

ESTIMATING VARIANCE COMPONENTS OF BINOMIAL FREQUENCIES

BU-368-M

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April, 1971

Abstract

A method is developed for obtaining unbiased estimators of variance components of binomial probabilities based upon using observed frequencies in familiar analyses of variance.

# ESTIMATING VARIANCE COMPONENTS OF BINOMIAL FREQUENCIES

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## Summary

A method for obtaining unbiased estimators of variance components of binomial probabilities is developed from the analysis of variance of observed frequencies, without the need for any transformations.

## 1. Introduction

Data arising from biological sources are often measured as percentages or frequencies. Examples are mortality in poultry (Lush et al. [1948]) and in turkey embryos (Bogyo and Becker [1965]), hatchability in poultry (Crittenden et al. [1957]), conception rate in the artificial breeding of dairy cattle (Shannon and Searle [1962], and reproduction in beef cattle (Temple [1966] and Wythe [1970]). In all these cases the variable of interest is a rate or frequency, which we call  $p$ , that can take values in the  $(0, 1)$  interval. As such it can also be interpreted as a binomial probability, namely the probability of the occurrence of some event, for example, the probability that a turkey embryo will die, that an egg will hatch or that a cow will conceive.

Suppose  $\hat{p}$  represents an observed frequency from data of the nature just described. Analyses of variance of such observed frequencies are usually considered in terms of transformed frequencies, using a transformation designed to stabilize the variance of  $\hat{p}$ . Such, for example, is the purpose of the familiar

transformations  $\sqrt{\hat{p}}$  and  $\sin^{-1}\sqrt{\hat{p}}$  (e.g. Bartlett [1936]) and of improvements thereto of the nature  $\sqrt{\hat{p} + 3/8n}$  considered by Anscombe [1947]. Detailed discussion of the analysis of variance of observed frequencies  $\hat{p}$  and of these transformations of them is given by Gabriel [1963], and estimation of variance components when using the  $\sin^{-1}\sqrt{\hat{p}}$  transformation is considered by Bogyo and Becker [1965]. In contrast we present a method for obtaining unbiased estimators of variance components directly from analyses of variance of the  $\hat{p}$ 's as they stand, without transformation. The method can be used with unbalanced data and yields unbiased estimators of variance components of the  $p$ 's.

## 2. Preliminaries

Let  $\hat{p}$  be a sample binomial frequency based on  $n$  observations (e.g., a fraction  $\hat{p}$  of  $n$  turkey embryos dying). Then, for  $E_s$  representing expectation over repeated sampling of  $n$  observations, it is well known that the mean and variance of  $\hat{p}$  are

$$E_s(\hat{p}) = p \quad (1)$$

and

$$E_s(\hat{p} - p)^2 = pq/n \quad (2)$$

where

$$q = 1 - p, \quad (3)$$

$p$  being the population value of the underlying binomial distribution. Hence

$$E_s(\hat{p}^2) = pq/n + p^2, \quad (4)$$

so that from (1) and (4)

$$E_s\left(\frac{\hat{p}q}{n-1}\right) = \frac{pq}{n}. \quad (5)$$

These results are used frequently in the sequel, especially (5). And the notation  $E_p$  is used for expectation <sup>over binomial sampling</sup> to distinguish it from expectation over other sampling that will be discussed.

### 3. Estimating a simple variance

In the artificial breeding of dairy cattle ampules of diluted semen are administered to cows over a wide geographical region by many different technicians. Suppose in a sample of technicians who have administered semen from some particular bull that the observed conception rate based on  $n_i$  cows receiving that semen is  $\hat{p}_i$  for the  $i$ -th technician,  $i = 1, 2, \dots, a$ . Then, if  $p_i$  is the corresponding population conception rate,  $\hat{p}_i$  follows a binomial distribution, which we write as

$$\hat{p}_i | p_i \sim B(n_i, p_i), \quad (6)$$

the conditional distribution of  $\hat{p}_i$  given  $p_i$ . Now, corresponding to the random sample of technicians we consider  $p_i$  as a random variable with mean  $\pi$ , the true conception rate for the bull concerned, and variance  $\sigma^2$ . For this we might write the model

$$p_i = \pi + e_i \quad (7)$$

with  $E_p(p_i) = \pi$  and  $E_p(p_i - \pi)^2 = \sigma^2$ ,  $E_p$  representing expectation over repeated sampling of the  $p_i$ 's (in this example, of the technicians).

