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**SUPER-EXTREMAL PROCESSES
AND THE ARGMAX PROCESS**

by

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Keywords: random closed set, random upper semicontinuous function, super extremal process, argmax, Poisson process, closed set valued process.

Abstract

The theory of classical vector-valued extremal processes is extended to *super-extremal processes*, $\mathbf{Y} = \{\mathbf{Y}_t, t > 0\}$. At any $t > 0$, \mathbf{Y}_t is a random upper semicontinuous function on a locally compact, separable, metric space T . General path properties of \mathbf{Y} are discussed and it is shown that \mathbf{Y} is Markov with state-space $US(T)$. For each $t > 0$, \mathbf{Y}_t is associated.

For compact, metric T , we consider the *argmax* process $\mathbf{M} = \{M_t, t > 0\}$, where $M_t = \{\tau \in T, Y_t(\tau) = \vee_{\zeta \in T} Y_t(\zeta)\}$. \mathbf{M} is a closed set-valued random process, and its path properties are investigated in this general setting. We also study super extremal processes \mathbf{Y} , which have for each $t > 0$, \mathbf{Y}_t a.s. continuous.

SUPER-EXTREMAL PROCESSES AND THE ARGMAX PROCESS

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1 Introduction

In this paper, the classical theory of multivariate extremal processes is extended to function space (i.e. upper semicontinuous functions) valued extremal processes. A *super-extremal process* $\mathbf{Y} = \{\mathbf{Y}_t, t > 0\}$, has the property that for each $t > 0$, \mathbf{Y}_t is a random element of $US(T)$, the space of non-negative, real-valued, upper semicontinuous (usc) functions on a locally compact, separable metric space T . Alternatively, \mathbf{Y} may be viewed as a *sup-measure* valued random process (cf. Vervaat(1988)) with the super-extremal process as its *sup-derivative* valued process. When T is a compact, metric space, the *argmax process* $\mathbf{M} = \{M_t, t > 0\}$, is defined as $M_t = \{\tau \in T : \forall \zeta \in T Y_t(\zeta) = Y_t(\tau)\}$, for each $t > 0$, and for each $t > 0$, M_t is a random element of $\mathcal{F}(T)$, the space of closed subsets of T .

Our original interest in these processes arose from investigations into extensions of static random utility models of continuous choice (see Resnick and Roy(1991), and the references therein) to dynamic models. In the context of dynamic continuous choice, T represents the set of alternatives, \mathbf{Y}_t is interpreted as the random utility at time t for elements of T , and M_t is the random set of utility maximizing alternatives. In this paper, we develop the properties of \mathbf{Y} and \mathbf{M} in a general setting, and applications to dynamic choice modeling are taken up in Resnick and Roy(1991).

The paper progresses as follows. The topological preliminaries for the space $US(T)$ are set up in section 2. In section 3, the super-extremal process \mathbf{Y} is defined and its path properties are developed. We find that many of the properties of finite-dimensional vector valued extremal processes generalize quite nicely to the infinite dimensional case of super-extremal processes. In particular, \mathbf{Y} is Markov, has a version in $D((0, \infty), US(T))$, and for each $t > 0$, \mathbf{Y}_t is sup-infinitely divisible (cf. Norberg(1986)). Then in section 4, with T compact metric, the path properties of the argmax process \mathbf{M} are developed. For future applications to optimization models, general conditions for M_t to be a singleton set are provided. Miscellaneous results about \mathbf{Y} and \mathbf{Y}_t , in different settings

are collected in section 5. In Section 5.1, we show that sup-infinately divisible random elements of $US(T)$ (e.g. Y_t) are associated. In Section 5.2, with T compact, metric, we construct super-extremal processes X , such that for each $t > 0$, X_t is a random element of $C(T)$, the space of non-negative continous functions over T . We find that X has a version in $D((0, \infty), C(T))$. Finally in section 5.3, we demonstrate that for each $t > 0$, Y_t has a spectral representation. Appendix A.I verifies the measurability requirements for some sets utilized in the paper, and A.II describes the Hausdorff metric and stochastic convergence on $\mathcal{F}(T)$.

An index to the notation used in this paper is in Appendix A.III. Note that inequalities on functions and vectors should be interpreted componentwise. Hence $f \leq g$ means $f(\tau) \leq g(\tau)$, $\forall \tau \in T$, $f \vee g$ means the function $f \vee g(\tau) = f(\tau) \vee g(\tau)$ with domain T .

2 The Super-Extremal Process

Suppose $(\Omega, \mathcal{A}, \mathbf{P})$ is a complete probability space, and T is a locally compact, separable metric space, with countable dense subset D_T , and metric d . $\mathcal{B}(T)$ denotes the Borel σ -algebra on T . $US(T)$ is the space of upper semicontinuous functions from $T \mapsto [0, \infty]$, endowed with the sup-vague topology (cf. Vervaat(1988)). The *sup-vague* topology has basis sets of the form

$$\begin{aligned} \{f \in US(T) & : \bigvee_{t \in K} f(t) < x\} \\ \{f \in US(T) & : \bigvee_{t \in G} f(t) > x\} \end{aligned}$$

where $K \in \mathcal{K}(T)$, the compact subsets of T , and $G \in \mathcal{G}(T)$, the open subsets of T .

Henceforth, for any measurable $B \subseteq T$ and $f \in US(T)$ we use the notation

$$f^\vee(B) := \bigvee_{\tau \in B} f(\tau).$$

For $f \in US(T)$, f^\vee is a *regular maxitive capacity* (cf. Norberg(1986)) or *sup-measure* (cf. Ver-

vaat(1988)). By regular we mean that

$$f^\vee(B) = \vee_{K \subset B} f^\vee(K) = \wedge_{G \supset B} f^\vee(G); \quad G \in \mathcal{G}(T), K \in \mathcal{K}(T).$$

Convergence in the sup-vague topology (denoted by sup- \mathcal{K} convergence) is defined as follows : For $\{f_n\}, f \in US(T)$, $f_n \rightarrow f$ sup- \mathcal{K} iff

$$\begin{aligned} \limsup_n f_n^\vee(K) &\leq f^\vee(K), \quad K \in \mathcal{K}(T) \\ \liminf_n f_n^\vee(G) &\geq f^\vee(G), \quad G \in \mathcal{G}(T) \end{aligned}$$

$\mathcal{B}(US(T))$ denotes the usual Borel σ -algebra on $US(T)$, i.e. the σ -algebra generated by open sets. $US(T)$ is compact, separable and metrizable (cf. Dolecki, Salinetti, and Wets(1983), Norberg(1986)). Let $US_0(T) = US(T) - \{\mathbf{0}\}$, i.e. $US(T)$ punctured by removal of the function identically zero on T . Obviously, $US_0(T)$ is not compact.

For $\epsilon > 0$ and $B \subseteq T$ define

$$B^{\geq \epsilon} = \{f \in US(T) : f^\vee(B) \geq \epsilon\}.$$

The usual method of proving compactness for a set $A \subseteq US_0(T)$ is to check whether A is closed in $US(T)$ and $A \cap \{\mathbf{0}\} = \emptyset$. It is easily observed that $K^{\geq \epsilon}$ satisfies these criteria. The set $B^{> \epsilon}$ is defined similarly with obvious modifications. For $K \in \mathcal{K}(T)$, $K^{> \epsilon}$ is measurable and relatively compact in $US_0(T)$. We observe that if T is not compact, then $T^{\geq \epsilon}$ need not be closed in T . As an example, let $T = [0, \infty)$ and let $F(x) = 1 - e^{-x}$, $x \geq 0$ and set $f(x) = \epsilon F(x)$, so that $f^\vee(T) = \epsilon$. Define

$$g_n(x) = f(x)1_{[n, \infty)}(x)$$

so that each g_n is usc on T . We claim $g_n \rightarrow 0$ in $US(T)$, since for compact $K \subset [0, I] \subset T$

$$\limsup_{n \rightarrow \infty} g_n^\vee(K) \leq \limsup_{n \rightarrow \infty} g_n^\vee([0, I]) = 0$$

and for any open set $G \subset T$, $\liminf_{n \rightarrow \infty} g_n^\vee(G) \geq 0$. So $g_n \in T^{\geq \epsilon}$ (since $g_n^\vee(T) = f^\vee(T) = \epsilon$), $g_n \rightarrow 0$, but since $0^\vee(T) = 0$, we have $0 \notin T^{\geq \epsilon}$.

If $(\Omega, \mathcal{A}, \mathbf{P})$ is a complete probability space, we say that the map $\xi : \Omega \rightarrow US(T)$ is a random usc function if it is a random element of $(US(T), \mathcal{B}(US(T)))$. This means $\xi^{-1}(\mathcal{B}(US(T))) \subseteq \mathcal{A}$.

3 Construction of the Super-Extremal Process

We begin with a definition of a super-extremal process. Start by letting

$$N = \sum_{k \geq 1} \varepsilon_{(t_k, \eta_k)}$$

be a Poisson random measure (PRM) on $[0, \infty) \times US_0(T)$ with mean measure μ (see, for example, Resnick (1987)) such that μ is Radon (finite on compact sets) on $[0, \infty) \times US_0(T)$. We assume for all $t > 0$

$$\mu([0, t] \times \{f \in US(T) : f^\vee(K) = \infty\}) = 0, \forall K \in \mathcal{K}(T) \quad (1)$$

$$\mu([0, t] \times US_0(T)) = \infty \quad (2)$$

$$\mu(\{t\} \times \cdot) = 0. \quad (3)$$

The PRM N is time-homogeneous, if there exists a Radon measure ν on $US_0(T)$, such that for $A \in \mathcal{B}(US_0(T))$

$$\mu([0, t] \times A) = t\nu(A).$$

and ν satisfies the analogues of (1) and (2) above.

