SUBEXPONENTIAL TAILS AND LONG-RANGE DEPENDENCE: CLUSTERED BEHAVIORS OF EXTREME VALUES

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SUBEXPONENTIAL TAILS AND LONG-RANGE DEPENDENCE: CLUSTERED BEHAVIORS OF EXTREME VALUES

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I study extreme values from certain stationary infinitely divisible (SID) processes with subexponential tails. These processes are represented as stochastic integrals of a nonrandom function composed with iterated transformations with respect to some random measure. In this ergodic theory approach, I concentrate on conservative dynamics which generate long memories of various strength. I prove functional theorems both in the space of sup measures and in the space of Càdlàg functions for such SID processes.

In a subexponential stochastic system, extremal events are often caused by a unique dominating movement. However, under long range dependence, I demonstrate that extremes in a SID process may come from multiple large values.

Extremes for a class of a symmetric stable random fields are thoroughly studied. For such tails, only under moderate long range dependence, a unique large value determines the extremal behavior and the limits have the Fréchet distribution, and the non-Fréchet limits under stronger long memory are also characterized. However, a moderately long range dependent setting is not sufficient to guarantee “heuristic of a single big value”. In this moderate zone, as the subexponential tails become light enough, i.e. dropping into the Gumbel maximum domain of attraction, extremes may automatically depend on multiple large values of the driving noise.

In addition, I show that extremal clusters can be quite intricate and they result from the interactions between the flows and the noise. In the extremal limit theorems of sup-measures, the limits are based on random upper continuous functions supported on stable regenerative sets. If the tails are heavy enough, e.g. power-law tails and lognormal tails, extremal clusters share the common feature that each stable regenerative set supports one value. However, for semiexponential tails, a new shape arises, where each stable regenerative set supports a random panoply of varying extremes. In the presence of long memory, fine heterogeneities within the subexponential distributions can be manifested.

BIOGRAPHICAL SKETCH

Zaoli Chen was born in 1994 in Huaibei, a once glorious city in east China for coal industry. His interest in math began in high school, where he was accidentally selected to be the math course representative in his class. He then became a math major at University of Science and Technology of China. After graduation, his good luck led him to the Department of Mathematics at Cornell University to pursue his doctoral degree. He was even more fortunate to be chosen by Professor Gennady Samorodnitky, who guided him towards applied probability. Zaoli will join the Department of Mathematics and Statistics at University of Ottawa as a postdoctoral researcher in the August of 2021.

To my grandfather Jiaming Zhang, who enlightened my teenage days with physics and mathematics.

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CHAPTER 1

INTRODUCTION

1.1 A Glimpse of Extreme Value Theory

Extreme value theory (EVT) studies the pattern of dominantly large values in a stochastic system. For example, let $\{X_t\}_{t\in\mathbb{Z}}$ be an independent and identically distributed (*i.i.d.*) sequence of random variables, and let $M_n := \bigvee_{t=0}^{n} X_t$ for $n \in \mathbb{N}$ be the partial maximums. A natural question in EVT is to characterize the possible nondegenerate limit distributions of the normalized maximums,

$$\frac{M_n - b_n}{a_n} \Rightarrow F, \tag{1.1}$$

where $a_n, b_n \in \mathbb{R}$ are normalizing constants. The limit distribution F is referred to as a extreme value distribution. If we specify $F(x)$ up to a composition $F(ax+b)$ for some $a > 0$ and $b \in \mathbb{R}$, then there are three classes of distributions.

(i) $\alpha-$Fréchet distribution: $\Phi_\alpha(x) = \exp\{-x^{-\alpha}\}\mathbb{1}_{\{x>0\}}$ for some $\alpha > 0$.

(ii) $\alpha-$Weibull distribution: $\Psi_\alpha(x) = \exp\{-(-x)^\alpha\}\mathbb{1}_{\{x\leqslant 0\}} + \mathbb{1}_{\{x>0\}}$ for some $\alpha > 0$.

(iii) Gumbel distribution: $\Lambda(x) = \exp\{-e^{-x}\}$ for all $x \in \mathbb{R}$.

This cornerstone result was developed by Fréchet (1927), Fisher and Tippett (1928), von Mises (1936) and the seminal work Gnedenko (1943). The monograph Gumbel (1958) is the first monograph on EVT, which summarizes the early theory and shows engineering applications.

Modern EVT covers a wider scope and extremes values may arise from multivariate, functional, or dependent time series.The superb book Leadbetter et al.

(1983) discusses extremes from stationary processes. Probabilistic and statistical aspects of EVT are systematically formulated in Resnick (1987), Beirlant et al. (2004), de Haan and Ferreira (2006) and Falk (2019). Statistical methodologies are further developed to meet the practical demands from environmental sciences, operation research, financial engineering and other fields. See, *e.g.*, de Haan and de Ronde (1998),de Haan et al. (1994), Katz et al. (2002), and Naveau et al. (2005).

Random fields contribute to another fruitful pool of topics, whose "time" parameters typically live in spaces of dimensionality more than one. The theoretical analysis of extremes from random fields are often related to other fields in mathematics, *e.g.* Riemannian geometry and algebraic topology. The applications vary from oceanography to neuroimaging. See Adler and Taylor (2007), Adler et al. (2010), and Azaïs and Wschebor (2009).

In recent years, we have also seen the active interactions between EVT and other disciplines of probability that are related to statistical mechanics. For example, extremes of Gaussian free fields are analyzed in Ding and Zeitouni (2014) and Bramson et al. (2016). For the extremes from branching Brownian motions, see Arguin et al. (2011) and Arguin et al. (2013).

1.2 Subexponential Distributions

In this dissertation, we are mainly interested in extreme value properties of subexponential distributions. We adopt the definition in Chistyakov (1964), *i.e.*, a random variable X with distribution $H(x) = \mathbb{P}\{X \leqslant x\}, x \in \mathbb{R}$ is *subexponential* if

$$\lim_{x\to\infty} \frac{\overline{H * H}(x)}{\overline{H}(x)} = 2, \tag{1.2}$$

where $\overline{H}(x) = 1 - H(x)$ is the right tail and $*$ denotes the convolution. A consequence of (1.2) is

$$\mathbb{E}e^{\epsilon X} = \infty, \quad \text{for all } \epsilon > 0, \tag{1.3}$$

which intuitively illustrates subexponentiality. The condition (1.3) rules out many common exponentially-tailed distributions, especially the sub-Gaussian and Poisson distributions. By contrast, the subexponential distributions are heavy-tailed so that there is a much higher probability of exceeding significantly large values. Such wild randomness builds a natural bridge between heavy-tailed phenomena and EVT. Studies have been made by various researchers. We refer to the monographs Embrechts et al. (1997), Adler et al. (1998), Resnick (2007) and Kulik and Soulier (2020) for deeper studies. Below we review some essentially important subexponential distributions in this research.

The heaviest tail for us is the power law decay, for which only partial moments exist. A distribution H is *regularly varying* of tail index $\alpha > 0$ if

$$\overline{H}(x) = x^{-\alpha}L(x) \quad \text{as } x \to \infty, \tag{1.4}$$

where $L(x)$ is a slowly varying function, *i.e.* $\lim_{x\to\infty} L(\lambda x)/L(x) = 1$ for any $\lambda > 0$. Such a distribution belongs to the $\alpha-$Fréchet maximum domain of attration. Regular variations are widely used in EVT, see de Haan (1970), Resnick (1987). They are also useful in other fields of mathematics, *e.g.* complex analysis, see the encyclopedia Bingham et al. (1987).

We would like to mention a special family called $\alpha-$*stable* distributions among the regularly varying distributions. An $\alpha-$stable distribution H is automatically regularly varying with index α, *i.e.*

$$\overline{H}(x) \sim c_\alpha x^{-\alpha} \quad \text{as } x \to \infty, \tag{1.5}$$

for some $c_\alpha > 0$. It is called stable because for any $n \geqslant 2$ there exists some $d_n \in \mathbb{R}$ so that

$$\underbrace{H * \cdots * H}_{n \text{ times}}(x) = H(n^{-1/\alpha}x + d_n). \tag{1.6}$$

Such distributions play a fundamental role in probability theory because they characterize all possible (nondegenerate) non-Gaussian limits in central limit theorems. The monographs Zolotarev (1986) and Samorodnitsky and Taqqu (1994) includes probabilistic aspects of stable distributions. The recent book Nolan (2020) highlights many practical uses of stable distributions.

Certain subexponential distributions lie in the Gumbel maximum domain of attraction. A classical result is that they have all finite moments, and hence their tails are lighter than the regularly varying tails. However, it is intrinsically hard to parameterize all these distributions simultaneously. Nonetheless, we are interested in the family of *lognormal* tails

$$\overline{H}(x) \sim cx^{\beta}(\log x)^{\xi} \exp\{-\lambda(\log x)^{\gamma}\} \quad \text{as } x \to \infty \tag{1.7}$$

where $c, \lambda > 0, \beta, \xi \in \mathbb{R}$ and $\gamma > 1$, and the family of semi-exponential tails

$$\overline{H}(x) \sim \exp\{-x^{\alpha}L(x)\}, \tag{1.8}$$

where $0 < \alpha < 1$ and $L(x)$ is slowly varying. The above two kinds are helpful to understand moderately heavy-tailed system. For example, in mathematical finance, geometric Brownian motions have lognormal margins. More discussions on finance and insurance are available in Mikosch and Nagaev (1998) and the references therein. Fine analysis of such distributions, especially the large deviation probabilities, can be found in numerous publications. In these theoretical studies, it is quite common to synthesize techniques from EVT and theory of random walks. See Borovkov (2000), Borovkov and Mogul'skiĭ (2006), Borovkov and Borovkov (2008), and Mikosch and Yslas (2020).

1.3 Stationary Infinitely Divisible Processes

This dissertation is about extremes from stable regenerative models. It is necessary to consider first the structures of stationary infinitely divisible (SID) processes from an ergodic theory perspective.

1.3.1 Infinitely Divisible Dynamics

Let T be an arbitrary parameter space, a stochastic process $\mathbf{X} = \{X_t\}_{t \in T}$ is called *infinitely divisible* if for any $n \in \mathbb{N}$, there exists a process $\{X_t^{(0)}\}_{t \in T}$ such that

$$\{X_t\}_{t \in T} \overset{\mathrm{d}}{=} \sum_{i=1}^{n} \{X_t^{(i)}\}_{t \in T}, \tag{1.9}$$

where $\{X_t^{(i)}\}_{t \in T}, i = 1, \ldots, n$ are *i.i.d.* copies of $\{X_t^{(0)}\}_{t \in T}$. Many important stochastic processes are infinitely divisible, *e.g.* Lévy processes, Lévy driven Ornstein-Uhlenbeck processes, fractional stable processes, *etc.* For general references, see Maruyama (1970) and Rosiński (2007). For a comprehensive study of finitely dimensional infinitely divsibile distributions, see Sato (2013).

Under mild conditions, an infinitely divisible process admit stochastic integration representations, see Rajput and Rosiński (1989). Reversely, this integration approach enables us to generate SID processes whose structures are flexible to manipulate. Henceforth, we take $T = \mathbb{Z}$ and sketch this method. We defer the detailed construction to Section 2.1. First, let $(E, \mathcal{E}, \mu)$ be a $\sigma-$finite measure space on which $\theta : E \to E$ is a measure preserving transformation (bijective and bi-measurable). Second, let M be a homogeneous and infinitely divisible random measure on $(E, \mathcal{E})$ controlled by μ. Choose any $A \in \mathcal{E}$ satisfies $0 < \mu(A) < \infty$, then the process $\mathbf{X} = \{X_{\mathbf{t}}\}_{\mathbf{t}\in\mathbb{Z}^d}$ defined by

$$X_{\mathbf{t}} = \int_E \mathbb{1}_A \circ \theta^t(x) M(dx) \quad \text{for all } t \in \mathbb{Z} \tag{1.10}$$

is SID. Diverse combinations of the dynamical systems and the random measures make this mechanism consisting of various SID processes.

From (1.10), we see that the dependence structure of $\mathbf{X}$ is inherited from the underlying dynamical system $(E, \mathcal{E}, \mu, \theta)$. More specifically, it is the behavior of the return "times"

$$\{t \in \mathbb{Z} : \theta^t(y) \in A\} \quad \text{for } y \in E$$

under control measure μ determines the strength of memory in $\mathbf{X}$. The more frequent returns, the stronger dependence. In fact, the dynamical system $(E, \mathcal{E}, \mu, \Theta)$ splits into disjoint sub-systems. The "frenquency" of return times of each subsystem has a unique feature. This is based on the positive-null and the Hopf decompositions in ergodic theory, see Section 2.2 for the detail.

Back to (1.10), we can decompose $\mathbf{X}$ into three independent SID processes:

$$\{X_t\}_{t\in\mathbb{Z}} \overset{\mathrm{d}}{=} \{X_t^{(\mathrm{p})}\}_{t\in\mathbb{Z}} + \{X_t^{(\mathrm{d})}\}_{t\in\mathbb{Z}} + \{X_t^{(\mathrm{cn})}\}_{t\in\mathbb{Z}}. \tag{1.11}$$

The first process $\{X_{\mathbf{t}}^{(\mathrm{p})}\}_{\mathbf{t}\in\mathbb{Z}^d}$ is driven by a positive flow and has the strongest dependence. A typical example is the identical process, and hence it is less interesting by comparing with the rest two processes.

The second process $\{X_{\mathbf{t}}^{(\mathrm{d})}\}_{\mathbf{t}\in\mathbb{Z}^d}$ is produced by a dissipative flow. In particular, such flows can generate infinitely divisible moving average processes, which have been extensively studied and used. The classical monograph Brockwell and Davis (1991) offers wealthy information on such processes. More variants on the moving averages and their applications can be found in Barndorff-Nielsen (2011) and the references therein.

The third process $\{X_{\mathbf{t}}^{(\mathrm{cn})}\}_{\mathbf{t}\in\mathbb{Z}^d}$ is generated by a conservative flow of null type, which are far lesser investigated than the second kind. Due to the limited research, it is impossible to offer a prototypical process. However, conservativeness can leads to sufficiently strong long-range dependence, which can further leads to strikingly unclassical patterns of extreme values, see Samorodnitsky (2004). We will focus on certain "solvable" conservative flows and their related SID processes, see the following Section 1.3.2. More EVT aspects of such processes are introduced in Section 1.4.2. The dissertation only cover a tip of an iceberg and there are much more fascinating phenomena to be explored.

To sum up, we point out that the ergodic theory approach has been applied to classify several kinds of SID processes. See Rosiński (1995), Rosiński (2000) and Samorodnitsky (2005) for stationary $\alpha-$stable processes, and Roy (2007) for stationary Poisson processes.

1.3.2 Models Related to Stable Regenerative Sets

In this section, we shall illustrate the main features of the conservative dynamics and the corresponding SID process that are primarily important in the sequel. We intend to highlight the main spirit and leave the details to Sections 3.2, 4.2.3, and 5.2.2 respectively.

Let $(E, \mathcal{E}, \mu)$ be a $\sigma-$finite infinite measure space and $\theta : E \to E$ be an measure preserving transformation. Choose a set $A \in \mathcal{E}$ with $0 < \mu(A) < \infty$ such that $\sum_{t=0}^{\infty} \mathbb{1}_A \circ \theta^t = \infty$ almost everywhere on A. Equivalently, this set A is recurrent. We are interested in such a set A whose return times exhibit discrete fractal behaviors, *i.e.* self-similarities in large scale, see Barlow and Taylor (1992). So one of the key assumptions is

$$\mu(A \cap \{\varphi > t\}) = t^{-\beta} L(t), \tag{1.12}$$

where $\beta \in (0,1)$, L is slowly varying and $\varphi(y) = \inf\{t \geqslant 1 : \theta^t(y) \in A\}$ for $y \in A$ is the first visit time of A. By rescaling the measure μ, we can assume that $\mu(A \cap \{\varphi > t\})$ is the tail of some discrete random variable. Hence, the probabilistic renewal theory enables us to analyze the regenerative pattern of the return times. Such dynamical systems and reference sets do exist. See the interval maps in Thaler (2000), or the occupation times of markov chains in Darling and Kac (1957). More generalized theory can be found in Aaronson (1997).

We typically care about the model $\mathbf{X} = \{\mathbf{X_t}\}_{\mathbf{t}\in\mathbb{Z}}$, which is a SID process in the form of (1.10) with the set A satisfies (1.12). This process has the marginal distribution determined by the random measure and dependence structure is parameterized by the return times. The smaller β, the more susceptible the flow is to escape the set A, and the weaker the long range dependence in $\mathbf{X}$. The scaling limit of return times is often related to the $\beta-$stable regenerative set, see Fitzsimmons and Taksar (1988). This is random closed subset of $[0, \infty)$ which is stationary and has Hausdorff dimension $a.s.$ equals to β.

Limit theorems based on such models of both light and heavy tails have been investigated in several different aspects. In Kalikow (1981) and Grabinsky (1984), the mixing properties of Possion processes driven by Markovian dynamics are studied. When the margins are $\alpha-$stable, the mixing properties are available in Rosiński and Samorodnitsky (1996). When the margins are generally regularly varying, the sample covariances and autocovariances are studied in Resnick et al. (2000) and Owada (2016) , the central limit theorems can be found in Owada and Samorodnitsky (2015a) and Bai et al. (2020), and some large deviation probabilities are investigated in Fasen and Roy (2016).

1.4 Functional Extremal Limit Theorems

1.4.1 Short Range Dependence

We consider a stationary process $\mathbf{X} = \{X_t\}_{t\in\mathbb{Z}}$. For convenience, we denote by $X_{n,0} \geqslant X_{n,1} \geqslant \cdots \geqslant X_{n,n}$ the order statistics of $\{X_0, \ldots, X_n\}$. In this section, we only look at short range dependent processes. The main results of this dissertation will be introduced in the next section.

Let $\mathbf{X}^{(0)} = \{X_t^{(0)}\}_{t\in\mathbb{Z}}$ be the *i.i.d.* process such that $X_0^{(0)} \overset{\mathrm{d}}{=} X_0$. By assuming certain regularities of the tail $\mathbb{P}\{X_0 > x\}$, the classical EVT tells that

$$\mathcal{N}_n^{(0)} = \sum_{t=0}^{n} \delta_{\left(X_t^{(0)} - b_n^{(0)}\right) / a_n^{(0)}} \Rightarrow \mathcal{N}_\infty^{(0)}. \tag{1.13}$$

Here δ_x denote the unit point mass on $x \in \mathbb{R}$, $a_n^{(0)}$ and $b_n^{(0)}$ are normalising constants, $\mathcal{N}_\infty$ is some Poisson point processes on $(-\infty, \infty]$, and " $\Rightarrow$ " usually denotes the vague convergence of Radon measures, see Resnick (1987).

Heuristically speaking, if the stationary process $\mathbf{X}$ is of sufficiently weak dependence, then the extremes should cluster together and should asymptomatic behave as if they were calculated from *i.i.d.* samples, and we can consider the point processes convergence. Namely, set

$$\mathcal{N}_n = \sum_{t=0}^{n} \delta_{\left(X_t - b_n^{(0)}\right) / a_n^{(0)}} \quad \text{for each } n \in \mathbb{N}. \tag{1.14}$$

Then under the anti-clustering conditions $D(u_n)$ and $D'(u_n)$, see Leadbetter et al. (1983) or Embrechts et al. (1997), we shall have $\mathcal{N}_n \Rightarrow \mathcal{N}_\infty^{(0)}$, which implies the joint convergence of

$$\left(\left(X_{n,0} - b_n^{(0)}\right)/a_n^{(0)}, \ldots, \left(X_{n,m} - b_n^{(0)}\right)/a_n^{(0)}\right) \quad \text{for any } m \in \mathbb{N}.$$

If the dependence in $\mathbf{X}$ gets stronger, the way dependence acts on extreme values can be viewed as a shrink of effective sample size. Leadbetter introduced the notion called *extremal index* $\theta \in (0,1)$ to describe it, see Leadbetter et al. (1983). Namely, we replace the $a_n^{(0)}$ and $b_n^{(0)}$ by $a_{\lfloor \theta n \rfloor}^{(0)}$ and $b_{\lfloor \theta n \rfloor}^{(0)}$ respectively in (1.14), and with limit other than $\mathcal{N}_\infty^{(0)}$. The reciprocal θ^{-1} can be regarded as the mean cluster size of the extreme values. The new limit is often of a clustered nature that reflects the local dependence structure of $\mathbf{X}$, see Hsing et al. (1988).

The theory of extremal index work successfully on many moving average processes, see Rootzén (1978), Davis and Resnick (1985), and Fasen (2005). Other types of extremal limit theorems of moving averages can be found in Davis and Resnick (1988), and Davis and Resnick (1996), Fasen (2006), and Fasen (2009).

1.4.2 Long Range Dependence

In this dissertation, our main interests are SID processes of vanishing extremal indices. For such a process, it is associated with an positive sequence $\theta_n \to 0$ such that extreme values occur in clusters of mean sizes $\theta_n^{-1} \to \infty$. Equivalently speaking, there is a partition of $\{0, \ldots, n\}$ into roughly $\lfloor n\theta_n \rfloor$ many blocks whose mean sizes are all approximately θ_n^{-1}. We call this phenomenon *long range dependence*(LRD). Compared with the previous section, several significant changes many occur.

First, for SID processes driven by conservative flows of form (1.10), the clustering may become so dense such that the convergence (1.14) to clustered Poisson random measure do not hold, see Resnick and Samorodnitsky (2004). Instead, we consider the set functionals called *random sup-measures*, *i.e.*

$$\mathcal{M}_n(B) = \max_{t \in nB} X_t, \quad n \in \mathbb{N}, B \in \mathcal{B}(\mathbb{R}_+). \tag{1.15}$$

The theory of sup-measures are developed in O'Brien et al. (1990). It is powerful tool to study stable regenerative models and the limit functional is closely related to the stable regenerative sets, see Lacaux and Samorodnitsky (2016).

Second, as the dependence gets even stronger, we expect the limit objects may stop to be of classical extreme value distribution, *i.e.* multiple phase may appear if we have extremely long range dependence. We refer to Samorodnitsky (2004) and Samorodnitsky and Wang (2019) to see the stable regenerative models with regularly varying tails.

We are also interested in studying the convergence of the extremal processes

$$\left\{ \frac{\max_{0 \leqslant i \leqslant nt} X_i - b_n}{a_n} \right\}_{t \in \mathbb{R}_+} \Rightarrow \{\mathbb{M}(t)\}_{t \in \mathbb{R}_+} \tag{1.16}$$

preferably in the space $D(\mathbb{R}_+)$ equipped with the J_1-topology. For the stable regenerative models with regularly varying tails, such convergences are established in Owada and Samorodnitsky (2015b).

The general theory of Fréchet extremal processes, which are limits for stationary regularly varying processes, have been extensively studied. For the spectral representation, see Stoev and Taqqu (2005), Kabluchko (2009), Kabluchko and Stoev (2012) and Kabluchko and Stoev (2016). For the ergodic properties, see Stoev (2008) and Stoev and Wang (2010).

1.5 Outline of Dissertation

We will concentrate on two objects: (i) $\alpha-$stable random fields $\{X_{\mathbf{t}}\}_{\mathbf{t}\in\mathbb{Z}^d}$ driven by multiparameter conservative flows; (ii) stable regenerative models whose margins are subexponential and lie in the Gumbel maximum domain of attraction. We will establish functional convergences in spaces of sup-measures and extremal processes.

Chapter 2 consists of certain essential details in Chapter 1.

Chapter 3 is based on the following article that has been published. Samorodnitsky, Gennady; Chen, Zaoli, Extreme Value Theory for Long Range Dependent Stable Random Fields, *Journal of Theoretical Probability* 33 (2020), 1894–1918. Both authors contributed equally to the article.

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CHAPTER 2

PRELIMINARIES

2.1 Infinite Divisibile Random Measures

In this section, we will show that (1.10) determines a SID process. We need first to introduce the characteristic triple of an infinitely divisible process, which is the key to understand the finitely dimensional distributions (FDDs). Let us start with three technical notions.

(i) For any parameter space T, $\mathbb{R}^{(T)}$ denotes set

$$\left\{\mathbf{x} \in \mathbb{R}^T : x_t = 0 \text{ for all but finitely many } t \in T\right\}.$$

(ii) The *truncation function* $[\![\cdot]\!]$ is defined as

$$[\![x]\!] = x \cdot \mathbb{1}_{\{-1 \leqslant x \leqslant 1\}} + \mathbb{1}_{\{x > 1\}} - \mathbb{1}_{\{x < -1\}} \quad \text{for any number } x \in \mathbb{R},$$

$$[\![\boldsymbol{x}]\!] = ([\![x_1]\!], \ldots, [\![x_d]\!]) \quad \text{for any vector } \boldsymbol{x} = (x_1, \ldots, x_d) \in \mathbb{R}^d.$$

(iii) A measure $\boldsymbol{\nu}$ on $\mathbb{R}^T$ for any parameter space T is called a *Lévy measure* if it satisfies two conditions. First,

$$\int_{\mathbb{R}^T} [\![x_t]\!]^2 \, \boldsymbol{\nu}(d\mathbf{x}) < \infty \qquad \text{for any } t \in T.$$

If T is countable, then the second condition is

$$\boldsymbol{\nu}\left(\mathbf{x} \in \mathbb{R}^T : x_t = 0 \text{ for all } t \in T\right) = 0.$$

Otherwise, for every countable subset $T_1 \subset T$ such that

$$\boldsymbol{\nu}\left(\mathbf{x} \in \mathbb{R}^T : x_t = 0 \text{ for all } t \in T_1\right) > 0,$$

we require that there exists $t_0 \in T_1^c$ such that

$$\boldsymbol{\nu}\left(\mathbf{x} \in \mathbb{R}^T : x_t = 0 \text{ for all } t \in T_1, x(t_0) \neq 0\right) > 0.$$

The following proposition characterizes the FDDs, see Theorem 3.1.7 in Samorodnitsky (2016) for the proof.

Proposition 2.1.1. *A stochastic process* $\{X_t\}_{t\in T}$ *is infinitely divisible if and only if there exists a uniquely deterministic triple* $(\boldsymbol{\Sigma}, \boldsymbol{\nu}, \boldsymbol{b})$ *such that for every* $\boldsymbol{\theta} \in \mathbb{R}^{(T)}$,

$$\begin{aligned} &\mathbb{E}\exp\left\{ i \sum_{t\in\mathbb{T}} \boldsymbol{\theta}_t X_t \right\} \\ =& \exp\left\{ -\frac{1}{2}\boldsymbol{\theta}^T\boldsymbol{\Sigma}\boldsymbol{\theta} + \int_{\mathbb{R}^T} \left(e^{i(\boldsymbol{\theta},\boldsymbol{x})} - 1 - i(\boldsymbol{\theta}, [\![\boldsymbol{x}]\!])\right) \boldsymbol{\nu}(d\boldsymbol{x}) + i(\boldsymbol{\theta}, \boldsymbol{b}) \right\}, \end{aligned}$$

where $\boldsymbol{\Sigma} = (\boldsymbol{\Sigma}(s,t) : s,t \in T)$ *is a nonnegative definite function on* T, $\boldsymbol{\nu}$ *is a Lévy measure on* $\mathbb{R}^T$, *and* $\boldsymbol{b} \in \mathbb{R}^T$ *is a vector.*

Now we begins the second part in this section. Let $(E, \mathcal{E}, \mu)$ be a $\sigma-$finite measure space. We denote by $\mathcal{E}_+$ the collection of finitely positive measurable sets $\{B \in \mathcal{E} : 0 < \mu(B) < \infty\}$. Let (σ^2, ν, b) be the characteristic triple of an infinitely divisible distribution on $\mathbb{R}$. A *homogeneous and infinitely divisible random measure* M on $(E, \mathcal{E})$ with control measure μ and density (σ^2, ν, b) is a infinitely divisible process $\{M(B)\}_{B\in\mathcal{E}_+}$. We specify the latter process by its characteristic triple $(\boldsymbol{\Sigma}, \boldsymbol{\nu}, \boldsymbol{b})$. Namely,

$$\begin{aligned} &\boldsymbol{\Sigma}(B_1, B_2) = \sigma^2\mu(B_1 \cap B_2) \quad \text{for any } B_1, B_2 \in \mathcal{E}_+, \\ &\boldsymbol{b}(B) = b\mu(B) \quad \text{for any } B_1, B_2 \in \mathcal{E}_+, \\ &\boldsymbol{\nu} = (\mu \times \nu) \circ \Phi^{-1}, \end{aligned}$$

where $\mu \times \nu$ is the product measure on $E \times (\mathbb{R} \setminus \{0\})$ and Φ is the map

$$\Phi : E \times (\mathbb{R} \setminus \{0\}) \to \mathbb{R}^{\mathcal{E}_+}, \quad \Phi(s,x)(B) = x\mathbb{1}_{\{s\in B\}} \quad \text{for any } B \in \mathcal{E}_+.$$

The triple $(\boldsymbol{\Sigma}, \boldsymbol{\nu}, \boldsymbol{b})$ is well-defined, *i.e.* it meets the requirements in Proposition 2.1.1, see Section 3.2 in Samorodnitsky (2016).

We are now ready to show (1.10) guarantees stationarity. It suffices to check that the FDDs are invariant under shifts. That is, for any $n \in \mathbb{N}$, $a_1, \ldots, a_n \in \mathbb{R}$ and $t_1, \ldots, t_n \in \mathbb{Z}$, the value of the characteristic function

$$\mathbb{E} \exp \left\{ i \left(a_1 X_{t_1+s} + \cdots + a_n X_{t_n+s} \right) \right\} \tag{2.1}$$

is independent of the choice $s \in \mathbb{Z}$. A direct computation leads to that

$$\begin{aligned}
&\mathbb{E} \exp \left\{ i \left(a_1 X_{t_1+s} + \cdots + a_n X_{t_n+s} \right) \right\} \\
&= \exp \Bigg\{ -\frac{\sigma^2}{2} \sum_{j,k=1}^{n} a_j a_k \mu \circ \theta^{-s} \left(\theta^{-t_j}(A) \cap \theta^{-t_k}(A) \right) \\
&\quad + \int_E \mu \circ \theta^{-s}(dy) \\
&\quad \int_{\mathbb{R}\setminus\{0\}} \nu(dx) \Bigg(\exp \left\{ ix \sum_{j=1}^{n} a_j \mathbb{1}_{\left\{ y \in \theta^{-t_j}(A) \right\}} \right\} - 1 - i \sum_{j=1}^{n} a_j \left\llbracket x \mathbb{1}_{\left\{ y \in \theta^{-t_j}(A) \right\}} \right\rrbracket \Bigg) \\
&\quad + b \sum_{j=1}^{n} \mu \circ \theta^{-s} \left(\theta^{-t_j}(A) \right) \Bigg\}.
\end{aligned}$$

Since the transformation θ is measure preserving, we immediately get the desired conclusion.

2.2 Elements of Ergodic Theory

Throughout this section, we assume $(E, \mathcal{E}, \mu)$ is a $\sigma-$finite measure space and $\theta : E \to E$ is a measure preserving and bi-measurable transformation. We will introduce several terminologies and two decomposition theorems to establish (1.11).

A set $A \in \mathcal{E}$ is *invariant* if $\mu(A \Delta \theta^{-1}(A)) = 0$ where Δ denotes the symmetric difference among two sets. If the only invariant sets are either of $\mu(A) = 0$ or of $\mu(A \Delta E) = 0$, then we say θ is ergodic.

If there exists a probability measure P on $(E, \mathcal{E})$ that is equivalent to μ and is also preserved by θ, then we say the map θ is *positive*. In general,$(E, \mathcal{E})$ contains both positive and nonpositive parts, and we can distinguish them. The following result is taken from Theorem 2.4.8 in Samorodnitsky (2016).

Proposition 2.2.1. *Given a dynamical system* $(E, \mathcal{E}, \mu, \theta)$*, there is a partition of* E *into invariant sets* $\mathcal{P}(\theta)$ *and* $\mathcal{N}(\theta)$ *such that*

(i) θ *is positive on* $\mathcal{P}(\theta)$*,*

(ii) $\mathcal{N}(\theta)$ *contains no invariant set* A *with* $0 < \mu(A) < \infty$*.*

This decomposition if unique up to modulus μ*, i.e. if* $E = \mathcal{P}_1(\theta) \cup \mathcal{N}_1(\theta)$ *is another partition such that* $\mathcal{P}_1(\theta)$ *and* $\mathcal{N}_1(\theta)$ *satisfy* (i) *and* (ii) *respectively, then* $\mu(\mathcal{P}(\theta)\Delta\mathcal{P}_1(\theta)) = \mu(\mathcal{N}(\theta)\Delta\mathcal{N}_1(\theta)) = 0$.

Now we begin to introduce the second decomposition. The dynamical system $(E, \mathcal{E}, \mu, \theta)$ is called *conservative* if

$$\sum_{n=1}^{\infty} \mathbb{1}_A \circ \theta^n = \infty \quad \text{almost everywhere on } A$$

for every $A \in \mathcal{E}$ with $\mu(A) > 0$. A set $W \in \mathcal{E}$ with $\mu(W) > 0$ is called *wandering* if $\mu(\theta^{-n}(A) \cap \theta^{-m}(A)) = 0$ for any distinct $n, m \in \mathbb{N}_0$. Clearly, whenever a wandering set exists, $(E, \mathcal{E}, \mu, \theta)$ can not be conservative. A stronger result is the following Hopf decomposition, see Theorem 2.4.3 in Samorodnitsky (2016) for the proof.

Proposition 2.2.2. *Given a dynamical system* $(E, \mathcal{E}, \mu, \theta)$*, there is a partition of* E *into invariant sets* $\mathcal{C}(\theta)$ *and* $\mathcal{D}(\theta)$ *such that*

(i) $\mathcal{C}(\theta)$ *contains no subset of positive measure that is wandering,*

(ii) $\mathcal{D}(\theta) = \cup_{n=-\infty}^{\infty} \theta^n(W)$ *for some wandering set* W*.*

This decomposition if unique up to modulus μ*, i.e. if* $E = \mathcal{C}_1(\theta) \cup \mathcal{D}_1(\theta)$ *is another partition such that* $\mathcal{C}_1(\theta)$ *and* $\mathcal{D}_1(\theta)$ *satisfy* (i) *and* (ii) *respectively, then* $\mu(\mathcal{C}(\theta)\Delta\mathcal{C}_1(\theta)) = \mu(\mathcal{D}(\theta)\Delta\mathcal{D}_1(\theta)) = 0$*.*

We say $(E, \mathcal{E}, \mu, \theta)$ is called *dissipative* if $\mu(\mathcal{C}(\theta)) = 0$. It is direct to note that $mathcalP(\theta) \subset mathcalC(\theta)$, since the positive part can not admit wandering sets.

Finally, we derive (1.11) by setting for each $t \in \mathbb{Z}$ that

$$
\begin{aligned}
X_t^{(\mathrm{p})} &= \int_{\mathcal{P}(\theta)} \mathbb{1}_A \circ \theta^t(x) M(dx), \\
X_t^{(\mathrm{d})} &= \int_{\mathcal{D}(\theta)} \mathbb{1}_A \circ \theta^t(x) M(dx), \\
X_t^{(\mathrm{cn})} &= \int_{\mathcal{C}(\theta)\setminus\mathcal{P}(\theta)} \mathbb{1}_A \circ \theta^t(x) M(dx).
\end{aligned}
$$

CHAPTER 3

EXTREME VALUE THEORY FOR LONG RANGE DEPENDENT STABLE RANDOM FIELDS

3.1 Introduction

Extreme value theorems describe the limiting behaviour of the largest values in increasingly large collections of random variables. The classical extremal theorems, beginning with Fisher and Tippett (1928) and Gnedenko (1943), deal with the extremes of i.i.d. (independent and identically distributed) random variables. The modern extreme value theory techniques allow us to study the extremes of dependent sequences; see Leadbetter et al. (1983) and the expositions in Coles (2001) and de Haan and Ferreira (2006). The effect of dependence on extreme values can be restricted to a loss in the effective sample size, through the extremal index of the sequence. When the dependence is sufficiently long, more significant changes in extreme value may occur; see e.g. Samorodnitsky (2004), Owada and Samorodnitsky (2015b). The present paper aims to contribute to our understanding of the effect of memory on extremes when the time is of dimension larger than 1, i.e. for random fields.

We consider a discrete time stationary random field $\mathbf{X} = \big(X_{\mathbf{t}},\, \mathbf{t} \in \mathbb{Z}^d\big)$. For $\mathbf{n} = (n_1, \ldots, n_d) \in \mathbb{N}^d$ we would like to study the extremes of the random field over growing hypercubes of the type

$$[\mathbf{0}, \mathbf{n}] = \big\{\mathbf{0} \leq \mathbf{k} \leq \mathbf{n}\big\},\ \mathbf{n} \to \infty\,,$$

where $\mathbf{0}$ is the vector with zero coordinates, the notation $\mathbf{s} \leq \mathbf{t}$ for vectors $\mathbf{s} = (s_1, \ldots, s_d)$ and $\mathbf{t} = (t_1, \ldots, t_d)$ means that $s_i \leq t_i$ for all $i = 1, \ldots, d$, and the notation $\mathbf{n} \to \infty$ means that all d components of the vector $\mathbf{n}$ tend to infinity. Denote

$$M_{\mathbf{n}} = \max_{\mathbf{0} \leq \mathbf{k} \leq \mathbf{n}} X_{\mathbf{t}} \,.$$

What limit theorems does the array $(M_{\mathbf{n}})$ satisfy? It was shown by Leadbetter and Rootzén (1998) that under appropriate strong mixing conditions, only the classical three types of limiting distributions (Gumbel, Fréchet and Weibull) may appear (even when forcing $\mathbf{n} \to \infty$ along a monotone curve). In the case when the marginal distributions of the field $\mathbf{X}$ have regularly varying tails, this allows only the Fréchet distribution as a limit.

In this paper we will discuss only random fields with regularly varying tails, in which case the experience from the classical extreme value theory tells us to look for limit theorems for the type

$$\frac{1}{b_{\mathbf{n}}} M_{\mathbf{n}} \Rightarrow Y \quad \text{as } \mathbf{n} \to \infty \tag{3.1}$$

for some nondegenerate random variable Y. The regular variation of the marginal distributions means that

$$P(X(\mathbf{0}) > x) = x^{-\alpha} L(x), \ \alpha > 0, \ L \text{ slowly varying}, \tag{3.2}$$

see e.g. Resnick (1987). Notice that the assumption is only on the right tail of the distribution since, in most cases, one does not expect a limit theorem for the partial maxima as in (3.1) to be affected by the left tail of $X(\mathbf{0})$.

If the random field $\mathbf{X}$ consists of i.i.d. random variables satisfying the regular variation condition (3.2), then the classical extreme value theory tells us that the convergence in (3.1) holds if we choose

$$b_{\mathbf{n}} = \inf\bigl\{x > 0 : \ P(X(\mathbf{0}) > x) \leq (n_1 \cdots n_d)^{-1}\bigr\} \,, \tag{3.3}$$

in which case the limiting random variable Y has the standard Fréchet distribution. We are interested in understanding how the spatial dependence in the random field $\mathbf{X}$ affects the scaling in and the distribution of the limit not only in (3.1), but in its functional versions, which can be stated in different spaces, for example in the space $D(\mathbb{R}_+^d)$ of right continuous, with limits along monotone paths, functions (see Straf (1972)), or in the space of random sup measures $\mathcal{M}(\mathbb{R}_+^d)$; see O'Brien et al. (1990). We will describe the relevant spaces below.

If the time is one-dimensional, and the memory in the stationary process is short, then the standard normalization (3.3) is still the appropriate one, and the limits both in (3.1) and its functional versions change only through a change in a multiplicative constant; see Samorodnitsky (2016) and references therein. However, when the memory becomes sufficiently long, both the order of magnitude of the normalization in the limit theorems changes, and the nature of the limit changes as well; see Samorodnitsky (2004) and Owada and Samorodnitsky (2015b). Furthermore, the limit may even stop having the Fréchet distribution (or Fréchet marginal distributions, in the functions limit theorems); see Samorodnitsky and Wang (2019). It is reasonable to expect that similar phenomena happen for random fields, but because it is harder to quantify how long the memory is when the time is not one-dimensional, less is known in this case.

In this paper we will concentrate on the case where the random field $\mathbf{X}$ is a symmetric α-stable (SαS) random field, $0 < \alpha < 2$. Recall that this means that every finite linear combination of the values of the values of the random field has a one-dimensional SαS distribution, i.e. has a characteristic function of the form $\exp\{-\sigma^\alpha|\theta|^\alpha\}$, $\theta \in \mathbb{R}$, where $\sigma \in [0, \infty)$ is a scale parameter that depends on the linear combination; see Samorodnitsky and Taqqu (1994). The marginal distributions of SαS random fields satisfy the regular variation assumption (3.2) with $0 < \alpha < 2$ that coincides with the index of stability. In this case a series of results on the relation between the sizes of the extremes of stationary SαS random fiels and certain ergodic-theoretical properties of the Lévy measures of these fields is due to Parthanil Roy and his coworkers; see Roy and Samorodnitsky (2008), Chakrabarty and Roy (2013), Sarkar and Roy (2016). These results are made possible because of the connection between the structure of the SαS random fields and ergodic theory established by Rosiński (2000).

This paper contributes to understanding the extremal limit theorems for SαS random fields and their connection to the dynamics of the Lévy measures. In this sense our paper is related to the ideas of Rosiński (2000). However, we will restrict ourselves to certain Markov flows. This will allow us to avoid, to a large extent, the language of ergodic theory, and state everything in purely probabilistic terms. There is not doubt, however, that our results could be extended to more general dynamical systems acting on the Lévy measures of SαS random fields. The generality in which work is sufficient to demonstrate the new phenomena that may arise in extremal limit theorems for random fields with long range dependence. We will exhibit new types of limits, some of which will have non-Fréchet distributions, both in the space of random sup measures and in the space $D(\mathbb{R}_+^d)$.

This paper is organized as follows. In Section 3.2 we introduce the class of stationary symmetric α-stable random fields we will study in this paper. In Section 3.3 we provide some background on random closed sets and random sup measures, and describe the limiting random sup measure that appears as the weak limit the extremal theorem in Section 3.4. Finally, in Section 3.5 we prove versions of our extremal limit theorems in the space $D(\mathbb{R}_+^d)$.

Notation: For a function g on an arbitrary set with values in a linear space we denote the set of zeros of g by $\mathcal{Z}(g)$. Arithmetic operations involved vectors are performed component-wise. Thus, if $\mathbf{x} = (x_1, \ldots, x_d)$ and $\mathbf{y} = (y_1, \ldots, y_d)$, then, say, $\mathbf{xy} = (x_1y_1, \ldots, x_dy_d)$. This extends to sets: if $A = A_1 \times \cdots \times A_d$, then $\mathbf{x}A = x_1A_1 \times \cdots \times x_dA_d$.

3.2 A SαS random field with long range dependence

We start with a construction of a family of stationary SαS random fields, $0 < \alpha < 2$, whose memory has a natural finite-dimensional parameterization. It is an extension to random fields of models considered before in the case of one-dimensional time; see e.g. Resnick et al. (2000), Samorodnitsky (2004), Owada and Samorodnitsky (2015a,b), Owada (2016) and Lacaux and Samorodnitsky (2016).

We start with d σ-finite, infinite measures on $\big(\mathbb{Z}^{\mathbb{N}_0}, \mathcal{B}(\mathbb{Z}^{\mathbb{N}_0})\big)$ defined by

$$\mu_i := \sum_{k\in\mathbb{Z}} \pi_k^{(i)} P_k^{(i)} \,, \tag{3.4}$$

where for $i = 1, \ldots, d$, $P_k^{(i)}$ is the law of an irreducible aperiodic null-recurrent Markov chain $(Y_n^{(i)})_{n\geq 0}$ on $\mathbb{Z}$ starting at $Y_0^{(i)} = k \in \mathbb{Z}$. Further, $(\pi_k^{(i)})_{k\in\mathbb{Z}}$ is its unique (infinite) invariant measure satisfying $\pi_0^{(i)} = 1$. Given this invariant measure, we can extend the probability measures $P_k^{(i)}$ from measures on $\mathbb{Z}^{\mathbb{N}_0}$ to measures on $\mathbb{Z}^{\mathbb{Z}}$ which, in turn, allows us to extend the measure μ_i in (3.4) to $\mathbb{Z}^{\mathbb{Z}}$ as well. We will keep using the same notation as in (3.4).

We will work with the product space

$$(E, \mathcal{E}) = \big(\mathbb{Z}^{\mathbb{Z}} \times \cdot \times \mathbb{Z}^{\mathbb{Z}},\ \mathcal{B}(\mathbb{Z}^{\mathbb{Z}}) \times \cdot \times \mathcal{B}(\mathbb{Z}^{\mathbb{Z}})\big)$$

of d copies of $\big(\mathbb{Z}^{\mathbb{Z}}, \mathcal{B}(\mathbb{Z}^{\mathbb{Z}})\big)$, on which we put the product, σ-finite, infinite, measure

$$\mu = \mu_1 \times \cdot \times \mu_d \, .$$

The key assumption is a regular variation assumption on the return times of the Markov chains $(Y_n^{(i)})_{n\geq 0}$, $i = 1, \ldots, d$. For $\mathbf{x} = (\ldots, x_{-1}, x_0, x_1, x_2 \ldots) \in \mathbb{Z}^{\mathbb{Z}}$ we define the first return time to the origin by $\varphi(\mathbf{x}) = \inf\{n \geq 1 : x_n = 0\}$. We assume that for $i = 1, \ldots, d$ we have

$$P_0^{(i)}(\varphi > n) \in RV_{-\beta_i} \tag{3.5}$$

for some $0 < \beta_i < 1$. This implies that

$$\begin{aligned} &\mu_i \left(\{\mathbf{x} : \, x_k = 0 \text{ for some } k = 0, 1, \ldots, n\} \right) \\ \sim &\sum_{k=1}^{n} P_0^{(i)}(\varphi > k) \sim (1 - \beta_i)^{-1} n P_0^{(i)}(\varphi > n) \in RV_{1-\beta_i} \, . \end{aligned} \tag{3.6}$$

See Resnick et al. (2000).

On $\mathbb{Z}^{\mathbb{Z}}$ there is a natural left shift operator

$$T\big((\ldots, x_{-1}, x_0, x_1, x_2 \ldots)\big) = (\ldots, x_0, x_1, x_2, x_3 \ldots) \, .$$

It is naturally extended to a group action of $\mathbb{Z}^d$ on E as follows. Writing an element $\mathbf{x} \in E$ as $\mathbf{x} = (\mathbf{x}^{(1)}, \ldots, \mathbf{x}^{(d)})$ with $\mathbf{x}^{(i)} = (\ldots, x_{-1}^{(i)}, x_0^{(i)}, x_1^{(i)}, x_2^{(i)} \ldots)\big) \in \mathbb{Z}^{\mathbb{Z}}$ for $i = 1, \ldots, d$, we set for $\mathbf{n} = (n_1, \ldots, n_d) \in \mathbb{Z}^d$,

$$T^{\mathbf{n}}\mathbf{x} = (T^{n_1}\mathbf{x}^{(1)}, \ldots, T^{n_d}\mathbf{x}^{(d)}) \in E \, . \tag{3.7}$$

Even though we are using the same notation T for operators acting on different spaces, the meaning will always be clear from the context. Note that each individual left shift T on $\big(\mathbb{Z}^{\mathbb{N}_0}, \mathcal{B}(\mathbb{Z}^{\mathbb{N}_0}), \mu_j\big)$ is measure preserving (because each $(\pi_i^{(j)})_{i\in\mathbb{Z}}$ is an invariant measure.) It is also conservative and ergodic by Theorem 4.5.3 in

Aaronson (1997). Therefore, the group action $\mathcal{T} = \{T^{\mathbf{n}} : \mathbf{n} \in \mathbb{Z}^d\}$ is conservative, ergodic and measure preserving on $(E, \mathcal{E}, \mu)$.

Equipped with a measure preserving group action on the space $(E, \mathcal{E})$ we can now define a stationary symmetric α-stable random field by

$$X_{\mathbf{n}} = \int_E f \circ T^{\mathbf{n}}(\mathbf{x}) \, M(d\mathbf{x}), \quad \mathbf{n} \in \mathbb{Z}^d \,, \tag{3.8}$$

where M is a $S\alpha S$ random measure on $(E, \mathcal{E})$ with control measure μ, and

$$f(\mathbf{x}) = \mathbb{1}(\mathbf{x}^{(i)} \in A, \, i = 1, \ldots, d), \; \mathbf{x} = (\mathbf{x}^{(1)}, \ldots, \mathbf{x}^{(d)}) \,. \tag{3.9}$$

where $A = \{\mathbf{x} \in \mathbb{Z}^{\mathbb{Z}} : x_0 = 0\}$. Clearly, $f \in L^\alpha(\mu)$, which guarantees that the integral in (3.8) is well defined. We refer the reader to Samorodnitsky and Taqqu (1994) for general information on stable processes and integrals with respect to stable measures, and to Rosiński (2000) on more details on stationary stable random fields and their representations.

The random field model defined by (3.8) is attractive because the key parameters involve in its definition have a clear intuitive meaning: the index of stability $0 < \alpha < 2$ is responsible for the heaviness of the tails, while $0 < \beta_i < 1$, $i = 1, \ldots, d$ (defined in (3.5)) are responsible for the "length of the memory". The latter claim is not immediately obvious, but its (informal) validity will become clearer in the sequel.

