

A MODULI SPACE OF MARKED HYPERBOLIC  
STRUCTURES FOR BIG SURFACES

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# A MODULI SPACE OF MARKED HYPERBOLIC STRUCTURES FOR BIG SURFACES

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We introduce the moduli space of marked, complete, Nielsen-convex hyperbolic structures on a surface of negative, but not necessarily finite, Euler characteristic. The emphasis is on the case in which the surface is of infinite type, the aim being to study the mapping class group of such a surface via its action on this marked moduli space. In this dissertation, we lay the foundations for the study of the marked moduli space and this action. We define a topology on the marked moduli space and describe it in various ways, each reminiscent of the Teichmüller space. We prove that it reduces to the usual Teichmüller space in case the surface is of finite type. We prove that the above natural action of the mapping class group on the marked moduli space by change of marking is continuous. We show that the marked moduli space is contractible. The proof is uniform for all surfaces. We also investigate the geometry of the marked moduli space. We show that for certain classes of surfaces, the marked moduli space does not admit any mapping class group invariant metrics.

## BIOGRAPHICAL SKETCH

Chaitanya Tappu was born and raised in the city of Pune near the west coast of India. He finished his schooling in Bharatiya Vidya Bhavan's Sulochana Natu Vidya Mandir and junior college at the Modern College of Arts, Science and Commerce, Shivajinagar. Tappu, as he is known to friends and colleagues, moved to Bengaluru to pursue undergraduate studies at the Indian Institute of Science and graduated with a major in Mathematics and a minor in Physics. He has since been a doctoral student in the Department of Mathematics at Cornell University. After receiving a PhD in Mathematics from Cornell, Tappu will continue mathematics research as a postdoctoral fellow at the Instituto de Matemática e Estatística at the Universidade de São Paulo.

॥श्री ॥

सदगुरुश्रीअनिरुद्धार्पणमस्तु ।

*sadguruśrīaniruddhārpaṇamastu ।*

अनिरुद्धा तुझा मी किती ऋणी झालो ।

*aniruddhā tujhā mī kitī ṛṇī jhālo ।*

गुरुपौर्णिमा शके १९४७ ।

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# CHAPTER 1

## INTRODUCTION

### 1.1 Motivations

The Teichmüller space of a fixed surface of finite type can be thought of as the moduli space of either marked Riemann surface structures or marked complete hyperbolic structures on the surface. The two viewpoints are equivalent due to the Koebe–Poincaré uniformisation theorem, the Killing–Hopf theorem, and the fact that the isometries of the hyperbolic plane are exactly its biholomorphisms. The mapping class group of the surface acts on the Teichmüller space by change of marking. This is an action by isometries, and in fact, the mapping class group is (up to index two) all of the isometry group of Teichmüller space by Royden’s theorem. This action is an important instance of geometric group theory, that is, the idea of studying groups via their actions on geometric spaces.

Moduli of marked complete hyperbolic structures appear in the work of Fricke and Klein ([FK97]). Moduli of complex structures had already been studied earlier by Riemann. Teichmüller and Ahlfors–Bers ([AB60],[Ber60a], [Ahl60]) later studied them using quasiconformal mappings. Hyperbolic geometry was used by Nielsen ([Nie27]) to study surface homeomorphisms, their dynamics and their mapping classes. This circle of ideas ripened in the 1970s, when the above action was exploited successfully by William Thurston ([Thu88]) and Lipman Bers ([Ber78]) to provide a classification (now called the Nielsen–Thurston classification) of mapping classes into three types: finite order, reducible and pseudo-Anosov. The Teichmüller space and its geometry have been studied extensively in the last several decades (see the brief survey [Pap07]). The Teichmüller space is known to possess a number of mapping class group invariant metrics with characteristics of nonpositive curvature in the large scale. A large number of fascinating results produced since are premised on this action and

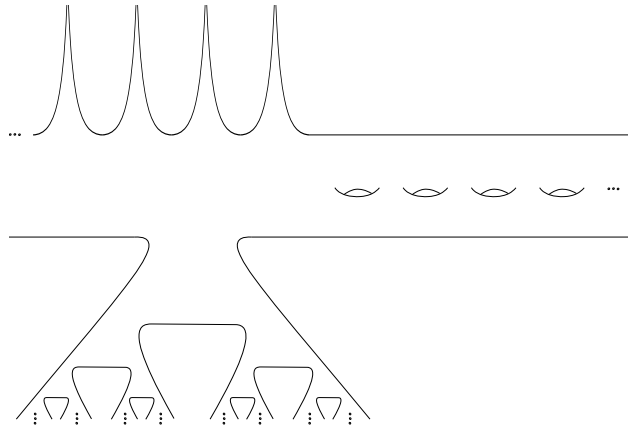


Figure 1.1: An example of a surface of infinite type: Arya's favourite surface.

this geometry, such as the geometric classification of mapping tori ([Thu98]), the solution to the Nielsen realisation problem ([Ker83]), et cetera. Today there are numerous accounts that expost these results in detail, ([FLP79], its English translation [FLP<sup>+</sup>12], [Abi80], [FM11], [Hub06], [Hub16], [Hub22]).

Additionally, there are a number of related combinatorial complexes, such as the curve graph or the pants graph, which can be used to study the coarse geometry of the mapping class group via its action on them. Teichmüller space is also connected to rich examples of dynamics. This includes inhomogeneous smooth dynamics of  $\mathrm{SL}(2, \mathbb{R})$  on the the space of quadratic differentials, which is the cotangent bundle of Teichmüller space; dynamics of Kleinian groups on three dimensional hyperbolic space and Patterson–Sullivan theory; as well as dynamics of polynomials and rational functions on the complex projective line.

The above serves as our inspiration in this dissertation. We introduce the moduli space  $\mathcal{T}(S)$  of marked, complete, Nielsen-convex hyperbolic structures on a surface  $S$  of negative Euler characteristic. We are especially interested in studying  $\mathcal{T}(S)$  when  $S$  is a surface of infinite type. See Figure 1.1 for an example. We also refer to  $\mathcal{T}(S)$  as the *marked moduli space*, and to a point in it as a *marked hyperbolic structure*. This dissertation develops the foundations of the theory of the marked moduli spaces of such surfaces.

Surfaces have been studied for so long that it is quite difficult to trace the history of the notion, which has evolved considerably and been refined over centuries. They have been studied both topologically as well as geometrically, both intrinsically as well as within an ambient Euclidean space. Compact surfaces were already classified up to homeomorphism before the dust was settling on the precise phrasing of the definition of a manifold (*Mannigfaltigkeit* in German and *variété* in French). The first complete proof of this classification has been attributed to Brauer ([Bra21]) in a historical survey ([GX13, Appendix D]). Kerékjártó classified noncompact surfaces shortly after in the book ([Ker23]), albeit with some gaps that were filled in later by Richards ([Ric63]).

Surfaces of infinite type and their mapping class groups (called *big mapping class groups*) appear naturally in various other contexts in geometry, topology and group theory, such as stable homology of mapping class groups, braided Thompson’s group  $BV$ , leaves of foliations of three dimensional manifolds, monodromy of fibred manifolds, et cetera ([AV20]). The study of big mapping class groups has seen a surge in popularity in the last decade, frequently attributed to Calegari’s 2009 blog post ([Cal09]). Since then, there has since been rapid progress on the fronts of the algebraic and topological properties ([PV18]), coarse geometry ([MR23]), actions on combinatorial complexes ([Bav16], [AFP17]), and other facets of big mapping class groups (see the survey [AV20]). This dissertation resides in the niche of real hyperbolic geometry and Teichmüller theory.

## 1.2 Summary of Results

As above, let  $S$  be a connected, oriented surface with negative Euler characteristic, or equivalently, a nonabelian fundamental group. The Euler characteristic need not be finite. As mentioned above, our emphasis is on the case in which  $S$  is an *surface of infinite type*, that is, its Euler characteristic is  $\chi(S) = -\infty$ . We now define the set  $\mathcal{T}(S)$ .

**Definition 1.1** (The set underlying the marked moduli space).

$$\mathcal{T}(S) := \left\{ (X, f) \left| \begin{array}{l} X \text{ is a complete Nielsen-convex hyperbolic surface} \\ f: S \rightarrow X \text{ is a homeomorphism} \end{array} \right. \right\} / \sim \quad (1.1)$$

where  $(X_1, f_1) \sim (X_2, f_2)$  if there is an isometry  $\varphi: X_1 \rightarrow X_2$  homotopic to  $f_2 \circ f_1^{-1}$ .

As is standard in finite type Teichmüller theory, we call the homeomorphism  $f$  the *marking map*. The notion of Nielsen-convexity is discussed and characterised in Section 3.1 and Proposition 3.1.

The *mapping class group* of the surface  $S$  is  $\text{MCG}(S) = \text{Homeo}^+(S) / \text{Homeo}_0(S)$ . It acts on  $\mathcal{T}(S)$  by change of marking (see Proposition 4.4). We denote the action of the mapping class  $[\psi]$  by the function  $A_{[\psi]}$  from  $\mathcal{T}(S)$  to itself. The action is defined precisely in equation (4.2) below.

The main results of this dissertation are the following.

**Theorem 1.2.** *The set  $\mathcal{T}(S)$  has a geometrically defined topology which can be described in various ways (see Definition 4.9, Proposition 4.10, Theorem 4.15, Theorem 4.20, Corollary 4.21, and Theorem 6.9).*

**Theorem 1.3.** *The marked moduli space  $\mathcal{T}(S)$  is Hausdorff and completely metrisable (see Proposition 4.10 and Corollary 4.26).*

**Theorem 1.4.** *The marked moduli space  $\mathcal{T}(S)$  agrees with the Teichmüller space (and its usual topology) when  $S$  is of finite type (see Corollary 4.16).*

**Theorem 1.5.** *The action function  $A$  is continuous (see Theorem 5.6) and  $\text{MCG}(S)$  acts on  $\mathcal{T}(S)$  by homeomorphisms (see Corollary 5.7).*

**Theorem 1.6.** *The marked moduli space  $\mathcal{T}(S)$  is contractible (see Theorem 6.11).*

**Theorem 1.7.** *If  $S$  belongs to one of certain classes of surfaces, then the marked moduli space  $\mathcal{T}(S)$  admits no  $\text{MCG}(S)$ -invariant metric (see Theorem 7.7 and Corollary 7.8).*

The contents of Sections 2.2 to 2.4, Sections 3.1 to 3.4, Sections 4.1 to 4.3, and Chapter 5 are available in preprint form on the arXiv ([Tap23]).

## CHAPTER 2

# PRELIMINARIES

In this chapter, we recall some basic facts of surface topology, hyperbolic geometry, complex analysis and algebraic topology, allowing us to fix notation of various objects used throughout. This material is not new, and we provide references to well known expositions in each section. The reader familiar with these subjects may well skip this chapter.

### 2.1 Surfaces of Infinite Type

References for this section are [Ric63] and the survey chapter [AV20, Sections 2–4]. For us, a *surface* is a two dimensional real topological manifold, that is, a Hausdorff and paracompact topological space which is locally homeomorphic to  $\mathbb{R}^2$ . A surface with boundary is defined the same way, except the local model is the closed half plane. In this dissertation, we will define the marked moduli space only for connected surfaces without boundary, although we recall the reduced Teichmüller space for surfaces with boundary of finite type in Section 2.8 (Quasiconformal Teichmüller Space). All surfaces in consideration are oriented, and all homeomorphisms between surfaces (including isometries between hyperbolic surfaces or locally quasiconformal homeomorphisms between Riemann surfaces) are orientation preserving. In fact, homeomorphisms between one dimensional manifolds are also orientation preserving, and we suppress mention of orientation.

Surfaces have been classified up to homeomorphism, and here we recall all the invariants of homeomorphism of surfaces. Note that it is enough to classify connected surfaces, since an arbitrary surface is the disjoint union of its connected components. Closed surfaces are homeomorphic to the connect sum of a sphere with a number of tori. This number of tori is an invariant, called the *genus* and typically denoted by the letter  $g$ . One can attach this invariant to compact surfaces with boundary as well. To do so, note that the

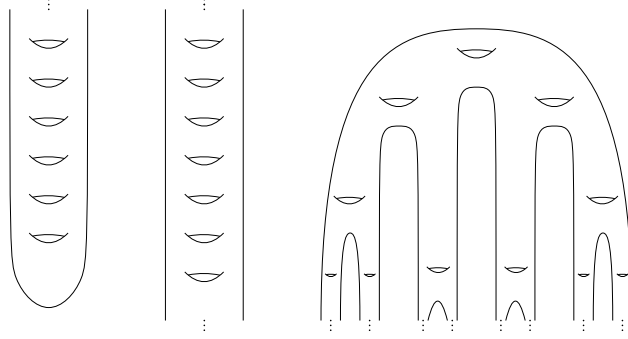


Figure 2.1: Examples of surfaces of infinite type. From left to right: Loch Ness monster surface, Jacob’s ladder surface, and the blooming Cantor tree surface. The Loch Ness monster has one end (going toward the top), Jacob’s ladder has two ends (going toward the top and bottom), and the blooming Cantor tree has infinitely many ends. All three have infinite genus. Figure taken from [AV20].

boundary of such a surface is the disjoint union of finitely many components, all of which are homeomorphic to the circle. Capping off each circle component of the boundary by attaching a disk gives a closed surface, whose genus is, by definition, the genus of the original surface with boundary. As an aside, we remark that the *number of boundary components* is also an invariant of compact surfaces with boundary and typically denoted by the letter  $b$ . The genus of an arbitrary (possibly noncompact) surface is the supremum of the genera of its compact subsurfaces with boundary, and thus takes values in  $\mathbb{N} \cup \{\infty\}$ . A surface is called *planar* if its genus is zero. Another invariant is the *space of ends*, denoted by  $\text{Ends}$ . Roughly, an end of a surface is a way to leave compact subsets of the surface and go to infinity.  $\text{Ends}$  is a compact, metrisable, totally disconnected space, and thus realisable as a closed subspace of the Cantor space. An end is called *planar* if it has a neighbourhood in the surface which is a planar surface. If not, then every neighbourhood of the end in the surface is a nonplanar surface and the end is called *nonplanar*, or is said to be *accumulated by genus*. The set of nonplanar ends  $\text{Ends}_{\text{np}}$  is a closed subset of  $\text{Ends}$  (and therefore is also compact, metrisable and totally disconnected space). The pair of spaces  $(\text{Ends}, \text{Ends}_{\text{np}})$  is an invariant. The genus and the spaces of ends satisfy a compatibility condition.

**Proposition 2.1.** *For a surface  $S$ , we have*

$$g(S) = \infty \text{ if and only if } \text{Ends}_{\text{np}}(S) \neq \emptyset \quad (2.1)$$

The above are exactly all the invariants required to classify surfaces up to homeomorphism.

**Theorem 2.2** ([Ker23], [Ric63, Theorem 1], Classification). *Two (connected, oriented) surfaces  $S, S'$  are homeomorphic if and only if  $g(S) = g(S')$  and  $(\text{Ends}(S), \text{Ends}_{\text{np}}(S)) \cong (\text{Ends}(S'), \text{Ends}_{\text{np}}(S'))$ .*

They are also all realisable as long as the compatibility condition Equation (2.1) is satisfied.

**Theorem 2.3** ([Ric63, Theorem 2], Realisation). *Let  $G \in \mathbb{N} \cup \{\infty\}$ , let  $E$  be a compact, metrisable, totally disconnected space, and let  $E' \subset E$  be a closed subset. Suppose  $G$  and the pair  $(E, E')$  satisfy the compatibility Equation (2.1). Then there is a (connected, oriented) surface  $S$  such that  $g(S) = G$  and  $(\text{Ends}(S), \text{Ends}_{\text{np}}(S)) \cong (E, E')$ .*

An isolated and planar end is called a *puncture*. Each puncture has a neighbourhood homeomorphic to the standard open disk with the origin removed. Each end of a surface is a puncture if and only if  $\text{Ends}_{\text{np}}$  is empty and  $\text{Ends}$  is a finite discrete space, which is then totally determined by its cardinality. In this case, the number of punctures is a (finite) nonnegative integer, and this invariant is typically denoted by the letter  $p$ . The final invariant we recall is the *Euler characteristic*  $\chi(S) = \dim_{\mathbb{Q}}(H_0(S); \mathbb{Q}) - \dim_{\mathbb{Q}}(H_1(S); \mathbb{Q}) + \dim_{\mathbb{Q}}(H_2(S); \mathbb{Q})$ . Note that since the zeroth Betti number is one for the connected surface  $S$  and second Betti number is at most one, the above sum makes sense and takes values in  $\{-\infty\} \cup \mathbb{Z}_{\leq 2}$ . Note that the fundamental group  $\pi_1(S)$  is abelian if and only if  $\chi(S) \geq 0$ .

**Definition 2.4** (Finite Type and Infinite Type). *A surface  $S$  is of finite type if  $\chi(S) > -\infty$ , and is of infinite type if  $\chi(S) = -\infty$ .*

**Proposition 2.5** (Alternate characterisations).

- *A surface  $S$  is of finite type if and only if it has finite genus and finite end space.*
- *A surface  $S$  is of finite type if and only if its fundamental group is finitely generated.*

Finally, a surface with boundary of finite type is one with finite genus  $g$ , a finite end space with finite number of punctures  $p$  and a finite number of circle boundary components  $b$ . This surface is denoted  $\Sigma_{g,p}^b$ .

We recall some facts about the mapping class group of a surface.

**Definition 2.6** (Mapping Class Group). *The mapping class group of a surface  $S$  is the group  $\text{MCG}(S) := \text{Homeo}^+(S) / \text{Homeo}_0(S)$ .*

The group  $\text{Homeo}^+(S)$  is a topological group with the compact-open topology, which agrees, for any metric on  $S$ , with the topology of uniform convergence on compact subsets. The subgroup  $\text{Homeo}_0(S)$  of homeomorphisms isotopic to the identity is a closed subgroup. Hence the quotient  $\text{MCG}(S)$  is a topological group with the quotient topology. If  $S$  is of finite type, then  $\text{MCG}(S)$  is a discrete, finitely presented group.

However, if  $S$  is of infinite type,  $\text{MCG}(S)$  is uncountable (and so not finitely or even countably generated), not even compactly generated, and not locally compact (and so certainly not discrete). However,  $\text{MCG}(S)$  is a Polish topological group, that is, it is separable and completely metrisable. In fact, it is homeomorphic to the Baire space. Analogues of generating sets of finite type mapping class groups can serve as topological generating sets of big mapping class groups ([PV18]). Classical geometric group theory works for finitely generated groups, and there has been considerable recent work in developing the foundations of a geometric group theory of big mapping class groups ([MR23]).

## 2.2 Plane Hyperbolic Geometry

A reference for this section is [And06, Chapters 1–2]. The *hyperbolic plane* may be described as either the upper half plane or the unit disk in the Riemann sphere, and is equipped with the Poincaré metric and the usual orientation. These two models of the hyperbolic plane are isometric. We denote the hyperbolic plane by  $\mathbb{H}^2$ . It has a *circle at infinity* which is the real projective line in the upper half plane model or the standard unit circle in the unit disk model. We denote the circle at infinity by  $S^1$ . It inherits an orientation as the boundary of the closed disk  $\mathbb{H}^2 \cup S^1$ , and, being a topologically a circle, also possesses a corresponding circular order. We fix, once and for all, a positively oriented triple of distinct points on  $S^1$ , such as  $(0, 1, \infty)$  in the real projective line model.

The group  $\mathrm{PSL}(2, \mathbb{R})$  of  $2 \times 2$  real matrices of determinant 1, modulo  $\pm I$ , acts by fractional linear transformations on the upper half plane. This action induces an isomorphism between  $\mathrm{PSL}(2, \mathbb{R})$  and the group of (orientation preserving) isometries of  $\mathbb{H}^2$ . Each isometry of  $\mathbb{H}^2$  extends to a (an orientation preserving) homeomorphism of  $S^1$ . In fact, this action of  $\mathrm{PSL}(2, \mathbb{R})$  on the circle at infinity is simply the faithful action by linear fractional transformations on the real projective line. It induces an isomorphism between  $\mathrm{PSL}(2, \mathbb{R})$  and a subgroup of  $\mathrm{Homeo}^+(S^1)$ , which is the group of (orientation preserving) homeomorphisms of the circle. Moreover,  $\mathrm{PSL}(2, \mathbb{R})$  acts freely and transitively on positively oriented triples of distinct points in  $S^1$ . For a positively oriented triple  $(a, b, c)$  of distinct points in  $S^1$ , we denote by  $M(a, b, c)$  the unique element of  $\mathrm{PSL}(2, \mathbb{R})$  that maps 0 to  $a$ , 1 to  $b$  and  $\infty$  to  $c$ ; The function  $(a, b, c) \mapsto M(a, b, c)$  is continuous. In particular, the faithfulness of this action means that an isometry of  $\mathbb{H}^2$  is determined by its induced homeomorphism of the circle at infinity. We will freely use the same symbol to denote an element of  $\mathrm{PSL}(2, \mathbb{R})$  whether it is viewed as an isometry of  $\mathbb{H}^2$  or a homeomorphism of  $S^1$ .

Even though the trace of an element  $\gamma \in \mathrm{PSL}(2, \mathbb{R})$  is not well defined, the absolute value

of the trace  $|\operatorname{tr}(\gamma)|$  is well defined. The absolute trace  $|\operatorname{tr} \cdot|: \operatorname{PSL}(2, \mathbb{R}) \rightarrow \mathbb{R}_{\geq 0}$  is a continuous function. A nontrivial element of  $\operatorname{PSL}(2, \mathbb{R})$  is one of exactly three types depending on the number of points it fixes in  $\mathbb{H}^2 \cup S^1$ , and these correspond with the absolute value of the trace. Namely, a nontrivial element is *elliptic* iff it has exactly one fixed point, which is in  $\mathbb{H}^2$ , or equivalently has absolute trace less than 2, is *parabolic* iff it has exactly one fixed point, which is in  $S^1$ , or equivalently has absolute trace equal to 2, or is *hyperbolic* iff it has exactly two fixed points, both in  $S^1$ , or equivalently has absolute trace greater than 2. For a hyperbolic  $\gamma$ , we denote by  $\gamma_\infty$ , the unique attracting fixed point of  $\gamma$  on  $S^1$ , also known as its *sink*; the function  $\gamma \mapsto \gamma_\infty$  is continuous. Then  $(\gamma^{-1})_\infty$  is the unique repelling fixed point of  $\gamma$  on  $S^1$ , and is called its *source*. Further,  $\gamma$  has a positive translation distance  $2 \cosh^{-1} |\operatorname{tr}(\gamma)|$  which is achieved on the geodesic joining its source and sink. Further, note that  $\operatorname{Homeo}^+(S^1)$  is a topological group with the compact-open topology, which agrees, for any metric on  $S^1$ , with the topology of uniform convergence. Then  $\operatorname{PSL}(2, \mathbb{R})$  is a closed and embedded subgroup of  $\operatorname{Homeo}^+(S^1)$ .

## 2.3 Complete Hyperbolic Surfaces

A reference for this section is [Kat92, Chapters 2–3]. Let  $X$  be a complete (and connected and oriented) *hyperbolic surface*. For us, a ‘complete’ hyperbolic surface always means a ‘geodesically complete’ hyperbolic surface, as opposed to merely a metrically complete hyperbolic surface, unless otherwise specified. We denote by  $d_X$  the *distance function* on  $X$ , and by  $\operatorname{inj}_X$  the *injectivity radius* as a function on  $X$ , which is well known to be a continuous function. Being a complete hyperbolic surface,  $X$  has a (Riemannian locally isometric, orientation preserving) universal cover  $p_X: \mathbb{H}^2 \rightarrow X$ , and any other such cover is of the form  $p_X \circ \sigma$  for some  $\sigma \in \operatorname{PSL}(2, \mathbb{R})$ . We denote the deck group of the universal cover  $p_X$  by  $\Gamma_X$ , which is a group of isometries of  $\mathbb{H}^2$  acting freely and properly discontinuously on  $\mathbb{H}^2$ .

Alternatively, as a subgroup of  $\mathrm{PSL}(2, \mathbb{R})$ ,  $\Gamma_X$  is a torsion-free discrete group, also known as a torsion-free *Fuchsian* group. The universal cover induces an isometry between  $\Gamma_X \backslash \mathbb{H}^2$  and  $X$ . For any  $x \in X$ ,  $\mathrm{inj}_X(x)$  is the radius of the largest disk centred at  $x$  which is evenly covered by the universal cover. Thus if  $d_X(x, y) < \mathrm{inj}_X(x)$ , then there is a unique geodesic segment from  $x$  to  $y$ .

The deck group  $\Gamma_X$ , being a subgroup of  $\mathrm{PSL}(2, \mathbb{R})$ , acts on the circle at infinity  $S^1$  as well. As a group acting freely and properly discontinuously by isometries on  $\mathbb{H}^2$ ,  $\Gamma_X$  cannot contain any elliptic elements. Thus all the nontrivial elements of  $\Gamma_X$  are either hyperbolic or parabolic. The discreteness of  $\Gamma_X$  implies that each element of  $\Gamma_X$  has a root which is *primitive*, that is, not a nontrivial power of another element of  $\Gamma_X$ . If  $\Gamma_X$  is nontrivial, and not a cyclic group generated by a parabolic element, then it must have at least one hyperbolic element. Any two hyperbolic elements of  $\Gamma_X$  which have the same sink also have the same source, and are in fact both positive powers of some hyperbolic in  $\Gamma_X$ . We denote by  $(\Gamma_X)_\infty$  the set of sinks of all the hyperbolic elements of  $\Gamma_X$ , and it is easily seen to be  $\Gamma_X$ -invariant.

The set of accumulation points in  $\mathbb{H}^2 \cup S^1$  of the  $\Gamma_X$ -orbit of some (any) point in  $\mathbb{H}^2$  is a closed subset of  $S^1$ , known as the *limit set*  $\Lambda_X$  of  $\Gamma_X$ . If  $\Lambda_X$  is nonempty, it is the smallest nonempty  $\Gamma_X$ -invariant closed subset of  $S^1$ . Thus  $\Lambda_X$  is the closure of  $(\Gamma_X)_\infty$  in  $S^1$ , if  $(\Gamma_X)_\infty$  is nonempty.  $\Gamma_X$  acts freely and properly discontinuously on  $S^1 \setminus \Lambda_X$ , and the quotient  $\Gamma_X \backslash (S^1 \setminus \Lambda_X)$  is called the *ideal boundary*  $I(X)$  of  $X$  (see [Ber63, p334] or [Ber64, §1.3] or [Hub06, Section 3.7]). The convex hull  $\mathrm{CH}(\Lambda_X)$  of  $\Lambda_X$  in  $\mathbb{H}^2$  is called the *Nielsen convex region* of  $\Gamma_X$ , easily seen to be  $\Gamma_X$ -invariant, and its quotient by  $\Gamma_X$  is the *convex core*  $C(X)$  of  $X$ . The Fuchsian group  $\Gamma_X$  is said to be *of the first kind* if its limit set  $\Lambda_X$  is the entire circle at infinity.

## 2.4 Algebraic Topology of Hyperbolic Surfaces

A reference for this section is [Hat02, Section 1.3]. As before, let  $X$  be a complete (and connected and oriented) hyperbolic surface. Given the universal cover  $p_X: \mathbb{H}^2 \rightarrow X$ , a basepoint  $x \in X$  and a choice of its lift  $\tilde{x} \in \mathbb{H}^2$ , or equivalently, given a pointed universal cover  $p_X: (\mathbb{H}^2, \tilde{x}) \rightarrow (X, x)$ , there is an isomorphism  $\varphi_X: \pi_1(X, x) \rightarrow \Gamma_X$ , known as the *holonomy representation*, and it is described as follows. If  $\alpha$  is an oriented closed curve in  $X$  based at  $x$ , then  $\varphi_X[\alpha]$  is the unique deck transformation, called the *holonomy around  $\alpha$* , that maps  $\tilde{x}$  to the endpoint of the lift of  $\alpha$  starting at  $\tilde{x}$ . If  $\tilde{\alpha}$  is the bi-infinite lift of  $\alpha$  passing through  $\tilde{x}$ , then the deck transformation  $\varphi[\alpha]$  acts on  $\tilde{\alpha}$  by translation. Note that the isomorphism  $\varphi_X$  depends not only on the choice of the basepoint, but also on the choice of lift  $\tilde{x}$ , and it changes with these choices as follows. Suppose  $x' \in X$  is another basepoint and  $\tilde{x}' \in \mathbb{H}^2$  is a choice of its lift with the corresponding holonomy representation  $\varphi'_X: \pi_1(X, x') \rightarrow \Gamma_X$ . Then the two holonomy representations satisfy the relation  $\varphi'_X = \varphi_X \circ c_\beta$ . Here  $\beta$  is the projection  $p(\tilde{\beta})$  of a continuous path  $\tilde{\beta}$  from  $\tilde{x}$  to  $\tilde{x}'$ , and  $c_\beta$  is ‘conjugation by  $\beta$ ’, that is,  $c_\beta[\alpha'] = [\beta \cdot \alpha' \cdot \bar{\beta}]$  for all  $[\alpha'] \in \pi_1(X, x')$ . If  $\alpha$ , based at  $x$ , is homotopic to  $\alpha'$ , based at  $x'$ , and  $\beta$  is the track of  $x$  under this homotopy, then  $[\alpha] = c_\beta[\alpha']$ . Combining these two observations, we see that  $\varphi_X[\alpha] = \varphi'_X[\alpha']$ . In other words, if two closed curves are homotopic, then the holonomies around them (with respect to appropriate choices of lifts of basepoints under the same universal cover) are equal. Conversely, if the holonomies around two curves are equal, then the curves are free homotopic.

If  $X, Y$  are complete hyperbolic surfaces, and  $p_X: \mathbb{H}^2 \rightarrow X$  and  $p_Y: \mathbb{H}^2 \rightarrow Y$  are their universal covers, then any homeomorphism  $f: X \rightarrow Y$  lifts to a homeomorphism  $\tilde{f}: \mathbb{H}^2 \rightarrow \mathbb{H}^2$ . This induces an isomorphism  $f_*: \Gamma_X \rightarrow \Gamma_Y$  between the two deck groups, which is ‘conjugation by  $\tilde{f}$ ’, that is,  $f_*(\gamma) = \tilde{f} \circ \gamma \circ \tilde{f}^{-1}$  for all  $\gamma \in \Gamma_X$ . Note that the isomorphism  $f_*$  depends on the choice of the lift  $\tilde{f}$ , but we denote it by  $f_*$ , suppressing the lift from the notation. We also write the above relation as  $\tilde{f} \circ \gamma = f_*(\gamma) \circ \tilde{f}$  for all  $\gamma \in \Gamma_X$ . Collecting

all these equalities together, we have an equality of sets  $\tilde{f} \circ \Gamma_X = \Gamma_Y \circ \tilde{f}$ . This set is the set of all lifts of  $f$ . Conversely, if  $F: \mathbb{H}^2 \rightarrow \mathbb{H}^2$  satisfies  $F \circ \Gamma_X = \Gamma_Y \circ F$ , then  $F$  descends to a homeomorphism from  $X$  to  $Y$ . In this situation, we say that  $\tilde{f}$  (or  $F$ ) is *equivariant* under the action of the deck groups. If  $\varphi_X$  and  $\varphi_Y$  (with respect to basepoints  $\tilde{y} = \tilde{f}(\tilde{x})$  and  $y = p_Y(\tilde{y})$ ) are the holonomy representations of  $X$  and  $Y$  respectively, then we have  $f_*(\varphi_X[\alpha]) = \varphi_Y(f_*[\alpha])$ . Here the  $f_*$  on the left hand side is the isomorphism between the deck groups, whereas on the right hand side,  $f_*$  is the  $\pi_1$  functor. In other words, if  $\gamma \in \Gamma_X$  is the holonomy around the oriented closed  $\alpha \subset X$ , then  $f_*(\gamma) \in \Gamma_Y$  is the holonomy around the oriented closed curve  $f(\alpha)$ .

## 2.5 Riemann Surfaces and Hyperbolic Surfaces

A reference for this section is [IT92, Chapters 1–2]. Let  $X$  be a *Riemann surface*. Its universal cover is simply connected and therefore, by the uniformisation theorem, isomorphic to exactly one of the  $\mathbb{CP}^1$ ,  $\mathbb{C}$  or  $\mathbb{H}^2$ . Throughout this dissertation, we will only consider Riemann surfaces whose universal cover is isomorphic to  $\mathbb{H}^2$ .<sup>1</sup> The group  $\Gamma_X$  of deck transformations of the universal cover  $\mathbb{H}^2 \rightarrow X$  is a subgroup of the biholomorphism group of  $\mathbb{H}^2$ . However, the biholomorphism group of  $\mathbb{H}^2$  is exactly the isometry group of  $\mathbb{H}^2$ . Therefore the Poincaré metric of  $\mathbb{H}^2$  descends to a hyperbolic metric on  $X$ . We denote the metric tensor of this hyperbolic metric by  $\rho_X$  and the corresponding line element by  $d\rho_X$ . Since the Poincaré metric of  $\mathbb{H}^2$  is complete and is conformal with its complex structure, so is that of  $X$ . We say that  $X$  has been uniformised as  $\Gamma_X \backslash \mathbb{H}^2$ , and the uniformised metric on  $X$  is  $\rho_X$ .