Define the *super-extremal process* $\mathbf{Y} = \{\mathbf{Y}_t, t > 0\}$ by

$$\mathbf{Y}_t := \bigvee_{t_k \leq t} \eta_k \quad (4)$$

(cf. Resnick(1987)). μ is often called the *sup-Levy* or *exponent* measure of \mathbf{Y} . As shown below in Theorem 2.1, the super-extremal process \mathbf{Y} is for fixed $t > 0$, a T -indexed stochastic process

$\mathbf{Y}_t = \{Y_t(\tau), \tau \in T\}$, which is a random element of the function space $US_0(T)$. Also define

$$Y_{st} = \bigvee_{s < t_k \leq t} \eta_k.$$

For $B \in \mathcal{B}(T)$, the process $Y^\vee(B) = \{Y_t^\vee(B), t > 0\}$ is a classical univariate extremal process, since

$$Y_t^\vee(B) = \bigvee_{t_k \leq t} \eta_k^\vee(B)$$

and

$$\sum_{k \geq 1} \varepsilon_{(t_k, \eta_k^\vee(B))}$$

is a PRM on $[0, \infty) \times (0, \infty]$ with mean measure of $[0, t] \times (x, \infty] = \mu([0, t] \times \{f \in US_0(T) : f^\vee(B) > x\})$ (cf. Resnick(1987, pp.180)).

The following theorem collects some facts about the super-extremal process $\mathbf{Y} = \{\mathbf{Y}_t, t > 0\}$. Following Whitt(1980), for a topological space S , let $D((0, \infty), S)$ denote the space of functions with domain $(0, \infty)$ and range S , which are right continuous and have left limits in S . Give $D((0, \infty), S)$ the Skorohod- J_1 topology when S is a separable metric space.

Theorem 3.1 $\mathbf{Y} = \{\mathbf{Y}_t, t > 0\}$ is a super-extremal process. Then

- (a) There exists Ω_* with $\mathbf{P}[\Omega_*] = 1$, such that for any fixed $t > 0$ and $\omega \in \Omega_*$, $\tau \rightarrow Y_t(\tau, \omega)$ is usc in τ . A version of $\{\mathbf{Y}_t, t > 0\}$ exists (also called $\{\mathbf{Y}_t, t > 0\}$) such that for each fixed $t > 0$, \mathbf{Y}_t is a random element of $US(T)$. Furthermore for any $B \in \mathcal{B}(T)$, $Y_t^\vee(B)$ is a random variable. For any fixed $t > 0$, $(\tau, \omega) \mapsto Y_t(\tau, \omega)$ is $\mathcal{B}(T) \times \mathcal{A}/\mathcal{B}((0, \infty))$ measurable.
- (b) \mathbf{Y} is stochastically continuous in the sup-vague topology.
- (c) There is a version of \mathbf{Y} which is a random element of $D((0, \infty), US(T))$. The map $(t, \omega) \mapsto \mathbf{Y}_t(\omega)$ is $\mathcal{B}((0, \infty)) \times \mathcal{A}/\mathcal{B}(US(T))$ measurable.

(d) $\mathbf{Y} = \{Y_t, t > 0\}$ is Markov with state-space $US_0(T)$ and its transition probabilities are determined by $(0 < s < t, h \in US_0(T), K_i \in \mathcal{K}(T), x_i \geq 0, i = 1, \dots, m)$:

$$\begin{aligned} & \mathbf{P}[Y_t^\vee(K_i) \leq x_i, i = 1, \dots, m | \mathbf{Y}_s = h] \\ &= \begin{cases} 0, & h^\vee(K_i) > x_i, \text{ some } i \in \{1, \dots, m\} \\ \exp(-\mu((s, t] \times \{f : f^\vee(K_i) \leq x_i, i = 1, \dots, m\}^c)) & \text{otherwise} \end{cases} \end{aligned}$$

Proof(a): Since T is a locally compact, separable metric space, there exists a sequence of compacta $K_j \subset \mathcal{K}(T)$ such that $K_j \subset K_{j+1}^\circ \uparrow T$, where K_{j+1}° denotes the interior of K_{j+1} . Since for every $j, K_j^{\geq \epsilon}$ is compact, for $\epsilon > 0$ we have for a fixed positive integer m

$$\mathbf{E}[N([0, m] \times K_j^{\geq \epsilon})] = \mu([0, m] \times K_j^{\geq \epsilon}) < \infty.$$

This implies for any $j > 0$

$$\mathbf{P}[N([0, m] \times K_j^{\geq n^{-1}}) < \infty] = 1. \quad (5)$$

Consequently for all $n > 0$, the sets $\Omega_{n,j}^{(m)} := \{\omega : N(\omega, [0, m] \times K_j^{\geq n^{-1}}) < \infty\}$ satisfy $\mathbf{P}[\Omega_{n,j}^{(m)}] = 1$.

For fixed $t \in [0, m]$ and $\omega \in \cap_{n,j} \Omega_{n,j}^{(m)}$ we show $\tau \mapsto Y_t(\tau, \omega)$ is usc. Fix $\tau_0 \in T$. Then there exists j_0 such that $\tau_0 \in K_{j_0}^\circ$ and to show upper semicontinuity we consider two cases.

Case 1 : If $Y_t(\tau_0, \omega) > 0$, then there exists n_0 such that $Y_t(\tau_0, \omega) \geq n_0^{-1}$. Since $\omega \in \cap_n \Omega_{n,j}^{(m)} \subset \Omega_{n_0, j_0}^{(m)}$ it follows that

$$\begin{aligned} \limsup_{\tau \rightarrow \tau_0} Y_t(\tau, \omega) &= \limsup_{\tau \rightarrow \tau_0} \bigvee_{t_k \leq t} \eta_k(\tau, \omega) \\ &= \limsup_{\tau \rightarrow \tau_0} \left[\left(\bigvee_{\substack{\eta_k^\vee(K_{j_0}, \omega) \geq n_0^{-1} \\ t_k \leq t}} \eta_k(\tau, \omega) \right) \vee \left(\bigvee_{\substack{\eta_k^\vee(K_{j_0}, \omega) < n_0^{-1} \\ t_k \leq t}} \eta_k(\tau, \omega) \right) \right] \\ &\leq \limsup_{\tau \rightarrow \tau_0} \left(\bigvee_{\substack{\eta_k^\vee(K_{j_0}, \omega) \geq n_0^{-1} \\ t_k \leq t}} \eta_k(\tau, \omega) \right) \vee n_0^{-1}. \end{aligned}$$

Since $\omega \in \Omega_{n_0, j_0}^{(m)}$, we deduce that $\vee_{\{\eta_k^\vee(K_{j_0}, \omega) \geq n_0^{-1}, t_k \leq t\}} \eta_k(\cdot, \omega)$ is the maximum of *finitely* many functions in $US_0(T)$ and hence is in $US_0(T)$. Thus

$$\begin{aligned} \limsup_{\tau \rightarrow \tau_0} Y_t(\tau, \omega) &\leq \left(\bigvee_{\substack{\eta_k^\vee(K_{j_0}, \omega) \geq n_0^{-1} \\ t_k \leq t}} \eta_k(\tau_0, \omega) \right) \vee n_0^{-1} \leq \bigvee_{t_k \leq t} \eta_k(\tau_0, \omega) \vee n_0^{-1} \\ &= Y_t(\tau_0, \omega) \vee n_0^{-1} = Y_t(\tau_0, \omega) \end{aligned}$$

since n_0 was chosen to satisfy $Y_t(\tau_0, \omega) \geq n_0^{-1}$.

Case 2 : Suppose $Y_t(\tau_0, \omega) = 0$ where $\omega \in \cap_{n,j} \Omega_{n,j}^{(m)}$. Then for any n

$$\limsup_{\tau \rightarrow \tau_0} Y_t(\tau, \omega) \leq \left(\bigvee_{\substack{\eta_k^\vee(K_{j_0}, \omega) \geq n^{-1} \\ t_k \leq t}} \eta_k(\tau_0, \omega) \right) \vee n^{-1} \leq n^{-1}.$$

and since n is arbitrary $\limsup_{\tau \rightarrow \tau_0} Y_t(\tau, \omega) = Y_t(\tau_0, \omega) = 0$.

For either Case 1 or Case 2, we have shown for $\omega \in \cap_{n,j} \Omega_{n,j}^{(m)} =: \Omega^{(m)}$

$$\limsup_{\tau \rightarrow \tau_0} Y_t(\tau, \omega) \leq Y_t(\tau_0, \omega).$$

Hence for any $t \in (0, m]$, $\tau \mapsto Y_t(\tau, \cdot)$ is usc with probability one. So on $\Omega_* = \cap_{m=1}^\infty \Omega^{(m)}$ we have for any $t > 0$, that $\tau \rightarrow Y_t(\tau, \omega)$ is usc and since $\mathbf{P}[\Omega_*] = 1$, we have proved the first part of (a).

Since

$$[Y_t \in \{f : f^\vee(K) < x\}] = [Y_t^\vee(K) < x] = [N([0, t] \times \{f : f^\vee(K) \geq x\}) = 0] \in \mathcal{A},$$

we have for fixed $t > 0$, $Y_t(\cdot)$ is a random element of $US(T)$. Also from the completeness of the probability space and Proposition 3.2, Norberg(1986), we conclude for $B \in \mathcal{B}(T)$, $Y_t^\vee(B)$ is a random variable. The $\mathcal{B}(T) \times \mathcal{A}/\mathcal{B}((0, \infty))$ measurability of \mathbf{Y}_t for fixed $t > 0$ is justified in Resnick & Roy(1991).

(b) Let $t_n \rightarrow t > 0$ and we need to show $Y_{t_n} \xrightarrow{\mathbf{P}} Y_t$ in the sup-vague topology. Equivalently we need to show that for any subsequence $\{t_{n''}\} \subset \{t_n\}$, there exists a further subsequence $\{t_{n'}\} \subset \{t_{n''}\}$ such that $Y_{t_{n'}} \rightarrow Y_t$ almost surely. Let $\mathcal{K}_0 \subset \mathcal{K}(T)$ be a countable collection of compacta such that T is locally \mathcal{K}_0 in the sense of Vervaat(1988), section 3; i.e. for any $\tau \in T$ and any open $G \ni \tau$ there exists $K \in \mathcal{K}_0$ such that $\tau \in K_0 \subset K \subset G$. Also let \mathcal{G}_0 be a countable base in T . From Theorem 5.3, Vervaat(1988) it suffices to check that almost surely

$$((Y_{t_{n'}}^\vee(G), Y_{t_{n'}}^\vee(K)), K \in \mathcal{K}_0, G \in \mathcal{G}_0) \rightarrow ((Y_t^\vee(G), Y_t^\vee(K)), K \in \mathcal{K}_0, G \in \mathcal{G}_0). \quad (6)$$

However, since for $B \in \mathcal{B}(T)$, $Y_t^\vee(B) = \{Y_t^\vee(B), t > 0\}$ is a one-dimensional extremal process, we have that $Y_t^\vee(B)$ is stochastically continuous (cf. Resnick(1987),page 180) Hence

$$((Y_t^\vee(G), Y_t^\vee(K)), K \in \mathcal{K}_0, G \in \mathcal{G}_0)$$

is a stochastically continuous \mathbf{R}^∞ valued stochastic process. So given $\{t_{n''}\}$, there exists $\{t_{n'}\} \subset \{t_{n''}\}$ such that in \mathbf{R}^∞ we have (6) holding almost surely. This suffices for the desired result.