The following array of positive numbers will play the crucial role in the extremal limit theorems in this paper. Denote for $n = 1, 2, \ldots$ and $i = 1, \ldots, d$,

$$b_n^{(i)} = \big(\mu_i \left(\{\mathbf{x} : \, x_k = 0 \text{ for some } k = 0, 1, \ldots, n\}\right)\big)^{1/\alpha} \,,$$

and let

$$b_{\mathbf{n}} = \prod_{i=1}^{d} b_{n_i}^{(i)}, \; \mathbf{n} = (n_1, \ldots, n_d) \in \mathbb{N}_0^d \,. \tag{3.10}$$

Then $b_{\mathbf{n}}^{\alpha} = \mu(B_{\mathbf{n}})$, where

$$B_{\mathbf{n}} = \{\mathbf{x} = (\mathbf{x}^{(1)}, \ldots, \mathbf{x}^{(d)}) \in E : \\ x_{k_i}^{(i)} = 0 \ \text{ for some } \ 0 \le k_i \le n_i, \ \text{each } i = 1, \ldots, d\} .$$

Therefore, we can define, for each $\mathbf{n} \in \mathbb{N}_0^d$, a probability measure $\eta_{\mathbf{n}}$ on $(E, \mathcal{E})$ by

$$\eta_{\mathbf{n}}(\cdot) = b_{\mathbf{n}}^{-\alpha} \mu(\cdot \ \cap B_{\mathbf{n}}) . \tag{3.11}$$

This probability measure allows us to represent the restriction of the stationary $S\alpha S$ random field $\mathbf{X}$ in (3.8) to the hypercube $[\mathbf{0}, \mathbf{n}] = \{\mathbf{0} \le \mathbf{k} \le \mathbf{n}\}$ as a series, described below, and that we will find useful in the sequel. It is useful to note also that the measure $\eta_{\mathbf{n}}$ is the product measure of d probability measures on $\big(\mathbb{Z}^{\mathbb{Z}}, \mathcal{B}(\mathbb{Z}^{\mathbb{Z}})\big)$: $\eta_{\mathbf{n}} = \eta_{n_1}^{(1)} \times \cdot \times \eta_{n_d}^{(d)}$ for $\mathbf{n} = (n_1, \ldots, n_d) \in \mathbb{N}_0^d$, where for $i = 1, \ldots, d$ and $n \ge 0$,

$$\eta_n^{(i)}(\cdot) = (b_n^{(i)})^{-\alpha} \mu_i\big(\cdot \ \cap \{\mathbf{x} \in \mathbb{Z}^{\mathbb{Z}} : \, x_k = 0 \text{ for some } 0 \le k \le n\}\big) . \tag{3.12}$$

The restriction of the stationary $S\alpha S$ random field $\mathbf{X}$ in (3.8) to the hypercube $[\mathbf{0}, \mathbf{n}]$ admits, in law, the series representation

$$X_{\mathbf{k}} = b_{\mathbf{n}} C_{\alpha}^{1/\alpha} \sum_{j=1}^{\infty} \epsilon_j \Gamma_j^{-1/\alpha} \mathbb{1}_{A^d} \circ T^{\mathbf{k}}(U_{j,\mathbf{n}}), \ \mathbf{0} \le \mathbf{k} \le \mathbf{n} , \tag{3.13}$$

with $A^d = A \times \cdot \times A$ the direct product of d copies of A and A is in (3.9), where the constant C_α is the tail constant of the α-stable random variable:

$$C_{\alpha} = \left(\int_0^{\infty} x^{-\alpha} \sin x dx \right)^{-1} = \begin{cases} \frac{1-\alpha}{\Gamma(2-\alpha)\cos(\pi\alpha/2)} & \alpha \ne 1 \\ 2/\pi & \alpha = 1 \end{cases} .$$

Furthermore, $\{\epsilon_j\}$ is a i.i.d. sequence of Rademacher random variables, $\{\Gamma_j\}$ is the sequence of the arrival times of a unit rate Poisson process on $(0, \infty)$, and $\{U_{j,\mathbf{n}}\}$ are i.i.d. E-valued random elements with common law $\eta_{\mathbf{n}}$. The sequence $\{\epsilon_j\}$, $\{\Gamma_j\}$ and $\{U_{j,\mathbf{n}}\}$ are independent. See Samorodnitsky and Taqqu (1994) for details.

3.3 Stable regenerative sets and random sup measures

In this section we describe the limiting object one obtains in an extremal limit theorem from the random field $\mathbf{X}$ of the previous section. We start with a bit of technical background information on random closed sets and random sup measures. The reader should consult Molchanov (2017) for more details.

Let $\mathbb{E}$ be a locally compact and second countable Hausdorff topological space (it will be $\mathbb{R}^d$ or $[0,1]^d$ in our case). We denote by $\mathcal{G}, \mathcal{F}, \mathcal{K}$ the families of open, closed, compact sets of $\mathbb{E}$, respectively. The Fell topology on the space $\mathcal{F}$ of closed sets has a subbasis consisting of the sets

$$\mathcal{F}_G = \{F \in \mathcal{F} : F \cap G \neq \emptyset\},\ G \in \mathcal{G}$$
$$\mathcal{F}^K = \{F \in \mathcal{F} : F \cap K = \emptyset\},\ K \in \mathcal{K}\,.$$

The Fell topology is metrizable and compact.

A random closed set is a measurable mapping from a probability space to $\mathcal{F}$ equipped with the Borel σ-field $\mathcal{B}(\mathcal{F})$ generated by the Fell topology. A specific random closed set in $\mathbb{R}$, the so-called stable regenerative set, is the key for describing the main results of this paper.

For $0 < \beta < 1$ let $(L_\beta(t),\, t \geq 0)$ be the standard β-stable subordinator. That is, it is an increasing Lévy process with Laplace transform $\mathbb{E}e^{-\theta L_\beta(t)} = e^{-t\theta^\beta}$, $\theta \geq 0$. The β-stable regenerative set is defined to be the closure of the range of the β-subordinator, viewed as a random closed set of $\mathbb{R}$:

$$R_\beta \overset{d}{:=} \overline{\{L_\beta(t)\,, t \geq 0\}}\,. \tag{3.14}$$

See e.g. Fitzsimmons and Taksar (1988). Products of shifted stable regenerative sets produce random closed subsets of $\mathbb{R}^d$ as follows.

For $0 < \beta_i < 1$, $i = 1, \dots, d$, let $R^{(i)}_{\beta_i}$, $i = 1, \dots, d$ be independent β_i-stable regenerative sets. Let $v^{(i)} > 0$, $i = 1, \dots, d$, and denote $\tilde{R}^{(i)}_{\beta_i} = v^{(i)} + R^{(i)}_{\beta_i}$. Then

$$\tilde{R}_\beta := \prod_{i=1}^{d} \tilde{R}^{(i)}_{\beta_i} \tag{3.15}$$

is a random closed subset of $\mathbb{R}^d$. Such random closed sets have interesting intersection properties. The following proposition follows from Lemma 3.1 of Samorodnitsky and Wang (2019).

Proposition 3.3.1. *Let* $\{\tilde{R}_{\beta,j}\}_{j\geq 1}$ *be independent random closed sets in* $\mathbb{R}^d$ *as defined by* (3.15)*. Suppose that the corresponding shift vectors* $(v^{(i)}_j, i = 1, \dots, d)_{j\geq 1}$ *satisfy* $v^{(i)}_{j_1} \neq v^{(i)}_{j_2}$ *if* $j_1 \neq j_2$ *for each* $i = 1, \dots, d$*. Then for any* $m = 1, 2, \dots,$

$$P(\cap_{j=1}^{m} \tilde{R}_{\beta,j} \neq \emptyset) = 0 \text{ or } 1\,.$$

The probability is equal to 1 if and only if $m < \min_{i=1,\dots,d}(1-\beta_i)^{-1}$.

The next object to define is a sup measure. For simplicity we take $\mathbb{E}$ be the space $[0,1]^d$ or $\mathbb{R}^d$. The details of the presentation below can be found in O'Brien et al. (1990). A map $m : \mathcal{G} \to [0,\infty]$ a called sup measure if $m(\emptyset) = 0$ and for an arbitrary collection of open sets $\{G_\gamma\}$ we have $m(\cup_\gamma G_\gamma) = \sup_\gamma m(G_\gamma)$.

The sup derivative $d^\vee m$ of a sup measure m is defined by

$$d^\vee m(t) := \inf_{t\in G} m(G), \quad G \in \mathcal{G}\,. \tag{3.16}$$

It is automatically an upper semi-continuous function. Conversely, for any function $f : \mathbb{E} \to [0,\infty]$, its sup integral $i^\vee f$ is defined as

$$i^\vee f(G) := \sup_{t\in G} f(t), \quad G \in \mathcal{G}\,. \tag{3.17}$$

If f is upper semi-continuous, then $f = d^\vee i^\vee f$. Furthermore, $m = i^\vee d^\vee m$, and one can use (3.17) to extend the domain of a sup measures to sets that are not necessarily open, by setting

$$m(B) := \sup_{t \in B} d^{\vee} m(t), \quad B \subset \mathbb{E}\,.$$

On the space SM of all sup measures we introduce a topology, the so-called sup vague topology, by saying that a sequence $\{m_n\}$ of sup measures converges to a sup measure m if

$$\limsup_{n\to\infty} m_n(K) \le m(K) \text{ for all } K \in \mathcal{K} \ \text{ and } \ \liminf_{n\to\infty} m_n(G) \ge m(G) \text{ for all } G \in \mathcal{G}.$$

The space of sup measures with sup vague topology is compact and metrizable; see Theorem 2.4. in Norberg (1990), and we will often use the notation $\mathcal{M}(\mathbb{E})$ for the space of sup measures on $\mathbb{E}$.

A random sup measure is a measurable map from a probability space into SM equipped with the Borel σ-field induced by the sup vague topology. For a random sup measure η, a continuity set is an open set G such that $\eta(G) = \eta(\bar{G})$ (the closure of G) a.s., and a useful criterion for weak convergence in the sup vague topology of random sup-measures is as follows. Let $\{\eta_n\}_{n\ge1}$ be a sequence of random sup measures, and η a random sup measure. Then $\eta_n \Rightarrow \eta$ if and only if

$$(\eta_n(B_1), \ldots, \eta_n(B_m)) \Rightarrow (\eta(B_1), \ldots, \eta(B_m)) \tag{3.18}$$

for arbitrary disjoint open rectangles $B_1, \ldots, B_m$ in $\mathbb{E}$ that are continuity sets for η.

We are now ready to construct the random sup measure that will appear as the limit in the extremal limit theorem in the space SM of the next section. We will define this measure through its sup derivative, which is a random upper semi-continuous function. Let $0 < \beta_i < 1$, $i = 1, \ldots, d$. We start with d independent families of i.i.d. β_i-stable regenerative sets $\{R^{(i)}_{\beta_i,j}\}_{j\ge1}$, $i = 1, \ldots, d$. Furthermore, let $(U_{\alpha,j}, V_{\beta,j})_{j\geq 1}$ be a measurable enumeration of the points of a Poisson point process on $\mathbb{R} \times \mathbb{R}^d$, independent of the stable regenerative sets, with the mean measure

$$\alpha u^{-1-\alpha} du \prod_{i=1}^{d} (1-\beta_i) v_i^{-\beta_i} dv_i, \quad u, v_1, \ldots, v_d > 0\,.$$

Then the triples $(U_{\alpha,j}, V_{\beta,j}, R_{\beta,j})_{j\geq 1}$ form a Poisson point process on $\mathbb{R}\times\mathbb{R}^d\times\mathcal{F}(\mathbb{R}^d)$ with the mean measure

$$\alpha u^{-1-\alpha} du \left(\prod_{i=1}^{d} (1-\beta_i) v_i^{-\beta_i} dv_i \right) d\tilde{P}_\beta, \quad u, v_1, \ldots, v_d > 0\,. \tag{3.19}$$

Here $\tilde{P}_\beta$ is a probability measure on $\mathcal{F}(\mathbb{R})^d$ defined by

$$\tilde{P}_\beta = (P_{\beta_1} \times \cdots \times P_{\beta_d}) \circ H^{-1}\,,$$

with P_β being the law of the β-stable regenerative set, in (3.14), and $H : (\mathcal{F}(\mathbb{R}))^d \to \mathcal{F}(\mathbb{R}^d)$ is defined by

$$H(F_1, \ldots, F_d) = F_1 \times \cdots \times F_d\,.$$

Let

$$\eta_{\alpha,\beta}(\mathbf{t}) = \sum_{j=1}^{\infty} U_{\alpha,j} \mathbb{1}_{\{\mathbf{t}\in V_{\beta,j}+R_{\beta,j}\}} \ \ \mathbf{t} \in \mathbb{R}^d\,. \tag{3.20}$$

Several observations are in order. First of all, by Proposition 3.3.1, on event of probability 1, for each $\mathbf{t}$ the series in (3.20) has less than

$$\ell(\beta) := \min_{i=1,\ldots,d} (1-\beta_i)^{-1}$$

non-zero terms, so there are no convergence issues. On the same event the function defined by (3.20) is upper semi-continuous. Indeed, for any finite ℓ the function

$$\sum_{j=1}^{\ell} U_{\alpha,j} \mathbb{1}_{\{\mathbf{t}\in V_{\beta,j}+R_{\beta,j}\}} \ \ \mathbf{t} \in \mathbb{R}^d$$

is upper semi-continuous since each terms in this finite sum is upper semi-continuous due to the fact that each shifted product of stable regenerative sets is a closed set. Moreover, it is easy to check that, on each compact set, the uniform distance between this function and that defined in (3.20), goes to zero as $\ell \to \infty$; see p. 10 in Samorodnitsky and Wang (2019).

We now define a random sup measure as the sup integral of the random upper semi-continuous function in (3.20), and we will use the same notation, $\eta_{\alpha,\beta}$, for this sup measure. That is,

$$\eta_{\alpha,\beta}(B) = \sup_{t\in B} \sum_{j=1}^{\infty} U_{\alpha,j} \mathbb{1}_{\{\mathbf{t}\in V_{\beta,j}+R_{\beta,j}\}}, \quad B \in \mathcal{B}(\mathbb{R}^d)\,. \tag{3.21}$$

Remark 3.3.1. The random sup measure $\eta_{\alpha,\beta}$ defined by (3.21) is stationary, in the sense that for every $\mathbf{x} \geq \mathbf{0}$, $\eta_{\alpha,\beta}(\cdot + \mathbf{x}) \overset{\mathrm{d}}{=} \eta_{\alpha,\beta}$. This follows from the shift invariance of the law the random upper semi-continuous function in (3.20) as in Proposition 3.2 in Samorodnitsky and Wang (2019) dealing with the case $d = 1$. The argument in that proposition also shows that the random sup measure $\eta_{\alpha,\beta}$ is self-similar, in the sense that for any $c_1 > 0, \ldots, c_d > 0$,

$$\eta_{\alpha,\beta} \circ p_{c_1,\ldots,c_d} \overset{\mathrm{d}}{=} \prod_{i=1}^{d} c_i^{(1-\beta_i)/\alpha} \eta_{\alpha,\beta}\,,$$

where $p_{c_1,\ldots,c_d} : \mathbb{R}^d \to \mathbb{R}^d$ is the multiplication functional $p_{c_1,\ldots,c_d}(t_1, \ldots, t_d) = (c_1 t_1, \ldots, c_d t_d)$.

Importantly,, $\eta_{\alpha,\beta}$ is a Fréchet random sup measure if and only if the sets $(V_{\beta,j} + R_{\beta,j})$, $j = 1, 2, \ldots$ are a.s. disjoint. According to Proposition 3.3.1, a necessary and sufficient condition for this is $\beta_i \leq 1/2$ for some $i = 1, \ldots, d$.

The restriction of the random sup measure $\eta_{\alpha,\beta}$ in (3.21) to the hypercube $[\mathbf{0}, \mathbb{1}]$ has a somewhat more convenient representation. Let $\{R^{(i)}_{\beta_i,j}\}_{j\geq 1}$, $i = 1, \ldots, d$ be as stable regenerative sets as above, and let $\{V_j^{(i)}\}_{j\geq 1}$ be d independent families of i.i.d. random variables on $[0,1]$ with distributions given by

$$P(V_1^{(i)} \leq x) := x^{1-\beta_i}, \quad x \in [0,1] . \tag{3.22}$$

Let now $\{\Gamma_j\}$ be the sequence of the arrival times of a unit rate Poisson process on $(0,\infty)$. Assume that the families $\{V_j^{(1)}\}_{j\geq 1}, \ldots, \{V_j^{(d)}\}_{j\geq 1}$, $\{R_{\beta_1,j}^{(1)}\}_{j\geq 1}, \ldots, \{R_{\beta_d,j}^{(d)}\}_{j\geq 1}$ and the Poisson process are independent. Denoting

$$\tilde{R}_{\beta_i,j}^{(i)} = V_j^{(i)} + R_{\beta_i,j}^{(i)}, \ \ 1 \leq i \leq d, \ j \geq 1$$

and

$$\tilde{R}_{\beta,j} = \prod_{i=1}^{d} \tilde{R}_{\beta_i,j}^{(i)} \subset \mathbb{R}^d ,$$

an alternative representation for the random upper semi-continuous function in (3.20) restricted to $[\mathbf{0},\mathbb{1}]$ is

$$\eta_{\alpha,\beta}(\mathbf{t}) = \sum_{j=1}^{\infty} \Gamma_j^{-1/\alpha} \mathbb{1}_{\{\mathbf{t}\in \tilde{R}_{\beta,j}\}}, \quad \mathbf{t} \in [0,1]^d , \tag{3.23}$$

with the corresponding change in (3.21).

3.4 Convergence of the random sup measures

In this section we establish the first functional extremal theorem for the stationary random field $\mathbf{X}$ in (3.8). The random field naturally induces a family of random sup-measures $\{\eta_{\mathbf{n}}\}_{\mathbf{n}\in\mathbb{N}^d}$ by

$$\eta_{\mathbf{n}}(B) := \max_{\mathbf{k}/\mathbf{n}\in B} X_{\mathbf{k}}, \quad B \in \mathcal{B}([0,\infty)^d) . \tag{3.24}$$

In the following theorem we prove an extremal theorem in the space of the random sup measures.

Theorem 3.4.1. *For all* $0 < \alpha < 2$ *and* $0 < \beta_i < 1$, $i = 1, \ldots, d$,

$$\frac{1}{b_{\mathbf{n}}} \eta_{\mathbf{n}} \Rightarrow \left(\frac{C_\alpha}{2} \right)^{1/\alpha} \eta_{\alpha,\beta}, \quad \mathbf{n} \to \infty \,, \tag{3.25}$$

where $\eta_{\alpha,\beta}$ *is the random sup-measure defined in* (3.21). *The weak convergence holds in the space of sup measures* $\mathcal{M}(\mathbb{R}^d)$ *equipped with the sup vague topology.*

To simplify the notation, we will show the weak convergence in $\mathcal{M}([\mathbf{0}, \mathbb{1}])$. Note that by (3.13) we can represent, in law, the sup measure in the left hand side of (3.25) as

$$\frac{1}{b_{\mathbf{n}}} \eta_{\mathbf{n}}(B) = \max_{\mathbf{k}/\mathbf{n} \in B} C_\alpha^{1/\alpha} \sum_{j=1}^{\infty} \epsilon_j \Gamma_j^{-1/\alpha} \mathbb{1}_{A^d} \circ T^{\mathbf{k}}(U_{j,\mathbf{n}}), \ B \in \mathcal{B}([0,1]^d) \,. \tag{3.26}$$

As it is often done, we prove Theorem 3.4.1 via a truncation argument. We fix an $\ell \in \mathbb{N}$ and construct a truncated random sup-measure $\eta_{\mathbf{n},\ell}$ so that

$$\frac{1}{b_{\mathbf{n}}} \eta_{\mathbf{n},\ell}(B) = \max_{\mathbf{k}/\mathbf{n} \in B} C_\alpha^{1/\alpha} \sum_{j=1}^{\ell} \epsilon_j \Gamma_j^{-1/\alpha} \mathbb{1}_{A^d} \circ T^{\mathbf{k}}(U_{j,\mathbf{n}}), \ B \in \mathcal{B}([0,1]^d) \,. \tag{3.27}$$

Note that we can write

$$U_{j,\mathbf{n}}(\mathbf{k}) = \big(U_{j,n_1}^{(1)}(k_1), \ldots, U_{j,n_d}^{(d)}(k_d) \big)$$

for $\mathbf{n} = (n_1, \ldots, n_d)$ and $\mathbf{k} = (k_1, \ldots, k_d)$, with independent components in the right hand side, where $U_{j,n}^{(i)}$ has the law $\eta^{(i)}$ given in (3.12), $i = 1, \ldots, d$. Therefore, the set of zeroes of $U_{j,\mathbf{n}}$ satisfies

$$\mathcal{Z}(U_{j,\mathbf{n}}) = \mathcal{Z}(U_{j,n_1}^{(1)}) \times \cdots \times \mathcal{Z}(U_{j,n_d}^{(d)}) \,.$$

To proceed, we need to introduce new notation. Let $S \subset \mathbb{N}$. We set

$$\hat{I}_{S,n}^{(i)} = \cap_{j \in S} \mathcal{Z}(U_{j,n}^{(i)}), \ i = 1, \ldots, d, \ n \geq 1, \ \hat{I}_{S,\mathbf{n}} = \cap_{j \in S} \mathcal{Z}(U_{j,\mathbf{n}}), \ \mathbf{n} \in \mathbb{N}^d,$$

$$I_S^{(i)} = \cap_{j \in S} \tilde{R}_{\beta_i,j}^{(i)}, \ i = 1, \ldots, d, \ I_S = \cap_{j \in S} \tilde{R}_{\beta,j} \,.$$

At this stage the random objects described above do not need to be defined on the same probability space. We need the following extension of Theorem 5.4 of Samorodnitsky and Wang (2019).

Proposition 3.4.1.

$$\left(\frac{1}{\mathbf{n}}\hat{I}_{S,\mathbf{n}}\right)_{S\subset\{1,\dots,\ell\}} \Rightarrow (I_S)_{S\subset\{1,\dots,\ell\}},\ \mathbf{n}\to\infty\,,$$

in $\left(\mathcal{F}([\mathbf{0},\mathbb{1}])\right)^{2^\ell}$.

Proof. By Theorem 5.4 of Samorodnitsky and Wang (2019), for each $i = 1,\dots,d$ and $S\subset\{1,\dots,\ell\}$,

$$\frac{1}{n}\left[\hat{I}^{(i)}_{S,n}\cap[0,1]\right] \Rightarrow I^{(i)}_S\cap[0,1], \quad n\to\infty\,,$$

in the sense of weak convergence of random closed sets. By Corollaries 1.7.13 and 1.7.14 in Molchanov (2017) applied to rectangles of the type $\prod_{i=1}^d [a_i,b_i]$, $0\le a_i\le b_i\le 1$, $i=1,\dots,d$, we conclude that for every $S\subset\{1,\dots,\ell\}$,

$$\frac{1}{\mathbf{n}}\hat{I}_{S,\mathbf{n}} \Rightarrow I_S,\ \mathbf{n}\to\infty\,,$$

$(\mathcal{F}([\mathbf{0},\mathbb{1}])$. By Theorem 2.1 (ii) in Samorodnitsky and Wang (2019), this implies the joint convergence in the proposition. □

For $S\subset\{1,\dots,\ell\}$ we define now

$$\hat{I}^*_{S,\mathbf{n}} = \hat{I}_{S,n}\cap\left(\bigcup_{j\in\{1,\dots,\ell\}\setminus S}\mathcal{Z}(U_{j,\mathbf{n}})\right)^c\,, \tag{3.28}$$

the set of times where only the Markov chains corresponding to $j\in S$ reach 0. Similarly we define

$$I^*_S = I_S\cap\left(\bigcup_{j\in\{1,\dots,\ell\}\setminus S}\tilde{R}_{\beta,j}\right)^c\,, \tag{3.29}$$

As in the case of the one-dimensional time, for large $\mathbf{n}$ the sets $\hat{I}^*_{S,\mathbf{n}}$ and I^*_S are likely to be alike.

Lemma 3.4.1. *For an open rectangle $B \subset [0,1]^d$, let $H_{\mathbf{n}}(B)$ be the event*

$$H_{\mathbf{n}}(B) := \bigcup_{S\subset\{1,\dots,\ell\}} \left(\left\{ \frac{\mathbb{1}}{\mathbf{n}} \hat{I}_{S,\mathbf{n}} \cap B \neq \emptyset \right\} \cap \left\{ \frac{\mathbb{1}}{\mathbf{n}} \hat{I}^*_{S,\mathbf{n}} \cap B = \emptyset \right\} \right) \tag{3.30}$$

Then, $\lim_{\mathbf{n}\to\infty} P(H_{\mathbf{n}}(B)) = 0$.

Proof. Write $B = B_1 \times \cdots \times B_d$, with $B_1, \dots, B_d$ open rectangles in $[0,1]$. Denoting

$$\hat{I}^{(i)*}_{S,n} := \hat{I}^{(i)}_{S,n} \cap \left(\bigcup_{j\in\{1,\dots,\ell\}\setminus S} \mathcal{Z}(U^{(i)}_{j,n}) \right)^c, \ i = 1, \dots, d, \ S \subset \{1, \dots, \ell\}, \ n = 1, 2, \dots,$$

we have

$$H_{\mathbf{n}}(B) \subset \bigcup_{S\subset\{1,\dots,\ell\}} \bigcup_{i=1,\dots,d} \left(\left\{ \frac{1}{n_i} \hat{I}^{(i)}_{S,n_i} \cap B_i \neq \emptyset \right\} \cap \left\{ \frac{1}{n_i} \hat{I}^{(i)*}_{S,n_i} \cap B_i = \emptyset \right\} \right).$$

The right hand side above is a finite union events, and the probability of each one is asymptoticly vanishing by Lemma 5.5 in Samorodnitsky and Wang (2019). □

Remark 3.4.1. The argument of Lemma 5.5 in Samorodnitsky and Wang (2019) shows also the following version of the lemma: let

$$H^*_{\mathbf{n}} = \bigcup_{a_i>0, i=1,\dots,d} H_{\mathbf{n}} \left(\prod_{i=1}^d (0, a_i) \right).$$

Then $\lim_{\mathbf{n}\to\infty} P(H^*_{\mathbf{n}}) = 0$. We will find this formulation useful in the sequel.

We are now ready to prove convergence of the truncated random sup-measures.

Proposition 3.4.2. *Let $\ell \geq 1$, and define a random sup-measure $\eta_{\alpha,\beta,\ell}$ by*

$$\eta_{\alpha,\beta,\ell}(B) = \sup_{\mathbf{t}\in B} \sum_{j=1}^{\ell} \Gamma_j^{-1/\alpha} \mathbb{1}_{\{\mathbf{t}\in\tilde{R}_{\beta,j}\}}, \quad B \in \mathcal{B}([0,1]^d). \tag{3.31}$$

Then

$$\frac{1}{b_{\mathbf{n}}} \eta_{\mathbf{n},\ell} \Rightarrow \left(\frac{C_\alpha}{2} \right)^{1/\alpha} \eta_{\alpha,\beta,\ell}, \quad \mathbf{n} \to \infty$$

in the space of sup measures $\mathcal{M}([\mathbf{0}, \mathbb{1}])$ equipped with the sup vague topology.

Proof. We start by observing that an alternative expression for the random sup-measure $\eta_{\alpha,\beta,\ell}$ is

$$\eta_{\alpha,\beta,\ell}(B) = \max_{S\subset\{1,\dots,\ell\}} \mathbb{1}_{\{I_S\cap B\neq\emptyset\}} \sum_{j\in S} \Gamma_j^{-1/\alpha} \,. \tag{3.32}$$

Since stable subordinators do not hit fixed points, by (3.18) it suffices to show that for any m disjoint open rectangles $B_r = \prod_{i=1}^d (a_i^{(r)}, b_i^{(r)})$, $r = 1,\dots,m$ in $[0,1]^d$, we have a convergence of random vectors:

$$\frac{1}{b_{\mathbf{n}}}\left(\eta_{\mathbf{n},\ell}(B_1),\dots,\eta_{\mathbf{n},\ell}(B_m)\right) \Rightarrow \left(\frac{C_\alpha}{2}\right)^{1/\alpha} \left(\eta_{\alpha,\beta,\ell}(B_1),\dots,\eta_{\alpha,\beta,\ell}(B_m)\right) \,.$$

It is clear that for any $r = 1,\dots,m$, on the compliment of the event $H_{\mathbf{n}}(B_r)$,

$$\max_{\mathbf{k}/\mathbf{n}\in B_r} \sum_{j=1}^{\ell} \epsilon_j \Gamma_j^{-1/\alpha} \mathbb{1}_{A^d} \circ T^{\mathbf{k}}(U_{j,\mathbf{n}}) = \max_{S\subset\{1,\dots,\ell\}} \mathbb{1}_{\{(\mathbb{1}/\mathbf{n})\hat{I}_{S,\mathbf{n}}\cap B_r\neq\emptyset\}} \sum_{j\in S} \epsilon_j \Gamma_j^{-1/\alpha} \,.$$

Since $\hat{I}_{S,\mathbf{n}}$ is decreasing as the set S increases, we can choose, for a fixed S, the set $S' = \{j\in S : \epsilon_j = 1\}$ to obtain

$$\begin{aligned}
&\max_{S\subset\{1,\dots,\ell\}} \mathbb{1}_{\{(\mathbb{1}/\mathbf{n})\hat{I}_{S,\mathbf{n}}\cap B_r\neq\emptyset\}} \sum_{j\in S} \epsilon_j \Gamma_j^{-1/\alpha} \\
&= \max_{S\subset\{1,\dots,\ell\}} \mathbb{1}_{\{(\mathbb{1}/\mathbf{n})\hat{I}_{S,\mathbf{n}}\cap B_r\neq\emptyset\}} \sum_{j\in S} \mathbb{1}(\epsilon_j = 1)\Gamma_j^{-1/\alpha} \,.
\end{aligned}$$

Hence, on the compliment of the event $H_{\mathbf{n}}(B_1)\cup\dots\cup H_{\mathbf{n}}(B_m)$,

$$\begin{aligned}
&\frac{1}{b_{\mathbf{n}}}\left(\eta_{\mathbf{n},\ell}(B_1),\dots,\eta_{\mathbf{n},\ell}(B_{\mathbf{n}})\right) \\
&= C_\alpha^{1/\alpha} \left(\max_{S\subset\{1,\dots,\ell\}} \mathbb{1}_{\{(\mathbb{1}/\mathbf{n})\hat{I}_{S,\mathbf{n}}\cap B_r\neq\emptyset\}} \sum_{j\in S} \mathbb{1}_{\{\epsilon_j=1\}} \Gamma_j^{-1/\alpha} \right)_{r=1,\dots,m} \,.
\end{aligned}$$

By Proposition 3.4.1 the random vector in the right hand side converges weakly as $\mathbf{n}\to\infty$ to the random vector

$$C_\alpha^{1/\alpha} \left(\max_{S\subset\{1,\dots,\ell\}} \mathbb{1}_{\{I_S\cap B_r\neq\emptyset\}} \sum_{j\in S} \mathbb{1}_{\{\epsilon_j=1\}} \Gamma_j^{-1/\alpha} \right)_{r=1,\dots,m} \,.$$

Since, by Lemma 3.4.1, the event $H_{\mathbf{n}}(B_1) \cup \cdots \cup H_{\mathbf{n}}(B_m)$ has an asymptotically vanishing probability, the random vector

$$\frac{1}{b_{\mathbf{n}}} \left(\eta_{\mathbf{n},\ell}(B_1), \ldots, \eta_{\mathbf{n},\ell}(B_m) \right)$$

converges weakly to the same limit. The claim of the proposition follows by noticing that the thinned Poisson random measure $(\mathbb{1}_{\{\epsilon_j=1\}}\Gamma_j^{-1/\alpha})_{j\geq 1}$ has the same law as $(2^{-1/\alpha}\Gamma_j^{-1/\alpha})_{j\geq 1}$ and using (3.32). $\square$

We now deal with the part of the random sup measure in Theorem 3.4.1 that is left after the truncation procedure above. The following proposition is crucial.

Proposition 3.4.3. *For all $\delta > 0$,*

$$\lim_{\ell\to\infty} \limsup_{\mathbf{n}\to\infty} P\left(\max_{\mathbf{0}\leq\mathbf{k}\leq\mathbf{n}} \left| \sum_{j=\ell+1}^{\infty} \epsilon_j \Gamma_j^{-1/\alpha} \mathbb{1}_{A^d} \circ T^{\mathbf{k}}(U_{j,\mathbf{n}}) \right| > \delta \right) = 0\,. \tag{3.33}$$

Proof. Clearly,

$$\begin{aligned} & P\left(\max_{\mathbf{0}\leq\mathbf{k}\leq\mathbf{n}} \left| \sum_{j=\ell+1}^{\infty} \epsilon_j \Gamma_j^{-1/\alpha} \mathbb{1}_{A^d} \circ T^{\mathbf{k}}(U_{j,\mathbf{n}}) \right| > \delta \right) \qquad (3.34) \\ \leq & P\left(\max_{\mathbf{0}\leq\mathbf{k}\leq\mathbf{n}} \left| \sum_{j=\ell+1}^{\infty} \epsilon_j \Gamma_j^{-1/\alpha} \mathbb{1}(\Gamma_j > b_{\mathbf{n}}^{\alpha}) \mathbb{1}_{A^d} \circ T^{\mathbf{k}}(U_{j,\mathbf{n}}) \right| > \delta/2 \right) \\ + & P\left(\max_{\mathbf{0}\leq\mathbf{k}\leq\mathbf{n}} \left| \sum_{j=\ell+1}^{\infty} \epsilon_j \Gamma_j^{-1/\alpha} \mathbb{1}(\Gamma_j \leq b_{\mathbf{n}}^{\alpha}) \mathbb{1}_{A^d} \circ T^{\mathbf{k}}(U_{j,\mathbf{n}}) \right| > \delta/2 \right)\,. \end{aligned}$$

By symmetry,

$$\begin{aligned} & P\left(\max_{\mathbf{0}\leq\mathbf{k}\leq\mathbf{n}} \left| \sum_{j=\ell+1}^{\infty} \epsilon_j \Gamma_j^{-1/\alpha} \mathbb{1}(\Gamma_j > b_{\mathbf{n}}^{\alpha}) \mathbb{1}_{A^d} \circ T^{\mathbf{k}}(U_{j,\mathbf{n}}) \right| > \delta/2 \right) \\ \leq & \, 2P\left(\max_{\mathbf{0}\leq\mathbf{k}\leq\mathbf{n}} \left| \sum_{j=1}^{\infty} \epsilon_j \Gamma_j^{-1/\alpha} \mathbb{1}(\Gamma_j > b_{\mathbf{n}}^{\alpha}) \mathbb{1}_{A^d} \circ T^{\mathbf{k}}(U_{j,\mathbf{n}}) \right| > \delta/2 \right)\,. \end{aligned}$$

The sum in the right hand side is a representation, in law, of the restriction to the set $\mathbf{0} \leq \mathbf{k} \leq \mathbf{n}$ of the random field $(b_{\mathbf{n}}^{-1} Y_{\mathbf{k}},\, \mathbf{k} \in \mathbb{Z}^d)$, where $(Y_{\mathbf{k}},\, \mathbf{k} \in \mathbb{Z}^d)$ is a stationary symmetric infinitely divisible random field defined, similarly to the original stationary symmetric α-stable random field in (3.8), by

$$Y_{\mathbf{k}} = \int_E f \circ T^{\mathbf{k}}(\mathbf{x})\, \tilde{M}(d\mathbf{x}), \quad \mathbf{k} \in \mathbb{Z}^d\,, \tag{3.35}$$

with the distinction that the local Lévy measure $\tilde{\rho}$ of the symmetric infinitely divisible random measure in (3.35) has the density $\alpha|x|^{-(\alpha+1)}$ restricted to $|x| \le 1$. See Chapter 3 in Samorodnitsky (2016). In particular, each $Y_{\mathbf{k}}$ has a Lévy measure with a bounded support and, hence, has (faster than) exponentially fast decaying tails. See e.g. Sato (1999) Chapter 5. We conclude by the regular variation (3.6) of the factors in (3.10) that for $\mathbf{n} = (n_1, \ldots, n_d)$,

$$P\left(\max_{\mathbf{0}\le\mathbf{k}\le\mathbf{n}} \left| \sum_{j=\ell+1}^{\infty} \epsilon_j \Gamma_j^{-1/\alpha} \mathbb{1}(\Gamma_j > b_{\mathbf{n}}^{\alpha}) \mathbb{1}_{A^d} \circ T^{\mathbf{k}}(U_{j,\mathbf{n}}) \right| > \delta/2 \right)$$

$$\le 2P\big(\max_{\mathbf{0}\le\mathbf{k}\le\mathbf{n}} |Y_{\mathbf{k}}| > b_{\mathbf{n}}(\delta/2)\big) \le 2\prod_{i=1}^{d}(1+n_i) P\big(|Y_{\mathbf{0}}| > b_{\mathbf{n}}(\delta/2)\big) \to 0$$

as $\mathbf{n} \to \infty$. Therefore, the claim of the proposition will follow once we prove that

$$\lim_{\ell\to\infty} \limsup_{\mathbf{n}\to\infty} P\left(\max_{\mathbf{0}\le\mathbf{k}\le\mathbf{n}} \left| \sum_{j=\ell+1}^{\infty} \epsilon_j \Gamma_j^{-1/\alpha} \mathbb{1}(\Gamma_j \le b_{\mathbf{n}}^{\alpha}) \mathbb{1}_{A^d} \circ T^{\mathbf{k}}(U_{j,\mathbf{n}}) \right| > \delta \right) = 0\,. \tag{3.36}$$

To this end, let $M > 0$ and set $D_\ell^M := \{\Gamma_{\ell+1} \ge M\}$. By the Strong Law of Large Numbers, $\lim_{\ell\to\infty} P(D_\ell^M) = 1$. Therefore, we may replace the probability in (3.36) by

$$P\left(\left\{\max_{\mathbf{0}\le\mathbf{k}\le\mathbf{n}} \left| \sum_{j=\ell+1}^{\infty} \epsilon_j \Gamma_j^{-1/\alpha} \mathbb{1}(\Gamma_j \le b_{\mathbf{n}}^{\alpha}) \mathbb{1}_{A^d} \circ T^{\mathbf{k}}(U_{j,\mathbf{n}}) \right| > \delta \right\} \cap D_\ell^M \right)\,.$$

The above quantity does not exceed

$$\sum_{\mathbf{0}\le\mathbf{k}\le\mathbf{n}} P\left(\left\{ \left| \sum_{j=\ell+1}^{\infty} \epsilon_j \Gamma_j^{-1/\alpha} \mathbb{1}(\Gamma_j \le b_{\mathbf{n}}^{\alpha}) \mathbb{1}_{A^d} \circ T^{\mathbf{k}}(U_{j,\mathbf{n}}) \right| > \delta \right\} \cap D_\ell^M \right) \tag{3.37}$$

$$= \prod_{i=1}^{d}(n_i+1) P\left(\left\{ \left| \sum_{j=\ell+1}^{\infty} \epsilon_j \Gamma_j^{-1/\alpha} \mathbb{1}(\Gamma_j \le b_{\mathbf{n}}^{\alpha}) \mathbb{1}_{A^d}(U_{j,\mathbf{n}}) \right| > \delta \right\} \cap D_\ell^M \right)\,,$$

since the probabilities in the sum in (3.37) do not depend on $\mathbf{k}$. Using symmetry, we have

$$
\begin{aligned}
&P\left(\left\{\left|\sum_{j=\ell+1}^{\infty} \epsilon_j \Gamma_j^{-1/\alpha} \mathbb{1}(\Gamma_j \le b_{\mathbf{n}}^{\alpha}) \mathbb{1}_{A^d}(U_{j,\mathbf{n}})\right| > \delta\right\} \cap D_\ell^M\right) \\
\le &P\left(\left|\sum_{j=\ell+1}^{\infty} \epsilon_j \Gamma_j^{-1/\alpha} \mathbb{1}(M \le \Gamma_j \le b_{\mathbf{n}}^{\alpha}) \mathbb{1}_{A^d}(U_{j,\mathbf{n}})\right| > \delta\right) \\
\le &2P\left(\left|\sum_{j=1}^{\infty} \epsilon_j \Gamma_j^{-1/\alpha} \mathbb{1}(M \le \Gamma_j \le b_{\mathbf{n}}^{\alpha}) \mathbb{1}_{A^d}(U_{j,\mathbf{n}})\right| > \delta\right). \qquad (3.38)
\end{aligned}
$$

Notice that the Poisson point process $(b_{\mathbf{n}}^{-\alpha} \Gamma_j \mathbb{1}_{A^d}(U_{j,\mathbf{n}}))_j$ has the same law as the Poisson point proces $(\Gamma_j)_j$. Therefore, the probability in (3.38) coincides with

$$
\begin{aligned}
&P\left(b_{\mathbf{n}}^{-1}\left|\sum_{j=1}^{\infty} \epsilon_j \Gamma_j^{-1/\alpha} \mathbb{1}_{\{M b_{\mathbf{n}}^{-\alpha} \le \Gamma_j \le 1\}}\right| > \delta\right) \\
\le &P\left(\left|b_{\mathbf{n}}^{-1} \sum_{j=j_M+1}^{\infty} \epsilon_j \Gamma_j^{-1/\alpha} \mathbb{1}_{\{M b_{\mathbf{n}}^{-\alpha} \le \Gamma_j \le 1\}}\right| > \delta/2\right) \\
\le &b_{\mathbf{n}}^{-p} (\delta/2)^{-p} E\left|\sum_{j=j_M+1}^{\infty} \epsilon_j \Gamma_j^{-1/\alpha} \mathbb{1}_{\{M b_{\mathbf{n}}^{-\alpha} \le \Gamma_j \le 1\}}\right|^p
\end{aligned}
$$

for any $p > 0$, where $j_M = \lceil M^{1/\alpha} \delta/2 \rceil$. If $p > 0$ is large enough, then by the regular variation (3.6),

$$
\lim_{\mathbf{n}\to\infty} \prod_{i=1}^{d} (n_i + 1) b_{\mathbf{n}}^{-p} \to 0\,.
$$

Therefore, (3.36) will follow once we check that for any $p > 0$ we can take $M > 0$ large enough so that

$$
\limsup_{\mathbf{n}\to\infty} E\left|\sum_{j=j_M+1}^{\infty} \epsilon_j \Gamma_j^{-1/\alpha} \mathbb{1}_{\{M b_{\mathbf{n}}^{-\alpha} \le \Gamma_j \le 1\}}\right|^p < \infty\,. \qquad (3.39)
$$

Let us take $p = 2k$, an even integer. By the Khintchine inequality (see e.g. (A.1) in Nualart (1995)), there is a constant $c_p \in (0, \infty)$ such that

$$
\begin{aligned}
E\left|\sum_{j=j_M+1}^{\infty} \epsilon_j \Gamma_j^{-1/\alpha} \mathbb{1}_{\{M b_{\mathbf{n}}^{-\alpha} \le \Gamma_j \le 1\}}\right|^p &\le c_p E\left(\sum_{j=j_M+1}^{\infty} \left(\Gamma_j^{-1/\alpha} \mathbb{1}_{\{M b_{\mathbf{n}}^{-\alpha} \le \Gamma_j \le 1\}}\right)^2\right)^{p/2} \\
\le c_p E\left(\sum_{j=j_M+1}^{\infty} \Gamma_j^{-2/\alpha}\right)^k &\le c_p \left(\sum_{j=j_M+1}^{\infty} \left(E(\Gamma_j^{-2k/\alpha})\right)^{1/k}\right)^k.
\end{aligned}
$$

The claim (3.39) now follows since $E\big(\Gamma_j^{-2k/\alpha}\big) < \infty$ for $j > 2k/\alpha$, and

$$E\big(\Gamma_j^{-2k/\alpha}\big) \sim j^{-2k/\alpha},\ j \to \infty\,.$$

$\square$

We are now ready to finish the proof of Theorem 3.4.1.

Proof of Theorem 3.4.1 . Once again, by (3.18) it suffices to show that for any m disjoint open rectangles $B_r = \prod_{i=1}^d (a_i^{(r)}, b_i^{(r)})$, $r = 1, \ldots, m$ in $[0,1]^d$, we have

$$\frac{1}{b_{\mathbf{n}}}\left(\eta_{\mathbf{n}}(B_1), \ldots, \eta_{\mathbf{n},\ell}(B_m)\right) \Rightarrow \left(\frac{C_\alpha}{2}\right)^{1/\alpha}\left(\eta_{\alpha,\beta}(B_1), \ldots, \eta_{\alpha,\beta,\ell}(B_m)\right)\,.$$

Since by Proposition 3.4.2

$$\frac{1}{b_{\mathbf{n}}}\left(\eta_{\mathbf{n},\ell}(B_1), \ldots, \eta_{\mathbf{n},\ell}(B_m)\right) \Rightarrow \left(\frac{C_\alpha}{2}\right)^{1/\alpha}\left(\eta_{\alpha,\beta,\ell}(B_1), \ldots, \eta_{\alpha,\beta,\ell}(B_m)\right)\,,$$

and $\eta_{\alpha,\beta,\ell}$ increases to $\eta_{\alpha,\beta}$ almost surely, we can use the "convergence together" argument, as in Theorem 3.2 of Billingsley (1999). To this end we need to check that for each $r = 1, \ldots, m$ and any $\epsilon > 0$,

$$\lim_{\ell\to\infty}\limsup_{\mathbf{n}\to\infty} P\left(\frac{1}{b_{\mathbf{n}}}\left|\eta_{\mathbf{n}}(B_r) - \eta_{\mathbf{n},\ell}(B_r)\right| > \epsilon\right) = 0\,.$$

This is, however, an immediate conclusion from Proposition 3.4.3. $\square$

Remark 3.4.2. Note that the limiting sup measure in Theorem 3.4.1 is Fréchet only when $\beta_i \le 1/2$ for some $i = 1, \ldots, d$. See Remark 3.3.1.

The stationary random field $\mathbf{X}$ in (3.8) induces another family of random sup measures $\{\tilde\eta_{\mathbf{n}}\}_{\mathbf{n}\in\mathbb{N}^d}$ via

$$\tilde\eta_{\mathbf{n}}(B) := \max_{\mathbf{k}/\mathbf{n}\in B} |X_{\mathbf{k}}|, \quad B \in \mathcal{B}([0,\infty)^d)\,. \tag{3.40}$$

This family of random sup-measures satisfies the following analogue of Theorem 3.4.1.

