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TECHNICAL REPORT NO. 1022

August 1992

PERCENTILE BOUNDS AND PREDICATION LIMITS FOR THE INVERSE GAUSSIAN DISTRIBUTION

by

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Summary & Conclusions—

In spite of the percentile of Inverse Gaussian distribution is a implicit function, a mathematical programming approach is used to derive the s–confidence bounds for the percentiles from the s–confidence intervals of its parameters. The exact 2–sided predication limits for future observations are also obtained by applying this approach. An example is presented for illustrative purpose.

1. INTRODUCTION

The estimate of percentiles is useful for reliability evaluation. The cumulative distribution function of Inverse Gaussian distribution is expressed in terms of the standard normal distribution and cannot inverted as a explicit function. In this paper, a exact 2–sided s–confidence bounds for the percentiles of Inverse Gaussian distribution is constructed from the confidence intervals of its parameters by modeling the problem as a mathematical programming.

Chhikara & Guttman [3] obtained exact predication intervals for a single future observation of the Inverse Gaussian distribution from both frequency and Bayes viewpoint. Padgett [6] proposed an approximate predication interval for the mean of future observations from the Inverse Gaussian distribution. The approach for constructing percentile bounds can also be used to obtain the predication limits for the Inverse Gaussian distribution is presented.

Notation

MLE	Maximum likelihood estimator
ξ, λ	parameters of Inverse Gaussian distribution
IG(ξ, λ)	Inverse Gaussian distribution with parameters ξ, λ
$\Phi(\cdot)$	cdf of standard normal distribution
t_p	pth percentile
$\hat{(\cdot)}$	MLE of $(\cdot)$
$\underline{(\cdot)}$, $\overline{(\cdot)}$	lower and upper s–confidence intervals of $(\cdot)$
$F^{\nu 1}_{\nu 2,\gamma}$	γ percentile of F distribution with df (ν1, ν2)
$\chi_{\nu,\gamma}$	γ percentile of χ distribution with df ν
γ_i, α	confidence level, $0 \le \gamma_i$, $\alpha \le 1$

p_ℓ, p_u lower and upper predication probability

$[\ell(y), u(y)]$ predication limits

Other, standard notation is given in " Information for Readers & Authors " at rear of each issue.

2. THE PERCENTILE BOUNDS

The pdf of Inverse Gaussian distribution is considered as the following form with parameters ξ and λ:

$$f(t;\, \xi,\, \lambda) = [\, \lambda/(2\pi t^3)^{1/2}].\exp[\, -\lambda(t-\xi)^2/2\xi^2 t\,], \quad t>0, \;\; \xi, \lambda > 0. \tag{1}$$

The distribution function of eq.(1) is given by

$$F(t;\, \xi,\, \lambda) = 1 - \Phi[\sqrt{\lambda/t}\cdot(1 - \frac{t}{\xi})] + \exp(\frac{2\lambda}{\xi})\cdot\Phi[-\sqrt{\lambda/t}\,(1 + \frac{t}{\xi})]. \tag{2}$$

The pth percentile t_p of Inverse Gaussian distribution is the solution of

$$F(t_p;\, \xi,\, \lambda) - p = 0. \tag{3}$$

We denote this solution as $t_p(\xi, \lambda)$ to indicate its dependent on ξ and λ.

Suppose that ξ is known, then let $\mathbf{X} = \mathbf{T} / \xi$. In figure 1, we plot $x_p = t_p/\xi$ against $\sigma = \lambda/\xi$ for various p. It can be seen that x_p is not monotone with respect to σ. Similarly, suppose that λ is known, and let $\mathbf{Y} = \mathbf{T} / \lambda$, we plot $y_p = t_p/\lambda$ against $\theta = \xi/\lambda$ as shown in figure 2. Note that the y_p is monotone increasing with respect to θ. Consequently, there is no equivariant confidence set for the percentiles of Inverse Gaussian distribution [5].

