UNBOUNDED MULTILINEAR MULTIPLIERS ADAPTED TO LARGE SUBSPACES AND ESTIMATES FOR DEGENERATE SIMPLEX OPERATORS

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We prove in Chapter 1 that for every integer $n \geq 3$ the n-sublinear map

$$nC^{\vec{\alpha}} : (f_1, ..., f_n) \mapsto \sup_M \left| \int_{\vec{\xi}\cdot\vec{\alpha}>0, \xi_n<M} \left[\prod_{j=1}^{n} \hat{f}_j(\xi_j) e^{2\pi i x \xi_j} \right] d\vec{\xi} \right|$$

satisfies no L^p estimates provided $\vec{\alpha} \in \mathbb{R}^n$ satisfies $\alpha_j^{-1} = \alpha + q_j$ for some $\vec{q} \in \mathbb{Q}^n$ with distinct entries and $\alpha \in \mathbb{R}$ with $q_j \neq -\alpha$ for all $1 \leq j \leq n$. Furthermore, if $n \geq 5$ and $\vec{\alpha} \in \mathbb{R}^n$ satisfies $\alpha_j^{-1} = q_j + \alpha q_j^2$ for some $\vec{q} \in \mathbb{Q}^n$ with distinct, non-zero entires such that $q_j \alpha \neq -1$ for all $1 \leq j \leq n$, it is shown that there is a symbol $m : \mathbb{R}^n \to \mathbb{C}$ adapted to the hyperplane $\Gamma^{\vec{a}} = \left\{ \vec{\xi} \in \mathbb{R}^n : \sum_{j=1}^n \xi_j \cdot a_j = 0 \right\}$ and supported in $\left\{ \vec{\xi} : dist(\vec{\xi}, \Gamma^{\vec{\alpha}}) \lesssim 1 \right\}$ for which the associated n-linear multiplier also satisfies no L^p estimates. Next, we construct a Hörmander-Marcinkiewicz symbol $\Pi : \mathbb{R}^2 \to \mathbb{R}$, which is a paraproduct of (ϕ, ψ) type, such that the trilinear operator T_m whose symbol m is given by $m(\xi_1, \xi_2, \xi_3) = sgn(\xi_1 + \xi_2)\Pi(\xi_2, \xi_3)$ satisfies no L^p estimates. Finally, we establish a converse to a theorem of C. Muscalu, T. Tao, and C. Thiele concerning estimates for multipliers with subspace singularities of dimension at least half of the total space dimension using Riesz kernels in the spirit of C. Muscalu's recent work. Specifically, for every pair of integers $(\mathfrak{d}, n)$ s.t. $\frac{n}{2} + \frac{3}{2} \leq \mathfrak{d} < n$ we construct an explicit collection $\mathfrak{C}$ of uncountably many $\mathfrak{d}$-dimensional non-degenerate subspaces of $\mathbb{R}^n$ such that for each $\Gamma \in \mathcal{C}$ there

is an associated symbol m_Γ adapted to Γ in the Mikhlin-Hörmander sense and supported in $\left\{\vec{\xi} : dist(\vec{\xi}, \Gamma) \lesssim 1\right\}$ for which the associated multilinear multiplier T_{m_Γ} is unbounded.

In Chapter 2, we consider for each $2 < p \leq \infty$ the space $W_p(\mathbb{R}) = \left\{f \in L^p(\mathbb{R}) : \hat{f} \in L^{p'}(\mathbb{R})\right\}$ with norm $||f_2||_{W_{p_2}(\mathbb{R})} = ||\hat{f}_2||_{L^{p'_2}(\mathbb{R})}$. Letting $\Gamma = \{\xi_1 + \xi_2 = 0\} \subset \mathbb{R}^2$ and $a_1, a_2 : \mathbb{R}^2 \to \mathbb{C}$ satisfy

$$\left|\partial^{\vec{\alpha}} a_j \left(\vec{\xi}\right)\right| \lesssim_{\vec{\alpha}} \frac{1}{dist(\vec{\xi}, \Gamma)^{|\vec{\alpha}|}}$$

for each $j \in \{1, 2\}$ and sufficiently many multi-indices $\vec{\alpha} \in (\mathbb{N} \bigcup \{0\})^2$, we construct a time-frequency framework to show that the degenerate trilinear simplex multiplier defined for any $(f_1, f_2, f_3) \in \mathcal{S}(\mathbb{R})^3$ by the formula

$$B[a_1, a_2] : (f_1, f_2, f_3) \to \int_{\mathbb{R}^3} a_1(\xi_1, \xi_2) a_2(\xi_2, \xi_3) \left[\prod_{j=1}^{3} \hat{f}_j(\xi_j) e^{2\pi i x \xi_j}\right] d\xi_1 d\xi_2 d\xi_3$$

maps $L^{p_1}(\mathbb{R}) \times W_{p_2}(\mathbb{R}) \times L^{p_3}(\mathbb{R})$ into $L^{\frac{1}{\frac{1}{p_1}+\frac{1}{p_2}+\frac{1}{p_3}}}(\mathbb{R})$ provided

$$1 < p_1, p_3 < \infty, 2 < p_2 < \infty, \frac{1}{p_1} + \frac{1}{p_2} < 1, \frac{1}{p_2} + \frac{1}{p_3} < 1.$$

Unlike non-degenerate simplex multipliers, $B[a_1, a_2]$ cannot be written as a global paracomposition modulo harmless error terms. Nonetheless, $B[a_1, a_2]$ can be written as a local paracomposition, and we show this is enough to obtain generalized restricted type mixed estimates. Mixed Marcinkiewicz interpolation then finishes the argument.

Lastly, in Chapter 3, we turn our attention to proving a wide range of L^p estimates for two so-called semi-degenerate simplex multipliers defined on tuples of Schwartz functions by the following maps:

$$\begin{aligned} C^{1,1,-2} : (f_1, f_2, f_3) &\mapsto \int_{\mathbb{R}^3} \hat{f}_1(\xi_1)\hat{f}_2(\xi_2)\hat{f}_3(\xi_3)e^{2\pi ix(\xi_1+\xi_2-2\xi_3)}d\vec{\xi} \\ C^{1,1,1,-2} : (f_1, f_2, f_3, f_4) &\mapsto \int_{\mathbb{R}^4} \hat{f}_1(\xi_1)\hat{f}_2(\xi_2)\hat{f}_3(\xi_3)\hat{f}_4(\xi_4)e^{2\pi ix(\xi_1+\xi_2+\xi_3-2\xi_4)}d\vec{\xi}. \end{aligned}$$

We also construct elementary counterexamples to show that our target L^p ranges for both $C^{1,1,-2}$ and $C^{1,1,1,-2}$ are sharp. A crucial ingredient in our analysis of the semi-degenerate setting is a novel l^1-based energy estimate.

BIOGRAPHICAL SKETCH

The author was both in Norfolk, VA, USA on March 7, 1989. He graduated from Princeton University in 2011 with a Bachelor of Arts degree in mathematics under the direction of Elliott Lieb and began doctoral studies at Cornell University the same year. Awarded a Hutchinson Fellowship in 2013, he then pursued research in multilinear harmonic analysis under the direction of his advisor Professor Camil Muscalu.

His time at Cornell resulted in two publications. One paper extended previous REU work on the Casimir Effect and was published with Benjamin Steinhurst in 2013 as a chapter in the *Contemporary Mathematics* book series of the AMS. The other paper resolved a problem involving mixed estimates using Christ-Kiselev-Paley decompositions, generalized Rubio de Francia inequalities, and the Bi-Carleson operator published in the *Journal of Mathematical Analysis and Applications* in 2015. See [9, 10] for details.

To my wife, whose unrelenting support made this dissertation possible

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CHAPTER 1

UNBOUNDED MULTILINEAR MULTIPLIERS ADAPTED TO LARGE SUBSPACES

1.1 Introduction

Camil Muscalu, Terry Tao, and Christoph Thiele prove in [15] that a maximal variant of the BHT called the Bi-Carleson operator defined *a priori* for any pair of Schwartz functions f_1, f_2 by

$$BiC(f_1, f_2)(x) = \sup_{N \in \mathbb{R}} \left| \int_{\xi_1 < \xi_2 < N} \hat{f}_1(\xi_1) \hat{f}_2(\xi_2) e^{2\pi i x(\xi_1 + \xi_2)} d\xi_1 d\xi_2 \right|$$

extends to a continuous map $L^{p_1}(\mathbb{R}) \times L^{p_2}(\mathbb{R}) \to L^{\frac{p_1 p_2}{p_1 + p_2}}(\mathbb{R})$ for all (p_1, p_2) such that $1 < p_1, p_2 \leq \infty$ and $0 < \frac{1}{p_1} + \frac{1}{p_2} < 3/2$. While the Bi-Carleson is non-degenerate, it is well known that multi-(sub)linear Fourier multipliers with degenerate singularities may satisfy no L^p estimates. To be precise, we recall the following two definitions:

Definition 1. *[18] A $\mathfrak{d}-$dimensional subspace $\Gamma \subset \mathbb{R}^n$ is said to be non-degenerate provided*

$$\tilde{\Gamma} := \left\{ (\xi_1, ..., \xi_{n+1}) : (\xi_1, ..., \xi_n) \in \Gamma, \sum_{j=1}^{n+1} \xi_j = 0 \right\} \subset \mathbb{R}^{n+1}$$

is a graph over the variables $(\xi_{i_1}, ..., \xi_{i_\mathfrak{d}})$ for every chain $1 \leq i_1 < ... < i_\mathfrak{d} \leq n+1$. A subspace $\Gamma \subset \mathbb{R}^n$ is said to be degenerate if it is not non-degenerate.

Definition 2. *An n-(sub)linear operator T defined* a priori *on* $\mathcal{S}(\mathbb{R})^n$ *satisfies no* L^p *estimates provided there does not exist any* $n-$*tuple* $(p_1, ..., p_n) \in [1, \infty]^n$ *for which threre is a constant* $C_{T,\vec{p}}$ *such that*

$$\left|\left|T\left(\vec{f}\right)\right|\right|_{\frac{1}{\sum_{j=1}^{n}\frac{1}{p_j}}} \leq C_{T,\vec{p}} \prod_{j=1}^{n} ||f_j||_{p_j}$$

for all n-tuples of functions $(f_1, ..., f_n) \in \mathcal{S}(\mathbb{R})^n$.

For example, [19] shows that the degenerate operator

$$\widetilde{BiC}(f_1, f_2)(x) = \sup_{M \in \mathbb{R}} \left| \int_{\xi_1+\xi_2<0, \xi_2<M} \hat{f}_1(\xi_1)\hat{f}_2(\xi_2)e^{2\pi i x(\xi_1+\xi_2)}d\xi_1 d\xi_2 \right|$$

satisfies no L^p estimates even though the Hilbert transform of a product of functions $H_2 : (f, g) \mapsto H(f \cdot g)$ is a multiplier with degenerate singularity $\{\xi_1 + \xi_2 = 0\}$ that still maps into L^p for $1 < p < \infty$. Degenerate simplex multipliers for which no L^p estimates hold do nonetheless have weaker mixed estimates, see [9]. Our first result in this paper is to exhibit L^p unboundedness results for the most natural generalization of BiC to higher dimensions. Specifically, we prove

Theorem 1. *Fix a dimension* $n \geq 3$ *along with* $\vec{\alpha} \in \mathbb{R}^n$ *satisfying* $\alpha_j^{-1} = \alpha + q_j$ *for some* $\vec{q} \in \mathbb{Q}^n$ *and* $\alpha \in \mathbb{R}$ *such that* $q_i \neq q_j$ *whenever* $i \neq j$ *and* $q_j \neq -\alpha$ *for all* $1 \leq j \leq n$. *Then the operator defined a priori on* $\mathcal{S}(\mathbb{R})^n$ *by the formula*

$$nC^{\vec{\alpha}}(\vec{f})(x) = \sup_{M \in \mathbb{R}} \left| \int_{\vec{\xi}\cdot\vec{\alpha}>0, \xi_n<M} \left[\prod_{j=1}^{n} \hat{f}_j(\xi_j)e^{2\pi i x \xi_j}\right] \xi_1 ... d\xi_n \right|.$$

satisfies no L^p *estimates.*

Because of [16], this result is new only in 3 dimensions. Moreover, the most natural generalization of BHT to higher dimensions is the $n-$linear Hilbert transform, for which negative results have already been obtained when $n = 3$ by C. Demeter in [3] for target exponent below $\frac{1}{3}\left(1 + \frac{\log_6 2}{1+\log_6 2}\right)$. Our next result is

Theorem 2. *There exists a Hörmander-Marcinkiewicz symbol* $a : \mathbb{R}^2 \to \mathbb{R}$ *of* (ϕ, ψ) *type, i.e.* a *is sum of tensor products that are* ϕ *type in the first index and* ψ *type in the second index satisfying* $|\partial^{\vec{\alpha}} a(\xi)| \leq \frac{C_{\vec{\alpha}}}{dist(\vec{\xi},\vec{0})^{|\vec{\alpha}|}}$*, such that the trilinear operator* T_m *whose symbol* m *is given by* $m(\xi_1, \xi_2, \xi_3) = sgn(\xi_1+\xi_2)a(\xi_2, \xi_3)$ *satisfies no* L^p *estimates.*

To make sense of the remaining statements, we shall need

Definition 3. *[16] For each subspace* $\Gamma \subset \mathbb{R}^d$ *let*

$$\mathcal{M}_\Gamma(\mathbb{R}^d) = \left\{ m : \mathbb{R}^d \to \mathbb{R} : \forall\ \vec{\alpha} \in (\mathbb{N} \cup \{0\})^d\ \exists C_{\vec{\alpha}} \text{ such that } \left|\partial^{\vec{\alpha}} m(\vec{\xi})\right| \leq \frac{C_{\vec{\alpha}}}{dist(\vec{\xi}, \Gamma)^{|\vec{\alpha}|}} \right\}.$$

Definition 4. *For* $m \in L^\infty(\mathbb{R}^d)$ *let* T_m *be the operator defined* a priori *on d-tuples of Schwartz functions* $\vec{f} = (f_1, \cdots, f_d)$ *by the formula*

$$T_m(\vec{f})(x) := \int_{\mathbb{R}^d} m(\vec{\xi}) \left[\prod_{j=1}^d \hat{f}_j(\xi_j) e^{2\pi i x \xi_j} \right] d\vec{\xi}.$$

The following is already known:

Theorem 3 ([16])**.** *For any two generic non-degenerate subspaces* $\Gamma_1, \Gamma_2 \subseteq \mathbb{R}^n$ *of maximal dimension* $n-1$*, there are symbol* $m_1 \in \mathcal{M}_{\Gamma_1}(\mathbb{R}^n)$ *and* $m_2 \in \mathcal{M}_{\Gamma_2}(\mathbb{R}^n)$ *for which at least one of the associated n-linear operators* T_{m_1}, T_{m_2} *does not satisfy any* L^p *estimates.*

Our next result gives an uncountable collection of subspaces with adapted symbols satisfying no L^p estimates:

Theorem 4. *Let* $n \geq 5$ *and* $\vec{\alpha} \in \mathbb{R}^n$ *satisfy* $\alpha_j^{-1} = q_j + \alpha q_j^2$ *for some* $\vec{q} \in \mathbb{Q}^n$ *with distinct, non-zero entires such that* $q_j \alpha \neq -1$ *for all* $1 \leq j \leq n$*. Then there exists* $m \in \mathcal{M}_{\Gamma^{\vec{\alpha}}}(\mathbb{R}^n)$ *such that* T_m *satisfies no* L^p *estimates.*

This statement should be viewed as a microlocal version of one of the main results in [16] that the intersection of two generic half-spaces does not produce a bounded operator on any L^p space. Indeed, unboundedness of multipliers given by intersections of generic half-spaces implies that for a given $\vec{p}$ there is a symbol adapted in the Mikhlin-Hörmander sense either to one hyperplane or a symbol adapted to the other hyperplane for which at least one of the two multipliers satisfies no L^p estimates, i.e.

$$T_{1_{\Gamma^{\vec{\alpha}} \cap \Gamma^{\vec{\beta}}}} = T_{m_1} + T_{m_2},$$

where $m_1 \in \mathcal{M}_{\Gamma^{\vec{\alpha}}}(\mathbb{R}^d)$ and $m_2 \in \mathcal{M}_{\Gamma^{\vec{\beta}}}(\mathbb{R}^d)$. Of course, if T does not satisfy any L^p estimates than either T_{m_1} or T_{m_2} does not satisfy estimates for a given $\vec{p} = (p_1, ..., p_d)$. In fact, C. Muscalu's arguments show that at least one of T_{m_1} and T_{m_2} must satisfy no L^p estimates. Hence, the main benefit of our calculations is to exhibit unbounded hyperplane-adapted multipliers for an explicit collection of uncountably many hyperplanes.

C. Muscalu strengthens Theorem 3 in [16] by showing that generic subspaces of codimension smaller than around $\sqrt{n}$ also satisfy no L^p estimates:

Theorem 5 ([16]). *Let $n \geq 5$. For any $1 < p_1, ..., p_n \leq \infty$ and $0 < p < \infty$ with $1/p_1 + ... + 1/p_n = 1/p$, for any integer $\mathfrak{d}$ satisfying*

$$n - \left(\frac{n-1}{2}\right)^{\frac{1}{2}} < \mathfrak{d} \leq n - 1,$$

and for any two non-degenerate subspaces $\Gamma_1, \Gamma_2 \subset \mathbb{R}^n$ with $dim(\Gamma_1) = dim(\Gamma_2) = \mathfrak{d}$, there are symbols $m_1 \in \mathcal{M}_{\Gamma_1}(\mathbb{R}^n)$ and $m_2 \in \mathcal{M}_{\Gamma_2}(\mathbb{R}^n)$ for which at least one of the associated $n-$linear operators T_{m_1}, T_{m_2} do not map $L^{p_1} \times \cdots \times L^{p_n}$ into L^p.

In the positive direction, C. Muscalu et al. have established L^p estimates for Mikhlin-Hörmander multipliers adapted to non-degenerate subspace singularities of dimension at most roughly half of the total dimension; among other results, they have

Theorem 6 ([18]). *Let $\Gamma \subset \mathbb{R}^n$ be a non-degenerate subspace of dimension $\mathfrak{d}$ where*

$$0 \leq \mathfrak{d} < \frac{n+1}{2}$$

and furthermore suppose $m \in \mathcal{M}_\Gamma(\mathbb{R}^n)$. Then T_m maps $L^{p_1}(\mathbb{R}) \times \cdots \times L^{p_n}(\mathbb{R}) \to L^{p_{n+1}}(\mathbb{R})$ for all $\sum_{j=1}^{n+1} \frac{1}{p_j} = 1$ and $1 \leq p_j < \infty$ for all $j \in \{1, ..., n+1\}$.

Our last theorem should be viewed as a converse to Theorem 6. Adopting the strategy of [16], we use Riesz kernels to prove the existence of uncountably many non-degenerate subspaces of dimension roughly half of the total spatial dimension such that for each subspace there is a Mikhlin-Hörmander multiplier adapted to that subspace that satisfies no L^p estimates. We have the following formal statement:

Theorem 7. *Let $n, \mathfrak{d} \in \mathbb{N}$ satisfy $\frac{n+3}{2} \leq \mathfrak{d} < n$ and $n \geq 5$. Then there is an uncountable collection $\mathfrak{C}$ of $\mathfrak{d}-$ dimensional non-degenerate subspace $\Gamma \subset \mathbb{R}^n$ such that for each $\Gamma \in \mathfrak{C}$ there is an associated symbol m_Γ adapted to Γ in the Mikhlin-Hormander sense for which the associated multilinear multiplier T_{m_Γ} is unbounded.*

1.1.1 Open Questions

These theorems leave open the question of L^p estimates for Mikhlin-Hörmander multipliers with non-degenerate singularities of dimension $\mathfrak{d} \in [\frac{n+1}{2}, \frac{n+3}{2})$. Hence, for each $n \geq 3$ there is a unique dimension $\mathfrak{d}(n) = \lceil \frac{n+1}{2} \rceil$ for which $n-$linear Mikhlin-Hörmander multipliers adapted to singularities of dimension $\mathfrak{d}(n)$ do not have guaranteed L^p estimates via Theorem 6 and for which Theorem 7 provides no multiplier counterexamples. It is therefore likely that new ideas are required to understand the behavior of subspace-adapted multipliers in this range.

Furthermore, our arguments only show the existence of particular subspaces of small dimension for which there exist unbounded multipliers. It is easy to see that for a *generic* choice of non-degenerate subspace no counterexamples can be constructed using our methods with codimension larger than roughly the square root of the total space dimension.

Question 1. *Can one prove Theorems 1 and 4 only assuming $\vec{\alpha} \in \mathbb{R}^n$ with distinct non-zero entries?*

Question 2. *For a given $n \geq 3$ is there a non-degenerate subspace $\Gamma \subset \mathbb{R}^n$ of dimension $\lceil \frac{n+1}{2} \rceil$ and Mikhlin-Hörmander multiplier $m : \mathbb{R}^n \to \mathbb{R}$ adapted to Γ for which no L^p estimates are satisfied?*

Question 3. *If the answer to Question 2 is yes, can one produce generic counterexamples having singularities of the smallest possible size? That is, for a given $n \geq 3$ and* generic *choice of non-degenerate subspace Γ of dimension $\lceil \frac{n+1}{2} \rceil$ can one construct a Mikhlin-Hörmander multiplier $m : \mathbb{R}^n \to \mathbb{R}$ adapted to Γ for which no L^p estimates are satisfied?*

1.1.2 Method of Proof and Organization

The Gaussian chirps appearing throughout this paper were originally devised to show unboundedness for particular multilinear multipliers related to AKNS expansions in [19] and were later applied in [16] to construct unbounded multipliers given by the intersection of two generic hyperplanes and to generate unbounded multipliers with singularities of small codimension using Riesz transforms. In addition, a lower bound on the size of Bohr sets will enable us to extend counterexamples featuring hyperplanes with normal vectors in $\mathbb{R}(\mathbb{Q}^n)$ to the non-trivial irrational setting.

The organization of this chapter is as follows:

§2 proves Theorem 1.

§3 proves Theorem 2.

§4 proves Theorem 4.

§5 proves Theorem 7.

1.1.3 Notation

In an abuse of notation, the author has not used the principal value symbol $p.v.$ for the many singular integrals appearing throughout this paper. Where necessary, the reader should always take such integrals in the principal value sense. Moreover, one should interpret $\lesssim$ to mean $\leq C$ where the constant C depends on inessential parameters, which should be clear from context. The symbol $\mathcal{S}(\mathbb{R})$ will always denote the collection of Schwartz functions on the real line.

1.2 Higher-Dimensional Generalizations of the Bi-Carleson Operator

Theorem 1. *Fix a dimension $n \geq 3$ along with $\vec{\alpha} \in \mathbb{R}^n$ satisfying $\alpha_j^{-1} = \alpha + q_j$ for some $\vec{q} \in \mathbb{Q}^n$ and $\alpha \in \mathbb{R}$ such that $q_i \neq q_j$ whenever $i \neq j$ and $q_j \neq -\alpha$ for all $1 \leq j \leq n$. Then the operator defined a priori on $\mathcal{S}(\mathbb{R})^n$ by the formula*

$$dC^{\vec{\alpha}}(\vec{f})(x) = \sup_{M \in \mathbb{R}} \left| \int_{\vec{\xi}\cdot\vec{\alpha}>0, \xi_n < M} \left[\prod_{j=1}^{n} \hat{f}_j(\xi_j) e^{2\pi i x \xi_j} \right] \xi_1 ... d\xi_n \right| .$$

satisfies no L^p estimates.

Setting $\alpha = 0$, we obtain

Corollary 1. *Let $n \geq 3$ and $\vec{q} \in \mathbb{Q}^n$ with distinct non-zero entires. Then the operator defined a priori on $\mathcal{S}(\mathbb{R})^n$ by the formula*

$$dC^{\vec{\alpha}}(\vec{f})(x) = \sup_{M \in \mathbb{R}} \left| \int_{\vec{\xi}\cdot\vec{\alpha}>0, \xi_n < M} \left[\prod_{j=1}^{n} \hat{f}_j(\xi_j) e^{2\pi i x \xi_j} \right] \xi_1 ... d\xi_n \right| .$$

satisfies no L^p estimates.

To prove Theorem 1, we shall use an elementary fact concerning the existence of orthogonal rational vectors:

Lemma 1. *Fix $n \geq 3$. Suppose $\vec{\alpha} \in \mathbb{R}^n$ satisfies for all $j \in \{1, 2, 3\}$*

$$\alpha_j^{-1} = \alpha + q_j$$

for some $\alpha \in \mathbb{R}$ *and* $\vec{q} \in \mathbb{Q}^n$ *with distinct entries such that* $q_j \neq -\alpha$ *for all* $1 \leq j \leq n$*. Then there exists a non-trivial solution* $\vec{\#} \in \mathbb{R}^n$ *to the system*

$$\sum_{j=1}^{n} \#_j \alpha_j = \sum_{j=1}^{n} \#_j \alpha_j^2 = 0$$

with the additional property that $\alpha_j^2 \#_j \in \mathbb{Q}$ *for all* $1 \leq j \leq n$*.*

Proof. Introduce $\tilde{\#}_j = \#_j \alpha_j^2$. Then we want to show that there exists $\vec{\tilde{\#}} \in \mathbb{Q}^n$ such that

$$\sum_{j=1}^{n} \tilde{\#}_j = \sum_{j=1}^{n} \tilde{\#}_j \alpha_j^{-1} \quad = \quad 0.$$

Moreover, $\vec{\tilde{\#}} \in \mathbb{Q}^n$ is a solution iff it is orthogonal to $\vec{1}$ and $\vec{\alpha}^{-1}$. Hence, if $\#_j = 0$ for $3 < j \leq n$, any 3-tuple $(\#_1, \#_2, \#_3)$ parallel to the symbolic determinant

$$\det \begin{bmatrix} \vec{e}_1 & \vec{e}_2 & \vec{e}_3 \\ 1 & 1 & 1 \\ \alpha_1^{-1} & \alpha_2^{-1} & \alpha_3^{-1} \end{bmatrix} = \det \begin{bmatrix} \vec{e}_1 & \vec{e}_2 & \vec{e}_3 \\ 1 & 1 & 1 \\ q_1 & q_2 & q_3 \end{bmatrix} \in \mathbb{Q}^3$$

is a solution.

□

Note that if $\alpha \in \mathbb{R} \cap \mathbb{Q}^c$, then $\vec{\alpha} \notin \mathbb{R}(\mathbb{Q}^n)$. We shall also need to include the following brief detour:

1.2.1 Bohr Sets

Fix $S \subset \mathbb{R}, 0 < \rho \leq \frac{1}{2}$, and $N \in \mathbb{N}$. Define the Bohr set $Bohr_N(S, \rho)$ by

$$Bohr_N(S,\rho) := \left\{ n \in \mathbb{Z} \cap [1,N] : \sup_{\xi \in S} ||\xi \cdot n||_{\mathbb{R}/\mathbb{Z}} < \rho \right\}.$$

Lemma 2. *Let* $S \subset \mathbb{R}$ *with* $|S| < \infty, \rho \in (0, \frac{1}{2}]$, *and* $N \in \mathbb{N}$. *Then* $|Bohr_N(S,\rho))| \geq N\rho^{|S|} - 1$.

Proof. The proof is a straightforward adaptation of Lemma 4.22 from *Additive Combinatorics* by Terry Tao and Van Vu [25]. Letting $\mathcal{L}^{|S|}$ denote $|S|-$dimensional Lebesgue measure on $\mathbb{T}^{|S|}$ and $\{\xi_1, ..., \xi_{|S|}\}$ be an enumeration of the elements in S, we have for all $n \in \{1, ..., N\}$

$$\mathcal{L}^{|S|} \left\{ \vec{\theta} \in \mathbb{T}^{|S|} : ||\xi_i \cdot n - \theta_i||_{\mathbb{R}/\mathbb{Z}} < \rho \ \forall \ i \in \{1, ..., |S|\} \right\} = 2^{|S|}\rho^{|S|}.$$

It follows that

$$\begin{aligned} N2^{|S|}\rho^{|S|} &= \sum_{n=1}^{N} \mathcal{L}^{|S|} \left| \left\{ \vec{\theta} : ||\xi_i \cdot n - \theta_i||_{\mathbb{R}/\mathbb{Z}} < \rho \ \forall \ i \in \{1, ..., n\} \right\} \right| \\ &= \int_{\mathbb{T}^{|S|}} \sum_{n=1}^{N} 1_{\left\{ \vec{\theta} \in \mathbb{T}^{|S|} : ||\xi_i \cdot n - \theta_i||_{\mathbb{R}/\mathbb{Z}} < \rho \ \forall \ i \in \{1, ..., |S|\} \right\}} (\vec{\theta}) d\vec{\theta}. \end{aligned}$$

Therefore, there exists $\vec{\theta}_* \in \mathbb{T}^n$ for which

$$\sum_{n=1}^{N} 1_{\left\{ \vec{\theta} \in \mathbb{T}^{|S|} : ||\xi_i \cdot n - \theta_i||_{\mathbb{R}/\mathbb{Z}} < \rho \ \forall \ i \in \{1, ..., |S|\} \right\}} (\vec{\theta}_*) \geq N2^{|S|}\rho^{|S|}.$$

Then $S_N := \{n \in \{1, ..., N\} : ||\xi_i \cdot n - \theta_{*,i}||_{\mathbb{R}/\mathbb{Z}} < \rho \ \forall \ i \in \{1, ..., |S|\}\}$ satisfies $|S_N| \geq N2^{|S|}\rho^{|S|}$. However, for every $(n_1, n_2) \in S_N \times S_N$, the triangle inequality yields

$$||\xi_i \cdot (n_1 - n_2)||_{\mathbb{R}/\mathbb{Z}} < 2\rho \ \forall \ i \in \{1, ..., |S|\}.$$

Hence, $\# \{n \in [1, N] \cap \mathbb{Z} : \sup_{\xi \in S} ||\xi \cdot n||_{\mathbb{R}/\mathbb{Z}} < 2\rho \} \geq N2^{|S|}\rho^{|S|} - 1$. Let $\rho \mapsto \rho/2$.

□

Fix $S \subset \mathbb{R}, 0 < \rho \leq \frac{1}{2}$, and $N \in \mathbb{N}$. Define the Bohr set $Bohr_N(S, \rho)$ by

$$Bohr_N(S, \rho) := \left\{ n \in \mathbb{Z} \cap [1, N] : \sup_{\xi \in S} ||\xi \cdot n||_{\mathbb{R}/\mathbb{Z}} < \rho \right\}.$$

Proof. **1.2.2 PART I: The Rational Case**

Assume $\vec{\alpha} \in \mathbb{Q}^n$ so that $\alpha = 0$. Then dilate $\vec{\alpha}$ by some suitably large integer to ensure $\vec{\alpha} \in \mathbb{Z}^n$ and assume WLOG that $\alpha_n > 0$. Indeed, the proof will easily be seen to hold with minor adjustments for the case $\alpha_d < 0$. Let $\phi \in \mathcal{S}(\mathbb{R})$ satisfy $\phi \geq 0$, $\phi(0) \neq 0$, and have compact Fourier support in $[-\frac{1}{2}, \frac{1}{2}]$. Fix $N \in \mathbb{N}$ and let $A \in \mathbb{Z}$ be chosen independent of N and sufficiently large. What sufficiently large means will be determined later. Construct for each $1 \leq j \leq n$ the function

$$f^{N,A,\#}(x) := \sum_{-N \leq m \leq N} \phi(x - Am)e^{2\pi i A \# m x} =: \sum_{-N \leq m \leq N} f_m^{N,A,\#}(x),$$

where we choose $\vec{\#} \in \mathbb{Z}^n$ such that $\vec{\#} \cdot \vec{\alpha} = \vec{\#} \cdot \vec{\alpha^2} = 0$ and $\#_n > 0$. [Here, $\vec{\alpha^2} = \vec{\alpha} \wedge \vec{\alpha} := (\alpha_1^2, ..., \alpha_n^2)$.] One may always choose $\vec{\#}$ satisfying the above conditions because of our assumptions on $\vec{\alpha}$. So for each $n_0 \in [-N/3, N/3]$ fix $x \in \left[An_0, An_0 + \frac{c_{\vec{\alpha}}}{A}\right]$ for some $c_{\vec{\alpha}} << 1$ to be determined and set $M(n_0) = A\#_n n_0 + 5\#_n$. Inserting $\vec{f}^{N,A,\vec{\#}}$ yields

$$
\begin{aligned}
&\left| nC^{\vec{\alpha}}(\vec{f}^{N,A,\vec{\#}})(x) \right| \\
\geq &\left| \sum_{-N \leq m_1,\ldots,m_n \leq N} \int_{\vec{\xi}\cdot\vec{\alpha}>0, \xi_n < M(n_0)} \left[\prod_{j=1}^{n} \mathcal{F}\left[\phi(\cdot - Am_j)e^{2\pi i A\#_j m_j \cdot}\right](\xi_j) \right] e^{2\pi i x(\sum_{j=1}^{n}\xi_j)} d\vec{\xi} \right| \\
= &\left| \sum_{-N \leq m_1,\ldots,m_n \leq N} T_{\vec{\alpha},n_0}\left(\left\{ f_{m_j}^{N,A,\$_j} \right\}_{j=1}^{n} \right)(x) \right|.
\end{aligned}
$$

The frequency restriction $\xi_n < M(n)$ combined with the compact Fourier support of ϕ ensures that all and only those terms $\vec{m}$ corresponding to $m_n \leq n_0$ contribute non-zero summands. Discretizing the kernel representation of our operator therefore yields

$$
\sum_{k\in\mathbb{Z}} \quad \sum_{-N \leq m_1,\ldots,m_n \leq N : m_n \leq n_0} \int_{A(k-\frac{1}{2})}^{A(k+\frac{1}{2})} \prod_{j=1}^{n} \left[\phi(x - Am_j - \alpha_j t)e^{2\pi i A\#_j m_j (x-\alpha_j t)}\right] \frac{dt}{t}.
$$

The main contribution for fixed k derives from the terms $\vec{m}$ for which $n_0 - m_j(n_0,k) - \alpha_j k = 0$ for each $j \in \{1, ..., n\}$. Whenever this is the case, however, use the conditions $\vec{\#}\cdot\vec{\alpha} = \vec{\#}\cdot\vec{\alpha^2} = 0$ to arrive at a lower bound

$$
\begin{aligned}
&Re\left[e^{-2\pi i(\sum_{j=1}^{n}\#_j)n_0 x} \int_{A(k-\frac{1}{2})}^{A(k+\frac{1}{2})} \prod_{j=1}^{n} \left[\phi(x - Am_j - \alpha_j t)e^{2\pi i A\#_j m_j (x-\alpha_j t)}\right] \frac{dt}{t} \right] \Bigg|_{m_j = n_0 - \alpha_j k} \\
\gtrsim\ & \frac{1}{Ak}.
\end{aligned}
$$

For fixed $k \in \mathbb{Z}$, of course it may be the case that no such vector $\vec{m}$ satisfies the desired inequality. There are two possible reasons for this: either $k < 0$ so the additional constant $m_n \leq n_0$ must be violated, or there exists some coordinate

$j : 1 \leq j \leq n$ satisfying $n_0 - \alpha_j k \notin [1, N]$. However, if $k \geq 0$ and for each $j : 1 \leq j \leq n$ the condition $n_0 - \alpha_j k \in [1, N]$ is satisfied, then $\exists \vec{m} \in [[1, N] \cap \mathbb{N}]^n$ such that $n_0 - m_j - \alpha_j k = 0$ for all $j \in \{1, ..., n\}$ and the desired bound is found. In particular, suppose we further impose the condition $n_0 \in [N/2, 2N/3]$. Then for all $k \in \left[1, \frac{N/3}{\max_{1 \leq j \leq n}\{|\alpha_j|\}}\right] \cap \mathbb{N}$ such a vector $\vec{m}$ exists.

Lemma 3. *To prove Theorem 7, it suffices to show* $\exists c_{\vec{\alpha}} > 0$ *such that for every* $x \in \bigcup_{n_0 \in \mathbb{Z} \cap [N/2, N]} [An_0, An_0 + \frac{c_{\vec{\alpha}}}{A}]$ *and* $k \in \left[1, \frac{N/3}{\max_{1 \leq j \leq n}\{|\alpha_j|\}}\right]$

$$T^k_{\vec{\alpha}, n_0}\left(\left\{f^{N,A,\#_j}\right\}_{j=1}^n\right)(x)$$

$$:= \sum_{-N \leq m_1, ..., m_n \leq N : m_n \leq n_0} \int_{A(k-\frac{1}{2})}^{A(k+\frac{1}{2})} \prod_{j=1}^n \phi(x - Am_j - \alpha_j t) e^{2\pi i A \#_j m_j (x - \alpha_j t)} \frac{dt}{t}$$

satisfies

$$\left| Re\left[e^{-2\pi i A (\sum_{j=1}^n \#_j) n_0 x} T^k\left(\left\{f^{N,A,\#_j}\right\}_{j=1}^n\right)(x)\right]\right| \gtrsim \frac{1}{Ak}.$$

Proof. The first claim is that $T^k\left(\left\{f^{N,A,\#_j}\right\}_{j=1}^n\right)$ decays rapidly in $|k|$ when $k \notin [1, N]$. For $k \in [1, N] \cap \left[1, \frac{N/3}{\max_{1 \leq j \leq n}\{|\alpha_j|\}}\right]^c$, we may content ourselves with the cheapest possible upper bound

$$\left|T^k(f_1^N, ..., f_n^N)(x)\right| \lesssim \frac{1}{Ak}.$$

Summing over all $k \simeq N$ yields a $O(1)$ bound. For $k \geq N$, it is enough to observe

$$
\begin{aligned}
& \sum_{k \geq N} \left| T^k \left(\left\{ f^{N,A,\#_j} \right\}_{j=1}^n \right)(x) \right| \\
\leq & \sum_{k \geq N} \sum_{1 \leq m_1,...,m_n \leq N : m_n \leq n_0} \left| \int_{A(k-\frac{1}{2})}^{A(k+\frac{1}{2})} \prod_{j=1}^n \left[\phi(x - Am_j - \alpha_j t) e^{2\pi i \#_j m_j (x - \alpha_j t)} \right] \frac{dt}{t} \right| \\
\lesssim_A & \sum_{k \geq N} \sum_{1 \leq m_1,...,m_n \leq N} \prod_{j=1}^n \frac{1}{1 + |n_0 - m_j - \alpha_j k|^N} \\
\lesssim_A & \sum_{k \geq N} \sum_{1 \leq m_n \leq N} \frac{1}{1 + |n_0 - m_n - \alpha_n k|^N} \\
\lesssim_A & \sum_{1 \leq m_n \leq N} \frac{1}{1 + |n_0 - m_n|^{N-1}} \\
\lesssim_A & \ 1.
\end{aligned}
$$

Moreover, replacing the sum over $k \geq N$ with the sum $k \leq 0$ yields the same estimate. Therefore,

$$
\begin{aligned}
& Re \left[e^{-2\pi i A (\sum_{j=1}^n \#_j) n_0 x} T \left(\left\{ f^{N,A,\#_j} \right\}_{j=1}^n \right)(x) \right] \\
\geq & \sum_{1 \leq k \lesssim_{A,\vec{\alpha}} N} \frac{1}{Ak} - \left| \sum_{k \geq N} T_k(f_1^N, ..., f_n^N)(x) + \sum_{k \leq 0} T_k(f_1^N, ..., f_n^N)(x) + \sum_{k \simeq_{A,\vec{\alpha}} N} T_k(f_1^N, .., f_n^N)(x) \right| \\
\gtrsim & \ \frac{\log(N)}{A}.
\end{aligned}
$$

Using the point-wise bound $\left| T \left(\left\{ f^{N,A,\#_j} \right\}_{j=1}^n \right)(x) \right| \gtrsim \frac{\log(N)}{A} 1_{S_N}(x)$, where

$$
S_N = \bigcup_{N/2 \leq n_0 \leq N} \left[An_0, An_0 + \frac{c_{\vec{\alpha}}}{A} \right]
$$

together with $|S_N| \simeq_{\vec{\alpha},A} N$, we may conclude

$$
\left| \left| T(f_1^N, ..., f_n^N) \right| \right|_{\frac{1}{\sum_{i=1}^N \frac{1}{p_i}}} \gtrsim_A \log(N) N^{\sum_{i=1}^n \frac{1}{p_i}} >> N^{\sum_{i=1}^n \frac{1}{p_i}} \simeq \prod_{i=1}^n ||f_i||_{p_i}.
$$

Taking N arbitrarily large finishes the proof. $\square$

The remaining portion of §3 is dedicated to proving Lemma 3. We must take a little care in understanding those terms for which the ϕ arguments are relatively small yet oscillation may be introduced with respect to x or t. For fixed $n_0 \in [-N/3, N/3]$ and $k \in \left[1, \frac{N/3}{\max_{1\le j\le n}\{|\alpha_j\}}\right]$, let $m_j(n_0, k) := n_0 - \alpha_j k$. We now proceed to organize the sum over $\vec{m} \in [1, N]^n$ around this "core" vector into two sets: small perturbations and large perturbations. Essentially, integration by parts together with the integrality of $\vec{m}$ and $\vec{\alpha}$ will allow us to handle those terms arising from the small perturbations successfully.

Small Perturbations: $\sum_{j=1}^{n} \#_j \Delta_j \alpha_j = 0$

This contribution cannot be subsumed as error. As before, let $\vec{m}(n_0, k) = n_0 - \vec{\alpha}k$ be the unperturbed initial state. Let Δ_j satisfy $\tilde{m}_j = m_j(n_0, k) + \Delta_j$ for each $j \in \{1, ..., n\}$ and $|\Delta_j| \le 2 \cdot max_{1\le j\le n}\{|\alpha_j|\}$. Then the contribution of the perturbed summand is

$$Re\left[e^{-2\pi i A(\sum_{j=1}^{n} \#_j) n_0 x} \int_{A(k-\frac{1}{2})}^{A(k+\frac{1}{2})} \prod_{j=1}^{n} \left[\phi(x - Am_j - \alpha_j t) e^{2\pi i A \#_j m_j (x-\alpha_j t)}\right] \frac{dt}{t}\right].$$

Because $x \in [An_0, An_0 + \frac{c_{\vec{\alpha}}}{A}]$, we may rewrite $x = An_0 + \theta_x$, where $|\theta_x| \le \frac{c_{\vec{\alpha}}}{A}$, and use the integrality condition:

$$
\begin{aligned}
& Re\left[e^{-2\pi i A(\sum_{j=1}^n \#_j)n_0 x}\int_{A(k-\frac{1}{2})}^{A(k+\frac{1}{2})}\prod_{j=1}^n \phi(x-Am_j-\alpha_j t)e^{2\pi i A(\sum_{j=1}^n \#_j m_j)(x-\alpha_j t)}\frac{dt}{t}\right] \\
= \quad & Re\left[e^{2\pi i A(\sum_{j=1}^n \#_j \Delta_j)x}\int_{A(k-\frac{1}{2})}^{A(k+\frac{1}{2})}\prod_{j=1}^n \phi(x-Am_j-\alpha_j t)\frac{dt}{t}\right] \\
\geq \quad & Re\left[e^{2\pi i (\sum_{j=1}^n \#_j \Delta_j)c_{\vec{\alpha}}}\int_{A(k-\frac{1}{2})}^{A(k+\frac{1}{2})}\prod_{j=1}^n \phi(x-Am_j-\alpha_j t)\frac{dt}{t}\right] \\
\gtrsim \quad & \frac{1}{Ak}.
\end{aligned}
$$

This term is acceptable as an additional piece of the main contribution provided we take $c_{\vec{\alpha}}$ sufficiently small.

Small Perturbations: $\sum_{j=1}^n \#_j \Delta_j \alpha_j \neq 0$

There may be many small perturbations of $\vec{m}$, say $\tilde{\vec{m}}$, for which $\sum_{j=1}^n \#_j \alpha_j \tilde{m}_j \neq 0$ and yet $0 \in [n_0 - \tilde{m}_j - \alpha_j(l+\frac{1}{2}), n_0 - \tilde{m}_j - \alpha_j(l-\frac{1}{2})]$ for all j where $\tilde{m}_j = m_j + \Delta_j$ for all $j \in \{1, ..., n\}$ and

$$\max_{1\leq j\leq n} |\Delta_j| \leq \max_{1\leq j\leq n} |\alpha_j|.$$

Putting absolute values inside the integral for these terms would therefore yield unacceptable error terms on the same order as the main contribution. Instead, we need to observe

$$\left|\int_{A(k-\frac{1}{2})}^{A(k+\frac{1}{2})}\prod_{j=1}^n \phi(x-Am_j-\alpha_j t)e^{-2\pi i A(\sum_{j=1}^n \#_j \tilde{m}_j \alpha_j)t}\frac{dt}{t}\right| \lesssim \frac{1}{A^2 k}.$$

To show this, it suffices to prove

$$\left|\int_{A(k-\frac{1}{2})}^{A(k+\frac{1}{2})}\prod_{j=1}^{n}\phi(x-Am_j-\alpha_j t)e^{-2\pi i\tilde{A}t}\frac{dt}{t}\right|\lesssim_{\vec{\alpha}}\frac{1}{|\tilde{A}A|k}.$$

Indeed, using integration by parts, we have

$$\begin{aligned}
&\left|\int_{A(k-\frac{1}{2})}^{A(k+\frac{1}{2})}\prod_{j=1}^{n}\phi(x-Am_j-\alpha_j t)e^{-2\pi i\tilde{A}t}\frac{dt}{t}\right| \\
= \quad & \frac{1}{2\pi|\tilde{A}|}\left|\int_{A(k-\frac{1}{2})}^{A(k+\frac{1}{2})}\prod_{j=1}^{n}\phi(x-Am_j-\alpha_j t)\frac{d}{dt}\left[e^{-2\pi i\tilde{A}t}\right]\frac{dt}{t}\right| \\
\lesssim \quad & \frac{1}{|\tilde{A}|}\left[\frac{c}{Ak}+\left|\int_{A(k-\frac{1}{2})}^{A(k+\frac{1}{2})}\frac{d}{dt}\left[\prod_{j=1}^{n}\phi(x-Am_j-\alpha_j t)\right]e^{-2\pi i\tilde{A}t}\frac{dt}{t}\right|\right] \\
+ \quad & \frac{1}{|\tilde{A}|}\left|\int_{A(k-\frac{1}{2})}^{A(k+\frac{1}{2})}\prod_{j=1}^{n}\phi(x-Am_j-\alpha_j t)e^{-2\pi i\tilde{A}t}\frac{dt}{t^2}\right| \\
\lesssim \quad & \frac{1}{|\tilde{A}|Ak}.
\end{aligned}$$

As there are $O_{\vec{\alpha}}(1)$ many small perturbations of $\vec{m}(n_0,k)$, those perturbations satisfying the additional property that $\sum_{j=1}^{n}\#_j\Delta_j\alpha_j\neq 0$ may be subsumed as error upon taking A sufficiently large.

Large Perturbations

Fix n_0, k. Restrict attention to all vectors $\tilde{\vec{m}}$ s.t. $\exists$ index $1\leq j_*\leq n$ for which $\tilde{m}_{j_*}=m_{j_*}(n_0,k)+\Delta_{j_*}$ and $|\Delta_{j_*}|\geq 2\cdot max_{1\leq j\leq n}\{|\alpha_j|\}$. Then

$$|1_{[A(k-\frac{1}{2}),A(k+\frac{1}{2})]}(t)\phi(x-Am_{j_*}-A\Delta_{j_*}-\alpha_{j_*}t)|\lesssim_{\phi}\frac{1}{A^N}\frac{1}{1+|\Delta_{j_*}|^N}.$$

Therefore, the total contribution of large perturbations can be majorized by

$$\frac{C}{A^N}\sum_{\vec{\Delta}\in\mathbb{Z}^n}\prod_{j=1}^{n}\frac{1}{1+|\Delta_j|^N}\frac{1}{k}\lesssim_{N,\vec{\alpha}}\frac{1}{A^N k}.$$

Again, by taking A sufficiently large independent of N, this contribution becomes an error term.

We have now proven Lemma 3 and therefore Theorem 7 when $\vec{\alpha}\in\mathbb{Q}^d$.

1.2.3 PART II: The Irrational Case

Fix $N, A\in\mathbb{N}$. Construct for each $1\leq j\leq n$ the function

$$f^{N,A,\#}(x)=\sum_{-N\leq m\leq N}\phi(x-A\alpha_j m)e^{2\pi i A\#\alpha_j m x}=\sum_{-N\leq m\leq N}f_m^{N,A,\#}(x).$$

Set $S=\{1/\alpha_1,...,1/\alpha_n\}$, $\rho=1/A^2$, and $Bohr_{c(\vec{\alpha})N}(S,\rho)=\left\{N_1,...,N_{|Bohr_{c(\vec{\alpha})N}(S,\rho)|}\right\}$ for some constant $0<c_{\vec{\alpha}}<<1$. Then $|Bohr_{c(\vec{\alpha})N}(S,\rho)|\simeq_{\vec{\alpha},A}N$ by Lemma 2. Moreover, setting $\mathfrak{N}_j^{n_0}$ equal to the closest integer to $\alpha_j^{-1}n_0$ for each $j\in\{1,...,n\}$ and $n_0\in Bohr_{c(\vec{\alpha})N}(S,\rho)$ ensures $m_j=n_0-\alpha_j k$ can be approximately solved in the sense that setting $m_j=-k+\mathfrak{N}_j^{n_0}$ ensures $|n_0-\alpha_j m_j-\alpha_j k|\lesssim_{\vec{\alpha}}\frac{1}{A^2}$. Moreover, by choosing $c(\vec{\alpha})$ small enough we may assume $m_j=-k+\mathcal{N}_j^{n_0}\in[-N,N]$ for all $n_0\in Bohr_{c(\vec{\alpha})N}(S,\rho)$ and $|k|\lesssim N$. Restrict our attention to $x\in\Omega:=\bigcup_{n_0\in Bohr_{c(\vec{\alpha})N}(S,\rho)}[An_0,An_0+c_{\vec{\alpha}}/A]$. Therefore, $|\Omega|\gtrsim_{A,\vec{\alpha}}N$. Now we are ready to investigate the main contribution. To this end, assume $t=\tilde{t}+Ak$ where $|\tilde{t}|\leq\frac{A}{2}$ and write down

$$
\begin{aligned}
& \sum_{j=1}^{n} \#_j(m_j(n_0,k))(x-\alpha_j t) \\
= & \sum_{j=1}^{n} \#_j(n_0-\alpha_j k+\delta(n_0,j))(x-\alpha_j t) \\
= & \left[\sum_{j=1}^{n} \#_j\right] n_0 x + \sum_{j=1}^{n} \#_j(\delta(n_0,j))(x-\alpha_j t) \\
= & \left[\sum_{j=1}^{n} \#_j\right] n_0 x + \sum_{j=1}^{n} A\#_j\delta(n_0,j)n_0 + \sum_{j=1}^{n} \#_j\delta(n_0,j)(\theta_x-\alpha_j\tilde{t}-A\alpha_j k) \\
= & C(x) - \sum_{j=1}^{n} \#_j\delta(n_0,j)\alpha_j\tilde{t} - \sum_{j=1}^{n} A\#_j\delta(n_0,j)\alpha_j k
\end{aligned}
$$

where $Re\left[C(x)-\left[\sum_{j=1}^{n}\#_j\right]n_0x-\sum_{j=1}A\#_j\delta(n_0,j)n_0\right] \gtrsim 1$ for all $|x-An_0| \lesssim_{\vec{\alpha},A} 1$. Moreover,

$$
\sum_{j=1}^{n} \#_j\delta(n_0,j)\alpha_j k = \sum_{j=1}^{n} \#_j(n_0-\alpha_j\mathfrak{N}_j^{n_0})\alpha_j k = -\sum_{j=1}^{n} \#_j\alpha_j^2\mathfrak{N}_j^{n_0} k \in \mathbb{Z}.
$$

Hence, provided $A \in \mathbb{Z}$,

$$
e^{2\pi iA\sum_{j=1}^{n}\#_j(m_j(n_0,k))(x-\alpha_j t)} = e^{2\pi iA(C(n_0,x)-\sum_{j=1}^{n}\#_j\delta(n_0,j)\alpha_j\tilde{t})}.
$$

Using $|\delta(n_0,j)| \lesssim_{\vec{\alpha}} \frac{1}{A^2}$ gives an acceptable main contribution, i.e.

$$
\begin{aligned}
& Re\left[e^{-2\pi iA\left[\sum_{j=1}^{n}\#_j n_0 x + A\sum_{j=1}^{n}\#_j\delta(n_0,j)n_0\right]}\times\right. \\
& \left.\int_{A(k-\frac{1}{2})}^{A(k+\frac{1}{2})} \prod_{j=1}^{n} \phi(x-Am_j-\alpha_j t)e^{2\pi iA(\sum_{j=1}^{n}\#_j m_j)(x-\alpha_j t)}\frac{dt}{t}\right] \\
= & Re\left[\int_{-\frac{A}{2}}^{\frac{A}{2}} \prod_{j=1}^{n} \phi(\theta_x - A\delta(n_0,j)+\alpha_j t)e^{-2\pi iA\left[\sum_{j=1}^{n}\#_j\delta(n_0,j)\alpha_j t\right]}\frac{dt}{t+Ak}\right] \\
\gtrsim & \frac{1}{Ak}.
\end{aligned}
$$

Small Perturbations: $\sum_{j=1}^{n} \#_j \Delta_j \alpha_j^2 = 0$

As in the rational case, small perturbation with the above cancellation property cannot be subsumed as error. So, let $\tilde{m}_j = m_j(n_0,k) + \Delta_j$ with $\max_{1\le j\le n} |\Delta_j| < 2\max_{1\le j\le n} |\alpha_j|$, then observe

$$\begin{aligned}
& \sum_{j=1}^{n} \#_j \alpha_j (m_j(n_0,k) + \Delta_j)(x - \alpha_j t) \\
= & \sum_{j=1}^{n} \#_j (n_0 - \alpha_j k + \delta(n_0, j) + \alpha_j \Delta_j)(x - \alpha_j t) \\
= & \left[\sum_{j=1}^{n} \#_j\right] n_0 x + \sum_{j=1}^{n} \#_j (\delta(n_0,j) + \alpha_j \Delta_j)(x - \alpha_j t) \\
= & \left[\sum_{j=1}^{n} \#_j\right] n_0 x + \sum_{j=1}^{n} \#_j (\delta(n_0,j) + \alpha_j \Delta_j)(x - \alpha_j \tilde{t} - A\alpha_j k).
\end{aligned}$$

Because $x \in [An_0, An_0 + \frac{c_{\vec{\alpha}}}{A}]$, we may rewrite $x = An_0 + \theta_x$, where $|\theta_x| \le \frac{c_{\vec{\alpha}}}{A}$, and again use integrality observe

$$e^{2\pi i A \sum_{j=1}^{n} \#_j (m_j(n_0,k))(x-\alpha_j t)} = e^{2\pi i A (C(x) - \sum_{j=1}^{n} \#_j \delta(n_0,j)\alpha_j \tilde{t} + \sum_{j=1}^{n} \#_j \Delta_j \alpha_j x)}.$$

We are not quite content with the additional factor $e^{2\pi i \left[\sum_{j=1}^{n} \#_j \alpha_j \Delta_j\right] x}$, as we do not at first glance have good control on the sign of its real part, say. However, because $|x - An_0| \lesssim_{\vec{\alpha}} \frac{1}{A}$, it clearly suffices to obtain good control on the sign of the real part of

$$e^{2\pi i A^2 \left[\sum_{j=1}^{n} \#_j \alpha_j \Delta_j\right] n_0} = e^{2\pi i A^2 \left[\sum_{j=1}^{n} \#_j \alpha_j \Delta_j\right](\alpha_j \mathcal{N}_j^{n_0} + \delta(n_0,j))} = e^{2\pi i A^2 \left[\sum_{j=1}^{n} \#_j \alpha_j \Delta_j\right] \delta(n_0,j)}.$$

Since $|\delta(n_0,j)| \lesssim_{\vec{\alpha}} \frac{1}{A^2}$, we do in fact have good control. Hence, small perturbations satisfying $\sum_{j=1}^{n} \#_j \Delta_j \alpha_j^2 = 0$ always reinforce the main contribution.

Small Perturbations: $\sum_{j=1}^{n} \#_j \Delta_j \alpha_j^2 \neq 0$

Use $|\delta(n_0, j)| \leq \frac{1}{A^2}$ combined with the integration by parts from before to produce acceptable error terms.

1.2.4 Large Perturbations

This case is handled using the same argument as before, so the details are omitted.

$\square$

1.3 Symbols of Type $sgn(\xi_1+\xi_2)\Pi(\xi_2,\xi_3)$ and Maximal Variants

Recall

Theorem 2. *There exists a Hörmander-Marcinkiewicz symbol* $a : \mathbb{R}^2 \to \mathbb{R}$ *of* (ϕ, ψ) *type, i.e. a is sum of tensor products that are* ϕ *type in the first index and* ψ *type in the second index satisfying* $|\partial^{\vec{\alpha}} a(\xi)| \leq \frac{C_{\vec{\alpha}}}{dist(\vec{\xi},\vec{0})^{|\vec{\alpha}|}}$, *such that the trilinear operator* T_m *whose symbol* m *is given by* $m(\xi_1, \xi_2, \xi_3) = sgn(\xi_1+\xi_2)a(\xi_2, \xi_3)$ *satisfies no* L^p *estimates.*

Remark: This negative result is a strengthening of Muscalu, Tao, and Thiele's observation in [19], where the symbol m is taken to be $m(\xi_1, \xi_2, \xi_3) = sgn(\xi_1 + \xi_2)sgn(\xi_2 + \xi_3)$. Morally speaking, the sgn multiplier cannot be combined with even nice symbols involving indices outside the sgn to yield a bounded operator.

Corollary 2. *There exists two families of Schwartz functions* $\{\phi_k\}_{k\in\mathbb{Z}}, \{\tilde{\phi}_k\}_{k\in\mathbb{Z}}$ *with uniformly bounded* L^1 *norms satisfying supp* $\hat{\phi}_k, \hat{\tilde{\phi}}_k \subset [-2^k, 2^k]$ *such that the maximal bi-sublinear operator given by*

$$S_1\left(\{\tilde{\phi}_k\}, \{\phi_k\}\right) : (f_1, f_2) \mapsto \sup_{k\in\mathbb{Z}} \left| H(f_1 * \tilde{\phi}_k, f_2 * \phi_k)\right|$$

satisfies no L^p *estimates.*

Proof. For a contradiction, assume every pair of families $\{\tilde{\phi}_k\}$ and $\{\phi_k\}$ satisfying the conditions of the corollary are bounded on some Lebesgue tuple (p_1, p_2) satisfying $1 < p_1, p_2, \frac{p_1 p_2}{p_1+p_2} < \infty$ depending on $\{\tilde{\phi}_k\}$ and $\{\phi_k\}$. We proceed to show multipliers of the form $m(\xi_1, \xi_2, \xi_3) = sgn(\xi_1 + \xi_2)a(\xi_2, \xi_3)$ where $a : \mathbb{R}^2 \to \mathbb{R}$ is a Mikhlin-Hörmander symbol in $\mathbb{R}^2$ would satisfy some estimates. WLOG, decompose for some bounded sequences $\{c_k\}_{k\in\mathbb{Z}}, \{d_k\}_{k\in\mathbb{Z}}, \{e_k\}_{k\in\mathbb{Z}}$

$$\begin{aligned} & T_m(f_1, f_2, f_3) \\ = & \sum_{k\in\mathbb{Z}} c_k H(f_1 f_2 * \psi_k) f_3 * \psi_k + d_k H(f_1 f_2 * \phi_k) f_3 * \psi_k + e_k H(f_1 f_2 * \psi_k) f_3 * \phi_k \\ := & I + II + III, \end{aligned}$$

where $\{\psi_k\}$ is another family of uniformly L^1 bounded Schwartz functions with *supp* $\hat{\psi}_k \subset [-2^{k+1}, -2^{k-1}] \cup [2^{k-1}, 2^{k+1}]$. It suffices to prove estimates for I, II, III separately. Handling the contribution from I is immediate from from Cauchy-Schwarz, basic vector-valued inequalities, and the standard square functions estimates. By dualizing, we see that estimating II is essentially the same as estimating III. To this end, we write down

$$
\begin{aligned}
&\int_{\mathbb{R}} \sum_{k\in\mathbb{Z}} \left[d_k H \left[f_1 \cdot f_2 * \phi_k\right] f_3 * \psi_k\right] f_4 dx \\
= &\int_{\mathbb{R}} \sum_{k\in\mathbb{Z}} \left[d_k H \left[f_1 * \tilde{\phi}_k \cdot f_2 * \phi_k\right] f_3 * \psi_k\right] f_4 dx \\
+ &\int_{\mathbb{R}} \sum_{k\in\mathbb{Z}} \sum_{l \gtrsim k} d_k H \left[f_1 * \tilde{\psi}_l \cdot f_2 * \phi_k\right] f_3 * \psi_k f_4 dx \\
= &\; II_a + II_b.
\end{aligned}
$$

Then $II_a = \int_{\mathbb{R}} \sum_{k\in\mathbb{Z}} d_k H \left[f_1 * \tilde{\phi}_k \cdot f_2 * \phi_k\right] f_3 * \psi_k f_4 * \tilde{\psi}_k dx$ which satisfies estimates by our assumption. For II_b, we may again break the sum into two subsums:

$$
\begin{aligned}
II_b \;=\; &\int_{\mathbb{R}} \sum_{k\in\mathbb{Z}} \sum_{l \simeq k} d_k H \left[f_1 * \tilde{\psi}_l \cdot f_2 * \phi_k\right] f_3 * \psi_k f_4 dx \\
+ &\int_{\mathbb{R}} \sum_{k\in\mathbb{Z}} \sum_{l >> k} d_k H \left[f_1 * \tilde{\psi}_l \cdot f_2 * \phi_k\right] f_3 * \psi_k f_4 dx \\
:= &\; II_{b,1} + II_{b,2}.
\end{aligned}
$$

Estimates for $II_{b,1}$ follows immediately by Cauchy-Schwarz, vector-valued inequalities, and routine estimates for square and maximal functions. Moreover,

$$
\begin{aligned}
II_{b,2} \;=\; &\int_{\mathbb{R}} \sum_{l\in\mathbb{Z}} \sum_{k << l} d_k H \left[f_1 * \tilde{\psi}_l \cdot f_2 * \phi_k\right] f_3 * \psi_k f_4 * \tilde{\tilde{\psi}}_l dx \\
= &\int_{\mathbb{R}} \sum_{l\in\mathbb{Z}} \sum_{k << l} d_k f_1 * \tilde{\psi}_l \cdot f_2 * \phi_k f_3 * \psi_k f_4 * \tilde{\tilde{\psi}}_l dx \\
= &\int_{\mathbb{R}} \sum_{l\in\mathbb{Z}} \sum_{k \in\mathbb{Z}} d_k f_1 * \tilde{\psi}_l \cdot f_2 * \phi_k f_3 * \psi_k f_4 * \tilde{\tilde{\psi}}_l dx \\
- &\int_{\mathbb{R}} \sum_{l\in\mathbb{Z}} \sum_{k \simeq l} d_k f_1 * \tilde{\psi}_l \cdot f_2 * \phi_k f_3 * \psi_k f_4 * \tilde{\tilde{\psi}}_l dx \\
- &\int_{\mathbb{R}} \sum_{l\in\mathbb{Z}} \sum_{k >> l} d_k f_1 * \tilde{\psi}_l \cdot f_2 * \phi_k f_3 * \psi_k f_4 * \tilde{\tilde{\psi}}_l dx \\
:= &\; II_{b,2,1} - II_{b,2,2} - II_{b,2,3}.
\end{aligned}
$$

Estimates for $II_{b,2,1}$ are immediate from the paraproduct theory. Bounds for $II_{b,2,2}$ follows from routine estimates for square and maximal functions. Lastly, $II_{b,2,3} = 0$ on account of the various frequency supports and the assumption $k >> l$.

□

Corollary 3. *For each tuple (p_1, p_2) such that $1 < p_1, p_2, \frac{p_1 p_2}{p_1+p_2} < \infty$ there exists two families of Schwartz functions $\{\psi_k\}, \{\tilde{\psi}_k\}$ with uniformly bounded L^1 norms satisfying supp $\hat{\psi}_k \subset [-2^{k+1}, -2^{k-2}] \cup [2^{k-2}, 2^{k+1}]$ such that the maximal bi-sublinear operator given by*

$$S_2\left(\{\phi_k\}, \{\tilde{\phi}_k\}\right) : (f_1, f_2) \mapsto \sup_{l \in \mathbb{Z}} \left| \sum_{k \leq l} H(f_1 * \tilde{\psi}_k f_2 * \psi_k) \right|$$

does not continuously map $L^{p_1}(\mathbb{R}) \times L^{p_2}(\mathbb{R}) \to L^{\frac{p_1 p_2}{p_1+p_2}}(\mathbb{R})$.

Proof. For a contradiction, suppose S_2 were bounded for some Lebesgue type (p_1, p_2) satisfying $1 < p_1, p_2, \frac{p_1 p_2}{p_1+p_2} < \infty$ for all admissible families $\{\tilde{\psi}_k\}, \{\psi_k\}$. It suffices to observe for any decomposition $\mathbb{R} = \bigcup_{i \in \mathcal{I}} C_i$

$$\int_{\mathbb{R}} \sum_{i \in \mathcal{I}} 1_{C_i} H(f_1 * \tilde{\phi}_i, f_2 * \phi_i) f_3 \; dx \;\; \simeq \;\; \int_{\mathbb{R}} \sum_{i \in \mathcal{I}} \sum_{k_1, k_2 \leq i} 1_{C_i} H(f_1 * \tilde{\psi}_{k_1}, f_2 * \psi_{k_2}) f_3 \; dx,$$

where the two families $\{\tilde{\psi}\}$ and $\{\psi\}$ satisfy the uniform L^1 property in addition to *supp* $\hat{\psi}_k$, *supp* $\hat{\tilde{\psi}}_k \subset [-2^{k+1}, -2^{k-1}] \cup [2^{k-1}, 2^{k+1}]$. By assumption, the diagonal terms $k_1 = k_2$ will satisfy estimates, so we are left handling

$$\int_{\mathbb{R}} \sum_{i \in \mathcal{I}} \sum_{k_1 \neq k_2 \leq i} 1_{C_i} H(f_1 * \tilde{\psi}_{k_1}, f_2 * \psi_{k_2}) f_3 \; dx := I.$$

By the frequency support assumptions, if $|k-l| \geq 2$, then $H(f_1 * \tilde{\psi}_k \cdot f_2 * \psi_l) = f_1 * \tilde{\psi}_k \cdot f_2 * \psi_l$. Therefore, we may further decompose the above sum into the following parts:

$$\begin{aligned} I &= \int_{\mathbb{R}} \sum_{i \in \mathcal{I}} \sum_{(k_1,k_2):|k_1-k_2|=1, k_1,k_2 \leq i} 1_{C_i} \cdot H(f_1 * \tilde{\psi}_{k_1} \cdot f_2 * \psi_{k_2}) \cdot f_3 \; dx \\ &+ \int_{\mathbb{R}} \sum_{i \in \mathcal{I}} \sum_{(k_1,k_2):|k_1-k_2|\geq 2, k_1,k_2 \leq i} 1_{C_i} \cdot f_1 * \tilde{\psi}_{k_1} \cdot f_2 * \psi_{k_2} \cdot f_3 \; dx \\ &:= I_a + I_b. \end{aligned}$$

Estimating I_b is straightforward. Indeed, it is easy to see for $f_3 \in L^{p_3}(\mathbb{R})$ such that $||f_3||_{p_3} = 1$ and $\sum_{j=1}^3 \frac{1}{p_i} = 1$ that $|I_b| \lesssim ||\mathfrak{M}(f_1)||_{p_1} ||\mathfrak{M}(f_2)||_{p_2} + ||\mathcal{S}(f_1)||_{p_1} ||\mathcal{S}(f_2)||_{p_2}$ where $\mathfrak{M}(f)(x) := \sup_{k_1,k_2} \left| \sum_{k_1 \leq k \leq k_2} f * \psi_k(x) \right|$ and $\mathcal{S}$ is the Littlewood-Paley square function. Moreover, as $\mathfrak{M}$ maps L^p into L^p for all $1 < p < \infty$, it suffices to prove estimates for I_a. We may rewrite

$$\begin{aligned} & I_a \\ = & \int_{\mathbb{R}} \sum_{i \in \mathcal{I}} \sum_{k_1 \leq i} 1_{C_i} \left(\left[H(f_1 * \tilde{\psi}_{k_1} \cdot f_2 * \psi_{k_1-1}) \right] + H \left[f_1 * \tilde{\psi}_{k_1-1} \cdot f_2 * \psi_{k_1} \right] \right) f_3 \; dx \\ = & \int_{\mathbb{R}} \sum_{i \in \mathcal{I}} \sum_{k_1 \leq i} 1_{C_i} H \left[f_1 * (\tilde{\psi}_{k_1} + \tilde{\psi}_{k_1-1}) f_2 * (\psi_{k_1} + \psi_{k_1-1}) \right] dx \\ - & \int_{\mathbb{R}} \sum_{i \in \mathcal{I}} \sum_{k_1 \leq i} 1_{C_i} \left(H \left[f_1 * \tilde{\psi}_{k_1-1} f_2 * \psi_{k_1} \right] + H \left[f_1 * \tilde{\psi}_{k_1-1} f_2 * \psi_{k_1-1} \right] \right) f_3 \; dx. \end{aligned}$$

However, by our hypothesis, each of the three main terms can be bounded, and hence $\sup_{k \in \mathbb{Z}} \left| H(f_1 * \tilde{\phi}_k \cdot f_2 * \phi_k) \right|$ would satisfy estimates. $\square$

In fact, it is not hard to prove directly that for specific choices of $\{\phi_k\}$ and $\{\psi_k\}$ obeying the uniform L^1 conditions and support restrictions *supp* $\hat{\phi}_k \subset [-2^k, 2^k]$ and *supp* $\hat{\psi}_k \subset [2^{k-1}, 2^{k+1}]$ the maps

$$(f_1, f_2) \mapsto \sup_{k\in\mathbb{Z}} |H(f_1 * \phi_k \cdot f_2 * \psi_k)|$$

as well as

$$(f_1, f_2) \mapsto \sup_{k\in\mathbb{Z}} \left| \sum_{l\leq k} H\left(f_1 * \psi_k, f_2 * \psi_k\right) \right|$$

satisfy no L^p estimates. We now prove Theorem 2.

Proof. Fix $N, M \in \mathbb{N}$. Let $A = 2^M$ and choose $\phi \in \mathcal{S}(\mathbb{R})$ with compact Fourier support inside $[-1/4, 1/4]$ such that $\phi(0) \neq 0$. Then construct the functions

$$\begin{aligned} f_1^{N,A}(x) &= \sum_{1\leq m\leq N} \phi(x - Am)e^{-2\pi i 2^m x} := \sum_{1\leq m\leq N} f_{1,m}^{N,A}(x) \\ f_2^{N,A}(x) &= \sum_{1\leq m\leq N} \phi(x - Am)e^{2\pi i 2^m x} := \sum_{1\leq m\leq N} f_{2,m}^{N,A}(x) \\ f_3^{N,A}(x) &= \sum_{1\leq m\leq N} \phi(x - Am)e^{2\pi i 2^m x} := \sum_{1\leq m\leq N} f_{3,m}^{N,A}(x). \end{aligned}$$

Let $S_N := \bigcup_{n\in\mathbb{Z}\cap[N/2,2N/3]}[An, An+1]$. The claim is that $|T_m(f_1^N, f_2^N, f_3^N)(x)1_{S_N}(x)| \gtrsim_A \log(N)1_{S_N}(x)$ for sufficiently large choice of A independent of N, from which we immediately deduce that $||T_m(f_1^N, f_2^N, f_3^N)||_p \gtrsim \log(N)N^{\frac{1}{p_1}+\frac{1}{p_2}}$ whereas $\prod_{i=1}^2 ||f_i^N||_{p_i} \simeq N^{\frac{1}{p_1}+\frac{1}{p_2}}$. Taking N arbitrarily large yields the theorem. For each $k \in \mathbb{N}$ choose $\phi_k, \psi_k \in \mathcal{S}(\mathbb{R})$ satisfying $1_{[-2^{k-1},2^{k-1}]} \leq \hat{\phi}_k \leq 1_{[-2^k,2^k]}$ and $1_{[2^{k-1/4},2^{k+1/4}]} \leq \hat{\psi}_k \leq 1_{[2^{k-\frac{1}{2}},2^{k+\frac{1}{2}}]}$ in addition to the properties

$$\left|\left[\frac{d}{d\xi}\right]^\alpha \hat{\psi}_k(\xi)\right|, \left|\left[\frac{d}{d\xi}\right]^\alpha \hat{\phi}_k(\xi)\right| \leq \frac{C_\alpha}{2^{k\alpha}} \;\; \forall\, \vec{\alpha} \in 0 \cup \mathbb{N}.$$

Take $a(\xi_2, \xi_3) = \sum_{k\in\mathbb{N}} \hat{\phi}_k(\xi_2)\hat{\psi}_k(\xi_3)$ and observe that for $1 \leq n_2, n_3 \leq N$ whenever $n_3 << n_2$

$$
\begin{aligned}
&T_m(f_1^A, f_2^A, f_3^A)(x) \\
:= &\sum_{k\in\mathbb{N}} \sum_{1\le m_1,m_2,m_3\le N} \int_{\xi_1+\xi_2>0} \left[\prod_{j=1}^{3} \hat{f}^A_{j,m_j}(\xi_j) e^{2\pi i x\xi_j}\right] \hat{\phi}_k(\xi_2)\hat{\psi}_k(\xi_3) d\vec{\xi} \\
:= &\sum_{k\in\mathbb{Z}} \sum_{1\le m_1,m_2,m_3\le N} T^k_m(f^{N,A}_{1,m_1}, f^A_{2,m_2}, f^A_{3,m_3})(x).
\end{aligned}
$$

By construction, *supp* $\hat{f}^A_{2,n}$, *supp* $\hat{f}^A_{3,n} \subset [2^{M+n}-1/4, 2^{M+n}+1/4]$. Hence, the only tuples (m_1, m_2, m_3) for which $T^k_m(f^A_{1,m_1}, f^A_{2,m_2}, f^A_{3,m_3}) \not\equiv 0$ must satisfy $m_2 \le m_3$. Moreover, $m_2 \le m_3 - 2$ together with $T^k_m(f^{N,A}_{1,m_1}, f^A_{2,m_2}, f^A_{3,m_3})(x) \not\equiv 0$ ensures

$$
\begin{aligned}
&T^k_m(f^{N,A}_{1,m_1}, f^A_{2,m_2}, f^A_{3,m_3})(x) \\
= &\left(\int_{\mathbb{R}} \prod_{j=1}^{2} \phi(x - Am_j - t) e^{2\pi i(-1)^j 2^{m_j} t} \frac{dt}{t}\right) \left(\phi(x - Am_3) e^{2\pi i(2^{m_2}+2^{m_3}-2^{m_1})x}\right).
\end{aligned}
$$

To save space, say $A << B$ provided $A \le B - 2$. Moreover, it is easy to see that those terms corresponding to $m_2 \in \{m_3 - 1, m_3\}$ and $m_2 \not< \not< m_3$ can be satisfactorily estimated. Indeed, this portion is writable as $\sum_{k\in\mathbb{N}} H(f_1 \cdot f_2 * \tilde{\psi}_k) \cdot f_3 * \psi_k$, which can again be handled by Cauchy-Schwarz, the Fefferman-Stein inequality, and routine square function estimates. Hence, it suffices to produce a log-type pointwise blow-up for the remaining terms:

$$
\begin{aligned}
&\tilde{T}_m(f_1^A, f_2^A, f_3^A)(x) e^{2\pi i(-2^{m_2}-2^{m_3}+2^{m_1})x} \\
:= &\sum_{1\le m_2\le N} \sum_{m_1\le m_2} \sum_{m_3>>m_2} \int_{\mathbb{R}} \prod_{j=1}^{2} \phi(x - Am_j - t) e^{2\pi i(-1)^j 2^{m_j} t} \frac{dt}{t} \times \phi(x - Am_3).
\end{aligned}
$$

1.3.1 Main Contribution

Fix $x \in [An_0, An_0 + 1]$. Due to the Schwartz decay of ϕ, one expects the main term to arise from the cases where $m_3 = n_0$, in which case the corresponding sum over the (m_1, m_2) will be $\sum_{m_2 << n_0} \sum_{m_1 \le m_2}$. Moreover, of these terms, one expects the largest component to arise from the terms where $m_1 \simeq m_2$. Thus, at least heuristically, we are able to produce a quantity $O(\log(N))$ on a set of size $O(N)$ as claimed.

