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CONSTRUCTION OF CONFOUNDED MIXED FACTORIAL AND MIXED LATTICE DESIGNS[2]

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ABSTRACT

This paper extends methods of constructing confounded and lattice designs for the symmetrical factorial to the asymmetrical or mixed factorial. Beside providing unifying techniques some enumeration results are also given along with a fully discussed non-trivial example. In bringing about this extension and unification the usual Galois field theory was modified to ring theory; cyclic collineations as used in ℓ-restrictional lattice designs were adapted to the mixed factorial case.

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1. INTRODUCTION

As pointed out by White and Hultquist [1965] the construction of confounding plans for mixed factorials breaks down when the Galois field approach is taken. They generalized this approach to mixed factorials by presenting a technique of combining elements from distinct prime fields. Raktoe [1968] presented a new and equivalent method of combining elements from k istinct prime fields and in a recent paper [1969] he extended this method to k finite fields, not necessarily associated with distinct primes but these may be prime powered. White and Hultquist [1965] in solving the confounding problem in mixed factorials did not follow the exact same approach as done in the symmetrical factorial case, as for example mentioned in Kempthorne [1952]. For a list of references concerning the problem of confounding in mixed factorials or symmetrical factorials, the reader is referred to White and Hultquist's paper.

The aim of this paper is:

(i) To discuss the construction of confounding plans for the mixed factorial using the classical incomplete block design approach.

(ii) To present some enumeration results on the number of confounding plans for given block sizes.

(iii) To generalize the construction of lattice designs to the "mixed lattice" case in the sense of Raktoe [1967].

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2. PRELIMINARIES AND BACKGROUND

We adopt the notation and results obtained by Raktoe [1969]. In summarized form these are:

(i) $GF(s_1=p_1^{n_1}), GF(s_2=p_2^{n_2}), \cdots, GF(s_k=p_k^{n_k})$ are distinct prime or prime powered fields, in the sense that the p_j's are distinct.

(ii) $P_1(x), P_2(x), \cdots, P_k(x)$ are the prime irreducible polynomials used in constructing $GF(s_1), GF(s_2), \cdots, GF(s_k)$ respectively.

(iii) $R(p)$ is the commutative ring of integers modulus $p = \prod_{j=1}^{k} p_j$.

(iv) $I(b_j)$ is the ideal in $R(p)$ generated by $b_j = 1 + p_j\left(\prod_{i\neq j}^{k} p_i - p_j\right)^{-1}$, $j=1,2,\cdots,k$.

(v) $P_j^*(x) = (b_j)(P_j(x))$ is the polynomial over $I(b_j)$ corresponding to $P_j(x)$, $j=1,2,\cdots,k$.

(vi) $R(x,p)$ is the ring of polynomials over $R(p)$.

(vii) $R(x,b_j) \subset R(x,p)$, is the ring of polynomials over $I(b_j)$, $j=1,2,\cdots,k$.

(viii) $R(x,b_j,P_j^*(x))$ is the residue class ring of $R(x,b_j)$ modulus $P_j^*(x)$, $j=1,2,\cdots,k$.

(ix) $R(x,p,P_1^*(x),P_2^*(x),\cdots,P_k^*(x))$ is the residue class ring of $R(x,p)$ modulus $P_1^*(x)$, modulus $P_2^*(x)$, $\cdots$, modulus $P_k^*(x)$.

(x) $R(x,p,P_1^*(x),P_2^*(x),\cdots,P_k^*(x)) = \sum_{j=1}^{\oplus} R(x,b_j,P_j^*(x))$.

(xi) Definition: $\mu_j \in GF(s_j)$, $\mu_{j*} \in GF(s_{j*})$ and $r \in R(x,p,P_1^*(x),\cdots,P_k^*(x))$

$$j \neq j^* : \mu_j + \mu_{j*} = [(b_j)(\mu_j)] + [(b_{j*})(\mu_{j*})]$$

$$\mu_j \cdot \mu_{j*} = [(b_j)(\mu_j)] \cdot [(b_{j*})(\mu_{j*})] = 0 \quad .$$

$$r + \mu_j = r + [(b_j)(\mu_j)]$$

$$r \cdot \mu_j = r \cdot [(b_j)(\mu_j)] \ .$$

(xii) $R(x,p,P_1^*(x),\cdots,P_k^*(x)) = \sum_{j=1\oplus} GF(s_j)$.