Theorem 3.4.2. *For all* $0 < \alpha < 2$ *and* $0 < \beta_i < 1$, $i = 1, \ldots, d$,

$$\frac{1}{b_{\mathbf{n}}} \tilde{\eta}_{\mathbf{n}} \Rightarrow \left(\frac{C_\alpha}{2} \right)^{1/\alpha} \max\big(\eta^{(1)}_{\alpha,\beta},\, \eta^{(2)}_{\alpha,\beta} \big), \quad \mathbf{n} \to \infty\,, \tag{3.41}$$

where $\eta^{(1)}_{\alpha,\beta}$ *and* $\eta^{(2)}_{\alpha,\beta}$ *are two independent copies of the random sup measure defined in* (3.20)*. The weak convergence holds in the space of sup measures* $\mathcal{M}(\mathbb{R}^d)$ *equipped with the sup vague topology.*

Proof. Once again, we will show the weak convergence in $\mathcal{M}([\mathbf{0}, \mathbb{1}])$. We continue using the notation of the proof of Theorem 3.4.1. The same argument as in the that proof works once we show that, in the obvious notation, for any $\ell = 1, 2, \ldots$,

$$\frac{1}{b_{\mathbf{n}}} \tilde{\eta}_{\mathbf{n},\ell} \Rightarrow \frac{C_\alpha^{1/\alpha}}{2} \max\big(\eta^{(1)}_{\alpha,\beta,\ell},\, \eta^{(2)}_{\alpha,\beta,\ell} \big), \quad \mathbf{n} \to \infty\,, \tag{3.42}$$

where

$$\frac{1}{b_{\mathbf{n}}} \tilde{\eta}_{\mathbf{n},\ell}(B) = \max_{\mathbf{k}/\mathbf{n} \in B} \left| \sum_{j=1}^{\ell} \epsilon_j \Gamma_j^{-1/\alpha} \mathbb{1}_{A^d} \circ T^{\mathbf{k}}(U_{j,\mathbf{n}}) \right|, \ B \in \mathcal{B}([0,1]^d)\,. \tag{3.43}$$

Outside the vanishing event $H_n(B_1) \cup \cdots \cup H_n(B_m)$ we now have

$$\begin{aligned} &\frac{1}{b_{\mathbf{n}}} \left(\tilde{\eta}_{\mathbf{n},\ell}(B_1), \ldots, \tilde{\eta}_{\mathbf{n},\ell}(B_{\mathbf{n}}) \right) \\ &= C_\alpha^{1/\alpha} \Bigg(\max \Bigg[\max_{S \subset \{1,\ldots,\ell\}} \mathbb{1}_{\{(\mathbb{1}/\mathbf{n}) \hat{I}_{S,\mathbf{n}} \cap B_r \neq \emptyset\}} \sum_{j \in S} \mathbb{1}_{\{\epsilon_j = 1\}} \Gamma_j^{-1/\alpha}, \\ &\qquad \max_{S \subset \{1,\ldots,\ell\}} \mathbb{1}_{\{(\mathbb{1}/\mathbf{n}) \hat{I}_{S,\mathbf{n}} \cap B_r \neq \emptyset\}} \sum_{j \in S} \mathbb{1}_{\{\epsilon_j = -1\}} \Gamma_j^{-1/\alpha} \Bigg] \Bigg)_{r=1,\ldots,m}. \end{aligned}$$

Using again Proposition 3.4.1 and Lemma 3.4.1 we conclude that, as $\mathbf{n} \to \infty$,

$$\begin{aligned} &\frac{1}{b_{\mathbf{n}}} \left(\tilde{\eta}_{\mathbf{n},\ell}(B_1), \ldots, \tilde{\eta}_{\mathbf{n},\ell}(B_{\mathbf{n}}) \right) \\ &\Rightarrow C_\alpha^{1/\alpha} \Bigg(\max \Bigg[\max_{S \subset \{1,\ldots,\ell\}} \mathbb{1}_{\{I_S \cap B_r \neq \emptyset\}} \sum_{j \in S} \mathbb{1}_{\{\epsilon_j = 1\}} \Gamma_j^{-1/\alpha}, \\ &\qquad \max_{S \subset \{1,\ldots,\ell\}} \mathbb{1}_{\{I_S \cap B_r \neq \emptyset\}} \sum_{j \in S} \mathbb{1}_{\{\epsilon_j = -1\}} \Gamma_j^{-1/\alpha} \Bigg] \Bigg)_{r=1,\ldots,m}. \end{aligned}$$

The statement of the theorem now follows since $(\mathbb{1}_{\{\epsilon_j=1\}}\Gamma_j^{-1/\alpha})_j$ and $(\mathbb{1}_{\{\epsilon_j=-1\}}\Gamma_j^{-1/\alpha})_j$ are two independent Poisson random measures, each with the same law as $(2^{-1/\alpha}\Gamma_j^{-1/\alpha})_{j\geq 1}$, and using (3.32). $\square$

Remark 3.4.3. It is interesting to observe that in the case $0 < \beta_i \leq 1/2$ for some $i = 1, \ldots, d$, the sup measure $\eta_{\alpha,\beta}$ is a Fréchet random sup measure, and so (3.25) can be reformulated as

$$\frac{1}{b_{\mathbf{n}}}\tilde{\eta}_{\mathbf{n}} \Rightarrow C_\alpha^{1/\alpha}\eta_{\alpha,\beta}, \quad \mathbf{n} \to \infty\,. \tag{3.44}$$

See Remark 3.3.1. However, if $0 < \beta_i > 1/2$ for all $i = 1, \ldots, d$, then the random sup measure $\eta_{\alpha,\beta}$ is not max-stable, so (3.44) is no longer a valid statement of Theorem 3.4.2.

3.5 Convergence of the partial maxima processes

In this section we prove another version of a functional extremal theorem for the stationary random field $\mathbf{X}$ in (3.8). This time we will be working in the space $D(\mathbb{R}_+^d)$, and the limit will itself be a random field. The random field $\mathbf{X}$ induces an array of partial maxima random fields $\{M_{\mathbf{n}}\}$ by

$$M_{\mathbf{n}}(\mathbf{t}) := \max_{\mathbf{0}\leq\mathbf{k}\leq\mathbf{nt}} X_{\mathbf{k}}, \quad \mathbf{t} \in \mathbb{R}_+^d\,.$$

The random sup measure $\eta_{\alpha,\beta}$ in (3.20) also induces a random field $W_{\alpha,\beta}$, by

$$W_{\alpha,\beta}(\mathbf{t}) := \eta_{\alpha,\beta}([\mathbf{0},\mathbf{t}]), \quad \mathbf{t} \in \mathbb{R}_+^d\,.$$

Remark 3.5.1. It follows immediately from Remark 3.3.1 that the random field $\big(W_{\alpha,\beta}(\mathbf{t}),\, \mathbf{t} \in \mathbb{R}_+^d\big)$ is self-similar, in the sense that for any $c_1 > 0, \ldots, c_d > 0$

$$\big(W_{\alpha,\beta}((c_1t_1, \ldots, c_dt_d)),\, \mathbf{t} \in \mathbb{R}_+^d\big) \overset{\mathrm{d}}{=} \left(\prod_{i=1}^d c_i^{(1-\beta_i)/\alpha} W_{\alpha,\beta}(\mathbf{t}),\, \mathbf{t} \in \mathbb{R}_+^d\right).$$

This is, of course, what a multivariate version of Lamperti's theorem requires from the limit in any functional extremal theorem; see e.g. Theorem 8.1.5 in Samorodnitsky (2016).

The following functional extremal theorem is the main result of this section.

Theorem 3.5.1. *For all* $0 < \alpha < 2$ *and* $0 < \beta_i < 1$, $i = 1, \ldots, d$,

$$\left(\frac{1}{b_{\mathbf{n}}} M_{\mathbf{n}}(\mathbf{t}),\, t \in \mathbb{R}_+^d\right) \Rightarrow \left(\left(\frac{C_\alpha}{2}\right)^{1/\alpha} W_{\alpha,\beta}(\mathbf{t}),\, \mathbf{t} \in \mathbb{R}_+^d\right) \tag{3.45}$$

in the Skorohod J_1 *topology on the space* $D(\mathbb{R}_+^d)$.

Proof. The usual reference for multiparameter weak convergence is Straf (1972). For our purposes there is little difference between the properties of weak convergence in $D(\mathbb{R}_+^d)$ for $d = 1$ and $d > 1$. We will show weak convergence in $D([\mathbf{0}, \mathbb{1}])$, and we use the series representation (3.13). By (3.26) we can write, in law,

$$\frac{1}{b_{\mathbf{n}}} M_{\mathbf{n}}(\mathbf{t}) = \max_{\mathbf{0} \le \mathbf{k}/\mathbf{n} \le \mathbf{t}} C_\alpha^{1/\alpha} \sum_{j=1}^{\infty} \epsilon_j \Gamma_j^{-1/\alpha} \mathbb{1}_{A^d} \circ T^{\mathbf{k}}(U_{j,\mathbf{n}}),\ \mathbf{t} \in [\mathbf{0}, \mathbb{1}]\,. \tag{3.46}$$

We again use a truncation argument. For $\ell \in \mathbb{N}$, we define random fields $M_{\mathbf{n},\ell}$ by

$$\frac{1}{b_{\mathbf{n}}} M_{\mathbf{n},\ell}(\mathbf{t}) = \max_{\mathbf{0} \le \mathbf{k}/\mathbf{n} \le \mathbf{t}} C_\alpha^{1/\alpha} \sum_{j=1}^{\ell} \epsilon_j \Gamma_j^{-1/\alpha} \mathbb{1}_{A^d} \circ T^{\mathbf{k}}(U_{j,\mathbf{n}}),\ \mathbf{t} \in [\mathbf{0}, \mathbb{1}]\,. \tag{3.47}$$

Similarly, starting with the truncated random sup measure $\eta_{\alpha,\beta,\ell}$ we define a random field $W_{\alpha,\beta,\ell}$ by

$$W_{\alpha,\beta,\ell}(\mathbf{t}) := \eta_{\alpha,\beta,\ell}([\mathbf{0}, \mathbf{t}]), \quad \mathbf{t} \in [\mathbf{0}, \mathbb{1}]\,.$$

We start by proving that

$$\left(\frac{1}{b_{\mathbf{n}}} M_{\mathbf{n},\ell}(\mathbf{t}),\, t \in \mathbb{R}_+^d\right) \Rightarrow \left(\left(\frac{C_\alpha}{2}\right)^{1/\alpha} W_{\alpha,\beta,\ell}(\mathbf{t}),\, \mathbf{t} \in [\mathbf{0}, \mathbb{1}]\right)\,. \tag{3.48}$$

Note that the representation (3.32) can be written, in law, as

$$W_{\alpha,\beta,\ell}(\mathbf{t}) = \max_{S\subset\{1,\dots,\ell\}} \mathbb{1}_{\{I_S\cap[\mathbf{0},\mathbf{t}]\neq\emptyset\}}\, 2^{1/\alpha}\sum_{j\in S}\mathbb{1}_{\{\epsilon_j=1\}}\Gamma_j^{-1/\alpha}, \quad \mathbf{t}\in[0,1]^d\,. \tag{3.49}$$

Furthermore, from the argument in Proposition 3.4.2 we know that outside of an event $A_{\mathbf{n}}$ whose probability goes to zero as $\mathbf{n}\to\infty$, the random field $\big((1/b_{\mathbf{n}})M_{\mathbf{n},\ell}(\mathbf{t}),\, \mathbf{t}\in[\mathbf{0},\mathbb{1}]\big)$ coincides with the random field

$$\max_{S\subset\{1,\dots,\ell\}} \mathbb{1}_{\{(\mathbb{1}/\mathbf{n})\hat{I}_{S,\mathbf{n}}\cap[\mathbf{0},\mathbf{t}]\neq\emptyset\}}C_\alpha^{1/\alpha}\sum_{j\in S}\mathbb{1}_{\{\epsilon_j=1\}}\Gamma_j^{-1/\alpha}, \quad \mathbf{t}\in[\mathbf{0},\mathbb{1}]\,;$$

see Remark 3.4.1. Therefore, (3.48) will follows once we prove that

$$\begin{aligned}&\left(\max_{S\subset\{1,\dots,\ell\}} \mathbb{1}_{\{(\mathbb{1}/\mathbf{n})\hat{I}_{S,\mathbf{n}}\cap[\mathbf{0},\mathbf{t}]\neq\emptyset\}}\sum_{j\in S}\mathbb{1}_{\{\epsilon_j=1\}}\Gamma_j^{-1/\alpha},\, \mathbf{t}\in[\mathbf{0},\mathbb{1}]\right)\\ \Rightarrow &\left(\max_{S\subset\{1,\dots,\ell\}} \mathbb{1}_{\{I_S\cap[\mathbf{0},\mathbf{t}]\neq\emptyset\}}\sum_{j\in S}\mathbb{1}_{\{\epsilon_j=1\}}\Gamma_j^{-1/\alpha},\, \mathbf{t}\in[\mathbf{0},\mathbb{1}]\right)\,.\end{aligned} \tag{3.50}$$

Since the Fell topology on $\mathcal{F}([0,1]^d)$ is separable and metrizable (see Salinetti and Wets (1981)), by the Skorohod representation theorem, we can find a common probability space for $\big(\hat{I}_{S,\mathbf{n}},\, S\subset\{1,\dots,\ell\}\big)$ and $\big(I_s,\, S\subset\{1,\dots,\ell\}\big)$ such that the convergence in Proposition 3.4.1 becomes the almost sure convergence. On that probability space we will prove a.s. convergence in (3.50).

For $i=1,\dots,d$ and $S\subset\{1,\dots,\ell\}$ denote

$$t_{S,\mathbf{n}}^{(i)} = \inf\big\{t>0:\, t\in n_i^{-1}\hat{I}_{S,n_i}^{(i)}\big\}, \quad t_S^{(i)} = \inf\big\{t>0:\, t\in I_S^{(i)}\big\}\,.$$

Then the a.s. convergence in Proposition 3.4.1 implies that $t_{S,\mathbf{n}}^{(i)}\to t_S^{(i)}$ a.s. as $\mathbf{n}\to\infty$ for every $i=1,\dots,d$ and $S\subset\{1,\dots,\ell\}$. If we denote $\mathbf{t}_{S,\mathbf{n}} = \big(t_{S,\mathbf{n}}^{(1)},\dots,t_{S,\mathbf{n}}^{(d)}\big)$ and $\mathbf{t}_S = \big(t_S^{(1)},\dots,t_S^{(d)}\big)$ for $S\subset\{1,\dots,\ell\}$, then $\mathbf{t}_{S,\mathbf{n}}\to\mathbf{t}_S$ a.s. as $\mathbf{n}\to\infty$. Since stable subordinators do not hit fixed points, the 2^ℓ points $\mathbf{t}_S, S\subset\{1,\dots,\ell\}$ are distinct. Furthermore, given $(\epsilon_j,\Gamma_j)_{j\in S}$, these points determine the realization of the random field in the right hand side of (3.50), while the 2^ℓ points $\mathbf{t}_{S,\mathbf{n}}, S\subset$

$\{1,\ldots,\ell\}$ determine the realization of the random field in the left hand side of (3.50). Therefore, any homeomorphism of $[0,1]^d$ onto itself that fixes the origin and moves $\mathbf{t}_{S,\mathbf{n}}$ to $\mathbf{t}_S$ for each $S\subset\{1,\ldots,\ell\}$ makes the values of the field in the left hand side of (3.50) equal to the values of the field in the right hand side of (3.50). The convergence $\mathbf{t}_{S,\mathbf{n}}\to\mathbf{t}_S$ for each S guarantees that these homeomorphisms can be chosen to converge to the identity in the supremum norm. Hence a.s. convergence in (3.50).

To complete the proof we use, once again, the "convergence together" argument in Theorem 3.2 of Billingsley (1999). The first step to this end is to show that

$$\big(W_{\alpha,\beta,\ell}(\mathbf{t}),\,\mathbf{t}\in[\mathbf{0},\mathbb{1}]\big)\Rightarrow\big(W_{\alpha,\beta}(\mathbf{t}),\,\mathbf{t}\in[\mathbf{0},\mathbb{1}]\big) \tag{3.51}$$

as $\ell\to\infty$ in the Skorohod J_1 topology on the space $D([\mathbf{0},\mathbb{1}])$. Since we can represent the random field in the right hand side of (3.51), in law, as

$$W_{\alpha,\beta}(\mathbf{t})=\sup_{S\subset\mathbb{N}}\mathbb{1}_{\{I_S\cap[\mathbf{0},\mathbf{t}]\neq\emptyset\}}\,2^{1/\alpha}\sum_{j\in S}\mathbb{1}_{\{\epsilon_j=1\}}\Gamma_j^{-1/\alpha},\quad \mathbf{t}\in[\mathbf{0},\mathbb{1}]\,, \tag{3.52}$$

we will use the representations in law, (3.49) and (3.52), and prove a.s. convergence of the random fields in the right hand sides of these representations. Furthermore, we will show this a.s. convergence in the uniform distance. To this end, fix $\mathbf{t}\in[\mathbf{0},\mathbb{1}]$ and note that by Proposition 3.3.1, with probability 1, any set $S\subset\mathbb{N}$ such that $I_S\cap[\mathbf{0},\mathbf{t}]\neq\emptyset$ must be of cardinality smaller than $\min_{i=1,\ldots,d}(1-\beta_i)^{-1}$. Furthermore, for any such S, the set $S\cap\{1,\ldots,\ell\}$ contributes to the maximum in the right hand side in (3.49). Therefore,

$$\begin{aligned}0\le&\sup_{S\subset\mathbb{N}}\mathbb{1}_{\{I_S\cap[\mathbf{0},\mathbf{t}]\neq\emptyset\}}\,2^{1/\alpha}\sum_{j\in S}\mathbb{1}_{\{\epsilon_j=1\}}\Gamma_j^{-1/\alpha}\\&-\max_{S\subset\{1,\ldots,\ell\}}\mathbb{1}_{\{I_S\cap[\mathbf{0},\mathbf{t}]\neq\emptyset\}}\,2^{1/\alpha}\sum_{j\in S}\mathbb{1}_{\{\epsilon_j=1\}}\Gamma_j^{-1/\alpha}\le 2^{1/\alpha}\min_{i=1,\ldots,d}(1-\beta_i)^{-1}\Gamma_{\ell+1}^{-1/\alpha}\,.\end{aligned}$$

That is,

$$\begin{aligned}\sup_{\mathbf{t}\in[\mathbf{0},\mathbb{1}]} \Bigg| &\sup_{S\subset\mathbb{N}} \mathbb{1}_{\{I_S\cap[\mathbf{0},\mathbf{t}]\neq\emptyset\}}\, 2^{1/\alpha} \sum_{j\in S} \mathbb{1}_{\{\epsilon_j=1\}}\Gamma_j^{-1/\alpha} \\ &- \max_{S\subset\{1,\dots,\ell\}} \mathbb{1}_{\{I_S\cap[\mathbf{0},\mathbf{t}]\neq\emptyset\}}\, 2^{1/\alpha} \sum_{j\in S} \mathbb{1}_{\{\epsilon_j=1\}}\Gamma_j^{-1/\alpha} \Bigg| \\ \le\, & 2^{1/\alpha} \min_{i=1,\dots,d} (1-\beta_i)^{-1}\Gamma_{\ell+1}^{-1/\alpha} \to 0\end{aligned}$$

as $\ell \to \infty$, proving the a.s. convergence in the uniform distance.

For the second ingredient in the "convergence together" argument we use again the uniform distance and prove that for any $\epsilon > 0$,

$$\lim_{\ell\to\infty} \limsup_{\mathbf{n}\to\infty} P\left(\frac{1}{b_{\mathbf{n}}} \sup_{\mathbf{t}\in[\mathbf{0},\mathbb{1}]} |M_{\mathbf{n}}(\mathbf{t}) - M_{\mathbf{n},\ell}(\mathbf{t})| > \epsilon \right) = 0, .$$

By (3.46) and (3.47) it is enough to prove that

$$\lim_{\ell\to\infty} \limsup_{\mathbf{n}\to\infty} P\left(\sup_{\mathbf{t}\in[\mathbf{0},\mathbb{1}]} \max_{\mathbf{0}\le\mathbf{k}/\mathbf{n}\le\mathbf{t}} \left| \sum_{j=\ell+1}^{\infty} \epsilon_j \Gamma_j^{-1/\alpha} \mathbb{1}_{A^d} \circ T^{\mathbf{k}}(U_{j,\mathbf{n}}) \right| > \epsilon \right) = 0\,.$$

This is, however, an immediate consequence of Proposition 3.4.3. □

Theorem 3.5.1 has a natural counterpart for the partial maxima of the absolute values of the random field (3.8). It can be obtained along the same lines as we obtained Theorem 3.4.2. We omit the argument.

Theorem 3.5.2. *Let* $0 < \alpha < 2$ *and* $0 < \beta_i < 1$, $i = 1, \dots, d$. *Define*

$$\tilde{M}_{\mathbf{n}}(\mathbf{t}) := \max_{\mathbf{0}\le\mathbf{k}\le\mathbf{n}\mathbf{t}} |X_{\mathbf{k}}|, \quad \mathbf{t}\in\mathbb{R}_+^d\,.$$

Then

$$\left(\frac{1}{b_{\mathbf{n}}} \tilde{M}_{\mathbf{n}}(\mathbf{t}),\, \mathbf{t}\in\mathbb{R}_+^d\right) \Rightarrow \left(\left(\frac{C_\alpha}{2}\right)^{1/\alpha} \max\big(W_{\alpha,\beta}^{(1)}(\mathbf{t}), W_{\alpha,\beta}^{(2)}(\mathbf{t})\big),\, \mathbf{t}\in\mathbb{R}_+^d \right) \tag{3.53}$$

in the Skorohod J_1 *topology on the space* $D(\mathbb{R}_+^d)$. *Here* $\big(W_{\alpha,\beta}^{(1)}(\mathbf{t}),\, \mathbf{t}\in\mathbb{R}_+^d\big)$ *and* $\big(W_{\alpha,\beta}^{(2)}(\mathbf{t}),\, \mathbf{t}\in\mathbb{R}_+^d\big)$ *are two independent copies of the limiting process in Theorem 3.5.1.*

Remark 3.5.2. The structure of the limit in Theorem 3.5.2 together with Remark 3.5.1 immediately implies that the random field in the right hand side of (3.53) is self-similar. Furthermore, as in Remark 3.4.3, in the case $0 < \beta_i \leq 1/2$ for some $i = 1, \ldots, d$, and only in that case, the limiting random field in Theorem 3.5.2 is Fréchet. In this case an alternative way of stating the theorem is

$$\left(\frac{1}{b_{\mathbf{n}}} \tilde{M}_{\mathbf{n}}(\mathbf{t}), \, \mathbf{t} \in \mathbb{R}_+^d \right) \Rightarrow \left(C_\alpha^{1/\alpha} W_{\alpha,\beta}(\mathbf{t}), \, \mathbf{t} \in \mathbb{R}_+^d \right) .$$

CHAPTER 4

EXTREMAL CLUSTERING UNDER MODERATE LONG RANGE DEPENDENCE AND MODERATELY HEAVY TAILS

4.1 Introduction

This paper is about a very unusual clustering of extreme values that can occur in certain types of stationary stochastic processes with long range dependence. It is useful to recall the basic definitions of the classical extreme value theory. A distribution H on $\mathbb{R}$ is in a maximum domain of attraction if there is a positive sequence (a_n) and a real sequence (b_n) such that the law of $(M_n^{(0)} - b_n)/a_n$ converges weakly as $n \to \infty$ to a nondegenerate distribution G. Here $M_n^{(0)} = \max(Y_1, \ldots, Y_n)$ is the largest among n i.i.d. random variables $Y_1, Y_2, \ldots$ with a common distribution H. The distribution G is then, automatically, of the form $G(x) = G_\gamma(Ax+B)$, $x \in \mathbb{R}$ for some $A > 0, B \in \mathbb{R}$, and some $\gamma \in \mathbb{R}$. The "standard" distributions G_γ are Fréchet $G_\gamma(x) = \exp\{-x^{-1/\gamma}\}$, $x \geq 0$ if $\gamma > 0$, Gumbel $G_0(x) = \exp\{-e^{-x}\}$, $x \in \mathbb{R}$, and Weibull $G_\gamma(x) = \exp\{-(-x)^{-1/\gamma}\}$, $x \leq 0$ if $\gamma < 0$. See e.g. de Haan and Ferreira (2006).

The extreme values of an i.i.d. sequence, obviously, do not cluster. If, on the other hand, $X_1, X_2, \ldots$ is a stationary sequence with a common marginal distribution H, its extreme values may exhibit a clustering phenomenon. This is a well studied topic in the extreme value theory, where a numerical measure of clustering, *the extremal index*, goes back to Leadbetter (1983). Let $M_n = \max(X_1, \ldots, X_n)$, $n \geq 1$. As above, we denote by $M_n^{(0)}$ the largest of the first n *i.i.d.* observations $Y_1, Y_2, \ldots$ with the same marginal distribution H. Then the stationary sequence $X_1, X_2, \ldots$ has extremal index θ if for some nondegenerate distribution G we have both

$$(M_n^{(0)} - b_n)/a_n \Rightarrow G \ \text{ and } \ (M_n - b_n)/a_n \Rightarrow G^\theta \tag{4.1}$$

as $n \to \infty$; see e.g. de Haan and Ferreira (2006). An extremal index, if it exists, is in the range $0 < \theta \leq 1$. Note that the first statement in (4.1) can be rewritten in the form

$$(M_{[n\theta]}^{(0)} - b_n)/a_n \Rightarrow G^\theta \,,$$

which, in conjuction with the second statement in (4.1), says that the largest among the first n observations from the stationary sequence is "similar" to the largest of the first $[n\theta]$ observations from the corresponding i.i.d. sequence. In fact, in most cases a stationary sequence satisfying (4.1) also satisfies

$$(M_n - b_{[n\theta]})/a_{[n\theta]} \Rightarrow G \,, \tag{4.2}$$

which emphasizes the similarity between M_n and $M_{[n\theta]}^{(0)}$ even more. It is also a part of the folklore in extreme value theory that the extremal index can be interpreted as the reciprocal of "the expected extremal cluster size", though we will introduce neither the exact definition of this object nor the conditions under which this interpretation is valid; see e.g. Embrechts et al. (1997). If we denote such expected extremal cluster size by κ, then an alternative expression of (4.2) is

$$(M_n - b_{[n/\kappa]})/a_{[n/\kappa]} \Rightarrow G \,. \tag{4.3}$$

A special situation occurs when the stationary sequence $X_1, X_2, \ldots$ exhibits long range dependence with respect to its extremes (see Samorodnitsky (2016)). In this case (4.1) may hold with $\theta = 0$ which, of course, only says that the centering and scaling of M_n should be changed to obtain a nondegenerate limit. Intuitively, the extremes cluster so much that the extremal clusters become unbounded and one expects that (4.3) should be replaced by

$$(M_n - b_{m_n})/a_{m_n} \Rightarrow G \,, \tag{4.4}$$

where $m_n = [n/\kappa_n]$, and now κ_n is "the expected extremal cluster size" among the first n observations. This would allow $\kappa_n \to \infty$ and, therefore, $m_n = o(n)$ as $n \to \infty$. Hence, a change in the order of magnitude of scaling and/or centering for the partial maximum M_n.

In fact, it has turned out that (4.4) holds for certain stationary infinitely divisible processes with regularly varying tails. In this case the marginal distributions are in the Fréchet maximum domain of attraction ($\gamma > 0$), which involves no centering ($b_n \equiv 0$); see Samorodnitsky (2004) and Lacaux and Samorodnitsky (2016). In this setup the sequence (m_n) in (4.4) was not obtained via the relation $m_n = [n/\kappa_n]$ but, rather, turned out to be a direct ingredient in the memory in the system. It is important to mention that in these cases an important distinction has appeared between "moderate" long range dependence and "extreme" long range dependence. In the former case the weak limit in (4.4) is, up to shifting and scaling, the standard Fréchet distribution, while in the latter case the limit is not one of the classical extreme value distributions; see Samorodnitsky and Wang (2019). In the former case an extreme of the process due to a single large value of the underlying noise, consistent with the "heuristic of a single big jump" for extreme events and large deviations of heavy tailed systems (e.g. Rhee et al. (2019)). On the other hand, in the latter case this heuristic fails.

Our goal in this paper is to understand how the extremes of a long mempry stationary process cluster when the marginal tails are still heavy (so that the "heuristic of a single big jump" is still often the first guidance one has), but less heavy than the regularly varying tails considered earlier. The natural class of such marginal distributions is the class of *subexponential distributions.* Recall that a distribution H is subexponential if

$$\lim_{x\to\infty} \frac{\overline{H*H}(x)}{\overline{H}(x)} = 2\,, \tag{4.5}$$

where $\overline{H}(x) = 1-H(x)$ (Chistyakov (1964)). Distributions with a regularly varying right tail are, of course, subexponential, but we are interested in the subexponential distributions whose tails are lighter than any regularly varying tail. Specifically, we are interested in the subexponential distributions in the maximum domain of attraction of the Gumbel distribution G_0. We will give the exact assumptions on the marginal tails in the sequel.

A surprising conclusion of our results that, while (4.4) does continue to hold in the class of long memory stationary proceses with certain subexponential distributions in the maximum domain of attraction of the Gumbel distribution G_0 as the marginal distributions, (4.4) breaks down for such distributions once their tails become light enough. This may happen even when the memory is only "moderate long memory" (to be defined precisely in the sequel.) The moderate long memory case is the only one we consider in this paper. It turns out that when the tails become light enough, the centering in the left hand side of (4.4) acquires another term, of a smaller order than b_{m_n} (but of a larger order than a_{m_n}.) This term arises because the "single big jump" heuristic breaks down once again. That is, that heuristic may break down not only when the memory is too long, but also when the tails are too light (while remaining subexponential, hence heavy!)

The paper is organized as follows. In Section 4.2 we review some essential facts on subexponential distributions in the Gumbel maximum domain of attraction, random closed sets, null-recurrent Markov chains underlying the infinitely divisible dynamics in our model and random sup-measures, and introduce the limiting random sup-measure later appearing in the main result. In Section 4.3, we introduce the stationary infinitely divisible processes we are considering and list the assumptions we are imposing. The main results, the extremal limit theorems in the space of random sup-measures and in the space of cadlag functions are stated and proved in Section 4.4. This sections also contains two natural examples. The appendices 4.5.1 and 4.5.2 contain several arguments and verifications needed in the earlier parts of the paper.

We will throughout use the following standard notation. For any set S, we denote by $\#S$ its cardinality. For any measure μ on $\mathbb{R}$, $\overline{\mu}(x) := \mu(x, \infty)$. Particularly, $\overline{H}$ stands for the tail probability if H is a distribution function. For a non-decreasing function $J : \mathbb{R} \to \mathbb{R}$, the generalized inverse of J is defined as $J^{\leftarrow}(y) = \inf\{s : J(s) \geqslant y\}$.

Let $\{a_n\}_{n\in\mathbb{N}}$ and $\{b_n\}_{n\in\mathbb{N}}$ be two positive sequences. We describe the asymptotic relations between them by writing:

- $a_n \sim b_n$ if $\lim_{n\to\infty} a_n/b_n = 1$,
- $a_n \lesssim b_n$ if there exist $C > 0$ such that $a_n \leqslant Cb_n$ for large enough n, and analogously with $a_n \gtrsim b_n$,
- $a_n \asymp b_n$ if both $a_n \lesssim b_n$ and $a_n \gtrsim b_n$.

If $\{A_n\}_{n\in\mathbb{N}}$ and $\{B_n\}_{n\in\mathbb{N}}$ are two sequences of positive random variables, we will write

- $A_n = o_P(B_n)$ if $A_n/B_n \to 0$ in probability,
- $A_n \lesssim_P B_n$ if (A_n/B_n) is tight, and analogously with $A_n \gtrsim_P B_n$.

4.2 Preliminaries

This section is of the background nature. It collects a number of mostly well-known notions and results needed in this paper.

4.2.1 Subexponential distributions in the Gumbel maximum domain of attraction

Most of the material quoted in this section is in Resnick (1987); see also Goldie and Resnick (1988). Subexonentiality requires the distribution to have a support which is unbounded on the right, so we only consider such distributions.

A distribution H is in the maximum domain of attraction of the Gumbel distribution if and only if $G = (1/\overline{H})^{\leftarrow}$ is Π-varying. For a non-negative and non-decreasing function V to be Π-varying, it is required that there exist functions $a(t) > 0, b(t) \in \mathbb{R}$ such that for $x > 0$

$$\lim_{t\to\infty} \frac{V(tx) - b(t)}{a(t)} = \log x. \tag{4.6}$$

Alternatively, H is in the maximum domain of attraction of the Gumbel distribution if and only if there exist $x_0 \in \mathbb{R}$ and $c(x) \to c > 0$ as $x \to \infty$ such that for $x_0 < x < \infty$

$$\overline{H}(x) = c(x) \exp\left\{ - \int_{x_0}^{x} \frac{1}{h(u)} du \right\} \tag{4.7}$$

where h (the so-called auxilliary function) is an absolutely continuous positive function on (x_0, ∞) with density h' satsifying $\lim_{u\to\infty} h'(u) = 0$. This function h necessarily satisfies $h(x) = o(x)$ as $x \to \infty$, and subexponentiality of H additionally requires $\lim_{u\to\infty} h(u) = \infty$. For a distribution H satisfying (4.7), the centering and scaling required for the convergence $(M_n^{(0)} - b_n)/a_n \to G_0$ can be chosen as

$$b_n = \left(\frac{1}{1-H}\right)^{\leftarrow}(n), \;\; a_n = h(b_n)\,.$$

Under the above notation, we also need the following facts in the sequel.

(i) The function $1/\overline{H}$ is Γ-varying. In particular, for any other tail distribution $\overline{H_1} \sim \overline{H}$, we have

$$\lim_{t\to\infty} \frac{\overline{H_1}(t)}{\overline{H_1}(t + xh(t))} = e^x \quad \text{for all } x \in \mathbb{R}. \tag{4.8}$$

(ii) If one replaces the function c in (4.7) by an asymptotically equivalent function, and denotes the new normalizing sequences by $(\tilde{a}_n)$ and $(\tilde{b}_n)$, then

$$\lim_{n\to\infty} \frac{b_n - \tilde{b}_n}{a_n} = 0\,. \tag{4.9}$$

4.2.2 Random closed sets

We use the notation $\mathcal{G}, \mathcal{F}$ and $\mathcal{K}$ for the families of open, closed and compact sets of $[0,1]$ or $[0,\infty)$ (depending on the context), respectively. Details on most of the material in this section can be found in Molchanov (2017).

The Fell topology $\mathcal{B}(\mathcal{F})$ on $\mathcal{F}$ is the topology generated by the subbasis

$$\mathcal{F}_G = \{F \in \mathcal{F} : F \cap G \neq \emptyset\}\,, \quad G \in \mathcal{G},$$
$$\mathcal{F}^K = \{F \in \mathcal{F} : F \cap K = \emptyset\}\,, \quad K \in \mathcal{K}.$$

The Fell topology is metrizable and compact. A random closed set is a measurable mapping from a probability space to $\mathcal{B}(\mathcal{F})$.

For $\beta \in (0,1)$, let $\{L_\beta(t)\}_{t\geqslant 0}$ be the standard β-stable subordinator, i.e. the increasing Lévy process with the Laplace transform $\mathbb{E}e^{-\theta L_\beta(t)} = e^{-t\theta^\beta}, \theta \geqslant 0$. We define the β-stable regenerative set Z to be the closure of the range of a β-subordinator, i.e.,

$$Z = \overline{\{L_\beta(t) : t \geqslant 0\}}. \tag{4.10}$$

It is a random closed subset of $[0,\infty)$. Much of the discussion in this paper revolves around a sequence of i.i.d. random closed subsets of $[0,1]$ defined as follows.

Let $\{V_j\}_{j\geqslant 1}$ be a family of i.i.d. random variables on $[0,1]$ with

$$\mathbb{P}(V_1 \leqslant x) = x^{1-\beta}, \quad x \in [0,1]\,, \tag{4.11}$$

independent of an i.i.d. sequence $\{Z_j\}_{j\geqslant 1}$ of β-stable regenerative sets. We define

$$\overline{R_j} = (V_j + Z_j) \cap [0,1]. \tag{4.12}$$

That is, each $\overline{R_j}$ is the restriction of a shifted stable regenerative set to $[0,1]$. It is, clearly, non-empty.

4.2.3 Null-recurrent Markov chains

We describe now some ergodic theoretic notions associated with certain null recurrent Markov chains. Our main reference here is Aaronson (1997). Let $\{Y_t\}_{t\in\mathbb{Z}}$ be an irreducible, aperiodic, and null recurrent Markov chain on $\mathbb{Z}$, and denote by $(\pi_i)_{i\in\mathbb{Z}}$ its unique invariant measure satisfying $\pi_0 = 1$. If $(E,\mathcal{E})$ is the "path space" $(\mathbb{Z}^\mathbb{Z}, \mathcal{B}(\mathbb{Z}^\mathbb{Z}))$, we can define an infinite σ-finite measure on $(E,\mathcal{E})$ by

$$\mu(\cdot) := \sum_{i\in\mathbb{Z}} \pi_i \mathbb{P}_i(\cdot)\,, \tag{4.13}$$

where $\mathbb{P}_i$ is the probability law of $\{Y_t\}_{t\in\mathbb{Z}}$ on $(E, \mathcal{E})$ given $Y_0 = i$. The left shift operator on E by θ defined by

$$\theta : (\ldots, y_0, y_1, y_2, \ldots) \mapsto (\ldots, y_1, y_2, y_3, \ldots) \tag{4.14}$$

is a measure preserving, conservative and ergodic operator on $(E, \mathcal{E}, \mu)$. See Harris and Robbins (1953). For $n \in \mathbb{Z}$ let

$$A_n := \{y \in E : y_n = 0\}. \tag{4.15}$$

The *wandering rate sequence* $\{w_n\}_{n\in\mathbb{N}}$ is then defined as

$$w_n := \mu\left(\bigcup_{t=0}^{n} A_t\right), \quad n \in \mathbb{N}. \tag{4.16}$$

Define the first visit time to state 0 by

$$\varphi(y) := \inf\{t \geqslant 1 : y_t = 0\}, y \in E. \tag{4.17}$$

The Markov chains we consider satisfy the following assumption.

Assumption 4.2.1. *There exists $\beta \in (0,1)$ and a slowly varying function L such that*

$$\overline{F}(n) := \mathbb{P}_0\{\varphi > n\} = n^{-\beta}L(n) \in RV_{-\beta}\,. \tag{4.18}$$

Furthermore,

$$\sup_{n\geqslant 0} \frac{n\mathbb{P}_0\{\varphi = n\}}{\overline{F}(n)} < \infty\,. \tag{4.19}$$

Remark 4.2.1. Clearly, $w_n = \mu\{A_0 \setminus \cup_{t=1}^{n} A_t\} + \sum_{t=0}^{n} \mu\{\varphi = t\}$ and $\mu\{A_0 \setminus \cup_{t=1}^{n} A_t\} \to 0$ due to the recurrence. Thus, under the Assumption 4.2.1,

$$w_n \sim \sum_{t=1}^{n} \mu\{\varphi = t\} = \sum_{t=1}^{n} \mathbb{P}_0\{\varphi \geqslant t\} \sim \frac{n^{1-\beta}L(n)}{1-\beta} \in \mathrm{RV}_{1-\beta} \tag{4.20}$$

The penultimate equality follows from $\mu\{\varphi = t\} = \mathbb{P}_0(\varphi \geqslant t)$ for any $t \in \mathbb{N}$, which can be proved by induction, see Resnick et al. (2000) Lemma 3.3 (3.9). The last asymptotic relation is direct consequence of the Karamata theorem.

The times a sequence $y \in E$ visits state 0 under certain conditional versions of the measure μ are of crucial importance for us. Specifically, for each $n \in \mathbb{N}$, let

$$\mu_n(B) := \frac{\mu\left(B \cap \bigcup_{t=0}^{n} A_t\right)}{w_n}, \quad B \in \mathcal{E}, \tag{4.21}$$

and let $\{Y^{(j,n)}\}_{j\in\mathbb{N}}$ be a family of i.i.d. random elements in E with law μ_n. For each j we set

$$I_{j,n} := \left\{0 \leqslant t \leqslant n : Y_t^{(j,n)} = 0\right\}. \tag{4.22}$$

We further define

$$\widehat{I}_{1,n} := I_{1,n}, \tag{4.23}$$

$$\widehat{I}_{j,n} := I_{j,n} \cap \bigcap_{i=1}^{j-1} I_{i,n}^c, \quad j \geqslant 2. \tag{4.24}$$

The facts mentioned below are in Samorodnitsky and Wang (2019). First, by Theorem 5.4 *ibid.*,

$$\left(\frac{1}{n} I_{j,n}\right)_{1\leqslant j\leqslant m} \Rightarrow (\overline{R_j})_{1\leqslant j\leqslant m},\ m = 1, 2, \ldots \tag{4.25}$$

weakly in the space of random closed subsets of $[0,1]$, where $\overline{R_j}$ is defined in (4.12). In particular,

$$\lim_{n\to\infty} \mathbb{P}\left(I_{j,n} \cap nG \neq \emptyset\right) = \mathbb{P}\left(\overline{R_j} \cap G \neq \emptyset\right) > 0 \text{ for any } G \in \mathcal{G}([0,1]). \tag{4.26}$$

If, in addition, $0 < \beta < 1/2$, then for any two distinct $j_1, j_2 \in \mathbb{N}$

$$\lim_{n\to\infty} \mathbb{P}\left(I_{j_1,n} \cap I_{j_2,n} \neq \emptyset\right) = 0. \tag{4.27}$$

Therefore, for any $m \in \mathbb{N}$,

$$\lim_{n\to\infty} \mathbb{P}\left(I_{j,n} = \widehat{I}_{j,n},\ j = 1, \ldots, m.\right) = 1. \tag{4.28}$$

In the sequel we will need estimates of how quickly the intersection probability in (4.27) and certain related probabilities converge to zero. For an open interval $T \subset [0,1]$ we define

$$p_{n,T} := \mathbb{P}\left(I_{1,n} \cap I_{2,n} \cap nT \neq \emptyset\right), \tag{4.29}$$

$$\overline{p}_{n,T} := \mathbb{P}\left(I_{1,n} \cap I_{2,n} \cap nT \neq \emptyset \middle| Y^{(1,n)}\right), \tag{4.30}$$

with the latter probability being random. Clearly, $p_{n,T} = \mathbb{E}\overline{p}_{n,T}$. The following theorem may be of independent interest. It is proved in Appendix 4.5.1.

Theorem 4.2.1. *Under Assumption 4.2.1 with $0 < \beta < 1/2$, for any open interval T,*

(i)

$$p_{n,T} \asymp \frac{n^\beta}{w_n L(n)}. \tag{4.31}$$

(ii) For any $C > 0$, there exists $c > 0$ such that for every $n \geq 1$

$$\mathbb{P}\left(\overline{p}_{n,T} \geqslant \frac{c n^\beta \log n}{w_n L(n)}\right) \leqslant n^{-C}. \tag{4.32}$$

(iii) For any $\gamma > (1-2\beta)^{-1}$ and $\epsilon > 0$, there exists $c > 0$ such that

$$\liminf_{n\to\infty} \mathbb{P}\left(\overline{p}_{n,T} \geqslant \frac{c n^\beta}{w_n L(n)} \cdot \frac{L\left((\log n)^\gamma\right)}{(\log n)^{\gamma\beta}} \,\Big|\, I_{1,n} \cap nT \neq \emptyset\right) \geqslant 1 - \epsilon. \tag{4.33}$$

It follows immediately from part (iii) of the theorem that

$$\overline{p}_{n,[0,1]} \gtrsim_P \frac{n^\beta}{w_n L(n)} \cdot \frac{L\left((\log n)^\gamma\right)}{(\log n)^{\gamma\beta}}. \tag{4.34}$$

for any $\gamma > (1-2\beta)^{-1}$.

4.2.4 Random Sup-Measures

We continue to use the notation of Subsection 4.2.2. Our main reference is O'Brien et al. (1990); note that our sup-measures take values in $\overline{\mathbb{R}} = [-\infty, \infty]$.

A sup-measure is a mapping $m : \mathcal{G} \to \overline{\mathbb{R}}$ such that $m(\emptyset) = -\infty$ and $m(\cup_\alpha G_\alpha) = \vee_\alpha m(G_\alpha)$ for an aribtrary collection (G_α) of open sets. The sup-derivative $d^\vee m$ of m is defined by

$$d^\vee m(t) = \bigwedge_{t \in G} m(G),$$

it is an upper semicontinuous $\overline{\mathbb{R}}$-valued function of t. Given an $\overline{\mathbb{R}}$-valued function f, the sup-integral of f defined by

$$i^\vee f(G) = \bigvee_{t \in G} f(t), \quad G \in \mathcal{G};$$

it is automatically a sup-measure. It is always true that $m = i^\vee d^\vee m$, and we can extend the domain of a sup-measure to all Borel sets via

$$m(B) = \bigvee_{t \in B} d^\vee(t), \quad B \text{ Borel.} \tag{4.35}$$

On the collection SM of all sup-measures one defines the sup-vague topology, in which a sequence of sup-measures $\{m_n\}_{n \geqslant 1}$ converges to a sup-measure m if and only if

$$\limsup_{n \to \infty} m_n(K) \leqslant m(K), \quad \text{for each } K \in \mathcal{K},$$

$$\liminf_{n \to \infty} m_n(G) \geqslant m(G), \quad \text{for each } G \in \mathcal{G}.$$

The space SM equipped with sup-vague topology is compact and metrizable.

A random sup-measure M is a measurable mapping from a probability space to SM. For a random sup-measure M, let $\mathscr{I}(M)$ be the collection of continuity intervals of M, defined by

$$\mathscr{I}(M) = \{I \text{ an open interval}: \ M(I) = M(\text{clos } I) \text{ a.s.}\}.$$

If $\{M_n\}_{n\geqslant 1}$ and M are random sup-measures, then $M_n \Rightarrow M$ if and only if

$$(M_n(I_1), \ldots, M_n(I_m)) \Rightarrow (M(I_1), \ldots, M(I_m)) \tag{4.36}$$

for arbitrary disjoint intervals $I_1, \ldots, I_m \in \mathscr{I}(M)$.

We now define a family of random sup-measures that will arise naturally in the sequel. Let $\beta \in (0, 1/2)$ and consider a Poisson point process on $\mathbb{R} \times \mathbb{R}_+ \times \mathcal{F}(\mathbb{R}_+)$ with mean measure

$$e^{-u}du(1-\beta)v^{-\beta}dv\, dP_\beta\,,$$

where P_β is the law of the β-stable regenerative set in (4.10). Let $(U_j, V_j^*, Z_j)_{j\in\mathbb{N}}$ be a measurable enumeration of points of this Poisson point process, and denote

$$R_j = V_j^* + Z_j, \quad j \in \mathbb{N}. \tag{4.37}$$

Since $\beta \in (0, 1/2)$, we have

$$\mathbb{P}(R_1 \cap R_2 = \emptyset) = 1\,; \tag{4.38}$$

see Lemma 3.1 in Samorodnitsky and Wang (2019). It follows immediately that, on an event of probability 1, the function $\eta : \mathbb{R}_+ \to \overline{\mathbb{R}}$ defined by

$$\eta(t) = \bigvee_{j=1}^{\infty} U_j 1_{\{t\in R_j\}}$$

is an upper semicontinuous function. Hence, it is the sup-derivative of the random sup-measure

$$\mathcal{M}(B) = \bigvee_{j=1}^{\infty} U_j 1_{\{B\cap R_j \neq \emptyset\}}\,. \tag{4.39}$$

This measure is stationary, i.e.

$$\mathcal{M}(r+\cdot) \overset{\mathrm{d}}{=} \mathcal{M}(\cdot) \tag{4.40}$$

for any $r \geqslant 0$. This easily follows from that each $R_j \in \mathcal{F}(\mathbb{R}_+)$ is stationary, see Proposition 4.1 (c) in Lacaux and Samorodnitsky (2016). Moreover, we claim that M is self-affine, i.e.

$$\mathcal{M}(a\cdot) \overset{\mathrm{d}}{=} \mathcal{M}(\cdot) + (1-\beta)\log a \tag{4.41}$$

for any $a > 0$. Indeed, note that

$$\begin{aligned}\mathcal{M}(aB) &= \bigvee_{j=1}^{\infty} U_j 1_{\{B \cap (a^{-1}V_j^* + a^{-1}Z_j) \neq \emptyset\}} \\ &\overset{\mathrm{d}}{=} \bigvee_{j=1}^{\infty} U_j 1_{\{B \cap (a^{-1}V_j^* + Z_j) \neq \emptyset\}} \\ &\overset{\mathrm{d}}{=} \bigvee_{j=1}^{\infty} (U_j + (1-\beta)\log a) 1_{\{B \cap (V_j^* + Z_j) \neq \emptyset\}} = \mathcal{M}(B) + (1-\beta)\log a\,;\end{aligned}$$

see e.g. Proposition 4.1(b) in Lacaux and Samorodnitsky (2016) for the first distributional equality, while the second one holds because both the points $(U_j, aV_j^*, Z_j)_{j\in\mathbb{N}}$ and the points $(U_j + (1-\beta)\log a, V_j^*, Z_j)_{j\in\mathbb{N}}$ form a Poisson point process with mean measure

$$a^{1-\beta} e^{-u} du (1-\beta) v^{-\beta} dv\, dP_\beta\,.$$

Suppose that $\{X_t\}_{t\in\mathbb{Z}}$ is a stationary process. It induces naturally a sequence of random sup-measures by setting for $n \in \mathbb{N}$

$$M_n(B) = \max_{t\in nB} X_t, \quad B \in \mathcal{B}(\mathbb{R}_+)\,. \tag{4.42}$$

If $\mathcal{M}$ is the random sup measure in (4.39), then the weak convergence

$$\frac{M_n(\cdot) - b_n}{a_n} \Rightarrow \mathcal{M}(\cdot)$$

in the space of random sup measures on $[0,1]$ for some (a_n, b_n) guarantees also this weak convergence in the space of random sup measures on $\mathbb{R}_+$. Furthermore, every open interval is a continuity interval for $\mathcal{M}$, since stable regenerative sets do not hit fixed points. By (4.36)

$$\left(\frac{M_n(I_1) - b_n}{a_n}, \ldots, \frac{M_n(I_m) - b_n}{a_n}\right) \Rightarrow (\mathcal{M}(I_1), \ldots, \mathcal{M}(I_m)) \tag{4.43}$$

for arbitrary disjoint open intervals $I_1, \ldots, I_m$ in $[0,1]$ is necessary and sufficient for weak convergence to $\mathcal{M}$.

