e\bar{e})(Y_N\bar{N})$	$\square$	$1/2$	-1	$-$
$LY_N\bar{N}$	$\square$	$-1/2$	0	$+$
$\bar{e}Y_e\tilde{Y}_N M_N \bar{N}$	$\mathbf{1}$	1	-2	$+$
$(Y_e\bar{e})(\tilde{Y}_N M_N \tilde{Y}_N)(Y_e\bar{e})$	$\mathbf{1}$	2	-2	$+$
$L(Y_N \tilde{M}_N Y_N)L$	$\square\square$	-1	2	$+$
$M_N\bar{N}\bar{N}$	$\mathbf{1}$	0	-2	$+$

Table 5.6: A complete list of holomorphic flavor singlets involving Y_N and M_N. We indicate the lepton number of the fields only, not counting that "carried" by the spurion M_N.

As shown in appendix 5.A, the complete list of holomorphic flavor singlets involving Y_N, M_N or $\bar{N}$ is that given in Table 5.6, where we denote the matrix of cofactors of a matrix Y as $\tilde{Y} \equiv (\det Y)\, Y^{-1}$. From these flavor singlets, only one of the three renormalizable lepton number violating superpotential terms of (5.2), $\lambda LL\bar{e}$, can be constructed:

$$W_{\rm LNV}^{(\rm hol)} = \frac{1}{2\Lambda_R}\, w\,(LL)\left(\tilde{Y}_N M_N \tilde{Y}_N\right)(Y_e\bar{e})\ , \tag{5.28}$$

where w is an unknown $\mathcal{O}(1)$ coefficient.

In addition, as shown in appendix 5.B, bilinear superpotential terms, and in particular the lepton-number violating term LH_u, can be generated nonholomorpically after SUSY breaking. As we saw before in the absence of neutrino masses, a $\mathbb{Z}_3^L$ symmetry ensures that lepton number is preserved mod 3, forbidding this term. However, while the $\mathbb{Z}_3^L$ symmetry is not broken by Y_N, it is broken by M_N, which is charged under $\mathbb{Z}_3^L$. Therefore, bilinear lepton-number violating terms are allowed, though they necessarily involve at least one factor of $\mu_N \sim M_R/\Lambda_R$.

The non-holomorphic corrections to the superpotential take the form:

$$W_{\rm LNV}^{(\rm non-hol)} = m_{\rm soft} [\mathcal{V}^\dagger]^a L_a H_u \; , \tag{5.29}$$

where there are two potentially leading contributions to the dimensionless spurion $\mathcal{V}$:

$$\mathcal{V}_a^{(1)} = \frac{1}{\Lambda_R} \varepsilon_{abc} \left[\tilde{Y}_N^\dagger\right]_i^b [M_N^\dagger]^{ij} \, [Y_N]_j^c \quad , \quad \mathcal{V}_a^{(2)} = \frac{1}{\Lambda_R} \varepsilon_{abc} \left[Y_e Y_e^\dagger\right]_d^b \left[Y_N M_N^\dagger Y_N\right]^{cd} \; . \tag{5.30}$$

$\mathcal{V}^{(2)}$ contains more spurions, but if $Y_N \ll 1$ then the presence of the additional Y_e spurions can be easily compensated by the omission of one Y_N insertion, especially at large $\tan\beta$.

The corresponding B-term can also be generated, and takes the form:

$$\mathcal{L}_{\rm soft} \supset m_{\rm soft}^2 [\mathcal{V}^\dagger]^a \tilde{L}_a H_u + h.c. \, , \tag{5.31}$$

This will lead to a left-handed sneutrino VEV

$$\langle L_a \rangle \sim -v_u \, \mathcal{V}_a \, , \tag{5.32}$$

up to an unknown $\mathcal{O}(1)$ coefficient. Inserting this VEV into the canonical Kähler potential $L^\dagger L$, we obtain the gaugino/lepton mixing

$$\mathcal{L} \supset -v_u \, \lambda \, (\mathcal{V}^\dagger L) + c.c. \, . \tag{5.33}$$

This mixing is of approximately the same order as the lepton/higgsino mixing arising from (5.29). Lepton number violation can also appear in the Kähler potential,

$$K_{\rm LNV} \sim [\mathcal{V}^\dagger]^a L_a H_d^\dagger + h.c. \, , \tag{5.34}$$

and in the correspond soft mass term. This will lead to further gaugino/lepton mixing, but proportional to v_d instead of v_u.

In the presence of R-parity violation it is not always simple to define which linear combination of the four fields L_i, H_d is the Higgs, and which are leptons [229]. The physical effects of R-parity violation arise from a basis independent misalignment of the different mixings between the lepton and Higgs superfields. In our case there are several mixing terms, and cancellations can occur. As supersymmetric sources of bilinear lepton-number violation can be eliminated by the field redefinition $L \to L - \mathcal{V} H_d$, these cancellations will depend on the mechanism of supersymmetry breaking.

Indeed, some cancellation may naturally occur in gauge-mediated supersymmetry breaking models, since, due to the flavor-blind nature of gauge interactions, SUSY breaking effects are flavor universal, up to RGE running and subleading corrections induced by the supersymmetric sources of flavor-breaking. We do not, however, assume a particular mechanism for SUSY breaking, and thus will take the mixings (5.29) and (5.33) to be representative without substantial cancellation. Any such cancelation will only make the lepton-number violating effects smaller, and so ignoring such a possibility is a conservative assumption.

The mixing (5.33) can lead to additional contributions to the left-handed neutrino masses via a weak-scale see-saw mechanism. We find

$$\delta m_\nu \sim \frac{\mathcal{V}^2 v_u^2}{m_\lambda} \,. \tag{5.35}$$

Imposing $|\delta m_\nu| \lesssim 1$ eV, we obtain an upper bound

$$\mathcal{V} \lesssim 2 \times 10^{-6} \left(\frac{m_\lambda}{100 \text{ GeV}}\right)^{1/2} \tag{5.36}$$

Proton decay, however, will impose a much stronger bound on $\mathcal{V}$, and consequently the weak see-saw contribution to the left-handed neutrino masses will be negligible.

In the above discussion, we have focused on the see-saw mechanism for generating small neutrino masses. If we instead integrate out the heavy neutrinos and consider the theory below the scale M_R, only one combination of the Y_N and M_N spurions, $Y_N M_N^{-1} Y_N^T$, is relevant for neutrino mass generation. If we ignored all other spurions built from Y_N and M_N, taking a viewpoint that is agnostic about the high-scale mechanism for neutrino mass generation, we would obtain a theory for low-energy lepton-number violation which is more restrictive than that considered above. We have also neglected the effects of RGE running below the scale M_R. While such effects can be significant in detailed numerical calculations [230], they will not substantially alter our order of magnitude estimates.

5.6 Constraints from proton decay

In combination with the baryon-number violating interactions studied in section 5.3 and section 5.4, the lepton-number violating interactions enumerated in section 5.5 will lead to a finite proton lifetime. The strongest constraint on the proton lifetime comes from the bound [231]

$$\tau_{p\to\pi^0 e^+} \geq 8.2\times 10^{33}\ \text{yrs}\,. \tag{5.37}$$

However, this bound only constrains the partial lifetime for the particular final state $\pi^0\, e^+$. For other final states, the partial lifetime bounds are weaker, often substantially [221].

As we show below, MFV SUSY has a strong preference for final states with positive strangeness. Such decay modes are also strongly constrained [221, 232]:

$$\begin{aligned}
\tau_{p\to e^+ K^0} \geq 1.0\times 10^{33}\ \text{yrs} \quad &, \quad \tau_{n\to e^- K^+} \geq 3.2\times 10^{31}\ \text{yrs}\,,\\
\tau_{p\to \mu^+ K^0} \geq 1.3\times 10^{33}\ \text{yrs} \quad &, \quad \tau_{n\to \mu^- K^+} \geq 5.7\times 10^{31}\ \text{yrs}\,,\\
\tau_{p\to \nu K^+} \geq 2.3\times 10^{33}\ \text{yrs} \quad &, \quad \tau_{n\to \nu K^0} \geq 1.3\times 10^{32}\ \text{yrs}\,,
\end{aligned} \tag{5.38}$$

where we also show the (weaker) limits on bound-neutron partial lifetimes. There are similar bounds on some three-body decays of the form $N \to \ell + \pi + K$.

Before discussing the constraints arising from these bounds, we first estimate the size of the coefficients of the lepton-number violating operators. We use the generic parametrization of the neutrino Yukawa couplings of Casas and Ibarra [233]:

$$Y_N^T = \frac{1}{v_u}\,\mathrm{diag}\left(\sqrt{M_{R1}},\sqrt{M_{R2}},\sqrt{M_{R3}}\right)\, R\,\mathrm{diag}\left(\sqrt{m_{\nu 1}},\sqrt{m_{\nu 2}},\sqrt{m_{\nu 3}}\right)\, U^\dagger\,, \quad (5.39)$$

where R is a complex orthogonal matrix describing mixing among the right-handed neutrinos, U is the left handed neutrino mixing matrix giving rise to atmospheric and solar neutrino oscillations, and M_{Ri} and $m_{\nu i}$ $(i = 1, 2, 3)$ are the heavy right-handed neutrino masses and the light left-handed neutrino masses, respectively. The mixing angles in U are large and the elements of U non-hierarchical.

Since R and the right-handed neutrino masses cannot be measured at low energies, we will assume a generic flavor-structure for Y_N. For simplicity we will assume that the right-handed neutrinos have masses of the same magnitude, and that the left-handed neutrinos also have roughly equal masses of order 0.1 eV, with order-one neutrino mixing angles. Substantially lighter neutrino masses would imply a more hierarchical spectrum, with small Yukawa couplings Y_N and consequently more suppressed lepton-number violation, whereas substantially heavier neutrino masses begin to conflict with cosmological bounds.

The neutrino Yukawa coupling is then approximately

$$Y_N \sim \frac{\sqrt{M_R\, m_\nu}}{v_u}\,, \quad (5.40)$$

where we assume that the entire Y_N matrix has elements of this order. The $LL\bar{e}$ coupling is therefore

$$\lambda_{ijk} \sim \frac{M_R^3\, m_\nu^2}{\Lambda_R\, v_u^4}\, y_k^{(e)}\,, \quad (5.41)$$

whereas the $\mathcal{V}$ spurions are

$$\mathcal{V}_i^{(1)} \sim \frac{M_R^{\frac{5}{2}} m_\nu^{\frac{3}{2}}}{\Lambda_R\, v_u^3}\,, \quad \mathcal{V}_{e,\mu}^{(2)} \sim \frac{M_R^2\, m_\nu}{\Lambda_R\, v_u^2}\, y_\tau^2\,, \quad \mathcal{V}_\tau^{(2)} \sim \frac{M_R^2\, m_\nu}{\Lambda_R\, v_u^2}\, y_\mu^2\,. \quad (5.42)$$

Note that

$$\lambda_{ijk} \sim y_k^{(e)} Y_N \, \mathcal{V}^{(1)} \,, \tag{5.43}$$

up to flavor structure. Therefore, due to the smallness of the Yukawa couplings, the $LL\bar{e}$ superpotential term will be a subdominant source of lepton-number violation.

We now search for the largest possible nucleon decay diagram. The simplest diagrams for nucleon decay to a meson and a lepton are those shown in Fig. 5.4, where the squark emits a chargino or neutralino, which mixes into an outgoing charged lepton or neutrino, respectively, via (5.33).[8] Requiring the external quarks to be light, with at most one strange quark, it is straightforward to check that the leading diagram for charged lepton emission involves a tds vertex with $\tilde{t} \to d$ flavor changing at the chargino vertex, whereas the leading diagram for neutrino emission also involves a tds vertex, but with $\tilde{t} \to \tilde{u}$ mass mixing on the squark line.

The neutrino diagram has an additional flavor suppression of order $y_b^2/2$ relative to the charged-lepton diagram. However, the latter diagram, which leads to $n \to K^+\mu^-$ decay, suffers from a chiral suppression, as we illustrate in Fig. 5.4. The suppression occurs because the right to right chargino propagator is roughly $\not{p}/m_{\tilde{C}}^2$, leading to an additional suppression of at least $\sim m_p/m_{\tilde{C}}$ relative to the right to left propagator. This chiral suppression is not present in the $p \to K^+\bar{\nu}$ diagram. Combined with the stronger partial lifetime bound for this decay mode, the latter diagram will give the strongest constraints.

The amplitude is

$$\mathcal{M}_{p\to K^+\bar{\nu}} \sim \frac{\lambda^3 \, m_d \, m_s \, m_b^2}{2 \, m_t^3 \, m_{\tilde{N}}} \left(\frac{\tilde{\Lambda}}{m_{\tilde{q}}} \right)^2 \, \mathcal{V} \tan^4 \beta \,. \tag{5.44}$$

up to order-one mixing angles and gauge couplings, where $\tilde{\Lambda}^2$ is a hadronic matrix element. We will take $\tilde{\Lambda} \sim 250$ MeV, in rough agreement with lattice computa-

[8]The lepton/higgsino mixing (5.29) gives another contribution to this mixing of a similar form.

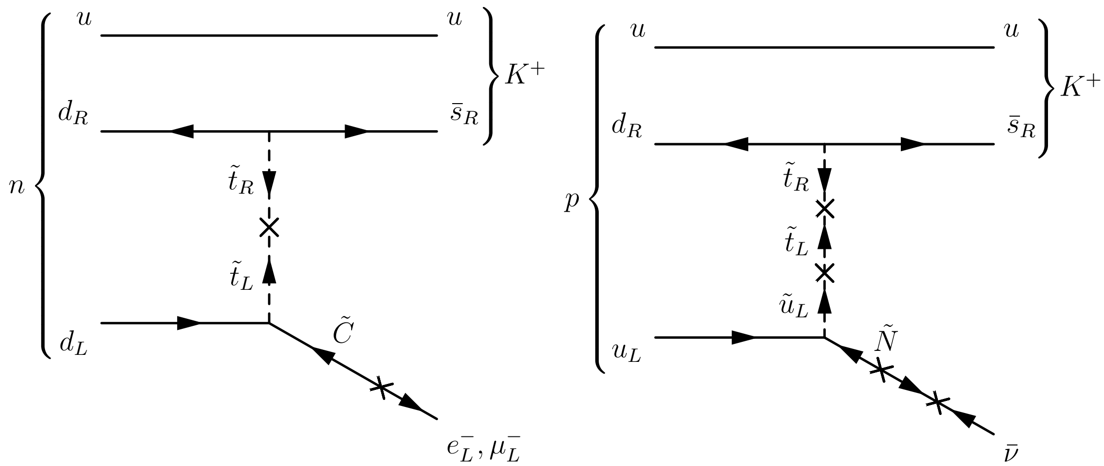


Figure 5.4: The leading charged (left) and neutral (right) flavor-changing diagrams for $n \to \ell^- K^+$ and $p \to K^+ \bar{\nu}$ nucleon decay, respectively. Arrows indicate chirality. The charged flavor-changing diagram has less flavor suppression, but suffers from a chiral suppression due to the right $\to$ right chargino propagator.

tions [234, 235]. The width is

$$\Gamma \sim \frac{m_p}{8\pi} |\mathcal{M}|^2 . \tag{5.45}$$

Comparing with the experimental bound (5.38), we obtain

$$\mathcal{V} \tan^4 \beta \lesssim (3 \times 10^{-14}) \left(\frac{m_{\tilde{q}}}{100 \text{ GeV}} \right)^2 \left(\frac{m_{\tilde{N}}}{100 \text{ GeV}} \right) . \tag{5.46}$$

For sufficiently large $\tan\beta$, we have $\mathcal{V}^{(2)} \gg \mathcal{V}^{(1)}$ and $\mathcal{V}^{(2)}$ gives the dominant contribution to $\mathcal{V}$. Using $m_\nu = 0.1$ eV, we then obtain the upper bound on M_R

$$M_R \lesssim (3 \times 10^7 \text{ GeV}) \left(\frac{10}{\tan\beta} \right)^3 \left(\frac{m_{\tilde{q},\tilde{N}}}{100 \text{ GeV}} \right)^{3/2} \left(\frac{\Lambda_R}{10^{16} \text{ GeV}} \right)^{1/2} . \tag{5.47}$$

One can check that $\mathcal{V}^{(1)}$ gives a weaker bound than this as long as

$$\tan\beta \gtrsim 6 \left(\frac{m_{\tilde{q},\chi}}{1 \text{ TeV}} \right)^{3/14} \left(\frac{\Lambda_R}{10^{16} \text{ GeV}} \right)^{1/14} . \tag{5.48}$$

Thus, for $\Lambda_R = 10^{16}$ GeV and $m_{\tilde{q},\tilde{N}} \lesssim 1$ TeV, $\mathcal{V}^{(2)}$ is dominant for $\tan\beta \gtrsim 6$, whereas for $\tan\beta \lesssim 6$, $\mathcal{V}^{(1)}$ is dominant for sufficiently large superpartner masses. The bound on M_R, including both contributions, is illustrated in Fig. 5.5.

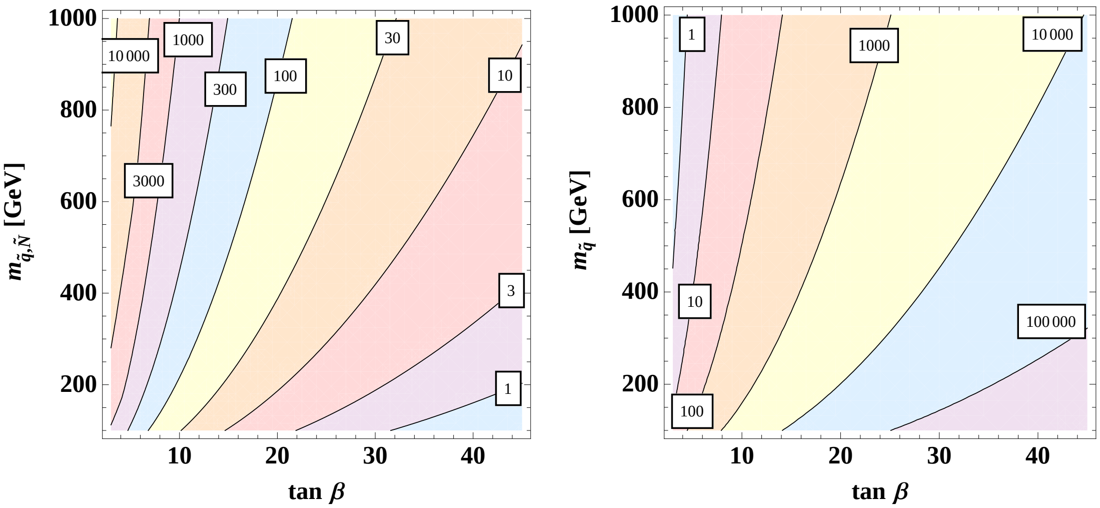


Figure 5.5: Left: the upper bound on M_R due to the nonobservation of nucleon decay, in units of 10^6 GeV. For this plot, we have fixed $\Lambda_R = 10^{16}$ GeV and $m_\nu = 0.1$ eV. Near the left edge, the dominant constraint comes from the $\mathcal{V}^{(1)}$ spurion; elsewhere $\mathcal{V}^{(2)}$ is dominant. Right: the approximate lower bound on $m_{3/2}$, in KeV, due to the nonobservation of $p \to K^+\tilde{G}$.

The bound on M_R depends strongly on Λ_R. For instance, if $\Lambda_R \sim 10$ TeV, the bound (5.47) is reduced by six orders of magnitude. If the right-handed neutrinos are sufficiently light, they could be produced at colliders, though the Yukawa couplings are necessarily very small, so that such a scenario is unlikely to be excluded in the near future.

If the gravitino is sufficiently light, proton decay can proceed via the baryon-number violating vertex (5.4) alone, without lepton number violation [236]. In particular, the gravitino is derivatively coupled to chiral superfields [237]:

$$\mathcal{L}_{\rm int} = -\frac{1}{\sqrt{3}\, m_{3/2}\, M_{\rm pl}}\, \bar{\psi}_L \gamma^\mu \gamma^\nu (\partial_\mu \tilde{G})(D_\nu \phi) + c.c.\,, \tag{5.49}$$

where $\tilde{G}$ is the gravitino, (ϕ, ψ) is any chiral superfield, and $M_{\rm pl}$ is the reduced Planck mass. If kinematically allowed, the decay $p \to K^+\tilde{G}$ will proceed via the diagram in Fig. 5.6, with the width

$$\Gamma \sim \frac{m_p}{8\pi} \left(\frac{\tilde{\Lambda}}{m_{\tilde{q}}}\right)^4 \left(\frac{\Lambda^2}{\sqrt{3} m_{3/2} M_{\rm pl}}\right)^2 \frac{\lambda^6 m_d^2 m_s^2 m_b^4}{4 m_t^8} \tan^8 \beta\,, \tag{5.50}$$

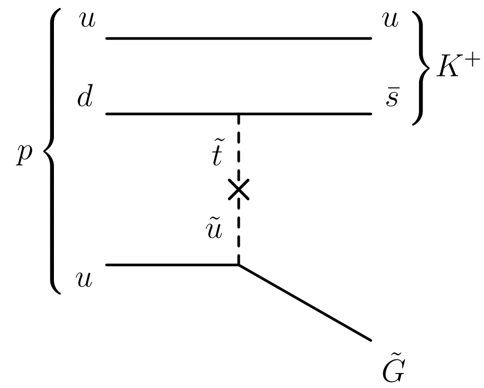


Figure 5.6: The leading contribution to $p \to K^+\tilde{G}$ decay.

where we use the same matrix element as above, replacing the momentum insertions with a characteristic energy scale, Λ.

While we are unaware of a direct search for $p \to K^+\tilde{G}$, for a very light gravitino $p \to K^+\nu$ gives the same experimental signature. If we conservatively assume that the $p \to K^+\nu$ bound (5.38) applies to $p \to K^+\tilde{G}$ decays for any gravitino mass, we obtain an approximate lower bound on $m_{3/2}$:

$$m_{3/2} \gtrsim (300 \text{ KeV}) \left(\frac{300 \text{ GeV}}{m_{\tilde{q}}}\right)^2 \left(\frac{\tan\beta}{10}\right)^4 , \qquad (5.51)$$

where we take $\Lambda \sim \tilde{\Lambda} \sim 250$ MeV. This bound is illustrated in Fig. 5.5.

5.7 LSP decay and LHC phenomenology

The phenomenology of MFV SUSY models will be very different from the R-parity conserving MSSM, and is distinctive among R-parity violating theories. In this section, we attempt to explore the general phenomenological features of these models. The results depend on the spectrum, and we will not attempt to exhaustively enumerate all possibilities, instead focusing on the general features for various LSPs.

We will not assume that the LSP is electrically and color neutral; since it decays there is no particular motivation for that requirement. Thus the LSP could be either a squark, a slepton, a neutralino, a chargino, or the gluino. However, MFV places restrictions on the squark and slepton masses. In particular, the mass matrix for up-type squarks must be of the form

$$M_{\tilde{U}}^2 = \begin{pmatrix} m_{\tilde{Q}}^2\,(1+\alpha_u Y_u Y_u^\dagger + \alpha_d Y_d Y_d^\dagger) + d_{u,L} & A_u\,Y_u \\ A_u^\star\,Y_u^\dagger & m_{\tilde{u}}^2\,(1+\beta_u Y_u^\dagger Y_u) + d_{u,R} \end{pmatrix} + \dots\,, \tag{5.52}$$

where the omitted terms are higher-order in the Yukawa couplings, A_u is some combination of holomorphic parameters specifying the left-right mixing (coming from the Yukawa couplings and A-terms), $\alpha_{u,d}$ and β_u are non-holomorphic parameters coming from the left and right-handed squark masses, respectivley, and $d_{u,L}$ and $d_{u,R}$ are the flavor-universal D-term contributions to the squark masses.

Naturalness, in this context, indicates that $\alpha_{u,d}$ and β_u should be order-one numbers, whereas $m_{\tilde{Q}}$, $m_{\tilde{u}}$, and A_u are of order m_{soft}. Thus, the leading deviations from universality will involve only the $\mathcal{O}(1)$ top Yukawa coupling, and, in particular, it is very easy to make one of the stops very light. Since other non-universal terms are suppressed by Yukawa couplings and/or CKM factors, the remaining squarks are expected to be nearly degenerate. A similar argument applies to down-type squarks, where the bottom squark can be made light. In the charged slepton sector, the leading non-universal term comes from the y_τ suppressed left/right mixing, implying a nearly degenerate spectrum, except at very large $\tan\beta$. The sneutrinos will be even more degenerate, since this left/right term is absent, and the leading non-universality comes from y_τ^2 suppressed soft-mass corrections.

Thus, it is very natural for the stop or the sbottom to be the LSP. A stau (or tau sneutrino) LSP, however, typically implies a nearly degenerate spectrum, and is somewhat less natural in this context. Other squarks or sleptons are not likely to be the LSP.

Since the largest R-parity violating operator is in the quark sector, the most

Figure 5.7: The leading diagrams for stop (left) and left-handed sbottom (right) LSP decay. A right-handed sbottom decays similarly, without the mass insertion.

interesting scenario is when the LSP is the stop or the sbottom. We consider the stop LSP case in detail. The direct decay of the stop is given by the diagram in Fig. 5.7. The partial widths $\Gamma(\tilde{t} \to \bar{d}_i \bar{d}_j)$ are given by

$$\Gamma_{ij} \sim \frac{m_{\tilde{t}}}{8\pi} \sin^2\theta_{\tilde{t}} |\lambda''_{3ij}|^2 \,, \tag{5.53}$$

where $\theta_{\tilde{t}}$ is the stop mixing angle. To estimate the lifetime numerically, we use the renormalized quark masses at a scale $m_t \sim v \sim 174$ GeV, which are approximately [238, 239]:

$$\begin{aligned} m_u \sim 1.2 \text{ MeV} \quad &, \quad m_c \sim 600 \text{ MeV} \;,\; m_t \sim v \sim 174 \text{ GeV} \,, \\ m_d \sim 3 \text{ MeV} \quad &, \quad m_s \sim 50 \text{ MeV} \;,\; m_b \sim 2.8 \text{ GeV} \,, \end{aligned} \tag{5.54}$$

Using these masses to compute the relevant Yukawa couplings, we find a lifetime

$$\tau_{\tilde{t}} \sim (2\ \mu\text{m}) \left(\frac{10}{\tan\beta}\right)^4 \left(\frac{300 \text{ GeV}}{m_{\tilde{t}}}\right) \left(\frac{1}{2\sin^2\theta_{\tilde{t}}}\right) . \tag{5.55}$$

Thus no displaced vertices are expected except for very small values of $\tan\beta$ and a very light LSP. The decay length of the stop LSP is shown in Fig. 5.8.