We seek an estimator of  $\sigma^2$  from the data  $\hat{p}_i$  and  $n_i$  for  $i = 1, \dots, a$ .

An "obvious" candidate to consider is

$$MS' = \frac{\sum_{i=1}^a \hat{p}_i^2 - \left( \sum_{i=1}^a \hat{p}_i \right)^2 / a}{a - 1} \quad (8)$$

In taking the expected value of (8) in the form

$$E[(a - 1)MS] = E_p E_s [(a - 1)MS]$$

first we consider the expectation represented by  $E_s$ :

$$E_s [(a - 1)MS] = \sum_{i=1}^a E_s (\hat{p}_i^2) - \left[ \sum_{i=1}^a \hat{p}_i^2 + \sum_{i \neq i'}^a E_s (\hat{p}_i \hat{p}_{i'}) \right] / a. \quad (9)$$

Now, by the nature of the observations on which  $\hat{p}_i$  and  $\hat{p}_{i'}$  (for  $i \neq i'$ ) are based, they are distributed independently so far as binomial sampling is concerned. Hence, using (1),

$$E_s (\hat{p}_i \hat{p}_{i'}) = p_i p_{i'}, \quad \text{for } i \neq i'. \quad (10)$$

Using (10) and (4) in (9) then gives

$$\begin{aligned} E_s [(a - 1)MS] &= \sum_{i=1}^a (p_i q_i / n_i + p_i^2) - \left[ \sum_{i=1}^a (p_i q_i / n_i + p_i^2) + \sum_{i \neq i'}^a p_i p_{i'} \right] / a \\ &= \sum_{i=1}^a p_i^2 - \left( \sum_{i=1}^a p_i \right)^2 / a + (1 - 1/a) \sum_{i=1}^a p_i q_i / n_i. \end{aligned} \quad (11)$$

Now define

$$C_1 = \frac{1}{a} \sum_{i=1}^a \frac{\hat{p}_i \hat{q}_i}{n_i - 1}. \quad (12)$$

Then, from (11) and (12), making use of (5), we have

$$E_s [(a - 1)(MS - C_1)] = \sum_{i=1}^a p_i^2 - \left( \sum_{i=1}^a p_i \right)^2 / a.$$

Taking the expected value of this with respect to the distribution of the  $p_i$ 's then gives, using (7),

$$E[(a - 1) (MS - C_1)] = E_P \left[ \sum_{i=1}^a p_i^2 - \frac{(\sum_{i=1}^a p_i)^2}{a} \right] = (a - 1)\sigma^2.$$

Hence an unbiased estimator of  $\sigma^2$  is

$$\begin{aligned} \hat{\sigma}^2 &= MS - C_1 \\ &= \frac{\sum_{i=1}^a \hat{p}_i^2 - (\sum_{i=1}^a \hat{p}_i)^2/a}{a - 1} - \frac{1}{a} \sum_{i=1}^a \frac{\hat{p}_i \hat{q}_i}{n_i - 1}. \end{aligned} \quad (13)$$

$C_1$  of (12) might conveniently be described as a correction for binomial sampling. An illustration of its use is given in Shannon and Searle [1962]. We extend it here to other analysis of variance situations, omitting details since they are based on the same principles as those used in deriving (13). Note that (13) holds true for any values of  $n_i$ , so long as each of them exceeds unity.