The restriction of the sup measure $\mathcal{M}$ to subsets of $[0,1]$ has representation somewhat more transparent than (4.39). Let $\{V_j\}_{j \geqslant 1}$ be a family of i.i.d. random variables on $[0,1]$ with the law (4.11). Let $\{Z_j\}_{j \geqslant 1}$ be a family of i.i.d. β-stable regenerative sets in (4.10). Finally, let $\{\Gamma_j\}_{j \geqslant 1}$ be the sequence of arrival times of a unit rate Poisson processes on $(0, \infty)$. We assume that all three sequences are independent. Then the points $(-\log \Gamma_j, V_j, Z_j)_{j \in \mathbb{N}}$ form a Poisson point process on $\mathbb{R} \times [0,1] \times \mathcal{F}(\mathbb{R}_+)$ whose mean measure is the mean measure of restriction of the Poisson point process $(U_j, V_j^*, Z_j)_{j \in \mathbb{N}}$ to $\mathbb{R} \times [0,1] \times \mathcal{F}(\mathbb{R}_+)$. Therefore, if we define i.i.d. random nonempty compact sets by (4.12), then the following representation in law holds:

$$\mathcal{M}(B) = \bigvee_{t \in B} -\log \Gamma_j 1_{\{B \cap \overline{R_j} \neq \emptyset\}}, \; B \in \mathcal{B}([0,1])\,. \tag{4.44}$$

4.3 A family of stationary subexponential infinitely divisible processes

We now define a family of stationary infinitely divisible processes for which we will establish extremal limit theorems. Our processes will be of the form

$$X_n = \int_E f \circ \theta^n(y) \, M(dy), \; n \in \mathbb{Z} \,, \tag{4.45}$$

where θ is the left shift operator on $E = \mathbb{Z}^{\mathbb{Z}}$ in (4.14) and M is an infinitely divisible random measure on $(E, \mathcal{E})$ with a constant local characteristic triple (σ^2, ν, b) and control measure μ in (4.13), associated with an invariant measure of an irreducible, aperiodic, and null recurrent Markov chain on $\mathbb{Z}$; see Chapter 3 in Samorodnitsky (2016) for details in infinitely divisible random measures and integrals with respect to such measures. The function f must satisfy certain integrability conditions; if it does, the process $\{X_n\}_{n\in\mathbb{Z}}$ is automatically stationary, because the left shift θ preservses the control measure μ. In the sequel we will assume, for simplicity, that f is the indicator function

$$f(y) = \mathbb{1}_{\{y_0=0\}}(y) \;\; \text{for} \;\; y = (\ldots, y_0, y_1, y_2, \ldots) \,, \tag{4.46}$$

but the results of this paper will undoubtedly hold for a more general class of functions f. The indicator function f in (4.46) automatically satisfies the integrability conditions and, in this case, each X_n is an infinitely divisible random variable with a characteristic triple (σ^2, ν, b); see Section 7 in Sato (2013). The key assumption we will impose in the sequel is that the distribution $(\nu(1,\infty))^{-1}\nu(\cdot \cap (1,\infty))$ is subexponential, from which it immediately follows that

$$\mathbb{P}(X_n > x) \sim \nu(x, \infty) := \bar{\nu}(x) \;\; \text{as} \; x \to \infty \tag{4.47}$$

and, in particular, X_n has a subexponential distribution; see Embrechts et al. (1979). We will, in fact, impose a number of additional assumptions on the Lévy measure ν. These assumptions will guarantee that the tail of X_n is light enough to be in the maximum domain of attraction of the Gumbel distribution. On the other hand, they will also guarantee that this tail is not "too light".

Assumption 4.3.1. *The distribution* $(\nu(1,\infty))^{-1}\nu(\cdot\cap(1,\infty))$ *is both subexponential and in the maximum domain of attraction of the Gumbel distribution. Furthermore, there is a distribution* $H_\#$ *satisfying* $\bar{\nu}(x)\sim a\overline{H_\#}$ *for* $a>0$, *and which satisfies* (4.7) *with* $c\equiv 1$, *i.e.*

$$\overline{H_\#}(x)=\exp\left(-\int_{x_0}^{x}\frac{1}{h(u)}du\right)\ \text{for } x>x_0\,, \tag{4.48}$$

and the auxiliary function h *with* $h'>0$ *on* (x_0,∞), *and such that*

$$\lim_{b\downarrow 1}\limsup_{x\to\infty}\frac{h(bx)}{h(x)}=1\,. \tag{4.49}$$

Denoting

$$G(x):=\left(\frac{1}{1-H_\#}\right)^{\leftarrow}(x),\ x\geq 0\,, \tag{4.50}$$

we assume that the function G *is of the form*

$$G(x)=\exp\left\{\int_e^x\frac{\zeta(u)}{u\log u}du\right\},\ x>x_1,\quad \text{for some } x_1>e, \tag{4.51}$$

where ζ *satisfies the following assumptions.*

(B1) ζ *is roughly increasing, i.e.,*

$$\zeta(x)\asymp\sup_{[1,x]}\zeta(u).$$

(B2) There exists some $\delta>0$ *such that*

$$(\log\log u)^\delta\ll\zeta(u)\lesssim\frac{\log u}{\log\log u}.$$

(B3) For small $\rho>0$,

$$\zeta(x^{1-\rho/\log\log x})\gtrsim\zeta(x)\,.$$

(B4) For any $c > 0$,

$$\liminf_{x\to\infty} \int_{x^{1-c/\zeta(x)}}^{x} \frac{\zeta(u)}{u\log u} du > 0.$$

We check in Appendix 4.5.2 below that the following two important classes of Lévy measures with subexponential tails satisfy Assumption 4.3.1.

Example 4.3.1 (lognormal-type tails)**.**

$$\overline{\nu}(x) \sim cx^{\eta}(\log x)^{\xi} \exp\left(-\lambda(\log x)^{\gamma}\right) \quad \textit{as } x \to \infty$$

for some $\gamma > 1$, $\lambda, c > 0$ and $\eta, \xi \in \mathbb{R}$.

Example 4.3.2 (super-lognormal-type tails)**.**

$$\overline{\nu}(x) \sim cx^{\eta}(\log x)^{\xi} \exp\left(\lambda(\log x)^{\gamma}\right) \exp\left(-\rho \exp\left(\mu(\log x)^{\alpha}\right)\right) \quad \textit{as } x \to \infty$$

for some $\alpha \in (0,1)$, $c, \mu, \rho > 0$ and $\eta, \xi, \lambda, \gamma \in \mathbb{R}$.

Remark 4.3.1. *The semi-exponential-type tails such as $\overline{\nu}(x) \sim \exp(-x^{\alpha})$, $0 < \alpha < 1$, unfortunately, do not satisfy the assumptions and, hence, are excluded from our analysis.*

The following proposition, proved in Appendix 4.5.2, lists certain properties of Lévy measures satisfying Assumption 4.3.1. We will find these properties useful in the sequel. Let G and $\delta > 0$ as in Assumption 4.3.1 ($B2$).

Proposition 4.3.1. *(i) $G(x) \gg \exp\left\{(\log\log u)^{1+\delta}/(1+\delta)\right\}$.*

(ii) For any $\alpha_1 > \alpha_2 > 0$,

$$\frac{G(x^{\alpha_1})}{G(x^{\alpha_2})} \gg \exp\left\{(\log\alpha_1 - \log\alpha_2)(\log\log x)^{\delta}\right\} . \tag{4.52}$$

(iii) For any $H_i \in RV_{\alpha_i}$, $i = 1, 2$, $\alpha_1 > \alpha_2 > 0$, for any $b < \log\alpha_1 - \log\alpha_2$,

$$\frac{h \circ G(H_1(x))}{h \circ G(H_2(x))} \gg \exp\left\{b(\log\log x)^{\delta}\right\} . \tag{4.53}$$

(iv) For any $\alpha \neq 0$,

$$|G(x(\log x)^{\alpha}) - G(x)| \asymp (\log\log x) h \circ G(x)\,. \tag{4.54}$$

(v) For all sufficiently small $\rho > 0$,

$$\min_{1 \leqslant j \leqslant \rho \log x / \zeta(x)} \frac{G(x) - G\left(x2^{-j}\right)}{j} \gtrsim j h \circ G(x)\,. \tag{4.55}$$

4.4 Extremal limit theorems

Let $(X_t)_{t\in\mathbb{Z}}$ be a stationary infinitely divisible process (4.45), associated with an irreducible, aperiodic, and null recurrent Markov chain on $\mathbb{Z}$. Recall that we assume that the function f is the indicator function (4.46). Our main result in this section is a limit theorem for the sequence of random sup-measures defined by the process via (4.42). The Lévy measure ν of the infinitely divisible random measure M in (4.45) is assumed to satisfy Assumption 4.3.1. We denote

$$V(x) = (1/\bar{\nu})^{\leftarrow}(x),\ x > 0\,. \tag{4.56}$$

The Markov chain underlying the process $(X_t)_{t\in\mathbb{Z}}$ is assumed to satisfy Assumption 4.2.1. We define for $n = 1, 2, \ldots$

$$b_n = V(w_n) + V\big(1/\overline{F}(n)\big),\quad a_n = h \circ V(w_n)\,, \tag{4.57}$$

with w_n is the wandering rate in (4.16), F is the first return time law in (4.18), and h the auxiliary function in (4.7).

Theorem 4.4.1. *Assume that Assumption 4.3.1 holds, and that Assumption 4.2.1 is satisfied with* $0 < \beta < 1/2$. *If* $(a_n), (b_n)$ *are given by* (4.57), *then*

$$\frac{M_n(\cdot) - b_n}{a_n} \Rightarrow \mathcal{M}(\cdot) \tag{4.58}$$

weakly in the space of sup-measures on $[0,\infty)$*, where* (M_n) *are the random sup-measures in* (4.42)*, and the limiting random sup-measure* $\mathcal{M}$ *is given by* (4.39)*.*

There is a natural counterpart of Theorem 4.4.1 that establishes an extremal limit theorem in a function space. Recall that a standard Gumbel extremal process is a nondecreasing process $\big(X_{\mathrm{G}}(t),\, t>0\big)$ satisfying

$$\mathbb{P}\big(X_{\mathrm{G}}(t_i)\le x_i,\, i=1,\dots,k\big)=\exp\left\{-\sum_{i=1}^{k}(t_i-t_{i-1})e^{-x_i}\right\}$$

for $0<t_1<\ldots<t_k$ and $x_1\le\ldots\le x_k$. The process is continuous in probability and has a version in $D(0,\infty)=\cap_{\varepsilon>0}D[\varepsilon,\infty)$; see Resnick and Rubinovitch (1973). It is immediate from the definition of the random sup-measure $\mathcal{M}$ in (4.39) that

$$\big(\mathcal{M}([0,t]),\, t>0\big)\overset{\mathrm{d}}{=}\big(X_{\mathrm{G}}(t^{1-\beta})\,;t>0\big)\,, \tag{4.59}$$

see also Lacaux and Samorodnitsky (2016). Note that the finite-dimensional convergence part in the following theorem already follows from Theorem 4.4.1.

Theorem 4.4.2. *Under the assumptions of Theorem 4.4.1,*

$$\left(\frac{\max_{s\le nt}X_s-b_n}{a_n},\, t>0\right)\Rightarrow\big(X_{\mathrm{G}}(t^{1-\beta}),\, t>0\big) \tag{4.60}$$

weakly in the Skorohod J_1 *topology on* $D(0,\infty)$*.*

Remark 4.4.1. Let us return to the discussion in the introduction of this paper and compare the statement of Theorems 4.4.1 and 4.4.2 to the "expected behavior" of the extreme values presented in (4.4). The results of Lacaux and Samorodnitsky (2016) and Samorodnitsky and Wang (2019) in the case of regularly varying tails suggest that $m_n=w_n$, and the centering and the normalization in (4.4) do not appear to be consistent with Theorema 4.4.1 and 4.4.2 due to the presence of an extra term $V\big(1/\overline{F}(n)\big)$ in the centering sequence. It turns out, however, that as long as the tails of the process $(X_t)_{t\in\mathbb{Z}}$ are "not too light" we have

$$\lim_{n\to\infty} \frac{V\big(1/\overline{F}(n)\big)}{a_n} = 0\,, \tag{4.61}$$

and so (4.4) does predict the correct centering and the normalization. Once the tails of the process become lighter, however, (4.61) may fail, and a different centering becomes necessary. We can see this phenomenon on Examples 4.3.1 and 4.3.2. In fact, for the lognormal-type tails of Example 4.3.1 the relation (4.61) holds, while for the super-lognormal-type tails of Example 4.3.2, (4.61) holds if $0 < \alpha < 1/2$ and fails if $1/2 < \alpha < 1$. These claims are verified in Appendix 4.5.2.

Remark 4.4.2. When $\beta \in (1/2, 1)$, the (shifted) stable regenerative sets $\{R_j\}_{j\in\mathbb{N}}$ in (4.37) start to intersect and our methods break down. This "aggregated" regime was first studied in Samorodnitsky and Wang (2019), where the marginal tail $\mathbb{P}\{X_0 > x\}$ is regularly varying, and non-Fréchet limits appear.

Under the Assumption 4.3.1, we conjecture that non-Gumbel limits will emerge if $\beta \in (1/2, 1)$. Moreover, if the margin is heavy enough, then weak convergence of sup-measures is conjectured to preserve the form

$$\frac{M_n(\cdot) - V(w_n)}{h \circ V(w_n)} \Rightarrow \mathcal{M}(\cdot),$$

which implies the vanishing extremal index $\theta = 0$. We leave this problem to a future investigation.

We will prove the two theorems in the remainder of this section, beginning with Theorem 4.4.1. We start with a preliminary analysis that will split the proof into several steps. First of all, by (4.43), we need to prove that for arbitrarily disjoint open intervals $I_1, \ldots, I_m$ in $[0,1]$,

$$\left(\frac{M_n(I_i) - b_n}{a_n}\right)_{i=1,\ldots,m} \Rightarrow \big(\mathcal{M}(I_i)\big)_{i=1,\ldots,m} \tag{4.62}$$

weakly in $\mathbb{R}^m$. Note, further, that for any $0 < \varepsilon < a$ the function V in (4.56) satisfies

$$G\big(x(a-\varepsilon)\big) \leq V(x) \leq G\big(x(a+\varepsilon)\big) \tag{4.63}$$

for all x large enough.

Next, we decompose the stationary process $(X_t)_{t\in\mathbb{Z}}$ as follows. Let $M^{(1)}$ and $M^{(2)}$ be two independent infinitely divisible random measures on $(E,\mathcal{E})$, both with with the same control measure μ as the measure M in (4.45). With (σ^2,ν,b) being the local characteristic triple of M, we set the local characteristic triple of $M^{(1)}$ to be $(0,[\nu]_{(x_0,\infty)},b^{(1)})$, and the local characteristic triple of $M^{(2)}$ to be $(\sigma^2,[\nu]_{(-\infty,x_0]},b-b^{(1)})$, with x_0 as in (4.48) and

$$b^{(1)} = \left(\int_{-\infty}^{-1} -1 + \int_{-1}^{1} x + \int_{1}^{\infty} 1\right) [\nu]_{(x_0,\infty)}(dx). \tag{4.64}$$

If we define for each $t \in \mathbb{Z}$

$$X_t^{(1)} = \int_E f\circ\theta^t(y)\, M^{(1)}(dy), \quad X_t^{(2)} = \int_E f\circ\theta^t(y)\, M^{(2)}(dy)\,, \tag{4.65}$$

then $\{X_t^{(i)}\}_{t\in\mathbb{Z}}, i=1,2$ are two independent stationary infinitely divisible processes such that $\{X_t\}_{t\in\mathbb{Z}} \overset{\mathrm{d}}{=} \{X_t^{(1)}+X_t^{(2)}\}_{t\in\mathbb{Z}}$. For $i=1,2$ we let $M_n^{(i)}(\cdot)$ be the random sup-measure defined for $\{X_t^{(i)}\}_{t\in\mathbb{Z}}$ as in (4.42). The following proposition shows that $M_n^{(2)}$ is asymptotically negiligible with our scaling.

Proposition 4.4.1. $M_n^{(2)}([0,1])/a_n \to 0$ *as* $n\to\infty$.

Proof. Since the Lévy measure of $X_0^{(2)}$ is bounded on the right, $\mathbb{P}(X_0^{(2)} > r) = o(e^{-cr})$ for any $c>0$ see e.g. Theorem 26.1 in Sato (2013). Using the fact that $\zeta(x)\to\infty$ we use (4.63) and part (i) of Proposition 4.3.1 to see that for any $p>0$, for all large n,

$$\begin{aligned} a_n \geq h \circ G(aw_n/2) &= \frac{G(aw_n/2)\zeta(aw_n/2)}{\log(aw_n/2)} \\ &\gg \frac{G(aw_n/2)}{\log(aw_n/2)} \gg (\log w_n)^p \,, \end{aligned}$$

therefore, taking $p > 1$ we have for any $\epsilon > 0$

$$\begin{aligned} \mathbb{P}\left\{M_n^{(2)}([0,1]) > \epsilon a_n\right\} \leq & n\mathbb{P}\left\{X_0^{(2)} > \epsilon(\log w_n)^p\right\} \\ = & o\left(\exp\left\{-\epsilon(\log w_n)^p\right\}\right) \to 0 \end{aligned}$$

by (4.20). □

Proposition 4.4.1 implies that, in order to show (4.62), we need to prove that

$$\left(\frac{M_n^{(1)}(I_i) - b_n}{a_n}\right)_{i=1,\ldots,m} \Rightarrow \left(\mathcal{M}(I_i)\right)_{i=1,\ldots,m} . \tag{4.66}$$

With some abuse of notation, we now carry out as follows.

(i) Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space that supports an family of i.i.d. random closed sets $\{\overline{R_j}\}_{j\in\mathbb{N}}$ distributed as in (4.12).

(ii) By the Skorohod representation theorem, we proceed to assume that on the space $(\Omega, \mathcal{F}, \mathbb{P})$ live random elements $\{Y^{(j,n)}\}_{j,n\in\mathbb{N}}$ such that

  (ii.1) for each $n \in \mathbb{N}$, $\{Y^{(j,n)}\}_{j\in\mathbb{N}}$ are i.i.d. distributed with law μ_n in (4.21);

  (ii.2) for each $j \in \mathbb{N}$, $\{Y^{(j,n)}\}_{n\in\mathbb{N}}$ are dependent such that

$$\frac{1}{n} I_{j,n} \to \overline{R_j} \quad \text{a.s. as} \;\; n \to \infty \;\; \text{for each} \;\; j \in \mathbb{N}, \tag{4.67}$$

  with $I_{j,n}$ defined by (4.22).

(iii) Enlarge $(\Omega, \mathcal{F}, \mathbb{P})$ such that it supports the arrival times $\{\Gamma_j\}_{j\in\mathbb{N}}$ of a unit rate Poisson process on $\mathbb{R}_+$, which is independent of $\{I_{j,n}, \overline{R_j}\}_{j,n\in\mathbb{N}}$.

Remark 4.4.3. In general, it is risky to assume any sort of independence when applying the Skorohod representation theorem. However, due to that $\overline{R_j}$'s are indenpendent in (4.25), we are allowed to assume (ii.1). Hence, for each n, $\{I_{j,n}\}_{j\in\mathbb{N}}$ is a family of i.i.d. random closed set.

The following series reperesentation of the process $\{X_t^{(1)}\}_{t\in\mathbb{Z}}$ is the key for our argument, which will be verified in appendix 4.5.2. For each $n \in \mathbb{N}$,

$$\left(X_t^{(1)}\right)_{0\leqslant t\leqslant n} \overset{\mathrm{d}}{=} \left(\sum_{j=1}^{\infty} \tilde{V}\big(w_n/\Gamma_j\big)\mathbb{1}_{\{t\in I_{j,n}\}}\right)_{0\leqslant t\leqslant n}, \tag{4.68}$$

where

$$\tilde{V}(y) = \begin{cases} V(y) & \text{for } y > 1/\bar{\nu}(x_0) \\ 0 & \text{otherwise} \end{cases}. \tag{4.69}$$

Remark 4.4.4. The wandering rate w_n enters into sight because $\left(X_t^{(1)}\right)_{0\leqslant t\leqslant n}$ arises as stochastic integrals supported on $\cup_{t=0}^{n} A_n$, which has measure w_n.

When proving (4.66) we will simply assume that the process $\{X_t^{(1)}\}_{t\in\mathbb{Z}}$ is given by the right hand side of (4.68). Furthermore, for notational simplicity we will drop the "tilde" over V in the sequel, while keeping in mind that it vanishes for small values of the argument, as in (4.69). We now state several propositions that will prove (4.66).

For $k \in \mathbb{N}$ we define, in the notation of (4.23) and (4.24),

$$M_{n,(k)}(B) = \max_{t\in nB\cap\widehat{I}_{k,n}} \sum_{j=1}^{\infty} V\big(w_n/\Gamma_j\big)\mathbb{1}_{\{t\in I_{j,n}\}},$$

$$\mathcal{M}_{(k)}(B) = \begin{cases} -\log\Gamma_k & \text{if} \quad \overline{R_k}\cap B \neq \emptyset \\ -\infty & \text{otherwise} \end{cases}.$$

Proposition 4.4.2. *For each $k \in \mathbb{N}$ and each open interval I in* $[0, 1]$,

$$\frac{M_{n,(k)}(I) - b_n}{a_n} \xrightarrow{P} \mathcal{M}_{(k)}(I). \tag{4.70}$$

We define, further, for $K \in \mathbb{N}$,

$$M_{n,K}(B) = \bigvee_{k=1}^{K} M_{n,(k)}(B),$$
$$\mathcal{M}_K(B) = \bigvee_{k=1}^{K} \mathcal{M}_{(k)}(B).$$

It follows from Proposition 4.4.2 that for each K and each open interval I in $[0,1]$,

$$\frac{M_{n,K}(I) - b_n}{a_n} \xrightarrow{P} \mathcal{M}_K(I)\,. \tag{4.71}$$

Since it is also clear that for any open interval I in $[0,1]$, as $K \to \infty$,

$$\mathcal{M}_K(I) \;\longrightarrow \mathcal{M}(I) \;\text{ a.s.}$$

if the limting sup-measure $\mathcal{M}$ is defined on the same probability space $(\Omega, \mathcal{F}, \mathbb{P})$ by (4.44), then the only remaining step to establish (4.66) is the following claim.

Proposition 4.4.3. *For any open interval I in $[0,1]$ and $\epsilon > 0$,*

$$\lim_{K\to\infty} \lim_{n\to\infty} \mathbb{P}\left(|M_{n,K}(I) - M_n(I)| \geqslant \epsilon\right) = 0\,. \tag{4.72}$$

We now prove Propositions 4.4.2 and 4.4.3.

Proof of Proposition 4.4.2. We will only consider the case $I = (a,b)$ for some $0 \leqslant a < b \leqslant 1$, the other cases being similar. Further, it is enough to prove the proposition for $k = 1$. Indeed, for any $k \in \mathbb{N}$, (4.27) implies that

$$\lim_{n\to\infty} \mathbb{P}\{I_{j,n} \cap I_{k,n} = \emptyset, j = 1, \ldots, k-1\} = 1,$$

so that

$$\lim_{n\to\infty} \mathbb{P}\Bigg\{ M_{n,(k)}(I) = V\left(\frac{w_n}{\Gamma_k}\right) \mathbb{1}_{\{I \cap I_{k,n} \neq \emptyset\}} + \max_{t \in I_{k,n}} \sum_{j>k} V\left(\frac{w_n}{\Gamma_j}\right) \mathbb{1}_{\{t \in I_{j,n}\}} \Bigg\} = 1.$$

Hence the proof used for $k = 1$ can formally carry over to any other k.

Since the normalized tail $\bar{\nu}$ is in the Gumbel maximum domain of attraction, the function V is π-varying, so by (4.6) and (4.9) we have

$$\frac{V(w_n/\Gamma_1) - V(w_n)}{h \circ V(w_n)} \longrightarrow -\log \Gamma_1 \quad \text{a.s. as } n \to \infty. \tag{4.73}$$

Furthermore, by (4.67)

$$\mathbb{1}_{\{I_{1,n} \cap nI \neq \emptyset\}} \longrightarrow \mathbb{1}_{\{\overline{R_1} \cap I \neq \emptyset\}} \quad \text{a.s. as } n \to \infty.$$

Since $h \circ V(w_n) = o(V(w_n))$, we have

$$\frac{V\left(w_n/\Gamma_1\right) \mathbb{1}_{\{I_{1,n} \cap nI \neq \emptyset\}} - V(w_n)}{h \circ V(w_n)} \longrightarrow \mathcal{M}_{(1)}(I) \quad \text{a.s.}.$$

If we denote

$$S_{n,(1)}(I) = M_{n,(1)}(I) - V\big(w_n/\Gamma_1\big) \mathbb{1}_{\{I_{1,n} \cap nI \neq \emptyset\}}\,,$$

then the claim of the proposition will follow from the following two statements:

$$\limsup_{n\to\infty} \frac{S_{n,(1)}(I) - V\big(1/\bar{F}(n)\big)}{h \circ V(w_n)} \leqslant 0 \quad \text{in probability} \tag{4.74}$$

and

$$\mathbb{P}\left(\liminf_{n\to\infty} \frac{S_{n,(1)}(I) - V\big(1/\bar{F}(n)\big)}{h \circ V(w_n)} \geqslant 0 \,\middle|\, \overline{R_1} \cap I \neq \emptyset\right) = 1\,, \tag{4.75}$$

which we proceed to prove. We start with (4.74). Note that

$$0 \leqslant S_{n,(1)}(I) \leqslant \max_{t \in I_{1,n}} \sum_{j=2}^{\infty} V\big(w_n/\Gamma_j\big) \mathbb{1}_{\{t \in I_{j,n}\}} =: S_{n,(1)}.$$

Let $c_1, c_2 > 0$ be positive constants to be determined later and write $A_{c_1,n} =$

$c_1 \log n/\bar{F}(n)$. Then

$$\begin{aligned}
&\mathbb{P}\left(S_{n,(1)} \geqslant V\left(A_{c_1,n}\right) + c_2 h \circ V\left(A_{c_1,n}\right)\right) \\
\leqslant &A_{c_1,n} \cdot \mathbb{P}\left(\sum_{j=2}^{\infty} V\big(w_n/\Gamma_j\big) \mathbb{1}_{\{0 \in I_{j,n}\}} \geqslant V\left(A_{c_1,n}\right) + c_2 h \circ V\left(A_{c_1,n}\right)\right) \\
&+ \mathbb{P}\left(\# I_{1,n} > A_{c_1,n}\right) \\
\leqslant &A_{c_1,n} \cdot \mathbb{P}\left(X_0 \geqslant V\left(A_{c_1,n}\right) + c_2 h \circ V\left(A_{c_1,n}\right)\right) + \mathbb{P}\left(\# I_{1,n} > A_{c_1,n}\right) \\
=: &A_{c_1,n} \cdot B_1 + B_2.
\end{aligned}$$

In the first inequality, we use the fact that different $I_{j,n}$'s are i.i.d. and each $I_{j,n}$ is stationary, i.e. $\mathbb{P}\{t \in I_{j,n}\} = \mathbb{P}\{0 \in I_{j,n}\}$ for all $0 \leqslant t \leqslant n$. By (4.18) and (4.102), $B_2 \to 0$ if c_1 is large enough. We further note that $A_{c_1,n} \sim (\mathbb{P}\{X_0 > V(A_{c_1,n})\})^{-1}$, combining it with the fact (4.8), it follows that $A_{c_1,n} \cdot B_1 \to e^{-c_2}$. Therefore, fix any $\epsilon \in (0,1)$, we can choose $c_1, c_2 > 0$ such that

$$\mathbb{P}\left(S_{n,(1)} \leqslant V\left(A_{c_1,n}\right) + c_2 h \circ V\left(A_{c_1,n}\right)\right) \geqslant 1 - \epsilon.$$

The claim (4.74) follows because by (4.9) and parts (ii), (iii) of Proposition 4.3.1,

$$\begin{aligned}
V(A_{c_1,n}) - V\big(1/\bar{F}(n)\big) &= G(A_{c_1,n}) - G\big(1/\bar{F}(n)\big) \\
&\quad + o(h \circ G(A_{c_1,n})) + o\left(h \circ G\big(1/\bar{F}(n)\big)\right) \\
&\lesssim (\log\log n) h \circ G\big(1/\bar{F}(n)\big) \\
&\quad + o(h \circ G(A_{c_1,n})) + o\left(h \circ G\big(1/\bar{F}(n)\big)\right) \\
&= o(h \circ V(w_n)) \,.
\end{aligned}$$

We now prove (4.75). Let $\Omega_1 = \{I_{1,n} \cap nI \neq \emptyset\}$. For a fixed $\omega_1 \in \Omega_1$ we view $\left\{\mathbb{1}_{\{I_{1,n}(\omega_1) \cap I_{j,n} \cap nI \neq \emptyset\}} : j = 2, 3, \ldots\right\}$ as a Bernoulli sequence with the success probability is $\overline{p}_{n,T}(\omega_1)$ in (4.30). By Theorem 4.2.1 (ii) and (iii), for every $0 < \epsilon < 1$

and $\gamma > (1-2\beta)^{-1}$ we can choose $c_1, c_2 > 0$ such that the event

$$D_1 := \left\{ \frac{c_1 L((\log n)^\gamma)}{w_n \bar{F}(n) (\log n)^{\gamma\beta}} \leqslant \overline{p}_{n,T} \leqslant \frac{c_2 \log n}{w_n \bar{F}(n)} \right\}$$

satisfies $\mathbb{P}\left(D_1 \,|\, \Omega_1\right) \geqslant 1-\epsilon$ for all n large enough.

For $\omega_1 \in \Omega_1$ we denote $j_1 = j_1(\omega_1) = \inf\{j \geqslant 2 : I_{j,n} \cap I_{1,n}(\omega_1) \cap nI \neq \emptyset\}$ and note that $S_{n,(1)}(I) \geqslant V(w_n/\Gamma_{j_1})$. Therefore, for any $c_3 > 0$ we have

$$\begin{aligned}
&\mathbb{P}\left(S_{n,(1)}(I) \geqslant V\left(\frac{c_1}{c_3} \cdot \frac{L((\log n)^\gamma)}{\bar{F}(n)(\log n)^{\gamma\beta}}\right) \,\Big|\, \overline{R_1} \cap I \neq \emptyset \right) \\
\geqslant &\mathbb{P}\left(D_1 \cap \left\{ V(w_n/\Gamma_{j_1}) \geqslant V\left(\frac{c_1}{c_3} \cdot \frac{L((\log n)^\gamma)}{\bar{F}(n)(\log n)^{\gamma\beta}}\right) \right\} \,\Big|\, \overline{R_1} \cap I \neq \emptyset \right) \\
\geqslant &\mathbb{P}\left(D_1 \cap \left\{ \Gamma_{j_1} \leqslant c_3 (\overline{p}_{n,T})^{-1} \right\} \,\Big|\, \overline{R_1} \cap I \neq \emptyset \right) \\
\geqslant &\mathbb{P}\left(D_1 \cap \left\{ j_1 \leqslant (c_3/2) (\overline{p}_{n,T})^{-1} \right\} \,\Big|\, \overline{R_1} \cap I \neq \emptyset \right) - \epsilon \\
\geqslant &\mathbb{P}\big(D_1 \,|\, \overline{R_1} \cap I \neq \emptyset \big) - 2\epsilon \geqslant \mathbb{P}\big(D_1 \,|\, \Omega_1 \big) - 3\epsilon \geqslant 1 - 4\epsilon,
\end{aligned}$$

for large n, where the 3rd inequality follows from the law of large numbers, the 4th inequality follows from the Markov inequaliy if c_3 large enough, and the penultimate inequality follows from (4.67). Since we can take $\epsilon > 0$ as small as we wish, it is enough to show that

$$\left| V\left(\frac{c_1}{c_3} \cdot \frac{L((\log n)^\gamma)}{\bar{F}(n)(\log n)^{\gamma\beta}}\right) - V\left(1/\bar{F}(n)\right) \right| = o\left(h \circ V(w_n)\right) . \tag{4.76}$$

To this end, choose any $\alpha > \gamma\beta$ and note that for large n, by parts (iii) and (iv) of Proposition 4.3.1, the expression in the left-hand side does not exceed

$$\begin{aligned}
&V\left(1/\bar{F}(n)\right) - V\left((\log n)^{-\alpha}/\bar{F}(n)\right) \\
\lesssim &(\log\log n)\, h \circ V\left(1/\bar{F}(n)\right) \ll h \circ V(w_n),
\end{aligned}$$

as required. □

Proof of Proposition 4.4.3. We start by fixing a small constant ρ and setting

$$k_n = \left\lfloor \frac{\rho \log w_n}{\zeta(w_n)} \right\rfloor . \tag{4.77}$$

The first step is to establish the following claim, that shows that for large k, $M_{n,(k)}(I)$ is not likely to become the overall maximum $M_n(I)$.

$$\lim_{i_0\to\infty,\, K\to\infty} \limsup_{n\to\infty} \mathbb{P}\left(\max_{2^{i_0}\leqslant k<2^{k_n}} M_{n,(k)}(I) > M_{n,K}(I) \right) = 0 \,. \tag{4.78}$$

To this end, observe that, by Proposition 4.4.2, for any $\epsilon \in (0,1)$ we can choose $C_1 > 0$ large enough so that for all K large enough,

$$\lim_{n\to\infty} \mathbb{P}\left(M_{n,K}(I) \geqslant b_n - C_1 a_n \right) \geqslant 1-\epsilon \,.$$

Next, for $c > 0$ let $A_{c,n} = c \log n / \bar{F}(n)$. For $i_0 \leqslant j \leqslant k_n - 1$, let H_j be the event

$$\bigcap_{k=2^j}^{2^{j+1}-1} \left\{ M_{n,(k)}(I) \leqslant V\big(2w_n/k\big) + V\left(A_{c,n}\right) + 2jh\circ V(A_{c,n}) \right\} .$$

We claim that, given $0 < \epsilon < 1$, we can find c, $C_2 > 0$ such that

$$\lim_{i_0\to\infty} \liminf_{n\to\infty} \mathbb{P}\left(\bigcap_{j=i_0}^{k_n-1} H_j \right) \geqslant 1-\epsilon. \tag{4.79}$$

Assuming, for a moment, that this is true, the claim (4.78) will follow once we check that for all j and n large enough,

$$V\left(w_n\right) - V\big(w_n/2^{j-1}\big) - C_1 a_n > V(A_{c,n}) + 2jh\circ V(A_{c,n}) - V(1/\bar{F}(n)). \tag{4.80}$$

Indeed, by (4.9) and part (v) of Proposition 4.3.1,

$$\begin{aligned} & V(w_n) - V\big(w_n/2^{j-1}\big) - C_1 a_n \\ =& G(w_n) - G\big(w_n/2^{j-1}\big) - (C_1 + o(1)) h\circ G(w_n) \\ \gtrsim & \, jh\circ G(w_n) \,, \end{aligned}$$

while by part (iv) of Proposition 4.3.1,

$$\begin{aligned}
&V(A_{c,n}) + 2jh \circ V(A_{c,n}) - V(1/\bar{F}(n)) \\
=&G(A_{c,n}) + 2jh \circ G(A_{c,n}) - G(1/\bar{F}(n)) + o(h \circ G(A_{c,n})) \\
\lesssim&(j + \log\log n)h \circ V(A_{c,n}) \, .
\end{aligned}$$

By part (iii) of Proposition 4.3.1 this gives (4.80), and, hence, (4.78), so we now prove (4.79). Switching to the complements, we will show that

$$\lim_{k_0 \to \infty} \limsup_{n \to \infty} \mathbb{P}\left(\bigcup_{j=i_0}^{k_n - 1} H_j^c \right) \leqslant \epsilon \, . \tag{4.81}$$

Recall that

$$M_{n,(k)} \leqslant V\big(w_n/\Gamma_k\big) + S_{n,(k)}, \quad S_{n,(k)} := \max_{t \in I_{k,n}} \sum_{j=k+1}^{\infty} V\big(w_n/\Gamma_j\big) 1_{\{t \in I_{j,n}\}}.$$

Therefore, for each $i_0 \leqslant j < k_n$, $H_j^c \subseteq \bigcup_{k=2^j}^{2^{j+1}-1} (U_k \cup D_k \cup L_k)$ with

$$\begin{aligned}
&U_k = \{\Gamma_k \leqslant k/2\} \, , \\
&D_k = \{\# I_{k,n} > A_{c,n}\} \, , \\
&L_k = \big\{ S_{n,(k)} \geqslant V(A_{c,n}) + 2jh \circ V(A_{c,n}), \, \# I_{k,n} \leqslant A_{c,n} \big\} \, .
\end{aligned}$$

Trivially,

$$\sum_{k=1}^{\infty} \mathbb{P}(U_k) < \infty \, , \tag{4.82}$$

and, if c is large enough, then by (4.102) we also have

$$\sum_{k=1}^{\infty} \mathbb{P}(D_k) < \infty \, . \tag{4.83}$$

Next, as $\mathbb{P}(X_0^{(1)} > x) \sim \overline{\nu}(x)$ by subexponentiality and $(A_{c,n})^{-1} \asymp \overline{\nu}(V(A_{c,n}))$, we have for $2^j \leqslant k < 2^{j+1}$,

$$\begin{aligned}
\mathbb{P}(L_k) \leqslant & A_{c,n} \cdot \mathbb{P}\left(X_0^{(1)} \geqslant V(A_{c,n}) + 2jh \circ V(A_{c,n})\right) \\
\lesssim & \frac{\overline{\nu}\left(V(A_{c,n}) + 2jh \circ V(A_{c,n})\right)}{\overline{\nu}\left(V(A_{c,n})\right)} \\
\lesssim & \exp\left\{-\int_0^{2j} \frac{h \circ V(A_{c,n})}{h\big[V(A_{c,n}) + uh \circ V(A_{c,n})\big]} du\right\} \\
\lesssim & \exp\left\{-\frac{2jh \circ V(A_{c,n})}{h\big[V(A_{c,n}) + 2(k_n-1)h \circ V(A_{c,n})\big]}\right\}.
\end{aligned}$$

Note that by Assumption 4.3.1 $(B1)$, for some constant C, for large n,

$$\begin{aligned}
2(k_n-1)h \circ V(A_{c,n}) \sim & 2k_n h \circ G(A_{c,n}) \\
\sim & \frac{2\rho \log w_n}{\zeta(w_n)} \cdot \frac{\zeta(A_{c,n})}{\log A_{c,n}} V(A_{c,n}) \leqslant C\rho V(A_{c,n}),
\end{aligned}$$

so we can choose ρ small enough so that

$$\mathbb{P}(L_k) \leqslant \exp\left\{-2j \frac{h \circ V(A_{c,n})}{h\big[(1+C\rho)V(A_{c,n})\big]}\right\} \leqslant e^{-j}$$

because h is assumed to satisfy (4.49). It follows that

$$\sum_{k=1}^{\infty} \mathbb{P}(L_k) < \infty$$

which, tohether with (4.82) and (4.83), proves (4.81), so we have established (4.78). Now the claim of Proposition 4.4.3 will follow from the following statetement that we prove next.

We claim that, with k_n given, once again, by (4.77),

$$\lim_{K\to\infty} \limsup_{n\to\infty} \mathbb{P}\left(\max_{k\geqslant 2^{k_n}} M_{n,(k)} \leqslant M_{n,K}(I)\right) = 1.$$

Since $b_n \sim G(w_n)$ and $a_n = o(b_n)$, by (4.71) it is enough to show that for some $\eta \in (0,1)$,

$$\lim_{n\to\infty} \mathbb{P}\left(\max_{k\geqslant 2^{k_n}} M_{n,(k)} \leqslant \eta G(w_n)\right) = 1. \tag{4.84}$$

To this end, choose $0 < r < (1-2\beta)/2$ and write

$$\max_{k \geqslant 2^{k_n}} M_{n,(k)} \leqslant \max_{0 \leqslant t \leqslant n} \sum_{j=2^{k_n}}^{\lfloor n^r \rfloor} V\big(w_n/\Gamma_j\big) \mathbb{1}_{\{t \in I_{j,n}\}} + \max_{0 \leqslant t \leqslant n} \sum_{j=\lfloor n^r \rfloor + 1}^{\infty} V\big(w_n/\Gamma_j\big) \mathbb{1}_{\{t \in I_{j,n}\}} =: T_{1,n} + T_{2,n} \,.$$

By the choice of r,

$$\mathbb{P}\left(\max_{0 \leqslant t \leqslant n} \sum_{j=2^{k_n}}^{\lfloor n^r \rfloor} \mathbb{1}_{\{t \in I_{j,n}\}} \geqslant 2 \right) \leqslant n \mathbb{P}\left(\sum_{j=1}^{\lfloor n^r \rfloor} \mathbb{1}_{\{0 \in I_{j,n}\}} \geqslant 2 \right) \lesssim \frac{n^{2r+1}}{w_n^2} \to 0.$$

Therefore, with probability increasing to 1,

$$T_{1,n} \leqslant V\big(w_n/\Gamma_{2^{k_n}}\big) \lesssim G\big(w_n/2^{k_n-1}\big) \,.$$

By Assumption 4.3.1 ($B4$),

$$\limsup_{n \to \infty} \frac{G(w_n/2^{k_n-1})}{G(w_n)} < 1 \,,$$

so (4.84) will be established once we prove that for any $\epsilon > 0$,

$$\lim_{n \to \infty} \mathbb{P}\left(T_{2.n} > \epsilon G(w_n)\right) = 0 \,.$$

The latter statement will follow from the following claim:

$$\lim_{n \to \infty} n \cdot \mathbb{P}\left(\sum_{\Gamma_j > n^r} V\big(w_n/\Gamma_j\big) \mathbb{1}_{\{0 \in I_{j,n}\}} > \epsilon G(w_n) \right) = 0 \,.$$

Since for some $s > 0$ $V(x) \leq G(sx)$ for all x, we will prove instead that

$$\lim_{n \to \infty} n \cdot \mathbb{P}\left(\sum_{\Gamma_j > n^r} G\big(s w_n/\Gamma_j\big) \mathbb{1}_{\{0 \in I_{j,n}\}} > \epsilon G(w_n) \right) = 0 \,. \tag{4.85}$$

To this end, denote for $n \in \mathbb{N}$,

$$\widetilde{x}_n = \epsilon G(w_n), \quad x_n = G(s w_n/n^r), \quad m_0 = \lfloor \widetilde{x}_n / x_n \rfloor,$$

and define $H_n : (x_0, \infty)^n \to \mathbb{R}$ by

$$H_n(z_1, \ldots, z_n) = \int_0^{z_1} \frac{du}{h(u)} + \cdots + \int_0^{z_n} \frac{du}{h(u)} := q(z_1) + \cdots + q(z_n) \,.$$

Let N_n be a Poisson random variable with mean $s(1 - n^r s^{-1} w_n^{-1})$ (positive for large n). If $\{\xi_i\}_{i=1}^{\infty}$ is a family of i.i.d. random variables independent of N_n with distribution equal to normalized $H_\#$ restricted to the interval (x_0, x_n). Then

$$\sum_{\Gamma_j > n^r} G\big(sw_n/\Gamma_j\big) \mathbb{1}_{\{0 \in I_{j,n}\}} \overset{\mathrm{d}}{=} \sum_{i=1}^{N_n} \xi_i \,, \tag{4.86}$$

so that

$$\begin{aligned} &\mathbb{P}\left(\sum_{\Gamma_j > n^r} G\big(sw_n/\Gamma_j\big) \mathbb{1}_{\{0 \in I_{j,n}\}} > \widetilde{x}_n \right) \\ &= \sum_{d=1}^{\infty} \mathbb{P}\left(N_n = m_0 + d\right) \mathbb{P}\left(\sum_{i=1}^{m_0+d} \xi_i > \widetilde{x}_n \right) =: \sum_{d=1}^{\infty} B_d \cdot Q_d \,. \end{aligned} \tag{4.87}$$

Clearly,

$$B_d \leqslant s^{m_0+d}/(m_0+d)! \,. \tag{4.88}$$

On the other hand, for some constant $c > 0$,

$$\begin{aligned} Q_d &= \int_{(x_0,x_n)^{m_0+d}} \mathbb{1}_{\{\sum_{i=1}^{m_0+d} z_i > \widetilde{x}_n\}} \prod_{i=1}^{m_0+d} P_{\xi_i}(dz_i) \\ &\leqslant c^{m_0+d} \int_{(x_0,x_n)^{m_0+d}} \mathbb{1}_{\{\sum_{i=1}^{m_0+d} z_i > \widetilde{x}_n\}} \prod_{i=1}^{m_0+d} H_\#(dz_i) \\ &= c^{m_0+d} \int_{(x_0,x_n)^{m_0+d}} \mathbb{1}_{\{\sum_{i=1}^{m_0+d} z_i > \widetilde{x}_n\}} \prod_{i=1}^{m_0+d} \exp\{-q(z_i)\} q'(z_i)\, dz_i \\ &\leqslant \left(cq(x_n)\right)^{m_0+d} \exp\left(- \inf \left\{ \sum_{i=1}^{m_0+d} q(z_i) : \sum_{i=1}^{m_0+d} z_i > \widetilde{x}_n, x_0 < z_i < x_n \right\} \right) \,. \end{aligned}$$

To evaluate the infimum inside the above exponential, note that that the function H_n is increasing and concave in all of its variables. Hence its infimum is achieved at a boundary point which will have, say, k_d coordinates equal to x_n, $m_0 + d - k_d - 1$ coordinates equal to x_0, and a final coordinate that makes the sum of all coordinates equal to $\widetilde{x}_n$, for the smallest possible value of k_d that makes it possible. That means that

$$\inf\left\{\sum_{i=1}^{m_0+d} q(z_i) : \sum_{i=1}^{m_0+d} z_i > \widetilde{x}_n, x_0 < z_i < x_n\right\} \geq k_d q(x_n)\,. \tag{4.89}$$

Clearly,

$$k_d = \left[\left\lceil \frac{\widetilde{x}_n - x_n - (m_0+d-1)x_0}{x_n - x_0} \right\rceil\right]_+ \geqslant \frac{\widetilde{x}_n - x_n}{2(x_n - x_0)} \tag{4.90}$$

if

$$d \leqslant \frac{2}{x_0}(\widetilde{x}_n - x_n) - m_0 - 1\,. \tag{4.91}$$

Notice that by (4.88), the part of the sum in (4.87) corresponding to d outside of the above range does not exceed, for large n,

$$\sum_{d \geqslant \widetilde{x}_n/x_0} s^{m_0+d}/(m_0+d)! = o(1/n)$$

by part (i) of Proposition 4.3.1. On the other hand, for large n, for d in the range (4.91), $k_d \geq m_0/3$ by (4.90). Therefore, the part of the sum in (4.87) corresponding to d in the range (4.91) can be bounded by

$$\sum_{d=1}^{\infty} \frac{\left(csq(x_n)\right)^{m_0+d}}{(m_0+d)!} \exp\left\{-m_0 q(x_n)/3\right\} = \exp\bigl\{-q(x_n)(m_0/3 - cs)\bigr\}\,.$$

Since $m_0 \gg \log\log n \to \infty$ by part (ii) of Proposition 4.3.1, and

$$q(x_n) = -\log \overline{H_{\#}}(G(sw_n/n^r)) \sim (1-\beta-r)\log n\,,$$

the part of the sum in (4.87) corresponding to d in the range (4.91) is also $o(1/n)$, proving (4.85) and, hence, completing the proof of Proposition 4.4.3. $\square$

Proof of Theorem 4.4.2. We need to prove that for any fixed $0 < T_1 < T_2 < \infty$,

$$\left(\frac{\max_{s \leq nt} X_s - b_n}{a_n},\, T_1 \leqslant t \leqslant T_2\right) \Rightarrow \left(X_{\mathrm{G}}(t^{1-\beta}),\, T_1 \leqslant t \leqslant T_2\right)$$

weakly in the Skorohod J_1 topology on $D[T_1, T_2]$, and without loss of generality we assume that $T_2 \leq 1$. Accordng to (4.59) and Proposition 4.4.1, is the same as proving

$$\left(\frac{M_n^{(1)}([0,t]) - b_n}{a_n}, \, T_1 \leqslant t \leqslant T_2 \right) \Rightarrow \big(\mathcal{M}([0,t]), \, T_1 \leqslant t \leqslant T_2 \big) \tag{4.92}$$

in the same space. We construct all the random objects in (4.58) on the same probability space as in the proof of Theorem 4.4.1 and prove a.s convergence in $D[T_1, T_2]$. In the course of the proof of the latter theorem we have shown that for every $\varepsilon > 0$ there is $K \geq 1$ such that

$$\limsup_{n\to\infty} \mathbb{P}\Big[\big(M_n^{(1)}([0,t]), \, T_1 \leqslant t \leqslant T_2 \big) \neq \big(M_{n,K}([0,t]), \, T_1 \leqslant t \leqslant T_2 \big) \Big] \leq \varepsilon \, .$$

Since, clearly,

$$\lim_{K\to\infty} \mathbb{P}\Big[\big(\mathcal{M}_K([0,t]), \, T_1 \leqslant t \leqslant T_2 \big) = \big(\mathcal{M}([0,t]), \, T_1 \leqslant t \leqslant T_2 \big) \Big] = 1 \, .$$

(4.92) will follow once we prove that for every $K = 1, 2, \ldots$

$$\left(\frac{M_{n,K}([0,t]) - b_n}{a_n}, \, T_1 \leqslant t \leqslant T_2 \right) \to \big(\mathcal{M}_K([0,t]), \, T_1 \leqslant t \leqslant T_2 \big)$$

a.s. in $D[T_1, T_2]$ as $n \to \infty$. The stochastic process in the right hand side may take the value $-\infty$; the probability of this converges to zero as $K \to \infty$. For non-decreasing functions the value of $-\infty$ introduces no difficulties in the J_1 topology if one interpretes $(-\infty) - (-\infty)$ as zero. The assumption $0 < \beta < 1/2$ implies that the stable regenerative sets $(\overline{R}_j)$ are a.s. disjoint, so the latter statement will follows from

$$\left(\frac{M_{n,(k)}([0,t]) - b_n}{a_n}, \, T_1 \leqslant t \leqslant T_2 \right) \to \big(\mathcal{M}_{(k)}([0,t]), \, T_1 \leqslant t \leqslant T_2 \big)$$

a.s. in $D[T_1, T_2]$ for every $k \geqslant 1$ and, as before, it is enough to consider the case $k = 1$. As in the proof of Proposition 4.4.2, we only need to check that

$$\left(\frac{V\left(w_n/\Gamma_1\right) \mathbb{1}_{\{I_{1,n} \cap [0,nt] \neq \emptyset\}} - V(w_n)}{h \circ V(w_n)}, \, T_1 \leqslant t \leqslant T_2 \right) \tag{4.93}$$

$$\to \big(\mathcal{M}_{(1)}([0,t]), \, T_1 \leqslant t \leqslant T_2 \big)$$

a.s. However, it follows from (4.67) that, a.s.,

$$\inf\{I_{1,n}/n\} \to \inf\{\overline{R}_1\}\,.$$

Together with (4.73) this establishes (4.93), as required. $\square$

4.5 Appendices

4.5.1 Random Walks with Regularly Varying Tails

Among the major goals of this appendix is to prove Theorem 4.2.1. We start with recalling certain results on the ranges of the random walks from Barlow and Taylor (1992) and Samorodnitsky and Wang (2019). We consider a random walk $\{S_n\}_{n\geqslant 0}$ with $\mathbb{N}_0$-valued steps $\{\xi_n\}$ whose distribution F satisfies Assumption 4.2.1. Recall the standard notions

(a) the *range* $A = \{S_n : n = 0, 1, 2, \ldots\}$,

(b) the *sojourn time* in a set $W \subset \mathbb{N}_0$ up to time k, $T_W(k) = \#\{0 \leqslant n \leqslant k : S_n \in W\}$ for $k \in \mathbb{N} \cup \{\infty\}$.