Now let $x = An_0 + \theta_x$ where $0 \le \theta_x < 1$. The main contribution to $T(f_1^{N,A}, f_2^{N,A}, f_3^{N,A})$ when $m_3 = n_0, m_2 = m_1$ is given by the formula

$$\begin{aligned}
& \sum_{m_1 << n_0} T_m\left(f^{N,A}_{1,m_1}, f^{N,A}_{2,m_1}, f^{N,A}_{3,n_0}\right)(x) \\
= & \left(\sum_{m_1 << n_0} \int_{\mathbb{R}} \phi^2(An_0 + \theta_x - Am_1 - t)\frac{dt}{t}\right)\left(\phi(\theta_x)e^{2\pi i 2^{n_0}\theta_x}\right) \\
= & \left(\sum_{m_1 << n_0} \sum_{l \in \mathbb{Z}} \int_{A(l-\frac{1}{2})}^{A(l+\frac{1}{2})} \phi^2(An_0 + \theta_x - Am_1 - t)\frac{dt}{t}\right)\left(\phi(\theta_x)e^{2\pi i 2^{n_0}\theta_x}\right) \\
:= & \sum_{m_1 << n_0} \sum_{l \in \mathbb{Z}} T^l_m\left(f^{N,A}_{1,m_1}, f^{N,A}_{2,m_1}, f^{A}_{3,n_0}\right)(x).
\end{aligned}$$

Lemma 4. *To prove Theorem 2, it suffices to show that for sufficiently large $M \in \mathbb{Z}$ and each $l \in \mathbb{Z} \cap [2, N/4]$*

$$Re\left[e^{-2\pi i A 2^{n_0} x} \sum_{m_1 << n_0} T^l_m\left(f^{N,A}_{1,m_1}, f^{N,A}_{2,m_1}, f^{N,A}_{3,n_0}\right)(x)\right] 1_{S_N}(x) \gtrsim \frac{1}{Al}.$$

Indeed, assuming the claim, it follows that $\forall x \in S_N$

$$\begin{aligned}
& Re\left[e^{-2\pi i A 2^{n_0} x} \sum_{m_1 << n_0} T_m\left(f_{1,m_1}^{N,A}, f_{2,m_1}^{N,A}, f_{3,n_0}^{N,A}\right)(x)\right] \\
= \;& Re\left[e^{-2\pi i A 2^{n_0} x} \sum_{m_1 << n_0} \sum_{l \in \mathbb{Z}} T_m^l\left(f_{1,m_1}^{N,A}, f_{2,m_1}^{N,A}, f_{3,n_0}^{N,A}\right)(x)\right] \\
= \;& Re\left[e^{-2\pi i A 2^{n_0} x} \sum_{m_1 << n_0} \sum_{l \in \mathbb{Z} \cap [2, N/4]} T_m^l\left(f_{1,m_1}^{N,A}, f_{2,m_1}^{N,A}, f_{3,n_0}^{N,A}\right)(x)\right] \\
+ \;& Re\left[e^{-2\pi i A 2^{n_0} x} \sum_{m_1 << n_0} \sum_{l \in \mathbb{Z} \cap [2, N/4]^c} T_m^l\left(f_{1,m_1}^{N,A}, f_{2,m_1}^{N,A}, f_{3,n_0}^{N,A}\right)(x)\right] \\
\geq \;& C_{A,\alpha} \log(N) - \sum_{m_1 << n_0} \left| \sum_{l \in \mathbb{Z} \cap [2, N/4]^c} T_m^l\left(f_{1,m_1}^{N,A}, f_{2,m_1}^{N,A}, f_{3,n_0}^{N,A}\right)(x)\right|.
\end{aligned}$$

We further break apart the last sum as follows:

$$\begin{aligned}
& \sum_{m_1 << n_0} \left| \sum_{l \in \mathbb{Z} \cap [\alpha, N/4]^c} T_m^l\left(f_{1,m_1}^{N,A}, f_{2,m_1}^{N,A}, f_{3,n_0}^{A}\right)(x)\right| \\
= \;& \sum_{m_1 << n_0} \left| \sum_{l \in \mathbb{Z} \cap (N/4, 100N]} T_m^l\left(f_{1,m_1}^{N,A}, f_{2,m_1}^{N,A}, f_{3,n_0}^{N,A}\right)(x)\right| \\
+ \;& \sum_{m_1 << n_0} \left| \sum_{l \in \mathbb{Z} \cap (100N, \infty)} T_m^l\left(f_{1,m_1}^{N,A}, f_{2,m_1}^{N,A}, f_{3,n_0}^{N,A}\right)(x)\right| \\
+ \;& \sum_{m_1 << n_0} \left| \sum_{l \in \mathbb{Z} \cap [0,2)} T_m^l\left(f_{1,m_1}^{N,A}, f_{2,m_1}^{N,A}, f_{3,n_0}^{A}\right)(x)\right| \\
+ \;& \sum_{m_1 << n_0} \left| \sum_{l \in \mathbb{Z} \cap (-\infty,0)} T_m^l\left(f_{1,m_1}^{N,A}, f_{2,m_1}^{N,A}, f_{3,n_0}^{A}\right)(x)\right| \\
:= \;& I + II + III + IV.
\end{aligned}$$

To bound $I(x)$, note $|T_m^l\left(f_{1,m_1}^{N,A}, f_{2,m_1}^{N,A}, f_{3,n_0}^{N,A}\right)(x)| \lesssim \frac{1}{(1+A^2|n_0-m_1-l|^2)l}$ which ensures

$$| \ I(x) \ | \lesssim \sum_{m_1 << n_0} \sum_{l \in \mathbb{Z} \cap (N/4, 100N]} \frac{1}{(1 + A^2 |n_0 - m_1 - l|^2) \cdot l} \lesssim 1.$$

For $II(x)$, note that by the restrictions placed on n_0 and m_1

$$| \ II(x) \ | \lesssim \sum_{m_1 << n_0} \sum_{l \in \mathbb{Z} \cap (100N, \infty)} \frac{1}{1 + A^3 |n_0 - m_1 - l|^3} \lesssim \sum_{1 \leq m_1 \leq N} \frac{1}{1 + A^3 |n_0 - m_1|^2} \lesssim 1.$$

Term $III(x)$ is estimated by $T_m^l \left(f_{1,m_1}^{N,A}, f_{2,m_1}^{N,A}, f_{3,n_0}^{N,A} \right)(x)| \lesssim 1$. Indeed, it is trivial for $l \neq 0$. If $l = 0$, then one only needs

$$\begin{aligned}
& |T_m \left(f_{1,m_1}^{N,A}, f_{2,m_1}^{N,A}, f_{3,n_0}^{A} \right)(x)| \\
\lesssim \quad & \int_{-\frac{A}{2}}^{\frac{A}{2}} \left| \phi^2(An_0 + \theta_x - Am_1 + t) - \phi^2(An_0 + \theta_x - Am_1 - t) \right| \frac{dt}{t} \\
\lesssim_A \quad & \frac{1}{1 + A^2 |n_0 - m_1|^2}
\end{aligned}$$

since $\phi^2 \in \mathcal{S}(\mathbb{R})$ hence its derivative is uniformly bounded and ϕ^2 is Lipschitz. Lastly, it is clear that

$$| \ IV(x) \ | \lesssim \sum_{m_1 << n_0} \sum_{l \in \mathbb{Z} \cap (-\infty, 0)} \frac{1}{A^3 |n_0 - m_1 - l|^3} \lesssim \sum_{m_1 << n_0} \frac{1}{A^3 |n_0 - m_1|^2} \lesssim 1.$$

Hence, to prove Theorem 2 assuming the claim, it is enough to handle two remaining error terms $E_I(x)$ and $E_{II}(x)$ where

$$\begin{aligned}
E_I(x) \quad &:= \quad \sum_{1 \leq m_3 \leq N} \sum_{m_2 << m_3} \sum_{m_1 < m_2} T_m(f_{1,m_1}^{N,A}, f_{2,m_2}^{N,A}, f_{3,m_3}^{N,A})(x) \\
E_{II}(x) \quad &:= \quad \sum_{m_3 \neq m_0} \sum_{m_2 << m_3} \sum_{m_1 = m_2} T_m(f_{1,m_1}^{N,A}, f_{2,m_2}^{N,A}, f_{3,m_3}^{N,A})(x).
\end{aligned}$$

We estimate $E_I(x), E_{II}(x)$ separately:

$$
\begin{aligned}
&|E_I(x)| \\
\leq\ & \sum_{1\leq m_3\leq N}\sum_{m_2<<m_3}\sum_{m_1<m_2}\left|\int_{\mathbb{R}}\phi(x-Am_1-t)\phi(x-Am_2-t)e^{2\pi i(2^{m_1}-2^{m_2})t}\frac{dt}{t}\right. \\
\times\ & \left.\phi(x-Am_3)\right| \\
\leq\ & \sum_{1\leq m_3\leq N}\sum_{m_2<<m_3}\sum_{m_1<m_2}\phi(x-Am_1)\phi(x-Am_2)\phi(x-Am_3) \\
\lesssim\ & \sum_{1\leq m_3\leq N}\sum_{m_1<m_2}\sum_{m_2<<m_3}\frac{1}{1+A^2|n_0-m_1|^2}\frac{1}{1+A^2|n_0-m_2|^2}\frac{1}{1+A^2|n_0-m_3|^2} \\
\lesssim\ & 1.
\end{aligned}
$$

The remaining error will require A to be sufficiently large:

$$
\begin{aligned}
&|E_{II}(x)| \\
=\ & \left|\sum_{m_3\neq n_0}\sum_{m_1=m_2}\sum_{m_2<<m_3}\int_{\mathbb{R}}\left[\prod_{j=1}^{2}\phi(x-Am_j-t)\right]e^{-2\pi i2^{m_1}(x-t)}e^{2\pi i2^{m_2}(x-t)}\frac{dt}{t}\right. \\
\times\ & \left.\phi(x-Am_3)e^{2\pi i2^{m_3}x}\right| \\
=\ & \left|\sum_{m_3\neq n_0}\sum_{m_1<<m_3}H\left[\phi^2\right](x-Am_1)\cdot\phi(x-Am_3)e^{2\pi i2^{m_3}x}\right| \\
\lesssim\ & \sum_{m_3\neq n_0}\sum_{1\leq m_1\leq N}\frac{1}{1+A|n_0-m_1|}\frac{1}{1+A^2|n_0-m_3|^2} \\
\lesssim\ & \frac{\log(N)}{A^3}.
\end{aligned}
$$

We can therefore choose $M\in\mathbb{N}$ large enough to achieve the desired point-wise lower bound for the main contribution.

Lastly, to prove the claim, write down $\forall\ x\in S_N$ and $\forall\ l\in\mathbb{Z}\cap[2,N/4]$

$$Re\left[e^{-2\pi i 2^{n_0}x}\sum_{m_1<<n_0} T_m^l\left(f_{1,m_1}^{N,A}, f_{2,m_1}^{N,A}, f_{3,n_0}^{N,A}\right)(x)\right]$$
$$\gtrsim \quad Re\left[\left(\sum_{m_1<<n_0}\int_{A(l-\frac{1}{2})}^{A(l+\frac{1}{2})}\phi^2(An_0+\theta_x-Am_1-t)\frac{dt}{t}\right)\right].$$

Assuming $n_0 - N/4 \leq m_1 \leq n_0 - 2$ and $l = n_0 - m_1$ note that $\int_{A(l-\frac{1}{2})}^{A(l+\frac{1}{2})}\phi^2(An_0 + \theta_x - Am_1 - t)\frac{dt}{t} \gtrsim \frac{1}{Al}$. As $n_0 \in [N/2, 2N/3]$, $1 \leq m_1 \leq N$, $m_1 << n_0$, $l \in [2, N/4]$ satisfy our constraints, and the remaining terms in the sum are all positive, the claim is true and, hence, Theorem 2 has been shown.

□

1.4 Unboundedness for Hyperplane Symbols in Dimension $n \geq 5$

Definition 5. *Fix $\Phi, \Psi \in \mathcal{S}(\mathbb{R})$ such that $1_{[-1/2,1/2]} \leq \Phi \leq 1_{[-1,1]}$ and $1_{[2,\infty)} \leq \Psi \leq 1_{[1,\infty)}$. For every pair of distinct hyperplanes $(\Gamma^{\vec{\alpha}}, \Gamma^{\vec{\beta}}) \subset \mathbb{R}^n$ and symbol $m : \mathbb{R}^n \to \mathbb{C}$, the (Φ, Ψ)-localization of m near $\Gamma^{\vec{\alpha}}$ away from $\Gamma^{\vec{\beta}}$ is the symbol defined by*

$$m[\vec{\alpha}, \vec{\beta}, \Phi, \Psi](\vec{\xi}) = \Phi(\vec{\alpha}\cdot\vec{\xi})\Psi(\vec{\beta}\cdot\vec{\xi})m(\vec{\xi}) \quad \forall\vec{\xi} \in \mathbb{R}^n.$$

By construction, $m[\vec{\alpha}, \vec{\beta}, \Phi, \Psi](\vec{\xi}) \in \mathcal{M}_{\Gamma^{\vec{\alpha}}}(\mathbb{R}^n) \cap \mathcal{M}_{\Gamma^{\vec{\beta}}}(\mathbb{R}^n)$ is supported inside $\{dist(\vec{\xi}, \Gamma^{\vec{\alpha}}) \lesssim 1\}$ and if $\vec{\xi} \in \{dist(\vec{\xi}, \Gamma^{\vec{\alpha}}) \lesssim 1\} \cap \{dist(\vec{\xi}, \Gamma^{\vec{\beta}}) \gtrsim dist(\vec{\xi}, \Gamma^{\vec{\alpha}})\}$, then $m_{\Gamma^{\vec{\alpha}},\Gamma^{\vec{\beta}}}(\vec{\xi}) = m(\vec{\xi})$.

Our main result in this section is

Theorem 8. *Fix $n \geq 5$. Let $\vec{\alpha}, \vec{\beta} \in \mathbb{R}^n$ satisfy $\alpha_j = \beta_j z_j$ for some $\vec{z} \in \mathbb{Z}^n$. Assume there exists $\vec{\#} \in \mathbb{R}^n$ s.t.*

$$\begin{aligned}\sum_{j=1}^n \#_j \alpha_j = \sum_{j=1}^n \#_j \alpha_j^2 = \sum_{j=1}^n \#_j \beta_j = \sum_{j=1}^n \#_j \beta_j^2 &= 0 \\ \sum_{j=1}^n \#_j \alpha_j \beta_j &\neq 0\end{aligned}$$

with the additional property that $\#_j \alpha_j^2 \in \mathbb{Q}$ for all $1 \leq j \leq n$. Moreover, suppose K_1, K_2 are two real-valued kernels for which $\hat{K}_1, \hat{K}_2 \in \mathcal{M}_{\{0\}}(\mathbb{R})$, K_1, K_2 are odd, there exist $C_1, C_2 > 0$ so that

$$\begin{aligned}K_1(\xi) \geq 0 &\qquad \forall \xi > 0 \\ K_1(\xi) \geq C_1/\xi &\quad \forall \xi \geq 1 \\ \liminf_{s\to\infty} Im[\check{K}_2](s) = C_2.&\end{aligned}$$

Then every (Φ, Ψ)-localization of $\hat{K}_1(\vec{\alpha} \cdot \vec{\xi})\hat{K}_2(\vec{\beta} \cdot \vec{\xi}) : \mathbb{R}^2 \to \mathbb{C}$ gives rise to a multiplier $T_{\hat{K}_1(\vec{\alpha}\cdot\vec{\xi})\hat{K}_2(\vec{\beta}\cdot\vec{\xi})[\vec{\alpha},\vec{\beta},\Phi,\Psi]}$ satisfying no L^p estimates.

Lemma 5. *Let $n \geq 5$ and $\vec{\alpha} \in \mathbb{R}^n$ satisfy $\alpha_j^{-1} = q_j \alpha + q_j^2$ for some $\vec{q} \in \mathbb{Q}^n$ with distinct, non-zero entries such that $q_j \alpha \neq -1$ for all $1 \leq j \leq n$. Moreover, let $\beta_j = \alpha_j q_j^{-1}$. Then there exists a non-trivial solution $\vec{\#} \in \mathbb{R}^n$ to the system*

$$\begin{aligned}\sum_{j=1}^n \#_j \alpha_j = \sum_{j=1}^n \#_j \alpha_j^2 = \sum_{j=1}^n \#_j \beta_j = \sum_{j=1}^n \#_j \beta_j^2 &= 0 \\ \sum_{j=1}^n \#_j \alpha_j \beta_j &\neq 0\end{aligned}$$

with the property that $\#_j \alpha_j^2 \in \mathbb{Q}$ for all $1 \leq j \leq n$.

Proof. Set $\tilde{\#}_j = \#_j \alpha_j^2$ for all $1 \leq j \leq n$. We seek non-trivial $\vec{\tilde{\#}} \in \mathbb{Q}^n$ satisfying

$$\begin{aligned} \sum_{j=1}^{n} \tilde{\#}_j \alpha_j^{-1} = \sum_{j=1}^{n} \tilde{\#}_j (q_j \alpha + q_j^2) = 0; \quad \sum_{j=1}^{n} \tilde{\#}_j &= 0 \\ \sum_{j=1}^{n} \tilde{\#}_j \alpha_j^{-2} \beta_j = \sum_{j=1}^{n} \tilde{\#}_j (q_j \alpha + q_j^2) q_j^{-1} = 0; \quad \sum_{j=1}^{n} \tilde{\#}_j \alpha_j^{-2} \beta_j^2 = \sum_{j=1}^{n} \tilde{\#}_j q_j^{-2} &= 0 \\ \sum_{j=1}^{n} \tilde{\#}_j \alpha_j^{-1} \beta_j = \sum_{j=1}^{n} \tilde{\#}_j q_j^{-1} &\neq 0. \end{aligned}$$

Requiring $\sum_{j=1}^{n} \tilde{\#}_j q_j^m = 0$ for all $m \in \{-2, 0, 1, 2\}$, we may for $n \geq 5$ find $\vec{\tilde{\#}}_j \in \mathbb{Q}^n$ by the Gram-Schmidt process for which $\sum_{j=1}^{n} \tilde{\#}_j q_j^{-1} \neq 0$.

□

Combining Theorem 8 and Lemma 5, we obtain

Theorem 4. *Let $n \geq 5$ and $\vec{\alpha} \in \mathbb{R}^n$ satisfy $\alpha_j^{-1} = q_j + \alpha q_j^2$ for some $\vec{q} \in \mathbb{Q}^n$ with distinct, non-zero entires such that $q_j \alpha \neq -1$ for all $1 \leq j \leq n$. Then there exists $m \in \mathcal{M}_{\Gamma^{\vec{\alpha}}}(\mathbb{R}^n)$ supported in $\left\{\vec{\xi} : dist(\vec{\xi}, \Gamma^{\vec{\alpha}}) \lesssim 1\right\}$ such that T_m satisfies no L^p estimates.*

Note that if $\alpha \in \mathbb{R} \cap \mathbb{Q}^c$, then $\vec{\alpha} \notin \mathbb{R}(\mathbb{Q}^n)$. Setting $\alpha = 0$ yields that for any $\vec{q} \in \mathbb{Q}^n$ with distinct, non-zero entries, there exists $m \in \mathcal{M}_{\Gamma^{\vec{q}}}(\mathbb{R}^n)$ such that T_m satisfies no L^p estimates. Hence, we have

Corollary 4 (Generic $n-$linear Hilbert transform satisfies no L^p estimates)**.** *Let $n \geq 5$ and $\alpha_j = j$ for all $1 \leq j \leq n$. Then there exists $m \in \mathcal{M}_{\Gamma^{\vec{\alpha}}}(\mathbb{R}^n)$ supported in $\left\{\vec{\xi} : dist(\vec{\xi}, \Gamma^{\vec{\alpha}}) \lesssim 1\right\}$ such that T_m satisfies no L^p estimates.*

Proof. *(Theorem 8).*

1.4.1 PART 1: The Rational Case

Assuming $\vec{\alpha}, \vec{\beta} \in \mathbb{Q}^n$, we may always dilate $\vec{\alpha}, \vec{\beta}$ by sufficiently large integers to guarantee $\vec{\alpha}, \vec{\beta} \in \mathbb{Z}^n$. Then our assumption is that there exists $\vec{\#} \in \mathbb{Z}^n$ solving the system

$$\begin{aligned} \sum_{j=1}^{n} \#_j \alpha_j = \sum_{j=1}^{n} \#_j \alpha_j^2 = \sum_{j=1}^{n} \#_j \beta_j = \sum_{j=1}^{n} \#_j \beta_j^2 &= 0 \\ \sum_{j=1}^{n} \#_j \alpha_j \beta_j &\neq 0. \end{aligned}$$

For $A \in \mathbb{Z}^+$ and $j \in \{1, ..., d\}$, construct for each $1 \leq j \leq n$ the function

$$f^{N,A,\#_j}(x) = \sum_{-N \leq m \leq N} \phi(x - Am) e^{2\pi i A \#_j m x},$$

where we now wish to choose $\phi \in \mathcal{S}(\mathbb{R})$ satisfying *supp* $\hat{\phi} \subset [-1, 1]$, $\phi \geq 0$, $\phi(0) \neq 0$, and ϕ is symmetric about the origin. This is easily done by taking one's favorite non-trivial real-valued non-negative smooth function, symmetric about the origin with compact support in $[-\frac{1}{2}, \frac{1}{2}]$, say Φ. Then one need only set $\hat{\phi} = \Phi * \Phi$ to observe

$$\phi := \mathcal{F}^{-1}(\Phi * \Phi) = \check{\Phi}^2 \geq 0$$

with $\phi(0) = ||\Phi||_{L^1}^2 > 0$ in addition to the desired Fourier support and symmetry properties. Moreover, we want to choose $m \in \mathcal{M}_{\vec{\alpha}}(\mathbb{R}^d)$ to satisfy the property that is identically equal to $1_{\vec{\xi}\cdot\vec{\alpha}\geq 0}1_{\vec{\xi}\cdot\vec{\beta}\leq 0}$ in some neighborhood of the hyperplane $\Gamma^{\vec{\alpha}}$ away from the singularity $\Gamma^{\vec{\beta}}$. That is, m is supported inside $\{dist(\vec{\xi},\Gamma^{\vec{\alpha}}) \lesssim 1\}$ and if $\vec{\xi} \in \{dist(\vec{\xi},\Gamma^{\vec{\alpha}}) \lesssim 1\} \bigcap \{dist(\vec{\xi},\Gamma^{\vec{\beta}}) \gtrsim dist(\vec{\xi},\Gamma^{\vec{\alpha}})\}$, then $m(\vec{\xi}) = 1_{\vec{\xi}\cdot\vec{\alpha}\geq 0}(\vec{\xi})1_{\vec{\xi}\cdot\vec{\beta}\leq 0}(\vec{\xi})$. The parameter A is to be taken sufficiently large to give us sparseness in frequency and time, which enables to assume that the only $\vec{m}$ which may appears in the integrand of the kernel representation of T are *precisely* those for which both $\sum_{j=1}^{n} \#_j m_j \alpha_j = 0$ and $\sum_{j=1}^{n} \#_j m_j \beta_j < 0$. Moreover, assuming $\sum_{j=1}^{n} \#_j m_j \beta_j < 0$ for each such $\vec{m}$, the multiplier $m(\vec{\xi})$ is identically $1_{\vec{\xi}\cdot\vec{\alpha}\geq 0}1_{\vec{\xi}\cdot\vec{\beta}\leq 0}$ when restricted to the domain $\prod_{j=1}^{n}[A\#_j m_j - 1, A\#_j m_j + 1]$. Recall that the n-linear operator given by

$$T(f_1,,...,f_n)(x) := p.v. \int_{\mathbb{R}^2} \prod_{j=1}^{n} f_j(x-\alpha_j t - \beta_j s) K_1(t) K_2(s) ds\ dt$$

has the Fourier representation (defined on Schwartz functions, say) given by

$$\int_{\mathbb{R}^n} \hat{K}_1(\vec{\alpha}\cdot\vec{\xi})\hat{K}_2(\vec{\beta}\cdot\vec{\xi}) \left[\prod_{j=1}^{n} \hat{f}_j(\xi_j) e^{2\pi i \xi_j x}\right] d\vec{\xi}.$$

Thus, we may rewrite $T_{\hat{K}_1(\vec{\alpha}\cdot\vec{\xi})\hat{K}_2(\vec{\beta}\cdot\vec{\xi})[\vec{\alpha},\vec{\beta},\Phi,\Psi]}$ acting on $\vec{f}^{N,A,\vec{\#}}$ as

$$\begin{aligned} & T_{\hat{K}_1(\vec{\alpha}\cdot\vec{\xi})\hat{K}_2(\vec{\beta}\cdot\vec{\xi})[\vec{\alpha},\vec{\beta},\Phi,\Psi]}\left(\vec{f}^{N,A,\vec{\#}}\right)(x) \\ = & \sum_{(k,l)\in\mathbb{Z}^2} \sum_{\vec{m}\in\mathbb{M}} \int_{A(k-\frac{1}{2})}^{A(k+\frac{1}{2})} \int_{A(l-\frac{1}{2})}^{A(l+\frac{1}{2})} \left[\prod_{j=1}^{n} \phi(x - Am_j - \alpha_j t - \beta_j s) e^{2\pi i A \#_j m_j (x-\alpha_j t - \beta_j s)}\right] \\ \times & \quad K_1(s) K_2(t) ds dt, \end{aligned}$$

where

$$\mathbb{M} = \left\{\vec{m} \in \mathbb{Z}^n : -N \leq m_j \leq N\ \forall\ j \in \{1,...,n\},\ \sum_{j=1}^{n} m_j \#_j \alpha_j = 0,\ \sum_{j=1}^{n} m_j \#_j \beta_j < 0\right\}.$$

Lemma 6. *Let*

$$T^{k,0}_{\hat{K}_1(\vec{\alpha}\cdot\vec{\xi})\hat{K}_2(\vec{\beta}\cdot\vec{\xi})[\vec{\alpha},\vec{\beta},\Phi,\Psi]}\left(\left\{f^{N,A,\#_j}_{m_j}\right\}_{j=1}^{n}\right)(x)$$
$$:= \int_{A(k-\frac{1}{2})}^{A(k+\frac{1}{2})}\int_{-\frac{A}{2}}^{\frac{A}{2}}\left[\prod_{j=1}^{n} f^{N,A,\#_j}_{m_j}(x-\alpha_j t-\beta_j s)\right]K_1(s)K_2(t)\ dsdt.$$

To prove Theorem 4, it suffices to show $\exists c_{\vec{\alpha},\vec{\beta}} > 0$ *such that for every* $x \in \left[An_0, An_0 + \frac{c_{\vec{\alpha},\vec{\beta}}}{A}\right]$, $k \in \left[k_0, \frac{N}{3\max_{1\le j\le n}\{|\alpha_j|\}}\right]$, *and* $n_0 \in [-N/3, N/3]$, *the following estimate holds:*

$$\sum_{1\le k\le \frac{N}{C_{\vec{\alpha},\vec{\beta}}}} Im\left[e^{-2\pi i A\left(\sum_{j=1}^{n}\#_j\right)n_0 x}T^{(k,0)}_{\hat{K}_1(\vec{\alpha}\cdot\vec{\xi})\hat{K}_2(\vec{\beta}\cdot\vec{\xi})[\vec{\alpha},\vec{\beta},\Phi,\Psi]}\left(\left\{f^{N,A,\#_j}_{m_j}\right\}_{j=1}^{n}\right)(x)\right] \gtrsim \frac{\log(N)}{A}.$$

Before proving the lemma, we verify the lower bound in the above display.

1.4.2 Main Contribution

The condition for the term corresponding to $\vec{m}$ to be in our sum is $0 \le A\sum_{j=1}^{n}\#_j m_j\alpha_j \lesssim 1$, which gives that $\sum_{j=1}^{n}\#_j m_j \alpha_j = 0$ by the integrality of $\vec{\#}, \vec{\alpha}$ and $\vec{m}$. Now choose any $c_{\vec{\alpha},\vec{\beta}} \in \mathbb{N}$ such that $c_{\vec{\alpha},\vec{\beta}} > 10(max_{j\in\{1,\dots,n\}}\{|\alpha_j|\} + max_{j\in\{1,\dots,n\}}\{|\beta_j|\})$ and fix $x \in \left[An_0, An_0 + \frac{c_{\vec{\alpha},\vec{\beta}}}{A}\right]$ for some $n_0 \in [-N/3, N/3]$. We expect the largest contribution to come from the terms where $n_0 - m_j - \alpha_j k - \beta_j l = 0$ for each $j \in \{1, ..., n\}$. Of course, by the frequency restrictions imposed by $0 \le \vec{\xi}\cdot\vec{\alpha} \lesssim 1$ and $\vec{\xi}\cdot\vec{\beta} \le 0$, we have that $l = 0$ and $k > 1$ for A sufficiently large. To this end, construct $\vec{m}_{k,l} := n_0 - \vec{\alpha}k - \vec{\beta}l$ for each $1 \le k \le \frac{N}{C_{\vec{\alpha},\vec{\beta}}}$. Then, observe

$$\sum_{j=1}^{n} \#_j m_j \alpha_j = \sum_{j=1}^{n} \#_j (n_0 - \alpha_j k - \beta_j l)\alpha_j = 0$$

must be satisfied, which, provided $\sum_{j=1}^{n} \alpha_j \beta_j \#_j \neq 0$, requires $l = 0$. For the condition $\sum_{j=1}^{n} \#_j m_j \beta_j < 0$, we require $k \geq 1$. (Take A sufficiently large to guarantee that all and only those $k \geq 1$ arise in the in sum over k, l corresponding to $m_j(k,l) = n_0 - \alpha_j k - \beta_j l$.) The main contribution can be expressed in this new notation as

$$\sum_{1 \leq k \leq \frac{N}{C_{\vec{\alpha},\vec{\beta}}}} T^{k,0}_{\hat{K}_1(\vec{\alpha}\cdot\vec{\xi})\hat{K}_2(\vec{\beta}\cdot\vec{\xi})[\vec{\alpha},\vec{\beta},\Phi,\Psi]} \left(\left\{ f^{N,A,\#_j}_{m_j(n_0,k,0)} \right\}_{j=1}^{n} \right)(x)$$

Let $\sum_{j=1}^{n} \#_j \alpha_j \beta_j =: \mathfrak{C} > 0$. Fix a single term in the above sum, set $C(x) = e^{-2\pi i A (\sum_{i=1}^{n} \#_i) n_0 x}$ and compute using Lemma 8

$$\begin{aligned}
& Im\left[C(x) \cdot T^{k,0}_{\hat{K}_1(\vec{\alpha}\cdot\vec{\xi})\hat{K}_2(\vec{\beta}\cdot\vec{\xi})[\vec{\alpha},\vec{\beta},\Phi,\Psi]} \left(\left\{ f^{N,A,\#_j}_{m_j(n_0,k,0)} \right\}_{j=1}^{n} \right)(x)\right] \\
= \; & Im\left[C(x) \cdot \int_{A(k-\frac{1}{2})}^{A(k+\frac{1}{2})} \int_{A(l-\frac{1}{2})}^{A(l+\frac{1}{2})} \left[\prod_{j=1}^{n} \phi(x - Am_j - \alpha_j t - \beta_j s) e^{2\pi i A \#_j m_j (x - \alpha_j t - \beta_j s)}\right]\right. \\
\times \; & K_1(s) K_2(t) ds dt]|_{m_j = n_0 - \alpha_j k} \\
= \; & Im\left[\int_{-\frac{A}{2}}^{\frac{A}{2}} \int_{-\frac{A}{2}}^{\frac{A}{2}} \prod_{j=1}^{n} [\phi(\theta_x - \alpha_j t - \beta_j s)] e^{2\pi i A \mathfrak{C} k s} K_1(s) K_2(t + Ak) ds dt\right] \\
\gtrsim \; & \frac{D_{\phi,\vec{\alpha},A,C_1,C_2}}{Ak} \\
\gtrsim \; & \frac{D_{\phi,\vec{\alpha},\infty,C_1,C_2}}{Ak}.
\end{aligned}$$

Summing over $1 \leq k \leq \frac{N}{c_{\vec{\alpha},\vec{\beta}}}$ yields a total contribution $\sim \frac{\log(N)}{A}$. Therefore, setting

$$\Re(n_0) = \left[\bigcup_{k \in \left[1, \frac{N}{3\max_{1\le j\le n}\{|\alpha_j|\}}\right]} \{k\} \times \{0\} \times \{\vec{m}(n_0,k,0)\} \right]^c \bigcap (\mathbb{Z} \times \mathbb{Z} \times \mathbb{M}),$$

we are left with satisfactorily estimating

$$Im\left[e^{-2\pi i A(\sum_{j=1}^n \#_j) n_0 x} \sum_{(k,l,\vec{m}) \in \Re(n_0)} T^{k,l}_{\hat{K}_1(\vec{\alpha}\cdot\vec{\xi})\hat{K}_2(\vec{\beta}\cdot\vec{\xi})[\vec{\alpha},\vec{\beta},\Phi,\Psi]} \left(\left\{ f^{N,A,\#_j}_{m_j} \right\}_{j=1}^n \right)(x) \right].$$

The rest of this section is dedicated to proving Lemma 6.

1.4.3 Small Perturbations

CASE: $1 \le k \lesssim N$

For each $n_0 \in [N/2, 2N/3]$, $k \in \left[1, \frac{N}{3\max_{1\le j\le n}\{|\alpha_j|\}}\right]$, and $l \in \mathbb{Z}$ consider those $\vec{m} \in \mathbb{M}$ satisfying $m_j = m_j(n_0,k,l) + \Delta_j$ where $0 < \sup_{1\le j\le n} |\Delta_j| \le max_{1\le j\le n}\{|\alpha_j| + |\beta_j|\}$ along with $\sum_{j=1}^n \#_j m_j \alpha_j = 0$ and $\sum_{j=1}^n \#_j m_j \beta_j < 0$. For each $(k,l) \in \mathbb{Z}\times\mathbb{Z}$, denote this collection by $\mathbb{M}^S_{n_0,k,l}$ and define the collection of large perturbations by $\mathbb{M}^L_{n_0,k,l}$ by the relation

$$\mathbb{M} \;=\; \mathbb{M}^S_{n_0,k,l} \coprod \mathbb{M}^L_{n_0,k,l} \coprod (\{\vec{m}(n_0,k,0)\} \cap \mathbb{M})$$

Hence, for every $n_0 \in \mathbb{Z}$

$$
\begin{aligned}
\mathbb{Z}\times\mathbb{Z}\times\mathbb{M} &= \left[\bigcup_{(k,l)\in\mathbb{Z}^2}\{k\}\times\{l\}\times\mathbb{M}^S_{n_0,k,l}\right]\coprod\left[\bigcup_{(k,l)\in\mathbb{Z}^2}\{k\}\times\{l\}\times\mathbb{M}^L_{n_0,k,l}\right]\\
&\coprod\left[\bigcup_{k\in\mathbb{Z}}\{k\}\times\{0\}\times\{\vec{m}(n_0,k,0)\cap\mathbb{M}\}\right].
\end{aligned}
$$

Now observe for every $\vec{m}\in\mathbb{M}^S_{n_0,k,l}$

$$
\begin{aligned}
&\sum_{j=1}^{n}\#_j m_j(x-\alpha_j t-\beta_j s)\\
=&\sum_{j=1}^{n}\#_j m_j(x-\beta_j s)\\
=&\sum_{j=1}^{n}\#_j m_j(n_0,k,l)(x-\beta_j s)+\sum_{j=1}^{n}\#_j\Delta_j(x-\beta_j s)\\
=&\left(\sum_{j=1}^{n}\#_j\alpha_j\beta_j\right)ks+\left(\sum_{j=1}^{n}\#_j(\Delta_j+n_0)\right)x-\left(\sum_{j=1}^{n}\#_j\Delta_j\beta_j\right)s.
\end{aligned}
$$

Therefore, setting $C(x,\vec{\Delta})=e^{-2\pi iA\left[\sum_{j=1}^{n}\#_j(\Delta_j+n_0)\right]x}$, it follows that

$$
\begin{aligned}
&C_{\vec{\Delta}}(x)\cdot T^{k,l}_{\hat{K}_1(\vec{\alpha}\cdot\vec{\xi})\hat{K}_2(\vec{\beta}\cdot\vec{\xi})[\vec{\alpha},\vec{\beta},\Phi,\Psi]}\left(\left\{f^{N,A,\#_j}_{j,m_j}\right\}_{j=1}^{n}\right)(x)\\
=&\int_{A(k-\frac{1}{2})}^{A(k+\frac{1}{2})}\int_{A(l-\frac{1}{2})}^{A(l+\frac{1}{2})}\left[\prod_{j=1}^{n}[\phi(x-Am_j-\alpha_j t-\beta_j s)]\right]e^{2\pi iA\left[\sum_{j=1}^{n}\#_j(\alpha_j k-\Delta_j)\beta_j\right]s}\\
\times&\ K_1(s)K_2(t)dsdt.
\end{aligned}
$$

Fix $\epsilon>0$ and choose $c_{\vec{\alpha},\vec{\beta}}$ sufficiently small to ensure that $\forall x\in\left[An_0,An_0+\frac{c_{\vec{\alpha},\vec{\beta}}}{A}\right]$

$$
Re\left[e^{2\pi iA\sum_{j=1}^{n}\#_j\Delta_j x}\right]=Re\left[e^{2\pi IA\left(\sum_{j=1}^{n}\#_j\Delta_j\right)\theta_x}\right]>\gamma\ for\ some\ \gamma:|\gamma-1|<\epsilon<<1.
$$

Then $Im\left[e^{2\pi i A\left(\sum_{j=1}^{n}\#_j\Delta_j\right)x}\right] < \sqrt{1-\gamma^2}$. Fixing n_0, k and note

$$\begin{aligned}
& Im\left[e^{-2\pi i A\sum_{j=1}^{n}\#_j n_0 x}T^{(k,l)}_{\hat{K}_1(\vec{\alpha}\cdot\vec{\xi})\hat{K}_2(\vec{\beta}\cdot\vec{\xi})[\vec{\alpha},\vec{\beta},\Phi,\Psi]}(\vec{f}^{N,A,\vec{\#}}_{\vec{m}})(x)\right] \\
\geq \quad & \gamma Im\left[\int_{A(k-\frac{1}{2})}^{A(k+\frac{1}{2})}\int_{A(l-\frac{1}{2})}^{A(l+\frac{1}{2})}\left[\prod_{j=1}^{n}\phi(x-Am_j-\alpha_j t-\beta_j s)e^{2\pi i A\#_j(\alpha_j k-\Delta_j)\beta_j s}\right]\right. \\
\times \quad & K_1(s)K_2(t)dsdt] \\
- \quad & \sqrt{1-\gamma^2}\left|\int_{A(k-\frac{1}{2})}^{A(k+\frac{1}{2})}\int_{A(l-\frac{1}{2})}^{A(l+\frac{1}{2})}\left[\prod_{j=1}^{n}\phi(x-Am_j-\alpha_j t-\beta_j s)\right]\right. \\
\times \quad & \left. e^{2\pi i A\left[\sum_{j=1}^{n}\#_j(\alpha_j k-\Delta_j)\beta_j\right]s}K_1(s)K_2(t)dsdt\right| \\
\geq \quad & -\sqrt{1-\gamma^2}\left|\int_{A(k-\frac{1}{2})}^{A(k+\frac{1}{2})}\int_{A(l-\frac{1}{2})}^{A(l+\frac{1}{2})}\left[\prod_{j=1}^{n}\phi(x-Am_j-\alpha_j t-\beta_j s)\right]\right. \\
\times \quad & \left. e^{2\pi i A\left[\sum_{j=1}^{n}\#_j(\alpha_j k-\Delta_j)\beta_j\right]s}K_1(s)K_2(t)dsdt\right|
\end{aligned}$$

Then summing the above lower bound over all $l \in \mathbb{Z}$ and $\vec{m} \in \mathbb{M}^S_{n_0,k,l}$ yields for all sufficiently large k and sufficiently small $\epsilon(A)$ a lower bound of $-\frac{1}{A^2 k}$. Indeed, this is true for the following 3 reasons: Lemma 8 guarantees that $\gamma Im[\cdot] \geq 0$; for fixed (n_0,k,l), $\#\left[\mathbb{M}^S_{n_0,k,l}\right] = O(1)$; for fixed (n_0,k), the collection $\mathbb{M}^S_{n_0,k,l}$ is empty for all but $O(1)$ distinct $l \in \mathbb{Z}$. To observe this last claim, write down $\vec{m} = \vec{m}(n_0,k,l)+\vec{\Delta} \in \mathbb{M}$. Then $\sum_{j=1}^{n}\#_j m_j\alpha_j = 0$ implies $-\mathfrak{C}l+\sum_{j=1}^{n}\#_j\alpha_j\Delta_j = 0$. Therefore, $|l| \lesssim_{\vec{\alpha}} 1$, and the small perturbations $\mathbb{M}^S_{n_0,k,l}$ produce an acceptable error when $1 \leq k \lesssim N$.

CASE: $k << 0$

If $\sum_{j=1}^{n}\#_j m_j\beta_j < 0$ and $\vec{m} \in \mathbb{M}^S_{n_0,k,l}$ for some $k << 0$, then

$$0 > \sum_{j=1}^{n}\#_j m_j\beta_j = -\mathfrak{C}k + \sum_{j=1}^{n}\#_j\Delta_j\beta_j = -\mathfrak{C}k + O(1).$$

Therefore, $\mathbb{M}^S_{n_0,k,l} = \emptyset$ for all $k << 0$.

CASE: $|k| \lesssim 1$

The total contribution is $O(1)$, which is acceptable error in light of the main contribution $O(\log(N))$.

CASE: $k \simeq N$

The total contribution is $O(1)$, which is acceptable error in light of the main contribution.

CASE: $k >> N$

If $\sum_{j=1}^{n} \#_j m_j \beta_j < 0$ and $\vec{m} \in \mathbb{M}^S_{n_0,k,l}$ for some $k >> N$, then

$$m_j = m_j(n_0, k, l) + \Delta_j = n_0 - \alpha_j k - \beta_j l + \Delta_j.$$

From previous considerations, $|l| = O(1)$. Therefore, $|m_j| \geq |\alpha_j k| - n_0 - |\beta_j l| >> N$, which contradicts $\vec{m} \in \mathbb{M}$. Hence, $\mathbb{M}^S_{n_0,k,l} = \emptyset$ for all $k >> N$.

1.4.4 Large Perturbations

It now suffices to bound the contribution of $\mathbb{M}^L_{n_0,k,l}$.

CASE: $k < 0$

For each $(k, l) \in \mathbb{Z} \times \mathbb{Z}$, $\exists j_* \in \{1, ..., n\}$ satisfying

$$|m_{j_*} - n_0 - \alpha_{j_*}k - \beta_{j_*}l| \geq \frac{|l|}{5n \cdot \sup_{1\leq j\leq n} |\#_j \alpha_j|}.$$

The proof is an easy contradiction argument. If $|m_j - n_0 - \alpha_j k - \beta_j l| < \frac{|l|}{5n \cdot \sup_{1\leq j\leq n} |\#_j \alpha_j|}$ for each $j \in \{1, ..., n\}$,

$$\sum_{j=1}^{n} \#_j m_j \alpha_j = \sum_{j=1}^{n} \#_j (m_j - n_0 - \alpha_j k - \beta_j l)\alpha_j + \sum_{j=1}^{n} \#_j (n_0 + \alpha_j k + \beta_j l)\alpha_j := I + II.$$

By assumption, $|\ I\ | \leq n \left[\sup_{1\leq j\leq n} |\#_j \alpha_j|\right] \frac{|l|}{5n \cdot \sup_{1\leq j\leq n} |\#_j \alpha_j|} = \frac{|l|}{5}$. Moreover, $II \geq l$. Therefore, $\sum_{j=1}^{n} \#_j m_j \alpha_j \geq \frac{4|l|}{5}$, which contradicts the frequency restrictions imposed by the multiplier. Therefore, one is free to extract the decay $\frac{1}{l^C}$ for any $C >> 1$. Similarly, for each $(k, l) \in \mathbb{Z}^- \times \mathbb{Z}$, $\exists j_* \in \{1, ..., n\}$ such that

$$|m_{j_*} - n_0 - \alpha_{j_*}k - \beta_{j_*}l| \geq \frac{|k|}{5n \cdot \sup_{1\leq j\leq n} |\#_j \beta_j|}.$$

Indeed, suppose $|m_j - n_0 - \alpha_j k| < \frac{|k|}{5n}$ for all $j \in \{1, ..., n\}$. Then $\sum_{j=1}^{n} m_j \#_j \beta_j \leq 0$ and yet

$$\left(\sum_{j=1}^{n} \#_j m_j \beta_j\right) \geq -\frac{4k}{5} \left(\sum_{j=1}^{n} \#_j \alpha_j \beta_j\right) > 0$$

This contradicts the restriction that $\sum_{j=1}^{n} \#_j m_j \beta_j < 0$. If $k = 0$, then Hence, summing over all $\vec{m} \in \mathbb{M}^L_{n_0,k,l}$ along with $(k, l) \in \mathbb{Z}^- \times \mathbb{Z}$ yields an acceptable error term, i.e.

$$\sum_{k\leq 0}\sum_{l\in\mathbb{Z}}\sum_{\vec{m}\in\mathbb{M}^L_{n_0,k,l}}\left|T^{k,l}_{\hat{K}_1(\vec{\alpha}\cdot\vec{\xi})\hat{K}_2(\vec{\beta}\cdot\vec{\xi})[\vec{\alpha},\vec{\beta},\Phi,\Psi]}\left(\vec{f}^{N,A,\vec{\#}}_{\vec{m}}\right)(x)\right|$$

$$\leq \sum_{k\leq 0}\sum_{l\in\mathbb{Z}}\sum_{\vec{m}\in\mathbb{M}}\left|\int_{A(k-\frac{1}{2})}^{A(k+\frac{1}{2})}\int_{A(l-\frac{1}{2})}^{A(l+\frac{1}{2})}\left[\prod_{j=1}^{n}\phi(x-Am_j-\alpha_j t-\beta_j s)e^{2\pi i A\#_j m_j(x-\alpha_j t-\beta_j s)}\right]\right.$$

$$\times \quad K_1(s)K_2(t)dsdt|$$

$$\lesssim \sum_{k,l\in\mathbb{Z}}\sum_{\vec{n}\in\mathbb{Z}^n}\frac{1}{(1+|l|^{\tilde{C}})(1+|k|^{\tilde{C}})}\prod_{j=1}^{n}\frac{1}{1+|n_j|^{\tilde{C}}}$$

$$\lesssim 1.$$

CASE: $1\leq k\lesssim N$

This contribution is slightly more delicate and will feature the same $\log(N)$ growth as the main contribution. Note that $\sum_{j=1}^{n}\#_j\alpha_j m_j=0$ requires the existence of some index $j_*\in\{1,...,n\}$ for which

$$|m_{j_*}-n_0-\alpha_{j_*}k-\beta_{j_*}l|\geq\frac{|\ l\ |}{5n\cdot\sup_{1\leq j\leq n}|\#_j\alpha_j|}.$$

Hence, the total contribution will be $O\left(\frac{\log(N)}{A^M}\right)$, which is acceptable in light of the $O\left(\frac{\log(N)}{A}\right)$ contribution of the main terms by taking a large enough absolute constant A.

CASE: $k\simeq N$

As before, we shall have $O\left(\frac{1}{1+l^{\tilde{C}}}\right)$ decay. The summation over $k\simeq N$ is harmless owing to $\sum_{k\simeq N}\frac{1}{k}\simeq 1$.

CASE: $k >> N$

The decay is $O\left(\frac{1}{l^{\bar{C}}}\right) \cdot O\left(\frac{1}{|k|^{\bar{C}}}\right)$. This concludes the estimates for

$$Im\left[e^{-2\pi iA\left(\sum_{j=1}^{n} \#_j\right)n_0 x} T^{k,l}_{\hat{K}_1(\vec{\alpha}\cdot\vec{\xi})\hat{K}_2(\vec{\beta}\cdot\vec{\xi})[\vec{\alpha},\vec{\beta},\Phi,\Psi]}\left(\vec{f}^{N,A,\vec{\#}}_{\vec{m}}\right)(x)\right]$$

and hence the proofs of Lemma 3 and Theorem 4 in the rational case.

1.4.5 PART 2: The Irrational Case

Let $\vec{\alpha}, \vec{\beta} \in \mathbb{R}^n$ satisfy $\beta_j = z_j\alpha_j$ for all $1 \leq j \leq n$ where $\vec{z} \in \mathbb{Z}^n$. Assume there exists $\vec{\#} \in \mathbb{R}^n$ such that

$$\begin{aligned} \sum_{j=1}^{n} \#_j\alpha_j = \sum_{j=1}^{n} \#_j\alpha_j^2 = \sum_{j=1}^{n} \#_j\beta_j = \sum_{j=1}^{n} \#_j\beta_j^2 &= 0 \\ \sum_{j=1}^{n} \#_j\alpha_j\beta_j &\neq 0 \end{aligned}$$

in addition to the condition $\#_j\alpha_j^2 \in \mathbb{Q}$. By dilating $\vec{\#}$, we may assume $\#_j\alpha_j^2 \in \mathbb{Z}$ for all $1 \leq j \leq n$ and hence $\#_j\alpha_j\beta_j, \#_j\beta_j^2 \in \mathbb{Z}$ for all $1 \leq j \leq n$. For $A \in \mathbb{Z}^+$ and $j \in \{1, ..., d\}$, construct the functions

$$f^{N,A,\#,\alpha}(x) = \sum_{-N \leq m \leq N} \phi(x - A\alpha m)e^{2\pi iA\#\alpha mx}.$$

Moreover, set $S(\vec{\alpha}) = \{1/\alpha_1, ..., 1/\alpha_n\}, \rho = 1/A^2$, and $Bohr_{c(\vec{\alpha})N}(S(\vec{\alpha}), \rho) = \{N_1, ..., N_{|Bohr_{c(\vec{\alpha})N}(S(\alpha),\rho)|}\}$. Just as before, $\left|Bohr_{c(\vec{\alpha})N}(S(\vec{\alpha}), \rho)\right| \simeq_A N$. Fix $n_0 \in Bohr_{c(\vec{\alpha})N}(S(\vec{\alpha}), \rho)$ and set $\mathcal{N}_j^{n_0}$ equal to the closest integer to $\alpha_j^{-1}n_0$ for each $1 \leq j \leq n$. Set

$$\Omega := \bigcup_{n_0 \in Bohr_{c(\vec{\alpha})N}(S(\vec{\alpha}),\rho)} [An_0, An_0 + c_{\vec{\alpha},\vec{\beta}}/A].$$

Then the theorem in the irrational case will follow from the pointwise estimate

$$\left| T_{\hat{K}_1(\vec{\alpha}\cdot)\hat{K}_2(\vec{\beta}\cdot)[\vec{\alpha},\vec{\beta},\Phi,\Psi]} \left(\left\{ f_{m_j}^{N,A,\#_j,\alpha_j} \right\}_{j=1}^{n} \right)(x) \right| \gtrsim_A \log(N) 1_\Omega(x) \; \forall \; x \in \mathbb{R}.$$

To justify the claim, let us first calculate the main contribution. To this end, assume $t = \tilde{t} + Ak$, where $|\tilde{t}| \leq A/2$ and observe for $1 \leq k \leq \frac{N}{C_{\vec{\alpha},\vec{\beta}}}$

$$\begin{aligned} & \sum_{j=1}^{n} \#_j \alpha_j (-k + \mathcal{N}_j^{n_0})(x - \alpha_j t - \beta_j s) \\ = \; & \sum_{j=1}^{n} \#_j (n_0 - \alpha_j k + \delta(n_0, j))(x - \alpha_j \tilde{t} - A\alpha_j k - \beta_j s) \\ = \; & \sum_{j=1}^{n} \#_j n_0 x + \sum_{j=1}^{n} A \#_j \delta(n_0, j) n_0 + \sum_{j=1}^{n} \#_j \delta(n_0, j)(\theta_x - \alpha_j \tilde{t} - A\alpha_j k - \beta_j s) + \mathfrak{C} ks \\ = \; & D(x) - \sum_{j=1}^{n} \#_j \delta(n_0, j) \alpha_j \tilde{t} + \left[\mathfrak{C} k - \sum_{j=1}^{n} \#_j \delta(n_0, j) \beta_j \right] s - A \sum_{j=1}^{n} \#_j \delta(n_0, j) \alpha_j k, \end{aligned}$$

where $D(x) := \sum_{j=1}^{n} \#_j n_0 x + \sum_{j=1}^{n} A \#_j \delta(n_0, j) n_0 + \sum_{j=1}^{n} \#_j \delta(n_0, j) \theta_x$. Moreover,

$$A \sum_{j=1}^{n} \#_j \delta(n_0, j) \alpha_j k = A \sum_{j=1}^{n} \#_j \alpha_j (n_0 - \alpha_j \mathcal{N}_j^{n_0}) \in \mathbb{Z}.$$

Therefore, letting $m_j(n_0, k) = -k + \mathcal{N}_j^{n_0}$ and $\sum_{j=1}^{n} \#_j \alpha_j \beta_j = \mathcal{C}$,

$$e^{2\pi i A \sum_{j=1}^{n} \#_j \alpha_j m_j(n_0,k)(x - \alpha_j t - \beta_j s)} = e^{2\pi i A D(n_0,x)} e^{-2\pi i A \sum_{j=1}^{n} \#_j \delta(n_0,j) \alpha_j \tilde{t}} e^{2\pi i A \left[\mathfrak{C} k - \sum_{j=1}^{n} \#_j \delta(n_0,j) \beta_j \right] s}.$$

Using $|\delta(n_0, j)| \leq_{\vec{\alpha}} \frac{1}{A^2}$, we may deduce for every $|x - n_0| = |\theta_x| \lesssim_{A,\{\vec{\alpha}\},\{\vec{\beta}\}} 1$ the lower bound

$$Im\left[e^{-2\pi iAD(x)}T^{k,0}_{\hat{K}_1(\vec{\alpha}\cdot\vec{\xi})\hat{K}_2(\vec{\beta}\cdot\vec{\xi})[\vec{\alpha},\vec{\beta},\Phi,\Psi]}\left(\left\{f^{N,A,\#_j,\alpha_j}_{m_j(n_0,k)}\right\}_{j=1}^{n}\right)(x)\right]\gtrsim\frac{1}{Ak}.$$

1.4.6 Small Perturbations

CASE: $1\le k\lesssim N$

For each $n_0\in[c_AN,N]\cap Bohr_N(S,\rho(\vec{\alpha}))$ and $k\in[1,d_{\vec{\alpha},A}N]$, Letting $m_j(n_0,k,l)=-k+z_jl+\mathcal{N}_j^{n_0}$ and using $\sum_{j=1}^{n}\#_j\alpha_j^2m_j=0$ for all $\vec{m}\in\mathbb{M}$ yields for all $\vec{m}=\vec{m}_j(n_0,k,l)+\Delta_j\in\mathbb{M}$

$$\begin{aligned}
&\sum_{j=1}^{n}\#_j\alpha_j(m_j(n_0,k,l)+\Delta_j)(x-\alpha_jt-\beta_js)\\
=\ &\sum_{j=1}^{n}\#_j\alpha_j(-k-z_jl+\mathcal{N}_j^{n_0})(x-\beta_j\tilde{s}-A\beta_jl)+\sum_{j=1}^{n}\#_j\alpha_j\Delta_j(x-\beta_j\tilde{s}-A\beta_jl)\\
=\ &\sum_{j=1}^{n}\#_j\alpha_j\left[\mathcal{N}_j^{n_0}-z_jl+\Delta_j\right]x+\left[\mathcal{C}k-\sum_{j=1}^{n}\#_j\alpha_j(\Delta_j+\mathcal{N}_j^{n_0})\beta_j\right]\tilde{s}+Z\\
=\ &I+II+Z,
\end{aligned}$$

where $Z\in\mathbb{Z}$. For term I, we may note $\sum_{j=1}^{n}\#_j\alpha_jz_j=\sum_{j=1}^{n}\#_j\beta_j=0$ as well as

$$\sum_{j=1}^{n}\#_j\alpha_j\Delta_jx=\sum_{j=1}^{n}\#_j\alpha_j\Delta_j(A(\alpha_j\mathcal{N}_j^{n_0}+\delta(n_0,j))+\theta_x)=Z_1+\sum_{j=1}^{n}\#_j\alpha_j\delta_j(A\delta(n_0,j)+\theta_x),$$

where $Z_1\in\mathbb{Z}$. This remainder is acceptable using $A|\delta(n_0,j)|,|\theta_x|\lesssim\frac{1}{A}$. To handle term II, rewrite

$$\sum_{j=1}^{n} \#_j \alpha_j \mathcal{N}_j^{n_0} \beta_j \tilde{s} = \sum_{j=1}^{n} \#_j (n_0 + \delta(n_0, j)) \beta_j \tilde{s} = \sum_{j=1}^{n} \#_j \delta(n_0, j) \beta_j \tilde{s}.$$

Therefore, $II = [\mathfrak{C}k + O(1/A^2)]\tilde{s}$, and applying Lemma 3 gives a satisfactory lower bound of $\frac{C}{Ak}$ for

$$Im \Bigg[e^{-2\pi i A \sum_{j=1}^{n} \#_j \alpha_j \mathcal{N}_j^{n_0} x} \int_{A(k-\frac{1}{2})}^{A(k+\frac{1}{2})} \int_{A(l-\frac{1}{2})}^{A(l+\frac{1}{2})} \prod_{j=1}^{n} \phi(x - Am_j - \alpha_j t - \beta_j s)$$
$$\times \quad e^{2\pi i A \#_j m_j (x - \alpha_j t - \beta_j s)} K_1(s) K_2(t) ds dt \Bigg]$$

1.4.7 Large Perturbations

CASE: $k < 0, l \in \mathbb{Z}, m \in \mathbb{M}$

For each $(k,l) \in \mathbb{Z} \times \mathbb{Z}$, $\exists j_* \in \{1, ..., n\}$ satisfying

$$|\alpha_{j_*} m_{j_*} - n_0 - \alpha_{j_*} k - \beta_{j_*} l| \geq \frac{|l|}{5n \cdot \sup_{1 \leq j \leq n} |\#_j \alpha_j|}.$$

Indeed, if not, then $|\alpha_{j_*} m_{j_*} - n_0 - \alpha_{j_*} k - \beta_{j_*} l| \leq \frac{|l|}{5n \cdot \sup_{1 \leq j \leq n} |\#_j \alpha_j|}$ for all $1 \leq j \leq n$, and so

$$\begin{aligned} \sum_{j=1}^{n} \#_j \alpha_j^2 m_j &= \sum_{j=1}^{n} \#_j \alpha_j (\alpha_j m_j - n_0 - \alpha_j k - \beta_j l) + \sum_{j=1}^{n} \#_j \alpha_j (n_0 - \alpha_j k - \beta_j l) \\ &= I + II. \end{aligned}$$

By assumption, $|I| \leq \frac{|l|}{5}$. Moreover, $II = \sum_{j=1}^{n} \#_j \alpha_j (n_0 + \alpha_j k + \beta_j l) \geq l$, which contradicts the restriction $\sum_{j=1}^{n} \#_j \alpha_j^2 m_j = 0$ for all $\vec{m} \in \mathbb{M}$. Similarly, if $k < 0$, $\exists j_* \in \{1, ..., n\}$ such that

$$|\alpha_{j_*} m_{j_*} - n_0 - \alpha_{j_*} k - \beta_{j_*} l| \geq \frac{|k|}{5n \cdot \sup_{1 \leq j \leq n} |\#_j \beta_j|}.$$

The interested reader may easily check that the collection of large perturbations yields an acceptable error to the main contribution.

□

1.4.8 Lemma

Lemma 7. *Fix $\phi \in \mathcal{S}(\mathbb{R})$, $A > 0$ and let K_1, K_2 satisfy the usual conditions. Then there exists k_0 such that for all $k \geq k_0(\phi, \alpha, \beta, A, K_1, K_2)$,*

$$Im \left[\int_{-A}^{A} \int_{-A}^{A} \phi(\alpha t + \beta s) e^{2\pi i A s k} K_1(t + Ak) K_2(s) ds dt \right] \gtrsim \frac{D_{\phi,\alpha,A,C_1,C_2}}{Ak},$$

where $D_{\phi,\alpha,A,C_1,C_2} := C_1 \cdot C_2 \cdot \int_{-A}^{A} \phi(\alpha t) dt$.

Proof. The proof is a straightforward application of elementary decay estimates and integration by parts. First, note that is suffices to assume $C_1 = K_1(Ak)$ and $C_2 = Im \left[\check{K}_2(Ak) \right]$ and then prove

$$\lim_{k \to \infty} \left\| Im \left[\int_{-A^2 k}^{A^2 k} \phi \left(\alpha t + \frac{\beta s}{Ak} \right) e^{2\pi i s} \frac{1}{Ak} K_2 \left(\frac{s}{Ak} \right) ds \right] - Im \left[\check{K}_2(Ak) \right] \phi(\alpha t) \right\|_{L_t^\infty([-A,A])}$$

equals 0. Assuming the claim, choose $k_0(A)$ large enough to ensure the relevant difference is uniformly bounded by $O(1/A^2)$. Then

$$
\begin{aligned}
&\left| Im\left[\int_{-A}^{A}\int_{-A^2k}^{A^2k}\phi\left(\alpha t+\frac{\beta s}{Ak}\right)e^{2\pi i s}\frac{1}{Ak}K_2\left(\frac{s}{Ak}\right)ds\ K_1(t+Ak)dt-\frac{D_A}{Ak}\right]\right| \\
\leq\ &\left| Im\left[\int_{-A}^{A}\left(\int_{-A^2k}^{A^2k}\phi\left(\alpha t+\frac{\beta s}{Ak}\right)e^{2\pi i s}\frac{1}{Ak}K_2\left(\frac{s}{Ak}\right)ds-C_2\phi(\alpha t)\right)K_1(t+Ak)dt\right]\right| \\
+\ &\left| Im\left[C_2\left(\int_{-A}^{A}\phi(\alpha t)\left(K_1(t+Ak)-K_1(Ak)\right)dt\right)\right]\right|.
\end{aligned}
$$

We have easy estimates for both $\mid I \mid \lesssim \frac{1}{A^2k}$ and

$$
\mid II \mid \leq c\int_{-A}^{A}|\phi(\alpha t)|\frac{|t|}{(Ak)^2}dt \lesssim \frac{1}{(Ak)^2}.
$$

Hence, the lemma follows once we show the claim. To this end, break up the interior integral into three pieces:

$$
\begin{aligned}
Im\left[\int_{-A^2k}^{A^2k}\phi\left(\alpha t+\frac{\beta s}{Ak}\right)e^{2\pi i s}\frac{1}{Ak}K_2\left(\frac{s}{Ak}\right)ds\right] &= Im\left[\left(\int_{-A^2k}^{-1}+\int_{-1}^{1}+\int_{1}^{A^2k}\right)...\right] \\
&:= I_a^{k,t}+I_b^{k,t}+I_c^{k,t}.
\end{aligned}
$$

It is easy to see that $I_b^{k,t} = Im\left[\int_{-1}^{1}\phi\left(\alpha t+\frac{\beta s}{Ak}\right)e^{2\pi i s}\frac{1}{Ak}K_2\left(\frac{s}{Ak}\right)ds\right] \to \phi(\alpha t)Im\left[\int_{-1}^{1}e^{2\pi i s}\frac{1}{Ak}K_2\left(\frac{s}{Ak}\right)ds\right]$ as $k\to\infty$ uniformly in $t\in[-A,A]$ as desired. Moreover, a simple integration by parts argument on terms $I_a^{k,t}$ and $I_c^{k,t}$ yields

$$
\begin{aligned}
I_c^{k,t} &= Im\left[\frac{1}{2\pi i}\int_{1}^{A^2k}\phi\left(\alpha t+\frac{\beta s}{Ak}\right)\frac{d}{ds}\left(e^{2\pi i s}\right)\frac{1}{Ak}K_2\left(\frac{s}{Ak}\right)ds\right] \\
&= Im\left[\frac{1}{2\pi i}\left[\frac{\phi(\alpha t+\beta A)K_2(A)}{Ak}-\phi\left(\alpha t+\frac{\beta}{Ak}\right)\frac{K_2\left(\frac{1}{Ak}\right)}{Ak}\right.\right. \\
&- \frac{\beta}{Ak}\int_{1}^{A^2k}\phi'\left(\alpha t+\frac{\beta s}{Ak}\right)e^{2\pi i s}\frac{1}{Ak}K_2\left(\frac{s}{Ak}\right)ds \\
&- \left.\left.\int_{1}^{A^2k}\phi\left(\alpha t+\frac{\beta s}{Ak}\right)e^{2\pi i s}\frac{1}{(Ak)^2}K_2'\left(\frac{s}{Ak}\right)ds\right]\right].
\end{aligned}
$$

However,

$$\lim_{k\to\infty}\left|\left|\int_{A^2k}^{\infty}\phi\left(\alpha t+\frac{\beta s}{Ak}\right)e^{2\pi is}\frac{1}{(Ak)^2}K_2'\left(\frac{s}{Ak}\right)\right|\right|_{L_t^\infty([-A,A])}=0$$

from the pointwise estimate $\left|\frac{1}{(Ak)^2}K_2'\left(\frac{s}{Ak}\right)\right| \lesssim \frac{1}{s^2}$. Moreover, using $\left|\phi(\alpha t+\frac{\beta s}{Ak})-\phi(\alpha t)\right| \lesssim \left(\frac{\beta s}{Ak}\right)^{\frac{1}{2}}$ yields

$$\lim_{k\to\infty}\left|\left|\int_{1}^{\infty}\left[\phi\left(\alpha t+\frac{\beta s}{Ak}\right)-\phi\left(\alpha t\right)\right]e^{2\pi is}\frac{1}{(Ak)^2}K_2'\left(\frac{s}{Ak}\right)ds\right|\right|_{L_t^\infty([-A,A])}=0.$$

Again using the uniform smoothness of ϕ and oddness of K_2,

$$\lim_{k\to\infty}\left|\left|I_c^{k,t}+Im\left[\frac{1}{2\pi i}\left[\phi(\alpha t)\frac{K_2(\frac{1}{Ak})}{Ak}+\phi(\alpha t)\int_1^\infty e^{2\pi is}\frac{1}{(Ak)^2}K_2'\left(\frac{s}{Ak}\right)ds\right]\right]\right|\right|_{L_t^\infty([-A,A])}$$
$$\lim_{k\to\infty}\left|\left|I_a^{k,t}+Im\left[\frac{1}{2\pi i}\left[\phi(\alpha t)\frac{K_2(\frac{1}{Ak})}{Ak}+\phi(\alpha t)\int_{-\infty}^{-1} e^{2\pi is}\frac{1}{(Ak)^2}K_2'\left(\frac{s}{Ak}\right)ds\right]\right]\right|\right|_{L_t^\infty([-A,A])}$$

both vanish. Therefore, it is enough to show

$$\begin{aligned}&Im\left[\frac{1}{2\pi i}\left[-\frac{2K_2\left(\frac{1}{Ak}\right)}{Ak}-\int_{\mathbb{R}\cap[-1,1]^c}e^{2\pi is}\frac{1}{(Ak)^2}K_2'\left(\frac{s}{Ak}\right)ds\right]\right.\\&+\left.\int_{-1}^{1}e^{2\pi is}\frac{1}{Ak}K_2\left(\frac{s}{Ak}\right)ds-\check{K}_2(Ak)\right]=0.\end{aligned}$$

However, this is immediate via integrating by parts $\int_{\mathbb{R}-[-1,1]}e^{2\pi is}\frac{1}{(Ak)^2}K_2'\left(\frac{s}{Ak}\right)ds$ over its two disjoint regions to rewrite the LHS as $\int_{\mathbb{R}}e^{2\pi is}\frac{1}{Ak}K_2\left(\frac{s}{Ak}\right)ds-\check{K}_2(Ak)$ and then perform the change of variable $s\mapsto Aks$.

□

The following statement is an immediate corollary of Lemma 7:

Lemma 8. *Fix $\phi \in \mathcal{S}(\mathbb{R})$, $A > 0$, and K_1, K_2 satisfying the usual conditions. Then there exists k_0 such that for all $k \geq k_0(\phi, \vec{\alpha}, \vec{\beta}, A, K_1, K_2)$*

$$Im \left[\int_{-A}^{A} \int_{-A}^{A} \prod_{j=1}^{n} \phi(\alpha_j t + \beta_j s) e^{2\pi i A s k} K_1(s) K_2(t + Ak) ds dt \right] \gtrsim \frac{D_{\phi,\vec{\alpha},A,C_1,C_2}}{Ak}$$

where $D_{\phi,\vec{\alpha},A,C_1,C_2} := C_1 \cdot C_2 \left[\int_{-A}^{A} \prod_{j=1}^{n} \phi(\alpha_j t) dt \right]$.

Proof. Same as before. □

1.5 Symbols Adapted to Subspaces $\Gamma \subset \mathbb{R}^n$ satisfying $dim\ \Gamma \geq \frac{n}{2} + \frac{3}{2}$

Definition 6. *Fix $\Phi, \Psi \in \mathcal{S}(\mathbb{R})$ such that $1_{[-1/2,1/2]} \leq \Phi \leq 1_{[-1,1]}$ and $1_{[2,\infty)} \leq \Psi \leq 1_{[1,\infty)}$. Let $\Gamma\left(\{\vec{\alpha}^m\}_{m=1}^d\right) = \bigcap_{m=1}^d \left\{\vec{\xi} \cdot \vec{\alpha}^m = 0\right\}$ and $\Gamma\left(\{\vec{\beta}^m\}_{m=1}^d\right) = \bigcap_{m=1}^d \left\{\vec{\xi} \cdot \vec{\beta}^m = 0\right\}$. For every symbol $m : \mathbb{R}^n \to \mathbb{C}$, the $(\Phi, \Psi)-$localization of m near $\Gamma\left(\{\vec{\alpha}^m\}_{m=1}^d\right)$ away from $\Gamma\left(\{\beta^m\}_{m=1}^d\right)$ is the symbol defined by*

$$m[\{\vec{\alpha}^m\}, \{\vec{\beta}^m\}, \Phi, \Psi](\vec{\xi}) = \left[\prod_m \Phi(\vec{\alpha}^m \cdot \vec{\xi}) \right] \left[\prod_m{}' \Psi(\vec{\beta}^m \cdot \vec{\xi}) \right] m(\vec{\xi}) \quad \forall \vec{\xi} \in \mathbb{R}^n,$$

where the primed product means one multiplies only over those $m \in \{1, ..., d\}$ for which $\vec{\beta}^m \neq \vec{0}$.

Our main result in this section is

Theorem 9. *Let $n \geq 5, d \geq 1$. Let $\left\{\alpha_j^m\right\}_{1 \leq j \leq n; 1 \leq m \leq d}, \left\{\beta_j^m\right\}_{1 \leq j \leq n; 1 \leq m \leq d} \in \mathbb{R}^{nd}$ be given. Suppose there exists a vector $\vec{a} \in \mathbb{R}^n$ such that $\alpha_j^m = a_j q_j^m$ and $\beta_j^1 = a_j r_j$ where $q_j^m, r_j \in \mathbb{Q}$ for all $1 \leq j \leq n$ and $1 \leq m \leq d$. Furthermore, assume there are $\vec{\#} \in \mathbb{R}^n$ and $\mathfrak{C} > 0$ such that*

$$\begin{aligned} \sum_{j=1}^n \#_j \alpha_j^n = \sum_{j=1}^n \#_j \beta_j^n &= 0 && \forall\ n \in \{1, ..., d\} \\ \sum_{j=1}^n \#_j \alpha_j^n \alpha_j^m = \sum_{j=1}^n \#_j \beta_j^n \beta_j^m &= 0 && \forall\ n, m \in \{1, ..., d\} \\ \sum_{j=1}^n \#_j \alpha_j^n \beta_j^m &= \mathfrak{C}(\vec{\#}) \cdot \delta_{n,1} \delta_{m,1} && \forall\ n, m \in \{1, ..., d\}, \end{aligned}$$

where $\delta : \{1, ..., d\} \times \{1, ..., d\} \to \{0, 1\}$ is the Kronecker delta function, $\mathfrak{C} > 0$, and $\#_j a_j^2 \in \mathbb{Q}$ for all $1 \leq j \leq n$. Moreover, let $K_d(s) = \frac{s_1}{|\vec{s}|^{d+1}}$ be the first $d-$dimensional Riesz kernel. Then every (Φ, Ψ)-localization of $\hat{K}_d(A\vec{\xi})\hat{K}_d(B\vec{\xi}) : \mathbb{R}^n \to \mathbb{C}$ gives rise to a multilinear multiplier

$$T_{\hat{K}_d(A\vec{\xi})\hat{K}_d(B\vec{\xi})[\{\vec{\alpha}^m\}, \{\vec{\beta}^m\}, \Phi, \Psi]}$$

which satisfies no L^p estimates.

Remark: The notation q_j^m does not mean $(q_j)^m$. We should consider q_j^m as an doubly-indexed quantity. In the following lemma, $q_j^m = (q_j)^m$!

Lemma 9. *Let $n \geq 5$, $d \geq 1$, and $2d + 3 \leq n$. Suppose $\vec{a} \in \mathbb{R}^n$ satisfies $a_j^{-1} = q_j + \alpha q_j^2$ and $\beta_j = a_j q_j^{-1}$ for some $\alpha \in \mathbb{R}$ and $\vec{q} \in \mathbb{Q}^n$ with distinct, non-zero entries such that $q_j \alpha \neq -1$ for all $1 \leq j \leq n$. Let $\alpha_j^n = a_j q_j^n$ for $2 \leq n \leq d$ and $\alpha_j^1 = a_j$. Then there are $\vec{\#} \in \mathbb{R}^n$ and $\mathfrak{C} > 0$ such that*

$$\sum_{j=1}^{n} \#_j \alpha_j^n = \sum_{j=1}^{n} \#_j \alpha_j^n \alpha_j^m = \sum_{j=1}^{n} \#_j \beta_j^1 = \sum_{j=1}^{n} \#_j \left[\beta_j^1\right]^2 = 0$$

$$\sum_{j=1}^{n} \#_j \alpha_j^m \beta_j^1 = \mathfrak{C}\delta_{1,m}$$

with the additional property that $\#_j a_j^2 \in \mathbb{Q}$ *for all* $1 \leq j \leq d$.

Proof. It suffices to prove there is a rational solution $\vec{\tilde{\#}} \in \mathbb{Q}^n$ to the system

$$\sum_{j=1}^{n} \#_j \alpha_j^1 = \sum_{j=1}^{n} \tilde{\#}_j a_j^{-2} a_j = \sum_{j=1}^{n} \tilde{\#}_j (q_j + \alpha q_j^2) = 0$$

$$\sum_{j=1}^{n} \#_j \alpha_j^m = \sum_{j=1}^{n} \#_j a_j q_j^m = \sum_{j=1}^{n} \tilde{\#}_j a_j^{-1} q_j^m = \sum_{j=1}^{n} \tilde{\#}_j (q_j + \alpha q_j^2) q_j^m = 0 \quad (2 \leq m \leq d)$$

$$\sum_{j=1}^{n} \#_j \beta_j^1 = \sum_{j=1}^{n} \tilde{\#}_j a_j^{-2} a_j q_j^{-1} = \sum_{j=1}^{n} \tilde{\#}_j (q_j + \alpha q_j^2) q_j^{-1} = 0$$

$$\sum_{j=1}^{n} \#_j \left[\alpha_j^1\right]^2 = \sum_{j=1}^{n} \#_j a_j^2 = \sum_{j=1}^{n} \tilde{\#}_j = 0$$

$$\sum_{j=1}^{n} \#_j \alpha_j^m \alpha_j^n = \sum_{j=1}^{n} \#_j a_j^2 q_j^m q_j^n = \sum_{j=1}^{n} \tilde{\#}_j q_j^{m+n} = 0 \quad (2 \leq n, m \leq d)$$

$$\sum_{j=1}^{n} \#_j \alpha_j^1 \alpha_j^m = \sum_{j=1}^{n} \#_j a_j^2 q_j^m = \sum_{j=1}^{n} \tilde{\#}_j q_j^m = 0 \quad (2 \leq m \leq d)$$

$$\sum_{j=1}^{n} \#_j \left[\beta_j^1\right]^2 = \sum_{j=1}^{n} \tilde{\#}_j q_j^{-2} = 0$$

$$\sum_{j=1}^{n} \#_j \beta_j^1 \alpha_j^m = \sum_{j=1}^{n} \#_j a_j^2 q_j^m q_j^{-1} = \sum_{j=1}^{n} \tilde{\#}_j q_j^{m-1} = 0 \quad (2 \leq m \leq d)$$

$$\sum_{j=1}^{n} \#_j \alpha_j^1 \beta_j^1 = \sum_{j=1}^{n} \tilde{\#}_j q_j^{-1} \neq 0.$$

Because $q_i \neq q_j$ whenever $i \neq j$ and $2d+3 \leq n$, one can choose non-trivial $\vec{\tilde{\#}} \in \mathbb{Q}^n$ to ensure $\sum_{j=1}^{n} \tilde{\#}_j q_j^m = 0$ for all $m \in \{-2, 0, 1, 2, ..., 2d\}$ and $\sum_{j=1}^{n} \tilde{\#}_j q_j^{-1} \neq 0$. □

Using Theorem 9 and Lemma 9, we obtain

Theorem 10. *Let* $n, \mathfrak{d} \in \mathbb{N}$ *satisfy* $\frac{n+3}{2} \leq \mathfrak{d} < n$ *and* $n \geq 5$. *Furthermore, let* $a_j^{-1} = q_j + \alpha q_j^2$ *for some* $\alpha \in \mathbb{R}$ *and* $\vec{q} \in \mathbb{Q}^n$ *with distinct, non-zero entries such that* $q_j \alpha \neq -1$ *for all* $1 \leq j \leq n$. *Let* $\alpha_j^n = a_j q_j^n$ *for* $2 \leq n \leq d$ *and* $\alpha_j^1 = a_j$. *Moreover, let*

$$\Gamma = \bigcap_{m=1}^{d} \left\{ \vec{\xi} \cdot \vec{\alpha}^m = 0 \right\} \subset \mathbb{R}^n.$$

Then there exists a symbol m_Γ *adapted to* Γ *in the Mikhlin-Hörmander sense and supported in* $\left\{ \vec{\xi} : dist(\vec{\xi}, \Gamma) \lesssim 1 \right\}$ *for which the associated multilinear multiplier* T_{m_Γ} *is unbounded.*

Proposition 1. *Let* $n \geq d+1$. *Fix* $\vec{q} \in \mathbb{Q}^n$ *and* $\alpha \in \mathbb{R}$. *Let* $a_j^{-1} = q_j + \alpha q_j^2$ *with distinct, non-zero entries such that* $q_j \alpha \neq -1$ *for all* $1 \leq j \leq n$. *Let* $\alpha_j^n = a_j q_j^n$ *for* $2 \leq n \leq d$ *and* $\alpha_j^1 = a_j$. *Let*

$$\Gamma(\alpha, \vec{q}) := \bigcap_{m=1}^{d} \left\{ \vec{\xi} \in \mathbb{R}^n \;\middle|\; \vec{\xi} \cdot \vec{a}^m(\alpha, \vec{q}) = 0 \right\}.$$

Then the number of distinct subspaces in the collection $\{\Gamma(\alpha, \vec{q})\}_{|\alpha| \leq \epsilon}$ *is uncountable for every* $\epsilon > 0$.

Proof. Because $\Gamma(0, \vec{q}) = \left[Span \left\{ \vec{q}^{-1}, \vec{q}, \vec{q} \wedge \vec{q}, ..., \wedge^{d-1} \vec{q} \right\} \right]_\perp$,

$$\Gamma(0, \vec{q})_\perp = \left[\left[Span \left\{ \vec{q}^{-1}, \vec{q}, \vec{q} \wedge \vec{q}, ..., \wedge^{d-1} \vec{q} \right\} \right]_\perp \right]_\perp = Span \left\{ \vec{q}^{-1}, \vec{q}, \vec{q} \wedge \vec{q}, ..., \wedge^{d-1} \vec{q} \right\}.$$

As $n \geq d+1$ and $\vec{q} \in \mathbb{Q}^n$ has distinct non-zero entries, $\vec{1} \notin Span \left\{ \vec{q}^{-1}, \vec{q}, \vec{q} \wedge \vec{q}, ..., \wedge^{d-1} \vec{q} \right\}$ so that $d := dist(\vec{1}, \Gamma(0, \vec{q})_\perp) > 0$. Therefore,

$\frac{d}{d\alpha}\vec{a}(\alpha) = -\frac{q_j^2}{(q_j+\alpha q_j^2)^2}$. Therefore, $\frac{d}{d\alpha}\vec{a}(\alpha)\big|_{\alpha=0} \propto \vec{1}$. It follows that for small enough ϵ depending on $\vec{q}$, $\vec{a}(\epsilon) \notin \Gamma(0,\vec{q})_\perp$. Therefore, letting

$$D_\epsilon(\alpha) := dist(\vec{a}(\epsilon), \Gamma(\alpha,\vec{q})_\perp),$$

$D_\epsilon(0) \neq 0, D_\epsilon(\epsilon) = 0$. Therefore, $Im\,[D_\epsilon([0,\epsilon])] \supset [0,\delta)$ for some $\delta > 0$. In particular, D_ϵ takes on uncountably many values, and so $\{\Gamma(\alpha,\vec{q})_\perp\}_{|\alpha|\leq\epsilon}$ and $\{\Gamma(\alpha,\vec{q})\}_{|\alpha|\leq\epsilon}$ must be uncountable.

□

Proposition 2. *Let $n \geq d+1$. Fix $\vec{q} \in \mathbb{Q}^n$ and $\alpha \in \mathbb{R}$. Let $a_j^{-1} = q_j + \alpha q_j^2$ with distinct, non-zero entries such that $q_j\alpha \neq -1$ for all $1 \leq j \leq n$. Let $\alpha_j^n = a_j q_j^n$ for $2 \leq n \leq d$ and $\alpha_j^1 = a_j$. Let*

$$\Gamma(\alpha,\vec{q}) := \bigcap_{m=1}^{d} \left\{\vec{\xi} \in \mathbb{R}^n \;\middle|\; \vec{\xi}\cdot\vec{a}^m(\alpha,\vec{q}) = 0\right\}.$$

Then, for each $\vec{q} \in \mathbb{Q}^n$ with distinct non-zero entries, there exists $\epsilon > 0$ such that $\Gamma(\alpha,\vec{q})$ is non-degenerate for all $|\alpha| \leq \epsilon$ in the sense of Muscalu, Tao, and Thiele [18], i.e.

$$\tilde{\Gamma} := \left\{(\xi_1,, ..., \xi_{n+1}) \in \mathbb{R}^{n+1} : (\xi_1, ..., \xi_n) \in \Gamma, \sum_{j=1}^{n+1} \xi_j = 0\right\}$$

is a graph over the variables $\xi_{i_1}, ..., \xi_{i_{n-d}}$ for every $1 \leq i_1 < i_2 < ... < i_{n-d} \leq n+1$.

Proof. First observe that for $\vec{v} \in \mathbb{Z}^n$, set $\Gamma^{\vec{v}} := \left\{\vec{\xi} \in \mathbb{R}^n : \sum_{j=1}^n \xi_j v_j = 0\right\}$. Let $\vec{\alpha}^j = \wedge^j \vec{\gamma}$ for $j \in \{1, ..., d\}$ where $\vec{\gamma} \in \mathbb{Z}^n$ satisfies $\gamma_i \neq \gamma_j$ for all $i \neq j$. Then the subspace $\Gamma := \bigcap_{j=1}^{d} \Gamma^{\vec{\alpha}^j} \subset \mathbb{R}^n$ is a non-degenerate subspace. Indeed, let $\mathcal{M} \in M_{d+1,n+1}(\mathbb{Z})$ be given by

$$\mathcal{M} = \begin{bmatrix} 1 & 1 & .... & 1 & 1 \\ \gamma_1^1 & \gamma_2^1 & ... & \gamma_n^1 & 0 \\ \gamma_1^2 & \gamma_2^2 & ... & \gamma_n^2 & 0 \\ \vdots & \vdots & \dots & \vdots & \vdots \\ \gamma_1^d & \gamma_2^d & ... & \gamma_n^d & 0 \end{bmatrix}.$$

Then $\vec{\xi} \in \tilde{\Gamma} \subset \mathbb{R}^{n+1}$ iff $\mathcal{M}\vec{\xi} = \vec{0}$. Suppose $n-d$ distinct indices $\mathcal{I} \subset (i_1, i_2, ..., i_{n-d}) \subset (1, 2, ..., n+1)$ have been chosen. Form $\mathcal{M}_{\mathcal{I}^c} \in M_{d+1,d+1}$ by deleting those columns with indices in $\mathcal{I}$. Furthermore, let $\vec{\xi}_{\mathcal{I}^c} \in \mathbb{R}^{n-d}$ be given by deleting all indices in $\mathcal{I}$. Then for every $\vec{\xi} \in \mathbb{R}^{n+1}$, $\mathcal{M}\vec{\xi} = \mathcal{M}_{\mathcal{I}}\vec{\xi}_{\mathcal{I}} + \mathcal{M}_{\mathcal{I}^c}\vec{\xi}_{\mathcal{I}^c}$ and for every $\xi \in \tilde{\Gamma}$,

$$\mathcal{M}_{\mathcal{I}}\vec{\xi}_{\mathcal{I}} = -\mathcal{M}_{\mathcal{I}^c}\vec{\xi}_{\mathcal{I}^c}.$$

However, $\mathcal{M}_{\mathcal{I}^c}$ must take one of the following forms:

$$\begin{bmatrix} 1 & 1 & ... & 1 \\ \gamma_{j_1}^1 & \gamma_{j_2}^1 & ... & \gamma_{j_{d+1}}^1 \\ \gamma_{j_1}^2 & \gamma_{j_2}^2 & ... & \gamma_{j_{d+1}}^2 \\ \vdots & \vdots & \dots & \vdots \\ \gamma_{j_1}^d & \gamma_{j_2}^d & ... & \gamma_{j_{d+1}}^d \end{bmatrix} \quad or \quad \begin{bmatrix} 1 & 1 & .... & 1 & 1 \\ \gamma_{j_1}^1 & \gamma_{j_2}^1 & ... & \gamma_{j_d}^1 & 0 \\ \gamma_{j_1}^2 & \gamma_{j_2}^2 & ... & \gamma_{j_d}^2 & 0 \\ \vdots & \vdots & \dots & \vdots & \vdots \\ \gamma_{j_1}^d & \gamma_{j_2}^d & ... & \gamma_{j_d}^d & 0 \end{bmatrix}$$

for some $\vec{j} = (j_1, ..., j_d) : 1 \leq j_1 \leq ... \leq j_d$. In either case, $\mathcal{M}_{\mathcal{I}^c}$ is invertible by the assumption $\gamma_i \neq \gamma_j$ whenever $i \neq j$. Hence, there is a well-defined mapping $\gamma : (\xi_{i_1}, ..., \xi_{i_{n-d}}) \to (\xi_1, ..., \xi_{n+1})$ expressing $\tilde{\Gamma}$ as a graph over the variables

$(\xi_{i_1}, ..., \xi_{i_{n-d}})$. In the limit as $\alpha \to 0$, $\vec{\alpha} \to q_j^{-1}$. Therefore, $\vec{\alpha}^1 \to \vec{q}^{-1}$ and $\vec{\alpha}^n \to \vec{q}^{n-1}$ for $2 \leq n \leq d$. The matrix

$$\mathcal{M}(\vec{q}) := \begin{bmatrix} 1 & 1 & .... & 1 & 1 \\ q_1^{-1} & q_2^{-1} & ... & q_d^{-1} & 0 \\ q_1 & q_2 & ... & q_d & 0 \\ \vdots & \vdots & \dots & \vdots & \vdots \\ q_1^{d-1} & q_2^{d-1} & ... & q_d^{d-1} & 0 \end{bmatrix},$$

has non-zero determinant for all its minors, which ensures $\Gamma(0, \vec{q})$ is non-degenerate. Moreover, as non-degeneracy holds at $\alpha = 0$, it continuous to hold for all $|\alpha| \leq \epsilon(\vec{q})$.