(c) We first check that almost all paths of $\{Y_t, t > 0\}$ are right-continuous in the sup-vague topology. Observe that for each $\tau \in T$, $Y_s(\tau)$ is non-decreasing in s so for each $t > 0$

$$\lim_{s \downarrow t} Y_s(\tau) =: Y_{t+}(\tau)$$

exists. By Vervaat(1988, pp.18), $Y_{t+} = \{Y_{t+}(\tau), \tau \in T\}$ is in $US(T)$ and $Y_s \rightarrow Y_{t+}$ a.s. in the sup-vague topology, and for any $K \in \mathcal{K}(T)$, with probability one, $Y_s^\vee(K) \rightarrow Y_{t+}^\vee(K)$. Let \mathcal{K}_0 be countable and locally T . Since $\{Y_u^\vee(K), u > 0\}$ is a one-dimensional extremal process, it is right continuous on a set Ω_K of measure 1 so that $\{(Y_u^\vee, u > 0), K \in \mathcal{K}_0\}$ is right continuous on $\bigcap_{K \in \mathcal{K}_0} \Omega_K = \Omega_\cap$, which is a set of probability 1. So on Ω_\cap

$$\lim_{s \downarrow t} (Y_s^\vee(K), K \in \mathcal{K}_0) = (Y_t^\vee(K), K \in \mathcal{K}_0)$$

and thus

$$\mathbf{P}[(Y_t^\vee(K), K \in \mathcal{K}_0) = (Y_{t+}^\vee(K), K \in \mathcal{K}_0), \forall t > 0] = 1.$$

From the definition of upper semicontinuity we conclude $\mathbf{P}[\mathbf{Y}_t = \mathbf{Y}_{t+}, \forall t > 0] = 1$ which gives the desired right continuity.

We now show left-hand limits exist. For $s_n \uparrow t$, define the random sup-measure ($G \in \mathcal{G}(T)$)

$$Y_{t-}^\vee(G) = \bigvee_n Y_{s_n}^\vee(G)$$

(cf. Vervaat(1988), Theorem 6.2(a)). Let \mathbf{Y}_{t-} be the random sup-derivative of Y_{t-}^\vee , so that $\mathbf{P}[\mathbf{Y}_{t-} \in US(T)] = 1$ (cf. Vervaat(1988), section 11). Since $Y_{s_n}^\vee \rightarrow Y_{t-}^\vee$ in the vague topology on the space of sup-measures (cf. Vervaat(1988), section 3), we get that $\mathbf{Y}_{s_n} \rightarrow \mathbf{Y}_{t-}$ in the sup-vague topology on $US(T)$.

The joint measurability statement in (c) follows as usual from right-continuity.

Now modify \mathbf{Y} on a \mathbf{P} -null set such that $t \mapsto \mathbf{Y}_t(\omega)$ is in $D((0, \infty), US(T))$ for all $\omega \in \Omega$. \mathbf{Y}_t is a random element of $US(T)$, for any $t > 0$, and $\mathbf{Y}(\omega)$ is in $D((0, \infty), US(T))$ for all $\omega \in \Omega$. Since the Borel σ -field on $D((0, \infty), US(T))$ coincides with the Kolmogorov σ -field on $D((0, \infty), US(T))$ generated by the finite-dimensional (co-ordinate) projection maps on $D((0, \infty), US(T))$ (cf. Whitt(1980, Lemma 2.7)), we get that \mathbf{Y} is a random element of $D((0, \infty), US(T))$.

(d) For $0 < s < t$

$$\mathbf{Y}_t = \mathbf{Y}_s \vee \mathbf{Y}_{st}$$

where \mathbf{Y}_t and \mathbf{Y}_{st} are independent. Hence $\mathbf{Y} = \{\mathbf{Y}_t, t > 0\}$ is Markov with state space $US_0(T)$.

The transition probabilities are obtained by noting

$$\mathbf{P}[Y_t^\vee(K_i) \leq x_i, i = 1, \dots, m | \mathbf{Y}_s = h] = \mathbf{P}[h^\vee(K_i) \vee Y_{st}^\vee(K_i) \leq x_i, i = 1, \dots, m]. \bullet$$

Remark 3.1: Let ξ be a random element of $US(T)$. Then ξ is *sup-infinitely divisible* iff for each $n > 1$, there exists an i.i.d collection $\{\xi_i\}_{i=1}^n$, of random elements of $US(T)$, such that $\xi \stackrel{d}{=} \bigvee_{i=1}^n \xi_i$

(cf. Norberg(1986)). Then from the definition of the super-extremal process in (4), and Theorem 6.1, Norberg(1986), it follows that for each $t > 0$, \mathbf{Y}_t is sup-infinitely divisible.

Remark 3.2: The following distributional formulae can be derived from the definition of the super-extremal process \mathbf{Y} :

(i) For any $t > 0$, $\{G_j\}_{j=1}^n \in \mathcal{G}(T)$ and $x_j > 0$, $j = 1, \dots, n$

$$\mathbf{P}\left[\bigcap_{j=1}^n \{Y_t^\vee(G_j) \leq x_j\}\right] = \exp(-\mu([0, t] \times \cup_{j=1}^n \{f \in US_0(T) : f^\vee(G_j) > x_j\})).$$

(ii) For $h \in US_0(T)$ and $t > 0$

$$\mathbf{P}[\mathbf{Y}_t \leq h] = \exp(-\mu([0, t] \times \{f \leq h\}^c))$$

(The fact that $\{f \leq h\}^c \in \mathcal{B}(US(T))$ may be readily checked, and is left to the reader).

(iii) For $G \in \mathcal{G}(T)$, $(Y_t^\vee(G), t > 0)$ is a one-dimensional extremal process and hence for $0 < t_1 < \dots < t_n$ and $x_i > 0$, $i = 1, \dots, n$

$$\begin{aligned} & \mathbf{P}\left[\bigcap_{i=1}^n \{Y_{t_i}^\vee(G) \leq x_i\}\right] \\ &= e^{(-\mu([0, t_1] \times \{f \in US_0(T) : f^\vee(G) > \wedge_{i=1}^n x_i\}))} e^{(-\mu((t_1, t_2] \times \{f \in US_0(T) : f^\vee(G) > \wedge_{i=2}^n x_i\}))} \dots \\ & \dots e^{(-\mu((t_{n-1}, t_n] \times \{f \in US_0(T) : f^\vee(G) > x_n\}))} \end{aligned}$$

4 The Argmax Process

In this section, T is a compact metric space with metric d , and we have a fixed version of the super-extremal process $\mathbf{Y} = \{\mathbf{Y}_t, t > 0\}$ defined in (4), which is a random element of $D((0, \infty), US(T))$. Recall $\mathcal{F} = \mathcal{F}(T)$ is the class of closed subsets of T , and note that $\mathcal{F}(T) = \mathcal{K}(T)$. $\mathcal{F}(T)$ is given the *Fell* or *hit-miss* or *vague* topology, by declaring the following collection as sub-basis sets of the topology :

$$\{F \in \mathcal{F}(T) : F \cap K = \phi\}, \tag{7}$$

$$\{F \in \mathcal{F}(T) : F \cap G \neq \phi\} \quad (8)$$

for $K \in \mathcal{K}(T)$, $G \in \mathcal{G}(T)$. Since T is compact, this topology coincides with the topology generated by the Hausdorff metric, d_H (see definition in Appendix A.II) on $\mathcal{F}(T)$ (cf. Vervaat(1988)). $\mathcal{F}(T)$ is a compact, metric space in the vague topology. Convergence of sequences in the vague topology has the following characterization : For $\{F_n\}_{n \geq 1}$, $F \in \mathcal{F}(T)$, F_n vaguely converges to F , i.e. $F_n \xrightarrow{v} F$ as $n \rightarrow \infty$ iff

$$F \cap K = \phi \Rightarrow F_n \cap K = \phi \text{ eventually, } K \in \mathcal{K}(T) \quad (9)$$

$$F \cap G \neq \phi \Rightarrow F_n \cap G \neq \phi \text{ eventually, } G \in \mathcal{G}(T). \quad (10)$$

Equivalently, $F_n \xrightarrow{v} F$ iff $d_H(F_n, F) \rightarrow 0$.

The *upper topology* (cf. Castaing and Valadier(1977), Vervaat(1988)) on $\mathcal{F}(T)$, is generated by taking the collection of sets in (7) above as sub-basis. Convergence in the upper topology is characterized by (9) above, and denoted by $F_n \xrightarrow{u} F$. Equivalently $F_n \xrightarrow{u} F$ iff $d^u(F_n, F) \rightarrow 0$, where $d^u(F_1, F_2) = \sup_{x \in F_1} d(x, F_2)$ is the upper Hausdorff distance Cf. A.II). From the definition, we have $d^u(F_1, F_2) < \epsilon$ means $F_1 \subset F_2^\epsilon$, where F_2^ϵ is the ϵ -swelling or neighborhood of F_2 . Thus we always have $F_n \xrightarrow{u} T$, so limits are not unique and the upper topology is not Hausdorff. Call a function $H : S \rightarrow \mathcal{F}(T)$ (where S is a separable metric space) *upper continuous* iff H is continuous with respect to the upper topology on $\mathcal{F}(T)$: for any sequence $s_n \rightarrow s$ in S , $K \in \mathcal{F}(T)$

$$H(s) \cap K = \phi \Rightarrow H(s_n) \cap K = \phi \text{ eventually,} \quad (11)$$

which is equivalent to the sets $\{s \in S : H(s) \cap K = \phi\}$ being open in S . Finally we note that if $\tau_n \in T$, $n \geq 0$, we have $\{\tau_n\} \xrightarrow{u} \{\tau\}$ iff $d(\tau_n, \tau) \rightarrow 0$.

Similarly, the *lower topology* on $\mathcal{F}(T)$ is generated by taking the collection in (8) above as sub-basis. Convergence in the lower topology is characterized by (10) above, and is denoted by $F_n \xrightarrow{l} F$.