The following properties are well-known; see e.g. Appendix A in Samorodnitsky and Wang (2019). As $n \to \infty$,

$$\mathbb{E}_0\, T_{[0,n]}(\infty) \sim \frac{n^{\beta}}{\Gamma(1+\beta)\Gamma(1-\beta)L(n)}\,, \tag{4.94}$$

$$\mathbb{P}_0\left(A \cap \{n\} \neq \emptyset\right) \sim \frac{n^{\beta-1}\mathbb{P}(\xi_1 > 0)}{\Gamma(\beta)\Gamma(1-\beta)L(n)}\,. \tag{4.95}$$

For $W \subset \mathbb{Z}$ we denote by $D(W) := \{x - y : x, y \in W\}$ its *difference set*. For every $\delta \in (0,1)$, there exists $c_0 = c_0(\delta) > 0$ such that for every W and every

$k \in \mathbb{N} \cup \{\infty\}$ with $0 < \mathbb{E}_0(T_{D(W)}(k)) < \infty$, we have

$$\mathbb{P}_x\left(T_W(k) \geqslant c\,\mathbb{E}_0(T_{D(W)}(k))\right) \leqslant e^{-c\delta} \text{ for each } \ x \in W \text{ and } c > c_0\,, \tag{4.96}$$

see e.g. Lemma 3.1 in Pruitt and Taylor (1969). Choose $W = W_n = \{0, 1, \ldots, n\}$ and note S_n has non-negative steps, so $T_{D(W_n)}(k) = T_{F_n}(k)$ for all k under $\mathbb{P}_0$. Thus, by (4.94),

$$\mathbb{E}_0 T_{D(W_n)}(\infty) = \mathbb{E}_0 T_{W_n}(\infty) \lesssim \frac{n^\beta}{L(n)}.$$

Therefore, choosing in (4.96) $\delta = 1/2$, we see that for any $C > 0$ we can choose $c > 0$ so that for all $n \in \mathbb{N}$,

$$\mathbb{P}_0\left(\#(A \cap [0,n]) \geqslant \frac{cn^\beta \log n}{L(n)}\right) \leqslant \mathbb{P}_0\left(T_{W_n}(\infty) \geqslant \frac{cn^\beta \log n}{L(n)}\right) \leqslant n^{-C}. \tag{4.97}$$

In particular, for a sufficiently large c, for each n we have

$$\mathbb{P}_0\left(\#(A \cap [0, 2^n]) \geqslant \frac{cn2^{\beta n}}{L(2^n)}\right) \leqslant e^{-n}\,. \tag{4.98}$$

Lemma 4.5.1. *Assume that Assumption 4.2.1 holds and* $S_0 = 0$*. Then*

$$\limsup_{n_0 \to \infty} \sup_{n > n_0} \max_{0 \leqslant k \leqslant n-1} \max_{m \in \mathbb{Z}} \frac{\#(A \cap [m, m+2^k) \cap [2^{n_0}, 2^n))}{n2^{\beta k}/L(2^k)} < \infty \quad a.s. \tag{4.99}$$

Proof. Let c be such that (4.98) holds. Then

$$\begin{aligned}
&\mathbb{P}\left(\sup_{n>n_0} \max_{0 \leqslant k \leqslant n-1} \max_{m \in \mathbb{Z}} \frac{\#(A \cap [m, m+2^k) \cap [2^{n_0}, 2^n))}{n2^{\beta k}/L(2^k)} \geq c\right) \\
&\leq \sum_{n > n_0} 2^{n+1} \max_{0 \leqslant k \leqslant n-1} \max_{m \in \mathbb{Z}} \mathbb{P}\left(\#(A \cap [m, m+2^k) \cap [2^{n_0}, 2^n)) \geq \frac{cn2^{\beta k}}{L(2^k)}\right) \\
&\leq \sum_{n > n_0} 2^{n+1}\, \mathbb{P}_0\left(\#(A \cap [0, 2^n)) \geqslant \frac{cn2^{\beta n}}{L(2^n)}\right) \leq \sum_{n > n_0} 2(2/e)^n\,,
\end{aligned}$$

where the penultimate inequality is derived by conditioning on the entry point $\min\left(A \cap [m, m+2^k) \cap [2^{n_0}, 2^n)\right)$, and then applying the Markov property. Since the last expression is summable in n_0, (4.99) follows by the first Borel-Cantelli Lemma. $\square$

Lemma 4.5.2. *Assume that Assumption 4.2.1 holds and $S_0 = 0$. For any $\eta, \gamma > 0$,*

$$\#\left\{k : S_k \leqslant \eta n,\, \xi_k \geqslant (\log n)^\gamma\right\} \gtrsim_P \frac{n^\beta L\left((\log n)^\gamma\right)}{(\log n)^{\gamma\beta} L(n)}\,. \tag{4.100}$$

Proof. Let $N_t = \max\{k : S_k \leqslant t\} + 1$, $t \geq 0$. Then for each $x > 0$, as $m \to \infty$,

$$\mathbb{P}\left(\overline{F}(m) N_m \geqslant x^{-\beta}\right) \to J_\beta(x),$$

where J_β is an $\mathbb{R}_+$-supported strictly β-stable distribution; see e.g. Feller (1966) XI.5 (5.6). Therefore, for any $\epsilon > 0$ we can choose $c > 0$ so small that with $m_n = \lceil c n^\beta / L(n) \rceil$, and by (4.18), we have $\liminf_{n\to\infty} P(B_n) > 1 - \epsilon$ for the events $B_n = \{N_{\eta n} \geqslant m_n\}$, $n \geq 1$. Consider also the events

$$D_n = \left\{ \frac{1}{m_n} \sum_{k=1}^{m_n} 1_{\{\xi_k > (\log n)^\gamma\}} < \overline{F}\left((\log n)^\gamma\right)/2 \right\},\ n = 1, 2, \ldots\,.$$

By Chebyshev's inequality and (4.18), as $n \to \infty$,

$$\begin{aligned}\mathbb{P}(D_n) \lesssim & \frac{m_n \cdot \operatorname{var}(1_{\{\xi_k > (\log n)^\gamma\}})}{\left(m_n \overline{F}\left((\log n)^\gamma\right)\right)^2} \\ \sim & \frac{1}{m_n \overline{F}\left((\log n)^\gamma\right)} \lesssim \frac{L(n)}{n^\beta} \cdot \frac{(\log n)^{\gamma\beta}}{L\left((\log n)^\gamma\right)} \to 0\,.\end{aligned}$$

Hence $\liminf_{n\to\infty} \mathbb{P}(B_n \cap D_n^c) \geqslant 1 - \epsilon$. However, on the event $B_n \cap D_n^c$

$$\#\left\{k : S_k \leqslant \eta n,\, \xi_k \geqslant (\log n)^\gamma\right\} \geqslant \frac{c}{2} \frac{n^\beta L\left((\log n)^\gamma\right)}{(\log n)^{\gamma\beta} L(n)}\,,$$

leading to the desired conclusion. $\square$

Proof of Theorem 4.2.1 (i). We use the notation $T = (a, b)$ throughout the proof. Let $\{Y_t^{(1)}\}_{t\in\mathbb{Z}}$ and $\{Y_t^{(2)}\}_{t\in\mathbb{Z}}$ be i.i.d. Markov chains on $\mathbb{Z}$ starting at 0, satisfying Assumption 4.2.1. The simultaneous visit times of the two chains to 0,

$$\varphi_j^* = \inf\{n \geqslant \varphi_{j-1}^* + 1 : Y_n^{(1)} = Y_n^{(2)} = 0\},\ j = 1, 2, \dots$$

with $\varphi_0^* = 0$ and $\varphi_j^* = \infty$ if $\varphi_{j-1}^* = \infty$, form a terminating, i.e. transient, renewal process. Indeed, by the independence among two chains and (4.95),

$$\begin{aligned} u_n &=: \mathbb{P}\{\text{There is a simultaneous renewal at time } n\} \\ &= \mathbb{P}\{Y_n^{(1)} = 0\}\mathbb{P}\{Y_n^{(2)} = 0\} \in \mathrm{RV}_{2\beta-2}. \end{aligned}$$

Since $\beta < 1/2$, the renewal sequence $\{u_n\}_{n\geqslant 1}$ satisfy $\sum_{n=1}^{\infty} u_n < \infty$, which implies the transience.

We denote by $\overline{F^*}$ the tail distribution of $\varphi^* := \varphi_1^*$. By the last entrance decomposition,

$$\begin{aligned} p_{n,T} &= \sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor} \mathbb{P}\{k \in I_{1,n} \cap I_{2,n}\} \\ &\qquad \cdot \mathbb{P}\left\{\text{No simultaneous returns after } k \text{ in } nT \,\middle|\, Y_k^{(1)} = Y_k^{(2)} = 0\right\} \\ &= \sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor} \frac{1}{w_n^2}\overline{F^*}(\lfloor nb\rfloor - k) \asymp \frac{n}{w_n^2} \end{aligned}$$

since $\overline{F^*}(\infty) > 0$. This proves (4.31).

In the remaining part of the proof we will use the following simple observation that allows us to use the basic facts about random walks with regularly varying tails describing earlier in this section. For a fixed $n \in \mathbb{N}$ we construct a random walk $\{S_k^{(n)}\}_{k\geqslant 0}$ by choosing the initial state distributed as $\min I_{1,n}$ and the steps with the distribution F in (4.18). Recall that

$$\mathbb{P}(\min I_{1,n} \le nx) = \frac{w_{[nx]}}{w_n} \quad \text{for } 0 < x < 1. \tag{4.101}$$

The range A_n of $\{S_k^{(n)}\}_{k\geqslant 0}$, obviously, satisfies

$$A_n \cap [0,m] \overset{\mathrm{d}}{=} I_{1,n} \cap [0,m] \text{ for all } m \le n\,.$$

By conditioning on $S_0^{(n)}$ and using (4.97), we see that for any $C > 0$ we can choose $c > 0$ so that for all $n \in \mathbb{N}$,

$$\mathbb{P}\left(\#I_{1,n} \geqslant \frac{cn^\beta \log n}{L(n)}\right) \leqslant n^{-C}\,. \tag{4.102}$$

Proof of Theorem 4.2.1 (ii). For $c > 0$ denote $a_{c,n} := cn^\beta \log n / L(n)$ and consider the event $B_n = \{\#(I_{1,n} \cap nT) \leqslant a_{c,n}\}$. On B_n, the last entrance decomposition and the regular variation of w_n show that for large n,

$$\overline{p}_{n,T} \leqslant \frac{\#(I_{1,n} \cap nT)}{\min_{k\in nT} w_k} \leqslant \frac{2}{a^{1-\beta}} \frac{a_{c,n}}{w_n}.$$

Therefore it suffices to show that $\mathbb{P}(B_n^c) \leqslant n^{-C}$ if c is large enough. This, however, follows immediately from (4.102).

Proof of Theorem 4.2.1 (iii). By the measure preserving property of the shift θ it is enough to consider intervals of the form $T = (0,b)$. Let $v_1 < v_2 < v_3 < \cdots$ be the enumeration of the points of $I_{1,n}$ in the increasing order. We construct a subset of by $I_{1,n}$

$$I_{1,n,\gamma} := \{v_i \in I_{1,n} : v_{i+1} - v_i \geqslant (\log n)^\gamma\} \tag{4.103}$$

(not including the last point in $I_{1,n}$.) For an $\omega_1 \in \{I_{1,n} \cap nT \neq \emptyset\}$, a lower bound for $\overline{p}_{n,T}(\omega_1)$ is derived below, where for typographical convenice we use the notation

$\mathbb{P}_2$ to denote the probability measure associated with $Y^{(2,n)}$.

$$\begin{aligned}
\overline{p}_{n,T}(\omega_1) &\geqslant \mathbb{P}_2\left(I_{1,n,\gamma}(\omega_1) \cap I_{2,n} \cap nT \neq \emptyset\right) \\
&\geqslant \sum_{u \in I_{1,n,\gamma}(\omega_1) \cap nT} \mathbb{P}_2\left(u = \max(I_{1,n,\gamma}(\omega_1) \cap I_{2,n})\right) \\
&= \frac{1}{w_n} \sum_{u \in I_{1,n,\gamma}(\omega_1) \cap nT} \mathbb{P}_2\left(I_{1,n,\gamma}(\omega_1) \cap I_{2,n} \cap (u,\infty) \cap nT = \emptyset \,\middle|\, u \in I_{2,n}\right) .
\end{aligned}$$

Now the claim of part (iii) of the theorem follows from the following two statements.

For any $\epsilon \in (0,1)$, there exists $c = c(\epsilon) > 0$ such that for all $\gamma > 0$

$$\liminf_{n\to\infty} \mathbb{P}\left(\#(I_{1,n,\gamma} \cap nT) \geqslant d_n \,\middle|\, I_{1,n} \cap nT \neq \emptyset\right) \geqslant 1 - \epsilon , \tag{4.104}$$

where

$$d_n = \frac{cn^{\beta} L\left((\log n)^{\gamma}\right)}{(\log n)^{\gamma\beta} L(n)} .$$

Further we claim that, if $\gamma > (1-2\beta)^{-1}$, then for every $0 < \epsilon < 1$ there is an event B with $\mathbb{P}(B) > 1-\epsilon$ such that for every $w_1 \in B$,

$$\sup_{u \in I_{1,n,\gamma}(\omega_1) \cap nT} \mathbb{P}_2\left(I_{1,n,\gamma}(\omega_1) \cap I_{2,n} \cap (u,\infty) \cap nT \neq \emptyset \,\middle|\, u \in I_{2,n}\right) = o_P(1) . \tag{4.105}$$

These two statements are proved in the remainder of this section.

Fix $\epsilon \in (0,1)$. For any $\eta \in (0,1)$ we have by (4.25) and (4.11),

$$\begin{aligned}
&\mathbb{P}\left(\frac{1}{n} I_{1,n} \cap \eta nT \neq \emptyset \,\middle|\, \frac{1}{n} I_{1,n} \cap nT \neq \emptyset\right) \\
\to &\mathbb{P}\left(\overline{R_1} \cap \eta T \neq \emptyset \,\middle|\, \overline{R_1} \cap T \neq \emptyset\right) = \eta^{1-\beta}.
\end{aligned}$$

Therefore, if η is sufficiently close to 1,

$$\liminf_{n\to\infty} \mathbb{P}\left(\frac{1}{n} I_{1,n} \cap \eta nT \neq \emptyset \,\middle|\, \frac{1}{n} I_{1,n} \cap nT \neq \emptyset\right) \geqslant \sqrt{1-\epsilon}. \tag{4.106}$$

Note that

$$
\begin{aligned}
&\mathbb{P}\left(\#(I_{1,n,\gamma} \cap nT) \geqslant d_n \,\middle|\, \frac{1}{n} I_{1,n} \cap \eta nT \neq \emptyset\right)\\
=&\sum_{i\in\eta nT} P\big(S_0 = i \big| S_0 \in \eta nT\big)\, \mathbb{P}_i\left(\#\{k : \xi_k \geqslant (\log n)^\gamma, S_k \leqslant nb - i\} \geqslant d_n\right)\\
\geqslant&\mathbb{P}_0\left(\#\{k : \xi_k \geqslant (\log n)^\gamma, S_k \leqslant \lfloor n(1-\eta)b\rfloor\} \geqslant d_n\right) .
\end{aligned}
$$

We conclude by Lemma 4.5.2 that a fixed $\eta \in (0,1)$ for which (4.106) holds, we can choose c such that

$$\liminf_{n\to\infty} \mathbb{P}\left(\#(I_{1,n,\gamma} \cap nT) \geqslant d_n \,\middle|\, I_{1,n} \cap \eta nT \neq \emptyset\right) \geqslant \sqrt{1-\epsilon}\,. \tag{4.107}$$

Clearly, (4.106) and (4.107) give us (4.104), so it remains to prove (4.105).

Let k_1 and k_2 be such that $2^{k_1} \leqslant (\log n)^\gamma < 2^{k_1+1}$ and $2^{k_2-1} \leqslant n < 2^{k_2}$. Let $u \in I_{1,n,\gamma}(\omega_1) \cap nT$ and denote by $\overline{q}_n(u|\omega_1)$ the probability in the left hand side of (4.105). We have

$$
\begin{aligned}
&\overline{q}_n(u|\omega_1)\\
\leqslant&\sum_{k=k_1}^{k_2} \mathbb{P}\left(s \in I_{2,n} \text{ for some } s \in [u+2^k, u+2^{k+1}) \cap I_{1,n}(\omega_1) \,\middle|\, u \in I_{2,n}\right)\\
\leqslant&\sum_{k=k_1}^{k_2} \#\left([u+2^k, u+2^{k+1}) \cap I_{1,n}(\omega_1)\right) \cdot \max_{i\in[u+2^k,u+2^{k+1})} \mathbb{P}\left(i \in I_{2,n} \,|\, u \in I_{2,n}\right).
\end{aligned}
$$

Fix any $\epsilon \in (0,1)$. By (4.101) and Lemma 4.5.1, there is $C > 0$ and an event B with probability higher than $1-\epsilon$ such that for all n large enough, all $w_1 \in B$, all $u \in I_{1,n,\gamma}(\omega_1) \cap nT$ and all $k \geq k_1$,

$$\#\left([u+2^k, u+2^{k+1}) \cap I_{1,n}(\omega_1)\right) \leqslant C \log n \frac{2^{\beta k}}{L(2^k)}\,. \tag{4.108}$$

Further, by (4.95),

$$\sup_{u\geq 0} \max_{i\in[u+2^k,u+2^{k+1})} \mathbb{P}\left(i \in I_{2,n} \,|\, u \in I_{2,n}\right) \lesssim \frac{2^{-(1-\beta)k}}{L(2^k)}\,. \tag{4.109}$$

Combining (4.108), (4.109), and Potter's bounds, we see that for any $w_1 \in B$

$$\max_{u \in I_{1,n,\gamma}(\omega_1) \cap nT} \overline{q}_n(u|\omega_1) \lesssim \log n \sum_{k=k_1}^{k_2} \frac{2^{-(1-2\beta)k}}{(L(2^k))^2} \lesssim (\log n)^{1+\alpha\gamma}$$

for any $\alpha > 2\beta - 1$. By the choice of γ, we can select α in such a way that $1 + \alpha\gamma < 0$. This proves (4.105).

4.5.2 Supplements to Sections 4.3 and 4.4

We start by checking the series representation.

Proof of (4.68). Fix any $n \in \mathbb{N}$. We claim first that the right hand side of (4.68) is compound Poisson. Indeed, by (4.69), it in distribution equals to

$$\sum_{j=1}^{N_n} \left(\tilde{V}\left(\frac{1}{\overline{\nu}(x_0) U_j} \right) \mathbb{1}_{\{t \in I_{j;n}\}} \right)_{0 \leqslant t \leqslant n},$$

where N_n is Poisson with mean $\overline{\nu}(x_0) w_n$, $\{U_j\}_{j \in \mathbb{N}}$ are i.i.d. with U_1 uniformly distributed over $[0,1]$, and these two families are independent of $\{I_{j,n}\}_{j \in \mathbb{N}}$. Thus we see it is compound Poisson with characteristic function

$$\exp\left\{ \overline{\nu}(x_0) w_n \left(\widehat{P_n} - 1 \right) \right\}. \tag{4.110}$$

Here $\widehat{P_n}$ is the characteristic function of the law P_n given by

$$\left(\mu_n \times \frac{[\nu]_{(x_0,\infty)}}{\overline{\nu}(x_0)} \right) \circ H_n^{-1};$$

$$H_n : (\cup_{t=0}^n A_t) \times \mathbb{R} \to \mathbb{R}^{n+1}, \quad (y, x) \mapsto (\mathbb{1}_{\{y_t = 0\}} x)_{0 \leqslant t \leqslant n}.$$

It remains to check the characteristic function of left hand side of (4.68). Apply the Theorem 3.3.10 in Samorodnitsky (2016) to compute the weak characteristic triple, a tedious calculation shows

$$\mathbb{E}\exp\left\{i\sum_{t=0}^{n}\theta_t X_t^{(1)}\right\} = \exp\left\{w_n\overline{\nu}(x_0)\int_{\mathbb{R}^{n+1}}\left(e^{i\langle\theta,z\rangle}-1\right)P_n(dz)\right\}. \tag{4.111}$$

for any $\theta = (\theta_t)_{0\leqslant t\leqslant n} \in \mathbb{R}^{n+1}$. The match between (4.110) and (4.111) implies (4.68).

$\square$

We next check that the lognormal-type tails of Example 4.3.1 satisfy Assumption 4.3.1. The fact that $(\nu(1,\infty))^{-1}\nu(\cdot\cap(1,\infty))$ is a subexponential distribution follows from Theorem 4.1.17 in Samorodnitsky (2016). Next, let

$$\overline{H_\#}(x) = c_1 x^{\beta}(\log x)^{\xi}\exp\left(-\lambda(\log x)^{\gamma}\right)$$

for $x > x_0$ that is large enough so that this function is decreasing and c_1 is such that $\overline{H_\#}(x_0) = 1$. That is, (4.48) holds with

$$h(x) = \left(\frac{\lambda\gamma(\log x)^{\gamma-1}}{x} - \frac{\xi}{x\log x} - \frac{\beta}{x}\right)^{-1}.$$

Regular variation of h is clear, and so are the eventual positivity of h' and the fact that $\lim_{x\to\infty}h'(x) = 0$. In particular, $H_\#$ is in the maximum domain of attraction of the Gumbel distribution. Next, by the implicit function theorem, G is, for large values of the argument, of the form (4.51). The relation $\overline{H_\#}\circ G(x) = x^{-1}$ for $x > 1/\overline{H_\#}(x_0) := x_1$ means, in this case, that

$$c_1 G(x)^{\beta}(\log G(x))^{\xi}\exp\left(-\lambda(\log G(x))^{\gamma}\right) = x^{-1},$$

so $\log G(x) \sim (\log x/\lambda)^{1/\gamma}$ as $x\to\infty$. Denoting $g(x) = G'(x)$ we also have, for $x > x_1$,

$$\beta\frac{g(x)}{G(x)} + \xi\frac{g(x)}{G(x)\log G(x)} - \lambda\gamma\frac{(\log G(x))^{\gamma-1}g(x)}{G(x)} = -\frac{1}{x},$$

so that, as $x \to \infty$,

$$\zeta(x) = \frac{g(x)}{G(x)} x \log x \sim \gamma^{-1} \lambda^{-1/\gamma} (\log x)^{1/\gamma} \,. \tag{4.112}$$

The assumptions $(B1)$-$(B4)$ follow from (4.112).

Next we check that the super-lognormal-type tails of Example 4.3.2 satisfy Assumption 4.3.1. Once again, the fact that $(\nu(1,\infty))^{-1}\nu(\cdot \cap (1,\infty))$ is a subexponential distribution follows from Theorem 4.1.17 in Samorodnitsky (2016). Now we set

$$\overline{H_\#}(x) = c_1 x^\beta (\log x)^\xi \exp\left(\lambda (\log x)^\gamma\right) \exp\left(-\rho \exp\left(\mu (\log x)^\alpha\right)\right)$$

for or $x > x_0$ and appropriate x_0, c_1, and (4.48) holds with

$$h(x) = \left(\frac{\rho\alpha\mu(\log x)^{\alpha-1} \exp\left(\mu(\log x)^\alpha\right)}{x} - \frac{\lambda\gamma(\log x)^{\gamma-1}}{x} - \frac{\xi}{x \log x} - \frac{\beta}{x} \right)^{-1} .$$

All of the arguments we used in the previous example still work. In this case we have

$$\exp\left(\mu(\log G(x))^\alpha\right) \sim \log x/\rho \quad \text{as } x \to \infty,$$

so also $\log G(x) \sim (\log\log x/\mu)^{1/\alpha}$ as $x \to \infty$. Since

$$\begin{aligned} \beta \frac{g(x)}{G(x)} + \xi \frac{g(x)}{G(x)\log G(x)} &+ \lambda\gamma \frac{(\log G(x))^{\gamma-1} g(x)}{G(x)} \\ &- \rho\mu\alpha \exp\left(\mu(\log x)^\alpha\right) \frac{(\log G(x))^{\alpha-1} g(x)}{G(x)} = -\frac{1}{x} \,, \end{aligned}$$

we conclude that

$$\zeta(x) = \frac{g(x)}{G(x)} x \log x \sim \alpha^{-1} \mu^{-1/\alpha} \left(\log\log x\right)^{(1-\alpha)/\alpha} \quad \text{as } x \to \infty. \tag{4.113}$$

As before, the assumptions $(B1)$-$(B4)$ follow from (4.113).

Proof of Proposition 4.3.1.

(i) and (ii) follow by direct integration. To show (iii), we note that the derivative g of G satisfies $h \circ G(x) = xg(x)$ for all large x. Therefore, for large x,

$$\begin{aligned}\frac{h \circ G(H_1(x))}{h \circ G(H_2(x))} &= \frac{G(H_1(x))}{G(H_2(x))} \cdot \frac{\zeta(H_1(x))}{\zeta(H_2(x))} \cdot \frac{\log(H_2(x))}{\log(H_1(x))} \\ \sim& \frac{G(H_1(x))}{G(H_2(x))} \cdot \frac{\zeta(H_1(x))}{\zeta(H_2(x))} \gg \exp\left\{b(\log\log x)^{\delta}\right\}\end{aligned}$$

by Assumption 4.3.1 ($B2$), Potter's bounds and direct integration.

For part (iv), we only conside the case $\alpha > 0$. When $\alpha < 0$, a similar argument works. Write

$$G\left(x(\log x)^{\alpha}\right) - G(x) = \int_1^{(\log x)^{\alpha}} \frac{G(ux)\zeta(ux)}{u \log(ux)} du\,.$$

Dividing this identity by $h \circ G(x) = G(x)\zeta(x)/\log x$ gives us

$$\frac{G\left(x(\log x)^{\alpha}\right) - G(x)}{h \circ G(x)} = \int_1^{(\log x)^{\alpha}} \frac{G(ux)}{G(x)} \cdot \frac{\zeta(ux)}{\zeta(x)} \cdot \frac{\log x}{\log(ux)} \cdot \frac{du}{u}\,. \tag{4.114}$$

Denote $I = [1, (\log x)^{\alpha}]$. Clearly, $\log x \sim \log(ux)$ uniformly over $u \in I$. Further, by Assumption 4.3.1 $(B1), (B3)$, we see that $\zeta(x) \asymp \zeta(ux)$ uniformly over $u \in I$. Finally, for $u \in I$, by Assumption 4.3.1 ($B2$),

$$\begin{aligned}1 \le \frac{G(ux)}{G(x)} &= \exp\left\{\int_x^{ux} \frac{\zeta(v)}{v \log v} dv\right\} \\ &\le \exp\left\{C \int_x^{ux} \frac{1}{v \log\log v} dv\right\} \\ &\le \exp\left\{C \int_x^{x(\log x)^{\alpha}} \frac{1}{v \log\log v} dv\right\} \to e^{\alpha C}\,,\end{aligned}$$

where C is a constant. The claim now follows from (4.114) since

$$\int_1^{(\log x)^{\alpha}} \frac{du}{u} = \alpha \log\log x\,.$$

The argument for (v) is similar to that for (iv), We start with

$$\frac{G(x) - G\left(x2^{-j}\right)}{h \circ G(x)} = \int_{2^{-j}}^{1} \frac{G(xu)}{G(x)} \cdot \frac{\zeta(xu)}{\zeta(x)} \cdot \frac{\log x}{\log(xu)} \cdot \frac{du}{u}. \tag{4.115}$$

Denoting now $I = \big[2^{-\rho \log x/\zeta(x)}, 1\big]$. Due to $\zeta \to \infty$, it is clear that $\log x \sim \log(xu)$ uniformly over $u \in I$. Furthermore, $x2^{-\rho \log x/\zeta(x)} \to \infty$, so by Assumption 4.3.1 $(B1)$, $(B2)$ and $(B3)$,

$$\frac{\zeta(xu)}{\zeta(x)} \gtrsim \frac{\zeta\left(x2^{-\rho \log x/\zeta(x)}\right)}{\zeta(x)} \gtrsim \frac{\zeta\left(x2^{-\rho \log x/(\log\log x)^{\delta}}\right)}{\zeta(x)} \gtrsim 1\,,$$

unformly over $u \in I$. Finally, for $u \in I$, by Assumption 4.3.1 $(B1)$, $(B2)$, for some constant C,

$$\begin{aligned}\frac{G(ux)}{G(x)} &\gtrsim \frac{G\left(x2^{-\rho \log x/\zeta(x)}\right)}{G(x)} = \exp\left(-\int_{x2^{-\rho \log x/\zeta(x)}}^{x} \frac{\zeta(u)}{u\log u}du\right) \\ &\geqslant \exp\left(-C\zeta(x)\int_{x2^{-\rho \log x/\zeta(x)}}^{x} \frac{du}{u\log u}\right) \geqslant \exp(-2\rho C)\end{aligned}$$

for all x large, unformly over $u \in I$. Therefore, by (4.115) and Assumption 4.3.1 $(B2)$,

$$\frac{G(x) - G\left(x2^{-j}\right)}{jh \circ G(x)} \gtrsim 1\,,$$

as required. □

We finish by checking the claims made in Remark 4.4.1, and we start with the lognormal-type tails of Example 4.3.1. To see that (4.61) holds, it is enough to check that for any $C > 0$,

$$\lim_{n\to\infty} \frac{G\big(1/\overline{F}(n)\big)}{h \circ G(cw_n)} = 0\,. \tag{4.116}$$

The ratio above is asymptotic to

$$\frac{G\big(1/\overline{F}(n)\big)}{G(cw_n)}\frac{\log w_n}{\zeta(Cw_n)} = \exp\left(\int_{w_n}^{1/\bar{F}(n)} \frac{\zeta(u)}{u\log u}du\right)\frac{\log w_n}{\zeta(Cw_n)}\,, \tag{4.117}$$

which converges to 0 as $n \to \infty$ by (4.112). Next, for the super-lognormal-type tails of Example 4.3.2 with $0 < \alpha < 1/2$ one checks that (4.61) holds in the same way as above, by using (4.113) instead of (4.112). Finally, to see that (4.61) fails when $1/2 < \alpha < 1$, one needs to prove that, in this case, for any $C > 0$ the limit in (4.116) is infinity instead of 0. To do so one uses, once again, (4.117). It is routine to see that the expression there converges to infinity by using (4.113).

CHAPTER 5

A NEW SHAPE OF EXTREMAL CLUSTERS FOR CERTAIN STATIONARY SEMI-EXPONENTIAL PROCESSES WITH MODERATE LONG RANGE DEPENDENCE

5.1 Introduction

In this paper, we study the extremes of certain stationary infinitely divisible processes with long range dependence. The marginal log-tails are, roughly, of the order $-x^{-\alpha}$ for some $0 < \alpha < 1$. We call such tails semi-exponential, and the exact definition and the assumptions are given in Section 5.3. Such marginal tails are in the Gumbel maximum domain of attraction, and the assumptions we impose will also guarantee that these tails are subexponential.

Extremal limit theorems for such processes are interesting from several points of view. First of all, how do the extreme values of such processes cluster? Second, to what extent does the "single large jump" heuristic hold for such processes? This principle usually governs both extreme values and large deviations of weakly dependent subexponential stochastic systems. Extremal clusters appear when the values of the process at distinct time points are tail (asymptotically) dependent; see Resnick (2007). The size of extremal clusters is a subject of long-standing interest, and the notion of extremal index due to Davis (1982) and Leadbetter (1983) is specifically designed to quantify this size. When the stationary process has long memory affecting the extreme values of the process in the sense of Samorodnitsky (2016), the extremal clusters may become so large that scaling is necessary to obtain a finite limit, and then it becomes possible to talk about the limiting shape of an extremal cluster. For certain classes of stationary infinitely divisible processes with subexponential tails, this limiting shape is a random fractal, specifically a stable regenerative set, supporting one extreme value. This has been shown by Lacaux and Samorodnitsky (2016) in the case when the process has regularly varying tails (which are, of course, in the Fréchet maximum domain of attraction) and by Chen and Samorodnitsky (2020) in the case when the process has certain marginal tails in the Gumbel maximum domain of attraction. The results of the latter paper required, however, that these marginal tails were not too light. In particular, Chen and Samorodnitsky (2020) allowed lognormal-like marginal tails but excluded semi-exponential marginal tails, whose limiting shape of the extremal clusters remained unknown. In this paper, we solve this problem and characterize the limiting shape of the extremal clusters. While the clusters are still supported by stable regenerative sets, a new random structure appears, in which related but different extreme values are placed in randomly chosen locations over each stable regenerative set.

The new shape of the extremal clusters is related to a failure of the "single large jump" principle: the extreme values of the process are caused by multiple large values of the underlying Poisson random measure. Recall that a distribution H on $[0,\infty)$ is called subexponential if in the usual notation for distributional tails,

$$\lim_{x\to\infty} \frac{\overline{H*H}(x)}{\overline{H}(x)} = 2. \tag{5.1}$$

"Single large jump" is a widely used heuristic for processes with subexponential tails; see *e.g.* Foss et al. (2007). This principle already fails in the case of heavier tails in the Gumbel maximum domain of attraction, see Chen and Samorodnitsky (2020). But in the case of semi-exponential tails, this failure is even more dramatic. In one of our limit theorems we obtain a new limiting process of two parameters. It can be viewed as a bridge between the standard Gumbel extremal process of Resnick and Rubinovitch (1973) and the time-changed extremal process of Chen and Samorodnitsky (2020). When the parameters of the new process tend to some of their boundary values, either of the latter two processes can be recovered.

The paper is organized as follows. Section 5.2 reviews the main notions and tools we will use throughout the paper: random closed sets, null recurrent Markov chains, distributions in the Gumbel domain of attraction and random sup-measures. Section 5.3 describes the stationary infinitely divisible process whose extremes are to be analyzed. We state the two main extremal limit theorems, establishing weak convergence in the spaces of sup-measures and càdlàg functions respectively. This section also contains most of the proofs. Several auxiliary proofs are postponed to the two Appendices.

The adjective "moderate" decorating the term "long range dependence" in the title of the paper is due to the restricted range $\beta \in (0, 1/2)$ of the parameter responsible for the long memory. What happens if memory becomes even longer, *i.e.* $\beta \in (1/2, 1)$, remains a subject of future investigations, and we expect to get limit theorems with non-Gumbel limits. When the marginal tails are regularly varying, non-Fréchet limits in this range of β are established in Samorodnitsky and Wang (2019).

The following notation will be used throughout the paper. We denote the set of natural numbers $\{1, 2, \ldots\}$ by $\mathbb{N}$. For a nondecreasing function H on $\mathbb{R}$, the inverse of H is defined by $H^{\leftarrow}(x) = \inf\{s : H(s) \geqslant x\}$, with the usual convention $\inf \emptyset = \infty$. Further, the tail of a measure ν on $\mathbb{R}$ is $\overline{\nu}(x) = \nu(x, \infty)$. In particular $\overline{F}(x)$ is the tail of a distribution F. We will use the following symbols when comparing positive sequences.

(a) $a_n \sim b_n$ if $\lim_{n\to\infty} a_n/b_n = 1$,

(b) $a_n \lesssim b_n$ if there exist $C > 0$ such that $a_n \leqslant Cb_n$ for large enough n, and analogously with $a_n \gtrsim b_n$,

(c) $a_n \asymp b_n$ if both $a_n \lesssim b_n$ and $a_n \gtrsim b_n$.

If $\{A_n\}_{n\in\mathbb{N}}$ and $\{B_n\}_{n\in\mathbb{N}}$ are two sequences of positive random variables, we write

(a) $A_n = o_P(B_n)$ if $A_n/B_n \to 0$ in probability,

(b) $A_n \lesssim_P B_n$ if (A_n/B_n) is tight, and analogously with $A_n \gtrsim_P B_n$.

5.2 Preliminaries

5.2.1 Random Closed Sets

Random closed sets play a key role in many parts of this paper, particularly in the description of the main limiting objects. This section is an overview of mostly well-known facts about random closed sets. Unless stated otherwise, these facts are taken from Molchanov (2017).

We work with an underlying space E, which will be either $[0, 1]$ or $\mathbb{R}_+$. We write $\mathcal{G}, \mathcal{F}$, $\mathcal{F}'$, $\mathcal{K}$ and $\mathcal{K}'$ for the family of open, closed, nonempty closed, compact and nonempty compact sets in E, respectively. If we want to emphasize the choice of E, we will use a notation of the type $\mathcal{F}([0, 1])$. For any $A \subset E$, we define

$$\mathcal{F}^A = \{F \in \mathcal{F} : F \cap A = \emptyset\}, \tag{5.2}$$

$$\mathcal{F}_A = \{F \in \mathcal{F} : F \cap A \neq \emptyset\}. \tag{5.3}$$

The *Fell* topology on $\mathcal{F}$ is generated by the sub-basis consisting of $\{\mathcal{F}_G : G \in \mathcal{G}\}$ and $\{\mathcal{F}^K : K \in \mathcal{K}\}$, under which the space $\mathcal{F}$ is compact and metrizable. In the case $E = [0,1]$, $\mathcal{F} = \mathcal{K}$ and the Fell topology on $\mathcal{F}$ agrees with the so-called *myopic* topology on $\mathcal{K}$. In particular, $\mathcal{F}'$, with the subspace topology, is metrizable by the Hausdorff metric

$$\rho_{\mathrm{H}}(F_1, F_2) := \max\left\{\sup_{x\in F_1} \rho(x, F_2), \sup_{x\in F_2} \rho(F_1, x)\right\}, \quad F_1, F_2 \in \mathcal{F}', \tag{5.4}$$

where $\rho(\cdot,\cdot)$ is the standard distance function

$$\rho(x, F) = \min\{|x-y| : y \in F\}. \tag{5.5}$$

We will also use another common distance function

$$\rho(F_1, F_2) = \min\{|x-y| : x \in F_1,\, y \in F_2\}. \tag{5.6}$$

For either choice of E, a *random closed set* is a measurable mapping from a probability space $(\Omega, \mathscr{F}, \mathbb{P})$ to $(\mathcal{F}, \mathcal{B}(\mathcal{F}))$. A sequence of random closed sets $\{F_n\}_{n\in\mathbb{N}}$ weakly converges to F if

$$\lim_{n\to\infty} \mathbb{P}\{F_n \cap A \neq \emptyset\} = \mathbb{P}\{F \cap A \neq \emptyset\}, \quad \text{for any } A \in \mathcal{A} \cap \mathfrak{S}_F. \tag{5.7}$$

Here, $\mathcal{A}$ is the collection of all finite unions of open intervals, and $\mathfrak{S}_F$ is the collection of all continuity sets of F:

$$\Big\{B \in \mathcal{B} : B \text{ is relatively compact and } \mathbb{P}\{F \cap \mathrm{cl}\ B \neq \emptyset\} = \mathbb{P}\{F \cap \mathrm{int}\ B \neq \emptyset\}\Big\}. \tag{5.8}$$

We now introduce the random closed sets of our primary concern. For a $\beta \in (0, 1/2)$, let $\{Z(t)\}_{t\in\mathbb{R}_+}$ be a standard β-stable subordinator, which is an increasing Lévy process with the Laplace transform

$$\mathbb{E}\exp\{-\theta Z(t)\} = \exp\left\{-t\theta^{\beta}\right\}, \quad \theta \in \mathbb{R}_+. \tag{5.9}$$

The *β-stable regenerative set* R is the closure of the range of $\{Z(t)\}_{t\in\mathbb{R}_+}$,

$$R = \mathrm{cl}\left\{Z(t) : t \in \mathbb{R}_+\right\}. \tag{5.10}$$

Next, take a random variable $Z^*(0)$ independent of $\{Z(t)\}_{t\in\mathbb{R}_+}$ with the distribution

$$\mathbb{P}\{Z^*(0) \leqslant x\} = x^{1-\beta}, \quad x \in [0,1]. \tag{5.11}$$

We define the process

$$Z^*(t) = Z^*(0) + Z(t), \quad t \in \mathbb{R}_+, \tag{5.12}$$

whose range induces another random closed set

$$R^* := \mathrm{cl}\,\{Z^*(t) : t \in \mathbb{R}_+\} = Z^*(0) + R. \tag{5.13}$$

Let m^ϕ be the measure associated with the dimension (or gauge) function

$$\phi(x) = x^\beta (\log|\log x|)^{1-\beta}. \tag{5.14}$$

According to Theorem 1 in Taylor and Wendel (1966), there is a finite positive constant c_β such that on an event of probability 1,

$$m^\phi(R \cap [0,t]) = c_\beta Z^{\leftarrow}(t) \ \text{ for all } t \in \mathbb{R}_+. \tag{5.15}$$

Note that $Z^{\leftarrow}(\cdot)$ is a standard Mittag-Leffler process, which is self-similar with exponent β and has continuous sample paths. An immediate consequence of (5.15) is that on the same event of probability 1,

$$m^\phi(R^* \cap [0,t]) = c_\beta Z^{*\leftarrow}(t) \ \text{ for all } t \in \mathbb{R}_+. \tag{5.16}$$

In the sequel, we will be mostly interested in the restriction

$$\overline{R^*} := R^* \cap [0,1] \tag{5.17}$$

of R^* to the unit interval. Furthermore, we will need to sample points from this restriction according to the normalized measure m^ϕ on it. We now set up a technical framework for doing so. Assuming the random set R^* is defined on some probability space $(\Omega, \mathscr{F}, \mathbb{P})$, let $\{U_i\}_{i\geqslant 1}$ be a sequence of *i.i.d.* standard uniform random variables defined on another probability space, say, $(\Omega_1, \mathscr{F}_1, \mathbb{P}_1)$. For $\omega \in \Omega$ define

$$\eta_\omega : [0,1] \to [0,1], \quad t \mapsto \frac{m^\phi(R^*(\omega) \cap [0,t])}{m^\phi(R^*(\omega) \cap [0,1])} \tag{5.18}$$

if the denominator is positive, while setting $\eta_\omega(t) \equiv t$ for the ω in the zero probability event that the denominator vanishes. We define

$$J_i(\omega, \omega_1) = \eta_\omega^{\leftarrow}(U_i(\omega_1)),\ i = 1, 2, \ldots,\ (\omega, \omega_1) \in (\Omega \times \Omega_1), \tag{5.19}$$

and view $\big(\{Z^{*\leftarrow}(t)\}_{t\in\mathbb{R}_+}, \overline{R^*}, \{J_i\} : i \in \mathbb{N}\}\big)$ as a random element of the space $C[0,\infty) \times \mathcal{F}([0,1]) \times \big(\mathcal{F}([0,1])\big)^\infty$, defined on the product probability space $\big(\Omega \times \Omega_1, \mathscr{F} \times \mathscr{F}_1, \mathbb{P} \times \mathbb{P}_1\big)$. The law of this random element will be very important in the sequel. It follows from (5.16) that for $\mathbb{P} - a.s.\ \omega \in \Omega$

$$\mathbb{P}_1\left\{\cup_{i\geqslant 1}\{J_i(\omega,\omega_1)\} \subset \overline{R^*}\right\} = \mathbb{P}_1\left\{\mathrm{cl}\left(\cup_{i\geqslant 1}\{J_i(\omega,\omega_1)\}\right) = \overline{R^*}\right\} = 1, \tag{5.20}$$

Therefore also

$$\mathbb{P}\left\{\cup_{i\geqslant 1}\{J_i(\omega,\omega_1)\} \subset \overline{R^*}\right\} = \mathbb{P}\left\{\mathrm{cl}\left(\cup_{i\geqslant 1}\{J_i(\omega,\omega_1)\}\right) = \overline{R^*}\right\} = 1. \tag{5.21}$$

5.2.2 Null Recurrent Markov Chains

We introduce certain null recurrent Markov chains from an ergodic theoretic perspective. More details can be found in Aaronson (1997).

Let $\{Y_t\}_{t\in\mathbb{Z}}$ be an irreducible, aperiodic, and null recurrent Markov chain on $\mathbb{Z}$. We specify a unique invariant measure $(\pi_i)_{i\in\mathbb{Z}}$ by taking $\pi_0 = 1$. Let $(E, \mathcal{E})$ be the path space $(\mathbb{Z}^\mathbb{Z}, \mathcal{B}(\mathbb{Z}^\mathbb{Z}))$ and $\mathbb{P}_i(\cdot)$ the law $\mathbb{P}\{\cdot \mid Y_0 = i\}$ induced by the trajectories of the Markov chain on E. Setting

$$\mu(\cdot) = \sum_{i\in\mathbb{Z}} \pi_i \mathbb{P}_i(\cdot), \tag{5.22}$$

$$\theta : E \to E, \quad (\cdots, y_0, y_1, y_2, \cdots) \mapsto (\cdots, y_1, y_2, y_3, \cdots), \tag{5.23}$$

makes $(E, \mathcal{E}, \mu, \theta)$ a measure preserving, conservative and ergodic dynamical system, see Harris and Robbins (1953).

We consider the *first visit time* to 0,

$$\varphi(y) = \inf\{t \geqslant 1 : y_t = 0\}, \quad y \in E, \tag{5.24}$$

and adopt the following assumption.

Assumption 5.2.1. *For some $\beta \in (0, 1/2)$ and slowly varying function L,*

$$\overline{F}(n) := \mathbb{P}_0\{\varphi > n\} = n^{-\beta} L(n), \tag{5.25}$$

$$\sup_{n\geqslant 0} \frac{n\mathbb{P}_0\{\varphi = n\}}{\overline{F}(n)} < \infty. \tag{5.26}$$

Remark 5.2.1. *The main results of the paper require the assumption (5.26), see Theorem B in Doney (1997).*

The *wandering rate sequence* $\{w_n\}_{n\geqslant 0}$ is defined by

$$w_n = \mu\left\{\cup_{k=0}^{n} A_k\right\} \quad \text{with} \quad A_n = \{y \in E : y_n = 0\}, \ n \in \mathbb{N}_0. \tag{5.27}$$

Under (5.25) it follows from Lemma 3.3 in Resnick et al. (2000) that

$$w_n \sim n^{1-\beta} L(n) \big/ (1 - \beta). \tag{5.28}$$

The extreme value analysis in this paper requires certain additional delicate details hidden in $(E, \mathcal{E}, \mu, \theta)$. For each $n \in \mathbb{N}_0$, we define a probability measure on E by

$$\mu_n(\cdot) = \mu\big(\cdot \cap \bigcup_{k=0}^{n} A_k\big) \big/ w_n. \tag{5.29}$$

Let $\{Y^{(k;n)}\}_{k\in\mathbb{N}_0}$ be *i.i.d.* random elements in E with law μ_n. We are interested in the (random) zero sets

$$I_{k;n} = \{0 \leqslant t \leqslant n : Y_t^{(k;n)} = 0\},\ k \in \mathbb{N}_0, \tag{5.30}$$

and their intersections. For fixed $n, k \in \mathbb{N}$ we define

$$j_{k,1;n} = \inf\{j > k : I_{j;n} \cap I_{k;n} \neq \emptyset\} \tag{5.31}$$

and continue inductively by setting for $i \geq 2$,

$$j_{k,i;n} = \inf\{j > j_{k,i-1;n} : I_{j;n} \cap I_{k;n} \neq \emptyset\} \tag{5.32}$$

if $j_{k,i-1;n} < \infty$ and $j_{k,i;n} = \infty$ otherwise. For $i \geq 1$, on the event $\{j_{k,i;n} < \infty\}$ we define

$$I_{k,i;n} = I_{k;n} \cap I_{j_{k,i;n};n}. \tag{5.33}$$

For $k \in \mathbb{N}$, consider the random probability

$$\overline{p}_{k;n} := \mathbb{P}\left\{I_{k;n} \cap I_{0;n} \neq \emptyset \,\middle|\, I_{k;n}\right\}, \tag{5.34}$$

and note that, conditionally on $I_{k;n}$, $j_{k,1;n} - k$ is geometrically distributed with success probability $\overline{p}_{k;n}$. The following theorem is interesting on its own right. We precede it with some notation. Let $\left(\{Z_k^{*\leftarrow}(t)\}_{t\in\mathbb{R}_+}, \overline{R_k^*}, \{J_{k,i}\} : i \in \mathbb{N}\right)$, $k = 1, 2, \ldots$ be *i.i.d.* copies of the random element $\left(\{Z^{*\leftarrow}(t)\}_{t\in\mathbb{R}_+}, \overline{R^*}, \{J_i\} : i \in \mathbb{N}\right)$ in $C[0,\infty) \times \mathcal{F}([0,1]) \times \left(\mathcal{F}([0,1])\right)^\infty$ constructed in Section 5.2.1. Let $\mathbf{\Gamma}_k$, $k = 1, 2, \ldots$ be an independent of them i.i.d. sequence of unit rate Poisson processes on $(0,\infty)$. That is, each $\mathbf{\Gamma}_k = \{\Gamma_{k,i}\}_{i\in\mathbb{N}}$ consists of the arrival times of a unit rate Poisson processes on $(0,\infty)$ (listing the points of $\mathbf{\Gamma}_k$ in the increasing order).