#### 4. The 1-way classification

Suppose data on egg hatchability (as considered by Crittenden et al. [1957], for example) consist of records for  $n_{ij}$  eggs from a random sample of  $N_i$  dams mated to the  $i$ 'th sire of a random sample of a sires. For the  $j$ 'th dam mated to the  $i$ 'th sire, we observe a proportion  $\hat{p}_{ij}$  of  $n_{ij}$  eggs hatching, corresponding to  $p_{ij}$  being the probability that an egg of the  $j$ 'th dam mated to the  $i$ 'th sire will hatch. Analogous to (6) we now have

$$\hat{p}_{ij} | p_{ij} \sim B(n_{ij}, p_{ij}) \quad (14)$$

and, similar to (7), use for  $p_{ij}$  the model

$$p_{ij} = \pi + \alpha_i + e_{ij} \quad (15)$$

for  $i = 1, \dots, a$  and  $j = 1, \dots, N_i$ , with  $N. = \sum_{i=1}^a N_i$ . The  $\alpha_i$  and  $e_{ij}$  are assumed to be random with zero means and variances  $\sigma_\alpha^2$  and  $\sigma_e^2$  respectively, with all covariances being zero. Although environmental factors such as different storage periods and different incubator positions for eggs from the same hen may affect hatchability, these factors are taken as being part of the error term for purposes of estimating variance components of the probabilities. Since these within full-sib group environmental factors are so highly, if not wholly, confounded with individual eggs it is difficult if not impossible to estimate the effect of these within group effects, as is evident in the study of Bingham et al. [1969].<sup>1/</sup>

Note that in (15) it is the true probabilities for which the variance component model is postulated, and not transformed frequencies such as are used, for example, in Cochran [1940] and Bogyo and Becker [1965]. They first use a transformation designed to stabilize the variance of the  $\hat{p}$ 's and then postulate a model for the transformed  $\hat{p}$ 's. Thus Cochran [1940, p. 342] writes "The unequal weighting may be removed by transforming... and assuming that the prediction formula is linear in the transformed scale." It is this assumption of linearity in the transformed scale that is avoided in our procedure, by assuming linearity for the true frequencies, as in (15). True it is, that linear models for probabilities may be questionable, although they are no more than the crutch upon which the variance components of interest are founded.

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<sup>1/</sup> We are grateful to a referee for bringing this point to our notice.

Nevertheless it is perhaps a little more reasonable to use these models directly on the probabilities as in (15) than on the transformed observed frequencies as do, for example, Cochran [1940] and Bogyo and Becker [1965].

Corresponding to (15), the between and within analysis of variance of the  $\hat{p}_{ij}$ 's has the sums of squares shown in Table 1.

Table 1: Analysis of variance for a 1-way classification of  $\hat{p}_{ij}$ 's.

Source	d.f.	Sums of Squares
Between	a-1	$SSB = \sum_{i=1}^a \left( \sum_{j=1}^{N_i} \hat{p}_{ij} \right)^2 / N_i - \left( \sum_{i=1}^a \sum_{j=1}^{N_i} \hat{p}_{ij} \right)^2 / N$
Within	$N_i - a$	$SSW = \sum_{i=1}^a \sum_{j=1}^{N_i} \hat{p}_{ij}^2 - \sum_{i=1}^a \left( \sum_{j=1}^{N_i} \hat{p}_{ij} \right)^2 / N_i$
Total	$N_i - 1$	$SST = \sum_{i=1}^a \sum_{j=1}^{N_i} \hat{p}_{ij}^2 - \left( \sum_{i=1}^a \sum_{j=1}^{N_i} \hat{p}_{ij} \right)^2 / N$

Using the corrections for binomial sampling

$$C_2 = \frac{1}{N} \sum_{i=1}^a \sum_{j=1}^{N_i} \frac{\hat{p}_{ij} \hat{q}_{ij}}{n_{ij} - 1} \quad (16)$$

and



$$c_3 = \sum_{i=1}^a \frac{1}{N_i} \sum_{j=1}^{N_i} \frac{\hat{p}_{ij} \hat{q}_{ij}}{n_{ij} - 1}, \quad (17)$$

manipulations similar to those of the preceding section yield

$$E_s[SSB - (c_3 - c_2)] = \sum_{i=1}^a \left( \sum_{j=1}^{N_i} p_{ij} \right)^2 / N_i - \left( \sum_{i=1}^a \sum_{j=1}^{N_i} p_{ij} \right)^2 / N.$$

and

$$E_s[SSW - (N \cdot c_1 - c_3)] = \sum_{i=1}^a \sum_{j=1}^{N_i} p_{ij}^2 - \sum_{i=1}^a \left( \sum_{j=1}^{N_i} p_{ij} \right)^2 / N_i.$$