Then $F_n \xrightarrow{l} F$ iff $d_l(F_n, F) \rightarrow 0$, where d_l is the lower Hausdorff distance (cf. Appendix (A.II)).

Analogously, we say H is *lower continuous* iff for any sequence $s_n \rightarrow s$ in S , $G \in \mathcal{G}(T)$

$$H(s) \cap G \neq \phi \Rightarrow H(s_n) \cap G \neq \phi \text{ eventually,} \quad (12)$$

which is equivalent to the sets $\{s \in S : H(s) \cap G \neq \phi\}$ being open in S . Therefore H is vaguely continuous iff H is upper and lower continuous. In Castaing and Valadier(1977), upper (lower) continuous functions were called upper (lower) semicontinuous.

Let $\mathcal{B}(\mathcal{F}(T))$ be the Borel σ -algebra generated by the open subsets of $\mathcal{F}(T)$. A random element of $(\mathcal{F}(T), \mathcal{B}(\mathcal{F}(T)))$ is a *random (closed) set*(cf. Castaing and Valadier(1977), Vervaat(1988)).

The *argmax functional* A_v is defined on $US(T)$ to be

$$A_v(f) := \{\tau \in T : f(\tau) = f^v(T)\} \quad (13)$$

$$= \{\tau \in T : f(\tau) \geq f^v(T)\} \quad (14)$$

$$= f^{-1}[f^v(T), \infty]. \quad (15)$$

From (15) and the upper semicontinuity of f we see that $A_v(f)$ is closed (cf. Resnick and Roy(1991)). Furthermore A_v is $\mathcal{B}(US(T))/\mathcal{B}(\mathcal{F}(T))$ measurable since for $K \in \mathcal{K}(T)$

$$\begin{aligned} A_v^{-1}\{F : F \cap K = \phi\} &= \{f : A_v(f) \cap K = \phi\} =: K^{(<)} \\ &= \{f : f^v(K) < f^v(T)\} \\ &= \bigcup_{r \in Q_+} \{f : f^v(K) < r\} \cap \{f : f^v(T) > r\} \end{aligned}$$

where Q_+ is the set of non-negative rational numbers. Since this expresses $A_v^{-1}\{F : F \cap K = \phi\}$ as a countable union of open sets in $US(T)$ we have $A_v^{-1}\{F : F \cap K = \phi\}$ is open and

$$\begin{aligned} (K^{(<)})^c &= A_v^{-1}\{F : F \cap K \neq \phi\} \\ &= \{f \in US(T) : A_v(f) \cap K \neq \phi\} \end{aligned}$$

is closed in $US(T)$. So we get that the argmax functional A_v on $US(T)$ is upper continuous. This is noted for future reference in the lemma below.

Lemma 4.1 *Let $A_v : US(T) \rightarrow \mathcal{F}(T)$ be the argmax functional defined above in (13–15). Then A_v is upper continuous.*

In general, the sets $\{f \in US(T) : A_v(f) \subseteq K\} =: K^{(>)} (K \in \mathcal{F}(T))$ are *not* closed (see Appendix A.I.1) implying that the sets $\{f \in US(T) : A_v(f) \cap G \neq \emptyset\} (G \in \mathcal{G}(T))$ are *not* open. Thus in general, A_v is *not* lower continuous. Consider the following example. Let f_n be a sequence of functions defined on $T = [0, 1]$ by

$$\begin{aligned} f_n(x) &= 1/2, x \in [0, 1/2) \\ &= 1/2 + 1/n, x \in [1/2, 1], \end{aligned}$$

and set $f(x) = 1/2, \forall x \in [0, 1]$. So $\{f_n\}_{n \geq 1}, f \in US(T)$ and as $n \rightarrow \infty, f_n \downarrow f$ in the sup-vague topology. Then

$$A_v(f_n) = \{x : f_n(x) = \bigvee_{y \in [0, 1]} f_n(y)\} = [1/2, 1] \in \mathcal{F}(T), \forall n,$$

and $A_v(f) = \{x : f(x) = \bigvee_{y \in [0, 1]} f(y)\} = [0, 1] \in \mathcal{F}(T)$. $A_v(f_n)$ does *not* vaguely converge to $A_v(f)$ and in fact, for all n , the Hausdorff lower distance between $A_v(f_n)$ and $A_v(f)$ is $1/2$. Note that $A_v(f_n)$ converges to $A_v(f)$ in the upper topology.

For a super-extremal process $\mathbf{Y} = \{Y_t, t > 0\}$ (cf. section 2.1) define the set-valued process $\mathbf{M} = \{M_t, t > 0\}$ by

$$M_t := A_v(\mathbf{Y}_t). \tag{16}$$

\mathbf{M} is called the *argmax process*. Here are the basic facts about \mathbf{M} .

Theorem 4.1 $\mathbf{M} = \{M_t, t > 0\}$ is the argmax process of a super-extremal process $\mathbf{Y} = \{Y_t, t > 0\}$.

(a) For each $t > 0, M_t$ is a random element of $\mathcal{F}(T)$.

(b) M is right upper continuous in $\mathcal{F}(T)$ with probability one.

(c) M is $\mathcal{B}(0, \infty) \times \mathcal{A}/\mathcal{B}(\mathcal{F}(T))$ measurable.

(d) M is stochastically upper continuous.

Proof: For (a), see Resnick and Roy(1991, section 2). From Lemma 4.1, and Theorem 3.1, we get (b) and (d). Also from Theorem 3.1(c), we know that Y is $\mathcal{B}((0, \infty)) \times \mathcal{A}/\mathcal{B}(US(T))$ measurable. Since for any $K \in \mathcal{K}(T)$, the set $K^{(\geq)} = K^{(>)} \cup K^{(=)} \in \mathcal{B}(US(T))$ (see Appendix A.I), we get

$$\{(t, \omega) : M_t(\omega) \cap K \neq \emptyset\} = \{(t, \omega) : Y_t(\omega) \in K^{(\geq)}\} \in \mathcal{B}((0, \infty)) \times \mathcal{A},$$

which is (c). •

From Theorem 4.1, we know that $M : (0, \infty) \times \Omega \rightarrow \mathcal{F}(T)$ is $\mathcal{B}((0, \infty)) \times \mathcal{A}/\mathcal{B}(\mathcal{F}(T))$ -measurable. Hence there exists a *measurable selection*, i.e. a measurable function $\mathbf{m} : (0, \infty) \times \Omega \rightarrow T$ which is $\mathcal{B}((0, \infty)) \times \mathcal{A}/\mathcal{B}(T)$ -measurable and for \mathbf{P} -a.a. $\omega \in \Omega$,

$$m_t(\omega) \in M_t(\omega).$$

(cf. Castaing and Valadier(1977, III.6)). Thus the *selection process* $\mathbf{m} = \{m_t, t \in (0, \infty)\}$ is a measurable stochastic process, which \mathbf{P} -a.s. picks out elements in M .

Define $US(T)_{SING}$ to be the functions in $US_0(T)$ which achieve their maxima at a unique point in T :

$$\begin{aligned} US(T)_{SING} &:= \bigcup_{\tau \in T} \{f \in US_0(T) : f^\vee(T) = f(\tau) > f(\tau'), \forall \tau' \in T - \{\tau\}\} \\ &= \bigcup_{\tau \in T} \{f \in US_0(T) : A_\vee(f) = \{\tau\}\}. \end{aligned}$$

To check that this set is measurable, let $\{B(d_m, r_n)\}$ be closed balls covering T , with centres $d_m \in D_T$ and rational radii r_n . Then

$$[US(T)_{SING}]^c = \bigcup_{r_n \in Q_+} \bigcup_{d_m \in D_T} B(d_m, r_n)^{(\neq)} \in \mathcal{B}(US(T)), \quad (17)$$

where $B(d_m, r_n)^{(\neq)} = \{f : A_\vee(f) \cap B(d_m, r_n) \neq \phi, A_\vee(f) \cap B(d_m, r_n)^c \neq \phi\}$. Since $B(d_m, r_n)^{(\neq)} \in \mathcal{B}(US(T))$ (see A.I.3), we have $US(T)_{SING} \in \mathcal{B}(US(T))$.

From the measurability of \mathbf{Y} (cf. Theorem 3.1), and (17) above

$$\mathbf{Y}^{-1}([US(T)_{SING}]^c) \in \mathcal{B}((0, \infty)) \times \mathcal{A},$$

and its projection into Ω

$$\text{Proj}_\Omega (\mathbf{Y}^{-1}([US(T)_{SING}]^c)) = \bigcup_{t>0} \{\omega : (t, \omega) \in \mathbf{Y}^{-1}([US(T)_{SING}]^c)\}$$

is \mathcal{A} -analytic (cf. Dellacherie and Meyer(1978,III.13)). The probability space $(\Omega, \mathcal{A}, \mathbf{P})$ is complete, implying

$$\text{Proj}_\Omega (\mathbf{Y}^{-1}([US(T)_{SING}]^c)) = \bigcup_{t>0} \{\omega : M_t(\omega) \text{ is not singleton} \} \in \mathcal{A}.$$

(cf. Dellacherie and Meyer(1978,III.33)). Therefore the set

$$SING_{\mathcal{F}(T)} := \bigcap_{t>0} \{\omega : M_t(\omega) \text{ is singleton} \} \quad (18)$$

$$= [\text{Proj}_\Omega (\mathbf{Y}^{-1}([US(T)_{SING}]^c))]^c \in \mathcal{A}. \quad (19)$$

Also, for any fixed $t > 0$,

$$SING_{\mathcal{F}(T)}(t) := \mathbf{Y}_t^{-1}(US(T)_{SING}) \quad (20)$$

$$= \{\omega : M_t(\omega) \text{ is singleton}\} \in \mathcal{A}. \quad (21)$$

When M is \mathbf{P} -a.s. singleton, its path properties are nicer as is described in the next theorem.

Theorem 4.2 *Suppose $\mathbf{P}[SING_{\mathcal{F}(T)}] = 1$; i.e. with probability 1, M_t is singleton for all $t > 0$.*

Then

(a) M is right continuous in $\mathcal{F}(T)$ with probability one.

(b) \mathbf{M} is stochastically continuous.

Furthermore for $\omega \in SING_{\mathcal{F}(T)}$:

(c) There is only one measurable selection process $\mathbf{m}(\omega) = \{m_t(\omega), t > 0\}$.

(d) $\{m_t(\omega), t > 0\}$ is right continuous and finally

(e) \mathbf{m} is stochastically continuous.