Theorem 5.2.1. *Under Assumption 5.2.1 there is a constant* $c_\infty \in (0,1)$ *such* *that for any* $K, m \in \mathbb{N}$

$$\left(\frac{w_n}{\vartheta_n}\overline{p}_{k;n}, \left(j_{k,i;n}\overline{p}_{k;n}\right)_{1\leqslant i\leqslant m}, \frac{1}{n}I_{k;n}, \left(I_{k,i;n}/n\right)_{1\leqslant i\leqslant m}\right)_{1\leqslant k\leqslant K} \tag{5.35}$$

$$\Rightarrow \left(c_\infty Z_k^{*\leftarrow}(1), \left(\Gamma_{k,i}\right)_{1\leqslant i\leqslant m}, \overline{R_k^*}, \{J_{k,i}\}_{1\leqslant i\leqslant m}\right)_{1\leqslant k\leqslant K}$$

as $n \to \infty$*, weakly in the space* $\left(\mathbb{R}_+^{1+m} \times (\mathcal{F}([0,1]))^{1+m}\right)^K$*, where*

$$\vartheta_n = \frac{(2-\beta)n^\beta}{\beta L(n)}, \ n \in \mathbb{N}. \tag{5.36}$$

This theorem is proved in Appendix 5.4.1, and so is the following proposition that establishes an exponential integrability of $\#I_{1,1;n}$ in both annealed and quenched situations.

Proposition 5.2.1. *(i) Let* $c_\infty \in (0,1)$ *be the constant in Theorem 5.2.1. Then*

$$\mathbb{P}\left\{I_{1,1;n} \geqslant m\right\} \leqslant (1-c_\infty)^m, \quad m = 1, 2, \ldots. \tag{5.37}$$

(ii) Let $I_{1;n}$ *be defined on* $(\Omega, \mathscr{F}, \mathbb{P})$*. Then for any* $\epsilon > 0$*, there is* $\delta = \delta(\epsilon) > 0$*,* $C = C(\epsilon) > 0$*, and an event* $\Omega_n \subset \Omega$ *satisfying* $\mathbb{P}\{\Omega_n\} \geqslant 1-\epsilon$ *such that*

$$\sup_{\omega\in\Omega_n} \mathbb{E}_\omega e^{\delta\cdot\#I_{1,1;n}} \leqslant C, \tag{5.38}$$

where $\mathbb{E}_\omega$ *denotes the conditional expectation* $\mathbb{E}\{\cdot \mid I_{1,n}(\omega)\}$*.*

The last proposition of this subsection is adapted from (A.3) and (A.9) of Chen and Samorodnitsky (2020).

Proposition 5.2.2. *For any* $p > 0$ *there is* $\mu_p < \infty$ *such that*

$$\sup_{n\geqslant 1} \mathbb{E}\left(\overline{F}(n)\#I_{1;n}\right)^p \leq \mu_p. \tag{5.39}$$

Further, for any $C > 0$ *there is* $c > 0$ *so that for all* $n \in \mathbb{N}$*,*

$$\mathbb{P}\left\{\#I_{1;n} \geqslant \frac{c\log n}{\overline{F}(n)}\right\} \leqslant n^{-C}. \tag{5.40}$$

5.2.3 Distributions in the Gumbel maximum domain of attraction

Recall that a distribution H, with an unbounded support on the right, is in the Gumbel maximum domain of attraction if and only if there exist $x_0 \in \mathbb{R}$ and $c(x) \to c > 0$ as $x \to \infty$ such that for $x_0 < x < \infty$

$$\overline{H}(x) = c(x) \exp\left\{ - \int_{x_0}^{x} \frac{1}{h(u)} du \right\} \tag{5.41}$$

where h (the so-called auxiliary function) is an absolutely continuous positive function on (x_0, ∞) with density h' satisfying $\lim_{u\to\infty} h'(u) = 0$; we refer the reader to Resnick (1987) and Goldie and Resnick (1988) for more details. The function h must satisfy $h(x) = o(x)$ as $x \to \infty$; if the distribution H is also subexponential, then its support is unbounded on the right and $\lim_{u\to\infty} h(u) = \infty$.

For a distribution H satisfying (5.41), the centering and scaling required for convergence in the extremal limit theorem can be chosen as

$$b_n = \left(\frac{1}{1-H} \right)^{\leftarrow} (n), \;\; a_n = h(b_n) \,.$$

We will often use the following fact: if one replaces the function $c(\cdot)$ in (5.41) by an asymptotically equivalent function, and denotes the new normalizing sequences by $(\tilde{a}_n)$ and $(\tilde{b}_n)$, then

$$\lim_{n\to\infty} \frac{b_n - \tilde{b}_n}{a_n} = 0, \;\; \lim_{n\to\infty} \frac{\tilde{a}_n}{a_n} = 1 \,. \tag{5.42}$$

5.2.4 Random Sup-Measures

We will deal with sup-measures taking values in $\overline{\mathbb{R}} = [-\infty, \infty]$. The main reference is O'Brien et al. (1990).

A sup-measure is a mapping $m : \mathcal{G} \to \overline{\mathbb{R}}$ such that $m(\emptyset) = -\infty$ and $m(\cup_\alpha G_\alpha) = \vee_\alpha m(G_\alpha)$ for an arbitrary collection (G_α) of open sets. The sup-derivative $d^\vee m$ of m is

$$d^\vee m(t) = \bigwedge_{t \in G} m(G);$$

it is automatically an upper semicontinuous $\overline{\mathbb{R}}$-valued function of t. Given any $\overline{\mathbb{R}}$-valued function f, the sup-integral of f

$$i^\vee f(G) = \bigvee_{t \in G} f(t), \quad G \in \mathcal{G}$$

is a sup-measure. The domain of a sup-measure can be extended to all Borel sets via

$$m(B) = \bigvee_{t \in B} d^\vee(t), \quad B \text{ Borel.}$$

The collection SM of sup-measures admits a natural metrizable sup-vague topology with its corresponding Borel measurability, which allows one to talk about random sup-measures. In particular, if $\{\mathcal{M}_n\}_{n \geqslant 1}$ and $\mathcal{M}$ are random sup-measures, then $\mathcal{M}_n \Rightarrow \mathcal{M}$ if and only if

$$(\mathcal{M}_n(I_1), \ldots, \mathcal{M}_n(I_m)) \Rightarrow (\mathcal{M}(I_1), \ldots, \mathcal{M}(I_m)) \tag{5.43}$$

for arbitrarily disjoint open intervals $I_1, \ldots, I_m$ such that $\mathbb{P}\{\mathcal{M}(I_i) = \mathcal{M}(\overline{I_i})\} = 1$ for all $i = 1, \ldots, m$.

A stochastic process $\{X_t\}_{t \in \mathbb{N}}$ induces a family of random sup-measures $\{\mathcal{M}_n(\cdot)\}_{n \geqslant 1}$ via

$$\mathcal{M}_n(B) := \max_{t \in nB} X_t, \quad B \in \mathcal{B}(E), \tag{5.44}$$

We now describe the limiting random sup-measures appearing in our main results. The construction is doubly stochastic.

First, let $\alpha \in (0, 1)$ and $\beta \in (0, 1/2)$, and denote by P_β the law of the closed range R of the β-stable subordinator in (5.10). Consider a Poisson point process

$\mathcal{N}$ on $\mathbb{R} \times \mathbb{R}_+ \times \mathcal{F}(\mathbb{R}_+)$ with mean measure

$$e^{-x}dx \times (1-\beta)y^{-\beta}dy \times dP_\beta,$$

defined on some probability space $(\Omega_1, \mathscr{F}_1, \mathbb{P}_1)$, and let $(X_k, Y_k, R_k)_{k\in\mathbb{N}}$ be a measurable enumeration of its points. For each point $(X_k, Y_k, R_k) = (X_k(\omega_1), Y_k(\omega_1), R_k(\omega_1))$ the dimension function ϕ in (5.14) produces a (random) measure

$$H_k(t, \omega_1) = m^\phi \left\{ \big(Y_k(\omega_1) + R_k(\omega_1)\big) \cap [0,t] \right\}, \quad t \in \mathbb{R}_+ \tag{5.45}$$

on $\mathbb{R}_+$. The second level of randomness is now introduced, conditionally on this point $(X_k(\omega_1), Y_k(\omega_1), R_k(\omega_1))$, via a Poisson point process $\mathbb{C}_k = \mathbb{C}_k(\omega_1)$ on $\mathbb{R}\times\mathbb{R}_+$ with the mean measure

$$\frac{C_{\alpha,\beta}}{c_\beta} \exp\left\{-C_{\alpha,\beta}\big(\lambda - X_k(\omega_1)\big)\right\} d\lambda \times dH_k(\cdot, \omega_1),$$

where

$$C_{\alpha,\beta} := \left(\frac{1-\beta}{\beta}\right)^{\frac{1}{\alpha}-1} > 1, \tag{5.46}$$

and c_β is the constant in (5.15). We assume that the point processes $(\mathbb{C}_k)$ live on some other probability space $(\Omega_2, \mathscr{F}_2, \mathbb{P}_2)$ and, conditionally on $\mathcal{N}$, are independent of each other. The overall probability space $(\Omega, \mathscr{F}, \mathbb{P})$ is the product space $(\Omega_1 \times \Omega_2, \mathscr{F}_1 \times \mathscr{F}_2, \mathbb{P}_1 \times \mathbb{P}_2)$.

For $k \in \mathbb{N}$, let $\{\Lambda_{k,i}, W_{k,i}\}_{i\in\mathbb{N}}$ be a measurable enumeration of the points of $\mathbb{C}_k$. We note that the points $\{W_{k,i}\}_{i\in\mathbb{N}}$ belong to the set $Y_k + R_k$ with probability 1. Consider a random sup-measure $\mathcal{M}$ on $\mathbb{R}_+$ given by

$$\mathcal{M}(B) = \sup_{k,i\in\mathbb{N}} \left\{\Lambda_{k,i} : W_{k,i} \in B\right\}, \quad B \in \mathbb{R}_+. \tag{5.47}$$

This sup-measure has certain invariance properties. First of all, it is clear that $\mathcal{M}$ can be represented in the form $\mathcal{M} = \varphi\big(\mathcal{N}(\omega_1), \omega_2\big)$ for some measurable function

φ, and the mean measure of $\mathcal{N}$ is invariant under positive shifts of the closed sets. Hence the stationarity of the $\mathcal{M}$:

$$\mathcal{M}(r+\cdot) \overset{\mathrm{d}}{=} \mathcal{M}(\cdot) \quad \text{for any} \quad r \geqslant 0;$$

cf. Proposition 4.3 in Lacaux and Samorodnitsky (2016). Next, $\mathcal{M}$ has a self-affiness property: for $a > 0$,

$$\mathcal{M}(a\cdot) \overset{\mathrm{d}}{=} \mathcal{M}(\cdot) + (1-\beta+\beta/C_{\alpha,\beta}) \log a,$$

which will be established in Remark 5.2.2 below.

The random sup-measure $\mathcal{M}$ naturally induces a stochastic process of independent interest by

$$\mathbb{M}(t) = \mathcal{M}([0,t]), \quad t \in (0,\infty). \tag{5.48}$$

This process is clearly nondecreasing, continuous in probability, and the sample paths are in $D(0,\infty) = \cap_{\epsilon>0} D[\epsilon,\infty)$. The finite-dimensional distributions can be read off the following proposition, which implies that $\mathbb{M}(t) \to -\infty$ as $t \downarrow 0$.

Proposition 5.2.3. *Let* $0 = t_0 < t_1 < \cdots < t_k < \infty$. *Then for any* $x_i \in \mathbb{R}, i = 1, \ldots, k$,

$$\mathbb{P}\left(\mathbb{M}\big((t_{i-1},t_i]\big) \leqslant x_i,\, i=1,\ldots,k\right) = \exp\Bigg\{-\Gamma\big(1-1/C_{\alpha,\beta}\big) \tag{5.49}$$
$$\int_0^\infty (1-\beta)y^{-\beta}\mathbb{E}\left[\sum_{i=1}^k e^{-C_{\alpha,\beta}x_i}\left[Z^{\leftarrow}\big((t_i-y)\big)_+ - Z^{\leftarrow}\big((t_{i-1}-y)_+\big)\right]\right]^{1/C_{\alpha,\beta}} dy\Bigg\}.$$

In particular, for any $t > 0$ *and* $x \in \mathbb{R}$ *we have*

$$\mathbb{P}\left\{\mathbb{M}(t) \leqslant x\right\} = \exp\left\{-K(\alpha,\beta)t^{1-\beta+\beta C_{\alpha,\beta}^{-1}}e^{-x}\right\}, \tag{5.50}$$

where

$$K(\alpha,\beta) = (1-\beta)\Gamma\big(1-1/C_{\alpha,\beta}\big)B\big(1-\beta, 1+\beta/C_{\alpha,\beta}\big)\mathbb{E}\big(Z^{\leftarrow}(1)\big)^{1/C_{\alpha,\beta}}.$$

Here $\Gamma(\cdot)$ *and* $B(\cdot,\cdot)$ *are, respectively, the Gamma function and the Beta function.*

Remark 5.2.2. Leting $t \downarrow 0$ in (5.50) shows that $\mathbb{M}(t) \to -\infty$ as $t \downarrow 0$. Next, let $a > 0$. By (5.49) and the self-similarity of $Z^{\leftarrow}$,

$$\begin{aligned}
&\mathbb{P}\left(\mathbb{M}\big((at_{i-1}, at_i]\big) \leqslant x_i,\, i = 1, \ldots, k\right) \\
=&\exp\Bigg\{-a^{\beta/C_{\alpha,\beta}}\Gamma\big(1 - 1/C_{\alpha,\beta}\big)\int_0^\infty (1-\beta)y^{-\beta} \\
&\qquad \mathbb{E}_1\left[\sum_{i=1}^k e^{-C_{\alpha,\beta}x_i}\big[Z^{\leftarrow}\big((t_i - y/a)\big)_+ - Z^{\leftarrow}\big((t_{i-1} - y/a)_+\big)\big]\right]^{1/C_{\alpha,\beta}} dy\Bigg\} \\
=&\exp\Bigg\{-a^{1-\beta+\beta/C_{\alpha,\beta}}\Gamma\big(1 - 1/C_{\alpha,\beta}\big)\int_0^\infty (1-\beta)y^{-\beta} \\
&\qquad \mathbb{E}_1\left[\sum_{i=1}^k e^{-C_{\alpha,\beta}x_i}\big[Z^{\leftarrow}\big((t_i - y)\big)_+ - Z^{\leftarrow}\big((t_{i-1} - y)_+\big)\big]\right]^{1/C_{\alpha,\beta}} dy\Bigg\} \\
=&\mathbb{P}\left(\mathbb{M}\big((t_{i-1}, t_i]\big) + (1 - \beta + \beta/C_{\alpha,\beta})\log a \leqslant x_i,\, i = 1, \ldots, k\right).
\end{aligned}$$

Hence the self-affiness of $\mathbb{M}$.

Proof of Proposition 5.2.3. We view the collection $\{\Lambda_{k,i}, W_{k,i}\}_{k,i\in\mathbb{N}}$ is a function of the points of the Poisson process $\mathcal{N}$ and their *i.i.d.* marks defined on $(\Omega_2, \mathscr{F}_2, \mathbb{P}_2)$. Since a marked Poisson process is still Poisson, and the event described in the left hand side of (5.49) is the event that the marked Poisson process has no points in a part of its domain, we see that

$$\begin{aligned}
&\mathbb{P}\left(\mathbb{M}\big((t_{i-1}, t_i]\big) \leqslant x_i,\, i = 1, \ldots, k\right) \\
=&\exp\Bigg\{-\int_{-\infty}^\infty e^{-z}dz\int_0^\infty (1-\beta)y^{-\beta}dy \\
&\qquad \mathbb{E}_1\Big[\mathbb{P}_2\big\{\mathbb{C}_{z,y}\big(\cup_{i=1}^k\big(x_i,\infty) \times (t_{i-1}, t_i]\big) > 0\big\}\Big]\Bigg\},
\end{aligned}$$

where, given a defined on $(\Omega_1, \mathscr{F}_1, \mathbb{P}_1)$ β-stable regenerative set R, the process $\mathbb{C}_{z,y}$ is a defined on $(\Omega_2, \mathscr{F}_2, \mathbb{P}_2)$ Poisson process on $\mathbb{R} \times \mathbb{R}_+$ with mean measure

$$\frac{C_{\alpha,\beta}}{c_\beta}\exp\big\{-C_{\alpha,\beta}\big(\lambda - z\big)\big\}\, d\lambda \times dm^\phi\left\{\big(y + R\big) \cap [0, \cdot]\right\}.$$

Therefore,

$$\begin{aligned}&\mathbb{P}_2\big\{\mathbb{C}_{z,y}\big(\cup_{i=1}^k\big(x_i,\infty)\times(t_{i-1},t_i]\big)>0\big\}\\ =&1-\mathbb{P}_2\big\{\mathbb{C}_{z,y}\big(\cup_{i=1}^k\big(x_i,\infty)\times(t_{i-1},t_i]\big)=0\big\}\\ =&1-\exp\left\{-c_\beta^{-1}\sum_{i=1}^k e^{-C_{\alpha,\beta}(x_i-z)}m^\phi\Big\{\big(y+R\big)\cap(t_{i-1},t_i]\Big\}\right\},\end{aligned}$$

and we conclude by (5.15) that

$$\begin{aligned}&\mathbb{E}_1\big[\mathbb{P}_2\big\{\mathbb{C}_{z,y}\big(\cup_{i=1}^k\big(x_i,\infty)\times(t_{i-1},t_i]\big)>0\big\}\\ =&\mathbb{E}_1\left(1-\exp\left\{-\sum_{i=1}^k e^{-C_{\alpha,\beta}(x_i-z)}\big[Z^{\leftarrow}\big((t_i-y)\big)_+-Z^{\leftarrow}\big((t_{i-1}-y)_+)\big]\right\}\right),\end{aligned}$$

and (5.49) follows by simple integration. Finally, using (5.49) with $k=1$ and $t_1=t, x_1=x$ we obtain

$$\mathbb{P}\left\{\mathbb{M}(t)\leqslant x\right\}=\exp\left\{-\Gamma\big(1-1/C_{\alpha,\beta}\big)e^{-x}\int_0^t(1-\beta)y^{-\beta}\mathbb{E}Z^{\leftarrow}(t-y)^{1/C_{\alpha,\beta}}dy\right\},$$

and (5.50) follows by the self-similarity of $Z^{\leftarrow}$ and simple integration. □

We can obtain an explicit representation of the restriction of the sup-measure $\mathcal{M}$ to $[0,1]$.

First, the restriction of the Poisson point process $(X_k,Y_k+R_k)_{k\in\mathbb{N}}$ to $\mathbb{R}\times\mathcal{K}'([0,1])$ (we only need to look at nonempty compact sets) can be represented as a Poisson point process $\mathcal{N}_0$ on $\mathbb{R}$ with the mean measure $e^{-x}dx$, marked by *i.i.d.* copies of the random closed set $\overline{R^*}$ in (5.17). The markings are independent of $\mathcal{N}_0$. The k^{th} copy $\overline{R_k^*}$ is associated via (5.16) with a shifted stable subordinator (Z_k^*) satisfying for $t\in[0,1]$,

$$m^\phi(\overline{R_k^*}\cap[0,t])=c_\beta Z_k^{*\leftarrow}(t).$$

Furthermore, the process $\mathcal{N}_0$ itself can be represented by the points $-\log\Gamma_k$, $k=1,2,\ldots$, where (Γ_k) form a standard unit rate Poisson process on $\mathbb{R}_+$.

Second, for a fixed k the mean measure of the Poisson point process $\mathbb{C}_k$ can be rewritten in the form

$$\begin{aligned}&\frac{C_{\alpha,\beta}}{c_\beta} m^\phi(\overline{R_k^*} \cap [0,1]) \exp\left\{-C_{\alpha,\beta}\big(\lambda - X_k(\omega_1)\big)\right\} d\lambda \times d\eta_k(\cdot,\omega_1)\\ =&C_{\alpha,\beta} Z_k^{*\leftarrow}(1) \exp\left\{-C_{\alpha,\beta}\big(\lambda - X_k(\omega_1)\big)\right\} d\lambda \times d\eta_k(\cdot,\omega_1),\end{aligned}$$

where $\eta_k(\cdot,\omega_1)$ is the (ω_1-dependent) probability measure (5.18) associated with (Z_k^*). Therefore, we can choose a measurable enumeration of the points of $\mathbb{C}_k$ by first selecting a measurable enumeration $\{\Lambda_{k,i}\}_{i\in\mathbb{N}}$ of the points of the Poisson point process on $\mathbb{R}$ with the mean measure

$$C_{\alpha,\beta} Z_k^{*\leftarrow}(1) \exp\left\{-C_{\alpha,\beta}\big(\lambda - X_k(\omega_1)\big)\right\} d\lambda$$

and then attaching to these points independent *i.i.d.* marks with the common law $\eta_k(\cdot,\omega_1)$. Since the former Poisson process is, once again, easily generated as a transformation of a unit rate Poisson process on $\mathbb{R}_+$ (say, $(\Gamma_{k,i})$), we conclude that we can choose a measurable enumeration of the points of $\mathbb{C}_k$ so that, in law,

$$(\Lambda_{k,i}, W_{k,i})_{k,i\in\mathbb{N}} = \left(-\log\Gamma_k + \frac{1}{C_{\alpha,\beta}}\Big(-\log\Gamma_{k,i} + \log Z_k^{*\leftarrow}(1)\Big), J_{k,i}\right)_{k,i\in\mathbb{N}}, \quad (5.51)$$

where $\{\Gamma_k\}_{k\in\mathbb{N}}$ is a unit rate Poisson process on $\mathbb{R}$ independent of the rest random elements that are defined in Theorem 5.2.1.

5.3 Extremal Limit Theorems for Stationary Semi-Exponential Processes

We focus on stationary infinitely divisible processes of form

$$X_t = \int_E 1_{A_0} \circ \theta^t(x) M(dx), \quad t \in \mathbb{Z}, \quad (5.52)$$

where $(E, \mathcal{E}, \mu, \theta)$ is the dynamical system described in (5.22)-(5.23), $A_0 = \{y \in E : y_0 = 0\}$, and M is an infinitely divisible random measure on $(E, \mathcal{E})$ with control measure μ and with constant local characteristic triple (σ^2, ν, b); see Samorodnitsky (2016). We note that the choice of the indicator function in (5.52) is mainly for convenience, and more general integrands could be considered.

The processes we consider have the marginal tails that are both subexponential and in the Gumbel maximum domain of attraction. They will also have a certain semi-exponential decay, Correspondingly, we choose the auxiliary function h in (5.41) to be of a specific type. The assumptions are imposed through the local Lévy measure ν of the infinitely divisible random measure.

Assumption 5.3.1. *For some* $\alpha \in (0,1)$ *and* $\gamma > 0$*,*

$$\overline{\nu}(x) := \nu\big((x,\infty)\big) \sim \gamma \overline{H}(x) := \gamma \exp\left\{-\int_1^x \frac{du}{u^{1-\alpha} L_\alpha(u)}\right\}, \ x \to \infty \tag{5.53}$$

for a differentiable slowly varying function L_α *such that*

$$L_\alpha'(x) = o\left(L_\alpha(x)/x\right). \tag{5.54}$$

For large values of the argument both $\overline{\nu}$ and $\overline{H}$ can be viewed as distributional tails. Automatically, these distributions are in the Gumbel maximum domain of attraction. Additionally, by Theorem 2 in Pitman (1980), these distributions are also subexponential. We conclude that the distribution of $X_0 \in D(\Lambda) \cap \mathcal{S}$ since $\mathbb{P}\{X_0 > x\} \sim \overline{\nu}(x)$, see Theorem 1 in Embrechts et al. (1979).

To state the main extremal limit theorems we define two functions by

$$V(x) = \big(1/\overline{\nu}\big)^{\leftarrow}(x), \quad h(x) = x^{1-\alpha} L_\alpha(x), \quad x \geqslant 1. \tag{5.55}$$

Some properties of these and other important functions are described in Proposition 5.4.4 in the Appendix B.

The normalizing constants in the extremal limit theorems are given by

$$b_n = V(w_n) + V(c_\infty \vartheta_n), \quad a_n = h \circ V(w_n), \tag{5.56}$$

where $\{w_n\}_{n\geqslant 1}$ is the wandering rate sequence in (5.27), $\{\vartheta_n\}_{n\geqslant 1}$ is the sequence in (5.36), and c_∞ is the constant in Theorem 5.2.1, given explicitly in (5.131).

The first extremal limit theorem establishes convergence in the space of random sup-measures.

Theorem 5.3.1. *Let $\{X_t\}_{t\in\mathbb{Z}}$ be the stationary infinitely divisible process defined by (5.52), such that* (5.53) *and* (5.54) *hold. Assume also that the dynamical system $(E, \mathcal{E}, \mu, \theta)$ satisfies the Assumptions 5.2.1. Then the random sup-measures defined by (5.44) satisfy*

$$\frac{\mathcal{M}_n(\cdot) - b_n}{a_n} \Rightarrow \mathcal{M}(\cdot) \quad \textit{in } SM([0,\infty)), \tag{5.57}$$

where $\mathcal{M}$ is given in (5.47).

The second extremal limit theorem establishes convergence in the functional space $D(0,\infty)$.

Theorem 5.3.2. *Under the assumptions of Theorem 5.3.1, let $\mathbb{M}_n(t) = \max_{i\leqslant nt} X_i$, $t \in \mathbb{R}_+$, $n = 1, 2, \ldots$. Then*

$$\left\{\frac{\mathbb{M}_n(t) - b_n}{a_n}\right\}_{t>0} \Rightarrow \{\mathbb{M}(t)\}_{t>0} \quad \textit{in } \big(D(0,\infty), J_1\big), \tag{5.58}$$

where $\{\mathbb{M}(t)\}_{t>0}$ is the stochastic process in (5.48).

Remark 5.3.1. It follows from (5.50) that the limiting process in Theorem 5.3.2 has one dimensional marginal distributions equal to the one dimensional marginal distributions of the process $\big(X_G\big(t^{1-\beta+\beta/C_{\alpha,\beta}}\big),\, t \geq 0\big)$, where $(X_G(t),\, t \geq 0)$ is a shifted Gumbel extremal process, i.e. a nondecreasing process satisfying

$$\mathbb{P}\big(X_G(t_i) \leq x_i,\, i = 1, \ldots, k\big) = \exp\left\{-K(\alpha,\beta)\sum_{i=1}^{k}(t_i - t_{i-1})e^{-x_i}\right\} \tag{5.59}$$

for $0 = t_0 < t_1 < \cdots < t_k$ and $x_1 \le x_2 \le \cdots \le x_k$.

We recall that in Chen and Samorodnitsky (2020), similar extremal limit theorems are proved for stationary processes with marginal tails heavier than the ones in Theorem 5.3.2. The limiting processes therein are, in distribution, the power time changes of the standard Gumbel extremal process. Analogous time-changed results for Fréchet extremal processes are showed in Owada and Samorodnitsky (2015b) and Lacaux and Samorodnitsky (2016).

Interestingly, the law of the limiting process in Theorem 5.3.2 is different from the law of the power time change of the Gumbel extremal process in (5.59). Indeed, if the two processes had the same law, we would have by (5.49) and (5.50), for $0 < t_1 < t_2$ and $x_1 \le x_2$,

$$
\begin{aligned}
&\exp\left\{-K(\alpha,\beta)\left[t_1^{1-\beta+\beta/C_{\alpha,\beta}}e^{-x_1} + (t_2^{1-\beta+\beta/C_{\alpha,\beta}} - t_1^{1-\beta+\beta/C_{\alpha,\beta}})e^{-x_2}\right]\right\}\\
=&\mathbb{P}\left(X_G\big(t_1^{1-\beta+\beta/C_{\alpha,\beta}}\big) \le x_1, X_G\big(t_2^{1-\beta+\beta/C_{\alpha,\beta}}\big) \le x_2\right)\\
=&\mathbb{P}\left(M(t_1) \le x_1,\, M\big((t_1,t_2] \le x_2\big)\right.\\
=&\exp\bigg\{-\Gamma\big(1-1/C_{\alpha,\beta}\big)\int_0^\infty (1-\beta)y^{-\beta}dy\,\mathbb{E}\Big[e^{-C_{\alpha,\beta}x_1}Z^{\leftarrow}\big((t_1-y)\big)_+\\
&\quad + e^{-C_{\alpha,\beta}x_2}\Big[Z^{\leftarrow}\big((t_2-y)\big)_+ - Z^{\leftarrow}(t_1-y)_+\Big]\Big]^{1/C_{\alpha,\beta}}\bigg\},
\end{aligned}
$$

and due to the connection between the constants in (5.49) and (5.50) this reduces to

$$
\begin{aligned}
&\mathbb{E}\int_0^\infty (1-\beta)y^{-\beta}dy\left(e^{-C_{\alpha,\beta}x_1}Z^{\leftarrow}\big((t_1-y)\big)_+\right)^{1/C_{\alpha,\beta}}\\
&+\mathbb{E}\int_0^\infty (1-\beta)y^{-\beta}dy\left[\left(e^{-C_{\alpha,\beta}x_2}Z^{\leftarrow}\big((t_2-y)\big)_+\right)^{1/C_{\alpha,\beta}}\right.\\
&\qquad\qquad\qquad\qquad\left.-\left(e^{-C_{\alpha,\beta}x_2}Z^{\leftarrow}\big((t_1-y)\big)_+\right)^{1/C_{\alpha,\beta}}\right]\\
&=\mathbb{E}\int_0^\infty (1-\beta)y^{-\beta}dy\Big[e^{-C_{\alpha,\beta}x_1}Z^{\leftarrow}\big((t_1-y)\big)_+\\
&\qquad\qquad\qquad\qquad+e^{-C_{\alpha,\beta}x_2}\big[Z^{\leftarrow}\big((t_2-y)\big)_+-Z^{\leftarrow}(t_1-y)_+\big]\Big]^{1/C_{\alpha,\beta}}.
\end{aligned}
$$

We argue that this is impossible since $C_{\alpha,\beta}>1$. Indeed, for every $t_1>0$ and $t_2,t_3>t_1$, a simple convexity argument shows that for $C>1$ we have

$$
t_2^C-t_1^C+t_3^C<(t_2-t_1+t_3)^C. \tag{5.60}
$$

Now apply (5.60) with

$$
\begin{aligned}
&t_1=e^{-x_2}\big[Z^{\leftarrow}\big((t_1-y)\big)_+\big]^{1/C_{\alpha,\beta}},\ t_2=e^{-x_1}\big[Z^{\leftarrow}\big((t_1-y)\big)_+\big]^{1/C_{\alpha,\beta}},\\
&t_3=e^{-x_2}\big[Z^{\leftarrow}\big((t_2-y)\big)_+\big]^{1/C_{\alpha,\beta}}.
\end{aligned}
$$

Remark 5.3.2. We will prove both theorems with the time domain restricted to the interval $[0,1]$. The general case is only notationally different.

As it is often done when analyzing the extremes of subexponential processes, we start by decomposing the process in (5.52) into a sum of two independent processes. One will collect the large Poissonian contributions of the original process and the other will collect the small such contributions. Note that, by (5.54), we can choose $x_0>0$ (which we assume to be 1 for notational simplicity) satisfying

$$
\left(\frac{1}{x^{1-\alpha}L_\alpha(x)}\right)'<0 \quad \text{for all } x>x_0, \tag{5.61}
$$

and we split the random measure M in (5.52) into a sum $M \overset{\mathrm{d}}{=} M^{(1)} + M^{(2)}$ of two independent infinitely divisible random measures $M^{(1)}$ and $M^{(2)}$ with the same control measure as M and with constant local characteristic $\big(0, [\nu]_{(x_0,\infty)}, 0\big)$ and $\big(\sigma^2, [\nu]_{(-\infty,x_0]}, b\big)$, respectively. We define two independent stationary infinitely divisible processes by

$$X_t^{(i)} = \int_E 1_{A_0} \circ \theta^t(x) M^{(i)}(dx), \quad t \in \mathbb{Z}, , i = 1, 2. \tag{5.62}$$

This gives us a desired decomposition

$$\{X_t\}_{t\in\mathbb{Z}} \overset{\mathrm{d}}{=} \{X_t^{(1)}\}_{t\in\mathbb{Z}} + \{X_t^{(2)}\}_{t\in\mathbb{Z}}. \tag{5.63}$$

By construction, the random variables $\{X_t^{(1)}\}_{t\in\mathbb{Z}}$ are compound Possion. For each $n \in \mathbb{N}$, it is convenient to take a series representation of $\{X_t^{(1)}\}_{0\leqslant t\leqslant n}$, which arranges the Poissonian jumps in the decreasing order. The representation uses crucially the zero sets $(I_{k;n})$ defined in (5.30). It follows from Corollary 3.4.2 in Samorodnitsky (2016) (see also (4.12) in Chen and Samorodnitsky (2020)) that

$$\left(X_t^{(1)}\right)_{0\leqslant t\leqslant n} \overset{\mathrm{d}}{=} \left(\sum_{j\geqslant 1} V_1\left(w_n/\Gamma_j\right) 1_{\{t\in I_{j;n}\}}\right)_{0\leqslant t\leqslant n}. \tag{5.64}$$

Here (Γ_j) are the ordered arrival times of a unit rate Poisson process on $\mathbb{R}_+$ independent of the *i.i.d.* zero sets $(I_{j;n})$. Furthermore, V_1 is a truncated function V in (5.55):

$$V_1(y) := \begin{cases} V(y), & \text{if } y > 1/\overline{\nu}(x_0) \\ 0, & \text{otherwise} \end{cases}.$$

For notational simplicity, we will hereafter use (5.64) with V instead of V_1. We keep in mind that this function vanishes in a neighborhood of 0.

In the sequel, we will view $\{X_t^{(1)}\}_{t\in\mathbb{Z}}$ as defined by the series in (5.64) and write

$$\mathcal{M}_n(B) = \max_{t\in nB}\left\{X_t^{(2)} + \sum_{j\geqslant 1} V\left(w_n/\Gamma_j\right) 1_{\{t\in I_{j;n}\}}\right\}, \quad B \in \mathcal{B}([0,1]). \tag{5.65}$$

The proofs of Theorems 5.3.1 and 5.3.2 use a number of random sup-measures related to (5.65) and we list them below. They use the random sets defined in (5.33). We also use the random sets

$$\widehat{I}_{k;n} = I_{k;n} \setminus \cup_{j=1}^{k-1} I_{j;n}, \ k \geqslant 1, \tag{5.66}$$
$$\widehat{I}_{k,i;n} = I_{k,i;n} \setminus \cup_{j=1}^{i-1} I_{k,j;n}, \ k, i \geqslant 1.$$

For $B \subset [0,1]$ and $n \in \mathbb{N}$, we define for $k, i, K \in \mathbb{N}$,

$$\mathcal{M}_{k,i;n}(B) = \max_{t \in \widehat{I}_{k,i;n} \cap nB} \left\{ X_t^{(2)} + \sum_{j \geqslant 1} V\left(w_n/\Gamma_j\right) 1_{\{t \in I_{j;n}\}} \right\}; \tag{5.67}$$

$$\mathcal{M}_{k;n}(B) = \max_{t \in \widehat{I}_{k;n} \cap nB} \left\{ X_t^{(2)} + \sum_{j \geqslant 1} V\left(w_n/\Gamma_j\right) 1_{\{t \in I_{j;n}\}} \right\}; \tag{5.68}$$

$$\mathcal{M}_{[K];n}(B) = \bigvee_{k=1}^{K} \mathcal{M}_{k;n}(B). \tag{5.69}$$

Furthermore, referring to the random sup-measure $\mathcal{M}$ in (5.47) we also define

$$\mathcal{M}_{k,i}(B) = \begin{cases} \Lambda_{k,i}, & W_{k,i} \in B \\ -\infty, & W_{k,i} \notin B \end{cases}, \quad k, i \in \mathbb{N}; \tag{5.70}$$

$$\mathcal{M}_{k}(B) = \bigvee_{i=1}^{\infty} \mathcal{M}_{k,i}(B), \quad k \in \mathbb{N}; \tag{5.71}$$

$$\mathcal{M}_{[K]}(B) = \bigvee_{k=1}^{K} \mathcal{M}_{k}(B), \quad K \in \mathbb{N}. \tag{5.72}$$

The proofs of Theorems 5.3.1 and 5.3.2 rely heavily on Theorem 5.2.1. By the Skorohod embedding we may assume that all random elements appearing in (5.65), (5.67), (5.68), and (5.69) are defined on a common probability space $(\Omega, \mathscr{F}, \mathbb{P})$ and the convergence in Theorem 5.2.1 holds as the *a.s.* convergence for these random elements. Furthermore, the random elements appearing as the limit in the right hand side of (5.35) are used to construct the points of the point processes $(\mathbb{C}_k)$

via (5.51). The random variables (Γ_j) are already naturally coupled via (5.65) and (5.51). This way the random sup-measure $\mathcal{M}$ is coupled to the random sup-measures $(\mathcal{M}_n)$. This setup will be in force for the duration of this section, and it follows from (5.43) that the following proposition suffices to prove Theorem 5.3.1.

Proposition 5.3.1. *For each open interval $B \subset [0,1]$,*

$$\frac{\mathcal{M}_n(B) - b_n}{a_n} \xrightarrow{P} \mathcal{M}(B). \tag{5.73}$$

This proposition is a consequence of the three statements below.

Proposition 5.3.2. *For each $k, i \in \mathbb{N}$ and each open interval $B \subset [0,1]$,*

$$\frac{\mathcal{M}_{k,i;n}(B) - b_n}{a_n} \xrightarrow{P} \mathcal{M}_{k,i}(B). \tag{5.74}$$

In the next two propositions, B is an open subset of $[0,1]$ and we use the notation $\Omega_B = \{\overline{R_1^*} \cap B \neq \emptyset\}$,

Proposition 5.3.3. *Let $\ell_n := \lfloor \rho \log n \rfloor$ with $\rho > 0$. If ρ is small enough, then*

$$\lim_{\ell_0 \to \infty} \liminf_{n \to \infty} \mathbb{P}\left\{ \mathcal{M}_{1;n}(B) > \bigvee_{i=2^{\ell_0}}^{2^{\ell_n}-1} \mathcal{M}_{1,i;n}([0,1]) \,\middle|\, \Omega_B \right\} = 1 \tag{5.75}$$

and

$$\lim_{\ell_0 \to \infty} \liminf_{n \to \infty} \mathbb{P}\left\{ \mathcal{M}_n(B) > \bigvee_{k=2^{\ell_0}}^{2^{\ell_n}-1} \mathcal{M}_{k;n}([0,1]) \right\} = 1. \tag{5.76}$$

Proposition 5.3.4. *For any $\rho > 0$, we have*

$$\lim_{n \to \infty} \mathbb{P}\left\{ \mathcal{M}_n(B) > \bigvee_{k \geqslant n^\rho} \mathcal{M}_{k;n}([0,1]) \right\} = 1 \tag{5.77}$$

and

$$\lim_{n \to \infty} \mathbb{P}\left\{ \mathcal{M}_{1;n}(B) > \bigvee_{i \geqslant n^\rho} \mathcal{M}_{1,i;n}([0,1]) \,\middle|\, \Omega_B \right\} = 1. \tag{5.78}$$

We start by showing how Proposition 5.3.1 follows from the three statements above.

Proof of Proposition 5.3.1. First, we note that by (5.75) and (5.78)

$$\lim_{m\to\infty}\liminf_{n\to\infty}\mathbb{P}\left\{\mathcal{M}_{1;n}(B)=\bigvee_{i=1}^{m}\mathcal{M}_{1,i;n}(B)\,\middle|\,\Omega_B\right\}=1.$$

Second, for almost every $\omega\in\Omega_B^c$, we also have $\overline{R_1^*}\cap\overline{B}=\emptyset$ because the stable regenerative set does not hit fixed points. Due to $I_{1;n}/n\to\overline{R_1^*}$ *a.s.*, we therefore see that for almost every $\omega\in\Omega_B^c$, $I_{1;n}\cap nB=\emptyset$ for all n large enough, and hence, $\mathcal{M}_{1;n}(B)=\mathcal{M}_{1,i;n}(B)=-\infty$ for all $i\in\mathbb{N}$. We deduce that

$$\lim_{m\to\infty}\liminf_{n\to\infty}\mathbb{P}\left\{\mathcal{M}_{1;n}(B)=\bigvee_{i=1}^{m}\mathcal{M}_{1,i;n}(B)\right\}=1.$$

This identity obviously remains valid if $\mathcal{M}_{1;n}$ and $\mathcal{M}_{1,i;n}$ are replaced by $\mathcal{M}_{k;n}$ and $\mathcal{M}_{k,i;n}$ respectively. Thus for any $K\in\mathbb{N}$,

$$\lim_{m\to\infty}\liminf_{n\to\infty}\mathbb{P}\left\{\mathcal{M}_{k;n}(B)=\bigvee_{i=1}^{m}\mathcal{M}_{k,i;n}(B),1\leqslant k\leqslant K\right\}=1.$$

We proceed to note from (5.76) and (5.77) that

$$\lim_{K\to\infty}\liminf_{n\to\infty}\mathbb{P}\left\{\mathcal{M}_n(B)=\mathcal{M}_{[K];n}(B)\right\}=1,$$

and conclude that

$$\lim_{\substack{K\to\infty\\ m\to\infty}}\liminf_{n\to\infty}\mathbb{P}\Big\{\bigvee_{\substack{1\leqslant k\leqslant K\\ 1\leqslant i\leqslant m}}\mathcal{M}_{k,i;n}(B)=\mathcal{M}_n(B)\Big\}=1. \tag{5.79}$$

Finally, we note that

$$\lim_{\substack{K\to\infty\\ m\to\infty}}\bigvee_{\substack{1\leqslant k\leqslant K\\ 1\leqslant i\leqslant m}}\mathcal{M}_{k,i}(B)=\mathcal{M}(B)\ \ a.s.. \tag{5.80}$$

By the standard "convergent together" argument, (5.73) follows from Proposition 5.3.2, (5.79) and (5.80). □

The proof of Theorem 5.3.1 is, therefore, complete apart from proving Propositions 5.3.2, 5.3.3 and 5.3.4 which we now commence.

Proof of Proposition 5.3.2. We consider $k = 1$. Recall that for any i, a.s.,

$$\frac{1}{n}(I_{1,1;n}, \ldots, I_{1,i;n}) \to \big(\{J_{1,1}\}, \ldots, \{J_{1,i}\}\big)$$

and $J_{1,1}, \ldots, J_{1,i}$ are distinct points. Hence, the sets $I_{1,1;n}, \ldots, I_{1,i;n}$ are disjoint for all sufficiently large n, so it is enough co consider the case $i = 1$.

Once again, since $I_{1,1;n}/n \to \{J_{1,1}\}$ *a.s.*, for almost every $\omega \in \{J_{1,1} \notin B\}$ we have $I_{1,1;n} \cap nB = \emptyset$ for all n large enough. Hence for such ω both sides of (5.74) are equal (to $-\infty$), and so we only need to show that

$$\frac{\mathcal{M}_{1,1;n}(B) - b_n}{a_n} \xrightarrow{P} \Lambda_{1,1} \quad \text{on} \quad \{J_{1,1} \in B\}. \tag{5.81}$$

To this end, observe that on the event $\{I_{1,1;n} \cap nB \neq \emptyset\}$,

$$\begin{aligned} \mathcal{M}_{1,1;n}(B) =& V\left(w_n/\Gamma_1\right) + V\left(w_n/\Gamma_{j_{1,1;n}}\right) \\ &+ \max_{t \in I_{1,1;n} \cap nB} \Big\{ X_t^{(2)} + \sum_{j > j_{1,1;n}} V\left(w_n/\Gamma_j\right) 1_{\{t \in I_{j;n}\}} \Big\}. \end{aligned}$$

First, it follows from (5.148) that as $n \to \infty$,

$$\frac{V\left(w_n/\Gamma_1\right) - V(w_n)}{a_n} \to -\log \Gamma_1.$$

Second, by the strong law of large numbers,

$$V\left(w_n/\Gamma_{j_{1,1;n}}\right) - V\left(w_n/j_{1,1;n}\right) = o\left(h \circ V\big(w_n/j_{1,1;n}\big)\right) \leq o\big(h \circ V(w_n)\big).$$

By Theorem 5.2.1

$$\begin{aligned} V\left(w_n/j_{1,1;n}\right)-V\left(w_n\overline{p}_{1;n}/\Gamma_{1,1}\right) &= o\left(h\circ V\left(w_n\overline{p}_{1;n}/\Gamma_{1,1}\right)\right) \\ &\le o\big(h\circ V(w_n)\big), \\ V\left(w_n\overline{p}_{1;n}/\Gamma_{1,1}\right)-V\left(c_\infty\vartheta_n Z_1^{*\leftarrow}(1)/\Gamma_{1,1}\right) &= o\left(h\circ V\left(c_\infty\vartheta_n Z_1^{*\leftarrow}(1)/\Gamma_{1,1}\right)\right) \\ &\le o\big(h\circ V(w_n)\big). \end{aligned}$$

Apply (5.148) again to get

$$\frac{V\left(c_\infty\vartheta_n Z_1^{*\leftarrow}(1)/\Gamma_{1,1}\right)-V(c_\infty\vartheta_n)}{h\circ V(c_\infty\vartheta_n)} = \log Z_1^{*\leftarrow}(1)-\log\Gamma_{1,1}+o(1).$$

Due to (5.147), we have

$$\frac{h\circ V(c_\infty\vartheta_n)}{h\circ V(w_n)} \sim \left(\frac{\log w_n}{\log\vartheta_n}\right)^{1/\alpha-1} \to \frac{1}{C_{\alpha,\beta}}.$$

This implies that

$$\frac{V\left(w_n/\Gamma_{j_{1,1;n}}\right)-V(c_\infty\vartheta_n)}{a_n} \to C_{\alpha,\beta}\big(-\log\Gamma_{1,1}+\log Z_1^{*\leftarrow}(1)\big), \tag{5.82}$$

and so,

$$\begin{aligned} &\frac{V\left(w_n/\Gamma_1\right)+V\left(w_n/\Gamma_{j_{1,1;n}}\right)-b_n}{a_n} \\ &\to -\log\Gamma_1+C_{\alpha,\beta}^{-1}\big(-\log\Gamma_{1,1}+\log Z_1^{*\leftarrow}(1)\big)=\Lambda_{1,1} \end{aligned}$$

by (5.51).

Finally, we notice that the cardinality $\#I_{1,1;n}$ is tight by Proposition 5.2.1 (i). Because a_n grows to infinity, we see that

$$\frac{\max_{t\in I_{1,1;n}\cap nB}\left\{X_t^{(2)}+\sum_{j>j_{1,1;n}}V\left(w_n/\Gamma_j\right)1_{\{t\in I_{j;n}\}}\right\}}{a_n} \xrightarrow{P} 0.$$

This proves (5.74). □

Proof of Proposition 5.3.3. It is convenient to assume that the random sets $\{I_{1;n}, n \in \mathbb{N}\}$ and $\overline{R_1^*}$ are defined a probability space $(\Omega, \mathscr{F}, \mathbb{P})$, while the remaining random elements are defined on another probability space, $(\Omega_1, \mathscr{F}_1, \mathbb{P}_1)$, so that the overall probability space is the product space. Clearly, Ω_B can be viewed as an element of $\mathscr{F}$.

