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PARTIALLY BALANCED DESIGNS AND PROPERTIES A AND B

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ABSTRACT

The following relations have been demonstrated in this paper: an incomplete block design with property A in the Kurkjian-Zelen sense, is a balanced factorial experiment in the Shah sense which in turn is a partially balanced incomplete block design having a "binary number association scheme" and conversely; likewise, a rectangular design which has property A in columns and property B in rows in the Zelen-Federer sense, is a balanced factorial rectangular experiment which in turn is a partially balanced rectangular design, and conversely. The establishment of these relations is useful in constructing block and row-column designs which are partially balanced and in constructing fractional replicates which are balanced.

1. INTRODUCTION AND SUMMARY

Kurkjian and Zelen [1963] introduced a structural property, which was designated as Property A, of the (block) incidence matrix N associated with a class of block and direct product experiment designs. Extending this idea, Zelen and Federer [1964] introduced two structural properties, which were designated as Property A and Property B, associated with experiment designs for two-way elimination of heterogeneity, such as a k-row by b-column rectangular experiment design, and direct product designs. Property A is associated with the column (block) incidence matrix N and Property B is associated with the row incidence matrix $\tilde{N}$. In the paper by Paik and Federer [1971], the incomplete block designs (IBD) possessing Property A were designated as PA type incomplete block designs and likewise, those two-way elimination of heterogeneity designs possessing both Property A and Property B were designated as PAB type designs. Also, these authors showed that, under factorial structure, PA type incomplete block designs are "balanced factorial experiments (BFE)", as defined by Shah [1960], and are partially balanced incomplete block (PBIB) designs, with relevant parameters.

In this paper, we show that every BFE is a PA type incomplete block design but a PBIB design is not always a balanced factorial incomplete block design (BFIBD) nor a PA type incomplete block design. We define a "binary number association scheme (BNAS)" and obtain a condition for which a PBIB design is a BFIBD or a PA type IBD. Also, we define n-ary partially balanced block (NPBB) designs which may have a specified treatment appearing an arbitrary number of times in a block, and discuss some properties of NPBB designs.

The relationships discussed herein may be summarized as follows:

(i) (PA and PB) $\rightleftarrows$ PAB
(ii) BFE $\rightleftarrows$ PA $\rightleftarrows$ (NPBB with BNAS)
(iii) PBIB without BNAS is not a BFE
(iv) PBIB $\rightarrow$ NPBB but not conversely
(v) BFIBD $\rightleftarrows$ (PBIB with BNAS)
(vi) (PA type IBD) $\rightleftarrows$ (PBIB with BNAS)
(vii) PAB $\rightleftarrows$ (NPBB in rows and in columns)
(viii) PAB $\rightarrow$ balanced factorial rectangular experiment (BFRE), but not conversely.

2. PRELIMINARIES

The following notation (after Kurkjian and Zelen [1963]) will be used:

1_{m_i} : $m_i \times 1$ column vector having all elements unity,

$J_{m_i} = 1_{m_i} 1'_{m_i}$: $m_i \times m_i$ matrix with all elements unity,

I_{m_i} : $m_i \times m_i$ identity matrix,

$$I_i^{x_i} = \begin{cases} 1_{m_i} & \text{if } x_i = 0 \\ I_{m_i} & \text{if } x_i = 1 \end{cases}, \qquad M_i = m_i I_{m_i} - J_{m_i}$$

$$D^{\delta_i} = \begin{cases} I_{m_i} & \text{if } \delta_i = 0 \\ J_{m_i} & \text{if } \delta_i = 1 \end{cases}, \qquad M_i^{x_i} = \begin{cases} 1'_m & \text{if } x_i = 0 \\ M_i & \text{if } x_i = 1 \end{cases}.$$

The direct product, or Kronecker product, of $M_i^{x_i}$ and $M_j^{x_j}$ will be written as $M_i^{x_i} \otimes M_j^{x_j}$ and in general, the joint direct product of n $M_i^{x_i}$ (i=1,2,...,n) will be written as $\prod_{i=1}^{n} \otimes M_i^{x_i}$.

Let there be v treatments, each replicated r times in b blocks of k plots each. Let $N = \|n_{ij}\|$, i=1,2,...,v; j=1,2,...,b, be the incidence matrix of the design, where n_{ij} (=0,1,2,...,) is equal to the number of times the i^{th} treatment occurs in the j^{th} block. The model assumed is

$$y_{ij} = \mu + t_i + b_j + \epsilon_{ij}, \tag{2.1}$$

where y_{ij} is the yield of the plot in the j^{th} block to which the i^{th} treatment is applied, μ is the overall effect, t_i is the effect of the i^{th} treatment,

b_j is the effect of the j^{th} block, and ϵ_{ij} is the experimental error. The effects μ, t_i, b_j, are assumed to be fixed constants, while the error ϵ_{ij}'s are assumed to be independent normal variates with mean zero and variance σ^2. Let T_i be the total yield of all plots having the i^{th} treatment, B_j be the total yield of all the plots of the j^{th} block and $\hat{t}_i$ be a solution for t_i in the normal equations. Further, denote the column vectors $(T_1, T_2, \ldots, T_v)'$, $(B_1, B_2, \ldots B_b)'$, $(t_1, t_2, \ldots, t_v)'$, and $(\hat{t}_1, \hat{t}_2, \ldots, \hat{t}_v)'$ by $\underline{T}$, $\underline{B}$, and $\underline{t}$ and $\hat{\underline{t}}$ respectively. It is known that the reduced normal equations for intra-block estimates of treatment effects are

$$C\hat{\underline{t}} = \underline{Q}, \tag{2.2}$$

where

$$C = rI_v - \frac{1}{k}NN', \text{ and} \tag{2.3}$$

$$\underline{Q} = \underline{T} - \frac{1}{k}N\underline{B}. \tag{2.4}$$

The matrix C defined in (2.3) will be called the C-matrix of the design. The solution of (2.2) is $\hat{\underline{t}} = C^+\underline{Q}$, where C^+ is a generalized inverse of C.