Actually,  $\rho_X$  is the only complete hyperbolic metric conformal with the complex structure

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<sup>1</sup>This excludes from consideration, the Riemann surfaces carrying a conformal complete flat metric (the complex plane, all tori, and the punctured complex plane, but not any other Riemann surface homeomorphic to an annulus) and the Riemann sphere, which carries a conformal spherical metric. We will eventually exclude other annuli and the hyperbolic plane as well.

of  $X$ . For, if  $\rho'_X$  is another such metric, then the lifted metric  $\widetilde{\rho'_X}$  on the universal cover  $\mathbb{H}^2$  is also hyperbolic, since the universal cover is a local isometry by definition. By the Killing–Hopf theorem, there is only one simply connected two dimensional hyperbolic manifold up to isometry, the hyperbolic plane. Therefore  $\widetilde{\rho'_X}$  is isometric to the (standard) Poincaré metric of  $\mathbb{H}^2$ . However, since  $\rho'_X$  is conformal with the complex structure of  $X$ ,  $\widetilde{\rho'_X}$  is also conformal with the (standard) complex structure of  $\mathbb{H}^2$ . Thus the above isometry is actually a biholomorphism of  $\mathbb{H}^2$ , and therefore an isometry of the Poincaré metric of  $\mathbb{H}^2$ . Hence  $\widetilde{\rho'_X}$  is actually equal the Poincaré metric of  $\mathbb{H}^2$ , and  $\rho'_X$  is just  $\rho_X$ .

Conversely, given a hyperbolic surface  $X$ , there is precisely one complex structure conformal to its hyperbolic metric. One way to prove this integrability is to solve a related Beltrami equation. Another way to see this is again that the universal cover of any complete hyperbolic surface is the hyperbolic plane, whose biholomorphism group and isometry group coincide. The local inverses to the universal cover form an atlas of holomorphic coordinate charts. This complex structure is the unique one conformal to the hyperbolic metric due to the Cauchy–Riemann equations.

## 2.6 Riemann Surfaces and Quasiconformal Mappings

A reference for this section is [Hub06, Chapter 4]. For a homeomorphism (orientation preserving)  $f: U \rightarrow V$  of plane domains, the complex dilatation and quasiconformal dilatation at a point  $q \in U$  are  $\frac{\partial f/\partial \bar{z}}{\partial f/\partial z}(q)$  and  $\frac{|\partial f/\partial z| + |\partial f/\partial \bar{z}|}{|\partial f/\partial z| - |\partial f/\partial \bar{z}|}(q)$  respectively. These quantities certainly make sense if  $f$  is continuously differentiable and nonsingular at  $q$ , and their values can be a complex number of modulus less than one or a finite real number at least one respectively. They measure the conformality of  $f$  at  $q$ , or failure thereof. Due to the Cauchy–Riemann equations, the homeomorphism  $f$  is conformal at  $q$  precisely when the complex dilatation is zero, or equivalently, quasiconformal dilatation is one. A quasiconformal homeomorphism

is, roughly, a homeomorphism that has a small failure of conformality. It is also convenient to require a much lower regularity of such homeomorphisms than smooth diffeomorphism. The homeomorphism  $f$  is called quasiconformal if it has locally square integrable distributional first partial derivatives, and there is a  $k \in [0, 1)$  such that  $|\partial f/\partial \bar{z}| \leq k |\partial f/\partial z|$  almost everywhere. Then the quasiconformal dilatation is at most  $T := \frac{1+k}{1-k}$  almost everywhere, and we say that  $f$  is a  $T$ -quasiconformal homeomorphism. A 1-quasiconformal homeomorphism is a biholomorphism, and a composition of  $T_1$ - and  $T_2$ -quasiconformal homeomorphisms is a  $(T_1 T_2)$ -quasiconformal homeomorphism. Thus the advantage of measuring quasiconformal dilatation rather than simply the partial derivative  $\partial f/\partial \bar{z}$  is that it is invariant under conformal mappings and therefore is well defined for Riemann surfaces in general. For  $k \in [0, 1)$  and  $T = \frac{1+k}{1-k}$ , a homeomorphism  $f: X \rightarrow Y$  of Riemann surfaces is  $T$ -quasiconformal if for every holomorphic coordinate charts  $z$  on  $X$  and  $w$  on  $Y$ ,  $w \circ f \circ z^{-1}$  is (using appropriate plane domains and codomains)  $T$ -quasiconformal. We drop the  $T$  if we do not wish to emphasise what its value is.

To make a coordinate free definition of the complex dilatation, we need to interpret the complex dilatation of  $w \circ f \circ z^{-1}$  at  $z(q)$  without reference to the coordinates, and use that as the complex dilatation of  $f$  at  $q$ . We again consider  $f$  to be continuously differentiable and nonsingular (and orientation preserving) at  $q \in X$ . Note that the derivative  $Df_q: T_q X \rightarrow T_{f(q)} Y$  is a real linear isomorphism between complex vector spaces of complex dimension one. There is a unique decomposition  $Df_q = \partial f_q + \bar{\partial} f_q$ , where  $\partial f_q$  is complex linear and  $\bar{\partial} f_q$  is complex antilinear. Then the *complex dilatation of  $f$  at  $q$* , or *Beltrami coefficient of  $f$  at  $q$* , is the complex antilinear self map of  $T_q X$  given by  $\mu_f(q) := (\partial f_q)^{-1} \circ (\bar{\partial} f_q)$ , defined without reference to coordinates. Clearly  $Df_q$  is complex linear if and only if  $\mu_f(q) = 0$ . Observe that the set  $L_*(T_q X)$  of complex antilinear maps of  $T_q X$  to itself is a complex vector space of complex dimension one. Since  $d\bar{z}_q$  and  $dz_q$  are respectively complex antilinear and complex linear isomorphisms of  $T_q X$  with  $\mathbb{C}$ ,  $(dz_q)^{-1} \circ d\bar{z}_q$  is an antilinear isomorphism of  $T_q X$ . We denote it by  $\frac{d\bar{z}_q}{dz_q}$ , and it is a basis vector for  $L_*(T_q X)$ . Then in these local

coordinates, we have  $\mu_f(q) = \frac{\partial(w \circ f)/\partial\bar{z}}{\partial(w \circ f)/\partial z}(q) \frac{d\bar{z}_q}{dz_q}$ , tying the coordinate free definition back to the earlier one. Any norm on  $T_q X$  (as a complex vector space) leads to the same operator norm on  $L_*(T_q X)$ . In this norm  $\left| \frac{d\bar{z}_q}{dz_q} \right| = 1$ , and hence  $|\mu_f(q)| = \left| \frac{\partial(w \circ f)/\partial\bar{z}}{\partial(w \circ f)/\partial z}(q) \right|$ , which is less than one since  $Df_q$  is orientation preserving. Thus the Beltrami coefficient at  $q$  lies in the unit ball of the complex vector space  $L_*(T_q X)$ . The *quasiconformal dilatation of  $f$  at  $q$*  is  $\kappa_f(q) = \frac{1 + |\mu_f(q)|}{1 - |\mu_f(q)|}$ , and the *global quasiconformal dilatation of  $f$*  is  $\operatorname{ess\,sup}_{q \in X} \kappa_f(q)$ . In general, if  $f$  is a  $T$ -quasiconformal homeomorphism but not necessarily continuously differentiable, then  $\mu_f$  and  $\kappa_f$  exist only almost everywhere on  $X$ , and we have  $|\mu_f| \leq k$  and  $|\kappa_f| \leq T$  almost everywhere. Note that  $f$  is biholomorphic if and only if  $\mu_f \equiv 0$ , or equivalently,  $\kappa_f \equiv 1$ . Thus the functions  $\mu_f$  and  $\kappa_f$  measure how far  $f$  is from being biholomorphic.

Any quasiconformal homeomorphism of  $\mathbb{H}^2$  extends to a homeomorphism of  $S^1$  (in fact a quasimetric homeomorphism). Since this extension is equivariant under the composition action of  $\operatorname{PSL}(2, \mathbb{R})$ , any quasiconformal homeomorphism between Riemann surfaces  $X$  and  $Y$  extends to a homeomorphism of their ideal boundaries  $I(X)$  and  $I(Y)$ . The *quasiconformal mapping class group* of a Riemann surface  $X$  is  $\operatorname{QMCG}(X) := \operatorname{QC}(X) / \operatorname{QC}_0(X)$  where  $\operatorname{QC}(X)$  is the group of quasiconformal self homeomorphisms of  $X$  and  $\operatorname{QC}_0(X)$  is the subgroup of homeomorphisms which are isotopic to the identity  $\operatorname{rel} I(X)$ .

## 2.7 Quasiconformal Beltrami Forms

References for this section are [Ahl66] and [Hub06, Sections 4.6–4.8]. According to our coordinate free definition,  $\mu_f$  is a section of the vector bundle  $L_*(TX)$  over  $X$ , of the complex antilinear bundle maps from  $TX$  to itself. Here  $TX$ , the tangent bundle of  $X$ , is a holomorphic one dimensional complex vector bundle. In general, a  $(-1, 1)$ -form  $\mu$  is a complex antilinear bundle map from  $TX$  to itself. That is, for any  $q \in X$ ,  $\mu(q): T_q X \rightarrow T_q X$  is

a complex antilinear map (not necessarily an isomorphism). If  $z$  is a holomorphic local coordinate on an open neighbourhood  $U$  of  $q$ , we have  $\mu(q) = \alpha(q) \frac{d\bar{z}_q}{dz_q}$  for some complex valued function  $\alpha$  on  $U$ . Here  $\alpha(q) = (dz_q) \circ \mu(q) \circ (d\bar{z}_q)^{-1}(1)$ . Thus we write  $\mu = \alpha \frac{d\bar{z}}{dz}$ , and we have  $|\mu| = |\alpha|$  as functions on  $U$ . Here, the bars on the left denote the norm on fibres of  $L_*(TX)$ , which is the unique operator norm as noted above. The  $(-1, 1)$ -form  $\mu$  is *measurable* if for every local coordinate  $z$  on  $X$ , the function  $\alpha$  is measurable. We view measurable forms up to almost everywhere equality. If  $\mu$  is a measurable  $(-1, 1)$ -form, then  $|\mu|$  is a measurable real valued function on  $X$ . We define the  $L^\infty$  norm of  $\mu$  to be that of  $|\mu|$ , that is,  $\|\mu\|_X = \operatorname{ess\,sup}_{q \in X} |\mu(q)|$ . We say that  $\mu$  is *bounded* if  $\|\mu\|_X < +\infty$ . The Banach space of bounded measurable  $(-1, 1)$ -forms is denoted  $\operatorname{bel}_{\text{qc}}(X)$ , and its unit ball  $\operatorname{Bel}_{\text{qc}}(X)$ . Elements of  $\operatorname{Bel}_{\text{qc}}(X)$  are called *quasiconformal Beltrami forms*. That is,  $\mu \in \operatorname{Bel}_{\text{qc}}(X)$  if and only if  $\|\mu\|_X < 1$ .

If  $f$  is a quasiconformal homeomorphism, then  $\mu_f$  is clearly a quasiconformal Beltrami form, and is called the *Beltrami coefficient* of  $f$ . This produces quasiconformal Beltrami forms from quasiconformal homeomorphisms. In fact, one can go the other way too, which is the content of the mapping theorem below. First, note that if  $g$  is a biholomorphism, then  $\mu_f = \mu_{g \circ f}$ , so the quasiconformal homeomorphism and its target are well defined only up to biholomorphism.

**Theorem 2.7** (Mapping Theorem for Domains). *If  $\mu$  is a measurable function on  $\mathbb{C}$  with  $\|\mu\|_{\mathbb{C}} < 1$ , then there is a unique quasiconformal homeomorphism  $f: \mathbb{C} \rightarrow \mathbb{C}$  fixing 0 and 1 whose complex dilatation is  $\mu$ .*

In particular, the uniqueness implies that if  $f, g$  are quasiconformal homeomorphisms of  $\mathbb{C}$  and have the same quasiconformal dilatation, then  $f \circ g^{-1}$  is a biholomorphism. The mapping theorem also holds if  $\mu$  is a measurable function given only on a domain  $U \subset \mathbb{C}$ , for then one can simply extend it to all of  $\mathbb{C}$  by setting it to be zero outside  $U$  and apply the above. If  $\mu$  is a measurable function on  $\mathbb{H}^2$  (as the upper half plane in  $\mathbb{C}$ ) with  $\|\mu\|_{\mathbb{H}^2} < 1$ , then one can

reflect  $\mu$  about  $S^1$  (as the real line  $\mathbb{R}$  in  $\mathbb{C}$ ) to extend it to all of  $\mathbb{C}$  and apply the above to get a unique *normalised* quasiconformal homeomorphism  $f^\mu$  of  $\mathbb{H}^2$  which (upon extending to a homeomorphism  $\partial f^\mu$  of  $S^1$ ) fixes 0, 1, and  $\infty$ . Given a quasiconformal Beltrami form  $\mu$  on an arbitrary Riemann surface  $X$ , one can apply the above on each holomorphic coordinate chart to obtain a new complex structure on the same topological surface  $X$ .

**Theorem 2.8** ([Hub06, Proposition and Definition 4.8], Quasiconformal Mapping Theorem).

*If  $\mu$  is a quasiconformal Beltrami form on  $X$ , then there is a unique complex structure  $X_\mu$ , called the integration of  $\mu$ , on the same topological surface  $X$  such that  $\text{id}_{X \rightarrow X_\mu}: X \rightarrow X_\mu$  is quasiconformal and its Beltrami coefficient is  $\mu$ . Moreover, if  $f: X \rightarrow Y$  is a quasiconformal homeomorphism with Beltrami coefficient  $\mu$ , then  $f: X_\mu \rightarrow Y$  is a biholomorphism.*

**Remark 2.9.** *The construction of  $X_\mu$  in the proof of the above theorem in [Hub06] does not use the full hypothesis that  $\mu$  is a bounded Beltrami form. It only uses the local boundedness of  $\mu$ , and therefore  $X_\mu$  exists even with a weaker hypothesis. See Theorem 3.18 below for the more general mapping theorem.*

Note that  $X_0 = X$ , where 0 is the zero Beltrami form.

## 2.8 Quasiconformal Teichmüller Space

References for this section are [Ber60b], [Ahl63], and [Hub06, Chapters 4, 6]. Given a Riemann surface  $X_0$ , its quasiconformal Teichmüller space is

**Definition 2.10** (Quasiconformal Teichmüller space).

$$\mathcal{T}_{\text{qc}}(X_0) := \left\{ (X, f) \left| \begin{array}{l} X \text{ is a Riemann surface} \\ f: X_0 \rightarrow X \text{ is a quasiconformal homeomorphism} \end{array} \right. \right\} / \sim \quad (2.2)$$

where  $(X_1, f_1) \sim (X_2, f_2)$  if there is a biholomorphism  $\varphi: X_1 \rightarrow X_2$  homotopic to  $f_2 \circ f_1^{-1}$  rel  $I(X_1)$ .

The quasiconformal Teichmüller space is a complete metric space with the Teichmüller metric. There is an obvious *integration function* from the unit ball of quasiconformal Beltrami forms to the quasiconformal Teichmüller space.

**Definition 2.11** (Quasiconformal integration function).  $\Phi_{\text{qc}}^{X_0}: \text{Bel}_{\text{qc}}(X_0) \rightarrow \mathcal{T}_{\text{qc}}(X_0)$  is defined by  $\Phi_{\text{qc}}^{X_0}(\mu) := [X_\mu, \text{id}_{X_0 \rightarrow X_\mu}]$ .

The integration function  $\Phi_{\text{qc}}^{X_0}$  is a quotient map. To describe its fibres, we first define the pullback operation on Beltrami forms. Suppose  $h: Z \rightarrow X$  is differentiable and nonsingular at  $q \in Z$ , and that  $\mu$  is a Beltrami form on  $X$  at  $h(q)$ . Denote  $M = (\text{id} + \mu) \circ Dh_q: T_q Z \rightarrow T_{h(q)} X$ , where  $\text{id}$  is the identity isomorphism of  $T_{h(q)} X$ .  $M$  is a real linear isomorphism, so decompose  $M$  into complex linear and complex antilinear maps  $M = \partial M + \bar{\partial} M$ . Then the *pullback Beltrami form*  $h^* \mu$  is  $(\partial M)^{-1} \circ (\bar{\partial} M)$ . Observe that if  $\mu$  is the Beltrami coefficient of some  $f$  at  $h(q)$ , then  $\bar{\partial} f_{h(q)} = \partial f_{h(q)} \circ \mu$ . Therefore  $Df_{h(q)} = \partial f_{h(q)} \circ (\text{id} + \mu)$  and so  $D(f \circ h)_q = \partial f_{h(q)} \circ M = \partial f_{h(q)} \circ \partial M + \partial f_{h(q)} \circ \bar{\partial} M$ . But this is precisely the decomposition of  $D(f \circ h)_q$ , since  $\partial f_{h(q)}$  is complex linear, and it cancels on computing the Beltrami form! Therefore  $\mu_{f \circ h}(q) = h^* \mu$ . Thus the Beltrami form of  $f \circ h$  depends only on  $Dh_q$  and  $\mu$ , and not on the entirety of  $Df$ . In other words,  $\mu_{f \circ h} = h^* \mu_f$ . Some more development shows that  $h^* \mu = (\partial h_q + \mu \circ \bar{\partial} h_q)^{-1} \circ (\bar{\partial} h_q + \mu \circ \partial h_q)$ . Thus  $h^*$  is actually a biholomorphism from the unit ball of  $L_*(T_{h(q)} X)$  to the unit ball of  $L_*(T_q Z)$ .

The fibres of  $\Phi_{\text{qc}}^{X_0}$  are orbits of the pullback operation by  $\text{QC}_0(X_0)$ . That is,  $\Phi_{\text{qc}}^{X_0}(\mu_1) = \Phi_{\text{qc}}^{X_0}(\mu_2)$  if and only if there is a  $\varphi \in \text{QC}_0(X_0)$  such that  $\varphi^*(\mu_2) = \mu_1$ . There is another description of the fibres. Fix a universal cover  $p_0: \mathbb{H}^2 \rightarrow X_0$ . Every quasiconformal Beltrami form  $\mu$  on  $X_0$  pulls back (or lifts) to a quasiconformal Beltrami form  $\tilde{\mu}$  on  $\mathbb{H}^2$ , and its normalised quasiconformal homeomorphism extends to a homeomorphism at infinity  $\partial f^{\tilde{\mu}}$  of  $S^1$ . Then  $\Phi_{\text{qc}}^{X_0}(\mu_1) = \Phi_{\text{qc}}^{X_0}(\mu_2)$  if and only if  $\partial f^{\tilde{\mu}_1} = \partial f^{\tilde{\mu}_2}$ .

Finally, we recall that if  $X_0$  and  $X'_0$  are quasiconformal Riemann surfaces, then their

spaces of quasiconformal Beltrami forms and Teichmüller spaces are homeomorphic. In particular, if  $h: X'_0 \rightarrow X_0$  is a quasiconformal homeomorphism, then we have a homeomorphism  $h^*: \mathcal{T}_{\text{qc}}(X_0) \rightarrow \mathcal{T}_{\text{qc}}(X'_0)$  given by  $h^*[X, f] = [X, f \circ h]$ . This commutes with pullback of Beltrami forms  $h^* \circ \Phi_{\text{qc}}^{X_0} = \Phi_{\text{qc}}^{X'_0} \circ h^*$ . Thus, there is one Teichmüller space for every quasiconformal type of Riemann surfaces. In particular, since any two closed homeomorphic Riemann surfaces are quasiconformal, there is only one Teichmüller space for each closed topological surface. Further, any two homeomorphic Riemann surfaces of (topologically) finite type and finite (hyperbolic) area are quasiconformal, and the Teichmüller space of this quasiconformal type is usually called the Teichmüller space of the surface of finite type.

**Definition 2.12** (Teichmüller space of surfaces of finite type). *Let  $S$  be a surface with finite negative Euler characteristic.*

$$\mathcal{T}(S) := \left\{ (X, f) \left| \begin{array}{l} X \text{ is a complete finite area hyperbolic surface} \\ f: S \rightarrow X \text{ is a homeomorphism} \end{array} \right. \right\} / \sim \quad (2.3)$$

where  $(X_1, f_1) \sim (X_2, f_2)$  if there is an isometry  $\varphi: X_1 \rightarrow X_2$  homotopic to  $f_2 \circ f_1^{-1}$ .

While points in the quasiconformal Teichmüller space record data on the ideal boundary, points in the Teichmüller space of surfaces of finite type do not, since the ideal boundary is empty. Therefore, the latter are also spaces of marked Fuchsian groups of a certain topological and geometric type. Even when there is ideal boundary, it is sometimes useful to forget the data on the ideal boundary in order to recover a space of marked Fuchsian groups of a certain topological and geometric type.

**Definition 2.13** (Reduced Teichmüller space of surfaces with boundary of finite type). *Let  $\Sigma_{g,p}^b$  be the surface with boundary of finite type and negative Euler characteristic, with genus  $g$ ,  $p$  punctures and  $b$  boundary components.*

$$T_{g,p}^b := \left\{ (X, f) \left| \begin{array}{l} X \text{ is a metrically complete finite area} \\ \text{hyperbolic surface with geodesic boundary} \\ f: \Sigma_{g,p}^b \rightarrow X \text{ is a homeomorphism} \end{array} \right. \right\} / \sim \quad (2.4)$$

where  $(X_1, f_1) \sim (X_2, f_2)$  if there is an isometry  $\varphi: X_1 \rightarrow X_2$  homotopic to  $f_2 \circ f_1^{-1}$ , preserving the geodesic boundary.

## CHAPTER 3

# TOOLS

In this chapter, we prove some key tools, recall others, and set up more notions which will be of use in the novel portions of the dissertation. In Section 3.1, we include a discussion of Nielsen-convexity, which appears in the definition of  $\mathcal{T}(S)$ . In Section 3.2, we carefully prove Proposition 3.2 (Homeomorphism at infinity), and in Section 3.3, we recall the Douady–Earle extension, which will be crucially used in Section 4.2 while defining the topology of the marked moduli space as well as in Section 6.2 in the proof of its contractibility. These as well as the existence of straight line homotopy, proved in detail in Section 3.4, are ingredients in Chapter 5 in the proof of continuity of the mapping class group action. In Section 3.5, we develop the same theory as in Sections 2.6 to 2.7, but for homeomorphisms which are only locally quasiconformal rather than (globally) quasiconformal. This theory is used to describe the topology of the marked moduli space in another way in Section 4.5, which is analogous to Section 2.8. It is further used to prove the contractibility of marked moduli space in Chapter 6.

### 3.1 Nielsen-convexity for Complete Hyperbolic Surfaces

We have adopted the term *Nielsen-convex* from work of Alessandrini, Liu, and others ([ALP<sup>+</sup>11, Definition 4.3]). They actually define a notion of Nielsen-convexity for hyperbolic surfaces which may or may not be complete. However, we are interested only in complete hyperbolic surfaces. So instead of recalling their original definition, we simply characterise Nielsen-convexity for complete hyperbolic surfaces in multiple different but equivalent ways in Proposition 3.1 (Nielsen-convexity) below. Note that in [ALP<sup>+</sup>11], the hyperbolic pair of

pants with three cusps is treated separately due to technical reasons concerning their definition of Nielsen-convexity. However, since it satisfies the conditions below, we will call it Nielsen-convex. Notation from Section 2.3 is used, and the objects in some of the conditions below are defined post the choice of a pointed universal cover.

**Proposition 3.1** (Nielsen-convexity). *Let  $X$  be a complete hyperbolic surface. Then the following are equivalent.*

1.  $X$  is Nielsen-convex according to [ALP<sup>+</sup>11, Definition 4.3].
2. The ideal boundary  $I(X)$  of  $X$  is empty.
3. The convex core  $C(X)$  of  $X$  equals  $X$ .
4. The limit set  $\Lambda_X$  of the deck group  $\Gamma_X$ , or equivalently, of the action of  $\pi_1(X)$  on the universal cover  $\mathbb{H}^2$ , is the entire circle at infinity  $S^1$ .
5. The set  $(\Gamma_X)_\infty$  of sinks of hyperbolic elements in  $\Gamma_X$  is dense in  $S^1$ .
6.  $X$  is isometric to  $\Gamma_X \backslash \mathbb{H}^2$  for some torsion-free Fuchsian group  $\Gamma_X$  of the first kind.
7.  $X$  can be constructed by gluing hyperbolic pairs of pants (possibly with cusps) along their boundary components.
8.  $X$  has no visible ends.

*Proof.* The equivalence of the first and the third conditions is [ALP<sup>+</sup>11, Proposition 4.6] (applied to complete hyperbolic surfaces), and that of the first and the seventh conditions is part of [ALP<sup>+</sup>11, Theorem 4.5]. The equivalence of the third and the fourth conditions follows easily from the definitions in Section 2.3 above. Indeed, since  $C(X) = \Gamma_X \backslash \text{CH}(\Lambda_X)$  and  $X = \Gamma_X \backslash \mathbb{H}^2$ ,  $C(X) = X$  if and only if  $\text{CH}(\Lambda_X) = \mathbb{H}^2$ , which is equivalent to  $\Lambda_X = S^1$ . The sixth condition is really just a restatement of the fourth, and the equivalence of the second and the fourth conditions also follows straight from definitions. To see the equivalence of the fourth and the fifth conditions, first observe that neither the hyperbolic plane nor its

quotient by a single parabolic satisfy either of the two conditions. Thus  $\Gamma_X$  contains at least one hyperbolic. The set  $(\Gamma_X)_\infty$  is nonempty, and hence dense in  $\Lambda_X$ . Thus  $\Lambda_X = S^1$  if and only if  $(\Gamma_X)_\infty$  is dense in  $S^1$ , establishing the equivalence. For a discussion of visible ends, see [Ba19, Section 2]. The equivalence of the third and the eighth conditions above follows easily from Lemma 2.2 there.  $\square$

We also say that a Riemann surface  $X$  is Nielsen-convex if, when uniformised as a complete hyperbolic surface, it is Nielsen-convex according to Proposition 3.1 above. Clearly the notion of Nielsen-convexity is invariant under isometry of hyperbolic surfaces and biholomorphism of Riemann surfaces.

## 3.2 The Homeomorphism at Infinity

In this section, we provide a detailed proof of the Proposition 3.2 (Homeomorphism at infinity), which is actually a special case of [Thu86, Proposition 5.3]. In that paper, Thurston used it to reduce the earthquake theorem for general hyperbolic surfaces to a version of the earthquake theorem for their universal covers (that is, the hyperbolic plane). We will use homeomorphisms at infinity repeatedly, especially in the definition of the topology of the marked moduli space.

**Proposition 3.2** (Homeomorphism at infinity). *Let  $X, Y$  be complete hyperbolic surfaces. Fix universal covers  $p_X: \mathbb{H}^2 \rightarrow X$ ,  $p_Y: \mathbb{H}^2 \rightarrow Y$ , and let  $\Gamma_X, \Gamma_Y$  be the respective deck groups. Let  $f: X \rightarrow Y$  be a homeomorphism with a lift  $\tilde{f}: \mathbb{H}^2 \rightarrow \mathbb{H}^2$  to the universal covers, which induces an isomorphism  $f_*: \Gamma_X \rightarrow \Gamma_Y$  of deck groups. Assume that both  $X$  and  $Y$  are Nielsen-convex. Then*

1. *The lift  $\tilde{f}$  extends to a homeomorphism at infinity  $\partial\tilde{f}: S^1 \rightarrow S^1$ .*

2. The function  $\hat{f}$  which equals  $\tilde{f}$  on  $\mathbb{H}^2$  and  $\partial\tilde{f}$  on  $S^1$  is a homeomorphism of  $\mathbb{H}^2 \cup S^1$ .
3. The homeomorphism  $\partial\tilde{f}$  is equivariant under the action of the deck groups. That is, for every  $\gamma \in \Gamma_X$ , we have  $(\partial\tilde{f}) \circ \gamma = f_*(\gamma) \circ (\partial\tilde{f})$ . Thus in  $\text{Homeo}^+(S^1)$ , the left coset  $(\partial\tilde{f}) \circ \Gamma_X$  equals the right coset  $\Gamma_Y \circ (\partial\tilde{f})$ .
4. If  $f_t$  is an isotopy which lifts to an isotopy  $\tilde{f}_t$ , then  $\partial\tilde{f}_0 = \partial\tilde{f}_1$ .
5. Suppose  $Z$  is another complete, Nielsen-convex hyperbolic surface with a universal cover  $p_Z: \mathbb{H}^2 \rightarrow Z$ , and  $g: Y \rightarrow Z$  is a homeomorphism with lift  $\tilde{g}$ . Then  $\partial(\tilde{g} \circ \tilde{f}) = (\partial\tilde{g}) \circ (\partial\tilde{f})$ .
6. If  $\sigma \in \text{PSL}(2, \mathbb{R})$ , then  $\sigma \circ \tilde{f}$  also extends to a homeomorphism at infinity, and  $\partial(\sigma \circ \tilde{f}) = \sigma \circ (\partial\tilde{f})$ .

**Remark 3.3.** Note that if  $X, Y$  are closed surfaces, then  $\tilde{f}$  is a quasiconformal homeomorphism and so extends to a (quasisymmetric) homeomorphism of  $S^1$ . However, if  $X, Y$  are surfaces of infinite type,  $\tilde{f}$  may not be a quasiconformal homeomorphism. The proposition asserts that nevertheless, it extends to a (not necessarily quasisymmetric) homeomorphism of  $S^1$ .

**Remark 3.4.** Note that the homeomorphism  $\partial\tilde{f}$  depends on the choice of the lift  $\tilde{f}$ . The other choices of the lift of  $f$  are  $\gamma \circ \tilde{f}$  for  $\gamma \in \Gamma_Y$  or  $\tilde{f} \circ \gamma$  for  $\gamma \in \Gamma_X$ , leading to the homeomorphism at infinity  $\gamma \circ (\partial\tilde{f})$  or  $(\partial\tilde{f}) \circ \gamma$  by part (6) of the proposition.

The outline of the proof is as follows. First, in Lemma 3.5, we prove that  $f_*$  preserves the type (hyperbolic or parabolic) of elements of the deck group. Next, in Definition 3.6, we define  $\partial\tilde{f}$  on the set  $(\Gamma_X)_\infty$  of sinks of hyperbolic elements of  $\Gamma_X$ , which is dense in  $S^1$ , and give an alternate characterisation in Lemma 3.8. Then, in Lemma 3.9, we show that  $\partial\tilde{f}$  is monotonic, that is, it preserves that circular order of points in  $S^1$ . In Lemma 3.10 we show that  $\partial\tilde{f}$  has no jump discontinuities, allowing us to extend it to a unique homeomorphism of  $S^1$ . Finally, in Lemma 3.11, we show that the extension yields a homeomorphism of  $\mathbb{H}^2 \cup S^1$ .

**Lemma 3.5.** *Keep the hypotheses of Proposition 3.2. Then  $f_*$  is a type preserving isomorphism. That is, for every  $\gamma \in \Gamma_X$ ,  $f_*(\gamma) \in \Gamma_Y$  is hyperbolic (resp. parabolic) if  $\gamma$  is hyperbolic (resp. parabolic).*

*Proof.* It is enough to prove that  $f_*$  preserves the type of primitive elements, since powers of hyperbolic and parabolic elements of  $\mathrm{PSL}(2, \mathbb{R})$  are hyperbolic and parabolic respectively, and every element of  $\Gamma_X$  and  $\Gamma_Y$  has a root which is a primitive element. Suppose, for the sake of contradiction, that there is a primitive element  $\gamma \in \Gamma_X$  such that  $\gamma$  and  $f_*(\gamma)$  are not of the same type. Note that group isomorphisms map primitive elements to primitive elements and so  $f_*(\gamma)$  is also a primitive element in  $\Gamma_Y$ . As the torsion-free Fuchsian groups  $\Gamma_X$  and  $\Gamma_Y$  cannot have elliptics, one of  $\gamma$  and  $f_*(\gamma)$  is hyperbolic and the other is parabolic. Replacing  $f$  with  $f^{-1}$  if necessary, assume that  $\gamma$  is hyperbolic and  $f_*(\gamma)$  is parabolic. The idea of the proof here is that the parabolic  $f_*(\gamma)$  is the holonomy around a simple closed curve which is peripheral, that is, one of the components of its complement is topologically an annulus. Since  $f$  is a homeomorphism, the same then holds for the hyperbolic  $\gamma$ , which contradicts the Nielsen-convexity of  $X$ . We provide details below, carefully constructing these annuli.