Observe that, for $\omega \in \Omega_B$ the random variables $(J_{1,i})$ are *i.i.d.* on $(\Omega_1, \mathscr{F}_1, \mathbb{P}_1)$ and each has a positive $\mathbb{P}_1$-probability to be in B. By Theorem 5.2.1, the $\mathbb{P}_1$-probability that each $I_{1,i;n}$ intersects nB is bounded away from zero for all large n. Therefore, we can choose K_1 so large that with

$$A_{B;n}^{(1)} = \{I_{1,i;n} \cap nB \neq \emptyset \text{ for some } 1 \leqslant i \leqslant K_1\}$$

we have

$$\liminf_{n\to\infty} \mathbb{P}\big\{A_{B;n}^{(1)} \,\big|\, \Omega_B\big\} \geqslant 1 - \epsilon/2.$$

It is clear that on $A_{B;n}^{(1)}$,

$$\mathcal{M}_{1;n}(B) \geqslant V\left(w_n/\Gamma_1\right) + V\left(w_n/\Gamma_{j_{1,K_1;n}}\right) + O_P(1). \tag{5.83}$$

Using once again (5.148) and arguing as in the proof of Proposition 5.3.2, if we choose $c_1 > 0$ sufficiently large, then we can make the probability

$$\mathbb{P}\left\{V\left(w_n/\Gamma_{j_{1,K_1;n}}\right) \geqslant V(\vartheta_n) - c_1 h \circ V(\vartheta_n)\right\}$$

arbitrarily close to 1 as $n \to \infty$. Hence for some large $c_1 > 0$, there is a sequence of subsets of $A_{B;n}^{(2)} \subseteq A_{B;n}^{(1)}$ that satisfy

$$\liminf_{n\to\infty} \mathbb{P}\{A_{B;n}^{(2)} \,|\, \Omega_B\} \geqslant 1 - \epsilon, \tag{5.84}$$

$$\liminf_{n\to\infty} \inf_{\omega \in A_{B;n}^{(2)}} \mathbb{P}_\omega\Big\{V\left(w_n/\Gamma_{j_{1,K_1;n}}\right) \geqslant V(\vartheta_n) - c_1 h \circ V(\vartheta_n)\Big\} \geqslant 1 - \epsilon, \tag{5.85}$$

with $\mathbb{P}_\omega(\cdot) = \mathbb{P}\{\cdot \,|\, I_{1;n}, n \in \mathbb{N}, \overline{R_1^*}\}$.

Next, for any ℓ,

$$\begin{aligned}\bigvee_{i=2^\ell}^{2^{\ell+1}-1}\mathcal{M}_{1,i;n}([0,1]) &\leqslant V\left(w_n/\Gamma_1\right)+V\left(w_n/\Gamma_{j_{1,2^\ell;n}}\right)\\ &\quad+\max\left\{X_t^{(2)}+\sum_{j>j_{1,2^\ell;n}}V\left(w_n/\Gamma_j\right)1_{\{t\in I_{j;n}\}}:t\in\cup_{i=2^\ell}^{2^{\ell+1}-1}\widehat{I}_{1,i;n}\right\}\\ &\leqslant_{\mathrm{st}} V\left(w_n/\Gamma_1\right)+V\left(w_n/\Gamma_{j_{1,2^\ell;n}}\right)+\max\left\{X_t^{(0)}\,:\,1\leqslant t\leqslant\sum_{i=2^\ell}^{2^{\ell+1}-1}\#I_{1,i;n}\right\},\end{aligned}$$

where $\{X_t^{(0)}\}_{t\in\mathbb{Z}}$ are *i.i.d.* with $X_0^{(0)}\overset{\mathrm{d}}{=}X_0$, which are also independent of the rest two random variables on the right hand side. It suffices to derive suitable upper bounds for the last two terms.

Take $A_{B;n}^{(3)}=\Omega_n$ as defined in Proposition 5.2.1 (ii). We then note that

$$\liminf_{n\to\infty}\mathbb{P}\left\{A_{B;n}^{(3)}\,\middle|\,\Omega_B\right\}\geqslant 1-\epsilon \tag{5.86}$$

and we can choose $c_2=c_2(\epsilon)$ so that

$$\lim_{\ell_0\to\infty}\liminf_{n\to\infty}\inf_{\omega\in A_{B;n}^{(3)}}\mathbb{P}_\omega\left\{\frac{\sum_{i=2^\ell}^{2^{\ell+1}-1}\#I_{1,i;n}(\omega)}{2^\ell}\leqslant c_2,\,\ell_0\leqslant\ell\leqslant\ell_n\right\}=1. \tag{5.87}$$

Write $v_\ell=V(c_2 2^\ell)$. From the facts that $\mathbb{P}\{X_0>x\}\sim\overline{\nu}(x)$ and that V is the inverse of $1/\overline{\nu}$, we have for any positive constant c_3,

$$\begin{aligned}&\mathbb{P}\left\{\max_{1\leqslant t\leqslant c_2 2^\ell}X_t^{(0)}>v_\ell+c_3\ell h(v_\ell)\right\}\leqslant c_2 2^\ell\cdot\mathbb{P}\left\{X_0\geqslant v_\ell+c_3\ell h(v_\ell)\right\}\\ \lesssim&\frac{\overline{H}\left(v_\ell+c_3\ell h(v_\ell)\right)}{\overline{H}\left(v_\ell\right)}=\exp\left\{-\int_0^1\frac{c_3\ell h(v_\ell)\cdot du}{h\left(v_\ell+c_3\ell h(v_\ell)\cdot u\right)}\right\},\quad\text{as }\ell\to\infty.\end{aligned}$$

By (5.54), h is eventually increasing, and we use (5.146) and (5.147) to verify that for large ℓ the integral in the exponent is at least

$$\frac{h(v_\ell)c_3\ell}{h\left(v_\ell+c_3\ell h(v_\ell)\right)}\sim c_3\left(\frac{\alpha\log 2}{\alpha\log 2+c_3}\right)^{1-\alpha}\ell.$$

It follows that

$$\lim_{\ell_0\to\infty}\limsup_{n\to\infty}\sum_{\ell=\ell_0}^{\ell_n}\mathbb{P}\left\{\max_{1\leqslant t\leqslant c_2 2^\ell}X_t^{(0)}>v_\ell+c_3\ell h(v_\ell)\right\}=0. \tag{5.88}$$

Recalling the Skorohod embedding of the convergence in Theorem 5.2.1, we see that we can choose an event $A_B^{(4)}$ and some $c_4 = c_4(\epsilon) > 0$ such that

$$\mathbb{P}\left\{A_B^{(4)}\,\middle|\,\Omega_B\right\} \geqslant 1 - \epsilon, \quad \sup_{n\geqslant 1}\sup_{\omega\in A_B^{(4)}} \frac{w_n}{\vartheta_n}\overline{p}_{1;n}(\omega) \leqslant c_4. \tag{5.89}$$

The upper bound on $\overline{p}_{1;n}$ in (5.89) guarantees the uniform convergence

$$\sup_{\omega\in A_B^{(4)}}\sup_{\lambda\leq 1/2}\left|\mathbb{E}_\omega e^{\lambda j_{1,1;n}\overline{p}_{1;n}} - \mathbb{E}e^{\lambda\Gamma_{1,1}}\right| \to 0,$$

as in the argument for (5.140). Since under $\mathbb{P}_\omega$ the product $j_{1,2^\ell;n}\cdot\overline{p}_{1;n} - 1$ is the sum of 2^ℓ independent copies of $j_{1,1;n}\cdot\overline{p}_{1;n} - 1$, the exponential Markov inequality tells us that

$$\lim_{\ell_0\to\infty}\limsup_{n\to\infty}\sup_{\omega\in A_B^{(4)}}\sum_{\ell=\ell_0}^{\ell_n}\mathbb{P}_\omega\left\{j_{1,2^\ell;n}\cdot\overline{p}_{1;n} \leqslant 2^{\ell-1}\right\} = 0. \tag{5.90}$$

Combining (5.89) with (5.90) gives us

$$\lim_{\ell_0\to\infty}\limsup_{n\to\infty}\sup_{\omega\in A_{B;n}^{(4)}}\sum_{\ell=\ell_0}^{\ell_n}\mathbb{P}_\omega\left\{w_n/j_{1,2^\ell;n} \geqslant c_4\vartheta_n/2^{\ell-1}\right\} = 0 \tag{5.91}$$

and, therefore,

$$\lim_{\ell_0\to\infty}\limsup_{n\to\infty}\inf_{\omega\in A_B^{(4)}}\mathbb{P}_\omega\left\{V\left(w_n/\Gamma_{1,2^\ell;n}\right) \leqslant V\left(c_4\vartheta_n/2^\ell\right),\, \ell_0 \leqslant \ell \leqslant \ell_n\right\} = 1. \tag{5.92}$$

Set

$$A_{B;n} = A_{B;n}^{(2)} \cap A_{B;n}^{(3)} \cap A_B^{(4)}, \quad n \in \mathbb{N}.$$

so that

$$\liminf_{n\to\infty}\mathbb{P}\left\{A_{B;n}\,\middle|\,\Omega_B\right\} \geqslant 1 - 3\epsilon,. \tag{5.93}$$

Using the constants defined above, we set for $\ell = \ell_0, \ldots, \ell_n$

$$T_{1,\ell} := V(\vartheta_n) - V\left(c_4\vartheta_n 2^{-\ell}\right),$$

$$T_{2,\ell} := c_1 h \circ V(\vartheta_n) + v_\ell + c_3\ell h(v_\ell).$$

It is elementary to check that

$$\begin{aligned}\lim_{\ell_0\to\infty}\limsup_{n\to\infty}\inf_{\omega\in A_{B;n}}\mathbb{P}_\omega\left\{\mathcal{M}_{1,n}(B)>\bigvee_{i=2^{\ell_0}}^{2^{\ell_n}-1}\mathcal{M}_{1,i;n}([0,1])\right\}\qquad(5.94)\\ \geq-\epsilon+\lim_{\ell_0\to\infty}\limsup_{n\to\infty}\mathbb{1}\Big(T_{1,\ell}>T_{2,\ell}\;\text{ for all }\ell=\ell_0,\ldots,\ell_n\Big).\end{aligned}$$

However, by (5.145), (5.143), (5.147), once we take $0<\rho<\beta$, we see that, uniformly in ℓ,

$$\begin{aligned}T_{1,\ell}&=G(\vartheta_n)-G\left(c_4\vartheta_n 2^{-\ell}\right)-o\left(h\circ G(\vartheta_n)\right)\\&=\int_{c_4 2^{-\ell}}^{1}\frac{h\circ G(\vartheta_n u)}{u}du-o\left(h\circ G(\vartheta_n)\right)\gtrsim\ell(\log n)^{1/\alpha-1}\mathscr{L}(\log n).\end{aligned}$$

On the other hand, by (5.145) and (5.147), uniformly in ℓ,

$$T_{2,\ell}\lesssim\left(\ell^{1/\alpha}+(\log n)^{1/\alpha-1}\right)\mathscr{L}(\log n).$$

As long as ρ is small enough, we see that the indicator function in the right hand side of (5.94) is equal to 1 for all n large enough, and so

$$\lim_{\ell_0\to\infty}\limsup_{n\to\infty}\inf_{\omega\in A_{B;n}}\mathbb{P}_\omega\left\{\mathcal{M}_{1,n}(B)>\bigvee_{i=2^{\ell_0}}^{2^{\ell_n}-1}\mathcal{M}_{1,i;n}([0,1])\right\}\geqslant 1-\epsilon.\qquad(5.95)$$

It follows from (5.95) and (5.93) that

$$\lim_{\ell_0\to\infty}\liminf_{n\to\infty}\mathbb{P}\left\{\mathcal{M}_{1;n}(B)>\bigvee_{i=2^{\ell_0}}^{2^{\ell_n}-1}\mathcal{M}_{1,i;n}([0,1])\,\middle|\,\Omega_B\right\}\geq 1-4\epsilon.$$

Letting $\epsilon\to 0$ establishes (5.75).

We proceed now to prove (5.76). Since $\lim_{n\to\infty}\mathbb{P}\{I_{1;n}\cap nB\neq\emptyset\}=\mathbb{P}\{\overline{R_1^*}\cap B\neq\emptyset\}>0$, it follows that

$$\lim_{K\to\infty}\lim_{n\to\infty}\mathbb{P}\left\{I_{k;n}\cap nB\neq\emptyset\text{ for some }1\leqslant k\leqslant K\right\}=1,$$

Therefore, repeating the argument used to find a lower bound on $\mathcal{M}_{1;n}(B)$ in the proof of (5.75) shows that for any $\epsilon > 0$ we have, outside of an event of probability ϵ,

$$\mathcal{M}_n(B) \geqslant V(w_n/\Gamma_1) + V(\vartheta_n) - O_P(h \circ V(\vartheta_n)). \tag{5.96}$$

Next, continuing to use the notation of the proof of (5.75),

$$\mathcal{M}_{k;n}([0,1]) \leqslant_{\mathrm{st}} V\left(w_n/\Gamma_k\right) + \max\left\{X_t^{(0)} : t \in I_{k;n}\right\}, \quad k \in \mathbb{N},$$

with the process $\{X_t^{(0)}\}$ depending on k, even though our notation does not show it. For a large constant $c_5 > 0$, let $v_{\ell;n} = V\left(c_5 2^\ell \big/ \overline{F}(n)\right)$, where F is the law of the first hitting time φ in (5.25). For a small positive constant c_6, we thus have

$$\begin{aligned}
&\mathbb{P}\Bigg\{ \bigcup_{\ell=\ell_0}^{\ell_n} \Bigg\{ \bigvee_{k=2^\ell}^{2^{\ell+1}-1} \mathcal{M}_{k;n}([0,1]) \geqslant V\left(w_n/2^{\ell-1}\right) + v_{\ell;n} + c_6 \ell h(v_{\ell;n}) \Bigg\}\Bigg\} \\
&\leqslant \sum_{\ell=\ell_0}^{\ell_n} \mathbb{P}\left\{\Gamma_{2^\ell} < 2^{\ell-1}\right\} + \sum_{\ell=\ell_0}^{\ell_n} \mathbb{P}\left\{ 2^{-\ell} \sum_{k=2^\ell}^{2^{\ell+1}-1} \# I_{k;n} > c_5/\overline{F}(n) \right\} \\
&\quad + \sum_{\ell=\ell_0}^{\ell_n} \mathbb{P}\Bigg\{ \max_{1 \leqslant t \leqslant c_5 2^\ell \big/ \overline{F}(n)} X_t^{(0)} \geqslant v_{\ell;n} + c_6 \ell h(v_{\ell;n}) \Bigg\} =: S_{1.n} + S_{2,n} + S_{3,n}.
\end{aligned}$$

We emphasize that the process $\{X_t^{(0)}\}$ in $S_{3,n}$ is a concatenation of 3 different processes with the same marginal distribution. Only the marginal distribution is relevant in the subsequent calculation.

An exponential Markov inequality immediately shows that

$$\lim_{\ell_0 \to \infty} \lim_{n \to \infty} S_{1.n} = 0. \tag{5.97}$$

Next, in the notation of (5.39), choosing $c_5 \geq 2\mu_1$ we have by the Chebyshev inequality

$$\begin{aligned}
&\mathbb{P}\left\{2^{-\ell}\sum_{k=2^\ell}^{2^{\ell+1}-1}\#I_{k;n} > c_5/\overline{F}(n)\right\}\\
\leq&\mathbb{P}\left\{2^{-\ell}\sum_{k=2^\ell}^{2^{\ell+1}-1}\Big(\#I_{k;n}\overline{F}(n)-\mathbb{E}\big(\#I_{1;n}\overline{F}(n)\big)\Big) > c_5/2\right\}\\
\leq&c2^{-\ell}\mathrm{Var}\big(\#I_{1;n}\overline{F}(n)\big)\leq c2^{-\ell}\mu_2,
\end{aligned}$$

which implies that

$$\lim_{\ell_0\to\infty}\limsup_{n\to\infty}S_{2,n}=0. \tag{5.98}$$

Finally, the statement

$$\lim_{\ell_0\to\infty}\limsup_{n\to\infty}S_{3,n}=0. \tag{5.99}$$

follows the same way as in the proof of (5.88). By (5.97), (5.98) and (5.99), we note that

$$\lim_{\ell_0\to\infty}\limsup_{n\to\infty}\mathbb{P}\Bigg\{\bigcap_{\ell=\ell_0}^{\ell_n}\Bigg\{\bigvee_{k=2^\ell}^{2^{\ell+1}-1}\mathcal{M}_{k;n}([0,1])\leqslant V\left(w_n/2^{\ell-1}\right)+v_{\ell;n}+c_6\ell h(v_{\ell;n})\Bigg\}\Bigg\}=1.$$

For $\ell_0\leqslant\ell\leqslant\ell_n$ let

$$\begin{aligned}
T_{1,\ell}&:=V(w_n/\Gamma_1)-V(w_n2^{-\ell+1}),\\
T_{2,\ell}&:=v_{\ell;n}+2c_7\ell h(v_{\ell;n})-V(\vartheta_n),
\end{aligned}$$

so that for any $\epsilon>0$

$$\mathbb{P}\left\{\mathcal{M}_n(B)>\bigvee_{k=2^{\ell_0}}^{2^{\ell_n}-1}\mathcal{M}_{k;n}([0,1])\right\}\geqslant\mathbb{1}\big(T_{1,\ell}>T_{2,\ell}\text{ for all }\ell=\ell_0,\dots,\ell_n\big)-\epsilon$$

for all large n. As in the proof of (5.75), for any $\epsilon>0$, on an event of probability converging to 1, the following two inequalities holds uniformly in ℓ.

$$\begin{aligned}
T_{1,\ell}\geqslant&\frac{\log 2(1-\beta-\rho\log 2)^{1/\alpha-1}}{\alpha}\ell(\log n)^{1/\alpha-1}\mathscr{L}(\log n),\\
T_{2,\ell}\leqslant&\frac{(1+\epsilon)(\log 2+c_7)\big((\beta+\rho\log 2)\big)^{1/\alpha-1}}{\alpha}\ell(\log n)^{1/\alpha-1}\mathscr{L}(\log n).
\end{aligned}$$

If c_7, ρ and ϵ are chosen to be small enough, we have $T_{1,\ell} > T_{2,\ell}$ uniformly in ℓ for large n. Thus

$$\lim_{\ell_0\to\infty} \liminf_{n\to\infty} \mathbb{P}\left\{ \mathcal{M}_n(B) > \bigvee_{k=2^{\ell_0}}^{2^{\ell_n}-1} \mathcal{M}_{k;n}([0,1]) \right\} \geq 1-\epsilon,$$

and (5.76) follows by letting $\epsilon \to 0$. □

Proof of Proposition 5.3.4. Due to the lower bound (5.96), the claim (5.77) will follow once we show that

$$\lim_{n\to\infty} \mathbb{P}\left\{ V(w_n/\Gamma_1) + V(\vartheta_n) - o(V(w_n)) > \bigvee_{\Gamma_k > n^\rho} \mathcal{M}_{k;n}([0,1]) \right\} = 1. \tag{5.100}$$

It suffices to consider the case $\rho < r_1$ as defined in Lemma 5.4.3 (i). Removing an event of probability ϵ ensures that Γ_1 is bounded from above by a constant. Modifying, if necessary, $o(V(w_n))$ shows that (5.100) will follow once we check that

$$\lim_{n\to\infty} \mathbb{P}\left\{ V(w_n) + V(\vartheta_n) - o(V(w_n)) > \bigvee_{\Gamma_k > n^\rho} \mathcal{M}_{k;n}([0,1]) \right\} = 1.$$

Furthermore, since the support of the marginal Lévy measure of the process $X_0^{(2)}$ is bounded on the right, the process has marginal distributional tails lighter than exponentially light; see Section 26 in Sato (2013). It follows that

$$\max_{0\leqslant t\leqslant n} X_t^{(2)} = o_P(\log n) = o_P(V(w_n))$$

as $n \to \infty$. Therefore, it is enough to prove that

$$\lim_{n\to\infty} \mathbb{P}\left\{ \bigvee_{\Gamma_k > n^\rho} \max_{t\in I_{k;n}} \sum_{j\geq k} V\left(w_n/\Gamma_j\right) 1_{\{t\in I_{j;n}\}} < V(w_n) + V(\vartheta_n) - o(V(w_n)) \right\} = 1. \tag{5.101}$$

We prove (5.101) through a series of steps.

Let $\tilde{\psi}$ be the function defined in Lemma 5.4.3. We start by proving that for any $0 < r < 1-\beta$ and any $\epsilon > 0$,

$$\begin{aligned}\mathbb{P}\Bigg\{ \sum_{\Gamma_j > n^r} V\left(w_n/\Gamma_j\right) 1_{\{0 \in I_{j;n}\}} + V\left(w_n/n^r\right) \qquad\qquad (5.102)\\ > V(w_n) + V(\vartheta_n) - o(V(w_n)) \Bigg\} \leqslant \exp\Big\{(-\tilde{\psi}(r) + \epsilon)\log n\Big\}\end{aligned}$$

for all large n.

For this purpose denote

$$\begin{aligned} z_n &= z_n(r) = V(w_n) + V(\vartheta_n) - V(w_n/n^r) - o(V(w_n)),\\ \overline{z}_n &= \overline{z}_n(r) = V(w_n/n^r).\end{aligned}$$

Recalling the partition $0 = r_0 < r_2 < \cdots$ of the interval $(0, 1-\beta)$ defined in Lemma 5.4.3, we see that $r \in (r_m.r_{m+1}]$ for some $m = 0, 1, 2\ldots$, so by the same Lemma 5.4.3, for large n, the probability in (5.102) is

$$\begin{aligned} \mathbb{P}\Bigg\{ \sum_{\Gamma_j > n^r} V\left(w_n/\Gamma_j\right) 1_{\{0 \in I_{j;n}\}} > z_n(r) \Bigg\}\\ \leqslant \overline{z}_n(r)^{\gamma_m} \left(\overline{H}(\overline{z}_n(r))\right)^m \overline{H}(z_n(r) - m\overline{z}_n(r)). \qquad (5.103)\end{aligned}$$

It follows from (5.146) that

$$\overline{z}_n(r)^{\gamma_m} \lesssim \left((\log n)^{1/\alpha} \mathscr{L}(\log n)\right)^{\gamma_m}. \qquad (5.104)$$

Next, by Karamata's theorem, (5.146) and (5.147)

$$\begin{aligned}\int_1^{\overline{z}_n(r)} \frac{du}{u^{1-\alpha} L_\alpha(x)} &\sim \frac{\overline{z}_n(r)}{\alpha h(\overline{z}_n(r))}\\ &= \frac{1}{\alpha} \cdot \frac{V(w_n/n^r)}{h \circ V(w_n/n^r)} \sim \log(w_n/n^r) \sim (1-\beta-r)\log n. \qquad (5.105)\end{aligned}$$

Let

$$\theta = \left((1-\beta)^{1/\alpha} + \beta^{1/\alpha} - (m+1)(1-\beta-r)^{1/\alpha}\right)^\alpha;$$

due to the range of r this is a well defined power of a positive number. It follows by (5.146) that

$$z_n - m\bar{z}_n \sim V(n^\theta),$$

Therefore, by (5.146) and (5.147),

$$\int_1^{z_n - m\bar{z}_n} \frac{du}{u^{1-\alpha} L_\alpha(u)} \sim \frac{z_n - m\bar{z}_n}{\alpha h(z_n - m\bar{z}_n)} \sim \frac{V(n^\theta)}{\alpha h \circ V(n^\theta)} \sim \theta \log n. \tag{5.106}$$

Putting (5.103), (5.104), (5.105) and (5.106) together establishes (5.102).

Next, we prove that (5.101) holds if $\rho \in (r_m, r_{m+1}]$ for any sufficiently large m. Indeed, by (5.102),

$$\begin{aligned}
&\mathbb{P}\left\{ \bigvee_{\Gamma_k > n^\rho} \max_{t \in I_{k;n}} \sum_{j \geq k} V\left(w_n/\Gamma_j\right) 1_{\{t \in I_{j;n}\}} \geqslant V(w_n) + V(\vartheta_n) - o(V(w_n)) \right\} \\
\leqslant & n\mathbb{P}\left\{ \sum_{\Gamma_j > n^r} V\left(\frac{w_n}{\Gamma_j}\right) 1_{\{0 \in I_{j;n}\}} + V(w_n/n^r) \geqslant V(w_n) + V(\vartheta_n) - o(V(w_n)) \right\} \\
\leqslant & n \exp\left\{ \left(-\widetilde{\psi}(r) + \epsilon \right) \log n \right\}
\end{aligned}$$

for large n. Since $\widetilde{\psi}(r) \to \infty$ as $r \to 1 - \beta$, the claim follows.

Given that (5.101) holds if $\rho \in (r_m, r_{m+1}]$ for any sufficiently large m, the fact that (5.101) also holds for any $0 < \rho < r_1$ will follow once we prove that for any such ρ and any m,

$$\begin{aligned}
\lim_{n \to \infty} \mathbb{P}\Bigg\{ & \bigvee_{n^\rho < \Gamma_k \leqslant n^{r_m}} \max_{t \in I_{k;n}} \sum_{j \geq k} V\left(w_n/\Gamma_j\right) 1_{\{t \in I_{j;n}\}} \\
& < V(w_n) + V(\vartheta_n) - o(V(w_n)) \Bigg\} = 1.
\end{aligned}$$

We will prove the above statement by constructing two increasing sequences,

$$\{i_s\}_{0 \leqslant s \leqslant m} \subseteq \mathbb{N} \quad \text{and} \quad \{\rho_{i_s}\}_{0 \leqslant s \leqslant m} \subseteq \mathbb{R}_+,$$

with $i_0 := 0$, $\rho_0 := \rho$, $\rho_{i_s} \in (r_s, r_{s+1})$ for $s = 1, \ldots, m$ such that for every such s,

$$\lim_{n\to\infty} \mathbb{P}\Biggl\{ \bigvee_{n^{\rho_{i_{s-1}}} < \Gamma_k \leqslant n^{\rho_{i_s}}} \max_{t\in I_{k;n}} \sum_{j\geq k} V\left(w_n/\Gamma_j\right) 1_{\{t\in I_{j;n}\}} \tag{5.107}$$

$$\geqslant V(w_n) + V(\vartheta_n) - o(V(w_n)) \Biggr\} = 0.$$

We will describe the construction in the case $m = 1$. The case of a general m is similar.

Let $\delta_0 = \widetilde{\psi}(\rho_0) - (\rho_0 + \beta) > 0$, by Lemma 5.4.3 (ii). So $\widetilde{\psi}(\rho_0) > \rho_0 + \beta + 2\delta_0/3$, and so using (5.102) and the notation that follows it,

$$\mathbb{P}\Biggl\{ \sum_{\Gamma_j > n^{\rho_0}} V\left(w_n/\Gamma_j\right) 1_{\{0\in I_{j;n}\}} \geqslant z_n(\rho_0) \Biggr\} = o\left(n^{-\rho_0-\beta-2\delta_0/3}\right). \tag{5.108}$$

Take $\rho_1 = \rho_0 + \delta_0/3$, then for any $c > 0$,

$$\begin{aligned}
&\mathbb{P}\Biggl\{ \bigvee_{n^{\rho_0} < \Gamma_k \leqslant n^{\rho_1}} \max_{t\in I_{k;n}} \sum_{j\geq k} V\left(w_n/\Gamma_j\right) 1_{\{t\in I_{j;n}\}} \geqslant V(w_n) + V(\vartheta_n) - o(V(w_n)) \Biggr\} \\
\leqslant &\mathbb{P}\bigl\{ \#\{k : n^{\rho_0} < \Gamma_k \leqslant n^{\rho_1}\} > 2n^{\rho_1} \bigr\} \\
+ &\mathbb{P}\Bigl\{ \# I_{k;n} \geqslant \frac{cn^\beta \log n}{L(n)} \ \text{ for some } k \text{ with } n^{\rho_0} < \Gamma_k \leqslant n^{\rho_1} \Bigr\} \\
+ &2n^{\rho_1} \frac{cn^\beta \log n}{L(n)} \cdot \mathbb{P}\Biggl\{ \sum_{\Gamma_j > n^{\rho_0}} V\left(w_n/\Gamma_j\right) 1_{\{0\in I_{j;n}\}} > z_n(\rho_0) \Biggr\}.
\end{aligned}$$

The first term in the right hand side vanishes in the limit by the law of large numbers. If c is large enough, by Proposition 5.2.2 so does the second term. The same is true for the third term by (5.108). Notice that, if $\rho_1 > r_1$, we set $i_1 = 1$ and replace ρ_1 with $\min(\rho_1, r_2)$ to complete the construction of the two sequences. Otherwise, we inductively define, for as long as $\rho_i \leqslant r_1$,

$$\delta_i = \widetilde{\psi}(\rho_i) - (\rho_i + \beta), \quad \rho_{i+1} = \rho_i + \frac{i+1}{i+3}\delta_{i+1}, \quad i \geqslant 1. \tag{5.109}$$

The same argument as above then shows that for every i,

$$\mathbb{P}\left\{ \bigvee_{n^{\rho_i} < \Gamma_k \leqslant n^{\rho_{i+1}}} \max_{t \in I_{k;n}} \sum_{j \geq k} V\left(w_n/\Gamma_j\right) 1_{\{t \in I_{j;n}\}} \geqslant V(w_n) + V(\vartheta_n) - o(V(w_n)) \right\} \to 0.$$

We claim that this inductive procedure must end after finitely many steps. That is, $\rho_{i_1} > r_1$ for some $i_1 \in \mathbb{N}$, in which case we replace ρ_{i_1} with $\min(\rho_{i_1}, r_2)$ and, once again, complete the construction.

Indeed, if the inductive procedure did not terminate, it would follow that $\delta_i \to 0$, so $\rho_i \uparrow \rho_* \leqslant r_1$ satisfying $\widetilde{\psi}(\rho_*-) = \rho_* + \beta$. Since $\widetilde{\psi}$ is constant on $(0, r_1)$, this is easily seen to contradict the property $r_1 < 1/2 - \beta$ established in Lemma 5.4.3 (i).

This completes the proof of (5.77).

We proceed to prove (5.78). Let $\epsilon > 0$. For $c, C > 0$ we set $\Omega_n = \Omega_n^{(1)} \cap \Omega_n^{(2)}$, where

$$\begin{aligned} \Omega_n^{(1)} &= \Big\{\omega \in \Omega : \min\big(\Gamma_{j_{1,k;n}}(\omega), j_{1,k;n}(\omega)\big) \geqslant c w_n k/\vartheta_n \ \text{ for all } k \in \mathbb{N}\Big\}, \quad n \in \mathbb{N}; \\ \Omega_n^{(2)} &= \Big\{\#I_{1,k;n}(\omega) < C \log n \text{ for all } 1 \leqslant k \leqslant \vartheta_n/c\Big\}, \quad n \in \mathbb{N}. \end{aligned}$$

If $\delta > 0$ is such that V vanishes on $(0, \delta]$, then on $\Omega_n^{(1)}$ we have

$$\#\{k : \Gamma_{j_{1,k;n}} \leqslant w_n/\delta\} \leqslant \vartheta_n/(c\delta).$$

We claim that we can choose $c = c(\epsilon)$ small enough and $C = C(\epsilon)$ large enough so that

$$\liminf_{n \to \infty} \mathbb{P}\big\{\Omega_n^{(i)}\big\} \geqslant 1 - \epsilon, \quad i = 1, 2. \tag{5.110}$$

The fact that this is true for $i = 1$ if c is small enough follows as in (5.91) and the law of large numbers, for $i = 2$ it follows for large enough C by Proposition 5.2.1 (ii).

With c chosen above we denote

$$\overline{\mathcal{M}}_{1,k;n} = \max_{t \in I_{1,k;n}} \left\{ V\left(w_n/\Gamma_{j_{1,k;n}}\right) + \sum_{j \geqslant j_{1,k;n}+1} V\left(w_n/\Gamma_j\right) 1_{\{t \in I_{j;n}\}} \right\}, \quad k \in \mathbb{N};$$

$$\mathcal{M}'_{1,k;n} = \max_{t \in I_{1,k;n}} \left\{ V\left(\vartheta_n/(ck)\right) + \sum_{\Gamma_j > ckw_n/\vartheta_n} V\left(w_n/\Gamma_j\right) 1_{\{t \in I_{j;n}\}} \right\}, \quad k \in \mathbb{N},$$

so that on each event Ω_n, we have $\overline{\mathcal{M}}_{1,k;n} \leqslant \mathcal{M}'_{1,k;n}$, $k \geqslant 1$. Since $c_\infty < 1$, it follows from (5.82) that on the event Ω_B,

$$V\left(w_n/\Gamma_{j_{1,1;n}}\right) \geqslant V(\vartheta_n) - o_P(V(\vartheta_n)).$$

Since $\epsilon > 0$ can be taken as small as we wish, (5.78) will follow from the following claim:

$$\lim_{n\to\infty} \mathbb{P}\left\{ \max_{\lfloor n^\rho \rfloor \leq k \leq \lceil \vartheta_n/(c\delta) \rceil} \mathcal{M}'_{1,k;n} < V(\vartheta_n) - o(V(\vartheta_n)) \,\middle|\, \Omega_n \right\} = 1. \tag{5.111}$$

We start by constructing an increasing sequence $\rho_i \uparrow \beta, i \in \mathbb{N}_0$ with $\rho_0 = \rho$ such that for every i

$$\mathbb{P}\left\{ \max_{\lfloor n^{\rho_i} \rfloor \leqslant k \leqslant \lceil n^{\rho_{i+1}} \rceil} \mathcal{M}'_{1,k;n} \geqslant V(\vartheta_n) - o(V(\vartheta_n)) \,\middle|\, \Omega_n \right\} \to 0, \quad n \to \infty. \tag{5.112}$$

To this end, we define inductively for $i = 0, 1, \ldots$

$$\delta_i = \left(\beta^{1/\alpha} - (\beta - \rho_i)^{1/\alpha}\right)^\alpha - \rho_i, \quad \rho_{i+1} = \rho_i + \frac{i+1}{i+3}\delta_i; \tag{5.113}$$

note that $\delta_i > 0$ for all i. We have

$$\begin{aligned}
&\mathbb{P}\left\{ V\left(\vartheta_n/(cn^{\rho_i})\right) + \sum_{\Gamma_j > cn^{\rho_i} w_n/\vartheta_n} V\left(w_n/\Gamma_j\right) 1_{\{0 \in I_{j;n}\}} \geqslant V(\vartheta_n) - o(V(\vartheta_n)) \right\} \\
\leqslant &\mathbb{P}\left\{ \sum_{j=0}^{\infty} V\left(w_n/\Gamma_j\right) 1_{\{0 \in I_{j;n}\}} \geqslant V(\vartheta_n) - o(V(\vartheta_n)) - V\left(\vartheta_n/(cn^{\rho_i})\right) \right\} \\
\lesssim &\overline{H}\Big(V(\vartheta_n) - V\left(\vartheta_n/(cn^{\rho_i})\right) - o\big(V(\vartheta_n)\big) \Big),
\end{aligned}$$

and by (5.146),

$$\log\Big[\overline{H}\Big(V(\vartheta_n) - V\left(\vartheta_n/(cn^{\rho_i})\right) - o\big(V(\vartheta_n)\big)\Big)\Big] \sim -\left(\beta^{1/\alpha} - (\beta-\rho_i)^{1/\alpha}\right)^{\alpha}\log n.$$

Therefore, by the first part of (5.113), for large n we have

$$\overline{H}\Big(V(\vartheta_n) - V\left(\vartheta_n/(cn^{\rho_i})\right) - o\big(V(\vartheta_n)\big)\Big) \leqslant \exp\left\{-\left(\rho_i + \frac{i+2}{i+3}\delta_i\right)\log n\right\}$$

and, hence, since $\Omega_n \supseteq \Omega_n^{(2)}$,

$$\begin{aligned}&\mathbb{P}\left\{\max_{\lfloor n^{\rho_i}\rfloor\leqslant k\leqslant\lceil n^{\rho_{i+1}}\rceil}\mathcal{M}'_{1,k;n} \geqslant V(\vartheta_n) - o(V(\vartheta_n))\,\middle|\,\Omega_n\right\}\\ \lesssim & n^{\rho_{i+1}}\cdot C\log n\cdot\exp\left\{-\left(\rho_i + \frac{i+2}{i+3}\delta_i\right)\log n\right\}\\ =& C\log n\cdot\exp\left\{-\frac{\delta_i}{i+3}\log n\right\}\to 0\end{aligned}$$

as $n\to\infty$. Therefore, (5.112) follows, and (5.111) will be established once we show that for all i large enough,

$$\mathbb{P}\left\{\max_{\lfloor n^{\rho_i}\rfloor\leqslant k\leqslant\lceil \vartheta_n/c\rceil}\mathcal{M}'_{1,k;n} \geqslant V(\vartheta_n) - o(V(\vartheta_n))\,\middle|\,\Omega_n\right\}\to 0, \quad n\to\infty. \tag{5.114}$$

Note that

$$\sum_{\Gamma_j > cn^{\rho_i}w_n/\vartheta_n} V\left(w_n/\Gamma_j\right)1_{\{0\in I_{j;n}\}} \overset{\mathrm{d}}{=} \sum_{j=1}^{N_n}\overline{\xi_j},$$

where N_n is a Poisson random variable with mean $\overline{\nu}(x_0) - cn^{\rho_i}/\vartheta_n$, and $\{\overline{\xi}_i\}_{i\geqslant 1}$ is a family of *i.i.d.* random variables independent of N_n, and the law of $\overline{\xi}_1$ is the restriction of ν to $\big(1, V(\vartheta_n c^{-1}n^{-\rho_i})\big)$, normalized to be a probability measure. Here we have used the fact that we assume $x_0 = 1$ in (5.61).

Let

$$m = m(i) = \left\lceil\frac{V(\vartheta_n) - o(V(\vartheta_n)) - V(\vartheta_n c^{-1}n^{-\rho_i})}{V(\vartheta_n c^{-1}n^{-\rho_i})}\right\rceil;$$

$m(i)$ also depends on n, but by (5.146) it is bounded as a function of n. By Lemma

5.4.4 (i) we have for some large numbers $\gamma_1(i)$, $\gamma_2(i)$,

$$\begin{aligned}
&\mathbb{P}\left\{ \sum_{\Gamma_j > cn^{\rho_i} w_n/\vartheta_n} V\left(w_n/\Gamma_j\right) 1_{\{0 \in I_{j;n}\}} > V(\vartheta_n) - o(V(\vartheta_n)) - V(\vartheta_n c^{-1} n^{-\rho_i}) \right\} \\
&\leqslant \sum_{s=m}^{\infty} \frac{1}{s!} \mathbb{P}\left\{ \sum_{j=1}^{s} \overline{\xi}_j \geqslant V(\vartheta_n) - o(V(\vartheta_n)) - V(\vartheta_n c^{-1} n^{-\rho_i}) \right\} \\
&\leqslant \sum_{s=m}^{\infty} \frac{1}{s!} \gamma_1(i)^s \left(V(\vartheta_n c^{-1} n^{-\rho_i})\right)^{\gamma_2(i)} \left(\overline{H}(V(\vartheta_n c^{-1} n^{-\rho_i}))\right)^{m(i)} \\
&\qquad \cdot \overline{H}\big(V(\vartheta_n) - o(V(\vartheta_n)) - mV(\vartheta_n c^{-1} n^{-\rho_i})\big) \\
&\lesssim \left(V(\vartheta_n c^{-1} n^{-\rho_i})\right)^{\gamma_2(i)} \left(\overline{H}(V(\vartheta_n c^{-1} n^{-\rho_i}))\right)^{m(i)} \lesssim (\log n)^{\gamma_2(i)/\alpha} \big(\vartheta_n n^{-\rho_i}\big)^{-m(i)},
\end{aligned}$$

where we have once again used (5.146). Furthermore,

$$m(i) \to \frac{\beta^{1/\alpha} - (\beta - \rho_i)^{1/\alpha}}{(\beta - \rho_i)^{1/\alpha}}$$

as $n \to \infty$, so that for large n, $\big(\vartheta_n n^{-\rho_i}\big)^{-m(i)} \leqslant n^{-C_1}$ for

$$C_1 = (\beta - \rho_i) \frac{\beta^{1/\alpha} - (\beta - \rho_i)^{1/\alpha}}{(\beta - \rho_i)^{1/\alpha}}.$$

Since $\rho_i \to \beta$ as $i \to \infty$ and $\alpha < 1$, C_1 will become arbitrarily large for large i. Taking into account the bound on the cardinality of $I_{t,k;n}$ on the event Ω_n, we see that for i such that $C_1 > \beta$ and for large n,

$$\begin{aligned}
&\mathbb{P}\left\{ \max_{\lfloor n^{\rho_i} \rfloor \leqslant k \leqslant \lceil \vartheta_n / c \rceil} \mathcal{M}'_{1,k;n} \geqslant V(\vartheta_n) - o(V(\vartheta_n)) \,\middle|\, \Omega_n \right\} \\
&\lesssim \vartheta_n \log n \cdot n^{-C_1} \to 0
\end{aligned}$$

as $n \to \infty$. This establishes (5.114) and, hence, completes the proof. □

Proof of Theorem 5.3.2. Recall that the random sup-measure $\mathcal{M}$ is coupled to the random sup-measures $(\mathcal{M}_n)$ as described when proving Theorem 5.3.1. Therefore, it suffices to show that for any fixed $0 < T_1 < T_2 \leqslant 1$,

$$\left\{ \frac{\mathbb{M}_n(t) - b_n}{a_n} \right\}_{t \in [T_1, T_2]} \xrightarrow{P} \{\mathbb{M}(t)\}_{t \in [T_1, T_2]} \quad \text{in } \big(D[T_1, T_2], J_1\big). \tag{5.115}$$

For the duration of the proof we fix a small $\epsilon > 0$.

We start by observing that by the construction of $\mathbb{M}(t)$ in (5.47), (5.51) and (5.48), there exists $N = N(\epsilon) > 0$ such that the event

$$\Omega^{(1)} := \left\{ \mathbb{M}(t) = \max_{1 \leqslant k,i \leqslant N} \left\{ \Lambda_{k,i} : W_{k,i} \in [0,t] \right\} \text{ for } t \in [T_1, T_2] \right\} \tag{5.116}$$

satisfies $\mathbb{P}\{\Omega^{(1)}\} \geqslant 1 - \epsilon$. Consider the random finite set $\mathcal{T} = \{W_{k,i} : 1 \leqslant k, i \leqslant N\} \cup \{0, T_1, T_2\}$. Since the random measure η_ω in (5.18) is a.s. atomless, it follows that there is $\delta > 0$ such that the event

$$\Omega^{(2)} := \left\{ |t_1 - t_2| \geqslant \delta \text{ for all distinct } t_1, t_2 \in \mathcal{T} \right\} \tag{5.117}$$

satisfies $\mathbb{P}\{\Omega^{(2)}\} \geqslant 1 - \epsilon$. Then on $\Omega^{(1)} \cap \Omega^{(2)}$ the process $\{\mathbb{M}(t) : t \in [T_1, T_2]\}$ has nondecreasing piecewise constant sample paths, with at most N^2 jumps, and each two jump points are separated by at least δ. Let $T_1 = W_{k_0,i_0} < W_{k_1,i_1} < \cdots < W_{k_\ell,i_\ell} \leqslant T_2$ be the enumeration of the jumps.

We saw when establishing (5.81) with $B = [0,1]$, that for any $k, i \in \mathbb{N}$,

$$\max_{t \in I_{k,i;n}} \left| \frac{X(t) - b_n}{a_n} - \Lambda_{k,i} \right| \xrightarrow{P} 0 \quad \text{as } n \to \infty.$$

Therefore by decreasing δ if necessary, the event

$$\Omega_n^{(3)} := \left\{ \left| \frac{\mathbb{M}_n([0,T_i]) - b_n}{a_n} - \mathbb{M}(T_i) \right| \leqslant \delta,\ i = 1, 2, \right.$$
$$\left. \bigvee_{k,i=1}^{N} \max_{t \in I_{k,i;n}} \left| \frac{X(t) - b_n}{a_n} - \Lambda_{k,i} \right| \leqslant \delta \right\}$$

satisfies $\mathbb{P}\{\Omega_n^{(3)}\} \geqslant 1 - \epsilon$ for all large n. Furthermore, by Proposition 5.3.3 and Proposition 5.3.4, for N large enough, the event

$$\Omega_n^{(4)} := \left\{ \mathbb{M}_n(t) = \max_{1 \leqslant k,i \leqslant N} \max_{t \in I_{k,i;n} \cap [0,nt]} X(t) \text{ for all } t \in [T_1, T_2]. \right\} \tag{5.118}$$

satisfies $\mathbb{P}\{\Omega_n^{(4)}\} \geqslant 1-\epsilon$ for all large n. It is clear that we can select the same N in (5.116) and (5.118). Finally, since for every k, i

$$I_{k,i;n}/n \longrightarrow \{W_{k,i}\} \text{ a.s.}$$

with respect to the Hausdorff metric as $n \to \infty$, the event

$$\Omega_n^{(5)} := \left\{ \max_{1\leqslant k,i\leqslant N} \max_{t\in I_{k,i;n}} |t/n - W_{k,i}| \leqslant \delta/2 \right\} \tag{5.119}$$

satisfies $\mathbb{P}\{\Omega_n^{(5)}\} \geqslant 1-\epsilon$ for all large n.

Fix $\omega \in \Omega^{(1)} \cap \Omega^{(2)} \cap \Omega_n^{(3)} \cap \Omega_n^{(4)} \cap \Omega_n^{(5)}$. We construct a function $e_n : [T_1, T_2] \to [T_1, T_2]$ as follows. First set

$$e_n(t) = \begin{cases} T_1 & t = T_1 \\ W_{k_s,i_s} & t = \min\bigl(I_{k_s,i_s;n}/n\bigr) \text{ for } s = 1, \ldots, \ell \\ T_2, & t = T_2 \end{cases}$$

and note that due to the choice of ω this an increasing function. We extend it to a continuous increasing map from $[T_1, T_2]$ onto $[T_1, T_2]$ by linear interpolation. Then

$$\sup_{t\in[T_1,T_2]} |e_n(t) - \mathrm{id}(t)| = \bigvee_{s=1}^{\ell} \left|\min\bigl(I_{k_s,i_s;n}/n\bigr) - W_{k_s,i_s}\right| \leqslant \delta/2$$

so that

$$\begin{aligned}
&d_{J_1}\left(\left\{ \frac{\mathbb{M}_n(t) - b_n}{a_n} \right\}_{t\in[T_1,T_2]}, \{\mathbb{M}(t)\}_{t\in[T_1,T_2]} \right) \\
\leqslant & \sup_{t\in[T_1,T_2]} |e_n(t) - \mathrm{id}(t)| \bigvee \sup_{t\in[T_1,T_2]} \left| \frac{\mathbb{M}_n(t) - b_n}{a_n} - \mathbb{M}(e_n(t)) \right| \\
\leqslant & \frac{\delta}{2} \vee \left(\bigvee_{i=1}^{2} \left| \frac{\mathbb{M}_n([0,T_i]) - b_n}{a_n} - \mathbb{M}(T_i) \right| \right) \vee \left(\bigvee_{s=1}^{\ell} \max_{t\in I_{k_s,i_s;n}} \left| \frac{X_t - b_n}{a_n} - \Lambda_{k_s,i_s} \right| \right) \\
\leqslant & \delta.
\end{aligned}$$

Since $\delta > 0$ is arbitrary, while

$$\liminf_{n\to\infty} \mathbb{P}\left\{ \Omega^{(1)} \cap \Omega^{(2)} \Omega_n^{(3)} \cap \Omega_n^{(4)} \cap \Omega_n^{(5)} \right\} \geqslant 1 - 5\epsilon,$$

and $\epsilon > 0$ is also arbitrary, (5.115) follows. □

5.4 Appendices

5.4.1 Ranges and Local Times

This appendix is largely devoted to proving Theorem 5.2.1. Unless otherwise stated, F denotes the distribution in Assumption 5.2.1.

Let us begin by looking closely at the set $I_{0;n}$ in (5.30). By the definition of the probability measure μ_n, the random function $Y^{(k;n)}$ has its first zero coordinate between 0 and n, and the subsequent zero coordinates appear whenever the Markov chain returns back to zero. Therefore, $I_{0;n}$ is, in distribution, the restriction to $\{0, \ldots, n\}$ of the range of a random walk with the step distribution F, starting at a random position in $\{0, \ldots, n\}$. The following definition formalizes this type of random walks, which we will study in this appendix.

Assumption 5.4.1. *For $n \in \mathbb{N}_0$, $\mathbf{S}^{(n)} = \{S_t^{(n)}\}_{t \in \mathbb{N}_0}$ is a random walk such that*

(a) the initial position $S_0^{(n)}$ has the law of $\min\{0 \leqslant t \leqslant n : Y_t^{(n)} = 0\}$*, where $\{Y_t^{(n)}\}_{t \in \mathbb{Z}}$ has the law μ_n in (5.29);*

(b) the steps $\{\xi_t\}_{t \in \mathbb{N}}$ are i.i.d. with distribution F, and are independent of $S_0^{(n)}$.

That is, the different random walks in Assumption 5.4.1 only differ in the law of the initial position. Hereafter, $\{\{S_t^{(k;n)}\}_{t \in \mathbb{N}_0}\}_{k \in \mathbb{N}_0}$ denotes a collection of *i.i.d.* copies of $\mathbf{S}^{(n)}$. For the duration of this section we work with the sets $I_{k;n}$ in (5.30) written as

$$I_{k;n} = \left\{ S_t^{(k;n)} : t \in \mathbb{N}_0 \right\} \cap \{0, \ldots, n\}, \quad n, k \in \mathbb{N}_0. \tag{5.120}$$

By the definition these are nonempty random sets.

We will need several facts about these random walks, and we list these facts in the proposition below. Most of them are well known. In the sequel we use the notation $\min A$ ($\max A$) to denote the smallest (largest) point in a discrete set A.