Then, taking expectations over the  $p_{ij}$ -distribution, using (15), gives

$$E_p E_s[SSB - (c_3 - c_2)] = (a - 1)(N_0 \sigma_\alpha^2 + \sigma_e^2)$$

where

$$N_0 = (N - \sum_{i=1}^a N_i^2 / N) / (a - 1),$$

$$\text{and } E_p E_s[SSW - (N \cdot c_2 - c_3)] = (N - a) \sigma_e^2.$$

Hence unbiased estimators of the variance components are, using the mean squares available from Table 1, namely  $MSB = SSB / (a - 1)$  and  $MSW = SSW / (N - a)$ ,

$$\hat{\sigma}_e^2 = MSW - \frac{N \cdot c_2 - c_3}{N - a}$$

and

$$\hat{\sigma}_\alpha^2 = \frac{MSB - \hat{\sigma}_e^2}{N_0} = \frac{c_3 - c_2}{N_0(a - 1)}$$

$$= \frac{MSB - MSW}{N_0} + \frac{1}{N_0} \left[ \frac{N \cdot c_2 - c_3}{N - a} - \frac{c_3 - c_2}{a - 1} \right]$$

$$= \frac{MSB - MSW}{N_0} + \frac{(N - 1)(a c_2 - c_3)}{(N - a)(a - 1)}$$

The first terms in each of  $\hat{\sigma}_e^2$  and  $\hat{\sigma}_\alpha^2$  are the usual estimators for non-binomial situations; the second terms take account of the binomial sampling.

When the  $N_i$  values are equal (the same number of dams mated to each sire),  $N_i = N$  say, for all  $i$ , then  $N_0 = N$  also, and  $aC_2 = C_3$ . The estimators then reduce to

$$\hat{\sigma}_e^2 = MSW - C_2 \quad \text{and} \quad \hat{\sigma}_\alpha^2 = (MSB - MSW)/N, \quad (18)$$

with only  $\hat{\sigma}_e^2$  being affected by the binomial sampling.

#### 5. The 2-way crossed classification

Cochran [1940] discusses percentages of unfit ears of corn. Suppose we have<sup>a</sup> a sample of a crosses of 2 varieties of corn tested in a randomized complete block experiment of  $b$  blocks. If  $\hat{p}_{ij}$  is the observed fraction of unfit ears from among  $n_{ij}$  ears in the  $i$ -th cross in the  $j$ -th block, we have  $\hat{p}_{ij} | p_{ij} \sim B(n_{ij}, p_{ij})$  just as in (14). Only now, instead of (15) we use the model

$$p_{ij} = \pi + \alpha_i + \beta_j + e_{ij} \quad (19)$$

for  $i = 1, \dots, a$  and  $j = 1, \dots, b$ , where  $\alpha_i$  and  $\beta_j$  are random effects, with zero means as have the residual  $e_{ij}$ -terms, the variances being  $\sigma_\alpha^2$ ,  $\sigma_\beta^2$  and  $\sigma_e^2$  respectively. All covariances are assumed zero. The customary analysis of variance, made on the  $\hat{p}_{ij}$ 's, has the sums of squares shown in Table 2.

Table 2: Analysis of variance for a 2-way crossed classification of  $\hat{p}_{ij}$ 's .