Proof : (a) Define $\Omega_{ruc} := \{\omega : \mathbf{M}(\omega) \text{ is right upper continuous}\}$ whence $\mathbf{P}[\Omega_{ruc}] = 1$. Then for $\omega \in \Omega_{ruc} \cap SING_{\mathcal{F}(T)}$, we know that $\mathbf{M}(\omega)$ is right upper continuous, i.e. for $t_n \downarrow t$, $d^u(M_{t_n}(\omega), M_t(\omega)) \rightarrow 0$. But since $M_t(\omega)$ and $\{M_{t_n}(\omega)\}$ are all singleton, $d^u = d_l = d_H$ (see A.II). Thus $d_H(M_{t_n}(\omega), M_t(\omega)) \rightarrow 0$, which is the desired conclusion.

(b) From Theorem 4.1(d), \mathbf{M} is stochastically upper continuous, and therefore if $t_n \rightarrow t$ and any $\epsilon > 0$

$$0 = \lim_{n \rightarrow \infty} \mathbf{P}[d^u(M_{t_n}, M_t) > \epsilon] = \lim_{n \rightarrow \infty} \mathbf{P}[d_H(M_{t_n}, M_t) > \epsilon].$$

(c) For $\omega \in SING_{\mathcal{F}(T)}$ we have for all $t > 0$, $M_t(\omega) = m_t(\omega)$.

(d) and (e) : For single point sets, convergence with respect to d^u or d_H is equivalent to point-wise convergence. •

Remark 4.1 : In general, if singleton sets in $\mathcal{F}(T)$ upper converge to a possibly multivalued limit in $\mathcal{F}(T)$, then it does *not* imply that the selections converge, when treated as T -valued functions : Let $T = [-1, 1]$ and define a sequence in $\mathcal{F}([-1, 1])$ as follows,

$$F_n = \begin{cases} \{-1\}, & \text{if } n \text{ is odd,} \\ \{1\}, & \text{if } n \text{ is even,} \end{cases}$$

and $F = \{-1, 1\} \in \mathcal{F}([-1, 1])$. Then $d^u(F_n, F) \rightarrow 0$ but $d_l(F_n, F) = 2$, all n . No matter how the selections m_n, m are defined, it is never true that $m_n \rightarrow m$.

However it is true, that if a sequence of singletons in $\mathcal{F}(T)$ vaguely upper or lower converge to a *singleton limit* in $\mathcal{F}(T)$, then that sequence is vaguely convergent in $\mathcal{F}(T)$, and the corresponding selections are convergent as T -valued functions.

Now we characterize super-extremal processes whose usc sample paths (at each $t > 0$) have unique maxima. This will facilitate the construction of optimization models in applications where it is often useful to have singleton sets of parameters maximizing a random objective function.

Theorem 4.3 *Suppose $M = \{M_t, t > 0\}$ is the argmax process of a super-extremal process $Y = \{Y_t, t > 0\}$ with exponent measure μ satisfying for every t that the measure on $(0, \infty]$*

$$\mu([0, t] \times \{f \in US(T) : f^\vee(T) \in \cdot\})$$

is atomless. Then

(a) **P**-almost surely M_t is singleton for all $t > 0$, i.e. $\mathbf{P}[SING_{\mathcal{F}(T)}] = 1$, if and only if

$$\mu([0, \infty) \times [US(T)_{SING}]^c) = 0.$$

(b) M_t is **P**-a.s. singleton for any $t > 0$, i.e. $\mathbf{P}[SING_{\mathcal{F}(T)}(t)] = 1$, if and only if

$$\mu([0, t] \times [US(T)_{SING}]^c) = 0 \text{ in which case } \mathbf{P}[\cap_{s \leq t} SING_{\mathcal{F}(T)}(s)] = 1.$$

Proof: (a) Suppose first that $\mu([0, \infty) \times [US(T)_{SING}]^c) = 0$. Then for any $t > 0$,

$$\mathbf{P}[A] = \mathbf{P}[N([0, t] \times [US(T)_{SING}]^c) = 0] = 1$$

and we may write on $[N([0, t] \times [US(T)_{SING}]^c) = 0]$ that

$$Y_t = \left(\bigvee_{\substack{t_k \leq t, \eta_k^\vee(T) > \epsilon \\ \eta_k \in US(T)_{SING}}} \eta_k \right) \vee \left(\bigvee_{\substack{t_k \leq t, \eta_k^\vee(T) \leq \epsilon \\ \eta_k \in US(T)_{SING}}} \eta_k \right)$$

so that on the set $S_\epsilon(t) = [N([0, t] \times \{f : f^\vee(T) > \epsilon\}) > 0]$ we have

$$\mathbf{Y}_t = \left(\bigvee_{\substack{t_k \leq t, \eta_k^\vee(T) > \epsilon \\ \eta_k \in US(T)_{SING}}} \eta_k \right)$$

and the maximum is over a finite number of functions. Note

$$\begin{aligned} \lim_{\epsilon \downarrow 0} \mathbf{P}[S_\epsilon(t)] &= \lim_{\epsilon \downarrow 0} \mathbf{P}[N([0, t] \times \{f : f^\vee(T) > \epsilon\}) > 0] \\ &= \mathbf{P}[N([0, t] \times US_0(T)) > 0] = 1. \end{aligned}$$

The condition that $\mu([0, t] \times \{f \in US(T) : f^\vee(T) \in \cdot\})$ be atomless guarantees

$$\mathbf{P}[C] = \mathbf{P}[\eta_k^\vee(T) \neq \eta_l^\vee(T), \forall k, l \text{ with } t_k \vee t_l \leq t] = 1.$$

So for $0 < \delta < t$

$$\mathbf{P}[\cap_{\delta \leq s \leq t} \{M_s \text{ is singleton}\}] = \mathbf{P}[\cap_{\delta \leq s \leq t} \{M_s \text{ is singleton}\} \cap S_\epsilon(\delta) \cap C \cap A] + g_\epsilon(\delta)$$

where $g_\epsilon(\delta) \leq \mathbf{P}[S_\epsilon(\delta)^c] \rightarrow 0$ as $\epsilon \downarrow 0$. On $S_\epsilon(\delta) \cap C \cap A$ we have for any $s \in [\delta, t]$ that \mathbf{Y}_s is the maximum of a finite number of functions, each having a unique maximum and such that no two of these functions have the same maximum. So

$$S_\epsilon(\delta) \cap C \cap A \subset \bigcap_{\delta \leq s \leq t} [M_s \text{ is singleton}] = \bigcap_{\delta \leq s \leq t} SING_{\mathcal{F}(T)}(s)$$

and since $\mathbf{P}[S_\epsilon(\delta)] \rightarrow 1$ as $\epsilon \downarrow 0$ we get $\mathbf{P}[\cap_{\delta \leq s \leq t} SING_{\mathcal{F}(T)}(s)] = 1$. Letting first $\delta \downarrow 0$ and then $t \rightarrow \infty$ yields $\mathbf{P}[SING_{\mathcal{F}(T)}] = 1$ as desired.

Conversely, suppose $\mathbf{P}[SING_{\mathcal{F}(T)}] = 1$. For any $t > 0$ and any closed ball $B_{mn} = B(d_m, r_n) \subseteq T$, with centre $d_m \in D_T$ and rational radii r_n , the sets $B_{mn}^{(>)}, B_{mn}^{(<)}, B_{mn}^{(=)} \in \mathcal{B}(US(T))$ (see A.I.1–A.I.3).

Then, for any $B_{mn} \in \mathcal{K}(T)$ and any $t > 0$, define the random variables

$$X_t(B_{mn}^{(>)}) = \vee_{t_i \leq t} \eta_i^\vee(T) \mathbf{1}_{[\eta_i \in B_{mn}^{(>)}]}, \quad (22)$$

$$X_t(B_{mn}^{(<)}) = \bigvee_{t_i \leq t} \eta_i^\vee(T) \mathbf{1}_{[\eta_i \in B_{mn}^{(<)}]}, \quad (23)$$

$$X_t(B_{mn}^{(=)}) = \bigvee_{t_i \leq t} \eta_i^\vee(T) \mathbf{1}_{[\eta_i \in B_{mn}^{(=)}]}. \quad (24)$$

The complete randomness of the PRM N , implies that $X_t(B_{mn}^{(>)})$, $X_t(B_{mn}^{(<)})$ and $X_t(B_{mn}^{(=)})$ are independent. Also the atomless condition on μ ensures that no two of the random variables $X_t(B_{mn}^{(>)})$, $X_t(B_{mn}^{(<)})$ and $X_t(B_{mn}^{(=)})$ are equal with probability one.

Since $\mathbf{P}[SING_{\mathcal{F}(T)}] = 1$, implies for any $t > 0$, $\mathbf{P}[SING_{\mathcal{F}(T)}(t)] = 1$, from Eddy and Trader(1982), we get for any $B_{mn} \in \mathcal{K}(T)$

$$\mathbf{P}[M_t \cap B_{mn} \neq \phi] = \mathbf{P}[M_t \subseteq B_{mn}].$$

This implies

$$\begin{aligned} 0 &= \mathbf{P}[X_t(B_{mn}^{(=)}) > X_t(B_{mn}^{(>)}) \vee X_t(B_{mn}^{(<)})] \\ &= \int_{(0, \infty)} e^{-\mu_t(\{f \in B_{mn}^{(>)} : f^\vee(T) > x\})} e^{-\mu_t(\{f \in B_{mn}^{(<)} : f^\vee(T) > x\})} d[e^{-\mu_t(\{f \in B_{mn}^{(=)} : f^\vee(T) > x\})}] \\ &= - \int_{(0, \infty)} e^{-\mu_t(\{f \in US_0(T) : f^\vee(T) > x\})} d[\mu_t(\{f \in B_{mn}^{(=)} : f^\vee(T) > x\})]. \end{aligned}$$

Since $\mu_t(\{f \in US_0(T) : f^\vee(T) > x\}) \in [0, \infty)$ and is monotone, non-increasing in x , any $x > 0$ implies $\mu_t(\{f \in B_{mn}^{(=)} : f^\vee(T) > x, \text{ all } x \in (0, \infty)\}) = 0$. Therefore

$$\begin{aligned} \mu_t([US(T)_{SING}]^c) &= \mu_t \left(\bigcup_{r_n \in Q_+} \bigcup_{d_m \in D_T} B_{mn}^{(=)} \right) \\ &\leq \sum_{r_n \in Q_+} \sum_{d_m \in D_T} \mu_t(B_{mn}^{(=)}) = 0, \end{aligned}$$

whence $\mu_t([US(T)_{SING}]^c) = 0$ for any fixed $t > 0$. Then

$$\mu([0, t] \times [US(T)_{SING}]^c, \text{ any } t > 0) = \mu \left(\bigcup_{t \in (0, \infty)} \{[0, t] \times [US(T)_{SING}]^c\} \right)$$

and from monotone convergence

$$= \lim_{t \uparrow \infty} \mu_t([US(T)_{SING}]^c) = 0.$$

The proof of (b) proceeds as above, but for a fixed $t > 0$. •

Remark 4.2: Note Theorem 4.3(b) characterizes sup-infinitely divisible random elements of $US(T)$ with atomless sup-Lévy measures which have singleton argmax sets.