The MLE of ξ and λ are given by Chhikara and Folks [2] as

$$\hat{\xi} = \frac{1}{n}\sum_{i}^{n} T_i \tag{4}$$

$$\hat{\lambda} = \left[\frac{1}{n}\sum_{i}^{n}\left(\frac{1}{T_i} - \frac{1}{\hat{\xi}}\right)\right]^{-1} \tag{5}$$

Their $(1-\gamma)$ level confidence intervals are given by

$$[\,\underline{\xi}, \overline{\xi}\,] = \left[\left(\frac{1}{\hat{\xi}} + \sqrt{\frac{F^{1}_{n-1,1-r}}{(n-1)\hat{\lambda}\hat{\xi}}}\right)^{-1}, \left(\frac{1}{\hat{\xi}} - \sqrt{\frac{F^{1}_{n-1,1-r}}{(n-1)\hat{\lambda}\hat{\xi}}}\right)^{-1}\right] \tag{6}$$

$$[\,\underline{\lambda}, \overline{\lambda}\,] = \left[\frac{\hat{\lambda}\chi_{n-1,r/2}}{n}, \frac{\lambda\chi_{n-1,1-r/2}}{n}\right] \tag{7}$$

respectively.

The following results give the exact confidence bounds for t_p given in eq.(3).

Theorem 1: The $(1-\gamma_1-\gamma_2+\gamma_1\gamma_2)$ level confidence bounds for t_p in eq.(3) is given by the interval

$$[\,\underline{t}_p, \overline{t}_p\,] = \left[\inf_{\underline{\lambda}\leq\lambda\leq\overline{\lambda}} t_p(\underline{\xi}, \lambda), \sup_{\underline{\lambda}\leq\lambda\leq\overline{\lambda}} t_p(\overline{\xi}, \lambda)\right] \tag{8}$$

for each p, where $\underline{\xi}$, $\overline{\xi}$ are given in eq.(6) with γ replaced by γ_1; and $\underline{\lambda}$, $\overline{\lambda}$ are given in eq.(7) with γ replaced by γ_2.

Proof: See Appendix

Since $t_p(\xi, \lambda)$ in eq.(3) is a implicit function, in practice, we may be more convenient in solving

$$\text{Min.(Max.) } f = t_p \qquad (9)$$

subject to

$$F(t_p;\ \xi, \lambda) - p = 0$$
$$\underline{\lambda} \le \lambda \le \overline{\lambda}$$
$$0 \le t_p$$
$$\xi = \underline{\xi}\ (\ \xi = \overline{\xi}\ )$$

There are many mathematical programming software can be used to solve eq.(9), such as the Optimization toolbox/*MATLAB* [7], etc

3. THE PREDICATION LIMITS

Suppose that $Y_1, Y_2, \cdots, Y_m$ are future random sample from Inverse Gaussian distribution $IG(\xi, \lambda)$. Then $\overline{Y}_m$, the mean of future observations, has $IG(\xi, m\lambda)$distribution. We want to construct a $100(1 - \alpha)^2\%$ exact 2–sided predication interval $[\ \ell(\mathbf{y}), u(\mathbf{y})\ ]$ for $\overline{Y}_m$. These two–sided predication limits $\ell(\mathbf{y})$ and $u(\mathbf{y})$ are given by

$$\int_{-\infty}^{\ell(\mathbf{y})} f(x;\ \xi, m\lambda)\, dx = \alpha - \frac{\alpha^2}{2} \qquad (10)$$

and

$$\int_{-\infty}^{u(\mathbf{y})} f(x;\ \xi, m\lambda)\, dx = 1 - \alpha + \frac{\alpha^2}{2}, \qquad (11)$$

where $f(x;, \xi, m\lambda)$ is the pdf of $IG(\xi, m\lambda)$.

The predication limits [$\ell(y)$, $u(y)$] is equivalent to find the lower bound for the percentile $p_\ell = \alpha - \alpha^2/2$ and the upper bound for the percentile $p_u = 1 - \alpha + \alpha^2/2$. In this case, we need only one–sided confidence interval for ξ. That is,

$$\text{Min.(Max.) } f = \ell(y) \text{ (} u(y) \text{)} \tag{12}$$

subject to

$$F(y_{p_\ell}; \xi, m\lambda) - p_\ell = 0 \text{ (} F(y_{p_u}; \xi, m\lambda) - p_u = 0 \text{)}$$

$$\underline{\lambda} \le \lambda \le \overline{\lambda}$$

$$0 \le \ell(y) \text{ (} u(y) \text{)}$$

$$\xi = \underline{\xi} \text{ (} \overline{\xi} \text{)}$$

where $P[\ \lambda \le \lambda \le \lambda\ ] = 1 - \alpha$, $P[\ \xi \ge \underline{\xi}\ ] = 1 - \alpha$ and $P[\ \xi \le \overline{\xi}\ ] = 1 - \alpha$.

4. ILLUSTRATIVE EXAMPLE

The data of this example consist of active repair time (in hours) for an airborne communication transceiver. The data set were used in Hsieh [4].