There is a horodisk in  $\mathbb{H}^2$  about the fixed point of  $f_*(\gamma)$  which projects, under  $p_Y$ , to a cusp neighbourhood  $U$  of an end  $e$  of  $Y$ . See Figure 3.1. Let  $L \subset \mathbb{H}^2$  be the axis of the hyperbolic element  $\gamma$  and  $l \subset X$  be its projection under  $p_X$ . Then  $l$  is a closed geodesic of  $X$  and hence compact. Thus its image  $f(l)$  is also compact. Hence there exists a horocyclic oriented closed curve  $\alpha \subset U \subset Y$ , in a sufficiently small neighbourhood of the end  $e$ , that is disjoint from  $f(l)$ , and such that the holonomy around  $\alpha$  is  $f_*(\gamma)$ . Since  $f_*(\gamma)$  is a primitive element,  $\alpha$  is a simple closed curve. Further,  $\alpha$  cuts  $Y$  into two components, one of which is (topologically) an annulus  $A_1 \subset U$ , and the other contains  $f(l)$ . Since  $f$  is a homeomorphism,  $f^{-1}(\alpha)$  cuts  $X$  into two components, one of which is the annulus  $A_2 = f^{-1}(A_1)$ , and the other contains  $l$ .

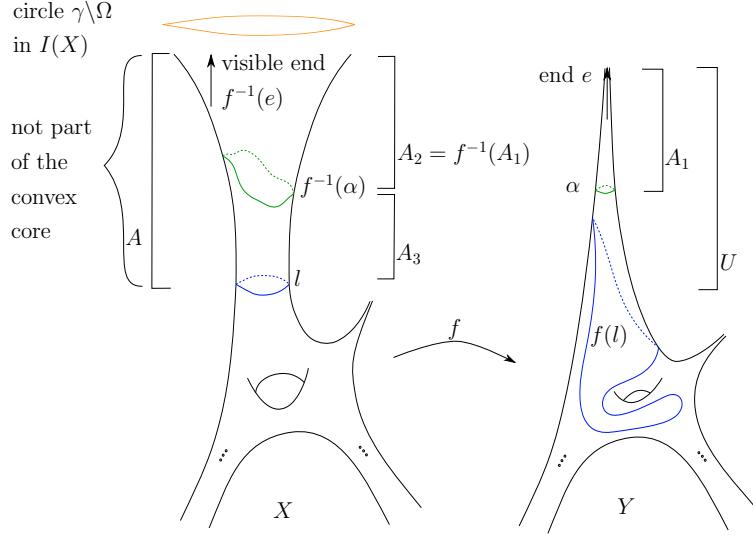


Figure 3.1:  $f_*$  is a type preserving isomorphism.

Now  $l$  and  $f^{-1}(\alpha)$  are closed curves in the same free homotopy class, since the holonomy around each is  $\gamma$ . The geodesic representative in this class is  $l$ , which is a simple curve because  $f^{-1}(\alpha)$  is also a simple curve. Further,  $l$  and  $f^{-1}(\alpha)$  are disjoint because  $f(l)$  and  $\alpha$  are disjoint. Thus  $l$  and  $f^{-1}(\alpha)$  are disjoint and isotopic, and hence bound an annulus  $A_3 \subset X \setminus A_2$ . Thus  $l$  cuts  $X$  into two components, one of which is  $A = A_2 \cup f^{-1}(\alpha) \cup A_3$ , an annulus. The closure of  $A$  is a complete hyperbolic surface with geodesic boundary  $l$ . Topologically it is an annulus with one boundary component, and thus is a hyperbolic funnel. That is, it is the quotient of the half plane in  $\mathbb{H}^2$  bounded by  $L$  by the isometry  $\gamma$  translating along  $L$ . But then the cyclic subgroup generated by  $\gamma$  acts freely and properly discontinuously on the arc  $\Omega$  of the circle at infinity cut out by  $L$ , and elements of  $\Gamma_X$  outside this subgroup move  $\Omega$  off of itself. Therefore  $\Omega$  is contained in the complement of the limit set  $\Lambda_X$ , and its quotient by  $\gamma$  is a circle. Hence the ideal boundary  $I(X)$  is nonempty, and contains a circle component  $\gamma \setminus \Omega$ . Another way to view this situation is that  $L$  is part of the boundary of the convex hull  $\text{CH}(\Lambda_X)$ , and therefore  $l$  is part of the boundary of the convex core of  $X$ . Hence  $A$  is not part of the convex core. Due to Proposition 3.1, this contradicts the hypothesis that  $X$  is a complete, Nielsen-convex hyperbolic surface. We conclude that  $f_*$ , the induced map between deck groups, is type preserving.  $\square$

Let  $(\Gamma_X)_\infty$  and  $(\Gamma_Y)_\infty$  be the sets of sinks of all the hyperbolic elements of  $\Gamma_X$  and  $\Gamma_Y$  respectively. Now we define  $\partial\tilde{f}$  on  $(\Gamma_X)_\infty$ .

**Definition 3.6.** *Suppose  $q \in (\Gamma_X)_\infty$ , and  $\gamma \in \Gamma_X$  is a hyperbolic element such that  $q = \gamma_\infty$ . By Lemma 3.5,  $f_*$  is type preserving, so  $f_*(\gamma)$  is also a hyperbolic element. We define  $\partial\tilde{f}(q)$  to be the sink  $(f_*(\gamma))_\infty$ .*

**Lemma 3.7.** *For each  $q \in (\Gamma_X)_\infty$ ,  $\partial\tilde{f}(q)$  is a well defined element of  $(\Gamma_Y)_\infty$ . Thus the definition above yields a well defined function  $\partial\tilde{f}: (\Gamma_X)_\infty \rightarrow (\Gamma_Y)_\infty$ , and it is a bijection.*

*Proof.* We need to show that  $\partial\tilde{f}(q)$  is independent of the choice of the hyperbolic element  $\gamma \in \Gamma_X$  in the definition of  $\partial\tilde{f}$ . Let  $\gamma'$  be another hyperbolic element in the Fuchsian group  $\Gamma_X$  such that  $\gamma'_\infty = q$ . Then  $\gamma, \gamma'$  are in fact positive powers of some hyperbolic element  $\gamma'' \in \Gamma_X$ . Thus  $f_*(\gamma)$  and  $f_*(\gamma')$  are positive powers of  $f_*(\gamma'')$  and hence all three have the same sink. Thus  $\partial\tilde{f}(q)$  is unambiguously defined and is a sink of a hyperbolic element of  $\Gamma_Y$ , so lies in  $(\Gamma_Y)_\infty$ . For the second assertion, note that  $\partial(\tilde{f}^{-1})$  is an inverse, and hence the function  $\partial\tilde{f}$  so defined is a bijection from  $(\Gamma_X)_\infty$  to  $(\Gamma_Y)_\infty$ .  $\square$

There is another way of viewing the function  $\partial\tilde{f}$  (so far defined on the subset  $(\Gamma_X)_\infty$ ) as follows. If  $L$  is an oriented geodesic line whose forward endpoint is  $q \in (\Gamma_X)_\infty$ , then  $\partial\tilde{f}(q)$  is the forward endpoint of  $\tilde{f}(L)$ .

**Lemma 3.8.** *Keep the hypotheses of Proposition 3.2. Let  $\gamma_\infty$  be the sink of a hyperbolic element  $\gamma \in \Gamma_X$ , and suppose  $L$  is an oriented geodesic line in  $\mathbb{H}^2$  whose forward endpoint on the circle at infinity is  $\gamma_\infty$ . Then the oriented curve  $\tilde{f}(L)$  has forward endpoint  $(f_*(\gamma))_\infty$  on the circle at infinity.*

*Proof.* Let  $L'$  be the axis of  $\gamma$ , oriented from its source  $(\gamma^{-1})_\infty$  to its sink  $\gamma_\infty$ . First we prove the lemma for the oriented geodesic line  $L'$ . To do so, note that  $\gamma$  acts by translation along  $L'$ . Hence  $f_*(\gamma)$  acts by translation along  $\tilde{f}(L')$ . Now the images of any point in  $\mathbb{H}^2$

under larger and larger positive powers of  $f_*(\gamma)$  limit to its attracting fixed point  $(f_*(\gamma))_\infty$ . Therefore the forward endpoint on the circle at infinity of  $\tilde{f}(L')$  is the sink  $(f_*(\gamma))_\infty$ .

Now we tackle the case of an arbitrary oriented geodesic line  $L$  with forward endpoint  $\gamma_\infty$ . The idea of the proof here is that  $L$  is asymptotic to  $L'$ , and thus the projection  $p_X(L)$  is eventually a bounded distance from the closed curve  $p_X(L')$ . Due to uniform continuity of  $f$  on compact sets, actually the lift  $\tilde{f}(L)$  of  $f(p_X(L))$  is also eventually at a bounded distance from the lift  $\tilde{f}(L')$  of  $f(p_X(L'))$ . Thus  $\tilde{f}(L)$  has the same forward endpoint as  $\tilde{f}(L')$ , which we have shown above to be  $(f_*(\gamma))_\infty$ . We provide the details below, carefully lifting  $f(p_X(L))$  and  $f(p_X(L'))$  to  $\tilde{f}(L)$  and  $\tilde{f}(L')$  respectively.

Let  $\tilde{K}$  denote the closed 1-neighbourhood of the axis  $L'$  in  $\mathbb{H}^2$ , that is, the set of points of distance at most 1 from  $L'$ . Let  $K = p_X(\tilde{K})$ , so that  $K$  is the closed 1-neighbourhood of the closed geodesic  $p_X(L')$  in  $X$ . The set  $K$  is compact, and so is its image  $f(K) \subset Y$ . Since injectivity radius is a continuous function on any Riemannian manifold, the injectivity radius of points in  $f(K)$  is bounded below by a number  $\varepsilon > 0$ . Also since  $K$  is compact,  $f|_K$  is uniformly continuous. Let  $\delta > 0$  be such that for all  $x, y \in K$ ,  $d_X(x, y) < \delta$  implies  $d_Y(f(x), f(y)) < \varepsilon$ . Reducing  $\delta$  if necessary, assume that  $\delta \leq 1$ . Since  $L$  and  $L'$  are oriented geodesic lines with the same forward endpoint, they are asymptotic. Let  $L_1$  be a subray of  $L$  that is within the  $\delta$ -neighbourhood of  $L'$ .

Take an arbitrary point  $\tilde{y} \in L_1$ . There exists a point  $\tilde{x} \in L'$  such that  $d_{\mathbb{H}^2}(\tilde{x}, \tilde{y}) < \delta$ . Since  $\delta \leq 1$ , we have  $d_{\mathbb{H}^2}(\tilde{x}, \tilde{y}) < 1$  and so  $\tilde{x}, \tilde{y} \in \tilde{K}$ . See Figure 3.2. Let  $\alpha$  be the geodesic segment from  $\tilde{x}$  to  $\tilde{y}$ , and let  $x = p_X(\tilde{x})$ ,  $y = p_X(\tilde{y})$ . Since  $p_X$  is a Riemannian local isometry,  $d_X(x, y) \leq d_{\mathbb{H}^2}(\tilde{x}, \tilde{y}) < \delta$ . Hence by uniform continuity as above,  $d_Y(f(x), f(y)) < \varepsilon$ . In fact, the diameter of  $p_X(\alpha)$  is less than  $\delta$  so  $f(p_X(\alpha))$  is a curve that lies entirely in the  $\varepsilon$ -neighbourhood of the point  $f(x)$  in  $Y$ . Since the injectivity radius at  $f(x)$  is at least  $\varepsilon$ , this  $\varepsilon$ -neighbourhood of  $f(x)$  in  $Y$  is evenly covered by the universal cover  $p_Y$ , and one component of its  $p_Y$ -preimage is the  $\varepsilon$ -neighbourhood of  $\tilde{f}(\tilde{x})$  in  $\mathbb{H}^2$ . Hence  $f(p_X(\alpha))$  lifts to

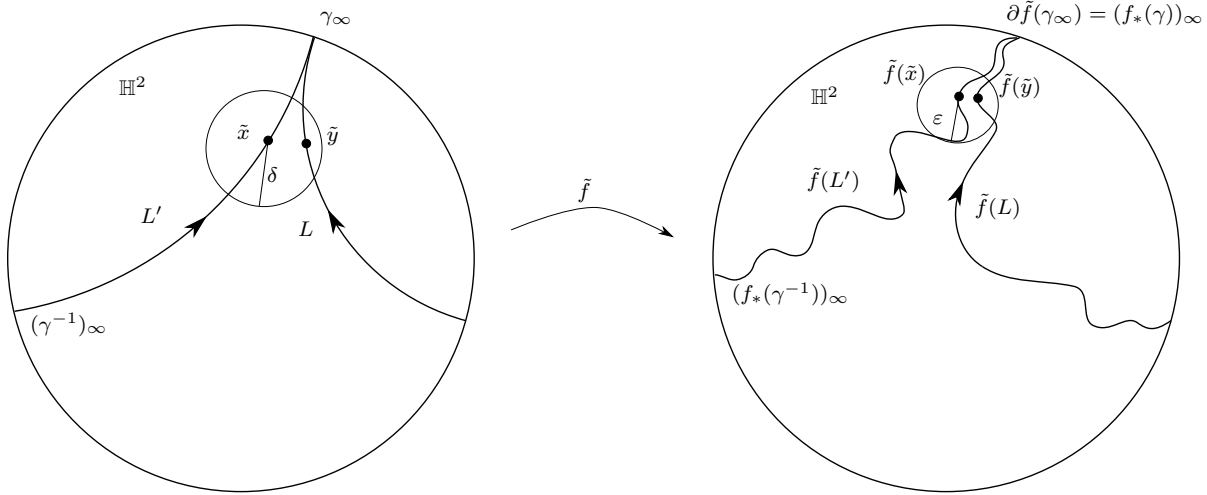


Figure 3.2: The forward endpoint of  $\tilde{f}(L)$  is  $(f_*(\gamma))_\infty$ .

a curve contained in this  $\varepsilon$ -neighbourhood of  $\tilde{f}(\tilde{x})$ . But the endpoint of this lifted curve is  $\tilde{f}(\tilde{y})$ , so  $d_{\mathbb{H}^2}(\tilde{f}(\tilde{x}), \tilde{f}(\tilde{y})) < \varepsilon$ . But  $\tilde{f}(\tilde{y})$  is an arbitrary point on  $\tilde{f}(L_1)$  and  $\tilde{f}(\tilde{x})$  lies on  $\tilde{f}(L')$ , so we conclude that the curve  $\tilde{f}(L_1)$  is at a bounded distance from  $\tilde{f}(L')$ . Therefore  $\tilde{f}(L_1)$  limits to a point on  $S^1$  and moreover, the forward endpoints of  $\tilde{f}(L_1)$  and  $\tilde{f}(L')$  are the same, namely  $(f_*(\gamma))_\infty$ . Thus  $\tilde{f}(L)$  has forward endpoint  $(f_*(\gamma))_\infty$  on the circle at infinity.  $\square$

**Lemma 3.9.** *Keep the hypotheses of Proposition 3.2, and consider the function  $\partial\tilde{f}$  of Definition 3.6, defined on the subset  $(\Gamma_X)_\infty$ . Then  $\partial\tilde{f}$  is monotonic, that is, it preserves the circular order on  $S^1$ . In particular, if  $a, b, c \in (\Gamma_X)_\infty$  and  $(a, b, c)$  is a positively oriented triple, then so is the triple  $(\partial\tilde{f}(a), \partial\tilde{f}(b), \partial\tilde{f}(c))$ .*

*Proof.* Suppose  $a, b, c \in (\Gamma_X)_\infty$  such that  $(a, b, c)$  is a positively oriented triple. Let  $L_1, L_2, L_3$  be oriented geodesic lines joining  $a$  to  $b$ ,  $b$  to  $c$  and  $c$  to  $a$  respectively, and let  $T$  be the ideal geodesic triangle with vertices  $a, b, c$ . Then by Lemma 3.8,  $\tilde{f}$  maps  $T$  to an ideal triangle  $\tilde{f}(T)$  (whose sides  $\tilde{f}(L_1), \tilde{f}(L_2), \tilde{f}(L_3)$  are not necessarily geodesics) with vertices at  $\partial\tilde{f}(a), \partial\tilde{f}(b), \partial\tilde{f}(c)$ . Now  $L_1, L_2, L_3$  form the boundary of the triangle  $T$ , with its induced border orientation. Since  $f$  is orientation preserving and hence so is  $\tilde{f}$ , we conclude that the orientation of  $\tilde{f}(L_1) \cup \tilde{f}(L_2) \cup \tilde{f}(L_3)$  matches the orientation induced as the boundary of the triangle  $\tilde{f}(T)$ . Therefore the vertices  $(\partial\tilde{f}(a), \partial\tilde{f}(b), \partial\tilde{f}(c))$  of the image triangle  $\tilde{f}(T)$  form

a positively oriented triple. This concludes the proof of monotonicity of  $\partial\tilde{f}$ .  $\square$

**Lemma 3.10.** *Keep the hypotheses of Proposition 3.2, and consider the function  $\partial\tilde{f}$  of Definition 3.6, defined on the subset  $(\Gamma_X)_\infty$ . Then  $\partial\tilde{f}$  can be extended uniquely to a homeomorphism of  $S^1$ , hence part (1) of the proposition.*

*Proof.* Note that  $(\Gamma_X)_\infty$  and  $(\Gamma_Y)_\infty$  are both dense in  $S^1$  by Proposition 3.1, since  $X$  and  $Y$  are complete, Nielsen-convex hyperbolic surfaces. Since  $\partial\tilde{f}$  is monotonic, it can be extended uniquely and continuously to the closure  $\overline{(\Gamma_X)_\infty}$ , unless there are jump discontinuities. However the image  $\partial\tilde{f}((\Gamma_X)_\infty) = (\Gamma_Y)_\infty$  is dense in  $S^1$ , and therefore there cannot be any jump discontinuities. So  $\partial\tilde{f}$  can be extended uniquely and continuously to the closure  $\overline{(\Gamma_X)_\infty}$ , which is  $S^1$ . Thus the extension is defined over all of  $S^1$ , and  $\partial\tilde{f}: S^1 \rightarrow S^1$  is continuous. The function  $\partial\tilde{f}$  is monotonic because  $\partial\tilde{f}|_{(\Gamma_X)_\infty}$  is monotonic, and hence injective. The image  $\partial\tilde{f}(S^1)$  is compact and hence closed in  $S^1$ , but contains the dense set  $(\Gamma_Y)_\infty$ , and so is all of  $S^1$ , making  $\partial\tilde{f}$  surjective. Thus  $\partial\tilde{f}$  is a continuous bijection between one dimensional manifolds, so is a homeomorphism of  $S^1$ , proving part (1) of the proposition.  $\square$

**Lemma 3.11.** *Keep the hypotheses of Proposition 3.2, and consider the function  $\partial\tilde{f}$  of Lemma 3.10. Let  $\hat{f}$  be the function which equals  $\tilde{f}$  on  $\mathbb{H}^2$  and  $\partial\tilde{f}$  on  $S^1$ . Then  $\hat{f}$  is a homeomorphism of  $\mathbb{H}^2 \cup S^1$ , that is, part (2) of the proposition.*

*Proof.* Note that  $\hat{f}$  is already continuous on the open set  $\mathbb{H}^2$  of  $\mathbb{H}^2 \cup S^1$  and has a continuous inverse on its image  $\mathbb{H}^2$ , being equal to the homeomorphism  $\tilde{f}$  there. It remains to show that  $\hat{f}$  is continuous at points of  $S^1$ , so that its inverse, being equal to  $\widehat{f^{-1}}$  will also be continuous at points of  $S^1$ , concluding the proof. Toward that end, let  $q$  be an arbitrary point on  $S^1$ , at which we prove the continuity of  $\hat{f}$ . For three distinct points  $a, b, c \in S^1$ , we denote by  $H(a, b; c)$  the union of the open hyperbolic half plane on the  $c$  side of the geodesic joining  $a$  and  $b$ , and the open arc containing  $c$  of the circle at infinity cut out

by  $a$  and  $b$ . A basis of neighbourhoods at  $\hat{f}(q)$  in  $\mathbb{H}^2 \cup S^1$  is the set  $\{H(a, b; \hat{f}(q)) \mid a, b \in (\Gamma_Y)_\infty, \text{ and } a, b, \hat{f}(q) \text{ are distinct}\}$ , since  $(\Gamma_Y)_\infty$  is dense in  $S^1$  by Proposition 3.1.

Let  $a, b$  be arbitrary distinct points in  $(\Gamma_Y)_\infty$ . We need to show that  $\hat{f}^{-1}H(a, b; \hat{f}(q))$  is an open set in  $\mathbb{H}^2 \cup S^1$  containing  $q$ . By Lemma 3.8 applied to  $\tilde{f}^{-1}$ , the  $\hat{f}$ -preimage of the geodesic line joining  $a$  and  $b$  (including the endpoints  $a$  and  $b$ ) is a curve embedded in  $\mathbb{H}^2 \cup S^1$  joining  $\partial\tilde{f}^{-1}(a)$  and  $\partial\tilde{f}^{-1}(b)$  whose interior lies entirely in  $\mathbb{H}^2$ . This curve separates  $\mathbb{H}^2 \cup S^1$  into two open sets, and we denote by  $U$  the one which contains  $q$ . Since  $\tilde{f}$  is a homeomorphism, it is clear that  $\tilde{f}^{-1}(H(a, b; \hat{f}(q)) \cap \mathbb{H}^2) = U \cap \mathbb{H}^2$ . Similarly since  $\partial\tilde{f}$  is a circular order preserving homeomorphism, we have  $\partial\tilde{f}^{-1}(H(a, b; \hat{f}(q)) \cap S^1) = U \cap S^1$ . Then it is clear that  $\hat{f}^{-1}H(a, b; \hat{f}(q)) = U$ , which is an open set containing  $q$ , concluding the proof of part (2) of the proposition.  $\square$

*Proof of Proposition 3.2 parts (3)–(6).* The function  $\partial\tilde{f}$  is equivariant because it is a continuous extension of the equivariant map  $\tilde{f}$  to the closure of its domain. Since  $f_*(\gamma)$  ranges over all of  $\Gamma_Y$  as  $\gamma$  ranges over  $\Gamma_X$ , we collect the two sides of the equality  $(\partial\tilde{f}) \circ \gamma = f_*(\gamma) \circ (\partial\tilde{f})$  into an equality of sets  $(\partial\tilde{f}) \circ \Gamma_X = \Gamma_Y \circ (\partial\tilde{f})$  in  $\text{PSL}(2, \mathbb{R})$ , proving part (3). Next, we prove part (4). If  $f_t$  is an isotopy which lifts to an isotopy  $\tilde{f}_t$ , then we have  $f_{0*} = f_{1*}$ , and hence for every hyperbolic  $\gamma \in \Gamma_X$ ,  $\partial\tilde{f}_0(\gamma_\infty) = f_{0*}(\gamma)_\infty = f_{1*}(\gamma)_\infty = \partial\tilde{f}_1(\gamma_\infty)$ , using Definition 3.6. Therefore the continuous functions  $\partial\tilde{f}_0$  and  $\partial\tilde{f}_1$  agree on the dense set  $(\Gamma_X)_\infty$ , and hence are equal. Next, we prove part (5). We compute, using Definition 3.6, that for every hyperbolic  $\gamma \in \Gamma_X$ ,  $\partial(\tilde{g} \circ \tilde{f})(\gamma_\infty) = ((g \circ f)_*(\gamma))_\infty = (g_*(f_*(\gamma)))_\infty = \partial\tilde{g}((f_*(\gamma))_\infty) = (\partial\tilde{g}) \circ (\partial\tilde{f})(\gamma_\infty)$ . Again the continuous functions  $\partial(\tilde{g} \circ \tilde{f})$  and  $(\partial\tilde{g}) \circ (\partial\tilde{f})$  agree on the dense set  $(\Gamma_X)_\infty$ , and hence are equal. For the final part (6), note that  $\sigma \in \text{PSL}(2, \mathbb{R})$ , as an isometry of  $\mathbb{H}^2$ , also extends to a homeomorphism of  $S^1$ , which we are denoting by the same symbol  $\sigma$  as mentioned in Section 2.2. Therefore  $\sigma \circ \tilde{f}$  extends to a homeomorphism at infinity, and  $\partial(\sigma \circ \tilde{f}) = \sigma \circ (\partial\tilde{f})$ .  $\square$

### 3.3 The Douady–Earle Extension

Here we recall the *Douady–Earle extension* from [DE86]. The Douady–Earle extension is a construction that extends homeomorphisms of  $S^1$  to homeomorphisms of  $\mathbb{H}^2 \cup S^1$  in a conformally natural way. This means that the extension of an element of  $\mathrm{PSL}(2, \mathbb{R})$  as a homeomorphism of  $S^1$  is the same element, now viewed as a homeomorphism of  $\mathbb{H}^2$ ; and that the extension respects composition on the left or right side by elements of  $\mathrm{PSL}(2, \mathbb{R})$ . However, note that this construction does not yield a group homomorphism. More precisely,

**Theorem 3.12** (see [DE86, Sections 3–4]). *One can associate to every homeomorphism  $f$  of the circle  $S^1$ , a homeomorphism  $\mathrm{DE}(f)$  of the disc  $\mathbb{H}^2 \cup S^1$  which is real analytic on  $\mathbb{H}^2$ , such that all the following hold:*

1. *For every  $f \in \mathrm{Homeo}^+(S^1)$ , we have  $\mathrm{DE}(f)|_{S^1} = f$ . Thus  $\mathrm{DE}(f)$  extends  $f$ .*
2.  $\mathrm{DE}(\mathrm{id}_{S^1}) = \mathrm{id}_{\mathbb{H}^2 \cup S^1}$ .
3. (Conformal naturality) *For every  $f \in \mathrm{Homeo}^+(S^1)$  and every  $\sigma_1, \sigma_2 \in \mathrm{PSL}(2, \mathbb{R})$ , we have  $\mathrm{DE}(\sigma_1 \circ f \circ \sigma_2) = \sigma_1 \circ \mathrm{DE}(f) \circ \sigma_2$ .*
4. *The function  $\mathrm{DE}: \mathrm{Homeo}^+(S^1) \rightarrow \mathrm{Homeo}^+(\mathbb{H}^2 \cup S^1)$  is continuous, where  $\mathrm{Homeo}^+(\mathbb{H}^2 \cup S^1)$  has the compact-open topology, or equivalently, the topology of uniform convergence on compact sets).*
5. *The function  $\mathrm{DE}: \mathrm{Homeo}^+(S^1) \rightarrow \mathrm{Diff}(\mathbb{H}^2)$  is continuous, where  $\mathrm{Diff}(\mathbb{H}^2)$  has the  $C^\infty$  topology.*

### 3.4 Straight line homotopy

We will have occasion to use a straight line homotopy between two maps from a topological space into a hyperbolic surface, and we explain this term in this section in Definition 3.13

(Straight line homotopy). We provide a detailed proof of the fact that any two maps to a hyperbolic surface are connected by a unique straight line homotopy if they are sufficiently close in Proposition 3.14.

**Definition 3.13** (Straight line homotopy). *Let  $Z$  be a topological space,  $X$  a complete hyperbolic surface and  $h_0, h_1: Z \rightarrow X$  be continuous functions. We say that a homotopy  $h_t$  between  $h_0$  and  $h_1$  is a straight line homotopy if for every  $q \in Z$ , there is a unique shortest geodesic segment in  $X$  from  $h_0(q)$  to  $h_1(q)$ , and that  $t \mapsto h_t(q)$  is a constant speed parametrisation of this geodesic segment.*

It is clear that if a straight line homotopy exists between two maps, then it is unique, although this uniqueness will not have any consequences for our purpose. We are concerned only with the existence of the straight line homotopy. The main thrust of the proposition below is that  $h_t(q)$  as described above is actually jointly continuous in  $q$  and  $t$ , that is, the function  $h: Z \times [0, 1] \rightarrow X$  given by  $h(q, t) := h_t(q)$  is continuous with respect to the product topology on  $Z \times [0, 1]$ .

**Proposition 3.14** (Straight line homotopy on hyperbolic surfaces). *Use the notation of Definition 3.13 (Straight line homotopy), and assume that*

$$\text{for each } q \in Z, \text{ we have } d_X(h_0(q), h_1(q)) < \text{inj}_X(h_0(q)) \quad (3.1)$$

*Then there is a straight line homotopy between  $h_0$  and  $h_1$ .*

*Proof.* For each  $q \in Z$ , since  $h_1(q)$  is contained in the disk centred at  $h_0(q)$  of radius equal to its injectivity radius, there is a unique geodesic segment joining  $h_0(q)$  to  $h_1(q)$  (possibly degenerate of zero length, a degeneracy that occurs exactly when  $h_1(q)$  coincides with  $h_0(q)$ ). So we can define  $h_t(q)$  by declaring that  $t \mapsto h_t(q)$  parametrises this geodesic segment with constant speed  $d_X(h_0(q), h_1(q))$ . It remains to be shown that the function  $h$  is continuous.

First we prove continuity in the case when the hyperbolic surface is  $\mathbb{H}^2$ . Note that the injectivity radius at any point on the hyperbolic plane is infinite, so the inequality (3.1) is always satisfied for any  $Z, h_0, h_1$ . Let  $T\mathbb{H}^2$  be the tangent bundle of  $\mathbb{H}^2$  and  $\pi: T\mathbb{H}^2 \rightarrow \mathbb{H}^2$  be the natural projection. The function  $\Theta: T\mathbb{H}^2 \rightarrow \mathbb{H}^2 \times \mathbb{H}^2$  given by  $\Theta(v) = (\pi(v), \exp(v))$  is continuous because of continuous dependence of solutions of ordinary differential equations on initial values, and bijective because there is a unique geodesic segment between any pair of points on  $\mathbb{H}^2$ . Thus  $\Theta$  is a continuous bijection between manifolds of the same dimension, and hence is a homeomorphism. Then the required straight line homotopy is  $h_t(q) = \exp(t\Theta^{-1}(h_0(q), h_1(q)))$ , and  $h$  is clearly continuous.

Now we tackle the case of an arbitrary hyperbolic surface  $X$ . Suppose that  $p_X: \mathbb{H}^2 \rightarrow X$  is a universal cover, and let  $q_0 \in Z$  be an arbitrary point. We construct an open neighbourhood  $U$  of  $q_0$  in  $Z$  as follows, and then show that  $h$  is continuous over  $U \times [0, 1]$ . The disk  $B$  of radius  $\text{inj}_X(h_0(q_0))$  centred at  $h_0(q_0)$  is evenly covered by the universal cover, so the function  $p_X$  has a local inverse  $P: B \xrightarrow{\sim} \tilde{B}$ . That is,  $P$  maps  $B$  diffeomorphically onto a disk  $\tilde{B} \subset \mathbb{H}^2$ . Now  $p_X$  and  $P$  are Riemannian isometries, so they map Riemannian geodesics to Riemannian geodesics. However, note that  $p_X$  and  $P$  are not necessarily metric isometries; they may not preserve the restriction to  $B$  of the distance function  $d_X$ .

Let  $U$  be the set of all points  $q \in Z$  such that  $h_0(q), h_1(q) \in B$  and  $d_{\mathbb{H}^2}(P \circ h_0(q), P \circ h_1(q)) < \text{inj}_X(h_0(q))$ , which contains  $q_0$  due to hypothesis and the construction of  $B$  and  $P$ . To show that  $U$  is open, consider the sets  $U_0 = h_0^{-1}B$  and  $U_1 = h_1^{-1}B$ , which are open because  $h_0$  and  $h_1$  are continuous functions. Therefore  $P \circ h_0$  and  $P \circ h_1$  are functions defined on  $U_0 \cap U_1$ , and are continuous on this domain of definition. Thus the real valued function  $\eta(q) = \text{inj}_X(h_0(q)) - d_{\mathbb{H}^2}(P \circ h_0(q), P \circ h_1(q))$  is defined and continuous on  $U_0 \cap U_1$ , and the preimage  $U = \eta^{-1}(0, +\infty)$  is an open set in  $Z$ . Since  $q_0 \in U$  as noted above,  $U$  is an open neighbourhood of  $q_0$  in  $Z$ .