Consider a factorial experiment with n factors $F_1, F_2, \ldots, F_n$, where factor F_i has m_i levels for $i=1,2,\ldots,n$, there being $v = \prod_{i=1}^{n} m_i$ treatments. Kurkjian and Zelen [1963] introduced a structural property of the design which was related to the block (or column) incidence matrix N of the design. This structural property was termed Property A and may be defined as follows: A block design will be said to have Property A, or will be called a PA type block design, if

$$NN' = \sum_{s=0}^{n} \left\{ \sum_{\delta_1+\delta_2+\ldots+\delta_n=s} h(\delta_1, \delta_2, \ldots, \delta_n) \prod_{i=1}^{n} \otimes D_i^{\delta_i} \right\} \tag{A}$$

where δ_i=0 or 1 for $i=1,2,\ldots,n$, and $h(\delta_1, \delta_2, \ldots, \delta_n)$ are constants. In this case, we obtain the following solution for equation (2.2):

(2.5) $$\hat{\underline{t}} = \sum_{s=1}^{n} \left\{ \sum_{x_1+x_2+\cdots+x_n=s} \frac{\prod_{i=1}^{n} \otimes I_i^{x_i} M_i^{x_i}}{rv\theta(x_1,x_2,\ldots,x_n)} \right\} \underline{Q} \; .$$

where $\theta(x_1,x_2,\ldots,x_n)$ is the efficiency factor associated with the estimate of generalized interaction $F_1^{x_1} F_2^{x_2} \ldots F_n^{x_n}$ and

(2.6) $$r\theta(x_1,x_2,\ldots,x_n)$$

$$= \sum_{s=0}^{n-1} \left\{ \sum_{\delta_1+\delta_2+\cdots+\delta_n=s} g(\delta_1,\delta_2,\ldots,\delta_n) \prod_{i=1}^{n} m_i^{(1-x_i)\delta_i}(1-x_i\delta_i) \right\}$$

for $g(0,0,\ldots,0) = r - \frac{1}{k}h(0,0,\ldots,0)$, $g(\delta_1,\delta_2,\ldots,\delta_n) = -\frac{1}{k}h(\delta_1,\delta_2,\ldots,\delta_n)$ if $(\delta_1,\delta_2,\ldots,\delta_n) \neq 0$.

In a k-row by b-column rectangular experiment design with $v = \prod_{i=1}^{n} m_i$ treatments being replicated r times each, suppose the column incidence matrix N has Property A and the row incidence matrix $\tilde{N}$ of the experiment design has Property B as introduced by Zelen and Federer [1964], i.e.,

(B) $$\tilde{N}\tilde{N}' = \sum_{s=0}^{n} \left\{ \sum_{\delta_1+\delta_2+\cdots+\delta_n=s} \tilde{h}(\delta_1,\delta_2,\ldots,\delta_n) \prod_{i=1}^{n} \otimes D_i^{\delta_i} \right\} ,$$

where $\tilde{h}(\delta_1,\delta_2,\ldots,\delta_n)$ are constants, then the kxb rectangular design will be said to have Property A and Property B or will be called a PAB type kxb rectangular design. It is known that, in a kxb rectangular design with v treatments being replicated r times each, the reduced normal equations for estimating the treatment effects may be written as

$$(2.7) \qquad \tilde{C}\hat{\underline{t}} = \tilde{\underline{Q}} ,$$

where

$$(2.8) \qquad \tilde{C} = rI_v - \frac{1}{k}NN' - \frac{1}{b}\tilde{N}\tilde{N}' + J_v(r/v), \text{ and}$$

$$(2.9) \qquad \tilde{\underline{Q}} = \underline{T} - \frac{1}{k}N\underline{B} - \frac{1}{b}\tilde{N}\underline{R} + 1_v(g/v),$$

where $\underline{R} = (R_1, R_2, \ldots, R_k)'$, R_s = total yield of all the plots of the s^{th} row; g = total yield of all the plots in the experiment. The solution of (2.7) is $\hat{t} = \tilde{C}^+\tilde{\underline{Q}}$, where $\tilde{C}^+$ is a generalized inverse of matrix $\tilde{C}$. In the PAB type design,

$$(2.10) \qquad \tilde{C}^+ = \sum_{s=1}^{n} \left\{ \sum_{x_1+x_2+\ldots+x_n=s} \frac{\prod_{i=1}^{n} \otimes I_i^{x_i} M_i^{x_i}}{rv\tilde{\theta}(x_1,x_2,\ldots,x_n)} \right\}$$