□

Combining Theorem 10 with Propositions 1 and 2 finally yields the takeaway result:

Theorem 7. *Let $n, \mathfrak{d} \in \mathbb{N}$ satisfy $\frac{n+3}{2} \leq \mathfrak{d} < n$ and $n \geq 5$. Then there is an uncountable collection $\mathfrak{C}$ of $\mathfrak{d}-$ dimensional non-degenerate subspaces $\Gamma \subset \mathbb{R}^n$ such that for each $\Gamma \in \mathfrak{C}$ there is an associated symbol m_Γ adapted to Γ in the Mikhlin-Hörmander sense for which the associated multilinear multiplier T_{m_Γ} is unbounded.*

Proof. [Theorem 9]

1.5.1 PART 1: The Rational Case

Note that our assumption in the case $\alpha = 0$ is equivalent to the superficially weaker assumption that $A = (\vec{\alpha}_1, ..., \vec{\alpha}_n) \in (\mathbb{Q}^d)^n$ and $B = \left(\vec{\beta}_1, ..., \vec{\beta}_n\right) \in (\mathbb{Q}^d)^n$ satisfy

$$(\vec{\alpha}^1 \wedge \vec{\beta}^1) \notin Span\left\{\vec{\alpha}^1, \vec{\beta}^1, \left.(\vec{\alpha}^n \wedge \vec{\beta}^m)\right|_{(n,m):\delta_{n,1}\delta_{m,1}=0}, \left.(\vec{\alpha}^n \wedge \vec{\alpha}^m)\right|_{(n,m)}, \left.(\vec{\beta}^n \wedge \vec{\beta}^m)\right|_{(n,m)}\right\}.$$

Alternatively, $A = \left(\vec{\alpha}^1, ..., \vec{\alpha}^d\right), B = \left(\vec{\beta}^1, ..., \vec{\beta}^d\right) \in (\mathbb{Q}^n)^d$ under the obvious identification $\left(\mathbb{Q}^d\right)^n \simeq (\mathbb{Q}^n)^d \simeq \mathbb{Q}^{dn}$. By dilating if necessary, we shall assume $A, B \in \mathbb{Z}^{nd}$. Indeed, by the dimensionality constraints and spanning condition, we are assured by the Gram-Schmidt process of finding of vector $\vec{\#} \in \mathbb{R}^n$ for which the orthogonality constraints (*) are satisfied:

$$\begin{aligned}
\sum_{j=1}^{n} \#_j \alpha_j^n = \sum_{j=1}^{n} \#_j \beta_j^n &= 0 && \forall\, n \in \{1, ..., d\} \\
\sum_{j=1}^{n} \#_j \alpha_j^n \alpha_j^m = \sum_{j=1}^{n} \#_j \beta_j^n \beta_j^m &= 0 && \forall\, n, m \in \{1, ..., d\} \\
\sum_{j=1}^{n} \#_j \alpha_j^n \beta_j^m &= \mathfrak{C}(\vec{\#}) \cdot \delta_{n,1}\delta_{m,1} && \forall\, n, m \in \{1, ..., d\},
\end{aligned}$$

where $\delta : \{1, ..., d\} \times \{1, ..., d\} \to \{0, 1\}$ is the usual Kronecker delta function and $\mathfrak{C} \in \mathbb{R}^c \cap \{0\}^c$. In fact, one can always restrict $\vec{\#} \in \mathbb{Z}^n$. Indeed, because $A, B \in \mathbb{Z}^{dn}$, we may perform the Gramm-Schmidt process to form an orthogonal basis $\{\vec{\gamma}^1, ..., \vec{\gamma}^p\}$ for $\mathcal{S} := Span\left\{\vec{1}, \vec{\alpha}^1, \vec{\beta}^1, \left.(\vec{\alpha}^n \wedge \vec{\beta}^m)\right|_{(n,m):\delta_{n,1}\delta_{m,1}=0}, \left.(\vec{\alpha}^n \wedge \vec{\alpha}^m)\right|_{(n,m)}, \left.(\vec{\beta}^n \wedge \vec{\beta}^m)\right|_{(n,m)}\right\}$ such that $\vec{\gamma}^j \in \mathbb{Q}^n$ for every $j \in \{1, ..., p\}$. Moreover, $\vec{\alpha}^1 \wedge \vec{\beta}^1$ is not in the span, so we can find an element in $\mathbb{Q}^n$ orthogonal to $\mathcal{S}$ by setting

$$\vec{\#} = \vec{\alpha}^1 \wedge \vec{\beta}^1 - \sum_{j=1}^{p} \frac{\vec{\gamma}^j \langle \vec{\gamma}^j, \vec{\alpha}^1 \wedge \vec{\beta}^1 \rangle}{\langle \vec{\gamma}^j, \vec{\gamma}^j \rangle}$$

for which the constraints $(*)$ are satisfied. By a signed dilation, we may take $\vec{\#} \in \mathbb{Z}^n$ so that $\mathcal{C} \in \mathbb{Z}^+$ without loss of generality.

Recall $f^{N,A,\#}(x) := \sum_{-N \leq m \leq N} \phi(x - Am) e^{2\pi i Am \# x} = \sum_{-N \leq m \leq N} f_m^{N,A\#}(x)$, where ϕ satisfies the same properties as the ϕ appearing in Theorem 4. Fix $\vec{m}$ and assume $m_j = n_0 - \vec{\alpha_j} \cdot \vec{k} - \vec{\beta_j} \cdot \vec{l}$ for some $\vec{k}$ and $\vec{l}$ for all $j \in \{1, ..., n\}$. Then for appropriate choices for ϕ and A, the frequency support assumptions ensure

$$\sum_{j=1}^{n} \#_j \alpha_j^k m_j = 0 \ \ \forall 1 \leq k \leq d; \quad \sum_{j=1}^{n} \#_j \beta_j^1 m_j > 0$$

provided

$$T_{\hat{K}_d(A\cdot)\hat{K}_d(B\cdot)[\{\vec{\alpha}^m\},\{\vec{\beta}^m\},\Phi,\Psi]} \left(\left\{ f_{m_j}^{N,A\#_j} \right\}_{j=1}^{n} \right) \not\equiv 0.$$

Indeed, by construction,

$$\hat{K}_d(A\cdot)\hat{K}_d(B\cdot)[\{\vec{\alpha}^m\},\{\vec{\beta}^m\},\Phi,\Psi](\vec{\xi}) = \hat{K}_d(A\vec{\xi})\hat{K}_d(B\vec{\xi}) m_1\left(\vec{\xi}\right) m_2 \left(\vec{\xi}\right),$$

where the symbols m_1 and m_2 are Mikhlin-Hörmander adapted to $\bigcap_{j=1}^d \{\vec{\xi} : dist(\vec{\xi}, \Gamma^{\vec{\alpha}^j}) \lesssim 1\}$ and $\{\vec{\xi} : \xi \cdot \vec{\beta}^1 \gtrsim 1\}$ and are also identically equal to 1 on $\bigcap_{j=1}^d \{\vec{\xi} : dist(\vec{\xi}, \Gamma^{\vec{\alpha}^j}) \lesssim 1\}$ and $\left\{\vec{\xi} : \vec{\xi} \cdot \vec{\beta}^1 \gtrsim 1\right\}$ respectively. Therefore, setting

$$\mathbb{M} := \left\{ \vec{m} \in \mathbb{Z}^n \cap [-N, N]^n \middle| \sum_{j=1}^{n} \#_j m_j \beta_j^1 > 0 \ ; \ \sum_{j=1}^{n} \#_j m_j \alpha_j^k = 0 \ \forall \ 1 \leq k \leq d \right\},$$

$$\begin{aligned} & T_{\hat{K}_d(A\cdot)\hat{K}_d(B\cdot)[\{\vec{\alpha}^m\},\{\vec{\beta}^m\},\Phi,\Psi]} \left(\left\{ f^{N,A,\#_j} \right\}_{j=1}^{n} \right) \\ = & \sum_{\vec{m} \in \mathbb{M}} T_{\hat{K}_d(A\cdot)\hat{K}_d(B\cdot)[\{\vec{\alpha}^m\},\{\vec{\beta}^m\},\Phi,\Psi]} \left(\left\{ f_{m_j}^{N,A,\#_j} \right\}_{j=1}^{n} \right). \end{aligned}$$

Furthermore, note

$$
\begin{aligned}
& T_{\hat{K}_d(A\cdot)\hat{K}_d(B\cdot)[\{\vec{\alpha}^m\},\{\vec{\beta}^m\},\Phi,\Psi]}\left(\left\{f^{N,A,\#_j}\right\}_{j=1}^n\right)(x) \\
= & \sum_{\vec{m}\in\mathbb{M}}\int_{\mathbb{R}^{2d}}\left[\prod_{j=1}^n \phi(x-Am_j-\vec{\alpha}_j\cdot\vec{t}-\vec{\beta}_j\cdot\vec{s})e^{2\pi iA\#_j m_j(x-\vec{\alpha}_j\cdot\vec{t}-\vec{\beta}_j\cdot\vec{s})}\right]\frac{s_1}{|\vec{s}|^{d+1}}\frac{t_1}{|\vec{t}|^{d+1}}d\vec{t}d\vec{s} \\
= & \sum_{\vec{m}\in\mathbb{M}}\sum_{\vec{l},\vec{k}\in\mathbb{Z}^d}\int_{A(l_1-\frac{1}{2})}^{A(l_1+\frac{1}{2})}\dots\int_{A(k_d-\frac{1}{2})}^{A(k_d+\frac{1}{2})}\left[\prod_{j=1}^n \phi(x-Am_j-\vec{\alpha}_j\cdot\vec{t}-\vec{\beta}_j\cdot\vec{s})e^{2\pi iA\#_j m_j(x-\vec{\alpha}_j\cdot\vec{t}-\vec{\beta}_j\cdot\vec{s})}\right] \\
\times & \frac{s_1}{|\vec{s}|^{d+1}}\frac{t_1}{|\vec{t}|^{d+1}}d\vec{t}d\vec{s}.
\end{aligned}
$$

Large terms arise whenever $x \in \left[An_0, An_0 + \frac{c_{\vec{\alpha},\vec{\beta}}}{A}\right]$ and there exists $\vec{m}\in[-N,N]^n\cap\mathbb{Z}^n$ together with $\vec{k}\in\mathbb{Z}^d$ satisfying the constraints $k_1 \gtrsim 1$, $|\vec{k}|\lesssim N$ and

$$n_0 - m_j(n_0,\vec{k},\vec{0}) - \vec{\alpha}_j\cdot\vec{k} = 0 \ \forall\ j\in\{1,...,n\}.$$

In fact, we shall prove that it is enough to produce a lower bound for these large terms because the remainder may be subsumed as error:

Lemma 10. *To prove Theorem 7, it suffices to show* $\exists c_{\left\{\vec{\alpha}^j,\vec{\beta}^j\right\}_{j\in\{1,\dots,d\}}}$ *with the property that for every* $x \in \left[An_0, An_0 + \frac{c_{\{\vec{\alpha}^j,\vec{\beta}^j\}_{j\in\{1,\dots,d\}}}}{A}\right]$, $k \in \left[k_0, \frac{N}{3d\max_{(j_1,j_2)\in[1,\dots,n]\times[1,\dots,d]}\left\{|\alpha_{j_1}^{j_2}|\right\}}\right]$, *and* $n_0\in[-N/3,N/3]$

$$
\begin{aligned}
& Im\left[e^{-2\pi iA\left(\sum_{j=1}^n \#_j\right)n_0 x}\sum_{\vec{z}\in\mathbb{Z}^d:\vec{m}(n_0,\vec{z},\vec{0})\in\mathbb{M},z_1=k_1}\right. \\
& \left. T^{\vec{k},\vec{0}}_{\hat{K}_d(A\cdot)\hat{K}_d(B\cdot)[\{\vec{\alpha}^m\},\{\vec{\beta}^m\},\Phi,\Psi]}\left(\left\{f^{N,A,\#_j}_{m_j(n_0,\vec{k},\vec{0})}\right\}_{j=1}^n\right)(x)\right] \\
\gtrsim & \frac{1}{A^d k_1}.
\end{aligned}
$$

1.5.2 Main Contribution: $\vec{k} : k_1 \gtrsim 1, \vec{l} = \vec{0}$

Before proving the lemma, we verify the lower bound in the above display. Note that if $n_0 \in [-N/3, N/3]$ and $\vec{k} \in \mathbb{Z}^d \cap \left[k_0, \frac{N}{3d \max_{(j_1,j_2)\in[1,\dots,d]\times[1,\dots,n]}\left\{|\alpha_{j_1}^{j_2}|\right\}}\right]^d$, then $m_j(\vec{k},\vec{0}) := n_0 - \vec{\alpha}_j \cdot k \ \forall \ j \in \{1,...,n\} \in \mathbb{Z}^n \cap [-N,N]^n$. For $x \in \left[An_0, An_0 + \frac{c_{\{\vec{\alpha}^j,\vec{\beta}^j\}_{j\in\{1,\dots,d\}}}}{A}\right]$, let $\theta_x = \lfloor x \rfloor$. Evaluating the diagonal contribution yields

$$
\begin{aligned}
& e^{-2\pi i A\left(\sum_{j=1}^n \#_j\right)n_0 x} T^{\vec{k},\vec{0}}_{\hat{K}_d(A\cdot)\hat{K}_d(B\cdot)[\{\vec{\alpha}^m\},\{\vec{\beta}^m\},\Phi,\Psi]}\left(\left\{f^{N,A,\#_j}_{m_j(n_0,k)}\right\}_{j=1}^n\right)(x) \\
= & \int_{[-\frac{A}{2},\frac{A}{2}]^d}\int_{A(k_1-\frac{1}{2})}^{A(k_1+\frac{1}{2})}\cdots\int_{A(k_d-\frac{1}{2})}^{A(k_d+\frac{1}{2})} e^{2\pi i A\left(\sum_{j=1}^d \#_j \alpha_j^1\beta_j^1\right)s_1 k_1}\frac{s_1 t_1}{|\vec{s}|^{d+1}|\vec{t}|^{d+1}}d\vec{t}d\vec{s} \\
= & \int_{-\frac{A^2\mathfrak{C}k_1}{2}}^{\frac{A^2\mathfrak{C}k_1}{2}}\cdots\int_{-\frac{A^2\mathfrak{C}k_1}{2}}^{\frac{A^2\mathfrak{C}k_1}{2}}\int_{-\frac{A}{2}}^{\frac{A}{2}}\cdots\int_{-\frac{A}{2}}^{\frac{A}{2}}\prod_{j=1}^n \phi\left(\theta_x + \vec{\alpha}_j\cdot\vec{t} + \vec{\beta}_j\cdot\frac{\vec{s}}{A\mathfrak{C}k_1}\right) \\
\times & \ e^{2\pi i s_1}\frac{s_1(t_1 + Ak_1)}{|\vec{s}|^{d+1}|\vec{t}+A\vec{k}|^{d+1}}d\vec{t}d\vec{s}.
\end{aligned}
$$

To show the desired inequality, it suffices to prove

$$
\begin{aligned}
& \lim_{k_1\to\infty}\int_{-A^2\mathfrak{C}k_1/2}^{A^2\mathfrak{C}k_1/2}\cdots\int_{-A^2\mathfrak{C}k_1/2}^{A^2\mathfrak{C}k_1/2}\prod_{j=1}^n\phi\left(\theta_x + \vec{\alpha}_j\cdot\vec{t}+\vec{\beta}_j\cdot\frac{\vec{s}}{A\mathfrak{C}k_1}\right)e^{2\pi i s_1}\frac{s_1}{|\vec{s}|^{d+1}}d\vec{s} \\
= & \left[\prod_{j=1}^n \phi\left(\theta_x + \vec{\alpha}_j\cdot\vec{t}\right)\right]\int_{\mathbb{R}}\cdots\int_{\mathbb{R}} e^{2\pi i s_1}\frac{s_1}{|\vec{s}|^{d+1}}d\vec{s}
\end{aligned}
$$

uniformly in $t \in [-\frac{A}{2},\frac{A}{2}]^d$. Indeed, the RHS of the uniform limit is integrated with respect to $\vec{t}$ and summed over all appropriate vectors $\vec{k}$ to give

$$
\left[\int_{\mathbb{R}^d} e^{2\pi i s_1}\frac{s_1 d\vec{s}}{|\vec{s}|^{d+1}}\right]\times\left[\sum_{\vec{z}\in\mathbb{Z}^d: z_1=k_1,|z_2|,..,|z_d|\lesssim k_1}\frac{1}{|Az_1|^d}\right]\simeq\frac{1}{A^d k_1}.
$$

Moreover,

$$
\left[\frac{Ak_1}{A^{d+1}}\right]\left[\sum_{\vec{z}\in\mathbb{Z}^d:z_1=k_1,|z_2|,..,|z_d|\lesssim k_1}\frac{1}{|Az_1|^d}\right]\simeq\left[\frac{Ak_1}{A^{d+1}}\right]\sum_{\vec{\nu}\in\mathbb{Z}^{d-1}:|\vec{\nu}|\gtrsim k_1}\frac{1}{|\vec{\nu}|^{d+1}}\simeq\frac{1}{A^d k_1}.
$$

To prove the uniform limit, it suffices to control

$$
\begin{aligned}
& \left| \int_{-A^2k_1/2}^{A^2k_1/2} ... \int_{-A^2k_1/2}^{A^2k_1/2} \left[\prod_{j=1}^{n} \phi\left(\theta_x + \vec{\alpha}_j \cdot \vec{t} + \vec{\beta}_j \cdot \frac{\vec{s}}{A\mathfrak{C}k_1}\right) - \prod_{j=1}^{n} \phi\left(\theta_x + \vec{\alpha}_j \cdot \vec{t}\right) \right] \right. \\
\times & \left. e^{2\pi i s_1} \frac{s_1}{|\vec{s}|^{d+1}} d\vec{s} \right| \\
\leq & \left| \int_{-A^2k_1/2}^{-1} \int_{-A^2k_1/2}^{A^2k_1/2} ... \int_{-A^2k_1/2}^{A^2k_1/2} \left[\prod_{j=1}^{n} \phi\left(\theta_x + \vec{\alpha}_j \cdot \vec{t} + \vec{\beta}_j \cdot \frac{\vec{s}}{A\mathfrak{C}k_1}\right) - \prod_{j=1}^{n} \phi\left(\theta_x + \vec{\alpha}_j \cdot \vec{t}\right) \right] \right. \\
\times & \left. e^{2\pi i s_1} \frac{s_1}{|\vec{s}|^{d+1}} d\vec{s} \right| \\
+ & \left| \int_{-1}^{1} \int_{-A^2k_1/2}^{A^2k_1/2} ... \int_{-A^2k_1/2}^{A^2k_1/2} \left[\prod_{j=1}^{n} \phi\left(\theta_x + \vec{\alpha}_j \cdot \vec{t} + \vec{\beta}_j \cdot \frac{\vec{s}}{A\mathfrak{C}k_1}\right) - \prod_{j=1}^{n} \phi\left(\theta_x + \vec{\alpha}_j \cdot \vec{t}\right) \right] \right. \\
\times & \left. e^{2\pi i s_1} \frac{s_1}{|\vec{s}|^{d+1}} d\vec{s} \right| \\
+ & \left| \int_{1}^{A^2k_1/2} \int_{-A^2k_1/2}^{A^2k_1/2} ... \int_{-A^2k_1/2}^{A^2k_1/2} \left[\prod_{j=1}^{n} \phi\left(\theta_x + \vec{\alpha}_j \cdot \vec{t} + \vec{\beta}_j \cdot \frac{\vec{s}}{A\mathfrak{C}k_1}\right) - \prod_{j=1}^{n} \phi\left(\theta_x + \vec{\alpha}_j \cdot \vec{t}\right) \right] \right. \\
\times & \left. e^{2\pi i s_1} \frac{s_1}{|\vec{s}|^{d+1}} d\vec{s} \right| \\
:= & \; I + II + III.
\end{aligned}
$$

Bounding Term II

Because $\phi \in \mathcal{S}(\mathbb{R})$,

$$
\left| \prod_{j=1}^{n} \phi\left(\theta_x + \vec{\alpha}_j \cdot \vec{t} + \vec{\beta}_j \cdot \frac{\vec{s}}{A\mathfrak{C}k_1}\right) - \prod_{j=1}^{n} \phi\left(\theta_x + \vec{\alpha}_j \cdot \vec{t}\right) \right| \lesssim \frac{|\vec{s}|}{k_1}
$$

uniformly in $t \in \mathbb{R}$. Moreover, splitting the integrals in term II by

$$
\int_{-A^2k/2}^{A^2k/2} ... \int_{-A^2k/2}^{A^2k/2} = \left(\int_{-A^2k/2}^{-1} + \int_{-1}^{1} + \int_{1}^{A^2k_{\frac{1}{2}}} \right) ... \left(\int_{-A^2k/2}^{-1} + \int_{-1}^{1} + \int_{1}^{A^2k_{\frac{1}{2}}} \right)
$$

and bringing the modulus inside the integrals reduces the estimate to controlling only two types of terms:

$$\frac{1}{k_1}\int_{[-1,1]^d}\frac{d\vec{s}}{|\vec{s}|^{d-1}} \lesssim \frac{1}{k_1}$$
$$\frac{1}{k_1}\int_{[-A^2k_1/2,A^2k_1/2]^{d-1}}\frac{d\vec{s}}{1+|\vec{s}|^{d-1}} \lesssim \frac{\log(k_1)}{k_1}.$$

These estimate are trivial and together show the uniform limit of term II.

Bounding Terms I, III

Integrating terms I and III by parts in s_1 and then bringing the absolute values crudely inside the resulting integrals yields a quantity $O_A(1/|k_1|)$. Therefore, the limit

$$\lim_{k_1\to\infty}\int_{-A^2\mathfrak{C}k_1/2}^{A^2\mathfrak{C}k_1/2}\cdots\int_{-A^2\mathfrak{C}k_1/2}^{A^2\mathfrak{C}k_1/2}\prod_{j=1}^{n}\phi\left(\theta_x+\vec{\alpha}_j\cdot\vec{t}+\vec{\beta}_j\cdot\frac{\vec{s}}{A\mathfrak{C}k_1}\right)e^{2\pi i s_1}\frac{s_1}{|\vec{s}|^{d+1}}d\vec{s}$$
$$= \left[\prod_{j=1}^{n}\phi\left(\theta_x+\vec{\alpha}_j\cdot\vec{t}\right)\right]\int_{\mathbb{R}}\cdots\int_{\mathbb{R}}e^{2\pi i s_1}\frac{s_1}{|\vec{s}|^{d+1}}d\vec{s}$$

is uniform for $t\in[-A/2,A/2]^d$ and the reduction to the lemma follows. To prove the theorem, it therefore suffices to show Lemma 10. To this end, we consider several cases.

1.5.3 Small Perturbations

CASE: $\vec{k} : 1 \lesssim k_1 \lesssim N, \vec{l} = \vec{0}$

By the proceeding calculation, it suffices to bound the integrals for the cases where $\vec{l} = \vec{0}$. Moreover, without loss of generality, $m_j = n_0 - \vec{\alpha}_j \cdot \vec{k} + \Delta_j \ \forall \ j \in \{1, ..., n\}$ where $\max_{1 \leq j \leq n} |\Delta_j| \leq 2d \max_{1 \leq j \leq n} \max_{1 \leq k \leq d} \left\{ |\alpha_j^k|, |\beta_j^k| \right\} := C_{A,B}$. (The restriction $|\Delta_j| \lesssim_{A,B} 1$ arises because summing over those $\vec{m} \in \mathbb{M}$ which are not contained in this collection yields an acceptable error term.) For each $\vec{k} : 1 \lesssim k_1 \lesssim N$, we may restrict our attention to those $\vec{m}$ in

$$\begin{aligned} \mathbb{M}_{\vec{k}}^S \ &:= \ \Big\{ \vec{m} \in \mathbb{M} \Big| m_j = n_0 - \vec{\alpha}_j \cdot \vec{k} + \Delta_j \ \ \forall \ j \in \{1, ..., n\} \\ &; \quad \sum_{j=1}^n \#_j \Delta_j \vec{\alpha}_j = \vec{0} \quad ; \quad \sup_{1 \leq j \leq n} |\Delta_j| \leq C_{A,B} \Big\}. \end{aligned}$$

Indeed, in this case,

$$\begin{aligned} & \sum_{j=1}^n \#_j m_j (x - \vec{\alpha}_j \cdot \vec{t} - \vec{\beta}_j \cdot \vec{s}) \\ = \ & \sum_{j=1}^n \#_j (n_0 - \vec{\alpha}_j \cdot \vec{k} + \Delta_j)(x - \vec{\alpha}_j \cdot \vec{t} - \vec{\beta}_j \cdot \vec{s}) \\ = \ & \left(\sum_{j=1}^n \#_j \right) n_0 x + s_1 k_1 - \left(\sum_{j=1}^n \#_j \Delta_j \vec{\alpha}_j \right) \cdot \vec{t} - \left(\sum_{j=1}^n \#_j \Delta_j \vec{\beta}_j \right) \cdot \vec{s} \\ = \ & \sum_{j=1}^n \#_j n_0 x + \sum_{j=1}^n \#_j \Delta_j \theta_x + s_1 k_1 - \left(\sum_{j=1}^n \#_j \Delta_j \vec{\beta}_j \right) \cdot \vec{s} + Z. \end{aligned}$$

Hence, setting $C_{\vec{\Delta}}(x) = e^{2\pi i A \left[\sum_{j=1}^n \#_j (n_0 x + \Delta_j \theta_x) \right]}$,

$$
\begin{aligned}
& \int_{-\frac{A}{2}}^{\frac{A}{2}} \cdots \int_{-\frac{A}{2}}^{\frac{A}{2}} \int_{A(k_1-\frac{1}{2})}^{A(k_1+\frac{1}{2})} \cdots \int_{A(k_d-\frac{1}{2})}^{A(k_d+\frac{1}{2})} \left[\prod_{j=1}^{n} \phi(x - Am_j - \vec{\alpha}_j \cdot \vec{t} - \vec{\beta}_j \cdot \vec{s}) \right. \\
& \left. e^{2\pi i A \#_j m_j (x - \vec{\alpha}_j \cdot \vec{t} - \vec{\beta}_j \cdot \vec{s})} \right] \frac{s_1}{|\vec{s}|^{d+1}} \frac{t_1}{|\vec{t}|^{d+1}} d\vec{t} d\vec{s} \\
= \quad & C_{\vec{\Delta}}(x) \int_{-\frac{A}{2}}^{\frac{A}{2}} \cdots \int_{-\frac{A}{2}}^{\frac{A}{2}} \left[\prod_{j=1}^{n} \phi(\theta_x - \vec{\alpha}_j \cdot \vec{t} - \vec{\beta}_j \cdot \vec{s}) \right] e^{2\pi i A \mathfrak{C} s_1 k_1} \\
\times \quad & e^{-A\left(\sum_{j=1}^{d} \#_j \Delta_j \vec{\beta}_j\right) \cdot \vec{s}} \frac{s_1}{|\vec{s}|^{d+1}} \frac{t_1 + Ak_1}{|\vec{t} + A\vec{k}|^{d+1}} d\vec{t} d\vec{s}.
\end{aligned}
$$

As in the proof of Theorem 4, we can control the argument of the above display by the requirement $|\theta_x| \lesssim_{\vec{\alpha},\vec{\beta},A} 1$.

CASE: $\vec{k} : 1 \lesssim k_1 \lesssim N, \vec{l} \neq \vec{0}$

These terms will be subsumed as error by integration by parts (twice) with respect to s_1. Let $\vec{m}(n_0, \vec{k}, \vec{l})$ be given component-wise by $m_j(n_0, \vec{k}, \vec{l}) = n_0 - \vec{\alpha}_j \cdot \vec{k} - \vec{\beta}_j \cdot \vec{l} + \Delta_j$. Furthermore, assume $\max_{1 \leq j \leq n} |\Delta_j| \lesssim 1$. Then we need to estimate

$$
\begin{aligned}
& \left| \left[\int_{[-\frac{A^2 \mathfrak{C} k_1}{2}, \frac{A^2 \mathfrak{C} k_1}{2}]^d} \int_{[-\frac{A}{2}, \frac{A}{2}]^d} e^{2\pi i s_1} \frac{(s_1 + A^2 \mathfrak{C} k_1 l_1) \left[\prod_{j=1}^{n} \phi\left(\theta_x - A\Delta_j + \vec{\alpha}_j \cdot \vec{t} + \vec{\beta}_j \cdot \frac{\vec{s}}{A \mathfrak{C} k_1} \right) \right]}{|\vec{s} + A^2 \mathfrak{C} k_1 \vec{l}|^{d+1}} \cdot \theta(\vec{t}) \cdot \frac{\cdot t_1 + Ak_1}{|\vec{t} + A\vec{k}|^{d+1}} d\vec{t} d\vec{s} \right] \right| \\
& \lesssim I + III + III + IV,
\end{aligned}
$$

where

$$I = \left|\left[\int_{[-\frac{A^2\mathfrak{C}k_1}{2},\frac{A^2\mathfrak{C}k_1}{2}]^{d-1}}\int_{[-\frac{A}{2},\frac{A}{2}]^d}\frac{A^2\mathfrak{C}k_1l_1}{|(A^2\mathfrak{C}k_{\frac{1}{2}},\vec{s}+A^2\mathfrak{C}k_1\vec{l})|^{d+1}}\right.\right.$$

$$\left.\left.\left[\prod_{j=1}^{n}\phi\left(\theta_x - A\Delta_j + \vec{\alpha}_j\cdot\vec{t}+\vec{\beta}_j\cdot\left(\frac{A}{2},\frac{\vec{s}}{A\mathfrak{C}k_1}\right)\right)\right]\cdot\theta(\vec{t})\cdot\frac{t_1+Ak_1}{|\vec{t}+A\vec{k}|^{d+1}}d\vec{t}d\vec{s}\right]\right|$$

$$II = \left|\left[\int_{[-\frac{A^2\mathfrak{C}k_1}{2},\frac{A^2\mathfrak{C}k_1}{2}]^{d}}\int_{[-\frac{A}{2},\frac{A}{2}]^d}\right.\right.$$

$$\left.\left.e^{2\pi is_1}\frac{\left[\prod_{j=1}^{n}\phi\left(\theta_x - A\Delta_j + \vec{\alpha}_j\cdot\vec{t}+\vec{\beta}_j\cdot\frac{\vec{s}}{A\mathfrak{C}k_1}\right)\right]}{|\vec{s}+A^2\mathfrak{C}k_1\vec{l}|^{d+1}}\cdot\theta(\vec{t})\cdot\frac{t_1+Ak_1}{|\vec{t}+A\vec{k}|^{d+1}}d\vec{t}d\vec{s}\right]\right|$$

$$III = \left|\left[\int_{[-\frac{A^2\mathfrak{C}k_1}{2},\frac{A^2\mathfrak{C}k_1}{2}]^{d}}\int_{[-\frac{A}{2},\frac{A}{2}]^d}e^{2\pi is_1}\frac{(s_1+A^2\mathfrak{C}k_1l_1)^2}{|\vec{s}+A^2\mathfrak{C}k_1\vec{l}|^{d+3}}\right.\right.$$

$$\left.\left.\left[\prod_{j=1}^{n}\phi\left(\theta_x - A\Delta_j + \vec{\alpha}_j\cdot\vec{t}+\vec{\beta}_j\cdot\frac{\vec{s}}{A\mathfrak{C}k_1}\right)\right]\cdot\theta(\vec{t})\cdot\frac{t_1+Ak_1}{|\vec{t}+A\vec{k}|^{d+1}}d\vec{t}d\vec{s}\right]\right|$$

$$IV = \left|\left[\int_{[-\frac{A^2\mathfrak{C}k_1}{2},\frac{A^2\mathfrak{C}k_1}{2}]^{d}}\int_{[-\frac{A}{2},\frac{A}{2}]^d}e^{2\pi is_1}(s_1+A^2\mathfrak{C}k_1l_1)\right.\right.$$

$$\times\left.\left.\frac{\frac{d}{ds_1}\left[\prod_{j=1}^{n}\phi\left(\theta_x - A\Delta_j + \vec{\alpha}_j\cdot\vec{t}+\vec{\beta}_j\cdot\frac{\vec{s}}{A\mathfrak{C}k_1}\right)\right]}{|\vec{s}+A^2\mathfrak{C}k_1\vec{l}|^{d+1}}\cdot\theta(\vec{t})\cdot\frac{t_1+Ak_1}{|\vec{t}+A\vec{k}|^{d+1}}d\vec{t}d\vec{s}\right]\right|.$$

Summing over all vectors $\vec{k}\in\mathbb{Z}^d : k_1$ is fixed and $\vec{m}\in\mathbb{M}$ yields a total contribution at most

$$C\cdot\sum_{\vec{l}\in\mathbb{Z}^d:|l_1|\lesssim 1}\;\sum_{\vec{\Delta}\in\mathbb{Z}^d:|\vec{\Delta}|\lesssim 1}\frac{1}{k_1^2}\frac{1}{1+|\vec{\Delta}|^2}\frac{|l_1|}{1+|\vec{l}|^{d+1}}\lesssim\frac{1}{k_1^2}.$$

The quadratic decay in k_1 clearly yields an acceptable error to the main contribution.

1.5.4 Large Perturbations

CASE: $\vec{k} : 1 \leq k_1 \lesssim N, \vec{l} = \vec{0}$

If $\max_{1\leq j\leq n} |\Delta_j| \gtrsim 1$, then an arbitrary decay of $\frac{1}{A^{\tilde{N}}}$ may be attached to $\frac{1}{|k_1|}$, which is acceptable whenever $1 \leq k_1 \lesssim N$ upon taking A sufficiently large. Further details are left to the reader.

CASE: $k_1 << 0, \vec{l} = \vec{0}$

Let $\vec{m} \in \mathbb{M}$, $\vec{t} \in \prod_{j=1}^{d}[A(k_j - \frac{1}{2}), A(k_j + \frac{1}{2})]$, and $\vec{s} \in \prod_{j=1}^{d}[A(l_j - \frac{1}{2}), A(l_j + \frac{1}{2})]$. Then there exists $j \in \{1, ..., n\}$ satisfying

$$|An_0 - Am_j - \vec{\alpha}_j \cdot \vec{t} - \vec{\beta}_j \cdot \vec{s}| \geq \frac{Ak_1}{5n \cdot \sup_{1\leq j\leq n} |\#_j \beta_j^1|}.$$

Indeed, suppose not. Then

$$\begin{aligned} & A\sum_{j=1}^{n} \#_j m_j \beta_j^1 \\ = & \sum_{j=1}^{n} \#_j(-An_0 + Am_j + \vec{\alpha}_j \cdot \vec{t} + \vec{\beta}_j \cdot \vec{s})\beta_j^1 + \sum_{j=1}^{n} \#_j(An_0 - \vec{\alpha}_j \cdot \vec{t} - \vec{\beta}_j \cdot \vec{s})\beta_j^1 \\ \leq & -A\mathfrak{C}k_1 + \frac{Ak_1}{5} + O(A) < 0, \end{aligned}$$

which would violate the assumption $\vec{m} \in \mathbb{M}$. If one iterates the proceeding uniform argument, it is a simple matter to use $|\theta_x| \gtrsim Ak_1$ to extract $O\left(|k_1|^{-\tilde{N}}\right)$decay, which is summable .

CASE: $k_1 >> N, \vec{l} = \vec{0}$

The same argument as in the proceeding section yields $O\left(|k_1|^{-\tilde{N}}\right)$ decay. Summability then ensures this contribution can be subsumed as error. Further details are left to the reader.

CASE: $|l_1| >> 1$

It suffices to observe that for every $\vec{m} \in \mathbb{M}$, $\vec{t} \in \prod_{j=1}^{d}[A(l_j - \frac{1}{2}), A(l_j + \frac{1}{2})]$, and $\vec{s} \in \prod_{j=1}^{d}[A(k_j - \frac{1}{2}), A(k_j + \frac{1}{2})]$, there exists $j \in \{1, ..., n\}$ satisfying

$$|An_0 - Am_j - \vec{\alpha}_j \cdot \vec{t} - \vec{\beta}_j \cdot \vec{s}| \geq \frac{Al_1}{5n \cdot \sup_{1 \leq j \leq n} |\#_j \alpha_j^1|}.$$

Indeed, suppose not. Then

$$\begin{aligned}
& A\left|\sum_{j=1}^{n} \#_j m_j \alpha_j^1\right| \\
= & \left|\sum_{j=1}^{n} \#_j(-An_0 + Am_j + \vec{\alpha}_j \cdot \vec{t} + \vec{\beta}_j \cdot \vec{s})\alpha_j^1 + \sum_{j=1}^{n} \#_j(An_0 - \vec{\alpha}_j \cdot \vec{t} - \vec{\beta}_j \cdot \vec{s})\alpha_j^1\right| \\
\geq & A\mathfrak{C}l_1 - Al_1/5 + O(A) > 0,
\end{aligned}$$

which would again violate the assumption $\vec{m} \in \mathbb{M}$.. This extra $O(|l|^{-\tilde{N}})$ decay enables us to use the same integration by parts argument as before to deduce the following estimate $\forall x \in \mathbb{R}$:

$$\begin{aligned}
& \sum_{\vec{z} \in \mathbb{M}: z_1 = k_1} \sum_{\vec{l} \neq \vec{0}} \sum_{\vec{m} \in \mathbb{M}_S(n_0, \vec{k}, \vec{l})} \left| T^{\vec{z}, \vec{l}} \left(\left\{ f_{j, m_j}^{\vec{N}, A, \#_j} \right\}_{j=1}^{n} \right)(x) \right| \\
\lesssim & \sum_{\vec{l} \in \mathbb{Z}^d : \vec{l} \neq 0} \sum_{\vec{\Delta} \in \mathbb{Z}^d} \frac{1}{k_1^2} \sum_{1 + |\vec{\Delta}|^{\tilde{N}}} \frac{|l_1|}{1 + |\vec{l}|^{d+1}} \frac{1}{1 + |l_1|^{\tilde{N}}} \lesssim \frac{1}{k_1^2}
\end{aligned}$$

1.5.5 PART 2: The Irrational Case

Let $\{\alpha_j^m\}_{1\leq j\leq n;1\leq m\leq d} \in \mathbb{R}^{nd}$ be given by $\alpha_j^m = a_j q_j^m$ and $\beta_j^1 = a_j r_j$ for all components $1 \leq j \leq n, 1 \leq m \leq d$. Furthermore, assume there are $\vec{\#} \in \mathbb{R}^n$ and $\mathfrak{C} > 0$ such that

$$\begin{aligned}\sum_{j=1}^n \#_j \alpha_j^m = \sum_{j=1}^n \#_j \alpha_j^m \alpha_j^l = \sum_{j=1}^n \#_j \beta_j^1 = \sum_{j=1}^n \#_j \left[\beta_j^1\right]^2 &= 0 \quad (1 \leq m, l \leq d) \\ \sum_{j=1}^n \#_j \alpha_j^m \beta_j^1 &= \mathfrak{C}\delta_{1,m} \quad (1 \leq m \leq d)\end{aligned}$$

with the additional property that $\#_j a_j^2 \in \mathbb{Q}$ for all $1 \leq j \leq n$. By dilating, we shall assume $q_j^m, r_j, \#_j a_j^2 \in \mathbb{Z}$. For $A \in \mathbb{Z}^+$ and $j \in \{1, ..., d\}$, construct the functions

$$f_j^{N,A,\vec{\#}}(x) := \sum_{-N\leq m\leq N} \phi(x - Aa_j m) e^{2\pi i A \#_j a_j m x} = \sum_{-N\leq m\leq N} f_m^{N,A,\#}(x).$$

As in the rational case, let

$$\mathbb{M} := \left\{ \vec{m} \in \mathbb{Z}^n \cap [-N, N]^n \,\middle|\, \sum_{j=1}^n \#_j a_j m_j \beta_j^1 > 0; \sum_{j=1}^n \#_j a_j m_j \alpha_j^k = 0 \; \forall \; 1 \leq k \leq d \right\}$$

and note

$$\vec{m} \notin \mathbb{M} \implies T_{\hat{K}_d(A\cdot)\hat{K}_d(B\cdot)[\{\vec{\alpha}^m\},\{\vec{\beta}^m\},\Phi,\Psi]}\left(\left\{ f_{m_j}^{N,A\#_j} \right\}_{j=1}^n\right) \equiv 0$$

for sufficiently large constant A (only depending on the matrices $\{\vec{\alpha}_j\}, \left\{\vec{\beta}_j\right\}$. As usual, set $S(\vec{a}) = \left\{a_1^{-1}, ..., a_n^{-1}\right\}$, choose $\rho = 1/A^2$, and let $Bohr_{c(\vec{a})N}(S(\vec{a}), \rho) = \left\{N_1, ..., N_{|Bohr_{c(\vec{a})N}(S(\vec{a}),\rho)}\right\}$. Then $|Bohr_{c(\vec{a})N}(S(\vec{\alpha}), \rho)| \simeq_A N$. Construct

$$\Omega = \bigcup_{n_0 \in Bohr_{c(\vec{a})N}(S(\vec{a}),\rho)} \left[An_0 - \frac{c_{\{\vec{\alpha}_j\},\{\vec{\beta}_j\}}}{A}, An_0 + \frac{c_{\{\vec{\alpha}_j\},\{\vec{\beta}_j\}}}{A} \right].$$

Then $|\Omega| \gtrsim_A N$ for small enough implicit constant in the definition of $\tilde{N}$. Consequently, it suffices to show the pointwise estimate

$$\left| T_{\hat{K}_d(A\cdot)\hat{K}_d(B\cdot)[\{\vec{\alpha}^m\},\{\vec{\beta}^m\},\Phi,\Psi]} \left(\left\{ f_{m_j}^{N,A\#_j} \right\}_{j=1}^{n} \right)(x) \right| \gtrsim_A \log(N) 1_{\Omega}(x) \ \forall x \in \mathbb{R}.$$

1.5.6 Main Contribution: $\vec{k} : k_1 \gtrsim 1; \vec{l} = \vec{0}$

First note that we are able to approximately solve

$$n_0 - a_j m_j(n_0, \vec{k}, \vec{0}) - \vec{\alpha}_j \cdot \vec{k} \simeq 0 \ \forall \ 1 \leq j \leq n.$$

Indeed, for any $n_0 \in Bohr_{c(\vec{a})N}(S(\vec{a}), \rho)$, define $m_j(n_0, \vec{k}, \vec{0}) = \mathcal{N}_j^{n_0} - \vec{q}_j \cdot \vec{k} \in \mathbb{Z}$, where $\mathcal{N}_j^{n_0}$ is the closest integer to n_0/a_j. By construction,

$$|n_0 - a_j m_j(n_0, \vec{k}, \vec{0}) - \vec{\alpha}_j \cdot \vec{k}| = \delta(n_0, j) \lesssim 1/A^2$$

and for sufficiently small $c(\vec{a})$ we have $\mathcal{N}_j^{n_0} - \vec{q}_j \cdot \vec{k} \in [-N, N]$ for all $n_0 \in Bohr_{c(\vec{a})N}(S(\vec{a}), \rho)$ and $|\vec{k}| \lesssim N$. Moreover, it is simple to observe for $\vec{m} \in \mathbb{M}$

$$
\begin{aligned}
&\sum_{j=1}^{n} \#_j a_j(\mathcal{N}_j^{n_0} - \vec{q}_j \cdot \vec{k})(x - \vec{\alpha}_j \cdot \vec{t} - \vec{\beta}_j \cdot \vec{s}) \\
= &\sum_{j=1}^{n} \#_j a_j \mathcal{N}_j^{n}(x - \vec{\beta}_j \cdot \vec{s}) + \mathfrak{C}k_1 s_1 \\
= &\sum_{j=1}^{n} \#_j (n_0 + \delta(n_0, j))(x - \vec{\beta}_j \cdot \vec{s}) + \mathfrak{C}k_1 s_1 \\
= &\sum_{j=1}^{n} \#_j n_0 x + \left[\mathfrak{C}k_1 s_1 - \sum_{j=1}^{n} \#_j \delta(n_0, j)\vec{\beta}_j \cdot \vec{s} \right].
\end{aligned}
$$

We may run the same $t-$uniform argument as before to deduce an acceptable main contribution. Details are left to the reader.

Small Perturbations

A quick glance at the rational case shows that $\vec{l} = 0$ is the only potential difficulty. However, as in the proof of the irrational case of Theorem 4, we may rewrite

$$
\begin{aligned}
&\sum_{j=1}^{n} \#_j a_j \left(\mathcal{N}_j^{n_0} - \vec{q}_j \cdot \vec{k} + \Delta_j \right) x \\
= &\sum_{j=1}^{n} \#_j a_j \mathcal{N}_j^{n_0} x + \sum_{j=1}^{n} \#_j a_j \Delta_j \left(n_0 + \theta_x \right) \\
= &\sum_{j=1}^{n} \#_j a_j \mathcal{N}_j^{n_0} x + \sum_{j=1}^{n} \#_j a_j \Delta_j \left(a_j \mathcal{N}_j^{n_0} + \delta(n_0, j) + \theta_x \right) \\
= &\sum_{j=1}^{n} \#_j a_j \mathcal{N}_j^{n_0} x + \sum_{j=1}^{n} \#_j a_j \Delta_j \left(a_j \mathcal{N}_j^{n_0} + \delta(n_0, j) + \theta_x \right) + Z,
\end{aligned}
$$

where $Z \in \mathbb{Z}$. Because $|\delta(n_o, j)| \leq 1/A^2$ and $|\theta_x| \lesssim 1/A$, the argument of the above display is under good control. It is worth pointing out that our arguments do not require us to solve or even approximately solve up to some admissible error the relations

$$m_j(n_0, \vec{k}, \vec{l}) = n_0 - \vec{\alpha}_j \cdot \vec{k} - \vec{\beta}_j \cdot \vec{l} \ \ \forall\ 1 \leq j \leq n.$$

Indeed, we have seen that when $\vec{l} \neq 0$, the corresponding contribution can be handled by brutally placing mods inside its various constituent pieces. Specifically, we have the estimate

$$\sum_{\vec{k} \in \mathbb{Z}^d} \sum_{\vec{l} \neq \vec{0}} \sum_{\vec{m} \in \mathbb{M}_S(n_0, \vec{k}, \vec{l})} \left| T^{\vec{k}, \vec{l}}_{\hat{K}_d(A\cdot)\hat{K}_d(B\cdot)[\{\vec{\alpha}^m\}, \{\vec{\beta}^m\}, \Phi, \Psi]} \left(\left\{ f^{N, A\#_j}_{m_j} \right\}_{j=1}^{n} \right)(x) \right| \lesssim 1 \ \forall\ x \in \mathbb{R}.$$

Large Perturbations

This case is handled using the same argument as before, so the details are omitted.

$\square$

CHAPTER 2

MIXED ESTIMATES FOR DEGENERATE SIMPLEX OPERATORS

2.1 Introduction

Several recent papers have investigated multilinear singular integral operators from a time-frequency perspective, for example [11, 15, 18, 20, 21]. These objects arise naturally in asymptotic expansions of AKNS systems in [2], where, given $\vec{\epsilon} \in \mathbb{R}^n$, estimates from $\prod_{i=1}^n L^{p'_i}(\mathbb{R})$ into $L^{\frac{1}{\sum_{i=1}^n \frac{1}{p_i}}}(\mathbb{R})$ are sought for the $n-$linear operator

$$\mathcal{C}^{\vec{\epsilon}} : (f_1, ..., f_n) \mapsto \int_{\xi_1 < ... < \xi_n} \left[\prod_{j=1}^n f_j(\xi_j) e^{2\pi i x \epsilon_j \xi_j} \right] d\vec{\xi}.$$

It is also of interest to study $C^{\vec{\epsilon}}(f_1, ..., f_n)(x) := \mathcal{C}^{\vec{\epsilon}}(\hat{f}_1, ..., \hat{f}_n)(x)$, which C. Muscalu, T. Tao, C. Thiele, C. Fefferman, and others have observed satisfies no L^p estimates provided $\epsilon_i + \epsilon_{i+1} = 0$ for some $i \in \{1, ..., n-1\}$. Even in such degenerate cases, [9] proves $C^{\vec{\epsilon}}$ satisfies a wide range of mixed estimates. More precisely, we have

Definition 7. *Let* $W_{p_i}(\mathbb{R}) := \left\{ f \in L^{p_i}(\mathbb{R}) : \hat{f} \in L^{p'_i}(\mathbb{R}) \right\}$ *with* $||f||_{W_{p_i}(\mathbb{R})} := \left|\left| \hat{f} \right|\right|_{L^{p'_i}(\mathbb{R})}$. *In addition, let* $X_{p_i}(\mathbb{R}) \in \{L^{p_i}(\mathbb{R}), W_{p_i}(\mathbb{R})\}$ *for all* $i \in \{1, ..., n\}$. *An n-(sub)linear operator T satisfies the weak estimate* $\prod_{i=1}^n X_{p_i}(\mathbb{R}) \to L^{\frac{1}{\sum_{i=1}^n \frac{1}{p_i}}}(\mathbb{R})$ *whenever*

$$\left|\left| T(\vec{f}) \right|\right|_{L^{\frac{1}{\sum_{i=1}^n \frac{1}{p_i}}}(\mathbb{R})} \lesssim C_{\vec{p}} \prod_{i=1}^n ||f_i||_{X_{p_i}(\mathbb{R})}$$

for all $(f_1, ..., f_n) \in \mathcal{S}(\mathbb{R})^n$.

To obtain mixed estimates for $C^{-1,1,1}$ and $C^{-1,1,1}$, the arguments in [9] use martingale structure decompositions similar to those in [2, 17] combined with a Littlewood-Paley result for arbitrary intervals due to Rubio de Francia in [12] and the Bi-Carleson estimates of Muscalu, Tao, and Thiele from [15]. In addition, C. Muscalu and C. Benea have independently obtained in [1] the same estimates as a consequence of general vector-valued arguments. Our primary purpose in what follows is to establish a robust time-frequency framework tailored specifically to handle mixed estimates for general Hörmander-Mikhlin multipliers adapted to $\{\xi_1+\xi_2=0\}\subset\mathbb{R}^2$ as well as the degenerate trilinear operator $C^{-1,1,-1}$. Specifically, we shall prove the following:

Theorem 11. *Let $m:\mathbb{R}^2\to\mathbb{C}$ be adapted to $\Gamma=\{(\xi_1,\xi_2):\xi_1+\xi_2=0\}\subset\mathbb{R}^2$ in the Mikhlin-Hörmander sense $|\partial^{\vec{\alpha}}m(\vec{\xi})|\lesssim_{\vec{\alpha}}\frac{1}{dist(\vec{\xi},\Gamma)^{|\vec{\alpha}|}}$ for sufficiently many multi-indices $\vec{\alpha}$. Then $T_m:W_{p_1}(\mathbb{R})\times L^{p_2}(\mathbb{R})\to L^{\frac{p_1p_2}{p_1+p_2}}(\mathbb{R})$ whenever $\frac{1}{p_1}+\frac{1}{p_2}<1$ and $2<p_1\leq\infty$.*

Theorem 12. *Let $a_1,a_2\in\mathcal{M}_\Gamma(\mathbb{R}^2)$. Then the trilinear simplex multiplier defined on $\mathcal{S}(\mathbb{R})^3$ by the formula*

$$B[a_1,a_2]:(f_1,f_2,f_3)\to\int_{\mathbb{R}^3}a_1(\xi_1,\xi_2)a_2(\xi_2,\xi_3)\left[\prod_{j=1}^{3}\hat{f}_j(\xi_j)e^{2\pi ix\xi_j}\right]d\xi_1d\xi_2d\xi_3$$

maps $L^{p_1}(\mathbb{R})\times W_{p_2}(\mathbb{R})\times L^{p_3}(\mathbb{R})$ into $L^{\frac{1}{\frac{1}{p_1}+\frac{1}{p_2}+\frac{1}{p_3}}}(\mathbb{R})$ provided

$$1<p_1,p_2<\infty,\frac{1}{p_1}+\frac{1}{p_2}<1,\frac{1}{p_2}+\frac{1}{p_3}<1,2<p_2<\infty.$$

In particular, $B[a_1,a_2]$ has mixed estimates into $L^r(\mathbb{R})$ for all $\frac{1}{2}<r<\infty$.

2.1.1 Organization

In §2.2, we construct an explicit counterexample for generic multipliers of Hilbert transform type before obtaining generic mixed estimates for these and related objects in §2.3 and §2.4. In §2.5 and §2.6, we reprove and extend results from [9]. Then we move to the trilinear setting, where various mixed estimates for the Bad Biest and its generic versions are shown in §2.5-§2.16. In particular, §2.9 contains key mixed estimate interpolation results, which are then applied to reprove mixed estimates for operators of Hilbert transform type in §2.11-§2.12. In §2.15 and §2.16, we introduce the toy and main models necessary for understanding the generic Bad Biest. The arguments in §2.11-§2.16 are robust, and we hope they provide some insight for the reader into what is happening microlocally with degenerate simplex operators .

2.2 Counterexample for Symbols of Hilbert Transform Type

Theorem 13. *There exists a multiplier* $m : \mathbb{R}^2 \to \mathbb{C}$ *adapted to the singularity* $\Gamma = \{\xi_1 + \xi_2 = 0\}$ *satisfying*

$$\left|\partial^{\vec{\alpha}} m(\vec{\xi})\right| \leq \frac{C_{\vec{\alpha},m}}{|dist(\vec{\xi},\Gamma)|^{|\vec{\alpha}|}}$$

for all multi-indices $\vec{\alpha} \in (\mathbb{N} \cup \{0\})^2$ *and* $\vec{\xi} \in \mathbb{R}^2$ *such that*

$$T_m : (f_1, f_2) \mapsto \int_{\mathbb{R}^2} m(\xi_1,\xi_2)\hat{f}_1(\xi_1)\hat{f}_2(\xi_2)e^{2\pi i x(\xi_1+\xi_2)}d\xi_1 d\xi_2$$

(a priori defined for all $\vec{f} \in \mathcal{S}(\mathbb{R})^2$) satisfies no L^p estimates.

Proof. Pick $\Phi \in C^\infty([-1/2, 1/2])$ real, symmetric such that $\hat{\Phi}(0) > 0$. Then $\hat{\Phi}$ is also real, symmetric. Let $\Gamma >> 1$. Define a collection of frequency squares

$$\mathbb{Q} := \bigcup_{k \geq 8} \bigcup_{m \in \mathbb{Z}} \bigcup_{-2^{k-8} < \lambda < 2^{k-8}} \vec{Q}_{k,m,\lambda},$$

where for each $k \geq 8, m \in \mathbb{Z}, -2^{k-8} < \lambda < 2^{k-8}$

$$\begin{aligned} \vec{Q}_{k,m\lambda} &:= [m + \lambda 2^{-k} - 2^{-k-1}, m + \lambda 2^{-k} + 2^{-k-1}] \\ &\times [-m - \lambda 2^{-k} - 2^{-k-1} + \Gamma 2^{-k}, -m - \lambda 2^{-k} + 2^{-k-1} + \Gamma 2^{-k}]. \end{aligned}$$

Next, assign

$$\begin{aligned} \eta^1_{\vec{Q}_{k,m,\lambda}}(x) &:= 2^{-k}\hat{\Phi}(x2^{-k})e^{2\pi i(m+\lambda 2^{-k})x} \\ \eta^2_{\vec{Q}_{k,m,\lambda}}(x) &= 2^{-k}\hat{\Phi}(x2^{-k})e^{-2\pi i(m+\lambda 2^{-k}-\Gamma 2^{-k})x}e^{2\pi i \Gamma 2^{-k} m}. \end{aligned}$$

Let $m(\xi_1, \xi_2) = \sum_{k \geq 8} \sum_{|\vec{Q}|=2^{-k}} \hat{\eta}^1_{Q_1}(\xi_1)\hat{\eta}^2_{Q_2}(\xi_2) \in \mathcal{M}_{\{\xi_1+\xi_2=0\}}(\mathbb{R}^2)$. Moreover, letting $\epsilon = 1/100$, choose $\psi \in \mathcal{S}(\mathbb{R})$ satisfying

$$1_{[-1/2+\epsilon, 1/2-\epsilon]} \leq \hat{\psi} \leq 1_{[-1/2,1/2]}.$$

In addition, for each $N \in \mathbb{N}$, construct $f_1^N(x) = \sum_{1 \leq n \leq N} \psi(x-n)e^{2\pi i n x}$ and $f_2^N(x) = \sum_{1 \leq n \leq N} \psi(x-n)e^{-2\pi i n x}$. For a given $(c_{Q_1}, c_{Q_2}) = (m+\lambda 2^{-k}, -m-\lambda 2^{-k}+\Gamma 2^{-k})$ for which $[m + \lambda 2^{-k} - 2^{-k-1}, m + \lambda 2^{-k} + 2^{-k-1}] \cap [n_1 - 1/2, n_1 + 1/2] \neq \emptyset$ and $[m + \lambda 2^{-k} - \Gamma 2^{-k} - 2^{-k-1}, m + \lambda 2^{-k} - \Gamma 2^{-k} + 2^{-k-1}] \cap [n_2 - 1/2, n_2 + 1/2] \neq \emptyset$, then $m = n_1 = n_2$ for all $k \gtrsim C_\Gamma$, in which case

$$
\begin{aligned}
&[m+\lambda 2^{-k}-2^{-k-1}, m+\lambda 2^{-k}+2^{-k-1}] &&\subset [m-1/2+\epsilon, m+1/2-\epsilon] \\
&[m+\lambda 2^{-k}\Gamma 2^{-k}-2^{-k-1}, m+\lambda 2^{-k}-\Gamma 2^{-k}+2^{-k-1}] &&\subset [m-1/2+\epsilon, m+1/2-\epsilon].
\end{aligned}
$$

Therefore, for each $k \geq C_\Gamma$, we have

$$
\begin{aligned}
& T^k_m(f_1, f_2)(x) \\
= & \sum_{m\in\mathbb{Z}} \sum_{-2^{k-8}<\lambda<2^{k-8}} \sum_{1\leq n_1,n_2\leq N} \left(\psi(\cdot - n_1)e^{2\pi i n_1 \cdot}\right) * \eta^1_{\vec{Q}_{k,m,\lambda}}(x) \\
\times & \left(\psi(\cdot - n_2)e^{-2\pi i n_2 \cdot}\right) * \eta^2_{\vec{Q}_{k,m,\lambda}}(x) \\
= & \sum_{1\leq m\leq N} \sum_{-2^{k-8}<\lambda<2^{k-8}} \left(\psi(\cdot - m)e^{2\pi i m \cdot}\right) * \eta^1_{\vec{Q}_{k,m,\lambda}}(x) \left(\psi(\cdot - m)e^{-2\pi i m \cdot}\right) * \eta^2_{\vec{Q}_{k,m,\lambda}}(x) \\
= & \sum_{1\leq m\leq N} \sum_{-2^{k-8}<\lambda<2^{k-8}} 2^{-2k} \\
\times & \hat{\Phi}((x-n)2^k)e^{2\pi i(m+\lambda 2^{-k})(x-m)}\hat{\Phi}((x-m)2^k)e^{-2\pi i(m+\lambda 2^{-k}-\Gamma 2^{-k})(x-m)}e^{2\pi i \Gamma 2^{-k} m} \\
= & [2^{k-7}-1]2^{-2k} \sum_{1\leq m\leq N} (\hat{\Phi}((x-m)2^k)))^2 e^{2\pi i \Gamma 2^{-k} x}.
\end{aligned}
$$

By the assumption $\hat{\Phi}$ is real-valued with $\hat{\Phi}(0) > 0$, $|T^k_m(f_1, f_2)(x)| \gtrsim 1_{[1,N]}(x)$ for all $C_\Gamma \leq k \lesssim \log(N)$. Lastly, by picking $\Gamma = 100$, say,

$$
\textit{supp } \mathcal{F}(T^k_m(f^N_1, f^N_2)) \subset [99 \cdot 2^{-k}, 101 \cdot 2^{-k}].
$$

Letting $1 < p_1, p_2 < \infty$ satisfy $\frac{1}{p_1} + \frac{1}{p_2} < 1$, note

$$\begin{aligned}
||T_m(f_1^N, f_2^N)||_{\frac{p_1p_2}{p_1+p_2}} &= \left|\left|\sum_{k\geq 8} T_m^k(f_1^N, f_2^N)\right|\right|_{\frac{p_1p_2}{p_1+p_2}} \\
&\simeq \left|\left|\left(\sum_{k\geq 8} \left|T_m^k(f_1^N, f_2^N)\right|^2\right)^{1/2}\right|\right|_{\frac{p_1p_2}{p_1+p_2}} \\
&\gtrsim \log(N)^{1/2} N^{\frac{1}{p_1}+\frac{1}{p_2}}.
\end{aligned}$$

However, $||f_i||_{p_i} \simeq N^{1/p_i}$ for $i \in \{1, 2\}$, so taking N arbitrarily large finishes the theorem.

□

2.3 Mixed Estimates for Multipliers of Hilbert Transform Type

Theorem 11. *Let $m : \mathbb{R}^2 \to \mathbb{C}$ be adapted to $\Gamma = \{(\xi_1, \xi_2) : \xi_1 + \xi_2 = 0\} \subset \mathbb{R}^2$ in the Mikhlin-Hörmander sense $|\partial^{\vec{\alpha}} m(\vec{\xi})| \lesssim_{\vec{\alpha}} \frac{1}{dist(\vec{\xi},\Gamma)^{|\vec{\alpha}|}}$ for sufficiently many multi-indices $\vec{\alpha}$. Then $T_m : W_{p_1}(\mathbb{R}) \times L^{p_2}(\mathbb{R}) \to L^{\frac{p_1p_2}{p_1+p_2}}(\mathbb{R})$ whenever $\frac{1}{p_1} + \frac{1}{p_2} < 1$ and $2 < p_1 \leq \infty$.*

Proof. We carve the Hilbert transform a la the BHT and insert a martingale structure coming from the Christ-Kiselev decomposition. Let us recall the following standard result: there is a Whitney decomposition of the region $\mathcal{R} := \{\xi_1 + \xi_2 \geq 0\} \subset \mathbb{R}^2$ with boundary $\Gamma = \{\xi_1 + \xi_2 = 0\}$ into disjoint squares such that $\mathcal{R} = \coprod_{i\in\mathbb{Z}} Q_i$ and the Whitney property holds:

$$dist(Q_i, \Gamma) \simeq |Q_i|.$$

Having obtained this decomposition, we mollify the sum of characteristic functions of these disjoint Whitney squares and then expand the product of each these mollified characteristic functions with the multiplier m as a double Fourier series about the original frequency boxes. This enables us to rewrite T_m in the following manner:

$$\begin{aligned} T_m(f_1, f_2) &= \int_{\mathbb{R}^2} m(\xi_1, \xi_2)\hat{f}_1(\xi_1)\hat{f}_2(\xi_2)e^{2\pi i x(\xi_1+\xi_2)}d\xi_1 d\xi_2 \\ &= \int_{\mathbb{R}^2} (1_{\xi_1+\xi_2\geq 0} + 1_{\xi_1+\xi_2<0})m(\xi_1, \xi_2)\hat{f}_1(\xi_1)\hat{f}_2(\xi_2)e^{2\pi i x(\xi_1+\xi_2)}d\xi_1 d\xi_2 \\ &= \sum_{l_1, l_2\in\mathbb{Z}} \int_{\mathbb{R}^2} \eta_{Q_1}^{1,l_1}(\xi_1)\eta_{Q_2}^{2,l_2}(\xi_2)\hat{f}_1(\xi_1)\hat{f}_2(\xi_2)d\xi_1 d\xi_2 \\ &= \sum_{l_1, l_2\in\mathbb{Z}} f_1 * \eta_{Q_1}^{1,l_1} f_2 * \eta_{Q_2}^{2,l_2}. \end{aligned}$$

By using the polynomial decay in both parameters l_1, l_2 it is easy to see that to handle T_m it suffices to prove estimates for the generic symbol with Hilbert transform type symbol. Next, we need to introduce data coming from $\hat{f}_1$. To this end, we introduce another decomposition on top of the one already obtained. Define the distribution function $\gamma_{f_1} : \mathbb{R} \to (0, 1)$ given by

$$\gamma_{f_1}(x) := \frac{\int_{-\infty}^{x} |\hat{f}_1|^{p_1'} d\bar{x}}{||\hat{f}_1||_{p_1'}^{p_1'}}.$$

By a simple limiting argument, we may assuming $\{\hat{f}_1 = 0\} = \emptyset$, in which case we may partition the set of points $\xi_1 > -\xi_2$ based on the smallest dyadic interval that contains both ξ_1 and $-\xi_2$. Using γ_{f_1} we may transfer the dyadic structure of $[0, 1]$ back to $\mathbb{R}$ by taking preimages. So, for $m \geq 0$ and $0 \leq k \leq 2^m - 2$ set

$$E_k^m := \gamma_{f_1}^{-1}\left([2^{-m}k, 2^{-m}(k+1))\right)$$

with an obvious modification at the right end point corresponding to $k = 2^m - 1$. Lastly, we construct

$$E_{k,l}^m := \gamma_{f_1}^{-1}\left([2^{-m}k, 2^{-m}(k+1/2))\right) E_{k,r}^m := \gamma_{f_1}^{-1}\left([2^{-m}(k+1/2), 2^{-m}(k+1))\right)$$

with another obvious modification at the right end point of $E_{k,r}^m$ corresponding to $k = 2^m - 1$. Hence,

$$\begin{aligned} \left\{(\xi_1,\xi_2)\in\mathbb{R}^2 : \xi_1+\xi_2 \geq 0\right\} &= \coprod_{m\geq 0}\coprod_{k=0}^{2^m-1} -E_{k,l}^m \times E_{k,r}^m \bigcup \left\{(\xi,-\xi)\in\mathbb{R}^2 : \xi\in\mathbb{R}\right\} \\ \left\{(\xi_1,\xi_2)\in\mathbb{R}^2 : \xi_1+\xi_2 < 0\right\} &= \coprod_{m\geq 0}\coprod_{k=0}^{2^m-1} E_{k,l}^m \times -E_{k,r}^m. \end{aligned}$$

Armed with these two decompositions, we may rewrite $T_m(f_1, f_2)$ as indicated above and suppress the dependence on l_1, l_2 to obtain

$$\begin{aligned} T_m(f_1,f_2) &= \sum_{m\geq 0}\sum_{j=0}^{2^m-1}\sum_{\vec{Q}\in\mathbb{Q}_j^{m,1}} f_1 * \check{1}_{-E_{j,l}^m} * \eta_{Q_1}^1 f_2 * \check{1}_{E_{j,r}^m} * \eta_{Q_2}^2 \\ &+ \sum_{m\geq 0}\sum_{j=0}^{2^m-1}\sum_{\vec{Q}\in\mathbb{Q}_j^{m,2}} f_1 * \check{1}_{E_{j,l}^m} * \eta_{Q_1}^3 f_2 * \check{1}_{-E_{j,r}^m} * \eta_{Q_2}^4. \end{aligned}$$

WLOG, we may focus our attention exclusively on the first term. Then, we can decompose $\mathbb{Q}_j^{m,1} := \mathbb{Q}_j^m$ into $O(1)$ paraproducts. Upon dualizing, one can estimate the corresponding 3-form as follows:

$$\begin{aligned}
\Lambda_{T_m,1}(f_1,f_2,f_3) &:= \int_{\mathbb{R}} T_m(f_1,f_2)\cdot f_3\, dx \\
&= \int_{\mathbb{R}} \sum_{m\geq 0}\sum_{j=0}^{2^m-1}\sum_{\vec{Q}\in\mathbb{Q}_j^m} f_1 * \check{1}_{-E^m_{j,l}} * \eta_{-Q} f_2 * \check{1}_{E^m_{j,r}} * \eta_Q f_3 * \psi_{|Q|} dx \\
&= \int_{\mathbb{R}} \sum_{m\geq 0}\sum_{j=0}^{2^m-1}\sum_{\vec{Q}\in\mathbb{Q}_j^{m,1}} f_1 * \check{1}_{-E^m_{j,l}} * \eta_{-Q} f_2 * \check{1}_{E^m_{j,r}} * \eta_Q f_3 * \psi_{|Q|} dx \\
&+ \int_{\mathbb{R}} \sum_{m\geq 0}\sum_{j=0}^{2^m-1}\sum_{\vec{Q}\in\mathbb{Q}_j^{m,2}} f_1 * \check{1}_{-E^m_{j,l}} * \eta_{-Q} f_2 * \check{1}_{E^m_{j,r}} * \eta_Q f_3 * \psi_{|Q|} dx \\
&:= I + II.
\end{aligned}$$

The notation $\mathbb{Q}_j^{m,i}$ signifies that the set of cubes in question are lacunary in both in the ith and 3rd positions. Hence, the first term can be satisfactorily estimated by

$$\begin{aligned}
|I| &\leq \int_{\mathbb{R}} \sum_{m\geq 0}\sum_{j=0}^{2^m-1}\left(\sum_{\vec{Q}\in\mathbb{Q}_j^{m,1}} |f_1 * \check{1}_{-E^m_{j,l}} * \eta_{-Q}|^2\right)^{1/2} \sup_{\vec{Q}\in\mathbb{Q}_j^{m,1}} \left|f_2 * \check{1}_{E^m_{j,r}} * \eta_Q\right| \\
&\times \left(\sum_{\vec{Q}\in\mathbb{Q}_j^{m,1}} |f_3 * \psi_{|Q|}|^2\right)^{1/2} dx \\
&\leq \int_{\mathbb{R}} \sum_{m\geq 0}\left(\sum_{j=0}^{2^m-1}\sum_{\vec{Q}\in\mathbb{Q}_j^{m,1}} |f_1 * \check{1}_{-E^m_{j,l}} * \eta_{-Q}|^2\right)^{1/2} \left(\sum_{j=0}^{2^m-1}\sup_{\vec{Q}\in\mathbb{Q}_j^{m,1}} \left|f_2 * \check{1}_{E^m_{j,r}} * \eta_Q\right|^2\right)^{1/2} \\
&\times \left(\sum_{k\in\mathbb{Z}} |f_3 * \psi_k|^2\right)^{1/2} dx.
\end{aligned}$$

At this point, we can bring the sum over m outside the integral and then Hölderize. We may use vector-valued inequalities for CZO operators combined with the definition of the martingale structure to obtain the geometric decay $2^{m(1/2-1/p_1')}||\hat{f}_1\ ||_{p_1'}$. The second factor can be handled using Fefferman-Stein's maximal inequality. The third requires a simple Littlewood-Paley square function estimate. As the sum over all cubes in $\mathbb{Q}_j^{m,2}$ is similar to the sum over cubes in $\mathbb{Q}_j^{m,1}$, we omit the details. $\square$

The next result establishes mixed estimates for a maximal variant of T_m.

2.4 Mixed Estimates for Generic Degenerate Bi-Carleson Operators

Theorem 14. *For $m \in \mathcal{M}_{\{\xi_1+\xi_2=0\}}(\mathbb{R}^2)$, construct the operator $\widetilde{\mathcal{M}H_m}$ where for each $x \in \mathbb{R}$,*

$$\widetilde{\mathcal{M}H_m}(f_1, f_2)(x) := \sup_{N\in\mathbb{R}} \left| \int_{\xi_2<\xi_1<N} m(\xi_1, \xi_2)\hat{f}_1(\xi_1)\hat{f}_2(\xi_2)e^{2\pi i x(\xi_1-\xi_2)}d\xi_1 d\xi_2 \right|.$$

Then $\widetilde{\mathcal{M}H} : W_{p_1}(\mathbb{R}) \times L^{p_2}(\mathbb{R}) \to L^{\frac{p_1p_2}{p_1+p_2}}(\mathbb{R})$ provided $p_1 > 2$ and $\frac{1}{p_1} + \frac{1}{p_2} < 1$.

Remark: Recall $W_{p_1}(\mathbb{R}) := \{f \in L^{p_1}(\mathbb{R}) : \hat{f} \in L^{p_1'}(\mathbb{R})\}$.

Proof. It is important for the proof that ξ_1 is adjacent to the N over which the supremum is taken. This enables us to carve things a little more easily than in the other case. In fact, we shall reduce the study of the case when the function in the Wiener space is "opposite" the N to this first case using estimates for maximal paraproducts. Upon introducing two carvings, we have that

$$\widetilde{mH}(f_1,f_2)) = \sup_{N\in\mathbb{R}}\left|\sum_{m_1,m_2\geq 0}\sum_{j_1=0}^{2^{m_1}-1}\sum_{j_2=0}^{2^{m_2}-1}\sum_{Q\in\mathbb{Q}^{m_1,m_2}_{j_1,j_2}} f_1 * \check{1}_{E^{m}_{j_1,r}} * \check{1}_{E^{m_2}_{j_2,l}} * \eta_{-Q}\right.$$

$$\left.\times \quad f_2 * \check{1}_{-E^{m}_{j_1,l}} * \eta_Q \cdot 1_{\left\{N\in E^{m_2}_{j_2,r}\right\}}\right|.$$

Dualizing with $g : ||g||_{p'} = 1$ yields

$$\int_{\mathbb{R}}\sum_{m_1,m_2\geq 0}\sum_{j_1=0}^{2^{m_1}-1}\sum_{j_2=0}^{2^{m_2}-1}\sum_{Q\in\mathbb{Q}^{m_1}_{j_1}} f_1 * \check{1}_{E^{m}_{j_1,r}} * \check{1}_{E^{m_2}_{j_2,l}} * \eta_{-Q}(x)\cdot f_2 * \check{1}_{-E^{m}_{j_1,l}} * \eta_Q(x)$$

$$\times \quad \left(1_{\left\{N\in E^{m_2}_{j_2,r}\right\}} g\right) * \psi_{|Q|}(x)\ dx.$$

As before, we need to split $\mathbb{Q}^{m_1}_{j_1}$ into two disjoint collections labeled $\mathbb{Q}^{m_1,1}_{j_1}$ and $\mathbb{Q}^{m_1,2}_{j_1}$ where the first is lacunary in positions 1 and 3, and the second is lacunary with respect to positions 2 and 3. WLOG, we handle the sum only over $\mathbb{Q}^{m_1}_{j_1}$. This is done as follows:

$$\int_{\mathbb{R}}\widetilde{mH}(f_1,f_2)(x)g(x)dx$$

$$\lesssim \int_{\mathbb{R}}\left[\sum_{m_1,m_2\geq 0}\sum_{j_1=0}^{2^{m_1}-1}\sum_{j_2=0}^{2^{m_2}-1}\left(\sum_{Q\in\mathbb{Q}^{m_1,1}_{j_1}}|f_1 * \check{1}_{E^{m}_{j_1,r}} * \check{1}_{E^{m_2}_{j_2,l}} * \eta_{-Q}|^2\right)^{1/2}\right.$$

$$\left.\times \sup_{Q\in\mathbb{Q}^{m_1,1}_{j_1}}|f_2 * \check{1}_{-E^{m}_{j_1,l}} * \eta_Q|\left(\sum_{Q\in\mathbb{Q}^{m_1,1}_{j_1}}|(1_{\left\{N\in E^{m_2}_{j_2,r}\right\}}g) * \psi_{|Q|}|^2\right)^{1/2} dx\right].$$

Again, we fix the scale, and apply Cauchy-Schwarz separately in each index j_1, j_2. After using Holder, this forces us to estimate three separate factors:

The first takes the form

$$\left\|\left(\sum_{j_1=0}^{2^{m_1}-1}\sum_{j_2=0}^{2^{m_2}-1}\sum_{Q\in\mathbb{Q}_{j_1}^{m_1,1}}|f_1*\check{1}_{E_{j_1,r}^{m}}*\check{1}_{E_{j_2,l}^{m_2}}*\eta_{-Q}|^2\right)^{1/2}\right\|_{\frac{p_1p_2}{p_1+p_2}}$$

$$\lesssim \left\|\left(\sum_{j_1=0}^{2^{m_1}-1}\sum_{j_2=0}^{2^{m_2}-1}|f_1*\check{1}_{E_{j_1,r}^{m}}*\check{1}_{E_{j_2,l}^{m_2}}|^2\right)^{1/2}\right\|_{\frac{p_1p_2}{p_1+p_2}}$$

$$\lesssim 2^{\max\{m_1,m_2\}(1/2-1/p_1')}||\hat{f}_1\ ||_{p_1'}.$$

The second term has the same bound as before, i.e. $|\ II\ |\lesssim 2^{m_1(\max\{0,1/p_2-1/2\})}||f_2||_{p_2}$. Lastly, we estimate

$$\left\|\left(\sum_{j_2=0}^{2^{m_2}-1}\sum_{Q\in\mathbb{Q}_{j_1}^{m_1,1}}|(1_{\left\{N\in E_{j_2,r}^{m_2}\right\}}g)*\psi_{|Q|}|^2\right)^{1/2}\right\|_{p'}$$

$$\le \left\|\left(\sum_{j_2=0}^{2^{m_2}-1}\sum_{k\in\mathbb{Z}}|(1_{\left\{N\in E_{j_2,r}^{m_2}\right\}}g)*\psi_k|^2\right)^{1/2}\right\|_{p'}$$

$$\simeq \left\|\left(\sum_{k\in\mathbb{Z}}\left(\mathbb{E}_t\left|\left(\sum_{j_2=0}^{2^{m_2}-1}r_{j_2}(t)1_{\left\{N\in E_{j_2,r}^{m_2}\right\}}g\right)*\psi_k\right|\right)^2\right)^{1/2}\right\|_{p'}$$

$$\le \mathbb{E}_t\left\|\left(\sum_{k\in\mathbb{Z}}\left|\left(\sum_{j_2=0}^{2^{m_2}-1}r_{j_2}(t)1_{\left\{N\in E_{j_2,r}^{m_2}\right\}}g\right)*\psi_k\right|^2\right)^{1/2}\right\|_{p'}$$

$$\lesssim \mathbb{E}_t\left\|\sum_{j_2=0}^{2^{m_2}-1}r_{j_2}(t)1_{\left\{N\in E_{j_2,r}^{m_2}\right\}}g\right\|_{p'}$$

$$\le ||g||_{p'}\le 1.$$

Therefore, our sum is reduced to $\sum_{m_1,m_2\ge 0}2^{\max\{m_1,m_2\}(1/2-1/p_1')}2^{m_1(\max\{0,1/p_2-1/2+\epsilon\})}$. We split the sum and compute

$$\sum_{0\le m_1\le m_2} 2^{\max\{m_1,m_2\}(1/2-1/p_1')}2^{m_1(\max\{0,1/p_2-1/2+\epsilon\})}$$
$$= \sum_{0\le m_1\le m_2} 2^{m_2(1/2-1/p_1')}2^{m_1(\max\{0,1/p_2-1/2+\epsilon\})}$$
$$\lesssim \sum_{0\le m_2} m_2 2^{m_2(\max\{1/2,1/p_2+\epsilon\}+1/p_1-1)} \lesssim_\epsilon 1$$

for sufficiently small $\epsilon(p_1,p_2)>0$. For the remaining sum, observe

$$\sum_{m_1>m_2\ge 0} 2^{m_1(1/2-1/p_1')}2^{m_1(\max\{0,1/p_2-1/2+\epsilon\})}$$
$$= \sum_{m_1\ge 0} m_1 2^{m_1(1/2-1/p_1')}2^{m_1(\max\{0,1/p_2-1/2\})} \lesssim 1$$

for sufficiently small $\epsilon(p_1,p_2)>0$. □

Corollary 5. *Let the operator $\mathcal{M}H_1$ be given on functions $(f_1,f_2)\in\mathcal{S}(\mathbb{R})^2$ by the formula*

$$\mathcal{M}H_1(f_1,f_2)(x) := \sup_{N\in\mathbb{R}}\left|\int_{\xi_1<\xi_2<N}\hat{f}_1(\xi_1)\hat{f}_2(\xi_2)e^{2\pi ix(\xi_1-\xi_2)}d\xi_1 d\xi_2\right|.$$

Then $\mathcal{M}H_1 : W_{p_1}(\mathbb{R})\times L^{p_2}(\mathbb{R})\to L^{\frac{p_1p_2}{p_1+p_2}}(\mathbb{R})$ provided $p_1>2$ and $\frac{1}{p_1}+\frac{1}{p_2}<1$.