Here, n=46 and the MLE of the parameters are

$$\hat{\xi} = 3.6065 \text{ and } \hat{\lambda} = 1.6589$$

and the 97.5% confidence intervals of ξ and λ are computed from eq.(6) and (7) as [2.3911, 7.3639] and [0.9548, 2.5411] respectively.

The MLE and 95.0625% confidence bounds of t_p for various 100·pth percentiles are obtained by solving eq.(3) and eq.(9) as table 1. Note that the trajectory of λ is switching from $\underline{\lambda}$ to $\overline{\lambda}$ as p increasing. The trajectories of ξ and λ is to be expected from figure 1.

Consider the size of future observation are m=1 and 5 respectively. Then the various $100\cdot(1-\alpha)^2\%$ predication limits are obtained by eq.(12) as table 2. The trajectories of ξ and λ is also to be expected from figure 1 since t_p is monotone increasing with respect to λ as $p_\ell \leq 0.6$ and t_p is monotone decreasing with respect to $\lambda/\xi \geq 1.0$ as $p_u \geq 0.8$. Although the predication limits are wider than Padgett & Tsois, however, this approach guaranteed the desired coverage probability while Padgett and Tsoi's interval only gives an approximation.

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APPENDIX: PROOF OF THEOREM 1

By the definition of confidence bound in Bickel and Doksum [1], we only need to show that

$$P[\inf_{\underline{\lambda}\le\lambda\le\overline{\lambda}} t_p(\underline{\xi}, \lambda) \le t_p(\xi, \lambda) \le \sup_{\underline{\lambda}\le\lambda\le\overline{\lambda}} t_p(\overline{\xi}, \lambda)\,] \ge 1 - \gamma_1 - \gamma_2 + \gamma_1\gamma_2.$$

It follows that

$$\inf_{\Omega} t_p(\xi, \lambda) \le t_p(\xi, \lambda) \le \sup_{\Omega} t_p(\xi, \lambda) \text{ whenever } \underline{\xi} \le \xi \le \overline{\xi},\ \underline{\lambda} \le \lambda \le \overline{\lambda}$$

where $\Omega = \{ (\xi, \lambda)\!: \underline{\xi} \le \xi \le \overline{\xi},\ \underline{\lambda} \le \lambda \le \overline{\lambda} \}$, since the latter is a subset of the former. Thus, from the concept of probability, we have

$$P[\inf_{\Omega} t_p(\xi, \lambda) \le t_p(\xi, \lambda) \le \sup_{\Omega} t_p(\xi, \lambda)\,]$$

$$\ge \quad P[\,\underline{\xi} \le \xi \le \overline{\xi},\ \underline{\lambda} \le \lambda \le \overline{\lambda}\,]$$

$$= \quad 1 - \gamma_1 - \gamma_2 + \gamma_1\gamma_2 \text{ since } \hat{\xi} \text{ and } \hat{\lambda} \text{ are independent.}$$

From figure 2, it can be seen that $t_p(\xi, \lambda)$ is monotone increasing in ξ. Hence it follows that

$$\inf_{\Omega} t_p(\xi, \lambda) = \inf_{\underline{\lambda}\le\lambda\le\overline{\lambda}} t_p(\underline{\xi}, \lambda) \text{ and } \sup_{\Omega} t_p(\xi, \lambda) = \sup_{\underline{\lambda}\le\lambda\le\overline{\lambda}} t_p(\overline{\xi}, \lambda)\ .$$

□

Table 1

95.0625% confidence bounds for percentile t_p

100pth	MLE	Confidence bounds	Trajectories of ξ and λ
25	0.8939	[.512,1.418]	$[\underline{\xi}, \underline{\lambda}]$ & $[\overline{\xi}, \overline{\lambda}]$
50	1.7810	[1.098,3.123]	$[\underline{\xi}, \underline{\lambda}]$ & $[\overline{\xi}, \overline{\lambda}]$
75	4.0508	[2.601,7.712]	$[\underline{\xi}, \underline{\lambda}]$ & $[\overline{\xi}, \overline{\lambda}]$
90	8.5719	[5.069,17.878]	$[\underline{\xi}, \overline{\lambda}]$ & $[\overline{\xi}, 2.118]$
95	13.0877	[6.856,32.617]	$[\underline{\xi}, \overline{\lambda}]$ & $[\overline{\xi}, \underline{\lambda}]$