where

$$(2.11) \qquad r\tilde{\theta}(x_1,x_2,\ldots,x_n)$$

$$= \sum_{s=0}^{n-1} \left\{ \sum_{\delta_1+\delta_2+\ldots+\delta_n=s} \tilde{g}(\delta_1,\delta_2,\ldots,\delta_n) \prod_{i=1}^{n} m_i^{(1-x_i)\delta_i}(1-x_i\delta_i) \right\},$$

for $\tilde{g}(0,0,\ldots,0) = r - \frac{1}{k}h(0,0,\ldots,0) - \frac{1}{b}\tilde{h}(0,0,\ldots,0)$, $\tilde{g}(\delta_1,\delta_2,\ldots,\delta_n)$ $= -\frac{1}{k}h(\delta_1,\delta_2,\ldots,\delta_n) - \frac{1}{b}\tilde{h}(\delta_1,\delta_2,\ldots,\delta_n)$ if $(\delta_1,\delta_2,\ldots,\delta_n) \neq 0$.

Paik and Federer [1971] have proved that PA type incomplete block designs are BFE and PAB type rectangular designs achieve "complete balance" over each of the interactions in the sense of the following definitions given by Shah [1960]:

Definition 2.1. (Shah). "Complete balance" is achieved over an interaction if and only if all the normalized contrasts belonging to the same interaction are estimated with the same variance.

Definition 2.2. (Shah). An experiment will be called a balanced factorial experiment (BFE), if the following conditions are satisfied:

(a). Each of the treatments is replicated the same number of times.

(b). Each of the blocks has the same number of plots.

(c). Estimates of contrasts belonging to different interactions are uncorrelated with each other.

(d). "Complete balance" is achieved over each of the interactions.

Shah [1960] proved that a BFE is a PBIB design with relevant parameters. Therefore, we state that a PA type block design is a BFE and is a PBIB design with relevant parameters. Also, a PAB type rectangular design is a balanced factorial rectangular experiment (BFRE).

3. BALANCED FACTORIAL EXPERIMENTS AND PA TYPE BLOCK DESIGNS

We shall prove that every BFE is a PA type block design. Consider a BFE in n factors, $F_1, F_2, \ldots, F_n$ at $m_1, m_2, \ldots, m_n$ levels respectively, with each treatment combination replicated r times in b blocks of k plots each, there being $v = \prod_{i=1}^{n} m_i$ treatments. Let $L_{q_1 q_2 \ldots q_n}$, where q_i is 0 or 1 for $i = 1,2,\ldots,n$, be a $v \times \prod_{i=1}^{n} (m_i - 1)^{q_i}$ matrix formed by a complete set of $\prod_{i=1}^{n} (m_i - 1)^{q_i}$ normalized orthogonal vectors forming the generalized interaction $F_1^{q_1} F_2^{q_2} \ldots F_n^{q_n}$, then using the results of Shah [1958, 1960] and Paik and Federer [1971], the C-matrix defined in (2.2) is uniquely represented and given by

$$(3.1) \qquad C = r \sum_{s=1}^{n} \left\{ \sum_{q_1+q_2+\ldots+q_n=s} \theta(q_1,q_2,\ldots,q_n) L_{q_1 q_2 \ldots q_n} L'_{q_1 q_2 \ldots q_n} \right\},$$

which can also be written as

$$(3.2) \qquad C = r \left\| \sum_{s=1}^{n} \left\{ \sum_{q_1+q_2+\cdots+q_n=s} \theta(q_1,q_2,\ldots,q_n) f_{ij}^{(q_1,q_2,\ldots,q_n)} \right\} \right\|,$$

$$i,j = 1,2,\ldots,v,$$

where the $f_{ij}^{(q_1,q_2,\ldots,q_n)}$ are the elements of $L_{q_1 q_2\ldots,q_n} L'_{q_1 q_2\ldots,q_n}$ corresponding to the i^{th} row and j^{th} column. Under the assumed model (2.1), the variance-covariance matrix of a contrast set corresponding to the generalized interaction $F_1^{q_1} F_2^{q_2}\ldots F_n^{q_n}$ will be

$$(3.3) \qquad \sigma^2/r\theta(q_1,q_2,\ldots,q_n) I_{\prod_{i=1}^{n}(m_i-1)^{q_i}}$$

and since representation of the matrix C is unique, i.e., if $L_{q_1 q_2\ldots,q_n}$ and $L^*_{q_1 q_2\ldots q_n}$ are two matrices formed by two sets of $\prod_{i=1}^{n}(m_i-1)^{q_i}$ normalized orthogonal vectors forming the generalized interaction $F_1^{q_1} F_2^{q_2}\ldots F_n^{q_n}$, then

$$(3.4) \qquad L_{q_1 q_2\ldots q_n} L'_{q_1 q_2\ldots q_n} = L^*_{q_1 q_2\ldots q_n} L^{*\prime}_{q_1 q_2\ldots q_n}.$$

Define P_i to be an $m_i \times (m_i-1)$ matrix such that $P_i P_i' = I_{m_i-1}$ and $\mathbf{1}'_{m_i} P_i = 0$ for i=1,2,...,n, and define, for i=1,2,...,n,

$$(3.5) \qquad P_i^{q_i} = \begin{cases} \dfrac{1}{\sqrt{m_i}} \mathbf{1}_{m_i}, & \text{if } q_i = 0 \\ P_i, & \text{if } q_i = 1. \end{cases}$$