(xiii) G is the multiplicative group consisting of elements in $R(x,p,P_1^*(x),\cdots,P_k^*(x))$, which have multiplicative inverses.

(xiv) $M = \{z' = (z_{11},z_{12},\cdots,z_{1m_1}, z_{21},z_{22},\cdots,z_{2m_2}, \cdots, z_{k1},z_{k2},\cdots,z_{km_k}),$ $z_{ji} \in R(x,p,P_1^*(x),\cdots,P_k^*(x)),\ i=1,2,\cdots,m_j,\ j=1,2,\cdots,k\}$, is the module over $R(x,p,P_1^*(x),\cdots,P_k^*(x))$.

(xv) $T^* = \{z^{*\prime} = (z_{11}^*,z_{12}^*,\cdots,z_{1m_1}^*, z_{21}^*,z_{22}^*,\cdots,z_{2m_2}^*, \cdots, z_{k1}^*,z_{k2}^*,\cdots,z_{km_k}^*),$ $z_{ji}^* \in R(x,b_j,P_j^*(x)),\ i=1,2,\cdots,m_j,\ j=1,2,\cdots,k\}$ is the submodule of M of order $\alpha \left(= \prod_{j=1}^{k} s_j^{m_j}\right)$.

(xvi) $E^* = \{y^{*\prime} = (y_{11}^*,y_{12}^*,\cdots,y_{1m_1}^*, y_{21}^*,y_{22}^*,\cdots,y_{2m_2}^*, \cdots, y_{k1}^*,y^*_{k2},\cdots,y_{km_k}^*) \neq 0',$ $y_{ji}^* \in R(x,b_j,P_j^*(x))$ where $y^{*\prime}$ represents the class $\rho y^{*\prime}, \rho \in G\}$, is a subset of T of order $\beta \left(= \sum_{(i_1,i_2,\cdots,i_t)} \prod \frac{(s_{i_1}^{m_{i_1}}-1)}{(s_{i_1}-1)} \frac{(s_{i_2}^{m_{i_2}}-1)}{(s_{i_2}-1)} \cdots \frac{(s_{i_t}^{m_{i_t}}-1)}{(s_{i_t}-1)}\right)$, where $(i_1,i_2,\cdots,i_t)$ is a subset of $(1,2,\cdots,k)$.

We are now ready to proceed with the confounding problem in mixed factorials.

3. CONFOUNDING PLANS IN MIXED FACTORIALS

Consider the $\prod_{j=1}^{k} s_j^{m_j}$ mixed factorial , i.e. a factorial in which m_j factors are at s_j levels, these levels being marks of the field $GF(s_j)$, then we know that:

(a) $T = \{u' = (u_{11}, u_{12}, \cdots, u_{1m_1}, u_{21}, u_{22}, \cdots, u_{2m_2}, \cdots, u_{k1}, u_{k2}, \cdots, u_{km_k}),$ $u_{ji} \in GF(s_j),\ i=1,2,\cdots,m_j,\ j=1,2,\cdots,k\}$, is the classical way of writing out the $\alpha \left(= \prod_{j=1}^{k} s_j^{m_j}\right)$ treatment combinations in a mixed factorial.

(b) $E = \{v' = (v_{11}, v_{12}, \cdots, v_{1m_1}, v_{21}, v_{22}, \cdots, v_{2m_2}, \cdots, v_{k1}, v_{k2}, \cdots, v_{km_k}) \neq 0',$ $v_{ji} \in GF(s_j)$ such that v_j' represents the calss $\rho_j v_j'$, ρ_j being a non-zero mark of $GF(s_j)\}$, is the classical representation of the β effects in a mixed factorial.