Next, let  $q \in U$  be arbitrary. There is a unique geodesic segment  $\gamma$  from  $P \circ h_0(q)$  to

$P \circ h_1(q)$  in  $\tilde{B} \subset \mathbb{H}^2$ , parametrised by  $\gamma(t) = \exp(t\Theta^{-1}(P \circ h_0(q), P \circ h_1(q)))$  as above. Note that here the parameter  $t$  takes values in  $[0, 1]$ , and the parametrisation is at constant speed equal to the length  $l(\gamma)$  of  $\gamma$ . Now the length  $l(\gamma)$  is equal to the hyperbolic distance between the two endpoints  $P \circ h_0(q)$  and  $P \circ h_1(q)$ , and is less than  $\text{inj}_X(h_0(q))$  due to the definition of  $U$ . Since  $p_X$  is a Riemannian isometry and its inverse is  $P$ ,  $p_X$  maps  $\gamma$  to a geodesic segment  $p_X \circ \gamma$  from  $h_0(q)$  to  $h_1(q)$  in  $X$ , which has the same length  $l(p_X \circ \gamma) = l(\gamma)$ , and which is also parametrised at the same constant speed  $l(\gamma)$ . Since  $l(p_X \circ \gamma) = l(\gamma) < \text{inj}_X(h_0(q))$ , we conclude that  $p_X \circ \gamma$  is in fact the unique shortest geodesic segment from  $h_0(q)$  to  $h_1(q)$  in  $X$ , and so its length  $l(p_X \circ \gamma)$  equals the distance  $d_X(h_0(q), h_1(q))$ . But we defined  $h_t$  by declaring  $t \mapsto h_t(q)$  to be the parametrisation of this unique shortest geodesic segment at constant speed  $d_X(h_0(q), h_1(q))$ . Therefore

$$h(q, t) = h_t(q) = p_X \circ \exp(t\Theta^{-1}(P \circ h_0(q), P \circ h_1(q))) \quad (3.2)$$

As  $q \in U$  and  $t \in [0, 1]$  are arbitrary, this expression of  $h_t(q)$  holds for all  $(q, t) \in U \times [0, 1]$ .

Thus we have found an expression for  $h$  over the domain  $U \times [0, 1]$  which is built by composing continuous functions, homeomorphisms, diffeomorphisms, scalar multiplication in finite dimensional vector spaces, and functions into product spaces whose components are continuous. By standard point set topology,  $h$  is continuous over  $U \times [0, 1]$ , and in particular continuous at all points in  $\{q_0\} \times [0, 1] \subset Z \times [0, 1]$ . Since  $q_0 \in Z$  was arbitrary,  $h$  is continuous over all of  $Z \times [0, 1]$ , and therefore defines a straight line homotopy as claimed.  $\square$

**Remark 3.15.** *Proposition 3.14 holds for arbitrary Riemannian manifolds  $X$  if quantified suitably stringently. That is, in the inequality (3.1),  $\text{inj}_X(h_0(q))$  should be replaced by a suitable smaller quantity.*

## 3.5 Locally Quasiconformal Mappings

In this section, we develop analogues of the content of Sections 2.6 to 2.7 by relaxing the notions of quasiconformality. As remarked in those sections, quasiconformality is roughly a small failure in conformality. However, the bound on this small failure in conformality is uniform over the entire Riemann surface. We relax this by allowing the failure in conformality to be unbounded, while requiring it to be bounded locally. Locally quasiconformal homeomorphisms have appeared in literature for a few decades (see, for example, [Zho93] or [Zho13]).

Let  $X, Y$  be Riemann surfaces and  $f: X \rightarrow Y$  be a homeomorphism.

**Definition 3.16.**  *$f$  is called locally quasiconformal (lqc) if every point on  $X$  has an open neighbourhood  $U$  such that  $f: U \rightarrow f(U)$  is a quasiconformal homeomorphism.  $\text{LQC}(X)$  is the group of lqc homeomorphisms of  $X$ , and  $\text{LQC}_0(X)$  is the subgroup of homeomorphisms which are isotopic to the identity.*

We can define the complex dilatation or Beltrami coefficient  $\mu_f := (\partial f)^{-1} \circ (\bar{\partial} f)$  as well as quasiconformal dilatation  $\kappa_f := |\mu_f|$  of lqc homeomorphisms just as for quasiconformal homeomorphisms. As before,  $\mu_f$  is a measurable  $(-1, 1)$  form on  $X$ , and  $|\mu_f|$  and  $\kappa_f$  are measurable functions on  $X$ . The important distinction from quasiconformal homeomorphisms is that the the quasiconformal dilatation of an lqc homeomorphism, although locally bounded, is not required to be globally bounded. Consequently, the homeomorphism need not extend to the ideal boundaries of the Riemann surfaces. Every homeomorphism of smooth surfaces is isotopic to a diffeomorphism. Therefore there is an lqc homeomorphism between any two (topologically) homeomorphic Riemann surfaces. We note an important property of the Beltrami coefficient  $\mu_f$ . Every point in  $X$  has an open neighbourhood  $U$  such that  $f$  is a quasiconformal homeomorphism from  $U$  to  $f(U)$ . Hence  $\text{esssup}_{q \in U} |\mu_f(q)| < +\infty$ . If  $U$

is a regular open set, then clearly  $\operatorname{ess\,sup}_{q \in \overline{U}} |\mu_f(q)| < +\infty$  as well. Therefore every point has a compact neighbourhood on which  $|\mu_f|$  is bounded. Consequently,  $|\mu_f|$  is bounded on every compact set.

In general, for any measurable set  $K \subset X$  and a measurable  $(-1, 1)$ -form  $\mu$ , we define  $\|\mu\|_K := \operatorname{ess\,sup}_{q \in K} |\mu(q)|$ . A measurable  $(-1, 1)$ -form is *locally bounded* if for every compact subset  $K \subset X$ , we have  $\|\mu\|_K < \infty$ . The complex vector space of locally bounded measurable  $(-1, 1)$ -forms on  $X$  is denoted by  $\operatorname{bel}(X)$ . We endow  $\operatorname{bel}(X)$  with the structure of a Fréchet space as follows. Recall that a Fréchet space is a vector space endowed with a topology generated by a countable family of seminorms, which is completely metrisable via a translation invariant metric. In the present scenario,  $\|\cdot\|_K$  is a seminorm on  $\operatorname{bel}(X)$  for every compact  $K$ , and generates a topology on  $\operatorname{bel}(X)$ . Observe that whenever  $K \subset K'$ , we have  $\|\cdot\|_K \leq \|\cdot\|_{K'}$ , so that the topology generated by  $\|\cdot\|_{K'}$  is finer than the one generated by  $\|\cdot\|_K$ . The family  $\{\|\cdot\|_K\}_{K \subset X \text{ compact}}$  of all such seminorms generates a topology on  $\operatorname{bel}(X)$ , the finest of them all. Therefore if  $K_1 \subset K_2 \subset \dots \subset X$  is an exhaustion of  $X$  by compact sets, then the same topology is also generated by the countable family of seminorms  $\{\|\cdot\|_{K_n}\}_{n=1}^\infty$ . Further,  $d(\mu, \mu') := \sum_{n=1}^\infty \frac{\|\mu - \mu'\|_{K_n}}{2^n(1 + \|\mu - \mu'\|_{K_n})}$  is a complete metric inducing the same topology as the above family of seminorms, and is also translation invariant. Thus  $\operatorname{bel}(X)$  is a Fréchet space. Since there isn't one norm generating the topology (if  $X$  is not compact), we have to work harder to get an analogue of a unit ball. We say that  $\mu \in \operatorname{bel}(X)$  is *locally bounded away from 1* if for every compact set  $K \subset X$ , we have  $\|\mu\|_K < 1$ .

**Definition 3.17.** A Beltrami form is a measurable  $(-1, 1)$ -form which is locally bounded away from 1.  $\operatorname{Bel}(X)$  is the subspace of  $\operatorname{bel}(X)$  consisting of all Beltrami forms.

The Beltrami coefficient of an lqc homeomorphism is clearly a Beltrami form, and every Beltrami form arises so. The mapping theorem below is almost identical to Theorem 2.8. As remarked there, the construction of  $X_\mu$  in the proof only uses the fact that  $\mu$  is a Beltrami form, and not the stronger hypothesis of being a quasiconformal Beltrami form.

**Theorem 3.18** (Mapping Theorem). *If  $\mu$  is a Beltrami form on  $X$ , then there is a unique complex structure  $X_\mu$ , called the integration of  $\mu$ , on the same topological surface  $X$  such that  $\text{id}_{X \rightarrow X_\mu} : X \rightarrow X_\mu$  is locally quasiconformal and its Beltrami coefficient is  $\mu$ . Moreover, if  $f : X \rightarrow Y$  is a locally quasiconformal homeomorphism with Beltrami coefficient  $\mu$ , then  $f : X_\mu \rightarrow Y$  is a biholomorphism.*

*Proof.* The proof is identical to that of Theorem 2.8 (as given in [Hub06, Proposition and Definition 4.8]). The only difference is that  $\text{id}_{X \rightarrow X_\mu} : X \rightarrow X_\mu$  is not quasiconformal if  $\mu$  is not bounded. □

# CHAPTER 4

## THE MARKED MODULI SPACE

As in the Introduction, fix a connected, oriented surface  $S$  with negative Euler characteristic; our emphasis is on the case in which  $S$  is a surface of infinite type. We define the marked moduli space  $\mathcal{T}(S)$  in the first section of this chapter. In the four sections that follow, we describe its topology in four different ways analogous to Teichmüller space. We also prove that  $\mathcal{T}(S)$  is Hausdorff, completely metrisable and reduces to the usual Teichmüller space in case  $S$  is of finite type.

### 4.1 The Underlying Set and the Action

We recall the definition of the set  $\mathcal{T}(S)$  from Section 1.2.

**Definition 4.1** (The set underlying the marked moduli space).

$$\mathcal{T}(S) := \left\{ (X, f) \left| \begin{array}{l} X \text{ is a complete Nielsen-convex hyperbolic surface} \\ f: S \rightarrow X \text{ is a homeomorphism} \end{array} \right. \right\} / \sim \quad (4.1)$$

where  $(X_1, f_1) \sim (X_2, f_2)$  if there is an isometry  $\varphi: X_1 \rightarrow X_2$  homotopic to  $f_2 \circ f_1^{-1}$ .

As is standard in finite type Teichmüller theory, we call the homeomorphism  $f$  the *marking map*. The next proposition says that the theory of the marked moduli space is not a trivial theory.

**Proposition 4.2.** *The set  $\mathcal{T}(S)$  is nonempty.*

*Proof.* Take a topological pants decomposition  $\mathcal{P}$  of  $S$ , where a pant is a surface of zero genus,  $b$  boundary components and  $p$  punctures with  $b + p = 3$ . Such a decomposition exists since  $\chi(S) < 0$ . For each pant in the scheme  $\mathcal{P}$ , consider a hyperbolic pair of pants with

geodesic boundary, whose components are called *cuffs*, and with cusps at punctures. The geometry of the hyperbolic pairs of pants is chosen so that the lengths of any two cuffs that get glued in the scheme  $\mathcal{P}$  are equal. This allows the hyperbolic pairs of pants to be glued according to the scheme  $\mathcal{P}$  to produce a hyperbolic surface  $X$  homeomorphic to  $S$  via a marking map  $f: S \rightarrow X$ . If the cuff lengths are all bounded above as well as below by two fixed finite positive numbers, then the injectivity radius of points in  $X$  is bounded below by a fixed positive number, and therefore  $X$  is complete. Further, since  $X$  is a union of hyperbolic pairs of pants, Proposition 3.1 (Nielsen-convexity) asserts that  $X$  is a complete, Nielsen-convex hyperbolic surface. Therefore  $[X, f] \in \mathcal{T}(S)$ , and so  $\mathcal{T}(S)$  is nonempty.  $\square$

**Remark 4.3.** *Even if the cuff lengths are arbitrary, [Ba19, Theorem 5.1] asserts that the hyperbolic pairs of pants may be glued with particular choices of twists in such a way that the resulting surface  $X$  is geodesically complete.*

Observe that if  $S$  is a closed surface, then  $\mathcal{T}(S)$  as above reduces, as a set, to the usual quasiconformal Teichmüller space. This is because a closed hyperbolic surface has empty ideal boundary, and does not have visible ends (or any ends at all). This observation holds true for surfaces of finite type also. Indeed, if  $S$  is a surface of finite type, then in the usual definition of Teichmüller space, the hyperbolic surface  $X$  is constrained to have finite area (Definition 2.12, Teichmüller space of surfaces of finite type). Then each end of  $X$  has cusped geometry and so is not a visible end, which means that  $X$  has empty ideal boundary. Thus  $\mathcal{T}(S)$  is an extension of the usual Teichmüller space to surfaces of infinite type, as a set. In Corollary 4.16, we strengthen this observation by showing that  $\mathcal{T}(S)$  is an extension of the usual Teichmüller space as a topological space as well.

Recall from Section 2.1 the *mapping class group*  $\text{MCG}(S) = \text{Homeo}^+(S) / \text{Homeo}_0(S)$ . It is clear that  $\text{MCG}(S)$  acts on  $\mathcal{T}(S)$  by change of marking. We denote the action of the mapping class  $[\psi]$  by the function  $A_{[\psi]}$  from  $\mathcal{T}(S)$  to itself. The action is defined precisely in equation (4.2) below.

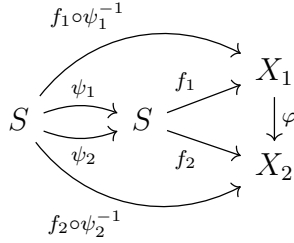


Figure 4.1: The action of  $\text{MCG}(S)$  on  $\mathcal{T}(S)$  is well defined.

**Proposition 4.4.** *There is a well defined group action  $A: \text{MCG}(S) \times \mathcal{T}(S) \rightarrow \mathcal{T}(S)$ , currying which yields, for every mapping class  $[\psi] \in \text{MCG}(S)$ , a function  $A_{[\psi]}: \mathcal{T}(S) \rightarrow \mathcal{T}(S)$ , defined by*

$$A_{[\psi]}[X, f] = A([\psi], [X, f]) := [X, f \circ \psi^{-1}] \quad (4.2)$$

*Proof.* We need to show that  $A([\psi], [X, f])$  is independent of two choices made in the definition of  $A$ , namely, the choice of the representative  $(X, f)$  of the marked hyperbolic structure, and the choice of the representative homeomorphism  $\psi$  of the mapping class. This follows easily from the diagram in Figure 4.1, which homotopy commutes.  $\square$

## 4.2 Topology via the Homeomorphisms at Infinity

In this section, we define the topology on the set  $\mathcal{T}(S)$  using homeomorphisms at infinity. The analogue in Teichmüller theory, the idea of representing marked Riemann surface structures by induced homeomorphisms at infinity, has existed since the early days of Teichmüller theory (see [Ahl63] and Section 2.8, Quasiconformal Teichmüller Space), although obtaining the topology from homeomorphisms at infinity is a delicate matter. To define the topology, fix a universal cover  $p: \mathbb{H}^2 \rightarrow S$  with deck group  $\Gamma \subset \text{PSL}(2, \mathbb{R})$ , which is a torsion-free Fuchsian group of the first kind. The universal cover  $p$  induces a homeomorphism between  $S$  and  $\Gamma \backslash \mathbb{H}^2$ , which is a complete Nielsen-convex hyperbolic surface by Proposition 3.1 (Nielsen-convexity). We define a topological space  $\mathcal{T}(p)$  associated to the universal cover  $p$ ,

and a bijective function  $\Phi_p: \mathcal{T}(S) \rightarrow \mathcal{T}(p)$ , which we then declare to be a homeomorphism, yielding a topology on  $\mathcal{T}(S)$ .

**Definition 4.5** (The space  $\mathcal{T}(p)$ ).

$$\begin{aligned} \mathcal{T}(p) &:= \mathrm{PSL}(2, \mathbb{R}) \backslash \tilde{\mathcal{T}}(p) \quad \text{where} \\ \tilde{\mathcal{T}}(p) &:= \{F \in \mathrm{Homeo}^+(S^1) \mid F \circ \Gamma \circ F^{-1} \subset \mathrm{PSL}(2, \mathbb{R})\} \end{aligned} \tag{4.3}$$

$\mathcal{T}(p)$  is naturally a Hausdorff topological space as follows.  $\tilde{\mathcal{T}}(p)$  inherits a topology as a subspace of  $\mathrm{Homeo}^+(S^1)$  with the compact-open topology.  $\mathcal{T}(p)$  has the quotient topology, where  $\mathrm{PSL}(2, \mathbb{R})$  acts on  $\tilde{\mathcal{T}}(p)$  by composition on the left. Since  $\mathrm{Homeo}^+(S^1)$  is a topological group and  $\mathrm{PSL}(2, \mathbb{R})$  is a closed subgroup, the right coset space  $\mathrm{Homeo}^+(S^1) \backslash \mathrm{PSL}(2, \mathbb{R})$  is Hausdorff (see [Mun00, p146, 7(d)]). Therefore its subspace  $\mathcal{T}(p)$  is also Hausdorff. We denote by  $\pi_p$  the quotient map  $\tilde{\mathcal{T}}(p) \rightarrow \mathcal{T}(p)$ .

**Definition 4.6** (The function  $\Phi_p$ ). *Suppose  $[X, f] \in \mathcal{T}(S)$  is a marked hyperbolic structure. Choose a universal cover  $p_X: \mathbb{H}^2 \rightarrow X$ . The marking map  $f: S \rightarrow X$  lifts to a homeomorphism  $\tilde{f}: \mathbb{H}^2 \rightarrow \mathbb{H}^2$ , which, by Proposition 3.2(1), extends to a homeomorphism at infinity  $\partial\tilde{f}: S^1 \rightarrow S^1$ . We define  $\Phi_p[X, f]$  to be the right coset  $[\partial\tilde{f}] = \mathrm{PSL}(2, \mathbb{R}) \circ (\partial\tilde{f})$  of  $\mathrm{PSL}(2, \mathbb{R})$  in  $\mathrm{Homeo}^+(S^1)$ .*

**Lemma 4.7.** *For each  $[X, f] \in \mathcal{T}(S)$ ,  $\Phi_p[X, f]$  is a well defined element of  $\mathcal{T}(p)$ . Thus the definition above yields a function  $\Phi_p: \mathcal{T}(S) \rightarrow \mathcal{T}(p)$ .*

*Proof.* First we need to show that  $\Phi_p[X, f]$  is independent of the three choices made in the definition of  $\Phi_p$ , namely, the choice of the representative  $(X, f)$  of the marked hyperbolic structure, the choice of the universal cover  $p_X$ , and the choice of the lift  $\tilde{f}$ . We treat these one at a time, and in reverse order.

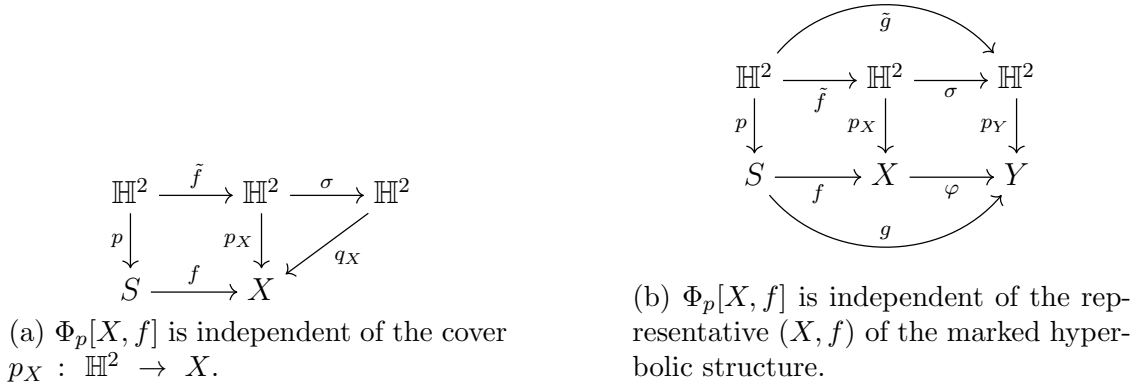


Figure 4.2:  $\Phi_p[X, f]$  is well defined.

1. The choice of the lift  $\tilde{f}$  of  $f$ : Any other lift of  $f$  is of the form  $\sigma \circ \tilde{f}$ , where  $\sigma \in \Gamma_X$  is a deck transformation. This extends to the homeomorphism at infinity  $\sigma \circ (\partial\tilde{f})$ , by Proposition 3.2(6). Since  $\sigma \in \text{PSL}(2, \mathbb{R})$ , we have an equality of right cosets  $\text{PSL}(2, \mathbb{R}) \circ (\sigma \circ (\partial\tilde{f})) = \text{PSL}(2, \mathbb{R}) \circ (\partial\tilde{f})$ . That is,  $[\partial(\sigma \circ \tilde{f})] = [\sigma \circ (\partial\tilde{f})] = [\partial\tilde{f}]$ .
2. The choice of the universal cover  $p_X : \mathbb{H}^2 \rightarrow X$ : Suppose that  $q_X : \mathbb{H}^2 \rightarrow X$  is another universal cover. Since universal covers are unique up to isometry, there is a  $\sigma \in \text{PSL}(2, \mathbb{R})$  such that  $p_X = q_X \circ \sigma$ . See the diagram in Figure 4.2a, which commutes. Then  $\sigma \circ \tilde{f}$  is a lift of  $f$  with respect to the universal covers  $p$  and  $q_X$ , which extends to the homeomorphism at infinity  $\sigma \circ (\partial\tilde{f})$ , by Proposition 3.2(6). Again we have  $[\partial(\sigma \circ \tilde{f})] = [\partial\tilde{f}]$ .
3. The choice of the representative  $(X, f)$  of the marked hyperbolic structure: Suppose  $(Y, g)$  is another representative of the same marked hyperbolic structure. Then there is an isometry  $\varphi : X \rightarrow Y$  such that  $\varphi \circ f$  is homotopic to  $g$ . Then  $\varphi$  lifts to an isometry  $\sigma \in \text{PSL}(2, \mathbb{R})$ . For a lift  $\tilde{f}$  of  $f$ ,  $\sigma \circ \tilde{f}$  is a lift of  $\varphi \circ f$ . The homotopy from  $\varphi \circ f$  to  $g$  lifts to a homotopy from  $\sigma \circ \tilde{f}$  to a lift  $\tilde{g}$  of  $g$ . See the diagram in Figure 4.2b, in which the squares commute and the top and bottom triangles homotopy commute. By Proposition 3.2, parts (4) and (6), we have  $\partial\tilde{g} = \partial(\sigma \circ \tilde{f}) = \sigma \circ (\partial\tilde{f})$ . Therefore again we have  $[\partial\tilde{g}] = [\sigma \circ (\partial\tilde{f})] = [\partial\tilde{f}]$ , so that  $\Phi_p[X, f]$  is a well defined right coset of  $\text{PSL}(2, \mathbb{R})$  in  $\text{Homeo}^+(S^1)$ .

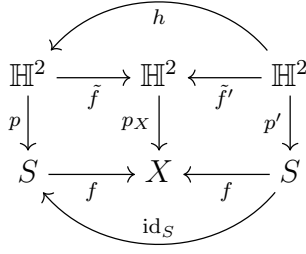
Next, to show that  $\Phi_p[X, f]$  is actually an element of the subspace  $\mathcal{T}(p)$  of the right coset space  $\mathrm{PSL}(2, \mathbb{R}) \setminus \mathrm{Homeo}^+(S^1)$ , we use Proposition 3.2(3) to obtain  $(\partial\tilde{f}) \circ \Gamma = \Gamma_X \circ (\partial\tilde{f})$ , or in other words,  $(\partial\tilde{f}) \circ \Gamma \circ (\partial\tilde{f})^{-1} = \Gamma_X \subset \mathrm{PSL}(2, \mathbb{R})$ . Therefore  $\partial\tilde{f} \in \tilde{\mathcal{T}}(p)$ , and  $[\partial\tilde{f}] \in \mathcal{T}(p)$ . So the codomain of  $\Phi_p$  is  $\mathcal{T}(p)$  indeed.  $\square$

**Lemma 4.8.**  $\Phi_p: \mathcal{T}(S) \rightarrow \mathcal{T}(p)$  is a bijection.

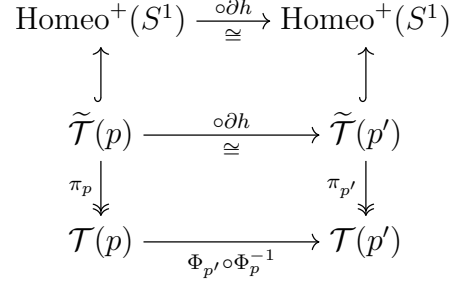
*Proof.* To see that  $\Phi_p$  is injective, suppose  $\Phi_p[X, f] = \Phi_p[Y, g]$ . Let  $\tilde{f}, \tilde{g}$  denote the lifts of  $f, g$  with respect to universal covers  $p_X: \mathbb{H}^2 \rightarrow X$  and  $p_Y: \mathbb{H}^2 \rightarrow Y$ . Note that  $(\partial\tilde{f}) \circ \Gamma \circ (\partial\tilde{f})^{-1} = \Gamma_X$  and  $(\partial\tilde{g}) \circ \Gamma \circ (\partial\tilde{g})^{-1} = \Gamma_Y$  by Proposition 3.2(3). Also,  $[\partial\tilde{g}] = [\partial\tilde{f}]$ , so there exists an element  $\sigma$  of  $\mathrm{PSL}(2, \mathbb{R})$  such that  $\partial\tilde{g} = \sigma \circ (\partial\tilde{f})$ . Thus we have  $\sigma \circ \Gamma_X = \sigma \circ (\partial\tilde{f}) \circ \Gamma \circ (\partial\tilde{f})^{-1} = (\partial\tilde{g}) \circ \Gamma \circ (\partial\tilde{f})^{-1} = \Gamma_Y \circ (\partial\tilde{g}) \circ (\partial\tilde{f})^{-1} = \Gamma_Y \circ \sigma$ . Thus  $\sigma$  is equivariant, not only as a homeomorphism of  $S^1$ , but also as an isometry of  $\mathbb{H}^2$ , and hence descends to an isometry  $\varphi: X \rightarrow Y$ . Further, for every  $\gamma \in \Gamma$ , we have  $(\varphi \circ f)_*(\gamma) = (\sigma \circ (\partial\tilde{f})) \circ \gamma \circ (\sigma \circ (\partial\tilde{f}))^{-1} = (\partial\tilde{g}) \circ \gamma \circ (\partial\tilde{g})^{-1} = g_*(\gamma)$ . Therefore  $(\varphi \circ f)_*$  and  $g_*$  are equal as isomorphisms from  $\Gamma$  to  $\Gamma_Y$ , and hence are equal as the  $\pi_1$  functor as well. Since  $S$  is a  $K(\pi_1(S), 1)$  classifying space,  $\varphi \circ f$  is homotopic to  $g$ . Therefore  $[X, f] = [Y, g]$  in  $\mathcal{T}(S)$ , so  $\Phi_p$  is injective.

To see that  $\Phi_p$  is surjective, suppose  $[F] \in \mathcal{T}(p)$ . Define  $\Gamma_X = F \circ \Gamma \circ F^{-1}$ . Since  $F \in \tilde{\mathcal{T}}(p)$ , we have  $\Gamma_X \subset \mathrm{PSL}(2, \mathbb{R})$ .  $\Gamma_X$  is discrete because conjugation by  $F$  is a homeomorphism of  $\mathrm{Homeo}^+(S^1)$ , and because  $\Gamma$  is also discrete.  $\Gamma_X$  is also torsion-free because it is isomorphic to  $\Gamma$ , which is torsion-free. Further,  $F$  maps the limit set of  $\Gamma$  to the limit set of  $\Gamma_X$ , which is therefore all of  $S^1$ . Hence  $X = \Gamma_X \setminus \mathbb{H}^2$  is a complete, Nielsen-convex hyperbolic surface by Proposition 3.1. The Douady–Earle extension (recall Theorem 3.12) of  $F$  is conformally natural and thus equivariant, hence descends to a homeomorphism  $f: S \rightarrow X$ . Clearly  $\Phi_p[X, f] = [F]$ , so  $\Phi_p$  is surjective. This completes the proof of bijectivity of  $\Phi_p$ .  $\square$

The above proposition allows us to define a topology on  $\mathcal{T}(S)$ . We then check that it actually does not depend on the choice of the universal cover  $p$  that we used in the definition.



(a) The identity lifts to the map  $h$  with respect to covers  $p'$  and  $p$ .



(b)  $\Phi_{p'} \circ \Phi_p^{-1}$  is induced by multiplication by  $\partial h$  on the right.

Figure 4.3:  $\Phi_{p'} \circ \Phi_p^{-1}$  is a homeomorphism.

**Definition 4.9** (Topology of the Marked Moduli Space). *The topology on  $\mathcal{T}(S)$  is defined by declaring the bijection  $\Phi_p: \mathcal{T}(S) \rightarrow \mathcal{T}(p)$  to be a homeomorphism. In other words, a subset  $U \subset \mathcal{T}(S)$  is open if and only if  $\Phi_p(U) \subset \mathcal{T}(p)$  is open.*

**Proposition 4.10** (The topology is well defined and Hausdorff). *If  $p': \mathbb{H}^2 \rightarrow S$  is another universal cover with deck group  $\Gamma' \subset \text{PSL}(2, \mathbb{R})$ , a torsion-free Fuchsian group of the first kind, then  $\Phi_{p'} \circ \Phi_p^{-1}: \mathcal{T}(p) \rightarrow \mathcal{T}(p')$  is a homeomorphism. Thus the topology on  $\mathcal{T}(S)$  does not depend on the choice of the universal cover  $p$ . Moreover,  $\mathcal{T}(S)$  is a Hausdorff space.*

*Proof.* The identity map of  $S$  lifts, with respect to the covers  $p'$  and  $p$ , to a map  $h: \mathbb{H}^2 \rightarrow \mathbb{H}^2$  such that  $p \circ h = p'$ . By Proposition 3.2(1),  $h$  extends to a homeomorphism at infinity  $\partial h$ . Note that  $h$  is not necessarily an isometry, because the two universal covers  $p$  and  $p'$  can possibly induce different hyperbolic metrics on  $S$ , although both are complete and Nielsen-convex by Proposition 3.1.

Let  $[X, f] \in \mathcal{T}(S)$  be a marked hyperbolic structure and  $p_X: \mathbb{H}^2 \rightarrow X$  be a universal cover. The marking map  $f$  lifts to  $\tilde{f}$  with respect to the covers  $p$  and  $p_X$ , so that  $\Phi_p[X, f] = [\partial\tilde{f}]$ . That is, the right  $\text{PSL}(2, \mathbb{R})$ -coset  $\Phi_p[X, f]$  is represented by the homeomorphism  $\partial\tilde{f}$ . Then the map  $\tilde{f}'$  defined by  $\tilde{f}' = \tilde{f} \circ h$  is a lift of  $f$  with respect to the covers  $p'$  and  $p_X$ . See the diagram in Figure 4.3a, which commutes. Thus  $\Phi_{p'}[X, f]$  is represented by  $\partial(\tilde{f} \circ h)$ , which equals  $(\partial\tilde{f}) \circ (\partial h)$  due to Proposition 3.2(5). Thus  $\Phi_{p'} \circ \Phi_p[\partial\tilde{f}] = [\partial\tilde{f}] \circ (\partial h)$ . In

other words,  $\Phi_{p'} \circ \Phi_p^{-1}: \mathcal{T}(p) \rightarrow \mathcal{T}(p')$  is induced simply by multiplication on the right by  $\partial h$ . Multiplication on the right by the fixed map  $\partial h$  is a homeomorphism of the topological group  $\text{Homeo}^+(S^1)$ , and it restricts to a homeomorphism from  $\tilde{\mathcal{T}}(p)$  to  $\tilde{\mathcal{T}}(p')$ , which descends to a homeomorphism  $\Phi_{p'} \circ \Phi_p^{-1}$  of the spaces of right cosets  $\mathcal{T}(p)$  and  $\mathcal{T}(p')$ . See the diagram in Figure 4.3b, which commutes, and in which the top two vertical arrows are embeddings and the bottom two vertical arrows are quotient maps.

Now we can show that the topology of  $\mathcal{T}(S)$  does not depend on the universal cover  $p$ . Indeed, for another universal cover  $p'$  and any subset  $U \subset \mathcal{T}(S)$ , the condition  $\Phi_p(U)$  is open in  $\mathcal{T}(p)$  is equivalent, since  $\Phi_{p'} \circ \Phi_p^{-1}$  is a homeomorphism, to the condition that  $\Phi_{p'}(U) = \Phi_{p'} \circ \Phi_p^{-1}(\Phi_p(U))$  is open in  $\mathcal{T}(p')$ .

Finally,  $\mathcal{T}(S)$  is Hausdorff because it is homeomorphic to  $\mathcal{T}(p)$ , which is Hausdorff as remarked after Definition 4.5. □

### 4.3 Topology via the Conjugacy Classes of $\text{PSL}(2, \mathbb{R})$ -Representations

In this section, we prove that the marked moduli space  $\mathcal{T}(S)$  embeds (topologically) into  $X(\pi_1(S), \text{PSL}(2, \mathbb{R})) = \text{Hom}(\pi_1(S), \text{PSL}(2, \mathbb{R})) / \text{PSL}(2, \mathbb{R})$ , the space of conjugacy classes of  $\text{PSL}(2, \mathbb{R})$ -representations of the fundamental group. Here the  $\text{PSL}(2, \mathbb{R})$ -representation space  $\text{Hom}(\pi_1(S), \text{PSL}(2, \mathbb{R}))$  has the topology of pointwise convergence. The quotient is by the conjugation action of  $\text{PSL}(2, \mathbb{R})$ , and the space  $X(\pi_1(S), \text{PSL}(2, \mathbb{R}))$  has the quotient topology. The embedding will imply that in case the surface  $S$  is of finite type, the topology on  $\mathcal{T}(S)$  coincides with the usual topology on Teichmüller space. The marked moduli space injects into the space  $X(\pi_1(S), \text{PSL}(2, \mathbb{R}))$  in the obvious way via the conjugacy class of a

holonomy representation. We describe the injection  $\Phi_{\text{at}}$  as follows. First fix a basepoint  $s \in S$ .