Note that in this case one does not expect a large number of top quarks in the final state, nor, of course, any missing energy. Roughly 90% of decays will go to bottom and strange quarks, about 8% to bottom plus down, and a few percent to down plus strange. These branching ratios are fixed by the flavor structure. Thus,

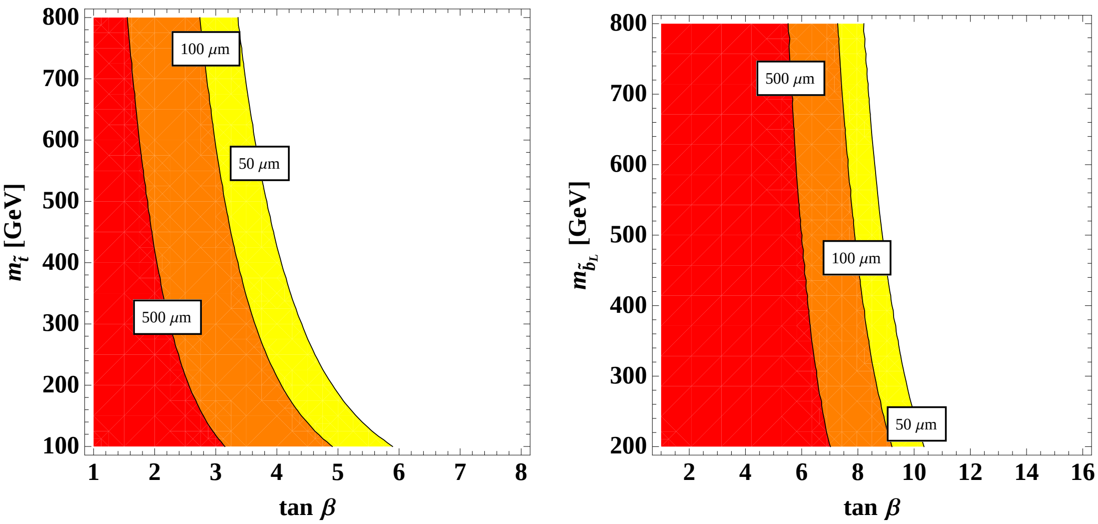


Figure 5.8: The decay length ($c\tau$) of a stop (or right-handed sbottom) (left) or left-hand sbottom (right) LSP, in units of μm. Displaced vertices are expected only for small $\tan\beta$ and a light LSP.

most of the events will contain b-quarks, and a generic signal for supersymmetry will be an overall increase in the number of events with b-jets, but with possible resonances in the jet spectrum at the squark masses. Since production of the superpartners would still be mainly through the R-parity conserving couplings, most SUSY events would actually end up with at least four jets, two of which are b-jets. Other superpartners will first decay to the stop. For example the neutralino is expected to decay to a stop plus charm as in Fig. 5.9. The neutralino lifetime for the case of a stop LSP is given by

$$\Gamma_{\tilde N} \sim \frac{m_{\tilde N}}{8\pi} g^2 \lambda^4 \frac{m_b^4}{m_t^4} \tan^4\beta \;\;,\;\; \tau_{\tilde N} \sim (10^{-19}\text{ s}) \left(\frac{10}{\tan\beta}\right)^4 \left(\frac{300\text{ GeV}}{m_{\tilde N}}\right) . \qquad (5.56)$$

Thus, absent a nearly-degenerate spectrum, the other superpartners are expected to be short-lived.

It is also possible for a bottom squark to be the LSP, decaying as shown in Fig. 5.7. For a right-handed sbottom, the lifetime is similar to that of a stop LSP lifetime, unless the decay is near threshold. The decay of a left-handed sbottom

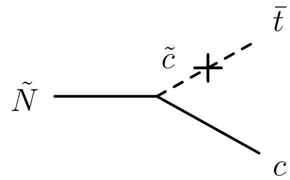


Figure 5.9: Neutralino NLSP decay.

LSP is further suppressed by a left-right mass insertion. In this case, the partial widths $\Gamma(\tilde{b}_L \to \bar{u}_i \bar{d}_j)$ are

$$\Gamma_{ij} \sim \frac{m_{\tilde{b}}}{8\pi}\, y_b^2\, |\lambda''_{ij3}|^2\,, \tag{5.57}$$

giving a total lifetime

$$\tau_{\tilde{b}_L} \sim (41\ \mu\text{m}) \left(\frac{10}{\tan\beta}\right)^6 \left(\frac{300\ \text{GeV}}{m_{\tilde{b}_L}}\right)\,. \tag{5.58}$$

Thus, displaced vertices are expected at low $\tan\beta$, as illustrated in Fig. 5.8. The phenomenology is distinct from that of a stop LSP: roughly 99% of decays will be to top and strange or top and down quarks, with less than one percent going to charm and strange quarks, and a small fraction to other final states. Thus, an increase in top quark production is expected, with most SUSY events containing at least two top-jets. However, fewer b-jets will be produced, except those arising from top decays.[9]

Otherwise, the LSP can be a chargino, a neutralino, or a slepton. Each of these will give a distinct phenomenology. Assuming that the LSP is a neutralino, its decay will be dominated by the diagram in Fig. 5.10. The width is approximately

$$\Gamma_{\tilde{N}} \sim \frac{m_{\tilde{N}}}{128\,\pi^3}\, |\lambda''_{tsb}|^2\,, \tag{5.59}$$

where we estimate a phase-space suppression of $1/16\pi^2$ for each additional final

[9]If $m_{\tilde{b}_L} \lesssim m_t$, the phenomenology will be different yet again, with displaced vertices more likely due the reduced width, but no extra top production.

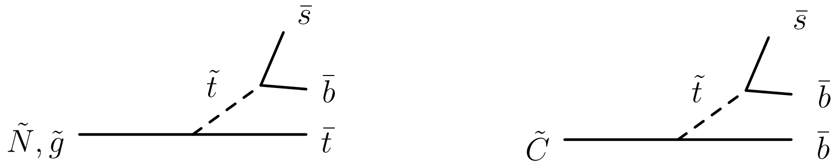


Figure 5.10: Neutralino/gluino (left) and chargino (right) LSP decays.

state particle. The lifetime is then

$$\tau_{\tilde{N}} \sim (12\ \mu\text{m}) \left(\frac{20}{\tan\beta}\right)^4 \left(\frac{300\ \text{GeV}}{m_{\tilde{N}}}\right) . \tag{5.60}$$

As shown in Fig. 5.11, this scenario is much more likely to produce displaced vertices, although they can still be avoided in a sizable region of parameter space. Thus, for the case of a neutralino LSP the expected signal of SUSY would be an increase in the top production cross section (since the LSP decay involves top quarks), including potentially same-sign tops, and possibly also displaced vertices for the lights jets. A gluino LSP would decay in a very similar fashion to a neutralino LSP, whereas a chargino LSP would have a similar lifetime, but would usually decay via two b-jets without a top quark, as shown in Fig. 5.10.

The case of a chargino LSP is very similar to that of a neutralino. The one significant difference, as can be seen from Fig. 5.10, is that in the chargino case we expect no top in the final state, and instead expect more b jets.

Finally, the LSP could be a slepton, mostly likely the lighter stau. This would probably be much easier to observe at the LHC. The leading decay of the stau would be a four-body decay involving top and bottom quarks, a light jet and either a lepton or missing energy, as shown in Fig. 5.12. Since it is a four-body decay, the NDA estimate for the width of the stau LSP is

$$\Gamma_{\tilde{\tau}} \sim \frac{m_{\tilde{\tau}}}{2048\pi^5} |\lambda''_{tsb}|^2 , \tag{5.61}$$

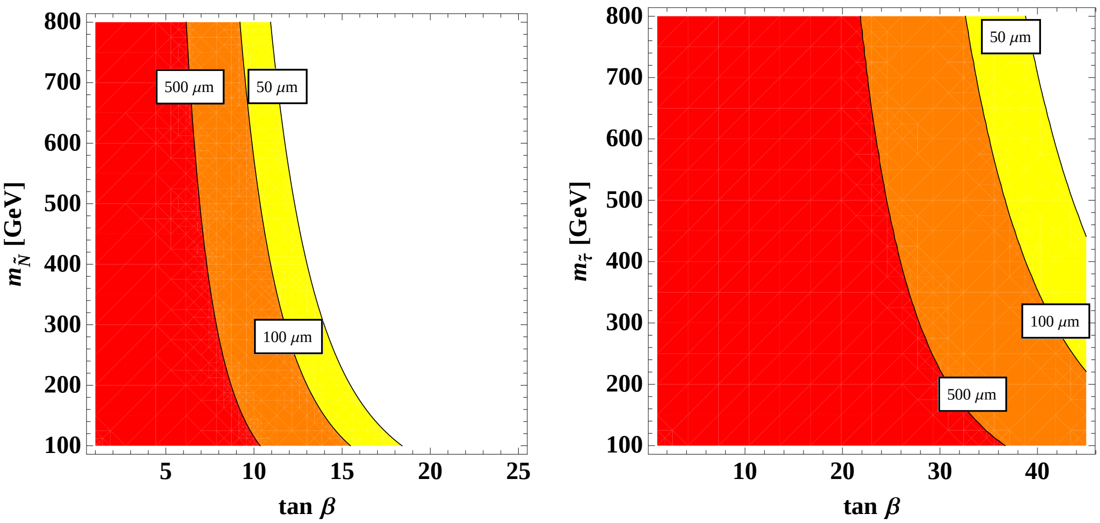


Figure 5.11: The decay length ($c\tau$) of a neutralino (left) or stau (right) LSP, in units of μm. For a neutralino LSP, displaced vertices can arise in a substantial region of parameter space, whereas for the stau, they are expected nearly everywhere.

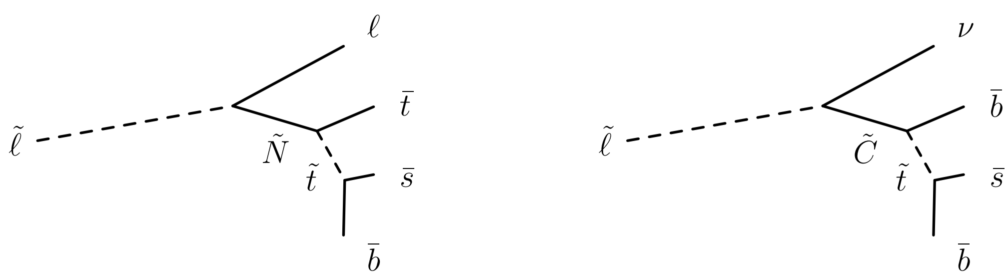


Figure 5.12: Slepton LSP decay without neutrinos (left) and with neutrinos (and thus missing energy) on the right.

with lifetime of order

$$\tau_{\tilde{\tau}} \sim (44\ \mu\text{m}) \left(\frac{45}{\tan\beta}\right)^4 \left(\frac{500\ \text{GeV}}{m_{\tilde{\tau}}}\right) . \tag{5.62}$$

Such long lifetimes will give displaced vertices in almost all of the relevant parameter space, as shown in Fig. 5.11. Thus the signal of SUSY in the case of a stau LSP would be events with displaced vertices, top and bottom quarks, and either a lepton or missing energy.

Current searches for R-parity violating supersymmetry are not very restrictive for MFV SUSY. The more restrictive searches look for leptons among the final state particles, and set bounds on the coupling λ': this is exactly the one vanishing in MFV SUSY. For the case of a stop LSP one could expect a resonance in the dijet searches; however the production cross section of the stop is typically about three orders of magnitudes smaller [240] than the experimental sensitivities both at the Tevatron [241] and at the LHC [242, 243].

The more relevant searches are the ones carried out by CMS [244] (and also by CDF [245]): here the R-parity violating decay of the gluino in the presence of a $\bar{u}\bar{d}\bar{d}$ coupling is considered by searching for a resonance in 3-jet final states, after appropriate kinematic cuts are introduced to separate potential SUSY events from QCD background. The most stringent CMS search (using 35 pb^{-1} of data) yields a bound on the gluino mass $m_{\tilde{g}} > 280$ GeV. However, we should emphasize that in these models the gluino does not play an essential role. Thus even if the gluino is in the TeV energy range the model could be completely natural. While these searches are very promising, an eventual null-result of this particular experiment would not remove the motivation for these theories, since this search relies on the production of a light gluino.

Another relevant search is for massive colored scalars in 4-jet events [246]. Here the four most energetic jets are paired up and a resonance in the average invariant masses of the two pairs is searched for. Stop pair production followed by decays to jets would contribute to this channel. The current bounds on the mass of a colored scalar octet using 2010 LHC data are in the $150 - 180$ GeV range. However, the production cross section for scalar triplets is smaller, and this bound will be substantially weakened or eliminated if applied to the stop. Better background rejection can be achieved using b-tagging, since almost all the stop quarks include at least one b-jet. A recent simulation [247] showed that such a search at the 14 TeV LHC will be able to discover stops decaying through the $\bar{u}\bar{d}\bar{d}$ coupling up to 650 GeV with 300 fb^{-1} data. A search for a lepton together with many jets has also been suggested [248]. This search could probe MFV SUSY if the LSP is a slepton, or if it decays to top quarks, which can produce a lepton in the final state.

Throughout this chapter we have been assuming a squark mass scale of order a few-hundred GeV. This is necessary to make SUSY a natural solution of the hierarchy problem. However, in this case the Higgs mass in the simplest MSSM-type extension will usually be too light. One needs an extension of the Higgs sector, for example to NMSSM-type models, to raise the Higgs mass over the 114 GeV LEP bound. Such an extension should not significantly alter the MFV structure of the theory. For example, while the $\mathbb{Z}_3$ symmetric version of the NMSSM has restricted couplings due to the (weakly broken) discrete symmetry, the superpotential (5.4) is $\mathbb{Z}_3$ invariant, leaving the essential features of our model intact.

One of the outstanding problems of the SM and the MSSM is the issue of baryogenesis. The Higgs mass is too high in both of these theories to account for the observed matter/antimatter asymmetry directly, and the leading explanation is baryogenesis via leptogenesis. In MFV SUSY, the appearance of the λ'' baryon number violating operator, (5.4), opens new possibilities for baryogenesis. Several scenarios that make use of this coupling have been proposed in [249–253]. For example the model of [253] would rely on out-of-equilibrium decays of the lightest neutralino $\tilde{N} \to \bar{u}\bar{d}\bar{d}$ and needs λ'' couplings in the $10^{-4} - 10^{-3}$ range.

Finally we comment on dark matter. One of the main motivations for R-parity is that it provides a stable heavy superpartner, which in many cases can be a candidate for a WIMP. In MFV SUSY we are obviously forgoing this possibility. However, this does not necessarily imply that there cannot be a good dark matter candidate in these models. While we are assuming the LSP within the SM superpartners to be the stop or another sparticle, the gravitino can still be lighter and be the real LSP. A gravitino dark matter scenario within R-parity violating SUSY has been advocated in [254]. There it was found that the leading decay of the gravitino is $\tilde{G} \to \gamma\nu$ (see Fig. 5.13) with a width of

$$\Gamma_{\tilde{G}} \sim \frac{1}{32\pi}|U_{\gamma\nu}|^2 \frac{m_{3/2}^3}{M_{Pl}^2}, \tag{5.63}$$

where $U_{\gamma\nu}$ is the photino-neutrino mixing due to the small sneutrino VEV. In our case the mixing is set by the spurion $\mathcal{V}$: $U_{\gamma\nu} \sim v_u \mathcal{V}/m_{\tilde{N}}$ where $m_{\tilde{N}}$ is a characteristic gaugino mass. Imposing the bound (5.46), we obtain a lower bound on the gravitino lifetime,

$$\tau_{\tilde{G}} \gtrsim (4 \times 10^{39}\ \text{yr}) \left(\frac{1\ \text{GeV}}{m_{3/2}}\right)^3 \left(\frac{300\ \text{GeV}}{m_{\tilde{q}}}\right)^4 \left(\frac{\tan\beta}{10}\right)^8 . \tag{5.64}$$

If the gravitino is heavier than $\sim$ 1 GeV it can decay to hadrons via the R-parity violating $\bar{u}\bar{d}\bar{d}$ vertex. While the exact decay mode will depend on what is kinematically available, for $m_{3/2} \gtrsim$ 10 GeV all decays are allowed, and the dominant diagram will be that shown in Fig. 5.13. The width for the illustrated three-body decay is

$$\Gamma_{\tilde{G}\to\bar{c}\bar{b}\bar{s}} \sim \frac{m_{3/2}^7}{128\pi^3(3M_{\text{pl}}^2)m_{\tilde{q}}^4} \left(\frac{\lambda^2\, m_c^2 m_s^2 m_b^2}{m_t^6}\right) \tan^4\beta\,. \tag{5.65}$$

Thus,

$$\tau_{\tilde{G}} \sim (3 \times 10^{16}\ \text{yrs}) \left(\frac{m_{\tilde{q}}}{300\ \text{GeV}}\right)^4 \left(\frac{10}{\tan\beta}\right)^4 \left(\frac{100\ \text{GeV}}{m_{3/2}}\right)^7 . \tag{5.66}$$

In either case a gravitino LSP is generically very long lived, with a lifetime much greater than the age of the universe. Thus, the gravitino is a dark matter candidate, though more study is needed to determine if it is a realistic one.

If the gravitino is the LSP, the NLSP can either decay to jets via the R-parity violating vertex, (5.4), or to the gravitino itself. The partial width for the simplest

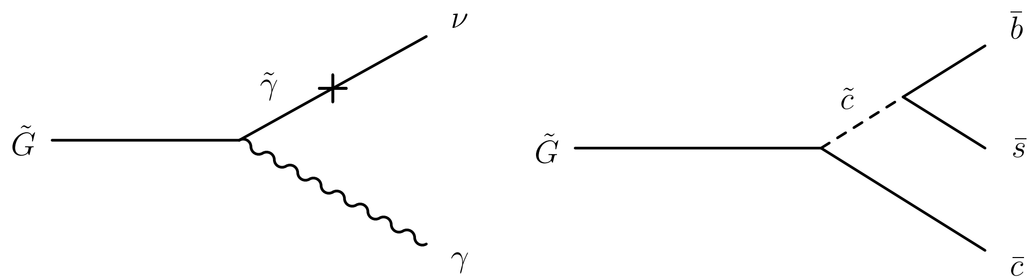


Figure 5.13: Gravitino decay via neutrino-photino mixing (left) for gravitinos below ~ 1 GeV, and to hadrons (right) for masses above ~ 1 GeV. The illustrated hadronic decay diagram (along with other decays arising from permutations of the *cbs* flavor labels) is dominant when kinematically allowed.

gravitino decay, e.g. $\tilde{t} \to t + \tilde{G}$, takes the form:

$$\Gamma \sim \frac{m_{\rm NLSP}^5}{24\pi m_{3/2}^2 M_{\rm pl}^2} \tag{5.67}$$

for a squark or slepton NLSP, with a similar expression in the case of a gaugino NLSP. Thus, the rate is enhanced for a lighter gravitino, and if we assume that $m_{3/2}$ saturates the lower bound (5.51), then we obtain a branching ratio:

$$\frac{\Gamma_{\tilde{t}\to t\tilde{G}}}{\Gamma_{\tilde{t}\to {\rm SM}}} \sim (7\times 10^{-10})\left(\frac{m_{\tilde{t}}}{300\ {\rm GeV}}\right)^8 \left(\frac{10}{\tan\beta}\right)^{12} \tag{5.68}$$

for a stop NLSP. Thus, the branching ratio is generically small, but depends strongly on the NLSP mass and on $\tan\beta$.[10] For other NLSPs, this branching ratio is enhanced, whereas it can always be suppressed by increasing $m_{3/2}$. Depending on all the parameters, NLSP to gravitino decays could generate a significant gravitino relic density, which is of cosmological interest. We defer further consideration of this interesting topic to a future work.

5.8 Conclusions

We have presented an alternative approach to R-parity in supersymmetric extensions of the standard model. We have shown that imposing minimal flavor violation

[10]For a very heavy NLSP at low $\tan\beta$, it is possible for gravitino decay to dominate, though not in a particularly promising region of parameter space.

in a manifestly supersymmetric way is powerful enough to reduce all baryon and lepton number violating amplitudes below current experimental bounds, while allowing a sufficiently rapid decay of the LSP such that no events with large missing transverse energy would be expected at the LHC.

The basic MFV assumption is that the only sources of flavor violation are the SM Yukawa coupling matrices $Y_{u,d,e}$. In a supersymmetric context these spurions should be treated as VEVs of chiral superfields. The flavor symmetry together with supersymmetry will pose very stringent restrictions on the low-energy effective Lagrangian, and R-parity will be an approximate accidental symmetry. The R-parity violating terms will be determined in terms of the flavor parameters of the theory, giving an underlying theory for these parameters. [11]

In the absence of neutrino masses only a single renormalizable R-parity violating flavor structure is allowed, and the proton is effectively stable, while $n - \bar{n}$ oscillations and dinucleon decay are sufficiently suppressed with mild restrictions on $\tan\beta$. In the presence of neutrino masses there are more R-parity violating spurions, including a cubic superpotential term, and quadratic Kähler and soft breaking terms. Proton decay will now place a mild bound on the right handed neutrino mass scale.

The phenomenology of the model depends strongly on the nature of the LSP. The most plausible candidate for the LSP is the stop, which can decay to two quarks via the R-parity violating superpotential term. If the LSP is a neutralino/chargino, the decay might include displaced vertices and top quarks, while a slepton LSP would most likely decay with displaced vertices, and might also involve missing energy. While the LSP is necessarily unstable in such models, a gravitino LSP is sufficiently long lived to be a dark matter candidate.

[11] For other theories of the R-parity violating terms see [255, 256].

There are a number of interesting directions for future work. The constraints on MFV SUSY arising from dinucleon decay are nontrivial, and a better understanding of the relevant hadronic matrix elements would help to establish a robust set of bounds on the parameter space of the model, as well as clarifying how the model can be probed using low energy observables. Detailed collider studies are needed to determine the cleanest experimental signatures of this model at the LHC, especially in light of the various possibilities for the LSP. Furthermore, the cosmological implications for baryogenesis and dark matter should be explored in detail.

Finally, possible UV completions of the model should be explored, see e.g. chapter 7. In R-parity conserving models, MFV is usually applied only to the SUSY breaking terms, which can be motivated by RGE evolution from flavor-universal soft terms, as in gauge mediation scenarios. In MFV SUSY, however, it is necessary to apply the MFV hypothesis to the superpotential as well, which cannot be similarly motivated. Nonetheless, an MFV structure can arise from weakly broken flavor symmetries, and constructing a well-motivated UV completion should prove to be an interesting challenge. If such a model can be found, it would give more information about the unknown flavor-singlet parameters.

MFV SUSY is a highly constrained theory, and its structure allows for a systematic approach to many problems. We outline several examples of this in these appendices. In appendix 5.A, we show that the form of the superpotential is highly constrained by systematically classifying holomorphic flavor singlets. In appendix 5.B, we examine the effect of supersymmetry breaking on arguments based on holomorphy. In appendix 5.C, we develop a heuristic scheme for estimating the flavor suppression of a given diagram, and in appendix 5.D we apply this technique to demonstrate that the diagrams presented in section 5.4 and section 5.6 are the leading contributions to low energy baryon-number violating observables. Finally, in appendix 5.E, we show that higher dimensional baryon and lepton-number violating operators are not dangerous for a sufficiently high cutoff $\Lambda \leq M_{GUT}$.

5.A Classifying holomorphic flavor singlets

To classify all terms which can appear in the superpotential, we now systematically construct all holomorphic flavor singlets, treating the spurions as holomorphic. In the quark sector, the irreducible holomorphic $\mathrm{SU}(3)_u \times \mathrm{SU}(3)_d$ singlets are $Y_u \bar{u}$, $Y_d \bar{d}$, $\det \bar{u}$, $\det \bar{d}$, and the flavor-singlet spurions $\det Y_{u,d}$. Ignoring the flavor singlet spurions, and combining $Y_u \bar{u}$ and $Y_d \bar{d}$ with Q to form $\mathrm{SU}(3)_Q$ singlets, it is straightforward to show that Table 5.3 contains a complete list of the irreducible $\mathrm{SU}(3)_Q \times \mathrm{SU}(3)_u \times \mathrm{SU}(3)_d$ singlets.

The lepton sector is more complicated. We first write down all possible holomorphic $\mathrm{SU}(3)_N$ singlets. Note that for any 3×3 matrix M

$$M^{ij} M^{kl} \varepsilon_{ikm} = \varepsilon^{jln} \tilde{M}_{nm} \,, \tag{5.69}$$

where $\tilde{M}$ is the matrix of cofactors, satisfying $\tilde{M} M = M \tilde{M} = (\det M) \mathbb{1}$. Thus, while in general a flavor singlet can contain an arbitrary number of ε-tensors, by repeated application of (5.69) we can reduce such a singlet to a form where no two M_N's, $\tilde{M}_N$'s, Y_N's, or $\tilde{Y}_N$'s are contracted with the same $\mathrm{SU}(3)_N$ ε-tensor, apart from factors of $\det Y_N$ and $\det M_N$. Since at most one $\bar{N}$ can contract with a given ε-tensor, the only surviving ε-tensors must be contracted as follows:

$$\tilde{M}_N^{ij} \tilde{Y}_a^k \bar{N}^l \epsilon_{jkl} = -\varepsilon_{abc} (\tilde{M}_N^{ij} Y_j^b)(Y_k^c \bar{N}^k) \,, \tag{5.70}$$

which is a reducible product of $\mathrm{SU}(3)_N$ singlets. Incorporating $Y_e \bar{e}$ and L, we obtain a relatively short list of irreducible $\mathrm{SU}(3)_N \times \mathrm{SU}(3)_e$ singlets, as shown in Table 5.7.

	$\mathrm{SU}(2)_L$	$\mathrm{U}(1)_Y$	$\mathrm{SU}(3)_L$	$\mathbb{Z}_2^R$
$\bar{N}M_N\bar{N}$	$\mathbf{1}$	0	$\mathbf{1}$	$+$
$Y_N\bar{N}$	$\mathbf{1}$	0	$\square$	$-$
$Y_e\bar{e}$	$\mathbf{1}$	1	$\square$	$-$
L	$\square$	$-1/2$	$\bar{\square}$	$-$
$\tilde{Y}_N M_N\bar{N}$	$\mathbf{1}$	0	$\bar{\square}$	$-$
$\tilde{Y}_N M_N\tilde{Y}_N$	$\mathbf{1}$	0	$\overline{\square\square}$	$+$
$Y_N\tilde{M}_N Y_N$	$\mathbf{1}$	0	$\square\square$	$+$

Table 5.7: The irreducible $\mathrm{SU}(3)_N \times \mathrm{SU}(3)_e$ singlets (we omit flavor-singlet spurions.)

The next step is to classify irreducible $\mathrm{SU}(3)_L$ singlets. Note that

$$(\tilde{Y}_N M_N \tilde{Y}_N)(Y_N \tilde{M}_N Y_N) = (\det Y_N)^2 (\det M_N)\mathbb{1}\,. \tag{5.71}$$

Thus, up to normalization, $\tilde{Y}_N M_N \tilde{Y}_N$ is the matrix of cofactors of $Y_N \tilde{M}_N Y_N$, and we can omit singlets containing more than one of either contracting with the same $\mathrm{SU}(3)_L$ ε-tensor. There is then a finite list of possible irreducible flavor singlets. Of these, some will be reducible due to the identities satisfied by Y_N and $\tilde{Y}_N$ and M_N and $\tilde{M}_N$. For instance, any contraction involving $Y_N\tilde{Y}_N$ or $\tilde{Y}_N Y_N$ is obviously reducible, since $Y_N\tilde{Y}_N = \tilde{Y}_N Y_N = (\det Y_N)\mathbb{1}$. Furthermore, certain ε-tensor contractions of Y_N with itself or $\tilde{Y}_N$ with itself will be reducible. In particular, we have

$$\begin{aligned}
(Y_N\bar{N})(Y_N\tilde{M}_N Y_N)(Y_N\bar{N}) &\sim \bar{N}\tilde{Y}_N\tilde{M}_N\tilde{Y}_N\bar{N} \\
&\sim (\bar{N}M_N\tilde{Y}_N)(\tilde{Y}_N M_N\bar{N}) - (\bar{N}M_N\bar{N})(\tilde{Y}_N M_N\tilde{Y}_N)\ , \\
(\tilde{Y}_N M_N\tilde{Y}_N)\tilde{Y}_N M_N\bar{N} &\sim (\det Y_N)(M_N\tilde{Y}_N)(M_N\bar{N})Y_N \ \sim\ (\det Y_N)Y_N\tilde{M}_N\tilde{Y}_N\bar{N} \\
&\sim (\det Y_N)(Y_N\tilde{M}_N Y_N)(Y_N\bar{N})\ ,
\end{aligned} \tag{5.72}$$

up to unimportant factors.