Under the operation of addition of $\left(\sum_{j=1}^{k} m_j\right)$-tuples it can be easily shown that:

(c) T is an Abelian group of order α and in fact $T = \sum_{j=1 \oplus} EG(m_j, s_j)$, where $EG(m_j, s_j)$ denotes the finite Euclidean geometry of dimension m_j over $GF(s_j)$, $j=1,2,\cdots,k$.

(d) E is a concrete representation of the Abelian group T of order α and in fact $E = \sum_{j=1 \oplus}^{k} PG(m_j-1, s_j)$, where $PG(m_j-1, s_j)$ denotes the finite projective geometry of dimension (m_j-1) over $GF(s_j)$, $j=1,2,\cdots,k$.

The following theorem can be established easily:

THEOREM 3.1. T and T* are isomorphic; also E and E* are isomorphic (so that any theorem derived for T and E have equivalent counterparts in T* and E* respectively and vice-versa).

In the following we will henceforth disregard the cumbersome representations T and E and we will work with T* and E* since all operations with the elements of T* and E* can be carried out within the module E* or the larger module M.

The concept of generalized interaction for the mixed factorial is an important one and we will define it as follows: If $y^{*\prime}$ and $y^{**\prime}$ are two elements of E* then the generalized interaction of $y^{*\prime}$ and $y^{**\prime}$ is the set B:

(e) $B = \{(\rho^* y^{*\prime} + \rho^{**} y^{**\prime}),\ \rho^* \text{ and } \rho^{**} \text{ are elements of } G\}$.

Another item especially useful in confounding is the concept of "level of an effect". First of all, the levels of m_j factors are elements of $R(x,b_j,P_j^*(x))$, $j=1,2,\cdots,k$. Now consider a set of ν factors with levels in $R(x,b_{i_1},P_{i_1}^*(x))$, $R(x,b_{i_2},P_{i_2}^*(x))$, $\cdots$, $R(x,b_{i_\nu},P_{i_\nu}^*(x))$ respectively, then the product of these ν factors will be defined to have levels in the ring $\sum_{i_h=1}^{\nu}\oplus R(x,b_{i_h},P_{i_h}^*(x))$. The g^{th} level of an effect (= element of E*) is the following set treatment combinations (= points of T*):

(f) $y_g^{*\prime} = \{z^{*\prime}$ such that $y^{*\prime} \cdot z^* = g$, g an element of the direct sum of the $R(x,b_j,P_j^*(x))$'s corresponding to factors present in $y^{*\prime}\}$. Here $y^{*\prime} \cdot z^*$ is ordinary vector multiplication with prime denoting transpose.

Now, following Federer and Raktoe [1965] and Raktoe [1967] let $\prod_{j=1}^{k} s_j^{m_j} = \left(\prod_{j=1}^{k} s_j^{r_j}\right) \cdot \left(\prod_{j=1}^{k} s_j^{m_j-r_j}\right)$ denote the incomplete block design with $\prod_{j=1}^{k} s_j^{m_j}$ treatment combinations in $\left(\prod_{j=1}^{k} s_j^{r_j}\right)$ blocks of $\prod_{j=1}^{k} s_j^{m_j - r_j}$ plots each, where $0 \le r_j \le m_j$, with not all r_j's simultaneously equal to 0 or equal to m_j, $j=1,2,\cdots,k$. The construction of such a design is equivalent to exhibiting r_1 generators (or independent points) from E_1^*, r_2 generators from E_2^*, etc., and finally r_k generators from E_k^*, where $E_j^* = \{0_1', 0_2', \cdots, 0_{j-1}', y_j, 0_{j+1}', \cdots, 0'\} \subset E^*$. The treatment combinations appearing in the blocks are then found considering combinations of levels of these generators. Before proceeding further it is a convenient place now to go through a complete example.

4. A COMPLETE AND NON-TRIVIAL EXAMPLE

Consider the $3^2 \times 4^3$ mixed factorial, i.e. with 2 factors at 3 levels and 3 factors at 4 levels, then we have, according to sections 2 and 3:

(i) $GF(3)$ and $GF(2^2)$ are the two distinct fields.