**Definition 4.11** (The function  $\Phi_{\text{at}}$ ). *Suppose  $[X, f] \in \mathcal{T}(S)$  is a marked hyperbolic structure. Take the basepoint on  $X$  to be  $x = f(s)$ , so that the marking map  $f: S \rightarrow X$  induces an isomorphism of fundamental groups  $f_*: \pi_1(S, s) \rightarrow \pi_1(X, x)$ . Choose a pointed universal cover  $p_X: (\mathbb{H}^2, \tilde{x}) \rightarrow (X, x)$  with deck group  $\Gamma_X \subset \text{PSL}(2, \mathbb{R})$ , and let  $\varphi_X: \pi_1(X, x) \xrightarrow{\cong} \Gamma_X \hookrightarrow \text{PSL}(2, \mathbb{R})$  be the corresponding holonomy representation. Composing the two group morphisms yields the representation  $\rho_{[X, f]} := \varphi_X \circ f_*: \pi_1(S, s) \rightarrow \text{PSL}(2, \mathbb{R})$ . We define  $\Phi_{\text{at}}[X, f]$  to be the conjugacy class  $[\rho_{[X, f]}]$  in  $X(\pi_1(S, s), \text{PSL}(2, \mathbb{R}))$ .*

**Lemma 4.12.** *For each  $[X, f] \in \mathcal{T}(S)$ ,  $\Phi_{\text{at}}[X, f]$  is a well defined conjugacy class of  $\text{PSL}(2, \mathbb{R})$ -representations of  $\pi_1(S, s)$ . Further,  $\Phi_{\text{at}}: \mathcal{T}(S) \rightarrow X(\pi_1(S, s), \text{PSL}(2, \mathbb{R}))$  is an injective function.*

*Proof.* First, we have to show that the conjugacy class  $[\rho_{[X, f]}]$  is independent of two choices made in the definition of  $\Psi_{\text{at}}$ , namely, the choice of the representative  $(X, f)$  of the marked hyperbolic structure, and the choice of the pointed universal cover  $p_X$ . We treat these one at a time, and in reverse order.

1. The choice of the pointed universal cover  $p_X: (\mathbb{H}^2, \tilde{x}) \rightarrow (X, x)$ : A different pointed universal cover is of the form  $p_X \circ \sigma^{-1}: (\mathbb{H}^2, \sigma(\tilde{x})) \rightarrow (X, x)$  for some  $\sigma \in \text{PSL}(2, \mathbb{R})$ . Let  $\rho'_{[X, f]}$  be the representation obtained using the new pointed universal cover. Suppose  $\gamma$  is a loop in  $S$  based at  $s$ , so that  $[\gamma] \in \pi_1(S, s)$ . Suppose that the lift of  $f(\gamma)$  to the universal cover, with respect the covering map  $p_X$ , starting at  $\tilde{x}$ , has endpoint  $\tilde{x}'$ , so that the deck transformation  $\rho_{[X, f]}[\gamma]$  maps  $\tilde{x}$  to  $\tilde{x}'$ . Then the lift of the same curve, with respect to the covering map  $p_X \circ \sigma^{-1}$ , starting at  $\sigma(\tilde{x})$  has endpoint  $\sigma(\tilde{x}')$ . Consequently, the deck transformation sending  $\sigma(\tilde{x})$  to  $\sigma(\tilde{x}')$  is  $\sigma \circ \rho_{[X, f]}[\gamma] \circ \sigma^{-1}$ . But this deck transformation is precisely the holonomy  $\rho'_{[X, f]}[\gamma]$  with respect to the new

pointed universal cover, so  $\rho'_{[X,f]}[\gamma] = \sigma \circ \rho_{[X,f]}[\gamma] \circ \sigma^{-1}$ . In other words, since this is true for all  $[\gamma] \in \pi_1(S, s)$ ,  $\rho'_{[X,f]}$  is the conjugate of  $\rho_{[X,f]}$  by  $\sigma \in \text{PSL}(2, \mathbb{R})$ , and so  $[\rho'_{[X,f]}] = [\rho_{[X,f]}]$ .

2. The choice of the representative  $(X, f)$  of the marked hyperbolic structure: Suppose that  $(Y, g)$  is another representative of the same marked hyperbolic structure. The choice of a pointed universal cover  $p_Y: (\mathbb{H}^2, \tilde{y}) \rightarrow (Y, y)$ , where  $y = g(s)$ , leads to a holonomy representation  $\varphi_Y: \pi_1(Y, y) \rightarrow \Gamma_Y \hookrightarrow \text{PSL}(2, \mathbb{R})$ . The equality  $[X, f] = [Y, g]$  means that there exists an isometry  $\psi: X \rightarrow Y$  such that  $g$  is homotopic to  $\psi \circ f$ . This homotopy lifts to universal covers as a homotopy from  $\tilde{g}$  to  $\tilde{\psi} \circ \tilde{f}$ , where  $\tilde{\psi}$ ,  $\tilde{f}$  and  $\tilde{g}$  are lifts of  $\psi$ ,  $f$  and  $g$  respectively, and let  $\tilde{s} \in \mathbb{H}^2$  be such that  $g(\tilde{s}) = \tilde{y}$ . Let the endpoint of the track of  $\tilde{s}$  under the lifted homotopy be  $\tilde{y}'$ , and let  $y' = p_Y(\tilde{y}') = \psi \circ f(s)$ . The pointed universal cover  $p: (\mathbb{H}^2, \tilde{y}') \rightarrow (Y, y')$  leads to a holonomy representation  $\varphi'_Y: \pi_1(Y, y') \rightarrow \Gamma_Y \hookrightarrow \text{PSL}(2, \mathbb{R})$ . For every closed curve  $\gamma$  in  $S$  based at  $s$ ,  $g(\gamma)$  is homotopic to  $\psi \circ f(\gamma)$  in  $Y$ . Since the holonomies around homotopic curves are equal,  $\varphi_Y(g_*[\gamma]) = \varphi'_Y((\psi \circ f)_*[\gamma])$  for every  $[\gamma] \in \pi_1(S, s)$ . Thus for every  $[\gamma] \in \pi_1(S, s)$ , we compute  $\rho_{[Y,g]}[\gamma] = \varphi_Y(g_*[\gamma]) = \varphi'_Y((\psi \circ f)_*[\gamma]) = \varphi'_Y(\psi_*(f_*[\gamma])) = \psi_*(\varphi_X(f_*[\gamma])) = \psi_*(\rho_{[X,f]}[\gamma])$ . Note that in the fourth equality,  $\psi_*$  denotes the  $\pi_1$  functor on the left and the isometry between deck groups induced by  $\tilde{\psi}$  on the right. However,  $\psi_*: \Gamma_X \rightarrow \Gamma_Y$  is conjugation by  $\tilde{\psi}$ , so  $\rho_{[Y,g]}[\gamma] = \psi_*(\rho_{[X,f]}[\gamma]) = \tilde{\psi} \circ \rho_{[X,f]}[\gamma] \circ \tilde{\psi}^{-1}$ . In words,  $\rho_{[Y,g]}[\gamma]$  is the conjugate of  $\rho_{[X,f]}[\gamma]$  by  $\tilde{\psi} \in \text{PSL}(2, \mathbb{R})$ . This holds for all  $[\gamma] \in \pi_1(S, s)$ , so we conclude that the representation  $\rho_{[Y,g]}$  is conjugate to  $\rho_{[X,f]}$  by some element of  $\text{PSL}(2, \mathbb{R})$ . That is,  $[\rho_{[Y,g]}] = [\rho_{[X,f]}]$ , so that  $\Phi_{\text{at}}[X, f]$  is a well defined conjugacy class of  $\text{PSL}(2, \mathbb{R})$ -representations.

Finally, we show that  $\Phi_{\text{at}}$  is injective. Suppose  $[X, f], [Y, g] \in \mathcal{T}(S)$  are marked hyperbolic structures such that  $[\rho_{[X,f]}] = [\rho_{[Y,g]}]$ . This means that there is an element  $\sigma \in \text{PSL}(2, \mathbb{R})$  such that  $\sigma \circ \rho_{[X,f]}[\gamma] \circ \sigma^{-1} = \rho_{[Y,g]}[\gamma]$  for every  $[\gamma] \in \pi_1(S, s)$ . Thus  $\sigma \circ \Gamma_X \circ \sigma^{-1} = \Gamma_Y$ ,

which we rearrange to  $\sigma \circ \Gamma_X = \Gamma_Y \circ \sigma$ , so  $\sigma$  is equivariant. Hence  $\sigma$  descends to an isometry  $\psi: X \rightarrow Y$ . Further, for every  $[\gamma] \in \pi_1(S, s)$ , we have  $\varphi_Y \circ g_*[\gamma] = \rho_{[Y, g]}[\gamma] = \sigma \circ \rho_{[X, f]}[\gamma] \circ \sigma^{-1} = \psi_* \circ \varphi_X \circ f_*[\gamma] = \varphi_Y \circ \psi_* \circ f_*[\gamma]$ . In the last equality,  $\psi_*$  denotes the isomorphism of deck groups on the left and the  $\pi_1$  functor on the right. Since  $\varphi_Y$  is an isomorphism,  $(\psi \circ f)_* = g_*$  as homomorphisms  $\pi_1(S, s) \rightarrow \pi_1(Y, y)$ . Since  $S$  is a  $K(\pi_1(S, s), 1)$  classifying space, we conclude that  $\psi \circ f$  is homotopic to  $g$ . Therefore  $[X, f] = [Y, g]$ , and  $\Phi_{\text{at}}$  is injective.  $\square$

Since the topology on  $\mathcal{T}(S)$  is defined using a universal cover  $p$  and the function  $\Phi_p$ , in order to prove  $\Phi_{\text{at}}$  is an embedding, we must compute  $\Phi_{\text{at}}$  in terms of  $\Phi_p$ . To that end, fix a pointed universal cover  $p: (\mathbb{H}^2, \tilde{s}) \rightarrow (S, s)$  with deck group  $\Gamma \subset \text{PSL}(2, \mathbb{R})$ , which is a torsion-free Fuchsian group of the first kind. This leads to a holonomy representation  $\varphi: \pi_1(S, s) \rightarrow \Gamma \hookrightarrow \text{PSL}(2, \mathbb{R})$  as before. For any marked hyperbolic structure  $[X, f] \in \mathcal{T}(S)$  with a pointed universal cover  $p_X: (\mathbb{H}^2, \tilde{x}) \rightarrow (X, x)$ , where  $x = f(s)$ , the marking map  $f$  lifts to a map  $\tilde{f}$ , which extends, by Proposition 3.2(1), to a homeomorphism at infinity  $\partial\tilde{f}$  representing  $\Phi_p[X, f]$ . The rough idea is that  $\Phi_{\text{at}}[X, f]$  is the conjugate of  $\varphi$  by  $\Phi_p[X, f]$ . As before, we have the representation  $\rho_{[X, f]} = \varphi_X \circ f_*$ . This is related to  $\partial\tilde{f}$  and  $\varphi$  as follows.

**Proposition 4.13** ( $\Phi_{\text{at}}$  in terms of  $\Phi_p$ ). *For any  $[\gamma] \in \pi_1(S, s)$ , we have*

$$\rho_{[X, f]}[\gamma] = (\partial\tilde{f}) \circ \varphi[\gamma] \circ (\partial\tilde{f})^{-1} \quad (4.4)$$

*In general, if  $\Phi_p[X, f]$  is represented by the homeomorphism  $F$ , then  $\Phi_{\text{at}}[X, f]$  is represented by the representation  $R$  which satisfies, for all  $[\gamma] \in \pi_1(S, s)$ ,*

$$R[\gamma] = F \circ \varphi[\gamma] \circ F^{-1} \quad (4.5)$$

*Proof.* We compute  $\rho_{[X, f]}[\gamma] = \varphi_X(f_*[\gamma]) = f_*(\varphi[\gamma]) = (\partial\tilde{f}) \circ \varphi[\gamma] \circ (\partial\tilde{f})^{-1}$  by Proposition 3.2(3). Note that in the second equality, the  $f_*$  on the left hand side is the  $\pi_1$  functor, whereas on the right hand side,  $f_*$  is the isomorphism between the deck groups  $\Gamma$

and  $\Gamma_X$  induced by  $\tilde{f}$ . Any other homeomorphism  $F$  representing  $\Phi_p[X, f]$  is of the form  $\sigma \circ (\partial\tilde{f})$  for some  $\sigma \in \text{PSL}(2, \mathbb{R})$ . Then for all  $[\gamma] \in \pi_1(S)$ , we have  $\sigma \circ \rho_{[X, f]}[\gamma] \circ \sigma^{-1} = \sigma \circ (\partial\tilde{f}) \circ \varphi[\gamma] \circ (\partial\tilde{f})^{-1} \circ \sigma^{-1} = (\sigma \circ (\partial\tilde{f})) \circ \varphi[\gamma] \circ (\sigma \circ (\partial\tilde{f}))^{-1} = F \circ \varphi[\gamma] \circ F^{-1} = R[\gamma]$ . Thus the representation  $R$  is conjugate to the representation  $\rho_{[X, f]}$ , and hence the conjugacy classes of  $\rho_{[X, f]}$  and  $R$  are equal. That is,  $R$  represents  $\Phi_{\text{at}}[X, f]$  and satisfies equation (4.5).  $\square$

We can also compute  $\Phi_p$  in terms of  $\Phi_{\text{at}}$ , by computing the image of the homeomorphism at infinity on the sinks of hyperbolic elements. Call an element  $[\gamma] \in \pi_1(S, s)$  *hyperbolic* if  $\varphi[\gamma]$  is a hyperbolic element of  $\text{PSL}(2, \mathbb{R})$ . In this case,  $R[\gamma] = f_*(\varphi[\gamma])$  is also hyperbolic by Lemma 3.5 and the fact that conjugates in  $\text{PSL}(2, \mathbb{R})$  have the same type. Further, a triple  $([\gamma_1], [\gamma_2], [\gamma_3])$  of hyperbolic elements in  $\pi_1(S, s)$  is *positively oriented* if the triple  $((\varphi[\gamma_1])_\infty, (\varphi[\gamma_2])_\infty, (\varphi[\gamma_3])_\infty)$  consists of distinct points in  $S^1$  and is positively oriented. In this case, the triple  $((R[\gamma_1])_\infty, (R[\gamma_2])_\infty, (R[\gamma_3])_\infty)$  is also positively oriented by equation (4.5). Thus the notions of hyperbolicity and positively oriented triple do not depend on the choice of the universal cover  $p$ , as long as its deck group  $\Gamma$  is of the first kind. The rough idea is that  $\Phi_p[X, f]$  maps sinks of the  $\varphi$  representation of hyperbolic elements of  $\pi_1(S, s)$  to the sinks of their  $\Phi_{\text{at}}[X, f]$  representation.

**Proposition 4.14** ( $\Phi_p$  in terms of  $\Phi_{\text{at}}$ ). *For any hyperbolic  $[\gamma] \in \pi_1(S, s)$ , we have*

$$\partial\tilde{f}((\varphi[\gamma])_\infty) = (\rho_{[X, f]}[\gamma])_\infty \quad (4.6)$$

*If general, if  $\Phi_{\text{at}}[X, f]$  is represented by the representation  $R$ , then  $\Phi_p[X, f]$  is represented by the homeomorphism  $F$  which satisfies, for all  $[\gamma] \in \pi_1(S, s)$ ,*

$$F((\varphi[\gamma])_\infty) = (R[\gamma])_\infty \quad (4.7)$$

*Proof.* To establish equation (4.6), we use Definition 3.6 and compute that for hyperbolic  $[\gamma] \in \pi_1(S, s)$ ,  $\partial\tilde{f}((\varphi[\gamma])_\infty) = (f_*(\varphi[\gamma]))_\infty = (\varphi_X(f_*[\gamma]))_\infty = (\rho_{[X, f]}[\gamma])_\infty$ . Note that in the second equality, the  $f_*$  on the left hand side is the isomorphism between the deck group  $\Gamma$

and  $\Gamma_X$  induced by  $\tilde{f}$ , whereas on the right hand side,  $f_*$  is the  $\pi_1$  functor. In the general case, since  $\Phi_{\text{at}}[X, f]$  is represented by  $R$ , it follows that  $R$  is conjugate to the representation  $\rho_{[X, f]}$  by some  $\sigma \in \text{PSL}(2, \mathbb{R})$ . That is, for every hyperbolic  $[\gamma] \in \pi_1(S, s)$ , we have  $R[\gamma] = \sigma \circ \rho_{[X, f]}[\gamma] \circ \sigma^{-1}$ . Therefore the sink  $(R[\gamma])_\infty$  is just the image under  $\sigma$  of the sink  $(\rho_{[X, f]}[\gamma])_\infty$ . In other words,  $(R[\gamma])_\infty = \sigma((\rho_{[X, f]}[\gamma])_\infty)$ , which equals, by equation (4.6),  $\sigma \circ (\partial \tilde{f})((\varphi[\gamma])_\infty)$ . Setting  $F = \sigma \circ (\partial \tilde{f})$ , we see that  $F$  also represents the same right coset of  $\text{PSL}(2, \mathbb{R})$  in  $\text{Homeo}^+(S^1)$  as that of  $(\partial \tilde{f})$ , and hence represents  $\Phi_p[X, f]$ . Thus we have established equation (4.7).  $\square$

**Theorem 4.15** (Topology via the conjugacy classes of  $\text{PSL}(2, \mathbb{R})$ -representations). *The function  $\Phi_{\text{at}}: \mathcal{T}(S) \rightarrow X(\pi_1(S, s), \text{PSL}(2, \mathbb{R}))$  is a topological embedding.*

*Proof.* We need to show that a sequence of marked hyperbolic structures  $[X_i, f_i]$  converges to  $[X, f]$  in  $\mathcal{T}(S)$  as  $i \rightarrow \infty$  if and only if the corresponding sequence of conjugacy classes  $\Phi_{\text{at}}[X_i, f_i]$  converges to  $\Phi_{\text{at}}[X, f]$  in  $X(\pi_1(S, s), \text{PSL}(2, \mathbb{R}))$  as  $i \rightarrow \infty$ . As before, fix a universal over  $p: \mathbb{H}^2 \rightarrow S$  with deck group  $\Gamma \subset \text{PSL}(2, \mathbb{R})$ , a torsion-free Fuchsian group of the first kind. By Definition 4.9,  $\Phi_p$  is a homeomorphism and so the convergence  $[X_i, f_i] \rightarrow [X, f]$  in  $\mathcal{T}(S)$  as  $i \rightarrow \infty$  is equivalent to the convergence  $\Phi_p[X_i, f_i] \rightarrow \Phi_p[X, f]$  in  $\mathcal{T}(p)$  as  $i \rightarrow \infty$ .

We begin with the ‘only if’ part. Suppose  $\Phi_p[X_i, f_i] \rightarrow \Phi_p[X, f]$  in  $\mathcal{T}(p)$  as  $i \rightarrow \infty$ . We can promote this to convergence in  $\tilde{\mathcal{T}}(p)$  by constructing a continuous section  $\Sigma_1: \mathcal{T}(p) \rightarrow \tilde{\mathcal{T}}(p)$ . For any  $[\hat{F}] \in \mathcal{T}(p)$ , let  $\Sigma_1([\hat{F}])$  be the unique homeomorphism of  $S^1$  in the right coset  $\text{PSL}(2, \mathbb{R}) \circ \hat{F}$  that fixes the three points  $0, 1, \infty$ . In fact,  $\Sigma_1([\hat{F}]) = M(\hat{F}(0), \hat{F}(1), \hat{F}(\infty))^{-1} \circ \hat{F}$ , where  $M(a, b, c)$  is the Möbius transformation mapping the triple  $(0, 1, \infty)$  to the triple  $(a, b, c)$ . Since  $M(a, b, c)$  is a continuous function of  $a, b, c \in S^1$ , evaluations at  $0, 1, \infty$  are continuous functions of  $\hat{F}$ , and compositions are continuous, we conclude that  $\Sigma_1$  is a continuous function. Therefore  $\Sigma_1(\Phi_p[X_i, f_i]) \rightarrow \Sigma_1(\Phi_p[X, f])$  as  $i \rightarrow \infty$ . Denoting  $\Sigma_1(\Phi_p[X_i, f_i])$  by  $F_i$  and  $\Sigma_1(\Phi_p[X, f])$  by  $F$ , we have  $F_i \rightarrow F$  in  $\tilde{\mathcal{T}}(p)$  as  $i \rightarrow \infty$ .

Note that, by Proposition 4.13,  $\Phi_{\text{at}}[X, f]$  is represented by the representation  $\rho$  which satisfies, for each  $[\gamma] \in \pi_1(S, s)$ , the relation  $\rho[\gamma] = F \circ \varphi[\gamma] \circ F^{-1}$ . Similarly,  $\Phi_{\text{at}}[X_i, f_i]$  is represented by the representation  $\rho_i$  which satisfies, for each  $[\gamma] \in \pi_1(S, s)$ , the relation  $\rho_i[\gamma] = (F_i) \circ \varphi[\gamma] \circ (F_i)^{-1}$ . This  $\rho_i[\gamma]$  converges, as  $i \rightarrow \infty$ , to  $F \circ \varphi[\gamma] \circ F^{-1} = \rho[\gamma]$ . Here we are using the fact that  $\text{PSL}(2, \mathbb{R})$  is an embedded subspace of  $\text{Homeo}^+(S^1)$ . On the level of representations, this means  $\rho_i \rightarrow \rho$  and on the level of conjugacy classes,  $[\rho_i] \rightarrow [\rho]$  as  $i \rightarrow \infty$ . In other words,  $\Phi_{\text{at}}[X_i, f_i] \rightarrow \Phi_{\text{at}}[X, f]$  in  $X(\pi_1(S, s), \text{PSL}(2, \mathbb{R}))$  as  $i \rightarrow \infty$  and therefore  $\Phi_{\text{at}}$  is continuous.

Next we prove the ‘if’ part. Suppose  $\Phi_{\text{at}}[X_i, f_i] \rightarrow \Phi_{\text{at}}[X, f]$  in  $X(\pi_1(S, s), \text{PSL}(2, \mathbb{R}))$  as  $i \rightarrow \infty$ . We construct a continuous section  $\Sigma_2: \Phi_{\text{at}}(\mathcal{T}(S)) \rightarrow \text{Hom}(\pi_1(S, s), \text{PSL}(2, \mathbb{R}))$  in order to promote the convergence of conjugacy classes of representations to convergence of representations on the nose. Let  $(\alpha_1, \alpha_2, \alpha_3)$  be a positively oriented triple of hyperbolic elements of  $\pi_1(S, s)$ , which exists since the set  $\Gamma_\infty$  is dense in  $S^1$  due to Proposition 3.1. Now, for any  $[\hat{\rho}] \in \Phi_{\text{at}}(\mathcal{T}(S))$ , let  $\Sigma_2[\hat{\rho}]$  be the unique representation that represents the same conjugacy class as  $\hat{\rho}$  and such that the sinks of  $\Sigma_2[\hat{\rho}](\alpha_1), \Sigma_2[\hat{\rho}](\alpha_2), \Sigma_2[\hat{\rho}](\alpha_3)$  are  $0, 1, \infty$  respectively. In fact  $\Sigma_2[\hat{\rho}]$  is the conjugate of  $\hat{\rho}$  by the Möbius transformation  $M((\hat{\rho}(\alpha_1))_\infty, (\hat{\rho}(\alpha_2))_\infty, (\hat{\rho}(\alpha_3))_\infty)^{-1}$ . Since  $M(a, b, c)$  is a continuous function of  $a, b, c \in S^1$ , evaluations at  $\alpha_1, \alpha_2, \alpha_3 \in \pi_1(S, s)$  are continuous functions of the representation  $\hat{\rho}$ , the sink of a hyperbolic element is a continuous function of the hyperbolic element, and compositions are continuous, we conclude that  $\Sigma_2$  is continuous. Therefore  $\Sigma_2(\Phi_{\text{at}}[X_i, f_i]) \rightarrow \Sigma_2(\Phi_{\text{at}}[X, f])$  as  $i \rightarrow \infty$ . Denoting  $\Sigma_2(\Phi_{\text{at}}[X_i, f_i])$  by  $\rho_i$  and  $\Sigma_2(\Phi_{\text{at}}[X, f])$  by  $\rho$ , we have  $\rho_i \rightarrow \rho$  in  $\text{Hom}(\pi_1(S, s), \text{PSL}(2, \mathbb{R}))$  as  $i \rightarrow \infty$ . In particular, this means that for all hyperbolic elements  $[\gamma] \in \pi_1(S, s)$ , we have  $\rho_i[\gamma] \rightarrow \rho[\gamma]$ , and so  $(\rho_i[\gamma])_\infty \rightarrow (\rho[\gamma])_\infty$  as  $i \rightarrow \infty$ .

Note that, by Proposition 4.14,  $\Phi_p[X, f]$  is represented by the homeomorphism  $F$  of  $S^1$ , which satisfies, for every hyperbolic  $[\gamma] \in \pi_1(S, s)$ , the relation  $F((\varphi[\gamma])_\infty) = (\rho[\gamma])_\infty$ . Similarly  $\Phi_p[X_i, f_i]$  is represented by the homeomorphism  $F_i$  of  $S^1$  which satisfies, for ev-

ery hyperbolic  $[\gamma] \in \pi_1(S, s)$ , the relation  $F_i((\varphi[\gamma])_\infty) = (\rho_i[\gamma])_\infty$ . To show that  $[X_i, f_i]$  converges to  $[X, f]$ , it is enough to prove that  $F_i$  converges to  $F$  as  $i \rightarrow \infty$ , in the topology of uniform convergence on  $S^1$  with respect to some path metric  $d$ , such as the visual metric based at  $\tilde{s} \in \mathbb{H}^2$ . This would imply that as  $i \rightarrow \infty$ ,  $\Phi_p[X_i, f_i] = [F_i]$  converges to  $\Phi_p[X, f] = [F]$ . Let  $\varepsilon > 0$  be given. Reducing  $\varepsilon$  if necessary, assume  $\varepsilon$  is less than half the  $d$ -length of  $S^1$ . We choose a finite set of points in  $(\Gamma_X)_\infty$  which is  $\frac{\varepsilon}{12}$ -dense in  $S^1$ . This is possible because  $S^1$  is compact and  $(\Gamma_X)_\infty$  is dense in  $S^1$  due to Proposition 3.1. Represent these points as sinks  $(\rho[\gamma_1])_\infty, (\rho[\gamma_2])_\infty, \dots, (\rho[\gamma_n])_\infty$  of holonomies around hyperbolic  $[\gamma_1], [\gamma_2], \dots, [\gamma_n] \in \pi_1(S, s)$ . Relabelling if necessary, and removing duplicates, assume that  $(\rho[\gamma_1])_\infty, (\rho[\gamma_2])_\infty, \dots, (\rho[\gamma_n])_\infty$  are distinct and in positive circular order. These points divide the circle at infinity into  $n$  intervals, each of length less than  $\frac{\varepsilon}{6}$ . Now for each  $j = 1, 2, \dots, n$ , we have  $(\varphi[\gamma_j])_\infty = F^{-1}(\rho[\gamma_j])_\infty$ . As  $F^{-1}$  is an orientation preserving homeomorphism, it follows that the sinks  $(\varphi[\gamma_1])_\infty, (\varphi[\gamma_2])_\infty, \dots, (\varphi[\gamma_n])_\infty$  are in positive circular order. Similarly for each  $i$  and each  $j = 1, 2, \dots, n$ , we have  $(\rho_i[\gamma_j])_\infty = F_i((\varphi[\gamma_j])_\infty)$ , and since  $F_i$  is an orientation preserving homeomorphism, it follows that the points  $(\rho_i[\gamma_1])_\infty, (\rho_i[\gamma_2])_\infty, \dots, (\rho_i[\gamma_n])_\infty$  are also in positive circular order. Since the sink of a hyperbolic element is a continuous function of the hyperbolic element, for each  $j = 1, 2, \dots, n$ , the convergence  $\rho_i[\gamma_j] \rightarrow \rho[\gamma_j]$  implies  $(\rho_i[\gamma_j])_\infty \rightarrow (\rho[\gamma_j])_\infty$  as  $i \rightarrow \infty$ . Therefore for all sufficiently large  $i$ , namely those larger than some index  $N$ , and for each of the finitely many indices  $j = 1, 2, \dots, n$ , we have  $d((\rho_i[\gamma_j])_\infty, (\rho[\gamma_j])_\infty) < \frac{\varepsilon}{6}$ . Further, for  $i > N$  and  $j = 1, 2, \dots, n$ , we have  $d((\rho_i[\gamma_j])_\infty, (\rho_i[\gamma_{j+1}])_\infty) \leq d((\rho_i[\gamma_j])_\infty, (\rho[\gamma_j])_\infty) + d((\rho[\gamma_j])_\infty, (\rho[\gamma_{j+1}])_\infty) + d((\rho[\gamma_{j+1}])_\infty, (\rho_i[\gamma_{j+1}])_\infty) < \frac{\varepsilon}{6} + \frac{\varepsilon}{6} + \frac{\varepsilon}{6} = \frac{\varepsilon}{2}$  (subscripts of the letter  $\gamma$  being modulo  $n$ ). Thus for  $i > N$ , the points  $(\rho_i[\gamma_1])_\infty, (\rho_i[\gamma_2])_\infty, \dots, (\rho_i[\gamma_n])_\infty$  divide  $S^1$  into intervals of length less than  $\frac{\varepsilon}{2}$ .

Suppose  $q \in S^1$  is an arbitrary point and  $i > N$  be an arbitrary index. Then for some  $j$ ,  $q$  belongs to an interval bounded by points  $(\varphi[\gamma_j])_\infty, (\varphi[\gamma_{j+1}])_\infty$ . Since  $F_i$  is an orientation preserving homeomorphism, it follows that  $F_i(q)$  is in the interval bounded by the points

$(\rho_i[\gamma_j])_\infty$  and  $(\rho_i[\gamma_{j+1}])_\infty$ . Therefore  $d(F_i(q), (\rho_i[\gamma_j])_\infty)$  is less than the length of this interval, which is less than  $\frac{\varepsilon}{2}$ . Similarly,  $F$  is an orientation preserving homeomorphism, so  $F(q)$  lies in the interval bounded by the points  $(\rho[\gamma_j])_\infty$  and  $(\rho[\gamma_{j+1}])_\infty$ , and hence  $d(F(q), (\rho[\gamma_j])_\infty)$  is less than the length of this interval, which is less than  $\frac{\varepsilon}{6}$ . Now we have  $d(F_i(q), F(q)) \leq d(F_i(q), (\rho_i[\gamma_j])_\infty) + d((\rho_i[\gamma_j])_\infty, (\rho[\gamma_j])_\infty) + d((\rho[\gamma_j])_\infty, F(q)) < \frac{\varepsilon}{2} + \frac{\varepsilon}{6} + \frac{\varepsilon}{6} < \varepsilon$ . Since  $q \in S^1$  was arbitrary, for every  $i > N$ , the maps  $F_i$  and  $F$  are uniformly  $\varepsilon$ -close. Since  $\varepsilon$  was an arbitrary positive number, we infer that  $F_i \rightarrow F$  as  $i \rightarrow \infty$  in the topology of uniform convergence. This concludes the ‘if’ part, that is,  $\Phi_{\text{at}}^{-1}: \Phi_{\text{at}}(\mathcal{T}(S)) \rightarrow \mathcal{T}(S)$  is continuous. Hence  $\Phi_{\text{at}}$  is an embedding.  $\square$

**Corollary 4.16.** *If  $S$  is a surface of finite type of negative Euler characteristic, then the topology on  $\mathcal{T}(S)$  agrees with the topology on Teichmüller space.*

*Proof.* Indeed, one of the ways to describe the Teichmüller space is as the space of conjugacy classes of all discrete faithful representations finite covolume of  $\pi_1(S, s)$  into  $\text{PSL}(2, \mathbb{R})$  ([FM11, Chapter 10]). In particular, the topology on the Teichmüller space is defined as a subspace of the full space of representation conjugacy classes  $X(\pi_1(S, s), \text{PSL}(2, \mathbb{R}))$ . But this is exactly the topology on  $\mathcal{T}(S)$  that we have defined above. Thus  $\mathcal{T}(S)$  reduces to the usual Teichmüller space in case  $S$  is a surface finite type.  $\square$

## 4.4 Topology via the Marked Length Spectrum

Let  $\mathcal{C}$  be the set of homotopy classes of *essential* (not nullhomotopic) and *nonperipheral* (not homotopic into a neighbourhood of a puncture) unoriented closed curves on  $S$ , and  $\mathcal{S} \subset \mathcal{C}$  be the set of classes represented by simple curves. Then  $\mathcal{C}$  is identified with the set of conjugacy classes of nontrivial cyclic subgroups of  $\pi_1(S)$  over which  $\pi_1(S)$  splits nontrivially. The Marked Length Spectrum (MLS) is a function which measures the length of the geodesic

representatives of each curve. In this section, we use it to describe the topology of the marked moduli space by proving that it is an embedding. This embedding is the start of Thurston's work on surfaces and the construction of the Thurston boundary of Teichmüller space, for surfaces of finite type.