Let

$$(3.6) \qquad L^*_{q_1 q_2\ldots q_n} = \prod_{i=1}^{n} \otimes P_i^{q_i};$$

then $L^*_{q_1 q_2 \cdots q_n}$ is a $v \times \prod_{i=1}^{n} (m_i - 1)^{q_i}$ matrix formed by a complete set of $\prod_{i=1}^{n} (m_i-1)^{q_i}$ normalized orthogonal vectors and we obtain:

$$(3.7) \qquad L^*_{q_1 q_2 \cdots q_n} L^{*\prime}_{q_1 q_2 \cdots q_n} = \left(\prod_{i=1}^{n} \otimes P_i^{q_i} \right) \left(\prod_{i=1}^{n} \otimes P_i^{q_i} \right)'$$

$$= \prod_{i=1}^{n} \otimes P_i^{q_i} \, (P_i^{q_i})' ,$$

and from (3.5)

$$(3.8) \qquad P_i^{q_i} (P_i^{q_i})' = \begin{cases} \frac{1}{m_i} J_{m_i}, & \text{if } q_i = 0 \\ I_{m_i} - \frac{1}{m_i} J_{m_i}, & \text{if } q_i = 1 . \end{cases}$$

Now, from (3.1), (3.4), (3.7) and (2.3), we obtain:

$$(3.9) \qquad NN' = kr \left(\prod_{i=1}^{n} \otimes I_{m_i} - \sum_{s=1}^{n} \left\{ \sum_{q_1 + q_2 + \cdots + q_n = s} \theta(q_1, q_2, \cdots, q_n) \cdot \prod_{i=1}^{n} \otimes P_i^{q_i} (P_i^{q_i})' \right\} \right) ,$$

and from (3.8), we know that the incidence matrix N has Property A.

<u>Theorem 3.1.</u> <u>Every</u> BFE <u>is a</u> PA <u>type block design and conversely</u>.

<u>Example 3.1.</u> In a BFE with $v = m_1 m_2$,

$$C = r \sum_{s=1}^{2} \left\{ \sum_{q_1 + q_2 = s} \theta(q_1, q_2) \prod_{i=1}^{2} \otimes P_i^{q_i} (P_i^{q_i})' \right\}$$

$$= r \left\{ \theta(0,1) P_1^0 (P_1^0)' \otimes P_2 P_2' + \theta(1,0) P_1 P_1' \otimes P_2^0 (P_2^0)' + \theta(1,1) P_1 P_1' \otimes P_2 P_2' \right\}$$

$$= r\left\{\theta(0,1)\left(\frac{1}{m_1}J_{m_1}\otimes I_{m_2} - \frac{1}{m_1m_2}J_{m_1m_2}\right) + \theta(1,0)\left(\frac{1}{m_2}I_{m_1}\otimes J_{m_2} - \frac{1}{m_1m_2}J_{m_1m_2}\right)\right.$$

$$\left. + \theta(1,1)\left(I_{m_1} - \frac{1}{m_1}J_{m_1}\right)\otimes\left(I_{m_2} - \frac{1}{m_2}J_{m_2}\right)\right\}$$

$$= r\theta(1,1)I_{m_1m_2} + r\left(\frac{\theta(1,0)}{m_2} - \frac{\theta(1,1)}{m_2}\right)I_{m_1}\otimes J_{m_2} + r\left(\frac{\theta(0,1)}{m_1} - \frac{\theta(1,1)}{m_1}\right)J_{m_1}\otimes I_{m_2}$$

$$+ r\left(\frac{\theta(1,1)}{m_1m_2} - \frac{\theta(0,1)}{m_1m_2} - \frac{\theta(1,0)}{m_1m_2}\right)J_{m_1m_2}.$$

Hence, $g(0,0) = r\theta(1,1)$, $g(0,1) = r\left(\frac{\theta(1,0)}{m_2} - \frac{\theta(1,1)}{m_2}\right)$, and $g(1,0) = r\left(\frac{\theta(0,1)}{m_1} - \frac{\theta(1,1)}{m_1}\right)$. Then, we obtain:

$$r\theta(1,1) = g(0,0),\ r\theta(1,0) = g(0,0) + m_2g(0,1),$$

and

$$r\theta(0,1) = g(0,0) + m_1g(1,0).$$

4. BALANCED FACTORIAL EXPERIMENTS AND PBIB DESIGNS

Consider a BFE with n factors as described in section 3. Two treatments are $p_1p_2\cdots p_n$ th associates, where $p_i = 1$, if the i^{th} factor occurs at the same level in both the treatments and $p_i = 0$ otherwise; $\lambda_{p_1p_2\cdots p_n}$ will denote the number of times these treatments occur together in a block. Then we have

$$(4.1)\qquad n_{p_1p_2\cdots p_n} = \sum_{s=0}^{n}\left\{\sum_{p_1+p_2+\cdots+p_n=s}\ \prod_{i=1}^{n}(m_i-1)^{1-p_i}\right\}.$$

This association is called the "binary number association scheme (BNAS)".

Note: The above association scheme was defined by Shah [1960], but the association scheme was not named. Also, the association scheme could be called the "factorial association scheme" but Srivastava and Anderson [1971] have used the name "factorial association scheme" in Multidimensional Partially Balanced designs in a more general sense.

