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ON BOUNDS IN ANSTREICHER'S MONOTONIC PROJECTIVE ALGORITHM

by

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ABSTRACT

Anstreicher's projective algorithm for fractional linear programming problems that guarantees monotonic decrease of the objective function uses bounds on the optimal value obtained in three ways. We demonstrate that one of these methods, using linear programming duality, always yields the strongest bounds.

1. <u>Introduction</u>

Anstreicher [1] has developed a variant of Karmarkar's projective algorithm [2] for linear programming that deals with a fractional linear programming problem

$$\text{(FLP)}\qquad \begin{aligned} \min\ & c^Tx/d^Tx \\ & Ax = 0 \\ & e^Tx = n \\ & x \geq 0, \end{aligned}$$

satisfying the assumptions:

1) $Ae = 0$, so that $x = e$ is feasible in (FLP) (here $e \in \mathbb{R}^n$ is a vector of ones);
2) d^Tx is positive and bounded away from zero on the feasible region of (FLP); and
3) a lower bound z^0 on the optimal value z^* of (FLP) is known.

In order to obtain bounds on z^* consider the parametric linear programming problem

$$\text{LP}(z)\qquad \begin{aligned} \min\ & (c-zd)^Tx \\ & Ax = 0 \\ & e^Tx = n \\ & x \geq 0. \end{aligned}$$

Clearly, if the optimal value of LP(z) is nonnegative, then $z^* \geq z$; moreover, this holds if there is a feasible solution to the dual problem

$$\text{DLP}(z) \qquad \begin{array}{ll} \max & n\lambda \\ & A^T y + e\lambda \leq c - zd \end{array}$$

with $\lambda \geq 0$.

Instead of considering LP(z) or its dual, Anstreicher uses three relaxations of LP(z); clearly, if any of these relaxations has nonnegative optimal value, then $z \leq z^*$. The first relaxation notes that the simplex $\{x \in \mathbb{R}^n: e^T x = n,\ x \geq 0\}$ is contained in the ball $\{x \in \mathbb{R}^n: e^T x = n,\ \|x-e\| \leq R\}$ where $R = \sqrt{n(n-1)}$. Thus the following problem is a relaxation of LP(z):

$$P_1(z) \qquad \begin{array}{rl} \theta_1(z) \equiv & \min(c-zd)^T x \\ & Ax = 0 \\ & e^T x = n \\ & \|x-e\| \leq R. \end{array}$$

Next, let P_A denote the orthogonal projection onto the null space of A, and let $c_q = P_A c$, $d_q = P_A d$. Since $Ax = 0$ implies $x = P_A x$ and $P_A^T = P_A$, the objective function of LP(z) can be replaced by $(c_q - zd_q)^T x$. Then by relaxing the constraints $Ax = 0$ we are led to

$$P_2(z) \qquad \begin{array}{rl} \theta_2(z) \equiv & \min(c_q - zd_q)^T x \\ & e^T x = n \\ & x \geq 0. \end{array}$$

Note that, if $\theta_2(z) \geq 0$ and A has full row rank, so that $P_A = I-A^T(AA^T)^{-1}A$, then $y = (AA^T)^{-1}A(c-zd)$, $\lambda = 0$ is feasible in DLP(z); thus the lower bound z is certified by duality. This is the bound introduced by Todd and Burrell [3] in the linear programming case.

Note also that the feasible region of $P_2(z)$ has n extreme points, $\{ne_j: 1 \leq j \leq n\}$. Thus we can express $\theta_2(z)$ alternatively as $\min_j\{ne_j^T(c_q-zd_q)\}$.

The third "relaxation" uses the value of (FLP) at $x = e$; call this $v = c^Te/d^Te$. The optimal solution of (FLP), say x^*, is clearly feasible in LP(z) and $P_1(z)$ and also satisfies $(c-vd)^Tx^* \leq 0$. Thus, if we define

$$
P_1^v(z) \qquad
\begin{aligned}
\theta_1^v(z) \equiv \min(c-zd)^T&x \\
Ax &= 0 \\
e^Tx &= n \\
(c-vd)^Tx &\leq 0 \\
\|x-e\| &\leq R,
\end{aligned}
$$

then $\theta_1^v(z) \geq 0$ implies $z \leq z^*$.

Since $x = e$ is feasible in all these problems, $\theta_1(v)$, $\theta_2(v)$ and $\theta_1^v(v)$ are all nonpositive. Hence, if we find some $z < v$ with $\theta_1(z)$ (or $\theta_2(z)$, $\theta_1^v(z)$) positive, we can perform a relatively simple search (see [1]) to obtain $z < \bar{z} \leq v$ with $\theta_1(\bar{z})$ (or $\theta_2(\bar{z})$, $\theta_1^v(\bar{z})$) equal to zero. This value $\bar{z}$ will then be our new bound.

Anstreicher uses a combination of these methods to generate bounds in his method. In particular, in case III of his section 5, he uses the bound given by $\theta_1^V(\cdot)$. In the next section we show that $\theta_2(\cdot)$ will always give the best bound. Thus bounds derived from duality and the method of Todd and Burrell [3] dominate those obtained from spherical or hemispherical relaxations.

Anstreicher's algorithm combines these bounds with a clever choice of direction to obtain an iterative scheme with the same guarantees of convergence promised by Karmarkar's method but also assuring that c^Tx^k/d^Tx^k is monotonically nonincreasing. For details, see [1]. This note is only concerned with generating bounds. However, it shows that cases I and II in section 5 of [1] suffice for the algorithm if $\bar{z}$ is updated using $\theta_2(\cdot)$ whenever possible.