Source	d.f.	Sums of Squares <sup>1</sup>
Crosses	a-1	$SSA = \sum_i (\sum_j \hat{p}_{ij})^2 / b - (\sum_i \sum_j \hat{p}_{ij})^2 / ab$
Blocks	b-1	$SSB = \sum_j (\sum_i \hat{p}_{ij})^2 / a - (\sum_i \sum_j \hat{p}_{ij})^2 / ab$
Residual	(a-1)(b-1)	$SSE = \sum_i \sum_j \hat{p}_{ij}^2 - (\sum_i \sum_j \hat{p}_{ij})^2 / ab - SSA - SSB$
Total	ab-1	$SST = \sum_i \sum_j \hat{p}_{ij}^2 - (\sum_i \sum_j \hat{p}_{ij})^2 / ab$

<sup>1</sup>Summation over i is for i = 1, ..., a and over j for j = 1, ..., b.

Using these sums of squares and the correction factor

$$C_4 = \frac{1}{ab} \sum_{i=1}^a \sum_{j=1}^b \frac{\hat{p}_{ij} \hat{q}_{ij}}{n_{ij} - 1}$$

it is not difficult to show, by the same methods as previously, that

$$E_p E_s [SSA - (a-1)C_4] = E_p (\sum_i p_{i.}^2 / b - p_{..}^2 / ab) = (a-1)(b\sigma_\alpha^2 + \sigma_e^2)$$

$$E_p E_s [SSB - (b-1)C_4] = E_p (\sum_j p_{.j}^2 / a - p_{..}^2 / ab) = (b-1)(a\sigma_\beta^2 + \sigma_e^2)$$

$$\begin{aligned} E_p E_s [SSE - (a-1)(b-1)C_4] &= E_p (\sum_i \sum_j p_{ij}^2 - \sum_i p_{i.}^2 / b - \sum_j p_{.j}^2 / a + p_{..}^2 / ab) \\ &= (a-1)(b-1)\sigma_e^2 . \end{aligned}$$

Unbiased estimators of the variance components provided by these results are, using the mean squares available from Table 2

$$\hat{\sigma}_e^2 = \text{MSE} - C_4$$

and

$$\hat{\sigma}_\alpha^2 = (\text{MSA} - \text{MSE})/b \quad \text{and} \quad \hat{\sigma}_\beta^2 = (\text{MSB} - \text{MSE})/a, \quad (20)$$

the latter two being unaffected by the binomial sampling.

### 6. The 2-way nested classification

Bogyo and Becker [1965] discuss data on pipping mortality in turkey embryos. Their basic datum can be considered as  $\hat{p}_{ijk}$ , the observed mortality rate among  $n_{ijk}$  embryos from the  $k$ -th dam mated to the  $j$ -th sire used in the  $i$ -th year. Here we have

$$\hat{p}_{ijk} | p_{ijk} \sim B(n_{ijk}, p_{ijk})$$

and use as a model for  $p_{ijk}$

$$p_{ijk} = \pi + \alpha_i + \beta_{ij} + e_{ijk} \quad (21)$$

where  $\alpha_i$  is a year effect,  $\beta_{ij}$  is the effect for the  $j$ -th sire used in the  $i$ -th year, these being random effects with zero means, as have the residual  $e_{ijk}$ -terms, the respective variances being  $\sigma_\alpha^2$ ,  $\sigma_\beta^2$  and  $\sigma_e^2$ . For a sample of  $t_0$  years, with  $b_i$  sires in the  $i$ -th year and  $N_{ij}$  dams mated/the  $j$ -th sire in that year, the analysis of variance of the  $\hat{p}_{ijk}$ 's has the sums of squares shown in Table

3. The T-terms there are

$$T_a = \sum_{i=1}^a \left( \sum_{j=1}^b \sum_{k=1}^{N_{ij}} \hat{p}_{ijk} \right)^2 / N_{i.}, \quad T_{ab} = \sum_{i=1}^a \sum_{j=1}^b \left( \sum_{k=1}^{N_{ij}} \hat{p}_{ijk} \right)^2 / N_{ij},$$

(22)

$$T_p = \sum_{i=1}^a \sum_{j=1}^b \sum_{k=1}^{N_{ij}} \hat{p}_{ijk}^2, \quad \text{and} \quad T_f = \left( \sum_{i=1}^a \sum_{j=1}^b \sum_{k=1}^{N_{ij}} \hat{p}_{ijk} \right)^2 / N_{..},$$

in accord with the notation used in Searle [1961].