Keeping these reductions in mind, it is straightforward to verify that Table 5.6

contains a complete list of $\mathrm{SU}(3)_L \times \mathrm{SU}(3)_e \times \mathrm{SU}(3)_N$ invariants, apart from $LY_e\bar{e}$, which appears in Table 5.3.

5.B Nonholomorphic operators from SUSY breaking

In the absence of supersymmetry breaking, the superpotential is constrained to be holomorphic, and only holomorphic combinations of spurions can appear there. We now explore the role of supersymmetry breaking in introducing nonholomorphic spurion combinations into the superpotential. To keep the discussion of supersymmetry breaking generic, we introduce a supersymmetry-breaking spurion X, a chiral superfield which acquires an F-term vev $\langle X\rangle_F = F$. We assume that X couples to the MSSM fields via nonrenormalizable operators, where the cutoff M is the messenger scale.

The resulting soft supersymmetry breaking terms will appear a scale $m_{\text{soft}} \sim F/M$. In particular, since we assume the absence of renormalizable couplings between X and the MSSM, the leading contributions to supersymmetry breaking come from the superpotential interactions

$$W_{\not{S}\text{USY}} \supset \frac{X}{M} A_{ijk}\Phi^i\Phi^j\Phi^k + \frac{X}{M} M_\lambda^{(i)}\mathrm{Tr}W_{(i)}^2 \tag{5.73}$$

and the Kähler potential interactions:

$$K_{\not{S}\text{USY}} \supset \frac{X^\dagger}{M}\tilde{\mu}_{ij}\Phi^i\Phi^j + \frac{X}{M}\tilde{J}_i^j\Phi^i\Phi_j^\dagger + \frac{X^\dagger X}{M^2}\tilde{B}_{ij}\Phi^i\Phi^j + c.c. + \frac{X^\dagger X}{M^2}\tilde{M}_i^j\Phi^i\Phi_j^\dagger \tag{5.74}$$

where nonholomorphic couplings are denoted with a tilde. The couplings A_{ijk} and M_λ generate A-terms and gaugino masses, whereas $\tilde{B}_{ij}$ and $\tilde{M}_i^j$ generate B-terms and soft-masses, $\tilde{\mu}_{ij}$ generates bilinear superpotential terms, and $\tilde{J}_i^j$ gives rise to a scalar/F-term mixing, the effects of which we discuss in detail below. In singlet extensions of the MSSM, including the NMSSM and see-saw models, supersymmetry breaking tadpoles can also arise:

$$K^{(\mathrm{tad})}_{\not{S}\mathrm{USY}} = \frac{X^\dagger X}{M}\tilde{E}_i\Phi^i \tag{5.75}$$

where the dimensionful coefficient is large, $F^2/M \sim M m_{\mathrm{soft}}^2 \gg m_{\mathrm{soft}}^3$. While these tadpoles are potentially problematic, whether they are generated and at what level will depend on the particular model of supersymmetry breaking. We will assume that they are suppressed by some mechanism, and will not consider them further.[12]

Thus, we conclude that A-terms are generated holomorphically, whereas the other soft terms are generated non-holomorphically. Furthermore, nonholomorphic bilinear couplings can appear in the superpotential at the scale m_{soft}. Non-holomorphic contributions to the A-terms and trilinear superpotential terms are suppressed. The leading contributions arise from the interactions

$$K_{\not{S}\mathrm{USY}} \supset \frac{X^\dagger}{M^2}\tilde{\lambda}_{ijk}\Phi^i\Phi^j\Phi^k + \frac{XX^\dagger}{M^3}\tilde{A}_{ijk}\Phi^i\Phi^j\Phi^k$$

which are suppressed by $\mathcal{O}\left(m_{\mathrm{soft}}/M\right)$ relative to the leading holomorphic contributions.

So far we have ignored the nonholomorphic scalar/F-term mixing $\tilde{J}_i^j$. We will show that these couplings give rise to nongeneric nonholomorphic contributions to the A-terms after a field redefinition, similar in form to (nonholomorphic) wave-function renormalization effects.

We first write the renormalizable superpotential and Kähler potential in the form:

$$\begin{aligned} W &= m_{\mathrm{soft}}\mu_{ij}\Phi^i\Phi^j + \lambda_{ijk}\Phi^i\Phi^j\Phi^k \\ K &= \tilde{K}_i^j\Phi^i\Phi_j^\dagger \end{aligned}$$

[12]For instance, a right-handed snuetrino tadpole is forbidden by $\mathbb{Z}_3^{(L)}$ in the case of Dirac neutrino masses ($M_N = 0$).

where $\tilde{K}_i^j$ is the Hermitean positive-definite Kähler metric. (Note that we cannot in general set $\tilde{K}_i^j = \delta_i^j$ by a field redefinition without introducing nonholomorphic couplings into the superpotential.) The scalar/F-term mixing can be eliminated by redefining

$$\Phi^i \to \Phi^i + \frac{X}{M}\tilde{P}_j^i \Phi^j$$

for $\tilde{P}_j^i = -\left[\tilde{K}^{-1}\right]_k^i \tilde{J}_j^k$. This redefinition produces additional A-terms of the form:

$$W_{\not{S}\mathrm{USY}} \supset \frac{X}{M}\left[\lambda_{ljk}\tilde{P}_i^l + \lambda_{ilk}\tilde{P}_j^l + \lambda_{ijl}\tilde{P}_k^l\right]\Phi^i\Phi^j\Phi^k$$

as well as corrections to the soft-masses and B-terms.

By contrast, writing the Kähler potential in the form

$$\tilde{K}_j^i = \delta_j^i + \tilde{k}_j^i$$

and assuming that $\tilde{k}_j^i$ is a subleading correction, we obtain similar nonholomorphic corrections to the superpotential itself (as well as the A-terms) upon moving to a canonical basis. Thus, we conclude that the qualitative effects of nonholomorphic scalar/F-term mixing are captured by nonholomorphic corrections to the Kähler potential, though $\tilde{J}_j^i$ leads to some additional "splitting" between the A-terms and superpotential terms.

5.C A heuristic estimation scheme

In section 5.4 and section 5.6, we estimated the dominant contribution to low-energy baryon-number violating processes by choosing the simplest diagrams and then finding the dominant flavor structure. The resulting diagrams were heavily suppressed by Yukawa couplings, CKM factors, and heavy propagators. Thus, in principle other diagrams could give competitive contributions. However, classifying all possible diagrams is a difficult task. Instead, we develop a scheme to estimate the flavor-suppression of a diagram based on its flavor structure alone. This will allow us to isolate potentially competitive diagrams, which can then be computed by more conventional means.

To do so, it is helpful to reinterpret a Feynman diagram for a candidate process in terms of the flow of "flavor," i.e. of $\mathrm{SU}(3)_Q \times \mathrm{SU}(3)_u \times \mathrm{SU}(3)_d$ charge. If quarks and squarks carry "flavor" and anti-quarks and anti-squarks carry "anti-flavor," then flavor can only be created or destroyed at baryon number violating vertices, such as (5.4). Otherwise, the rest of the diagram contains unbroken flavor lines, which either form closed loops or join to external quark lines.

Along flavor lines, flavor is altered through left $\leftrightarrow$ right mixing, charged CKM mixing, and neutral squark mass mixing, where each subprocess has an associated cost. In particular, for squarks, left $\leftrightarrow$ right mixing is suppressed by the associated Yukawa coupling, whereas charged-current flavor changing (on left-handed squarks) is CKM suppressed. FCNCs are suppressed by (5.18), and flavor changing of right-handed squarks is suppressed by the associated Yukawa couplings to convert them to left-handed squarks, together with the suppression for left-handed flavor changing.

If we assume similar suppressions for flavor-changing processes involving quarks, we obtain a useful heuristic estimate scheme for the MFV-dictated flavor-suppression of any given diagram. In particular, the least suppressed diagrams for a given process will involve a minimum number of baryon-number violating vertices, and a minimum of flavor changing. For each baryon number violating vertex (5.4), all three flavor lines should connect to external quarks; otherwise the diagram involves extra insertions of (5.4), and is subdominant.

Thus, we can estimate the amplitude for the diagram by specifying the flavor structure, by which we mean the flavors of the right-handed quarks/squarks connected to the baryon-number violating vertex (5.4), as well as the flavors of the external quarks on the flavor lines emanating from the BNV vertex. In addition to the vertex factor, the required charged and/or neutral flavor changes then come with a right $\to$ left Yukawa suppression, together with a CKM suppression for charged flavor-changing or a suppression of the form (5.17) for neutral flavor-changing, whereas quarks/squarks which do not change flavor receive no additional Yukawa suppression.

Given a flavor structure, the heuristic estimation scheme outlined above should give an approximate upper bound on the amplitude, once suppression from the superpartner propagators and loop suppression (if applicable) is accounted for. As the number of possible flavor structures is finite, and much smaller than the number of possible diagrams, it becomes straightforward to obtain an approximate upper bound on the amplitude for all relevant flavor structures.

If we can find a diagram with amplitude equal to the upper bound, then this diagram is probably the dominant contribution to the process in question. The simplest diagrams will often involve only squark flavor-changing, since otherwise additional W bosons are required. In this case, the heuristic scheme outlined above is essentially exact (up to unknown MFV coefficients, which are assumed to be order one). However, if quark flavor changing is involved, the amplitude is somewhat dependent on the details. In particular, while CKM suppression is still present, Yukawa suppression is less obvious. We now consider this point in detail.

For a light quark, the left $\leftrightarrow$ right propagator takes the approximate form m_q/E^2, where $E \sim \Lambda_{QCD} \gg m_q$ is the characteristic energy for the baryon-number violating process. By contrast, for a heavy quark ($m_q \gg E$) the left $\leftrightarrow$ right propagator will take the approximate form $1/m_q$. In either case, the contribution to the overall amplitude will be made dimensionless by a factor of $\sim E$ in the numerator, arising either from loop integrals or from a hadronic matrix element.

Thus, the overall left $\leftrightarrow$ right suppression appears to be only m_q/E and E/m_q for light and heavy quarks respectively, whereas (for light quarks), the assumed Yukawa suppression is much smaller. However, in general left $\leftrightarrow$ right mixing will be followed by charged flavor-changing — this is the reason for including it in the diagram — with an associated $g^2/M_W^2 = 2/v^2$ from the W boson propagator, where the dimensions will again be cancelled by factors of E. Counting one-half of the W propagator suppression. (the other end of W boson line will lead to flavor changing elsewhere in the diagram), we obtain a net suppression of approximately

$$\frac{m_q}{v/\sqrt{2}} \quad \text{or} \quad \frac{E^2}{m_q\, v/\sqrt{2}} \ , \tag{5.76}$$

for light and heavy quarks, respectively. Thus, for a light quark, the net suppression is the same as Yukawa suppression (for $\tan\beta = 1$), whereas for a heavy quark, the diagram is suppressed by an additional factor of $\sim (E/m_q)^2$. At large $\tan\beta$, the suppression is greater than Yukawa suppression for all quarks except for the up-quark, but the difference here is only $1/\sqrt{2}$, and is effectively negligible.

The above argument is more subtle in the case of a loop diagram, since q^2 within the loop may be much higher than Λ^2_{QCD}. Roughly, the net effect is to change the distinction between "light" and "heavy" quarks; for instance, if $q^2 \sim M_W^2$ within the loop, then only the top quark is "heavy." Yet more subtleties arise for flavor-changing neutral currents of right-handed quarks, since there are then more mass insertions than W vertices. However, the discrepancy is not very important if the mass insertions lie within a loop dominated by loop momentum $q^2 \gtrsim M_W^2$. Thus, the estimation scheme outlined above also applies qualitatively to quark flavor changing, where the Yukawa suppression now comes partly from W boson propagators and/or loop suppression. Although the exact amplitude will depend on the specifics, this heuristic scheme is a useful way to isolate the larger diagrams contributing to a process of interest.

5.D A systematic search for additional large diagrams

We now apply the estimation scheme developed in appendix 5.C to search for additional large diagrams which are potentially competitive with those considered in section 5.4 and section 5.6.

5.D.1 $n - \bar{n}$ oscillations

We first consider $n - \bar{n}$ oscillations. The amplitude must be built from two insertions of (5.4), each of which carries at least one second-generation down-type quark/squark, with all flavor lines connected to external quarks (there are no spectator quarks). As the external quarks are precisely two up-quarks and four down-quarks, the second and third generation quarks must all flavor change to first generation quarks. Furthermore, converting the two squarks into quarks requires the exchange of at least one gaugino or higgsino; any additional three or four-point interactions can only be present at one-loop or higher.

Due to the strong Yukawa suppression of the tree level amplitude (5.20), it is conceivable that one-loop amplitudes can be competitive with it. We now search for the largest such diagrams. In any $n-\bar{n}$ oscillation diagram of interest, the external quarks must all be first generation quarks. Thus, for a given flavor structure for the BNV vertex, we can estimate a minimum flavor suppression by assuming charged flavor-changing to first-generation quarks for each leg, since neutral flavor changing is never dominant over charged flavor changing in this context. The resulting flavor-dependent minimum suppression is shown in Table 5.8. It is straightforward to check that, for the assumed range $3 \lesssim \tan\beta \lesssim 45$, $tds \sim tbd$ gives the weakest suppression, whereas the next weakest, cbd, is $\lesssim 1/20$ as large.

There is only one possible one-loop diagram with two flavor-changing quarks (Fig. 5.14). Assuming that the dominant contribution to the loop integral occurs

	$s\,b$	$b\,d$	$d\,s$
u	$y_u y_s^2 y_b^2 \lambda^4/2$	$y_u y_b^2 y_d \lambda^4/2$	$y_u y_d y_s^2 \lambda^4/2$
c	$y_c^2 y_s^2 y_b^2 \lambda^6/2$	$y_c^2 y_b^2 y_d \lambda^4/2$	$y_c^2 y_d y_s^2 \lambda^4$
t	$y_s^2 y_b^2 \lambda^{10}/2$	$y_b^2 y_d \lambda^8/2$	$y_d y_s^2 \lambda^4$

Table 5.8: The minimum flavor-dependent suppression required for flavor changing to first generation quarks, where the rows and columns correspond to the flavors of the quarks/squarks attached to the BNV vertex.

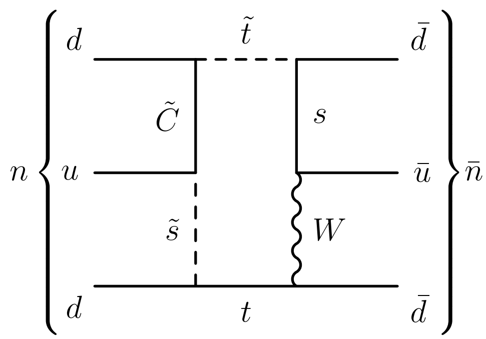


Figure 5.14: The leading one-loop contribution to $n - \bar{n}$ oscillation.

in the range $M_W^2 \lesssim q^2 \lesssim m_t^2$, we estimate:

$$\mathcal{M} \sim \frac{g^2}{16\pi^2}\tilde{\Lambda}\, t_\beta^5\, \lambda^8 \frac{m_d^2 m_s^4}{m_t^6} \left(\frac{\tilde{\Lambda}}{m_{\tilde{q}}}\right)^4 \left(\frac{\tilde{\Lambda}}{m_\chi}\right), \tag{5.77}$$

for two tds vertices, where the $\tan\beta$ dependence is less strong than our naive estimate because the strange-quark left $\leftrightarrow$ right mass insertion is not enhanced at large $\tan\beta$, unlike the corresponding Yukawa coupling. While (5.77) is competitive with (5.20) at $\tan\beta = 3$, it grows more slowly at large $\tan\beta$, and becomes subdominant. Other combinations of tds and tbd give a similar result. Since other flavor structures ought to lead to further suppression, we conclude that the tree-level result (5.20) is the dominant contribution to $n - \bar{n}$ oscillations at large $\tan\beta$, where the predicted oscillation time is closest to present experimental bounds.

5.D.2 Dinucleon decay

We now consider additional contributions to dinucleon decay. Conservation of electric charge requires that each up-type $\to$ down-type flavor change has a corresponding down-type $\to$ up-type flavor change, which can in principle occur on one of the "spectator" flavor lines (those not connected to the BNV vertices). However, each such occurrence is strongly suppressed – by about $g\Lambda_{\mathrm{QCD}}/M_W$ – due to the W boson propagator, since at most half of the propagator suppression accounts for necessary Yukawa suppression on the "primary" flavor lines (those connected to the BNV vertices), as discussed in appendix 5.C.

Keeping this suppression in mind, we can search for additional large diagrams by exhaustively cataloging the possible flavor structures for each BNV vertex, grouped together on the basis of the flavors of their external light quarks, estimating the suppression for each flavor structure according to the scheme of appendix 5.C. To find the largest diagrams, we find the least suppressed flavor structures for each set of external quarks, and then take the products of all pairs of these suppressions, bearing in mind that for final-state strangeness $|S| \geq 3$, two-body decays are not possible (leading to phase-space suppression), and appending a factor of $\sim g\Lambda_{\mathrm{QCD}}/M_W$ for each unit of net charge of the external quarks.

Besides the two diagrams already considered in section 5.4.2, such a search turns up no flavor structures with a lesser suppression for any $3 \lesssim \tan\beta \lesssim 45$. Thus we conclude that, to the extent to which the scheme of appendix 5.C is valid, the two dominant diagrams are the charged and neutral flavor-changing diagrams already considered.

5.D.3 Proton decay

Finally, we consider additional contributions to proton decay. In the quark sector, we require a single baryon number violating vertex (5.4), with a corresponding squark propagator suppression. Requiring that the external quarks be light with strangeness $|\Delta S| \leq 1$ and applying the method of appendix 5.C, we find that a tds vertex with $t \to d$ flavor-changing is the least suppressed, with $t \to u$ neutral flavor-changing competitive at large $\tan\beta$. These are the same flavor structure that were considered in section 5.6.

However, as argued in section 5.6, the charged-lepton diagram suffers from a chiral suppression. This will occur whenever the squark is up-type and undergoes charged flavor changing, emitting an ℓ^- (via mixing with the chargino), i.e. when the net-charge of the external quarks connected to the baryon-number violating vertex is -1, since charge conservation otherwise requires the exchange of a W boson with one of the spectator quarks, resulting in a comparable suppression, as disucussed in section 5.D.2. Accounting for the chiral suppression and reapplying the methods of appendix 5.C, we conclude that the neutral flavor-changing diagram considered in section 5.6 is always dominant.

As the bounds on $|\Delta S| = 0$ decays are somewhat stronger, one might be tempted to consider diagrams of this type. However, according to our estimation scheme, the largest $|\Delta S| = 0$ processes — tbd with $b \to u, d$ and $t \to d$ flavor changing or tds with $t \to d$ and $s \to u$ flavor changing — receive an additional flavor suppression of about $y_s\lambda$, or at least 10^{-2} for the assumed range $3 \lesssim \tan\beta \lesssim 45$. Consequently, $|\Delta S| = 1$ decays are strongly preferred, and their non-observation will lead to the strongest constraints.

5.E Higher dimensional operators

We now consider whether higher-dimensional operators can affect our conclusions. We first consider $|\Delta B| = 2$ processes. Lepton-number violating interactions are irrelevant, since they are strongly suppressed by Y_N and $\mu_N = M_N/\Lambda_R$. At dimension five, there is only one allowed baryon-number violating correction, which appears in the Kähler potential:

$$K^{(5)}_{BNV} = \frac{1}{\Lambda}(Y_u Y_u^\dagger + Y_d Y_d^\dagger) Q Q Y_d^\dagger \bar{d}^\dagger \,. \tag{5.78}$$

After integrating out the auxiliary fields, this term (combined with the $QY_d\bar{d}H_d$ Yukawa coupling), has a similar effect to a Q^3H_d superpotential term, but with at least two Y_d spurions, leading to a minimum Yukawa suppression of y_b^2. Together with the dimension-five $\sim v/\Lambda$ suppression and CKM suppression (of the same form as for (5.4)), it is straightforward to check that the vertex factor must be substantially smaller than any of those contributing to the dominant diagrams considered in section 5.4 — in the latter case we also include any additional suppression from flavor changing — so long as $\Lambda \gtrsim 10^{12}$ GeV.[13] Thus, for a GUT scale cutoff, such contributions are strongly subdominant, whereas dimension six and higher operators are sufficiently suppressed without any flavor suppression.

In the case of nucleon decay, higher-dimensional $|\Delta L| = 1$ operators are potentially dangerous. However, they necessarily come with a suppression of at least $\mu_N Y_N^2$ (ignoring flavor structure) in addition to their $\sim v/\Lambda$ cutoff suppression, and are therefore subdominant to the lepton-gaugino mixing induced by the $\mathcal{V}^{(2)}$ spurion. Thus, for a high cutoff, higher dimensional lepton-number violating operators can only be significant if they lead to an enhancement in the quark sector. Specifically, operators which violate lepton *and* baryon number can be dangerous, but these occur first at dimension six, both in the Kähler potential and the superpotential. Notably, the dangerous (R-parity even) dimension-five operators Q^3L, $\bar{u}\bar{u}\bar{d}\bar{e}$, and $\bar{u}\bar{d}\bar{d}\bar{N}$ are absent from the superpotential due to holomorphy constraints. Dimension six operators are not dangerous in this context, since the smallness of $\mathcal{V}$ spurion (cf. (5.46)) combined with cutoff suppression is sufficient to easily evade bounds on the proton lifetime.

[13] A more detailed analysis might reveal that an even lower cutoff is permissible.

CHAPTER 6

MESINO OSCILLATION IN MFV SUSY

R-parity violating supersymmetry in a Minimal Flavor Violation paradigm can produce same-sign dilepton signals via direct sbottom-LSP pair production.[1] Such signals arise when the sbottom hadronizes and the resulting mesino oscillates into an anti-mesino. The first bounds on the sbottom mass are placed in this scenario using current LHC results.

6.1 Introduction

The 2011 and 2012 data from the Large Hadron Collider (LHC) place severe constraints on natural R-parity conserving models of supersymmetry (SUSY) [257–259]. While such models are not excluded by the data, if they are to solve the hierarchy problem of the Standard Model (SM), they are forced to have either non-generic spectra where only third-generation squarks are light [260–265] or nearly degenerate particles, either in the form of stealth SUSY [266] or a squashed [267] spectrum. On the other hand, the stubborn agreement between SM predictions and observations in channels with large missing transverse energy (MET) cuts may indicate that the assumption of exact R-parity conservation is incorrect.

Models with R-parity violation (RPV) have been proposed since the early days of SUSY [203–209]. A possible connection between the problems of baryon and lepton number violation and large flavor changing operators was highlighted in chapter 5, see also [216, 217]. The assumption of Minimal Flavor Violation (MFV) has been shown to be sufficient to prevent both rapid nuclear decay (and other baryon-number violating processes) and large corrections to flavor observables in the B, D, and K systems. In models of MFV SUSY, sparticles are pair produced as

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in R-parity conserving models, while the lightest supersymmetric partner (LSP) is generally unstable on collider scales and will decay via the baryon-number violating $u^c d^c d^c$ superpotential term. In this chapter, we investigate one of the interesting scenarios that can arise in the model introduced in chapter 5.

In MFV SUSY models it is particularly compelling to consider the case when the LSP is a third generation squark: naturalness requires light third generation squarks in general, and we will see below that in the MFV scenario there is a high probability for this to be actually realized, due to the large top Yukawa coupling. The phenomenology of this scenario is also particularly rich, as the lifetime of an LSP stop or sbottom is long enough that the squark hadronizes to form a mesino by binding to a light quark pulled from the vacuum. It is, however, usually sufficiently short-lived to decay before reaching the detector. Observing squark production is challenging in this scenario, due to the lack of any obvious handles on the events, such as missing energy or displaced vertices. Instead we will make use of the idea of mesino-antimesino oscillations, following Sarid and Thomas [268]. We will demonstrate that sbottom-LSP pair production often allows for mesino-antimesino oscillations, which may lead to same-sign dilepton signals.

A sbottom LSP decays dominantly to a top quark and a strange quark, see chapter 5. If one of the sbottom mesinos oscillates before decaying, the tops will be of the same charge, and if both tops decay leptonically this leads to same-sign leptons. These events would also contain b quarks from the top decays, providing further handles on the event. Recently, CMS searched for such events and placed bounds on their cross sections [269]. We will show that this CMS search already places some bounds on sbottoms, which should improve significantly with more data.

The rest of this chapter is organized as follows. In section 2, we study the typical squark spectra in MFV SUSY scenarios and demonstrate that the stop and sbottom are most often the lightest squarks. In section 3, we present a calculation of the decay rate and oscillation time for a squark LSP in MFV SUSY and show that a significant oscillation probability is possible and occurs frequently. In section 4, we comment on the sensitivity of existing LHC searches to this scenario. We conclude in section 5. The details of the calculation of the mesino-antimesino oscillation rate is given in the appendix.

6.2 MFV Squark Spectra

In an MFV SUSY model the LSP decays and we are not restricted to models with a neutralino LSP, whereas the phenomenology of the model will depend on the identity of the LSP. In particular, the LSP can be colored and, as we consider below, can be a squark.

MFV requires that all flavor violation be proportional to the appropriate combination of Yukawa matrices, which are treated as spurions of the flavor symmetry. The squark mass matrices are then required to have the following form (cf. (5.52)):

$$M_{\tilde{u}}^2 = \begin{pmatrix} m_{\tilde{q}}^2 \mathbb{1} + (a_q + v_u^2) Y_u Y_u^\dagger + b_q Y_d Y_d^\dagger + D_{u_L} & A_u Y_u \\ A_u^* Y_u^\dagger & m_{\tilde{u}}^2 \mathbb{1} + (a_u + v_u^2) Y_u^\dagger Y_u + D_{u_R} \end{pmatrix}, \tag{6.1}$$

and

$$M_{\tilde{d}}^2 = \begin{pmatrix} m_{\tilde{q}}^2 \mathbb{1} + a_q Y_u Y_u^\dagger + (b_q + v_d^2) Y_d Y_d^\dagger + D_{d_L} & A_d Y_d \\ A_d^* Y_d^\dagger & m_{\tilde{d}}^2 \mathbb{1} + (a_d + v_d^2) Y_d^\dagger Y_d + D_{d_R} \end{pmatrix}. \tag{6.2}$$

The D terms are automatically flavor diagonal and given by

$$D_L = \left(T^3 - Q s_w^2\right) \cos(2\beta)\, m_Z^2\,, \qquad D_R = Q s_w^2 \cos(2\beta)\, m_Z^2\,, \tag{6.3}$$

where $Q = +2/3$ $(-1/3)$ and $T^3 = +1/2$ $(-1/2)$ for the up-type (down-type) squarks, s_w is the sine of the Weinberg angle, $\tan\beta$ is the ratio of the Higgs VEVs,

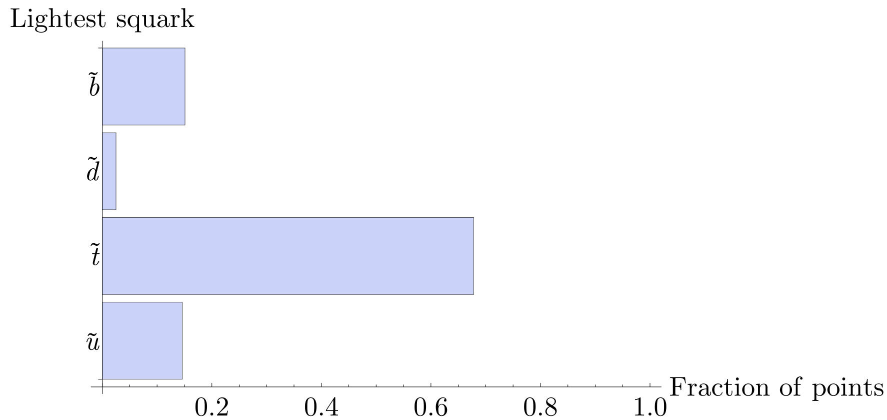

Figure 6.1: Distribution of lightest squark flavor over a random sampling of MFV SUSY parameter space.

and m_Z is the mass of the Z. The parameters m_i^2, a_i, b_i, and A_i arise from supersymmetry breaking, and we therefore expect them to be of order m_{soft}^2.