Suppose  $\alpha \in \mathcal{C}$  and  $[X, f] \in \mathcal{T}(S)$ . Then the holonomy around  $f(\alpha)$  is a hyperbolic element of  $\mathrm{PSL}(2, \mathbb{R})$ , and so the free homotopy class  $f(\alpha)$  contains a unique closed geodesic on  $X$ , whose length is denoted as  $\ell_{f(\alpha)}(X)$ .

**Definition 4.17** (Marked Length Spectrum (MLS), Simple MLS). *The Marked Length Spectrum is the function  $\ell_*: \mathcal{T}(S) \rightarrow \mathbb{R}_{>0}^{\mathcal{C}}$  defined as follows. For every  $\alpha \in \mathcal{C}$  and  $[X, f] \in \mathcal{T}(S)$ ,  $\ell_\alpha([X, f])$  is the length of the geodesic representative of  $f(\alpha)$  on  $X$ . The restriction of the Marked Length Spectrum to any subset of  $\mathcal{C}$  is also denoted by  $\ell_*$ , such as the Simple Marked Length Spectrum  $\ell_*: \mathcal{T}(S) \rightarrow \mathbb{R}_{>0}^{\mathcal{S}}$ .*

The Marked Length Spectrum is closely related to the Marked Absolute Trace Spectrum, which measures the absolute values of the traces of holonomies around each curve. Suppose  $\alpha \in \mathcal{C}$  and  $[X, f] \in \mathcal{T}(S)$ . Then  $\alpha$  is the conjugacy class of some nontrivial cyclic subgroup of  $\pi_1(S)$ , whose generator we denote by  $\gamma$ . The marked hyperbolic structure  $[X, f]$  has a holonomy representation  $\rho_{[X, f]}: \pi_1(S) \rightarrow \mathrm{PSL}(2, \mathbb{R})$ , which is defined only up to conjugation. Therefore  $\rho_{[X, f]}(\gamma) \in \mathrm{PSL}(2, \mathbb{R})$  is well defined up to conjugacy. But since the trace of a matrix is conjugacy invariant,  $|\mathrm{tr} \rho_{[X, f]}(\gamma)|$  is well defined. Since the holonomy  $\rho_{[X, f]}(\gamma)$  around  $f(\alpha)$  on  $X$  is a hyperbolic element of  $\mathrm{PSL}(2, \mathbb{R})$ , this absolute trace is greater than 2.

**Definition 4.18** (Marked Absolute Trace Spectrum). *The Marked Absolute Trace Spectrum is the function  $\tau_*: \mathcal{T}(S) \rightarrow \mathbb{R}_{>2}^{\mathcal{C}}$  defined as follows. For every  $\alpha \in \mathcal{C}$  and  $[X, f] \in \mathcal{T}(S)$ ,  $\tau_\alpha([X, f])$  is the absolute value of the trace of the holonomy around  $f(\alpha)$  on  $X$ . The restriction of the marked absolute trace spectrum to any subset of  $\mathcal{C}$  is also denoted by  $\tau_*$ .*

**Proposition 4.19.** *The marked length spectrum and the marked absolute trace spectrum satisfy the relation*

$$\ell_* = 2 \cosh^{-1} \tau_* \text{ and hence } \tau_* = \cosh(\ell_*/2) \quad (4.8)$$

Thus  $\ell_*$  is a topological embedding if and only if  $\tau_*$  is.

*Proof.* For any  $\alpha \in \mathcal{C}$  and  $[X, f] \in \mathcal{T}(S)$ , the length  $\ell_\alpha([X, f])$  equals the translation distance of holonomy around  $f(\alpha)$  on  $X$ , that is, the hyperbolic element  $\rho_{[X, f]}(\gamma) \in \mathrm{PSL}(2, \mathbb{R})$ . Thus  $\ell_\alpha([X, f]) = 2 \cosh^{-1}(|\mathrm{tr} \rho_{[X, f]}(\gamma)|) = 2 \cosh^{-1}(\tau_\alpha([X, f]))$ .

Since  $\cosh: \mathbb{R}_{>0} \rightarrow \mathbb{R}_{>2}$  is a homeomorphism,  $\ell_*$  is continuous if and only if  $\tau_*$  is. Moreover, for any sequence  $\mathcal{X}_i$  in  $\mathcal{T}(S)$ ,  $\ell_*(\mathcal{X}_i)$  converges if and only if  $\tau_*(\mathcal{X}_i)$  converges. Thus  $\ell_*$  is a topological embedding if and only if  $\tau_*$  is.  $\square$

If  $\Sigma = \Sigma_{g,p}^b$  is the surface with boundary of finite type with genus  $g$ ,  $p$  punctures, and  $b$  boundary components, then  $T_{g,p}^b$  denotes the reduced Teichmüller space consisting of marked hyperbolic structures on  $\Sigma_{g,p}^b$  with  $p$  cusped punctures and  $b$  geodesic boundary components. The topology of  $T_{g,p}^b$  is defined using the space of conjugacy classes of  $\mathrm{PSL}(2, \mathbb{R})$ -representations of  $\pi_1(\Sigma_{g,p}^b)$ , just as in Section 4.3. The sets of homotopy classes of curves  $\mathcal{C}(\Sigma)$  and  $\mathcal{S}(\Sigma)$  are defined similarly to  $\mathcal{C}(S)$  and  $\mathcal{S}(S)$ , but note that they include the boundary curves and their powers as well. By work of Keen ([Kee66]) and Luo ([Luo98, Corollary]) there is a finite subset  $F \subset \mathcal{S}(\Sigma)$  such that the restriction  $\tau_*: T_{g,p}^b \rightarrow \mathbb{R}_{>2}^F$  is an embedding (in fact a real analytic embedding). In view of Proposition 4.19,  $\ell_*: T_{g,p}^b \rightarrow \mathbb{R}_{>0}^F$  is also an embedding. It follows that  $\tau_*: T_{g,p}^b \rightarrow \mathbb{R}_{>2}^{\mathcal{S}(\Sigma)}$ ,  $\ell_*: T_{g,p}^b \rightarrow \mathbb{R}_{>0}^{\mathcal{S}(\Sigma)}$ ,  $\tau_*: T_{g,p}^b \rightarrow \mathbb{R}_{>2}^{\mathcal{C}(\Sigma)}$  and  $\ell_*: T_{g,p}^b \rightarrow \mathbb{R}_{>0}^{\mathcal{C}(\Sigma)}$  are also embeddings. We prove the same for  $\mathcal{T}(S)$  for any surface  $S$ , whether of finite or infinite type.

**Theorem 4.20** (Topology via the marked length spectrum). *The Marked Length Spectrum  $\ell_*: \mathcal{T}(S) \rightarrow \mathbb{R}_{>0}^{\mathcal{S}}$  is a topological embedding, and so is  $\ell_*: \mathcal{T}(S) \rightarrow \mathbb{R}_{>0}^{\mathcal{C}}$ .*

*Proof.* To show that  $\ell_*$  is continuous, we need to show that for every curve  $\alpha$  (simple or not),  $\ell_\alpha: \mathcal{T}(S) \rightarrow \mathbb{R}_{>0}$  is continuous. Now  $\alpha$  is the conjugacy class of some nontrivial cyclic subgroup of  $\pi_1(S)$ , whose generator we denote by  $\gamma$ . The marked hyperbolic structure  $[X, f]$  has holonomy representation  $\rho_{[X, f]}: \pi_1(S) \rightarrow \mathrm{PSL}(2, \mathbb{R})$ . We have  $\tau_\alpha([X, f]) = |\mathrm{tr} |(\rho_{[X, f]}(\gamma))|$ . Since the absolute trace is a continuous function on  $\mathrm{PSL}(2, \mathbb{R})$ ,  $\tau_\alpha: \mathcal{T}(S) \rightarrow \mathbb{R}_{>2}$  is continuous. Due to Equation (4.8),  $\ell_\alpha: \mathcal{T}(S) \rightarrow \mathbb{R}_{>0}$  is also continuous. Thus  $\tau_*: \mathcal{T}(S) \rightarrow \mathbb{R}_{>2}^{\mathcal{C}}$  and  $\ell_*: \mathcal{T}(S) \rightarrow \mathbb{R}_{>0}^{\mathcal{C}}$  are continuous, and so are  $\tau_*: \mathcal{T}(S) \rightarrow \mathbb{R}_{>2}^{\mathcal{S}}$  and  $\ell_*: \mathcal{T}(S) \rightarrow \mathbb{R}_{>0}^{\mathcal{S}}$ .

Now we show that  $\ell_*$ , or equivalently,  $\tau_*$ , is an embedding. Note that if  $\Pi: \mathbb{R}_{>0}^{\mathcal{C}} \rightarrow \mathbb{R}_{>0}^{\mathcal{S}}$  is the projection map picking out the coordinates corresponding to the subset  $\mathcal{S} \subset \mathcal{C}$  and discarding the rest, then  $\ell_* = \Pi \circ \ell_*$ , where  $\ell_*$  is the simple MLS on the left side and the full MLS on the right. Therefore it is enough to show that  $\ell_*: \mathcal{T}(S) \rightarrow \mathbb{R}_{>0}^{\mathcal{S}}$  is an embedding. To do so, we use the description of the  $\mathcal{T}(S)$  topology provided by representation space. Pick a positively oriented triple  $(\alpha_1, \alpha_2, \alpha_3)$  of hyperbolic elements in  $\pi_1(S)$ , such that the free homotopy class of each is represented by a simple curve. For  $[X, f] \in \mathcal{T}(S)$ , let  $\rho_{[X, f]}: \pi_1(S) \rightarrow \mathrm{PSL}(2, \mathbb{R})$  be the holonomy representation so that the sinks of these hyperbolic elements are  $\rho_{[X, f]}(\alpha_1)_\infty = 0$ ,  $\rho_{[X, f]}(\alpha_2)_\infty = 1$ , and  $\rho_{[X, f]}(\alpha_3)_\infty = \infty$ . Thus to show that  $\ell_*$  is an embedding, we need to show that if  $\ell_*[X_i, f_i] \rightarrow \ell_*[X, f]$ , then for every  $\gamma \in \pi_1(S)$ ,  $\rho_{[X_i, f_i]}(\gamma) \rightarrow \rho_{[X, f]}(\gamma)$  in  $\mathrm{PSL}(2, \mathbb{R})$ .

Suppose such a sequence  $[X_i, f_i]$ , its simple MLS limit  $[X, f]$ , and a  $\gamma \in \pi_1(S)$  are given. Pick any subsurface with boundary of finite type  $\Sigma = \Sigma_{g,p}^b \subset S$  which is *essential* ( $\Sigma \subset S$  is a closed subspace, and the boundary components of  $\Sigma$  are not nullhomotopic and are not homotopic into punctures) and contains representatives of  $\gamma, \alpha_1, \alpha_2, \alpha_3 \in \pi_1(S)$  (based at the basepoint of the fundamental group which we are suppressing in our notation). Then the inclusion of  $\Sigma$  in  $S$  is injective on fundamental groups, that is,  $\pi_1(\Sigma)$  is a subgroup of  $\pi_1(S)$ . For any  $[X', f'] \in \mathcal{T}(S)$ , let  $\Sigma^{f'}$  denote the metrically complete subsurface with geodesic boundary of  $X'$  isotopic to  $f'(\Sigma)$ . Then  $[\Sigma^{f'}, f'] \in T_{g,p}^b$  is a marked hyperbolic

structure with geodesic boundary and cusp ends, since all punctures of the Nielsen-convex hyperbolic surface  $X'$  have cusp geometry. Moreover, the holonomy representation  $\rho_{[\Sigma^{f'}, f']}$  is the restriction  $\rho_{[X', f']}|_{\pi_1(\Sigma)}$  of the holonomy representation of  $[X', f']$  to the subsurface  $\Sigma$ . In particular, the traces are equal as well and  $\ell_\alpha[\Sigma^{f'}, f'] = \ell_\alpha[X', f']$ .

Now let  $F \subset \mathcal{S}(\Sigma)$  be the finite set of simple curves as above. Since  $\mathcal{S}(\Sigma) \subset \mathcal{S}(S)$ , we have  $\ell_\alpha[\Sigma^{f_i}, f_i] = \ell_\alpha[X_i, f_i] \rightarrow \ell_\alpha[X, f] = \ell_\alpha[\Sigma^f, f]$  for every  $\alpha \in F$ . Since  $\ell_*: T_{g,p}^b \rightarrow \mathbb{R}_{>0}^F$  is an embedding, we conclude that  $[\Sigma^{f_i}, f_i] \rightarrow [\Sigma^f, f]$  in  $T_{g,p}^b$ . Therefore  $\rho_{[\Sigma^{f_i}, f_i]} \rightarrow \rho_{[\Sigma^f, f]}$  as representations. In particular,  $\rho_{[X_i, f_i]}(\gamma) = \rho_{[\Sigma^{f_i}, f_i]}(\gamma) \rightarrow \rho_{[\Sigma^f, f]}(\gamma) = \rho_{[X, f]}(\gamma)$ . Since  $\gamma \in \pi_1(S)$  was arbitrary, we conclude that  $[X_i, f_i] \rightarrow [X, f]$  in  $\mathcal{T}(S)$  as  $i \rightarrow \infty$ . Therefore  $\ell_*: \mathcal{T}(S) \rightarrow \mathbb{R}_{>0}^S$  is an embedding.  $\square$

By Proposition 4.19, we have the topology via the marked absolute trace spectrum.

**Corollary 4.21** (Topology via the marked absolute trace spectrum). *The Marked Absolute Trace Spectra  $\tau_*: \mathcal{T}(S) \rightarrow \mathbb{R}_{>2}^S$  and  $\tau_*: \mathcal{T}(S) \rightarrow \mathbb{R}_{>2}^C$  are topological embeddings.*

In fact, Luo's results go further, and characterise the image  $\tau_*(\mathcal{T}(S))$  as a closed subset of  $\mathbb{R}_{>2}^S$ . For  $\Sigma = \Sigma_{g,p}^b$ , define  $\mathcal{S}'(\Sigma)$  to be the set of homotopy classes of unoriented closed curves on  $\Sigma$  which are essential, nonperipheral and not *boundary parallel* (not homotopic into a neighbourhood of the boundary). A *coordinate function* on the product space  $\mathbb{R}_{>2}^{\mathcal{S}'(\Sigma)}$  is the projection onto any of the  $\mathbb{R}_{>2}$ 's in the product. A *polynomial function* on  $\mathbb{R}_{>2}^{\mathcal{S}'(\Sigma)}$  is a function given by a polynomial in some finite set of its coordinate functions. Note that polynomial functions are continuous. We recall the two main theorems of [Luo98]. To do so, we recall the notion of a surface of complexity one and fix some notation. A surface of *complexity one* is a  $\Sigma_{g,p}^b$  for the following set of invariants:

$$(g, p, b) \in I = \{(0, 0, 4), (0, 1, 3), (0, 2, 2), (0, 3, 1), (0, 4, 0), (1, 0, 1), (1, 1, 0)\} \quad (4.9)$$

The following theorem characterises the marked absolute trace spectra of surfaces of complexity one.

**Theorem 4.22** ([Luo98, Theorem 2]). *For a surface of complexity one, that is,  $\Sigma = \Sigma_{g,p}^b$ , where  $(g, p, b) \in I$  as in Equation (4.9), there exists a set  $\mathcal{P}(\Sigma)$  of polynomial functions on  $\mathbb{R}_{>2}^{\mathcal{S}'(\Sigma)}$  whose zero set is precisely the set of marked absolute trace spectra of marked hyperbolic structures in  $T_{g,p}^b$ . That is, a real valued function of the set of curves  $t \in \mathbb{R}_{>2}^{\mathcal{S}'(\Sigma)}$  is the marked absolute trace spectrum  $\tau_*(\mathcal{X})$  for a marked hyperbolic structure  $\mathcal{X} \in T_{g,p}^b$  if and only if  $p(t) = 0$  for every  $p \in \mathcal{P}(\Sigma)$ .*

The set of polynomials  $\mathcal{P}(\Sigma)$  is spelled out more explicitly in [Luo98], but we do not require the explicit polynomials. We require only the following corollary.

**Corollary 4.23.** *For a surface of complexity one, that is,  $\Sigma = \Sigma_{g,p}^b$ , where  $(g, p, b) \in I$  as in Equation (4.9), the image  $\tau_*(T_{g,p}^b)$  is a closed subset of  $\mathbb{R}_{>2}^{\mathcal{S}'(\Sigma)}$  and the image  $\ell_*(T_{g,p}^b)$  is a closed subset of  $\mathbb{R}_{>0}^{\mathcal{S}'(\Sigma)}$ .*

*Proof.* The image of  $\tau_*$  is characterised by Theorem 4.22 as intersection of the preimages of  $\{0\}$  by a set  $\mathcal{P}$  of continuous real valued functions. By Proposition 4.19, the same is true for  $\ell_*$ . □

An inclusion  $\iota: \Sigma \hookrightarrow S$  as an essential subsurface induces an inclusion of the set of curves  $\iota: \mathcal{S}'(\Sigma) \hookrightarrow \mathcal{S}$  and therefore a (continuous) projection map  $\Pi_\iota: \mathbb{R}_{>0}^{\mathcal{S}} \rightarrow \mathbb{R}_{>0}^{\mathcal{S}'(\Sigma)}$  picking out the coordinates corresponding to the subset  $\iota(\mathcal{S}'(\Sigma)) \subset \mathcal{S}$  and discarding the rest. The marked length spectrum of a marked hyperbolic structure on  $S$  always projects under  $\Pi_\iota$  to the marked length spectrum of a marked hyperbolic structure on  $\Sigma$ , which is just the restriction of the same marked hyperbolic structure to  $\Sigma$ . That is, for any  $\mathcal{X} \in \mathcal{T}(S)$  we have a restriction  $\iota^*(\mathcal{X}) \in T_{g,p}^b$  such that  $\ell_*(\iota^*(\mathcal{X})) = \Pi_\iota(\ell_*(\mathcal{X}))$ . The following theorem says that the converse is also true.

**Theorem 4.24** ([Luo98, Theorem 1]). *Suppose  $S$  is a surface of negative Euler characteristic, of finite type or infinite type. A function  $\mathcal{L}: \mathcal{S} \rightarrow \mathbb{R}_{>0}$  is the marked length spectrum of*

a marked hyperbolic structure on  $S$  if and only if for every surface of complexity one  $\Sigma$  and an inclusion  $\iota: \Sigma \hookrightarrow S$  as an essential subsurface, the restriction  $\Pi_\iota(\mathcal{L})$  is the marked length spectrum of a marked hyperbolic structure on  $\Sigma$ .

**Corollary 4.25.** *The image  $\ell_*(\mathcal{T}(S))$  is a closed subset of  $\mathbb{R}_{>0}^S$  and the image  $\tau_*(\mathcal{T}(S))$  is a closed subset of  $\mathbb{R}_{>2}^S$ .*

*Proof.*  $\ell_*(\mathcal{T}(S))$  is the intersection of preimages of closed subsets of  $\mathbb{R}_{>0}^{\mathcal{S}'(\Sigma)}$  under continuous functions  $\Pi_\iota$ , and therefore is closed. By Proposition 4.19, the same is true for  $\tau_*$ .  $\square$

**Corollary 4.26** (Topology is completely metrisable).  *$\mathcal{T}(S)$  is completely metrisable.*

*Proof.* The space  $\mathbb{R}_{>0}^S$  is a countable product of the completely metrisable space  $\mathbb{R}_{>0} \cong \mathbb{R}$ , and is therefore completely metrisable. Hence so is its closed subspace  $\ell_*(\mathcal{T}(S))$ . But since  $\ell_*$  is an embedding of  $\mathcal{T}(S)$  into  $\mathbb{R}_{>0}^S$ , it is a homeomorphism of  $\mathcal{T}(S)$  onto its image. Therefore  $\mathcal{T}(S)$  is completely metrisable.  $\square$

## 4.5 Topology via the Space of Nielsen-convex Beltrami Forms

In this section, we describe the topology of the marked moduli space as the quotient of the space of Nielsen-convex Beltrami forms by the action of the group of locally quasiconformal homeomorphisms isotopic to the identity by the pullback operation. We defer most of the proof to Chapter 6. This is exactly analogous to the quasiconformal Teichmüller space being the quotient of the unit ball in the Banach space of bounded Beltrami forms as in Section 2.8.

First, let  $X_0$  be a Riemann surface which is, when uniformised as a complete hyperbolic

surface, Nielsen-convex, and let  $f_0 : S \rightarrow X_0$  be a marking homeomorphism. We collect all the Beltrami forms on  $X_0$  whose integration is Nielsen-convex into a set.

**Definition 4.27** (Nielsen-convex Beltrami forms). *A Beltrami form  $\mu \in \text{Bel}(X_0)$  is said to be Nielsen-convex if its integration  $X_\mu$  is (when uniformised as a complete hyperbolic surface) a Nielsen-convex Riemann surface.  $\text{Bel}_{\text{nc}}(X_0)$  is the subspace of  $\text{bel}(X_0)$  consisting of all Nielsen-convex Beltrami forms.*

There is the integration function from the space of Nielsen-convex Beltrami forms to the marked moduli space.

**Definition 4.28** (Integration function).  $\Phi_{\text{nc}}^{X_0, f_0} : \text{Bel}_{\text{nc}}(X_0) \rightarrow \mathcal{T}(S)$  is defined by  $\Phi_{\text{nc}}^{X_0, f_0}(\mu) := [X_\mu, f_0]$ .

We drop the superscript  $X_0, f_0$  if there is no confusion. But before that, we first describe the relation between the integration functions corresponding to the different choices. Suppose  $X'_0$  is also a Nielsen-convex Riemann surface and  $f'_0 : S \rightarrow X'_0$  a marking homeomorphism. Let  $h := f_0 \circ (f'_0)^{-1}$ . Assume that  $h : X'_0 \rightarrow X_0$  is an lqc homeomorphism. The pullback operation by  $h$  gives a homeomorphism from  $\text{Bel}(X_0)$  to  $\text{Bel}(X'_0)$ , and restricts to a homeomorphism from  $\text{Bel}_{\text{nc}}(X_0)$  to  $\text{Bel}_{\text{nc}}(X'_0)$ .

**Proposition 4.29.**  $\Phi_{\text{nc}}^{X_0, f_0} = \Phi_{\text{nc}}^{X'_0, f'_0} \circ h^*$ .

*Proof.* Suppose  $\mu \in \text{Bel}_{\text{nc}}(X_0)$ , so that  $h^*\mu \in \text{Bel}_{\text{nc}}(X'_0)$ . As  $\mu$  is the Beltrami coefficient of  $\text{id}_{X_0 \rightarrow X_\mu} : X_0 \rightarrow X_\mu$ , by definition of the pullback operation,  $h^*\mu$  is the Beltrami coefficient of  $h : X'_0 \rightarrow X_\mu$ . Hence  $h : X'_{h^*\mu} \rightarrow X_\mu$  is a biholomorphism. But  $h = f_0 \circ (f'_0)^{-1}$  is the marking between the two marked Riemann surface structures  $[X_\mu, f_0]$  and  $[X'_{h^*\mu}, f'_0]$ , which are therefore equal. Since  $\mu \in \text{Bel}_{\text{nc}}(X_0)$  is arbitrary,  $\Phi_{\text{nc}}^{X_0, f_0} = \Phi_{\text{nc}}^{X'_0, f'_0} \circ h^*$ .  $\square$

Now fix the choice of  $X_0$  and  $f_0$  and drop the superscripts. The integration function  $\Phi_{\text{nc}}$  is a quotient map and its fibres are orbits of the pullback operation by  $\text{LQC}_0(X_0)$ . We prove

this here as sets. For the proof of the assertion as topological spaces, see Chapter 6 and Theorem 6.9.

**Proposition 4.30.**  $\Phi_{\text{nc}}$  is a set quotient function by the action of the group  $\text{LQC}_0(X_0)$  by the pullback operation.

*Proof.* First, observe that  $\Phi_{\text{nc}}$  is surjective. For any  $[X, f] \in \mathcal{T}(S)$ , the marking  $f \circ f_0^{-1}$  is isotopic to a diffeomorphism  $f': X_0 \rightarrow X$ , which is therefore an lqc homeomorphism. Since  $f': X_{\mu_{f'}} \rightarrow X$  is a biholomorphism and  $X_0$  is Nielsen-convex, so is  $X_{\mu_{f'}}$ . Therefore the Beltrami coefficient  $\mu_{f'}$  is a Nielsen-convex Beltrami form on  $X_0$ . Further, we have  $[X_{\mu_{f'}}, \text{id}_{X_0 \rightarrow X_{\mu_{f'}}} \circ f_0] = [X, f]$  in the marked moduli space, since the biholomorphism  $f': X_{\mu_{f'}} \rightarrow X$  is homotopic to the marking  $f \circ f_0^{-1}$ . Thus  $[X, f] = \Phi_{\text{nc}}(\mu_{f'})$ . Since  $[X, f] \in \mathcal{T}(S)$  is arbitrary,  $\Phi_{\text{nc}}$  is surjective.

If  $\varphi \in \text{LQC}_0(X_0)$  and  $\mu \in \text{Bel}_{\text{nc}}(X_0)$ , then  $\mu$  is the Beltrami coefficient of  $\text{id}_{X_0 \rightarrow X_\mu}: X_0 \rightarrow X_\mu$  and so by the definition of the pullback operation,  $\varphi^*(\mu)$  is the Beltrami coefficient of  $\varphi: X_0 \rightarrow X_\mu$ . Thus  $\varphi: X_{\varphi^*(\mu)} \rightarrow X_\mu$  is a biholomorphism. But this biholomorphism equals the marking  $\text{id}_{X_{\varphi^*(\mu)} \rightarrow X_\mu}: X_{\varphi^*(\mu)} \rightarrow X_\mu$  and so  $[X_{\varphi^*(\mu)}, f_0] = [X_\mu, f_0]$ . Thus  $\Phi_{\text{nc}}(\varphi^*(\mu)) = \Phi_{\text{nc}}(\mu)$ .

Conversely, if  $\Phi_{\text{nc}}(\mu) = \Phi_{\text{nc}}(\mu')$ , then  $[X_{\mu'}, \text{id}_{X_0 \rightarrow X_{\mu'}} \circ f_0] = [X_\mu, \text{id}_{X_0 \rightarrow X_\mu} \circ f_0]$ . Therefore the marking  $\text{id}_{X_{\mu'} \rightarrow X_\mu}: X_{\mu'} \rightarrow X_\mu$  (which, as a function, is just the identity function on the topological surface  $X_0$ ) is isotopic to a biholomorphism  $\varphi: X_{\mu'} \rightarrow X_\mu$ . But this means that  $\mu'$  is the Beltrami differential of the lqc homeomorphism  $\varphi: X_0 \rightarrow X_\mu$ , which is just the composition of lqc homeomorphisms first  $\varphi: X_0 \rightarrow X_0$  followed by  $\text{id}_{X_0 \rightarrow X_\mu}: X_0 \rightarrow X_\mu$ . By definition of the pullback operation,  $\mu' = \varphi^*(\mu)$  where  $\varphi$  is now an lqc homeomorphism from  $X_0$  to itself. Thus  $\Phi_{\text{nc}}$  is surjective, and its fibres are orbits of the pullback operation by  $\text{LQC}_0(X_0)$ .  $\square$

## CHAPTER 5

# CONTINUITY OF THE ACTION

Recall the action function  $A$ , and the action  $A_{[\psi]}$  of a mapping class  $[\psi]$ , from equation (4.2). Now that we have a well defined topology on  $\mathcal{T}(S)$ , we can ask if the action of  $\text{MCG}(S)$  on  $\mathcal{T}(S)$  is continuous, and we answer it in the affirmative in this chapter. We can also ask if the action representation  $\text{MCG}(S) \rightarrow \text{Homeo}(\mathcal{T}(S))$  is continuous. However, we have not defined any topology on the codomain. If  $S$  is a surface of infinite type, we do not expect  $\mathcal{T}(S)$  to be locally compact in general. So the compact-open topology may not be very useful. Instead we prove that the action function  $A: \text{MCG}(S) \times \mathcal{T}(S) \rightarrow \mathcal{T}(S)$  is continuous. It will follow that for each mapping class  $[\psi] \in \text{MCG}(S)$ , its action on  $\mathcal{T}(S)$ , that is, the function  $A_{[\psi]}: \mathcal{T}(S) \rightarrow \mathcal{T}(S)$ , is a homeomorphism. We remark that the continuity of the action function  $A$  is straightforward for the mapping class group of a surface of finite type acting on the Teichmüller space. In this case, the mapping class group is discrete, and so  $A$  is continuous if and only if for all  $[\psi] \in \text{MCG}(S)$ ,  $A_{[\psi]}$  is continuous. But this is true because  $A_{[\psi]}$  is an isometry. The continuity of  $A$  can be proven using any of the descriptions of the topology of  $\mathcal{T}(S)$  from Chapter 4, but we use the description using homeomorphisms at infinity in this chapter.

## 5.1 The Action in terms of the Homeomorphisms at Infinity

In order to prove continuity of  $A$ , we must compute  $A$  in terms of homeomorphisms at infinity, since the topology on  $\mathcal{T}(S)$  is defined via homeomorphisms at infinity. This is achieved by showing that mapping classes induce homeomorphisms at infinity as well. The idea of studying homeomorphisms of surfaces via the induced homeomorphisms at infinity

dates to work of Nielsen (see [HT85] and the references therein).

First we fix a universal cover  $p: \mathbb{H}^2 \rightarrow S$  with deck group  $\Gamma \subset \mathrm{PSL}(2, \mathbb{R})$ , a torsion-free Fuchsian group of the first kind as before. Recall that  $\Gamma$  is also naturally a subgroup of  $\mathrm{Homeo}^+(S^1)$ . Let  $N(\Gamma)$  be the normaliser of  $\Gamma$  in  $\mathrm{Homeo}^+(S^1)$  and let  $\pi_N$  be the quotient map  $N(\Gamma) \rightarrow N(\Gamma)/\Gamma$ . We now define a function  $\Psi_p: \mathrm{MCG}(S) \rightarrow N(\Gamma)/\Gamma$  which will enable us to compute the action function in terms of homeomorphisms at infinity.

**Definition 5.1** (The function  $\Psi_p$ ). *Suppose  $[\psi] \in \mathrm{MCG}(S)$  is a mapping class. The homeomorphism  $\psi$  lifts to  $\tilde{\psi}: \mathbb{H}^2 \rightarrow \mathbb{H}^2$  which, by Proposition 3.2(1), extends to a homeomorphism at infinity  $\partial\tilde{\psi}: S^1 \rightarrow S^1$ . We define  $\Psi_p[\psi]$  to be the right coset  $[\partial\tilde{\psi}] = \Gamma \circ (\partial\tilde{\psi})$  of  $\Gamma$  in  $\mathrm{Homeo}^+(S^1)$ .*

**Lemma 5.2.** *For each  $[\psi] \in \mathrm{MCG}(S)$ ,  $\Psi_p[\psi]$  is a well defined element of  $N(\Gamma)/\Gamma$ . Thus the definition above yields a function  $\Psi_p: \mathrm{MCG}(S) \rightarrow N(\Gamma)/\Gamma$ .*

*Proof.* First, we need to show that  $\Psi_p[\psi]$  is independent of two choices made in the definition of  $\Psi_p$ , namely, the choice of the representative homeomorphism  $\psi$  of the mapping class, and the choice of its lift  $\tilde{\psi}$ . We treat these one at a time, and in reverse order.

1. The choice of the lift  $\tilde{\psi}$  of  $\psi$ : Any other lift of  $\psi$  is of the form  $\sigma \circ \tilde{\psi}$ , where  $\sigma \in \Gamma$  is a deck transformation. This extends to the homeomorphism at infinity  $\sigma \circ (\partial\tilde{\psi})$ , by Proposition 3.2(6). Since  $\sigma \in \Gamma$ , we have an equality of right cosets  $\Gamma \circ (\sigma \circ (\partial\tilde{\psi})) = \Gamma \circ (\partial\tilde{\psi})$ . That is,  $[\partial(\sigma \circ \tilde{\psi})] = [\sigma \circ (\partial\tilde{\psi})] = [\partial\tilde{\psi}]$ .
2. The choice of the representative  $\psi$  of the mapping class  $[\psi]$ : Suppose  $\psi'$  is another representative of the same mapping class. Then  $\psi$  is homotopic to  $\psi'$ . This homotopy lifts to a homotopy between  $\tilde{\psi}$  and a lift  $\tilde{\psi}'$  of  $\psi'$ . By Proposition 3.2(4), we have  $\partial\tilde{\psi}' = \partial\tilde{\psi}$ , so  $[\partial\tilde{\psi}'] = [\partial\tilde{\psi}]$ , so that  $\Psi_p[\psi]$  is a well defined right coset of  $\Gamma$  in  $\mathrm{Homeo}^+(S^1)$ .