Proof. By the triangle inequality,

$$\mathcal{M}H_1(f_1,f_2)\le C(f_1)\cdot C(f_2)+\widetilde{\mathcal{M}H_1}(f_1,f_2).$$

It suffices to apply the L^p triangle inequality, Hölder's inequality, the Carleson estimates, and Theorem 14.

□

2.5 Mixed Estimates for $sgn(\xi_1+\xi_2)sgn(\xi_2+\xi_3)$

Theorem 15. *$T^{sgn,sgn}$ be defined initially on functions $(f_1, f_2, f_3) \in \mathcal{S}(\mathbb{R})^3$ by*

$$T^{sgn,sgn}(f_1,f_2,f_3)(x) := \int_{\mathbb{R}^3} sgn(\xi_1+\xi_2)sgn(\xi_2+\xi_3)\hat{f}_1(\xi_1)\hat{f}_2(\xi_2)\hat{f}_3(\xi_3)e^{2\pi i x(\xi_1+\xi_2+\xi_3)}d\vec{\xi}.$$

Then $T^{sgn,sgn}$ is a bounded operator from $L^{p_1}(\mathbb{R}) \times W_{p_2}(\mathbb{R}) \times L^{p_3}(\mathbb{R})$ into $L^{\frac{1}{\frac{1}{p_1}+\frac{1}{p_2}+\frac{1}{p_3}}}(\mathbb{R})$ provided the following exponent conditions hold: $\frac{1}{p_1}+\frac{1}{p_2}<1, \frac{1}{p_2}+\frac{1}{p_3}<1, \frac{1}{p_1}+\frac{1}{p_2}+\frac{1}{p_3}<\frac{3}{2}, 1<p_1<\infty, 2<p_2\leq\infty,$ and $1<p_3<\infty$.

Remark: In fact, using a more complicated argument, one can show that $\frac{1}{p_1}+\frac{1}{p_2}+\frac{1}{p_3}<\frac{3}{2}$ is not necessary for the above mixed estimates to hold. Theorem 15 is true even if one removes the requirement $\frac{1}{p_1}+\frac{1}{p_2}+\frac{1}{p_3}<1$ altogether.

Proof. Introduce the function $\mu_{f_2}:\mathbb{R}\to(0,1)$ given by

$$\mu_{f_2}(x) := \frac{\int_{-\infty}^{x}|\hat{f}_2(\bar{x})|^{p_2'}d\bar{x}}{||\hat{f}_2||_{p_2'}^{p_2'}}$$

and construct the following family of sets: for each $m\geq 0$ and $0\leq k\leq 2^m-1$,

$$\begin{aligned} E_k^m &= \mu_{f_2}^{-1}([2^{-m}k, 2^{-m}(k+1))) \\ E_{k,left}^m &= \mu_{f_2}^{-1}([2^{-m}k, 2^{-m}(k+1/2))) \\ E_{k,right}^m &= \mu_{f_2}^{-1}([2^{-m}(k+1/2), 2^{-m}(k+1))). \end{aligned}$$

Just as before,

$$\mathbb{R}^2 \supset \{\xi_1 + \xi_2 \geq 0\} = \coprod_{m \geq 0} \coprod_{k=0}^{2^m-1} -E^m_{k,left} \times E^m_{k,right} \bigcup \{\xi, -\xi) \in \mathbb{R}^2 : \xi \in \mathbb{R}\} .$$

Hence, modulo harmless difference terms, which satisfy the desired estimates,

$$\begin{aligned} T^{sgn,sgn}(f_1, f_2, f_3)(x) &\simeq \sum_{m,m' \geq 0} \sum_{k=0}^{2^m-1} \sum_{l=0}^{2^{m'}-1} f_1 * \check{1}_{-E^m_{k,left}}(x) \\ &\times \; f_2 * \check{1}_{E^m_{k,right}} * \check{1}_{-E^{m'}_{l,left}}(x) \cdot f_3 * \check{1}_{E^{m'}_{l,right}}(x). \end{aligned}$$

Next, perform Cauchy-Schwarz in both k, l for fixed m, m'. If $\frac{1}{p_1} + \frac{1}{p_2} + \frac{1}{p_3} \leq 1$, then we may use the triangle inequality to pull out the sum over m, m'. If $\frac{1}{p_1} + \frac{1}{p_2} + \frac{1}{p_3} > 1$, then set $p = \frac{1}{\frac{1}{p_1} + \frac{1}{p_2} + \frac{1}{p_3}}$ and use

$$\begin{aligned} & ||T^{sgn,sgn}(f_1, f_2, f_3)||^p_{L^p(\mathbb{R})} \\ \leq & \sum_{m,m' \geq 0} \left|\left| \left(\sum_{k=0}^{2^m-1} \left| f_1 * \check{1}_{-E^m_{k,left}} \right|^2 \right)^{1/2} \left(\sum_{k=0}^{2^m-1} \sum_{l=0}^{2^{m'}-1} \left| f_2 * \check{1}_{E^m_{k,right}} * \check{1}_{-E^{m'}_{l,left}} \right|^2 \right)^{1/2} \right.\right. \\ & \left.\left. \times \left(\sum_{l=0}^{2^{m'}-1} \left| f_3 * \check{1}_{E^{m'}_{l,right}} \right|^2 \right)^{1/2} \right|\right|^p_{L^p(\mathbb{R})} . \end{aligned}$$

In the quasi-Banach case, one may apply the generalized Hölder's inequality, generalized Rubio de Francia estimates, and the martingale structure mass decomposition to bound the last expression from above by

$$\begin{aligned} & \left[\sum_{m,m' \geq 0} \left| 2^{m \max\{0, 1/p_1 - 1/2 + \epsilon\}} 2^{\max\{m,m'\}(1/2 - 1/p_1')} 2^{m' \max\{0, 1/p_3 - 1/2 + \epsilon\}} \right|^p \right] \\ \times \; & ||f_1||^p_{p_1} ||\hat{f}_2||^p_{p_2'} ||f_3||^p_{p_3}. \end{aligned}$$

As before, we split the sum into two parts corresponding to $\sum_{m,m'\geq 0} = \sum_{0\leq m\leq m'} + \sum_{0\leq m'<m}$. This time, it is easy to see that both sums are summable for small enough $\epsilon(p_1,p_2,p_3)>0$ by the assumption $\frac{1}{p_1}+\frac{1}{p_2}+\frac{1}{p_3}<1$.

□

2.6 Mixed Estimates for Multipliers of $sgn(\xi_1+\xi_2)a(\xi_2,\xi_3)$ Type

Theorem 16. *Let $a \in \mathcal{M}_{\{\xi_1+\xi_2\}}(\mathbb{R}^2)$ and construct the trilinear operator $T^{sgn,a}$ defined initially on functions $(f_1,f_2,f_3)\in\mathcal{S}(\mathbb{R})^3$ by*

$$T^{sgn,a}(f_1,f_2,f_3)(x) := \int_{\mathbb{R}^3} sgn(\xi_1+\xi_2)a(\xi_2,\xi_3)\hat{f}_1(\xi_1)\hat{f}_2(\xi_2)\hat{f}_3(\xi_3)e^{2\pi i x(\xi_1+\xi_2+\xi_3)}d\xi_1 d\xi_2 d\xi_3.$$

Then $T^{sgn,a}$ is a bounded operator from $L^{p_1}(\mathbb{R})\times W_{p_2}(\mathbb{R})\times L^{p_3}(\mathbb{R})$ into $L^{\frac{1}{\frac{1}{p_1}+\frac{1}{p_2}+\frac{1}{p_3}}}(\mathbb{R})$ provided $1<p_1<\infty, 2<p_2\leq\infty, 1<p_3<\infty$, and $\frac{1}{p_1}+\frac{1}{p_2}+\frac{1}{p_3}<1$.

Proof. We begin by noting

$$\begin{aligned} m(\xi_1,\xi_2,\xi_3) &= sgn(\xi_1+\xi_2)a(\xi_2,\xi_3) = sgn(\xi_2+\xi_3)a(\xi_2,\xi_3)(1_{\xi_2+\xi_3\leq 0}+1_{\xi_2+\xi_3>0}) \\ &:= a_I + a_{II}. \end{aligned}$$

It will suffice to bound a_I. First set up two different carvings for $1_{\xi_2<-\xi_3}$. Introduce

$$a(\xi_2,\xi_3)1_{\{\xi_2+\xi_3\leq 0\}}(\xi_2,\xi_3) = \sum_{(\sigma,\sigma')\in\{0,\frac{1}{3},\frac{2}{3}\}^2}\sum_{\kappa,\kappa'}\sum_{\vec{P}\in\mathbb{P}} c(a)_\kappa \tilde{c}(a)_{\kappa'}\hat{\eta}^{\sigma,\kappa}_{P_1,1}(\xi_2)\hat{\eta}^{\sigma',\kappa'}_{P_2,2}(\xi_3) \quad (a.e).$$

It follows that

$$
\begin{aligned}
& \Lambda_{T^{sgn,a}}(f_0, f_1, f_2, f_3) \\
:= & \int_{\mathbb{R}} T^{sgn,a}(f_0, f_1, f_2)(x) f_3(x) dx \\
= & \sum_{(\sigma,\sigma')\in\{0,\frac{1}{3},\frac{2}{3}\}^2} \sum_{m,m'} \sum_{k=0}^{2^m-1} \sum_{k'=0}^{2^{m'}-1} \sum_{\kappa,\kappa'\in\mathbb{Z}} \sum_{\vec{P}\in\mathbb{P}} c_\kappa \tilde{c}_{\kappa'} \\
\times & \int_{\mathbb{R}} f_0 f_1 * \check{1}_{-E^m_{k,left}} f_2 * \check{1}_{E^m_{k,right}} * \eta^{\sigma,\kappa}_{P_1,1} * \check{1}_{E^{m'}_{l,left}} f_3 * \eta^{\sigma',\kappa'}_{P_2,2} * \check{1}_{-E^{m'}_{l,right}} dx.
\end{aligned}
$$

Because of rapid coefficient decay, it suffices to prove satisfactory estimates for

$$
\left| \sum_{m,m'} \sum_{k,l} \sum_{P\in\mathbb{P}^{m'}_l} \int_{\mathbb{R}} f_0 f_1 * \check{1}_{-E^m_{k,left}} f_2 * \check{1}_{E^m_{k,right}} * \eta^{\sigma,\kappa,\lambda}_{P_1,1} * \check{1}_{E^{m'}_{l,left}} f_3 * \eta^{\sigma',\kappa',\lambda'}_{P_2,2} * \check{1}_{-E^{m'}_{l,right}} dx \right|
$$

with a bound that is independent of $\sigma, \sigma', \kappa, \kappa'$. To this end, set $\mathbb{P}^{m'}_l = \left\{ \vec{P} \in \mathbb{P} : (P_1, P_2) \cap (-E^{m'}_{l,left}, E^{m'}_{l,right}) \neq \emptyset \right\}$. We organize our collection of $\mathbb{P}^{m'}_l$ into $O(1)$ disjoint paraproducts, which are either lacunary in the first index, in which case we say the paraproduct is type A, or lacunary in the 2nd index, in which case we say the paraproduct is type B. For notational simplicity, we write $\eta^{\sigma,\kappa}_{P_1,1}$ simply as $\eta_{P_1,1}$.

2.6.1 Estimates for $\mathbb{P}^{m'}_l[A]$

Let $\mathbb{P}^{m'}_l[A] \subset \mathbb{P}^{m'}_l$ be a type A praraproduct. Then we may majorize $\left| \Lambda^{m,m',\mathbb{P}[A]}_{T^{sgn,a}}(f_0, f_1, f_2, f_3) \right|$ by a rapidly decaying sum over expressions of the form

$$
\begin{aligned}
&\sum_{m,m'\geq 0}\left|\int_{\mathbb{R}}\sum_{k,l}\sum_{\vec{P}\in\mathbb{P}_l^{m'}[A]}(f_0 f_1 * \check{1}_{-E^m_{k,left}}) * \psi^{lac,\vec{P}}_{|P|}\right.\\
\times\quad & \left. f_2 * \check{1}_{E^m_{k,right}} * \check{1}_{-E^{m'}_{l,left}} * \eta_{P_1,1}\cdot f_3 * \check{1}_{E^{m'}_{l,right}}\ dx\right|\\
\leq\quad & \sum_{m,m'\geq 0}\int_{\mathbb{R}}\left(\sum_{k}\sum_{\vec{P}\in\mathbb{P}_l^{m'}[A]}\left|(f_0 f_1 * \check{1}_{-E^m_{k,left}}) * \psi^{lac,\vec{P}}_{|P|}\right|^2\right)^{1/2}\\
\times\quad & \left(\sum_{k,l}\sum_{\vec{P}\in\mathbb{P}_l^{m'}[A]}\left|f_2 * \check{1}_{E^m_{k,right}} * \check{1}_{-E^{m'}_{l,left}} * \eta_{P_1}\right|^2\right)^{1/2}\left(\sum_{l}\left|f_3 * \check{1}_{E^{m'}_{l,right}}\right|^2\right)^{1/2}\ dx.
\end{aligned}
$$

We may now invoke Hölder's inequality, use the vector-valued CZO estimate and generalized Rubio de Francia estimate for the first factor, use the martingale structure definitions to extract exponential decay over m, m' from the second factor, and apply the generalized Rubio de Francia estimate for the third factor.

2.6.2 Estimates for $\mathbb{P}_l^{m'}[B]$

Let $\mathbb{P}_l^{m'}[B] \subset \mathbb{P}_l^{m'}$ be a collection of type B frequencies. Then we may majorize $\left|\Lambda^{m,m',\mathbb{P}[B]}_{T^{sgn,a}}(f_0, f_1, f_2, f_3)\right|$ by a rapidly decaying sum over expressions of the form

$$
\begin{aligned}
&\left|\int_{\mathbb{R}}\sum_{k,l}\sum_{\vec{P}\in\mathbb{P}_l^{m'}[B]}(f_0 f_1 * \check{1}_{-E^m_{k,left}}) * \psi^{lac,\vec{P}}_{|P|}\right.\\
\times\quad & \left. f_2 * \check{1}_{E^m_{k,right}} * \check{1}_{-E^{m'}_{l,left}} f_3 * \check{1}_{E^{m'}_{l,right}} * \eta_{P_2,2}\ dx\right|\\
\leq\quad & \int_{\mathbb{R}}\left(\sum_{k}\sum_{\vec{P}\in\mathbb{P}_l^{m'}[B]}\left|(f_0 f_1 * \check{1}_{-E^m_{k,left}}) * \psi^{lac,\vec{P}}_{|P|}\right|^2\right)^{1/2}\\
\times\quad & \left(\sum_{k,l}\left|f_2 * \check{1}_{E^m_{k,right}} * \check{1}_{-E^{m'}_{l,left}}\right|^2\right)^{1/2}\left(\sum_{l}\sum_{\vec{P}\in\mathbb{P}_l^{m'}[B]}\left|f_3 * \check{1}_{E^{m'}_{l,right}} * \eta_{P_2,2}\right|^2\right)^{1/2}\ dx.
\end{aligned}
$$

We may now Hölderize, use the vector-valued CZO estimate and generalized Rubio de Francia estimate for the third factor, use the martingale structure to extract exponential decay over m, m' from the second factor, and apply the generalized Rubio de Francia estimate for the first factor.

$\square$

2.7 WLW-Type Estimates for $B[a_1, a_2]$

Theorem 17. *Let* $a_1, a_2 \in \mathcal{M}_{\{\xi_1+\xi_2\}}(\mathbb{R}^2)$ *and construct the trilinear operator* $B[a_1, a_2]$ *defined on* $(f_1, f_2, f_3) \in \mathcal{S}(\mathbb{R})^3$ *by the formula*

$$\int_{\mathbb{R}^3} a_1(\xi_1, \xi_2) a_2(\xi_2, \xi_3) \hat{f}_1(\xi_1) \hat{f}_2(\xi_3) \hat{f}_3(\xi_3) e^{2\pi i x(\xi_1+\xi_2+\xi_3)} d\xi_1 d\xi_2 d\xi_3.$$

Then $B[a_1, a_2]$ *extends to a bounded operator from* $W_{p_1}(\mathbb{R}) \times L^{p_2}(\mathbb{R}) \times W_{p_3}(\mathbb{R}) \rightarrow L^{\frac{1}{\frac{1}{p_1}+\frac{1}{p_2}+\frac{1}{p_3}}}(\mathbb{R})$ *for all* $\frac{1}{p_1} + \frac{1}{p_2} + \frac{1}{p_3} < 1$ *and* $p_1, p_3 > 2$. *Specifically,* T^{a_1, a_2} *can be defined on all* (f_1, f_2, f_3) *such that* $\hat{f}_1 \in L^{p_1'}(\mathbb{R}), f_2 \in L^{p_2}(\mathbb{R})$ *and* $\hat{f}_3 \in L^{p_3'}(\mathbb{R})$ *in such a way that*

$$||B[a_1, a_2](f_1, f_2, f_3)||_{L^{\frac{1}{\sum \frac{1}{p_1}+\frac{1}{p_2}+\frac{1}{p_3}}}(\mathbb{R})} \lesssim_{p_1, p_2, p_3} ||\hat{f}_1||_{L^{p_1'}(\mathbb{R})} ||f_2||_{L^{p_2}(\mathbb{R})} ||\hat{f}_3||_{L^{p_3'}(\mathbb{R})}.$$

Proof. As usual, we shall use a Christ-Kiselev-Paley decomposition, Whitney decomposition for $\Gamma = \{\xi_1 + \xi_2 = 0\}$, and the generalized Rubio de Francia inequality. The interaction of the mass decomposition of $\hat{f}_3$ with the geometric decomposition of Γ can be handled via vector-valued inequalities for C-Z operators.

To begin, carve

$$
\begin{aligned}
& a_1(\xi_1,\xi_2)a_2(\xi_2,\xi_3) \\
= \quad & a_1(\xi_1,\xi_2)a_2(\xi_2,\xi_3)\left[1_{0<\xi_1+\xi_2}1_{\xi_2+\xi_3\leq 0} + 1_{0<\xi_2+\xi_3}1_{\xi_1+\xi_2\leq 0}\right. \\
+ \quad & \left. 1_{0<\xi_2+\xi_3}1_{0<\xi_1+\xi_2} + 1_{\xi_2+\xi_3\leq 0}1_{0<\xi_1+\xi_2})\right] \\
:= \quad & a_I + a_{II} + a_{III} + a_{IV}.
\end{aligned}
$$

It will suffice to bound a_I. Now, set up two carvings for $1_{-\xi_1<\xi_2}$. First, introduce the function $\mu_{f_1} : \mathbb{R} \to (0,1)$ given by

$$
\mu_{f_1}(x) := \frac{\int_{-\infty}^{x} |\hat{f}_1(\bar{x})|^{p_1'} d\bar{x}}{||\hat{f}_1||_{p_1'}^{p_1'}}
$$

and construct the following family of sets: for each $m \in \mathbb{N}^+ \bigcup \{0\}$ and $0 \leq k \leq 2^m - 1$,

$$
\begin{aligned}
F_k^m \quad & := \quad \mu_{f_1}^{-1}\left([2^{-m}k, 2^{-m}(k+1))\right) \\
F_{k,left}^m \quad & := \quad \mu_{f_1}^{-1}\left([2^{-m}k, 2^{-m}(k+1/2))\right) \\
F_{k,right}^m \quad & := \quad \mu_{f_1}^{-1}\left([2^{-m}(k+1/2), 2^{-m}(k+1))\right).
\end{aligned}
$$

Next, introduce the function $\gamma_{f_3} : \mathbb{R} \to (0,1)$ given by

$$
\gamma_{f_3}(x) := \frac{\int_{-\infty}^{x} |\hat{f}_3(\bar{x})|^{p_3'} d\bar{x}}{||\hat{f}_3||_{p_3'}^{p_3'}}
$$

and construct the following family of sets: for each $m \in \mathbb{N}^+ \bigcup \{0\}$ and $0 \leq k \leq 2^m - 1$,

$$
\begin{aligned}
E_k^m &:= \gamma_{f_3}^{-1}\left([2^{-m}k, 2^{-m}(k+1))\right)\\
E_{k,left}^m &:= \gamma_{f_3}^{-1}\left([2^{-m}k, 2^{-m}(k+1/2))\right)\\
E_{k,right}^m &:= \gamma_{f_3}^{-1}\left([2^{-m}(k+1/2), 2^{-m}(k+1))\right).
\end{aligned}
$$

Hence, for each $m \geq 0$, $\mathbb{R} = \coprod_{k=0}^{2^m-1} E_k^m$. Moreover, $\{\vec{\xi} \in \mathbb{R}^2 : -\xi_1 < \xi_2\} = \coprod_{m=0}^{\infty}\coprod_{k=0}^{2^m-1} -E_{k,left}^m \times E_{k,right}^m$. Furthermore, we choose of Whitney decomposition for $\{-\xi_1 < \xi_2\} \subset \mathbb{R}^2$, namely $\{P_j\}_{j\in\mathbb{Z}}$ satisfying for every $j \in \mathbb{Z}$

$$
|side(P_j)| \simeq dist(P_j, \Gamma_2).
$$

Performing the same tricks as in the standard discretization of the BHT (adapted instead to the degenerate line Γ_2) produces

$$
a_1(\xi_1,\xi_2)1_{\{-\xi_1<\xi_2\}}(\xi_1,\xi_2) = \sum_{(\gamma,\gamma')\in\{0,\frac{1}{3},\frac{2}{3}\}^2}\sum_{k,k'\in\mathbb{Z}}\sum_{\vec{Q}\in\mathbb{Q}^{\gamma,\gamma'}} c_k\tilde{c}_{k'}\hat{\eta}_{Q_1,1}^{\gamma,k}(\xi_1)\hat{\eta}_{Q_2,2}^{\gamma',k'}(\xi_2),
$$

where the sequences $\{c_m\}_{m\in\mathbb{Z}}, \{\tilde{c}_m\}_{\tilde{m}\in\mathbb{Z}}$ are both rapidly decaying. Similarly, we have

$$
a_2(\xi_2,\xi_3)1_{\{\xi_2<-\xi_3\}}(\xi_2,\xi_3) = \sum_{(\sigma,\sigma')\in\{0,\frac{1}{3},\frac{2}{3}\}^2}\sum_{l,l'\in\mathbb{Z}}\sum_{\vec{P}\in\mathbb{P}^{\sigma,\sigma'}} d_l\tilde{d}_{l'}\hat{\eta}_{P_1,1}^{\sigma,l}(\xi_2)\hat{\eta}_{P_2,2}^{\sigma',l'}(\xi_3).
$$

Suppressing dependence on $\sigma,\sigma',\gamma,\gamma',k,k',l,l'$, it suffices to show estimates for the form $\Lambda_{T^{a_1,a_2}}$ given by

$$
\begin{aligned}
& \Lambda_{B[a_1,a_2]}(f_0,f_1,f_2,f_3) \\
= & \sum_{\vec{Q}\in\mathbb{Q}}\sum_{\vec{P}\in\mathbb{P}}\int_{\mathbb{R}} f_0\cdot f_1*\eta_{Q_1,1}\cdot f_2*\eta_{Q_2,2}*\eta_{P_1,1}\cdot f_3*\eta_{P_2,2}\,dx \\
= & \sum_{m\geq 0}\sum_{k=0}^{2^m-1}\sum_{(\vec{Q},\vec{P}):|\vec{Q}|>|\vec{P}|}\int_{\mathbb{R}} f_0\cdot f_1*\eta_{Q_1,1}\cdot f_2*\eta_{Q_2,2}*\eta_{P_1,1}*\check{1}_{E^m_{k,left}} \\
\times & f_3*\eta_{P_2,2}*\check{1}_{-E^m_{k,right}}\,dx \\
+ & \sum_{m\geq 0}\sum_{k=0}^{2^m-1}\sum_{(\vec{Q},\vec{P}):|\vec{Q}|\leq|\vec{P}|}\int_{\mathbb{R}} f_0\cdot f_1*\eta_{Q_1,1}*\check{1}_{-F^m_{k,left}}\cdot f_2*\eta_{Q_2,2}*\eta_{P_1,1}*\check{1}_{F^m_{k,right}} \\
\times & f_3*\eta_{P_2,2}\,dx \\
= & \Lambda_{B[a_1,a_2]_1}(f_0,f_1,f_2,f_3)+\Lambda_{B[a_1,a_2]_2}(f_0,f_1,f_2,f_3).
\end{aligned}
$$

By symmetry and the L^p estimates from [18], it suffices to prove there exists $C>0$ such that for all $m\geq 0$

$$
\begin{aligned}
& \Lambda^m_{B[a_1,a_2]_1}(f_0,f_1,f_2,f_3) \\
:= & \Bigg|\sum_{k=0}^{2^m-1}\sum_{(\vec{Q},\vec{P}):\vec{P}\in\mathbb{P}^m_k,|\vec{Q}|>>|\vec{P}|}\int_{\mathbb{R}} f_0\cdot f_1*\eta_{Q_1,1}\cdot f_2*\eta_{Q_2,2}*\eta_{P_1,1}*\check{1}_{E^m_{k,left}} \\
\times & f_3*\eta_{P_2,2}*\check{1}_{-E^m_{k,right}}\,dx\Bigg| \\
\lesssim_{p_1,p_2,p_3} & 2^{-Cm}||f_1||_{p_1}||f_2||_{p_2}||\hat{f}_3||_{p_3'}.
\end{aligned}
$$

To this end, rewrite $\Lambda^m_{B[a_1,a_2]_1}(f_0,f_1,f_2,f_3)$ as

$$
\begin{aligned}
& \sum_{k=0}^{2^m-1}\sum_{(\vec{Q},\vec{P}):\vec{P}\in\mathbb{P}^m_k,|\vec{Q}|>|\vec{P}|}\int_{\mathbb{R}}(f_0*\eta_{-\tilde{Q}_1,0}\cdot f_1*\eta_{Q_1,1})*\psi_{|P|}\cdot f_2*\eta_{Q_2,2}*\eta_{P_1,1}*\check{1}_{E^m_{k,left}} \\
\times\; & f_3*\eta_{Q_3,3}*\eta_{P_2,2}*\check{1}_{-E^m_{k,right}}\,dx.
\end{aligned}
$$

Recall that a paraproduct $\Pi = \{\vec{P}\}$ is type A provided $\{P_1\}_{\vec{P}\in\Pi}$ is lacunary, $\Pi = \left\{\vec{P}\right\}$ is Type B provided $\{P_2\}_{\vec{P}\in\Pi}$ is lacunary. For each $m \geq 0$ and $0 \leq k < 2^m - 1$, let $\mathbb{P}_k^m = \left\{(P_1, P_2) \in \mathbb{P} : \vec{P} \cap \left(E^m_{k,keft} \times -E^m_{k,right}\right) \neq \emptyset\right\}$. We may clearly split each $\mathbb{P}_k^m$ into $O(1)$ disjoint collections of paraproducts of types A and B. Hence, there is a splitting of $\tilde{\Lambda}^m_{T_1^{a_1,a_2}}(f_0, f_1, f_2, f_3)$ into a sum of two terms in the obvious way so that $|\tilde{\Lambda}^m_{T_1^{a_1,a_2}}(f_0, f_1, f_2, f_3)| \leq |\tilde{\Lambda}^{m,A}_{T_1^{a_1,a_2}}(f_0, f_1, f_2, f_3)| + |\tilde{\Lambda}^{m,B}_{T_1^{a_1,a_2}}(f_0, f_1, f_2, f_3)|$. Furthermore, for each $m \geq 0$ and $0 \leq k \leq 2^m - 1$, construct

$$\mathbb{Q}_k^m := \left\{(Q_1, Q_2) \in \mathbb{Q} : (Q_2 \times -Q_2) \cap \left(E^m_{k,left} \times -E^m_{k,right}\right) \neq \emptyset\right\}.$$

It is easy to see that for fixed m, k, there are $O(1)$ intervals Q_2 of a given size and hence $O(1)$ many cubes $\vec{Q}$ of a given size in $\mathbb{Q}_k^m$. The boxes $Q_2 \times -Q_2$ in $\mathbb{Q}_k^m$ are also essentially nested, so that $\{Q_1\}_{\vec{Q}\in\mathbb{Q}_k^m}$ is a Littlewood-Paley collection. We shall also need the following technical results:

Lemma 11. *Let $F, G, H \in \mathcal{S}(\mathbb{R})$. Moreover, let $\eta^{\vec{Q}}_{P_1,1}, \eta_{P_2,2}$ be functions which are fourier-localized onto the intervals P_1 and P_2 and such that the standard uniform decay properties hold. Moreover, suppose $P_1 + P_2 \subset [c|P|, C|P|]$. Then*

$$\int_{\mathbb{R}} F \cdot G * \eta^{\vec{Q}}_{P_1,1} \cdot H * \eta_{P_2,2}\, dx = \sum_{k\in\mathbb{Z}} c_k^{\vec{Q},\vec{P}} \int_{\mathbb{R}} F * \eta^{\vec{Q},\vec{P},k}_{|P|,1} \cdot G \cdot H * \eta^k_{P_2,2}\, dx,$$

where $\eta^{\vec{Q},\vec{P},k}_{|P|,1}, \eta^k_{P_2,2}$ are both L^1-normalized bump functions courier-adapted to $[c|P|, C|P|]$ and P_2 respectively and the sequence $c_k^{\vec{Q},\vec{P}}$ decays uniformly in the parameters $\vec{Q}$ and $\vec{P}$, i.e. we have an implicit constant such that

$$\left|c_k^{\vec{Q},\vec{P}}\right| \lesssim_N \frac{1}{1 + |k|^N}.$$

Proof. By assumption, we may include a function $\eta_{|P|,0}$ in the left hand side such that

$$\begin{aligned}
&\int_{\mathbb{R}} F(x) \cdot G * \eta^{\vec{Q}}_{P_1,1}(x) \cdot H * \eta_{P_2,2}(x)\ dx \\
= &\int_{\mathbb{R}} F * \eta_{|P|,0}(x) \cdot G * \eta^{\vec{Q}}_{P_1,1}(x) \cdot H * \eta_{P_2,2}(x)\ dx \\
= &\int_{\mathbb{R}^3} \delta(\xi_1+\xi_2+\xi_3)\hat{F}(\xi_1)\hat{\eta}_{|P|,0}(\xi_1)\hat{G}(\xi_2)\hat{\eta}^{\vec{Q}}_{P_1,1}(\xi_2)\hat{H}(\xi_3)\hat{\eta}_{P_2,2}(\xi_3)\ d\xi_1 d\xi_2 d\xi_3.
\end{aligned}$$

We may expand $\hat{\eta}^{\vec{Q}}_{P_1,1}(\xi_2)$ as a Fourier series by

$$\hat{\eta}^{\vec{Q}}_{P_1,1}(\xi_2) = \sum_{k\in\mathbb{Z}} c_k^{\vec{Q},\vec{P}} \hat{\eta}^k_{P_1,1}(\xi_2),$$

so that expanding $\hat{\eta}^k_{P_1}(-\xi_1-\xi_3)$ in double Fourier Series on $-[c|P|, C|P|] \times -P_2$ yields

$$\begin{aligned}
LHS \ = &\sum_{k\in\mathbb{Z}} \int_{\mathbb{R}^3} \delta(\xi_1+\xi_2+\xi_3) c_k^{\vec{Q},\vec{P}} \hat{F}(\xi_1)\hat{\eta}_{|P|,0}(\xi_1)\hat{G}(\xi_2)\hat{\eta}^k_{P_1,1}(\xi_2) \\
\times \ &\hat{H}(\xi_3)\hat{\eta}_{P_2,2}(\xi_3)\ d\xi_1 d\xi_2 d\xi_3 \\
= &\sum_{k\in\mathbb{Z}} \int_{\mathbb{R}^3} \delta(\xi_1+\xi_2+\xi_3) c_k^{\vec{Q},\vec{P}} \hat{F}(\xi_1)\hat{\eta}_{|P|,0}(\xi_1)\hat{G}(\xi_2)\hat{\eta}^k_{P_1,1}(-\xi_1-\xi_3) \\
\times \ &\hat{H}(\xi_3)\hat{\eta}_{P_2,2}(\xi_3)\ d\xi_1 d\xi_2 d\xi_3 \\
= &\sum_{k\in\mathbb{Z}} \sum_{l\in\mathbb{Z}} \int_{\mathbb{R}^3} \delta(\xi_1+\xi_2+\xi_3) c_k^{\vec{Q},\vec{P}} d_l^k \hat{F}(\xi_1)\hat{\eta}^{k,l}_{|P|,0}(\xi_1)\hat{G}(\xi_2) \\
\times \ &\hat{H}(\xi_3)\hat{\eta}^{k,l}_{P_2,2}(\xi_3)\ d\xi_1 d\xi_2 d\xi_3,
\end{aligned}$$

where $|d_l^k| \lesssim \frac{1}{1+|l|^N}$ uniformly in k. We may rewrite the above as

$$\sum_{k\in\mathbb{Z}} \sum_{l\in\mathbb{Z}} \tilde{c}_k \tilde{d}_l \int_{\mathbb{R}^3} \delta(\xi_1+\xi_2+\xi_3) \overline{c}_k^{\vec{Q},\vec{P}} \overline{d}_l^k \hat{F}(\xi_1)\hat{\eta}^{k,l}_{|P|,0}(\xi_1)\hat{G}(\xi_2)\hat{H}(\xi_3)\hat{\eta}^{k,l}_{P_2,2}(\xi_3)\ d\xi_1 d\xi_2 d\xi_3.$$

Defining $\hat{\tilde{\eta}}^{\vec{Q},\vec{P},k,l}_{|P|,0}(\xi_1) = \bar{c}^{\vec{Q},\vec{P}}_k \bar{d}^k_l \hat{\eta}^{k,l}_{|P|,0}(\xi_1)$ gives the lemma, once we inject $\mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}$.

$\square$

Lemma 12. *For each $k \in \mathbb{K}$, let*

$$m_k(\xi_1, \xi_2) := \sum_{\vec{P} \in \mathbb{P}} \eta_{P_1,1,k}(\xi_1)\eta_{P_2,2,k}(\xi_2)$$

be a generic multiplier of Hilbert transform type. Then, one has the following estimates for the maximal bi-sub-linear operator defined as $M_{\{m_k\}} : (f_1, f_2) \mapsto \sup_{k \in \mathbb{K}} |T_{m_k}(f_1, f_2)|$:

$$||M_{\{m_k\}_{k\in\mathbb{K}}}(f_1, f_2)||_{\frac{p_1 p_2}{p_1+p_2}} \lesssim_{\epsilon, p_1, p_2} |\mathbb{K}|^{\epsilon} ||f_1||_{p_1} ||\hat{f}_2||_{p_2'}$$

for all $f_1 \in L^{p_1}(\mathbb{R})$ and $f_2 \in W_{p_2}(\mathbb{R})$ satisfying $\frac{1}{p_1} + \frac{1}{p_2} < 1$ and $2 < p_2 \leq \infty$.

Proof. For each $k \in \mathbb{K}$, we introduce the same carving as before. So, WLOG,

$$T_{\{m_k\}}(f_1, f_2) = \sum_{m \geq 0} \sum_{k=0}^{2^m-1} \sum_{\vec{P} \in \mathbb{P}} f_1 * \eta_{P_1,1,k} * \check{1}_{E^m_{k,left}} f_2 * \eta_{P_2,2,k} * \check{1}_{-E^m_{k,right}}.$$

Therefore, we may dualize and obtain for the disjoint collection $\{S_k\}_{k \in \mathbb{K}}$

$$\begin{aligned}
& ||M_{\{m_k\}}(f_1, f_2)||_{\frac{p_1 p_2}{p_1+p_2}} \\
\simeq \quad & \int_{\mathbb{R}} \sum_{k \in \mathbb{K}} g 1_{S_k} \sum_{m \geq 0} \sum_{\lambda=0}^{2^m-1} \sum_{\vec{P} \in \mathbb{P}} f_1 * \eta_{P_1,1,k} * \check{1}_{E^m_{\lambda,left}} f_2 * \eta_{P_2,2,k} * \check{1}_{-E^m_{\lambda,right}} \, dx \\
= \quad & \int_{\mathbb{R}} \sum_{k \in \mathbb{K}} g 1_{S_k} \sum_{m \geq 0} \sum_{\lambda=0}^{2^m-1} \sum_{\vec{P} \in \mathbb{P}^m_k[A]} f_1 * \eta_{P_1,1,k} * \check{1}_{E^m_{\lambda,left}} f_2 * \eta_{P_2,2,k} * \check{1}_{-E^m_{\lambda,right}} \, dx \\
+ \quad & \int_{\mathbb{R}} \sum_{k \in \mathbb{K}} g 1_{S_k} \sum_{m \geq 0} \sum_{\lambda=0}^{2^m-1} \sum_{\vec{P} \in \mathbb{P}^m_k[B]} f_1 * \eta_{P_1,1,k} * \check{1}_{E^m_{\lambda,left}} f_2 * \eta_{P_2,2,k} * \check{1}_{-E^m_{\lambda,right}} \, dx \\
:= \quad & \sum_{m \geq 0} I^m_A + I^m_B.
\end{aligned}$$

It suffices to show that for each $m \geq 0$ we have

$$I^m_A + I^m_B \lesssim_{\epsilon, p_1, p_2} |\mathbb{K}|^{\epsilon} 2^{-Cm} ||f_1||_{p_1} ||\hat{f}_2||_{p_2'}.$$

Recall $\mathbb{P}^m_k[A]$ is a paraproduct of type A and so $\{P_1\}_{\vec{P} \in \mathbb{P}^m_k[A]}$ lacunary, while $\mathbb{P}^m_k[B]$ is a paraproduct of type B and so $\{P_2\}_{\vec{P} \in \mathbb{P}^m_k[B]}$ is lacunary. By symmetry, it suffices to handle I^m_B. To this end, note

$$
\begin{aligned}
&|I_B^m| \\
= &\left|\sum_{\lambda=0}^{2^m-1}\sum_{\vec{P}\in\mathbb{P}_\lambda^m[B]}\int_{\mathbb{R}}\sum_{k\in\mathbb{K}}(g1_{S_k})*\eta_{|P|,k}f_1*\check{1}_{E^m_{\lambda,left}}f_2*\eta_{P_2,2,k}*\check{1}_{-E^m_{\lambda,right}}\,dx\right| \\
\leq &\sum_{\lambda=0}^{2^m-1}\sum_{k\in\mathbb{K}}\int_{\mathbb{R}}\left(\sum_{\vec{P}\in\mathbb{P}_\lambda^m[B]}\left|(g1_{S_k})*\eta_{|P|,k}\right|^2\right)^{1/2}|f_1*\check{1}_{E^m_{\lambda,left}}| \\
\times &\left(\sum_{\vec{P}\in\mathbb{P}_\lambda^m[B]}|f_2*\eta_{P_2,2,k}*\check{1}_{-E^m_{\lambda,right}}|^2\right)^{1/2}\,dx \\
\leq &\sum_{\lambda=0}^{2^m-1}\sum_{k\in\mathbb{K}}\int_{\mathbb{R}}\left(\sum_{\vec{P}\in\mathbb{P}_\lambda^m[B]}\left|(g1_{S_k})*\eta_{|P|}\right|^2\right)^{1/2}|f_1*\check{1}_{E^m_{\lambda,left}}| \\
\times &\left(\sum_{\vec{P}\in\mathbb{P}_\lambda^m[B]}|f_2*\eta_{P_2,2}*\check{1}_{-E^m_{\lambda,right}}|^2\right)^{1/2}\,dx \\
\leq &\sum_{\lambda=0}^{2^m-1}\sum_{k\in\mathbb{K}}\int_{\mathbb{R}}\left|\tilde{H}\left((g1_{S_k})\right)\right|\cdot|f_1*\check{1}_{E^m_{\lambda,left}}|\cdot\left|\tilde{H}^\lambda\left(f_2*\check{1}_{-E^m_{\lambda,right}}\right)\right|dx \\
\leq &\int_{\mathbb{R}}\sum_{k\in\mathbb{K}}\left|\tilde{H}\left(g1_{S_k}\right)\right|\cdot\left(\sum_\lambda\left|f_1*\check{1}_{E^m_{\lambda,left}}\right|^2\right)^{1/2} \\
\times &\left(\sum_\lambda\left|\tilde{H}^\lambda\left(f_2*\check{1}_{-E^m_{\lambda,right}}\right)\right|^2\right)^{1/2}dx \\
\leq &\left\|\sum_{k\in\mathbb{K}}\left|\tilde{H}(g1_{S_k})\right|\right\|_{p_0}\left\|\left(\sum_\lambda\left|f_1*\check{1}_{E^m_{\lambda,left}}\right|^2\right)^{1/2}\right\|_{p_1} \\
\times &\left\|\left(\sum_\lambda\left|\tilde{H}^\lambda\left(f_2*\check{1}_{-E^m_{\lambda,right}}\right)\right|^2\right)^{1/2}\right\|_{p_2}.
\end{aligned}
$$

As the second and third factors have already been handled by previous estimates, it suffices to understand the first. We first perform discrete Holder, which gives us that

$$\left|\left|\sum_{k\in\mathbb{K}}\left|\tilde{H}(g1_{S_k})\right|\right|\right|_{p_0} \leq |\mathbb{K}|^{\epsilon}\left|\left|\left(\sum_{k\in\mathbb{K}}\left|\tilde{H}(g1_{S_k})\right|^{\frac{1}{1-\epsilon}}\right)^{1-\epsilon}\right|\right|_{p_0}$$

$$\lesssim_{\epsilon} |\mathbb{K}|^{\epsilon}\left|\left|\left(\sum_{k\in\mathbb{K}}|g1_{S_k})|^{\frac{1}{1-\epsilon}}\right)^{1-\epsilon}\right|\right|_{p_0}$$

$$\leq |\mathbb{K}|^{\epsilon}.$$

As before, the second and third factor give us a combined $2^{m\left(\frac{1}{p_1}-\frac{1}{p_2'}\right)}||f_1||_{p_1}||\hat{f}_2||_{p_2'}$.

□

We now use Lemmata 11 and 12 to conclude

$$\begin{aligned}
& \tilde{\Lambda}^m_{B[a_1,a_2]_1}(f_0,f_1,f_2,f_3) \\
:= & \left|\int_{\mathbb{R}}\sum_k\sum_{\vec{Q}\in\mathbb{Q}}\sum_{\vec{P}\in\mathbb{P}^m_k:|\vec{Q}|>>|\vec{P}|}(f_0*\eta_{-Q_1}f_1*\eta_{Q_1})*\eta_{|P|}f_2*\eta_{Q_2}*\eta_{P_1}*\check{1}_{E^m_{k,left}}\right. \\
\times & \left. f_3*\eta_{P_2}*\check{1}_{-E^m_{k,right}}\ dx\right| \\
= & \left|\int_{\mathbb{R}}\sum_k\sum_{\vec{Q}\in\mathbb{Q}^m_k}\sum_{\vec{P}\in\mathbb{P}^m_k:|\vec{Q}|>>|\vec{P}|}(f_0*\eta_{-Q_1}f_1*\eta_{Q_1})*\eta_{|P|}f_2*\eta_{Q_2}*\eta_{P_1}*\check{1}_{E^m_{k,left}}\right. \\
\times & \left. f_3*\eta_{P_2}*\check{1}_{-E^m_{k,right}}\ dx\right| \\
\lesssim & \left|\int_{\mathbb{R}}\sum_k\sum_{\vec{Q}\in\mathbb{Q}^m_k}\sum_{\vec{P}\in\mathbb{P}^m_k:|\vec{Q}|>>|\vec{P}|}(f_0*\eta_{-Q_1}f_1*\eta_{Q_1})*\eta^{\vec{Q}}_{|P|}f_2*\check{1}_{E^m_{k,left}}\right. \\
\times & \left. f_3*\eta_{P_2}*\check{1}_{-E^m_{k,right}}\ dx\right| \\
\leq & \int_{\mathbb{R}}\sum_k\left(\sum_{\vec{P}\in\mathbb{P}^m_k}\left|\sum_{|\vec{Q}|>>|\vec{P}|}(f_0*\eta_{-Q_1}f_1*\eta_{Q_1})*\eta^{\vec{Q}}_{|P|}\right|^2\right)^{1/2} \\
\times & |f_2*\check{1}_{E^m_{k,left}}|\left(\sum_{\vec{P}\in\mathbb{P}^m_k}\left|f_3*\eta_{P_2}*\check{1}_{-E^m_{k,right}}\right|^2\right)^{1/2}\ dx.
\end{aligned}$$

The last expression in the above display is majorized by

$$\leq \int_{\mathbb{R}} \sup_k \left[\left(\sum_{\vec{P}\in\mathbb{P}_k^m} \left| \sum_{\vec{Q}\in\mathbb{Q}_k^m:|\vec{Q}|>>|\vec{P}|} (f_0 * \eta_{-Q_1} f_1 * \eta_{Q_1}) * \eta_{|P|}^{\vec{Q}} \right|^2 \right)^{1/2} \right]$$

$$\times \left(\sum_k |f_2 * \check{1}_{E^m_{k,left}}|^2 \right)^{1/2} \left(\sum_k \sum_{\vec{P}\in\mathbb{P}_k^m} \left| f_3 * \eta_{P_2} * \check{1}_{-E^m_{k,right}} \right|^2 \right)^{1/2} dx,$$

where the inequality in the third line arises from the fact that one has implicitly used triangle inequality on a countable number of terms with rapidly decaying coefficients arising from two applications of Fourier Series as described in Lemma 11. Finish by using Hölder's inequality, linearizing with Rademacher functions, and applying Lemma 12.

□

2.8 Counterexample for a Bilinear Operator related to $B[a_1, a_2]$

From the proceeding proofs, it is clear that the estimate T^{a_1,a_2} : $L^{p_1}(\mathbb{R}) \times W_{p_2}(\mathbb{R}) \times L^{p_3}(\mathbb{R}) \rightarrow L^{\frac{1}{\frac{1}{p_1}+\frac{1}{p_2}+\frac{1}{p_3}}}(\mathbb{R})$ would hold if $m(\xi_1, \xi_2) := \sum_{\vec{P}\in\mathbb{P}} \sum_{\lambda\in\mathbb{Z}} \eta_{-P_1}(\xi_1)\eta_{P_1}(\xi_2)\hat{\psi}^{\vec{P}}_\lambda(\xi_1 + \xi_2)$ (where $\{P\}$ is a lacunary sequence) is a generic bounded multilinear multiplier and $\sup_{k\in\mathbb{K}} |T_{m_k}(\cdot, \cdot)|$ has an operatorial bound growing $O_\epsilon(|\mathbb{K}|^\epsilon)$ for every ϵ. However, the next proposition states that m_k can be chosen to satisfy no L^p estimates.

Proposition 3. *Let $P_k = [2^k - 2^{k-1}, 2^k + 2^{k-1}]$ for all $k \in \mathbb{Z}$. Then there exists a collection $\{\eta^\lambda_k\}_{(k,\lambda)\in\mathbb{Z}^2}$, where each $\hat{\eta}^\lambda_k$ is uniformly adapted to P_λ in $k \in \mathbb{Z}$ so that the bilinear operator given by*

$$\mathcal{B} : (f_1, f_2) \mapsto \sum_{k\in\mathbb{Z}} \sum_{\lambda\in\mathbb{Z}} \left[f_1 * \eta^k f_2 * \widetilde{\eta^k} \right] * \eta_k^\lambda$$

satisfies no L^p estimates.

Proof. Set $\eta^k \equiv \eta_k^\lambda \equiv 0$ if $k < 0$. For $k \geq 0$, set $\hat{\eta}^k(\xi) = 2^{-k}\tilde{1}(2^{-k}\xi)$ for some $\tilde{1} \in C^\infty(\mathbb{R})$ satisfying $1_{[7/8,9/8]}(\xi) \leq \tilde{1}(\xi) \leq 1_{[3/4,3/2]}(\xi)$. Furthermore, set $\hat{\eta}_k^{-k_0}(\xi) = e^{2\pi i 2^{-k_0}k}\chi(2^{k_0}(\xi - 2^{-k_0}))$ where $\check{\chi} \geq 1_{[-1,1]}$ and *supp* $\chi \subset [-1/2, 1/2]$.

Fix $N \in \mathbb{N}$. Choose $f_1^N = \sum_{1\leq n\leq N} e^{2\pi i 2^n x}\phi_1(x - n), f_2^N = \sum_{1\leq n\leq N} e^{-2\pi i 2^n x}\phi_2(x - n)$ where ϕ_1, ϕ_2 have Fourier support inside $[-1/8, 1/8]$ so that $\phi_1\phi_2$ has Fourier support inside $[-1/4, 1/4]$. Moreover, choose ϕ_1, ϕ_2 to ensure $\phi_1\phi_2$ has flat Fourier transform on $[-1/16, 1/16]$. Then the contribution at a given lacunary scale $k_0 : 1 \lesssim k_0 \lesssim \log(N)$ is

$$\mathcal{B}_{k_0}(f_1^N, f_2^N)(x) := \sum_{k\in\mathbb{Z}} \left[f_1^N * \eta^k f_2^N * \bar{\eta}^k \right] * \eta_k^{-k_0} = e^{2\pi i 2^{-k_0}x} \sum_{1\leq n\leq N} 2^{-k_0}\check{\chi}(2^{-k_0}(x - n)).$$

Hence, $|\mathcal{B}_{k_0}(f_1^N, f_2^N)(x)| \gtrsim 1_{[1,N]}(x)$. The Littlewood-Paley equivalence then yields

$$\left|\left| \sum_{k\in\mathbb{Z}} \mathcal{B}_k(f_1^N, f_2^N) \right|\right|_{\frac{p_1p_2}{p_1+p_2}} \simeq \left|\left| \left(\sum_{k\in\mathbb{Z}} \left|\mathcal{B}_k(f_1^N, f_2^N)\right|^2 \right)^{1/2} \right|\right|_{\frac{p_1p_2}{p_1+p_2}} \simeq \log(N)^{1/2} N^{\frac{1}{p_1}+\frac{1}{p_2}},$$

whereas $||f_1^N||_{p_1} \simeq N^{1/p_1}$ and $||f_2^N||_{p_2} \simeq N^{1/p_2}$. Taking N arbitrarily large proves the proposition.

□

Hence, our frequency discretization will need some refinement. To this end, we begin construction on a time-frequency framework.

2.9 Generalized Restricted Type Mixed Estimates

Let us begin by recalling the setup and notation taken from [26].

Definition 8. *For each measurable subset $E \subset \mathbb{R}$ with finite measure let $X(E) = \{f : |f| \leq 1_E \text{ a.e.}\}$ with respect to Lebesgue measure.*

Definition 9. *A multisublinear form is of restricted type $\alpha = (\alpha_1, ..., \alpha_n$ with $0 \leq \alpha_j \leq 1$ if there exists a constant C such that for each tuple $E = (E_1, ..., E_n)$ of measurable subsets of $\mathbb{R}$ and for each tuple $f = (f_1, ..., f_n)$ with $f_j \in X(E_j)$, we have*

$$|\Lambda(f)| := |\Lambda(f_1, ..., f_n)| \leq C|E|^\alpha$$

where $|E|^\alpha = \prod_j |E_j|^{\alpha_j}$.

Definition 10. *Let α be an $n-$tuple of real numbers and assume $\alpha_j \leq 1$ for all j. An $n-$sublinear form is called generalized restricted type α if there is a constant C such that for all tuples $E = (E_1, ..., E_n)$ there is an index j and a major subset $\tilde{E}_j$ of E_j such that for all tuples $f = (f_1, ..., f_n)$ with $f_k \in X(E_k)$ for all k and in addition $f_j \in X(\tilde{E}_j)$ we have*

$$|\Lambda(f_1, ..., f_n)| \leq C|E|^\alpha.$$

From the standpoint of multilinear Marcinkiewicz interpolation, we may in fact allow the exceptional set $\tilde{E}_j$ in the above description to depend not just on the choice of $E = (E_1, ..., E_n)$ but also on the choice of f_k for all $k \neq j$. In the case when

$j = n$, say, this would mean there exists a constant C such that for each $(E_1, ..., E_n)$ and all $(f_1, ..., f_{n-1})$ with $f_k \in X(E_k)$ for all $k \le n-1$, there exists a major subset $\tilde{E}_n(f_1, ..., f_{n-1})$ for which $|\Lambda(f_1, ..., f_n)| \le C|E|^\alpha$ for all $f_n \in X(\tilde{E}_n(f_1, ..., f_{n-1}))$.

To prove mixed type estimates for the given multi-linear form Λ , it suffices by Marcinekiewicz interpolation to obtain weak type mixed estimates. The weak type statement is that for every tuple $(E_1, ..., E_n)$ and collection of functions $f_j : \mathbb{R} \to \mathbb{C}$ satisfying $|f_j| \le 1_{E_j}, |\Lambda(\vec{f})| \lesssim_{\vec{p}} \prod_{j=1}^n |E_j|^{\frac{1}{p_j}}$. In the mixed setting, the usual condition $\sum_{j=1}^n \beta_j = 1$ is replaced by $\sum_{j\neq i}^n \beta_j = \beta_i$, supposing the mixed index falls on the *ith* position. Details of the Marcinekiewicz interpolation in this "mixed" setting are provided in the following two lemmas.

2.9.1 Marcinkiewicz Interpolation Lemmas

The proofs of both results are essentially the same as C. Thiele arguments in [26].

Lemma 13. *Let Λ be a multi-linear form which satisfies $|\Lambda(\vec{f})| \le C \prod_{j=1}^n |E_j|^{\alpha_j}$ with uniformly bounded constant C for all tuples $\vec{\alpha} = (\alpha_1, ..., \alpha_n)$ in some neighborhood of $\vec{\beta} = (\beta_1, ..., \beta_n)$ where $\sum_{j\neq i}^n \alpha_j = \alpha_i$, $0 < \alpha_j < 1$ for all $j \in \{1, .., n\}$ and assume, in addition, $\beta_i > \frac{1}{2}$. Then for all $\vec{f} = (f_1, ..., f_n)$,*

$$\left|\Lambda(\vec{f})\right| \lesssim C \prod_{j=1}^n ||f_j||_{\frac{1}{\beta_j}}.$$

Proof. Without loss of generality, suppose $f_j \ge 0$ for all indices $1 \le j \le n$. For each f_j appearing in the tuple $\vec{f}$, we construct 2 sequences of subsets of $\mathbb{R}$ denoted by $\left\{\tilde{F}_k^j\right\}_{k\in\mathbb{Z}}$ and $\left\{F_k^j\right\}_{k\in\mathbb{Z}}$ with the following properties:

$$\begin{aligned} |\tilde{F}^j_k| &= |F^j_k| = 2^k \\ \tilde{F}^j_k &\supset F^j_{k-1} \\ essinf\ \{f_j(x) : x \in \tilde{F}^j_k\} &\geq esssup\ \{f_j(x) : x \in F^j_k\} \\ essinf\ \{f_j(x) : x \in F^j_k\} &\geq esssup\ \{f_j(x) : x \notin F^j_k \cup \tilde{F}^j_k\} \\ \left\{F^j_k\right\}_{k\in\mathbb{Z}} & \quad is\ a\ partition. \end{aligned}$$

For each index $j \in \{1, .., n\}$ introduce the splitting $f_j(x) = \sum_{k\in\mathbb{Z}} f_j(x) 1_{F^j_k}(x)$ and note by multi-linearity of Γ

$$|\Lambda(\vec{f})| \leq C \sum_{k_1,...,k_n\in\mathbb{Z}} \prod_{j=1}^{n} \left[\left|esssup\{f_j(x) : x \in F^j_{k_j}\}\right| 2^{k_j\alpha_j}\right]$$

where C is the uniform constant appearing in the statement of the lemma and $\vec{\alpha}$ is a tuple in a neighborhood of $\vec{\beta}$ for which mixed weak type estimates hold. For fixed $(k_1, ..., k_n)$, we wish to choose $\vec{\alpha}$ in such a way as to guarantee for $k := \sum_{j=1}^{n} |k_j|$

$$\sum_{j=1}^{n} \alpha_j k_j = \sum_{j=1}^{n} \beta_j k_j - \epsilon \max\{|k_1 - k|, |k_2 - k|, ..., |k_i + k|, ..., |k_n - k|\}.$$

Indeed, $\vec{\alpha}$ and $\vec{\beta}$ are both subject to the restructions $\sum_{j\neq i}^{n} \alpha_j - \alpha_i = \sum_{j\neq i} \beta_j - \beta_i = 0$. Therefore, $(\vec{\alpha} - \vec{\beta}) \cdot (1, ..., -1, ..., 1) = 0$ and our conclusion is one can always choose $\vec{\alpha}(\vec{k})$ to satisfy $|\vec{\alpha} - \vec{\beta}| < \delta$ and $(\vec{\alpha} - \vec{\beta}) \cdot \vec{k} = -\epsilon \max\{|k_1 - k|, |k_2 - k|, ..., |k_i + k|, ..., |k_n - k|\}$ for some $\epsilon(\delta)$, where k is the average of the $k_i s$. Therefore, with this choice of $\vec{\alpha}(\vec{k})$,

$$|\Lambda(\vec{f})| \leq C 2^{-\epsilon \max\{|k_1 k|, |k_2 - k|, ..., |k_i + k|, ..., |k_n - k|\}} \prod_{j=1}^{n} \left[\left|esssup\{f_j(x) : x \in F^j_{k_j}\}\right| 2^{k_j\beta_j}\right].$$

Introduce $\tilde{k}_1 := k_1 - k, \tilde{k}_2 = k_2 - k, ..., \tilde{k}_i = k_i + k, ..., \tilde{k}_n = k_n - k$ to bound the above expression as

$$\sum_{\tilde{k}_1,...,\tilde{k}_{n-1}\in\mathbb{Z}} \sum_{k\in\mathbb{Z}} C2^{-\epsilon\max\{|\tilde{k}_1|,...,|\tilde{k}_{n-1}|\}} \prod_{j=1}^{n} \left[\left|esssup\{f_j(x) : x \in F^j_{\tilde{k}_j\pm k}\}\right| 2^{(\tilde{k}_j\pm k)\beta_j}\right].$$

Now apply Hölder's inequality in k with the following computation valid when $2\beta_i \geq 1$:

$$\begin{aligned}
&\sum_k \left[\left|esssup\{f_j(x) : x \in F^j_{\tilde{k}_j\pm k}\}\right| 2^{(\tilde{k}_j\pm k)\beta_j}\right] \\
\leq\ & \left(\sum_k \left(\left|esssup\{f_j(x) : x \in F^j_{\tilde{k}_j\pm k}\}\right| 2^{(\tilde{k}_j\pm k)\beta_j}\right)^{\frac{1}{2\beta_i}}\right)^{2\beta_i} \\
\leq\ & \prod_{j=1}^{n} \left(\sum_{k_j} \left|esssup\{f_j(x) : x \in F^j_{k_j}\}\right|^{\frac{1}{\beta_j}} 2^{k_j}\right)^{\beta_j} \\
\leq\ & \prod_{j=1}^{n} ||f_j||_{\frac{1}{\beta_j}}.
\end{aligned}$$

In the last line, we implicitly used for each $1 \leq j \leq n$

$$\begin{aligned}
&\left(\sum_{k_j} \left|esssup\{f_j(x) : x \in F^j_{k_j}\}\right|^{\frac{1}{\beta_j}} 2^{k_j}\right)^{\beta_j} \\
\leq\ & \left(\sum_{k_j} \left|essinf\ \{f_j(x) : x \in F^j_{k_j-1}\}\right|^{\frac{1}{\beta_j}} 2^{k_j}\right)^{\beta_j} \\
\lesssim\ & \left(\sum_{k_j} ||f_j 1_{F^j_{k_j}}||^{\frac{1}{\beta_j}}_{\frac{1}{\beta_j}}\right)^{\beta_j} \\
\leq\ & ||f_j||_{\frac{1}{\beta_j}}.
\end{aligned}$$

$\square$

Lemma 14. *Fix $n \geq 2$. Assume Λ is of generalized restricted weak type β where $\sum_{j \neq i} \beta_j = \beta_i$ for some $i \neq n$, $\beta_i \geq \frac{1}{2}, \beta_k > 0$ for all $k < n$ and $\beta_n \leq 0$. Assume Λ is also of generalized restricted type α for all α in a neighborhood of β satisfying $\sum_{j \neq i} \alpha_j = \alpha_i$ where $\alpha_i \geq \frac{1}{2}$.Then the dual form (in the nth function) T satisfies*

$$||T(f_1, ..., f_{n-1})||_{\frac{1}{1-\beta_n}} \leq C \prod_{j=1}^{n} ||f_j||_{1/\beta_j}$$

where C depends only on the constants appearing the generalized restricted type estimates near β.

Proof. Fix $f_1, ..., f_{n-1}$. By pre- and post composing with measure preserving transformation, we may assume that $|f_j|$as well as $|T(f_1, ..., f_n))|$ are supported in $[0, \infty)$ and non increasing. We write

$$\begin{aligned}
||T(f)||_p^p &= \int |T(f)(x)|^p dx \\
&\leq \sum_{k \in \mathbb{Z}} \left(2^{-k} \int_{2^k}^{2^{k+1}} Tf(x) dx \right)^p 2^k \\
&\lesssim \sum_{k \in \mathbb{Z}} 2^{k(1-p)} \left(\sum_{k_{\cdot}, ..., k_{n-1}} \int_{2^k}^{2^{k+1}} T(f_j 1_{[2^{k_j}, 2^{k_j+1})}) dx \right)^{1/p} .
\end{aligned}$$

Setting $E_n = (0, 2^{k+1})$, we see that

$$\int_{2^k}^{2^{k+1}} Tg(x) dx \leq \int |Tg(x)| 1_{\tilde{E}_n} dx$$

for every tuple g and every major subset $\tilde{E}_n$ of E_n. Therefore, by the generalized restricted type estimate,

$$||T(f)||_p^p \leq \sum_k 2^{k(1-p)} \left(\sum_{k_1,\ldots,k_{n-1}} 2^{k\alpha_n} \prod_{j=1}^{n-1} f_j(2^{k_j}) 2^{k_j \alpha_j} \right)^p .$$

Using the freedom to choose α_j for each tuple $(k_1, ..., k_{n-1})$, we obtain for $p = \frac{1}{1-\beta_n}$

$$\begin{aligned}
||T(f)||_p^p &\leq \sum_k 2^{k(1-p)} \left(\sum_{k_1,\ldots,k_{n-1}} 2^{-\epsilon \max|k_j \pm k|} 2^{k\beta_n} \prod_{j=1}^{n-1} f_j(2^{k_j}) 2^{k_j\beta_j} \right)^p \\
&= \sum_k 2^{k(1-p)} \left(\sum_{\tilde{k}_1,\ldots,\tilde{k}_{n-1}} 2^{-\epsilon \max|\tilde{k}_j|} 2^{k\beta_n} \prod_{j=1}^{n-1} f_j(2^{\tilde{k}_j \pm k}) 2^{(\tilde{k}_j \pm k)\beta_j} \right)^p \\
&\leq \sum_k \sum_{\tilde{k}_1,\ldots,\tilde{k}_{n-1}} 2^{-\tilde{\epsilon} \max|\tilde{k}_j|} \left(\prod_{j=1}^{n-1} f_j(2^{\tilde{k}_j \pm k}) 2^{(\tilde{k}_j \pm k)\beta_j} \right)^p \\
&\lesssim \sum_{\tilde{k}_1,\ldots,\tilde{k}_{n-1}} 2^{-\tilde{\epsilon} \max|\tilde{k}_j|} \left(\sum_k \left(\prod_{j=1}^{n-1} f_j(2^{\tilde{k}_j \pm k}) 2^{(\tilde{k}_j \pm k)\beta_j} \right)^{\frac{1}{2\beta_i - \beta_n}} \right)^{p(2\beta_i - \beta_n)} \\
&\leq \sum_{\tilde{k}_1,\ldots,\tilde{k}_{n-1}} 2^{-\tilde{\epsilon} \max|\tilde{k}_j|} \left(\prod_{j=1}^{n} ||f_j||_{1/\beta_j} \right)^p \\
&\lesssim \left(\prod_{j=1}^{n} ||f_j||_{1/\beta_j} \right)^p .
\end{aligned}$$

The condition $\beta_i \geq 1/2$ was crucial in the line before Hölder's inequality was applied.

□

2.10 Essential Time-Frequency Definitions

With the mixed estimate interpolation framework now complete, we may now introduce some more essential definitions and then prove number of results with increasing complexity concerning degenerate multilinear symbols. For reader's convenience, we include the definitions that will be used extensively in the remainder of this work.

Definition 11. *Let $n \geq 1$ and $\sigma \in \{0, \frac{1}{3}, \frac{2}{3}\}^n$. We define the shifted $n-$dyadic mesh $D = D^n_\sigma$ to be the collection of cubes of the form*

$$D^n_\sigma := \left\{2^j(k + (0,1)^n + (-1)^j\sigma) : j \in \mathbb{Z}, k \in \mathbb{Z}^n\right\}$$

Observe that for every cube Q, there exists a shifted dyadic cube Q' such that $Q \subseteq \frac{7}{10}Q'$ and $|Q'| \sim |Q|$; this property clearly follows from verifying the $n = 1$ case. The constant $\frac{7}{10}$ is not especially important here.

Definition 12. *A subset D' of a shifted $n-$dyadic grid D is called sparse, if for any two cubes Q, Q' in D with $Q \neq Q'$ we have $|Q| < |Q'|$ implies $|10^9 Q| < |Q'|$ and $|Q| = |Q'|$ implies $10^9 Q \cap 10^9 Q' = \emptyset$.*

It is immediate from the above definition that any subset of a shifted $n-$dyadic grid can be split into $O(C^n)$ sparse subsets.

Definition 13. *For a given spatial interval I, let $\tilde{\chi}_I(x) := \left(1 + \left(\frac{|x - x_I|}{|I|}\right)^2\right)^{1/2}$, where x_I is the center of I.*

Definition 14. *Let $P = (I_P, \omega_P)$ be a tile. A wave packet on P is a function Φ_P which has Fourier support in $\frac{9}{10}\omega_P$ and obeys the estimate*

$$|\Phi_P(x)| \lesssim_M |I_P|^{-1/2}\tilde{\chi}_{I_P}^M(x)$$

for some fixed large integer M. Therefore, Φ_P is L^2 normalized and adapted to the Heisenberg box (I_P, ω_P).

We next introduce the tile ordering $<$ from [21], which is in the spirit of Fefferman or Lacey and Thiele, but different inasmuch as P' and P do not have to intersect.

Definition 15. *Let $\sigma = (\sigma_1, \sigma_2, \sigma_3) \in \{0, \frac{1}{3}, \frac{2}{3}\}^3$, and let $1 \le i \le 3$. An $i-$tile with shift σ_i is a rectangle $P = (I_P, \omega_P)$ with area 1 and with $I_P \in D_0^1, \omega_P \in D_{\sigma_i}^1$. A tri-tile with shift σ is a 3-tuple $\vec{P} = (P_1, P_2, P_3)$ such that each P_i is an $i-$tile with shift σ_i, and the $I_{P_i} = I_{\vec{P}}$ are independent of i. The frequency cube $Q_{\vec{P}}$ of a tri-tile is defined to be $\prod_{i=1}^3 \omega_{P_i}$.*

Definition 16. *A set $\mathbb{P}$ of tri-tiles is called sparse, if all the tri-tiles in $\mathbb{P}$ have the same shift σ and the set of frequency cubes $\{Q_{\vec{P}} = (\omega_{P_1}, \omega_{P_2}, \omega_{P_3}) : \vec{P} \in \mathbb{P}\}$ is sparse.*

Definition 17. *Let P and P' be tiles. We write $P' < P$ if $I_{P'} \subsetneq I_P$ and $3\omega_P \subseteq 3\omega_{P'}$, and $P' \le P$ if $P' < P$ or $P' = P$. We write $P' \lesssim P$ if $I_{P'} \subseteq I_P$ and $10^7\omega_P \subseteq 10^7\omega_{P'}$. We write $P' \lesssim' P$ of $P' \lesssim P$ and $P' \not\le P$.*

Definition 18. *A collection $\mathbb{P}$ of tri-tiles is said to have rank 1 if one has the following properties for all $\vec{P}, \vec{P}' \in \mathbb{P}$:*

If $\vec{P} \neq \vec{P}'$, then $P_j \neq P_j'$ for all $j = 1, 2, 3$.

If $P_j' \le P_j$ for some $j = 1, 2, 3$, then $P_i' \lesssim P_i$ for all $1 \le i \le 3$.

If we further assume that $|I_{\vec{P}'}| > 10^9 |I_{\vec{P}}|$, then $P_i' \lesssim' P_i$ for all $i \neq j$.

Definition 19. *For any* $1 \leq j \leq 3$ *and tri-tile* $\vec{P}_T \in \mathbb{P}$*, define a* j*-tree with top* $\vec{P}_T$ *to be a collection of tri-tiles* $T \subset \mathbb{P}$ *such that*

$$P_j \leq P_{T,j} \text{ } for \text{ } all \text{ } \vec{P} \in T,$$

where $P_{T,j}$ *is the jth component of* $\vec{P}_T$*. We write* I_T *and* $\omega_{T,j}$ *for* $I_{\vec{P}_T}$ *and* $\omega_{P_{T,j}}$ *respectively. We say that* T *is a tree if it is a* $j-$*tree for some* $1 \leq j \leq 3$*.*

We do not require T to contain its top $\vec{P}_T$.

Definition 20. *Let* $1 \leq j \leq 3$*. Two trees* T, T' *are strongly* $j-$*disjoint if*

$P_j \neq P'_j$ *for all* $\vec{P} \in T, \vec{P}' \in T'$

Whenever $\vec{P} \in T, \vec{P}' \in T'$ *satisfy* $2\omega_{P_j} \cap 2\omega_{P'_j} \neq \emptyset$*, then* $I_{\vec{P}'} \cap I_T = \emptyset$*, and similarly with* T *and* T' *reversed.*

Note that if T and T' are strongly $j-$disjoint, then $I_{\vec{P}} \times 2\omega_{P_j} \cap I_{\vec{P}'} \times 2\omega_{P'_j} = \emptyset$ for all $\vec{P} \in T, \vec{P}' \in T'$.

Definition 21. *Let* ω_1 *and* ω_2 *be intervals. Then write* $\omega_1 \subset\subset \omega_2$ *provided* $|\omega_1| << |\omega_2|$ *for some sufficiently large absolute constant and* $\omega_1 \subset \omega_2$*.*

2.11 Mixed Estimates for the Scale-1 Hilbert Transform in the Plane

With these preliminaries out of the way, we now state and prove

Proposition 4. *Let* $\Gamma = \{\xi_1 + \xi_2 = 0\} \subset \mathbb{R}^2$. *Let* $m_0 : \mathbb{R}^2 \to \mathbb{C}$ *be supported in* $dist(\vec{\xi}, \Gamma) \simeq 1$ *with the additional property that* $|\partial^{\vec{\alpha}} m_0(\vec{\xi})| \lesssim_{\vec{\alpha}} 1$ *for sufficiently many multi-indices* $\vec{\alpha}$. *Then* $T_{m_0} : L^{p_1}(\mathbb{R}) \times W_{p_2}(\mathbb{R}) \to L^{\frac{p_1 p_2}{p_1+p_2}}(\mathbb{R})$ *provided* $\frac{1}{p_1} + \frac{1}{p_2} < 1, 2 < p_2 < \infty$.

Proof. By standard discretization arguments, see [17], it suffices to prove restricted weak-type estimates uniform in neighborhoods near the points $(1/2, 1/2, 0), (0, 1/2, 1/2), (1, 0, 0)$ and $(0, 0, 1)$ for the 3−form defined by

$$\Lambda_{T_0}(f_1, f_2, f_3) = \sum_{\vec{P} \in \mathbb{P}} \frac{1}{|I_{\vec{P}}|} \langle f_1, \Phi_{P_1,1} \rangle \langle f_2, \Phi_{P_2,2} \rangle \langle f_3, \Phi^{lac}_{|P|,3} \rangle$$

where $\mathbb{P}$ is a scale-1 collection of tiles. In particular, it suffices to show that for every (E_1, E_2, E_3) such that $E_j \subset \mathbb{R}$ is measurable for each $j \in \{1, 2, 3\}$ and (f_1, f_2, f_3) satisfying $f_j \in X(E_j)$ for $j \in \{1, 3\}$ and $|\hat{f}_2| \leq 1_{E_2}$, there exists $E_1'(\vec{f})$ a major subset of E_1 such that

$$|\Lambda(f_1 1_{E_1'}, f_2, f_3)| \lesssim_{\vec{p}} |E_1|^{1/p_1} |E_2|^{1-1/p_2} |E_3|^{1-1/p_1-1/p_2}$$

for (p_1, p_2, p_3) in neighborhoods of $(1/2, 1/2, 0), (1, 0, 0), (0, 0, 1)$ and a similar statement for $(0, 1/2, 1/2)$, except with the exceptional set attached to the 1st index. Let $\vec{P}_{n_1,n_2,n_3} = \mathbb{P}_{n_1,1} \cap \mathbb{P}_{n_2,2} \cap \mathbb{P}_{n_3,3}$ where for each $j \in \{1, 2, 3\}$ and $n \gtrsim 1$

$$\mathbb{P}_{n,j} := \left\{ \vec{P} \in \mathbb{P} : \frac{|\langle f_j, \Phi_{P_j,j} \rangle|}{|I_{\vec{P}}|^{1/2}} \simeq 2^{-n} \right\}.$$

Moreover, let $\Omega = \{M 1_{E_1} \geq C|E_1|\} \cap \left\{M f_2 \geq C|E_2|^{1/2}\right\}$ and set $\mathbb{P}^d = \left\{\vec{P} \in \mathbb{P} : 1 + dist(I_{\vec{P}}, \Omega^c)/|I_{\vec{P}}| \simeq 2^d\right\}$. Lastly, define $\mathbb{P}^d_{n_1,n_2,n_3} = \mathbb{P}^d \cap \mathbb{P}_{n_1,n_2,n_3}$. By triangle inequality and Cauchy-Schwarz,

$$\begin{aligned}
&|\Lambda(f_1, f_2, f_3)| \\
\leq\ & \sum_{d\geq 0} \sum_{n_1,n_2,n_3} \sum_{\vec{P}\in\mathbb{P}^d_{n_1,n_2,n_3}} \left[\sup_{\vec{P}\in\mathbb{P}_{n_1,n_2,n_3}} \frac{|\langle f_1, \Phi_{P_1}\rangle|}{|I_{\vec{P}}|^{1/2}}\right] \left[\sup_{\vec{P}\in\mathbb{P}_{n_1,n_2,n_3}} \frac{|\langle f_2, \Phi_{P_2}\rangle|}{|I_{\vec{P}}|^{1/2}}\right] \\
\times\ & \left[\sup_{\vec{P}\in\mathbb{P}_{n_1,n_2,n_3}} \frac{|\langle f_3, \Phi_{|\vec{P}|}\rangle|}{|I_{\vec{P}}|^{1/2}}\right] |I_{\vec{P}}| \\
\lesssim\ & \sum_{d\geq 0} \sum_{n_1\geq N_1(d), n_2\geq N_2(d), n_3\geq N_3(d)} 2^{-n_1}2^{-n_2}2^{-n_3} \sum_{\vec{P}\in\mathbb{P}^d_{n_1,n_2,n_3}} |I_{\vec{P}}|.
\end{aligned}$$

To prove $T_0 : L^{p_1}(\mathbb{R}) \times W_{p_2}(\mathbb{R}) \to L^{\frac{p_1p_2}{p_1+p_2}}(\mathbb{R})$, it suffices by weak type interpolation to Each $\mathbb{P}_{n_1,n_2,n_3}$ is a collection of scale 1 tiles. The sum over the spatial lengths of all tiles in this collection can be estimated in two ways:

$$\sum_{\vec{P}\in\mathbb{P}_{n_1,n_2,n_3}} |I_{\vec{P}}| \lesssim \{2^{2n_1}|E_1|, 2^{2n_2}|E_2|\}$$

However, this is not enough to get summability over all three parameters n_1, n_2, n_3. We must provide another estimate into the above sum which makes use of n_3, which is achieved using information about how many tiles may stack on top of each other.

Proposition 5.

$$\#_{n_1,n_2,n_3} := \sup_{|I_{\vec{P}}|=1} \left|\left\{\vec{Q} \in \mathbb{P}_{n_1,n_2,n_3} : I_{\vec{Q}} = I_{\vec{P}}\right\}\right| \lesssim 2^{n_2}||\hat{f}_1||_1$$

Proof. Because $\vec{P} \in \mathbb{P}_{n_1,n_2,n_3}, 2^{-n_2} \lesssim \frac{|\langle f_2,\Phi_{P_2}\rangle|}{|I_{P_2}|^{1/2}}$. Let the supremum be attained by some interval I_{P_0}. Then observe

$$\#_{n_1,n_2,n_3} 2^{-n_2} \lesssim \sum_{\vec{Q}\in\mathbb{P}_{n_1,n_2,n_3}} \frac{|\langle f_2, \Phi_{Q_2}\rangle|}{|I_{Q_2}|^{1/2}} \lesssim \sum_{\vec{Q}\in\mathbb{P}_{n_1,n_2,n_3}: I_{\vec{Q}}=I_{P_0}} \langle |\hat{f}_2|, \tilde{1}_{\omega_{Q_2}}\rangle \leq ||\hat{f}_2||_1.$$

□

It follows that $\sum_{\vec{P}\in\mathbb{P}_{n_1,n_2,n_3}} |I_{\vec{P}}| \lesssim 2^{n_2}||\hat{f}_2||_1 \cdot \sum_{I\in\mathbb{I}_{n_1,n_2,n_3}} | I |$ where

$$\mathbb{I}_{n_1,n_2,n_3} := \{I \in \mathbb{D} : \exists \vec{P} \in \mathbb{P}_{n_1,n_2,n_3} \ s.t. \ I = I_{\vec{P}}\}$$

and $\mathbb{D}$ is the collection of dyadic intervals. Moreover, for every $I \in \mathbb{I}_{n_1,n_2,n_3}$,

$$\sum_{I\in\mathbb{I}_{n_1,n_2,n_3}} |I| \leq \sum_{I\in\mathbb{I}_{n_3}} |I| \lesssim 2^{n_3} \sum_{I\in\mathbb{I}_{n_3}} \langle 1_{E_3}, \tilde{1}_I\rangle \lesssim 2^{n_3}|E_3|.$$

Putting it all together, we have the additional estimate $\sum_{\vec{P}\in\mathbb{P}_{n_1,n_2,n_3}} |I_{\vec{P}}| \lesssim 2^{n_2}2^{n_3}|E_2||E_3|$, which enables us to write down for any $(\theta_1, \theta_2, \theta_3)$ subject to the requirement $0 \leq \theta_1, \theta_2, \theta_3 \leq 1$ and $\theta_1 + \theta_2 + \theta_3 = 1$

$$|\Lambda(f_1, f_2, f_3)| \lesssim \sum_{n_1,n_2,n_3} 2^{-n_1(1-2\theta_1)}2^{-n_2(1-\theta_2-2\theta_3)}2^{-n_3(1-\theta_2)}|E_1|^{\theta_1}|E_2|^{\theta_2+\theta_3}|E_3|^{\theta_2}.$$

For summability, we must impose the additional requirement that $\theta_1 < 1/2$ and $\theta_2 + 2\theta_3 < 1$.

2.11.1 Restricted Type Estimates

By rescaling, we may assume $|E_3| = 1$. Note that the natural size restrictions are then $2^{-n_1} \lesssim 2^d|E_1|^\alpha$ for any $0 \leq \alpha \leq 1$, $2^{-n_2} \lesssim 2^d|E_2|^{1/2}$, and $2^{-n_3} \lesssim 2^{-\tilde{N}d}$. Fixing $(\theta_1, \theta_2, \theta_3)$ satisfying $0 \leq \theta_1, \theta_2, \theta_3 \leq 1$ and $\theta_1 + \theta_2 + \theta_3 = 1$ with $\theta_1 < 1/2$, the summation gives

$$\begin{aligned} &|\Lambda(f_1, f_2, f_3)| \\ \lesssim \ &\sum_{d\geq 0} \sum_{n_1\geq N_1(d)} \sum_{n_2\geq N_2(d)} \sum_{n_3\geq N_3(d)} 2^{-n_1(1-2\theta_1)}2^{-n_2(1-\theta_2-2\theta_3)}2^{-n_3(1-\theta_2)}|E_1|^{\theta_1}|E_2|^{\theta_2+\theta_3} \end{aligned}$$

Provided $\theta_1 < 1/2, \theta_2 < 1, \theta_2 + 2\theta_3 < 1$, we have the upper bound

$$|\Lambda(f_1, f_2, f_3)| \lesssim \min\left\{|E_1|^{\theta_1}, |E_1|^{1-\theta_1}\right\}|E_2|^{1/2+\theta_2/2}.$$

To produce restricted weak type estimates in a neighborhood of $(1/2, 1/2, 0)$, set $\theta_1 = 1/2 - \epsilon, \theta_2 = \epsilon, \theta_3 = 1/2 - \epsilon$. To do the same in neighborhoods of $(1, 0, 0)$ and $(0, 0, 1)$ use $\theta_1 = 2\epsilon, \theta_2 = 1 - 3\epsilon, \theta_3 = \epsilon$. By interpolation, it suffices to prove estimates in a neighborhood of $(0, 1/2, 1/2)$. To this end, assume $|E_1| = 1$ and the exceptional set Ω attached to f_1 satisfies

$$\Omega \supset \left\{Mf_2 \geq C|E_2|^{1/2}\right\} \bigcup \left\{M1_{E_3} \geq C|E_3|\right\}.$$

As before, this exceptional set will have an acceptable size provided C is sufficiently large. The natural size restrictions are $2^{-n_3} \lesssim 2^d|E_3|^\alpha$ for any $0 \leq \alpha \leq 1$, $2^{-n_2} \lesssim 2^d|E_2|^{1/2}$, and $2^{-n_1} \lesssim 2^{-\tilde{N}d}$. A similar calculation then yields for $\theta_1 < 1/2, \theta_2 < 1, \theta_2 + 2\theta_3 < 1$,

$$|\Lambda(f_1 1_{\Omega^c}, f_2, f_3)| \lesssim |E_2|^{1/2+\theta_2/2} \min\left\{|E_3|^{\theta_2}, |E_3|^{1-\theta_2}\right\}.$$

Choosing $\theta_1 = 1/2 - \epsilon, \theta_2 = 3\epsilon, \theta_3 = 1/2 - 2\epsilon$ yields the desired estimate for $\Lambda(f_1, f_2, f_3)$ near $(0, 1/2, 1/2)$ and therefore, by interpolation, produces the desired mixed estimates for T_{m_0}.