Table 2

Predication limits for m=1 and 5

	$100\cdot(1-\alpha)^2\%$	$[\ell(y), u(y)]$	Trajectories of ξ and λ	Padgett & Tsoi's predication limits
	90.25	[.226, 25.28]	$[\underline{\xi}, \underline{\lambda}]$ & $[\overline{\xi}, \underline{\lambda}]$	[.423, 30.75]
m=1	95.0625	[.168, 48.19]	$[\underline{\xi}, \underline{\lambda}]$ & $[\overline{\xi}, \underline{\lambda}]$	[.318, 41.03]
	99.0025	[.099, 189.79]	$[\underline{\xi}, \underline{\lambda}]$ & $[\overline{\xi}, \underline{\lambda}]$	[.193, 67.53]
	90.25	[.744, 17.46]	$[\underline{\xi}, \underline{\lambda}]$ & $[\overline{\xi}, \underline{\lambda}]$	[1.107, 11.75]
m=5	95.0625	[.590, 27.68]	$[\underline{\xi}, \underline{\lambda}]$ & $[\overline{\xi}, \underline{\lambda}]$	[.907, 14.34]
	99.0025	[.382, 79.68]	$[\underline{\xi}, \underline{\lambda}]$ & $[\overline{\xi}, \underline{\lambda}]$	[.624, 20.84]