In the above BFE, suppose that model (2.1) is assumed. Using (2.3) and (3.2), let

$$C = rI_v - \frac{1}{k}NN' = r\left\| \sum_{s=1}^{n} \left\{ \sum_{q_1+\dots q_n=s} \theta(q_1,q_2,\dots,q_n) f_{ij}^{(q_1,q_2,\dots,q_n)} \right\| , \quad i,j=1,2,\dots,v, \tag{4.2}$$

where $f_{ij}^{(q_1,q_2,\dots,q_n)}$ is the element of $L^*_{q_1q_2\dots q_n} L^{*\prime}_{q_1q_2\dots q_n}$ corresponding to the i^{th} row and j^{th} column. Shah [1958] showed that $f_{ij}^{(q_1,q_2,\dots q_n)}$ depends only on the associates, say $p_1p_2\dots p_n$, between the i^{th} and j^{th} treatment. Let us denote this by f_p^q, where $p = \sum_{h=1}^{n} p_h 2^{n-h}$, $q = \sum_{h=1}^{n} q_h 2^{n-h}$, $p = m = 2^n$ denotes all levels equal $(i = j)$, and f_m^q is a diagonal element. Then, from (4.2)

$$r\sum_{q=1}^{m} \theta_q f_m^q = r - \frac{r}{k}, \text{ where } \theta_q = \theta(q_1,q_2,\dots,q_n) \text{ such that}$$

$$q = \sum_{h=1}^{n} q_h 2^{n-h}$$

and

$$r\sum_{q=1}^{m} \theta_q f_p^q = -\frac{\lambda_p}{k}, \; p \neq m, \tag{4.3}$$

where $\lambda_p = \lambda_{p_1 p_2 \cdots p_n}$ such that $p = \sum_{h=1}^{n} p_h 2^{n-h}$ and λ_p equals the number of times two p^{th} associate treatments occur together.

From (4.3), we obtain the following expression:

$$(4.4)\quad r \begin{bmatrix} f_0^0 & f_0^1 & \cdots & f_0^m \\ f_1^0 & f_1^1 & \cdots & f_1^m \\ & \cdots & \cdot & \\ f_m^0 & f_m^1 & \cdots & f_m^m \end{bmatrix} \begin{bmatrix} \theta_0 \\ \theta_1 \\ \vdots \\ \theta_m \end{bmatrix} = -\frac{1}{k} \begin{bmatrix} \lambda_0 \\ \lambda_1 \\ \vdots \\ \lambda_m \end{bmatrix},$$

where λ_m = r-rk, $f_p^0 = \frac{1}{v}$ for $p=0,1,\ldots,m$, and θ_0 is a dummy parameter always equal to zero, introduced to simplify the inverse relation. Shah [1958] also obtained the following relation for (4.4):

$$(4.5)\qquad r \prod_{i=1}^{n} \otimes F_i(1) \cdot \underline{\theta} = -\frac{1}{k}\underline{\lambda},$$

where $F_i(1) = \frac{1}{m_i}\begin{pmatrix} 1 & -1 \\ 1 & m_i-1 \end{pmatrix}$, $\underline{\theta} = (\theta_0, \theta_1, \ldots, \theta_m)'$, and $\underline{\lambda} = (\lambda_0, \lambda_1, \ldots, \lambda_m)'$.

Since $\theta_0 = 0$, θ_m = r-rk, equation (4.4) can be written as:

$$(4.6)\quad r \begin{bmatrix} f_0^1 & f_0^2 & \cdots & f_0^m \\ f_1^1 & f_1^2 & \cdots & f_1^m \\ & \cdots & \cdot & \\ f_m^1 & f_m^2 & \cdots & f_m^m \end{bmatrix} \begin{bmatrix} \theta_1 \\ \theta_2 \\ \vdots \\ \theta_m \end{bmatrix} = -\frac{1}{k} \begin{bmatrix} \lambda_0 \\ \lambda_1 \\ \vdots \\ \lambda_{m-1} \\ r-rk \end{bmatrix}.$$

For convenience, let us denote this as follows:

$$rF^* \underline{\theta}^+ = -\frac{1}{k}\underline{\lambda}.$$

Consider a PBIB design having parameters $v = \prod_{i=1}^{n} m_i$, b, r, k, and $\lambda_0, \lambda_1, \ldots, \lambda_m$, then we may obtain equation (4.6) for the unknown variables $\theta_1, \theta_2, \ldots, \theta_m$. Suppose

$$\rho(F^*) = \rho(F^*, \lambda) = m, \tag{4.7}$$

where $\rho(F^*)$ denotes the rank of the matrix F^*, then we may obtain a unique solution for $\underline{\theta}^+$. Furthermore, if the above PBIB design is connected, none of θ_i, i=1,2,...,m, are zero. Hence, the above PBIB design is a balanced factorial incomplete block design (BFIBD) and is a PA type incomplete block design. Now, we proceed to prove relation (4.7). That $\rho(F^*) = m$, is clear from (4.4).

Next, since f_p^q only depends upon the p^{th} associates between two treatments in the $L_q L_q'$, where p replaces $(p_1, p_2, \ldots, p_n)$ and q replaces $(q_1, q_2, \ldots, q_n)$ such that $p = \sum_{h=1}^{n} p_h 2^{n-h}$ and $q = \sum_{h=1}^{n} q_h 2^{n-h}$ respectively, the number of f_p^q in any row or column of the $L_q L_q'$ is n_p or $n_{p_1 p_2 \ldots p_n}$ and since $1_v'(L_q L_q') = 0$, we have

$$\sum_{p=0}^{m} n_p f_p^q = 0 \quad \text{for } q=1,2,\ldots,m\ , \tag{4.8}$$

or

$$\sum_{p=0}^{m-1} n_p f_p^q = -\ f_m^q \quad \text{for } q=1,2,\ldots,m. \tag{4.9}$$

Also, it is known that

$$\sum_{p=0}^{m-1} n_p \lambda_p = r - rk. \tag{4.10}$$

Thus, from (4.9) and (4.10), we have proved that $\rho(F^*) = \rho(F^*, \underline{\lambda})$.