Proposition 5.4.1. *(i) For every $k \in \mathbb{N}_0$,*

$$\left(\frac{\min I_{k;n}}{n}, \frac{I_{k;n}}{n}\right) \Rightarrow \left(Z^*(0), \overline{R^*}\right) \quad \textit{weakly in} \;\; [0,1] \times \mathcal{F}([0,1]) \tag{5.121}$$

as $n \to \infty$, where $Z^*(0)$ and $\overline{R^*}$ are connected by (5.12), (5.13) *and (5.17).*

(ii) If A_0 denotes the full range $\{S_0^{(0)}, S_1^{(0)}, \ldots\}$ of $\mathbf{S}^{(0)}$, then

$$\mathbb{P}\left\{n \in A_0\right\} \sim \frac{n^{\beta-1}\overline{F}(0)}{\Gamma(\beta)\Gamma(1-\beta)L(n)} \quad \textit{as } n \to \infty \tag{5.122}$$

and

$$\limsup_{n_0\to\infty} \sup_{n>n_0} \max_{0\leqslant k\leqslant n-1} \max_{m\in\mathbb{Z}} \frac{\#\big(A_0 \cap [m, m+2^k) \cap [2^{n_0}, 2^n)\big)}{n/\overline{F}(2^k)} < \infty \quad \textit{a.s.} \tag{5.123}$$

(iii) For any $\gamma, \eta > 0$

$$\#\left\{k : S_k^{(0)} \leqslant \eta n, \xi_k \geqslant (\log n)^\gamma\right\} \xrightarrow{P} \infty, \quad n \to \infty. \tag{5.124}$$

(iv) Let $\{Z(t)\}_{t\in\mathbb{R}_+}$ and $\{Z^*(t)\}_{t\in\mathbb{R}_+}$ be given by (5.9) *and (5.12), respectively. Then with ϑ_n given by (5.36),*

$$\left\{\frac{1}{\vartheta_n} S_{\lfloor nt\rfloor}^{(0)\leftarrow}\right\}_{t\in\mathbb{R}_+} \Rightarrow \{Z^{\leftarrow}(t)\}_{t\in\mathbb{R}_+} \quad \textit{in } \big(D(\mathbb{R}_+), J_1\big), \tag{5.125}$$

$$\left\{\frac{1}{\vartheta_n} S_{\lfloor nt\rfloor}^{(n)\leftarrow}\right\}_{t\in\mathbb{R}_+} \Rightarrow \{Z^{*\leftarrow}(t)\}_{t\in\mathbb{R}_+} \quad \textit{in } \big(D(\mathbb{R}_+), J_1\big). \tag{5.126}$$

Proof. (i): The claim follows from Theorem 5.4 in Samorodnitsky and Wang (2019).

(ii): See (A.2) and Lemma A.1 in Appendix A in Chen and Samorodnitsky (2020).

(iii): See Lemma A.2 in Appendix A in Chen and Samorodnitsky (2020).

(iv): We first show (5.125). For a sequence $\{c_n\}$ satisfying $n\overline{F}(c_n) \sim 1/\Gamma(1-\beta)$ we have

$$\left\{\frac{1}{c_n} S^{(0)}_{\lfloor nt \rfloor}\right\}_{t\in\mathbb{R}_+} \Rightarrow \{Z(t)\}_{t\in\mathbb{R}_+} \quad \text{in } \big(D(\mathbb{R}_+), J_1\big); \tag{5.127}$$

see *e.g.* Theorem 4.5.3 in Whitt (2002), and it follows that (5.125) holds in the M_1-topology (see Whitt (1971)). Because the process $\{Z^{\leftarrow}(t)\}_{t\in\mathbb{R}_+}$ is *a.s.* continuous, the convergence also holds in the J_1-topology, see Section 12.4 in Whitt (2002). We combine this argument with part (i) of the proposition to get (5.126). □

We will occasionally drop the superscript on our random walk whenever the discussion depends only on the step distribution of the walk. Denoting for $A \subset \mathbb{Z}$ and $a \in A$

$$\mathbb{P}_a\{\mathbf{S} \text{ escapes } A\} = \mathbb{P}\{\mathbf{S} \text{ does not hit } A \setminus \{a\} \,\big|\, S_0 = a\},$$

we define the capacity of A by

$$\operatorname{cap}(A) = \sum_{a\in A} \mathbb{P}_a\{\mathbf{S} \text{ escapes } A\}. \tag{5.128}$$

It is well known that $\operatorname{cap}(A_1) \leqslant \operatorname{cap}(A_2)$ if $A_1 \subset A_2$, and $\operatorname{cap}(A_1 \cup A_2) \leqslant \operatorname{cap}(A_1) + \operatorname{cap}(A_2)$ for any A_1, A_2; see Spitzer (1964).

For $0 \leqslant m_1 < m_2 \leqslant \infty$ we consider the range

$$A_0(m_1, m_2) = \{S^{(0)}_{m_1}, \ldots, S^{(0)}_{m_2-1}\},$$

so that $A_0 = A_0(0,\infty)$ is the full range.

Proposition 5.4.2. *(i) Fix $\gamma > (1-2\beta)^{-1}$, and let $\mathbf{S} = \{S_t\}_{t\in\mathbb{N}_0}$ be independent of A_0. Then*

$$\max_{0\le j\le n-(\log n)^\gamma} \mathbb{P}\Big(\mathbf{S} \ \text{hits} \ A_0 \cap \big\{j + \lceil(\log n)^\gamma\rceil, \ldots, n\big\} \big| S_0 = j, A_0\Big) \to 0 \tag{5.129}$$

a.s. as $n \to \infty$*. In particular, if for* $n = 1, 2, \ldots, V_{1;n}$ *and* $V_{1;n}$ *are measurable nonempty subsets of* $A_0 \cap \{0, \ldots, n\}$ *with* $\min V_{2;n} - \max V_{1;n} \geqslant (\log n)^{\gamma}$*, then*

$$\frac{1}{\# V_{1;n}} \Big(cap\,(V_{1;n} \cup V_{2;n}) - cap\,(V_{1;n}) - cap\,(V_{2;n}) \Big) \to 0 \quad a.s.. \tag{5.130}$$

(ii) Let $\widetilde{\mathbf{S}}^{(0)}$ *be an independent copy of* $\mathbf{S}^{(0)}$*, with ranges denoted by* $\widetilde{A}_0(\cdot, \cdot)$*. Then*

$$c_{\infty} := \mathbb{P} \left\{ A_0 \cap \widetilde{A}_0 = \{0\} \right\} \in (0, 1). \tag{5.131}$$

Furthermore,

$$\frac{cap\,(A_0(0, n))}{n} \longrightarrow c_{\infty} \quad a.s. \ as\ n \to \infty. \tag{5.132}$$

Proof. (i): For (5.129) we write for $0 \leq j \leq n - (\log n)^{\gamma}$,

$$\begin{aligned} & P\Big(\mathbf{S} \ \text{ hits } \ A_0 \cap \{ j + \lceil (\log n)^{\gamma} \rceil, \ldots, n \} \big| S_0 = j, A_0 \Big) \\ \leqslant & \sum_{k = \lfloor \log_2 \lceil (\log n)^{\gamma} \rceil \rfloor}^{\lceil \log_2 n \rceil} \# \Big(A_0 \cap [j + 2^k, j + 2^{k+1}) \Big) \cdot \max_{m \geq 2^k} \mathbb{P}_0 \left\{ \mathbf{S} \text{ hits } m \right\}. \end{aligned}$$

By (5.123), there is an *a.s.* finite constant $B_1 = B_1(A_0)$ such that the first term in the sum does not exceed $B_1 \log n / \bar{F}(2^k)$, while by (5.122), the second term does not exceed $B_2 2^{(\beta-1)k} / L(2^k)$ for some finite constant B_2. Therefore, for any $\epsilon > 0$, the sum above can be bounded by

$$B_1 B_2 \sum_{k = \lfloor \log_2 (\log n)^{\gamma} \rfloor}^{\infty} \frac{\log n}{\overline{F}(2^k)} \cdot \frac{2^{(\beta-1)k}}{L(2^k)} \lesssim (\log n)^{1 + (2\beta - 1 + \epsilon)\gamma}.$$

Choosing $0 < \epsilon < 1 - 2\beta - \gamma^{-1}$ proves (5.129).

For (5.130) we enumerate, for a fixed n and $i = 1, 2$, $V_{i;n}$ from left to right as

$$\{ v_{i,1}, \ldots, v_{i,n_i} \}.$$

Then

$$\begin{aligned}\operatorname{cap}(V_{1;n} \cup V_{2;n}) &= \sum_{j=1}^{n_1} \mathbb{P}_{v_{1,j}} \left\{ \mathbf{S} \text{ escapes } V_{1;n} \cup V_{2;n} \middle| A_0 \right\} + \operatorname{cap}(V_{2;n}) \\ &= \operatorname{cap}(V_{1;n}) - \sum_{j=1}^{n_1} q_j + \operatorname{cap}(V_{2;n}),\end{aligned}$$

where in the obvious notation

$$q_j := \mathbb{P}\left\{ \mathbf{S} \text{ escapes } V_{1;n} \text{ but hits } V_{2;n} \middle| A_0, S_0 = v_{1,j} \right\}.$$

Now (5.130) follows from (5.129).

(ii): Note that by (5.122),

$$\mathbb{P}\{k \in A_0 \cap \widetilde{A}_0\} = \mathbb{P}\{k \in A_0\} \cdot \mathbb{P}\{k \in \widetilde{A}_0\} \in \mathrm{RV}_{2\beta-2},$$

so it is a summable sequence. This implies (5.131). To prove (5.132), we observe that the array $\{\operatorname{cap}(A_0(m_1, m_2))\}_{m_1 < m_2}$ forms a stationary and subadditive family, so by the subadditive ergodic theorem (see Theorem 5 in Kingman (1968)) we have

$$\frac{\operatorname{cap}(A_0(0,n))}{n} \to \Upsilon \quad a.s.$$

for some random variable $0 \leqslant \Upsilon \leqslant 1$. Since the invariant σ-field associated with the array is clearly trivial, it follows from Theorem 3 *ibid.* that Υ is a constant. It remains to show that the constant is equal to c_∞. We have

$$\Upsilon = \lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n} \mathbb{P}\left\{ \widetilde{A}_0 + S_i^{(0)} \bigcap A_0(i,n) = \{i\} \right\}$$

a.s., hence

$$\Upsilon = \lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n} \mathbb{P}\left\{ \widetilde{A}_0 \cap A_0(0, n-i) = \{0\} \right\} \quad a.s.$$

This is the arithmetic mean of a sequence that converges to c_∞, so $\Upsilon = c_\infty$ *a.s.*. □

Proposition 5.4.3. *For $n \in \mathbb{N}$, let*

$$\overline{A_n} = \{S_t^{(n)} : t = 0, 1, 2, \ldots\} \cap \{0, \ldots, n\}.$$

Let $\{Z^*(t)\}_{t\in\mathbb{R}_+}$, $\overline{R^*}$ and ϑ_n be as in (5.12), (5.17) and (5.36) *respectively.*

(i) As $n \to \infty$

$$\left(\left\{\frac{1}{\vartheta_n} S^{(n)\leftarrow}_{\lfloor nt \rfloor}\right\}_{t\in\mathbb{R}_+}, \frac{\overline{A_n}}{n}\right) \Rightarrow \left(\{Z^{*\leftarrow}(t)\}_{t\in\mathbb{R}_+}, \overline{R^*}\right) \quad \textit{in } \big(D(\mathbb{R}_+), J_1\big)\times\mathcal{F}([0,1]).$$

(ii) As $n \to \infty$

$$\left\{\frac{cap\left(\overline{A_n} \cap \{\lceil nx \rceil, \dots, \lfloor ny \rfloor\}\right)}{\vartheta_n}, 0 \leqslant x < y \leqslant 1\right\}$$

$$\Rightarrow \{c_\infty \left(Z^{*\leftarrow}(y) - Z^{*\leftarrow}(x)\right), 0 \leqslant x < y \leqslant 1\}$$

in finite-dimensional distributions.

Proof. (i): Since the marginal convergence has been established in Proposition 5.4.1 (i) and (iv), the tightness is automatic. It suffices, therefore, to show uniqueness of subsequential weak limits. Suppose that for some subsequence $\{n_k\}$,

$$\left(\frac{1}{\vartheta_{n_k}}\left\{S^{(n_k)\leftarrow}_{\lfloor n_k t \rfloor}\right\}_{t\in\mathbb{R}_+}, \frac{1}{n_k}\overline{A_{n_k}}\right) \Rightarrow \left(\{Z^\kappa(t)\}_{t\in\mathbb{R}_+}, R^\kappa\right)$$

for some Z^κ and R^κ. We will prove that, necessarily,

$$\left(\{Z^\kappa(t)\}_{t\in\mathbb{R}_+}, R^\kappa\right) \overset{\mathrm{d}}{=} \left(\{Z^{*\leftarrow}(t)\}_{t\in\mathbb{R}}, \overline{R^*}\right). \tag{5.133}$$

To this end, recall that the π-systems

$$\mathcal{C}_D = \left\{\{x(t_i) > a_i : t_i \geqslant 0, a_i \in \mathbb{R}, 1 \leqslant i \leqslant \ell\} : \ell \in \mathbb{N}\right\}$$

and

$$\mathcal{C}_\mathcal{F} = \left\{\mathcal{F}^T : T \ \text{ a finite union of open intervals}\right\}$$

generate the respective σ-fields in $D(\mathbb{R}_+)$ and $\mathcal{F}$, so it is enough to check that the laws of $\left(\{Z^\kappa(t)\}_{t\in\mathbb{R}_+}, R^\kappa\right)$ and $\left(\{Z^*(t)\}_{t\in\mathbb{R}_+}, \overline{R^*}\right)$ agree on $\mathcal{C}_D \times \mathcal{C}_\mathcal{F}$. That is, for any $\ell_1, \ell_2 \in \mathbb{N}_0$, $t_i > 0, a_i \in \mathbb{R}$, $i = 1, \dots, \ell_1$ and disjoint open intervals $T_1, \dots, T_{\ell_2}$, we have

$$\mathbb{P}\Big\{\bigcap_{\substack{1\leqslant i\leqslant \ell_1\\ 1\leqslant j\leqslant \ell_2}} \{R^\kappa \cap T_j \neq \emptyset, Z^\kappa(t_i) > a_i\}\Big\} = \mathbb{P}\Big\{\bigcap_{\substack{1\leqslant i\leqslant \ell_1\\ 1\leqslant j\leqslant \ell_2}} \{\overline{R^*} \cap T_j \neq \emptyset, Z^{*\leftarrow}(t_i) > a_i\}\Big\}.$$

Denote $T_1 = (c_1, d_1), \dots, T_{\ell_2} = (c_{\ell_2}, d_{\ell_2})$. We have

$$\begin{aligned}
&\mathbb{P}\Big\{ \bigcap_{\substack{1\leqslant i\leqslant \ell_1\\ 1\leqslant j\leqslant \ell_2}} \{Z^\kappa(t_i) > a_i, R^\kappa \cap T_j \neq \emptyset\} \Big\} \\
=&\lim_{k\to\infty} \mathbb{P}\Big\{ \bigcap_{\substack{1\leqslant i\leqslant \ell_1\\ 1\leqslant j\leqslant \ell_2}} \Big\{ \frac{S^{(n_k)\leftarrow}_{\lfloor n_k t_i \rfloor}}{\vartheta_{n_k}} > a_i, \overline{A_{n_k}} \cap n_k T_j \neq \emptyset \Big\} \Big\} \\
=&\lim_{k\to\infty} \mathbb{P}\Big\{ \bigcap_{\substack{1\leqslant i\leqslant \ell_1\\ 1\leqslant j\leqslant \ell_2}} \Big\{ \frac{S^{(n_k)\leftarrow}_{\lfloor n_k t_i \rfloor}}{\vartheta_{n_k}} > a_i, \frac{S^{(n_k)\leftarrow}_{\lfloor n_k d_j \rfloor} - S^{(n_k)\leftarrow}_{\lceil n_k c_j \rceil}}{\vartheta_{n_k}} > 0 \Big\} \Big\} \\
=&\mathbb{P}\Big\{ \bigcap_{\substack{1\leqslant i\leqslant \ell_1\\ 1\leqslant j\leqslant \ell_2}} \{Z^{*\leftarrow}(t_i) > a_i, Z^{*\leftarrow}(d_j) - Z^{*\leftarrow}(c_j) > 0\} \Big\} \\
=&\mathbb{P}\Big\{ \bigcap_{\substack{1\leqslant i\leqslant \ell_1\\ 1\leqslant j\leqslant \ell_2}} \big\{Z^{*\leftarrow}(t_i) > a_i, \overline{R^*} \cap T_j \neq \emptyset\big\} \Big\},
\end{aligned}$$

as long as we can justify the penultimate equality.

Since each $Z^{*\leftarrow}(t_i)$ is a continuous random variable, by the Portmanteau Theorem we only need to check that

$$\lim_{\varepsilon\to 0} \limsup_{n\to\infty} \mathbb{P}\Big\{ 0 < \frac{S^{(n)\leftarrow}_{\lfloor nd \rfloor} - S^{(n)\leftarrow}_{\lceil nc \rceil}}{\vartheta_n} < \varepsilon \Big\} = 0$$

for any $0 \leq c < d \leq 1$. This follows from the marginal convergence given in Proposition 5.4.1 (i).

(ii): By the Skorokhod embedding theorem, the convergence in part (i) of the proposition holds as the *a.s.* convergence on some probability space, which will again be denoted by $(\Omega, \mathscr{F}, \mathbb{P})$ for typographical convenience. Consider the partition $\Omega = \Omega_1 \cup \Omega_2$ with

$$\Omega_1 := \big\{\omega \in \Omega : \overline{R^*}(\omega) \cap (x, y) = \emptyset\big\}, \quad \Omega_2 := \big\{\omega \in \Omega : \overline{R^*}(\omega) \cap (x, y) \neq \emptyset\big\}.$$

We will show that the required convergence holds in probability on both Ω_1 and Ω_2.

Since $\overline{R^*}$ does not hit fixed points, we have $Z^{*\leftarrow}(y) - Z^{*\leftarrow}(x) = 0$ *a.s.* on Ω_1. Furthermore, we can write for some null event $\Omega_{0,1}$,

$$\Omega_1 = \Omega_{0,1} \cup \bigcup_{k\geqslant 1} \Omega_1^k, \quad \Omega_1^k := \left\{\rho\left(\overline{R^*}(\omega), [x,y]\right) \geqslant 1/k\right\},$$

where ρ is defined in (5.6). Since $\frac{1}{n}\overline{A_n} \longrightarrow \overline{R^s}$ a.s. in the Fell topology, the convergence holds in the Hausdorff metric ρ_{H} as well, so on each Ω_1^k,

$$\overline{A_n} \cap \{\lceil nx\rceil, \ldots, \lfloor ny\rfloor\} = \emptyset$$

for all n large enough. Hence the required convergence holds *a.s.* on Ω_1.

We now consider the event Ω_2. Let

$$\tau_1 = \inf\left\{\overline{R^*} \cap [x,y]\right\}, \quad \tau_2 = \sup\left\{\overline{R^*} \cap [\tau_1, y]\right\} \quad (\text{both } = y \text{ if } \overline{R^*} \cap [x,y] = \emptyset).$$

For some null event $\Omega_{0,2}$ we can write

$$\Omega_2 = \Omega_{0,2} \cup \bigcup_{k\geqslant 1} \Omega_2^k, \quad \Omega_2^k := \{\tau_2(\omega) - \tau_1(\omega) \geqslant 1/k\}.$$

On Ω_2, by the strong Markov property,

$$\mathrm{cap}\Big(\overline{A_n} \cap \{\lceil nx\rceil, \ldots, \lfloor ny\rfloor\}\Big) \stackrel{\mathrm{d}}{=} \mathrm{cap}\Big(A_0\Big(0, S^{(n)\leftarrow}_{\lfloor ny\rfloor} - S^{(n)\leftarrow}_{\lceil nx\rceil}\Big)\Big).$$

Once again, since $\frac{1}{n}\overline{A_n} \longrightarrow \overline{R^s}$ *a.s.* in the Hausdorf metric. So on each Ω_2^k, we have $S^{(n)\leftarrow}_{\lfloor ny\rfloor} - S^{(n)\leftarrow}_{\lceil nx\rceil} \to \infty$ *a.s.*. It follows by Proposition 5.4.2 (ii) that

$$\frac{\mathrm{cap}\left(\overline{A_n} \cap \{\lceil nx\rceil, \ldots, \lfloor ny\rfloor\}\right)}{S^{(n)\leftarrow}_{\lfloor ny\rfloor} - S^{(n)\leftarrow}_{\lceil nx\rceil}} \longrightarrow c_\infty$$

in probability on each Ω_2^k, hence also on the entire Ω_2. Finally,

$$\begin{aligned}&\frac{\mathrm{cap}\left(\overline{A_n} \cap \{\lceil nx\rceil, \ldots, \lfloor ny\rfloor\}\right)}{\vartheta_n}\\ =&\frac{\mathrm{cap}\left(\overline{A_n} \cap \{\lceil nx\rceil, \ldots, \lfloor ny\rfloor\}\right)}{S^{(n)\leftarrow}_{\lfloor ny\rfloor} - S^{(n)\leftarrow}_{\lceil nx\rceil}} \frac{S^{(n)\leftarrow}_{\lfloor ny\rfloor} - S^{(n)\leftarrow}_{\lceil nx\rceil}}{\vartheta_n} \to c_\infty\left(Z^{*\leftarrow}(y) - Z^{*\leftarrow}(x)\right)\end{aligned}$$

in probability on Ω_2. □

We proceed with an important lemma. Switching back to the terminology of Subsection 5.2.2, we suppose that the random elements $\{Y^{(k;n)}\}_{k\in\mathbb{N}}$ are defined on a probability space $(\Omega, \mathscr{F}, \mathbb{P})$, while $\{Y^{(0;n)}\}$ is defined on a different probability space, and the entire system is defined on the product probability space. We will use the notation $\mathbb{P}_\omega$ for the quenched (conditional) probability (computed with respect to $\{Y^{(0;n)}\}$). When needed in the sequel, the notion of quenched probability may change, and we will always specify its precise meaning.

Lemma 5.4.1. *For any $K \in \mathbb{N}$ and $\epsilon > 0$ there exists a sequence of events $\{\Omega^{\epsilon}_{[K];n}\}_{n\geqslant 1}$ in Ω satisfying*

$$\liminf_{n\to\infty} \mathbb{P}\left\{\Omega^{\epsilon}_{[K];n}\right\} > 1-\epsilon, \tag{5.134}$$

with the following properties.

(i) There exists $C = C(\epsilon) > 0$ such that for all $1 \leqslant k \leqslant K$ and all large n,

$$C^{-1}\frac{\vartheta_n}{w_n} \leqslant \overline{p}_{k;n} \leqslant C\frac{\vartheta_n}{w_n} \quad \text{on } \Omega^{\epsilon}_{[K];n}. \tag{5.135}$$

(ii) For all $1 \leqslant k_1 \neq k_2 \leqslant K$, $I_{k_1;n} \cap I_{k_2;n} = \emptyset$ on $\Omega^{\epsilon}_{[K];n}$.

(iii) For $1 \leqslant k_1 \leqslant k_2 \leqslant K$ and $n \geqslant 1$, set

$$\overline{p}_{[k_1:k_2];n} = \mathbb{P}\left\{I_{0;n} \cap (I_{k_1;n} \cup \dots \cup I_{k_2;n}) \neq \emptyset \,\middle|\, I_{k_1;n} \dots I_{k_2;n}\right\}.$$

Then

$$\overline{p}_{[k_1:k_2];n} = \sum_{k=k_1}^{k_2} \overline{p}_{k;n} - o\left(\frac{\vartheta_n}{w_n}\right) \quad \text{on } \Omega^{\epsilon}_{[K];n}. \tag{5.136}$$

Proof. The Skorohod embedding argument we have just used shows that there is a probability space (once again denoted by $(\Omega, \mathscr{F}, \mathbb{P})$) on which, for each $1 \leqslant k \leqslant K$,

$$\left(\left\{\frac{1}{\vartheta_n} S^{(k;n)\leftarrow}_{\lfloor nt\rfloor}\right\}_{t\in\mathbb{R}_+}, \frac{1}{n} I_{k;n}\right) \longrightarrow \left(\{Z^{*\leftarrow}_k(t)\}_{t\in\mathbb{R}_+}, \overline{R^{*}_k}\right) \tag{5.137}$$

a.s. in $\big(D(\mathbb{R}_+), J_1\big) \times \mathcal{F}([0,1])$ and

$$\frac{\operatorname{cap}\left(I_{k;n} \cap \{\lceil nx \rceil, \ldots, \lfloor ny \rfloor\}\right)}{\vartheta_n} \longrightarrow c_\infty \left(Z_k^{*\leftarrow}(y) - Z_k^{*\leftarrow}(x)\right) \tag{5.138}$$

in probability for all $0 \leqslant x \leqslant y \leqslant 1$. In the remainder of the proof we work on this probability space. We spell out the argument in the case $K = 2$; the general case can be treated similarly.

(i): The last visit decomposition shows that

$$\overline{p}_{k;n} = \operatorname{cap}(I_{k;n})/w_n, \quad k = 1, 2.$$

For $\varepsilon > 0$ choose $C_1 > 0$ so large that $\mathbb{P}\left(C_1^{-1} \leqslant Z^{*\leftarrow}(1) \leqslant C_1\right) > 1 - \epsilon/2$. Letting

$$\Omega^\epsilon_{[2];n} = \left\{c_\infty C_1^{-1} \leqslant \operatorname{cap}(I_{k;n})/\vartheta_n \leqslant c_\infty C_1,\, k = 1, 2\right\},$$

we see by (5.138) with $x = 0, y = 1$ that (5.134) holds. Then (5.135) holds with $C = C_1/c_\infty$.

(ii): Since $\mathbb{P}(I_{1;n} \cap I_{2;n} \neq \emptyset) \to 0$, we can make the events $\Omega^\epsilon_{[2];n}$ slightly smaller so that (5.134) still holds and the condition of (ii) also holds.

(iii): Recall that $\overline{R_1^*}$ and $\overline{R_2^*}$ intersect only on a null set. It follows from (5.137) that $\liminf \rho\left(I_{1;n}, I_{2;n}\right)/n > 0$ a.s. on $\Omega^\epsilon_{[2];n}$. After removing from $\Omega^\epsilon_{[2];n}$ the null set, The claim (5.136) now follows from (5.129), where the initial point j is the leftmost point in $I_{1;n} \cup I_{2;n}$ that is in $I_{0;n}$, and A_0 the extension to the left of that among $I_{1;n}, I_{2;n}$ which does not contain j. □

Lemma 5.4.2. *For any $K, m \in \mathbb{N}$, we have*

$$\lim_{n\to\infty} \mathbb{P}\left\{\textit{the numbers } \{j_{k,i;n}\}_{\substack{1\leqslant k\leqslant K\\ 1\leqslant i\leqslant m}} \textit{ are all different}\right\} = 1. \tag{5.139}$$

Proof. Again, we only spell out the argument in the case $K = 2$ and $m = 1$. For $0 < \epsilon < 1$ let $\Omega^\epsilon_{[K];n}$ and $C = C(\epsilon) > 0$ be as in Lemma 5.4.1. We have for a large

$a > 0$

$$
\begin{aligned}
&\mathbb{P}\left\{j_{1,1;n} = j_{2,1;n}\right\} \\
\leqslant &\mathbb{P}\left\{j_{1,1;n} = j_{2,1;n} \leqslant aC\frac{w_n}{\vartheta_n}\right\} + \mathbb{P}\left\{j_{1,1;n} > aC\frac{w_n}{\vartheta_n}\right\}.
\end{aligned}
$$

Letting now $\mathbb{P}_\omega$ be the quenched probability given $\{Y^{(1;n)}\}$, we recall that, with respect to $\mathbb{P}_\omega$, $j_{1,1;n}$ is geometrically distributed with success probability $\overline{p}_{1;n}$. Therefore,

$$
\begin{aligned}
&\mathbb{P}\left\{j_{1,1;n} > aC\frac{w_n}{\vartheta_n}\right\} = \int_\Omega \mathbb{P}_\omega\left\{j_{1,1;n} > aC\frac{w_n}{\vartheta_n}\right\}\mathbb{P}(d\omega) \\
\leq &\epsilon + \int_{\Omega^\epsilon_{[K];n}} \mathbb{P}_\omega\left\{j_{1,1;n} > aC\frac{w_n}{\vartheta_n}\right\}\mathbb{P}(d\omega) \\
\leq &\epsilon + \left(1 - C^{-1}\frac{\vartheta_n}{w_n}\right)^{aCw_n/\vartheta_n - 1} \to \epsilon + e^{-a}.
\end{aligned}
$$

On the other hand, by the inclusion-exclusion formula and Lemma 5.4.1 (iii),

$$
\begin{aligned}
&\mathbb{P}\left\{j_{1,1;n} = j_{2,1;n} \leqslant aC\frac{w_n}{\vartheta_n}\right\} \\
\leqslant &\epsilon + \frac{aCw_n}{\vartheta_n}\int_{\Omega^\epsilon_{[2];n}} \mathbb{P}_\omega\left\{I_{0;n}\cap I_{1;n} \neq \emptyset,\ I_{0;n}\cap I_{2;n} \neq \emptyset\right\}\mathbb{P}(d\omega) \\
\leqslant &\epsilon + \frac{aCw_n}{\vartheta_n}\sup_{\omega\in\Omega^\epsilon_{[2];n}} \mathbb{P}_\omega\left\{I_{0;n}\cap I_{1;n} \neq \emptyset,\ I_{0;n}\cap I_{2;n} \neq \emptyset\right\} \to \epsilon.
\end{aligned}
$$

Letting first $\epsilon \to 0$ and then $a \to \infty$ concludes the argument. □

Proof of Theorem 5.2.1. For notational simplicity we consider the case $K = m = 2$. Our method easily carries over to arbitrary K and m. We will once again use the Skorohod embedding and assume that (5.137) and (5.138) hold. Then $I_{k;n}/n \to \overline{R^*_k}$ *a.s.* and by Proposition 5.4.3 (ii), $w_n\overline{p}_{k;n}/\vartheta_n \to c_\infty Z^{*\leftarrow}_k(1)$ *a.s.* as well. We consider now the remaining components in (5.35).

Let $\mathbb{P}_\omega$ be the quenched probability given $\{Y^{(k;n)}\}$, $k = 1, 2$. To handle the second component in (5.35), it is enough to show that, for *a.s.* $\omega \in \Omega$,

$$\left(j_{1,1;n}\overline{p}_{1;n}, j_{2,1;n}\overline{p}_{2;n}\right) \Rightarrow (\Gamma_{1,1}, \Gamma_{2,1}) \tag{5.140}$$

under $\mathbb{P}_\omega$. For $0 < \epsilon < 1$, let $\Omega^{\epsilon}_{[2];n}$ be the event in Lemma 5.4.1. Let $x_1, x_2 > 0$. On $\Omega^{\epsilon}_{[2];n}$, for any subsequence (n_m) over which $\left(\overline{p}_{2;n_m}\right)^{-1} x_2 \geqslant \left(\overline{p}_{1;n_m}\right)^{-1} x_1$ we have

$$\begin{aligned}
&\mathbb{P}_\omega\left\{j_{1,1;n_m}\overline{p}_{1;n_m} \geqslant x_1,\, j_{2,1;n_m}\overline{p}_{2;n_m} \geqslant x_2\right\}\\
&= \left(1 - \overline{p}_{[1:2];n_m}\right)^{\lfloor (\overline{p}_{1;n_m})^{-1}x_1\rfloor - 1} \cdot \left(1 - \overline{p}_{2;n_m}\right)^{\lfloor (\overline{p}_{2;n_m})^{-1}x_2 - (\overline{p}_{1;n_m})^{-1}x_1\rfloor}\\
&= \left(\prod_{k=1}^{2}(1 - \overline{p}_{k;n_m}) + o\left(\frac{\vartheta_{n_m}}{w_{n_m}}\right)\right)^{\lfloor (\overline{p}_{1;n_m})^{-1}x_1\rfloor - 1}\\
&\quad \cdot \left(1 - \overline{p}_{2;n_m}\right)^{\lfloor (\overline{p}_{2;n_m})^{-1}x_2 - (\overline{p}_{1;n_m})^{-1}x_1\rfloor}\\
&= \left(1 + o(1)\right)\prod_{k=1}^{2}\left(1 - \overline{p}_{k;n_m} + o\left(\frac{\vartheta_{n_m}}{w_{n_m}}\right)\right)^{(\overline{p}_{k;n_m})^{-1}x_k} \to e^{-(x_1+x_2)}
\end{aligned}$$

as $m \to \infty$. The same is true for any subsequence (n_m) over which $\left(\overline{p}_{2;n_m}\right)^{-1} x_2 < \left(\overline{p}_{1;n_m}\right)^{-1} x_1$. We thus see that

$$\mathbb{P}_\omega\left\{j_{1,1;n_m}\overline{p}_{1;n_m} \geqslant x_1,\, j_{2,1;n_m}\overline{p}_{2;n_m} \geqslant x_2\right\} \to e^{-(x_1+x_2)}$$

over $\liminf \Omega^{\epsilon}_{[2];n}$ for every $0 < \epsilon < 1$ and, hence, also on an event of probability 1. Since this is true for all $x_1, x_2 > 0$, (5.140) follows.

We now consider the last component in (5.35). By Lemma 5.4.2 we only need to prove the following statement. Consider ℓ disjoint open intervals in $(0, 1)$, $\{B_i = (x_i, y_i) : i = 1, \ldots, \ell\}$. Then for any $\epsilon, \delta > 0$, there exists a sequence of events $\{\Omega'_n\}_{n\in\mathbb{N}}$ in Ω such that $\liminf_{n\to\infty} \mathbb{P}\{\Omega'_n\} \geqslant 1 - \epsilon$ and

$$\sup_{\omega\in\Omega'_n}\left|\mathbb{P}_\omega\left\{\bigcap_{r=1}^{\ell}\left\{\frac{1}{n}I_{1,1;n} \cap B_r \neq \emptyset\right\}\right\} - \mathbb{P}_1\left\{\bigcap_{r=1}^{\ell}\{J_{1,1} \in B_r\}\right\}\right| \leqslant \delta \tag{5.141}$$

where $\mathbb{P}_\omega$ is the quenched probability given $Y^{(1;n)}$ and $\mathbb{P}_1$ is the probability associated with an independent standard uniform random variable. We treat the cases $\ell = 1$ and $\ell \geqslant 2$ separately.

Suppose first that $\ell = 1$. By (5.16),

$$\mathbb{P}_1 \left\{J_{1,1} \in B_1\right\} = \frac{Z_1^{*\leftarrow}(y_1) - Z_1^{*\leftarrow}(x_1)}{Z_1^{*\leftarrow}(1)},$$

while by the last exit decomposition,

$$\mathbb{P}_\omega \left\{\frac{I_{1,1;n}}{n} \cap B_1 \neq \emptyset\right\} = \frac{\operatorname{cap}\big(I_{1;n}(\omega) \cap (nx_1, ny_1)\big)}{\operatorname{cap}(I_{1;n}(\omega))}.$$

Therefore, we can take $\Omega'_n = \Omega$ and (5.141) follows by (5.138).

If $\ell \geqslant 2$, then the second probability in (5.141) vanishes. Furthermore,

$$\begin{aligned} &\mathbb{P}_\omega \left\{I_{1,1;n} \cap nB_1 \neq \emptyset, \ldots, I_{1,1;n} \cap B_\ell \neq \emptyset\right\} \\ \leqslant &\mathbb{P}_\omega \left\{I_{1,1;n} \cap nB_1 \neq \emptyset, I_{1,1;n} \cap nB_2 \neq \emptyset\right\}. \end{aligned}$$

Letting $\delta = x_2 - y_1 > 0$ we have by the strong Markov property,

$$\begin{aligned} &\mathbb{P}\left\{I_{1,1;n} \cap nB_1 \neq \emptyset, I_{1,1;n} \cap nB_2 \neq \emptyset\right\} \\ \leqslant &\mathbb{P}\left\{A_0, \widetilde{A}_0 \;\text{ have a common point} > n\delta\right\} \to 0 \end{aligned}$$

as $n \to \infty$ by (5.122), we immediately obtain (5.141). ☐

Proof of Proposition 5.2.1. (i): The claim (5.37) follows from the obvious fact that

$$\# I_{1,1;n} \leqslant \#(A_0 \cap \widetilde{A}_0)$$

and the latter cardinality has the geometric distribution with success probability $1 - c_\infty$ (with c_∞ defined in (5.131)).

(ii): The claim follows from (i) by the Markov inequality. ☐

5.4.2 Additional auxiliary results

This section contains several auxiliary results that are essential in the main proofs. We start with describing certain useful properties of the functions V and h in (5.55) and some related functions. Let

$$G(x) = \left(1\Big/\gamma\overline{H}\right)^{\leftarrow}(x), \quad x \geqslant 1/\gamma; \tag{5.142}$$

notice that by the inverse function theorem,

$$xG'(x) = h \circ G(x). \tag{5.143}$$

Furthermore, by the Karamata theorem the function

$$x \mapsto \int_1^x \frac{du}{u^{1-\alpha} L_\alpha(u)}$$

is regularly varying at infinity with exponent α, so its inverse is regularly varying with exponent $1/\alpha$. Therefore, the function

$$\mathscr{L}(x) = x^{-1/\alpha} \left(\int_1^x \frac{du}{u^{1-\alpha} L_\alpha(u)} \right)^{\leftarrow} \tag{5.144}$$

is slowly varying.

Proposition 5.4.4. *The functions V, h, G and $\mathscr{L}$ have the following properties at infinity:*

$$G(x) - V(x) = o\big(h \circ G(x)\big). \tag{5.145}$$

$$V(x) \sim G(x) \sim (\log x)^{1/\alpha} \mathscr{L}(\log x), \tag{5.146}$$

$$h \circ V(x) \sim h \circ G(x) \sim \alpha^{-1} (\log x)^{1/\alpha - 1} \mathscr{L}(\log x). \tag{5.147}$$

Furthermore, for every $t > 0$ the function V satisfies

$$\lim_{x \to \infty} \frac{V(tx) - V(x)}{h \circ V(x)} = \log t. \tag{5.148}$$

Proof. The statement (5.145) follows from the properties of the tails in the Gumbel domain of attraction; see *e.g.* (2.4) in Chen and Samorodnitsky (2020). This now implies the first asymptotic equivalencies in (5.146) and (5.147). Since

$$G(x) = \big(\log(x\gamma)\big)^{1/\alpha} \mathscr{L}\big(\log(x\gamma)\big),$$

the second asymptotic equivalence in (5.146) follows from the regular variation. Further, by Karamata's theorem,

$$L_\alpha\big(G(x)\big) \sim \big(\mathscr{L}(\log x)\big)^\alpha / \alpha,$$

and the second asymptotic equivalence in (5.147) follows as well.

The version of the statement (5.148) with V replaced by G follows easily from the definition of G, and by (5.145) we may replace G by V. $\square$

We proceed with two lemmas used in the proof of Proposition 5.3.4. The first lemma is purely analytical, and we omit a straightforward proof.

Lemma 5.4.3. *(i) The function*

$$\psi(r) = \frac{(1-\beta)^{1/\alpha} + \beta^{1/\alpha}}{(1-\beta-r)^{1/\alpha}} - 1, \ 0 \le r < 1-\beta$$

is increasing to infinity. Furthermore, the numbers r_m defined by $\psi(r_m) = m$ satisfy $r_m < m/(m+1) - \beta$ for $m \geqslant 1$.

(ii) The function

$$\widetilde{\psi}(r) = (1-\beta-r)\big[\lfloor \psi(r) \rfloor + \big(\psi(r) - \lfloor \psi(r) \rfloor\big)^\alpha\big], \ 0 \le r < 1-\beta$$

is increasing and satisfies $\widetilde{\psi}(r) > r + \beta$ on $(0, 1-\beta)$. Finally, $\widetilde{\psi}(r) \to \infty$ as $r \to (1-\beta)^-$.

The next lemma is essential for Proposition 5.3.4.

Lemma 5.4.4. *Fix any* $m \in \mathbb{N}_0$.

(i) For $1 < b < \infty$ *such that* $\nu(1,b) > 0$ *let* $\{\xi_i\}_{i\in\mathbb{N}}$ *be i.i.d. random variables whose law is the restriction of* ν *to* $(1,b)$ *normalized to a probability measure. Then for any* $m \in \mathbb{N}_0$ *there are* $c_m, \gamma_m \geqslant 0$ *depending on* m *only such that for any* $y \in \big(mb, (m+1)b\big]$ *and* $d \geqslant m+1$,

$$\mathbb{P}\left\{\sum_{i=1}^{d} \xi_i \geqslant y\right\} \leqslant c_m^d b^{\gamma_m} \big(\overline{H}(b)\big)^m \overline{H}(y - mb). \tag{5.149}$$

(ii) For $0 < r < 1 - \beta$ *consider sequences* $z_n = V(w_n) + V(\vartheta_n) - V(w_n/n^r) - o\,(V(w_n))$ *and* $\overline{z}_n = V(w_n/n^r)$, $n \geqslant 1$. *Let* (r_m) *be as in Lemma 5.4.3. Then for any* $m \geqslant 1$ *there is* $\gamma_m > 0$ *such that for any* $r \in (r_m, r_{m+1}]$,

$$\lim_{n\to\infty} \frac{\mathbb{P}\left\{\sum_{\Gamma_j > n^r} V\left(w_n/\Gamma_j\right) 1_{\{0\in I_{j;n}\}} \geqslant z_n\right\}}{\overline{z}_n^{\gamma_m} \big(\overline{H}(\overline{z}_n)\big)^m \overline{H}(z_n - m\overline{z}_n)} = 0. \tag{5.150}$$

Proof. (i): Recall that $\overline{H}(x) = \exp\{-q(x)\}$ for $q \geq x_0$ with q increasing and concave, and $xq'(x) \leqslant q(x)$ for $x \geqslant x_1$, for some $x_1 \geqslant x_0$. Therefore, we can extend q in the obvious way from the range $[x_1, \infty)$ to an increasing and concave function on $[0, \infty)$, that vanishes at the origin. We work with this redefined H, while keeping the original notation H. There clearly is $C \geq 1$ so that $\mathbb{P}\{\xi_1 > x\} \leqslant C\overline{H}(x)$ for all $x > 0$. Since $\overline{H}$ is the tail of a subexponential distribution, there is $c_0 > 0$ such that, in the usual notation for the convolution power, for all $y > 0$

$$\mathbb{P}\left\{\sum_{i=1}^{d} \xi_i \geqslant y\right\} \leqslant C^d \overline{H^{*d}}(y) \leqslant c_0^d \overline{H}(y) \quad \text{for all } d \in \mathbb{N}, \tag{5.151}$$

see Proposition 4.1.10 in Samorodnitsky (2016). This gives (5.149) in the case of $m = 0$ and all $d \geqslant 1$ (with $\gamma_0 = 0$).

We proceed in the inductive manner. Assume that (5.149) holds for all $0 \leqslant m \leqslant m_0$ and all $d \geq m+1$. We first consider the case $m = m_0 + 1$ and $d = m + 1$.

Let H_b be the restriction of H to $(1, b)$. We still have

$$\mathbb{P}\{\xi_1 > x\} \leqslant (C/\|H_b\|)\overline{H_b}(x) \quad \text{for all } x > 0.$$

Therefore, for $y > 0$

$$\begin{aligned}
&\mathbb{P}\left\{\sum_{i=1}^{d} \xi_i \geqslant y\right\} \leqslant (C/\|H_b\|)^d \overline{H_b^{*d}}(y) \\
=&(C/\|H_b\|)^d \int_{(0,b)^d} 1_{\{\sum_{i=1}^{d} z_i > y\}} \prod_{i=1}^{d} \exp\{-q(z_i)\} q'(z_i) dz_i \\
\leqslant&(C/\|H_b\|)^d \big(q(b)\big)^{m+1} \\
&\cdot \exp\left\{-\inf\left\{\sum_{i=1}^{m+1} q(z_i) : \sum_{i=1}^{m+1} z_i > y,\, 0 < z_1, \ldots, z_{m+1} \leqslant b\right\}\right\}.
\end{aligned}$$

Since $q(\cdot)$ is increasing and concave, for $y \in \big(mb, (m+1)b\big]$, the infimum is achieved at, say, $z_1 = \cdots = z_m = b$, $z_{m+1} = y - mb$. Since for $b > 1$, $q(b) \leq C_1 b^{2\alpha}$ for some $C_1 > 0$, this establishes (5.149) in the case $d = m + 1$ with γ_m and c_m that must be at least 2α and CC_1, correspondingly. Their final values will be set in the sequel.

We continue to induct on d while keeping the same m. Assume, therefore, that (5.149) is valid for $d = m + 1, \ldots, m + \ell$, some $\ell \geqslant 1$. In the case $d = m + \ell + 1$ write for $y \in \big(mb, (m+1)b\big]$, in the obvious notation

$$\mathbb{P}\left\{\sum_{i=1}^{d} \xi_i \geqslant y\right\} = \int_1^b F_\xi(dz) \mathbb{P}\left\{\sum_{i=1}^{d-1} \xi_i \geqslant y - z\right\} =: T_1 + T_2,$$

where T_1 and T_2 are the integrals over $(1, y - mb]$ and $(y - mb, b)$, correspondingly. To estimate T_1, note that in this range $y - z > mb$, so we may use the inductive assumption over d to obtain

$$\mathbb{P}\left\{\sum_{i=1}^{d-1} \xi_i \geqslant y - z\right\} \leqslant c_m^{d-1} b^{\gamma_m} \left(\overline{H}(b)\right)^m \overline{H}\left(y - z - mb\right).$$

By (5.151)

$$\begin{aligned}\int_1^{y-mb} F_\xi(dz)\overline{H}\,(y-z-mb) \leqslant & C\int_1^{\infty} H(dz)\overline{H}\,(y-z-mb)\\ \leqslant & (c_0^2 C)\overline{H}\,(y-mb)\,.\end{aligned}$$

It follows that

$$T_1 \leqslant (c_0^2 C)\, c_m^{d-1} b^{\gamma_m} \left(\overline{H}(b)\right)^m \overline{H}(y-mb). \tag{5.152}$$

To estimate T_2, note that in this range $(m-1)b < y-z \leqslant mb$, and we use the inductive assumption over m to write

$$T_2 \leqslant c_{m-1}^{d-1} b^{\gamma_{m-1}} \left(\overline{H}(b)\right)^{m-1} \int_{y-mb}^{b} F_\xi(dz)\overline{H}\big(y-(m-1)b-z\big).$$

Using the same optimization under concavity argument as above shows that

$$\begin{aligned}&\int_{y-mb}^{b} F_\xi(dz)\overline{H}\big(y-(m-1)b-z\big) \leqslant C\overline{H^{*2}}\big(y-(m-1)b\big)\\ \leqslant & C\big(q(b)\big)^2 \overline{H}(b)\overline{H}(y-mb).\end{aligned}$$

Therefore,

$$T_2 \leqslant C c_{m-1}^{d-1} b^{\gamma_{m-1}} \big(q(b)\big)^2 \left(\overline{H}(b)\right)^m \overline{H}(y-mb). \tag{5.153}$$

It follows from (5.152) and (5.153) that to complete the inductive argument we only need to make the final selection of γ_m and c_m to be so large as to satisfy

$$c_m^d \geq (c_0^2 C) c_m^{d-1} + C c_{m-1}^{d-1}, \quad b^{\gamma_m} \geq b^{\gamma_{m-1}} \big(q(b)\big)^2.$$

Since this can clearly be done, this completes the proof of (5.149).

(ii): Note that

$$\sum_{\Gamma_j \geqslant n^r} V\left(w_n/\Gamma_j\right) 1_{\{0\in I_{j;n}\}} \overset{\mathrm{d}}{=} \sum_{i=1}^{N_n} \xi_i, \tag{5.154}$$

where N_n is a Poisson random variable with mean $\overline{\nu}(x_0) - n^r/w_n$, and $\{\xi_i\}_{i\geqslant 1}$ is a family of *i.i.d.* random variables independent of N_n, whose law is the measure ν

restricted to the interval $(x_0, \overline{z}_n)$ and normalized to a probability measure there. Because of the range of r and (5.146) we see that for large n the event $\{\sum_{i=1}^{d} \xi_i > z_n\}$ requires d to be at least $m+1$. Therefore, in the notation of the first part of the lemma, by (5.149),

$$\begin{aligned}
\mathbb{P}\left\{\sum_{i=1}^{N_n} \xi_i > z_n\right\} &= \sum_{d=m+1}^{\infty} \mathbb{P}\left\{\sum_{i=1}^{d} \xi_i > z_n\right\} \mathbb{P}\left\{N_n = d\right\} \\
&\leqslant \sum_{d=m+1}^{\infty} \frac{c_m^d \overline{z}_n^{\gamma_m} \left(\overline{H}(\overline{z}_n)\right)^m \overline{H}(z_n - m\overline{z}_n)}{d!} \\
&\leqslant e^{c_m} \overline{z}_n^{\gamma_m} \left(\overline{H}(\overline{z}_n)\right)^m \overline{H}(z_n - m\overline{z}_n),
\end{aligned}$$

Since $\overline{z}_n \to \infty$, using $\gamma_m + 1$ from the first part of the lemma as γ_m in (5.150) gives us (5.150). $\square$

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