Table 3. Analysis of variance for a 2-way nested classification of  $\hat{p}_{ijk}$ 's

Source	d.f.	Sum of Squares
Years	a - 1	SSA = $T_a - T_f$
Sires within years	b <sub>.</sub> - a	SSB/A = $T_{ab} - T_a$
Dams within sires (Residual)	N.. <sub>.</sub> - b.	SSE = $T_o - T_{ab}$
Total	N.. <sub>.</sub> - 1	SST = $T_o - T_f$

On defining

$$t_{ij} = \sum_{k=1}^{N_{ij}} \frac{\hat{p}_{ijk} \hat{q}_{ijk}}{n_{ijk} - 1}, \quad C_5 = t_{..} / N_{..},$$

$$C_6 = \sum_{i=1}^a (t_{i.} / N_{i.}) \quad \text{and} \quad C_7 = \sum_{i=1}^a \sum_{j=1}^b t_{ij} / N_{ij}$$

it will be found that for the sums of squares in Table 3

$$E_s[SSA - (C_6 - C_5)] = \tau_a - \tau_f$$

$$E_s[SSB/A - (C_7 - C_6)] = \tau_{ab} - \tau_a \quad (23)$$

$$E_s[SSE - (N..C_5 - C_7)] = \tau_o - \tau_{ab}$$

where the  $\tau$ 's are exactly the same functions of the  $p_{ijk}$ 's as the  $T$ 's of (22) are of the  $\hat{p}_{ijk}$ 's. Expectation of the right-hand sides of (23) over the  $p_{ijk}$ 's, using the model (21), therefore yields the customary expected sums of squares for the unbalanced 2-way nested classification given, for example, in Searle [1961]. From there, with

$$k_1 = \frac{a}{\sum_{i=1}^a N_{i.}^2} / N.., \quad k_3 = \frac{a}{\sum_{i=1}^a \sum_{j=1}^{b_i} N_{ij}^2} / N..$$

and

$$k_{12} = \frac{a}{\sum_{i=1}^a (\sum_{j=1}^{b_i} N_{ij}^2)} / N_{i.},$$

we therefore have

$$E_p E_s[SSA - (C_6 - C_5)] = (N.. - k_1)\sigma_\alpha^2 + (k_{12} - k_3)\sigma_\beta^2 + (a - 1)\sigma_e^2$$

$$E_p E_s[SSB/A - (C_7 - C_6)] = (N.. - k_{12})\sigma_\beta^2 + (b. - a)\sigma_e^2$$

$$E_p E_s[SSE - (N..C_5 - C_7)] = (N.. - b.)\sigma_e^2$$

Hence unbiased estimators of the variance components are obtained by solving the equations

$$(N.. - b.)\hat{\sigma}_e^2 = SSE - (N..C_5 - C_7)$$

$$(N.. - k_{12})\hat{\sigma}_\beta^2 + (b. - a)\hat{\sigma}_e^2 = SSB/A - (C_7 - C_6)$$

$$(N.. - k_1)\hat{\sigma}_\alpha^2 + (k_{12} - k_3)\hat{\sigma}_\beta^2 + (a - 1)\hat{\sigma}_e^2 = SSA - (C_6 - C_5)$$

When the data are balanced,  $N_{ij} = N$  for all  $i$  and  $j$ , and  $b_i = b$  for all  $i$ , these equations reduce to

$$\hat{\sigma}_e^2 = MSE - C_5$$

$$\hat{\sigma}_\alpha^2 = (MSB - MSE)/N \quad \text{and} \quad \hat{\sigma}_\beta^2 = (MSA - MSB)/bN,$$

in which we see that only the estimator of  $\sigma_e^2$  is affected by the binomial variation. The other estimators are not so affected. This is the same kind of situation as previously, as is evident in the estimators (18) and (20).

Extension of these methods to analyses of variance more complex than those considered here is clearly apparent.

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