Next, to show that  $\Psi_p[\psi]$  is actually an element of the subspace  $N(\Gamma)/\Gamma$  of the right coset space  $\Gamma \backslash \text{Homeo}^+(S^1)$ , we use Proposition 3.2(3) to obtain  $\Gamma \circ (\partial\tilde{\psi}) = (\partial\tilde{\psi}) \circ \Gamma$ , or in other words,  $(\partial\tilde{\psi}) \circ \Gamma \circ (\partial\tilde{\psi})^{-1} = \Gamma$ . Therefore  $\partial\tilde{\psi}$  lies in the normaliser  $N(\Gamma)$ , and  $[\partial\tilde{\psi}] \in N(\Gamma)/\Gamma$ . So the codomain of  $\Psi_p$  is  $N(\Gamma)/\Gamma$  indeed.  $\square$

Note that the left and right cosets of  $\Gamma$  by  $\partial\tilde{\psi}$  are equal, and our notation  $N(\Gamma)/\Gamma$  suggests left coset space. This is in keeping with the convention for group quotients by normal subgroups.

**Lemma 5.3.**  $\Psi_p: \text{MCG}(S) \rightarrow N(\Gamma)/\Gamma$  is a group isomorphism.

*Proof.* A similar argument as with  $\Phi_p$  in Lemma 4.8 shows that  $\Psi_p$  is a bijection. To see that  $\Psi_p$  is a group homomorphism, suppose that  $\psi_1, \psi_2 \in \text{Homeo}^+(S)$  are homeomorphisms with lifts  $\tilde{\psi}_1, \tilde{\psi}_2$  to the universal cover. Then  $\tilde{\psi}_1 \circ \tilde{\psi}_2$  is a lift of  $\psi_1 \circ \psi_2$ , and we compute  $\Psi_p([\psi_1] \cdot [\psi_2]) = \Psi_p[\psi_1 \circ \psi_2] = [\partial(\tilde{\psi}_1 \circ \tilde{\psi}_2)]$  which equals  $[(\partial\tilde{\psi}_1) \circ (\partial\tilde{\psi}_2)]$  by Proposition 3.2(5), and hence equals  $[\partial\tilde{\psi}_1] \circ [\partial\tilde{\psi}_2] = \Psi_p[\psi_1] \cdot \Psi_p[\psi_2]$ . Hence  $\Psi_p$  is a group homomorphism and, since it is a bijection as well, is a group isomorphism.  $\square$

We are ready to compute the action function in terms of homeomorphisms at infinity. The general idea is that  $\Phi_p(A([\psi], [X, f])) = \Phi_p[X, f] \circ (\Psi_p[\psi])^{-1}$ . However,  $\Phi_p[X, f]$  and  $\Psi_p[\psi]$  are right cosets of  $\text{PSL}(2, \mathbb{R})$  and  $\Gamma$  respectively in  $\text{Homeo}^+(S^1)$  and  $N(\Gamma)$  respectively. In order to avoid dealing with multiplication of cosets, we state the computation in the following manner:

**Proposition 5.4.** Suppose  $[X, f] \in \mathcal{T}(S)$  and  $[\psi] \in \text{MCG}(S)$ . If  $\Phi_p[X, f] = [F]$  and  $\Psi_p[\psi] = [G]$ , then we have

$$\Phi_p(A([\psi], [X, f])) = [F \circ G^{-1}] \quad (5.1)$$

*Proof.* Let  $\tilde{f}$  be a lift of the marking map  $f$ , and let  $\tilde{\psi}$  be a lift of the homeomorphism  $\psi$  of  $S$ . Then  $\Phi_p[X, f] = [\partial\tilde{f}]$  and  $\Psi_p[\psi] = [\partial\tilde{\psi}]$ . We have  $A([\psi], [X, f]) = [X, f \circ \psi^{-1}]$ . Then the

modified marking map  $f \circ \psi^{-1}$  lifts to the map  $\tilde{f} \circ \tilde{\psi}^{-1}$  which, by Proposition 3.2(5), extends to the homeomorphism at infinity  $\partial(\tilde{f} \circ \tilde{\psi}^{-1}) = (\partial\tilde{f}) \circ (\partial\tilde{\psi})^{-1}$ . Therefore  $\Phi_p(A([\psi], [X, f])) = \Phi_p[X, f \circ \psi^{-1}] = [(\partial\tilde{f}) \circ (\partial\tilde{\psi})^{-1}]$ . If  $F$  is any other representative of the right coset  $[\partial\tilde{f}]$  of  $\mathrm{PSL}(2, \mathbb{R})$  in  $\mathrm{Homeo}^+(S^1)$ , then  $F = \sigma \circ (\partial\tilde{f})$  for some  $\sigma \in \mathrm{PSL}(2, \mathbb{R})$ . Similarly if  $G$  is any other representative of the (left or right) coset  $[\partial\tilde{\psi}]$  of  $\Gamma$  in  $\mathrm{N}(\Gamma)$ , then  $G = (\partial\tilde{\psi}) \circ \gamma^{-1}$  for some  $\gamma \in \Gamma$ . Then  $F \circ G^{-1} = (\sigma \circ (\partial\tilde{f})) \circ ((\partial\tilde{\psi}) \circ \gamma^{-1})^{-1} = \sigma \circ (\partial\tilde{f}) \circ \gamma \circ (\partial\tilde{\psi})^{-1}$ , which equals, by Proposition 3.2(3),  $\sigma \circ f_*(\gamma) \circ (\partial\tilde{f}) \circ (\partial\tilde{\psi})^{-1}$ . Since  $\sigma, f_*(\gamma) \in \mathrm{PSL}(2, \mathbb{R})$ , we have  $[F \circ G^{-1}] = [(\partial\tilde{f}) \circ (\partial\tilde{\psi})^{-1}]$  as right cosets of  $\mathrm{PSL}(2, \mathbb{R})$  in  $\mathrm{Homeo}^+(S^1)$ . In other words,  $\Phi_p(A([\psi], [X, f])) = [F \circ G^{-1}]$ .  $\square$

Thus in terms of homeomorphisms at infinity, the action of the mapping class group is simply by multiplication on the right by the inverse. Since inversion and multiplication are continuous operations in the topological group  $\mathrm{Homeo}^+(S^1)$ , the formula  $(G, F) \mapsto F \circ G^{-1}$  defines a continuous function on  $\mathrm{N}(\Gamma) \times \tilde{\mathcal{T}}(p) \subset \mathrm{Homeo}^+(S^1) \times \mathrm{Homeo}^+(S^1)$ . This descends to the action function  $A$ , which is continuous as long as  $[\partial\tilde{\psi}]$  depends continuously on  $[\psi]$ . That is, we need to show that  $\Psi_p$  is continuous.

## 5.2 The Main Lemma

In this section, we finish relating the mapping class group to homeomorphisms at infinity and show that  $\Psi_p$  is an isomorphism of topological groups.

**Lemma 5.5.**  $\Psi_p: \mathrm{MCG}(S) \rightarrow \mathrm{N}(\Gamma)/\Gamma$  is a homeomorphism.

*Proof.* Fix a basepoint  $\tilde{s} \in \mathbb{H}^2$  and let  $s = p(\tilde{s})$ . Let  $d$  be any path metric on  $S^1$ , for example, the visual metric on  $S^1$  induced by the basepoint  $\tilde{s}$ . Since we have fixed a universal cover

$p: \mathbb{H}^2 \rightarrow S$ , we have access to the complete hyperbolic metric  $d_S$  on  $S$  and its injectivity radius function  $\text{inj}_S$ , which is a continuous function on  $S$ .

For the continuity of  $\Psi_p$ , we only need to show that  $\Psi_p$  is continuous at the identity  $[\text{id}_S] \in \text{MCG}(S)$ , since  $\Psi_p$  is a group isomorphism. Note that  $\Psi_p[\text{id}_S] = [\partial \text{id}_{\mathbb{H}^2}] = [\text{id}_{S^1}]$ . Let  $U$  be an open neighbourhood of  $[\text{id}_{S^1}]$  in  $\text{N}(\Gamma)/\Gamma$ . We have to show that  $\Psi_p^{-1}(U)$  is an open neighbourhood of  $[\text{id}_S]$  in  $\text{MCG}(S)$ . Since the quotient  $\pi_N: \text{N}(\Gamma) \rightarrow \text{N}(\Gamma)/\Gamma$  is continuous, the preimage  $\pi_N^{-1}(U)$  is open in  $\text{N}(\Gamma)$ , and hence is the intersection with  $\text{N}(\Gamma)$  of an open neighbourhood  $U_1$  of  $\text{id}_{S^1}$  in  $\text{Homeo}^+(S^1)$ . Therefore there is an  $\varepsilon$  such that the ball in  $\text{Homeo}^+(S^1)$  (in the metric of uniform convergence)

$$B_d(\text{id}_{S^1}, \varepsilon) = \{\theta \in \text{Homeo}^+(S^1) \mid d(\theta(q), q) < \varepsilon \text{ for all } q \in S^1\}$$

is contained in  $U_1$ . Reducing  $\varepsilon$  if necessary, assume that  $\varepsilon$  is less than half the  $d$ -length of  $S^1$ .

We will construct an open neighbourhood  $V_1$  of  $\text{id}_S$  in  $\text{Homeo}^+(S)$  and for all  $\psi \in V_1$ , construct a lift  $\tilde{\psi}$  such that  $\partial \tilde{\psi} \in B_d(\text{id}_{S^1}, \varepsilon)$ . Since  $B_d(\text{id}_{S^1}, \varepsilon) \subset U_1$ , this means that  $\Psi_p[\psi] = [\partial \tilde{\psi}] \in \pi_N(U_1 \cap \text{N}(\Gamma)) = U$ . Since quotienting by the group action of  $\text{Homeo}_0(S)$  is an open map, the projection  $V$  of  $V_1$  in  $\text{MCG}(S)$  is an open set. It follows that  $V$  is an open neighbourhood of  $[\text{id}_S]$  and is contained in  $\Psi_p^{-1}(U)$ . It will follow that  $\Psi_p$  is continuous at  $[\text{id}_S]$ .

To obtain the open neighbourhood  $V_1$ , we choose a finite set of points in  $(\Gamma)_\infty$  which is  $\frac{\varepsilon}{2}$ -dense in  $S^1$ . This is possible because  $S^1$  is compact and  $(\Gamma)_\infty$  is dense in  $S^1$  due to Proposition 3.1. Represent these points as sinks  $(\gamma_1)_\infty, (\gamma_2)_\infty, \dots, (\gamma_n)_\infty$  of hyperbolic elements  $\gamma_1, \gamma_2, \dots, \gamma_n \in \Gamma$ . Relabelling if necessary, and removing duplicates, assume that  $(\gamma_1)_\infty, (\gamma_2)_\infty, \dots, (\gamma_n)_\infty$  are distinct and in positive circular order. These points divide the circle at infinity into  $n$  intervals, each of length at most  $\varepsilon$ . For each  $j = 1, 2, \dots, n$ , choose an oriented closed curve  $\alpha_j$  in  $S$  based at the point  $s$  such that the holonomy around  $\alpha_j$  is

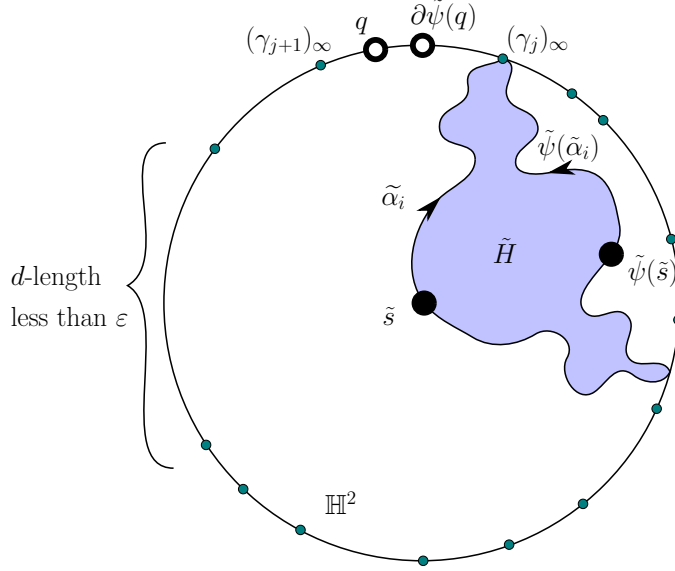


Figure 5.1: Lifts of the homeomorphisms in  $V_1$  and their extensions to the circle at infinity.

$\varphi[\alpha_j] = \gamma_j$ , where  $\varphi: \pi_1(S, s) \rightarrow \Gamma$  is the holonomy representation induced by the pointed universal cover  $p: (\mathbb{H}^2, \tilde{s}) \rightarrow (S, s)$ . Since the set  $\bigcup_{j=1}^n \alpha_j$  is a compact set, the continuous function  $\text{inj}_S$  has a minimum value  $\delta > 0$  on it. We define  $V_1$  to be the set

$$V_1 = \left\{ \psi \in \text{Homeo}^+(S) \mid d_S(\psi(q), q) < \delta \text{ for all } q \in \bigcup_{j=1}^n \alpha_j \right\}$$

which is a basic open neighbourhood of  $\text{id}_S$  of  $\text{Homeo}^+(S)$  in the topology of uniform convergence on compact sets.

Now let  $\psi \in V_1$ . Since  $d_S(\psi(s), s) < \delta \leq \text{inj}_S(s)$ , there is exactly one lift of  $\psi(s)$  in the  $\delta$ -neighbourhood of the lift  $\tilde{s}$  of  $s$ . Define  $\tilde{\psi}$  to be the unique lift of  $\psi$  that maps  $\tilde{s}$  to this point. Further, for each  $j = 1, 2, \dots, n$  and each  $q \in \alpha_j$ , since  $d_S(\psi(q), q) < \delta \leq \text{inj}_S(q)$ , there is a unique shortest geodesic segment from  $q$  to  $\psi(q)$ . Moving every point  $q$  along this geodesic segment at constant speed yields the straight line homotopy  $H$  from the curve  $\alpha_j$  to the curve  $\psi(\alpha_j)$  (recall Proposition 3.14). Therefore the holonomy around  $\psi(\alpha_j)$  equals the holonomy  $\gamma_j$  around  $\alpha_j$ , but it also equals  $\psi_*(\gamma_j)$ , using facts from Section 2.4. We infer that  $\psi_*(\gamma_j) = \gamma_j$ . Consequently,  $\partial\tilde{\psi}((\gamma_j)_\infty) = (\psi_*(\gamma_j))_\infty = (\gamma_j)_\infty$ , using Definition 3.6. Since  $j$  was arbitrary, this is true for all  $j = 1, 2, \dots, n$ . See Figure 5.1, which shows the lift  $\tilde{H}$  of the homotopy  $H$ . Here  $\tilde{H}$  is a homotopy from the (bi-infinite) lift  $\tilde{\alpha}_j$  of  $\alpha_j$  through  $\tilde{s}$  to the

lift  $\tilde{\psi}(\tilde{\alpha}_j)$  of  $\psi(\alpha_j)$  through  $\tilde{\psi}(\tilde{s})$ . Note that the forward endpoints of both  $\alpha_j$  and  $\tilde{\psi}(\tilde{\alpha}_j)$  are equal, and hence  $\partial\tilde{\psi}$  fixes  $(\gamma_j)_\infty$ .

Next, we show the membership  $\partial\tilde{\psi} \in B_d(\text{id}_{S^1}, \varepsilon)$  as promised. For any  $q \in S^1$ , if  $q = (\gamma_j)_\infty$  for some  $j$ , then trivially  $d(\partial\tilde{\psi}(q), q) = 0 < \varepsilon$ . Otherwise,  $q$  lies in an interval component of  $S^1 \setminus \{(\gamma_1)_\infty, (\gamma_2)_\infty, \dots, (\gamma_n)_\infty\}$  bounded by two sinks  $(\gamma_j)_\infty$  and  $(\gamma_{j+1})_\infty$  (subscripts of the letter  $\gamma$  being modulo  $n$ ). Since the homeomorphism  $\partial\tilde{\psi}$  preserves the circular order on  $S^1$  and fixes both  $(\gamma_j)_\infty$  and  $(\gamma_{j+1})_\infty$ , we conclude that  $\partial\tilde{\psi}(q)$  lies in the same interval as does  $q$ . Hence  $d(\partial\tilde{\psi}(q), q) < \varepsilon$ , as both lie in an interval of length less than  $\varepsilon$ . Thus  $d(\partial\tilde{\psi}(q), \text{id}_{S^1}(q)) < \varepsilon$  for all  $q \in S^1$ , and so  $\partial\tilde{\psi} \in B_d(\text{id}_{S^1}, \varepsilon)$ . This concludes the proof of continuity of  $\Psi_p$ .

We show that the inverse  $\Psi_p^{-1}: \text{N}(\Gamma)/\Gamma \rightarrow \text{MCG}(S)$  is continuous by describing it explicitly. Let  $\text{DE}: \text{Homeo}^+(S^1) \rightarrow \text{Homeo}^+(\mathbb{H}^2)$  denote the Douady-Earle extension restricted to  $\mathbb{H}^2$  (recall Theorem 3.12). Suppose  $G \in \text{N}(\Gamma) \subset \text{Homeo}^+(S^1)$  and  $\gamma \in \Gamma \subset \text{PSL}(2, \mathbb{R})$ . Then since  $G$  normalises  $\Gamma$  and  $\text{DE}$  is conformally natural, we have  $\text{DE}(G) \circ \gamma = \text{DE}(G \circ \gamma) = \text{DE}(\gamma' \circ G) = \gamma' \circ \text{DE}(G)$  for some  $\gamma' \in \Gamma$ . Note that in this chain of equalities, the  $\gamma$  and  $\gamma'$  within parentheses are homeomorphisms of  $S^1$ , those without are isometries of  $\mathbb{H}^2$ . Thus  $\text{DE}(G)$  normalises  $\Gamma$ , now viewed as a group of homeomorphisms of  $\mathbb{H}^2$ . We denote the normaliser of  $\Gamma$  in  $\text{Homeo}^+(\mathbb{H}^2)$  by  $\text{Homeo}_\Gamma^+(\mathbb{H}^2)$ . This is precisely the set of equivariant homeomorphisms of  $\mathbb{H}^2$ , or equivalently, the set of lifts of homeomorphisms of  $S$ . Let  $\pi: \text{Homeo}_\Gamma^+(\mathbb{H}^2) \rightarrow \text{Homeo}^+(S)$  be the function defined by  $\pi(\tilde{f}) = f$  for every homeomorphism  $f$  of  $S$  and every lift  $\tilde{f}$  of  $f$ , so that  $\pi \circ \text{DE}(G)$  is a homeomorphism of  $S$  for each  $G \in \text{N}(\Gamma)$ . Now each  $\gamma \in \Gamma$  is a deck transformation of the universal cover  $p: \mathbb{H}^2 \rightarrow S$ . Hence for each  $G \in \text{N}(\Gamma)$  and  $\gamma \in \Gamma$ , we have  $\pi \circ \text{DE}(G \circ \gamma) = \pi(\text{DE}(G) \circ \gamma) = \pi \circ \text{DE}(G)$ , and so  $\pi \circ \text{DE}$  descends to a function  $\overline{\pi\text{DE}}: \text{N}(\Gamma)/\Gamma \rightarrow \text{MCG}(S)$ . For  $[G] \in \text{N}(\Gamma)/\Gamma$ , a lift of  $\overline{\pi\text{DE}}[G]$  to  $\mathbb{H}^2$  is simply  $\text{DE}(G)$ , which extends to the homeomorphism at infinity  $G$ , so  $\Psi_p \circ \overline{\pi\text{DE}}[G] = [G]$ . As  $[G] \in \text{N}(\Gamma)/\Gamma$  was arbitrary,  $\Psi_p \circ \overline{\pi\text{DE}}$  is the identity on  $\text{N}(\Gamma)/\Gamma$ .

Hence  $\overline{\pi\text{DE}}$  is a right inverse to  $\Psi_p$ . But we already know that  $\Psi_p$  is a bijection, so in fact  $\overline{\pi\text{DE}} = \Psi_p^{-1}$ .

We show that  $\pi$  is continuous. Suppose  $G_i \rightarrow G$  as  $i \rightarrow \infty$  in  $\text{Homeo}_\Gamma^+(\mathbb{H}^2)$ , which has the topology of uniform convergence on compact sets. To show the  $\pi(G_i) \rightarrow \pi(G)$  as  $i \rightarrow \infty$  in  $\text{Homeo}^+(S)$  in the topology of uniform convergence on compact sets, we show the uniform convergence on an arbitrary compact set  $K \subset S$ . There exists a large enough closed disk  $D \subset \mathbb{H}^2$  such that  $K$  is contained in  $p(D)$ . Since  $D$  is compact,  $G_i|_D \rightarrow G|_D$  uniformly as  $i \rightarrow \infty$ . Now  $p$  is a Riemannian local isometry, and therefore contracts distances. Therefore  $\pi(G_i)|_K \rightarrow \pi(G)|_K$  uniformly as  $i \rightarrow \infty$ , and so  $\pi$  is continuous. Since  $\text{DE}$  is also a continuous function, the composition  $\pi \circ \text{DE}$  is continuous, and passing the quotient,  $\overline{\pi\text{DE}}$  is continuous. Therefore  $\Psi_p^{-1}$  is continuous, and  $\Psi_p$  is a homeomorphism.  $\square$

### 5.3 Proof of the Continuity of the Action

We are ready to prove the continuity of the action function  $A$ . The thrust of the argument is already proved in Lemma 5.5, we only need to finish up the point set topological details.

**Theorem 5.6.** *The action function  $A: \text{MCG}(S) \times \mathcal{T}(S) \rightarrow \mathcal{T}(S)$  given by equation (4.2) is continuous.*

*Proof.* As before, fix a universal cover  $p: \mathbb{H}^2 \rightarrow S$  with deck group  $\Gamma \subset \text{PSL}(2, \mathbb{R})$ , a torsion-free Fuchsian group of the first kind. See the diagram in Figure 5.2, which commutes. The proposition is that the function in the bottom row is continuous. This is equivalent to the continuity of the function in the third row just above it, because  $\Phi_p$  is a homeomorphism by definition, and  $\Psi_p$  is a homeomorphism by Lemma 5.5. The function in the top row is certainly continuous, because multiplication and inversion are continuous functions of the topological group  $\text{Homeo}^+(S^1)$ . Restricting to subspaces in the second row, we see that the

$$\begin{array}{ccc}
\text{Homeo}^+(S^1) \times \text{Homeo}^+(S^1) & \xrightarrow{(G, F) \mapsto F \circ G^{-1}} & \text{Homeo}^+(S^1) \\
\uparrow & & \uparrow \\
\text{N}(\Gamma) \times \tilde{\mathcal{T}}(p) & \xrightarrow{(G, F) \mapsto F \circ G^{-1}} & \tilde{\mathcal{T}}(p) \\
\pi_N \times \pi_p \downarrow & & \downarrow \pi_p \\
\text{N}(\Gamma)/\Gamma \times \mathcal{T}(p) & \xrightarrow{([\partial\tilde{\psi}], [\partial\tilde{f}]) \mapsto [(\partial\tilde{f}) \circ (\partial\tilde{\psi})^{-1}]} & \mathcal{T}(p) \\
\Psi_p \times \Phi_p \uparrow \cong & & \cong \downarrow \Phi_p^{-1} \\
\text{MCG}(S) \times \mathcal{T}(S) & \xrightarrow{([\psi], [X, f]) \mapsto [X, f \circ \psi^{-1}]} & \mathcal{T}(S) \\
& & A
\end{array}$$

Figure 5.2: Continuity of the action function  $A$ .

function in the second row is continuous as well. The quotient maps  $\pi_N: \text{N}(\Gamma) \rightarrow \text{N}(\Gamma)/\Gamma$  and  $\pi_p: \tilde{\mathcal{T}}(p) \rightarrow \mathcal{T}(p)$  are open, because they are quotients by the group actions of  $\Gamma$  and  $\text{PSL}(2, \mathbb{R})$  respectively. Therefore the product map  $\pi_N \times \pi_p$  is also open and surjective, hence a quotient map. By the universal property of quotients, the map in the second row descends to the function in the third row which is therefore continuous, as required. Thus the action function  $A$  is continuous.  $\square$

**Corollary 5.7.** *The mapping class group acts on the marked moduli space by homeomorphisms.*

*Proof.* Recall from equation (4.2) that for each fixed mapping class  $[\psi] \in \text{MCG}(S)$ , its action on the marked moduli space is given by the function  $A_{[\psi]}: \mathcal{T}(S) \rightarrow \mathcal{T}(S)$  which is  $A_{[\psi]}[X, f] = A([\psi], [X, f])$ . Then  $A_{[\psi]}$  is continuous by Theorem 5.6, and  $A_{[\psi^{-1}]}$  is evidently a continuous inverse, so  $A_{[\psi]}$  is a homeomorphism.  $\square$

# CHAPTER 6

## CONTRACTIBILITY OF THE MARKED MODULI SPACE

This chapter is devoted to proving that the marked moduli space is contractible. This is obvious if  $S$  is a surface of finite type once one proves that  $\mathcal{T}(S)$  is a cell of the appropriate dimension using Fenchel–Nielsen coordinates. In the analogous quasiconformal theory, it is known that every quasiconformal Teichmüller space is contractible, even if its basepoint Riemann surface is of infinite area. This was conjectured by Bers upon his definition of quasiconformal Teichmüller spaces ([Ber64]). Earle–Eells proved it for the universal Teichmüller space, that is,  $\mathcal{T}_{qc}(\mathbb{H}^2)$ , and Tukia proved it for Riemann surfaces of many (topologically) infinite types. Finally, Douady–Earle ([DE86]) proved the contractibility for every Riemann surface using a uniform proof.

We adapt the Douady–Earle proof to the setting of marked moduli spaces. Recall the integration function  $\Phi_{nc}: \text{Bel}_{nc}(X_0) \rightarrow \mathcal{T}(S)$  corresponding to a fixed Nielsen-convex Riemann surface  $X_0$  and a marking homeomorphism  $f_0: S \rightarrow X_0$ . We first prove that  $\Phi_{nc}$  is continuous in Section 6.1, and later in Section 6.2, prove that it is actually a quotient map. The key idea of the Douady–Earle proof is to construct a particular continuous section  $g_{nc}$  of  $\Phi_{nc}$ , using the Douady–Earle extension. Since  $\text{Bel}_{nc}(X_0)$  is contractible, its deformation retraction can be pushed down to one of  $\mathcal{T}(S)$  using  $\Phi_{nc}$  and  $g_{nc}$ , and we finish the proof in Section 6.3.

### 6.1 Continuity of the Integration Function

The work in this section is joint with Hiroshige Shiga. Recall that the metric tensor of the uniformised complete hyperbolic of a Riemann surface  $X$  is denoted by  $\rho_X$ , and the corresponding line element by  $d\rho_X$ . If  $Y \subset X$  is an open subsurface, then  $\rho_X$  restricts to

a metric of constant curvature  $-1$  on  $Y$ , which we still denote  $\rho_X$ . This is an incomplete metric unless  $Y = X$ . Note that the complete hyperbolic metric in the same conformal class would be  $\rho_Y$  according to this notation. Recall that for symmetric bilinear forms  $A$  and  $B$  on a vector space, we say  $A \geq B$  if  $A - B$  is positive definite.

We establish some estimates relating  $\rho_X$  to  $\rho_Y$ . The following is well known ([Abi80]).

**Theorem 6.1.** *For any Riemann surface  $X$  and open subsurface  $Y \subset X$ , we have  $\rho_Y \geq \rho_X$ .*

To have estimates in the other direction, first we do them on the hyperbolic plane, using the Poincaré disk model. For any  $0 < R \leq 1$ , let  $\mathbb{D}_R$  be the disk of radius (with respect to the Euclidean metric)  $R$  centred at the origin  $0$  in the complex plane, and  $z$  the standard coordinate. The Poincaré metric tensor for this disk is given by the formula  $\rho_{\mathbb{D}_R} = \frac{4R^2|dz|^2}{(R^2 - |z|^2)^2}$ . The metric tensor at  $0$  is therefore  $\frac{|dz|^2}{R^2}$ . Therefore if  $Y \subset \mathbb{D}_1$  contains  $\mathbb{D}_R$ , then  $\rho_Y \leq \rho_{\mathbb{D}_R} = \frac{|dz|^2}{R^2} = \frac{1}{R^2}\rho_{\mathbb{D}_1}$  at  $0$ . Let us recast this intrinsically in terms of the hyperbolic metric on  $\mathbb{D}_1$ . Recall that for any  $q \in \mathbb{D}_1$ , we have  $|z(q)| = \tanh(d(0, q)/2)$ . Therefore in  $\mathbb{D}_1$ ,  $\mathbb{D}_R$  is a ball centred at  $0$  of radius (with respect to the Poincaré metric)  $r$ , and  $R = \tanh(r/2)$ . Thus the coefficient above is  $1/R^2 = \coth^2(r/2)$ . We have proved the following.

**Lemma 6.2.** *If  $q$  is a point on the hyperbolic plane  $\mathbb{H}^2$ , and if  $Y \subset \mathbb{H}^2$  is an open subsurface containing the ball of radius  $r$  centred at  $q$ , then*

$$\rho_Y \leq \coth^2(r/2)\rho_{\mathbb{H}^2} \text{ at } q$$

**Lemma 6.3.** *Let  $X$  be a hyperbolic surface and  $q$  be a point on it. If  $Y \subset X$  is an open subsurface containing the ball of radius  $r$  centred at  $q$ , then*

$$\rho_Y \leq \coth^2(r/2)\rho_X \text{ at } q$$

*If  $K$  is a bounded set in  $X$  and  $\lambda > 1$ , then there is a precompact open subsurface  $Y_{K,\lambda} \subset X$  such that if  $Y \subset X$  is an open subsurface containing  $Y_{K,\lambda}$ , then*

$$\rho_Y \leq \lambda^2\rho_X \text{ on } K$$

*Proof.* Choose a universal cover  $p: \mathbb{H}^2 \rightarrow X$  and a point  $\tilde{q} \in p$  above  $q$ . Note that  $p$  is a local isometry, so  $dp_{\tilde{q}}: T_{\tilde{q}}\mathbb{H}^2 \rightarrow T_qX$  is a linear isometry. Let  $\tilde{Y} \subset \mathbb{H}^2$  be the component of  $p^{-1}Y$  containing  $\tilde{q}$ . Since  $Y$  contains the ball of radius  $r$  centred at  $q$ ,  $\tilde{Y}$  contains the ball of radius  $r$  centred at  $\tilde{q}$ . Then  $\tilde{Y}$  is an open connected subsurface of  $\mathbb{H}^2$  containing the ball of radius  $r$  centred at  $\tilde{q}$ , so  $\rho_{\tilde{Y}} \leq \rho_{\mathbb{H}^2}$  at  $\tilde{q}$ . However, since  $p|_{\tilde{Y}} \rightarrow Y$  is a covering, it is a local isometry. That is,  $dp_{\tilde{q}}: T_{\tilde{q}}\tilde{Y} \rightarrow T_qY$  is a linear isometry. Pushing forward the above inequality by  $dp_{\tilde{q}}$ , we get the first assertion of the lemma.

To prove the second assertion, note that  $\lim_{r \rightarrow +\infty} \coth^2(r/2) = 1$ , so there is an  $r > 0$  such that  $\coth(r/2) < \lambda$ . Take  $Y_{K,r}$  to be the  $r$ -neighbourhood of  $K$ . Since the  $r$ -neighbourhood of  $K$  contains the ball of radius  $r$  centred at any of its points, the first assertion says that  $\rho_{Y_{K,r}} \leq \lambda^2 \rho_X$ . Therefore, if  $Y$  contains  $Y_{K,r}$ , then  $\rho_Y \leq \rho_{Y_{K,r}} \leq \lambda^2 \rho_X$ .  $\square$

We recall Wolpert's theorem. For a Riemann surface  $X$  and an essential, nonperipheral closed curve  $\gamma$  on it, we denote the length of the unique geodesic in its homotopy class by  $\ell_\gamma(X)$ .