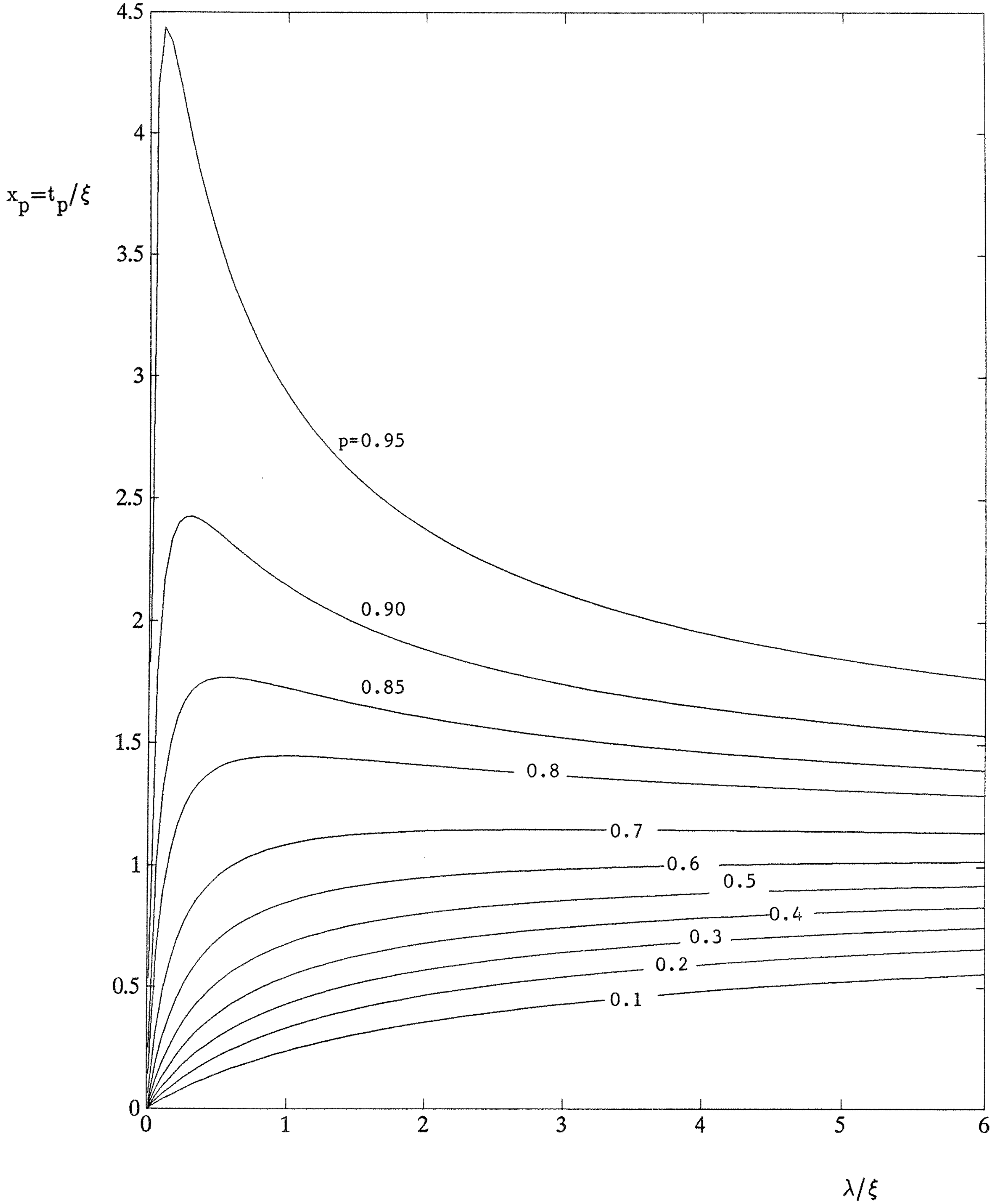

Figure 1: The pth percentiles as a function of λ/ξ

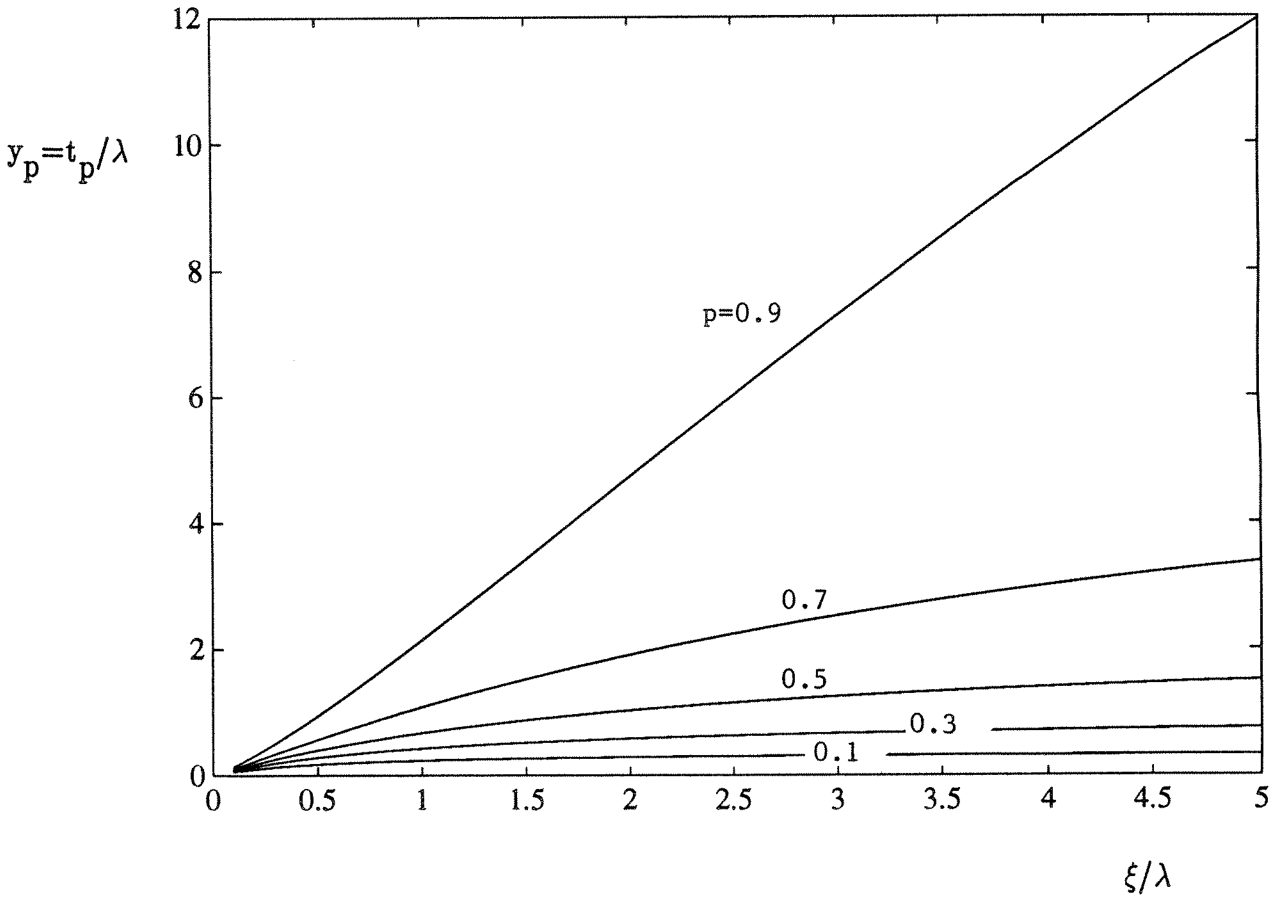

Figure 2: The pth percentiles as a function of ξ/λ