Given these constraints, we can perform a scan over the parameter space that determines the squark spectrum. We select random values for the undetermined dimension-two parameters uniformly in $[-m_{\text{soft}}^2, m_{\text{soft}}^2]$. For this scan, we choose $m_{\text{soft}} = 1$ TeV and $\tan\beta = 10$. The overall result is not very sensitive to this choice. We impose the constraint that the smallest eigenvalues of both squark mass matrices be greater than the top mass, $m_t \approx 175$ GeV. In general, left-right mixing is not too large and we therefore use notation where $\tilde{b}_L$ refers to the mass eigenstate of the sbottom that is mostly a left-handed sbottom. We also impose that the lightest stop-like squark be lighter than 500 GeV as demanded by naturalness. Under these conditions, the distribution of lightest squark flavors is given by Fig. 6.1.

We observe that roughly 85 % of parameter points have a third-generation lightest squark, out of which 15 % have a sbottom squark at the bottom of the spectrum. The large likelihood of a third generation lightest squark can be explained by the relatively large left-right mixing for this generation. This mixing tends to drive the mass of the lighter third generation squark down, making it more likely to be lightest overall. (There is also a significant contribution from the naturalness cut, since requiring one light stop tends to reduce the incidence of both stops being made heavy by a large positive a_q.) Note that at large $\tan\beta$ this effect is enhanced for the sbottoms, making it even more likely to get a sbottom LSP. It is therefore natural to consider signatures of a sbottom LSP in MFV SUSY and we do so from this point on.

6.3 Mesino oscillation in MFV SUSY

The MFV SUSY scenario offers a rich phenomenology due to the naturally small decay width of the LSP, a consequence of approximate R-parity conservation. The couplings are sufficiently small to yield LSP lifetimes that are longer than the timescales of SM short-distance physics, such as hadronization, yet often shorter than the timescales set by macroscopic distances in the LHC detectors. In this intermediate range, it can be difficult to construct observables that are not overwhelmed by SM background. If the LSP carries color, however, then it lives sufficiently long to hadronize, an intriguing possibility. This process can yield additional phenomena that allow for efficient selection of SUSY events.

The case of a sbottom LSP is particularly fruitful. If the gluino is heavy, the dominant SUSY production mode will be sbottom pair production. The dominant decay of the sbottom in MFV SUSY is to top and strange. The top has a leptonic decay mode, which already suppresses many SM backgrounds. As we show, the fact that the sbottom hardronizes allows for the possibility of sbottom oscillations, which lead, some fraction of the time, to same-sign lepton events.

While other squark flavors can also oscillate, this turns out to be parametrically rarer. In addition, up-type squark LSPs do not decay leptonically, precluding the possibility of a same-sign dilepton signature. We do not consider these other

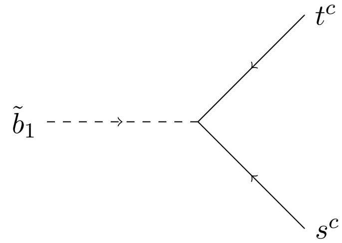

Figure 6.2: The leading diagram for the R-parity violating sbottom decay.

possibilities further in this chapter.

We also do not consider the case where gluino pair production is significant. This would lead to additional same-sign lepton events due to the Majorana nature of the gluino, providing a background to the case we are considering.

We begin by calculating the sbottom decay width. We denote the lightest sbottom mass eigenstate by $\tilde{b}$. Its decay width depends on an overall (generically order 1) coefficient that we denote by λ''. The Lagrangian terms that gives the decay are (cf. section 5.3):

$$\mathcal{L} = -(\lambda'')^* \epsilon_{ijk} \frac{m_q}{v c_\beta} U^D_{q_R,1} \frac{m_{u,j}}{v s_\beta} V_{j'j} \frac{m_{d,k}}{v c_\beta} \tilde{b}_1 u^{c\dagger}_{j'} d^{c\dagger}_k \sim -(\lambda'')^* V_{td} \frac{m_b m_s m_t}{v^3 c^2_\beta s_\beta} \tilde{b}_1 t^{c\dagger} s^{c\dagger}, \quad (6.4)$$

where $v = 174$ GeV, V is the CKM matrix, and we use $\tilde{b}_1$ to denote the lightest down-type squark, which we assume is predominantly sbottom-like. The mixing matrix U^D is defined such that

$$\tilde{q}_q = U^D_{q,i} \tilde{q}_i. \quad (6.5)$$

In this notation, $\tilde{q}_q$ are the squark flavor-basis fields in the mass basis of the quarks and $\tilde{q}_i$ are the squark mass-basis fields. The approximation is valid if the lightest sbottom is mostly right-handed. Otherwise, there is an additional suppression from the left-right mixing. The partial decay width can then be calculated using the diagram in Fig. 6.2. The result (neglecting the mass of the down quark in the

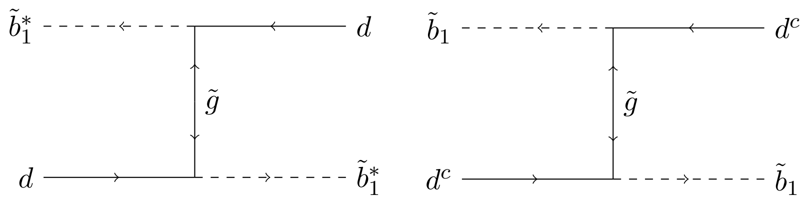

Figure 6.3: Diagrams for sbottom mesino oscillation mediated by a gluino. There are also similar diagrams mediated by neutralino exchange, which can become important if the gluino mass is very large.

phase space integral) is:

$$\Gamma = \sum_{j',k} \frac{1}{32\pi^2} \left| (\lambda'')^* \sum_{i,j,q} \epsilon_{ijk} \frac{m_q}{vc_\beta} U^D_{q_R,1} \frac{m_{u,j}}{vs_\beta} V_{j'j} \frac{m_{d,k}}{vc_\beta} \right|^2 m_{\tilde{b}} \left(1 - \frac{m^2_{u,j'}}{m^2_{\tilde{b}}}\right)^2 . \tag{6.6}$$

To gain some intuition about sbottom LSP decays, we now make a few approximations. The decay is dominated by $\tilde{b} \to t^c s^c$ provided there is sufficient phase space. In the interesting segment of parameter space, the LSP is made up almost entirely of some admixture of the left-handed and right-handed sbottom, so that the decay width is approximately:

$$\begin{aligned} \Gamma &\approx \frac{1}{32\pi^2} |\lambda''|^2 \sin^2\theta \frac{m_b^2 m_s^2 m_t^2}{v^6 c_\beta^4 s_\beta^2} |V_{td}|^2 m_{\tilde{b}} \left(1 - \frac{m_t^2}{m_{\tilde{b}}^2}\right)^2 \\ &\sim (2.6 \times 10^{-10}\ \text{GeV}) |\lambda''|^2 \sin^2\theta \left(\frac{t_\beta}{10}\right)^4 \left(\frac{m_{\tilde{b}}}{300\ \text{GeV}}\right), \end{aligned} \tag{6.7}$$

where θ is the left-right mixing squark mixing angle.

The sbottom decay rate is much less than the hadronization scale $\Lambda_{\text{QCD}} \sim 0.2$ GeV. Thus, the sbottom squark will hadronize before decaying to form fermionic mesino bound states $\tilde{B}_q = \tilde{b}^* q$ and $\tilde{B}^c = \tilde{b} q^c$. If $q = d, s$, then the mesino is neutral, opening up the possibility for mesino oscillations, first discussed in [268]. Since few details of the calculation of the oscillation rate were given in [268], we elaborate on it in appendix 6.A, explaining the necessary approximations. Our final result, eq. (6.30), is in broad agreement with that of [268], and we restate it here:

$$\Delta m = \omega = g_s^2 \left|(U^D_{d_L,1})^2 + (U^D_{d_R,1})^2\right| f_{\tilde{B}}^2 \left(1 - \frac{1}{N_c^2}\right) \frac{m_{\tilde{g}}}{m_{\tilde{g}}^2 - m_{\tilde{b}}^2}. \tag{6.8}$$

This result depends on the nature of the spectator quark. We can use MFV to approximate the ratio of the oscillation rates as we have $|U_M^{q1}| \propto |V_{tq}V_{tb}|$ for $M = L, R$. In this approximation, we get:

$$\frac{\omega_s}{\omega_d} \approx \left|\frac{V_{ts}}{V_{td}}\right|^2 \approx 23 \tag{6.9}$$

The dependence of this ratio on the dimension-two parameters of the squark mass matrix is generically very weak.

With this factor in mind, we consider oscillation of the sbottom-down mesino. The oscillation rate can be estimated by

$$\begin{aligned}\omega &\approx \frac{f_{\tilde{B}}^2}{2}\cos^2\theta\,|V_{td}V_{tb}^*|^2 \frac{m_t^4}{v^4 s_\beta^4}\frac{m_{\tilde{g}}}{m_{\tilde{g}}^2 - m_{\tilde{b}_1}^2} \\ &\sim (4\times 10^{-12}\text{ GeV})\left(\frac{f_{\tilde{B}}}{28.7\text{ MeV}}\right)^2 \cos^2\theta\left(\frac{1000\text{ GeV}}{m_{\tilde{g}}}\right).\end{aligned} \tag{6.10}$$

These results are not too far from the decay rates, eqs. (6.7), but with different parametric dependence. Thus, we expect some parts of parameter space where the oscillation rate is comparable to or larger than the decay rate, leading to appreciable mesino oscillations.

To get a better sense of how common such a phenomenon is, we define the oscillation parameter

$$x \equiv \frac{\Delta m}{\Gamma}. \tag{6.11}$$

The time-integrated probability for a sbottom mesino to oscillate into an anti-sbottom mesino before decaying is

$$p(x) \equiv P(\tilde{B} \to \tilde{B}^c) = \frac{x^2}{2(1+x^2)}. \tag{6.12}$$

The oscillation probability is small for $x \ll 1$ and becomes appreciable near $x \sim 1$, whereas for $x \gg 1$ the $\tilde{B}$ oscillates very rapidly, and the mesino contains an equal

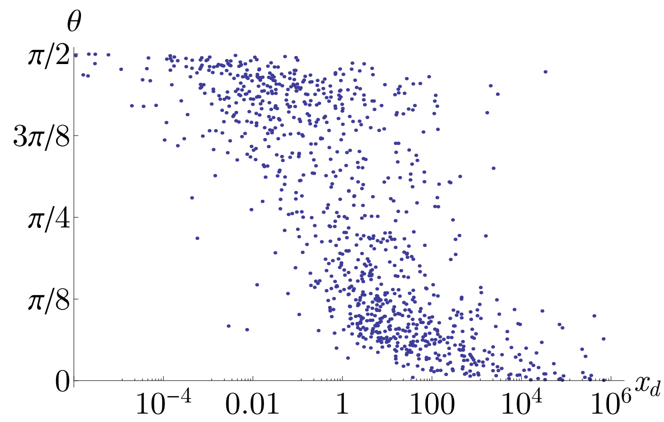

Figure 6.4: Oscillation parameter $x_d = \Delta m_{\tilde{B}_d}/\Gamma$ and left-right mixing angle θ resulting from a scan over parameter space, where $\theta = 0$ corresponds to a pure left-handed LSP.

mixture of sbottom and anti-sbottom components. We scan over parameter space using the same procedure as in section 6.2, selecting points with a sbottom LSP and calculating x_d (the $\tilde{B}_d$ oscillation parameter) and θ for each such point. The results of the scan are shown in Fig. 6.4. We observe that $x_d > 1$ in a significant portion of parameter space, particulary when the LSP is predominantly left-handed.

If the sbottom is the LSP and has a mesino oscillation time comparable to or larger than its lifetime, then there is a very distinct signature of direct sbottom pair-production. The sbottoms will hadronize and the resulting mesino may oscillate before decaying. The mesino must be neutral for oscillations to occur, which occurs when the spectator is a down or strange quark, or roughly half the time as estimated from the B system. If exactly one of the mesinos oscillates before decaying, then the resulting two halves of the final state will have the same charge. Furthermore, these final states each involve a top quark whose charge is easy to tag if it undergoes a leptonic decay. This final state has same-sign leptons, b jets, and a small, but non-negligible, amount of missing energy. The entire chain is illustrated in Fig. 6.5. The branching fraction for this mode is given by

$$\mathrm{Br}(\tilde{b}\tilde{b}^* \to bb\ell^\pm\ell^\pm) \;=\; \mathrm{Br}(W \to \ell\nu)^2 f(x_d, x_s) \approx f(x_d, x_s) \times 6.5\,\%,$$

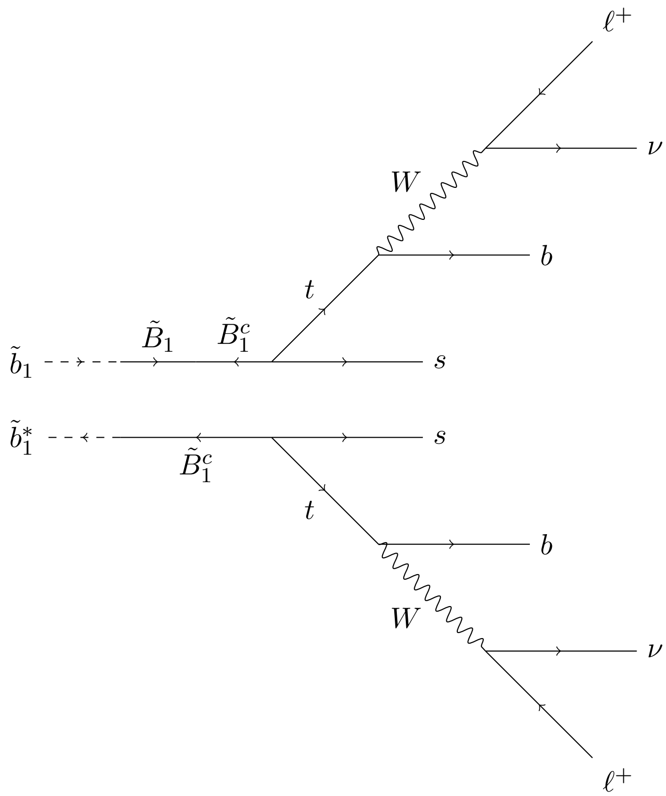

Figure 6.5: Diagram for R-parity violating sbottom decay that leads to same sign leptons.

$$f(x_d, x_s) = 2 \sum_{i=d,s} h_i p(x_i) \left(1 - \sum_{j=d,s} h_j p(x_j) \right), \tag{6.13}$$

where $f(x_d, x_s)$ denotes the probability that exactly one of the two mesinos oscillates. Here h_i is the fraction of sbottoms that form mesinos with spectator i and we use the effective leptonic rate for the W which includes leptonic tau decays. Note that for $x_i \gg 1$ the rate is maximal, and since $h_d + h_s \approx 1/2$ we have $f(x_d, x_s) \approx 3/8$. Despite the modest branching fraction, this decay mode will likely be the most sensitive channel for discovering a sbottom LSP in MFV SUSY.

6.4 Bounds from a CMS search

CMS already has a search [269] that is quite sensitive to the above decay chain. The same-sign dilepton and b jets search includes search regions with 0, 30 GeV and

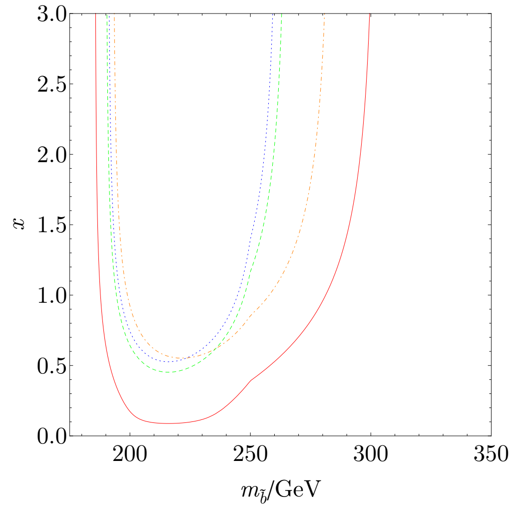

Figure 6.6: Recast CMS bounds on sbottom direct production in terms of the sbottom mass and x_d. Only the four most sensitive signal regions are shown: SR0 in dashed green, SR1 in dotted blue, SR2 in solid red, and SR4 in dash-dotted orange. The most conservative upper limit on the number of new physics events is used for each search region, though varying this number has little effect on the bounds.

50 GeV MET cuts, all of which can be sensitive to our scenario due to the neutrinos from leptonic top decays. The relevant bounds from this search are presented in Table 2 of [269]. In addition, they present efficiency fits for the various cuts in terms of parton-level objects, allowing for easy reinterpretation. In this section, we use this information to reinterpret their bounds in terms of MFV SUSY with a sbottom LSP, and comment on future prospects.

To obtain a bound, we generated $pp \to \tilde{b}\tilde{b}^*$ at 8 TeV using Pythia 8 with all showering and hadronization turned off. The events are decayed at the parton-level. The analysis cuts are applied using the efficiencies presented in section 6 of [269]. No mixing is introduced in event generation, but an x_d-dependent factor is applied to the final efficiency to account for the branching fraction to same-sign leptons. The cross-section for pair production is calculated at NLO using Prospino 2.1 [270, 271]. The resulting bounds are shown in Fig. 6.6. SR2, which counts only positively chaged same sign pairs, yields the strongest bound, since a presumed fluctuation in the data led to all observed same-sign events having negatively charged leptons. For a maximal same-sign branching fraction, the exclusion extends between 180 GeV and 305 GeV.

Note that obtaining same sign lepton events requires $x \gtrsim 1$. The reason that there is sensitivity in the small x_d region is due to the possibility of producing strange mesinos. Even for $x_d \ll 1$, it is possible to get $x_s > 1$.

6.5 Conclusions

MFV SUSY is a compelling new paradigm for exploring supersymmetry without R-parity that offers many new and challenging channels to explore at the LHC. A systematic study of the phenomenology of all plausible scenarios in this framework is required to ensure full sensitivity to weak-scale supersymmetry. We have explored one interesting scenario with a sbottom-like squark LSP.

Direct squark production will be essential for probing all possible corners of natural SUSY parameter space. Our work has demonstrated that the LHC can be sensitive to directly produced sbottom LSPs in the MFV SUSY scenario using the important fact that in this framework strongly-interacting LSPs will hadronize. Using a CMS search for same sign dileptons and b-jets, we have put a bound on sbottoms with masses between 180 and 305 GeV that undergo large mesino oscillations, which is a plausible scenario for the case with sbottom LSPs.

6.A Determination of the Mesino Oscillation Frequency

In this appendix, we present the details of the calculation of the mesino oscillation frequency, carefully listing all approximations as they enter. Throughout the appendix, we will assume a down quark spectator, but the results extend trivially to the strange quark case. We further denote the lightest squark by $\tilde{b}$ and assume that it is sbottom-like.

In the quark and squark mass basis, there are two combinations of sbottom and down quark that correspond to light mesino Weyl fermions:

$$\tilde{B}_1 \equiv \tilde{B} = \tilde{b}^* d\,, \qquad \tilde{B}_2 \equiv \tilde{B}^c = \tilde{b} d^c\,. \tag{6.14}$$

The most general quadratic Lagrangian for these mesino fields is given by:

$$\mathcal{L} = \frac{1}{2} m_{ij} \tilde{B}^i \tilde{B}^j + \text{h.c.}\,. \tag{6.15}$$

Before including corrections due to the gluino, the diagonal entries of m_{ij} vanish, and the two Weyl fermions combine to form a Dirac fermion. The mass, corresponding to the off-diagonal terms in (6.15), is given to leading order by

$$m_{12} = m_{\tilde{b}}\,. \tag{6.16}$$

The leading corrections are of order $\Lambda_{\rm QCD}$, which we neglect.

The diagonal elements m_{11} and m_{22}, corresponding to Majorana masses for $\tilde{B}$ and $\tilde{B}^c$, are not in general equal, and are generated at leading order by tree-level gluino exchange, leading to an oscillation between mesinos and antimesinos. The oscillation frequency is equal to the mass splitting between the two mass eigenstates, whose squared masses are the eigenvalues of $m^\dagger m$. We take m_{12} to be real by performing an appropriate field redefinition, in which case the eigenvalues of $m^\dagger m$ are given by

$$\frac{1}{2}\left(|m_{11}|^2 + |m_{22}|^2 + 2m_{12}^2 \pm \sqrt{(|m_{11}|^2 + |m_{22}|^2 + 2m_{12}^2)^2 - 4|m_{12}^2 - m_{11}m_{22}|^2}\right)\,. \tag{6.17}$$

To leading order in m_{11} and m_{22}, the resulting mass splitting is

$$\omega = \Delta m = |m_{11} + m_{22}^*|\,. \tag{6.18}$$

We work at leading order in the heavy squark approximation. Instead of determining m_{11} and m_{22} directly, we employ the simple and general formula:

$$\omega = \frac{1}{m_{12}} |\langle \bar{\tilde{B}}(\vec{0}, s)|\mathcal{H}_{\rm eff}(\vec{0})|\tilde{B}(\vec{0}, s)\rangle|\,, \tag{6.19}$$

for $\omega \ll m_{12}$, where $\mathcal{H}_{\text{eff}}(\vec{x})$ is the effective Hamiltonian density generated by integrating out the gluino and $|\tilde{B}(\vec{p},s)\rangle$ and $|\bar{\tilde{B}}(\vec{p},s)\rangle$ denote one-particle mesino and antimesino states, respectively, with momentum $\vec{p}$ and spin s with no sum over s. (We use the standard covariant normalization for one-particle momentum eigenstates, $\langle\vec{p}|\vec{q}\rangle = 2E_{\vec{p}}(2\pi)^3\delta^{(3)}(\vec{p}-\vec{q})$.) The effective Hamiltonian density from integrating out the gluino is:

$$\mathcal{H}_{\text{eff}} = \frac{C_L}{2}(\tilde{b}^* d)(\tilde{b}^* d) + \frac{C_R}{2}(\tilde{b} d^c)(\tilde{b} d^c) + h.c.\,, \tag{6.20}$$

for coefficients C_L and C_R to be determined, where the color indices are contracted as indicated by the parentheses. Thus,

$$\omega = \frac{1}{m_{\tilde{b}}}\left|\frac{C_L}{2}\langle\bar{\tilde{B}}|(\tilde{b}^* d)(\tilde{b}^* d)|\tilde{B}\rangle + \frac{C_R^*}{2}\langle\bar{\tilde{B}}|(\tilde{b}^* d^{c\dagger})(\tilde{b}^* d^{c\dagger})|\tilde{B}\rangle\right|\,. \tag{6.21}$$

The structure is very similar to (6.18), and indeed the two terms within the absolute value in (6.21) are precisely $m_{\tilde{b}}$ times the Majorana masses which appear in (6.18).

To determine the $C_{L,R}$, we compare the short-distance amplitudes for oscillation obtained using the MSSM Lagrangian and using the effective Hamiltonian in (6.20). The MSSM gluino exchange amplitudes $\mathcal{M}_L$ and $\mathcal{M}_R$ (Fig. 6.3) are given by

$$\begin{aligned}\mathcal{M}_L &= 2[g_s^2(U^{D*}_{d_L,1})^2]\left[\frac{m_{\tilde{g}}\delta^\alpha_\beta}{m_{\tilde{g}}^2 - m_{\tilde{b}_1}^2}\right]\left[t^a_{ij}t^a_{i'j'} + t^a_{ij'}t^a_{i'j}\right],\\ \mathcal{M}_R &= 2[g_s^2(U^{D}_{d_R,1})^2]\left[\frac{m_{\tilde{g}}\delta^\alpha_\beta}{m_{\tilde{g}}^2 - m_{\tilde{b}_1}^2}\right]\left[t^a_{ij}t^a_{i'j'} + t^a_{ij'}t^a_{i'j}\right],\end{aligned} \tag{6.22}$$

where we work in a basis where the gluino mass is real, and an overall factor of two arises since the gluino-quark-squark vertex comes with a factor of $\sqrt{2}$. The color factors in these amplitudes simplify to [12]:

$$t^a_{ij}t^a_{i'j'} + t^a_{ij'}t^a_{i'j} = \frac{1}{2}\left(\delta_{ij'}\delta_{i'j} + \delta_{ij}\delta_{i'j'} - \frac{1}{N_c}\delta_{ij'}\delta_{i'j} - \frac{1}{N_c}\delta_{ij}\delta_{i'j'}\right)\,. \tag{6.23}$$

The effective operators in (6.20) yield amplitudes:

$$\mathcal{M}'_{L,R} = C_{L,R}(\delta_{ij'}\delta_{i'j} + \delta_{ij}\delta_{i'j'})\,. \tag{6.24}$$

By demanding that $\mathcal{M}_L$ ($\mathcal{M}_R$) from (6.22) is equal to $\mathcal{M}'_L$ ($\mathcal{M}'_R$) from (6.24), we extract the coefficients C_L and C_R:

$$C_L = g_s^2 (U^{D*}_{d_L,1})^2 \frac{m_{\tilde{g}}}{m_{\tilde{g}}^2 - m_{\tilde{b}_1}^2} \left(1 - \frac{1}{N_c}\right), \qquad C_R = g_s^2 (U^{D}_{d_R,1})^2 \frac{m_{\tilde{g}}}{m_{\tilde{g}}^2 - m_{\tilde{b}_1}^2} \left(1 - \frac{1}{N_c}\right). \tag{6.25}$$

The same result can be obtained in the large $m_{\tilde{g}}$ limit by integrating out the gluino in the Lagrangian, neglecting the kinetic term.

As QCD is parity invariant, the hadronic matrix elements in (6.21) are equal. We estimate them using the vacuum insertion approximation. In this approximation, we insert the vacuum between the operators in all possible ways, giving [272, 273]:

$$\langle \bar{\tilde{B}} | (\tilde{b}_i^* d^i)(\tilde{b}_j^* d^j) | \tilde{B} \rangle \approx 2 \left[\langle \bar{\tilde{B}} | (\tilde{b}_i^* d^i) | 0 \rangle \langle 0 | (\tilde{b}_j^* d^j) | \tilde{B} \rangle + \langle \bar{\tilde{B}} | (\tilde{b}_i^* d^j) | 0 \rangle \langle 0 | (\tilde{b}_j^* d^i) | \tilde{B} \rangle \right], \tag{6.26}$$

where we indicate color indices explicitly, and there are two ways to obtain each of the terms, yielding a prefactor of 2. The contraction with the color-neutral external state kills the terms with $i \neq j$ in the second term. Exactly one in every N_c terms has $i = j$, so we get the relation:

$$\langle \bar{\tilde{B}} | (\tilde{b}_i^* d^j) | 0 \rangle \langle 0 | (\tilde{b}_j^* d^i) | \tilde{B} \rangle = \frac{1}{N_c} \langle \bar{\tilde{B}} | (\tilde{b}_i^* d^i) | 0 \rangle \langle 0 | (\tilde{b}_j^* d^j) | \tilde{B} \rangle. \tag{6.27}$$

Our result is thus:

$$\begin{aligned} \langle \bar{\tilde{B}} | (\tilde{b}^* d)(\tilde{b}^* d) | \tilde{B} \rangle &= \langle \bar{\tilde{B}} | (\tilde{b}^* d^{c\dagger})(\tilde{b}^* d^{c\dagger}) | \tilde{B} \rangle \\ &\approx 2 \frac{N_c + 1}{N_c} \langle \bar{\tilde{B}} | (\tilde{b}_i^* d^i) | 0 \rangle \langle 0 | (\tilde{b}_j^* d^j) | \tilde{B} \rangle \equiv 2 \frac{N_c + 1}{N_c} f_{\tilde{B}}^2 m_{\tilde{B}}. \end{aligned} \tag{6.28}$$

The mesino decay constant $f_{\tilde{B}}$ can be estimated using the B meson decay constant and assuming heavy quark symmetry. Up to threshold corrections, the relationship is given by [274]:

$$f_{\tilde{B}} = f_B \sqrt{\frac{m_b}{m_{\tilde{b}_1}}} \left(\frac{\alpha_s(m_b)}{\alpha_s(m_t)} \right)^{6/23} \left(\frac{\alpha_s(m_t)}{\alpha_s(m_{\tilde{b}_1})} \right)^{6/21}. \tag{6.29}$$

Using the latest values of $f_B = 190.6$ MeV [275, 276] and $\alpha_s(m_Z) = 0.1184$ [221], the $\overline{MS}$ quark masses $m_b = 4.19$ GeV and $m_t = 160$ GeV [221], as well as with a numerical solution to the NNNLO beta function for α_s [277], which we evaluate at $m_{\tilde{b}_1} = 300$ GeV, we find a value of $f_{\tilde{B}} = 28.7$ MeV.