□

2.12 Mixed Estimates for the Generic Hilbert Transform in the Plane

The proceeding argument uses the fact that each tree consists of only one tile and the number of scale-1 tiles stacking on top of each other is limited by the relevant size parameter. The problem with extending this line of argument to the general case is that for a given strongly-disjoint collection of trees $\mathbb{T}_{3,n_3}$, there is no reasonable bound for $\left|\left|\sum_{T\in\mathbb{T}_{3,n_3}} 1_{I_T}\right|\right|_\infty$. Hence, the proof has to move from stacking of trees to stacking of individual tiles above (and below) a certain *point* in time. Indeed, strong disjointness ensures that at each time the frequency projections of the relevant tiles with time concentration $I_{\vec{P}}$ intersecting a shared point are all disjoint. Now, we prove mixed estimates for generic bilinear degenerate symbols.

Theorem 18. *Let* $m:\mathbb{R}^2\to\mathbb{R}$ *be adapted to* $\Gamma=\{(\xi_1,\xi_2):\xi_1+\xi_2=0\}\subset\mathbb{R}^2$ *in the Mikhlin-Hörmander sense* $|\partial^{\vec{\alpha}}m(\vec{\xi})|\lesssim_{\vec{\alpha}}\frac{1}{dist(\vec{\xi},\Gamma)^{|\vec{\alpha}|}}$. *Then* $T_m:L^{p_1}(\mathbb{R})\times W_{p_2}(\mathbb{R})\to L^{\frac{p_1p_2}{p_1+p_2}}(\mathbb{R})$ *whenever*

$$\frac{1}{p_1}+\frac{1}{p_2}<1, 2<p_2<\infty.$$

Proof. The method for proving estimates for the model of $C^{1,1,-2}$ exploited the fact that we had estimates for low $BHT-$type variations mapping near L^1. This feature most naturally arises in the multilinear setting where one has multiple functions to work with. However, in the mixed setting, one does not have this luxury. The matter is not completely hopeless, as Hausdorff-Young variations below l^2 are bounded on suitably high L^p spaces, which is better than the pure L^p case where $r>2$ is a necessity for *any* estimates; however, the price one pays is that the mixed $r-$variation maps into L^p only when $p > r'$. Therefore, as r approaches 1, the permissible p values approach ∞. This places us somewhat far from the methods employed to handle $C^{1,1,-2}$ and $C^{1,1,1,-2}$. Instead, it is useful to fix the parameter n_3 which corresponds to a decomposition of $\mathbb{R}$ into disjoint intervals and then to define sizes in the other indices localized to this partition. The advantage of restricting to these time intervals will become apparent in the energy estimates. As should now be apparent, the index with respect to which we want to apply the low-variational estimates must be sharply localized. So, suppose $f_2 \in W_{p_2}(\mathbb{R})$. Then, modulo the usual details, it suffices to bound

$$\Lambda(f_1, f_2, f_3) := \sum_{\vec{Q} \in \mathbb{Q}_d} \frac{\langle f_1, \Phi^1_{Q_1} \rangle \langle f_2 * \eta_{\omega_{Q_2}}, \tilde{\Phi}^2_{Q_2} \rangle \langle f_3, \Phi^{3,lac}_{\vec{Q}} \rangle}{|I_{\vec{Q}}|^{1/2}}.$$

Setting

$$Size_3(f_3) = \sup_{I \subset \mathbb{R}} \frac{1}{|I|} \int_{\mathbb{R}} 1_{E_3}(x) \tilde{1}_I(x) dx,$$

we may obtain for each n_3 a collection of disjoint dyadic intervals $\mathbb{I}_{n_3}$ with the property that

$$\frac{1}{|I|} \int_I |f_3(x)| dx \gtrsim 2^{-n_3}$$

for every $I \in \mathbb{I}_{n_3}$ and such that each I is maximal with respect the set of all dyadic intervals enjoying the above property. Therefore, for every $I \in \mathbb{I}_{n_3}$ and $n_3 < m_3$, there exists a unique $J \in \mathbb{I}_{m_3}$ such that $I \subseteq J$. This filtration then gives rise to a partition of bi-tiles in the natural way: note that for every $d \geq 0$ there is an integer $n_0(d)$ such that for no integer $n_3 < n_0(d)$ is there a dyadic interval I with the property that $I \subset J$ for some $J \in \mathbb{I}_{n_3}$ and $1 + \frac{dist(I, \Omega^c)}{|I|} \simeq 2^d$. Indeed, if $I \subset J$ with $J \in \mathbb{I}_{n_3}$, then

$$
\begin{aligned}
2^{-n_3} &\lesssim \frac{1}{|J|}\int_{\mathbb{R}} 1_{E_3}(x)\tilde{1}_J(x)dx \\
&\lesssim 2^d \inf_{x\in 2^d J} M1_{E_3}(x) \\
&\leq 2^d \inf_{x\in 2^d I} M1_{E_3}(x) \\
&\leq 2^d \sup_{x\in \Omega^c} M1_{E_3}(x).
\end{aligned}
$$

It is routine to control the above display by an acceptable quantity by enlarging our exceptional set Ω. Therefore, we may start our decomposition at $n_3 \geq n_0(d)$. Let

$$
\begin{aligned}
\mathbb{Q}_{d,n_0(d)} &:= \left\{\vec{Q}\in\mathbb{Q} : I_{\vec{Q}} \subset \bigcup_{I\in\mathbb{I}_{n_0(d)}} I\right\} \\
\mathbb{Q}_{d,n_0(d)+1} &:= \left\{\vec{Q}\in\mathbb{Q}\cap\mathbb{Q}^c_{3,n_0(d)} : I_{\vec{Q}} \subset \bigcup_{I\in\mathbb{I}_{n_0(d)+1}} I\right\}.
\end{aligned}
$$

Inductively define $\mathbb{Q}_{d,m} := \left\{\vec{Q}\in\mathbb{Q}\cap\left[\bigcup_{n\leq m-1}\mathbb{Q}_{d,n}\right]^c : I_{\vec{Q}}\subset\bigcup_{I\in\mathbb{I}_m} I\right\}$. Therefore,

$$
\mathbb{Q} = \bigcup_{n\geq n_0(d)} \mathbb{Q}_{d,n}
$$

where the union is disjoint and for every tree T consisting of bi-tiles in $\mathbb{Q}_{3,n}$ sitting in some $I\in\mathbb{I}_n$

$$
\left(\frac{\sum_{\vec{Q}\in T}|\langle f_3, \Phi^{3,lac}_{\vec{Q}}\rangle|^2}{|I_T|}\right) \lesssim \sup_{\vec{Q}\in T}\frac{1}{|I_{\vec{Q}}|}\int_{\mathbb{R}}|f_3(x)|\tilde{1}_{I_{\vec{Q}}}(x)dx \lesssim 2^{-n}.
$$

Now, for each collection of disjoint dyadic intervals $\mathbb{I}_n$, we construct adapted sizes in the first and second indices. For a given collection $\mathbb{I}$ and collection of bi-tiles $\tilde{\mathbb{Q}}$

for which $I_{\vec{Q}} \subset \bigcup_{I \in \mathbb{I}} I$, let

$$Size_1^{\mathbb{I}}(\tilde{\mathbb{Q}}) := \sup_{T \subset \tilde{\mathbb{Q}} : I_T \subset \bigcup_{I \in \mathbb{I}} I} \frac{1}{|I_T|^{1/2}} \left(\sum_{Q \in T} |\langle f_1, \Phi^1_{Q_1} \rangle|^2 \right)^{1/2}.$$

Next, we define

$$Size_2^{\mathbb{I}}(\tilde{\mathbb{Q}}) := \sup_{T \subset \tilde{\mathbb{Q}} : I_T \subset \bigcup_{I \in \mathbb{I}_{n_3}} \mathbb{I}} \frac{1}{|I_T|^{1/2}} \left(\sum_{\vec{Q} \in T} \left| \langle f_2 * \eta_{\omega_{Q_2}}, \tilde{\Phi}^2_{Q_2} \rangle \right|^2 \right)^{1/2}.$$

This naturally gives rise to a decomposition of the tiles $\mathbb{Q}$ into a disjoint union of sub collections $\mathbb{Q}_{3,n_3}$, where each $\vec{Q} \in \mathbb{Q}_{3,n_3}$ has the property that $I_{\vec{Q}} \subset \bigcup_{I \in \mathbb{I}} I$.

$$\begin{aligned} & \sum_{\vec{Q} \in \mathbb{Q}_d} \frac{|\langle f_1, \Phi^1_{Q_1} \rangle| |\langle f_2 * \eta_{\omega_{Q_2}}, \tilde{\Phi}^2_{Q_2} \rangle| |\langle f_3, \Phi^{3,lac}_{\vec{Q}} \rangle|}{|I_{\vec{Q}}|^{1/2}} \\ = & \sum_{n_3} \sum_{\vec{Q} \in \mathbb{Q}_d \cap \mathbb{Q}_{3,n_3}} \frac{\langle f_1, \Phi^1_{Q_1} \rangle \langle f_2 * \eta_{\omega_{Q_2}}, \tilde{\Phi}^2_{Q_2} \rangle |\langle f_3, \Phi^{3,lac}_{\vec{Q}} \rangle|}{|I_{\vec{Q}}|^{1/2}}. \end{aligned}$$

As usual, we may break down each subcollection $\mathbb{Q}_{3,n_3}$ according to the same BHT type stopping time argument now done with respect to the new localized sizes. Clearly, the strongly disjoint trees can be grouped according to the interval $I \in \mathbb{I}_{n_3}$ containing the top of the tree I_T. Denote this collection of trees $\mathbb{T}_{1,n_1}(\mathbb{Q}_{3,n_3})[I]$ and the collection of bi-tiles $(\mathbb{Q}_{3,n_3})^{n_2}_{n_1}(I)$. Putting it all together therefore yields

$$\mathbb{Q} = \bigcup_{n_3} \mathbb{Q}_{3,n_3} = \bigcup_{n_3} \left[\bigcup_{n_1,n_2} (\mathbb{Q}_{3,n_3})_{n_1,n_2} \right] = \bigcup_{n_3} \left[\bigcup_{n_1,n_2} \bigcup_{I \in \mathbb{I}_{n_3}} (\mathbb{Q}_{3,n_3})^{n_2}_{n_1}(I) \right]$$

(One sorts the tiles on each interval $I \in \mathbb{I}_{n_3}$ separately.) Each set $(\mathbb{Q}_{3,n_3})_{n_1,n_2}(I)$ can be further decomposed into a collection of disjoint trees (modulo harmless modifications). Putting it all together yields

$$
\begin{aligned}
&\sum_{\vec{Q}\in\mathbb{Q}_d\cap\mathbb{Q}_{3,n_3}} \frac{|\langle f_1,\Phi^1_{Q_1}\rangle||\langle f_2*\eta_{\omega_{Q_2}},\tilde{\Phi}^2_{Q_2}\rangle||\langle f_3,\Phi^{3,lac}_{\vec{Q}}\rangle|}{|I_{\vec{Q}}|^{1/2}} \\
= &\sum_{\vec{Q}\in\mathbb{Q}_d\cap(\mathbb{Q}_{3,n_3})^{n_2}_{n_1}} \frac{|\langle f_1,\Phi^1_{Q_1}\rangle||\langle f_2*\eta_{\omega_{Q_2}},\tilde{\Phi}^2_{Q_2}\rangle||\langle f_3,\Phi^{3,lac}_{\vec{Q}}\rangle|}{|I_{\vec{Q}}|^{1/2}} \\
= &\sum_{I\in\mathbb{I}_{n_3}}\sum_{n_1,n_2}\sum_{\vec{Q}\in\mathbb{Q}_d\cap(\mathbb{Q}_{3,n_3})^{n_2}_{n_1}(I)} \frac{|\langle f_1,\Phi^1_{Q_1}\rangle||\langle f_2*\eta_{\omega_{Q_2}},\tilde{\Phi}^2_{Q_2}\rangle||\langle f_3,\Phi^{3,lac}_{\vec{Q}}\rangle|}{|I_{\vec{Q}}|^{1/2}}.
\end{aligned}
$$

2.12.1 Degenerate Tree Estimate

If T is a $2-tree$ in $(\mathbb{Q}_{3,n_3})_{n_1}(I)$, then

$$
\begin{aligned}
&\sum_{\vec{Q}\in T} \frac{|\langle f_1,\Phi^1_{Q_1}\rangle||\langle f_2*\eta_{\omega_{Q_2}},\tilde{\Phi}^2_{Q_2}\rangle||\langle f_3,\Phi^{3,lac}_{\vec{Q}}\rangle|}{|I_{\vec{Q}}|^{1/2}} \\
\leq\ &|I_T|\frac{\left(\sum_{\vec{Q}\in T}|\langle f_1,\Phi^1_{Q_1}\rangle|^2\right)^{1/2}}{|I_T|^{1/2}}\left[\sup_{\vec{Q}\in T}\frac{\langle f_2*\eta_{\omega_{Q_2}},\tilde{\Phi}^{2,\infty}_{Q_2}\rangle}{|I_{\vec{Q}}|}\right]\frac{\left(\sum_{\vec{Q}\in T}|\langle f_3,\Phi^{3,lac}_{\vec{Q}}\rangle|^2\right)^{1/2}}{|I_T|^{1/2}} \\
\lesssim\ &|I_T|2^{-n_1}2^{-n_2}2^{-n_3}.
\end{aligned}
$$

If T is a $1-tree$ in $(\mathbb{Q}_{3,n_3})^{n_2}(I)$, then

$$
\begin{aligned}
&\sum_{\vec{Q}\in T} \frac{|\langle f_1,\Phi^1_{Q_1}\rangle||\langle f_2*\eta_{\omega_{Q_2}},\tilde{\Phi}^2_{Q_2}\rangle||\langle f_3,\Phi^{3,lac}_{\vec{Q}}\rangle|}{|I_{\vec{Q}}|^{1/2}} \\
\leq\ &|I_T|\left[\sup_{\vec{Q}\in T}\frac{|\langle f_1,\Phi^{1,\infty}_{Q_1}\rangle|}{|I_{\vec{Q}}|}\right]\left(\frac{\sum_{\vec{Q}\in T}\langle f_2*\eta_{\omega_{Q_2}},\tilde{\Phi}^2_{Q_2}\rangle^2}{|I_T|}\right)^{1/2}\left(\frac{\sum_{\vec{Q}\in T}|\langle f_3,\Phi^{3,lac}_{\vec{Q}}\rangle|^2}{|I_T|}\right)^{1/2} \\
\lesssim\ &|I_T|2^{-n_1}2^{-n_2}2^{-n_3}.
\end{aligned}
$$

2.12.2 Degenerate Energy Estimate

It is routine to observe

$$\sum_{I\in\mathbb{I}_{n_3}}\sum_{T\in\mathbb{T}_{1,n_1}(\mathbb{Q}_{3,n_3})[I]}|I_T|\lesssim 2^{2n_1}|E_1|;\sum_{I\in\mathbb{I}_{n_3}}\sum_{T\in\mathbb{T}_{2,n_2}(\mathbb{Q}_{2,n_2})[I]}|I_T|\lesssim 2^{2n_2}|E_2|$$

This follows from the usual TT^* argument and the fact that collection $\bigcup_{I\in\mathbb{I}_{n_3}}\bigcup_{T\in\mathbb{T}_{1,n_1}(I)}\{T\},\bigcup_{I\in\mathbb{I}_{n_3}}\bigcup_{T\in\mathbb{T}_{2,n_2}(I)}\{T\}$ are both strongly-disjoint collections of trees. However, we have the following additional story:

$$\begin{aligned}
&\sum_{I\in\mathbb{I}_{n_3}}\sum_{T\in\mathbb{T}_{2,n_2}(\mathbb{Q}_{2,n_2})[I]}|I_T|\\
\lesssim\;& 2^{2n_2}\sum_{I\in\mathbb{I}_{n_3}}\sum_{T\in\mathbb{T}_{2,n_2}(\mathbb{Q}_{2,n_2})[I]}|\langle f_2*\eta_{\omega_{Q_2}},\tilde{\Phi}^2_{Q_2}\rangle|^2\\
\lesssim\;& 2^{2n_2}\sum_{I\in\mathbb{I}_{n_3}}\sum_{T\in\mathbb{T}_{2,n_2}(\mathbb{Q}_{2,n_2})[I]}|\langle f_2*\eta_{\omega_{Q_2}},\tilde{\Phi}^{2,\infty}_{Q_2}\rangle|\left[\sup_{T\in\mathbb{T}_{2,n_2}(\mathbb{Q}_{2,n_2})[I]}\sup_{\vec{Q}\in T}\left|\langle f_2,\Phi^{2,1}_{Q_2}\rangle\right|\right]\\
\lesssim\;& 2^{n_2}\sum_{I\in\mathbb{I}_{n_3}}\sum_{T\in\mathbb{T}_{2,n_2}(I)}\sum_{\vec{Q}\in T}\left|\left|f_2*\eta_{\omega_{Q_2}}\right|\right|_{L^\infty(\mathbb{R})}|I_{\vec{Q}}|\\
\leq\;& 2^{n_2}\sup_{I\in\mathbb{I}_{n_3}}\left[\int_{\mathbb{R}}\sum_{T\in\mathbb{T}_{2,n_2}(I)}\sum_{\vec{Q}\in T}\left|\left|\hat{f}_2\hat{\eta}_{\omega_{Q_2}}\right|\right|_{L^1(\mathbb{R})}1_{I_{\vec{Q}}}dx\right]\left[\sum_{I\in\mathbb{I}_{n_3}}|\ I\ |\right]\\
\lesssim\;& 2^{n_2}2^{n_3}||\hat{f}_2||_1|E_3|\\
=\;& 2^{n_2}2^{n_3}|E_2|\cdot|E_3|.
\end{aligned}$$

Therefore,

$$\begin{aligned}
&|\Lambda(f_1,f_2,f_3)|\\
\leq\;&\sum_{d,n_1,n_2,n_3}|\Lambda_{d,n_1,n_2,n_3}(f_1,f_2,f_3)|\\
\lesssim\;&\sum_{d\geq 0}\sum_{n_1\geq N_1(d)}\sum_{n_2\geq N_2(d)}\sum_{n_3\geq N_3(d)}2^{-n_1}2^{-n_2}2^{-n_3}\min\left\{2^{2n_1}|E_1|,2^{n_2}2^{n_3}|E_2|\cdot|E_3|,2^{2n_2}|E_2|\right\}.
\end{aligned}$$

2.12.3 Restricted Type Type Estimates

By scaling invariance, assume $|E_3| = 1$. By enlarging Ω if necessary, we may ensure for fixed $\alpha >> 1$ that

$$\Omega \supset \{M1_{E_1} \gtrsim |E_1|\} \bigcup \left\{Mf_2 \gtrsim |E_2|^{1/2}\right\}.$$

is an acceptable exceptional set large enough implicit constants. Note that the natural size restrictions are then $2^{-n_1} \lesssim 2^d |E_1|^\alpha$ for any $0 \leq \alpha \leq 1$, $2^{-n_2} \lesssim 2^d |E_2|^{1/2}$, and $2^{-n_3} \lesssim 2^{-\tilde{N}d}$. Fixing $(\theta_1, \theta_2, \theta_3)$ satisfying $0 \leq \theta_1, \theta_2, \theta_3 \leq 1$ and $\theta_1 + \theta_2 + \theta_3 = 1$ with $\theta_1 < 1/2$, the summation gives

$$\begin{aligned} & |\Lambda(f_1, f_2, f_3)| \\ \lesssim & \sum_{d \geq 0} \sum_{n_1 \geq N_1(d)} \sum_{n_2 \geq N_2(d)} \sum_{n_3 \geq N_3(d)} 2^{-n_1(1-2\theta_1)} 2^{-n_2(1-\theta_2-2\theta_3)} 2^{-n_3(1-\theta_2)} |E_1|^{\theta_1} |E_2|^{\theta_2+\theta_3} \end{aligned}$$

Provided $\theta_1 < 1/2, \theta_2 < 1, \theta_2 + 2\theta_3 < 1$, we have the upper bound

$$|\Lambda(f_1, f_2, f_3)| \lesssim \min\left\{|E_1|^{\theta_1}, |E_1|^{1-\theta_1}\right\} |E_2|^{1/2+\theta_2/2}.$$

To produce restricted weak type estimates in a neighborhood of $(1/2, 1/2, 0)$, set $\theta_1 = 1/2 - \epsilon, \theta_2 = \epsilon, \theta_3 = 1/2 - \epsilon$. To do the same in neighborhoods of $(1, 0, 0)$ and $(0, 0, 1)$ use $\theta_1 = 2\epsilon, \theta_2 = 1 - 3\epsilon, \theta_3 = \epsilon$. By interpolation, it suffices to prove estimates in a neighborhood of $(0, 1/2, 1/2)$. To this end, assume $|E_1| = 1$ and the exceptional set Ω attached to f_1 satisfies

$$\Omega \supset \left\{Mf_2 \gtrsim |E_2|^{1/2}\right\} \bigcup \{M1_{E_3} \gtrsim |E_3|\}.$$

As before, this exceptional set will have an acceptable size provided the implicit constants appearing in the above display are taken sufficiently large. The natural size restrictions are $2^{-n_3} \lesssim 2^d |E_3|^\alpha$ for any $0 \le \alpha \le 1$, $2^{-n_2} \lesssim 2^d |E_2|^{1/2}$, and $2^{-n_1} \lesssim 2^{-\tilde{N}d}$. A similar calculation as before yields for $\theta_1 < 1/2, \theta_2 < 1, \theta_2 + 2\theta_3 < 1$,

$$|\Lambda(f_1 1_{\Omega^c}, f_2, f_3)| \lesssim |E_2|^{1/2+\theta_2/2} \min\left\{|E_3|^{\theta_2}, |E_3|^{1-\theta_2}\right\}.$$

Choosing $\theta_1 = 1/2 - \epsilon, \theta_2 = 3\epsilon, \theta_3 = 1/2 - 2\epsilon$ yields the desired estimate for $\Lambda(f_1, f_2, f_3)$ near $(0, 1/2, 1/2)$ and therefore, by interpolation, the desired mixed estimates for T_m.

□

We should remark that the symmetry of trilinear-form associated to the BHT is not present in the degenerate case. Now, there is one "bad" index which does not provide frequency localization in the other indices. In other words, knowing the projection of a degenerate tri-tile onto the "bad" index does not uniquely determine the tri-tile. Also, note that estimates of the form $T_m : L^{p_1}(\mathbb{R}) \times W_{p_2}(\mathbb{R}) \to L^{\frac{p_1 p_2}{p_1+p_2}}(\mathbb{R})$ for $2 < p_2 \le p_1$ required the exceptional set to be attached to the function in the 1st slot, which does provide frequency localization for other indices and is therefore not "morally" speaking equivalent to the 3rd.

2.13 Generic Hilbert Transform Estimates $\Longrightarrow$ Bad Biest Estimates

Recall that for $\vec{\epsilon} \in (\mathbb{R} - \{0\})^n$,

$$C^{\vec{\epsilon}} : (f_1, ..., f_n) \mapsto \int_{\xi_1 < ... < \xi_n} \left[\prod_{j=1}^{n} \hat{f}_j(\xi_j) e^{2\pi i x \epsilon_j \xi_j} \right] \, d\vec{\xi}.$$

By a simple change of variable, it is easy to observe that $\tilde{C}^{-1,1,-1}$ defined by

$$\tilde{C}^{1,-1,1}(f_1, f_2, f_3)(x) = \int_{-\xi_1 < \xi_2 < -\xi_3} \hat{f}_1(\xi_1) \hat{f}_2(\xi_2) \hat{f}_3(\xi_3) e^{2\pi i x(\xi_1 + \xi_2 + \xi_3)} d\xi_1 d\xi_2 d\xi_3.$$

satisfies a given L^p estimate if and only if $C^{1,-1,1}$ satisfies the same L^p estimate. Our goal in this short section is to establish the following (somewhat informal) result:

Proposition 6. *Assume for every* $p_1, p_2 : 1 < p_1, p_2, \frac{p_1 p_2}{p_1 + p_2} < \infty$ *and* $m : \mathbb{R}^2 \to \mathbb{C}$ *satisfying (for sufficiently many multi-indices* $\vec{\alpha}$*)* $\left| \partial^{\vec{\alpha}} m(\vec{\xi}) \right| \leq \frac{C}{dist(\vec{\xi}, \Gamma)^{|\vec{\alpha}|}}$ *for all* $\vec{\xi} \in \mathbb{R}^2$ *and* $\Gamma = \{\xi \in \mathbb{R}^2 : \xi_1 + \xi_2 = 0\}$*, that the bilinear operator* T_m *initially defined on Schwartz functions* f_1, f_2 *by*

$$T_m(f_1, f_2)(x) = \int_{\mathbb{R}^2} m(\xi_1, \xi_2) \hat{f}_1(\xi_1) \hat{f}_2(\xi_2) e^{2\pi i x(\xi_1 + \xi_2)} d\xi_1 d\xi_2$$

satisfies the estimate $L^{p_1}(\mathbb{R}) \times L^{p_2}(\mathbb{R}) \to L^{\frac{p_1 p_2}{p_1 + p_2}}(\mathbb{R})$*. Moreover, assume the operatorial bound on* T_m *depends only on the constant* C *arising in the decay of the multi-derivatives of* m *and* (p_1, p_2)*. Then the bad Biest* $C^{1,-1,1}$ *would satisfy some* L^p *estimates.*

Hence, because the bad Biest $C^{1,-1,1}$ satisfies no L^p estimates, we obtain an indirect proof that the generic Hilbert transform of a product of two functions cannot satisfy all the L^p estimates enjoyed by the Hilbert transform of the product of two functions itself. However, the proof of Proposition 6 gives us little insight into what the unbounded multiplier looks like and which functions witness the unboundedness. Fortunately, we have already provided an explicit counterexample in §2.

Proof. It suffices to prove the conclusion with $\tilde{C}^{1,-1,1}$ instead of $C^{1,-1,1}$. To begin, we follow the lead of Muscalu, Tao, and Thiele in [21] and decompose $m(\xi_1, \xi_2, \xi_3) = 1_{-\xi_1<\xi_2<-\xi_3}(\xi_1, \xi_2, \xi_3)$ into a sum of multipliers $m = m_{\mathcal{R}_1} + m_{\mathcal{R}_2} + m_{\mathcal{R}_3}$ localized to 3 distinct regions

$$
\begin{aligned}
\mathcal{R}_1 &:= \left\{\vec{\xi} \in \mathbb{R}^3 : -\xi_1 < \xi_2 < -\xi_3,\ |\xi_1 + \xi_2| \simeq |\xi_2 + \xi_3|\right\} \\
\mathcal{R}_2 &:= \left\{\vec{\xi} \in \mathbb{R}^3 : -\xi_1 < \xi_2 < -\xi_3,\ |\xi_1 + \xi_2| << |\xi_2 + \xi_3|\right\} \\
\mathcal{R}_3 &:= \left\{\vec{\xi} \in \mathbb{R}^3 : -\xi_1 < \xi_2 < -\xi_3,\ |\xi_1 + \xi_2| >> |\xi_2 + \xi_3|\right\}
\end{aligned}
$$

where the precise values of the implicit constants appearing in the above construction are not important. It turns out that $m_{\mathcal{R}_1}$ satisfies a wide range of L^p estimates from [18], so to prove estimates for the Bad Biest, it would be enough to prove L^p estimates for generic $m_{\mathcal{R}_2}$, in which case generic $m_{\mathcal{R}_3}$ would also be bounded by symmetry. By standard techniques, it is possible to write the 4-form associated to a generic multiplier $m_{\mathcal{R}_2}$ as

$$
\begin{aligned}
&\Lambda_{m_{\mathcal{R}_2}}(f_1,f_2,f_3,f_4)\\
= &\sum_{|Q|<<|P|}\int_{\mathbb{R}} f_1*\eta_Q f_2*\eta_{-\tilde{Q}}*\eta_{-\tilde{P}} f_3*\eta_P f_4*\eta_{-P}\ dx\\
= &\sum_{|Q|<<|P|}\int_{\mathbb{R}} f_1*\eta_Q*\eta_P f_2*\eta_{-\tilde{Q}}*\eta_{-\tilde{P}} f_3*\eta_P f_4*\eta_{-P}\ dx\\
\simeq &\sum_{|Q_a|=|Q_b|<<|P|}\int_{\mathbb{R}} f_1*\eta_{Q_a}*\eta_P f_2*\eta_{-\tilde{Q}_a}*\eta_{-\tilde{P}} f_3*\eta_P*\eta_{Q_b} f_4*\eta_{-P}*\eta_{-\tilde{Q}_b}\ dx,
\end{aligned}
$$

where the last sum is over all intervals (Q_a, Q_b, P) with $|Q_a| = |Q_b| << |P|$ in some triple of shifted dyadic grids. Moreover, for any shifted dyadic interval, $\tilde{I}$ is the shifted dyadic interval of I, i.e. $\tilde{I} = I + C|I|$ for some fixed absolute constant $|C| >> 1$. We now want to use the fact that the $P-$sum can be eliminated because we have localized all the indices to frequency scales at most the frequency length of P. Therefore, we are left facing

$$
\begin{aligned}
&\Lambda_{m_2}(f_1,f_2,f_3,f_4)\\
\simeq &\sum_{Q_a,Q_b:|Q_a|=|Q_b|}\int_{\mathbb{R}} f_1*\eta_{Q_a} f_2*\eta_{-\tilde{Q}_a} f_3*\eta_{Q_b} f_4*\eta_{-\tilde{Q}_b}\ dx\\
= &\sum_{k\in\mathbb{Z}}\int_{\mathbb{R}}\left(\sum_{|Q_a|=2^k} f_1*\eta_{Q_a} f_2*\eta_{-\tilde{Q}_a}\right)\left(\sum_{|Q_b|=2^k} f_3*\eta_{Q_b} f_4*\eta_{-\tilde{Q}_b}\right)\ dx\\
\leq &\int_{\mathbb{R}}\left(\sum_{k\in\mathbb{Z}}\left|\sum_{|Q_a|=2^k} f_1*\eta_{Q_a} f_2*\eta_{-\tilde{Q}_a}\right|^2\right)^{1/2}\left(\sum_{k\in\mathbb{Z}}\left|\sum_{|Q_b|=2^k} f_3*\eta_{Q_b} f_4*\eta_{-\tilde{Q}_b}\right|^2\right)^{1/2} dx.
\end{aligned}
$$

At this point, it suffices to use Hölder's and Khintchine's inequalities along with the assumption that all multipliers of Hilbert transform type are bounded uniformly with respect to the decay parameters. This contradiction establishes the claim. □

2.14 LWL-Type Mixed Estimates for $B[a_1, a_2]$

Recall that we were able to prove $W_{p_1}(\mathbb{R}) \times L^{p_2}(\mathbb{R}) \times W_{p_3}(\mathbb{R})$-type estimates for $C^{-1,1,-1}$ and its generic versions using Christ-Kiselev-Paley decompositions, Whitney decompositions, and some elementary square and maximal function estimates. We also have many $LWL-$ type mixed estimates for $C^{-1,1,-1}$. It turns out that one needs to work a bit more to obtain LWL-type mixed estimates for generic versions of $C^{-1,1,-1}$. Our goal in this section is to show the following:

Theorem 12. *$B[a_1, a_2] : L^{p_1}(\mathbb{R}) \times W_{p_2}(\mathbb{R}) \times L^{p_3}(\mathbb{R}) \to L^{\frac{1}{\frac{1}{p_1}+\frac{1}{p_2}+\frac{1}{p_3}}}(\mathbb{R})$ provided*

$$1 < p_1, p_2 < \infty, \frac{1}{p_1} + \frac{1}{p_2} < 1, \frac{1}{p_2} + \frac{1}{p_3} < 1, 2 < p_2 < \infty.$$

In particular, $B[a_1, a_2]$ has mixed estimates into $L^r(\mathbb{R})$ for all $1/2 < r < \infty$.

Proof.

2.14.1 Discretization

To obtain mixed estimates for

$$B[a_1, a_2] : (f_1, f_2, f_3) \mapsto \int_{\mathbb{R}^3} a_1(\xi_1, \xi_2) a_2(\xi_2, \xi_3) \hat{f}_1(\xi_1) \hat{f}_2(\xi_2) \hat{f}_3(\xi_3) e^{2\pi i x(\xi_1+\xi_2+\xi_3)} d\vec{\xi},$$

we adopt the general philosophy applied in [21] to deal with the Biest, which is to carve the multiplier into various regions and then proceed to treat each region individually. These regions are

$$\begin{aligned}
\mathcal{R}_1^{1,1} &= \{-\xi_1 < \xi_2, \xi_2 < -\xi_3, |\xi_1+\xi_2| << |\xi_2+\xi_3|\} \\
\mathcal{R}_1^{2,1} &= \{-\xi_1 > \xi_2, \xi_2 < -\xi_3, |\xi_1+\xi_2| << |\xi_2+\xi_3|\} \\
\mathcal{R}_1^{1,2} &= \{-\xi_1 < \xi_2, \xi_2 > -\xi_3, |\xi_1+\xi_2| << |\xi_2+\xi_3|\} \\
\mathcal{R}_1^{2,2} &= \{-\xi_1 > \xi_2, \xi_2 > -\xi_3, |\xi_1+\xi_2| << |\xi_2+\xi_3|\} \\
\mathcal{R}_2^{1,1} &= \{-\xi_1 < \xi_2, \xi_2 < -\xi_3, |\xi_1+\xi_2| \simeq |\xi_2+\xi_3|\} \\
\mathcal{R}_2^{2,1} &= \{-\xi_1 > \xi_2, \xi_2 < -\xi_3, |\xi_1+\xi_2| \simeq |\xi_2+\xi_3|\} \\
\mathcal{R}_2^{1,2} &= \{-\xi_1 < \xi_2, \xi_2 > -\xi_3, |\xi_1+\xi_2| \simeq |\xi_2+\xi_3|\} \\
\mathcal{R}_2^{2,2} &= \{-\xi_1 > \xi_2, \xi_2 > -\xi_3, |\xi_1+\xi_2| \simeq |\xi_2+\xi_3|\} \\
\mathcal{R}_3^{1,1} &= \{-\xi_1 < \xi_2, \xi_2 < -\xi_3, |\xi_1+\xi_2| >> |\xi_2+\xi_3|\} \\
\mathcal{R}_3^{2,1} &= \{-\xi_1 > \xi_2, \xi_2 < -\xi_3, |\xi_1+\xi_2| >> |\xi_2+\xi_3|\} \\
\mathcal{R}_3^{1,2} &= \{-\xi_1 < \xi_2, \xi_2 > -\xi_3, |\xi_1+\xi_2| >> |\xi_2+\xi_3|\} \\
\mathcal{R}_3^{2,2} &= \{-\xi_1 > \xi_2, \xi_2 > -\xi_3, |\xi_1+\xi_2| >> |\xi_2+\xi_3|\}.
\end{aligned}$$

A wide range of L^p estimates exist for multipliers localized to $\mathcal{R}_2^{1,1}, \mathcal{R}_2^{2,1}, \mathcal{R}_2^{1,2}, \mathcal{R}_2^{2,2}$, where the multiplier is adapted to $\{-\xi_1 = \xi_2 = -\xi_3\}$. By results in [18], one checks the existence of (generalized) restricted weak type estimates near the desired extremal points in the collections E_0, E_1, E_2. Therefore, by symmetry, it will suffice to consider a generic multiplier of the form $\tilde{1}_{\mathcal{R}_1^{1,1}}(\xi_1,\xi_2,\xi_3)a_1(\xi_1,\xi_2)a_2(\xi_2,\xi_3)$, where $\tilde{1}_{\mathcal{R}_1^{1,1}} \equiv 1$ on a region of shape $\mathcal{R}_1^{1,1}$ and is supported inside a region of shape $\mathcal{R}_1^{1,1}$. Carving $a_1(\xi_1,\xi_2)1_{-\xi_1<\xi_2}(\xi_1,\xi_2)$ and $a_2(\xi_2,\xi_3)1_{\xi_2<-\xi_3}(\xi_2,\xi_3)$ using Whitney decompositions and then expanding bump functions adapted to Whitney cubes using double fourier series yields

$$a_1(\xi_1,\xi_2)1_{\{-\xi_1<\xi_2\}}(\xi_1,\xi_2) = \sum_{(\gamma,\gamma')\in\{0,\frac{1}{3},\frac{2}{3}\}^2}\sum_{k,k'\in\mathbb{Z}}\sum_{\vec{Q}\in\mathbb{Q}^{\gamma,\gamma'}} c_k\tilde{c}_{k'}\hat{\eta}^{\gamma,k}_{Q_1,1}(\xi_1)\hat{\eta}^{\gamma',k'}_{Q_2,2}(\xi_2)$$
$$a_2(\xi_2,\xi_3)1_{\{\xi_2<-\xi_3\}}(\xi_2,\xi_3) = \sum_{(\gamma,\gamma')\in\{0,\frac{1}{3},\frac{2}{3}\}^2}\sum_{k,k'\in\mathbb{Z}}\sum_{\vec{Q}\in\mathbb{Q}^{\gamma,\gamma'}} d_k\tilde{d}_{k'}\hat{\eta}^{\gamma,k}_{Q_1,1}(\xi_2)\hat{\eta}^{\gamma',k'}_{Q_2,2}(\xi_3),$$

where $Q_2 \simeq Q_1 + C|\vec{Q}|$ and $\hat{\eta}^i_I$ is a bump function adapted in the Hörmander-Mikhlin sense to the interval I. Hence, omitting the dependence on the shifts γ,γ' and oscillation parameters k,k', it suffices to handle for generic bumps functions $\hat{\eta}_{I,j}$ the symbol

$$\begin{aligned}
&\tilde{1}_{\mathcal{R}^{1,1}_1}(\xi_1,\xi_2,\xi_3)m(\xi_1,\xi_2,\xi_3)\\
=\;& \tilde{1}_{\mathcal{R}^{1,1}_1}(\xi_1,\xi_2,\xi_3)\sum\left[\sum_{\vec{Q}\in\mathcal{Q}}\sum_{\vec{P}\in\mathcal{P}}\hat{\eta}_{P_1,1}(\xi_1)\hat{\eta}_{P_2,2}(\xi_2)\hat{\eta}_{Q_1,3}(\xi_2)\hat{\eta}_{Q_2.4}(\xi_3)\right]\\
=\;& \tilde{1}_{\mathcal{R}^{1,1}_1}(\xi_1,\xi_2,\xi_3)\sum\left[\sum_{\vec{Q}\in\mathcal{Q}}\sum_{\vec{P}\in\mathcal{P}:|I_{\vec{P}}|\lesssim|I_{\vec{Q}}|}\hat{\eta}_{P_1,1}(\xi_1)\hat{\eta}_{P_2,2}(\xi_2)\hat{\eta}_{Q_1,3}(\xi_2)\hat{\eta}_{Q_2.4}(\xi_3)\right].
\end{aligned}$$

Furthermore, because $\tilde{1}_{\mathcal{R}^{1,1}_1}\equiv 1$ inside a region of the same shape, for large enough implicit constant

$$\begin{aligned}
&\tilde{1}_{\mathcal{R}^{1,1}_1}(\xi_1,\xi_2,\xi_3)\sum\left[\sum_{\vec{Q}\in\mathcal{Q}}\sum_{\vec{P}\in\mathcal{P}:|I_{\vec{P}}|<<|I_{\vec{Q}}|}\hat{\eta}_{P_1,1}(\xi_1)\hat{\eta}_{P_2,2}(\xi_2)\hat{\eta}_{Q_1,3}(\xi_2)\hat{\eta}_{Q_2.4}(\xi_3)\right]\\
=\;& \sum\left[\sum_{\vec{Q}\in\mathcal{Q}}\sum_{\vec{P}\in\mathcal{P}:|I_{\vec{P}}|<<|I_{\vec{Q}}|}\hat{\eta}_{P_1,1}(\xi_1)\hat{\eta}_{P_2,2}(\xi_2)\hat{\eta}_{Q_1,3}(\xi_2)\hat{\eta}_{Q_2.4}(\xi_3)\right].
\end{aligned}$$

Lastly, note

$$\tilde{1}_{\mathcal{R}_1^{1,1}}(\xi_1,\xi_2,\xi_3)\sum\left[\sum_{\vec{Q}\in\mathcal{Q}}\sum_{\vec{P}\in\mathcal{P}:|I_{\vec{P}}|\simeq|I_{\vec{Q}}|}\hat{\eta}_{P_1,1}(\xi_1)\hat{\eta}_{P_2,2}(\xi_2)\hat{\eta}_{Q_1,3}(\xi_2)\hat{\eta}_{Q_2.4}(\xi_3)\right]$$

is adapted in the Mikhlin-Hörmander sense to $\{-\xi_1=\xi_2=-\xi_3\}\subset\mathbb{R}^3$ and so satisfies L^p estimates. Therefore, it suffices to produce mixed estimates for generic symbols of the form

$$\sum_{\vec{Q}\in\mathcal{Q}}\sum_{\vec{P}\in\mathcal{P}:|I_{\vec{P}}|<<|I_{\vec{Q}}|}\hat{\eta}_{P_1,1}(\xi_1)\hat{\eta}_{P_2,2}(\xi_2)\hat{\eta}_{Q_1,3}(\xi_2)\hat{\eta}_{Q_2.4}(\xi_3).$$

Dualizing and completing yields the 4−form

$$\begin{aligned}&\Lambda(f_1,f_2,f_3,f_4\\ :=\ &\int_{\mathbb{R}}\sum_{\vec{Q}\in\mathcal{Q}}\sum_{\vec{P}\in\mathcal{P}:|I_{\vec{P}}|<<|I_{\vec{Q}}|}[f_4*\eta_{-P_1,4}f_1*\eta_{P_1,1}]*\eta^{lac}_{|\vec{Q}|,0}\cdot f_2*\eta_{P_2,2}*\eta_{Q_1,3}\cdot f_3*\eta_{Q_2,4}dx.\end{aligned}$$

Discretizing in time then gives that $\Lambda(f_1,f_2,f_3,f_4)$ can be written as a sum of expressions of the form

$$\begin{aligned}&\int_0^1\int_0^1\sum_{\vec{Q}\in\mathbb{Q}}\sum_{\vec{P}\in\mathbb{P}:\omega_{P_2}\supset\supset\omega_{Q_1}}\frac{1}{|I_{\vec{Q}}|^{1/2}|I_{\vec{P}}|^{1/2}}\langle f_4,\Phi^{\alpha}_{-P_1,4}\rangle\langle f_1,\Phi^{\alpha}_{P_1,1}\rangle\left\langle\Phi^{\alpha}_{P_3,5},\Phi^{\alpha'}_{Q_3,6}\right\rangle\\ \times\ &\langle f_2*\eta_{P_2,2},\Phi^{\alpha'}_{Q_1,2}\rangle\langle f_3,\Phi^{\alpha'}_{Q_2,3}\rangle d\alpha d\alpha',\end{aligned}$$

where $\mathbb{P}$ and $\mathbb{Q}$ are collections of tri-tiles adapted to the degenerate line $\{\xi_1+\xi_2=0\}$ when viewed only in their first two entries. In particular, for each $\vec{P}=(P_1,P_2,P_3)\in\mathbb{P}$ and $\vec{Q}=(Q_1,Q_2,Q_3)\in\mathbb{Q}$, $P_j=(I_{\vec{P}},\omega_{P_j})$, $Q_j=(I_{\vec{Q}},\omega_{Q_j})$ is a tile for $j\in\{1,2,3\}$, $\Phi_{T,j}$, is a wave-packet on the tile T for each $j\in\{1,2,3,4,5,6\}$, $\omega_{P_3}=[-|I_{\vec{P}}|^{-1}/2,|I_{\vec{P}}|^{-1}/2]$, and $\omega_{Q_3}=[c|I_{\vec{Q}}|^{-1},C|I_{\vec{Q}}|^{-1}]$ for some $0<c<C$ fixed.

2.15 Estimates for a Toy Model

It will be useful for us to first prove estimates for the simpler model $\Lambda^{k_0}_{Toy}(f_1, f_2, f_3, f_4)$ defined by

$$\sum_{\vec{Q}\in\mathbb{Q}} \sum_{\vec{P}\in\mathbb{P}:|I_{\vec{P}}|=2^{-k_0}|I_{\vec{Q}}|,\omega_{P_2}\supset\supset\omega_{Q_1}} \frac{\left|\langle f_4, \Phi_{-P_1,0}\rangle\langle f_1, \Phi_{P_1,1}\rangle\langle \tilde{1}_{I_{\vec{P}}}, \tilde{1}_{I_{\vec{Q}}}\rangle\langle f_2, \Phi_{Q_1,2}\rangle\langle f_3, \Phi_{Q_2,3}\rangle\right|}{|I_{\vec{Q}}||I_{\vec{P}}|}$$

with an operatorial norm that grows like 2^{2k_0}. As $k_0 \simeq 0$ corresponds to a model adapted to $\{-\xi_1 = \xi_2 = -\xi_3\} \subset \mathbb{R}^3$, the statement only needs to be proven for $k_0 >> 1$. To this end, observe that the main contribution to the above sum occurs for those pairs of tri-tiles $(\vec{Q}, \vec{P}) \in \mathbb{Q} \times \mathbb{P}$ for which $I_{\vec{P}} \subset I_{\vec{Q}}$. Indeed, we now make this heuristic rigorous by showing that it suffices to produce estimates for generic sums of the above form for which $I_{\vec{P}} \subset I_{\vec{Q}}$ also holds. Begin by noting that whenever $|I_{\vec{P}}| = 2^{-k_0}|I_{\vec{Q}}|$,

$$\langle \Phi^{n-l}_{|\vec{P}|}, \Phi^{lac}_{\vec{Q}}\rangle \lesssim \frac{1}{|I_{\vec{P}}|^{1/2}|I_{\vec{Q}}|^{1/2}}\langle \tilde{1}_{I_{\vec{P}}}, \tilde{1}_{I_{\vec{Q}}}\rangle \lesssim_N \frac{|I_{\vec{P}}|^{1/2}}{|I_{\vec{Q}}|^{1/2}} \frac{1}{1+\left(\frac{dist(I_{\vec{P}}, I_{\vec{Q}})}{|I_{\vec{Q}}|}\right)^N}.$$

Therefore, denote $\mathbb{P}_{\vec{Q}}(l) = \left\{\vec{P} \in \mathbb{P} : 1 + \frac{dist(I_{\vec{P}}, I_{\vec{Q}})}{|I_{\vec{Q}}|} \simeq 2^l\right\}$ and $\mathbb{Q}_{\vec{P}}(l) = \left\{\vec{Q} \in \mathbb{Q} : 1 + \frac{dist(I_{\vec{P}}, I_{\vec{Q}})}{|I_{\vec{Q}}|} \simeq 2^l\right\}$. Then

$$\begin{aligned}
&\sum_{\vec{Q}\in\mathbb{Q}} \sum_{\vec{P}\in\mathbb{P}:|I_{\vec{P}}|=2^{-k_0}|I_{\vec{Q}}|,\omega_{P_2}\supset\omega_{Q_1}} \frac{1}{|I_{\vec{Q}}|^{1/2}|I_{\vec{P}}|^{1/2}} \left|\langle f_4, \Phi_{-P_1,0}\rangle\langle f_1, \Phi_{P_1,1}\rangle \left\langle \Phi^{n-l}_{|\vec{P}|}, \Phi^{lac}_{|\vec{Q}|}\right\rangle\right. \\
\times\ & \left.\langle f_2, \Phi_{Q_1,2}\rangle\langle f_3, \Phi_{Q_2,3}\rangle\right| \\
\lesssim\ & \sum_{l\geq 0}\frac{1}{1+l^N}\sum_{\vec{P}\in\mathbb{P}}\sum_{\vec{Q}\in\mathbb{Q}_{\vec{P}}(l):|I_{\vec{Q}}|=2^{k_0}|I_{\vec{P}}|,\omega_{Q_1}\subset\omega_{P_2}} \frac{|\langle f_4, \Phi_{-P_1,0}\rangle\langle f_1, \Phi_{P_1,1}\rangle\langle f_2, \Phi_{Q_1,2}\rangle\langle f_3, \Phi_{Q_2,3}\rangle|}{|I_{\vec{Q}}|} \\
\lesssim\ & \sum_{l\geq 0}\frac{1+l^M}{1+l^N}\sum_{\vec{P}\in\mathbb{P}}\sum_{\vec{Q}\in\mathbb{Q}:|I_{\vec{Q}}|=2^{k_0}|I_{\vec{P}}|,\omega_{Q_1}\subset\omega_{P_2},I_{\vec{Q}}\supset I_{\vec{P}}} \frac{1}{|I_{\vec{Q}}|}\left|\langle f_4, \Phi_{-P_1,0}\rangle\langle f_1, \Phi_{P_1,1}\rangle\right. \\
\times\ & \left.\langle f_2, \Phi^l_{Q_1,2}\rangle\langle f_3, \Phi^l_{Q_2,3}\rangle\right|
\end{aligned}$$

for some $1 << M << N$. It therefore suffices to prove generic mixed estimates for $\tilde{\Lambda}^{k_0}_{Toy}(f_1, f_2, f_3, f_4)$ given by

$$\sum_{\vec{Q}\in\mathbb{Q}} \sum_{\vec{P}\in\mathbb{P}:|I_{\vec{P}}|=2^{-k_0}|I_{\vec{P}}|,\omega_{P_2}\supset\supset\omega_{Q_1},I_{\vec{P}}\subset I_{\vec{Q}}} \frac{1}{|I_{\vec{Q}}|}\left|\langle f_4, \Phi_{-P_1,0}\rangle\langle f_1, \Phi_{P_1,1}\rangle\langle f_2, \Phi_{Q_1,2}\rangle\langle f_3, \Phi_{Q_2,3}\rangle\right|.$$

2.15.1 Sizes and Energies

By scaling invariance, we shall assume $|E_4| = 1$. Moreover, let

$$\Omega := \{M1_{E_1} \geq C|E_1|\}\bigcup\{M1_{E_3} \geq C|E_3|\}\bigcup\left\{Mf_2 \geq C|E_2|^{1/2}\right\}.$$

As usual, choose $C \in \mathbb{R}$ large enough to ensure $|\Omega| \leq 1/2$ in which case $\tilde{E}_4 := E_4 \cap \Omega^c$ is a major subset of E_4.

Definition 22. *For* $d, \tilde{d} \in \mathbb{Z}_{\geq 0}$, *let* $\mathbb{P}^d = \left\{\vec{P} \in \mathbb{P} : 1 + \frac{dist(I_{\vec{P}}, \Omega^c)}{|I_{\vec{P}}|} \simeq 2^d\right\}; \mathbb{Q}^{\tilde{d}} := \left\{\vec{Q} \in \mathbb{Q} : 1 + \frac{dist(I_{\vec{Q}}, \Omega^c)}{|I_{\vec{Q}}|} \simeq 2^{\tilde{d}}\right\}$.

Definition 23. *For each measurable set* $E \subset \mathbb{R}$*, dyadic interval* $I \in \mathcal{D}$*, and* $j \in \{1, 4\}$*, set*

$$SIZE(E, I) := \frac{1}{|I|}\int_E \tilde{1}_I dx = \sup_{\omega \in \mathcal{D}: |\omega| \cdot |I| = 1} \left[\frac{1}{|I|} \sum_{\vec{P} \in \mathbb{P}: |I_{\vec{P}}| \lesssim 2^{-k_0}|I|, I_{\vec{P}} \subset I, \omega_{P_2} \supset \omega} \langle f_1, \Phi_{P_1,1} \rangle|^2 \right]^{1/2}.$$

Definition 24. *For each* $n_1 \in \mathbb{Z}$*, let* $\mathbb{I}^d_{n_1,1}$ *denote the set of dyadic intervals* $\{I\}$ *that are maximal with respect to the property*

$$SIZE(E_1, I) \quad \geq \quad 2^{-n_1}$$

Similarly, for each $n_4 \in \mathbb{Z}$*, let* $\mathbb{I}^d_{n_4,4}$ *denote the set of dyadic intervals* $\{I\}$ *that are maximal with respect to the property*

$$SIZE(E_4 \cap \Omega^c, I) \quad \geq \quad 2^{-n_4}.$$

Therefore, $\{\mathbb{I}_{n_1}\}_{n_1 \geq 0}$ and $\{\mathbb{I}_{n_4}\}_{n_4 \geq 0}$ generate two decompositions of $\mathbb{Q}$ using the recursive definitions

$$\mathbb{Q}_{n_1,1} \quad := \quad \left\{\vec{Q} \in \mathbb{Q} : I_{\vec{Q}} \subset I \in \mathbb{I}_{n_1,1}\right\} \cap \left[\bigcup_{N_1(d) \leq \tilde{n}_1 < n_1} \left\{\vec{Q} \in \mathbb{Q} : I_{\vec{Q}} \subset I \in \mathbb{I}_{\tilde{n}_1,1}\right\} \right]^c$$

$$\mathbb{Q}_{n_4,4} \quad := \quad \left\{\vec{Q} \in \mathbb{Q} : I_{\vec{Q}} \subset I \in \mathbb{I}_{n_4,4}\right\} \cap \left[\bigcup_{N_4(d) \leq \tilde{n}_4 < n_4} \left\{\vec{Q} \in \mathbb{Q} : I_{\vec{Q}} \subset I \in \mathbb{I}_{\tilde{n}_4,4}\right\} \right]^c.$$

Lastly, define

$$\mathbb{Q}^{\tilde{d}}_{n_1,n_4} := \mathbb{Q}_{n_1,1} \cap \mathbb{Q}_{n_4,4} \cap \mathbb{Q}^{\tilde{d}}.$$

It follows that $\mathbb{Q} = \bigcup_{\tilde{d}\geq 0}\bigcup_{n_1\geq N_1(\tilde{d})}\bigcup_{n_4\geq N_4(\tilde{d})}\mathbb{Q}^{\tilde{d}}_{n_1,n_4}$. Setting $\mathbb{I}_{n_1,n_4} = \{I\cap J : I\in\mathbb{I}_{n_1,1}, J\in\mathbb{I}_{n_4,4}\}$ and $\mathbb{Q}^{\tilde{d}}_{n_1,n_4}[I] = \left\{\vec{Q}\in\mathbb{Q}^{\tilde{d}}_{n_1,n_4} : I_{\vec{Q}}\subset I\right\}$, note

$$\mathbb{Q}^{\tilde{d}}_{n_1,n_4} = \bigcup_{I\in\mathbb{I}_{n_1,n_4}}\mathbb{Q}^{\tilde{d}}_{n_!,n_4}[I].$$

Definition 25. *A collection of (dyadic) intervals $\mathcal{P} := \{\omega_P\}$ is said to be lacunary provided about $\Omega\in\mathbb{R}$ provided*

$$\omega_P - \Omega_T \subset \left[|\omega_P|2^{-C_*}, |\omega_P|2^{C_*}\right]\ \forall\ \omega_P\in\mathcal{P}$$

for some fixed $C_* >> 1$.

Definition 26. *For a given $I\in\mathbb{I}_{n_1,n_4}$, $j\in\{2,3\}$, let*

$$SIZE^{\tilde{d}}_{n_1,n_4}(f_j, I) := \sup_{T:I_T\subset I,}\frac{1}{|I_T|}\left(\sum_{\vec{Q}\in T\cap\mathbb{Q}^{\tilde{d}}_{n_1,n_4}}|\langle f_j,\Phi_{Q_2}\rangle|^2\right)^{1/2}$$

generate another decomposition localized to I associated to the indices n_2, n_3 as follows:

$$\mathbb{Q}^{\tilde{d}}_{n_1,n_4}[I] = \bigcup_{n_2}\left[\mathbb{Q}^{\tilde{d}}_{n_1,n_4}[I]\right]_{n_2,2} = \bigcup_{n_3}\left[\mathbb{Q}_{n_1,n_4}[I]\right]_{n_3,3}.$$

Putting it all together using the standard BHT-type strongly disjoint tree decomposition from [21], say, we have tile decompositions for $\mathbb{Q}$:

$$\mathbb{Q} \;=\; \bigcup_{\tilde{d}\geq 0}\mathbb{Q}^{\tilde{d}} = \bigcup_{\tilde{d}\geq 0}\bigcup_{n_1,n_4}\bigcup_{I\in\mathbb{I}_{n_1,n_4}}\mathbb{Q}^{\tilde{d}}_{n_1,n_4}[I] = \bigcup_{\tilde{d}\geq 0}\bigcup_{n_1,n_4}\bigcup_{I\in\mathbb{I}_{n_1,n_4}}\bigcup_{n_2,n_3}\left[\mathbb{Q}^{\tilde{d}}_{n_1,n_4}[I]\right]^{n_3}_{n_2}$$

where

$$\left[\mathbb{Q}^{\tilde{d}}_{n_1,n_4}[I]\right]^{n_3}_{n_2} := \left[\mathbb{Q}^{\tilde{d}}_{n_1,n_4}[I]\right]_{n_2} \cap \left[\mathbb{Q}^{\tilde{d}}_{n_1,n_4}[I]\right]^{n_3} = \bigcup_{T_2 \in \mathcal{T}\left[\mathbb{Q}^{\tilde{d}}_{n_1,n_4}[I]\right]_{n_2}} \bigcup_{T_3 \in \mathcal{T}\left[\mathbb{Q}^{\tilde{d}}_{n_1,n_4}[I]\right]^{n_3}} T_2 \cap T_3$$

and the collection of trees $\mathcal{T}\left[\mathbb{Q}^{\tilde{d}}_{n_1,n_4}[I]\right]_{n_2}$ and $\mathcal{T}\left[\mathbb{Q}^{\tilde{d}}_{n_1,n_4}[I]\right]^{n_3}$ have subcollections $\mathcal{T}_*\left[\mathbb{Q}^{\tilde{d}}_{n_1,n_4}[I]\right]_{n_2}$ and $\mathcal{T}_*\left[\mathbb{Q}^{\tilde{d}}_{n_1,n_4}[I]\right]^{n_3}$ which are strongly $2-$ and $3-$disjoint respectively and for which the following energy-type estimates hold:

$$\sum_{T_2 \in \mathcal{T}_*\left[\mathbb{Q}^{\tilde{d}}_{n_1,n_4}[I]\right]_{n_2}} |I_{T_2}| \lesssim 2^{2n_2}|E_2|$$

$$\sum_{T_3 \in \mathcal{T}_*\left[\mathbb{Q}^{\tilde{d}}_{n_1,n_4}[I]\right]^{n_3}} |I_{T_3}| \lesssim 2^{2n_3}|E_3|.$$

To summarize, we have assembled a decomposition of the set $\mathbb{P} \times \mathbb{Q}$:

$$\mathbb{P} \times \mathbb{Q} = \bigcup_{\tilde{d} \geq 0} \bigcup_{n_1,n_4} \bigcup_{I \in \mathbb{I}_{n_1,n_4}} \bigcup_{n_2,n_3} \left(\mathbb{P} \times \left[\mathbb{Q}^{\tilde{d}}_{n_1,n_4}[I]\right]^{n_3}_{n_2} \right).$$

2.15.2 Splitting $\tilde{\Lambda}^{k_0}_{Toy}$

We now break apart our model as follows:

$$\begin{aligned}
&\tilde{\Lambda}^{k_0}_{Toy}(f_1,f_2,f_3,f_4) \\
= &\sum_{\vec{Q} \in \mathbb{Q}} \sum_{\vec{P} \in \mathbb{P} : |I_{\vec{P}}| = 2^{-k_0}|I_{\vec{Q}}|, \omega_{P_2} \supset\supset \omega_{Q_1}, I_{\vec{P}} \subset I_{\vec{Q}}} \frac{1}{|I_{\vec{Q}}|} |\langle f_4, \Phi_{P_1,4}\rangle \langle f_1, \Phi_{-P_1,1}\rangle \langle f_2, \Phi_{Q_1,2}\rangle \langle f_3, \Phi_{Q_2,3}\rangle| \\
= &\sum_{\tilde{d} \geq 0} \sum_{n_1,n_2,n_3,n_4} \sum_{I \in \mathbb{I}_{n_1,n_4}} \sum_{\vec{Q} \in \left[\mathbb{Q}^{\tilde{d}}_{n_1,n_4}[I]\right]^{n_3}_{n_2}} \sum_{\vec{P} \in \mathbb{P}^d, |I_{\vec{P}}| = 2^{-k_0}|I_{\vec{Q}}|, \omega_{P_2} \supset\supset \omega_{Q_1}, I_{\vec{P}} \subset I_{\vec{Q}}} \Lambda_{\vec{Q},\vec{P}}(f_1,f_2,f_3,f_4),
\end{aligned}$$

where

$$\Lambda_{\vec{Q},\vec{P}}(f_1,f_2,f_3,f_4) := \frac{1}{|I_{\vec{Q}}|}\left|\langle f_4, \Phi_{P_1,4}\rangle\langle f_1, \Phi_{-P_1,1}\rangle\langle f_2, \Phi_{Q_1,2}\rangle\langle f_3, \Phi_{Q_2,3}\rangle\right|.$$

2.15.3 Toy Model Tree Estimate

Lemma 15. *Let T be a $\mathbb{Q}-$tree satisfying $T \subset \left[[\mathbb{Q}^{\tilde{d}}_{n_1,n_4}[I]\right]^{n_3}_{n_2}$. Then*

$$\sum_{\vec{Q}\in T} \sum_{\vec{P}\in\mathbb{P}^d, |I_{\vec{P}}|=2^{-k_0}|I_{\vec{Q}}|, \omega_{P_2}\supset\supset\omega_{Q_1}, I_{\vec{P}}\subset I_{\vec{Q}}} \Lambda_{\vec{Q},\vec{P}}(f_1,f_2,f_3,f_4) \lesssim 2^{-n_1}2^{-n_2}2^{-n_3}2^{-n_4}|I_T|.$$

Proof. WLOG, assume $\{Q_1\}_{\vec{Q}\in T}$ are overlapping as the case when $\{Q_2\}_{\vec{Q}\in T}$ is overlapping is similar. Then it suffices to note that for large enough implicit constant in $|I_{\vec{P}}| << |I_{\vec{Q}}|$, the assumption that $\vec{Q}$ lie on a single tree ensures that

$$\{\omega_{P_2}\}_{\vec{P}\in\mathbb{P}:|I_{\vec{P}}|=2^{-k_0}|I_{\vec{Q}}|, \omega_{P_2}\supset\supset\omega_{Q_1}, I_{\vec{P}}\subset I_{\vec{Q}} \text{ for some } \vec{Q}\in T}$$

is overlapping. Therefore, $\{\omega_{P_1}\}_{\vec{P}\in\mathbb{P}:|I_{\vec{P}}|=2^{-k_0}|I_{\vec{Q}}|, \omega_{P_2}\supset\supset\omega_{Q_1}, I_{\vec{P}}\subset I_{\vec{Q}} \text{ for some } \vec{Q}\in T} := \Omega\{\mathcal{T}\}$ forms a lacunary sequence and setting

$$\mathbb{P}_{n_1,n_2} = \left\{\mathbb{P}\in\mathbb{P} : \exists\vec{Q}\in\mathbb{Q}_{n_1,n_2}, I_{\vec{Q}}\supset I_{\vec{P}}, |I| = 2^{k_0}|I_{\vec{P}}|\right\}$$

yields

$$
\begin{aligned}
&\sum_{\vec{Q}\in T}\ \sum_{\vec{P}\in\mathbb{P}:|I_{\vec{P}}|=2^{-k_0}|I_{\vec{Q}}|,\omega_{P_2}\supset\omega_{Q_1},I_{\vec{P}}\subset I_{\vec{Q}}} \Lambda_{\vec{Q},\vec{P}}(f_1,f_2,f_3,f_4)\\
\leq\ & \sup_{\vec{Q}\in T}\left[\frac{1}{|I_{\vec{Q}}|}\sum_{\vec{P}\in\mathbb{P}^d:I_{\vec{P}}\subset I_{\vec{Q}},\omega_{P_1}\in\Omega\{\mathcal{T}\}}|\langle f_4,\Phi_{-P_1,4}\rangle|^2\right]^{1/2}\\
\times\ & \left(\frac{1}{|I_T|}\sum_{\vec{P}\in\mathbb{P}^d:I_{\vec{P}}\subset I_T,\omega_{P_2}\in\Omega\{\mathcal{T}\}}|\langle f_1,\Phi_{-P_1,1}\rangle|^2\right)^{1/2}\\
\times\ & \left[\sup_{\vec{Q}\in T}\frac{|\langle f_2,\Phi^2_{Q_1}\rangle|}{|I_{\vec{Q}}|^{1/2}}\right]\left(\frac{1}{|I_T|}\sum_{\vec{Q}\in T_2}|\langle f_3,\Phi_{Q_2}\rangle|^2\right)^{1/2}\cdot|I_T|\\
\lesssim\ & \left[\sup_{\vec{P}\in\mathbb{P}_{n_1,n_4}}\frac{1}{|I_{\vec{P}}|}\int_{E_1}\tilde{1}_{I_{\vec{P}}}dx\right]\cdot\left[\sup_{\vec{P}\in\mathbb{P}_{n_1,n_4}}\frac{1}{|I_{\vec{P}}|}\int_{E_4\cap\Omega^c}\tilde{1}_{I_{\vec{P}}}dx\right]2^{-n_2}2^{-n_3}|I_T|\\
\lesssim\ & 2^{2k_0}2^{-n_2}2^{-n_3}\left[\sup_{I\in\mathbb{I}_{n_1,n_4}}\frac{1}{|I|}\int_{E_1}\tilde{1}_I dx\right]\cdot\left[\sup_{I\in\mathbb{I}_{n_1,n_4}}\frac{1}{|I|}\int_{E_4\cap\Omega^c}\tilde{1}_I dx\right]\\
\lesssim\ & 2^{2k_0}2^{-n_1}2^{-n_2}2^{-n_3}2^{-n_4}|I_T|.
\end{aligned}
$$

□

2.15.4 Toy Model Energy Estimate

The following lemma forms the core of our analysis and is one of the main reasons why we have mixed estimates.

Lemma 16. *The following estimate holds:*

$$\sum_{T_2\in\mathcal{T}_*\left[\mathbb{Q}^{\tilde{d}}_{n_1,n_4}[I]\right]_{n_2}}|I_{T_2}|\lesssim 2^{n_2}|E_2|\cdot|I|\,.$$

Proof. By definition, for every $T_2\in\mathcal{T}_*\left[\mathbb{Q}^{\tilde{d}}_{n_1,n_4}[I]\right]_{n_2}$, we have

$$|I_{T_2}| \lesssim 2^{2n_2} \sum_{\vec{Q} \in T_2} |\langle f_2, \Phi_{Q_1,2} \rangle|^2$$

where the tiles Q_2 are lacunary around some top frequency. Using strong 2−disjointness of the trees $\{T_2\}$ yields

$$\begin{aligned}
\sum_{T_2 \in \mathcal{T}_*\left[\mathbb{Q}^{\tilde{d}}_{n_1,n_4}[I]\right]_{n_2}} |I_{T_2}| &\lesssim 2^{2n_2} \sum_{T_2 \in \mathcal{T}_*\left[\mathbb{Q}^{\tilde{d}}_{n_1,n_4}[I]\right]_{n_2}} \sum_{\vec{Q} \in T_2} |\langle f_2, \Phi_{Q_1,2} \rangle|^2 \\
&\lesssim 2^{n_2} \sum_{T_2 \in \mathcal{T}_*\left[\mathbb{Q}^{\tilde{d}}_{n_1,n_4}[I]\right]_{n_2}} \sum_{\vec{Q} \in T_2} |\langle f_2, \Phi^{\infty}_{Q_1,2} \rangle| \\
&\leq 2^{n_2} \sum_{T_2 \in \mathcal{T}_*\left[\mathbb{Q}^{\tilde{d}}_{n_1,n_4}[I]\right]_{n_2}} \sum_{\vec{Q} \in T_2} \left|\left| f_2 * \eta_{\omega_{Q_1}} \right|\right|_{\infty} |I_{\vec{Q}}| \\
&\leq 2^{n_2} \int_I \sum_{T_2 \in \mathcal{T}_*\left[\mathbb{Q}^{\tilde{d}}_{n_1,n_4}[I]\right]_{n_2}} \sum_{\vec{Q} \in T_2} \left|\left| \hat{f}_2 1_{\omega_{Q_1}} \right|\right|_1 1_{I_{\vec{Q}}}(x) dx \\
&\lesssim 2^{n_2} |E_2| \cdot |I|.
\end{aligned}$$

□

We shall also need the following elementary result:

Lemma 17.

$$\begin{aligned}
\sum_{I \in \mathbb{I}_{n_1,n_4}} |I| &\leq \sum_{I \in \mathbb{I}_{n_1}} |I| \lesssim \left|\left\{ M 1_{E_1} \gtrsim 2^{-n_1} \right\}\right| \lesssim 2^{n_1} |E_1| \\
\sum_{I \in \mathbb{I}_{n_1,n_4}} |I| &\leq \sum_{I \in \mathbb{I}_{n_4}} |I| \lesssim \left|\left\{ M 1_{E_4} \gtrsim 2^{-n_4} \right\}\right| \lesssim 2^{n_4}.
\end{aligned}$$

Proof. Immediate from the definitions. □

Putting it all together, we find

$$\sum_{\vec{Q}\in\left[\mathbb{Q}^{\tilde{d}}_{n_1,n_4}[I]\right]^{n_3}_{n_2}} \sum_{\vec{P}:|I_{\vec{P}}|=2^{-k_0}|I_{\vec{Q}}|,\omega_{P_2}\supset\supset\omega_{Q_1},I_{\vec{P}}\subset I_{\vec{Q}}} \Lambda_{\vec{Q},\vec{P}}(f_1,f_2,f_3,f_4)$$
$$\lesssim 2^{-n_1}2^{-n_2}2^{-n_3}2^{-n_4}\min\{2^{2n_2}|E_2\cap I|, 2^{n_2}|E_2|\cdot|I|, 2^{2n_3}|E_3\cap I|\},$$

and using the two separate estimates for $\sum_{I\in\mathbb{I}_{n_1,n_4}}|I|$ yields

$$\begin{aligned}&\tilde{\Lambda}^{k_0}_{Toy}(f_1,f_2,f_3,f_4)\\ \lesssim &\sum_{\tilde{d}\geq 0}\sum_{n_1\geq N_1(d)}\sum_{n_2\geq N_2(\tilde{d})}\sum_{n_3\geq N_3(\tilde{d})}\sum_{n_4\geq N_4(d)} 2^{-n_1}2^{-n_2}2^{-n_3}2^{-n_4}\min\{2^{2n_2}|E_2|,\\ &2^{n_2}|E_2|\min\{2^{n_4},2^{n_1}|E_1|\}, 2^{2n_3}|E_3|\}.\end{aligned}$$

The natural size restrictions are easily seen to be $2^{-n_1}\lesssim\min\{1,2^d|E_1|\}$, $2^{-n_2}\lesssim\min\{2^{\tilde{d}}|E_2|^{1/2},2^{\tilde{d}}|E_2|\}$, $2^{-n_3}\lesssim\min\{1,2^{\tilde{d}}|E_3|\}$, $2^{-n_4}\lesssim 2^{-\tilde{N}d}$. Therefore, the above sum is summable, and a range of mixed estimates are available by interpolation. As the numerics are similar to that found in the main model, we postpone a detailed examination of exactly what these estimates are.

2.16 Main Model

Recall the main model $\Lambda(f_1,f_2,f_3,f_4)$ given by the formula

$$\begin{aligned}&\sum_{\vec{Q}\in\mathbb{Q}}\sum_{\vec{P}\in\mathbb{P}:|I_{\vec{P}}|<<|I_{\vec{Q}}|,\omega_{P_2}\supset\supset\omega_{Q_1}} \frac{1}{|I_{\vec{Q}}|^{1/2}|I_{\vec{P}}|^{1/2}}\langle f_4,\Phi_{-P_1,4}\rangle\langle f_1,\Phi_{P_1,1}\rangle\left\langle\Phi^{n-l}_{|\vec{P}|},\Phi^{lac}_{|\vec{Q}|}\right\rangle\\ \times\ &\langle f_2*\eta_{P_2,2},\Phi_{Q_1,2}\rangle\langle f_3,\Phi_{Q_2,3}\rangle.\end{aligned}$$

Before proceeding with details, let us pause to sketch the idea of the remaining proof. The main difficulty in bounding the above expression turns on the factor

$\langle f_2 * \eta_{P_2,2}, \Phi_{Q_2,1}\rangle$ because mixing $\vec{Q}$ and $\vec{P}$ in this way may work against the orthogonality introduced by $\Phi^{lac}_{|\vec{Q}|}$. Therefore, our goal is to split $\mathbb{Q}$ and $\mathbb{P}$ using a more cumbersome decomposition than in the toy model before. Then, we shall see rewrite the above expression localized to a single $\mathbb{Q}-$tree as a sum of two main pieces, say A and B. Then A is paracomposition, which gives us the desired orthogonality and therefore estimates. The remainder B can be written as an infinite sum over rapidly decaying pieces, each one of which can be reformulated as a toy model $\Lambda^{k_0}_{Toy}$ for some parameter $k_0 >> 1$ as before. Hence, our proceeding work will enable us to successfully estimate B, and with it, the main model Λ.

To begin, assume by scaling invariance that $|E_4| = 1$. Moreover, let

$$\Omega := \{M1_{E_1} \geq C|E_1|\} \bigcup \{M1_{E_3} \geq C|E_3|\} \bigcup \left\{Mf_2 \geq C|E_2|^{1/2}\right\}.$$

Recall $\mathbb{P}^d = \left\{\vec{P} \in \mathbb{P} : 1 + \frac{dist(I_{\vec{P}}, \Omega^c)}{|I_{\vec{P}}|} \simeq 2^d\right\}; \mathbb{Q}^{\tilde{d}} := \left\{\vec{Q} \in \mathbb{Q} : 1 + \frac{dist(I_{\vec{Q}}, \Omega^c)}{|I_{\vec{Q}}|} \simeq 2^{\tilde{d}}\right\}$.

Definition 27. *For each $\mathbb{Q}-$tree T with top frequency centered around c_{ω_T} in the 1st index, let $\Phi^{n-l,T}_{\vec{P}}(x) := \hat{\eta}_{P_2,2}(c_{\omega_T})\Phi^{n-l}_{|\vec{P}|}(x)$.*

Definition 28. *For each $\mathbb{Q}-$tree T, let $\mathbb{P}(T) := \left\{\vec{P} \in \mathbb{P} : \omega_{P_2} \supset\supset \omega_{Q_1} \text{ for some } \vec{Q} \in T\right\}$.*

It is important to observe that $\Omega(\mathbb{P}(T))_1 := \left\{\omega_{P_1} : \vec{P} \in \mathbb{P}(T)\right\}$ is lacunary about c_{ω_T} for every $\mathbb{Q}-$tree T.

Definition 29. *For every dyadic interval $I \in \mathcal{D}$ and measurable set $E \subset \mathbb{R}$, let*

$$SIZE(E, I) := \frac{1}{|I|}\int_E \tilde{1}_I(x)dx.$$

Definition 30. *For each $n_1 \geq 0$, let $\mathbb{I}_{n_1,1}$ denote the collection of dyadic intervals maximal with respect to the property*

$$SIZE(E_1, I) \geq 2^{-n_1}.$$

Similarly, for each $n_4 \geq 0$, let $\mathbb{I}_{n_4,4}$ denote those intervals maximal with respect to the property

$$SIZE(E_4, I) \geq 2^{-n_4}.$$

As before, $\{\mathbb{I}_{n_1,1}\}$ and $\{\mathbb{I}_{n_4,4}\}$ generate two decompositions of $\mathbb{P}$ using the recursive definitions

$$\begin{aligned} \mathbb{P}_{n_1,1} &:= \left\{\vec{P} \in \mathbb{P} : I_{\vec{P}} \subset I \in \mathbb{I}_{n_1,1}\right\} \cap \left[\bigcup_{\tilde{n}_1 < n_1} \left\{\vec{P} \in \mathbb{P} : I_{\vec{P}} \subset I \in \mathbb{I}_{\tilde{n}_1,1}\right\}\right]^c \\ \mathbb{P}_{n_4,4} &:= \left\{\vec{P} \in \mathbb{P} : I_{\vec{P}} \subset I \in \mathbb{I}_{n_4,4}\right\} \cap \left[\bigcup_{\tilde{n}_4 < n_4} \left\{\vec{P} \in \mathbb{P} : I_{\vec{P}} \subset I \in \mathbb{I}_{\tilde{n}_4,4}\right\}\right]^c \end{aligned}$$

Then define $\mathbb{P}^d_{n_1,n_4} := \mathbb{P}_{n_1,1} \cap \mathbb{P}_{n_4,4} \cap \mathbb{P}^d$.

Lastly, we need to introduce

Definition 31. *For any $\vec{Q} \in \mathbb{Q}$, let $\Phi_{Q_1,2,a}$, $\Phi_{Q_1,2,b}$, and $\Phi_{Q_1,2,c}$ be defined by the identities*

$$\begin{aligned} \hat{\Phi}_{Q_1,2,a}(\xi) &= \hat{\Phi}_{Q_1,2}(\xi) \\ \hat{\Phi}_{Q_1,2,b}(\xi) &= (\xi - c_{\omega_{Q_2}})|I_{\vec{Q}}|\hat{\Phi}_{Q_1,2}(\xi) \\ \hat{\Phi}_{Q_1,2,c}(\xi) &= (\xi - c_{\omega_{Q_2}})^2|I_{\vec{Q}}|^2\hat{\Phi}_{Q_1,2}(\xi). \end{aligned}$$

It is simple to check that $\Phi_{Q_1,2,a}, \Phi_{Q_2,2,b},\ \Phi_{Q_1,2,c}$ are all L^2-normalized wave packets on the tile Q_1.

Definition 32. *For a given $I \in \mathbb{I}_{n_1,n_4}$ and $j \in \{2,3\}$, let*

$$\begin{aligned}
& SIZE^{\tilde{d},2}_{n_1,n_4}(f_2, I) \\
= & \sup_{T\subset\mathbb{Q}^{\tilde{d}}_{n_1,n_4}: I_T\subset I} \frac{1}{|I_T|^{1/2}} \left[\left(\sum_{\vec{Q}\in T} |\langle f_2, \Phi_{Q_1,2,a}\rangle|^2 \right)^{1/2} \right. \\
+ & \left. \left(\sum_{\vec{Q}\in T} |\langle f_2, \Phi_{Q_1,2,b}\rangle|^2 \right)^{1/2} + \left(\sum_{\vec{Q}\in T} |\langle f_2, \Phi_{Q_1,2,c}\rangle|^2 \right)^{1/2} \right]
\end{aligned}$$

$$SIZE^{\tilde{d},3}_{n_1,n_4}(f_3, I) = \sup_{T\subset\mathbb{Q}^{\tilde{d}}_{n_1,n_4}: I_T\subset I} \frac{1}{|I_T|^{1/2}} \left(\sum_{\vec{Q}\in T} |\langle f_3, \Phi_{Q_2,3}\rangle|^2 \right)^{1/2}$$

where the supremum in $SIZE^{\tilde{d},2}_{n_1,n_4}$ is over all $2-$trees $T \subset \mathbb{Q}^{\tilde{d}}_{n_1,n_4}$ for which $I_T \subset \bigcup_{I\in\mathbb{I}_{n_1,n_4}} I$ and the supremum in $SIZE^{\tilde{d},3}_{n_1,n_4}$ is over all $1-$trees $T \subset \mathbb{Q}^{\tilde{d}}_{n_1,n_4}$ for which $I_T \subset \bigcup_{I\in\mathbb{I}_{n_1,n_4}} I$.