**Theorem 6.4** (Wolpert ([Wol79])). *If  $f: X \rightarrow Y$  is a  $\kappa$ -quasiconformal homeomorphism, then for every closed curve  $\gamma$  on  $X$ ,*

$$\kappa^{-1} \ell_\gamma(X) \leq \ell_{f(\gamma)}(Y) \leq \kappa \ell_\gamma(X)$$

**Proposition 6.5.**  $\Phi_{\text{nc}}$  is continuous.

*Proof.* We first show continuity at  $0 \in \text{Bel}_{\text{nc}}(X_0)$ . We need to show that  $\Phi_{\text{nc}}(\mu) \rightarrow \Phi_{\text{nc}}(0)$  as  $\mu \rightarrow 0$ . Since the marked length spectrum function  $\ell_*: \mathcal{T}(S) \rightarrow \mathbb{R}^c$  is an embedding by Theorem 4.20, we need to show that for every essential and nonperipheral homotopy class of closed curves  $\gamma$  on  $S$ ,  $\ell_\gamma(\Phi_{\text{nc}}(\mu)) \rightarrow \ell_\gamma(\Phi_{\text{nc}}(0))$  as  $\mu \rightarrow 0$ . Let  $\gamma$  be such a curve, set  $l = \ell_\gamma[X_0, f_0]$ , and suppose  $\varepsilon > 0$  is given. Choose  $\kappa, \lambda > 1$  such that  $\kappa\lambda < \frac{l}{l-\varepsilon}$ . Note that  $\frac{l}{l-\varepsilon} < \frac{l+\varepsilon}{l}$ . Thus  $l - \varepsilon < (\kappa\lambda)^{-1}l$  and  $(\kappa\lambda)l < l + \varepsilon$ .

The geodesic  $\gamma_0$  on  $X_0$  in the homotopy class  $f_0(\gamma)$  is a compact set, so by Lemma 6.3, there is a precompact open subsurface  $Y_0 \subset X_0$  such that  $\rho_{Y_0} \leq \lambda^2 \rho_{X_0}$  on  $\gamma_0$ . Therefore

$$\ell_{f_0(\gamma)}(Y_0) \leq \text{length}(\gamma_0, Y_0) = \int_{\gamma_0} d\rho_{Y_0} \leq \lambda \int_{\gamma_0} d\rho_{X_0} = \lambda \text{length}(\gamma_0, X_0) = \lambda \ell_{f_0(\gamma)}(X_0) = \lambda l$$

Similarly, since  $d\rho_{Y_0} \geq d\rho_{X_0}$ , we have  $\ell_{f_0(\gamma)}(Y_0) \geq \ell_{f_0(\gamma)}(X_0) = l$ . Thus  $l \leq \ell_{f_0(\gamma)}(Y_0) \leq \lambda l$ .

Let  $\mu \in \text{Bel}_{\text{nc}}(X_0)$  be such that  $\|\mu\|_{Y_0} < \frac{\kappa-1}{\kappa+1}$ . Let  $Y_\mu$  denote the image  $\text{id}_{X_0 \rightarrow X_\mu}(Y_0)$ . The bound on  $\mu$  ensures that  $\text{id}_{Y_0 \rightarrow Y_\mu}$  is  $\kappa$ -quasiconformal. By Wolpert's theorem,  $\kappa^{-1} \ell_{f_0(\gamma)}(Y_0) \leq \ell_{f_0(\gamma)}(Y_\mu) \leq \kappa \ell_{f_0(\gamma)}(Y_0)$ . Therefore  $\kappa^{-1} l \leq \ell_{f_0(\gamma)}(Y_\mu) \leq \kappa \lambda l$ .

Finally, since  $\lambda^{-1} \ell_{f_0(\gamma)}(X_\mu) \leq \ell_{f_0(\gamma)}(X_\mu) \leq \ell_{f_0(\gamma)}(Y_\mu)$ , we get  $(\kappa \lambda)^{-1} l \leq \ell_{f_0(\gamma)}(X_\mu) \leq (\kappa \lambda) l$ . In other words,  $(\kappa \lambda)^{-1} \ell_\gamma[X_0, f_0] \leq \ell_\gamma[X_\mu, f_0] \leq (\kappa \lambda) \ell_\gamma[X_0, f_0]$ . Thus we have shown that  $|\ell_\gamma(\Phi_{\text{nc}}(\mu)) - \ell_\gamma \Phi_{\text{nc}}(0)| < \varepsilon$ , and this is true whenever  $\|\mu\|_{Y_0} < \frac{\kappa-1}{\kappa+1}$ . Therefore  $\ell_\gamma \circ \Phi_{\text{nc}}$  is continuous, and this is true for every  $\gamma \in \mathcal{C}$ . We conclude that  $\Phi_{\text{nc}}$  is continuous at 0.

For continuity at an arbitrary  $\mu' \in \text{Bel}_{\text{nc}}(X_0)$ , recall from Proposition 4.29 that  $\Phi_{\text{nc}} = \Phi_{\text{nc}}^{X_0, f_0} = \Phi_{\text{nc}}^{X_{\mu'}, f_{\mu'}} \circ \varphi^*$ . Here  $\varphi = \text{id}_{X_{\mu'} \rightarrow X_0}$  is the inverse of the integration of the Beltrami coefficient  $\mu'$  and  $f_{\mu'} = \varphi \circ f_0$ . Also  $\varphi^*: \text{Bel}_{\text{nc}}(X_0) \rightarrow \text{Bel}_{\text{nc}}(X_{\mu'})$  is the pullback of Nielsen-convex Beltrami forms, and is a homeomorphism. Since  $\varphi^*(\mu') = 0$  and  $\Phi_{\text{nc}}^{X_{\mu'}, f_{\mu'}}$  is continuous at 0, we conclude  $\Phi_{\text{nc}}^{X_0, f_0}$  is continuous at  $\mu'$ .  $\square$

## 6.2 A Continuous Section of the Integration Map

Pick an arbitrary point in  $\mathcal{T}(S)$  and represent it as  $[X, f \circ f_0]$ , where  $f: X_0 \rightarrow X$  is an lqc homeomorphism. We can do this by precomposing the marking from  $S$  to  $X$  by  $(f_0)^{-1}$  and then isotoping it to be a diffeomorphism. The marking homeomorphism  $f$  lifts to homeomorphism  $\tilde{f}$  of universal covers, which extends to the homeomorphism  $\partial \tilde{f}$  of the boundary circle by Proposition 3.2. Let  $\phi := \text{DE}(\partial \tilde{f})$  be the Douady–Earle extension of this

homeomorphism. Since the Douady–Earle extension is conformally natural, this descends to a marking  $f^{\text{DE}}: X_0 \rightarrow X$  in the same isotopy class. By the Douady–Earle extension theorem Theorem 3.12,  $f^{\text{DE}}$  is a real analytic diffeomorphism and a canonical marking.

**Definition 6.6.** *The Douady–Earle marking is the marking homeomorphism  $f^{\text{DE}}: X_0 \rightarrow X$ .*

*The Douady–Earle section is the function  $g_{\text{nc}}$  defined by  $g_{\text{nc}}: \mathcal{T}(S) \rightarrow \text{Bel}_{\text{nc}}(X_0)$  to be  $g_{\text{nc}}[X, f] = \mu_{f^{\text{DE}}}$ .*

Note that that  $\Phi_{\text{nc}} \circ g_{\text{nc}}[X, f \circ f_0] = [X_{\mu_{f^{\text{DE}}}}, f_0]$ . But  $f^{\text{DE}}: X_{\mu_{f^{\text{DE}}}} \rightarrow X$  is a biholomorphism isotopic to the marking  $f$ . Clearly  $[X_{\mu_{f^{\text{DE}}}}, f_0] = [X, f \circ f_0]$ , so  $\Phi_{\text{nc}} \circ g_{\text{nc}}$  is the identity function on  $\mathcal{T}(S)$ . In other words,  $g_{\text{nc}}$  is a section of  $\Phi_{\text{nc}}$ .

In the proof of continuity of  $g_{\text{nc}}$  below, we will be required to compute Beltrami coefficients of a diffeomorphism in specific coordinates. We show the template of the computation now. Suppose  $f: X \rightarrow Y$  is a diffeomorphism and suppose  $p_X: \mathbb{H}^2 \rightarrow X$  and  $p_Y: \mathbb{H}^2 \rightarrow Y$  are universal covers. The map  $f$  lifts to a diffeomorphism  $\phi: \mathbb{H}^2 \rightarrow \mathbb{H}^2$ . Realise  $\mathbb{H}^2$  as a proper open subset of  $\mathbb{C}$ . Any simply connected proper subdomain of  $\mathbb{C}$  will do, but for concreteness one may assume it is the upper half plane.

To compute  $\mu_f$  we need holomorphic local coordinates on  $X$  and  $Y$ , and these are provided by local inverses of the universal coverings. Concretely, observe that for any simply connected open neighbourhood  $U \subset X$  of  $q$ , there is an injective holomorphic function, and hence a holomorphic local coordinate,  $z: U \rightarrow \mathbb{H}^2 \subset \mathbb{C}$ , which is a local inverse to  $p_X$ , such that  $z(q) = \tilde{q} \in \mathbb{C}$ . Here we are abusing notation;  $z$  is the standard coordinate on  $\mathbb{C}$  but we are also using the same letter to denote the above local coordinate on  $X$  around  $q$ . Similarly  $f(q)$  is a point in  $Y$  and  $\phi(\tilde{q}) \in \mathbb{H}^2 \subset \mathbb{C}$  is a point above it in the universal cover of  $Y$ . Again we have holomorphic local coordinate  $w: V \rightarrow \mathbb{C}$  around  $f(q)$ , which is a local inverse to  $p_Y$ .

Let  $q \in X$  be an arbitrary point and  $\tilde{q} \in \mathbb{H}^2 \subset \mathbb{C}$  be in the universal cover above  $q$ . We

compute that on the set  $z(U) \subset \mathbb{C}$ , we have  $w \circ f \circ z^{-1} = w \circ f \circ p_X = w \circ p_Y \circ \phi = \phi$ . That is, the coordinate representation of  $f$  is  $\phi$ . Hence we obtain the following.

**Lemma 6.7** (Computing Beltrami coefficients in coordinates).  $\mu_f(q) = \frac{\partial\phi/\partial\bar{z}}{\partial\phi/\partial z}(\tilde{q}) \frac{d\bar{z}_q}{dz_q}$ .

**Proposition 6.8.**  $g_{\text{nc}}$  is continuous.

*Proof.* To show that  $g_{\text{nc}}$  is continuous, we need to show that for all compact  $K \subset S$ ,  $g_{\text{nc}}: \mathcal{T}(S) \rightarrow (\text{Bel}_{\text{nc}}(X_0), \|\cdot\|_K)$  is continuous. Fix a universal cover  $p_0: \mathbb{H}^2 \rightarrow X_0$  and let  $\Gamma_0 \subset \text{PSL}(2, \mathbb{R})$  be its deck group. Let  $K$  be given, and let  $\varepsilon > 0$  be given. Take an arbitrary point  $[X, f \circ f_0] \in \mathcal{T}(S)$ , and a universal cover  $p_X: \mathbb{H}^2 \rightarrow X$  such that there is a lift  $\tilde{f}$  whose extension  $\partial\tilde{f}$  fixes  $0, 1, \infty$ . Pick a compact set  $\tilde{K} \subset \mathbb{H}^2$  so large that  $K \subset p_0(\tilde{K})$ . The ranges of  $\frac{\partial\phi}{\partial z}(\tilde{q})$  and  $\frac{\partial\phi}{\partial\bar{z}}(\tilde{q})$  as  $\tilde{q}$  ranges over  $\tilde{K}$  are some compact sets  $A, B$  contained in  $\mathbb{C} \setminus \{0\}, \mathbb{C}$  respectively.

Let  $\varepsilon_1$  be so small that whenever  $a_1 \in A, b_1 \in B$  and  $a \in \mathbb{C} \setminus \{0\}, b \in \mathbb{C}$  satisfy  $|a - a_1| < \varepsilon_1, |b - b_1| < \varepsilon_1$ , we have  $\left| \frac{b_1}{a_1} - \frac{b}{a} \right| < \varepsilon$ . Such  $\varepsilon_1$  exists by the uniform continuity of the continuous function  $(a, b) \mapsto \frac{b}{a}$  over the compact set  $A \times B \subset \mathbb{C}^2$ . Note that  $\text{DE}: \text{Homeo}^+(S^1) \rightarrow \text{Diff}^\infty(\mathbb{H}^2)$  is continuous. Therefore there is a neighbourhood  $V \subset \text{Homeo}^+(S^1)$  of  $\partial\tilde{f}$  such that whenever  $F_1 \in V$ , we have (setting  $\phi_1 = \text{DE}(F_1)$ ),  $\left| \frac{\partial\phi_1}{\partial z}(\tilde{q}) - \frac{\partial\phi}{\partial z}(\tilde{q}) \right| < \varepsilon_1$  and  $\left| \frac{\partial\phi_1}{\partial\bar{z}}(\tilde{q}) - \frac{\partial\phi}{\partial\bar{z}}(\tilde{q}) \right| < \varepsilon_1$  for every  $q \in \tilde{K}$ . Thus by the above, we have  $\left| \frac{\partial\phi_1/\partial\bar{z}}{\partial\phi_1/\partial z}(\tilde{q}) - \frac{\partial\phi/\partial\bar{z}}{\partial\phi/\partial z}(\tilde{q}) \right| < \varepsilon$ . Let  $U \subset V$  be the set of homeomorphisms fixing  $0, 1, \infty$  and conjugating  $\Gamma_0$  back into  $\text{PSL}(2, \mathbb{R})$ . Then  $\Phi_{p_0}(U)$  is a neighbourhood of  $[X, f]$  in  $\mathcal{T}(S)$ . For any point  $[X_1, f_1] \in U$ , we have  $\partial\tilde{f}_1 \in U \subset V$  (for the lift and extension which fixes  $0, 1, \infty$ ).

Now for any  $q \in K$ , there is some  $\tilde{q} \in \tilde{K}$  above  $q$ . Our computation above Lemma 6.7 shows that  $\mu_{f_1^{\text{DE}}}(q) = \frac{\partial\phi_1/\partial\bar{z}}{\partial\phi_1/\partial z}(\tilde{q}) \frac{d\bar{z}_q}{dz_q}$  and  $\mu_f(q) = \frac{\partial\phi/\partial\bar{z}}{\partial\phi/\partial z}(\tilde{q}) \frac{d\bar{z}_q}{dz_q}$ . Thus

$$\left| \left( \mu_{f_1^{\text{DE}}} - \mu_f \right) \right| (q) = \left| \frac{\partial\phi_1/\partial\bar{z}}{\partial\phi_1/\partial z}(\tilde{q}) \frac{d\bar{z}_q}{dz_q} - \frac{\partial\phi/\partial\bar{z}}{\partial\phi/\partial z}(\tilde{q}) \frac{d\bar{z}_q}{dz_q} \right| = \left| \frac{\partial\phi_1/\partial\bar{z}}{\partial\phi_1/\partial z}(\tilde{q}) - \frac{\partial\phi/\partial\bar{z}}{\partial\phi/\partial z}(\tilde{q}) \right| < \varepsilon$$

Since  $q \in K$  was arbitrary, we conclude

$$\|\mu_{f_1^{\text{DE}}} - \mu_{f^{\text{DE}}}\|_K = \operatorname{esssup}_{q \in K} \left| \left( \mu_{f_1^{\text{DE}}} - \mu_{f^{\text{DE}}} \right) (q) \right| \leq \varepsilon$$

In other words,  $\|g_{\text{nc}}[X_1, f_1] - g_{\text{nc}}[X, f]\|_K < \varepsilon$  for every  $[X_1, f_1] \in U$ . Therefore  $g_{\text{nc}}: \mathcal{T}(S) \rightarrow (\operatorname{Bel}_{\text{nc}}(X_0), \|\cdot\|_K)$  is continuous, as was to be shown.  $\square$

Finally, we finish proving that  $\Phi_{\text{nc}}$  is a quotient map. Recall that this gives another description of the topology of the marked moduli space

**Theorem 6.9.**  *$\Phi_{\text{nc}}$  is an open map, and therefore a quotient map.*

*Proof.* Suppose  $U \subset \operatorname{Bel}_{\text{nc}}(X_0)$  is an open set, and let  $V = \Phi_{\text{nc}}(U) \subset \mathcal{T}(S)$ . We wish to show that  $V$  is open. Consider the  $\operatorname{LQC}_0(X_0)$  orbit of  $U$ , that is, the union of the images of  $U$  under the pullback operation by lqc homeomorphisms of  $X_0$ . This orbit  $\operatorname{LQC}_0(X_0) \cdot U = \bigcup_{h \in \operatorname{LQC}_0(X_0)} h^*(U)$  is a union of open sets, and is therefore an open set itself, in  $\operatorname{Bel}_{\text{nc}}(X_0)$ . Recall from Proposition 4.30 that the fibres of  $\Phi_{\text{nc}}$  are precisely  $\operatorname{LQC}_0(X_0)$  orbits. Hence  $\operatorname{LQC}_0(X_0) \cdot U$  is a union of fibres of  $\Phi_{\text{nc}}$ . In particular, it is the union of the fibres of  $\Phi_{\text{nc}}$  over all points in the image  $V$ . For any  $\mathcal{X} \in V$ , its Douady–Earle section  $g_{\text{nc}}(\mathcal{X})$  gives a particular element of the fibre, and therefore  $g_{\text{nc}}(\mathcal{X}) \in U$ . Further, since  $g_{\text{nc}}$  is a section of  $\Phi_{\text{nc}}$ , we see that  $g_{\text{nc}}^{-1}(U) = V$ . Since  $g_{\text{nc}}$  is continuous by the Proposition 6.8 above,  $V$  is an open subset of  $\mathcal{T}(S)$ . Hence  $\Phi_{\text{nc}}$  is an open map. A surjective, continuous, and open map is a quotient map. Hence  $\Phi_{\text{nc}}$  is also a quotient map.  $\square$

$$\begin{array}{ccc}
& & \text{Bel}_{\text{nc}}(X_0) \hookrightarrow \text{bel}(X_0) \\
\text{LQC}_0(X_0) \subset & & \downarrow \Phi_{\text{nc}} \uparrow g_{\text{nc}} \\
& & \mathcal{T}(S)
\end{array}$$

Figure 6.1: Integration map and its section.

## 6.3 Proof of the Contractibility of the Marked Moduli Space

In this section, we finish the proof of the contractibility of the marked moduli space. As stated above, the idea is to adapt the Douady–Earle proof ([DE86]) of contractibility of Teichmüller space to the setting of marked moduli space.

**Proposition 6.10.**  *$\text{Bel}_{\text{nc}}(X_0)$  is a star shaped subset of  $\text{bel}(X_0)$  centred at the origin.*

*Proof.* If  $\mu \in \text{Bel}_{\text{nc}}(X_0)$  and  $t \in [0, 1)$  then  $\|t\mu\|_{X_0} \leq t < 1$ . Therefore  $\text{id}_{X_0 \rightarrow X_{t\mu}}$  is a quasiconformal homeomorphism, since its Beltrami form  $t\mu$  is globally bounded away from 1. Quasiconformal homeomorphisms extend to homeomorphisms of the ideal boundary. But the ideal boundary of the Nielsen-convex surface  $X_0$  is empty, and therefore so is the ideal boundary of  $X_{t\mu}$ . In other words,  $X_{t\mu}$  is Nielsen-convex and so  $t\mu \in \text{Bel}_{\text{nc}}(X_0)$ .  $\square$

**Theorem 6.11.** *The marked moduli space  $\mathcal{T}(S)$  is contractible.*

*Proof.* In Section 6.1 we prove that  $\Phi_{\text{nc}}$  is continuous and in Section 6.2 we construct the continuous section  $g_{\text{nc}}$ . See Figure 6.1. The formula  $H(\mathcal{X}, t) = \Phi_{\text{nc}}((1-t)g_{\text{nc}}(\mathcal{X}))$  defines a continuous function  $H: \mathcal{T}(S) \times [0, 1] \rightarrow \mathcal{T}(S)$ . For  $t = 0$ , we have  $H(\mathcal{X}, 0) = \Phi_{\text{nc}}((1-0)g_{\text{nc}}(\mathcal{X})) = \Phi_{\text{nc}} \circ g_{\text{nc}}(\mathcal{X}) = \mathcal{X}$ , since  $g_{\text{nc}}$  is a section of  $\Phi_{\text{nc}}$ . On the other hand, for  $t = 1$ , we have  $H(\mathcal{X}, 1) = \Phi_{\text{nc}}((1-1)g_{\text{nc}}(\mathcal{X})) = \Phi_{\text{nc}}(0) = [X_0, f_0]$ . Thus  $H$  is a deformation retraction of  $\mathcal{T}(S)$  onto the point  $[X_0, f_0]$ . Thus  $\mathcal{T}(S)$  is contractible.  $\square$

# CHAPTER 7

## MAPPING CLASS GROUP INVARIANT METRICS

The work in this chapter is joint with Assaf Bar-Natan. We now study the geometry of the marked moduli space. As mentioned in Section 2.8, the quasiconformal Teichmüller space is a complete metric space with the Teichmüller metric. The formula defining the Teichmüller metric can possibly take an infinite value when applied to two arbitrary marked complete Nielsen-convex hyperbolic structures on a surface of infinite type. Therefore, there is no Teichmüller metric on the marked moduli space, in the sense of a finite real valued distance function. We ask if there exists any mapping class group invariant metric at all, which is compatible with the topology of the marked moduli space. For certain classes of surfaces, the answer is surprisingly no.

### 7.1 Some Facts about the Action

Recall the definitions of *self-similar* surfaces from [MR23], *doubly pointed* surfaces from [APV21], *avenue* surfaces from [HQR22] and *translatable* surfaces from [SC24]. Also see [SC24, Remark 3.3] to compare all these definitions. If  $A$  is an isotopy class of a simple closed curve on  $S$ , then  $T_A \in \text{MCG}(S)$  denotes the *right handed Dehn twist* about  $A$ . We recall some basic facts about hyperbolic geometry and the action of the mapping class group on the surface and marked moduli space.

**Fact 7.1** (Lemma 2.5 in [APV21]). *On a hyperbolic surface, the number of closed geodesics of bounded length intersecting a compact set is finite.*

**Fact 7.2** (Lemma 2.6 in [APV21]). *The action of the isometry group of a complete hyperbolic surface with nonabelian fundamental group on the surface is properly discontinuous.*

The following three facts are well known in geometric topology circles, but we include

proofs here which do not depend on whether the surface is of finite type or infinite type, for the sake of completeness.

**Fact 7.3.** *A mapping class  $G$  fixes a marked hyperbolic structure  $[X, f]$  if and only if the mapping class  $G$  can be realised as an isometry of its hyperbolic surface  $X$ , that is, if and only if there is an isometry  $g$  of  $X$  such that  $G$  is the mapping class of the homeomorphism  $f^{-1} \circ g \circ f$  of  $S$ .*

*Proof.* For the if direction, note that  $G \cdot [X, f] = [X, g^{-1} \circ f]$  by the definition of the action of the mapping class group. However, note that  $g = f \circ (g^{-1} \circ f)^{-1}$  is an isometry of  $X$ , and therefore, by definition of the marked moduli space, pairs  $(X, f)$  and  $(X, g^{-1} \circ f)$  define the same marked hyperbolic structure. In other words,  $G$  fixes the marked hyperbolic structure  $[X, f]$ .

For the only if direction, suppose that  $\hat{g}$  is a homeomorphism of  $S$  in the mapping class  $G$ , and suppose that  $G$  fixes  $[X, f]$ . Since  $G \cdot [X, f] = [X, f \circ \hat{g}^{-1}]$ , we have  $[X, f \circ \hat{g}^{-1}] = [X, f]$ . By definition of the marked moduli space, the map  $f \circ (f \circ \hat{g}^{-1})^{-1} = f \circ \hat{g} \circ f^{-1}$  is homotopic to an isometry  $g : X \rightarrow X$ . Thus the homeomorphisms  $\hat{g}$  and  $f^{-1} \circ g \circ f$  are homotopic as well. Thus  $G$  is the mapping class of the homeomorphism  $f^{-1} \circ g \circ f$ , that is, it is realised as an isometry  $g$  of  $X$ . □

**Fact 7.4.** *If  $A$  is an isotopy class of essential, nonperipheral simple closed curves on a surface  $S$  with nonabelian fundamental group, then the Dehn twist  $T_A$  does not fix any marked hyperbolic structure on  $S$ .*

*Proof.* Since  $S$  has a nonabelian fundamental group, there exists an isotopy class  $B$  of simple closed curves on  $S$  such that the geometric intersection number between  $A$  and  $B$  is positive. Note that since  $T_A(A) = A$  and mapping classes preserve the geometric intersection number of isotopy classes of simple closed curves,  $A$  and  $T_A^n(B)$  also have positive intersection number,

for each  $n \in \mathbb{N}$ . Also note that the isotopy classes  $T_A^n(B)$  are all distinct, since  $A$  and  $B$  cannot be realised disjointly. Assume for contradiction that  $T_A$  fixes a marked hyperbolic structure  $[X, f]$ . Then by Fact 7.3,  $T_A$  is realised as an isometry  $g$  of  $X$ . Let  $a$  be a representative of  $A$  on  $X$  and let  $b$  be the geodesic representative of  $B$  on  $X$ . Then for every  $n \in \mathbb{N}$ ,  $g^n(b)$  is also a geodesic, and represents  $T_A^n(B)$ . Therefore the  $g^n(b)$  are all distinct simple closed geodesics on  $X$ . Moreover, each  $g^n(b)$  has the same length, equal to the length of  $b$ . However, each  $g^n(b)$  intersects  $a$ , a compact set. This contradicts Fact 7.1. We conclude that the Dehn twist  $T_A$  does not fix any marked hyperbolic structure on  $S$ .  $\square$

**Fact 7.5.** *If  $A$  is an isotopy class of simple closed curves on the surface  $S$  and  $G$  is any mapping class of homeomorphisms of  $S$ , then  $G \circ T_A \circ G^{-1} = T_{G(A)}$ .*

*Proof.* This is the content of Fact 3.7 on page 73 of [FM11]. We note that even though this book only considers surfaces of finite type, this fact holds equally well for surfaces of infinite type, and the same proof works.  $\square$

## 7.2 Infinite Order Isometry obstructs MCG Invariant

### Metrics

**Lemma 7.6** (Dehn twist moves infinitely symmetric points to zero distance away). *Let  $S$  be a connected, oriented surface of infinite type. Suppose  $\mathcal{X} = [X, f] \in \mathcal{T}(S)$  is a point in the marked moduli space whose underlying hyperbolic surface  $X$  admits an infinite order isometry  $g$ . Suppose  $A$  is a homotopy class of essential, nonperipheral simple closed curves on  $S$ , and  $d$  is an  $\text{MCG}(S)$ -invariant continuous pseudometric on  $\mathcal{T}(S)$ . Then  $d(T_A(\mathcal{X}), \mathcal{X}) = 0$ .*

*Proof.* By hypothesis,  $g : X \rightarrow X$  is an infinite order isometry. If  $g$  is an orientation reversing

isometry, we replace  $g$  by  $g^2$ , which is also an infinite order isometry and is orientation preserving. Thus we may assume that  $g$  is orientation preserving. Let the simple closed curve  $a$  on  $X$  represent the homotopy class  $f(A)$ , and define  $a_n = g^n(a_0)$ , so that  $a_{n+1} = g(a_n)$ , for every  $n \in \mathbb{N}$ . We claim that the sequence  $\{a_n\}_{n \in \mathbb{N}}$  of simple closed curves escapes every compact set of  $X$  as  $n \rightarrow \infty$ . That is, we claim that for every compact set  $K \subset X$ , there is an  $N \in \mathbb{N}$  such that for  $n > N$ ,  $a_n \subset X \setminus K$ . To see this, assume for contradiction that there is some compact set  $K$  and an infinite subsequence  $\{a_{n_k}\}_{k \in \mathbb{N}}$  such that  $a_{n_k}$  intersects  $K$  for each  $k \in \mathbb{N}$ . We may assume that  $K$  contains  $a$ , otherwise simply replace  $K$  with  $K \cup a$ , which is compact as  $a$  is a simple closed curve, and still satisfies the above property of intersecting each  $a_{n_k}$ . In particular,  $g^{n_k} K \cap K \neq \emptyset$  for each  $k \in \mathbb{N}$ . Note that the isometries  $g^{n_k}$ ,  $k \in \mathbb{N}$ , are all distinct, since our hypothesis is that  $g$  is of infinite order. Thus the set  $\{\varphi \in \text{Isom}(X) \mid \varphi K \cap K \neq \emptyset\}$  is infinite, a contradiction since the isometry group acts properly discontinuously on  $X$  by Fact 7.2. We conclude that the sequence  $\{a_n\}_{n \in \mathbb{N}}$  escapes every compact set of  $X$ .

The simple closed curves  $a_n$  define simple closed curves  $A_n$  up to isotopy on  $S$ , and the isometry  $g$  defines a mapping class  $G$  of homeomorphisms of  $S$ , via the identification  $f : S \rightarrow X$ . More precisely,  $A_n$  is the isotopy class of the simple closed curve  $f^{-1}(a_n)$ , and  $G$  is the mapping class of the homeomorphism  $f^{-1}gf$ . We make two preliminary observations about these mapping classes. Firstly, for large  $n$ ,  $T_{A_n}$  is supported on the complement of a large compact set, and thus  $T_{A_n} \rightarrow \text{id}$  in  $\text{MCG}(S)$  as  $n \rightarrow \infty$ . Since the mapping class group acts continuously on the marked moduli space by Theorem 1.5, we conclude that  $T_{A_n}(\mathcal{X}) \rightarrow \mathcal{X}$  in  $\mathcal{T}(S)$  as  $n \rightarrow \infty$ . Since  $d$  is a continuous pseudometric on  $\mathcal{T}(S)$ , we have  $d(T_{A_n}(\mathcal{X}), \mathcal{X}) \rightarrow d(\mathcal{X}, \mathcal{X}) = 0$  as  $n \rightarrow \infty$ . Secondly, observe that  $G(A_n) = A_{n+1}$ , and hence  $GT_{A_n}G^{-1} = T_{A_{n+1}}$  in  $\text{MCG}(S)$  by Fact 7.5.

Now we are ready to prove the theorem. For each  $n \in \mathbb{N}$ , we compute  $d(T_{A_n}(\mathcal{X}), \mathcal{X}) = d(GT_{A_n}(\mathcal{X}), G(\mathcal{X})) = d(T_{A_{n+1}}G(\mathcal{X}), G(\mathcal{X})) = d(T_{A_{n+1}}(\mathcal{X}), \mathcal{X})$ . Here the first equality uses

the invariance of  $d$  under the mapping class  $G$ , the second equality uses the conjugation property of Dehn twists, and the third equality is true because  $G$  fixes  $\mathcal{X}$ . Thus the function  $n \mapsto d(T_{A_n}(\mathcal{X}), \mathcal{X})$  is a constant function on  $\mathbb{N}$ , and its constant value is the nonzero real number  $d(T_A(\mathcal{X}), \mathcal{X})$ . Hence  $d(T_{A_n}(\mathcal{X}), \mathcal{X}) \rightarrow d(T_A(\mathcal{X}), \mathcal{X})$  as  $n \rightarrow \infty$ . However, as observed above, this limit is actually equal to 0. Therefore,  $d(T_A(\mathcal{X}), X) = 0$ .  $\square$

**Theorem 7.7** (Mapping class group invariant metrics). *Suppose  $X$  is a complete Nielsen-convex hyperbolic surface, homeomorphic to  $S$ , and which admits an infinite order isometry. Then there is no  $\text{MCG}(S)$ -invariant metric on  $\mathcal{T}(S)$  (compatible with its topology).*

*Proof.* Use the same notation as Lemma 7.6. Since the mapping class  $G$  can be realised as an isometry  $g$  on  $X$ , it fixes the marked hyperbolic structure  $\mathcal{X}$  by Fact 7.3. In other words,  $G(\mathcal{X}) = \mathcal{X}$ . However, Dehn twists cannot fix any marked hyperbolic structure by Fact 7.4. So  $T_A(\mathcal{X}) \neq \mathcal{X}$ . If  $d$  were an honest metric, then  $d(T_A(\mathcal{X}), \mathcal{X})$  would be different from zero, contradicting Lemma 7.6. Therefore, there does not exist any  $\text{MCG}(S)$ -invariant metric on  $\mathcal{T}(S)$ .  $\square$

**Corollary 7.8.** *If  $S$  is self-similar with no planar ends, or is translatable, then there is no  $\text{MCG}(S)$ -invariant metric on  $\mathcal{T}(S)$ .*

*Proof.* By [APV21] and [SC24], there is a hyperbolic surface homeomorphic to  $S$  which admits an infinite order isometry, so Theorem 7.7 applies.  $\square$

**Remark 7.9.** *The results in [APV21] state that if  $S$  has a nondisplaceable subsurface, then every hyperbolic surface homeomorphic to  $S$  has a finite isometry group. Therefore our argument does not apply and we cannot conclude that  $\mathcal{T}(S)$  has no  $\text{MCG}(S)$ -invariant metric. However, we do not know of a construction of such an invariant metric either, so the question of the existence remains open in general.*

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