Putting these pieces together, we arrive at our final expression:

$$\Delta m = g_s^2 \left|(U^D_{d_L,1})^2 + (U^D_{d_R,1})^2\right| f_{\tilde{B}}^2(m_{\tilde{b}_1}) \left(1 - \frac{1}{N_c^2}\right) \frac{m_{\tilde{g}}}{m_{\tilde{g}}^2 - m_{\tilde{b}_1}^2}\,, \tag{6.30}$$

with $f_{\tilde{B}}$ given by (6.29). This agrees with [268] up to a factor of 8 and the dependence on the CP-violating phase in the squark mixing matrix.[2] This result has some hadronic uncertainty, which we estimate to be of order 10 % based on estimates of the validity of the same approximations for the B meson systems.

[2]We thank Scott Thomas for correspondence on these issues.

CHAPTER 7

A COMPLETE MODEL FOR R-PARITY VIOLATION

We present a complete model whose low energy effective theory is the R-parity violating NMSSM[1] with a baryon number violating $\bar{u}\bar{d}\bar{d}$ vertex of the MFV SUSY form, leading to prompt LSP decay and evading the ever stronger LHC bounds on low-scale R-parity conserving supersymmetry. MFV flavor structure is enforced by gauging an $SU(3)$ flavor symmetry at high energies. After the flavor group is spontaneously broken, mass mixing between the standard model fields and heavy vector-like quarks and leptons induces hierarchical Yukawa couplings which depend on the mixing angles. The same mechanism generates the $\bar{u}\bar{d}\bar{d}$ coupling, explaining its shared structure. A discrete R-symmetry is imposed which forbids all other dangerous lepton and baryon-number violating operators (including Planck-suppressed operators) and simultaneously solves the μ problem. While flavor constraints require the flavor gauge bosons to be outside of the reach of the LHC, the vector-like top partners could lie below 1 TeV.

7.1 Introduction

Supersymmetry (SUSY) broken at the TeV scale has long been considered the leading candidate for a solution to the hierarchy problem of the Standard Model (SM). However, the first two years of LHC data do not contain any hints of the traditional signals of SUSY [278, 279], pushing the superpartner mass scale to uncomfortably high values in the simplest implementations of the theory, too high to solve the hierarchy problem without introducing other tunings. The recent discovery [1, 2] of the Higgs boson at around 126 GeV puts additional pressure on minimal SUSY: it is quite difficult to achieve such a heavy Higgs mass within the simplest models without tuning [280, 281]. If low-scale SUSY is nonetheless

[1]This chapter previously appeared as Csaba Csáki and Ben Heidenreich, "A Complete Model for R-parity Violation," arXiv:1302.0004 [hep-ph], and has been submitted to Physical Review D.

realized in nature, it is likely that one or more additional ingredients beyond the minimal version are present.

There are several known ways to avoid the direct superpartner searches, including raising the mass of the first two generation squarks and the gluino [261–263, 282] ("natural SUSY"), a compressed or stealthy spectrum [266, 283], an R-symmetric theory with Dirac gaugino masses [284–286], and R-parity violation, as in chapter 5 and also [287–292]. Similarly, the Higgs mass can be raised by extending the theory to the NMSSM, possibly by making the Higgs and the singlet composite [264, 265, 293], or by strengthening the Higgs quartic interaction by introducing additional gauge interactions [294–296]. In this chapter we focus on the scenario where the lightest superpartner (LSP) decays promptly via an R-parity violating (RPV) vertex, evading the bounds from direct superpartner searches. We then introduce an NMSSM singlet to raise the Higgs mass to the required 126 GeV value.

It has long been known that RPV [203–207] can significantly change the collider phenomenology of SUSY models without leading to excessive baryon (B) and lepton (L) number violation (for a review see [209]). This is most easily accomplished in models where either B or L is conserved to a very good approximation, since the most stringent constraints on these couplings arise from the nonobservation of proton decay, which generally requires both B and L to be violated. The remaining couplings are subject to the relatively weaker constraints on processes which only violate B or L individually, and can be large enough to have a substantial impact on collider signatures. A particularly interesting possibility is when the LSP is a third generation (stop or sbottom) squark, decaying via the RPV operator $\bar{u}\bar{d}\bar{d}$ as $\tilde{t} \to \bar{b}\bar{s}$ or $\tilde{b} \to \bar{t}\bar{s}$, which is very difficult to disentangle from the vast amount of

QCD background at the LHC, see chapters 5 and 6.[2] (For a recent attempt to distinguish these jets from the QCD background see [298].)

One of the principle objections to RPV models is aesthetic in nature: one needs to introduce a large number of additional small parameters, which, while technically natural, is usually not very appealing. One possible simplifying assumption is to employ the hypothesis of minimal flavor violation (MFV) [212–215]. In MFV models the only sources of flavor violation are the SM Yukawa couplings. If one applies this hypothesis (see also [216]) to the SUSY SM one obtains a robust prediction for the baryon-number violating RPV couplings: they will be related to the ordinary Yukawa couplings, as shown in chapter 5. Thus the BNV couplings for third generation quarks will be the largest, while those involving only light generations will be very strongly suppressed. The resulting simple model evades most direct LHC bounds while preserving naturalness of the Higgs mass, whereas the 126 GeV Higgs mass can be achieved by extending the model to the NMSSM.

However, MFV is only a spurion counting prescription, rather than a full-blown effective theory. It does not fix the overall coefficients of the RPV terms, and does not even fix the relative coefficients of the baryon number violating (BNV) and lepton number violating (LNV) operators. Moreover, it is not obvious a priori that a complete theory can be formulated that produces MFV SUSY as its low-energy effective theory and ensures that LNV operators are sufficiently suppressed to avoid proton decay. The aim of this chapter is to present a complete model that produces Yukawa-suppressed RPV terms in the low-energy effective theory. Since we want to explain the MFV structure of the entire effective Lagrangian, we will have to incorporate a full-fledged theory of flavor into the model. We assume that the flavor hierarchy arises due to (small) mixing with heavy vector-like quarks

[2] However, the gluino must be relatively heavy even in models with RPV, as decays to same sign tops will put a lower bound of order 700 GeV on the gluino mass, see for example [297].

and leptons. Upon integrating out these heavy fields, we obtain the SM flavor hierarchy as well as the Yukawa suppressed RPV terms. To ensure that only the operators compatible with MFV are generated, we will gauge an SU(3) subgroup of the $SU(3)^5$ spurious flavor symmetry of the standard model and impose a discrete symmetry to forbid other dangerous baryon and lepton number violating operators.

This chapter is organized as follows: in section 7.2 we first review how to obtain flavor hierarchies from mixing with heavy flavors. We then describe an anomaly-free gauged $SU(3)$ flavor symmetry which incorporates the heavy flavors, together with the flavor Higgs sector needed to spontaneously break this symmetry and introduce the required mass mixings to generate the SM Yukawa couplings. In section 7.3 we analyze all gauge-invariant operators that can lead to excessive baryon and lepton number violation, deriving experimental constraints on their couplings to determine which operators must be forbidden by a discrete symmetry. In section 7.4 we present an anomaly-free discrete symmetry which forbids all problematic operators and describe the allowed flavor Higgs potential, completing the model. In section 7.5 we consider the structure of the induced soft SUSY breaking terms and comment on the possibility that the third generation of heavy vector-like quarks could be within the range of the LHC. We conclude in section 7.6, presenting the details of our choice of a suitable anomaly-free discrete symmetry in an appendix.

7.2 The building blocks of the UV completed MFV SUSY

The MFV SUSY scenario, outlined in chapter 5, is an R-parity violating variant of the MSSM, with the superpotential

$$W = \mu H_u H_d + q Y_u \bar{u} H_u + q Y_d \bar{d} H_d + \ell Y_e \bar{e} H_d + \frac{1}{2} w'' (Y_u \bar{u})(Y_d \bar{d})(Y_d \bar{d}) \qquad (7.1)$$

and soft-terms with a minimal flavor-violating (MFV) structure. The Yukawa couplings, Y_u, Y_d, and Y_e, are holomorphic spurions charged under the $SU(3)_q \times SU(3)_{\bar{u}} \times SU(3)_{\bar{d}} \times SU(3)_\ell \times SU(3)_{\bar{e}}$ flavor symmetry. Unlike ordinary R-parity conserving MFV, MFV SUSY imposes relations between different *superpotential* couplings, and there is no RG mechanism for generating these relations, since the superpotential is not renormalized. Thus, to explain the form of the superpotential beyond the level of a spurion analysis, it is necessary to embed MFV SUSY within a high-scale model which naturally generates this flavor structure.

Another reason that MFV SUSY requires a UV completion is that, while the superpotential (7.1) is technically natural, it is not safe from Planck-suppressed corrections. For instance, the operator $\frac{1}{M_{\rm pl}} q^3 \ell$ may be generated by gravitational effects, whereas without an MFV structure this operator leads to rapid proton decay, as we show in section 7.3.2. Since global and/or spurious symmetries are generically broken by gravitational effects, to forbid this kind of operator we will ultimately require some additional gauge symmetry.

7.2.1 Yukawa hierarchies from mixing with heavy matter

One possibility would be to try to promote the entire (semi-simple) SM flavor symmetry $SU(3)_q \times SU(3)_{\bar{u}} \times SU(3)_{\bar{d}} \times SU(3)_\ell \times SU(3)_{\bar{e}}$ to a gauge symmetry, with the Yukawa couplings arising as vevs of superfields. However, in this case, the superpotential becomes nonrenormalizable, and in particular, the term

$$W = \frac{1}{\Lambda} q \Phi_u \bar{u} H_u \tag{7.2}$$

requires Φ_u to get a vev of the same order as the cutoff, due to the $\mathcal{O}(1)$ top Yukawa coupling. The resulting effective field theory will necessarily have a low cutoff and will need its own UV completion. This suggests that we must introduce additional massive matter fields, which generate the Yukawa couplings upon being integrated out. If the BNV couplings are generated along with the ordinary Yukawa couplings upon integrating out the heavy fields then this explains their related structure.

As an example consider a quark sector consisting of the usual light quarks $q, \bar{u}, \bar{d}$ together with three pairs of vector-like right-handed up and down quarks $U, \bar{U}$ and $D, \bar{D}$, where $\bar{U}$ and $\bar{D}$ share the same SM quantum numbers as $\bar{u}$ and $\bar{d}$ respectively. We assume the superpotential

$$W = \lambda_u q \bar{U} H_u + \lambda_d q \bar{D} H_d + \frac{1}{2}\lambda_{\text{bnv}} \bar{U}\bar{D}\bar{D} + U\mathcal{M}_u\bar{U} + D\mathcal{M}_d\bar{D} + U\mu_u\bar{u} + D\mu_d\bar{d}\,, \quad (7.3)$$

where $\lambda_{u,d}$ and λ_{bnv} are flavor-universal parameters while $\mathcal{M}_{u,d}$ and $\mu_{u,d}$ are in general 3×3 mass matrices. For $\mathcal{M} \gg \mu$, the low-energy effective theory will contain small effective Yukawa couplings for the chiral fields *and* an effective $\bar{u}\bar{d}\bar{d}$ BNV operator due to the mixing between $\bar{u}$ and $\bar{U}$ and between $\bar{d}$ and $\bar{D}$. At tree-level, one can integrate out the heavy fields using the U and D F-term conditions:

$$\bar{U} = -\mathcal{M}_u^{-1}\mu_u\bar{u}\,, \quad \bar{D} = -\mathcal{M}_d^{-1}\mu_d\bar{d}\,, \quad (7.4)$$

leading to the MFV SUSY superpotential (7.1) with $w'' = \lambda_{bnv}/(\lambda_u\lambda_d^2)$ and the Yukawa couplings

$$Y_x = \lambda_x \Upsilon_x \left(\mathbb{1} + \Upsilon_x^\dagger\Upsilon_x\right)^{-1/2}\,, \quad \Upsilon_x \equiv -\mathcal{M}_x^{-1}\mu_x\,, \quad (7.5)$$

for $x = u, d$.[1] This expression is readily understood by diagonalizing Υ_x. Each eigenvalue[2] σ_i of Υ_x corresponds to the tangent of the corresponding mixing angle between the SM field $\bar{u}$ or $\bar{d}$ and the vector-like partner $\bar{U}$ or $\bar{D}$. Since $\bar{U}$ and $\bar{D}$ couple directly to the Higgs with universal coupling $\lambda_{u,d}$, a small eigenvalue $\sigma_i \ll 1$ of Υ_x corresponds to a small Yukawa coupling $\lambda_x\sigma_i$, whereas a large eigenvalue

[1]The factor in parentheses arises upon canonically normalizing the Kähler potential after integrating out the heavy fields.

[2]More precisely singular value.

$\sigma_i \gg 1$ of Υ_x corresponds to a maximal Yukawa coupling λ_x, with a smooth transition between the two behaviors around $\sigma_i \sim \mathcal{O}(1)$.

We see that hierarchical Yukawa couplings can arise if the mass matrices $\mathcal{M}$ and/or μ have hierarchical eigenvalues, whereas w'' is order one so long as the flavor universal couplings $\lambda_{u,d}$ and λ_{bnv} are also order one. While other choices are possible, for the remainder of this chapter we will assume for simplicity that $\mu_{u,d}$ are flavor-universal parameters, so that all the flavor structure is generated by $\mathcal{M}_{u,d}$. This choice is motivated by the possibility of observable collider signatures, as it allows the vector-like third-generation partners to be relatively light, since the mass matrix for the vector-like generations takes the form:

$$M_x^2 = \mathcal{M}_x \mathcal{M}_x^\dagger + \mu_x \mu_x^\dagger = |\mu_x \lambda_x|^2 \left[Y_x Y_x^\dagger \right]^{-1} , \tag{7.6}$$

where the second equality follows in the case that μ_x is flavor-universal.

If $\lambda_{u,d} \lesssim 1$, then $\mathcal{M} \gg \mu$ will generate only small Yukawa couplings. In order to accommodate the $\mathcal{O}(1)$ top Yukawa coupling, one eigenvalue of $\mathcal{M}_u$, which we denote $\mathcal{M}_u^{(3)}$, should be smaller than μ_u. In this case one integrates out the fields $U^{(3)}$ and $\bar{u}^{(3)}$ at the scale μ, and $\bar{U}^{(3)}$ will remain in the spectrum with a Yukawa coupling of order λ_u, as discussed above. The mass scales in (7.3) implied by the observed Yukawa couplings are schematically illustrated in figure 7.1 for the case $\lambda_{u,d} \sim \tan\beta \sim 1$.

A similar construction for the lepton sector (with SM fields denoted by $\ell, \bar{e}$) has several possible variants, yielding somewhat different expressions for the neutrino masses. One possibility involves a set of three heavy vector-like RH charged leptons $E, \bar{E}$ and three RH neutrinos $\bar{N}$ with the superpotential

$$W = \lambda_e \ell \bar{E} H_d + \lambda_n \ell \bar{N} H_u + E \mathcal{M}_e \bar{E} + \frac{1}{2} \bar{N} \mathcal{M}_n \bar{N} + E \mu_e \bar{e} , \tag{7.7}$$

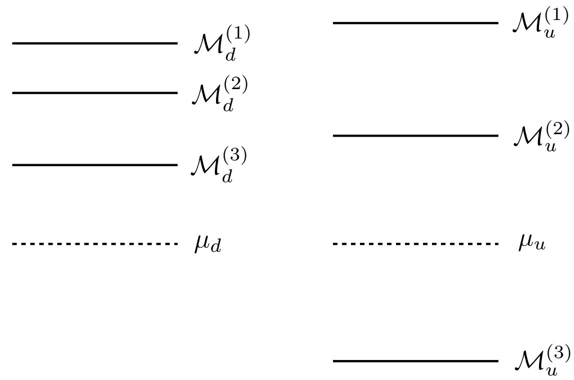

Figure 7.1: A schematic illustration of the relative scales of the eigenvalues of $\mathcal{M}$ vs. μ for down-type (left) and up-type (right) quarks for $\lambda_{u,d} \sim \tan\beta \sim 1$. When $\mu > \mathcal{M}$ the Yukawa coupling will be unsuppressed, while all other Yukawas are suppressed by a factor of $\mu/\mathcal{M}$.

which after integrating out the heavy fields yields just the SM Yukawa terms

$$W_{\text{eff}} = \ell Y_e \bar{e} H_d - \frac{1}{2}\lambda_n^2 (\ell H_u)\mathcal{M}_n^{-1}(\ell H_u) \;, \tag{7.8}$$

with Y_e given by (7.5).

Another possibility is to instead introduce three heavy lepton doublets $L, \bar{L}$ along with three RH neutrinos $\bar{n}$ and the superpotential

$$W = \lambda_e L\bar{e}H_d + \lambda_n L\bar{n}H_u + L\mathcal{M}_\ell \bar{L} + \frac{1}{2}\bar{n}\mathcal{M}_n\bar{n} + \ell\mu_\ell\bar{L} \;, \tag{7.9}$$

which gives rise to the effective superpotential:

$$W_{\text{eff}} = \ell Y_e \bar{e} H_d - \frac{1}{2}\frac{\lambda_n^2}{\lambda_e^2}(\ell H_u) Y_e \mathcal{M}_n^{-1} Y_e^T (\ell H_u) \,. \tag{7.10}$$

after integrating out the heavy fields, where now

$$Y_e = \lambda_e \left(\mathbb{1} + \Upsilon_\ell \Upsilon_\ell^\dagger\right)^{-1/2} \Upsilon_\ell \,, \quad \Upsilon_\ell \equiv -\mu_\ell \mathcal{M}_\ell^{-1} \,. \tag{7.11}$$

A third possibility, resulting in Dirac neutrino masses, is to introduce light RH neutrinos $\bar{n}$ together with vector-like pairs or RH charged leptons $E, \bar{E}$ and neutrinos $N, \bar{N}$. We then impose lepton number conservation, or (more minimally) a

$\mathbb{Z}_3$ symmetry taking $\{\ell, E, N\} \to \omega_3\{\ell, E, N\}$ and $\{\bar{e}, \bar{E}, \bar{n}, \bar{N}\} \to \omega_3^{-1}\{\bar{e}, \bar{E}, \bar{n}, \bar{N}\}$ where $\omega_k \equiv e^{2\pi i/k}$. The resulting model is closely analogous to the quark sector described above with the $\mathbb{Z}_3$ symmetry analogous to the $\mathbb{Z}_3$ center of $SU(3)_C$ (but without an analogue for $\bar{U}\bar{D}\bar{D}$). Due to this analogy, we omit further details.

7.2.2 Gauged flavor symmetries

There are two important features of the quark superpotential (7.3) which remain to be explained. Firstly, we must explain why the couplings $\lambda_{u,d}$ and λ_{bnv} are flavor universal, as this is needed to obtain the MFV SUSY superpotential after integrating out the heavy fields. Moreover, we must also explain the absence of other flavor universal couplings, such as $\bar{u}\bar{d}\bar{d}$ and $\ell\ell\bar{e}$, which lead to unsuppressed baryon and/or lepton number violation. Phrased differently, we have both a "flavor problem" (explaining the flavor structure of certain couplings) and a problem of accidental symmetries (explaining the absence of certain couplings). These problems are related to but not synonymous with the usual problems of flavor and baryon/lepton number violation in the MSSM.

In this subsection, we focus on the first of these two problems, returning to the second issue later on. A crucial observation is that all the marginal couplings are flavor universal. This suggests the presence of a spontaneously broken flavor symmetry, where the nontrivial flavor structure of the mass terms descends from a marginal coupling to a flavor-Higgs superfield. Non-universal contributions to marginal couplings can still descend from nonrenormalizable couplings to the flavor-Higgs field, but these are suppressed by v_F/Λ, where v_F is the scale of flavor symmetry breaking and Λ is the cutoff of the flavor-symmetric theory.

To avoid dangerous Goldstone modes ("familons") from the breaking of the flavor symmetry G — and also to protect G from gravitational effects — we choose to gauge it. We must therefore cancel the additional gauge anomalies $G^2U(1)_Y$

and G^3. While the former anomaly can be canceled by introducing additional "exotic" hypercharged matter, such fields are hard to remove from the low-energy spectrum and also hard to eventually embed into a GUT-like theory. We therefore wish to avoid introducing such exotic matter. It is surprisingly easy to achieve this if only a diagonal subgroup is gauged. A further benefit of introducing the minimum amount of additional gauge symmetries is the ability to write down a relatively simple yet suitable rich Higgs potential for the flavor sector, as we explore in sections 7.2.3 and 7.4. The simplest possibility is to gauge a diagonal $SU(3)_Q$ for quark flavor and a diagonal $SU(3)_L$ for lepton flavor. Once this is achieved, it is easy to take a single diagonal anomaly-free $SU(3)_F$ subgroup of the two to further simplify the model.

Examining the marginal couplings in (7.3), we conclude that q, $\bar{U}$, and $\bar{D}$ transform under a common $SU(3)_Q$ symmetry in the $\square$, $\bar{\square}$ and $\bar{\square}$ representations, respectively. If we also require the couplings $\mu_{u,d}$ to be flavor universal, then we conclude that $\bar{u}, \bar{d}$ and U, D occupy conjugate representations, whereas U, D and $\bar{U}, \bar{D}$ must occupy *the same* representation, otherwise $\mathcal{M}_{u,d}$ would also be flavor universal. Applying the same considerations in the lepton sector leads to the charge table:

	$SU(3)_C$	$SU(2)_L$	$U(1)_Y$	$SU(3)_Q$	$SU(3)_L$
q	$\square$	$\square$	$1/6$	$\square$	$\mathbf{1}$
$\bar{u}$	$\overline{\square}$	$\mathbf{1}$	$-2/3$	$\square$	$\mathbf{1}$
$\bar{d}$	$\overline{\square}$	$\mathbf{1}$	$1/3$	$\square$	$\mathbf{1}$
ℓ	$\mathbf{1}$	$\square$	$-1/2$	$\mathbf{1}$	$\square$
$\bar{e}$	$\mathbf{1}$	$\mathbf{1}$	1	$\mathbf{1}$	$\square$
$\bar{U}$	$\overline{\square}$	$\mathbf{1}$	$-2/3$	$\overline{\square}$	$\mathbf{1}$
U	$\square$	$\mathbf{1}$	$2/3$	$\overline{\square}$	$\mathbf{1}$
$\bar{D}$	$\overline{\square}$	$\mathbf{1}$	$1/3$	$\overline{\square}$	$\mathbf{1}$
D	$\square$	$\mathbf{1}$	$-1/3$	$\overline{\square}$	$\mathbf{1}$
$\bar{E}$	$\mathbf{1}$	$\mathbf{1}$	1	$\mathbf{1}$	$\overline{\square}$
E	$\mathbf{1}$	$\mathbf{1}$	-1	$\mathbf{1}$	$\overline{\square}$
$\bar{N}$	$\mathbf{1}$	$\mathbf{1}$	0	$\mathbf{1}$	$\overline{\square}$

(7.12)

Remarkably, all anomalies vanish, so there is no need to introduce exotic matter.

A variant of the lepton sector (also anomaly-free) with vector-like left-handed lepton doublets can be obtained by replacing the last three rows of the above table with

	$SU(3)_C$	$SU(2)_L$	$U(1)_Y$	$SU(3)_Q$	$SU(3)_L$
$\bar{L}$	$\mathbf{1}$	$\square$	$1/2$	$\mathbf{1}$	$\overline{\square}$
L	$\mathbf{1}$	$\square$	$-1/2$	$\mathbf{1}$	$\overline{\square}$
$\bar{n}$	$\mathbf{1}$	$\mathbf{1}$	0	$\mathbf{1}$	$\square$

(7.13)

A second variant of the lepton sector can be used if one wishes to obtain Dirac neutrino masses. In this case the lepton sector would contain the fields

	$SU(2)_L$	$U(1)_Y$	$SU(3)_L$	$\mathbb{Z}_3$
ℓ	$\square$	$-1/2$	$\square$	ω_3
$\bar{e}$	$\mathbf{1}$	1	$\square$	ω_3^{-1}
$\bar{n}$	$\mathbf{1}$	0	$\square$	ω_3^{-1}
$\bar{E}$	$\mathbf{1}$	1	$\overline{\square}$	ω_3^{-1}
E	$\mathbf{1}$	-1	$\overline{\square}$	ω_3
$\bar{N}$	$\mathbf{1}$	0	$\overline{\square}$	ω_3^{-1}
N	$\mathbf{1}$	0	$\overline{\square}$	ω_3

(7.14)

Here $\mathbb{Z}_3$ is a subgroup of lepton number which can be gauged to forbid Majorana neutrino masses as well as the most dangerous lepton number violating operators.

Note that all anomalies (including the discrete anomalies $SU(2)^2\mathbb{Z}_3$, $SU(3)_L^2\mathbb{Z}_3$ and $(\text{grav})^2\mathbb{Z}_3$) cancel.

Having chosen one of these simple anomaly-free spectra, there are two different straightforward embeddings of $SU(3)_F \subset SU(3)_Q \times SU(3)_L$: in one case all SM fields are $SU(3)_F$ fundamentals (the "standard embedding"), and in the other case the SM leptons are fundamentals while the quarks are anti-fundamentals (the "flipped embedding"). The standard embedding, which we focus on, could potentially arise in a GUT-like theory, since all SM matter fields have the same flavor quantum numbers. However, we will not pursue complete GUT-like models in this chapter, leaving this for future works [299].

7.2.3 The flavor Higgs sector and FCNCs

Given the matter content outlined above we still need to specify a flavor Higgs (flavon) sector that is capable of completely breaking the flavor symmetry and producing the superpotential of (7.3, 7.7). To produce the large masses for the $U, \bar{U}$ and $D, \bar{D}$ heavy quarks we require flavor Higgs fields $\Phi_{u,d}$ in the $\mathbf{6}$ (symmetric) representation of the $SU(3)_Q$ flavor symmetry. Since the anomalies of the matter fields all cancel, we assume that the flavor Higgs sector is vector-like, implying the existence of fields $\bar{\Phi}_{u,d}$ in the $\bar{\mathbf{6}}$ representation of $SU(3)_Q$ as well. We likewise require Higgs fields in the $\mathbf{6}$ and $\bar{\mathbf{6}}$ representations of $SU(3)_L$ to give masses to the heavy vector-like leptons and to generate a Majorana mass for the right-handed neutrinos. We label these fields as $\Phi_{e,\ell,n}$ or $\bar{\Phi}_{e,\ell,n}$ depending on whether they occupy a $\mathbf{6}$ or $\bar{\mathbf{6}}$ of $SU(3)_L$ and on which SM fields they give a mass to. Finally, it is convenient (though not strictly necessary) to replace the parameters $\mu_{u,d,e,\ell}$ with singlet Higgs fields $\phi_{u,d,e,\ell}$. These fields will become charged fields when we later introduce discrete symmetries, and thus will also require vector-like partners $\bar{\phi}_{u,d,e,\ell}$.