Remark 4.3: The atomless condition on the exponent measure in Theorem 4.3 cannot be weakened to get our characterization for singleton M_t . Consider the following example. Define

$$f(t) = \begin{cases} 1-t, & 0 \leq t \leq 1, \\ 0, & \text{otherwise} \end{cases}$$

so that $f \in US((-\infty, \infty))$. Let $N_0 = \sum_k \varepsilon_{(t_k, j_k)}$ be a PRM on $[0, 1] \times \{1, 1/2, 1/3, \dots\}$ with mean measure $\mu_0([0, t] \times B) = (t \wedge 1)\nu(B)$ where ν places mass 1 on each point $\{1, 1/2, 1/3, \dots\}$. Define $N = \sum_k \varepsilon_{(t_k, j_k + f(\cdot - t_k))}$ so that N is a PRM on $[0, \infty) \times US([0, 2])$ with mean measure

$$\mu([0, t] \times \cdot) = \int \int_{\{(s, y): y + f(\cdot - s) \in \cdot\}} 1_{[0, 1 \wedge t]}(s) ds \nu(dy).$$

Note $\mu([0, t] \times [US_{SING}([0, 2])]^c) = 0$. However $Y_1 = \bigvee_{t_k \leq 1} (j_k + f(\cdot - t_k))$ and $M_1 = \{t_k : j_k = \bigvee_{t_i \leq 1} j_i\}$ so that

$$\begin{aligned} \mathbf{P}[SING_{\mathcal{F}(T)}(1)] &= \mathbf{P}[M_1 \text{ is singleton}] \\ &= \sum_{k=1}^{\infty} \mathbf{P}[N_0([0, 1] \times (1/k, 1]) = 0, N_0([0, 1] \times \{1/k\}) = 1] \\ &= \sum_{k=1}^{\infty} e^{-(k-1)} e^{-1} = \frac{e^{-1}}{(1 - e^{-1})}. \end{aligned}$$

Finally we show that the joint evolution of $(M, Y^\vee(T))$ is Markov.

Theorem 4.4 *Y is a super-extremal process, and M is the $\mathcal{F}(T)$ -valued process defined above.*

Then the process $\{(M_t, Y_t^\vee(T)), t > 0\}$ is Markov.

Proof: Note that for $s, t > 0$

$$(M_{t+s}, Y_{t+s}^\vee(T)) = \begin{cases} (M_t, Y_t^\vee(T)) & \text{if } Y_t^\vee(T) > Y_{t,t+s}^\vee(T) \\ (M_{t,t+s}, Y_{t,t+s}^\vee(T)) & \text{if } Y_t^\vee(T) < Y_{t,t+s}^\vee(T) \end{cases}$$

where $M_{t,t+s} = \{\tau : Y_{t,t+s}(\tau) = Y_{t,t+s}^{\vee}(T)\}$. If $Y_t^{\vee}(T) = Y_{t,t+s}^{\vee}(T)$ then $M_{t+s} = M_t \cup \{\tau : Y_t^{\vee}(T) = Y_{t,t+s}(\tau)\}$, so in this case

$$(M_{t+s}, Y_{t+s}^{\vee}(T)) = (M_t \cup \{\tau : Y_t^{\vee}(T) = Y_{t,t+s}^{\vee}(\tau)\}, Y_t^{\vee}(T))$$

So we conclude that $(M_{t+s}, Y_{t+s}^{\vee}(T))$ can be written as a function of (M_t, Y_t^{\vee}) and quantities independent of (M_t, Y_t^{\vee}) and the Markov property follows. •

5 Complements

5.1 Dependence Properties

Here we note that for a super-extremal process \mathbf{Y} , defined by (4), for each $t > 0$, \mathbf{Y}_t is associated. T is a locally compact, separable metric space. $US(T)$ is partially ordered with respect to the ordering defined by pointwise comparisons i.e. $f, g \in US(T)$, then $f \leq g$ iff $f(\tau) \leq g(\tau), \forall \tau \in T$.

Definition: Suppose X is a random element of a partially ordered, locally compact, separable, metric space space E . Then X is *associated* iff for monotone, non-decreasing functions $g_i : E \rightarrow \mathbf{R}, i = 1, 2$ (i.e. if $f_1, f_2 \in E, f_1 \leq f_2$ then $g_i(f_1) \leq g_i(f_2), i = 1, 2$),

$$\text{Cov}(g_1(X), g_2(X)) \geq 0.$$

(cf. Esary, Proschan, Walkup(1967), Lindqvist(1988))

We recall from Burton and Waymire(1985), that the PRM $N = \sum_k \varepsilon_{(t_k, \eta_k)}$ on $E = [0, \infty) \times US_0(T)$ (cf. section 3) is associated. We show that \mathbf{Y}_t is associated by showing that it is an appropriately defined monotone function of N (cf. Resnick(1987)). To do this define a partial order on $M_p(E)$, the space of point measures on E , as follows : For $\mu, \nu \in M_p(E)$

$$\mu \leq \nu \text{ iff } \mu(A) \leq \nu(A), \forall A \in \mathcal{B}(E).$$

For any $m = \sum_k \varepsilon_{(t_k, \theta_k)} \in M_p(E)$, and any fixed $t > 0$, define a map $T_t : M_p(E) \rightarrow US_0(T)$ by

$$T_t m = \vee_{t_k \leq t} \theta_k.$$

Then $Y_t(\omega) = T_t N(\omega)$ is a monotone function of the associated random element N . Since non-decreasing functions of associated processes on partially-ordered spaces (eg. $US(T)$) are associated (cf. Resnick (1987), Lindqvist(1988)), we get Y_t is associated. It follows easily that for any collection of sets $B_i \in \mathcal{B}(T), i = 1, \dots, n$, the random vector $(Y_t^\vee(B_1), \dots, Y_t^\vee(B_n))$ is associated.

In particular, the argument above shows that sup-infinitely divisible processes in $US(T)$ or equivalently, infinitely divisible random sup-measures are associated.

5.2 Super-Extremal Processes and Continuity

In optimization models with uncertainty, it is often assumed that the objective function has continuous realizations over a compact set. Therefore, here we construct super-extremal processes which live on $C(T)$, the space of bounded, non-negative, continuous functions on T . We assume T is a compact, metric space, with countable dense subset D_T , and metric d . $C(T)$ is given the uniform topology, generated by the metric

$$d_c(f, g) = \sup_{\tau \in T} |f(\tau) - g(\tau)|, f, g \in C(T).$$

$C(T)$ is Polish in the uniform topology, and finite-dimensional open sets form a base for this topology. We denote by $\mathcal{B}(C(T))$, the usual Borel σ -algebra on $C(T)$, i.e. the σ -algebra generated by open sets. Let $C_0(T) = C(T) - \{\mathbf{0}\}$, i.e. $C(T)$ punctured by removal of the function identically zero on T .

In this setting, we proceed with the definition of a super-extremal process. Suppose

$$N_1 = \sum_{k \geq 1} \varepsilon_{(t_k, \xi_k)}$$

is a PRM on $[0, \infty) \times C_0(T)$ with Radon mean measure ρ on $[0, \infty) \times C_0(T)$, satisfying for all $t > 0$ the analogues of (1), (2), (3)

$$\rho([0, t] \times \{f \in C(T) : f^\vee(T) = \infty\}) = 0 \tag{25}$$

$$\rho([0, t] \times C_0(T)) = \infty \quad (26)$$

$$\rho(\{t\} \times \cdot) = 0. \quad (27)$$

Since in general, for $K \in \mathcal{K}(T)$, the sets $\{f \in C(T) : f^\vee(K) \geq \epsilon\}$ are not compact in the uniform topology, we require that ρ satisfy an additional condition, which is for any $\epsilon > 0$, $K \in \mathcal{K}(T)$, $t > 0$

$$\rho([0, t] \times \{f \in C(T) : f^\vee(K) \geq \epsilon\}) < \infty \quad (28)$$

(cf. Gine, Hahn, Vatan(1990)). Sometimes we write $\rho_t(\cdot) = \rho([0, t] \times \cdot)$.

The *super-extremal process* $\mathbf{X} = \{\mathbf{X}_t, t > 0\}$ is defined as

$$\mathbf{X}_t := \bigvee_{t_k \leq t} \xi_k \quad (29)$$

(cf. (4), section 3.1). We show below that for fixed $t > 0$ $\mathbf{X}_t = \{X_t(\tau), \tau \in T\}$, is a random element of $C(T)$.

As before, we note for $B \in \mathcal{B}(T)$, the process $\mathbf{X}^\vee(B) = \{X_t^\vee(B), t > 0\}$, defined by $X_t^\vee(B) := \bigvee_{t_k \leq t} \xi_k^\vee(B)$ is a classical univariate extremal process.

Theorem 5.1 $\mathbf{X} = \{\mathbf{X}_t, t > 0\}$ is a super-extremal process defined in (29). Then

- (a) There exists Ω_c with $\mathbf{P}[\Omega_c] = 1$, such that for any fixed $t > 0$ and $\omega \in \Omega_c$, $\tau \rightarrow X_t(\tau, \omega)$ is continuous in τ . A version of $\{\mathbf{X}_t, t > 0\}$ exists (also called $\{\mathbf{X}_t, t > 0\}$) such that for each fixed $t > 0$, \mathbf{X}_t is a random element of $C(T)$. Furthermore for any $B \in \mathcal{B}(T)$, $X_t^\vee(B)$ is a random variable and for any fixed $t > 0$, $(\tau, \omega) \mapsto X_t(\tau, \omega)$ is $\mathcal{B}(T) \times \mathcal{A}/\mathcal{B}((0, \infty))$ measurable.
- (b) There is a version of \mathbf{X} which is a random element of $D((0, \infty), C(T))$. The map $(t, \omega) \mapsto \mathbf{X}_t(\omega)$ is $\mathcal{B}((0, \infty)) \times \mathcal{A}/\mathcal{B}(C(T))$ measurable.
- (c) \mathbf{X} is stochastically continuous in the uniform topology.