(ii) $P_1(x) = x+1$ and $P_2(x) = x^2+x+1$ are the two irreducible polynomials used to construct $GF(3) = \{0,1,2\}$ and $GF(2^2) = \{0,1,x,x+1\}$.

(iii) $R(p) = R(6) = \{0,1,2,3,4,5\}$.

(iv) $I(b_1) = I(4) = \{0,4,2\}$ and $I(b_2) = I(3) = \{0,3\}$.

(v) $P_1^*(x) = 4x+4$, $P_2^*(x) = 3x^2+3x+3$.

(vi) $R(x,6) = \{a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0,\ a_i \in R(6)\}$.

(vii) $R(x,4) = \{d_n x^n + d_{n-1}x^{n-1} + \cdots + d_1 x + d_0,\ d_i \in I(4)\}$ and $R(x,3) = \{f_n x^n + f_{n-1}x^{n-1} + \cdots + f_1 x + f_0,\ f_i \in I(3)\}$. Note that $R(x,4) \subset R(x,6)$ and also $R(x,3) \subset R(x,6)\}$.

(viii) $R(x,4,4x+4) = \{0,4,2\}$, $R(x,3,x^2+x+1) = \{0,3,3x,3x+3\}$.

(ix) $R(x,6,4x+4,3x^2+3x+3) = \{0,1,2,3,4,5,3x,3x+1,3x+2,3x+3,3x+4,3x+5\}$.

(x) $R(x,6,4x+4,3x^2+3x+3) = R(x,4,4x+4) \oplus R(x,3,x^2+x+1)$.

(xi) If $u \in GF(3)$ and $u^* \in GF(2^2)$ and $r \in R(x,6,4x+4,3x^2+3x+3)$ then the rules of additions and multiplications are:

$$u + u^* = [(4)(u)] + [(3)(u^*)]$$

$$u \cdot u^* = [(4)(u)] \cdot [(3)(u^*)] = 0$$

$$r + u = r + [(4)(u)],\ r + u^* = r + [(3)(u^*)]$$

$$r \cdot u = r \cdot [(4)(u)],\ r \cdot u^* = r \cdot [(3)(u^*)] .$$

(xii) Hence: $R(x,6,4x+4,3x^2+3x+3) = GF(3) \oplus GF(2^2)$.

(xiii) $G = \{$elements with multiplicative inverses in $R(x,6,4x+4,3x^2+3x+3)\}$.

(xiv) $M = \{z' = (z_{11},z_{12},z_{21},z_{22},z_{23}),\ z_{ji} \in R(x,6,4x+4,3x^2+3x+3)\}$.

(xv) $T^* = \{z^{*\prime} = (z^*_{11},z^*_{12},z^*_{21},z^*_{22},z^*_{23}),\ z_{1i} \in R(x,4,4x+4),\ z_{2i} \in R(x,3,3x^2+3x+3)\}$. T^* is of order $3^2 \times 4^3 = 576$.

(xvi) $E^* = \{y^{*\prime} = (y^*_{11},y^*_{12},y^*_{21},y^*_{22},y^*_{23}) \neq 0',\ y^*_{1i} \in R(x,4,4x+4)$ and $y^*_{2i} \in R(x,3,3x^2+3x+3)$; $y^{*\prime}$ represents $\rho y^{*\prime}$, $\rho \in G\}$. Note that E^* is of order $\frac{(3^2-1)}{(3-1)} + \frac{(4^3-1)}{(4-1)} + \frac{(3^2-1)(4^3-1)}{(3-1)(4-1)} = 109$.

(a) $T = \{u' = (u_{11},u_{12},u_{21},u_{22},u_{23}),\ u_{1i} \in GF(3),\ i=1,2,\ u_{2i'} \in GF(2^2),$ $i'=1,2,3\}$. T is of order $3^2 \times 4^3 = 576$.