The flavor Higgs sector is then given by

	$SU(3)_Q$	$SU(3)_L$
$\Phi_{u,d}$	$\square\square$	$\mathbf{1}$
$\Phi_{e,n}$	$\mathbf{1}$	$\square\square$
$\phi_{u,d,e}$	$\mathbf{1}$	$\mathbf{1}$

(7.15)

for the case with vector-like RH leptons $E, \bar{E}$, where we only show those Higgs fields required to give masses to the matter fields (and not their vector-like partners). The superpotential is now:

$$\begin{aligned} W &= \lambda_u q \bar{U} H_u + \lambda_d q \bar{D} H_d + \lambda_n \ell \bar{N} H_u + \lambda_e \ell \bar{E} H_d + \lambda_b \bar{U}\bar{D}\bar{D} + \lambda_h S H_u H_d + \lambda_s S^3 \\ &\quad + \Phi_u U \bar{U} + \Phi_d D \bar{D} + \Phi_e E \bar{E} + \Phi_n \bar{N}^2 + \phi_u U \bar{u} + \phi_d D \bar{d} + \phi_e E \bar{e} \end{aligned} \tag{7.16}$$

where we introduce one or more NMSSM singlet fields S. The case with vector-like lepton doublets $L, \bar{L}$ is quite similar, except that $\bar{\Phi}_n$ generates the neutrino Majorana mass rather than Φ_n due to the difference in $SU(3)_L$ representations:

$$\begin{aligned} W &= \lambda_u q \bar{U} H_u + \lambda_d q \bar{D} H_d + \lambda_n L \bar{n} H_u + \lambda_e L \bar{e} H_d + \lambda_b \bar{U}\bar{D}\bar{D} + \lambda_h S H_u H_d + \lambda_s S^3 \\ &\quad + \Phi_u U \bar{U} + \Phi_d D \bar{D} + \Phi_\ell L \bar{L} + \bar{\Phi}_n \bar{n}^2 + \phi_u U \bar{u} + \phi_d D \bar{d} + \phi_\ell \bar{L} \ell \end{aligned} \tag{7.17}$$

We assume the presence of a Higgs potential which fixes all the moduli supersymmetrically and generates the required hierarchical Yukawa couplings. It is beyond the scope of this work to construct an explicit potential which does all of these things, but we can still impose minimum consistency requirements. To avoid pseudo-Goldstone bosons, we require a Higgs superpotential whose continuous symmetry group is precisely the (complexified) flavor gauge symmetry and no larger, and whose F-term conditions do not trivially set the vevs to zero. For instance, in the case of a single $\mathbf{6} \oplus \bar{\mathbf{6}}$ pair $\Phi, \bar{\Phi}$, the following potential meets all of these minimum requirements:

$$W = M \Phi \bar{\Phi} + \lambda \Phi^3 + \bar{\lambda} \bar{\Phi}^3 \,. \tag{7.18}$$

Although one can show that this potential generates no hierarchies, it should be possible to generate hierarchies from the analogous but richer potential arising from multiple $\mathbf{6} \oplus \bar{\mathbf{6}}$ pairs. However, we will not attempt to do so explicitly in this chapter.

The absence of flavor-changing neutral currents (FCNCs) beyond those predicted by the SM sets a lower bound on the scale at which the $SU(3)_F$ is Higgsed. In particular, the massive flavor gauge bosons generate the effective Kähler potential

$$K_{\text{eff}} \sim g_F^2 [M^2]^{-1}_{ab} (q^\dagger T^a q)(\bar{d}^\dagger T^b \bar{d}) + \ldots \tag{7.19}$$

where T^a denotes an $SU(3)_F$ generator, g_F the flavor gauge coupling, and M^2_{ab} the squared mass matrix for the flavor gauge bosons. Since we have only gauged a diagonal subgroup of the $SU(3)^3$ MFV flavor symmetry, this operator contributes directly to K–$\bar{K}$ mixing even if M^2_{ab} is $SU(3)_F$ invariant. Thus, we can only suppress FCNCs by raising the flavor Higgsing scale $M/g_F \sim \langle \Phi \rangle$.

Specifically, generic constraints on CP violating K–$\bar{K}$ mixing require the new physics scale to exceed approximately 5×10^5 TeV whereas generic constraints on CP conserving K–$\bar{K}$ mixing require the new physics scale to exceed approximately 3×10^4 TeV [300]. To avoid these constraints we conservatively require the flavor gauge bosons which interact with the down quark to be Higgsed at a scale 10^6 TeV or higher, preventing excessive contributions to either K–$\bar{K}$ or B–$\bar{B}$ mixing. This can be accomplished by taking the greatest eigenvalue of $\langle \Phi_d^{mn} \rangle$ — necessarily flavor-aligned with the down quark — to be at least 10^6 TeV. While B_s–$\bar{B}_s$ and (due to CKM mixing) D–$\bar{D}$ mixings can be mediated by other flavor gauge bosons, the constraints on these processes are much weaker, requiring a new physics scale of at least 6×10^2 TeV for B_s–$\bar{B}_s$ mixing and at least 6×10^3 TeV for D–$\bar{D}$ mixing. The relatively small hierarchy between the down and strange quark masses ensures that the next largest eigenvalue of $\langle \Phi_d^{mn} \rangle$ be not less than 10^4 TeV, easily satisfying these constraints.

Alternately, we can accommodate a much smaller $\langle \Phi_d \rangle$ vev if $SU(3)_F$ is completely broken at 10^6 TeV or higher by anarchic neutrino masses $\langle \Phi_n \rangle$ or by another flavor-Higgs field. However, if $\langle \Phi_u \rangle$ is the dominant source of $SU(3)_F$ breaking its largest eigenvalue must be substantially higher than this, due to the CKM misalignment between the up and down quarks. Because of this misalignment, certain dangerous flavor gauge bosons contributing to K–$\bar{K}$ mixing will only receive a mass at the scale of the second largest eigenvalue of $\langle \Phi_u \rangle$. Due to the large hierarchy between the charm quark and the up quark, this implies that the largest $\langle \Phi_u \rangle$ eigenvalue be at least 10^8 TeV in this situation.

Due to this and to the large hierarchy between the up and top quarks, an LHC accessible up-type $\bar{u}^3$, U_3 vector-like pair is somewhat better motivated than the down-type equivalent in this scenario, though either can be achieved in certain limits.

In principle the massive flavor-Higgs fields Φ, $\bar{\Phi}$ and ϕ, $\bar{\phi}$ can also contribute to FCNCs as well as the flavor gauge bosons. However, since their interactions invariably involve vector-like partners (such as U and D) with negligible overlap with the light quarks, such contributions are at least loop suppressed, if not more. Furthermore, the masses of the uneaten Higgs fields are a priori unrelated to the Higgsing scale,[3] and can in principle be made as heavy as necessary by choosing an appropriate Higgs potential. As such, we omit further discussion of this issue.

[3]Since we have employed the *super*-Higgs mechanism, there is one notable exception: the superpartners of the eaten Goldstone bosons acquire the same mass as the gauge bosons, since they complete the massive vector multiplet (along with the gaugino). However, we assume that any additional flavor violating effects due to the exchange of these fields are not much larger than those already captured by (7.19).

7.3 Dangerous lepton and baryon-number violating operators

The final missing component of our model is an explanation for the absence of dangerous superpotential terms which lead to excessive lepton number violation (LNV) or baryon number violation (BNV). For instance, in addition to the desired $\bar{U}\bar{D}\bar{D}$ superpotential operator, $SU(3)_F$ flavor gauge invariance also allows the dangerous operators $\bar{u}\bar{d}\bar{d}$ and $\ell\ell\bar{e}$, which lead to unsuppressed BNV and LNV, respectively. Dangerous LNV can also be generated by higher-dimensional Planck-suppressed operators, such as $\frac{1}{\Lambda}\Phi\ell\ell\bar{E}$ or $\frac{1}{\Lambda}\Phi L\ell\bar{e}$, and both LNV and BNV can be generated upon integrating out the heavy flavors, such as via the operators $\bar{N}U\bar{U}$ and UDD.

Our approach is to introduce a discrete gauge symmetry (see e.g. [301–303]), analogous to R-parity in the R-parity conserving MSSM, to forbid all problematic operators. Unlike its analogue, this discrete gauge symmetry is necessarily broken by the flavor Higgs fields, so there is no remnant in the low energy theory.

In this section, we aim to catalog the most dangerous operators in the high energy theory (both renormalizable and Planck-suppressed) which must be forbidden by this discrete symmetry. We do not attempt an exhaustive classification of all possible dangerous operators, since this list will depend on the flavor scale, superpartner masses, $\tan\beta$, and other details of the theory. Rather, we will list those operators which are obviously problematic, and which we will insist are forbidden by the discrete symmetry. Later, once we have chosen a discrete symmetry, we perform a more exhaustive search for LNV and BNV corrections.

7.3.1 BNV operators

We begin by discussing operators which violate baryon number only. The principle constraint on these operators is that they not induce too-rapid dinucleon decay.[4] For instance, if the low-energy effective BNV operator is $\bar{u}\bar{d}\bar{d}$, then applying the arguments of section 5.4.2 for a λ'' coupling with generic flavor structure, we see that if

$$\lambda''_{ijk} \lesssim 10^{-8} \;\; \text{for all } i, j, k \,, \tag{7.20}$$

then dinucleon decay is sufficiently suppressed, where the exact bound depends somewhat on the superpartner masses and other details. While the bound actually applies to the λ''_{uds} coupling, other couplings will be less strongly constrained, as will higher-dimensional BNV effective operators.

Any Planck-suppressed operator in the high energy theory is necessarily suppressed by at least $\langle\Phi\rangle/M_{\rm pl} \sim 10^{-10}$ if we assume a flavor scale of 10^6 TeV in compliance with FCNC constraints, as discussed above. Thus, Planck-suppressed operators which violate only baryon number are not dangerous, whereas the only possible renormalizable BNV operators are

$$W_{\rm BNV} = \bar{U}\bar{D}\bar{D} + \bar{u}\bar{d}\bar{d} + UDD \,. \tag{7.21}$$

The first of these operators leads to the MFV SUSY superpotential, as we have already shown, whereas the second leads to unsuppressed BNV in the low energy theory, and must be forbidden by the discrete symmetry. To determine the effect of the third operator, we must integrate out the heavy vector-like fields. Doing so in (7.3), we obtain:

$$U \to \frac{1}{\mu_u}(qH_u)\sqrt{1-\frac{Y_uY_u^\dagger}{|\lambda_u|^2}}Y_u + \frac{1}{2\mu_u}w''\left[\sqrt{1-\frac{Y_uY_u^\dagger}{|\lambda_u|^2}}Y_u\right](Y_d\bar{d})^2 \,, \tag{7.22}$$

[4] As in chapter 5, bounds on $n-\bar{n}$ oscillation typically provide a subleading constraint.

and an analogous expression for D. Thus, UDD generates the effective operator[5]

$$UDD \to \frac{1}{\mu_u \mu_d^2}\left[(qH_u)\sqrt{1-\frac{Y_u Y_u^\dagger}{|\lambda_u|^2}}Y_u\right]\left[(qH_d)\sqrt{1-\frac{Y_d Y_d^\dagger}{|\lambda_d|^2}}Y_d\right]^2 + \ldots \tag{7.23}$$

where the omitted terms conserve baryon number and/or are subleading. Thus, we obtain a BNV operator with a pseudo MFV SUSY structure, though not strictly MFV.[6] Due to this structure and the $(v_u/\mu_u)(v_d/\mu_d)^2$ suppression, this operator should not induce excessive dinucleon decay.

Thus, of all possible BNV operators in the high energy theory, we find that only one operator need be forbidden:

$$W_{\rm bad}^{(BNV)} = \bar{u}\bar{d}\bar{d}\,. \tag{7.24}$$

While other non-MFV operators (if present) could still contribute to proton decay in the presence of lepton number violation, this is a model-dependent question which we defer until we present a complete model in section 7.4.

7.3.2 Low energy constraints on LNV operators

We now discuss operators which violate lepton number, including both baryon number conserving and violating variants. These operators can be generated in three possible ways. They can be either directly generated in the high energy theory, induced by vevs of the flavor Higgs fields, or generated upon integrating out the vector-like flavors. In either of the first two cases, the resulting effective operators are either renormalizable or Planck suppressed, whereas the last mechanism

[5]Strictly speaking, introducing UDD will modify (7.22), but these modifications only generate very high dimensional corrections and/or affect the numerical prefactors of the low energy effective operators, and can therefore be ignored.

[6]Due to the presence of non-MFV terms in the superpotential, we must take the more general ansatz $Y_u = \mathrm{diag}(y_u, y_c, y_t)V_u$ and $Y_d = V_{CKM}\mathrm{diag}(y_d, y_s, y_b)V_d$, where V_u and V_d are in-principle arbitrary unitary matrices which can no longer be rotated away due to the reduced $SU(3)_F \subset SU(3)_q \times SU(3)_{\bar{u}} \times SU(3)_{\bar{d}}$ invariance; the combination $V_u V_d^\dagger$ then appears in (7.23).

will generate higher-dimensional operators with a lower cutoff. For the first two cases, it is expedient to classify all possible dangerous LNV corrections to the low energy effective theory that are either renormalizable or Planck suppressed and derive experimental bounds on these operators. These bounds can then be used to constrain the high-energy theory. We now present such a classification, returning to the question of LNV induced by integrating out the vector-like flavors later.

Assuming that the right-handed neutrinos are heavy, and therefore absent from the low energy effective theory, we find the following potentially dangerous corrections to the MFV SUSY effective superpotential:[7]

$$W_{\text{eff}}^{(\text{LNV})} = \bar{\mu}\ell H_u + \lambda\ell\ell\bar{e} + \lambda' q\ell\bar{d} + \frac{\tilde{\lambda}}{\Lambda}q^3\ell + \frac{\tilde{\lambda}'}{\Lambda}q\bar{u}\bar{e}H_d + \frac{\tilde{\lambda}''}{\Lambda}\bar{u}\bar{u}\bar{d}\bar{e} \tag{7.25}$$

where dimension-six operators are sufficiently suppressed to avoid too-rapid proton decay.

We now discuss the experimental constraints on these couplings from the nonobservation of proton decay. We will assume that $\bar{u}\bar{d}\bar{d}$ has the MFV SUSY form (7.1) to leading order along with MFV soft terms, whereas we take the lepton-number violating couplings to have a generic (non-MFV) flavor structure.

Bounds on bilinear LNV were discussed in detail in section 5.6, which in the present context gives[8]

$$w''\bar{\mu} \lesssim 4 \times 10^{-14} \frac{m_{\tilde{N}}}{\tan^3\beta}\left(\frac{m_{\tilde{N}}}{100\text{ GeV}}\right)\left(\frac{m_{\tilde{q}}}{100\text{ GeV}}\right)^2 \tag{7.26}$$

from the process shown in figure 7.2(a), where w'' is the MFV SUSY BNV parameter from (7.1).

The leading nucleon decay diagram induced by λ' is shown in figure 7.2(b). We

[7]We omit the NMSSM singlet S and the gauge invariant combination H_uH_d in favor of their vevs, as this simplification will not affect the resulting bounds.

[8]The bound given in section 5.6 constrains the corresponding B-term, and consequently has a slightly different $\tan\beta$ dependence.

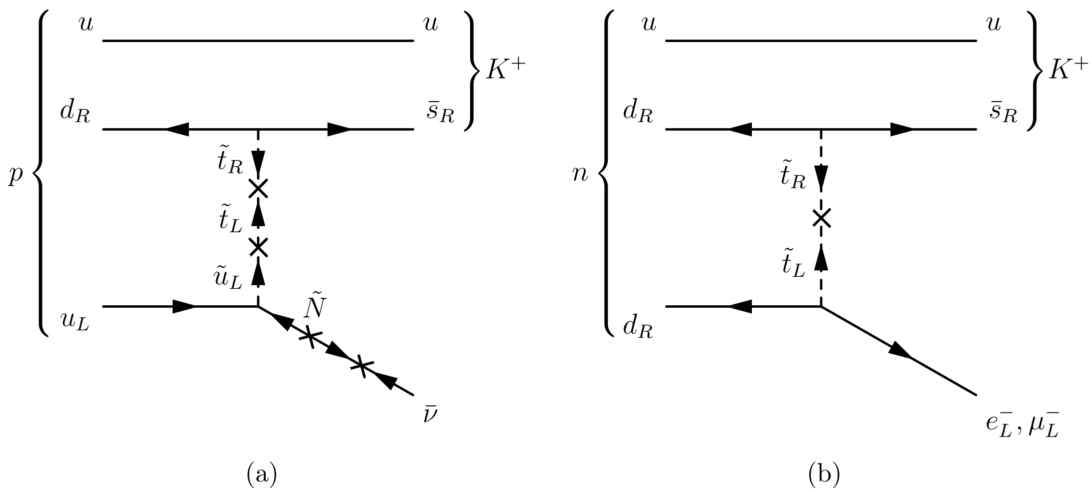

Figure 7.2: (a) The leading contribution to proton decay $p \to K^+\bar{\nu}$ constraining the bilinear RPV term $\bar{\mu}$, cf. figure 5.4. (b) The leading contribution to neutron decay yielding the strongest bound on the λ' vertex.

estimate the width as:

$$\Gamma_{n\to K^+\ell^-} \sim \frac{m_p}{8\pi}\left(w''\lambda'\,\frac{m_d m_s}{m_t^2}\left(\frac{\tilde{\Lambda}}{m_{\tilde{q}}}\right)^2\tan^2\beta\right)^2 \tag{7.27}$$

which leads to the bound:

$$w''\lambda' \lesssim 8\times 10^{-19}\frac{1}{\tan^2\beta}\left(\frac{m_{\tilde{q}}}{100\text{ GeV}}\right)^2 \tag{7.28}$$

for $\tilde{\Lambda} \sim 250$ GeV using the 5.7×10^{31} yrs experimental lower bound on the $n \to K^+\mu^-$ partial lifetime [221]. Similar considerations apply to the $\tilde{\lambda}'$ coupling upon inserting the H_d vev, giving the bound:

$$w''\tilde{\lambda}' \lesssim 0.05\,\frac{1}{\tan\beta}\left(\frac{m_{\tilde{q}}}{100\text{ GeV}}\right)^2\left(\frac{\Lambda}{10^{19}\text{ GeV}}\right). \tag{7.29}$$

The leading contribution to nucleon decay induced by λ comes from the loop diagram shown in figure 7.3(a) [304], which gives a neutrino/neutralino mass mixing of order

$$\Delta m_{\nu\tilde{N}} \sim \frac{m_\tau}{16\pi^2}\lambda\,. \tag{7.30}$$

Applying the bilinear LNV constraints from section 5.6, we obtain the bound:

$$w''\lambda \lesssim 6 \times 10^{-10} \frac{1}{\tan^4 \beta} \left(\frac{m_{\tilde{N}}}{100 \text{ GeV}}\right) \left(\frac{m_{\tilde{q}}}{100 \text{ GeV}}\right)^2 \tag{7.31}$$

However, the loop diagram vanishes if $\lambda_{ijj} = 0$ for all i, j, (e.g. for $\lambda_{ijk} \propto \epsilon_{ijk}$) if moreover the slepton masses are aligned with the charged lepton masses. In this case, the leading contribution to nucleon decay comes from the diagram shown in figure 7.3(b). The width for the four-body decay is approximately

$$\Gamma_{n \to K^+ \ell^- \bar{\nu}\bar{\nu}} \sim \frac{2\pi m_p^7}{(16\pi^2)^3} \left(\lambda \, w'' \frac{|V_{td}| m_d m_s}{m_t^2} \left(\frac{\tilde{\Lambda}}{m_{\tilde{q}}}\right)^2 \frac{\tan^2 \beta}{m_{\tilde{N}} m_{\tilde{\ell}}^2} \right)^2 . \tag{7.32}$$

While there is no direct bound on this decay mode, we assume a baseline sensitivity of at least 10^{30} yrs (which is similar to the bound on neutron disappearance [305].) We then obtain the bound:

$$w''\lambda \lesssim 1.3 \times 10^{-7} \frac{1}{\tan^2 \beta} \left(\frac{m_{\tilde{N}}}{100 \text{ GeV}}\right) \left(\frac{m_{\tilde{q}}}{100 \text{ GeV}}\right)^2 \left(\frac{m_{\tilde{\ell}}}{100 \text{ GeV}}\right)^2 , \tag{7.33}$$

for this special case.

The R-parity even couplings $\tilde{\lambda}$ and $\tilde{\lambda}''$ lead directly to proton decay independent of the BNV w'' coupling. For $\tilde{\lambda}$ the dominant diagram is shown in figure 7.4(a), with the width

$$\Gamma_{p \to K^+ \bar{\nu}} \sim \frac{m_p}{8\pi} \left(\frac{\tilde{\lambda} \tilde{\Lambda}^2}{16\pi^2 \Lambda \, m_{\text{soft}}} \right)^2 , \tag{7.34}$$

which gives the bound

$$\tilde{\lambda} \lesssim 4 \times 10^{-8} \left(\frac{m_{\text{soft}}}{100 \text{ GeV}}\right) \left(\frac{\Lambda}{10^{19} \text{ GeV}}\right) . \tag{7.35}$$

For $\tilde{\lambda}''$ there is more flavor suppression (see figure 7.4(b)), and we obtain the weaker bound:

$$\tilde{\lambda}'' \lesssim 10^{-4} \frac{1}{\tan \beta} \left(\frac{m_{\text{soft}}}{100 \text{ GeV}}\right) \left(\frac{\Lambda}{10^{19} \text{ GeV}}\right) . \tag{7.36}$$

We summarize the results of this section in Table 7.1.

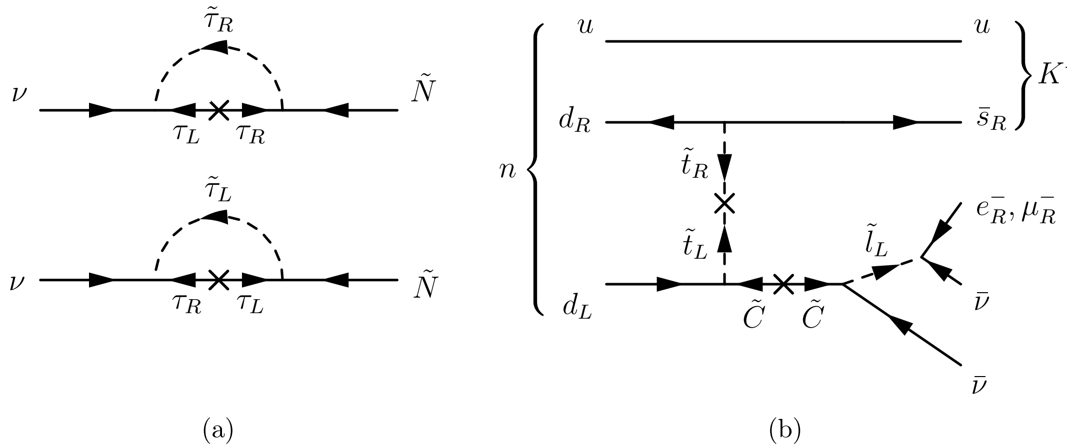

Figure 7.3: (a) Loop diagrams contributing to $\nu - \tilde{N}$ mixing using the λ vertex. Bounds will be obtained by including this mixing inside the diagram in figure 7.2(a). (b) The leading contribution to neutron decay using the λ vertex if $\lambda_{ijj} = 0$ for all i, j and the slepton masses are aligned with the lepton masses.

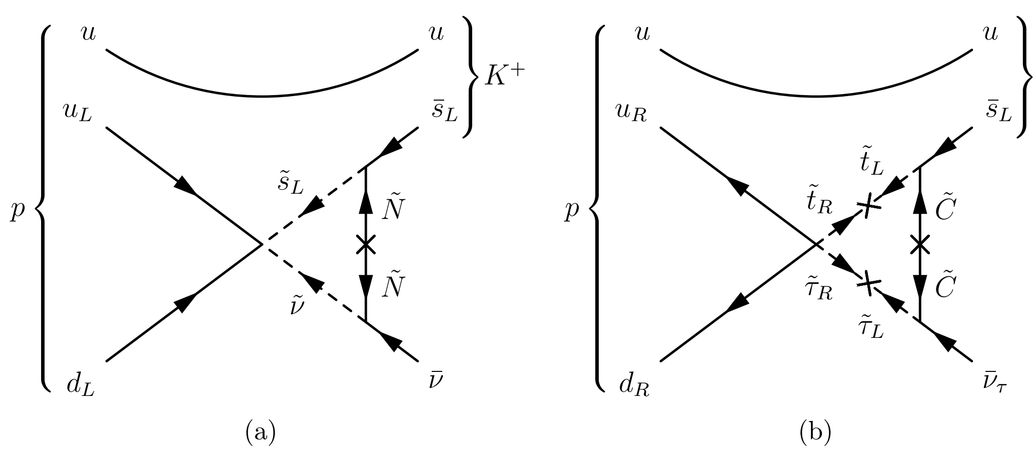

Figure 7.4: The leading contributions to proton decay from the higher dimensional R-parity even couplings (a) $\tilde{\lambda}$ and (b) $\tilde{\lambda}''$.

Operator	Bound	Eqn.	Fig.
$\lambda''\bar{u}\bar{d}\bar{d}$	$\lambda'' \lesssim 10^{-8}$	(7.20)	
$\bar{\mu}\ell H_u$	$w''\bar{\mu} \lesssim 4\times 10^{-14}\frac{m_{\tilde{N}}}{\tan^3\beta}\left(\frac{m_{\tilde{N}}}{100\text{ GeV}}\right)\left(\frac{m_{\tilde{q}}}{100\text{ GeV}}\right)^2$	(7.26)	7.2(a)
$\lambda\ell\ell\bar{e}$	$w''\lambda \lesssim 6\times 10^{-10}\frac{1}{\tan^4\beta}\left(\frac{m_{\tilde{N}}}{100\text{ GeV}}\right)\left(\frac{m_{\tilde{q}}}{100\text{ GeV}}\right)^2$	(7.31)	7.3(a)
$\lambda\epsilon_{ijk}\ell^i\ell^j\bar{e}^k$	$w''\lambda \lesssim 1.3\times 10^{-7}\frac{1}{\tan^2\beta}\left(\frac{m_{\tilde{N}}}{100\text{ GeV}}\right)\left(\frac{m_{\tilde{q}}}{100\text{ GeV}}\right)^2\left(\frac{m_{\tilde{\ell}}}{100\text{ GeV}}\right)^2$	(7.33)	7.3(b)
$\lambda' q\ell\bar{d}$	$w''\lambda' \lesssim 8\times 10^{-19}\frac{1}{\tan^2\beta}\left(\frac{m_{\tilde{q}}}{100\text{ GeV}}\right)^2$	(7.28)	7.2(b)
$\frac{\tilde{\lambda}}{\Lambda}q^3\ell$	$\tilde{\lambda} \lesssim 4\times 10^{-8}\left(\frac{m_{\text{soft}}}{100\text{ GeV}}\right)\left(\frac{\Lambda}{10^{19}\text{ GeV}}\right)$	(7.35)	7.4(a)
$\frac{\tilde{\lambda}'}{\Lambda}q\bar{u}\bar{e}H_d$	$w''\tilde{\lambda}' \lesssim 0.05\,\frac{1}{\tan\beta}\left(\frac{m_{\tilde{q}}}{100\text{ GeV}}\right)^2\left(\frac{\Lambda}{10^{19}\text{ GeV}}\right)$	(7.29)	
$\frac{\tilde{\lambda}''}{\Lambda}\bar{u}\bar{u}\bar{d}\bar{e}$	$\tilde{\lambda}'' \lesssim 10^{-4}\frac{1}{\tan\beta}\left(\frac{m_{\text{soft}}}{100\text{ GeV}}\right)\left(\frac{\Lambda}{10^{19}\text{ GeV}}\right)$	(7.36)	7.4(b)

Table 7.1: Summary of constraints on BNV and LNV corrections to the MFV SUSY superpotential with generic flavor structure, where w'' is the coefficient of the BNV operator $\frac{1}{2}w''(Y_u\bar{u})(Y_d\bar{d})(Y_d\bar{d})$, see (7.1).