This decomposes $\mathbb{Q}^{\tilde{d}}_{n_1,n_4}[I]$ into a union of trees corresponding to each of the 2 sizes for indices 2 and 3, i.e.

$$\begin{aligned}
\mathbb{Q}^{\tilde{d}}_{n_1,n_4}[I] &= \bigcup_{n_2} \mathbb{Q}^{\tilde{d}}_{n_1,n_4}[I]_{n_2,2} = \bigcup_{T_2\in\mathcal{T}\left\{\mathbb{Q}^{\tilde{d}}_{n_1,n_4}[I]_{n_2,2}\right\}} T_2 \\
\mathbb{Q}^{\tilde{d}}_{n_1,n_4}[I] &= \bigcup_{n_3} \mathbb{Q}^{\tilde{d}}_{n_1,n_4}[I]_{n_3,3} = \bigcup_{T_3\in\mathcal{T}\left\{\mathbb{Q}^{\tilde{d}}_{n_1,n_4}[I]_{n_3,3}\right\}} T_3.
\end{aligned}$$

Finish by setting

$$\left[\mathbb{Q}^{\tilde{d}}_{n_1,n_4}\right]^{n_3}_{n_2} := \bigcup_{I\in\mathbb{I}_{n_1,n_4}}\left[\left(\bigcup_{T_2\in\mathcal{T}\left\{\mathbb{Q}^{\tilde{d}}_{n_1,n_4}[I]_{n_2,2}\right\}} T_2\right)\bigcap\left(\bigcup_{T_3\in\mathcal{T}\left\{\mathbb{Q}^{\tilde{d}}_{n_1,n_4}[I]_{n_3,3\cdot}\right\}} T_3\right)\right].$$

2.16.1 Tree Localization

We have worked to decompose $\mathbb{Q}^{\tilde{d}}\times\mathbb{P}$ into a union of trees with useful properties. Let us now fix $\tilde{d}, n_1, n_4, n_0, I, n_2$ and let $T_2 \in \mathcal{T}\left\{\mathbb{Q}^{\tilde{d}}_{n_1,n_4}[I]_{n_2,2}\right\}$, say, and try to estimate $\Lambda_{T_2}(f_1,f_2,f_3,f_4)$ defined by

$$\sum_{\vec{Q}\in T_2}\sum_{\vec{P}\in\mathbb{P}(T_2)}\frac{1}{|I_{\vec{Q}}|^{1/2}|I_{\vec{P}}|^{1/2}}\langle f_4,\Phi_{-P_1,4}\rangle\langle f_1,\Phi_{P_1,1}\rangle\left\langle \Phi^{n-l}_{|\vec{P}|},\Phi^{lac}_{|\vec{Q}|}\right\rangle\langle f_2*\eta_{P_2,2},\Phi_{Q_1,2}\rangle\langle f_3,\Phi_{Q_2,3}\rangle.$$

Recall that T_2 is a $\mathbb{Q}-$tree comes equipped with a center frequency c_{ω_T}. Inspired by the argument of J. Jung in [8], we define $R^T_{\vec{P}}(\xi)$ by

$$\hat{\eta}_{P_2,2}(\xi) := \hat{\eta}_{P_2,2}(c_{\omega_T}) + \hat{\eta}^{(1)}(c_{\omega_T})(\xi - c_{\omega_T}) + \frac{1}{2}\hat{\eta}^{(2)}(c_{\omega_T})(\xi-c_{\omega_T})^2 + R^T_{\vec{P}}(\xi).$$

Now rewrite

$$\begin{aligned}
& \langle f_2*\eta_{P_2,2},\Phi_{Q_1}\rangle\\
= \;& \langle \hat{f}_2\hat{\eta}_{P_2,2},\hat{\Phi}_{Q_1,2}\rangle\\
= \;& \hat{\eta}_{P_2,2}(c_{\omega_T})\langle f_2,\Phi_{Q_1,2}\rangle + \hat{\eta}^{(1)}_{P_2,2}(c_{\omega_T})\langle \hat{f}_2,(\cdot - c_{\omega_T})\hat{\Phi}_{Q_1,2}\rangle\\
+ \;& \frac{\hat{\eta}^{(2)}_{P_2,2}(c_{\omega_T})}{2}\langle \hat{f}_2,(\cdot-c_{\omega_T})^2\hat{\Phi}_{Q_1,2}\rangle + \langle \hat{f}_2 R^T_{\vec{P}},\hat{\eta}_{Q_1,2}\rangle\\
= \;& I_a + I_b + I_c + II.
\end{aligned}$$

Therefore, $\Lambda(f_1,f_2,f_3,f_4) = \left[\Lambda_{I_a}+\Lambda_{I_b}+\Lambda_{I_c}\right](f_1,f_2,f_3,f_4) + \Lambda_{II}(f_1,f_2,f_3,f_4)$, where

$$
\begin{aligned}
&\Lambda_{I_a}(f_1,f_2,f_3,f_4)\\
= &\sum_{\vec{Q}\in T_2}\left\langle \sum_{\vec{P}\in\mathbb{P}(T_2)} \frac{\langle f_4,\Phi_{-P_1,4}\rangle\langle f_1,\Phi_{P_1,1}\rangle}{|I_{\vec{P}}|^{1/2}}\hat{\eta}_{P_2,2}(c_{\omega_T})\Phi^{n-l,T}_{\vec{P},5},\Phi^{lac}_{|\vec{Q}|}\right\rangle\\
\times &\frac{\langle f_2,\Phi_{Q_1,2}\rangle\langle f_3,\Phi_{Q_2,3}\rangle}{|I_{\vec{Q}}|^{1/2}}\\
&\Lambda_{I_b}(f_1,f_2,f_3,f_4)\\
= &\sum_{\vec{Q}\in T_2}\left\langle \sum_{\vec{P}\in\mathbb{P}(T_2)} \frac{\langle f_4,\Phi_{-P_1,4}\rangle\langle f_1,\Phi_{P_1,1}\rangle}{|I_{\vec{P}}|^{-1/2}}\hat{\eta}^{(1)}_{P_2,2}(c_{\omega_T})\Phi^{n-l,T}_{\vec{P},5},\Phi^{lac}_{|\vec{Q}|}\right\rangle\\
\times &\frac{\langle \hat{f}_2,(\cdot-c_{\omega_{T_2}})\hat{\Phi}_{Q_1,2}\rangle\langle f_3,\Phi_{Q_2,3}\rangle}{|I_{\vec{Q}}|^{1/2}}\\
&\Lambda_{I_c}(f_1,f_2,f_3,f_4)\\
= &\frac{1}{2}\sum_{\vec{Q}\in T_2}\left\langle \sum_{\vec{P}\in\mathbb{P}(T_2)} \frac{\langle f_4,\Phi_{-P_1,4}\rangle\langle f_1,\Phi_{P_1,1}\rangle}{|I_{\vec{P}}|^{-3/2}}\hat{\eta}^{(2)}_{P_2,2}(c_{\omega_T})\Phi^{n-l,T}_{\vec{P},5},\Phi^{lac}_{|\vec{Q}|}\right\rangle\\
\times &\frac{\langle \hat{f}_2,(\cdot-c_{\omega_{T_2}})^2\hat{\Phi}_{Q_1,2}\rangle\langle f_3,\Phi_{Q_2,3}\rangle}{|I_{\vec{Q}}|^{1/2}}
\end{aligned}
$$

and

$$
\begin{aligned}
&\Lambda_{II}(f_1,f_2,f_3,f_4)\\
= &\sum_{\vec{Q}\in T_2}\sum_{\vec{P}\in\mathbb{P}(T_2)}\frac{1}{|I_{\vec{Q}}|^{1/2}|I_{\vec{P}}|^{1/2}}\langle f_4,\Phi_{-P_1,4}\rangle\langle f_1,\Phi_{P_1,1}\rangle\left\langle\Phi^{n-l}_{|\vec{P}|},\Phi^{lac}_{|\vec{Q}|}\right\rangle\\
\times &\langle \hat{f}_2\cdot R^T_{\vec{P}}\cdot\hat{\eta}_{\omega_{Q_1}},\hat{\Phi}_{Q_1,2}\rangle\langle f_3,\Phi_{Q_2,3}\rangle.
\end{aligned}
$$

Furthermore, we may rewrite $(\xi-c_{\omega_{T_2}})=(\xi-c_{\omega_{Q_2}})+(c_{\omega_{Q_2}}-c_{\omega_{T_2}})$, in which case

$$
\begin{aligned}
& \Lambda_{I_b}(f_1, f_2, f_3, f_4) \\
= \quad & \sum_{\vec{Q}\in T_2} \frac{1}{|I_{\vec{Q}}|^{1/2}} \left\langle \sum_{\vec{P}\in\mathbb{P}(T_2)} \frac{\langle f_4, \Phi_{-P_1,4}\rangle\langle f_1, \Phi_{P_1,1}\rangle}{|I_{\vec{P}}|^{1/2}} \Phi^{n-l,T}_{|\vec{P}|,2}|I_{\vec{P}}|, \Phi^{lac}_{|\vec{Q}|} \right\rangle \\
\times \quad & \langle \hat{f}_2, (\cdot - c_{\omega_{Q_2}})\hat{\Phi}_{Q_1,2}\rangle\langle f_3, \Phi_{Q_2,3}\rangle \\
+ \quad & \sum_{\vec{Q}\in T_2} \frac{c_{\omega_{Q_2}} - c_{\omega_{T_2}}}{|I_{\vec{Q}}|^{1/2}} \left\langle \sum_{\vec{P}\in\mathbb{P}(T_2)} \frac{\langle f_4, \Phi_{-P_1,4}\rangle\langle f_1, \Phi_{P_1,1}\rangle}{|I_{\vec{P}}|^{1/2}} \Phi^{n-l,T}_{\vec{P},5}|I_{\vec{P}}|, \Phi^{lac}_{|\vec{Q}|} \right\rangle \\
\times \quad & \langle f_2, \Phi_{Q_1,2}\rangle\langle f_3, \Phi_{Q_2,3}\rangle \\
:= \quad & \Lambda_{I_{b,1}}(f_1, f_2, f_3, f_4) + \Lambda_{I_{b,2}}(f_1, f_2, f_3, f_4).
\end{aligned}
$$

By construction,

$$
\begin{aligned}
& \Lambda_{I_{b,1}}(f_1, f_2, f_3, f_4) \\
= \quad & \sum_{\vec{Q}\in T_2} \frac{1}{|I_{\vec{Q}}|^{1/2}} \left\langle \sum_{\vec{P}\in\mathbb{P}(T_2)} \frac{\langle f_4, \Phi_{-P_1,4}\rangle\langle f_1, \Phi_{P_1,1}\rangle}{|I_{\vec{P}}|^{1/2}} \Phi^{n-l,T}_{\vec{P},5}|I_{\vec{P}}|, \frac{\Phi^{lac}_{|\vec{Q}|}}{|I_{\vec{Q}}|} \right\rangle \langle f_2, \Phi_{Q_2,2,b}\rangle\langle f_3, \Phi_{Q_2,3}\rangle.
\end{aligned}
$$

Furthermore,

$$
\begin{aligned}
& \left|\Lambda_{I_{b,2}}(f_1, f_2, f_3, f_4)\right| \\
\leq \quad & \sum_{\vec{Q}\in T_2} \frac{1}{|I_{\vec{Q}}|^{1/2}} \left| \left\langle \sum_{\vec{P}\in\mathbb{P}(T_2)} \frac{\langle f_4, \Phi_{-P_1,4}\rangle\langle f_1, \Phi_{P_1,1}\rangle}{|I_{\vec{P}}|^{1/2}} \Phi^{n-l,T}_{\vec{P},5}|I_{\vec{P}}|, \frac{\Phi^{lac}_{|\vec{Q}|}}{|I_{\vec{Q}}|} \right\rangle \right| \left|\langle f_2, \Phi_{Q_1,2}\rangle\langle f_3, \Phi_{Q_2,3}\rangle\right|.
\end{aligned}
$$

To handle $\Lambda_{I_c}(f_1, f_2, f_3, f_4)$, we rewrite

$$
\begin{aligned}
(\xi - c_{\omega_{T_2}})^2 \quad &= \quad ((\xi - c_{\omega_{Q_2}}) + (c_{\omega_{Q_2}} - c_{\omega_{T_2}}))^2 \\
&= \quad (\xi - c_{\omega_{Q_2}})^2 + 2(\xi - c_{\omega_{Q_2}})(c_{\omega_{Q_2}} - c_{\omega_{T_2}}) + (c_{\omega_{Q_2}} - c_{\omega_{T_2}})^2.
\end{aligned}
$$

It is a straightforward matter to decompose $\Lambda_{I_c}(f_1, f_2, f_3, f_4)$ into three terms corresponding to the above display and then to bound each using previous observations. Each of the three terms can be majorized by a generic expression of the form

$$\sum_{\vec{Q}\in T_2} \frac{1}{|I_{\vec{Q}}|^{1/2}} \left| \left\langle \sum_{\vec{P}\in\mathbb{P}(T_2)} \frac{\langle f_4, \Phi_{-P_1,4}\rangle\langle f_1, \Phi_{P_1,1}\rangle}{|I_{\vec{P}}|^{1/2}} \tilde{\Phi}^{n-l,T}_{\vec{P}} |I_{\vec{P}}|^2, \frac{\Phi^{lac}_{|\vec{Q}|}}{|I_{\vec{Q}}|^2} \right\rangle \right| |\langle f_2, \Phi_{Q_1,2,j}\rangle\langle f_3, \Phi_{Q_2,3}\rangle| ,$$

where $j \in \{a, b, c\}$. In considering the remainder $R^T_{\vec{P}}(\xi)$, expand using Fourier series

$$R^T_{\vec{P}}(\xi)\hat{\eta}_{\omega_{Q_1}}(\xi) = \sum_{\lambda\in\mathbb{Z}} c^\lambda_{\vec{P},\vec{Q}} \hat{\eta}^\lambda_{\omega_{Q_1}}(\xi)$$

where the sequence $\left|c^\lambda_{\vec{P},\vec{Q}}\right| \lesssim \frac{|I_{\vec{P}}|^3}{|I_{\vec{Q}}|^3} \frac{1}{1+|\lambda|^{\tilde{N}}}$. Indeed, this is a straightforward consequence of the definition of $R^T_{\vec{P}}(\xi)$, and in the support of $R^T_{\vec{P}}(\xi)\hat{\eta}_{\omega_{Q_1}}(\xi)$, $|\xi - c_{\omega_T}| \lesssim \frac{1}{|I_{\vec{Q}}|}$. Plugging into our formula yields

$$\begin{aligned}
&\Lambda_{II}(f_1, f_2, f_3, f_4) \\
= &\sum_{\vec{Q}\in T_2} \sum_{\vec{P}\in\mathbb{P}^d_{n_1,n_4}\cap\mathbb{P}(T_2)} \sum_{\lambda\in\mathbb{Z}} \frac{c^\lambda_{\vec{P},\vec{Q}}}{|I_{\vec{Q}}|^{1/2}|I_{\vec{P}}|^{1/2}} \langle f_4, \Phi_{-P_1,4}\rangle\langle f_1, \Phi_{P_1,1}\rangle \left\langle \Phi^{n-l}_{|\vec{P}|}, \Phi^{lac}_{|\vec{Q}|} \right\rangle \\
\times &\langle f_2, \Phi^\lambda_{Q_1,2}\rangle\langle f_3, \Phi_{Q_2,3}\rangle
\end{aligned}$$

so that

$$\begin{aligned}
&|\Lambda_{II}(f_1, f_2, f_3, f_4)| \\
\leq &\sum_{\lambda\in\mathbb{Z}} \frac{1}{1+|\lambda|^{\tilde{N}}} \sum_{\vec{Q}\in T_2} \sum_{\vec{P}\in\mathbb{P}^d_{n_1,n_4}\cap\mathbb{P}(T_2)} \frac{|I_{\vec{P}}|^{5/2}}{|I_{\vec{Q}}|^{7/2}} \left| \langle f_4, \Phi_{-P_1,4}\rangle\langle f_1, \Phi_{P_1,1}\rangle \left\langle \Phi^{n-l}_{|\vec{P}|}, \Phi^{lac}_{|\vec{Q}|} \right\rangle \right. \\
\times &\left. \langle f_2, \Phi^\lambda_{Q_1,2}\rangle\langle f_3, \Phi_{Q_2,3}\rangle \right| .
\end{aligned}$$

At this stage, we do not need to analyze $\Lambda_{T_2,II}(f_1, f_2, f_3, f_4)$ any further.

2.16.2 Tree Estimates

Λ_{I_a} Tree Estimate

Lemma 18. *The following I_a type size estimate holds: for any $0<\theta<1$,*

$$\left(\sum_{\vec{Q}\in T_2}\left|\left\langle\sum_{\vec{P}\in\mathbb{P}:|I_{\vec{P}}|<<|I_{\vec{Q}}|}\frac{\langle f_1,\Phi_{-P_1,1}\rangle\langle f_4,\tilde{\Phi}_{P_1,4}\rangle}{|I_{\vec{P}}|^{1/2}}\Phi^{n-l,T}_{|\vec{P}|,a},\Phi^{lac}_{|\vec{Q}|}\right\rangle\right|^2\right)^{1/2}$$
$$\lesssim_\theta \left[\sup_{\vec{Q}\in\mathcal{T}_2}\frac{1}{|I_{\vec{Q}}|}\int_{E_1}\tilde{1}_{I_{\vec{Q}}}dx\right]^{1-\theta}\left[\sup_{\vec{Q}\in\mathcal{T}_2}\frac{1}{|I_{\vec{Q}}|}\int_{E_4\cap\Omega^c}\tilde{1}_{I_{\vec{Q}}}dx\right]^{\theta}.$$

Proof. We include the proof taken from [21] for the reader's convenience. For each $\vec{Q}\in T$, set

$$a^T_{\vec{Q}}=\left\langle\sum_{\vec{P}\in\mathbb{P}:|I_{\vec{P}}|<<|I_{\vec{Q}}|}\frac{\langle f_1,\Phi_{-P_1,1}\rangle\langle f_4,\tilde{\Phi}_{P_1,4}\rangle}{|I_{\vec{P}}|^{1/2}}\Phi^{n-l,T}_{|\vec{P}|,a},\Phi^{lac}_{|\vec{Q}|}\right\rangle.$$

By John-Nirenberg, it suffices to show

$$\left\|\left(\sum_{\vec{Q}\in T}|a^T_{\vec{Q}}|^2\frac{1_{I_{\vec{Q}}}}{|I_{\vec{Q}}|}\right)^{1/2}\right\|_{L^{1,\infty}(I_T)}\lesssim|I_T|\sup_{\vec{Q}\in T}\left[\frac{\int_{E_1}\tilde{1}_{I_{\vec{Q}}}}{|I_{\vec{Q}}|}\right]^{1-\theta}\left[\frac{\int_{E_4\cap\Omega^c}\tilde{1}_{I_{\vec{Q}}}dx}{|I_{\vec{Q}}|}\right]^{\theta}.$$

We may assume that T contains its top P_T, in which case we may reduce to

$$\left\|\left(\sum_{\vec{Q}\in T}|a^T_{\vec{Q}}|^2\frac{1_{I_{\vec{Q}}}}{|I_{\vec{Q}}|}\right)^{1/2}\right\|_{L^{1,\infty}(I_T)}\lesssim\left[\int_{E_1}\tilde{1}_{I_T}dx\right]^{1-\theta}\left[\int_{E_4\cap\Omega^c}\tilde{1}_{I_T}dx\right]^{\theta}.$$

Fix T. First consider the relatively easy case when f_1 vanishes on $5I_T$. In this case, we shall prove the stronger estimate

$$\begin{aligned} |a^T_{\vec{Q}}| &\lesssim |I_{\vec{Q}}|^{-1/2} \left[\int_{E_1} \tilde{1}_{I_{\vec{Q}}} dx\right]^{1-\theta} \left[\int_{E_4\cap\Omega^c} \tilde{1}_{I_{\vec{Q}}} dx\right]^{\theta} \\ &\lesssim |I_{\vec{Q}}|^{-1/2} \left(\frac{|I_{\vec{Q}}|}{|I_T|}\right)^{M(1-\theta)} \left[\int_{E_1} \tilde{1}_{I_T} dx\right]^{1-\theta} \left[\int_{E_4\cap\Omega^c} \tilde{1}_{I_T} dx\right]^{\theta}. \end{aligned}$$

The claim then follows by square-summing in $\vec{Q}$. To prove the claim, fix $\vec{Q} \in T$ and estimate

$$|a^T_{\vec{Q}}| \lesssim |I_{\vec{Q}}|^{-1/2} \sum_{\vec{P}\in\mathbb{P}(T):|I_{\vec{P}}|\lesssim|I_{\vec{Q}}|} |\langle f_1, \Phi_{-P_1,1}\rangle||\langle f_4, \Phi_{P_1,4}\rangle| \int_{\mathbb{R}} \frac{\tilde{1}_{I_{\vec{P}}}}{|I_{\vec{P}}|}\tilde{1}_{I_{\vec{Q}}} dx.$$

Interchanging the sum and the integral and applying Cauchy-Schwarz gives

$$|a^T_{\vec{Q}}| \lesssim |I_{\vec{Q}}|^{-1/2} \int_{\mathbb{R}} |S_1 f_1||S_4 f_4|\tilde{1}_{I_{\vec{Q}}} dx,$$

where each S_1, S_4 is a Calderon-Zygmund operator with localized estimates on $\tilde{1}_{I_{\vec{Q}}}$. Therefore, the claim is true and we are done when f_1 vanishes on $5I_T$. Similarly, we are done if f_4 vanishes on $5I_T$. Hence, it suffices to consider the case when both E_1, E_4 are supported inside $5I_T$. In this situation, it suffices to prove

$$\begin{aligned} &\left\| \left(\sum_{\vec{Q}\in T} \left| \left\langle \sum_{\vec{P}\in\mathbb{P}(T)} \frac{\langle f_1, \Phi_{-P_1,1}\rangle\langle f_4, \Phi_{P_1,4}\rangle}{|I_{\vec{P}}|} \Phi^{n-l}_{|\vec{P}|}, \Phi^{lac}_{\vec{Q}} \right\rangle \right|^2 \frac{\tilde{1}_{I_{\vec{Q}}}}{|I_{\vec{Q}}|} \right)^{1/2} \right\|_{L^{1,\infty}(\mathbb{R})} \\ \lesssim_\theta \quad & |E_1|^{1-\theta}|E_4\cap\Omega^c|^{\theta}. \end{aligned}$$

However, this follows from the $L^1 \to L^{1,\infty}$ estimates for Calderon-Zygmund operators together with the estimate $\sum_{\vec{P}\in\mathbb{P}(T)} |\langle f_1, \Phi_{-P_1,1}\rangle||\langle f_4, \Phi_{P_1,4}\rangle| \lesssim_\theta |E_1|^{1-\theta}|E_4 \cap \Omega^c|^\theta$.

$\square$

To handle $\Lambda_{I_a}(f_1, f_2, f_3, f_4)$, note that the $\mathbb{Q}-$tree T_2 must be overlapping in either the first or second index. Without loss of generality, assume that T_2 is overlapping in the 1st index. Then the Biest size estimate gives

$$\begin{aligned}
&|\Lambda_{T_2,I}(f_1, f_2, f_3, f_4)| \\
\leq\ & \left(\sum_{\vec{Q}\in T_2}\left|\left\langle \sum_{\vec{P}\in\mathbb{P}(T_2)} \frac{\langle f_1, \Phi_{-P_1,1}\rangle\langle f_4, \tilde{\Phi}_{P_1,4}\rangle}{|I_{\vec{P}}|^{1/2}}\Phi^{n-l,T}_{|\vec{P}|}, \Phi^{lac}_{|\vec{Q}|}\right\rangle\right|^2\right)^{1/2}\left[\sup_{\vec{Q}\in T_2}\frac{|\langle f_2, \Phi_{Q_1,2}\rangle|}{|I_{\vec{Q}}|^{1/2}}\right] \\
\times\ & \left(\sum_{\vec{Q}\in T_2}|\langle f_3, \Phi_{Q_2,3}\rangle|^2\right)^{1/2} \\
\lesssim_\theta\ & 2^{-n_1(1-\theta)}2^{-n_4\theta}2^{-n_2}2^{-n_3}|I_{T_2}|,
\end{aligned}$$

for any $0 < \theta < 1$.

$\Lambda_{I_b}, \Lambda_{I_c}$ **Tree Estimates**

To contend with $\Lambda_{I_b}(f_1, f_2, f_3, f_4)$ and $\Lambda_{I_c}(f_1, f_2, f_3, f_4)$, we verify

Lemma 19. *The following I_b, I_c type size estimate holds: for any $\epsilon > 0$ and $0 < \theta < 1$,*

$$\left(\sum_{\vec{Q} \in T_2} \left| \left\langle \sum_{\vec{P} \in \mathbb{P} : |I_{\vec{P}}| << |I_{\vec{Q}}|} \frac{1}{|I_{\vec{P}}|^{\epsilon}} \frac{\langle f_1, \Phi_{-P_1,1} \rangle \langle f_4, \tilde{\Phi}_{P_1,4} \rangle}{|I_{\vec{P}}|^{1/2}} \Phi^{n-l,T}_{|\vec{P}|}, \Phi^{lac}_{|\vec{Q}|} |I_{\vec{Q}}|^{\epsilon} \right\rangle \right|^2 \right)^{1/2}$$

$$\lesssim_{\epsilon,\theta} \quad \left[\sup_{\vec{Q} \in \mathcal{T}_2} \frac{1}{|I_{\vec{Q}}|} \int_{E_1} \tilde{1}_{I_{\vec{Q}}} dx \right]^{1-\theta} \left[\sup_{\vec{Q} \in \mathcal{T}_2} \frac{1}{|I_{\vec{Q}}|} \int_{E_4 \cap \Omega^c} \tilde{1}_{I_{\vec{Q}}} dx \right]^{\theta}.$$

Proof. The standard Biest size proof handles the cases when either E_1 or E_4 vanishes on $5I_T$. Hence, it suffices to assume $E_1, E_4 \subset 5I_T$ and show

$$\left\| \sum_{\vec{Q} \in T} \int_{\mathbb{R}} \left[\sum_{\vec{P} \in \mathbb{P}(T) : |I_{\vec{P}}| << |I_{\vec{Q}}|} \frac{|\langle f_1, \Phi_{-P_1,1} \rangle \langle f_4, \Phi_{P_1,4} \rangle|}{|I_{\vec{P}}|} \tilde{1}_{I_{\vec{P}}} \frac{\tilde{1}_{I_{\vec{Q}}}}{|I_{\vec{Q}}|^{1/2}} \right] dx \frac{1_{I_{\vec{Q}}}}{|I_{\vec{Q}}|^{1/2}} \frac{|I_{\vec{P}}|^{\epsilon}}{|I_{\vec{Q}}|^{\epsilon}} \right\|_{L^1(\mathbb{R})}$$

$$\lesssim_{\theta} \quad |E_1|^{1-\theta} |E_4 \cap \Omega^c|^{\theta}.$$

By the triangle inequality, it is enough to show

$$\left\| \sum_{\vec{P} \in \mathbb{P}(T)} \sum_{\vec{Q} \in T : |I_{\vec{Q}}| = 2^{-k_0} |I_{\vec{P}}|} \int_{\mathbb{R}} \frac{|\langle f_1, \Phi_{-P_1,1} \rangle \langle f_4, \Phi_{P_1,4} \rangle|}{|I_{\vec{P}}|} \tilde{1}_{I_{\vec{P}}} \frac{\tilde{1}_{I_{\vec{Q}}}}{|I_{\vec{Q}}|^{1/2}} dx \frac{1_{I_{\vec{Q}}}}{|I_{\vec{Q}}|^{1/2}} \right\|_{L^1(\mathbb{R})}$$

$$\lesssim_{\theta} \quad |E_1|^{1-\theta} |E_4 \cap \Omega^c|^{\theta}$$

with an implicit constant independent of $k_0 \geq 0$. However, this is immediate from the observation that the above display can be bounded by

$$\left\| \sum_{\vec{P} \in \mathbb{P}(T)} |\langle f_1, \Phi_{-P_1,1} \rangle \langle f_4, \Phi_{P_1,4} \rangle| \frac{\tilde{1}_{2^{k_0} I_{\vec{P}}}}{|2^{k_0} I_{\vec{P}}|} \right\|_{L^1(\mathbb{R})}$$

$$= \sum_{\vec{P} \in \mathbb{P}(T)} |\langle f_1, \Phi_{-P_1,1} \rangle \langle f_4, \Phi_{P_1,4} \rangle| \lesssim_{\theta} |E_1|^{1-\theta} |E_4 \cap \Omega^c|^{\theta}.$$

□

2.16.3 Energy Estimates

Before proceeding to proving generalized restricted type mixed estimates for Λ, we must state

Lemma 20.

$$\sum_{I \in \mathbb{I}_{n_1,n_4}} |I| \lesssim \min\{2^{n_1}|E_1|, 2^{n_4}\}.$$

Proof. Immediate from the construction of $\mathbb{I}_{n_1,n_4}$. □

Lemma 21. *The following energy estimate holds:*

$$\sum_{I \in \mathbb{I}_{n_1,n_4}} \sum_{T \in \mathcal{T}\left\{\mathbb{Q}^{\tilde{d}}_{n_1,n_4}[I]_{n_2,2}\right\}} |I_{T_2}| \lesssim_\epsilon \min\left\{2^{2n_2}|E_2|, 2^{n_2}|E_2| \min\{2^{n_1}|E_1|, 2^{n_4}\}\right\}.$$

Proof. The fact that

$$\sum_{I \in \mathbb{I}_{n_1,n_4}} \sum_{T \in \mathcal{T}\left\{\mathbb{Q}^{\tilde{d}}_{n_1,n_4}[I]_{n_2,2}\right\}} |I_{T_2}| \lesssim 2^{2n_2}|E_2|$$

is by now standard, and so its proof is omitted. Therefore, it suffices to establish

$$\sum_{I \in \mathbb{I}_{n_1,n_4}} \sum_{T \in \mathcal{T}\left\{\mathbb{Q}^{\tilde{d}}_{n_1,n_4}[I]_{n_2,2}\right\}} |I_{T_2}| \lesssim_\epsilon 2^{n_2}|E_2| \min\left\{2^{n_1}|E_1|, 2^{n_4}\right\}.$$

To this end, use an argument similar to that found in the toy model section to see

$$\begin{aligned} \sum_{T \in \mathcal{T}\left\{\mathbb{Q}^{\tilde{d}}_{n_1,n_4}[I]_{n_2,2}\right\}} |I_{T_2}| &\lesssim 2^{2n_2} \sum_{T \in \mathcal{T}\left\{\mathbb{Q}^{\tilde{d}}_{n_1,n_4}[I]_{n_2,2}\right\}} \sum_{\vec{Q} \in T_2} |\langle f_2, \Phi_{Q_1,2}\rangle|^2 \\ &\lesssim 2^{n_2}|E_2| \cdot |I|. \end{aligned}$$

Therefore, an application of the proceeding lemma yields the claim.

□

It is now straightforward to observe

Lemma 22. *For fixed* $\tilde{d} \geq 0, n_1 \geq N_1(\tilde{d}), n_2 \geq N_2(\tilde{d}), n_3 \geq N_3(\tilde{d}), n_4 \geq N_4(\tilde{d})$ *and any* $0 < \theta < 1$

$$\begin{aligned}
& \left| \sum_{\vec{Q} \in \left[\mathbb{Q}^{\tilde{d}}_{n_1,n_4}\right]^{n_3}_{n_2}} \sum_{\vec{P} \in \mathbb{P}^{d}_{n_1,n_4}} \Lambda_{\mathbb{Q},\mathbb{P}}(f_1, f_2, f_3, f_4) \right| \\
\lesssim_\theta \; & 2^{-n_1(1-\theta)} 2^{-n_2} 2^{-n_3} 2^{-n_4\theta} \min\left\{2^{2n_2}|E_2|, 2^{n_2}|E_2| \cdot \min\{2^{n_1}|E_1|, 2^{n_4}\}, 2^{2n_3}|E_3|\right\} \\
+ \; & \sum_{\lambda \in \mathbb{Z}} \frac{1}{1+|\lambda|^{\tilde{N}}} \sum_{\vec{Q} \in \left[\mathbb{Q}^{\tilde{d}}_{n_1,n_4}\right]^{n_3}_{n_2}} \sum_{\vec{P} \in \mathbb{P}: \omega_{P_2} \supset \supset \omega_{Q_1}} \frac{|I_{\vec{P}}|^2}{|I_{\vec{Q}}|^4} \left| \langle f_4, \Phi_{-P_1,4}\rangle \langle f_1, \Phi_{P_1,1}\rangle \left\langle \tilde{1}_{I_{\vec{P}}}, \tilde{1}_{I_{\vec{Q}}} \right\rangle \right. \\
\times \; & \left. \langle f_2, \Phi^{\lambda}_{Q_1,2}\rangle \langle f_3, \Phi_{Q_2,3}\rangle \right| .
\end{aligned}$$

Proof. Put together the sum decomposition with tree size estimates. □

An immediate consequence of the above lemma is

Corollary 6. *The following estimate holds:*

$$\begin{aligned}
& |\Lambda(f_1, f_2, f_3, f_4)| \\
\lesssim \; & \sum_{\tilde{d}, \vec{n} \geq \vec{N}(\tilde{d})} 2^{-n_1} 2^{-n_2} 2^{-n_3} 2^{-n_4} \min\left\{2^{2n_2}|E_2|, 2^{n_2}|E_2| \cdot \min\{2^{n_1}|E_1|, 2^{n_4}\}, 2^{2n_3}|E_3|\right\} \\
+ \; & \sum_{\lambda \in \mathbb{Z}} \frac{1}{1+|\lambda|^{\tilde{N}}} \sum_{\vec{Q} \in \mathbb{Q}} \sum_{\vec{P} \in \mathbb{P}: \omega_{P_2} \supset \supset \omega_{Q_1}} \frac{|I_{\vec{P}}|^2}{|I_{\vec{Q}}|^4} \left| \langle f_4, \Phi_{-P_1,4}\rangle \langle f_1, \Phi_{P_1,1}\rangle \left\langle \tilde{1}_{I_{\vec{P}}}, \tilde{1}_{I_{\vec{Q}}} \right\rangle \right. \\
\times \; & \left. \langle f_2, \Phi^{\lambda}_{Q_1,2}\rangle \langle f_3, \Phi_{Q_2,3}\rangle \right| \\
:= \; & \Lambda_I(f_1, f_2, f_3, f_4) + \Lambda_{II}(f_1, f_2, f_3, f_4).
\end{aligned}$$

Proof. Sum the previous estimate over $\tilde{d} \geq 0, n_1 \geq N_1(\tilde{d}), n_2 \geq N_2(\tilde{d}), n_3 \geq N_3(\tilde{d}), n_4 \geq N_4(\tilde{d})$ and use the triangle inequality. □

Note that $\Lambda_{II}(f_1, f_2, f_3, f_4)$ may be rewritten as

$$\sum_{\lambda \in \mathbb{Z}} \sum_{k_0 >> 1} \frac{2^{-3k_0}}{1+|\lambda|^{\tilde{N}}} \sum_{\vec{Q} \in \mathbb{Q}} \sum_{\vec{P} \in \mathbb{P} : |I_{\vec{P}}| = 2^{-k_0}|I_{\vec{Q}}|, \omega_{P_2} \supset\supset \omega_{Q_1}} \frac{1}{|I_{\vec{Q}}||I_{\vec{P}}|} |\langle f_4, \Phi_{-P_1,4}\rangle \langle f_1, \Phi_{P_1,1}\rangle$$
$$\times \left\langle \tilde{1}_{I_{\vec{P}}}, \tilde{1}_{I_{\vec{Q}}} \right\rangle \langle f_2, \Phi^{\lambda}_{Q_1,2}\rangle \langle f_3, \Phi_{Q_2,3}\rangle \Big| .$$

Therefore, to handle $\Lambda_{II}(f_1, f_2, f_3, f_4)$, it suffices to obtain restricted weak-type estimates for $\Lambda^{k_0}_{Toy}$ with operational bounds $O(2^{2k_0})$. However, this was already accomplished with the toy model decomposition.

2.16.4 Size Restrictions

It is routine to observe the size restrictions $2^{-n_1} \lesssim 2^{\tilde{d}}|E_1|, 2^{-n_2} \lesssim 2^{\tilde{d}}|E_2|, 2^{-n_3} \lesssim 2^{\tilde{d}}|E_3|, 2^{-n_4} \lesssim 2^{-\tilde{N}\tilde{d}}$, so the details are omitted.

2.16.5 Mixed Generalized Restricted Type Type Estimates for $\Lambda_I(f_1, f_2, f_3, f_4)$

It turns out that the global energy bound

$$\sum_{I \in \mathbb{I}_{n_1,n_4}} \sum_{T \in \mathcal{T}\left\{\mathbb{Q}^{\tilde{d}}_{n_1,n_4}[I]_{n_2,2}\right\}} |I_{T_2}| \lesssim 2^{2n_2}|E_2|$$

is not necessary to produce estimates. Putting it all together yields for any $0 \leq \theta_1, \theta_2 \leq 1$ satisfying $\theta_1 + \theta_2 = 1$ along with $0 < \theta < 1$ and $0 \leq \gamma \leq 1$ the upper bound

$$\begin{aligned}
&\Lambda_I(f_1,f_2,f_3,f_4)\\
\lesssim_\theta\ & \sum_{\tilde{d}\geq 0}\sum_{\vec{n}\geq \vec{N}(\tilde{d})} 2^{-n_1(1-\theta)}2^{-n_2}2^{-n_3}2^{-n_4\theta}\min\left\{2^{2n_2}|E_2|, 2^{n_2}|E_2|\cdot\min\{2^{n_1}|E_1|,2^{n_4}\}, 2^{2n_3}|E_3|\right\}\\
\leq\ & \sum_{\tilde{d}\geq 0}\sum_{\vec{n}\geq \vec{N}(\tilde{d})} 2^{-n_1(1-\theta)}2^{-n_2}2^{-n_3}2^{-n_4\theta}\left[2^{n_2}|E_2|\cdot 2^{n_1(1-\gamma)}|E_1|^{1-\gamma}2^{n_4\gamma}\right]^{\theta_1}\left[2^{2n_3}|E_3|\right]^{\theta_2}\\
=\ & \sum_{\tilde{d}\geq 0}\sum_{\vec{n}\geq \vec{N}(\tilde{d})} 2^{-n_1(1-\theta-\theta_1(1-\gamma))}2^{-n_2(1-\theta_1)}2^{-n_3(1-2\theta_2)}2^{-n_4(\theta-\theta_1\gamma)}|E_1|^{(1-\gamma)\theta_1}|E_2|^{\theta_1}|E_3|^{\theta_2}.
\end{aligned}$$

Therefore, the summability conditions are $\theta+\theta_1(1-\gamma),\theta_1<1, 2\theta_2<1$ and $\theta_1\gamma<\theta$, in which case the above display boils down to

$$\Lambda_I(f_1,f_2,f_3,f_4)\lesssim \min\{|E_1|^{(1-\gamma)\theta_1},|E_1|^{1-\theta}\}|E_2|^{1/2+\theta_1/2}\min\{|E_3|^{\theta_2},|E_3|^{1-\theta_2}\}.$$

Choosing $\theta_1=1/2+\epsilon,\theta_2=1/2-\epsilon,\theta=\gamma=1-\epsilon$ followed by $\theta_1=1/2-\epsilon,\theta_2=1/2+\epsilon,\theta=\gamma=\epsilon$ yields restricted weak-type estimates in neighborhoods near

$$E_0 = \left\{\left(0,\frac{1}{2},\frac{1}{2},0\right),\left(1,\frac{1}{2},\frac{1}{2},-1\right)\right\}.$$

Moreover, letting $\theta_1=1-\epsilon,\theta_2=\epsilon,\theta=\gamma=1-\epsilon$ yields estimates in neighborhoods near

$$E_2 = \{(0,1,1,0),(0,1,0,1)\}.$$

Lastly, letting $\theta_1=1-\epsilon,\theta_2=\epsilon,\theta=\gamma=\epsilon$ yields estimates in neighborhoods near

$$E_1 = \{(1,1,1,-1),(1,1,0,0)\}.$$

Now recall the generic toy model estimate

$$\begin{aligned}
&\Lambda^{k_0}_{Toy}(f_1,f_2,f_3,f_4)\\
\lesssim\ & 2^{2k_0}\sum_{\tilde{d},\vec{n}\geq\vec{N}(\tilde{d})} 2^{-n_1}2^{-n_2}2^{-n_3}2^{-n_4}\min\{2^{n_2}|E_2|\min\{2^{n_4},2^{n_1}|E_1|\},2^{2n_3}|E_3|\}
\end{aligned}$$

with size restrictions $2^{-n_1} \lesssim \min\{1, 2^d|E_1|\}$, $2^{-n_2} \lesssim \min\{2^{\tilde{d}}|E_2|^{1/2}, 2^{\tilde{d}}|E_2|\}, 2^{-n_3} \lesssim \min\{1, 2^{\tilde{d}}|E_3|\}, 2^{-n_4} \lesssim 2^{-\tilde{N}d}$. To estimate $\Lambda^{k_0}_{Toy}(f_1, f_2, f_3, f_4)$, it suffices to note

$$\begin{aligned} &\Lambda^{k_0}_{Toy}(f_1, f_2, f_3, f_4) \\ \lesssim &\sum_{\tilde{d}, \vec{n} \geq \vec{N}(\tilde{d})} 2^{-n_1(1-\theta_1(1-\gamma))} 2^{-n_2(1-\theta_1)} 2^{-n_3(1-2\theta_2)} 2^{-n_4(1-\gamma\theta_1)} |E_1|^{(1-\gamma)\theta_1} |E_2|^{\theta_1} |E_3|^{\theta_2}. \end{aligned}$$

It is a simple matter to see that (generalized) restricted weak type estimates are available in neighborhoods near E_0, E_1, E_2. Now interpolating and applying symmetry yields

$$B[a_1, a_2] : L^{p_1}(\mathbb{R}) \times W_{p_2}(\mathbb{R}) \times L^{p_3}(\mathbb{R}) \to L^{\frac{1}{\frac{1}{p_1}+\frac{1}{p_2}+\frac{1}{p_3}}}(\mathbb{R})$$

provided $\frac{1}{p_1} + \frac{1}{p_2} < 1, \frac{1}{p_2} + \frac{1}{p_3} < 1$, $2 < p_2 < \infty$. □

CHAPTER 3

L^P ESTIMATES FOR SEMI-DEGENERATE SIMPLEX OPERATORS

3.1 Introduction

We proved in Chapter 2 a wide range of mixed estimates for the generic trilinear degenerate simplex multiplier $B[a_1, a_2]$, which can be identified modulo harmless modifications in the simple case when $a_1 = a_2 = sgn$ with

$$C^{-1,1,-1} : (f_1, f_2, f_3) \mapsto \int_{\xi_1<\xi_2<\xi_3} \left[\prod_{j=1}^{3} \hat{f}_j(\xi_j) e^{2\pi i x (-1)^j \epsilon_j \xi_j} \right] d\vec{\xi}.$$

In this chapter, we shift our focus from mixed estimates for degenerate simplex operators to L^p estimates for semi-degenerate simplex operators. However, before delving into this new semi-degenerate situation, it is helpful to recall the mapping properties of a non-degenerate simplex multiplier called the Biest and given by the formula

$$C^{1,1,1} : (f_1, f_2, f_3) \mapsto \int_{\xi_1<\xi_2<\xi_3} \left[\prod_{j=1}^{3} \hat{f}_j(\xi_j) e^{2\pi i x \epsilon_j \xi_j} \right] d\vec{\xi}.$$

C. Muscalu, T. Tao, and C. Thiele use a robust time-frequency argument in [21] to show that the above operator satisfies a wide range of L^p estimates. More precisely, they have the following:

Theorem 19. *$C^{1,1,1} : L^{p_1}(\mathbb{R}) \times L^{p_2}(\mathbb{R}) \times L^{p_3}(\mathbb{R}) \to L^{p_4'}(\mathbb{R})$ as long as $(1/p_1, 1/p_2, 1/p_3, 1/p_4) \in \mathbb{D} \cap \mathbb{D}'$, $1 < p_1, p_2, p_3 \leq \infty$ and $0 < p_4' < \infty$, where $\mathbb{D}$ is the interior of the convex hull of the twelve points*

$$
\begin{aligned}
D_1 &= \left(1, \frac{1}{2}, 1, -\frac{3}{2}\right) \quad D_2 = \left(\frac{1}{2}, 1, 1, -\frac{3}{2}\right) \quad D_3 = \left(\frac{1}{2}, 1, -\frac{3}{2}, 1\right) \\
D_4 &= \left(1, \frac{1}{2}, -\frac{3}{2}, 1\right) \quad D_5 = \left(1, -\frac{1}{2}, 0, \frac{1}{2}\right) \quad D_6 = \left(1, -\frac{1}{2}, \frac{1}{2}, 0\right) \\
D_7 &= \left(\frac{1}{2}, -\frac{1}{2}, 0, 1\right) \quad D_8 = \left(\frac{1}{2}, -\frac{1}{2}, 1, 0\right) \quad D_9 = \left(-\frac{1}{2}, 1, 0, \frac{1}{2}\right) \\
D_{10} &= \left(-\frac{1}{2}, 1, \frac{1}{2}, 0\right) \quad D_{11} = \left(-\frac{1}{2}, \frac{1}{2}, 1, 0\right) \quad D_{12} = \left(-\frac{1}{2}, \frac{1}{2}, 0, 1\right)
\end{aligned}
$$

and $\mathbb{D}'$ is the interior of the convex hull of the collection $(D'_1, ..., D'_{12})$ where each D'_j is gotten from the corresponding D_j by swapping the 1st and 3rd positions. For instance, $D_2 = (1, 1, \frac{1}{2}, -\frac{3}{2})$.

One feature of these estimates is a certain dual index asymmetry. Indeed, for the dual index in positions 3 or 4, we may map near $L^{2/5}(\mathbb{R})$, while in positions 1 and 2 we only map near $L^{2/3}(\mathbb{R})$. Proving the $C^{1,1,1}$ estimates in [21] involves splitting $1_{\{\xi_1<\xi_2<\xi_3\}} = \tilde{1}_{\mathcal{R}_1} + \tilde{1}_{\mathcal{R}_2} + \tilde{1}_{\mathcal{R}_3}$ into a sum of three symbols localized to the regions $\mathcal{R}_1 = \{|\xi_1 - \xi_2| >> |\xi_2 - \xi_3|\}, \mathcal{R}_2 = \{|\xi_1 - \xi_2| \simeq |\xi_2 - \xi_3|\}, \mathcal{R}_3 = \{|\xi_1 - \xi_2| << |\xi_2 - \xi_3|\}$ respectively. More precisely, we take $\tilde{1}_{\mathcal{R}_1}$ to a supported inside a set of the form $\{\xi_1 < \xi_2 < \xi_3 : |\xi_1 - \xi_2| \geq C_1|\xi_2 - \xi_3|\}$ and identically equal to 1 on a set of the from $\{\xi_1 < \xi_2 < \xi_3 : |\xi_1 - \xi_2| \geq C_2|\xi_2 - \xi_3|\}$ for some constants $C_1 << C_2$. A similar statement then holds for both $\tilde{1}_{\mathcal{R}_2}$ and $\tilde{1}_{\mathcal{R}_3}$. As a wide range of L^p estimates hold for the multiplier with symbol $\tilde{1}_{\mathcal{R}_2}$, we focus on estimating the multipliers with symbols $\tilde{1}_{\mathcal{R}_1}$ and $\tilde{1}_{\mathcal{R}_2}$. By symmetry, it suffices to handle $\tilde{1}_{\mathcal{R}_1}$. It is (by now) standard to observe that $T_{\tilde{1}_{\mathcal{R}_1}}$ can be discretized in time and frequency to yield an average of models sums of type Λ_1, as clarified by

Definition 33. *A model of type Λ_1 is any 4-form writable as*

$$
\sum_{\vec{P} \in \mathbb{P}} \frac{\langle f_1, \Phi_{P_1,1}\rangle \langle f_4, \Phi_{P_2,4}\rangle}{|I_{\vec{P}}|^{1/2}} \left\langle \sum_{\vec{Q} \in \mathbb{Q}: |\omega_Q| << |\omega_P|} \frac{1}{|I_{\vec{Q}}|^{1/2}} \langle f_2, \Phi_{Q_1,2}\rangle \langle f_3, \Phi_{Q_2,3}\rangle \Phi_{Q_3,5}, \Phi_{P_2,0} \right\rangle,
$$

where $\mathbb{P}$ *is a rank-1 collection of tri-tiles for which* $(\omega_{P_1}, \omega_{P_2})$ *is adapted to* $\{-3\xi_1 = \xi_4\}$*, and* $\mathbb{Q}$ *is a rank-1 collection of tri-tiles for which* $(\omega_{Q_1}, \omega_{Q_2})$ *is adapted to* $\{\xi_1 = \xi_2\}$*.*

Restricted weak type estimates for the Biest model Λ_1 are then interpolated to yield the desired L^p estimates. For future use, we make the following official definitions:

Definition 34. *Let* $m : \mathbb{R}^n \to \mathbb{C}$*. Then define the multilinear multiplier* T_m *on* $(f_1, ..., f_n) \in \mathcal{S}(\mathbb{R})^n$ *to be*

$$T_m : (f_1, ..., f_n) \mapsto \int_{\mathbb{R}^n} m(\vec{\xi}) \prod_{j=1}^{n} \left[\hat{f}_j(\xi_j) e^{2\pi i x \xi_j} \right] d\vec{\xi}.$$

Definition 35. *For every* $\vec{\epsilon} \in \mathbb{R}^n$*, let* $C^{\vec{\epsilon}}$ *denote the* $n-$*linear operator defined for all* $(f_1, ..., f_n) \in \mathcal{S}(\mathbb{R})^n$ *by the formula*

$$C^{\vec{\epsilon}}(f_1, ..., f_n)(x) = \int_{\xi_1 < ... < \xi_n} \left[\prod_{j=1}^{n} \hat{f}_j(\xi_j) e^{2\pi i x \epsilon_j \xi_j} \right] d\vec{\xi}.$$

Definition 36. *For every* $\vec{\epsilon} \in \mathbb{R}^n$ *with only non-zero entries, let* $\tilde{C}^{\vec{\epsilon}}$ *denote the* $n-$*linear operator defined for all* $(f_1, ..., f_n) \in \mathcal{S}(\mathbb{R})^n$ *by the formula*

$$\tilde{C}^{\vec{\epsilon}}(f_1, ..., f_n)(x) = \int_{\frac{\xi_1}{\epsilon_1} < .... < \frac{\xi_n}{\epsilon_n}} \left[\prod_{j=1}^{n} \hat{f}_j(\xi_j) e^{2\pi i x \xi_j} \right] d\vec{\xi}.$$

By construction, for every $\vec{\epsilon} \in \mathbb{R}^n$ with non-zero entries, $\tilde{C}^{\vec{\epsilon}} = T_{1_{\frac{\xi_1}{\epsilon_1} < ... < \frac{\xi_n}{\epsilon_n}}}$ and, by a simple change of variables,

$$C^{\vec{\epsilon}}(f_1, .., f_n)(x) = \tilde{C}^{\vec{\epsilon}}(f_1(\epsilon_1 \cdot), ..., f_n(\epsilon_n \cdot))(x) \quad \forall x \in \mathbb{R} \quad \forall (f_1, .., f_n) \in \mathcal{S}(\mathbb{R})^n$$

so that $C^{\vec{\epsilon}}$ and $\tilde{C}^{\vec{\epsilon}}$ satisfy the same L^p estimates. We now introduce the following set of definitions:

Definition 37. *Let $\vec{\epsilon} \in \mathbb{R}^n$ satisfy the property that there exists a pair (i,j) such that $1 \leq i \leq j \leq n, j-i \in \{0,1\}$, and $\sum_{k=i}^{j} \epsilon_k = 0$. Then $\vec{\epsilon}$ is a degenerate tuple and $C^{\vec{\epsilon}}$ is a degenerate simplex multiplier.*

Definition 38. *Let $\vec{\epsilon} \in \mathbb{R}^n$ satisfy the property that there exists no pair $1 \leq i \leq j \leq n$ such that $\sum_{k=i}^{j} \epsilon_k = 0$. Then $\vec{\epsilon}$ is a fully non-degenerate tuple and $C^{\vec{\epsilon}}$ is a fully non-degenerate simplex multiplier.*

Definition 39. *Let $\vec{\epsilon} \in \mathbb{R}^n$ be a non-degenerate tuple for which there exists a pair (i,j) such that $1 \leq i < j \leq n$ and $\sum_{k=i}^{j} \epsilon_k = 0$. Then $\vec{\epsilon}$ is a semi-degenerate tuple and $C^{\vec{\epsilon}}$ is a semi-degenerate simplex multiplier.*

It is important to realize that for any fully non-degenerate $\vec{\epsilon} \in \mathbb{R}^3$ one can use the same argument as before to produce a variant of the Biest model, which still yields the same restricted type estimates as Λ_1. This is ultimately because the main ingredient needed for proving restricted weak-type estimates is geometric: namely, both tri-tile collections $\mathbb{P}$ and $\mathbb{Q}$ should be adapted to non-degenerate lines in $\mathbb{R}^2$, i.e. $l \notin \{\{\xi_1 = 0\}, \{\xi_2 = 0\}, \{\xi_1 + \xi_2 = 0\}\}$. The claim follows by noting that the localized regions of $C^{\epsilon_1,\epsilon_2,\epsilon_3}$ give rise to $\mathbb{P}$ and $\mathbb{Q}$ adapted to non-degenerate lines precisely when $\vec{\epsilon}$ is itself fully non-degenerate. In fact, we have from [22] the following

Theorem 20. *Fix $n \geq 1$ and let $\vec{\epsilon} \in \mathbb{R}^n$ be fully non-degenerate. Then $C^{\vec{\epsilon}}$ satisfies a wide range of L^p estimates.*

It is not hard to observe that $C^{1,1,-2}$ cannot give rise to a model of type Λ_1. Therefore, it is natural to ask whether L^p estimates hold in the semi-degenerate setting. As an initial foray, let us discuss one attractive feature of such simplex symbols: they can be broken into simpler pieces, as illustrated by

$$\begin{aligned} & \{\xi_1 < \xi_2 < -\xi_3/2\} \\ = & \{\xi_1 + \xi_2 < 2\xi_2 < -\xi_3\} \\ = & \{\xi_1 < \xi_2\} \bigcap \Big[(\{-\xi_3 < \xi_1 + \xi_2\} \cap \{\xi_1 < \xi_2\}) \bigcup \{\xi_1 + \xi_2 \leq -\xi_3 \leq 2\xi_2\}\Big]^c \\ = & \{\xi_1 < \xi_2\} \bigcap \Big[(\{\xi_1 + \xi_2 + \xi_3 > 0\} \cap \{\xi_1 < \xi_2\}) \bigcup (\{\xi_1 + \xi_2 + \xi_3 \leq 0\} \cap \{-\xi_3 \leq 2\xi_2\})\Big]^c. \end{aligned}$$

This elegant observation is due to C. Muscalu. Using $H^+ = T_{\{\xi > 0\}}$ and $H^- = T_{\{\xi \leq 0\}}$, the above decomposition yields the identity

$$\begin{aligned} & \tilde{C}^{1,1,-1/2}(f_1, f_2, f_3)(x) \\ = & \tilde{C}^{1,1}(f_1, f_2)(x) \cdot f_3(x) - H^+(\tilde{C}^{1,1}(f_1, f_2) \cdot f_3)(x) - H^-(f_1 \cdot \tilde{C}^{-1/2,1}(f_3, f_2))(x). \end{aligned}$$

Because each term on the RHS of the above display satisfies all interior Banach estimates, the same must be true for $\tilde{C}^{1,1,-1/2}$ and therefore $C^{1,1,-2}$. Given that $C^{1,1,-2}$ maps into $L^r(\mathbb{R})$ for all $1 < r < \infty$, it is tempting to ask whether such an object can map below $L^1(\mathbb{R})$, and if so, how low can the target exponent $r \geq \frac{1}{3}$ go. Our first result shows $r > 1/2$ is necessary for $C^{1,1,-2}$ to map into $L^r(\mathbb{R})$. Similarly, we have the identity

$$\begin{aligned} & \tilde{C}^{1,1,1,-1/2}(f_1, f_2, f_3, f_4)(x) \\ = & \tilde{C}^{1,1,1}(f_1, f_2, f_3)(x) f_4(x) - T_{\{\xi_1 < \xi_2 < \xi_3\} \cap \{\xi_2 + \xi_3 + \xi_4 > 0\}}(f_1, f_2, f_3, f_4)(x) \\ - & \tilde{C}^{1,1,-1}(f_1, f_2, \tilde{C}^{-1/2,1}(f_4, f_3))(x). \end{aligned}$$

Because both $T_{\{\xi_1<\xi_2<\xi_3\}\cap\{\xi_2+\xi_3+\xi_4>0\}}$ and $\tilde{C}^{1,1,-1}(f_1, f_2, \tilde{C}^{-1/2,1}(f_4, f_3))$ satisfy no L^p estimates, so any bounds for $\tilde{C}^{1,1,1,-1/2}$ must arise as a consequence of large destructive interference between these two unbounded terms. A natural question in light of these developments is whether the degeneracy condition is necessary for L^p estimates to fail. While not answering this question fully, we content ourselves in this section with establishing two principle results. The first is

Theorem 21. *$C^{1,1,-2} : L^{p_1}(\mathbb{R}) \times L^{p_2}(\mathbb{R}) \times L^{p_3}(\mathbb{R}) \to L^{p_4'}(\mathbb{R})$ provided $1 < p_1, p_2, p_3 \le \infty, 0 < p_4'$, and $(p_1, p_2, p_3, p_4) \in Int(Conv[\mathcal{A}]) \cap Int(Conv[\mathcal{A}'])$, where $\mathcal{A} = \{A_j\}_{j=1}^{9}$ is given by*

$$\begin{aligned}
A_1 &= \left(1, \frac{1}{2}, \frac{1}{2}, -1\right), A_2 = \left(\frac{1}{2}, \frac{1}{2}, 1, -1\right), A_3 = \left(\frac{1}{2}, 1, \frac{1}{2}, -1\right)\\
A_4 &= \left(-\frac{3}{2}, \frac{1}{2}, 1, 1\right), A_5 = \left(-\frac{3}{2}, 1, \frac{1}{2}, 1\right), A_6 = \left(\frac{1}{2}, -\frac{1}{2}, 1, 0\right)\\
A_7 &= \left(0, -\frac{1}{2}, 1, \frac{1}{2}\right), A_8 = \left(0, 1, -\frac{1}{2}, \frac{1}{2}\right), A_9 = \left(\frac{1}{2}, 1, -\frac{1}{2}, 0\right)
\end{aligned}$$

and $\mathcal{A}'$ denotes the collection $\{A_1', ...A_9'\}$ where each A_j' is gotten by the corresponding A_j by swapping the 1st and 3rd indices. For example, $A_2' = (1, \frac{1}{2}, \frac{1}{2}, -1)$. For any set of points $S \subset \mathbb{R}^n$, we use $Int(Conv([S]))$ to denote the interior of the convex hull of the set S.

To prove Theorem 7, we follow the standard procedure introduced in [21] of carving $1_{\xi_1<\xi_2<-\xi_3/2}$ into three localized pieces, discretizing each piece into a wave packet model, and then obtaining satisfactory estimates for each model. Central to our argument will be producing generalized restricted type estimates for models of type Λ_2, as clarified by

Definition 40. *A model of type Λ_2 is any 4-form writable as*

$$\sum_{\vec{P}\in\mathbb{P}} \frac{\langle f_1, \Phi_{P_1,1}\rangle\langle f_4, \Phi_{P_2,4}\rangle}{|I_{\vec{P}}|^{1/2}} \left\langle \int_0^1 \sum_{\vec{Q}\in\mathbb{Q}:\omega_{Q_3}\subset\subset\omega_{P_2}} \frac{1}{|I_{\vec{Q}}|^{1/2}} \langle f_3, \Phi^{\alpha}_{Q_1,3}\rangle\langle f_4, \Phi^{\alpha}_{Q_2,4}\rangle \Phi^{\alpha}_{Q_3,5} d\alpha, \Phi_{P_2,0} \right\rangle,$$

where $\mathbb{P}$ is a collection of tri-tiles for which $(\omega_{P_1}, \omega_{P_3})$ is adapted to $\{\xi_1+\xi_2=0\}$, $\Phi_{P_2,4}$ is lacunary about the origin at scale $|\omega_P|$, and $\mathbb{Q}$ is a rank-1 collection of tri-tiles for which $(\omega_{Q_1}, \omega_{Q_2})$ is adapted to $\{\xi_1=\xi_2\}$ and $\Phi^{\alpha}_{Q_1,3}$ is a wave packet on Q_1 uniformly for $\alpha\in[0,1]$. The implicit constant in the $\mathbb{Q}-$sum should be taken to be some sufficiently large absolute constant. Moreover, $\omega_{Q_2}\subset\subset\omega_{P_2}$ means $|\omega_{Q_2}| << |\omega_{P_2}|$ and $\omega_{Q_2}\subset\omega_{P_2}$.

Our second principle result is

Theorem 22. *$C^{1,1,1,-2}: L^{p_1}(\mathbb{R})\times L^{p_2}(\mathbb{R})\times L^{p_3}(\mathbb{R})\times L^{p_4}(\mathbb{R})\to L^{p_4'}(\mathbb{R})$ provided $1<p_1,p_2,p_3\le\infty, 0<p_4'<\infty$ and $(p_1,p_2,p_3,p_4)\in Int(Conv[\mathcal{B}])\cap Int(Conv[\mathcal{B}'])$, where $\mathcal{B}=\{B_j\}_{j=1}^{16}$ is given by*

$$\begin{aligned}
B_1 &= \left(1,1,\frac{1}{2},\frac{1}{2},-2\right) \quad B_2=\left(1,\frac{1}{2},\frac{1}{2},1,-2\right) \quad B_3=\left(1,\frac{1}{2},1,\frac{1}{2},-2\right)\\
B_4 &= \left(-2,1,\frac{1}{2},\frac{1}{2},1\right) \quad B_5=\left(-2,\frac{1}{2},1,\frac{1}{2},1\right) \quad B_6=\left(-2,\frac{1}{2},\frac{1}{2},1,1\right)\\
B_7 &= \left(0,-\frac{3}{2},\frac{1}{2},1,1\right) \quad B_8=\left(1,-\frac{3}{2},\frac{1}{2},1,0\right) \quad B_9=\left(0,-\frac{3}{2},1,\frac{1}{2},1\right)\\
B_{10} &= \left(1,-\frac{3}{2},1,\frac{1}{2},0\right) \quad B_{11}=\left(0,\frac{1}{2},-\frac{1}{2},1,0\right) \quad B_{12}=\left(\frac{1}{2},0,-\frac{1}{2},1,0\right)\\
B_{13} &= \left(0,0,-\frac{1}{2},1,\frac{1}{2}\right) \quad B_{14}=\left(0,\frac{1}{2},1,-\frac{1}{2},0\right) \quad B_{15}=\left(\frac{1}{2},0,1,-\frac{1}{2},0\right)\\
B_{16} &= \left(0,0,1,-\frac{1}{2},\frac{1}{2}\right)
\end{aligned}$$

and $\mathcal{B}'$ denotes the collection $\left\{B_j'\right\}_{j=1}^{16}$, where each B_j' is obtained from the corresponding B_j by the permutation $1\mapsto 1, 2\mapsto 4, 3\mapsto 2, 4\mapsto 3$. In particular, $B_3'=\left(1,1,\frac{1}{2},\frac{1}{2},-2\right)$. Moreover, $(1,\frac{2}{3},\frac{2}{3},\frac{2}{3}-2)\in\overline{Conv[\mathcal{B}]}\cap\overline{Conv[\mathcal{B}']}$ and $C^{1,1,1,-2}$ maps into $L^r(\mathbb{R})$ for all $\frac{1}{3}<r\le 1$.

Unlike $C^{1,1,-2}$, we do not a natural decomposition of $C^{1,1,1,-2}$ into simpler bounded operators. Hence, it is perhaps a little surprising that $C^{1,1,1,-2}$ satisfies any L^p estimates. At the end of the day, we are able to reduce matters to proving generalized restricted type estimates for models of Λ_3 type, as clarified by

Definition 41. *A model of type Λ_3 is any 5-form writable as*

$$
\begin{aligned}
&\Lambda_{\mathbb{P},\mathbb{Q},\mathbb{R}}(f_1, f_2, f_3, f_4, f_5) \\
= &\sum_{\vec{P}\in\mathbb{P}} \frac{1}{|I_{\vec{P}}|^{1/2}} \left\langle \sum_{\vec{R}\in\mathbb{R}:\widetilde{\omega_{R_1}}\supset\supset\omega_{P_2}} \frac{\langle f_1, \Phi_{R_1,1}\rangle\langle f_5, \Phi_{R_2,5}\rangle}{|I_{\vec{R}}|^{1/2}} \Phi^{n-l}_{R_3,0}, \Phi_{P_1,6} \right\rangle \langle f_2, \Phi_{P_2,2}\rangle \\
\times &\left\langle \int_0^1 BHT^{\alpha,\mathbb{Q}}_{\omega_{P_3}}(f_3, f_4) d\alpha, \Phi_{P_3,7} \right\rangle,
\end{aligned}
$$

where $\mathbb{P}, \mathbb{Q},$ and $\mathbb{R}$ are three tri-tile collections, $\mathbb{Q}$ is rank-1, and for each $\alpha \in [0,1]$,

$$
BHT^{\alpha,\mathbb{Q}}_{\omega_{P_3}}(f_3, f_4)(x) := \sum_{\vec{Q}\in\mathbb{Q}:\omega_{Q_3}\subset\subset\omega_{P_2}} \frac{1}{|I_{\vec{Q}}|^{1/2}} \langle f_3, \Phi^{\alpha}_{Q_1,3}\rangle\langle f_4, \Phi^{\alpha}_{Q_2,4}\rangle \Phi^{\alpha}_{Q_3,5}(x)
$$

where each wave packet $\Phi^{\alpha}_{Q_i,j(i)}$ for $i = 1, 2, 3$ has decay constant uniform in α.

An attractive feature of our main results is that the L^p target ranges for both $C^{1,1,-2}$ and $C^{1,1,1,-2}$ are the best possible. Indeed, that $C^{1,1,-2}$ cannot map below $L^{\frac{1}{2}}(\mathbb{R})$ and $C^{1,1,1,-2}$ cannot map below $L^{\frac{1}{3}}(\mathbb{R})$ follows from explicit counterexamples in §2. This sharpness is quite different from the fully non-degenerate setting, where the generic BHT model produces estimates only down to $L^{\frac{2}{3}+\epsilon}(\mathbb{R})$ and there are no known counterexamples at this time to rule out the BHT mapping all the way down to $L^{\frac{1}{2}+\epsilon}(\mathbb{R})$.

Theorems 21 and 22 also prompt another line of questioning; do we have the same L^p estimates if the corresponding symbols $1_{\{\xi_1<\xi_2<-\xi_3/2\}}$ and

$1_{\{\xi_1<\xi_2<\xi_3<-\xi_4/2\}}$ are respectively replaced with $m_1(\xi_1,\xi_2,\xi_3) = b_1(\xi_1,\xi_2)b_2(\xi_2,\xi_3)$ and $m_2(\xi_1,\xi_2,\xi_3,\xi_4) = c_1(\xi_1,\xi_2)c_2(\xi_3,\xi_3)c_3(\xi_3,\xi_4)$, where b_1, c_1, c_2 are adapted to $\{\xi_1 = \xi_2\}$, and b_2, c_3 are adapted to $\{\xi_1 = -\xi_2/2\}$? The answer is assuredly yes; however, the proofs in the generic case become longer, less reader-friendly, and tend to obscure the important points of the semi-degenerate analysis, and so the details of the argument are omitted. Nonetheless, we have all the tools necessary to carry out the proof and now provide the briefest possible sketch. Generic trilinear multipliers $m(\xi_1,\xi_2,\xi_3)$ of the form $b_1(\xi_1,\xi_2)\cdot b_2(\xi_2,\xi_3)$ may be reduced to models of type Λ_2 combined with error terms with even better mapping properties by following the arguments in [8]. Showing the same estimates for generic $4-$linear multipliers of the form $c_1(\xi_1,\xi_2)c_2(\xi_2,\xi_3)c_3(\xi_3,\xi_4)$ requires us to mimic our local discretization of the form associated to a regional piece of

$$B[a_1,a_2] : (f_1,f_2,f_3) \mapsto \int_{\mathbb{R}^3} a_1(\xi_1,\xi_2)a_2(\xi_2,\xi_3)\hat{f}_1(\xi_1)\hat{f}_2(\xi_2)\hat{f}_3(\xi_3)e^{2\pi i x(\xi_1+\xi_2+\xi_3)}d\xi_1 d\xi_2 d\xi_3$$

for *any* $a_1, a_2 : \mathbb{R}^2 \to \mathbb{C}$ adapted to the degenerate line $\{\xi_1+\xi_2=0\}$ and then deploy the l^1 energy boost from the proof of Theorem 21. Before describing the counterexamples and positive results in the semi-degenerate setting, we should say a bit about the bigger picture. What can be said for $C^{1,1,1,1,-2}$ or, for that matter, any $C^{\vec{\epsilon}}$ with $\vec{\epsilon} \in \mathbb{R}^n$ semi-degenerate? One expects that if any such simplex multiplier satisfied no L^p estimates, then $C^{1,1,1,-2}$ should fail to have L^p estimates. Indeed, as we see two bad best lurking in our natural decomposition of $C^{1,1,1,-2}$, it is reasonable to expect that matters cannot really deteriorate beyond such these dueling bad bests. As Theorem 22 ensures a wide range of estimates for $C^{1,1,1,-2}$, we are naturally led to

Conjecture 1. *Let $\vec{\epsilon} \in \mathbb{R}^n$ be semi-degenerate. Then $C^{\vec{\epsilon}}$ satisfies a wide range of*

L^p *estimates.*

Given Theorem 20 and the existence of generic mixed estimates for degenerate simplex multipliers, the resolution of Conjecture 1 in a certain sense completes the picture of simplex multiplier estimates.

3.2 Terry Lyons' Variational Estimate

Closely related to the $a.e.$ converge of the Fourier series of L^p functions are the fundamental estimates of Carleson and Hunt, which asserts that the map initially defined for $f \in \mathcal{S}(\mathbb{R})$ by the rule

$$C : f \mapsto \sup_{N \in \mathbb{R}} \left| \int_{(\infty, N]} \hat{f}(\xi) e^{2\pi i x \xi} d\xi \right|$$

can be extended to all of $L^p(\mathbb{R})$ and satisfies $||C(f)||_p \lesssim_p ||f||_p$ for all $1 < p < \infty$ and $f \in L^p(\mathbb{R})$. The variational Carleson estimates are a generalization of this result: for any $2 < \rho \leq \infty$,

$$\mathcal{C}^\rho : f \mapsto \sup_{k \in \mathbb{N}} \sup_{\xi_1 < \xi_2 < \ldots < \xi_k} \left(\sum_{n=1}^{k-1} \left| \int_{\xi_n < \eta < \xi_{n+1}} \hat{f}(\eta) e^{2\pi i x \eta} d\eta \right|^\rho \right)^{1/\rho}$$

extends to a map of $L^p(\mathbb{R}) \to L^p(\mathbb{R})$ for all $\rho' < p < \infty$. When $\rho = \infty$, we clearly recover the Carleson estimates. It is known via a direct counterexample appearing in the author's previous work that $\rho > 2$ is necessary for any L^p estimates to hold. In light of the variational Carleson story, it is natural to ask whether estimates hold for the variational Bi-Carleson, which is defined for variation exponent $0 < \rho \leq \infty$ and with domain $\mathcal{S}(\mathbb{R})^2$ to be

$$\mathcal{BC}^{\rho} : (f_1, f_2) \mapsto \sup_{k\in\mathbb{N}} \sup_{\xi_1<\xi_2<\ldots<\xi_k} \left(\sum_{n=1}^{k-1} \left| \int_{\xi_n<\eta_1<\eta_2<\xi_{n+1}} \hat{f}_1(\eta_1)\hat{f}_2(\eta_2) e^{2\pi i x(\eta_1+\eta_2)} d\eta_1 d\eta_2 \right|^{\rho} \right)^{1/\rho}.$$

If $\rho = \infty$, then $\mathcal{BC}^{\infty}$ is the Bi-Carleson operator, for which estimates were obtained in [15] and shown to coincide with the known BHT estimates $1 < p_1, p_2 \leq \infty$ and $0 < \frac{1}{p_1} + \frac{1}{p_2} < \frac{3}{2}$. Moreover, one expects $\mathcal{BC}^{1+\epsilon} : L^2(\mathbb{R}) \times L^2(\mathbb{R}) \to L^1(\mathbb{R})$ for every $\epsilon > 0$, provided the BHT behaves like a product. Indeed, in this simple case, we could deduce the desired $\rho = 1+\epsilon$ estimate by concatenating Cauchy-Schwarz with the variational Carleson estimate near $\rho = 2$. Interpolating between the $\rho = 1+\epsilon$ and $\rho = \infty$ cases would then yield

$$\mathcal{BC}^{\rho} : L^{p_1}(\mathbb{R}) \times L^{p_2}(\mathbb{R}) \to L^{\frac{p_1 p_2}{p_1+p_2}}(\mathbb{R})$$

for all $1 < p_1, p_2 < \infty$ such that $0 < \frac{1}{p_1} + \frac{1}{p_2} < 1 + \frac{1}{2\rho'}, \max\{\frac{1}{p_1}, \frac{1}{p_2}\} < \frac{1}{2} + \frac{1}{2\rho'}$. We next present a striking inequality due to Terry Lyons in [13], which provides a pointwise bound for trilinear simplex multipliers in terms of various powers of the variational Carleson and Bi-Carleson operators. For all $2 < r < 3$, we in fact have

$$\begin{aligned}
& C^{1,1,1}(f_1, f_2, f_3)(x) \leq [Var^r(f_1, f_2, f_3)(x)]^3 \quad where \\
& Var^r(f_1, f_2, f_3)(x) \\
:= \ & \mathcal{C}^r(f_1)(x) + \mathcal{C}^r(f_2)(x) + \mathcal{C}^r(f_3)(x) \\
+ \ & \left[\mathcal{BC}^{r/2}(f_1, f_2)(x)\right]^{1/2} + \left[\mathcal{BC}^{r/2}(f_2, f_3)(x)\right]^{1/2} + \left[\mathcal{BC}^{r/2}(f_1, f_3)(x)\right]^{1/2}.
\end{aligned}$$

Taking $r \simeq 3$ and $p_1 = p_2 = p_3 \simeq 3/2$ and using the variational Carleson and variational Bi-Carleson estimates gives the extremal mapping $L^{3/(2-\epsilon)} \times L^{3/(2-\epsilon)} \times L^{3/(2-\epsilon)} \to L^{1/(2-\epsilon)}$. By interpolation, one recovers all estimates in the convex hull of $\mathcal{S} := \mathcal{B} \cup (2/3 - \epsilon/3, 2/3 - \epsilon/3, 2/3 - \epsilon/3, -1 +$

ϵ), where $\mathcal{B}$ denotes the set of all interior Banach estimates, i.e. $\mathcal{B} = \left\{(1/p_1, 1/p_2, 1/p_3, 1-1/p_1-1/p_2-1/p_3) : 1 < p_1, p_2, p_3 < \infty, \frac{1}{p_1}+\frac{1}{p_2}+\frac{1}{p_3} < 1\right\}$. Our proof of Theorem 21 has the two-fold advantage of avoiding reliance on the variational Bi-Carleson/Carleson estimates and producing estimates beyond the convex hull of $\mathcal{S}$.

3.3 $C^{1,1,-2}$ and $C^{1,1,1,-2}$ Counterexamples

Proposition 7. *$C^{1,1,-2}$ does not map into $L^r(\mathbb{R})$ for $r \leq 1/2$.*

Proof. Fix $1 < p_1, p_2, p_3 \leq \infty$ with $\frac{1}{p_1}+\frac{1}{p_2}+\frac{1}{p_3} \geq 2$. Let $f_1 = f_2 = f_3 = \check{1}_{[-1,1]}$. Then $\prod_{j=1}^3 ||f_j||_{p_j} < \infty$. Note

$$
\begin{aligned}
& C^{1,1,-2}(f_1, f_2, f_3)(x) \\
= & \int_{-1<\xi_1<\xi_2<\xi_3<1} e^{2\pi i x(\xi_1+\xi_2-2\xi_3)} d\xi_1 d\xi_2 d\xi_3 \\
= & \frac{1}{2\pi i x}\int_{-1<\xi_2<\xi_3<1} \left[e^{2\pi i x(2\xi_2-2\xi_3)} - e^{2\pi i x(-1+\xi_2-2\xi_3)}\right] d\xi_2 d\xi_3 \\
= & \left[\frac{1}{2\pi i x}\right]^2 \int_{-1<\xi_3<1} \left[1/2 - e^{2\pi i x(-1-\xi_3)} - e^{2\pi i x(-2-2\xi_3)}/2 + e^{2\pi i x(-2-2\xi_3)}\right] d\xi_3 \\
= & \left[\frac{1}{2\pi i x}\right]^2 \int_{-1<\xi_3<1} \left[1/2 - e^{2\pi i x(-1-\xi_3)} - e^{2\pi i x(-2-2\xi_3)}/2\right] d\xi_3 \\
= & \left[\frac{1}{2\pi i x}\right]^2 + \left[\frac{1}{2\pi i x}\right]^3 \left[5/4 - e^{-4\pi i x} - e^{-8\pi i x}/4\right].
\end{aligned}
$$

Therefore, $C^{1,1,-2}(\vec{f})(x)$ decays like $\frac{1}{|x|^2}$ away from the origin ($|x| \gtrsim 1$) and so cannot belong to $L^r(\mathbb{R})$ for $r \leq 1/2$. If $p_i = 1$ for some $j \in \{1,2,3\}$, then one can instead take $f_1 = f_2 = f_3 = \mathcal{F}^{-1}[\phi]$ for some non-trivial, non-negative

$\phi \in C^{\infty}_{[-1,1]}(\mathbb{R})$ and use integration by parts to deduce the same quadratic decay as before.

□

The analogous statement for $C^{1,1,1,-2}$ is

Proposition 8. *$C^{1,1,1,-2}$ does not map into $L^r(\mathbb{R})$ for $r \leq \frac{1}{3}$.*

Proof. If $C^{1,1,1,-2}$ did map into $L^r(\mathbb{R})$ for some $r \leq \frac{1}{3}$, there would exist a 4-tuple (p_1, p_2, p_3, p_4) satisfying $1 \leq p_1, p_2, p_3, p_4 \leq \infty$ and $\frac{1}{p_1} + \frac{1}{p_2} + \frac{1}{p_3} + \frac{1}{p_4} \geq 3$ for which

$$\left|\left|C^{1,1,1,-2}(f_1, f_2, f_3, f_4)\right|\right|_{L^{\frac{1}{\frac{1}{p_1}+\frac{1}{p_2}+\frac{1}{p_3}+\frac{1}{p_4}}}(\mathbb{R})} \lesssim_{\vec{p}} ||f_1||_{L^{p_1}(\mathbb{R})}||f_2||_{L^{p_2}(\mathbb{R})}||f_3||_{L^{p_3}(\mathbb{R})}||f_4||_{L^{p_4}(\mathbb{R})}$$

for all $f_j \in L^{p_j}(\mathbb{R})$ and $j \in \{1, 2, 3, 4\}$.

CASE #1: $\frac{1}{p_2} + \frac{1}{p_3} + \frac{1}{p_4} > 2$. Then take $f_2 = f_3 = f_4 = \mathcal{F}^{-1}[\phi]$ along with $f_1^N = \mathcal{F}^{-1}\left[Dil^1_{N^{-1}} Tr_{-2N}\, \phi\right] = \mathcal{F}^{-1}[\phi](N^{-1}x)e^{-2\pi i 2x}$ where ϕ is again some non-trivial, non-negative function in $C^{\infty}_{[-1,1]}(\mathbb{R})$. Then for large enough N, $\mathcal{C}^{1,1,-2}(f_1^N, f_2, f_3, f_4) = f_1^N(x) C^{1,1,-2}(f_2, f_3, f_4)(x)$, and so

$$\left|\left|\mathcal{C}^{1,1,-2}(f_1^N, f_2, f_3, f_4)\right|\right|_{L^{\frac{1}{\frac{1}{p_1}+\frac{1}{p_2}+\frac{1}{p_3}+\frac{1}{p_4}}}(\mathbb{R})} \simeq N^{\frac{1}{p_1}+\frac{1}{p_2}+\frac{1}{p_3}+\frac{1}{p_4}-2},$$

whereas $||f_1^N||_{L^{p_1}(\mathbb{R})} \prod_{j=2}^4 ||f_j||_{L^{p_j}(\mathbb{R})} \simeq N^{1/p_1}$. Taking N arbitrarily large contradicts our original assumption.

CASE #2: $\frac{1}{p_2} + \frac{1}{p_3} + \frac{1}{p_4} = 2$. Then $p_1 = 1$. Setting $f_1^N(x) =$

$\mathcal{F}^{-1}\left[Dil^1_{N^{-1}}Tr_{-2N}\ \phi\right](x)\mathcal{F}^{-1}\left[1_{[-1,1]}\right](x)$ for the same ϕ as before ensures that $C^{1,1,1,-2}(f_1^N, f_2, f_3, f_4)(x) = f_1^N(x)C^{1,1,-2}(f_2, f_3, f_4)(x)$ for large enough N. Hence,

$$\left|\left|C^{1,1,1,-2}(f_1^N, f_2, f_3, f_4)\right|\right|_{L^{\frac{1}{\frac{1}{p_1}+\frac{1}{p_2}+\frac{1}{p_3}+\frac{1}{p_4}}}(\mathbb{R})} \simeq (\ln N)^3,$$

whereas $||f_1^N||_{L^{p_1}(\mathbb{R})}\prod_{j=2}^4 ||f_j||_{L^{p_j}(\mathbb{R})} \simeq \ln N$. Taking N arbitrarily large again contradicts our original assumption.