(d) $\mathbf{X} = \{\mathbf{X}_t, t \in (0, \infty)\}$ is Markov with state-space $C_0(T)$ and its transition probabilities are determined by $(0 < s < t, g \in C_0(T), \tau_i \in T, x_i \geq 0, i = 1, \dots, m)$:

$$\begin{aligned} & \mathbf{P}[\mathbf{X}_t(\tau_i) \leq x_i, i = 1, \dots, m | \mathbf{X}_s = g] \\ &= \begin{cases} 0, & g(\tau_i) > x_i, \text{ some } i \in \{1, \dots, m\} \\ \exp(-\rho((s, t] \times \{f : f(\tau_i) \leq x_i, i = 1, \dots, m\}^c)) & \text{otherwise} \end{cases} \end{aligned}$$

Proof(a): Almost-sure continuity of the paths of \mathbf{X}_t is proved by steps similar to those in the proof of Theorem 3.1(a). We sketch the argument.

From (28), for positive integers m and n , $\mathbf{E}[N([0, m] \times T^{\geq n^{-1}})] = \rho_m(T^{\geq n^{-1}}) < \infty$, which implies $\mathbf{P}[N([0, m] \times T^{\geq n^{-1}}) < \infty] = 1$. Consequently for all n , the sets $\Omega_n^{(m)} := \{\omega : N(\omega, [0, m] \times T^{\geq n^{-1}}) < \infty\}$ satisfy $\mathbf{P}[\Omega_n^{(m)}] = 1$.

For any $t \in [0, m]$, we need to show that $\tau \mapsto X_t(\tau)$ is a.s. continuous. Fix $\tau_0 \in T$, and distinguish two cases: $X_t(\tau_0, \omega) > 0$, and $X_t(\tau_0) = 0$. When $X_t(\tau_0, \omega) > 0$, an argument exactly as in Theorem 3.1(a), with T replacing K_{j_0} , shows that for $\omega \in \Omega_c := \bigcap_{m=1}^{\infty} \bigcap_n \Omega_n^{(m)}$, $\tau \mapsto X_t(\tau, \omega)$ is usc, and since $\mathbf{P}[\Omega_c] = 1$, for any $t > 0$, \mathbf{X}_t is a.s. usc. Since the pointwise suprema of continuous functions are lower semicontinuous (lsc), we get \mathbf{X}_t is both usc and lsc on Ω_c and hence continuous.

For any $x_i > 0, \tau_i \in T, i = 1, \dots, m$

$$[\mathbf{X}_t \in \bigcap_{i=1}^m \{f : f(\tau_i) \leq x_i\}] = \bigcap_{i=1}^m [X_t(\tau_i) \leq x_i] \in \mathcal{A},$$

which implies for fixed $t > 0$, $X_t(\cdot)$ is a random element of $C(T)$. Then it follows that for $B \in \mathcal{B}(T)$, $X_t^\vee(B)$ is a random variable.

The $\mathcal{B}(T) \times \mathcal{A}/\mathcal{B}((0, \infty))$ measurability of \mathbf{X}_t for fixed $t > 0$, follows from continuity.

(b) Initially, we check that almost all paths of $\{\mathbf{X}_t, t > 0\}$ are right-continuous in the uniform topology. Theorem 3.1(c) tells us that for $t_n \downarrow t$, $\mathbf{X}_{t_n} \rightarrow \mathbf{X}_t$ a.s. in the sup-vague topology. Since T is compact, and \mathbf{X}_t is a.s. continuous, from Theorem 7.2, Vervaat(1988) and Theorem 1, Beer(1982) we conclude $\mathbf{X}_{t_n} \rightarrow \mathbf{X}_t$ uniformly with probability one, which gives the a.s. right-continuity of \mathbf{X} .

Now consider the existence of left-hand limits. Since for any fixed $\tau \in T$, $\{X_t(\tau), t > 0\}$ is monotone, for $\omega \in \Omega_c$, and any $t > 0$ with $s_n \uparrow t$ define

$$X_{t-}(\tau, \omega) := \bigvee_{s_n < t} X_{s_n}(\tau, \omega).$$

Then $\mathbf{X}_{t-} = \{X_{t-}(\tau), \tau \in T\}$ is a.s. lsc, since the pointwise suprema of continuous functions is lsc. From Theorem 3.1(c), we know that there exists a random usc function \mathbf{Y}_{t-} such that $\mathbf{X}_{s_n} \uparrow \mathbf{Y}_{t-}$ a.s. in the sup-vague topology. Set $\Omega_{svl} = \{\omega : \mathbf{X}_{s_n}(\omega) \uparrow \mathbf{Y}_{t-}(\omega), \text{sup} - \mathcal{K}\}$ so that $\mathbf{P}[\Omega_{svl}] = 1$. Then for $\omega \in \Omega_c \cap \Omega_{svl}$, using the definition of \mathbf{Y}_{t-} as a sup-derivative (cf. Theorem 3.1(c)), for any $K \in \mathcal{K}(T)$, we have

$$Y_{t-}^\vee(K, \omega) = \bigvee_{s_n < t} \bigvee_{\tau \in K} X_{s_n}(\tau, \omega) = \bigvee_{\tau \in K} \bigvee_{s_n < t} X_{s_n}(\tau, \omega) = \bigvee_{\tau \in K} X_{t-}(\tau, \omega) = X_{t-}^\vee(K, \omega).$$

Therefore we may conclude $\mathbf{P}[X_{t-}^\vee(K) = Y_{t-}^\vee(K), \forall K \in \mathcal{K}(T)] = 1$ which implies $\mathbf{P}[\mathbf{X}_{t-} = \mathbf{Y}_{t-}] = 1$. Therefore \mathbf{X}_{t-} is a.s. continuous, and since it is bounded, Dini's lemma implies that \mathbf{X}_{s_n} converges uniformly to \mathbf{X}_{t-} with probability one.

Finally, proceeding as in the proof of Theorem 3.1(c), one finds that \mathbf{X} has a version which is a random element of $D((0, \infty), C(T))$.

The joint measurability of \mathbf{X} follows from right-continuity.

(c) Let $t_n \rightarrow t > 0$, we need to show $\mathbf{X}_{t_n} \xrightarrow{\mathbf{P}} \mathbf{X}_t$ in the uniform topology. Stochastic continuity from the right follows from the a.s. right continuity of \mathbf{X} in (b) above. From the left, for any $t_n \uparrow t$, an argument similar to that in Theorem 3.1(b), shows that $\mathbf{X}_{t_n} \xrightarrow{\mathbf{P}} \mathbf{X}_t$, in the inf-vague topology (cf. Dolecki, et.al.(1983)) on the space of non-negative lsc functions on T . Hence for any subsequence $\{\mathbf{X}_{t_{n''}}\} \subset \{\mathbf{X}_{t_n}\}$, there exists a further subsequence $\{\mathbf{X}_{t_{n'''}}\} \subset \{\mathbf{X}_{t_{n''}}\}$ such that $\mathbf{X}_{t_{n'''}} \uparrow \mathbf{X}_t$ almost surely in the inf-vague topology. Since $\mathbf{X}_t \in C_0(T)$, Theorem 1, Beer(1982), adapted to monotone increasing sequences of lsc functions inf-vague convergent to a bounded, continuous function implies $\mathbf{X}_{t_{n'''}} \uparrow \mathbf{X}_t$ uniformly, with probability one and the left stochastic continuity of \mathbf{X} follows.

(d) These are deduced as in Theorem 3.1(d). •

Remark 5.2.1: Note that for each $t > 0$, \mathbf{X}_t is a sup-infinitely divisible random element of $C(T)$ (cf. Gine, et.al(1990)).

Remark 5.2.2: For the super-extremal process \mathbf{X} above, the theory for the corresponding argmax process, $M_{ct} := A_\vee(\mathbf{X}_t)$, may be developed along the lines of section 4.

5.3 Spectral Representations

Now we show that the function space approach of the previous sections, to super extremal processes has an equivalent spectral function construction (cf. Balkema, de Haan, Karandikar(1990)). Set $\mathbf{R}_+ = [0, \infty)$. T is a locally, compact separable metric space. Lebesgue measure on the product σ -algebra $\mathcal{B}(\mathbf{R}_+) \times \mathcal{B}(\mathbf{R}_+)$, is denoted by λ^2 below.

Theorem 5.2 *\mathbf{Y} is a super-extremal process defined by (4), with sup-Levy measure μ . Then let $N' = \sum_k \varepsilon_{(u_k, v_k)}$ with points in \mathbf{R}_+^2 be PRM(λ^2). There exists a measurable function $\mathbf{f} := \mathbf{R}_+^2 \rightarrow \mathbf{R}_+ \times US_0(T)$, such that*

$$\mu = \lambda^2 \circ \mathbf{f}^{-1}.$$

Writing

$$\mathbf{f}(u, v) = (f_1(u, v), f_2(u, v))$$

we have that the process $\mathbf{Y}' = \{\mathbf{Y}'_t, t > 0\}$ defined by

$$\mathbf{Y}'_t := \bigvee_{f_1(u_k, v_k) \leq t} f_2(v_k) \tag{30}$$

satisfies $\mathbf{Y}' \stackrel{d}{=} \mathbf{Y}$ and has a version in $D((0, \infty), US(T))$. For each $t > 0$, \mathbf{Y}'_t is a sup-infinitely divisible random element of $US_0(T)$.