7.3.3 Directly induced lepton number violation

Based on the above constraints on corrections to the low energy theory, we now search for LNV operators in the high energy theory which can violate these constraints. In this subsection, we focus on operators which directly induce lepton number violation in the low energy theory, deferring consideration of LNV operators containing the heavy fields $U, D, E, \bar{N}$ or $\bar{n}, \bar{L}$ to the next section.

To select operators which are potentially relevant, we consider the reference point $\tan\beta = 10$, $m_{\text{soft}} = 300$ GeV and $w'' \sim 1$, with $\langle\Phi\rangle, \langle\bar{\Phi}\rangle \sim 10^6$ TeV and $\langle\phi\rangle \lesssim 10^3$ TeV in accordance with the Yukawa hierarchies. We then consider all possible gauge invariant operators which can generate the operators in (7.25) upon inserting the flavor Higgs vevs, accounting for the flavor structure induced by the mass mixings and retaining all operators which violate the experimental constraints for an order one coefficient and a cutoff of 10^{19} GeV. Since $\langle\phi\rangle/\Lambda \sim 10^{-10}$, dimension six operators are sufficiently suppressed except in the case of the ℓH_u coupling, and we can otherwise restrict our attention to dimension four and five operators.

The resulting list of dangerous gauge-invariant operators will depend on whether we choose the standard or flipped embedding of $SU(3)_Q \times SU(3)_L$ into $SU(3)_F$. We find that the following dangerous operators are common to the two cases:

$$W_{\text{bad}} = \ell\ell\bar{e} + \frac{1}{\Lambda}\Phi\ell\ell\bar{E} + \frac{1}{\Lambda}\Phi L\ell\bar{e} + \frac{1}{\Lambda}\Phi q\ell\bar{D} + \frac{1}{\Lambda}\bar{\Phi}qL\bar{D} + \frac{1}{\Lambda^2}\Phi^2\bar{\Phi}\ell H_u\,. \quad (7.37)$$

With the standard embedding we have the additional dangerous operators

$$W^{(\text{standard})}_{\text{bad}} = \left(1 + \frac{1}{\Lambda}\phi + \frac{1}{\Lambda}S\right)q\ell\bar{d} + \frac{1}{\Lambda}\Phi qL\bar{d} + \frac{1}{\Lambda}q\bar{u}\bar{e}H_d + \frac{1}{\Lambda}\bar{u}\bar{u}\bar{D}\bar{E} + \frac{1}{\Lambda}\bar{U}\bar{u}\bar{d}\bar{E} + \frac{1}{\Lambda}\bar{u}\bar{U}\bar{D}\bar{e}\,, \quad (7.38)$$

whereas with the flipped embedding, we have the additional dangerous operators

$$W^{(\text{flipped})}_{\text{bad}} = \left(1 + \frac{1}{\Lambda}\phi\right)qL\bar{d} + \frac{1}{\Lambda}\bar{\Phi}q\ell\bar{d} + \frac{1}{\Lambda}\bar{u}\bar{u}\bar{D}\bar{e} + \frac{1}{\Lambda}\bar{U}\bar{u}\bar{d}\bar{e} + \frac{1}{\Lambda}\bar{u}\bar{U}\bar{D}\bar{E}\,. \quad (7.39)$$

In each case, only some of these operators exist in a given theory, depending on which type of vector-like leptons are present.

These lists should be treated as representative only, since some operators on the list barely make the cut, such as $\frac{1}{\Lambda}\Phi\ell\ell\bar{E}$, and others barely miss it, such as $\frac{1}{\Lambda}\bar{U}\bar{U}\bar{d}\bar{e}$. Nonetheless, we will find that it is possible to forbid all of these operators (and many more besides) by choosing an appropriate discrete symmetry.

7.3.4 Lepton number violation mediated by heavy flavors

We now turn to the question of lepton number violation mediated by the heavy flavors, arising from LNV operators involving $U, D, E, \bar{N}$ or $U, D, \bar{L}, \bar{n}$ (depending on the theory). We have already considered a BNV operator of this type in (7.23), and we take the same approach in what follows, integrating out the heavy fields using the replacement (7.22), its analogue for D, and the replacements

$$E \;\to\; \frac{1}{\mu_e}(\ell H_d)\sqrt{1 - \frac{Y_e Y_e^\dagger}{|\lambda_e|^2}}\,Y_e$$

$$\bar{N} \;\to\; \frac{1}{\lambda_n}\frac{m_\nu}{v_u^2}(\ell H_u) \tag{7.40}$$

or

$$\begin{aligned} \bar{L} &\to \frac{1}{\mu_\ell}\sqrt{1-\frac{Y_e Y_e^\dagger}{|\lambda_e|^2}}\left[Y_e(\bar{e}H_d)+\frac{m_\nu}{v_u^2}(\ell H_u)H_u\right] \\ \bar{n} &\to \frac{\lambda_e}{\lambda_n}Y_e^{-1}\frac{m_\nu}{v_u^2}(\ell H_u) \end{aligned} \tag{7.41}$$

depending on which type of vector-like leptons are present, where m_ν is the left-handed neutrino Majorana mass matrix generated by the effective operator $\frac{1}{v_u^2}(\ell H_u)m_\nu(\ell H_u)$.

Thus, to find dangerous operators, in principle all we need do is to list all LNV dimension four and five operators in the high energy theory that we have not considered yet (those involving $U, D, E, \bar{L}, \bar{N}$ or $\bar{n}$), making the above replacements and then considering the consequences of the resulting effective operator for the low energy theory. This list contains a much wider variety of effective operators than those considered above, and it is very lengthy to derive explicit bounds for every possible operator. Instead, we develop a heuristic scheme to estimate which operators are likely dangerous.

Except in a few special cases where the high-energy operator is super-renormalizable after inserting the flavor Higgs vevs, the strongest bounds will come from inserting the electroweak Higgs vevs into the replacements (7.22, 7.40, 7.41), as this results in a lower-dimensional effective operator. Upon doing so for U, D, E or $\bar{L}$ insertions, we obtain one of the light lepton or quark superfields suppressed by a factor of the mass of the corresponding fermion divided by μ_u, μ_d, μ_e or μ_ℓ respectively, with a possible additional suppression for the third generation coming from the $\sqrt{1-\frac{Y_x Y_x^\dagger}{|\lambda_x|^2}}$ factor. In what follows, we assume that $\mu_x \gtrsim 1$ TeV for $x = u, d, \ell, e$, consistent with $\langle\Phi\rangle \sim 10^6$ TeV and the known Yukawa hierarchies.

For U and D there is an additional BNV term which can directly induce proton decay when inserted into a baryon-number conserving LNV operator. The resulting operator will be dimension five or higher, requiring a gaugino exchange loop to induce proton decay. This can be compared to the similar tree-level diagram involving squark exchange between the $\bar{u}\bar{d}\bar{d}$ MFV SUSY superpotential operator and the baryon-number conserving LNV operator. In place of the $m_q/\mu_{u,d}$ suppression from integrating out U or D, the loop diagram has a $\frac{g^2 m_{\text{soft}}}{16\pi^2 \mu_{u,d}}$ suppression, but otherwise a very similar structure. For $m_{\text{soft}} \sim 300$ GeV the loop diagram only dominates in place of the exchange of a "light" (u, d, s) squark. Since such diagrams are typically suppressed for other reasons, the loop diagram is usually subdominant.

Now consider $\bar{N}$ insertions. If we assume $m_\nu \sim 0.1$ eV then every such operator comes with a strong $m_\nu/v_u \sim 6 \times 10^{-13}$ suppression. However, since we assume $\langle\Phi\rangle \sim 10^6$ TeV, we require $\mathcal{M}_n \lesssim 10^6$ TeV which implies that $\lambda_n \lesssim 2 \times 10^{-3}$. Taking this into account, we find an overall suppression factor of about 3×10^{-10} for each $\bar{N}$ insertion. A similar argument applies to $\bar{n}$, except that the minimum suppression per $\bar{n}$ insertion is now only about 10^{-7} due to the factor of Y_e^{-1}.

We now proceed to classify all possible operators of dimension five or less based on the number of leptons and quarks they contain. We need not consider operators which violate lepton number by an even number, as this will not induce proton decay, so we can have either one or three leptons. Operators with three leptons cannot have any quarks due to the restriction on dimensionality, whereas operators with one lepton can have zero, two, or three quarks, where the latter also violate baryon number. In general operators in the high energy theory and the resulting effective operators in the low energy theory will have the same number of quarks and leptons, except that operators with two quarks and a lepton can also generate operators with three quarks and a lepton in the low energy theory through the insertion of the second term in (7.22).

We begin by considering operators with three leptons. Following the discussion in §7.3.2, we anticipate that a coupling of less than about 10^{-12} (roughly the bound on λ at our chosen reference point) is sufficient to suppress operators of this type to acceptable levels. Using this estimate, we find that none of the possible gauge invariant operators of this type (such as $\bar{N}^3, \bar{N}E\bar{E}$, etc.) are dangerous.

Next, we consider operators with one lepton and no quarks. It is straightforward to check that the only dangerous operators of this type are the RH neutrino tadpoles

$$W_{\rm bad}^{(L)} = \frac{1}{\Lambda}\Phi\bar{\Phi}^2\bar{N} + \frac{1}{\Lambda}\Phi^2\bar{\Phi}\bar{n} + \frac{1}{\Lambda^2}\Phi^4\bar{n} + \frac{1}{\Lambda^2}\Phi\bar{\Phi}^3\bar{n} \tag{7.42}$$

where as usual only some of these operators will appear in a given theory, depending on whether $\bar{N}$ or $\bar{n}$ is present. These tadpoles, which induce bilinear lepton number violation in the low energy theory, are a special case where dimension-six operators, such as $\frac{1}{\Lambda^2}\Phi^4\bar{n}$ can be (at least marginally) dangerous. Note that this operator differs from the analogous operator $\frac{1}{\Lambda^2}\bar{\Phi}^4\bar{N}$, which is small enough by about a factor of 10 to avoid experimental constraints; the difference lies in the different right-handed neutrino Yukawa couplings implied by the two models. In any case, the dimension-six contribution to the tadpole may be made sufficiently small by lowering the flavor scale to 5×10^5 TeV (still in reasonable agreement with flavor constraints), so it is in fact not very dangerous.

Next, we consider operators with one lepton and two quarks. Based on the discussion in §7.3.2, we anticipate that a coupling of less than about 10^{-20} (roughly an order of magnitude smaller than the bound on λ' at our chosen reference point, accounting for the possibility of the more strongly constrained $p \to K^+\bar{\nu}$ decay mode) is sufficient to suppress operators of this type to acceptable levels. The dangerous gauge-invariant operators will depend on whether we choose the standard or flipped embedding. The following dangerous operators are common to the two cases:

$$W_{\rm bad} = \frac{1}{\Lambda}\Phi\bar{n}U\bar{u} + \frac{1}{\Lambda}\bar{\Phi}q\bar{U}\bar{L} + \frac{1}{\Lambda}\bar{\Phi}U\bar{d}E + \frac{1}{\Lambda}\bar{\Phi}\bar{u}D\bar{E} + \frac{1}{\Lambda}\Phi\bar{u}D\bar{e} \tag{7.43}$$

In the first case, we obtain the additional dangerous operators:

$$W_{\rm bad}^{\rm (standard)} = \bar{N}U\bar{U} + \bar{N}D\bar{D} + U\bar{D}E + \bar{U}D\bar{E} + \frac{1}{\Lambda}\bar{\Phi}\bar{U}D\bar{e} + \frac{1}{\Lambda}\Phi q\bar{u}\bar{L} \tag{7.44}$$

whereas in the flipped case, we obtain the additional dangerous operators:

$$W_{\rm bad}^{\rm (flipped)} = \bar{n}U\bar{U} + \bar{n}D\bar{D} + \left(1 + \frac{1}{\Lambda}\phi\right) q\bar{u}\bar{L} + \left(1 + \frac{1}{\Lambda}\phi\right)\bar{U}D\bar{e} + \frac{1}{\Lambda}\Phi U\bar{D}E + \frac{1}{\Lambda}\Phi\bar{U}D\bar{E} \tag{7.45}$$

Finally, we consider operators with one lepton and three quarks, which are necessarily dimension five and require a loop to induce proton decay. Based on the discussion in §7.3.2, we expect that a coupling of less than 10^{-7} for a Planck scale cutoff (roughly the bound on $\tilde{\lambda}$ for our chosen parameters) is just sufficient to suppress proton decay to an acceptable level. Using this estimate, we obtain the following dangerous operators for the standard embedding

$$W_{\rm bad}^{\rm (standard)} = \frac{1}{\Lambda}q^2UE + \frac{1}{\Lambda}\bar{d}^2\bar{D}E\,, \tag{7.46}$$

and none in the flipped embedding.

7.4 A complete model using a discrete symmetry

Having enumerated the operators that are most likely to lead to proton decay or $\Delta B = 2$ processes (see (7.24), (7.37)–(7.39), (7.42)–(7.46)) we now search for a discrete symmetry which forbids these operators. In addition to these dangerous LNV and BNV corrections, we also aim to prevent the problematic cross-couplings between the electroweak and flavor Higgs sectors:

$$W_{\rm bad}^{\rm (cross)} = \mu_\phi\phi S + \phi H_u H_d + \phi S^2 + \phi^2 S + \Phi\bar{\Phi}S + \frac{1}{\Lambda}\Phi^3 S + \frac{1}{\Lambda}\bar{\Phi}^3 S\,, \tag{7.47}$$

which can lead to large dimensionful couplings in the Higgs potential and hence fine tunings. We can also solve the usual μ problem by forbidding the super-renormalizable operators

$$W_{\rm bad}^{(EW)} = \hat{\mu}^2 S + \mu H_u H_d + \mu_s S^2 \,. \tag{7.48}$$

On the other hand, the discrete symmetry will also constrain the flavor Higgs potential, potentially leading to accidental symmetries in the flavor Higgs sector. Such accidental symmetries will induce dangerous Goldstone modes which could mediate FCNCs. Remarkably, we will show that it is possible to choose a discrete symmetry which satisfies all of these constraints while allowing for a semi-realistic flavor Higgs potential without accidental symmetries.[9]

As this discrete symmetry is meant to constrain Planck suppressed as well as renormalizable couplings, it must be anomaly-free and gauged.[10] The discrete symmetry could be an ordinary symmetry or an R-symmetry. In the case of a discrete R-symmetry the superspace coordinate obtains a non-trivial phase η_θ under the discrete transformation, implying that gauginos are rotated by η_θ as well whereas the superpotential must pick up a phase $\eta_W = \eta_\theta^2$. In appendix 7.A we show that the anomaly cancellation conditions for the discrete symmetry together with the requirement that the operators in (7.47) are forbidden requires a discrete R-symmetry. Focusing on the case with $E, \bar{E}$ leptons and the regular embedding, we further argue that the smallest order choice for a discrete symmetry group forbidding all problematic operators while allowing for a semi-realistic flavor Higgs potential is a $\mathbb{Z}_{11}$ discrete R-symmetry, where we assume that the flavor Higgs sector is completely vector-like.

[9]Because this discrete symmetry is broken by the flavor Higgs fields, there is no remnant in the low-energy theory and it does not fit into the classification pursued in [306].

[10]Discrete gauge symmetries sometimes appear as remnants of a spontaneously broken continuous gauge symmetry, but they are well defined and distinct from discrete global symmetries even in the absence of such a mechanism.

	$\mathrm{SU}(3)_C$	$\mathrm{SU}(2)_L$	$U(1)_Y$	$\mathrm{SU}(3)_F$	$\mathbb{Z}_{11}[\omega_{11}^{-2}]$
q	$\square$	$\square$	$1/6$	$\square$	ω_{11}^3
$\bar{u}$	$\bar{\square}$	$\mathbf{1}$	$-2/3$	$\square$	ω_{11}^4
$\bar{d}$	$\bar{\square}$	$\mathbf{1}$	$1/3$	$\square$	ω_{11}^5
ℓ	$\mathbf{1}$	$\square$	$-1/2$	$\square$	ω_{11}^4
$\bar{e}$	$\mathbf{1}$	$\mathbf{1}$	1	$\square$	1
$\bar{U}$	$\bar{\square}$	$\mathbf{1}$	$-2/3$	$\bar{\square}$	ω_{11}^3
$\bar{D}$	$\bar{\square}$	$\mathbf{1}$	$1/3$	$\bar{\square}$	ω_{11}^3
$\bar{E}$	$\mathbf{1}$	$\mathbf{1}$	1	$\bar{\square}$	ω_{11}^2
U	$\square$	$\mathbf{1}$	$2/3$	$\bar{\square}$	ω_{11}
D	$\square$	$\mathbf{1}$	$-1/3$	$\bar{\square}$	ω_{11}^5
E	$\mathbf{1}$	$\mathbf{1}$	-1	$\bar{\square}$	ω_{11}^4
$\bar{N}$	$\mathbf{1}$	$\mathbf{1}$	0	$\bar{\square}$	ω_{11}^2
H_u	$\mathbf{1}$	$\square$	$1/2$	$\mathbf{1}$	ω_{11}^3
H_d	$\mathbf{1}$	$\square$	$-1/2$	$\mathbf{1}$	ω_{11}^3
S	$\mathbf{1}$	$\mathbf{1}$	0	$\mathbf{1}$	ω_{11}^3

Table 7.2: The "matter" fields of the complete model, where $\mathbb{Z}_k[\eta_W]$ denotes a $\mathbb{Z}_k$ discrete symmetry under which the superpotential picks up a phase η_W. (In specifying a discrete R-symmetry, it is unnecessary to specify η_θ if $\eta_W = \eta_\theta^2$ is given, since the two possible sign choices in taking the square root are related by $(-1)^F$.)

We now present an example of a complete model with a discrete $\mathbb{Z}_{11}$ R-symmetry. We choose $\eta_\theta = \omega_{11}^{-1} = e^{-2\pi i/11}$ without loss of generality, and thus $\eta_W = \omega_{11}^{-2}$. We then introduce the "matter" fields shown in table 7.2 and the flavor Higgs fields shown in table 7.3.

As shown in appendix 7.A, this model is anomaly free.[11] The most general renormalizable flavor Higgs superpotential allowed by the $\mathbb{Z}_{11}$ R-symmetry is:

$$\begin{aligned} W_{\text{Higgs}} = {} & M_u \Phi_u \bar{\Phi}_u + M_d \Phi_d \bar{\Phi}_d + M_e \Phi_e \bar{\Phi}_e + \lambda_1 \phi_u \Phi_d \bar{\Phi}_u + m_1 \phi_u \phi_e + m_2 \phi_d^2 \\ & + \lambda_2 \phi_e^2 \phi_d + \lambda_{ude} \Phi_u \Phi_d \Phi_e + \lambda_{eee} \Phi_e^3 + \bar{\lambda}_{udd} \bar{\Phi}_u \bar{\Phi}_d^2 + \bar{\lambda}_{dee} \bar{\Phi}_d \bar{\Phi}_e^2 \end{aligned} \tag{7.49}$$

[11]There is a naive $(\text{grav})^2\mathbb{Z}_{11}$ anomaly which can be cancelled by adding a second copy of the S field, but any hidden (e.g. SUSY-breaking) sector will contribute to this anomaly, as will the gravitino, so there is no clear constraint on the number of NMSSM singlets.

	$\mathrm{SU}(3)_F$	$\mathbb{Z}_{11}[\omega_{11}^{-2}]$
$\Phi_{u,n}$	$\square\square$	ω_{11}^5
Φ_d	$\square\square$	ω_{11}
Φ_e	$\square\square$	ω_{11}^3
$\bar{\Phi}_{u,n}$	$\overline{\square\square}$	ω_{11}^4
$\bar{\Phi}_d$	$\overline{\square\square}$	ω_{11}^8
$\bar{\Phi}_e$	$\overline{\square\square}$	ω_{11}^6
ϕ_u	$\mathbf{1}$	ω_{11}^4
ϕ_d	$\mathbf{1}$	ω_{11}^{-1}
ϕ_e	$\mathbf{1}$	ω_{11}^5

Table 7.3: The flavor Higgs sector of the complete model.

where Φ_u now stands for either Φ_u or Φ_n (which carry the same charges). One can check that this potential breaks all $U(1)$ global symmetries, and hence does not obviously lead to Goldstone modes. Although we may in general require more than one "flavor" of each type of Φ field to allow a suitably rich potential which can reproduce the flavor structure of the SM, the potential is likely sufficiently generic to also break any resulting non-abelian flavor symmetry, avoiding Goldstone's theorem. However, we will not study the flavor Higgs sector in detail, deferring this to a future work.

One can show that there are no further renormalizable superpotential couplings allowed by the $\mathbb{Z}_{11}$ R-symmetry beyond those in (7.16), (7.49). Performing a systematic search we find the following dimension-five lepton-number violating operators:

$$W_{\mathrm{LNV}}^{(5)} = \frac{1}{\Lambda}\bar{\Phi}_d \bar{N} D \bar{d} + \frac{1}{\Lambda}\phi_d \bar{N} D \bar{D} + \frac{1}{\Lambda}\phi_d \bar{U} D \bar{E} + \frac{1}{\Lambda} S \bar{N} U \bar{U} + \frac{1}{\Lambda} S \bar{N}^3 \tag{7.50}$$

One can check that these operators are more than sufficiently suppressed by a Planck scale cutoff and an order one coupling to avoid excessive proton decay for our chosen reference parameters of $\tan\beta = 10$, $m_{\mathrm{soft}} = 300$ GeV, $w'' = 1$, $\langle\Phi\rangle \sim 10^6$ TeV and $\langle\phi\rangle \lesssim 10^3$ TeV. Dimension six operators can also be significant if they contain at least three flavor Higgs fields. The most significant of these operators are

$$W^{(6)}_{\rm LNV} = \frac{1}{\Lambda^2}\bar{N}\bar{\Phi}_e\bar{\Phi}_u^3 + \frac{1}{\Lambda^2}\bar{N}\bar{\Phi}_u\Phi_d^3 + \dots \tag{7.51}$$

where the omitted terms generate subleading contributions to the $\bar{N}$ tadpole. One can show that these operators are also sufficiently suppressed for $\langle\Phi\rangle \sim 10^6$ TeV.

While we have not considered such operators above, higher-dimensional corrections to the Kähler potential can in principle lead to dangerous baryon and/or lepton number violation. Imposing the $\mathbb{Z}_{11}$ R-symmetry discussed above, the most significant of these corrections are:

$$K^{(5)}_{\rm LNV} = \frac{1}{\Lambda}\bar{U}\bar{E}\bar{d}^\dagger + \frac{1}{\Lambda}\bar{D}\bar{N}\bar{d}^\dagger + c.c. \tag{7.52}$$

One can check that these operators will not lead to too-rapid proton decay with a Planck-scale cutoff.

Planck suppressed operators can also contribute to the electroweak Higgs potential. In particular, we find the dimension-five contributions to the S tadpole:

$$W^{(5)}_{\rm EW} = \frac{1}{\Lambda}S\phi_d\Phi_d\bar{\Phi}_e + \frac{1}{\Lambda}S\phi_d\Phi_e\bar{\Phi}_u \tag{7.53}$$

However, one can check that these generate a tadpole of only about $(300\text{ GeV})^2$ for $\langle\Phi\rangle \sim 10^6$ TeV and $\langle\phi_d\rangle \sim 10^3$ TeV and thus will not cause a fine tuning of the electroweak scale, and can in fact facilitate electroweak symmetry breaking even in the absence of the SUSY breaking terms.

7.5 SUSY breaking and particle spectrum beyond the MSSM

In this section, we discuss supersymmetry breaking and its consequences for the low energy spectrum, as well as the possible effects of a light right-handed vector-like generation of quarks, such as can occur in our model.

We consider a supersymmetry breaking spurion X, a chiral superfield with an F-term vev $\langle X \rangle_F \sim F$, which couples to our model via higher-dimensional operators suppressed by a messenger scale M,[12] such that $m_{\text{soft}} \sim F/M$. In particular, we focus on the case of gravity mediation, where M is the Planck scale and X may be thought of as a hidden-sector field which couples to the SM sector via Planck-suppressed operators. We will show that, contrary to the usual situation where gravity mediation induces a flavor problem, the gauged flavor symmetry together with the $\mathbb{Z}_{11}$ gauged R-symmetry will protect against FCNCs, giving an MFV structure at leading order with corrections suppressed by $\langle \Phi \rangle / M \sim 10^{-10}$. Indeed, in this context gravity mediation is actually preferred, as lowering the messenger scale will eventually lead to subleading non-MFV corrections as the messenger scale approaches the flavor scale.

The soft SUSY-breaking squark masses for the right-handed up-type squarks are generated by the effective Kähler potential:

$$\int d^4\theta \left[\frac{X^\dagger X}{M^2} (a_1 \bar{u}^\dagger \bar{u} + a_2 \bar{U}^\dagger \bar{U}) \right] , \tag{7.54}$$

where both terms are $SU(3)_F$ universal due to the gauging of the flavor symmetry. Integrating out the heavy fields, we obtain the soft masses

$$\mathcal{L}_{\text{soft}} \supset m_{\text{soft}}^2 \tilde{\bar{u}}^\star \left(a_1 \mathbb{1} + \frac{a_2 - a_1}{|\lambda_u|^2} Y_u^\dagger Y_u \right) \tilde{\bar{u}} \, , \tag{7.55}$$

and likewise for the right-handed down-type squarks. Thus, the soft terms are MFV to leading order, though they are already non-universal in the high scale theory, even before accounting for the running between the flavor scale and the electroweak scale. Non-MFV corrections will arise from higher-dimensional operators involving the flavor Higgs fields, and will therefore be strongly suppressed for a Planck-scale cutoff.

[12] We assume that there are no renormalizable couplings to X in the high-scale flavor symmetric theory. If present, these could lead to flavon-mediated SUSY breaking which would generate non-MFV soft terms.

At first glance, the left-handed squark mass matrix appears to be universal at the flavor scale, arising from the effective Kähler potential

$$\int d^4\theta \left[\frac{X^\dagger X}{M^2} b_1 q q^\dagger\right] . \tag{7.56}$$

However, there are potentially important corrections upon integrating out the heavy vector-like generations coming from the effective Kähler potential

$$\int d^4\theta \left[\frac{X^\dagger X}{M^2}(b_2 U^\dagger U + b_3 D^\dagger D)\right] . \tag{7.57}$$

Upon integrating out the heavy fields we obtain the squark masses

$$b_1 m_{\rm soft}^2 \tilde{q}\tilde{q}^\star + b_2 m_{\rm soft}^2 \frac{v_u^2}{|\mu_u|^2} \tilde{u}_L Y_u Y_u^\dagger \left(1 - \frac{1}{|\lambda_u|^2} Y_u Y_u^\dagger\right) \tilde{u}_L^\star \\ + b_3 m_{\rm soft}^2 \frac{v_d^2}{|\mu_d|^2} \tilde{d}_L Y_d Y_d^\dagger \left(1 - \frac{1}{|\lambda_d|^2} Y_d Y_d^\dagger\right) \tilde{d}_L^\star , \tag{7.58}$$

so there are tree-level non-universal MFV contributions to the squared mass matrix suppressed by $(v/\mu)^2$, in addition to the usual RG corrections.