(b) $E = \{v' = (v_{11},v_{12},v_{21},v_{22},v_{23}) \neq 0',\ v_{1i} \in GF(3),\ v_{2i} \in GF(2^2)$, such that $v_1' = (v_{11},v_{12})$ represents the class $\rho_1 \cdot (v_{11},v_{12})$ and $v_2' = (v_{21},v_{22},v_{23})$ represents the class $\rho_2 \cdot (v_{21},v_{22},v_{23})$ with $\rho_1 \in GF(3)$ and $\rho_2 \in GF(2^2)\}$. E is of order 109.

(c) T is an Abelian group of order 576 under addition of 5-tuples and $T = EG(2,3) \oplus EG(3,2^2)$.

(d) E is a concrete representation of T such that $E = PG(1,3) \oplus PG(2,2^2)$.

Theorem 3.1: T is isomorphic to T* and E is isomorphic to E*.

(e) $B = \{(\rho^* y^{*\prime} + \rho^{**} y^{**\prime}),\ y^*$ and $y^{**} \in E^*,\ \rho^*$ and $\rho^{**} \in G\}$, where $y^{*\prime}$ and $y^{**\prime}$ and G are as above.

(f) $y^{*\prime}_g = \{z^{*\prime} \ni y^* \cdot z^{*\prime} = g\}$, for example: $(4,4,3,3,3)_5 = \{z^{*\prime} \ni 4z^*_{11} + 4z^*_{12} + 3z^*_{21} + 3z^*_{22} + 3z^*_{23} = 5\}$.

Consider the incomplete block design $3^2 \cdot 4^3 = (3^{r_1}.4^{r_2}) \cdot (3^{2-r_1}.4^{3-r_2})$, i.e. a design consisting of $(3^{r_1}.4^{r_2})$ blocks of $(3^{2-r_1}.4^{3-r_2})$ plots each with $0 \le r_1 \le 2$ and $0 \le r_2 \le 3$ excluding the cases $(r_1,r_2) = (0,0)$ and $(r_1,r_2) = (2,3)$. Needless to say that there are many possibilities here, e.g. $3^2 \cdot 4^3 = (3 \cdot 4^3) \cdot (3)$, $3^2 \cdot 4^3 = (4^3) \cdot (3^2)$, $3^2 \cdot 4^3 = (3 \cdot 4) \cdot (3 \cdot 4^2)$, etc. To indicate how to exhibit the generators for the design $3^2 \cdot 4^3 = (3 \cdot 4) \cdot (3 \cdot 4^2)$, we see that in this instance we <u>must</u> confound 1 generator from $E^*_1 = \{y^*_{11}, y^*_{12}, 0,0,0) \ne 0'\}$ and 1 generator from $E^*_2 = \{(0,0,y^*_{21}, y^*_{22}, y^*_{23}) \ne 0'\}$. Since E^*_1 is of order 4 and E^*_2 is of order 21 it follows that 84 choices are available in exhibiting such a pair of generators. The block constituents are found by considering combinations of levels (ℓ_1, ℓ_2) of generators, ℓ_1 from $R(x,4,4x+4)$ and ℓ_2 from $R(x,3,x^2+x+1)$ and then solving for the treatment combinations $z^{*\prime}$'s (or equivalently by considering a level ℓ of the resulting interaction of the generators, $\ell \in R(x,6,4x+4, 3x^2+3x+3)$, and then solving for the treatment combinations). Thus, for example, if a pair of generators was $[(4,4,0,0,0),(0,0,3x,3x+3)]$, then the block constituents are found most easily by considering the levels of the interaction

$(4,4,3x,3x+3)$, i.e. $(4,4,3x,3x+3)_{\ell}$, where $\ell \in R(x,6,4x+4,3x^2+3x+3)$. Each level would then represent the treatment combinations in a block, thus achieving the required 12 blocks.

In the next section we present some enumeration results on the number of possible confounding schemes.