While this discussion concerns only the initial iteration starting at $x = e$, Anstreicher shows that each iteration is equivalent to the first with modified data A, c and d. Thus it is sufficient to consider the first iteration.

2. The Results

We show that $\theta_2(\cdot)$ always yields a better bound than $\theta_1(\cdot)$ or $\theta_1^V(\cdot)$. While the first result is noted in Anstreicher [1], we give a short proof here also because the more difficult result uses similar reasoning.

Theorem 1. For any z, $\theta_1(z) \le \theta_2(z)$. Thus if $\bar{z}_1$ is a zero of θ_1, θ_2 has a zero $\bar{z}_2$ with $\bar{z}_2 \ge z_1$.

Theorem 2. Suppose $\bar{z}_1^V$ is a zero of θ_1^V. Then θ_2 has a zero $\bar{z}_2$ with $\bar{z}_2 \ge \bar{z}_1^V$.

The first step in the proofs is to express $\theta_1(z)$ and $\theta_1^V(z)$ as optimal values of problems without the constraint $Ax = 0$. For any vector u, let u_p, u_q and u_r denote the orthogonal projections of u onto the null spaces of $[A^T, e]^T$, A and e^T respectively. Since $Ae = 0$, it is easy to see that $(u_q)_r = u_p$ for any u.

The presence of the constraint $Ax = 0$ in both $P_1(z)$ and $P_1^V(z)$ implies that we can replace their objective functions by $(c_q - zd_q)^T x$; similarly, we can replace the constraint $(c-vd)^T x \leq 0$ by $(c_q - vd_q)^T x \leq 0$. Next, Anstreicher [1] shows that the optimal solutions of the two problems are of the form

$$\bar{x} = e - R(c_p - wd_p)/\|c_p - wd_p\|, \tag{1}$$

where $w = z$ in $P_1(z)$ and $w \geq z$ is the smallest value such that $(c_q - vd_q)^T \bar{x} \leq 0$ in $P_1^V(z)$. Since $(c_q - td_q)_r = c_p - td_p$ for any t, it follows that the constraint $Ax = 0$ can be omitted from the formulation of the two problems when c_q and d_q replace c and d. Thus we have

$$\bar{P}_1(z) \qquad \begin{aligned} \theta_1(z) \equiv \min (c_q - zd_q)^T x \\ e^T x = n \\ \|x - e\| \leq R, \end{aligned} \quad \text{and}$$

$$\bar{P}_1^V(z) \qquad \begin{aligned} \theta_1^V(z) \equiv \min (c_q - zd_q)^T x \\ e^T x = n \\ (c_q - vd_q)^T x \leq 0 \\ \|x - e\| \leq R. \end{aligned}$$

Proof of Theorem 1. From the above formulation, we see that $\theta_1(z)$ is the optimal value of a problem that is a relaxation of $P_2(z)$, since $e^T x = n$ and $x \geq 0$ imply $\|x-e\| \leq R$. Thus the first part of theorem 1 follows immediately. If $\theta_1(\bar{z}_1) = 0$, we deduce $\theta_2(\bar{z}_1) \geq 0$. Now $\bar{z}_1 \leq v$ and $\theta_2(v) \leq 0$; also θ_2 is piecewise-linear and concave. Hence $\theta_2(\bar{z}_2) = 0$ for some $\bar{z}_1 \leq \bar{z}_2 \leq v$, and the proof is complete.

Proof of Theorem 2. Note first that, if $\bar{z}_1^v$ is also a zero of θ_1, then the result follows from theorem 1. Hence we may assume that the optimal solution of $\bar{P}_1^v(\bar{z}_1^v)$ is $\bar{x}$ in (1), with $w > \bar{z}_1^v$, and that θ_1^v is monotonically decreasing everywhere--see [1].

Now $\theta_2(z) = \min_j\{ne_j^T(c_q - zd_q)\}$ is piecewise-linear and concave, and since $e^T d_q = e^T d > 0$, $\theta_2(z)$ is negative for large z. We distinguish two cases.

Case 1. θ_2 has a zero.

If θ_2 has at least two zeroes, we choose $\bar{z}_2$ as the rightmost one; otherwise, $\bar{z}_2$ is the unique zero. In either case, by considering $\theta_2(z)$ for z just greater than $\bar{z}_2$ we find k with $(ne_k)^T(c_q - \bar{z}_2 d_q) = 0$ and $ne_k^T d_q > 0$. Since $\bar{z}_2 \leq v$, we find that $x = ne_k$ satisfies $(c_q - vd_q)^T x = (c_q - \bar{z}_q d_q)^T x - (v - \bar{z}_2) d_q^T x \leq 0$. Hence x is feasible in $\bar{P}_1^v(\bar{z}_2)$, implying that $\theta_1^v(\bar{z}_2) \leq 0$. Because θ_1^v is monotonically decreasing, it follows that $\bar{z}_1^v \leq \bar{z}_2$ as desired.

Case 2. θ_2 has no zero.

Then, since $\theta_2(v) \leq 0$, θ_2 is negative everywhere. But as a piecewise-linear concave function with a finite number of pieces, it attains its maximum, either on a horizontal piece or where two pieces with opposite slopes meet. In the latter case we can combine the slopes to get zero, and thus in either case there is a vector u satisfying

$$u \geq 0, \quad c_q^T u < 0 \quad \text{and} \quad d_q^T u = 0.$$

Scale u if necessary so that $e^T u = n$. Then u is feasible in $\bar{P}_1^v(z)$ for all z, with a negative objective function value. Hence θ_1^v is negative everywhere, contradicting the hypothesis that $\bar{z}_1^v$ is a zero. Case 2 can therefore not occur, and the proof is complete.

References

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