Theorem 4.1. Every BFIBD is a PBIB with BNAS and conversely. Also, every PA type IBD is a PBIB with BNAS. and conversely.

In a PBIB design with $v = \prod_{i=1}^{n} m_i$, each of the $\lambda_{q_1 q_2 \ldots q_n}$ for all $q_i = 0$ or 1, does not always have a unique value, i.e., a PBIB design does not always have a "binary number association scheme".

Example 4.1. $v = 2\times 5$, $r = 2$, $k = 4$, $b = 5$.

b_1	b_2	b_3	b_4	b_5
(00)	(04)	(12)	(14)	(03)
(01)	(10)	(13)	(02)	(11)
(02)	(11)	(01)	(10)	(13)
(03)	(00)	(04)	(12)	(14)

This design is a triangular PBIB design with two associate classes. However, if we consider the BNAS, we may find $\lambda_{00} = 1$ or 0, $\lambda_{01} = 1$, and $\lambda_{11} = 6$, so the above PBIB design is not a BFE.

5. n-ary PARTIALLY BALANCED BLOCK DESIGNS (NPBB DESIGNS)

In the classical BIB design or PBIB design, no treatment appears more than once in a block. However, we may wish to apply some treatments more than once in a block. In such cases, the following definition of n-ary partially balanced block designs (NPBB (see Tocher [1952])) may be useful for application.

Definition 5.1. (i). The experimental material is divided into b blocks of k units each, some treatments may appear $n_{ij} = 0,1,2,\ldots,n-1$ = a specified number, times in the same block.

(ii). All treatments will be replicated the same number (say r) of times in the same number (say q) of blocks.

(iii). There can be established a relation of association between any two treatments satisfying the following requirements.

(a). Two treatments are either first associates, second associates,..., or u^{th} associates.

(b). Each treatment has exactly n_i i^{th} associates $(i=1,2,\ldots,u)$.

(c). Given any two treatments which are i^{th} associates, the number of treatments common to the j^{th} associate of the first and k^{th} associate of the second is p^i_{jk} and is independent of the pair of treatments we start with. Also $p^i_{jk} = p^i_{kj}$ $(i, j, k = 1,2,\ldots,u)$.

(iv). Two treatments which are i^{th} associates occur together the same number (say λ_i) of times. For example, if two treatments, t and t' say, are i^{th} associates and they are replicated r_{tj} and $r_{t'j}$ times in the j^{th} block respectively, and $r = \sum_{j=1}^{b} r_{tj}$ for $t = 1,2,\ldots,v$, then

$$\lambda_i = \sum_{j=1}^{b} r_{tj} r_{t'j}$$

is constant as long as two treatments t and t' are i^{th} associates. If $\lambda_1 = \lambda_2 = \ldots = \lambda_u$ = constant, the design may be called an n-ary balanced block design (NBB design).

(v). $r^* = \sum_{j=1}^{b} r^2_{tj}$ is constant for $t = 1,2,\ldots,v$.

The following relation between the parameters will hold:

(1). $vr = bk$

(2). $n_1 + n_2 + \ldots + n_u = v-1$.

Suppose that the t^{th} treatment is replicated r_{tj} times in the j^{th} block, then the t^{th} treatment appears together with others $r_{tj}(k-r_{tj})$ times in the same block, and $\lambda_i = \sum_j r_{tj} r_{t'j}$ times with each of its i^{th} associates if treatments t and t' are i^{th} associates. So,

$$\sum_{j=1}^{b} r_{tj}(k-r_{tj}) = \sum_{i=1}^{u} n_i\lambda_i \ .$$

Hence, we obtain:

(3). $n_1\lambda_1 + n_2\lambda_2 + \ldots + n_u\lambda_u = rk-r^*.$

Also,

(4). $p^i_{j1} + p^i_{j2} + \ldots + p^i_{ju} = \begin{cases} n_j-1 & \text{if } i = j \\ n_j & \text{if } i \neq j \end{cases}$

(5). $n_i p^i_{jk} = n_j p^j_{ik}$.

Now we define a BNAS for an NPBB design having $v = \prod_{i=1}^{n} m_i$ treatments applied in b blocks of k plots of each. In a factorial system of n factors $F_1, F_2, \ldots, F_n$ at $m_1, m_2, \ldots, m_n$ levels respectively, the two treatments are the $p_1p_2\cdots p_n{}^{th}$ associates, where $p_i = 1$, if the i^{th} factor occurs at the same level in both treatments and $p_i = 0$ otherwise; $\lambda_{p_1p_2\cdots p_n}$ will denote the number of times these treatments occur together in the same blocks. Suppose two treatments t and t' are $p_1p_2\cdots p_n{}^{th}$ associates and these treatments are replicated r_{tj} and $r_{t'j}$ times in the j^{th} block respectively, then

$$(5.1) \qquad \lambda_{p_1p_2\cdots p_n} = \sum_{j=1}^{b} r_{tj}r_{t'j} \ .$$

If $\lambda_{p_1p_2\cdots p_n}$ does not depend upon a particular pair of $p_1p_2\cdots p_n{}^{th}$ associates and if $r = \sum_{j=1}^{b} r_{tj}$ is a constant for $t = 1,2,\ldots,v$, then the above block design is an NPBB design with respect to the "binary number association scheme".