$\square$

3.4 $C^{1,1,-2}$ Estimates

Our goal in this section is to prove

Theorem 21. *$C^{1,1,-2} : L^{p_1}(\mathbb{R}) \times L^{p_2}(\mathbb{R}) \times L^{p_3}(\mathbb{R}) \to L^{p_4'}(\mathbb{R})$ provided $1 < p_1, p_2, p_3 \leq \infty, 0 < p_4'$, and $(p_1, p_2, p_3, p_4) \in Int(Conv[\mathcal{A}]) \cap Int(Conv[\mathcal{A}'])$, where $\mathcal{A} = \{A_j\}_{j=1}^9$ is given by*

$$\begin{aligned}
A_1 &= \left(1, \frac{1}{2}, \frac{1}{2}, -1\right), A_2 = \left(\frac{1}{2}, \frac{1}{2}, 1, -1\right), A_3 = \left(\frac{1}{2}, 1, \frac{1}{2}, -1\right)\\
A_4 &= \left(-\frac{3}{2}, \frac{1}{2}, 1, 1\right), A_5 = \left(-\frac{3}{2}, 1, \frac{1}{2}, 1\right), A_6 = \left(\frac{1}{2}, -\frac{1}{2}, 1, 0\right)\\
A_7 &= \left(0, -\frac{1}{2}, 1, \frac{1}{2}\right), A_8 = \left(0, 1, -\frac{1}{2}, \frac{1}{2}\right), A_9 = \left(\frac{1}{2}, 1, -\frac{1}{2}, 0\right)
\end{aligned}$$

and $\mathcal{A}'$ denotes the collection $\{A_1', ...A_9'\}$ where each A_j' is gotten by the corresponding A_j by swapping the 1st and 3rd indices. For example, $A_2' = (1, \frac{1}{2}, \frac{1}{2}, -1)$.

It is clear that $\mathcal{C}$ strictly contains the estimates obtained by interpolating between the diagonal

$$\Delta := \{(p,p,p,1-3p) : 1/3 \le p < 2/3\}$$

and the interior Banach estimates $\mathbb{B} = \{\vec{p} : 1 < p_j \le \infty \ \forall\ j \in \{1,2,3,4\}\}$, so our range is providing estimates not obtainable from Terry Lyon's estimate and the variational Bi-Carleson estimates. For instance, $C^{1,1,-2} : L^{p_1}(\mathbb{R}) \times L^{p_2}(\mathbb{R}) \times L^{p_3}(\mathbb{R}) \to L^{p_4'}(\mathbb{R})$ for tuples (p_1,p_2,p_3,p_4) in a small neighborhood of $(1,\frac{1}{2},\frac{1}{2},-1)$.

3.4.1 Reduction to the Model

Our analysis of the simplex multiplier $\tilde{C}^{1,1,-2}$ begins as in the Biest case by localizing the symbol $1_{\xi_1<\xi_2<-\xi_3/2}$ inside the three regions:

$$\begin{aligned}
\mathcal{R}_1 &= \{\xi_1 < \xi_2 < -\xi_3/2\} \cap \{|\xi_1 - \xi_2| << |\xi_2 + \xi_3/2|\} \\
\mathcal{R}_2 &= \{\xi_1 < \xi_2 < -\xi_3/2\} \cap \{|\xi_1 - \xi_2| \simeq |\xi_2 + \xi_3/2|\} \\
\mathcal{R}_3 &= \{\xi_1 < \xi_2 < -\xi_3/2\} \cap \{|\xi_1 - \xi_2| >> |\xi_2 + \xi_3/2|\}.
\end{aligned}$$

To this end, let us recall

$$1_{\{\xi_2<-\xi_3/2\}}(\xi_2,\xi_3) = \sum_{(\gamma,\gamma')\in\{0,\frac{1}{3},\frac{2}{3}\}^2} \sum_{k,k'\in\mathbb{Z}} \sum_{\vec{Q}\in\mathcal{Q}^{\gamma,\gamma'}} c_k \tilde{c}_{k'} \hat{\eta}^{\gamma,k}_{\omega_{Q_1},2}(\xi_2) \hat{\eta}^{\gamma',k'}_{\omega_{Q_2},3}(\xi_3)$$

where γ,γ' are dyadic shifts, k,k' are oscillation parameters, and each $\vec{Q} = (\omega_{Q_2},\omega_{Q_3})$ is a Whitney square for the set $\Gamma := \{\xi_2 = -\xi_3/2\}$ in the usual sense that the side-length of $\vec{Q}$ is proportional to $dist(\vec{Q},\Gamma)$. Similarly, we have

$$1_{\{\xi_1<\xi_2\}}(\xi_1,\xi_2) = \sum_{(\sigma,\sigma')\in\{0,\frac{1}{3},\frac{2}{3}\}^2} \sum_{l,l'\in\mathbb{Z}} \sum_{\vec{P}\in\mathcal{P}^{\gamma,\gamma'}} c_k \tilde{c}_{k'} \hat{\eta}^{\gamma,k}_{\omega_{P_1},1}(\xi_1) \hat{\eta}^{\gamma',k'}_{\omega_{P_2},0}(\xi_2)$$

where we have the same setup as before with the exception that each $\vec{P} = (\omega_{P_1}, \omega_{P_2})$ is a Whitney cube for the set $\tilde{\Gamma} := \{\xi_1 = \xi_2\}$. The main trick we want to use is that inside $\mathcal{R}_3$, say $\xi_1 < \xi_2 < -\xi_3/2$ holds iff $\xi_1 < -(\xi_2 + \xi_3); \xi_2 < -\frac{\xi_3}{2}$ holds. Therefore, setting

$$\begin{aligned}\tilde{1}_{\mathcal{R}_3}(\xi_1, \xi_2, \xi_3) &= \sum_{(\gamma,\gamma') \in \{0, \frac{1}{3}, \frac{2}{3}\}^2} \sum_{k,k' \in \mathbb{Z}} \sum_{\vec{Q} \in \mathcal{Q}^{\gamma,\gamma'}} c_k \tilde{c}_{k'} \hat{\eta}^{\sigma,l}_{\omega_{Q_1},2}(\xi_2) \hat{\eta}^{\sigma',l'}_{\omega_{Q_2},3}(\xi_3) \\ &\times \left[\sum_{(\sigma,\sigma') \in \{0, \frac{1}{3}, \frac{2}{3}\}^2} \sum_{l,l' \in \mathbb{Z}} \sum_{\vec{P} \in \mathcal{P}^{\gamma,\gamma'} : |\vec{P}| >> |\vec{Q}|} c_l \tilde{c}_{l'} \hat{\eta}^{\gamma,k}_{\omega_{P_1},1}(\xi_1) \hat{\eta}^{\gamma',k'}_{\omega_{P_2},0}(-(\xi_2 + \xi_3)) \right],\end{aligned}$$

it follows that for large enough implicit constant, $\tilde{1}_{\mathcal{R}_3}(\xi_1, \xi_2, \xi_3) \equiv 1$ on a set $\mathcal{R}_3^0$ and supported on a set of the same shape $\mathcal{R}_3^1$. We may similarly construct $\tilde{1}_{\mathcal{R}_1}$. Then putting it all together yields

$$\begin{aligned}& 1_{\xi_1<\xi_2<-\xi_3/2} \\ =\ & 1_{\xi_1<\xi_2<-\xi_3/2}(1 - \tilde{1}_{\mathcal{R}_1})(1 - \tilde{1}_{\mathcal{R}_2}) + 1_{\xi_1<\xi_2<-\xi_3/2}\tilde{1}_{\mathcal{R}_1} + 1_{\xi_1<\xi_2<-\xi_3/2}\tilde{1}_{\mathcal{R}_2} - 1_{\xi_1<\xi_2<-\xi_3/2}\tilde{1}_{\mathcal{R}_1}\tilde{1}_{\mathcal{R}_2} \\ :=\ & I + II + III + IV.\end{aligned}$$

It is straightforward to observe that I is Mikhlin-Hörmander symbol adapted to the set of shape $\mathcal{R}_2$ and $IV \equiv 0$ for large enough implicit constants. Therefore, it suffices to understand II and III. However, by construction,

$$\begin{aligned}1_{\xi_1<\xi_2<-\xi_3/2}\tilde{1}_{\mathcal{R}_1} &= \tilde{1}_{\mathcal{R}_1} \\ 1_{\xi_1<\xi_2<-\xi_3/2}\tilde{1}_{\mathcal{R}_3} &= \tilde{1}_{\mathcal{R}_3}\end{aligned}$$

so it suffices (morally speaking) to obtain estimate for $\tilde{1}_{\mathcal{R}_1}$ and $\tilde{1}_{\mathcal{R}_3}$. Moreover, by symmetry, it suffices (morally speaking) to obtain estimates for $\tilde{1}_{\mathcal{R}_3}$. Of course, we will need to write down estimates for symbols adapted to $\{\xi_1 = \xi_2 = -\xi_3/2\}$, but this argument will be postponed until later. To handle $\tilde{1}_{\mathcal{R}_3}$, we now wish to dualize by introducing f_4 and complete as follows:

$$\begin{aligned}
&\int_{\mathbb{R}} T_{\tilde{1}_{\mathcal{R}_3}}(f_1, f_2, f_3)(x) f_4(x) dx \\
= &\sum_{'} c(k, k', l, l') \int_{\mathbb{R}} f_1 * \eta^{\gamma,k}_{\omega_{P_1},1} \cdot \left[f_2 * \eta^{\sigma,l}_{\omega_{Q_1},2} f_3 * \eta^{\sigma',l'}_{\omega_{Q_2},3} \right] * \eta^{\gamma',k'}_{-\omega_{P_2},0} \cdot f_4 dx \\
= &\sum_{'} c(k, k', l, l') \int_{\mathbb{R}} f_1 * \eta^{\gamma,k}_{\omega_{P_1},1} \cdot \left[f_2 * \eta^{\sigma,l}_{\omega_{Q_1},2} f_3 * \eta^{\sigma',l'}_{\omega_{Q_2},3} \right] * \eta^{\gamma',k'}_{-\omega_{P_2},0} \cdot f_4 * \eta^{lac}_{|\vec{P}|,4} dx \\
= &\sum_{'} c(k, k', l, l') \int_{\mathbb{R}} \left(\left[f_1 * \eta^{\gamma,k}_{\omega_{P_1},1} f_4 * \eta^{lac}_{|\vec{P}|,4} \right] * \eta^{\gamma',k'}_{\omega_{P_2},0} \right) * \eta_{\omega_{Q_3},5} \cdot f_2 * \eta^{\sigma,l}_{\omega_{Q_1},2} \cdot f_3 * \eta^{\sigma',l'}_{\omega_{Q_2},3} dx,
\end{aligned}$$

where $\omega_{Q_3} \supset -\omega_{Q_1} - \omega_{Q_2}$. We are now pleased with the above integral expression and may proceed to discretize in time with respect to the $\mathcal{Q}$ and $\mathcal{P}$ Whitney cubes. The details required for this process are routine and so are omitted. At the end of the day,

$$\int_{\mathbb{R}} \left(\left[f_1 * \eta^{\gamma,k}_{\omega_{P_1},1} f_4 * \eta^{lac}_{|\omega_{\vec{P}}|,4} \right] * \eta^{\gamma',k'}_{\omega_{P_2},0} \right) * \eta_{\omega_{Q_3},5} \cdot f_2 * \eta^{\sigma,l}_{\omega_{Q_1},2} \cdot f_3 * \eta^{\sigma',l'}_{\omega_{Q_2},3} dx$$

can be written as

$$\begin{aligned}
&\sum_{'} \int_0^1 \int_0^1 \sum_{\vec{P} \in \mathbb{P}} \frac{1}{|I_{\vec{P}}|^{1/2}} \langle f_1, \Phi^{\alpha',\gamma,k}_{P_1,1} \rangle \langle f_4, \Phi^{\alpha'}_{P_4,4} \rangle \\
\times &\left\langle \sum_{\vec{Q} \in \mathbb{Q} : |\omega_Q| << |\omega_P|} \frac{1}{|I_{\vec{Q}}|^{1/2}} \langle f_2, \Phi^{\alpha,\sigma,l}_{Q_1,2} \rangle \langle f_3, \Phi^{\alpha,\sigma',l'}_{Q_2,3} \rangle \Phi^{\alpha}_{Q_3,5}, \Phi^{\alpha',\gamma',k'}_{P_2,0} \right\rangle d\alpha d\alpha',
\end{aligned}$$

where $\mathbb{Q}$ is a rank-1 collection of tri-tiles, where $\mathbb{P}$ is a collection of tri-tiles adapted to the degenerate line $\{\xi_1 + \xi_2 = 0\}$. Each tri-tile $\vec{P} = (P_1, P_2, P_3)$, where each $P_j = (I_{\vec{P}}, \omega_{P_j})$ is a tile. Moreover, each $\Phi_{P,j}$ is a wave-packet on the tile P for each $j \in \{0, 1, 2, 3, 4, 5\}$.

Definition 42. *Let $n \geq 1$ and $\sigma \in \{0, \frac{1}{3}, \frac{2}{3}\}^n$. We define the shifted $n-$dyadic mesh $D = D^n_\sigma$ to be the collection of cubes of the form*

$$D^n_\sigma := \left\{2^j(k + (0,1)^n + (-1)^j\sigma) : j \in \mathbb{Z}, k \in \mathbb{Z}^n\right\}$$

Observe that for every cube Q, there exists a shifted dyadic cube Q' such that $Q \subseteq \frac{7}{10}Q'$ and $|Q'| \sim |Q|$; this property clearly follows from verifying the $n = 1$ case. The constant $\frac{7}{10}$ is not especially important here.

Definition 43. *A subset D' of a shifted $n-$dyadic grid D is called sparse, if for any two cubes Q, Q' in D with $Q \neq Q'$ we have $|Q| < |Q'|$ implies $|10^9 Q| < |Q'|$ and $|Q| = |Q'|$ implies $10^9 Q \cap 10^9 Q' = \emptyset$.*

It is immediate from the above definition that any subset of a shifted $n-$dyadic grid can be split into $O(C^n)$ sparse subsets.

Definition 44. *For a given spatial interval I, let $\tilde{\chi}_I(x) := \left(1 + \left(\frac{|x-x_I|}{|I|}\right)^2\right)^{1/2}$, where x_I is the center of I.*

Definition 45. *Let $P = (I_P, \omega_P)$ be a tile. A wave packet on P is a function Φ_P which has Fourier support in $\frac{9}{10}\omega_P$ and obeys the estimate*

$$|\Phi_P(x)| \lesssim_M |I_P|^{-1/2} \tilde{\chi}^M_{I_P}(x)$$

for some fixed large integer M. Therefore, Φ_P is L^2 normalized and adapted to the Heisenberg box (I_P, ω_P).

We next introduce the tile ordering $<$ from [21], which is in the spirit of Fefferman or Lacey and Thiele, but different inasmuch as P' and P do not have to intersect.

Definition 46. *Let $\sigma = (\sigma_1, \sigma_2, \sigma_3) \in \{0, \frac{1}{3}, \frac{2}{3}\}^3$, and let $1 \leq i \leq 3$. An $i-$tile with shift σ_i is a rectangle $P = (I_P, \omega_P)$ with area 1 and with $I_P \in D_0^1, \omega_P \in D_{\sigma_i}^1$. A tri-tile with shift σ is a 3-tuple $\vec{P} = (P_1, P_2, P_3)$ such that each P_i is an $i-$tile with shift σ_i, and the $I_{P_i} = I_{\vec{P}}$ are independent of i. The frequency cube $Q_{\vec{P}}$ of a tri-tile is defined to be $\prod_{i=1}^3 \omega_{P_i}$.*

Definition 47. *A set $\mathbb{P}$ of tri-tiles is called sparse, if all the tri-tiles in $\mathbb{P}$ have the same shift σ and the set of frequency cubes $\{Q_{\vec{P}} = (\omega_{P_1}, \omega_{P_2}, \omega_{P_3}) : \vec{P} \in \mathbb{P}\}$ is sparse.*

Definition 48. *Let P and P' be tiles. We write $P' < P$ if $I_{P'} \subsetneq I_P$ and $3\omega_P \subseteq 3\omega_{P'}$, and $P' \leq P$ if $P' < P$ or $P' = P$. We write $P' \lesssim P$ if $I_{P'} \subseteq I_P$ and $10^7\omega_P \subseteq 10^7\omega_{P'}$. We write $P' \lesssim' P$ of $P' \lesssim P$ and $P' \nleq P$.*

Definition 49. *A collection $\mathbb{P}$ of tri-tiles is said to have rank 1 if one has the following properties for all $\vec{P}, \vec{P}' \in \mathbb{P}$:*

If $\vec{P} \neq \vec{P}'$, then $P_j \neq P_j'$ for all $j = 1, 2, 3$.

If $P_j' \leq P_j$ for some $j = 1, 2, 3$, then $P_i' \lesssim P_i$ for all $1 \leq i \leq 3$.

If we further assume that $|I_{\vec{P}'}| > 10^9 |I_{\vec{P}}|$, then $P_i' \lesssim' P_i$ for all $i \neq j$.

Due to the rapid decay of coefficients over the parameters k, k', l, l', it suffices to prove generalized restricted type estimates for models of type Λ_2 to prove generalized restricted type estimates for $T_{\tilde{1}_{\mathcal{R}_3}}$ with exceptional sets that are in fact allowed to depend on the dyadic shifts, tri-tile collections $\mathbb{P}$ and $\mathbb{Q}$, and wave packets arising from the tri-tile collections. Hence, the discretized and localized version of Theorem 21 is

Theorem 23. *Let $\sigma, \sigma' \in \{0, \frac{1}{3}, \frac{2}{3}\}^3$ be shifts, and let $\mathbb{P}, \mathbb{Q}$ be finite collections of tri-tiles with shifts σ, σ' respectively so that $\mathbb{Q}$ is rank 1. Define the form $\Lambda_{\mathbb{P},\mathbb{Q}}$ by*

$$\Lambda_{\mathbb{P},\mathbb{Q}}(f_1, f_2, f_3, f_4) := \sum_{\vec{P} \in \mathbb{P}} \frac{\langle f_1, \Phi_{P_1,1}\rangle \langle f_4, \Phi^{lac}_{P_4,4}\rangle \left\langle \int_0^1 BHT^{\alpha,\mathbb{Q}}_{\omega_{P_2}}(f_2, f_3) d\alpha, \Phi_{P_2,0} \right\rangle}{|I_{\vec{P}}|^{1/2}},$$

where the $\mathbb{P}$-sum is over all tri-tiles of the form $\vec{P} = (P_1, P_2, P_4), \Phi_{P_1,1}$ is a wave packet on $I_{\vec{P}} \times \omega_{P_1}$, $\Phi_{P_2,0}$ is a wave packet on $I_{\vec{P}} \times \omega_{P_2}$, $\Phi_{P_4,4}$ is a wave packet on $I_{\vec{P}} \times \omega^{lac}_{|\vec{P}|} := I_{\vec{P}} \times [c_1 |I_{\vec{P}}|^{-1}, c_2 |I_{\vec{P}}|^{-1}]$ for some absolute constants $c_1 << c_2$, and

$$BHT^{\alpha}_{\omega_{P_3}}(f_2, f_3)(x) := \sum_{\vec{Q} \in \mathbb{Q} : |\omega_Q| << |\omega_P|} \frac{1}{|I_{\vec{Q}}|^{1/2}} \langle f_2, \Phi^{\alpha}_{Q_1,2}\rangle \langle f_3, \Phi^{\alpha}_{Q_2,3}\rangle \Phi^{\alpha}_{Q_3,5}(x),$$

where for each $\alpha \in [0,1]$, the $\mathbb{Q}$-sum is over all tri-tiles of the form $\vec{Q} = (Q_1, Q_2, Q_3), \Phi^{\alpha}_{Q_1,2}$ is a wave packet on $I_{\vec{Q}} \times \omega_{Q_1}, \Phi^{\alpha}_{Q_2,3}$ is a wave packet on $I_{\vec{Q}} \times \omega_{Q_2}, \Phi^{\alpha}_{Q_3,5}$ is a wave packet on $I_{\vec{Q}} \times \omega_{Q_3}$. Then $\Lambda_{\mathbb{P},\mathbb{Q}}$ is restricted type α for all admissible tuples in $\alpha \in \mathcal{A}$, uniformly in the parameters

$$\sigma, \sigma', \mathbb{P}, \mathbb{Q}, \left\{\Phi_{P_i,j(i)}\right\}, \left\{\Phi^{\alpha}_{Q_i,j(i)}\right\}.$$

It is worth point out if $\vec{\alpha}$ has a bad index j, the restricted type estimate is *not* uniform in the sense that the major subset E'_j cannot be chosen independently of the parameters just mentioned.

As the major subset E'_j cannot be chosen independently of the wave packets

$\left\{\Phi^{\alpha}_{Q_i,j(i)}\right\}$, some care must be taken in deducing Theorem 21 from Theorem 23. For this reason, we isolate the following result:

Proposition 9. *To prove Theorem 21, it suffices to prove Theorem 23.*

Proof. We first consider trilinear multipliers with symbols adapted to $\{\xi_1 = \xi_2 = -\xi_3/2\}$ and prove the desired L^p estimates for $T_{\tilde{1}_{\mathcal{R}_2}}$, namely $T_{\tilde{1}_{\mathcal{R}_2}} : L^{p_1}(\mathbb{R}) \times L^{p_2}(\mathbb{R}) \times L^{p_3}(\mathbb{R}) \to L^{p_4'}(\mathbb{R})$ provided $1 < p_1, p_2, p_3 \leq \infty, 0 < p_4'$, and $(p_1, p_2, p_3, p_4) \in \mathcal{A} \cap \mathcal{A}'$, where $\mathcal{A}$ denotes the interior convex hull of the 9 points

$$\begin{aligned} A_1 &= \left(1, \frac{1}{2}, \frac{1}{2}, -1\right), A_2 = \left(\frac{1}{2}, \frac{1}{2}, 1, -1\right), A_3 = \left(\frac{1}{2}, 1, \frac{1}{2}, -1\right) \\ A_4 &= \left(-\frac{3}{2}, \frac{1}{2}, 1, 1\right), A_5 = \left(-\frac{3}{2}, 1, \frac{1}{2}, 1\right), A_6 = \left(\frac{1}{2}, -\frac{1}{2}, 1, 0\right) \\ A_7 &= \left(0, -\frac{1}{2}, 1, \frac{1}{2}\right), A_8 = \left(0, 1, -\frac{1}{2}, \frac{1}{2}\right), A_9 = \left(\frac{1}{2}, 1, -\frac{1}{2}, 0\right) \end{aligned}$$

and $\mathcal{A}'$ denotes the interior convex hull of the collection $(A_1', ...A_9')$ where each A_j' is gotten by the corresponding A_j by swapping the 1st and 3rd indices. By a standard discretization argument, it suffices to obtain restricted type estimates arbitrarily close to the extremal points in $\mathcal{A}$ for the 4−form

$$\sum_{\vec{P} \in \mathbb{P}} \frac{1}{|I_{\vec{P}}|} \langle f_1, \Phi_{P_1,1} \rangle \langle f_2, \Phi_{P_2,2} \rangle \langle f_3, \Phi_{P_3,3} \rangle \langle f_4, \Phi_{P_4,4} \rangle,$$

where $\mathbb{P} = (P_1, P_2, P_3, P_4)$ is a 4−tile, where each $\Phi_{P_j,j}$ is a wave packet on $P_j = (I_{\vec{P}}, \omega_{P_j})$ for $j = 1, 2, 3, 4$, for each $\vec{P} \in \mathbb{P}$, $Q_{\vec{P}} := (\omega_{P_1}, \omega_{P_2}, \omega_{P_3})$ is a Whitney cube with respect to $\{\xi_1 = \xi_2 = -\xi_3/2\}$, and $\omega_{P_4} = [c_1|I_{\vec{P}}|^{-1}, c_2|I_{\vec{P}}|^{-1}]$. Using a $BHT-$type tile decomposition, it is straightforward to obtain generalized restricted type estimates for all $\vec{\alpha}$ near the extremal points in $\mathcal{A}$, where the exceptional set can be taken independently of all the necessary parameters. By symmetry and fast coefficient decay, it shall therefore suffices by the proceeding discussion to prove that for every 4− tuple (E_1, E_2, E_3, E_4) of measurable subsets of $\mathbb{R}$, (f_1, f_2, f_3, f_4) satisfying $|f_j| \leq 1_{E_j}$ for $j = 1, 2, 3, 4$, and all $\vec{\alpha}$ in a small neighborhood of an extremal point $\vec{\beta} \in \mathcal{A}$ with bad index i, then there exists a major subset E_i' of E_i in the sense that $E_i' \subset E_i$ and $|E_i'| \geq |E_i|/2$ such that the following inequality holds for $\left\{f_j'\right\}_{j=1}^4$ where $f_j' := f_j$ if $j \neq i$ and $f_i' := f_i 1_{E_i'}$:

$$\left| \int_{\mathbb{R}} \int_0^1 T_{\tilde{1}_{\mathcal{R}_3}}^{\alpha',\gamma,\gamma',\sigma,\sigma',k,k',l,l'}(f_1', f_2', f_3')(x) d\alpha' f_4'(x) dx \right| \lesssim |E_1|^{\alpha_1} |E_2|^{\alpha_2} |E_3|^{\alpha_3} |E_4|^{\alpha_4},$$

where the implicit constant is uniform with respect to $\vec{\alpha}$ in a small neighborhood near but not containing the extremal point $\vec{\beta}$ and independent of $\alpha', \gamma, \gamma', \sigma, \sigma', k, k', l, l'$, and

$$\begin{aligned}
& T_{\tilde{1}_{\mathcal{R}_3}}^{\alpha',\gamma,\gamma',\sigma,\sigma',k,k',l,l'}(f_1', f_2', f_3')(x) \\
:= & \sum_{\vec{P} \in \mathbb{P}} \frac{1}{|I_{\vec{P}}|^{1/2}} \langle f_1', \Phi_{P_1,1}^{\alpha',\gamma,k} \rangle \\
\times & \left\langle \int_0^1 \sum_{\vec{Q} \in \mathbb{Q} : |\omega_Q| << |\omega_P|} \frac{1}{|I_{\vec{Q}}|^{1/2}} \langle f_2', \Phi_{Q_1,2}^{\alpha,\sigma,l} \rangle \langle f_3', \Phi_{Q_2,3}^{\alpha,\sigma',l'} \rangle \Phi_{Q_3,5}^{\alpha} d\alpha, \Phi_{P_2,0}^{\alpha',\gamma',k'} \right\rangle \overline{\Phi_{P_4,4}^{\alpha'}}(x) \\
:= & \sum_{\vec{P} \in \mathbb{P}} \frac{1}{|I_{\vec{P}}|^{1/2}} \langle f_1', \Phi_{P_1,1}^{\alpha',\gamma,k} \rangle \left\langle \int_0^1 BHT^{\alpha,\mathbb{Q}^{\tilde{d}}}(f_2, f_3) d\alpha, \Phi_{P_2,0}^{\alpha',\gamma',k'} \right\rangle \overline{\Phi_{P_4,4}^{\alpha'}}(x).
\end{aligned}$$

By Theorem 23, we know the required generalized restricted type estimates hold. Therefore, by interpolating weak type estimates, we know $T_{\tilde{1}_{\mathcal{R}_3}}$ satisfies the generalized restricted type estimate in the interior convex hull of $\mathcal{A}$. Moreover, by symmetry, $T_{\tilde{1}_{\mathcal{R}_1}}$ must satisfy all the generalized restricted type estimates in the interior convex hull of $\mathcal{A}'$. Using Marcinkiewicz interpolation as discussed in [26] and then combining the L^p estimates for $T_{\tilde{1}_{\mathcal{R}_1}}, T_{\tilde{1}_{\mathcal{R}_2}}$ and $T_{\tilde{1}_{\mathcal{R}_3}}$ yields Theorem 21.

$\square$

3.5 Generalized Restricted Type Estimates for $\Lambda_{\mathbb{P},\mathbb{Q}}$ near A_1, A_2, A_3

3.5.1 Tile Decomposition

Fix dyadic shifts σ, σ' and corresponding tri-tile collections $\mathbb{P}$ and $\mathbb{Q}$ once and for all. By assumption, $\mathbb{Q}$ is rank-1. Moreover, for convenience, we shall subsequently use f_j to denote f_j' for $j = 1, 2, 3, 4$ in Theorem 23 and assume by rescaling that $|E_4| = 1$ and the collections $\mathbb{P}, \mathbb{Q}$ are sparse. Next, set

$$\tilde{\Omega} = \{M1_{E_2} \gtrsim |E_2|\} \bigcup \{M1_{E_3} \gtrsim |E_3|\}.$$

Fix $\tilde{d} \geq 0$. Then let $\mathbb{Q}^{\tilde{d}} := \left\{ \vec{Q} \in \mathbb{Q} : 1 + \frac{dist(I_{\vec{Q}}, \tilde{\Omega}^c)}{|I_{\vec{Q}}|} \simeq 2^{\tilde{d}} \right\}$ and

$$\Omega_1^{\tilde{d}} := \left\{ M\left(\sum_{\vec{Q} \in \mathbb{Q}^{\tilde{d}}} \frac{\langle f_2, \Phi^{\alpha}_{Q_{1,2}} \rangle \langle f_3, \Phi^{\alpha}_{Q_{2,3}} \rangle \Phi_{Q_{3,6}}}{|I_{\vec{Q}}|^{1/2}} d\alpha \right) \gtrsim 2^{\tilde{d}} |E_2|^{1/2} |E_3|^{1/2} \right\}$$

$$\Omega_2^{\tilde{d}} := \left\{ M\left(\left[\int_0^1 \sum_{\vec{Q} \in \mathbb{Q}^{\tilde{d}}} \frac{|\langle f_2, \Phi^{\alpha}_{Q_{1,2}} \rangle \langle f_3, \Phi^{\alpha}_{Q_{2,3}} \rangle|}{|I_{\vec{Q}}|} \tilde{1}_{I_{\vec{Q}}} d\alpha \right]^2 \right) \gtrsim 2^{2\tilde{d}} |E_2||E_3| \right\}.$$

Lastly, construct

$$\Omega = \bigcup_{\tilde{d} \geq 0} \Omega_1^{\tilde{d}} \bigcup_{\tilde{d} \gtrsim 1} \Omega_2^{\tilde{d}} \bigcup \{M1_{E_1} \gtrsim |E_1|\} \bigcup \tilde{\Omega}.$$

Then for large enough implicit constants, $|\Omega| \leq 1/2$ and $\tilde{E}_4 := E_4 \cap \Omega^c$ is a major subset of E_4 since $|E_4| = 1$. Now let $\mathbb{P}^d := \left\{ \vec{P} \in \mathbb{P} : 1 + \frac{dist(I_{\vec{P}}, \Omega(\epsilon)^c)}{|I_{\vec{P}}|} \simeq 2^d \right\}$.

Assuming $|f_1| \leq 1_{E_1}, |f_2| \leq 1_{E_2}, |f_3| \leq 1_{E_3}, |f_4| \leq 1_{E_4 \cap \Omega^c}$, recall that our task in this section is to obtain the estimate $|\Lambda_{\mathbb{P},\mathbb{Q}}(f_1, f_2, f_3, f_4)| \lesssim |E_1|^{\alpha_1}|E_2|^{\alpha_2}|E_3|^{\alpha_3}$ for $(\alpha_1, \alpha_2, \alpha_3, \alpha_4)$ in a small neighborhood near an extremal point $\vec{\beta} \in \{A_1, A_2, A_3\}$. To decompose our tri-tiles, we shall first need to recall the following standard terminology from [21]:

It will be useful to have the following tree selection algorithm for $\mathbb{P}^d$ essentially from [21]:

Lemma 23. *Fix $d, \tilde{d} \geq 0$. Then there exist two decompositions of $\mathbb{P}^d$, namely $\bigcup_{n_1 \geq N_1(d)} \tilde{\mathbb{P}}^d_{n_1,1}$ and $\bigcup_{\mathfrak{d} \geq N_2(d,\tilde{d})} \mathbb{P}^{d,\tilde{d}}_{\mathfrak{d},2}$ such that $Size_1(f_1, \tilde{\mathbb{P}}^d_{n_1,1}) \lesssim 2^{-n_1}$ and $Size_2^{\tilde{d}}(f_2, f_3, \tilde{\mathbb{P}}^{d,\tilde{d}}_{\mathfrak{d},2}) \lesssim 2^{-\mathfrak{d}}$. Moreover, $\tilde{\mathbb{P}}^d_{n_1,1}$ and $\tilde{\mathbb{P}}^{d,\tilde{d}}_{\mathfrak{d},2}$ can each be written as a union of trees, i.e.*

$$\begin{aligned} \tilde{\mathbb{P}}^d_{n_1,1} &= \bigcup_{T \in \mathcal{T}^d_{n_1,1}} \bigcup_{\vec{P} \in T} \vec{P} \\ \tilde{\mathbb{P}}^{d,\tilde{d}}_{\mathfrak{d},2} &= \bigcup_{T \in \mathcal{T}^{d,\tilde{d}}_{\mathfrak{d},2}} \bigcup_{\vec{P} \in T} \vec{P}, \end{aligned}$$

such that

$$\begin{aligned} \sum_{T \in \mathcal{T}^d_{n_1,1}} |I_T| &\lesssim 2^{2n_1} \sum_{T \in \mathcal{T}^d_{n_1,1,*}} \sum_{\vec{P} \in T} |\langle f_1, \Phi_{P_1,1} \rangle|^2 \\ \sum_{T \in \mathcal{T}^{d,\tilde{d}}_{\mathfrak{d},2}} |I_T| &\lesssim 2^{2\mathfrak{d}} \sum_{T \in \mathcal{T}^{d,\tilde{d}}_{\mathfrak{d},2}} \sum_{\vec{P} \in T} \left| \left\langle \int_0^1 BHT^{\alpha,\mathbb{Q}^{\tilde{d}}}_{\omega_{P_2}}(f_2, f_3) d\alpha, \Phi_{P_2,0} \right\rangle \right|^2, \end{aligned}$$

where $\mathcal{T}^d_{n_1,1,*} \subset \mathcal{T}^d_{n_1,1}$ and $\mathcal{T}^{d,\tilde{d}}_{\mathfrak{d},2,*} \subset \mathcal{T}^{d,\tilde{d}}_{\mathfrak{d},2}$, each tree in $\mathcal{T}^d_{n_1,1,*}$ is a $2-$tree and each tree in $\mathcal{T}^{d,\tilde{d}}_{\mathfrak{d},2,*}$ is a $1-$tree, and the collections $\mathcal{T}^d_{n_1,1,*}, \mathcal{T}^{d,\tilde{d}}_{\mathfrak{d},2,*}$ can each be written as the union of two strongly 2-disjoint subcollections. We denote this last property by

$$\mathcal{T}^{d}_{n_1,1,*} = \mathcal{T}^{d}_{n_1,1,*,+} \bigcup \mathcal{T}^{d}_{n_1,1,*,-}$$
$$\mathcal{T}^{d,\tilde{d}}_{\mathfrak{d},2,*} = \mathcal{T}^{d,\tilde{d}}_{\mathfrak{d},2,*,+} \bigcup \mathcal{T}^{d,\tilde{d}}_{\mathfrak{d},2,*,-}.$$

Proof. We describe the procedure for producing the collection $\mathcal{T}_{n_1,1}$, as the decomposition into trees in the collection $\mathcal{T}^{d,\tilde{d}}_{\mathfrak{d},2}$ is very similar. Let $N_1(d)$ be the smallest integer for which $Size_1(f_1, \mathbb{P}^d) \geq 2^{-N_1(d)}$. We may assume without loss of generality that there are only finitely many tri-tiles in the collection $\mathbb{P}^d$, so $N_1(d)$ exists. Assume the collection $\tilde{\mathbb{P}}^d_{m,1}$ has already been constructed with all the desired properties for $m < n_1$. We now perform the following standard tile selection algorithm on the tri-tile collection $\mathbb{P}^d \cap \left[\bigcup_{N_1(d)\leq m<n_1} \tilde{\mathbb{P}}^d_{n_1,1}\right]^c$ to produce $\tilde{\mathbb{P}}^d_{n_1,1}$ with the desired properties. To this end, introduce the following notation: if P is a tile, let ξ_P denote the center of ω_P. If P and P' are tiles, we write $P' \lesssim^+ P$ if $P' \lesssim' P$ and $\xi_{P'} > \xi_P$, and $P' \lesssim^- P$ if $P' \lesssim' P$ and $\xi_{P'} < \xi_P$. Now consider the set of $2-$trees in $\mathbb{P}^d \cap \left[\bigcup_{N_1(d)\leq m<n_1} \tilde{\mathbb{P}}^d_{n_1,1}\right]^c$ which are upward in the sense that

$$P_j \lesssim^+ P_{T,j} \ for\ all\ \vec{P} \in T$$

and which satisfies $\sum_{\vec{P}\in T} |\langle f_1, \Phi_{P_1,1}\rangle|^2 \geq 2^{-2n_1-3}|I_T|$. If there are no trees with this property, terminate the algorithm. Otherwise, choose T among all such trees so that the center $\xi_{T,1}$ of $\omega_{P_{T},t}$ is maximal and that T is maximal with respect to the set inclusion. Moreover, let T' denote that $1-$tree

$$T' := \left\{\vec{P} \in \mathbb{P}^d \cap T^c : P_1 \leq P_{T,1}\right\}.$$

Now remove T and T' from $\mathbb{P}^d$. Then repeat the tile selection process with the remaining tri-tiles $\mathbb{P}^d \cap (T \cup T')^c$ until there are no more upward trees satisfying the size condition. Again, by our finiteness assumption, the algorithm terminates in a finite number of steps, producing trees $T_1, T_1', T_2, T_2', ..., T_M, T_M'$, where each T_j is a 2−tree and each T_j' is a 1−tree. Set

$$\mathcal{T}^d_{n_1,1,+} = \bigcup_{j=1}^{M} [T_j \cup T_j']$$
$$\mathcal{T}^d_{n_1,1,*,+} = \bigcup_{j=1}^{M} T_j.$$

The claim is now that the trees $T_1, ..., T_M$ are strongly 1− disjoint. Indeed, it is clear that $T_s \cap T_{s'} = \emptyset$ when $s \neq s'$. Therefore, we must have $P_1 \neq P_1'$ for all $\vec{P} \in T_s$, $\vec{P'} \in T_{s'}, s \neq s'$. Suppose for a contradiction that there were tri-tiles $\vec{P} \in T_s, \vec{P'} \in T_{s'}$ such that $2\omega_{P_1} \subsetneq 2\omega_{P_1'}$ and $I_{P_2'} \subset I_{T_s}$. By sparseness, we thus have $|\omega_{P_1'}| \geq 10^9 |\omega_{P_1}|$. Since $P_1 \lesssim^+ P_{T_s,1}$ and $P_1' \lesssim^+ P_{T_{s'},1}$, we thus see that $\xi_{P_{T_{s'},1}} < \xi_{P_{T_s},1}$. By our select algorithm, this implies $s < s'$. Also, since $|\omega_{P_1'}| \geq 10^9 |\omega_{P_1}|$, $I_{P_1'} \subset I_{T_s}$, and $P_1 \lesssim P_{T_s,1}$, it must be that $P_1' \leq P_{T_s,1}$. Since $s < s'$, this means that $\vec{P'} \in T_s'$. But T_s' and $T_{s'}$ are disjoint trees by construction, which is a contradiction. Now repeat the previous algorithm, but replace $\lesssim^+$ by $\lesssim^-$, so the trees T are downward pointing instead of upward pointing, and select the trees T so that the center $\xi_{T,j}$ is minimized rather than maximized. This yields two further collection of trees $\mathcal{T}^d_{n_1,1,-}$ and $\mathcal{T}^d_{n_1,1,*,-}$ such that for any 2−tree T consisting of unselected tiles

$$\sum_{\vec{P} \in T : P_2 \lesssim^- P_{T_2}} |\langle f_1, \Phi_{P_1,1} \rangle|^2 < 2^{-2n-3} |I_T|.$$

Letting $\mathcal{T}^d_{n_1,1} = \mathcal{T}^d_{n_1,1,+} \bigcup \mathcal{T}^d_{n_1,1,-}$ and $\mathcal{T}^d_{n_1,1,*} = \mathcal{T}^d_{n_1,1,*,+} \bigcup \mathcal{T}^d_{n_1,1,*,-}$, it follows that

$$Size_1\left(f_1, \mathbb{P}^d \cap \left[\bigcup_{N_1(d) \leq m \leq n_1} \mathcal{T}^d_{m,1}\right]^c\right) < 2^{-2(n_1+1)}.$$

□

Now define $\mathbb{P}^{d,\tilde{d}}_{n_1,\mathfrak{d}} = \tilde{\mathbb{P}}^d_{n_1,1} \cap \tilde{\mathbb{P}}^{d,\tilde{d}}_{\mathfrak{d},2}$. Hence, for each $\tilde{d} \geq 0$, we may decompose the space of $\mathbb{P}$ tri-tiles as

$$\mathbb{P} = \bigcup_{d \geq 0} \mathbb{P}^d = \bigcup_{d \geq 0} \bigcup_{n_1, \mathfrak{d}} \mathbb{P}^{d,\tilde{d}}_{n_1,\mathfrak{d}}.$$

Hence, we produce the joint decomposition $\mathbb{P} \times \mathbb{Q} = \bigcup_{\tilde{d} \geq 0} \bigcup_{d \geq 0} \mathbb{P}^{d,\tilde{d}}_{n_1,\mathfrak{d}} \times \mathbb{Q}^{\tilde{d}}$.

3.6 Tree Estimates

First, let T be a 2-tree consisting of bi-tiles in the collection $\mathbb{P}^{d,\tilde{d}}_{n_1,\mathfrak{d}}$. Then

$$\begin{aligned}
&\left| \sum_{\vec{P} \in T} \frac{\langle f_1, \Phi_{P_1,1}\rangle \langle f_4, \Phi^{lac}_{P_4,4}\rangle \left\langle \int_0^1 BHT^{\mathbb{Q}^{\alpha,\tilde{d}}}_{\omega_{P_2}}(f_2,f_3)d\alpha, \Phi_{P_2,0}\right\rangle}{|I_{\vec{P}}|^{1/2}} \right| \\
\leq\ & \frac{\left(\sum_{\vec{P}\in T} |\langle f_1, \Phi_{P_1,1}\rangle|^2\right)^{1/2}}{|I_T|^{1/2}} \cdot \frac{\left(\sum_{\vec{P}\in T} |\langle f_4, \Phi^{lac}_{\vec{P},4}\rangle|^2\right)^{1/2}}{|I_T|^{1/2}} \\
\times\ & \sup_{\vec{P}\in T}\left[\frac{\left|\left\langle \int_0^1 BHT^{\alpha,\mathbb{Q}^{\tilde{d}}}_{\omega_{P_2}}(f_2,f_3)d\alpha, \tilde{\Phi}^{\infty}_{P_2,0}\right\rangle\right|}{|I_{\vec{Q}}|}\right] |I_T| \\
\lesssim\ & 2^{-\tilde{N}d} 2^{-n_1} 2^{-\mathfrak{d}} |I_T|.
\end{aligned}$$

Now let $T \subset \mathbb{P}^{d,\tilde{d}}_{n_1,\mathfrak{d}}$ be a $1-$tree. Then

$$
\begin{aligned}
& \left| \sum_{\vec{P}\in T} \frac{\langle f_1, \Phi_{P_1}\rangle \langle f_4, \Phi^{lac}_{P_4,4}\rangle \left\langle \int_0^1 BHT^{\alpha,\mathbb{Q}^{\tilde{d}}}_{\omega_{P_2}}(f_2,f_3)d\alpha, \Phi_{P_2,0}\right\rangle}{|I_{\vec{P}}|^{1/2}} \right| \\
\lesssim & \left[\sup_{\vec{P}\in T} \frac{|\langle f_1, \Phi_{P_1,1}\rangle|}{|I_{\vec{P}}|^{1/2}} \right] \left(\sum_{\vec{P}\in T} \frac{|\langle f_4, \Phi^{lac}_{\vec{P},4}\rangle|^2}{|I_T|} \right)^{1/2} \\
\times & \left(\sum_{\vec{P}\in T} \frac{\left| \left\langle \int_0^1 BHT^{\alpha,\mathbb{Q}^{\tilde{d}}}_{\omega_{P_2}}(f_2,f_3)d\alpha, \Phi_{P_2,0}\right\rangle \right|^2}{|I_T|} \right)^{1/2} |I_T| \\
\lesssim & \; 2^{-\tilde{N}d} 2^{-n_1} 2^{-\mathfrak{d}} |I_T|.
\end{aligned}
$$

Lemma 24. *Let $\tilde{\mathbb{P}} \subset \mathbb{P}$ be any sub collection of tri-tiles. Then, for any $0<\theta<1$ and $1-$tree $T \subset \tilde{\mathbb{P}}$,*

$$
\begin{aligned}
& \left[\frac{1}{|I_T|} \sum_{\vec{P}\in T} \left| \left\langle \int_0^1 \sum_{\vec{Q}\in\mathbb{Q}^{\tilde{d}}:\omega_{Q_3}\supset\omega_P} \frac{1}{|I_{\vec{Q}}|^{1/2}} \langle f_2, \Phi^\alpha_{Q_1,2}\rangle \langle f_3, \Phi^\alpha_{Q_2,3}\rangle \Phi^\alpha_{Q_3,5}, \Phi_{P_2,0} \right\rangle \right|^2 \right]^{1/2} \\
\lesssim_\theta & \left[\sup_{\vec{P}\in\tilde{\mathbb{P}}} \frac{1}{|I_{\vec{P}}|} \int 1_{E_2} \tilde{1}_{I_{\vec{P}}} dx \right]^{\theta} \left[\sup_{\vec{P}\in\tilde{\mathbb{P}}} \frac{1}{|I_{\vec{P}}|} \int 1_{E_3} \tilde{1}_{I_{\vec{P}}} dx \right]^{1-\theta}.
\end{aligned}
$$

Proof. See [21]. □

3.6.1 Size Restrictions

Lemma 25. *Fix $d, \tilde{d}, n_1, \mathfrak{d}$ such that $\mathbb{P}^{d,\tilde{d}}_{n_1,\mathfrak{d}}$ is nonempty. Then*

$$
\begin{aligned}
2^{-n_1} & \lesssim 2^d |E_1| \\
2^{-\mathfrak{d}} & \lesssim 2^{-\tilde{\tilde{N}}(\tilde{d}-d)} |E_2|^{1/2} |E_3|^{1/2}.
\end{aligned}
$$

Proof. That $2^{-n_1} \lesssim 2^d|E_1|$ is standard, so the details are omitted. Therefore, it suffices to prove $2^{-\mathfrak{d}} \lesssim 2^{-\tilde{\tilde{N}}(\tilde{d}-d)}|E_2|^{1/2}|E_3|^{1/2}$.

CASE 1: It clearly suffices to show that for every $\tilde{\mathbb{P}}^{d,\tilde{d}}_{n_1,\delta}$-tree T

$$\frac{1}{|I_T|^{1/2}}\left(\sum_{\vec{P}\in T}\left|\left\langle\int_0^1 BHT^{\alpha,\mathbb{Q}^{\tilde{d}}}_{\omega_{P_2}}(f_2,f_3)d\alpha,\Phi_{P_2,0}\right\rangle\right|^2\right)^{1/2}\lesssim 2^d|E_2|^{1/2}|E_3|^{1/2}.$$

By the triangle inequality, the LHS of the above display is at most

$$\left(\sum_{\vec{P}\in T}\frac{\left|\left\langle\int_0^1 BHT^{\alpha,\mathbb{Q}^{\tilde{d}}}(f_2,f_3)d\alpha,\Phi_{P_2,3}\right\rangle\right|^2}{|I_T|}\right)^{1/2}$$
$$+\left(\sum_{\vec{P}\in T}\frac{\left|\left\langle\int_0^1 BHT^{\alpha,\mathbb{Q}^{\tilde{d}}}(f_2,f_3)-BHT^{\alpha,\mathbb{Q}^{\tilde{d}}}_{\omega_{P_2}}(f_2,f_3)d\alpha,\Phi_{P_2,3}\right\rangle\right|^2}{|I_T|}\right)^{1/2}.$$

Denote the first and second terms by I and II respectively. Then, by the John-Nirenberg inequality, $I\lesssim\sup_{\vec{P}\in T}\frac{1}{|I_{\vec{P}}|}\int_{\mathbb{R}}\left|\int_0^1 BHT^{\alpha,\mathbb{Q}^{\tilde{d}}}(f_2,f_3)d\alpha\right|\tilde{1}_{I_{\vec{P}}}dx\lesssim 2^d2^{\tilde{d}}|E_2|^{1/2}|E_3|^{1/2}$. This is clearly acceptable by the assumption $d\geq\tilde{d}$. Furthermore, $II\leq II_a+II_b$ where

$$II_a=\frac{1}{|I_T|^{1/2}}\left(\sum_{\vec{P}\in T}\left|\left\langle\int_0^1\sum_{\vec{Q}\in\mathbb{Q}^{\tilde{d}}:\omega_{Q_3}\supset\supset\omega_{P_2}}\frac{\langle f_2,\Phi^\alpha_{Q_1,2}\rangle\langle f_3,\Phi^\alpha_{Q_2,3}\rangle}{|I_{\vec{Q}}|^{1/2}}\Phi^\alpha_{\vec{Q}_3,5}d\alpha,\Phi_{P_2,0}\right\rangle\right|^2\right)^{1/2}$$
$$II_b=\frac{1}{|I_T|^{1/2}}\left(\sum_{\vec{P}\in T}\left|\left\langle\int_0^1\sum_{\vec{Q}\in\mathbb{Q}^{\tilde{d}}:|\omega_{Q_3}|\simeq|\omega_{P_2}|}\frac{\langle f_2,\Phi^\alpha_{Q_1,2}\rangle\langle f_3,\Phi^\alpha_{Q_2,3}\rangle}{|I_{\vec{Q}}|^{1/2}}\Phi^\alpha_{\vec{Q}_3,5}d\alpha,\Phi_{P_2,0}\right\rangle\right|^2\right)^{1/2}.$$

Using Fubini, interchanging the integral over α and the l^2 sum, and invoking the Biest size estimate from [21] gives $II_a\lesssim_\theta 2^d|E_2|^{1/2}|E_3|^{1/2}$. To handle term II_b, first assume the $\mathbb{Q}^{\tilde{d}}-$sum is further restricted to all $|\omega_{Q_3}| = |\omega_{P_2}|$ in which case

$$\begin{aligned} II_b &\lesssim \sup_{\vec{P}\in T} \frac{1}{|I_P|} \int_{\mathbb{R}} \left| \int_0^1 \sum_{\omega\in\Omega_2(T):|\omega|\le|\omega_{P_3}|} \sum_{\vec{Q}\in\mathbb{Q}^{\tilde{d}}:\omega_{Q_3}=\omega} \frac{\langle f_2, \Phi^\alpha_{Q_1,2}\rangle\langle f_3, \Phi^\alpha_{Q_2,3}\rangle}{|I_{\vec{Q}}|^{1/2}} \Phi^\alpha_{Q_3,5} d\alpha \right| \tilde{1}_{I_{\vec{P}}} dx \\ &\lesssim \int_0^1 \left[\sup_{\vec{P}\in T} \frac{1}{|I_P|} \int_{\mathbb{R}} \sum_{\omega\in\Omega_2(T):|\omega|\le|\omega_{P_3}|} \sum_{\vec{Q}\in\mathbb{Q}^{\tilde{d}}:\omega_{Q_3}=\omega} \frac{|\langle f_2, \Phi^\alpha_{Q_1,2}\rangle\langle f_3, \Phi^\alpha_{Q_2,3}\rangle|}{|I_{\vec{Q}}|} \tilde{1}_{I_{\vec{Q}}} \tilde{1}_{I_{\vec{P}}} dx \right] d\alpha \\ &\lesssim 2^d |E_2|^{1/2}|E_3|^{1/2}. \end{aligned}$$

Now suppose the $\mathbb{Q}^{\tilde{d}}$-sum in II_b is restricted to the collection $\left\{(\vec{P},\vec{Q}) : |\omega_{Q_3}| = 2^\lambda |\omega_{P_2}|\right\}$. Associate to each $\omega_{P_2} \in \Omega_2(T)$ a (σ'-shifted) dyadic interval of length $2^\gamma|\omega_{P_2}|$ denoted by $\overline{\omega_{P_2}}$ which intersects ω_{P_2}. Of course, if $\gamma \geq 0$, there are at most two choices for $\overline{\omega_{P_3}}$, while if $\gamma < 0$ and $|\gamma| \lesssim 1$, there will be $O(1)$ choices. By the triangle inequality, it is not hard to see that we can reduce our problem to obtaining estimates for expressions of the form

$$II_b^\gamma := \frac{1}{|I_T|^{1/2}} \left(\sum_{\vec{P}\in T} \left| \left\langle \int_0^1 \sum_{\vec{Q}\in\mathbb{Q}^{\tilde{d}}:\omega_{Q_3}=\overline{\omega_{P_2}}(\gamma)} \frac{\langle f_2, \Phi^\alpha_{Q_1,2}\rangle\langle f_3, \Phi^\alpha_{Q_2,3}\rangle}{|I_{\vec{Q}}|^{1/2}} \Phi^\alpha_{\vec{Q}_3,5} d\alpha, \Phi_{P_2,0} \right\rangle \right|^2 \right)^{1/2}$$

for all $|\gamma| \lesssim 1$. Moreover, the corresponding collection $\mathbb{Q}^{\tilde{d}}_\gamma(\Omega_2(T)) := \left\{\vec{Q} \in \mathbb{Q}^{\tilde{d}} : \omega_{Q_3} = \overline{\omega_{P_2}}(\gamma)\right\}_{\omega_{P_2}\in\Omega_2(T)}$ can be decomposed into the union of two sets

$$\mathbb{Q}^{\tilde{d}}_\gamma(\Omega_2(T)) = \mathbb{Q}^{\tilde{d}}_{\gamma,1}(\Omega_2(T)) \bigcup \mathbb{Q}^{\tilde{d}}_{\gamma,2}(\Omega_2(T))$$

where the frequencies in $\mathbb{Q}^{\tilde{d}}_{\gamma,1}(\Omega_2(T))$ are lacunary in the first index and the frequencies in $\mathbb{Q}^{\tilde{d}}_{\gamma,2}(\Omega_2(T))$ are lacunary in the second index. Indeed, because T is a $1-$tree,

$$dist(\omega_{Q_3}, c_{\omega_{T_2}}) \leq C_1|\omega_{Q_3}|$$

so that $dist(\omega_{Q_1}, -c_{\omega_{T_2}}) \leq C_2|\omega_{Q_1}|$ holds for all $\vec{Q} \in \mathbb{Q}^{\tilde{d}}(\Omega_2(T))$. Now set

$$\mathbb{Q}_1^{\tilde{d}}(\Omega_2(T)) = \left\{ \vec{Q} \in \mathbb{Q}^{\tilde{d}}(\Omega_2(T)) : dist(\omega_{Q_1}, -c_{\omega_{T_2}}) \geq |\omega_{Q_1}|/2 \right\}.$$

By construction, $\mathbb{Q}_1^{\tilde{d}}(\Omega_2(T))$ is lacunary and the complement in $\mathbb{Q}_1^{\tilde{d}}(\Omega_2(T))$ denoted by $\mathbb{Q}^{\tilde{d}}(\Omega_2(T))_2$ must lacunary in index 2 by the rank-1 property. Hence, without loss of generality, it suffices to worth only with$\mathbb{Q}_{\gamma,1}^{\tilde{d}}(\Omega_2(T))$ and bound II_b^{γ} by

$$\begin{aligned}
&\int_0^1 \frac{1}{|I_T|^{1/2}} \left(\sum_{\vec{P} \in T} \left| \left\langle \sum_{\vec{Q} \in \mathbb{Q}^{\tilde{d}} : \omega_{Q_3} = \overline{\omega_{P_2}}(\gamma)} \frac{\langle f_2, \Phi^{\alpha}_{Q_1,2} \rangle \langle f_3, \Phi^{\alpha}_{Q_2,3} \rangle}{|I_{\vec{Q}}|^{1/2}} \Phi^{\alpha}_{\vec{Q}_3,5}, \Phi_{P_2,0} \right\rangle \right|^2 \right)^{1/2} d\alpha \\
\lesssim_N \; & \sum_{l \in \mathbb{Z}} \frac{1}{1+|l|^N} \int_0^1 \frac{1}{|I_T|^{1/2}} \left(\sum_{\vec{Q} \in \mathbb{Q}_{\gamma,1}^{\tilde{d}}(\Omega_2(T)) : I_{\vec{Q}} \subset I_T + l \cdot I_T} \frac{|\langle f_2, \Phi^{\alpha}_{Q_1,2} \rangle|^2 |\langle f_3, \Phi^{\alpha}_{Q_2,3} \rangle|^2}{|I_{\vec{Q}}|} \right)^{1/2} d\alpha \\
\lesssim \; & \sum_{l \in \mathbb{Z}} \frac{2^{d/2}|E_3|^{1/2}}{1+|l|^N} \int_0^1 \frac{1}{|I_T|^{1/2}} \left(\sum_{\vec{Q} \in \mathbb{Q}_{\gamma,1}^{\tilde{d}}(\Omega_2(T)) : I_{\vec{Q}} \subset I_T + l \cdot I_T} |\langle f_2, \Phi^{\alpha}_{Q_1,2} \rangle|^2 \right)^{1/2} d\alpha \\
\lesssim \; & 2^d |E_2|^{1/2} |E_3|^{1/2}.
\end{aligned}$$

CASE 2: Now assume $\tilde{d} >> d$. It suffices to prove that for every $\mathbb{P}^{d,\tilde{d}}_{n_1,\delta}-$tree T and some $\tilde{\tilde{N}} >> 1$ that

$$\frac{1}{|I_T|^{1/2}} \left(\sum_{\vec{P} \in T} \left| \left\langle \int_0^1 BHT^{\alpha, \mathbb{Q}^{\tilde{d}}}_{\omega_{P_2}}(f_2, f_3) d\alpha, \Phi_{P_2,0} \right\rangle \right|^2 \right)^{1/2} \lesssim_{\tilde{\tilde{N}}} 2^{-\tilde{\tilde{N}}\tilde{d}} |E_2|^{1/2} |E_3|^{1/2}.$$

Here, we do not want to rewrite the BHT inside this sum as a difference of two other BHTs as before. Instead, we want to exploit the fact that the $\vec{Q}$ appearing inside the $\mathbb{P}$-sum have finer frequency localization and so have larger time intervals $I_{\vec{Q}}$. To this end, observe that whenever $(\vec{P}, \vec{Q}) \in \mathbb{P}^d \times \mathbb{Q}^{\tilde{d}}$ satisfy $|I_{\vec{P}}| \leq |I_{\vec{Q}}|$ and

$\tilde{d} >> d$, then $dist(I_{\vec{P}}, I_{\vec{Q}}) \gtrsim 2^{\tilde{d}}|I_{\vec{Q}}|$. This is because $\vec{P} \in \mathbb{P}^d$ implies $1+\frac{dist(I_{\vec{P}},\Omega^c)}{|I_{\vec{P}}|} \simeq 2^d$. Therefore, using $\Omega \supset \tilde{\Omega}$,

$$dist(I_{\vec{P}}, \tilde{\Omega}^c) \lesssim 2^d |I_{\vec{P}}|.$$

If the proposed inequality did not hold, then $dist(I_{\vec{Q}}, \tilde{\Omega}^c) \leq dist(I_{\vec{Q}}, I_{\vec{P}}) + |I_{\vec{P}}| + dist(I_{\vec{P}}, \tilde{\Omega}^c) << 2^{\tilde{d}}|I_{\vec{Q}}|$, which would violate the assumption $\vec{Q} \in \mathbb{Q}^{\tilde{d}}$. With this observation, we now can write down

$$\begin{aligned}
&\sum_{\vec{P}\in T}\left|\left\langle \int_0^1 BHT^{\alpha,\mathbb{Q}^{\tilde{d}}}_{\omega_{P_2}}(f_2,f_3)d\alpha, \Phi_{P_2,0}\right\rangle\right|^2 \\
=\ &\sum_{\vec{P}\in T}\left|\left\langle \int_0^1 \sum_{\vec{Q}\in\mathbb{Q}^{\tilde{d}}:\omega_{Q_3}\subset\subset\omega_{P_2}} \frac{\langle f_2,\Phi^{\alpha}_{Q_1,2}\rangle\langle f_3,\Phi^{\alpha}_{Q_2,3}\rangle}{|I_{\vec{Q}}|^{1/2}}\Phi^{\alpha}_{\vec{Q}_3,5}d\alpha, \Phi_{P_2,0}\right\rangle\right|^2 \\
\lesssim\ & 2^{-\tilde{N}\tilde{d}}\sum_{\omega\in\Omega\{T\}}\sum_{\vec{P}\in T:\omega_{P_2}=\omega}\left|\left\langle \int_0^1 \sum_{\vec{Q}\in\mathbb{Q}^{\tilde{d}}:\omega_{Q_3}\subset\subset\omega} \frac{\left|\langle f_2,\Phi^{\alpha}_{Q_1,2}\rangle\langle f_3,\Phi^{\alpha}_{Q_2,3}\rangle\right|}{|I_{\vec{Q}}|}\tilde{1}_{I_{\vec{Q}}}d\alpha, \tilde{1}_{I_{\vec{P}}}\right\rangle\right|^2 \\
\lesssim\ & 2^{-\tilde{N}\tilde{d}}\sum_{\omega\in\Omega\{T\}}\sum_{\vec{P}\in T:\omega_{P_2}=\omega}\left\|\int_0^1 \sum_{\vec{Q}\in\mathbb{Q}^{\tilde{d}}:\omega_{Q_3}\subset\subset\omega_{P_2}} \frac{\left|\langle f_2,\Phi^{\alpha}_{Q_1,2}\rangle\langle f_3,\Phi^{\alpha}_{Q_2,3}\rangle\right|}{|I_{\vec{Q}}|}d\alpha\tilde{1}_{I_{\vec{Q}}}\tilde{1}_{I_T}\right\|_2^2.
\end{aligned}$$

The last line of the above display is majorized by

$$2^{-\tilde{N}\tilde{d}}\left\|\int_0^1 \sum_{\vec{Q}\in\mathbb{Q}^{\tilde{d}}} \frac{\left|\langle f_2,\Phi^{\alpha}_{Q_1,2}\rangle\langle f_3,\Phi^{\alpha}_{Q_2,3}\rangle\right|}{|I_{\vec{Q}}|}\tilde{1}_{I_{\vec{Q}}}d\alpha\tilde{1}_{I_T}\right\|_2^2.$$

Therefore,

$$\frac{1}{|I_T|^{1/2}}\left(\sum_{\vec{P}\in T}\left|\left\langle BHT^{\mathbb{Q}^{\tilde{d}}}_{\omega_{P_2}}(f_2,f_3),\Phi_{P_{2,0}}\right\rangle\right|^2\right)^{1/2}$$

$$\lesssim \left(2^{-\tilde{N}\tilde{d}}\sup_{\vec{P}\in\mathbb{P}^{d,\tilde{d}}_{n_1,\mathfrak{d}}}\inf_{x\in I_{\vec{P}}} M\left(\left[\int_0^1\sum_{\vec{Q}\in\mathbb{Q}^{\tilde{d}}}\frac{\left|\langle f_2,\Phi^{\alpha}_{Q_1,2}\rangle\langle f_3,\Phi^{\alpha}_{Q_2,3}\rangle\right|}{|I_{\vec{Q}}|}\tilde{1}_{I_{\vec{Q}}}d\alpha\right]^2\right)(x)\right)^{1/2}$$

$$\lesssim 2^{-\tilde{N}\tilde{d}/2}2^{d/2}2^{\tilde{d}}|E_2|^{1/2}|E_3|^{1/2}$$

$$\lesssim 2^{-\tilde{N}\tilde{d}/4}|E_2|^{1/2}|E_3|^{1/2}.$$

By combining this estimate with the standard Biest size upper bound, we deduce the desired claim.

□

3.7 l^2 Energy Estimate

In preparation for the main semi-degenerate energy estimates, we first record a very useful inequality.

Lemma 26. *Fix $\alpha\in[0,1]$ and $\theta\in(0,1)$. Then*

$$\left|\left|BHT^{\alpha,\mathbb{Q}^{\tilde{d}}}(f_2,f_3)\right|\right|_2 \lesssim_\theta \left[|E_2|^{1/2}\sup_{\vec{Q}\in\mathbb{Q}^{\tilde{d}}}\frac{\int_{E_3}\tilde{1}_{I_{\vec{Q}}}dx}{|I_{\vec{Q}}|}\right]^{1-\theta}\cdot\left[|E_3|^{1/2}\sup_{\vec{Q}\in\mathbb{Q}^{\tilde{d}}}\frac{\int_{E_2}\tilde{1}_{I_{\vec{Q}}}dx}{|I_{\vec{Q}}|}\right]^{\theta}$$
$$\lesssim 2^{\tilde{d}}|E_2|^{\frac{1+\theta}{2}}|E_3|^{\frac{2-\theta}{2}},$$

where the implicit constant in the above display can be taken independently of α.

Proof. Apply the localized BHT size/energy estimate from [21] and use the definition of $\mathbb{Q}^{\tilde{d}}$. □

The localized energy bounds is a crucial ingredient in the proof of the following l^2 energy estimate:

Proposition 10. *Fix $d, \tilde{d} \geq 0$ along with $\mathfrak{d} \geq N_2(d, \tilde{d})$. Then for any $0 < \theta < 1$,*

$$\sum_{T \in \mathcal{T}_{\mathfrak{d},2}^{d,\tilde{d}}} |I_T| \lesssim_\theta 2^{2\mathfrak{d}} 2^{2\tilde{d}} |E_2|^{1+\theta} |E_3|^{2-\theta}.$$

Proof. Recall from Lemma 23 that

$$\sum_{T \in \mathcal{T}_{\mathfrak{d},2}^{d,\tilde{d}}} |I_T| \lesssim \sum_{T \in \mathcal{T}_{\mathfrak{d},2,*}^{d,\tilde{d}}} |I_T| = \sum_{T \in \mathcal{T}_{\mathfrak{d},2,*,+}^{d,\tilde{d}}} |I_T| + \sum_{T \in \mathcal{T}_{\mathfrak{d},2,*,-}^{d,\tilde{d}}} |I_T|.$$

We further decompose the trees in $\mathcal{T}_{\mathfrak{d},2,*,+}$ into the following union:

$$\mathcal{T}_{\mathfrak{d},2,*,+} = \mathcal{T}_{\mathfrak{d},2,*,+,I} \bigcup \mathcal{T}_{\mathfrak{d},2,*,+,II}$$

where

$$\begin{aligned}
\mathcal{T}_{\mathfrak{d},2,*,+,I} &= \left\{ T \in \mathcal{T}_{\mathfrak{d},2,*,+} : \sum_{\vec{P} \in T} \left| \left\langle \int_0^1 BHT^{\alpha,\mathbb{Q}^{\tilde{d}}}(f_2, f_3) d\alpha, \Phi_{P_2,0} \right\rangle \right|^2 \geq 2^{-2\mathfrak{d}-5} |I_T| \right\} \\
\mathcal{T}_{\mathfrak{d},2,*,+,II} &= \left\{ T \in \mathcal{T}_{\mathfrak{d},2,*,+} : \sum_{\vec{P} \in T} \left| \left\langle \int_0^1 BHT(f_2, f_3) - BHT_{\omega_P}^{\alpha,\mathbb{Q}^{\tilde{d}}} d\alpha, \Phi_{P_2,0} \right\rangle \right|^2 \geq 2^{-2\mathfrak{d}-5} |I_T| \right\}.
\end{aligned}$$

Similarly, we have the decomposition $\mathcal{T}_{\mathfrak{d},2,*,-} = \mathcal{T}_{\mathfrak{d},2,*,-,I} \bigcup \mathcal{T}_{\mathfrak{d},2,*,-,II}$. Therefore, putting it all together,

$$\sum_{T\in\mathcal{T}^{d,\tilde{d}}_{\mathfrak{d},2,*}}|I_T| \leq \sum_{T\in\mathcal{T}^{d,\tilde{d}}_{\mathfrak{d},2,*,+,I}}|I_T| + \sum_{T\in\mathcal{T}^{d,\tilde{d}}_{\mathfrak{d},2,*,+,II}}|I_T| + \sum_{T\in\mathcal{T}^{d,\tilde{d}}_{\mathfrak{d},2,*,-,I}}|I_T| + \sum_{T\in\mathcal{T}^{d,\tilde{d}}_{\mathfrak{d},2,*,-,II}}|I_T|.$$

Using the usual BHT energy calculation,

$$\begin{aligned}\sum_{T\in\mathcal{T}^{d,\tilde{d}}_{\mathfrak{d},2,*,+,I}}|I_T| &\lesssim 2^{2\mathfrak{d}}\sum_{T\in\mathcal{T}_{\mathfrak{d},2,*,+,I}}\sum_{\vec{P}\in T}\left|\left\langle\int_0^1 BHT^{\alpha,\mathbb{Q}^{\tilde{d}}}(f_2,f_3)d\alpha,\Phi_{P_2,0}\right\rangle\right|^2\\ &\lesssim 2^{2\mathfrak{d}}\left|\left|\int_0^1 BHT^{\alpha,\mathbb{Q}^{\tilde{d}}}(f_2,f_3)d\alpha\right|\right|_2^2\\ &\lesssim 2^{2\mathfrak{d}}2^{2\tilde{d}}|E_2|^{1+\theta}|E_3|^{2-\theta}.\end{aligned}$$

Moreover,

$$\begin{aligned}\sum_{T\in\mathcal{T}^{d,\tilde{d}}_{\mathfrak{d},2,*,+,II}}|I_T| &\lesssim 2^{2\mathfrak{d}}\sum_{T\in\mathcal{T}_{\mathfrak{d},2,*,+,II}}\sum_{\vec{P}\in T}\left|\left\langle\int_0^1 BHT^{\alpha,\mathbb{Q}^{\tilde{d}}}(f_2,f_3)-BHT^{\alpha,\mathbb{Q}^{\tilde{d}}}_{\omega_{P_2}}(f_2,f_3)d\alpha,\Phi_{P_2,0}\right\rangle\right|^2\\ &\lesssim 2^{2\mathfrak{d}}2^{2\tilde{d}}|E_2|^{1+\theta}|E_3|^{2-\theta}.\end{aligned}$$

The sums $\sum_{T\in\mathcal{T}^{d,\tilde{d}}_{\mathfrak{d},2,*,-,I}}|I_T|$ and $\sum_{T\in\mathcal{T}^{d,\tilde{d}}_{\mathfrak{d},2,*,-,II}}|I_T|$ are clearly handled similarly. □

3.8 l^1 Energy Boost

The standard BHT energy method involves obtaining l^2 estimates of the form

$$\sum|I_T| \lesssim 2^{2n}\sum_{T\in\mathcal{T}}\sum_{\vec{P}\in T}|\langle f,\Phi_P\rangle|^2 \lesssim 2^{2n}||f||_2^2,$$

where the trees $T \in \mathcal{T}$ are strongly disjoint. Because our tri-tile collection $\mathbb{P}$ is not rank-1, we are not able to pass the analysis directly to Biest arguments. Instead, we shall need an l^1-type energy estimate. Before stating this result precisely, we shall need to clarify terminology:

Definition 50. *For each $\mathfrak{b} \geq \mathfrak{d} \geq N_2(d, \tilde{d})$, let*

$$\mathbb{P}_{\mathfrak{d}}^{\mathfrak{b}} = \left\{ \vec{P} \in \tilde{\mathbb{P}}_{\mathfrak{d},2}^{d,\tilde{d}} : \left| \left\langle \int_0^1 BHT_{\omega_{P_2}}^{\alpha,\mathbb{Q}^{\tilde{d}}}(f_2, f_3) d\alpha, \Phi_{P_2,0}^1 \right\rangle \right| \simeq 2^{-\mathfrak{b}} \right\}.$$

Theorem 24. *For every $d, \tilde{d} \geq 0$, $\mathfrak{b} \geq \mathfrak{d} \geq N_2(d, \tilde{d})$ and $0 < \tilde{\epsilon} << 1$,*

$$\sum_{\vec{P} \in \mathbb{P}_{\mathfrak{d}}^{\mathfrak{b}}} |I_{\vec{P}}| \lesssim_{\tilde{\epsilon}} 2^{\frac{\mathfrak{b}}{1-\tilde{\epsilon}}} |E_2|^{1/2} |E_3|^{1/2}.$$

Before proving Theorem 24, we should explain why such an estimate is useful. Interpolating the l^2 energy bound

$$\sum_{T \in \mathcal{T}_{\lceil,2}^{d,\tilde{d}}} |I_T| \lesssim 2^{2\mathfrak{d}} |E_2|^{3/2} |E_3|^{3/2}$$

with the l^1 energy boost $\sum_{T \in \mathbb{T}_{\mathfrak{d},2}^{d,\tilde{d}}} |I_T| \lesssim 2^{\sim\mathfrak{d}} |E_2|^{1/2} |E_3|^{1/2}$ ensures

$$\sum_{T \in \mathbb{T}_{\mathfrak{d},2}^{d,\tilde{d}}} |I_T| \lesssim 2^{\sim 3/2\mathfrak{d}} |E_2||E_3|.$$

It follows that, modulo numerous details, we should have for every $0 \leq \theta_1, \theta_2 \leq 1$ such that $\theta_1 + \theta_2 = 1$

$$\begin{aligned} |\Lambda(f_1, f_2, f_3, f_4)| &\lesssim \sum_{n_1, n_4, \mathfrak{d} \geq 0} 2^{-n_1} 2^{-n_4} 2^{-\mathfrak{d}} \min \left\{ 2^{2n_1} |E_1|, 2^{\sim 3/2\mathfrak{d}} |E_2||E_3| \right\} \\ &\lesssim \sum_{n_1, n_4, \mathfrak{d} \geq 0} 2^{-n_1(1-2\theta_1)} 2^{-n_4} 2^{-\mathfrak{d}(1-(\sim 3\theta_2 \mathfrak{d}/2))} |E_1|^{\theta_1} |E_2|^{\theta_2} |E_3|^{\theta_2}. \end{aligned}$$

Choosing $\theta_1 \simeq 1/3, \theta_2 \simeq 2/3$ gives $|\Lambda(f_1, f_2, f_3, f_4)| \lesssim |E_1|^{\sim 2/3}|E_2|^{\sim 2/3}|E_3|^{\sim 2/3}$, which provides convincing evidence that $C^{1,1,-2}$ maps into $L^r(\mathbb{R})$ for all r in a small neighborhood near $\frac{1}{2}$. With this sketch in mind, it therefore remains to fill in the details.

We also record for later use

Corollary 7. *For every* $d, \tilde{d} \geq 0, \mathfrak{d} \geq N_2(d, \tilde{d})$, *and* $0 < \tilde{\epsilon} << 1$,

$$\sum_{T \in \mathcal{T}_{\mathfrak{d},2}^{d,\tilde{d}}} |I_T| \lesssim_{\tilde{\epsilon}} 2^{\frac{\mathfrak{d}}{1-\tilde{\epsilon}}} |E_1|^{1/2}|E_2|^{1/2}.$$

Proof.

$$\begin{aligned}
\sum_{T \in \mathcal{T}_{\mathfrak{d},2}^{d,\tilde{d}}} |I_T| &\lesssim 2^{2\mathfrak{d}} \sum_{T \in \mathcal{T}_{\mathfrak{d},2}^{d,\tilde{d}}} \sum_{\vec{P} \in T} \left| \left\langle \int_0^1 BHT_{\omega_{P_2}}^{\alpha, \mathbb{Q}^{\tilde{d}}}(f_2, f_3) d\alpha, \Phi_{P_2,0}^2 \right\rangle \right|^2 \\
&= 2^{2\mathfrak{d}} \sum_{\mathfrak{b} \geq \mathfrak{d}} \sum_{\vec{P} \in \mathbb{P}_{\mathfrak{d}}^{\mathfrak{b}}} \left| \left\langle \int_0^1 BHT_{\omega_{P_2}}^{\alpha, \mathbb{Q}^{\tilde{d}}}(f_2, f_3) d\alpha, \Phi_{P_2,0} \right\rangle \right|^2 \\
&\lesssim 2^{2\mathfrak{d}} \sum_{\mathfrak{b} \geq \mathfrak{d}} 2^{-2\mathfrak{b}} \sum_{\vec{P} \in \mathbb{P}_{\mathfrak{d}}^{\mathfrak{b}}} |I_{\vec{P}}| \\
&\lesssim 2^{2\mathfrak{d}} \sum_{\mathfrak{b} \geq \mathfrak{d}} 2^{-\left[2 - \frac{1}{1-\tilde{\epsilon}}\right]\mathfrak{b}} |E_1|^{1/2}|E_2|^{1/2} \\
&\lesssim 2^{\frac{\mathfrak{d}}{1-\tilde{\epsilon}}} |E_1|^{1/2}|E_2|^{1/2}.
\end{aligned}$$

□

Proof. [Theorem 24] Begin by using the definition of $\mathbb{P}_{\mathfrak{d}}^{\mathfrak{b}}$ and dualizing:

$$\begin{aligned}
\sum_{\vec{P} \in \mathbb{P}_{\mathfrak{d}}^{\mathfrak{b}}} |I_{\vec{P}}| &\simeq 2^{\mathfrak{b}} \sum_{\vec{P} \in \mathbb{P}_{\mathfrak{d}}^{\mathfrak{b}}} \left| \left\langle \int_0^1 BHT_{\omega_{P_2}}^{\alpha, \mathbb{Q}^{\tilde{d}}}(f_2, f_3), \Phi_{P_2,0}^{\infty} \right\rangle \right| \\
&:= 2^{\mathfrak{b}} \sum_{\vec{P} \in \mathbb{P}_{\mathfrak{d}}^{\mathfrak{b}}} h_{\vec{P}} \left\langle \int_0^1 BHT_{\omega_{P_2}}^{\alpha}(f_2, f_3) d\alpha, \tilde{1}_{I_{\vec{P}}} \right\rangle,
\end{aligned}$$

where $|h_{\vec{P}}| = 1$ for all $\vec{P} \in \mathbb{P}_{\mathfrak{d}}^{\mathfrak{b}}$. Rewriting the above display, we find

$$\begin{aligned} & 2^{-\mathfrak{b}} \sum_{\vec{P} \in \mathbb{P}_{\mathfrak{d}}^{\mathfrak{b}}} |I_{\vec{P}}| \\ \simeq & \int_0^1 \sum_{\vec{P} \in \tilde{\mathbb{P}}_{\mathfrak{b}}^{\mathfrak{d}}} \sum_{\vec{Q} \in \mathbb{Q}^{\tilde{d}} : \omega_{Q_3} \subset \omega_{P_2}} \frac{1}{|I_{\vec{Q}}|^{1/2}} \langle f_2, \Phi_{Q_1,2}^{\alpha} \rangle \langle f_3, \Phi_{Q_2,3}^{\alpha} \rangle \langle \Phi_{Q_3,5}^{\alpha}, h_{\vec{P}} \tilde{1}_{I_{\vec{P}}} \rangle d\alpha \\ = & \int_0^1 \sum_{\vec{Q} \in \mathbb{Q}^{\tilde{d}}} \frac{1}{|I_{\vec{Q}}|^{1/2}} \langle f_2, \Phi_{Q_1,1}^{\alpha} \rangle \langle f_3, \Phi_{Q_2,2}^{\alpha} \rangle \left\langle \Phi_{Q_3,5}^{\alpha}, \sum_{\vec{P} \in \mathbb{P}_{\mathfrak{d}}^{\mathfrak{b}} : \omega_{P_2} \supset \omega_{Q_3}} h_{\vec{P}} \tilde{1}_{I_{\vec{P}}} * \eta_{\omega_{P_2}} \right\rangle d\alpha. \end{aligned}$$

Observe that when the tiles are restricted to a single $\mathbb{Q}-$tree, the sum over $\vec{P} \in \mathbb{P}_{\mathfrak{d}}^{\mathfrak{b}}$ containing a frequency of the tree satisfies $\sum_{\vec{P} \in \mathbb{P}(T)} 1_{I_{\vec{P}}} \lesssim 1$. Moreover, $\left\| \sum_{\vec{P} \in \mathbb{P}_{\mathfrak{d}}^{\mathfrak{b}} : \omega_{P_2} \supset \omega_{Q_3}, I_{\vec{P}} \subset I_{\vec{Q}} \text{ for some } \vec{Q} \in \mathbb{Q}^{\tilde{d}}} h_{\vec{P}} \tilde{1}_{I_{\vec{P}}} * \eta_{\omega_{P_2}} \right\|_2^2 \lesssim \sum_{\vec{P} \in \tilde{\mathbb{P}}_{\mathfrak{d}}^{\mathfrak{b}}} |I_{\vec{P}}|$. Our hope is to bound $\left| \sum_{\vec{Q} \in \mathbb{Q}^{\tilde{d}}} \frac{1}{|I_{\vec{Q}}|^{1/2}} \langle f_2, \Phi_{Q_1,1}^{\alpha} \rangle \langle f_3, \Phi_{Q_2,2}^{\alpha} \rangle \left\langle \Phi_{Q_3,5}^{\alpha}, \sum_{\vec{P} \in \mathbb{P}_{\mathfrak{d}}^{\mathfrak{b}} : \omega_{P_2} \supset \omega_{Q_3}} h_{\vec{P}} \tilde{1}_{I_{\vec{P}}} * \eta_{\omega_{P_2}} \right\rangle \right|$ from above by

$$|E_2|^{\frac{1}{2}(1-\tilde{\epsilon})} |E_3|^{\frac{1}{2}(1-\tilde{\epsilon})} \left[\sum_{\vec{P} \in \mathbb{P}_{\mathfrak{d}}^{\mathfrak{b}}} |I_{\vec{P}}| \right]^{\tilde{\epsilon}}$$

with a bound uniform in $\alpha \in [0, 1]$. Then

$$\sum_{\vec{P} \in \mathbb{P}_{\mathfrak{b}}^{\mathfrak{d}}} |I_{\vec{P}}| \lesssim_{\tilde{\epsilon}} 2^{\mathfrak{b}} |E_2|^{\frac{1}{2}(1-\tilde{\epsilon})} |E_3|^{\frac{1}{2}(1-\tilde{\epsilon})} \left[\sum_{\vec{P} \in \mathbb{P}_{\mathfrak{d}}^{\mathfrak{b}}} |I_{\vec{P}}| \right]^{\tilde{\epsilon}}$$

easily implies

$$\sum_{\vec{P} \in \mathbb{P}_{\mathfrak{b}}^{\mathfrak{d}}} |I_{\vec{P}}| \lesssim_{\tilde{\epsilon}} 2^{\frac{\mathfrak{b}}{1-\tilde{\epsilon}}} |E_1|^{1/2} |E_2|^{1/2}.$$

3.8.1 $\Lambda^I_{\mathbb{P}^\flat_\mathfrak{d}}(f_2, f_3)$ Estimates

Our first reduction is to handle the model

$$\begin{aligned}
&\Lambda^I_{\mathbb{P}^\flat_\mathfrak{d}}(f_1, f_2) \\
:= \;& \int_0^1 \sum_{\vec{Q}\in\mathbb{Q}} \frac{1}{|I_{\vec{Q}}|^{1/2}} \langle f_2, \Phi^\alpha_{Q_1,1}\rangle \langle f_3, \Phi^\alpha_{Q_2,2}\rangle \\
\times \;& \left\langle \Phi^\alpha_{Q_3,5}, \sum_{\vec{P}\in\mathbb{P}^\flat_\mathfrak{d}:\omega_{P_2}\supset\omega_{Q_3}, |\omega_{P_2}|\simeq|\omega_{Q_3}|} h_{\vec{P}} \tilde{1}_{I_{\vec{P}}} * \eta_{\omega_{P_2}} \right\rangle d\alpha \\
= \;& \int_0^1 \sum_{\vec{P}\in\mathbb{P}^\flat_\mathfrak{d}} \sum_{0\le\lambda\le C} \sum_{\vec{Q}\in\mathbb{Q}:\omega_{Q_3}\subset\omega_{P_2}, |\omega_{Q_3}|=2^{-\lambda}|\omega_{P_2}|} \frac{1}{|I_{\vec{Q}}|^{1/2}} \langle f_2, \Phi^\alpha_{Q_1,1}\rangle \langle f_3, \Phi^\alpha_{Q_2,2}\rangle \\
\times \;& \left\langle \Phi^\alpha_{Q_3,5}, h_{\vec{P}} \tilde{1}_{I_{\vec{P}}} * \eta_{\omega_{P_2}} \right\rangle d\alpha.
\end{aligned}$$

By the well-distributed assumption on the time intervals of $I_{\vec{Q}}$ with intersecting frequencies, we may majorize the above display by a rapidly decaying sum of expressions of the form

$$\sum_{\vec{P}\in\mathbb{P}^\flat_\mathfrak{d}} \left| \langle f_2, \Phi_{\tilde{P}_1,6}\rangle \langle f_3, \Phi_{\tilde{P}_2,7}\rangle \right|,$$

where each $\left\{\tilde{P}_1, \vec{P}\in\mathbb{P}^\flat_\mathfrak{d}\right\}$ and $\left\{\tilde{P}_2 : \vec{P}\in\mathbb{P}^\flat_\mathfrak{d}\right\}$ form disjoint collections of tiles. The claim is that the above display can be bounded by an expression of the form $|E_2|^{\sim 1/2}|E_3|^{\sim 1/2}\left(\sum_{\vec{P}\in\mathbb{P}^\flat_\mathfrak{d}} |I_{\vec{P}}|\right)^{\tilde{\alpha}}$ for $0 < \tilde{\alpha} << 1$. This bound is easily achieved using the following trick.