5. ENUMERATION OF CONFOUNDING SCHEMES

For the symmetrical s^m factorial, let EG(m,s) and PG(m-1,s) be the corresponding finite Euclidean and finite projective geometries, then an incomplete block design consisting of s^r blocks of s^{m-r} plots each, requires the exhibition of an (r-1)-flat of PG(m-1,s) (for example see Raktoe [1967]). We know (e.g. see Mann [1949]) that the number of (r-1)-flats in PG(m-1,s) is given by the formula:

$$\text{(g)} \qquad \phi\Big((r-1),(m-1),s\Big) = \frac{(1+s+s^2+\cdots+s^{(m-1)})\ \cdots\cdots\ (s^{(r-1)}+s^2+\cdots+s^{(m-1)})}{(1+s+\cdots+s^{(r-1)})\ \cdots\cdots\ (s^{(r-2)}+s^{(r-1)})s^{(r-1)}}$$

Now, going back to our setting, let $\prod_{j=1}^{k} s_j^{m_j} = \left(\prod_{j=1}^{k} s_j^{r_j}\right).\left(\prod_{j=1}^{k} s_j^{m_j-r_j}\right)$ denote the incomplete block design, then we must choose from each E_j^* an (r_j-1)-flat since E_j^* is isomorphic to $PG(m_j-1,s_j)$, $j=1,2,\cdots,k$. From (g) it follows then, that we have precisely the following number of possible selections or confounding schemes:

$$\text{(h)} \qquad \prod_{j=1}^{k} \varphi\Big((r_j-1),(m_j-1),s_j\Big) \ .$$

Hence we have generalized the construction of confounding plans from the symmetrical factorial to the mixed factorial case, the first one being a special case now. The important contribution of the results in section 2 is that the Galois field approach used in the construction of symmetrical confounded designs has been generalized to the mixed factorial case. The contribution of the ideas of section 3 allows finite geometric results used in symmetrical factorials to be generalized to mixed factorials. In other words, we have one unified theory.

6. MIXED LATTICE DESIGNS

The ideas in this section will be highly correlated with the results of a paper by Raktoe [1967] and Federer and Raktoe [1965]. For the s^m symmetrical factorial the treatment combinations when identified with a set of $v = s^m$ varieties or treatments lead to designs known in the literature as pseudo-factorial of lattice designs. Elimination of block heterogeneity can be combined with these designs to produce in general ℓ-restrictional lattices. Using the notation of Raktoe [1967], we may define the ℓ-restrictional lattice design for the symmetric s^m case by writing $s^m = \prod_{i=1}^{\ell} s^{r_i}$, where $\sum_{i=1}^{\ell} r_i = m$ with the meaning that the s^m treatments are allocated to experimental units according to ℓ-restrictions, $\ell \leq m$. Since in any ℓ-restrictional $s^m = \prod_{i=1}^{\ell} s^{r_i}$ lattice design the pseudo-effects have no meaning we adopt the rule that with any such design we will associate that PG(m-1,s) such that $(s^m-1)/(s-1)$ and $(s^{r_i}-1)/(s-1)$, $i=1,2,\cdots,\ell$, are relatively prime. Thus for the $2^4 = 2^2 \cdot 2^2$ lattice square we would use the $PG(1,2^2)$ and not the PG(3,2). A balanced ℓ-restrictional symmetrical lattice design is a minimal set of confounding schemes such that each of the $(s^m-1)/(s-1)$ pseudo-effects is confounded an equal number of times in each of the ℓ-restrictions. Raktoe [1967] has shown that for balance we need $(s^m-1)/(s-1)$ arrangements generated by a cyclic collineation of order $(s^m-1)/(s-1)$. Let us now generalize the above concepts to the mixed lattice designs.

Let $\prod_{j=1}^{k} s_j^{m_j} = \left(\prod_{j=1}^{k} s_j^{r_{j1}}\right) \cdot \left(\prod_{j=1}^{k} s_j^{r_{j2}}\right) \cdots \left(\prod_{j=1}^{k} s_j^{r_{j\ell}}\right)$ denote the ℓ-restrictional mixed lattice design, where $\sum_{i=1}^{\ell} r_{ji} = m_j$, $j=1,2,\cdots,k$, then from the above and from section 5, we know that we must confound a k-tuple of flats $\big((r_{1i}-1)$-flat, $(r_{2i}-1)$-flat, $\cdots$, $(r_{ki}-1)$-falt$\big)$ in the i^{th} restriction, such that:

(i) the $(r_{ji}-1)$-flat is in E_j^*, $i=1,2,\cdots,\ell$, $j=1,2,\cdots,k$.