Any contrast belonging to the generalized interaction $F_1^{q_1} F_2^{q_2} \cdots F_n^{q_n}$ in the above design is estimated with variance

$$\sigma^2 / r\theta(q_1, q_2, \cdots, q_n), \tag{5.2}$$

where q_i is 0 or 1, then the relation between θ's and λ's is

$$r \prod_{i=1}^{n} \otimes F_i(1) \cdot \underline{\theta} = -\frac{1}{k}\underline{\lambda}, \tag{5.3}$$

where

$$F_i(1) = \frac{1}{m_i}\begin{pmatrix} 1 & -1 \\ 1 & m_i - 1 \end{pmatrix}, \tag{5.4}$$

$$\underline{\theta} = (\theta_0, \theta_1, \cdots, \theta_m)', \text{ and } \underline{\lambda} = (\lambda_0, \lambda_1, \cdots, \lambda_m)',$$

where θ_q and λ_p stand for $\theta(q_1, q_2, \cdots, q_n)$ and $\lambda_{p_1 p_2 \cdots p_n}$ such that $q = \sum_{h=1}^{n} q_h 2^{n-h}$ and $p = \sum_{h=1}^{n} p_h 2^{n-h}$ respectively and $\theta_0 = 0$ and

$$\lambda_m = \sum_{j=1}^{b} r_{tj}^2 - rk = r^* - rk .$$

Now we conclude the following:

Theorem 5.1. Any NPBB design having BNAS is a BFE and is a PA type block design, and conversely.

Example 5.1. Consider the following block design having $v = 2 \times 3$, $r = 4$, $k = 8$, and $b = 3$:

b_1	(00), (10), (01), (02), (11), (12), (00), (10).
b_2	(01), (11), (00), (11), (02), (10), (12), (01).
b_3	(02), (12), (12), (10), (00), (01), (11), (02).

In this design, treatment (00) is a $(1,0)^{th}$ associate with treatments (01) and (02); a $(0,1)^{th}$ associate with treatment (10); and a $(0,0)^{th}$ associate with treatments (11) and (12). Since $r_{(00),1} = 2$, $r_{(00),2} = 1$, $r_{(00),3} = 1$, and $r_{(01),1} = 1$, $r_{(01),2} = 2$, $r_{(01),3} = 1$, then $\lambda_{10} = 5$, and since $r_{(10),1} = 2$, $r_{(10),2} = 1$, $r_{(10),3} = 1$, then $\lambda_{01} = 6$, and lastly, since $r_{(11),1} = 1$, $r_{(11),2} = 2$, $r_{(11),3} = 1$, then $\lambda_{00} = 5$. Also, we obtain $r^* = 6$ and $\lambda_{11} = 6 - 32 = -26$. Hence, the above design is an NPBB design with $v = 2\times3$, $r = 4$, $k = 8$, $b = 3$ and $\lambda_{00} = 5$, $\lambda_{01} = 6$, $\lambda_{10} = 5$, $\lambda_{11} = -26$, is an NBFE, and is a PA type block design.

6. PAB TYPE RECTANGULAR DESIGNS AND n-ary PARTIALLY BALANCED RECTANGULAR DESIGNS

Consider a $k\times b$ rectangular design in n factors $F_1, F_2, \ldots, F_n$ at $m_1, m_2, \ldots, m_n$ levels having $v = \prod_{i=1}^{n} m_i$ treatments and each treatment being replicated r times. Suppose that the design is a BFE with respect to columns (a PA type column design) and any contrast belonging to the generalized interaction $F_1^{q_1} F_2^{q_2} \cdots F_n^{q_n}$ is estimated with variance

$$\sigma^2/r\theta(q_1, q_2, \ldots, q_n), \text{ where } q_i = 0 \text{ or } 1 \text{ for } i = 1,2,\ldots,n$$

or

$$(6.1) \qquad \sigma^2/r\theta_q, \text{ where } q = \sum_{h=1}^{n} q_h 2^{n-h},$$

then we may obtain the parameter vector λ from the following relation:

$$r \prod_{i=1}^{n} \otimes F_i(1) \cdot \underline{\theta} = -\frac{1}{k}\underline{\lambda} ,$$

where $\underline{\theta} = (0, \theta_1, \ldots, \theta_m)'$ and $\underline{\lambda} = (\lambda_0, \lambda_1, \ldots, \lambda_{m-1}, r-rk)'$. Also, suppose that the design is an NBFE with respect to rows, or a PA type row design, and any contrast belonging to generalized interaction $F_1^{q_1} F_2^{q_2} \ldots F_n^{q_n}$ is estimated with variance

$$\sigma^2/r\tilde{\theta}(q_1, q_2, \ldots, q_n), \text{ where } q_i = 0 \text{ or } 1 \text{ for } i = 1,2,\ldots,n,$$

or

$$\sigma^2/r\tilde{\theta}_q, \text{ where } q = \sum_{h=1}^{n} q_h 2^{n-h}; \tag{6.2}$$

then, we may obtain the parameter vector $\tilde{\underline{\lambda}}$ from the following relation:

$$r \prod_{i=1}^{n} \otimes F_i(1) \cdot \tilde{\underline{\theta}} = -\frac{1}{b}\tilde{\underline{\lambda}} ,$$

where $\tilde{\underline{\theta}} = (0, \tilde{\theta}_1, \ldots, \tilde{\theta}_m)'$, $\tilde{\underline{\lambda}} = (\tilde{\lambda}_0, \tilde{\lambda}_1, \ldots, \tilde{\lambda}_{m-1}, r^* - rk)'$.