The soft breaking A-terms will be holomorphic MFV to leading order. For example the effect of the $\bar{U}\bar{D}\bar{D}$ operator is

$$c_1 \int d^2\theta \frac{X}{M} \bar{U}\bar{D}\bar{D} \;\to\; c_1 \frac{m_{\rm soft}}{\lambda_u \lambda_d^2} (Y_u \tilde{\bar{u}})(Y_d \tilde{\bar{d}})(Y_d \tilde{\bar{d}}) . \tag{7.59}$$

Similarly, the A-terms corresponding to the ordinary Yukawa couplings are

$$c_2 \int d^2\theta \frac{X}{M} q \bar{U} H_u \;\to\; c_2 \frac{m_{\rm soft}}{\lambda_u} (\tilde{q} \hat{H}_u) Y_u \tilde{\bar{u}} , \tag{7.60}$$

where $\hat{H}_u$ denotes the scalar component of H_u. Certain non-holomorphic combinations of spurions can also appear in the A-terms:

$$c_3 \int d^2\theta \frac{X}{M} \phi U \bar{u} \;\to\; c_3 m_{\rm soft} \frac{\langle\phi\rangle}{\mu_u} (\tilde{q}\hat{H}_u) \left(1 - \frac{1}{|\lambda_u|^2} Y_u Y_u^\dagger\right) Y_u \tilde{\bar{u}} \\ + c_3 m_{\rm soft} \frac{w'' \langle\phi\rangle}{2\mu_u} \left[\left(1 - \frac{1}{|\lambda_u|^2} Y_u Y_u^\dagger\right) Y_u \tilde{\bar{u}}\right] \left[Y_d \tilde{\bar{d}}\right]^2 , \tag{7.61}$$

and likewise for the $\phi D\bar{d}$ A-term. Note that $\langle\phi\rangle/\mu \sim 1$, so these are non-holomorphic MFV corrections with order one coefficients, but they take a very particular form which was anticipated already in appendix 5.B and which is not in any way problematic.

However, there are additional sources of A-terms — some of which may be dangerous — from SUSY breaking terms of the form

$$c_4 \int d^2\theta \frac{X}{M} U\Phi\bar{U}\,. \tag{7.62}$$

Upon integrating out the heavy fields as usual we find

$$\mathcal{L}_{soft} \supset c_4 \frac{m_{\rm soft}}{\mu_u \lambda_u}(\tilde{q}\hat{H}_u)\sqrt{1-\frac{Y_u Y_u^\dagger}{|\lambda_u|^2}}\, Y_u \langle\Phi\rangle Y_u \tilde{\bar{u}} + c_4 w''(\ldots)\,. \tag{7.63}$$

If $\langle\Phi_u\rangle \propto \mathcal{M}_u$ then we get an additional MFV contribution of the same form as (7.61). However, in the model based on the gauged $\mathbb{Z}_{11}$ R-symmetry both Φ_u and Φ_n carry the same charges. This would not be problematic if the same linear combination of these fields were to appear in both the superpotential and the soft-terms, but there is no a priori reason for this to occur unless enforced by some symmetry principle. Conversely, if both combinations are allowed in the A-terms, then that A-term would contain a additional structure proportional to $Y_u M_N Y_u$ which deviates from the MFV form by an essentially arbitrary 3×3 symmetric matrix, contributing to off-diagonal holomorphic non-MFV squark masses (though still Yukawa suppressed). In order to forbid such contributions, one can for example introduce an additional $\mathbb{Z}_2$ discrete gauge symmetry, under which $\Phi_{u,d}, \bar{\Phi}_{u,d}$ and U, D are odd, and every other field is even. This $\mathbb{Z}_2$ is also anomaly free, and forbids the mixing of the Φ_u and Φ_n fields, but will also restrict the form of the general Higgs potential of (7.49). To avoid this problem, one can for instance introduce two copies of each Φ and ϕ Higgs field variant labeled $\Phi^\pm$, such that the $+$ and $-$ Higgs fields are respectively even and odd under the $\mathbb{Z}_2$. Thus, $\Phi_u \equiv \Phi_u^-$

will generate the up-sector Yukawa couplings, whereas $\Phi_n \equiv \Phi_u^+$ will generate the neutrino masses, and no mixing between the two is permitted by the $\mathbb{Z}_2$. (This extension of the Higgs sector allows a richer Higgs potential, which may in any case be needed to obtain the desired flavor structure.) Another possible solution is to choose the flipped embedding of $SU(3)_F \subset SU(3)_Q \times SU(3)_L$, so that the quark flavor structure is generated by $\bar{\Phi}$'s whereas that of the leptons is generated by Φ's.

Finally, we address the question of whether any of the additional particles in our model (beyond the NMSSM) could be within reach of the LHC and what their signals could be. As discussed in section 7.2.3, the constraints from FCNC's will force the flavor gauge bosons to be at $10^4 - 10^6$ TeV, well outside the LHC's range. Similarly, most of the heavy vector-like quarks $U, \bar{U}, D, \bar{D}$ will be too heavy for LHC energies, since their masses are determined by the same flavor Higgs VEVs $\Phi_{u,d}$ that contribute to the flavor gauge boson masses. However, in order for the top quark to have an $\mathcal{O}(1)$ Yukawa coupling, the corresponding $U, \bar{U}$ should have one eigenvalue $\mathcal{M}_u^{(3)}$ which should be comparable or smaller than the corresponding mixing term μ_u, which cannot itself exceed about 10 TeV in order to generate the large up/top hierarchy if the flavor scale is 10^6 TeV. These parameters are not strongly constrained by FCNC's, and could lie within the LHC energies. To study the phenomenology of the third generation up-type quarks we focus on their interactions, neglecting the other generations:

$$\mathcal{L} \supset \mu_u t_R^1 t_L^2 + \mathcal{M}_u t_R^2 t_L^2 + \lambda_u q^{(3)} t_R^1 H_u \ , \tag{7.64}$$

where we introduced the notation used in the little Higgs literature for top partners $\bar{u}^{(3)} = t_R^1, U^{(3)} = t_L^2, \bar{U}^{(3)} = t_R^2, q^{(3)} = (t_L^1, b_L)$ and $\mathcal{M}_u = \mathcal{M}_u^{(3)}$. The mass of the heavy vector partners is given by

$$m_T = \sqrt{\mu_u^2 + \mathcal{M}_u^2} \ , \tag{7.65}$$

with the mixing among the right handed quarks is given by the angle

$$\sin\alpha = \frac{\mu_u}{\sqrt{\mu_u^2 + \mathcal{M}_u^2}} \ . \tag{7.66}$$

The top quark mass is given by

$$m_t = \lambda_u \cos\alpha \frac{v_u}{\sqrt{2}} \ . \tag{7.67}$$

A mixing among the left handed top quarks is induced after EWSB and is given by the mixing angle

$$\sin\gamma = \frac{\lambda_u \mu_u v_u/\sqrt{2}}{\mu_u^2 + \mathcal{M}_u^2} = \frac{m_t}{m_T} \tan\alpha \ . \tag{7.68}$$

The mixing pattern is the same as for the heavy top partners in little Higgs models, and this will largely determine the phenomenology of these models. The main difference is that in our case the cancellation of the quadratic divergences is achieved via SUSY, rather than through the non-linearly realized SU(3) symmetry of the little Higgs models. However, this does not affect the phenomenology of the top partners. The couplings of the top partners to gauge bosons is discussed in detail in appendix A of [307]. Electroweak precision correction bounds from loops of the heavy vector-like top partners is around 450 GeV as long as the mixing angle α is not too small [307]. The direct production bounds from the 2011 dataset of 5 fb^{-1} is somewhat weaker, of order 350 GeV, while a more recent analysis puts a more stringent direct bound of 480 GeV on the mass of the top partners [308].[13] Thus we conclude that the third generation $U, \bar{U}$ states can be below 1 TeV and within the range of the 14 TeV LHC, but this need not be the case: they can be as heavy as 10 TeV for $\mathcal{M}_u^{(1)} \sim 10^6$ TeV.

[13]The superpartners of the top partners would just behave like heavy stops: their pair production cross section is very small, and they would then decay to the LSP and finally through the RPV term to jets.

7.6 Conclusions

We have presented a complete model which violates baryon number and R-parity in a controlled fashion, leading to prompt LSP decay and low energy energy signatures which evade the stringent LHC bounds on R-parity conserving supersymmetry broken at the electroweak scale. At the same time, our model solves the μ problem as well as the flavor problem of gravity mediated supersymmetry breaking, provides a potential explanation for the origin of flavor in the standard model, and is safe from Planck suppressed corrections.

We accomplish this by gauging an $SU(3)_F$ flavor symmetry at high energies and spontaneously breaking it. After integrating out the massive fields (vector-like right handed generations) the universal Yukawa couplings and $\bar{U}\bar{D}\bar{D}$ BNV coupling are simultaneously reduced to the low-energy hierarchical Yukawa couplings and a $\bar{u}\bar{d}\bar{d}$ R-parity violating BNV coupling of the MFV SUSY form.[14] We introduce a gauged discrete R-symmetry to forbid other sources of baryon number violation as well as excessive lepton number violation. This discrete symmetry also allows us to solve the μ problem by introducing NMSSM singlet(s) S and forbidding the super-renormalizable terms in the Higgs potential via the discrete symmetry. We exhibit an example of a $\mathbb{Z}_{11}$ discrete R-symmetry which accomplishes all of these goals while allowing a suitably rich potential for the flavor Higgs sector and protecting the model from dangerous Planck-suppressed corrections.

The gauged $SU(3)_F$ symmetry ensures that soft SUSY breaking terms are MFV to leading order, but with a non-universal structure which allows for flexibility in the low-energy superpartner spectrum. As FCNC constraints require a flavor scale of at about 10^6 TeV or higher, the flavor gauge bosons will be out of reach of the

[14] While we were concluding this project, two papers with similar models have been published [309, 310]. We thank Gordan Krnjaic for informing us ahead of time of the release of their paper and for useful discussions related to this chapter.

LHC. However, the third generation of right-handed vector-like up-type quarks must be much lighter than the flavor scale to generate the large up/top mass hierarchy, and could lie below 1 TeV. In this case it would have collider properties similar to the top partners in little Higgs models.

Since we have gauged only a single $SU(3)_F$ for both quarks and leptons, this kind of model (though not the exact model presented in §7.4) may be embeddable in an $SU(5)$-type GUT. We explore this possibility in a future work [299].

7.A Choosing the discrete symmetry

In this appendix, we search for an anomaly-free discrete symmetry which allows all of the necessary terms in the superpotential (7.16) or (7.17) while forbidding all of the problematic operators, (7.24), (7.37)–(7.39), (7.42)–(7.46), (7.47), and (7.48).

In particular, for simplicity we focus on the model with $E, \bar{E}$ leptons and the standard embedding of $SU(3)_F$ within $SU(3)_Q \times SU(3)_L$. Requiring that the superpotential (7.16) transforms as $W \to \eta_W W$, an arbitrary discrete symmetry of the theory must take the form shown in tables 7.4 and 7.5. Henceforward we make the simplifying assumption that the flavor Higgs sector is completely vector-like, i.e. that there exist fields $\bar{\Phi}_u$, $\bar{\Phi}_d$, etc. such that the mass terms $W_{\text{mass}} = M_u \Phi_u \bar{\Phi}_u + M_d \Phi_d \bar{\Phi}_d + \ldots$ can appear in the superpotential. This implies that the flavor Higgs sector makes no net contribution to the anomalies.

Discrete gauge symmetries are far less constrained than continuous gauge symmetries, since they lack cubic anomalies [311]. In fact, the only anomalies which must be cancelled for a discrete gauge symmetry are the $G^2\mathbb{Z}_k$ and $(\text{grav})^2\mathbb{Z}_k$ anomalies for all non-abelian gauge group factors G (precisely those anomalies which relate to gauge and gravitational instantons.) The cancelations of the

	$SU(3)_C$	$SU(2)_L$	$U(1)_Y$	$SU(3)_F$	$\mathbb{Z}_k[\eta_S^3]$
q	$\Box$	$\Box$	$1/6$	$\Box$	η_S
$\bar{u}$	$\bar{\Box}$	$\mathbf{1}$	$-2/3$	$\Box$	$\eta_{\bar{u}}$
$\bar{d}$	$\bar{\Box}$	$\mathbf{1}$	$1/3$	$\Box$	$\eta_{\bar{d}}$
ℓ	$\mathbf{1}$	$\Box$	$-1/2$	$\Box$	$\eta_S^2\eta_{\bar{E}}^{-1}$
$\bar{e}$	$\mathbf{1}$	$\mathbf{1}$	1	$\Box$	$\eta_{\bar{e}}$
$\bar{U}$	$\bar{\Box}$	$\mathbf{1}$	$-2/3$	$\bar{\Box}$	η_S
$\bar{D}$	$\bar{\Box}$	$\mathbf{1}$	$1/3$	$\bar{\Box}$	η_S
$\bar{E}$	$\mathbf{1}$	$\mathbf{1}$	1	$\bar{\Box}$	$\eta_{\bar{E}}$
U	$\Box$	$\mathbf{1}$	$2/3$	$\bar{\Box}$	η_U
D	$\Box$	$\mathbf{1}$	$-1/3$	$\bar{\Box}$	η_D
E	$\mathbf{1}$	$\mathbf{1}$	-1	$\bar{\Box}$	η_E
$\bar{N}$	$\mathbf{1}$	$\mathbf{1}$	0	$\bar{\Box}$	$\eta_{\bar{E}}$
H_u	$\mathbf{1}$	$\Box$	$1/2$	$\mathbf{1}$	η_S
H_d	$\mathbf{1}$	$\Box$	$-1/2$	$\mathbf{1}$	η_S
S	$\mathbf{1}$	$\mathbf{1}$	0	$\mathbf{1}$	η_S

Table 7.4: An arbitrary discrete symmetry which allows the superpotential (7.16) after mixing with an arbitrary subgroup of $U(1)_Y$ and the $\mathbb{Z}_3$ center of $SU(3)_C$, where $\mathbb{Z}_k[\eta_W]$ denotes a discrete R-symmetry under which $W \to \eta_W W$ (i.e. $\theta \to \eta_\theta \theta$ where $\eta_W = \eta_\theta^2$).

	$SU(3)_F$	$\mathbb{Z}_k[\eta_S^3]$
Φ_u	$\Box\Box$	$\eta_S^2\eta_U^{-1}$
Φ_d	$\Box\Box$	$\eta_S^2\eta_D^{-1}$
Φ_e	$\Box\Box$	$\eta_S^3\eta_E^{-1}\eta_{\bar{E}}^{-1}$
Φ_n	$\Box\Box$	$\eta_S^3\eta_{\bar{E}}^{-2}$
ϕ_u	1	$\eta_S^3\eta_U^{-1}\eta_{\bar{u}}^{-1}$
ϕ_d	1	$\eta_S^3\eta_D^{-1}\eta_{\bar{d}}^{-1}$
ϕ_e	1	$\eta_S^3\eta_E^{-1}\eta_{\bar{e}}^{-1}$

Table 7.5: The action of the discrete symmetry defined in table 7.4 on the flavor Higgs sector.

$SU(3)_C^2\,\mathbb{Z}_k$ and $SU(2)_L^2\,\mathbb{Z}_k$ anomalies impose the constraints

$$\eta_{\bar{u}}^3\eta_{\bar{d}}^3\eta_U^3\eta_D^3 = \eta_S^{15}\,, \quad \eta_{\bar{E}}^3 = \eta_S^2\ . \tag{7.69}$$

Assuming that the flavor Higgs sector is vector-like, the $SU(3)_F^2\,\mathbb{Z}_k$ anomaly together with the previous conditions requires

$$\eta_{\bar{e}}\eta_E = \eta_S^5\ . \tag{7.70}$$

Finally, cancellation of the $(\text{grav})^2\mathbb{Z}_k$ anomaly together with the previous conditions naively requires

$$\eta_S^{N_S-2} = 1\,, \tag{7.71}$$

where we now allow for an arbitrary number N_S of NMSSM singlets S and ignore any contribution from other hidden sectors of the theory. Such hidden sectors are inevitably present, however, as a truly complete theory will require a SUSY breaking sector, likely with R-charged gauginos, as well as a supergravity completion with an R-charged gravitino. Thus, while we can solve (7.71) by setting $N_S = 2$, the true anomaly constraint will depend on details of the hidden sector, and hence there is no clear constraint on N_S. It should be noted, however, that regardless of these details the true $(\text{grav})^2\mathbb{Z}_k$ anomaly can usually be cancelled by an appropriate choice of N_S.

The anomaly constraints (7.69) and (7.70) have no analogous caveats, and must be satisfied if no additional SM charged or flavored fields are added to the model. A general solution to these constraints can be parametrized by

$$\eta_{\bar{E}} = \alpha^2\,, \quad \eta_S = \alpha^3\,, \quad \eta_{\bar{e}} = \alpha^{15}\eta_E^{-1}\,, \quad \eta_{\bar{u}} = \omega_3^p\alpha^9\eta_U^{-1}\beta^{-1}\,, \quad \eta_{\bar{d}} = \omega_3^p\alpha^6\eta_D^{-1}\beta\,, \tag{7.72}$$

for phase factors α, β and an integer p, where $\eta_W = \eta_S^3 = \alpha^9$. Thus, $\eta_{\phi_e} = \alpha^{-6}$ and $\eta_{\bar{\phi}_e} = \alpha^{15}$ as a consequence of canceling the $SU(3)_F^2\,\mathbb{Z}_k$ anomaly. We wish to forbid the problematic cross couplings between the flavor and electroweak Higgs sectors (7.47), (7.48), which can lead to fine tuning of the electroweak scale. In particular, to forbid $\phi^2 S$ for $\phi \in \{\phi_e, \bar{\phi}_e\}$ we must require

$$\alpha^{18} \neq 1\,, \quad \alpha^{24} \neq 1\,. \tag{7.73}$$

Thus $\eta_W = \alpha^9 \neq 1$, and we require an R-symmetry.

One can check that these conditions imply that the couplings (7.48) are also forbidden, as are the remaining cross couplings in (7.47) involving only ϕ_e and $\bar{\phi}_e$. Suppose that ϕ is another flavor Higgs singlet in the theory with charge η_ϕ and conjugate field $\bar{\phi}$. To forbid the cross couplings (7.47) between $\phi, \bar{\phi}, \phi_e, \bar{\phi}_e$ and the electroweak Higgs sector (in particular $\phi_i \phi_j S$,) we must require:

$$\eta_\phi \notin \{\alpha^{-9}, \alpha^{-3}, \pm\alpha^3, \pm\alpha^6, \alpha^{12}, \alpha^{18}\}\,. \tag{7.74}$$

There are analogous constraints on the charge η_Φ of a flavor Higgs tensor Φ with conjugate field $\bar{\Phi}$ in order to forbid the cross couplings (7.47) as well as the $\bar{N}$ tadpole (7.42). Using $\eta_{\Phi_n} = \alpha^5$ and $\eta_{\bar{\Phi}_n} = \alpha^4$, we obtain the constraints

$$\eta_\Phi \notin \{\alpha^{-4}, \alpha^{-1}, \pm\alpha^{1/2}, \alpha^2, \omega_3^{\pm1}\alpha^2, \alpha^7, \omega_3^{\pm1}\alpha^7, \pm\alpha^8, \alpha^{11}\}\,. \tag{7.75}$$

These constraints limit the allowed charges of the flavor Higgs fields and hence the form of the Higgs potential. We impose minimum consistency requirements on the flavor Higgs potential: it must contain at least one Φ^3 and at least one $\bar{\Phi}^3$ operator (or else the F-term conditions set the fields to zero) and it must not have any accidental continuous symmetries. Given these requirements, we now search for the smallest possible discrete R-symmetry which allows an acceptable Higgs potential.

The lowest-order $\mathbb{Z}_k$ R-symmetries that do not contradict (7.73) are $k = 5, 7, 10, 11, 13, \ldots$, where $\mathbb{Z}_{10} \cong \mathbb{Z}_5 \times \mathbb{Z}_2$. For $k = 5$ and $k = 7$ one can check that the constraint (7.75) is so restrictive that $\eta_\Phi = \eta_{\Phi_n}$ necessarily, whereas Φ_n^3

and $\bar{\Phi}_n^3$ are forbidden by (7.73). For $k = 10$, we can choose either $\alpha = \omega_{10}$ or $\alpha = \omega_5$. In the former instance, we find that $\{\eta_\Phi\} \subset \{1, \omega_5^2, -1\}$, where $\{\eta_\Phi\}$ is a strict subset since the presence of all three variants will generate the $\bar{N}\Phi\bar{\Phi}^2$ tadpole. Since $\eta_{\Phi_n} = -1$, either $\{\eta_\Phi\} \subseteq \{1, -1\}$ or $\{\eta_\Phi\} \subseteq \{\omega_5^2, -1\}$, but in either case neither Φ^3 nor $\bar{\Phi}^3$ is permitted. For $\alpha = \omega_5$, we find $\eta_\Phi \subseteq \{1, \omega_{10}^3, -1, \omega_{10}^{-3}, \omega_{10}^{-1}\}$, but to avoid all $\bar{N}\Phi\bar{\Phi}^2$ tadpoles as well as $\Phi\bar{\Phi}S$ cross-couplings, we can have at most one additional variant of Φ beyond $\eta_{\Phi_n} = 1$, whereas no such pairing allows both Φ^3 and $\bar{\Phi}^3$ interactions.

The next lowest order possibility is $k = 11$, which we will show to be sufficient. We choose $\alpha = \omega_{11}$ so that $\eta_W = \omega_{11}^{-2}$. The above constraints dictate:

$$\{\eta_\Phi\} \subseteq \{\omega_{11}, \omega_{11}^3, \omega_{11}^4, \omega_{11}^5, \omega_{11}^{-2}\}\,. \tag{7.76}$$

However, one can show that to forbid all of the cross couplings (7.47) and the $\bar{N}\Phi\bar{\Phi}^2$ tadpole we must have $\{\eta_\Phi\} \subseteq \{\omega_{11}, \omega_{11}^3, \omega_{11}^5\}$, $\{\eta_\Phi\} \subseteq \{\omega_{11}^4, \omega_{11}^5\}$ or $\{\eta_\Phi\} \subseteq \{\omega_{11}^{-2}, \omega_{11}^5\}$. No Φ^3 or $\bar{\Phi}^3$ interactions are possible in the latter two cases, so we consider the first case. The possible cubic interactions are:

$$\begin{aligned} &\Phi^3 : \omega_{11} \cdot \omega_{11}^3 \cdot \omega_{11}^5 \ , \ \omega_{11}^3 \cdot \omega_{11}^3 \cdot \omega_{11}^3 \,, \\ &\bar{\Phi}^3 : \omega_{11}^8 \cdot \omega_{11}^8 \cdot \omega_{11}^4 \ , \ \omega_{11}^8 \cdot \omega_{11}^6 \cdot \omega_{11}^6 \,. \end{aligned} \tag{7.77}$$

Since $\eta_{\Phi_n} = \omega_{11}^5$, all three Φ variants must be present to generate both Φ^3 and $\bar{\Phi}^3$.

We also have

$$\{\eta_{\phi_i}\} \subseteq \{1, \omega_{11}^4, \omega_{11}^5, \omega_{11}^{-2}, \omega_{11}^{-1}\}\,. \tag{7.78}$$

Since $\eta_{\phi_e} = \omega_{11}^5$ and $\eta_{\bar{\phi}_e} = \omega_{11}^4$ these variants are always present, and one can show that the Higgs potential has an accidental $U(1)_R$ symmetry unless $\omega_{11}^{-1} \in \{\eta_{\phi_i}\}$ as well. The $\{1, \omega_{11}^{-2}\}$ variants are not necessary, but neither are they problematic. We will assume that they are absent for simplicity. Assuming $\{\eta_\Phi\} = \{\omega_{11}, \omega_{11}^3, \omega_{11}^5\}$ and $\{\eta_{\phi_i}\} = \{\omega_{11}^4, \omega_{11}^5, \omega_{11}^{-1}\}$, one can show that the flavor Higgs potential is free of accidental $U(1)$ symmetries, and that all of the cross couplings (7.47) and all $\bar{N}\Phi\bar{\Phi}^2$ tadpoles are forbidden, as is $(\ell H_u)\Phi^2\bar{\Phi}$.

Because we have assumed a $\mathbb{Z}_{11}$ discrete symmetry, we must set $p = 0$ in (7.72). Thus, $\eta_{\phi_u}\eta_{\phi_d} = \alpha^3$ and so $\beta = \eta_{\phi_u} \in \{\omega_{11}^4, \omega_{11}^{-1}\}$. We must also require $\eta_E \in \{\omega_{11}^2, \omega_{11}^4, \omega_{11}^6\}$. So far our discussion applies to both the standard and flipped embeddings of $SU(3)_F \subset SU(3)_Q \times SU(3)_L$. We now specialize to the standard embedding, which implies that $\eta_U, \eta_D \in \{\omega_{11}, \omega_{11}^3, \omega_{11}^5\}$. For the case $\beta = \omega_{11}^4$, we must have $\eta_U \neq \omega_{11}^3$ and $\eta_D \neq \omega_{11}$ to forbid $\bar{u}\bar{u}\bar{D}\bar{E}$ and $\phi q\ell\bar{d}$ respectively. To forbid $\bar{\Phi}\bar{u}D\bar{E}$ we require $\eta_U\eta_D^{-1} \in \{\omega_{11}^{-4}, \omega_{11}^{-2}, 1\}$ whereas to forbid $\bar{\Phi}U\bar{d}E$ and $\Phi\bar{u}D\bar{e}$ we require $\eta_U\eta_E\eta_D^{-1} \in \{\omega_{11}^7, \ldots, \omega_{11}^{10}, 1\}$. Since $\eta_E \in \{\omega_{11}^2, \omega_{11}^4, \omega_{11}^6\}$ this implies that $\eta_U\eta_D^{-1} \in \{\omega_{11}^{-4}, \omega_{11}^{-2}\}$ and $\eta_E \neq \omega_{11}^6$. Therefore $\eta_U = \omega_{11}$, but to forbid q^2UE we require $\eta_U\eta_E \neq \omega_{11}^3$, so $\eta_E = \omega_{11}^4$ and thus $\eta_U\eta_D^{-1} = \omega_{11}^{-4}$ so that $\omega_D = \omega_{11}^5$. This is the solution presented in section 7.4.

We also consider the case $\beta = \omega_{11}^{-1}$, so that $\eta_D \neq \omega_{11}^3$ to forbid $\phi q\ell\bar{d}$, whereas $\eta_U\eta_D^{-1} \in \{\omega_{11}^2, \omega_{11}^4\}$ to forbid $\bar{\Phi}\bar{u}D\bar{E}$ so that $\eta_D = \omega_{11}$. To forbid $\bar{\Phi}U\bar{d}E$ and $\Phi\bar{u}D\bar{e}$ we now require $\eta_U\eta_E \in \{\omega_{11}^2, \omega_{11}^3, \ldots, \omega_{11}^6\}$. However, the only possible solution is $\eta_U\eta_E = \omega_{11}^5$, i.e. $\eta_U = \omega_{11}^3$ and $\eta_E = \omega_{11}^2$.

We will not discuss this second model in detail, nor attempt to classify $\mathbb{Z}_{11}$ models with a flipped embedding. Note than we have not considered discrete gauge symmetries that are irreducible products, e.g. $\mathbb{Z}_5 \times \mathbb{Z}_5$. It would be interesting to determine if a "simpler" discrete gauge symmetry with the right properties can be found in this way.

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