Proof: Since λ^2 and μ are σ -finite, there exist measurable partitions $\{U_n \times V_n\}_{n=1}^\infty$ of \mathbf{R}_+^2 , and $\{W_n \times B_n\}_{n=1}^\infty$ of $\mathbf{R}_+ \times US_0(T)$ such that for each n

$$\mu(W_n \times B_n) = \lambda^2(U_n \times V_n) < \infty.$$

Also $\mathbf{R}_+ \times US_0(T)$ is a locally compact, separable metric space, which has cardinality of the continuum. Therefore $\mathbf{R}_+ \times US_0(T)$ and \mathbf{R}_+^2 are measurably isomorphic (cf. Dellacherie and Meyer(1978), Appendix III.80, Dudley(1989), Chapter 13). So for each n there exists a Borel isomorphism

$$\mathbf{f}_n = U_n \times V_n \rightarrow W_n \times B_n$$

(cf. Dudley(1989), Chapter 9), such that for any $W \times B \in \mathcal{B}(W_n) \times \mathcal{B}(B_n)$

$$\mu(W \times B) = \lambda^2(\mathbf{f}_n^{-1}(W \times B)).$$

Define

$$\mathbf{f} = (f_1, f_2) := \sum_{n=1}^{\infty} \mathbf{f}_n 1_{[U_n \times V_n]}$$

and then \mathbf{f} is a measurable isomorphism from \mathbf{R}_+^2 onto $\mathbf{R}_+ \times US_0(T)$ satisfying $\mu = \lambda^2 \circ \mathbf{f}^{-1}$. If N' is $\text{PRM}(\lambda^2)$ then $N' \circ \mathbf{f}^{-1}$ is $\text{PRM}(\lambda^2 \circ \mathbf{f}^{-1} = \mu)$ (Resnick (1987)).•

So given a λ^2 -measurable function $\mathbf{g} : \mathbf{R}_+^2 \rightarrow \mathbf{R}_+ \times US_0(T)$, such that the measure $\lambda^2 \circ \mathbf{g}^{-1}$ satisfies the analogues of (1), (2) and (3), one may always construct a super extremal process by the definition given in (30).

Finally, for completeness, we give spectral representations for some of the models described in sections 4 and 5.2, for compact, metric T , in the corollary below.

Corollary 5.1 (a) *Y is a super-extremal process defined by (4) and suppose $\mathbf{P}[SING_{\mathcal{F}(T)}] = 1$.*

Then the construction yields

$$\lambda^2(\{(u, v) : \mathbf{f}_2(u) \in [US(T)_{SING}]^c\}) = 0.$$

In this case, for each fixed $t > 0$, the argmax functional of the process \mathbf{Y}' defined in (30), $A_V(\mathbf{Y}'_t)$, is a.s. singleton for every $t > 0$.

(b) Suppose $\mathbf{X} = \{\mathbf{X}_t, t > 0\}$ is a super-extremal process defined by (29), with sup-Levy measure ρ . Then there exists a measurable function $\mathbf{f} = (f_1, f_2) : \mathbf{R}_+^2 \rightarrow \mathbf{R}_+ \times C_0(T)$, satisfying $\rho = \lambda^2 \circ \mathbf{f}^{-1}$, and a PRM $N_c = \sum_k \varepsilon_{(u_k, v_k)}$ with points in \mathbf{R}_+^2 and mean measure $\lambda^2 \circ \mathbf{f}^{-1}$, such that the process $\mathbf{X}' = \{\mathbf{X}'_t, t > 0\}$ defined by

$$\mathbf{X}'_t := \bigvee_{f_1(u_k, v_k) \leq t} \mathbf{f}_2(v_k) \quad (31)$$

satisfies $\mathbf{X}' \stackrel{d}{=} \mathbf{X}$, and has a version in $D((0, \infty), C(T))$. For each $t > 0$, \mathbf{X}'_t is a sup-infinitely divisible random element of $C_0(T)$.

Proof: (a) follow from the definitions, Theorem 4.3, and Theorem 5.2 above. For (b), it suffices to note that $C_0(T)$ is an uncountable, separable, metric space and then the results follow as in Theorem 5.2, using the facts about \mathbf{X} collected in Theorem 5.1. •

6 Concluding Remarks

Here we have developed the properties of super-extremal processes \mathbf{Y} and their argmax processes \mathbf{M} for the general case of sup-infinitely divisible \mathbf{Y}_t . In Resnick and Roy(1991), for applications to choice models, super-extremal processes with max-stable \mathbf{Y}_t 's are constructed. The argmax process \mathbf{M} in the latter setting, representing the random set of optimal alternatives, turns out to have several interesting properties.

7 Appendix

(A.I) **Measurability** : T is a compact, metric space. Recall from section 4 that the argmax functional A_v is $B(US(T))/B(\mathcal{F}(T))$ measurable. Hence for any $K \in \mathcal{K}(T)$

$$(A.I.1) \quad K^{(>)} = \{f \in US(T) : A_v(f) \subseteq K\} \in B(US(T))$$

$$(A.I.2) \quad K^{(<)} = \{f \in US(T) : A_v(f) \cap K = \phi\} \in B(US(T))$$

$$(A.I.3) \quad K^{(=)} = [K^{(>)} \cup K^{(<)}]^c \in B(US(T)).$$

(A.II) **The Hausdorff Metric and Convergence in Probability for Random Sets** :

T is compact with metric d . The *Hausdorff metric* on $\mathcal{F}(T)$ ($= \mathcal{K}(T)$) is defined as follows :

For $F_1, F_2 \in \mathcal{F}(T)$, set

$$d^u(F_1, F_2) = \sup_{x \in F_1} d(x, F_2) = \sup_{x \in F_1} \inf_{y \in F_2} d(x, y) \quad (32)$$

$$d_l(F_1, F_2) = d^u(F_2, F_1) = \sup_{x \in F_2} d(x, F_1) = \sup_{x \in F_2} \inf_{y \in F_1} d(x, y). \quad (33)$$

Then the Hausdorff distance between F_1 and F_2 (cf. Castaing and Valadier(1977)) is

$$d_H(F_1, F_2) = d^u(F_1, F_2) \vee d_l(F_1, F_2). \quad (34)$$

In general, neither d^u nor d_l are metrics, since

$$\begin{aligned} (1) \quad d^u(F_1, F_2) = 0 & \quad \not\Rightarrow F_1 = F_2 \\ (2) \quad d^u(F_1, F_2) \neq d^u(F_2, F_1) & \quad \text{in general,} \end{aligned}$$

with analogous statements for d_l . For singleton $F_1, F_2 \in \mathcal{F}(T)$, $d^u(F_1, F_2) = d_l(F_1, F_2) = d_H(F_1, F_2)$.

Also the functions $d_H, d^u, d_l : \mathcal{F}(T) \times \mathcal{F}(T) \rightarrow [0, \infty)$, are continuous in the vague, upper and lower topologies on $\mathcal{F}(T) \times \mathcal{F}(T)$, respectively. Since $\mathcal{F}(T)$ is separable, they are jointly measurable for the product σ -algebra. Thus for random sets F and K , the distances, $d_H(F, K)$, $d^u(F, K)$, and $d_l(F, K)$ are random variables.

A sequence of random sets $\{F_n\} \in \mathcal{F}(T)$ converges in probability to a random set $F \in \mathcal{F}(T)$ iff $d_H(F_n, F) \xrightarrow{\mathbf{P}} 0$ and likewise the sequence upper converges in probability iff $d^u(F_n, F) \xrightarrow{\mathbf{P}} 0$. The usual criterion for convergence in probability is applicable : F_n (upper) converges in probability iff for every sequence $\{n''\}$ there exists a further subsequence $\{n'\} \subset \{n''\}$ such that $(d^u(F_{n'}, F) \rightarrow 0)$ $d_H(F_{n'}, F) \rightarrow 0$ P-a.s.

(A.III) Summary of Notation :

- $Z_+ = \{1, 2, \dots\}$; Q_+ = set of positive rationals.
- D_T = countable dense subset of T .
- $\{B(d_m, r_n)\}_{\substack{d_m \in D_T \\ r_n \in Q_+}} = \{B_{mn}\}_{\substack{d_m \in D_T \\ r_n \in Q_+}}$ = collection of closed balls covering T with centres $d_m \in D_T$ and rational radii r_n .
- $\mathcal{G}(T), \mathcal{K}(T), \mathcal{F}(T)$ = collection of all non-empty open, compact, and closed subsets, respectively of T .
- $\mathcal{G}_0, \mathcal{K}_0, \mathcal{F}_0$ = countable basis for the vague topology on $\mathcal{G}(T), \mathcal{K}(T), \mathcal{F}(T)$, respectively.
- $US(T)$ = space of non-negative real valued usc functions on T .
- $US_0(T) = US(T) - \{\mathbf{0}\}$.
- $C(T)$ = space of bounded, non-negative real valued continuous functions on T .
- $C_0(T) = C(T) - \{\mathbf{0}\}$.
- $\mathbf{1}$ = function identically equal to 1 on T .
- $f^\vee(B) := \vee_{\tau \in B} f(\tau)$, $f \in US(T)$.
- $\{f \leq g\}^c = \cup_{\tau \in T} \{f \in US(T) : f(\tau) > g(\tau)\}$

- $T^{\geq \epsilon} = \{f \in US(T) : f^\vee(T) \geq \epsilon\} = \{f < \epsilon \mathbf{1}\}^c$
- $B^{\geq \epsilon} = \{f \in US(T) : f^\vee(B) \geq \epsilon\}, B \in \mathcal{B}(T)$.
- $A_\vee(f) = \{\tau \in T : f(\tau) = f^\vee(T)\}, f \in US(T)$.
- $US(T)_{SING} = \cup_{\tau \in T} \{f \in US_0(T) : A_\vee(f) = \{\tau\}\}$.
- $K^{(>)} = \{f \in US(T) : f^\vee(K) > f(s), \forall s \in K^c\} = \{f : A_\vee(f) \subseteq K\}, K \in \mathcal{K}(T)$.
- $K^{(<)} = \{f \in US(T) : f^\vee(K) < f^\vee(T)\} = \{f : A_\vee(f) \subset K^c\}, K \in \mathcal{K}(T)$.
- $K^{(=)} = [K^{(>)} \cup K^{(<)}]^c = [A_\vee(f) \cap K^c \neq \emptyset] \cap [A_\vee(f) \cap K \neq \emptyset], K \in \mathcal{K}(T)$.
- $\mu([0, t] \times \cdot) = \mu_t(\cdot); \mu((s, t] \times \cdot) = \mu_{st}(\cdot)$.
- $[f, g] = \{h \in US_0(T) : f(\tau) \leq h(\tau) \leq g(\tau), \forall \tau \in T\}, f, g \in US_0(T) \text{ and } f \leq g$.
- ' $\xrightarrow{\mathbf{P}}$ ' denotes convergence in probability.
- ' $\stackrel{d}{=}$ ' denotes equality in distribution.

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