In the above k×b rectangular design (from 2.8),

$$\begin{aligned} \tilde{C} &= rI_v - \frac{1}{k}NN' - \frac{1}{b}\tilde{N}\tilde{N}' + J_v\left(\frac{r}{v}\right) \\ &= \left(rI_v - \frac{1}{k}NN'\right) + \left(rI_v - \frac{1}{b}\tilde{N}\tilde{N}'\right) - r\left(I_v - \frac{1}{v}J_v\right). \end{aligned} \tag{6.3}$$

Then, the $\tilde{C}$-matrix can be written as follows:

$$\begin{aligned} \tilde{C} &= \sum_{q=1}^{m} r\theta_q L_q L_q' + \sum_{q=1}^{m} r\tilde{\theta}_q L_q L_q' - r \sum_{q=1}^{m} L_q L_q' \\ &= \sum_{q=1}^{m} r(\theta_q + \tilde{\theta}_q - 1) L_q L_q' \end{aligned} \tag{6.4}$$

$$= \sum_{q=1}^{m} r\theta_q^{*} L_q L_q' ,$$

where $\theta_q^* = \theta_q + \tilde{\theta}_q^{-1}$ and L_q is a $v \times \prod_{i=1}^{n} (m_i-1)^{q_i}$ (q_i = 0 or 1 and $q = \sum_{h=1}^{n} q_h 2^{n-h}$) matrix formed by a complete set of $\prod_{i=1}^{n} (m_i-1)^{q_i}$ normalized orthogonal vectors forming the generalized interaction $F_1^{q_1} F_2^{q_2} \ldots F_n^{q_n}$. Hence, the above design is a balanced factoral k×b rectangular experiment and is a PAB type k×b rectangular design. We now state the following theorem:

Theorem 6.1. If the design is an NPBB having BNAS with respect to rows and also to columns, then the design is BFRE and is a PAB type rectangular design.

From theorem 5.1 and the definition of a PAB type rectangular design, we obtain:

Theorem 6.2. Every PAB type rectangular design is an NPBB rectangular design having BNAS with respect to rows and to columns, and conversely.

Remark: A BFRE is not always a PAB type rectangular design nor an NPBB design having BNAS with respect to both rows and columns (see Kshirsager [1957]).

Example 6.1. Consider the design in Example 5.1 as a 3×8 rectangular arrangement.

(1). With respect to columns, $v = 2\times3$, $r = 4$, $k = 3$, $b = 8$, and $\lambda_{00} = 2$, $\lambda_{01} = 0$, $\lambda_{10} = 2$, and $\lambda_{11} = -8$.

Using the formula given by Shah [1960],

$$\underline{\theta} = -\frac{1}{rk} \prod_{i=1}^{m} \otimes G(m_i) \cdot \underline{\lambda} , \text{ where } G(m_i) = \begin{pmatrix} m_i-1 & 1 \\ -1 & 1 \end{pmatrix}$$

$$
= \frac{1}{12} \begin{bmatrix} 2 & 1 & 2 & 1 \\ -1 & 1 & -1 & 1 \\ -2 & -1 & 2 & 1 \\ 1 & -1 & -1 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 0 \\ 2 \\ -8 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 2/3 \\ 2/3 \end{bmatrix} .
$$

Hence,

$$
r\theta(0,1) = 4, \quad r\theta(1,0) = 8/3, \text{ and } r\theta(1,1) = 8/3 .
$$

(2). With respect to rows, $v = 2\times 3$, $r = 4$, $k = 8$, $b = 3$, and $\lambda_{00} = 5$, $\lambda_{01} = 6$, $\lambda_{10} = 5$, and $\lambda_{11} = -26$.

In this case

$$
\underline{\tilde{\theta}} = - \frac{1}{32} \begin{bmatrix} 2 & 1 & 2 & 1 \\ -1 & 1 & -1 & 1 \\ -2 & -1 & 2 & 1 \\ 1 & -1 & -1 & 1 \end{bmatrix} \begin{bmatrix} 5 \\ 6 \\ 5 \\ -26 \end{bmatrix} = \begin{bmatrix} 0 \\ 15/16 \\ 1 \\ 1 \end{bmatrix} .
$$

Hence,

$$
r\tilde{\theta}(0,1) = 15/4, \quad r\tilde{\theta}(1,0) = 4, \text{ and } r\tilde{\theta}(1,1) = 4 .
$$

(3). From (1) and (2), the above design is a "balanced factorial 3x8 rectangular experiment" and is a PAB type rectangular design having the following efficiency factors:

$$
\theta^*(0,1) = 15/16, \quad \theta^*(1,0) = 2/3, \text{ and } \theta^*(1,1) = 2/3 .
$$

γ'

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