(ii) the ℓ-tuple of flats $\big((r_{j1}-1)$-flat, $(r_{j2}-1)$-flat, $\cdots$, $(r_{j\ell}-1)$-flat$\big)$ exhaust E_j^*, $j=1,2,\cdots,k$.

These two conditions together imply then the fact that we have exhausted $E^* = \sum_{j=1}^{k}{}_{\oplus} E_j^*$.

Now, a balanced ℓ-restrictional lattice with respect to every point of E_j^* requires $(s_j^{m_j}-1)/(s_j-1)$ confounding schemes given by a collineation in E_j^* of order $(s_j^{m_j}-1)/(s_j-1)$, $j=1,2,\cdots,k$. Using these collineations we may construct $\prod_{j=1}^{k}(s_j^{m_j}-1)/(s_j-1)$ confounding schemes for our mixed ℓ-restrictional lattice design. This set of confounding schemes will be "<u>balanced</u>" in the sense that each point of E_j^* will be confounded λ_{ji} times in the i^{th} restriction, where $\lambda_{ji} = [(s_j^{r_{ji}}-1)/(s_j-1)] \cdot [\prod_{h\neq j}^{k}(s_h^{m_h}-1)/(s_h-1)]$, $i=1,2,\cdots,\ell$, $j=1,2,\cdots,k$, and all the remaining points of E^*, (i.e. $E^* - \bigcup_{j=1}^{k} E_j^*$) will be confounded

$\prod_{j=1}^{k} (s_j^{r_{ji}}-1)/(s_j-1)$ in the i^{th} restriction. Note that these remaining points of E^* are precisely the generalized interactions of points from the components of the k-tuple of flats $\big((r_{1i}-1)\text{-flat}, (r_{2i}-1)\text{-flat}, \cdots, (r_{ki}-1)\text{-flat}\big)$, $i=1,2,\cdots,\ell$.

To illustrate the results of this section let us pursue the example initiated in section 4. Consider the two-restrictional mixed lattice design $3^2 \times 4^3 = (3 \times 4) \cdot (3 \times 4^2)$, i.e. 576 treatment combinations to be allocated to a design with 12 rows and 48 columns. Here $r_{11}=1$, $r_{21}=1$, $r_{12}=1$, $r_{22}=2$ and $\ell=2$. We must confound a 2-tuple of flats, namely, a $\big((r_{11}-1)=0\text{-flat}, (r_{21}-1)=0\text{-flat}\big)$ in the 1^{st} restriction and a 2-tuple of flats $\big((r_{12}-1)=0\text{-flat}, (r_{22}-1)=1\text{-flat}\big)$ in the second restriction, where the $(r_{ji}-1)$-flat is in E_j^*, i=1,2 and j=1,2. The two collineations involved in this instance are obtained from Raktoe's [1967] paper, i.e.:

$$\begin{bmatrix} 0 & 4 \\ 4 & 4 \end{bmatrix} \text{ in } E_1^* \text{ and } \begin{bmatrix} 0 & 3 & 0 \\ 0 & 0 & 3 \\ 3x & 3x & 3x \end{bmatrix} \text{ in } E_2^*$$

The first one is of order 4 and the second one of order 21. Hence we may construct 84 confounding schemes, such that each point of E_1^* is confounded $\lambda_{11} = 21$ times in the first restriction and $\lambda_{12} = 21$ times in the second restriction; also each point of E_2^* is confounded $\lambda_{21} = 4$ times in the first restriction and $\lambda_{22} = 20$ times in the second restriction; finally every point of $(E^* - E_1^* \cup E_2^*)$ is confounded once in the first restriction and five times in the second restriction.

From the above developments we see that the ℓ-restrictional lattice design of the type s^m as discussed by Raktoe [1967] has been extended to the mixed case. One additional problem still remains concerning mixed lattices, namely, the extension of the analysis within the framework of the paper by Federer and Raktoe [1965]. This problem is currently under study by the authors.

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