Lemma 27. *Fix a collection of disjoint tiles $\tilde{\mathbb{P}}$ and assume for all $P\in\tilde{\mathbb{P}}, \Phi_P$ is an L^2-normalized wave packet on P satisfying supp $\hat{\Phi}_P\subset\omega_P$. Moreover, let*

$||f||_\infty \leq 1$. *Then for every* $\epsilon > 0$

$$\sum_{P \in \tilde{\mathbb{P}}} |\langle f, \Phi_P^{(2+\epsilon)'} \rangle|^{2+\epsilon} \lesssim_\epsilon ||f||_2^2.$$

Proof. Construct $\tilde{\mathbb{P}}_n := \left\{ P \in \tilde{\mathbb{P}} : |\langle f, \Phi_P^1 \rangle| \simeq 2^{-n} \right\}$. Hence, $\tilde{\mathbb{P}} = \bigcup_{n \geq 0} \tilde{\mathbb{P}}_n$ and

$$\begin{aligned}
\sum_{P \in \tilde{\mathbb{P}}} |\langle f, \Phi_P^{(2+\epsilon)'} \rangle|^{2+\epsilon} &= \sum_{n \geq 0} \sum_{P \in \tilde{\mathbb{P}}_n} |\langle f, \Phi_P^{(2+\epsilon)'} \rangle|^{2+\epsilon} \\
&\leq \sum_{n \geq 0} \left[\sup_{P \in \tilde{\mathbb{P}}_n} \left| \langle f, \Phi_P^1 \rangle \right| \right]^\epsilon \left[\sum_{P \in \tilde{\mathbb{P}}_n} |\langle f, \Phi_P \rangle|^2 \right] \\
&\lesssim \sum_{n \geq 0} 2^{-n\epsilon} ||f||_2^2 \\
&\lesssim_\epsilon ||f||_2^2.
\end{aligned}$$

□

We may now use the fact that both $\left\{ \tilde{P}_1 : \vec{P} \in \mathbb{P}_{\mathfrak{d}}^{\mathfrak{b}} \right\}$ and $\left\{ \tilde{P}_2 : \vec{P} \in \mathbb{P}_{\mathfrak{b}}^{\mathfrak{d}} \right\}$ are disjoint collections of tiles in order to deduce for small $\epsilon > 0$ and $\tilde{\epsilon} := \frac{\epsilon}{2+\epsilon}$ that

$$\begin{aligned}
& \sum_{\vec{P} \in \mathbb{P}_{\mathfrak{d}}^{\mathfrak{b}}} \left| \langle f_2, \Phi_{\tilde{P}_1,6} \rangle \langle f_3, \Phi_{\tilde{P}_2,7} \rangle \right| \\
= & \sum_{\vec{P} \in \mathbb{P}_{\mathfrak{d}}^{\mathfrak{b}}} \left| \langle f_2, \Phi_{\tilde{P}_1,6}^{(2+\epsilon)'} \rangle \langle f_3, \Phi_{\tilde{P}_2,7}^{(2+\epsilon)'} \rangle \right| \cdot |I_{\vec{P}}|^{\tilde{\epsilon}} \\
\leq & \left(\sum_{\vec{P} \in \mathbb{P}_{\mathfrak{d}}^{\mathfrak{b}}} \left| \langle f_2, \Phi_{\tilde{P}_1,6}^{(2+\epsilon)'} \rangle \right|^{2+\epsilon} \right)^{\frac{1}{2+\epsilon}} \left(\sum_{\vec{P} \in \mathbb{P}_{\mathfrak{d}}^{\mathfrak{b}}} \left| \langle f_3, \Phi_{\tilde{P}_2,7}^{(2+\epsilon)'} \rangle \right|^{2+\epsilon} \right)^{\frac{1}{2+\epsilon}} \left(\sum_{\vec{P} \in \mathbb{P}_{\mathfrak{d}}^{\mathfrak{b}}} |I_{\vec{P}}| \right)^{\tilde{\epsilon}} \\
\lesssim_\epsilon & |E_2|^{\frac{1}{2}(1-\tilde{\epsilon})} |E_3|^{\frac{1}{2}(1-\tilde{\epsilon})} \left(\sum_{\vec{P} \in \mathbb{P}_{\mathfrak{d}}^{\mathfrak{b}}} |I_{\vec{P}}| \right)^{\tilde{\epsilon}}.
\end{aligned}$$

3.8.2 $\Lambda^{II}_{\mathbb{P}^{\flat}_{\mathfrak{d}}}(f_1, f_2)$ Estimates

Our goal is now to estimate

$$\begin{aligned} & \Lambda^{II}_{\mathbb{P}^{\flat}_{\mathfrak{d}}}(f_2, f_3) \\ = & \int_0^1 \sum_{\vec{Q} \in \mathbb{Q}} \frac{1}{|I_{\vec{Q}}|^{1/2}} \langle f_2, \Phi^{\alpha}_{Q_1,2} \rangle \langle f_3, \Phi^{\alpha}_{Q_2,3} \rangle \left\langle \Phi^{\alpha}_{Q_3,5}, \sum_{\vec{P} \in \mathbb{P}^{\flat}_{\mathfrak{d}} : \omega_{P_2} \supset \omega_{Q_3}, |\omega_{P_2}| >> |\omega_{Q_3}|} h_{\vec{P}} \Phi^{\infty}_{\vec{P}} \right\rangle d\alpha \\ := & \int_0^1 \Lambda^{II,\alpha}_{\mathbb{P}^{\flat}_{\mathfrak{d}}}(f_2, f_3) d\alpha. \end{aligned}$$

To this end, we now consider $\alpha \in [0,1]$ to be a fixed and compute the 3rd index size and energy.

3.8.3 3-Size Bounds

Fix a $\mathbb{Q}^{\tilde{d}}-$tree T overlapping in either the 1st or 2nd index. Observe that since each $\eta_{\omega_{P_2}}$ can be chosen so that *supp* $\hat{\eta}_{\omega_{P_2}} \subset \left[c_{\omega_{P_2}} - \frac{19}{40}|\omega_{P_2}|, c_{\omega_{P_2}} + \frac{19}{40}|\omega_{P_2}|\right]$, the collection

$$\mathbb{P}^{\flat}_{\mathfrak{d}}(T) := \left\{ \vec{P} \in \mathbb{P}^{\flat}_{\mathfrak{d}} : \exists \vec{Q} \in T, \langle \Phi^{\alpha}_{Q_3,5}, \Phi^{\infty}_{P} * \eta_{\omega_{P_2}} \rangle \neq 0, |\omega_{P_2}| >> |\omega_{Q_3}| \right\}$$

consists of tiles with disjoint time projections. Indeed, it is easy to check that for large enough implicit constant (depending on the implicit constants in the definition of a tree), every tile $P_2 : \vec{P} \in \mathbb{P}^{\flat}_{\mathfrak{d}}(T)$ must contain the $\mathbb{Q}^{\tilde{d}}$-tree 's top frequency band ω_{T_3}, and $\mathbb{P}^{\flat}_{\mathfrak{d}}$ consists of disjoint tri-tiles. By the frequency restriction $\omega_{P_2} \supset \omega_{Q_3}$ and disjointness of the tiles $\left\{ P_2 : \vec{P} \in \mathbb{P}^{\flat}_{\mathfrak{d}} \right\}$, we observe

$$
\begin{aligned}
& \left(\frac{1}{|I_T|}\sum_{\vec{Q}\in T}\left|\left\langle \Phi^{\alpha}_{Q_3,5}, \sum_{\vec{P}\in\mathbb{P}^{\flat}_{\mathfrak{d}}:\omega_{P_2}\supset\omega_{Q_3}, |\omega_{Q_3}|>>|\omega_{P_3}|} h_{\vec{P}}\tilde{1}_{I_{\vec{P}}}\right\rangle\right|^2\right)^{1/2} \\
= & \left(\frac{1}{|I_T|}\sum_{\vec{Q}\in T}\left|\left\langle \Phi^{\alpha}_{Q_3,5}, \sum_{\vec{P}\in\mathbb{P}^{\flat}_{\mathfrak{d}}(T):|\omega_{P_2}|>>|\omega_{Q_3}|} h_{\vec{P}}\tilde{1}_{I_{\vec{P}}} * \eta_{\omega_{P_2}}\right\rangle\right|^2\right)^{1/2} \\
\lesssim & \left(\frac{1}{|I_T|}\sum_{\vec{Q}\in T}\left|\left\langle \Phi^{\alpha}_{Q_3,5}, \sum_{\vec{P}\in\mathbb{P}^{\flat}_{\mathfrak{d}}(T)} h_{\vec{P}}\tilde{1}_{I_{\vec{P}}} * \eta_{\omega_{P_2}}\right\rangle\right|^2\right)^{1/2} \\
+ & \left(\frac{1}{|I_T|}\sum_{\vec{Q}\in T}\left|\left\langle \Phi^{\alpha}_{Q_3,5}, \sum_{\vec{P}\in\mathbb{P}^{\flat}_{\mathfrak{d}}(T):|\omega_{P_2}|\simeq|\omega_{Q_3}|} h_{\vec{P}}\tilde{1}_{I_{\vec{P}}} * \eta_{\omega_{P_2}}\right\rangle\right|^2\right)^{1/2} \\
:= & \; I + II.
\end{aligned}
$$

For the last line, it is straightforward to check that for large enough implicit constant

$$
\left\{\vec{P}\in\mathbb{P}^{\flat}_{\mathfrak{d}}(T) : \exists\vec{Q}\in T, |\omega_{P_2}| << |\omega_{Q_3}|, \langle \Phi^{\alpha}_{Q_3,5}, \Phi^{\infty}_{\vec{P}} * \eta_{\omega_{P_2}}\rangle \neq 0\right\} = \emptyset.
$$

Our goal is then to show $I, II \lesssim 1$, in which case $2^{-n_3} \lesssim 1$. Term I may be estimated

$$
\begin{aligned}
I \lesssim & \sum_{l\in\mathbb{Z}}\left(\frac{1}{|I_T|}\sum_{\vec{Q}\in T}\left|\left\langle \Phi^{\alpha}_{Q_3,5}, \sum_{\vec{P}\in\mathbb{P}^{\flat}_{\mathfrak{d}}(T):I_{\vec{P}}\subset I_T+l} h_{\vec{P}}\tilde{1}_{I_{\vec{P}}} * \eta_{\omega_{P_2}}\right\rangle\right|^2\right)^{1/2} \\
= & \sum_{l\in\mathbb{Z}}\left(\frac{1}{|I_T|}\int_{\mathbb{R}}\left|\sum_{\vec{P}\in\mathbb{P}^{\flat}_{\mathfrak{d}}(T):I_{\vec{P}}\subset I_T+l}\tilde{\tilde{1}}_{I_{\vec{P}}}\right|^2\tilde{1}_{I_T}dx\right)^{1/2}.
\end{aligned}
$$

It clearly suffices to prove $\left(\frac{1}{|I_T|}\int_{\mathbb{R}}\left|\sum_{\vec{P}\in\mathbb{P}^{\flat}_{\mathfrak{d}}(T):I_{\vec{P}}\subset I_T+l}\tilde{\tilde{1}}_{I_{\vec{P}}}\right|^2\tilde{1}_{I_T}dx\right)^{1/2} \lesssim \frac{1}{1+l^N}$ for each $l\in\mathbb{Z}$ and some $N\geq 2$.

Lemma 28. *Let* $\tilde{1}_{I_{Q_1}}$ *and* $\tilde{1}_{I_{Q_2}}$ *be two rapidly decaying bump functions adapted to dyadic intervals* I_{Q_1} *and* I_{Q_2} *respectively (decaying at some polynomial rate* $\tilde{N}$ *away from* $c_{I_{Q_1}}$ *and* $c_{I_{Q_2}}$*). Then whenever* $|I_{Q_2}| \geq |I_{Q_1}|$*, there exists a rapidly decaying function* $\tilde{1}^N_{I_{Q_2}}$ *(decaying like* $1/|x|^N$ *away from* $c_{I_{Q_2}}$*) for some* $1 << N \lesssim \tilde{N}$ *such that*

$$\int_{\mathbb{R}} \tilde{1}_{I_{Q_1}} \tilde{1}_{I_{Q_2}} dx \lesssim_N \frac{|I_{Q_1}|}{1+\left[\frac{dist(I_{Q_1},I_{Q_2})}{|I_{Q_2}|}\right]^N} \lesssim \int_{\mathbb{R}} 1_{Q_1} \tilde{1}^N_{I_{Q_2}} dx.$$

Proof. The proof is straightforward and is therefore omitted. □

Lemma 29. *The following bound holds:*

$$\frac{1}{|I_T|}\int_{\mathbb{R}} \left| \sum_{\vec{P}\in\mathbb{P}^{\mathfrak{b}}_{\mathfrak{d}}(T):I_{\vec{P}}\subset I_T+l} \tilde{\tilde{1}}_{I_{\vec{P}}} \right|^2 \tilde{1}_{I_T} dx \lesssim \frac{1}{1+l^N}.$$

Proof.

$$\begin{aligned}
LHS &\lesssim \frac{1}{1+l^N}\frac{1}{|I_T|}\int_{\mathbb{R}} \sum_{\vec{P}_1,\vec{P}_2\in\mathbb{P}^{\mathfrak{b}}_{\mathfrak{d}}(T):I_{P_1},I_{P_2}\subset I_T+l|I_T|} \tilde{1}_{I_{P_1}}\tilde{1}_{I_{P_2}} dx \\
&\lesssim \frac{1}{1+l^N}\frac{1}{|I_T|}\int_{\mathbb{R}} \sum_{P_1,P_2\in\mathbb{P}^{\mathfrak{b}}_{\mathfrak{d}}(T):|I_{P_2}|\geq|I_{P_1}|:I_{P_1},I_{P_2}\subset I_T+l|I_T|} \tilde{1}_{I_{P_1}}\tilde{1}_{I_{P_2}} dx \\
&+ \frac{1}{1+l^N}\frac{1}{|I_T|}\int_{\mathbb{R}} \sum_{P_1,P_2\in\mathbb{P}^{\mathfrak{b}}_{\mathfrak{d}}(T):|I_{P_2}|<|I_{P_1}|:I_{P_1},I_{P_2}\subset I_T+l|I_T|} \tilde{1}_{I_{P_1}}\tilde{1}_{I_{P_2}} dx \\
&\lesssim \frac{1}{1+l^N}\frac{1}{|I_T|}\int_{\mathbb{R}} \sum_{P_2\in\mathbb{P}^{\mathfrak{b}}_{\mathfrak{d}}(T):I_{P_2}\subset I_T+l|I_T|} \tilde{1}^N_{I_{P_2}} dx \\
&+ \frac{1}{1+l^N}\frac{1}{|I_T|}\int_{\mathbb{R}} \sum_{P_1\in\mathbb{P}^{\mathfrak{b}}_{\mathfrak{d}}(T):I_{P_1}\subset I_T+l|I_T|} \tilde{1}^N_{I_{P_1}} dx \\
&\lesssim \frac{1}{1+l^N}.
\end{aligned}$$

□

Term II

For II, it suffices to observe

$$II \leq \sum_{|k|\leq C}\left(\frac{1}{|I_T|}\sum_{\vec{Q}\in T}\left|\left\langle \Phi^{\alpha}_{Q_3,5}, \sum_{\vec{P}\in\mathbb{P}^{\flat}_{\mathfrak{d}}(T):|\omega_{P_2}|=2^k|\omega_{Q_3}|} h_{\vec{P}}\tilde{1}_{I_{\vec{P}}} * \eta_{\omega_{P_2}}\right\rangle\right|^2\right)^{1/2}.$$

Therefore, for each scale, we compute

$$\sum_{\vec{P}\in T:|\omega_P|=2^{\lambda}}\left|\left\langle \Phi^{\alpha}_{P_3}, \sum_{\vec{P}\in\mathbb{P}^{\flat}_{\mathfrak{d}}(T):|\omega_{P_2}|=2^k|\omega_{Q_3}|} h_{\vec{P}}\tilde{1}_{I_{\vec{P}}} * \eta_{\omega_{P_2}}\right\rangle\right|^2 \lesssim \left\|\left[\sum_{\vec{Q}\in\mathbb{Q}_T:|\omega_Q|=2^{k+\lambda}} 1_{I_{\vec{P}}}\right]\tilde{1}_{I_T}\right\|_2^2.$$

Summing over all $\lambda\in\mathbb{Z}$ yields

$$\begin{aligned} II &\lesssim \frac{1}{|I_T|^{1/2}}\sum_{|k|\leq C}\left\|\left(\sum_{\lambda\in\mathbb{Z}}\left|\sum_{\vec{P}\in\mathbb{P}^{\flat}_{\mathfrak{d}}(T):|\omega_{P_2}|=2^{k+\lambda}} 1_{I_{\vec{P}}}\right|^2\right)^{1/2}\tilde{1}_{I_T}\right\|_2 \\ &\lesssim \frac{1}{|I_T|^{1/2}}\left\|\left(\sum_{\lambda\in\mathbb{Z}}\left[\sum_{\vec{P}\in\mathbb{P}^{\flat}_{\mathfrak{d}}(T):|\omega_{P_2}|=2^{\lambda}} 1_{I_{\vec{P}}}\right]^2\right)^{1/2}\tilde{1}_{I_T}\right\|_2 \lesssim 1. \end{aligned}$$

3.8.4 3-Energy Bound

Letting $\Phi^{\infty}_{\vec{Q}} := \tilde{1}_{I_{\vec{Q}}} * \eta_{\omega_Q}$ and

$$c_{Q_3} := 2^{-n_3}\overline{\left\langle \sum_{\vec{P}\in\mathbb{P}^{\flat}_{\mathfrak{d}}:\omega_{P_2}\supset\omega_{Q_3}} h_{\vec{P}}\tilde{1}_{I_{\vec{P}}}, \Phi^{\alpha}_{Q_3,5}\right\rangle}\cdot\left[\sum_{T\in\mathbb{T}}\sum_{\vec{Q}\in T}\left|\left\langle \sum_{\vec{P}\in\mathbb{P}^{\flat}_{\mathfrak{d}}:\omega_{P_2}\supset\omega_{Q_3}} h_{\vec{P}}\tilde{1}_{I_{\vec{P}}}, \Phi^{\alpha}_{Q_3,5}\right\rangle\right|^2\right]^{-1/2}$$

for all $\vec{Q} \in \bigcup_{T\in\mathbb{T}} T$, it follows that

$$\begin{aligned}
E_3 &\simeq 2^{-n_3}\left[\sum_{T\in\mathbb{T}}\sum_{\vec{Q}\in T}\left|\left\langle \sum_{\vec{P}\in\mathbb{P}^\flat_{\mathfrak{d}}:\omega_{P_2}\supset\omega_{Q_3}} h_{\vec{P}}\tilde{1}_{I_{\vec{P}}}, \Phi^{\alpha}_{Q_3,5}\right\rangle\right|^2\right]^{1/2} \\
&= \sum_{T\in\mathbb{T}}\sum_{\vec{Q}\in T}\left\langle \sum_{\vec{P}\in\mathbb{P}^\flat_{\mathfrak{d}}:\omega_{P_2}\supset\omega_{Q_3}} h_{\vec{P}}\tilde{1}_{I_{\vec{P}}}, c_{Q_3}\Phi^{\alpha}_{Q_3,5}\right\rangle \\
&= \sum_{\vec{P}\in\mathbb{P}^\flat_{\mathfrak{d}}}\left\langle h_{\vec{P}}\Phi^{\infty}_{\vec{P}}, \sum_{T\in\mathbb{T}}\sum_{\vec{Q}\in T:\omega_{Q_3}\subset\omega_{P_2}} c_{Q_3}\Phi^{\alpha}_{Q_3,5}\right\rangle \\
&= \left\langle \sum_{\vec{P}\in\mathbb{P}^\flat_{\mathfrak{d}}} h_{\vec{P}}\Phi^{\infty}_{\vec{P}}, \sum_{T\in\mathbb{T}}\sum_{\vec{Q}\in T} c_{Q_3}\Phi^{\alpha}_{Q_3,5}\right\rangle + \sum_{\vec{P}\in\mathbb{P}^\flat_{\mathfrak{d}}}\left\langle h_{\vec{P}}\Phi^{\infty}_{\vec{P}}, \sum_{T\in\mathbb{T}}\sum_{\vec{Q}\in T:\omega_{Q_3}\supseteq\omega_{P_2}} c_{Q_3}\Phi^{\alpha}_{Q_3,5}\right\rangle \\
&:= E_3^I + E_3^{II}.
\end{aligned}$$

From the definition, we have for all $\vec{Q} \in \bigcup_{T\in\mathbb{T}} T$

$$|c_{Q_3}| \simeq \left|\left\langle \sum_{\vec{P}\in\mathbb{P}^\flat_{\mathfrak{d}}:\omega_{P_2}\supset\omega_{Q_3}} h_{\vec{P}}\tilde{1}_{I_{\vec{P}}}, \Phi^{\alpha}_{Q_3,5}\right\rangle\right|\left[\sum_{T\in\mathbb{T}}|I_T|\right]^{-1/2}$$

and for any subtree $T' \subset T$, $\sum_{\vec{Q}\in T'}|c_{Q_3}|^2 \lesssim \frac{|I_{T'}|}{\sum_{T\in\mathbb{T}}|I_T|}$.

Lemma 30. *Assume $|h_{\vec{P}}| \leq 1$ for all $\vec{P} \in \mathbb{P}^\flat_{\mathfrak{d}}$. Then*

$$\left\|\sum_{\vec{P}\in\mathbb{P}^\flat_{\mathfrak{d}}} h_{\vec{P}}\Phi^{\infty}_{\vec{P}}\right\|_2 \lesssim \left(\sum_{\vec{P}\in\mathbb{P}^\flat_{\mathfrak{d}}}|I_{\vec{P}}|\right)^{1/2}.$$

Proof. It suffices to note

$$\left\|\sum_{\vec{P}\in\mathbb{P}^{\flat}_{\eth}} h_{\vec{P}}\Phi^{\infty}_{\vec{P}}\right\|_2^2 = \sum_{\vec{P},\tilde{\vec{P}}\in\mathbb{P}^{\flat}_{\eth}} h_{\vec{P}}h_{\tilde{\vec{P}}}\left\langle \Phi^{\infty}_{\vec{P}}, \Phi^{\infty}_{\tilde{\vec{P}}}\right\rangle$$
$$= \left(\sum_{|I_{\vec{P}}|>>|I_{\tilde{\vec{P}}}|} + \sum_{|I_{\vec{P}}|\simeq|I_{\tilde{\vec{P}}}|} + \sum_{|I_{\vec{P}}|<<|I_{\tilde{\vec{P}}}|}\right) h_{\vec{P}}h_{\tilde{\vec{P}}}\left\langle \Phi^{\infty}_{\vec{P}}, \Phi^{\infty}_{\tilde{\vec{P}}}\right\rangle$$
$$= I + II + III.$$

It is straightforward to observe $|II| \lesssim \sum_{\vec{P}\in\mathbb{P}^{\flat}_{\eth}} |I_{\vec{P}}|$. By symmetry, it suffices to handle term I and compute

$$|I| \leq \sum_{\vec{P}\in\mathbb{P}^{\flat}_{\eth}}\left[\sum_{\tilde{\vec{P}}\in\mathbb{P}^{\flat}_{\eth}:\omega_{\tilde{\vec{P}}}\supset\supset\omega_{\vec{P}}} \left|\left\langle \Phi^{\infty}_{\vec{P}}, \Phi^{\infty}_{\tilde{\vec{P}}}\right\rangle\right|\right] \lesssim \sum_{\vec{P}\in\mathbb{P}^{\flat}_{\eth}} |I_{\vec{P}}|.$$

□

3.8.5 E^I_3 Estimate

Using the fact that the trees $T \in \mathbb{T}$ form a strongly disjoint collection and that for any subtree $T' \subset T \in \mathbb{T}$, $\sum_{\vec{Q}\in T'} |c_{Q_3}|^2 \lesssim \frac{|I_{T'}|}{\sum_{T\in\mathbb{T}}|I_T|}$, we may deploy the standard BHT energy estimate from [21], say, to deduce

$$E^I_3 = \left|\left\langle \sum_{\vec{P}\in\mathbb{P}^{\flat}_{\eth}} h_{\vec{P}}\Phi^{\infty}_{\vec{P}}, \sum_{T\in\mathbb{T}}\sum_{\vec{Q}\in T} c_{Q_3}\Phi^{\alpha}_{Q_3,5}\right\rangle\right| \lesssim \left\|\sum_{\vec{P}\in\mathbb{P}^{\flat}_{\eth}} h_{\vec{P}}\Phi^{\infty}_{\vec{P}}\right\|_2 \lesssim \left(\sum_{\vec{P}\in\mathbb{P}^{\flat}_{\eth}} |I_{\vec{P}}|\right)^{1/2}.$$

Hence, it remains to obtain satisfactory bounds on E^{II}_3.

3.8.6 E_3^{II} Estimate

Now note

$$
\begin{aligned}
E_3^{II} &\lesssim \sum_{\vec{P}\in\mathbb{P}_{\mathfrak{d}}^{\mathfrak{b}}} \left| \left\langle h_{\vec{P}}\Phi_{\vec{P}}^{\infty}, \sum_{T\in\mathbb{T}} \sum_{\vec{Q}\in T:\omega_{Q_3}\supseteq\omega_{P_2}} c_{Q_3}\Phi_{Q_3,5}^{\alpha} \right\rangle \right| \\
&\lesssim \frac{\sum_{\vec{P}\in\mathbb{P}_{\mathfrak{d}}^{\mathfrak{b}}} |I_{\vec{P}}|}{\left(\sum_{T\in\mathbb{T}}|I_T|\right)^{1/2}} \left[\sup_{\vec{P}\in\mathbb{P}_{\mathfrak{d}}^{\mathfrak{b}}} \left\| \sum_{T\in\mathbb{T}} \sum_{\vec{Q}\in T:\omega_{Q_3}\supseteq\omega_{P_2}} 1_{I_{\vec{Q}}} \right\|_{L^{\infty}(\mathbb{R})} \right] \\
&\lesssim \frac{\sum_{\vec{P}\in\mathbb{P}_{\mathfrak{d}}^{\mathfrak{b}}} |I_{\vec{P}}|}{\left(\sum_{T\in\mathbb{T}}|I_T|\right)^{1/2}}.
\end{aligned}
$$

Putting our estimates for E_3^I and E_3^{II} together yields

$$
E_3 = 2^{-n_3}\left(\sum_{T\in\mathbb{T}}|I_T|\right)^{1/2} \simeq E_3^I + E_3^{II} \lesssim \left(\sum_{\vec{P}\in\mathbb{P}_{\mathfrak{d}}^{\mathfrak{b}}}|I_{\vec{P}}|\right)^{1/2} + \frac{\sum_{\vec{P}\in\mathbb{P}_{\mathfrak{d}}^{\mathfrak{b}}} |I_{\vec{P}}|}{\left(\sum_{T\in\mathbb{T}}|I_T|\right)^{1/2}}.
$$

CASE 1: $\sum_{T\in\mathbb{T}}|I_T| < \sum_{\vec{P}\in\tilde{\mathbb{P}}_{\mathfrak{d}}^{\mathfrak{d}}}|I_{\vec{P}}|$. Then $\sum_{T\in\mathbb{T}}|I_T| \lesssim 2^{n_3}\sum_{\vec{P}\in\mathbb{P}_{\mathfrak{d}}^{\mathfrak{b}}}|I_{\vec{P}}|$.

CASE 2: $\sum_{T\in\mathbb{T}}|I_T| \geq \sum_{\vec{P}\in\tilde{\mathbb{P}}_{\mathfrak{b}}^{\mathfrak{d}}}|I_{\vec{P}}|$. Then $\sum_{T\in\mathbb{T}}|I_T| \lesssim 2^{2n_3}\sum_{\vec{P}\in\mathbb{P}_{\mathfrak{d}}^{\mathfrak{b}}}|I_{\vec{P}}|$.

Recall the 3−size estimate $2^{-n_3} \lesssim 1$ so that in either case $\sum_{T\in\mathbb{T}}|I_T| \lesssim 2^{2n_3}\sum_{\vec{P}\in\mathbb{P}_{\mathfrak{d}}^{\mathfrak{b}}}|I_{\vec{P}}|$ and $E_3 \lesssim \left(\sum_{\vec{P}\in\mathbb{P}_{\mathfrak{d}}^{\mathfrak{b}}}|I_{\vec{P}}|\right)^{1/2}$. Moreover, $S_1, S_2, S_3 \lesssim 1$, $E_1 \lesssim |E_2|^{1/2}, E_2 \lesssim |E_3|^{1/2}$, where S_1, S_2, S_3 and E_1 and E_2 are the usual sizes and energies from [21]. Using the fundamental Size-Energy inequality, we have for any $0 \leq \theta_1, \theta_2, \theta_3 < 1$ such that $\theta_1 + \theta_2 + \theta_3 = 1$

$$
\left|\Lambda_{\mathbb{P}_{\mathfrak{d}}^{\mathfrak{b}}}^{II}(f_2, f_3)\right| \lesssim \prod_{i=1}^{3} E_j^{1-\theta_j} S_j^{\theta_j}.
$$

Therefore, we may pick $\theta_1, \theta_2 = \tilde{\epsilon}$ and $\theta_3 = 1 - 2\tilde{\epsilon}$ to deduce the claim

$$\sum_{\vec{P}\in\mathbb{P}^{\mathfrak{b}}_{\mathfrak{d}}} |I_{\vec{P}}| \lesssim 2^{\mathfrak{b}}|E_2|^{\frac{1}{2}(1-\tilde{\epsilon})}|E_3|^{\frac{1}{2}(1-\tilde{\epsilon})}\left[\sum_{\vec{P}\in\mathbb{P}^{\mathfrak{b}}_{\mathfrak{d}}}|I_{\vec{P}}|\right]^{\tilde{\epsilon}}.$$

3.9 Generalized Restricted Type Estimates near A_1, A_2, A_3

We now combine the proceeding results to prove generalized restricted type estimates uniform in small neighborhoods near each point in $\{A_1, A_2, A_3\}$. The decomposition

$$\mathbb{P}\times\mathbb{Q} = \bigcup_{d\geq 0}\bigcup_{\tilde{d}\geq 0}\tilde{\mathbb{P}}^{d,\tilde{d}}_{n_1,\mathfrak{d}}\times\mathbb{Q}^{\tilde{d}}$$

enables us to rewrite $\Lambda_{\mathbb{P},\mathbb{Q}}(f_1,f_2,f_3,f_4)$ as

$$\sum_{d,\tilde{d}\geq 0}\sum_{n_1\geq N_1(d)}\sum_{\mathfrak{d}\geq N_2(d,\tilde{d})}\sum_{\vec{P}\in\tilde{\mathbb{P}}^{d,\tilde{d}}_{n_1,\mathfrak{d}}}\frac{1}{|I_{\vec{P}}|^{1/2}}\langle f_1', \Phi^{\alpha',\gamma,k}_{P_1,1}\rangle\langle f_4,\Phi^{\alpha'}_{P_4,4}\rangle$$
$$\times\quad\left\langle\int_0^1 BHT^{\alpha,\mathbb{Q}^{\tilde{d}}}(f_2,f_3)d\alpha, \Phi^{\alpha',\gamma',k'}_{P_2,0}\right\rangle$$

For fixed $d,\tilde{d}\geq 0, n_1\geq N_1(d)$ and $\mathfrak{d}\geq N_2(d,\tilde{d})$, we may further rewrite

$$\sum_{\vec{P}\in\tilde{\mathbb{P}}^{d,\tilde{d}}_{n_1,\mathfrak{d}}}\frac{1}{|I_{\vec{P}}|^{1/2}}\langle f_1', \Phi^{\alpha',\gamma,k}_{P_1,1}\rangle\langle f_4,\Phi^{\alpha'}_{P_4,4}\rangle\left\langle\int_0^1 BHT^{\alpha,\mathbb{Q}^{\tilde{d}}}(f_2,f_3)d\alpha, \Phi^{\alpha',\gamma',k'}_{P_2,0}\right\rangle$$
$$= \sum_{T\in\mathcal{T}^{d}_{n_1,1}}\sum_{\vec{P}\in T\cap\tilde{\mathbb{P}}^{d,\tilde{d}}_{n_1,\mathfrak{d}}}\frac{1}{|I_{\vec{P}}|^{1/2}}\langle f_1', \Phi^{\alpha',\gamma,k}_{P_1,1}\rangle\langle f_4,\Phi^{\alpha'}_{P_4,4}\rangle\left\langle\int_0^1 BHT^{\alpha,\mathbb{Q}^{\tilde{d}}}(f_2,f_3)d\alpha, \Phi^{\alpha',\gamma',k'}_{P_2,0}\right\rangle$$
$$= \sum_{T\in\mathcal{T}^{d,\tilde{d}}_{\mathfrak{d},2}}\sum_{\vec{P}\in T\cap\tilde{\mathbb{P}}^{d,\tilde{d}}_{n_1,\mathfrak{d}}}\frac{1}{|I_{\vec{P}}|^{1/2}}\langle f_1', \Phi^{\alpha',\gamma,k}_{P_1,1}\rangle\langle f_4,\Phi^{\alpha'}_{P_4,4}\rangle\left\langle\int_0^1 BHT^{\alpha,\mathbb{Q}^{\tilde{d}}}(f_2,f_3)d\alpha, \Phi^{\alpha',\gamma',k'}_{P_2,0}\right\rangle.$$

Each tree in $\mathcal{T}^{d}_{n_1,1}$ is overlapping in either the 1st or 2nd index. Using the tree and energy estimates gives

$$\begin{aligned} & \left| \sum_{T \in \mathcal{T}^{d}_{n_1,1}} \sum_{\vec{P} \in T \cap \tilde{\mathbb{P}}^{d,\tilde{d}}_{n_1,\mathfrak{d}}} \frac{1}{|I_{\vec{P}}|^{1/2}} \langle f_1', \Phi^{\alpha',\gamma,k}_{P_1,1} \rangle \langle f_4, \Phi^{\alpha'}_{P_4,4} \rangle \left\langle \int_0^1 BHT^{\alpha,\mathbb{Q}^{\tilde{d}}}(f_2,f_3) d\alpha, \Phi^{\alpha',\gamma',k'}_{P_2,0} \right\rangle \right| \\ \lesssim \; & 2^{-\tilde{N}d} 2^{-n_1} 2^{-\mathfrak{d}} \sum_{T \in \mathcal{T}^{d}_{n_1,1}} |I_T| \\ \lesssim \; & 2^{-\tilde{N}d} 2^{-n_1} 2^{-\mathfrak{d}} \left[2^{2n_1} |E_1| \right]. \end{aligned}$$

Similarly, we have

$$\begin{aligned} & \left| \sum_{T \in \mathcal{T}^{d,\tilde{d}}_{\mathfrak{d},2}} \sum_{\vec{P} \in T \cap \tilde{\mathbb{P}}^{d,\tilde{d}}_{n_1,\mathfrak{d}}} \frac{1}{|I_{\vec{P}}|^{1/2}} \langle f_1', \Phi^{\alpha',\gamma,k}_{P_1,1} \rangle \langle f_4, \Phi^{\alpha'}_{P_4,4} \rangle \left\langle \int_0^1 BHT^{\alpha,\mathbb{Q}^{\tilde{d}}}(f_2,f_3) d\alpha, \Phi^{\alpha',\gamma',k'}_{P_2,0} \right\rangle \right| \\ \lesssim \; & 2^{-\tilde{N}d} 2^{-n_1} 2^{-\mathfrak{d}} \left[\sum_{T \in \mathcal{T}^{d,\tilde{d}}_{\mathfrak{d},2}} |I_T| \right] \\ \lesssim_\theta \; & 2^{-\tilde{N}d} 2^{-n_1} 2^{-\mathfrak{d}} \min \left\{ 2^{2\mathfrak{d}} 2^{2\tilde{d}} |E_2|^{1+\theta} |E_3|^{2-\theta}, 2^{\frac{\mathfrak{d}}{1-\tilde{\epsilon}}} |E_2|^{\frac{1}{2}} |E_3|^{\frac{1}{2}} \right\}. \end{aligned}$$

Hence, for any $(\theta_1, \theta_2, \theta_3)$ satisfying $0 \leq \theta_1, \theta_2, \theta_3 \leq 1$, one has

$$
\begin{aligned}
& |\Lambda_{\mathbb{P},\mathbb{Q}}(f_1,f_2,f_3,f_4 1_{\Omega'})| \\
\lesssim\ & \sum_{\tilde{d},d\geq 0}\sum_{n_1\geq N_1(d)}\sum_{\mathfrak{d}\geq N_2(d,\tilde{d})} 2^{-\tilde{N}d}2^{-n_1}2^{-\mathfrak{d}} \\
\times\ & \min\left\{2^{2n_1}|E_1|, 2^{2\mathfrak{d}}2^{2\tilde{d}}|E_2|^{1+\theta}|E_3|^{2-\theta}, 2^{\frac{\mathfrak{d}}{1-\tilde{\epsilon}}}|E_2|^{\frac{1}{2}}|E_3|^{\frac{1}{2}}\right\} \\
\lesssim\ & \sum_{\tilde{d},d\geq 0}\sum_{n_1\geq N_1(d)}\sum_{\mathfrak{d}\geq N_2(d,\tilde{d})} 2^{-\tilde{N}d}2^{2\theta_2\tilde{d}}2^{-n_1}2^{-\mathfrak{d}}2^{2\theta_1 n_1}|E_1|^{\theta_1}2^{2\theta_2\mathfrak{d}}|E_2|^{\theta_2(1+\theta)} \\
\times\ & |E_3|^{\theta_2(2-\theta)}2^{\theta_3\frac{\mathfrak{d}}{1-\tilde{\epsilon}}}|E_2|^{\frac{\theta_3}{2}}|E_3|^{\frac{\theta_3}{2}} \\
=\ & \sum_{\tilde{d},d\geq 0}\sum_{n_1\geq N_1(d)}\sum_{\mathfrak{d}\geq N_2(d,\tilde{d})} 2^{-\tilde{N}d}2^{2\theta_2\tilde{d}}2^{-n_1(1-2\theta_1)}2^{-\mathfrak{d}\left[1-2\theta_2-\frac{\theta_3}{1-\tilde{\epsilon}}\right]}|E_1|^{\theta_1}|E_2|^{\theta_2(1+\theta)+\frac{\theta_3}{2}} \\
\times\ & |E_3|^{\theta_2(2-\theta)+\frac{\theta_3}{2}}.
\end{aligned}
$$

Take $\tilde{\epsilon} \simeq 0$. To produce generalized restricted type estimates near $A_1 = \left(1, \frac{1}{2}, \frac{1}{2}, -1\right)$, set $\theta = \frac{1}{2}, \theta_1 \simeq 0, \theta_2 \simeq 0, \theta_3 \simeq 1$. For $A_2 = \left(\frac{1}{2}, \frac{1}{2}, 1, -1\right)$, set $\theta \simeq 0, \theta_1 \simeq \frac{1}{2}, \theta_2 \simeq \frac{1}{2}, \theta_3 \simeq 0$. Lastly, for $A_3 = \left(\frac{1}{2}, 1, \frac{1}{2} - 1\right)$, set $\theta \simeq 1, \theta_2 \simeq \frac{1}{2}, \theta_2 \simeq \frac{1}{2}, \theta_3 \simeq 0$.

3.10 Generalized Restricted Type Estimates near A_4, A_5

By rescaling, we may assume $|E_1| = 1$. Construct the exceptional set

$$
\tilde{\Omega} = \{M1_{E_2} \gtrsim |E_2|\} \bigcup \{M1_{E_3} \gtrsim |E_3|\}.
$$

Let $\mathbb{Q}^{\tilde{d}} := \left\{\vec{Q} \in \mathbb{Q} : 1 + \frac{dist(I_{\vec{Q}}, \tilde{\Omega}^c)}{|I_{\vec{Q}}|} \simeq 2^{\tilde{d}}\right\}$ and define

$$\Omega_1^{\tilde{d}} = \left\{ M\left[\int_0^1 BHT^{\alpha,\mathbb{Q}^{\tilde{d}}}(f_2,f_3)d\alpha\right] \gtrsim_\theta 2^{\tilde{d}}|E_2|^\theta|E_3|^{1-\theta}\right\}$$

$$\Omega_2^{\tilde{d}} = \left\{ M\left(\left[\int_0^1 \sum_{\vec{Q}\in\mathbb{Q}^{\tilde{d}}} \frac{|\langle f_2,\Phi^\alpha_{Q_1,2}\rangle\langle f_3,\Phi^\alpha_{Q_2,3}\rangle|}{|I_{\vec{Q}}|}\tilde{1}_{I_{\vec{Q}}}d\alpha\right]^2\right) \gtrsim_\theta 2^{2\tilde{d}}|E_2||E_3|\right\}.$$

Lastly, set

$$\Omega = \bigcup_{\tilde{d}\geq 0}\Omega_1^{\tilde{d}}\bigcup_{\tilde{d}\gtrsim 1}\Omega_2^{\tilde{d}}\bigcup\{M1_{E_4}\gtrsim |E_4|\}\bigcup\tilde{\Omega}.$$

Then for large enough implicit constants, $|\Omega|\leq 1/2$ and $\tilde{E}_1 := E_1\cap\Omega^c$ is a major subset of E_1 since $|E_1|=1$. The rest of the proof of Theorem 23 near $\{A_4,A_5\}$ proceeds exactly as before. As the end of the day, we have

$$\begin{aligned}
& |\Lambda_{\mathbb{P},\mathbb{Q}}(f_1 1_{\Omega'},f_2,f_3,f_4)| \\
\lesssim_{\tilde{\epsilon}} & \sum_{\tilde{d},d\geq 0}\sum_{n_1\geq N_1(d)}\sum_{\mathfrak{d}\geq N_2(d,\tilde{d})} 2^{-\tilde{N}d}2^{2\theta_2\tilde{d}}2^{-n_1(1-2\theta_1)}2^{-\mathfrak{d}\left[1-2\theta_2-\frac{\theta_3}{1-\tilde{\epsilon}}\right]}|E_2|^{\theta_2(1+\theta)+\frac{\theta_3}{2}} \\
\times & |E_3|^{\theta_2(2-\theta)+\frac{\theta_3}{2}}|E_4|.
\end{aligned}$$

Take $\tilde{\epsilon}\simeq 0$. We may then set $\theta\simeq 0,\theta_1\simeq\frac{1}{2},\theta_2\simeq\frac{1}{2}$ and $\theta_3\simeq 0$ to deduce the desired estimates near $A_4=\left(-\frac{3}{2},\frac{1}{2},1,1\right)$ and $\theta\simeq 0,\theta_2\simeq\frac{1}{2},\theta_3\simeq\frac{1}{2},\theta_3\simeq 0$ to deduce the estimates near $A_5=\left(-\frac{3}{2},1,\frac{1}{2},1\right)$.

3.11 Generalized Restricted Type Estimates near A_6,A_7,A_8,A_9

Recall that the trilinear simplex multiplier

$$\tilde{C}^{1,1,-1/2} : (f_1, f_2, f_3) \mapsto \int_{\xi_1<\xi_2<-\xi_3/2} \hat{f}_1(\xi_1)\hat{f}_2(\xi_2)\hat{f}_3(\xi_3)e^{2\pi i x(\xi_1+\xi_2+\xi_3)}d\xi_1 d\xi_2 d\xi_3$$

satisfies the identity

$$\begin{aligned}\tilde{C}^{1,1,-1/2}(f_1, f_2, f_3)(x) &= \tilde{C}^{1,1}(f_1, f_2)(x) \cdot f_3(x) - H^{+}(\tilde{C}^{1,1}(f_1, f_2) \cdot f_3)(x) \\ &- H^{-}(f_1 \cdot \tilde{C}^{-1/2,1}(f_3, f_2))(x).\end{aligned}$$

Therefore, the 3−adjoint denoted by $\tilde{C}^{1,1,-1/2}_{*,3}$ defined by the usual property

$$\int_{\mathbb{R}} \tilde{C}^{1,1,-1/2}(f_1, f_2, f_4)(x) f_3(x) dx = \int_{\mathbb{R}} \tilde{C}^{1,1,-1/2}_{*,3}(f_1, f_2, f_3)(x) f_4(x) dx$$

for all $(f_1, f_2, f_3, f_4) \in \mathcal{S}(\mathbb{R})^4$ is writable as

$$\begin{aligned}\tilde{C}^{1,1,-1/2}_{*,3}(f_1, f_2, f_3)(x) &= \tilde{C}^{1,1}(f_1, f_2)(x) \cdot f_3(x) - \tilde{C}^{1,1}(f_1, f_2)(x) \cdot H^{-}(f_3)(x) \\ &- \tilde{C}^{1,1}(f_1(-\cdot) \cdot H^{+}(f_3)(-\cdot), f_2)(-x).\end{aligned}$$

Indeed, it suffices to check the last term, which we claim is the $3-$adjoint of the map $(f_1, f_2, f_3) \mapsto -H^-(f_1 \cdot \tilde{C}^{-1/2,1}(f_3, f_2))(x)$. Indeed, we have

$$
\begin{aligned}
& \int_{\mathbb{R}} H^- \left[f_1 \cdot \tilde{C}^{-1/2,1}(f_4, f_2) \right] (x) f_3(x) dx \\
= & \left\langle H^- \left[f_1 \cdot \tilde{C}^{-1/2,1}(f_4, f_2) \right], \bar{f}_3 \right\rangle \\
= & \left\langle \tilde{C}^{-1/2,1}(f_4, f_2), \bar{f}_1 \cdot H^- \left[\bar{f}_3 \right] \right\rangle \\
= & \left\langle \mathcal{F} \left[\tilde{C}^{-1/2,1}(f_4, f_2) \right], \mathcal{F} \left[\bar{f}_1 \cdot H^- \left[\bar{f}_3 \right] \right] \right\rangle \\
= & \left\langle \mathcal{F} \left[\tilde{C}^{1,1} \left(\mathcal{F}^{-1} \left[\overline{\mathcal{F} \left[\bar{f}_1 \cdot H^- \left[\bar{f}_3 \right] \right]} \right] , f_2 \right) \right], \overline{\mathcal{F}(f_4)} \right\rangle \\
= & \left\langle \tilde{C}^{1,1} \left(\mathcal{F}^{-1} \left[\overline{\mathcal{F} \left[\bar{f}_1 \cdot H^- \left[\bar{f}_3 \right] \right]} \right], f_2 \right), \mathcal{F}^{-1} \left[\overline{\mathcal{F}(f_4)} \right] \right\rangle \\
= & \left\langle \tilde{C}^{1,1} \left(f_1(-\cdot) \cdot H^+ \left[f_3 \right] (-\cdot), f_2 \right), \bar{f}_4(-\cdot) \right\rangle \\
= & \int_{\mathbb{R}} \tilde{C}^{1,1} \left(f_1(-\cdot) \cdot H^+ \left[f_3 \right] (-\cdot), f_2 \right) (-x) f_4(x) dx.
\end{aligned}
$$

Hence, using the BHT and Hilbert transform estimates, $\tilde{C}_{*,3}$ maps into $L^r(\mathbb{R})$ for all $r \in (\frac{2}{3}, \infty)$. Therefore, generically speaking, we should not expect the adjoint models to map below $L^{\frac{2}{3}}(\mathbb{R})$. We now proceed to prove the generalized restricted type estimates near the points A_6, A_7, A_8, A_9, where the adjoint index is restricted to map into the above range. By symmetry, it will suffice to prove the estimate only near the points $A_8 = \left(0, 1, -\frac{1}{2}, \frac{1}{2}\right)$ and $A_9 = \left(\frac{1}{2}, 1 - \frac{1}{2}, 0\right)$, for which 3 is the adjoint index. Indeed, estimates near A_6, A_7 are obtained from estimates near A_8 and A_9 by interchanging the roles of f_2, f_3.

The adjoint situation is more complicated in the semi-degenerate case than in the fully non-degenerate one because one cannot simply flip the frequency inclusions to reduce to the situation where the exceptional set is associated with functions in the $2nd$ and $3rd$ index. This is ultimately because the paracomposition

$$\sum_{\vec{P}\in\mathbb{P}} \frac{1}{|I_{\vec{P}}|^{1/2}}\langle f_1, \Phi_{P_1,1}\rangle\langle f_4, \Phi^{lac}_{P_4,4}\rangle \left\langle \int_0^1 BHT^{\alpha,\mathbb{Q}}(f_2,f_3)d\alpha, \Phi_{P_2,0}\right\rangle$$

satisfies no restricted type estimates because $\mathbb{P}$ is not a rank-1 collection of ti-tiles. Moreover, if one tries to repeat the arguments for A_1, A_2, A_3, A_4, A_5 estimates, one cannot enlarge the exceptional set Ω to obtain good control over the averages of the $BHT^{\mathbb{Q}}-type$ operators on intervals much smaller than the time-lengths of the tiles in the $\mathbb{P}-$tree T, which may a much farther from Ω^c than the time-lengths of the corresponding $\mathbb{Q}-$tiles. The way around this obstruction is to decompose our collection of degenerate tri-tiles $\mathbb{P}$. To motivate our construction, we first focus on estimating standard tree sizes of the form

$$S(f_2,f_3,T) := \left[\frac{1}{|I_T|}\sum_{\vec{P}\in T}\left|\left\langle \sum_{\vec{Q}:\omega_{Q_3}\subset\omega_P} \frac{1}{|I_{\vec{Q}}|^{1/2}}\langle f_2,\Phi_{Q_1}\rangle\langle f_3,\Phi_{Q_2}\rangle\Phi_{Q_3}, \Phi_P\right\rangle\right|^2\right]^{1/2}.$$

So, fix a $\mathbb{P}-$tree T. Suppose for every $\vec{P}\in T$

$$I_{\vec{P}} \cap \left\{ M\left[\sum_{\vec{Q}\in\mathbb{Q}} \frac{1}{|I_{\vec{Q}}|^{1/2}}\langle f_2,\Phi_{Q_1}\rangle\langle f_3,\Phi_{Q_2}\rangle\Phi_{Q_3}\right] \geq 1\right\}^c \neq \emptyset.$$

Then, the John-Nirenberg inequality combined with the Biest size estimate implies $S(f_2,f_3,T)\lesssim 1$. Define

$$\mathbb{P}_1 = \left\{\vec{P}\in\mathbb{P} : I_{\vec{P}} \cap \left\{ M\left[\sum_{\vec{Q}\in\mathbb{Q}} \frac{1}{|I_{\vec{Q}}|^{1/2}}\langle f_2,\Phi_{Q_1}\rangle\langle f_3,\Phi_{Q_2}\rangle\Phi_{Q_3}\right] \geq 1\right\}^c \neq \emptyset\right\}.$$

Next let

$$\mathbb{P}_2 = \left\{ \vec{P} \in \mathbb{P} \cap \mathbb{P}_1^c \ : \ I_{\vec{P}} \cap \left\{ M \left[\sum_{\vec{Q} \in \mathbb{Q}} \frac{1}{|I_{\vec{Q}}|^{1/2}} \langle f_2, \Phi_{Q_1} \rangle \langle f_3, \Phi_{Q_2} \rangle \Phi_{Q_3} \right] \geq 2^1 \right\}^c \neq \emptyset \right\}.$$

Inductively construct

$$\mathbb{P}_k = \left\{ \vec{P} \in \mathbb{P} \cap \left(\bigcup_{\tilde{k} \leq k-1} \mathbb{P}_{\tilde{k}} \right)^c \ : \ I_{\vec{P}} \cap \left\{ M \left[\sum_{\vec{Q} \in \mathbb{Q}} \frac{1}{|I_{\vec{Q}}|^{1/2}} \langle f_2, \Phi_{Q_1} \rangle \langle f_3, \Phi_{Q_2} \rangle \Phi_{Q_3} \right] \geq 2^k \right\}^c \neq \emptyset \right\}.$$

By design, any $\mathbb{P}$-tree T satisfies $T = \bigcup_{k \in \mathbb{N}} T \cap \mathbb{P}_k$ and $\left| \bigcup_{\vec{P} \in \mathbb{P}_k} I_{\vec{P}} \right| \lesssim 2^{-k} |E_2|^{1/2} |E_3|^{1/2}$.

Lemma 31. *Let $T \in \mathbb{T}$ be a tree of lacunary tiles. Then the following adjoint size estimate holds:*

$$\begin{aligned} & \left[\frac{1}{|I_T|} \sum_{\vec{P} \in T \cap \mathbb{P}_k} \left| \left\langle \sum_{\omega_{Q_3} \subset \omega_P} \frac{1}{|I_{\vec{Q}}|^{1/2}} \langle f_2, \Phi_{Q_1} \rangle \langle f_3, \Phi_{Q_2} \rangle \Phi_{Q_3}, \Phi_P \right\rangle \right|^2 \right]^{1/2} \\ \lesssim & \left[\sup_{\vec{P} \in T} \frac{1}{|I_{\vec{P}}|} \int_{E_2} \tilde{1}_{I_{\vec{P}}} dx \right]^{\theta} \left[\sup_{\vec{P} \in T} \frac{1}{|I_{\vec{P}}|} \int_{E_3} \tilde{1}_{I_{\vec{P}}} dx \right]^{1-\theta} + 2^k \\ \lesssim & \ 2^k. \end{aligned}$$

Proof. By triangle inequality, it suffices to handle the sum

$$\begin{aligned} & \left[\frac{1}{|I_T|} \sum_{\vec{P} \in T \cap \mathbb{P}_k} \left| \left\langle \sum_{\vec{Q} \in \mathbb{Q}} \frac{1}{|I_{\vec{Q}}|^{1/2}} \langle f_2, \Phi_{Q_1} \rangle \langle f_3, \Phi_{Q_2} \rangle \Phi_{Q_3}, \Phi_P \right\rangle \right|^2 \right]^{1/2} \\ + & \left[\frac{1}{|I_T|} \sum_{\vec{P} \in T \cap \mathbb{P}_k} \left| \left\langle \sum_{\vec{Q} \in \mathbb{Q} : \omega_{Q_3} \supseteq \omega_P} \frac{1}{|I_{\vec{Q}}|^{1/2}} \langle f_2, \Phi_{Q_1} \rangle \langle f_3, \Phi_{Q_2} \rangle \Phi_{Q_3}, \Phi_P \right\rangle \right|^2 \right]^{1/2} \\ = & \ I + II. \end{aligned}$$

The Biest size estimate handles term II. For term I, use John-Nirenberg to observe

$$I \lesssim \sup_{\vec{P} \in T \cap \mathbb{P}_k} \frac{1}{|I_{\vec{P}}|} \int_{\mathbb{R}} \left| \sum_{\vec{Q} \in \mathbb{Q}} \frac{1}{|I_{\vec{Q}}|^{1/2}} \langle f_2, \Phi_{Q_1} \rangle \langle f_3, \Phi_{Q_2} \rangle \Phi_{Q_3} \right| \tilde{1}_{I_{\vec{P}}} dx.$$

Because $\vec{P} \in T \cap \mathbb{P}_k$, $I_{\vec{P}} \cap \left\{ M \left[\sum_{\vec{Q} \in \mathbb{Q}} \frac{1}{|I_{\vec{Q}}|^{1/2}} \langle f_2, \Phi_{Q_1} \rangle \langle f_3, \Phi_{Q_2} \rangle \Phi_{Q_3} \right] \geq 2^k \right\}^c \neq \emptyset$ and

$$\frac{1}{|I_{\vec{P}}|} \int_{\mathbb{R}} \left| \sum_{\vec{Q} \in \mathbb{Q}} \frac{1}{|I_{\vec{Q}}|^{1/2}} \langle f_2, \Phi_{Q_1} \rangle \langle f_3, \Phi_{Q_2} \rangle \Phi_{Q_3} \right| \tilde{1}_{I_{\vec{P}}} dx \lesssim 2^k.$$

This observation concludes the proof.

□

We must therefore contend with exponential growth in the sizes of the trees in our tile collections $\mathbb{P}_k$. What makes this growth acceptable is the observation that for every $k \geq 0$

$$\bigcup_{\vec{P} \in \mathbb{P}_k} I_{\vec{P}} \subset \left\{ M \left[\sum_{\vec{Q} \in \mathbb{Q}} \frac{1}{|I_{\vec{Q}}|^{1/2}} \langle f_2, \Phi_{Q_1} \rangle \langle f_3, \Phi_{Q_2} \rangle \Phi_{Q_3} \right] \gtrsim 2^k \right\}.$$

Therefore, $\left| \bigcup_{\vec{P} \in \mathbb{P}_k} I_{\vec{P}} \right| \lesssim 2^{-k} |E_2|^{1/2} |E_3|^{1/2}$ arises as a simple consequence of the $L^1 \to L^{1,\infty}$ bounds for the Hardy-Littlewood maximal function together with BHT estimates. Not surprisingly, it is the smallness of the support of the intervals in $\mathbb{P}_k$ that will allow us to accept the largeness of the tree sizes. Recall the standard tile decomposition:

Lemma 32. *There is a decomposition $\mathbb{P} = \bigcup_{n_4 \geq 0} \mathbb{P}_{n_4,4}$ into disjoint subcollections with the property that if $\mathbb{I}_{n_4,4} := \left\{ I_{\vec{P}} : \vec{P} \in \mathbb{P}_{n_4,4} \right\}$ then*

$$\begin{aligned} Size_1(f_4, \mathbb{P}_{n_4,4}) &\lesssim 2^{-n_4} \\ \sum_{I \in \mathbb{I}_{n_4,4}} |I| &\lesssim 2^{n_4} |E_4|. \end{aligned}$$

Proof. Initialize when $n_4 = 0$. Let $\mathbb{P}_{0,4} = \left\{ \vec{P} \in \mathbb{P} : I_{\vec{P}} \subset \{M 1_{E_4} \gtrsim 1\} \right\}$ and iteratively construct

$$\mathbb{P}_{n_4,4} := \left\{ \vec{P} \in \mathbb{P} \cap \left[\bigcup_{0 \le m < n_4} \mathbb{P}_{m,4} \right]^c : I_{\vec{P}} \subset \left\{ M 1_{E_4} \ge 2^{-n_4} \right\} \right\}.$$

By John-Nirenberg, it is easy to check the desired properties.

□

The next step is to fix $k \ge 0$ and perform a size-energy stopping time decomposition in $\mathbb{P}_k$.

Lemma 33. *Fix $\tilde{d}, k, n_4 \ge 0$. Let $\mathbb{I}_{k,n_4}$ be the collection of maximal (shifted) dyadic intervals in the collection*

$$\left\{ I \subset \left\{ M 1_{E_4} \ge 2^{-n_4} \right\} : \exists \vec{P} \in \mathbb{P}_k \ s.t. \ I = I_{\vec{P}} \right\}.$$

Then there exist two decompositions of $\mathbb{P}_k$, namely $\bigcup_{n_1 \ge 0} \tilde{\mathbb{P}}_{k,n_1,n_4,1}$ and $\bigcup_{\mathfrak{d} \ge -k} \tilde{\mathbb{P}}^{\tilde{d}}_{k,\mathfrak{d},2}$ such that $Size_1(f_1, \tilde{\mathbb{P}}_{k,n_1,n_4,1}) \lesssim 2^{-n_1}$ and $Size^{\tilde{d}}_2(f_2, f_3, \tilde{\mathbb{P}}^{d,\tilde{d}}_{\mathfrak{d},2}) \lesssim 2^{-\mathfrak{d}}$. Moreover, $\tilde{\mathbb{P}}^d_{n_1,1}$ and $\tilde{\mathbb{P}}^{d,\tilde{d}}_{\mathfrak{d},2}$ can each be written as a union of trees, i.e.

$$\begin{aligned} \tilde{\mathbb{P}}_{k,n_1,1} &= \bigcup_{T \in \mathcal{T}_{k,n_1,n_4,1}} \bigcup_{\vec{P} \in T} \vec{P} \\ \tilde{\mathbb{P}}^{\tilde{d}}_{k,\mathfrak{d},2} &= \bigcup_{T \in \mathcal{T}^{\tilde{d}}_{k,\mathfrak{d},2}} \bigcup_{\vec{P} \in T} \vec{P}, \end{aligned}$$

such that

$$\sum_{T\in\mathcal{T}^d_{k,n_1,n_4,1}} |I_T| \lesssim 2^{2n_1} \sum_{T\in\mathcal{T}^d_{k,n_1,n_4,1,*}} \sum_{\vec{P}\in T} |\langle f_1, \Phi_{P_1,1}\rangle|^2$$

$$\sum_{T\in\mathcal{T}^{\tilde{d}}_{k,\mathfrak{d},2}} |I_T| \lesssim 2^{2\mathfrak{d}} \sum_{T\in\mathcal{T}^{d,\tilde{d}}_{\mathfrak{d},2}} \sum_{\vec{P}\in T} \left| \left\langle \int_0^1 BHT^{\alpha,\mathbb{Q}^{\tilde{d}}}_{\omega_{P_2}}(f_2,f_3)d\alpha, \Phi_{P_2,0} \right\rangle \right|^2,$$

where $\mathcal{T}_{k,n_1,n_4,1,*} \subset \mathcal{T}_{k,n_1,n_4,1}$ *and* $\mathcal{T}^{\tilde{d}}_{k,\mathfrak{d},2,*} \subset \mathcal{T}^{\tilde{d}}_{k,\mathfrak{d},2}$*, each tree in* $\mathcal{T}_{k,n_1,1,*}$ *is a* $2-$*tree and each tree in* $\mathcal{T}^{\tilde{d}}_{k,\mathfrak{d},2,*}$ *is a* $1-$*tree, and the collections* $\mathcal{T}_{k,n_1,n_4,1,*}, \mathcal{T}^{\tilde{d}}_{k,\mathfrak{d},2,*}$ *can each be written as the union of two strongly* 2*-disjoint subcollections. We denote this property by*

$$\mathcal{T}_{k,n_1,1,*} = \mathcal{T}_{k,n_1,1,*,+} \bigcup \mathcal{T}_{k,\mathfrak{d},2,*,-}$$

$$\mathcal{T}^{\tilde{d}}_{k,\mathfrak{d},2,*} = \mathcal{T}^{\tilde{d}}_{k,\mathfrak{d},2,*,+} \bigcup \mathcal{T}^{\tilde{d}}_{k,\mathfrak{d},2,*,-}.$$

Lastly, $\left[\bigcup_{T\in\mathcal{T}_{k,n_1,n_4,1,*}} I_T\right] \cup \left[\bigcup_{T\in\mathcal{T}_{k,n_1,n_4,1,*}} I_T\right] \subset \bigcup_{I\in\mathbb{I}_{k,n_4}} I$ *.*

Proof. Apply the argument localized to each dyadic interval $I \in \mathbb{I}_k$ from Lemma 23 and use the John-Nirenberg inequality to impose the desired size restrictions. □

3.11.1 Energy Savings

We shall now use $\left|\bigcup_{\vec{P}\in\mathbb{P}_k} I_{\vec{P}}\right| \lesssim 2^{-k}|E_2|^{1/2}|E_3|^{1/2}$ to improve our standard energy estimate. So, fix $\epsilon_0 > 0$, quite small perhaps. We first produce an additional energy decay factor of $2^{-k\epsilon_0}$. Start with

$$\sum_{T\in\mathbb{T}_k} |I_T| \lesssim 2^{2\mathfrak{d}} \sum_{T\in\mathbb{T}_k} \sum_{\vec{P}\in T} \left| \left\langle \sum_{\omega_{Q_3}\subset\omega_P} \frac{1}{|I_{\vec{Q}}|^{1/2}} \langle f_2, \Phi_{Q_1}\rangle \langle f_3, \Phi_{Q_2}\rangle \Phi_{Q_3}, \Phi_P \right\rangle \right|^2.$$

The trick here is to again majorize the RHS of the above display by

$$2\cdot 2^{2\mathfrak{d}} \sum_{T\in\mathcal{T}^{\tilde{d}}_{k,\mathfrak{d},2,*}} \sum_{\vec{P}\in T} \left| \left\langle \sum_{\omega_{Q_3}\supseteq\omega_P} \frac{1}{|I_{\vec{Q}}|^{1/2}} \langle f_2,\Phi_{Q_1}\rangle\langle f_3,\Phi_{Q_2}\rangle\Phi_{Q_3},\Phi_P \right\rangle \right|^2$$
$$+\quad 2\cdot 2^{2\mathfrak{d}} \sum_{T\in\mathcal{T}^{\tilde{d}}_{k,\mathfrak{d},2,*}} \sum_{\vec{P}\in T} \left| \left\langle \sum_{\vec{Q}\in\mathbb{Q}} \frac{1}{|I_{\vec{Q}}|^{1/2}} \langle f_2,\Phi_{Q_1}\rangle\langle f_3,\Phi_{Q_2}\rangle\Phi_{Q_3},\Phi_P \right\rangle \right|^2.$$

Then using the localized Biest energy and standard BHT energy estimates,

$$I \lesssim 2^{2\mathfrak{d}} \sum_{I\in\mathbb{I}_{k,n_4}} ||f_2\tilde{1}_I||_4^2||f_3\tilde{1}_I||_4^2$$
$$II \lesssim 2^{2\mathfrak{d}} \sum_{I\in\mathbb{I}_{k,n_4}} ||BHT(f_2,f_3)\tilde{1}_I||_2^2.$$

Since $\sum_{I\in\mathbb{I}_{k,n_4}}|I| \lesssim 2^{-k}|E_2|^{1/2}|E_3|^{1/2}$,

$$\sum_{T\in\mathcal{T}^{\tilde{d}}_{k,\mathfrak{d},2,*}} |I_T| \lesssim 2^{2\mathfrak{d}}2^{-k/2}|E_2|^{1/2}|E_3|^{1/2}.$$

As $\sum_{T\in\mathcal{T}^{\tilde{d}}_{k,\mathfrak{d},2,*}}|I_T| \lesssim_\epsilon 2^{2\mathfrak{d}}2^{\tilde{d}}|E_2|^{2-\epsilon}$ and $\sum_{T\in\mathcal{T}^{\tilde{d}}_{k,\mathfrak{d},2,*}}|I_T| \lesssim 2^{\frac{\mathfrak{d}}{1-\tilde{\epsilon}}}|E_2|^{1/2}|E_3|^{1/2}$ (coming from the l^1 energy boost),

$$\sum_{T\in\mathcal{T}^{\tilde{d}}_{k,\mathfrak{d},2,*}} |I_T| \lesssim_{\epsilon,\tilde{\epsilon}} \min\left\{2^{2\mathfrak{d}}2^{-k/2}|E_2|^{1/2}|E_3|^{1/2}, 2^{2\mathfrak{d}}2^{\tilde{d}}|E_2|^{2-\epsilon}, 2^{\frac{\mathfrak{d}}{1-\tilde{\epsilon}}}|E_2|^{1/2}|E_3|^{1/2}\right\}$$

Similarly, we have $\sum_{T\in\mathcal{T}^{\tilde{d}}_{k,\mathfrak{d},2,*}}|I_T| \lesssim \min\{2^{2n_1}|E_1|, 2^{2n_1}2^{n_4}|E_4|\}$. Indeed, the localized BHT energy yields

$$\sum_{T\in\mathcal{T}_{k,n_1,n_2,1}} |I_T| \lesssim 2^{2n_1} \sum_{I\in\mathbb{I}_{k,n_1}} \left|\left|f_1\tilde{1}_I\right|\right|_2^2$$
$$\lesssim 2^{2n_1} \sum_{I\in\mathbb{I}_{k,n_1}} |I|$$
$$\lesssim 2^{2n_1}2^{n_4}|E_4|.$$

Putting it all together gives

$$\left|\Lambda_{\vec{\mathbb{P}},\vec{\mathbb{Q}}}(f_1, f_2, f_3 1_{\Omega^c}, f_4)\right|$$
$$\lesssim_{\epsilon,\tilde{\epsilon}} \sum_{\tilde{d},n_1,n_4,k\geq 0} \sum_{\mathfrak{d}\geq -k} 2^{-n_1}2^{-n_4}2^{-\mathfrak{d}}$$
$$\times \quad \min\left\{2^{2n_1}|E_1|, 2^{2n_1}2^{n_4}|E_4|, 2^{2\mathfrak{d}}2^{-k/2}|E_2|^{1/2}, 2^{2\mathfrak{d}}2^{\tilde{d}}|E_2|^{2-\epsilon}, 2^{\frac{\mathfrak{d}}{1-\tilde{\epsilon}}}|E_3|^{1/2}\right\}.$$

Take $\tilde{\epsilon} \simeq 0$. Using $0 \leq \theta_1, \theta_2, \theta_3, \theta_4, \theta_5 \leq 1$ to denote the weightings assigned to each term in the above minimum, we may deduce suitable weak type estimates in a neighborhood of $A_8 = (0, 1, -\frac{1}{2}, \frac{1}{2})$ by taking $\theta_1 \simeq 0, \theta_2 \simeq \frac{1}{2}, \theta_3 \simeq 0, \theta_4 \simeq \frac{1}{2}, \theta_5 \simeq 0$. For estimates near $A_9 = (\frac{1}{2}, 1, -\frac{1}{2}, 0)$, take $\theta_1 \simeq \frac{1}{2}, \theta_2 \simeq 0, \theta_3 \simeq 0, \theta_4 \simeq \frac{1}{2}, \theta_5 \simeq 0$. Some care has to be taken to ensure summability over $k \geq 0$. That this is possible is nonetheless straightforward, and so details are left to the reader. This concludes the proof of Theorem 21. □

3.12 $C^{1,1,1-2}$ Estimates

Our goal in this section is to prove

Theorem 22. *$C^{1,1,1,-2} : L^{p_1}(\mathbb{R}) \times L^{p_2}(\mathbb{R}) \times L^{p_3}(\mathbb{R}) \times L^{p_4}(\mathbb{R}) \to L^{p_4'}(\mathbb{R})$ provided $1 < p_1, p_2, p_3 \leq \infty, 0 < p_4' < \infty$ and $(p_1, p_2, p_3, p_4) \in Int(Conv[\mathcal{B}]) \cap Int(Conv[\mathcal{B}'])$,* *where* $\mathcal{B} = \{B_j\}_{j=1}^{16}$ *is given by*

$$
\begin{aligned}
B_1 &= \left(1, 1, \frac{1}{2}, \frac{1}{2}, -2\right) \quad B_2 = \left(1, \frac{1}{2}, \frac{1}{2}, 1, -2\right) \quad B_3 = \left(1, \frac{1}{2}, 1, \frac{1}{2}, -2\right) \\
B_4 &= \left(-2, 1, \frac{1}{2}, \frac{1}{2}, 1\right) \quad B_5 = \left(-2, \frac{1}{2}, 1, \frac{1}{2}, 1\right) \quad B_6 = \left(-2, \frac{1}{2}, \frac{1}{2}, 1, 1\right) \\
B_7 &= \left(0, -\frac{3}{2}, \frac{1}{2}, 1, 1\right) \quad B_8 = \left(1, -\frac{3}{2}, \frac{1}{2}, 1, 0\right) \quad B_9 = \left(0, -\frac{3}{2}, 1, \frac{1}{2}, 1\right) \\
B_{10} &= \left(1, -\frac{3}{2}, 1, \frac{1}{2}, 0\right) \quad B_{11} = \left(0, \frac{1}{2}, -\frac{1}{2}, 1, 0\right) \quad B_{12} = \left(\frac{1}{2}, 0, -\frac{1}{2}, 1, 0\right) \\
B_{13} &= \left(0, 0, -\frac{1}{2}, 1, \frac{1}{2}\right) \quad B_{14} = \left(0, \frac{1}{2}, 1, -\frac{1}{2}, 0\right) \quad B_{15} = \left(\frac{1}{2}, 0, 1, -\frac{1}{2}, 0\right) \\
B_{16} &= \left(0, 0, 1, -\frac{1}{2}, \frac{1}{2}\right)
\end{aligned}
$$

Let $\mathcal{B}'$ *denote the collection* $\left\{B_j'\right\}_{j=1}^{16}$*, where each* B_j' *is obtained from the corresponding* B_j *by the permutation* $1 \mapsto 1, 2 \mapsto 4, 3 \mapsto 2, 4 \mapsto 3$*. In particular,* $B_3' = \left(1, 1, \frac{1}{2}, \frac{1}{2}, -2\right)$*. Moreover,* $(1, \frac{2}{3}, \frac{2}{3}, \frac{2}{3} - 2) \in \overline{Conv[\mathcal{B}]} \cap \overline{Conv[\mathcal{B}']}$ *and* $C^{1,1,1,-2}$ *maps into* $L^r(\mathbb{R})$ *for all* $\frac{1}{3} < r \leq 1$*.*

3.12.1 Reduction to the Λ_3 Model

We proceed as in the proof of $C^{1,1,-2}$ estimates by decomposing $\left\{\xi_1 < \xi_2 < \xi_3 < -\frac{\xi_4}{2}\right\}$ into the following regions (viewed as subsets of $\left\{\xi_1 < \xi_2 < \xi_3 < -\frac{\xi_4}{2}\right\}$):

$$
\begin{aligned}
\mathcal{R}_0 &= \left\{|\xi_1-\xi_2| >> |\xi_2-\xi_3| \simeq \left|\xi_3+\frac{\xi_4}{2}\right|\right\} \\
\mathcal{R}_1 &= \left\{|\xi_1-\xi_2| >> |\xi_2-\xi_3| >> \left|\xi_3+\frac{\xi_4}{2}\right|\right\} \\
\mathcal{R}_2 &= \left\{|\xi_1-\xi_2| >> \left|\xi_3+\frac{\xi_4}{2}\right| >> |\xi_2-\xi_3|\right\} \\
\mathcal{R}_3 &= \left\{|\xi_1-\xi_2| \simeq |\xi_2-\xi_3| >> \left|\xi_3+\frac{\xi_4}{2}\right|\right\} \\
\mathcal{R}_4 &= \left\{|\xi_1-\xi_2| \simeq \left|\xi_3+\frac{\xi_4}{2}\right| >> |\xi_2-\xi_3|\right\} \\
\mathcal{R}_5 &= \left\{|\xi_1-\xi_2| \simeq |\xi_2-\xi_3| \simeq \left|\xi_3+\frac{\xi_4}{2}\right|\right\} \\
\mathcal{R}_6 &= \left\{\left|\xi_3+\frac{\xi_4}{2}\right| >> |\xi_2-\xi_3| >> |\xi_1-\xi_2|\right\} \\
\mathcal{R}_7 &= \left\{\left|\xi_2+\frac{\xi_4}{2}\right| >> |\xi_1-\xi_2| >> |\xi_2-\xi_3|\right\} \\
\mathcal{R}_8 &= \left\{\left|\xi_3+\frac{\xi_4}{2}\right| \simeq |\xi_2-\xi_3| >> |\xi_1-\xi_2|\right\} \\
\mathcal{R}_9 &= \left\{|\xi_3+\frac{\xi_4}{2}| >> |\xi_2-\xi_3| \simeq |\xi_1-\xi_2|\right\} \\
\mathcal{R}_{10} &= \{|\xi_2-\xi_3| >> |\xi_1-\xi_2|\} \cap \left\{|\xi_2-\xi_3| >> \left|\xi_3+\frac{\xi_4}{2}\right|\right\}.
\end{aligned}
$$

One expects that the most problematic regions are $\mathcal{R}_1$ and $\mathcal{R}_2$ and that, by symmetry, it should suffice to handle generalized restricted type estimates for the discretized model of generic symbols adapted to $\mathcal{R}_1$, say. In light of previous work with both $C^{-1,1,-1}$ and $C^{1,1,-2}$, it should come as no surprise that the most natural symbol $\tilde{1}_{\mathcal{R}_1}$ localized to $\mathcal{R}_1$ and identically equal to 1 on a subregion of the same shape cone can write down can be written as a sum of expressions all of the generic form

$$
\sum_{\vec{P}\in\mathcal{P}} \sum_{\vec{Q}\in\mathcal{Q}:|\vec{Q}|<<|\vec{P}|} \sum_{\vec{R}\in\mathcal{R}:\widetilde{\omega_{R_1}}\supset\supset\omega_{P_2}} \hat{\eta}_{R_1,1}(\xi_1)\hat{\eta}_{R_3,0}(\xi_2+\xi_3+\xi_4)\hat{\eta}_{P_2,2}(\xi_2)\hat{\eta}_{P_3,7}(\xi_3+\xi_4),
$$

where each $\vec{R} = (R_1, R_3)$ is a frequency square intersecting the line $\{\xi_2 = 0\}$, each $\vec{P} = (P_1, P_2)$ is a frequency square adapted to the line $\{\xi_1 + \xi_2 = 0\}$, and each $\vec{Q} = (Q_1, Q_2)$ is a frequency square adapted to $\{\xi_1 = -\xi_2/2\}$. As usual, we denote $\tilde{I} := I + C|I|$ for some large $C >> 1$. We may now dualize by introducing another function f_5 and then complete the resulting integral as follows:

$$
\begin{aligned}
& \Lambda(f_1, f_2, f_3, f_4, f_5) \\
:= & \int_{\mathbb{R}} \sum_{\vec{P}\in\mathcal{P}} \sum_{\vec{Q}\in\mathcal{Q}:|\vec{Q}|<<|\vec{P}|} \sum_{\vec{R}\in\mathcal{R}:\widetilde{\omega}_{R_1}\supset\supset\omega_{P_2}} f_1 * \eta_{R_1,1} \\
\times & \left[f_2 * \eta_{P_2,2}\left[f_3 * \eta_{Q_1,3} f_4 * \eta_{Q_2,4}\right] * \eta_{P_3,7}\right] * \eta_{-R_3,0} f_5 dx \\
= & \int_{\mathbb{R}} \sum_{\vec{P}\in\mathcal{P}} \sum_{\vec{Q}\in\mathcal{Q}:|\vec{Q}|<<|\vec{P}|} \sum_{\vec{R}\in\mathcal{R}:\widetilde{\omega}_{R_1}\supset\supset\omega_{P_2}} \left[f_1 * \eta_{R_1,1} f_5 * \eta_{R_2,5}\right] * \tilde{\eta}_{R_3,0} \\
\times & f_2 * \eta_{P_2,2}\left[f_3 * \eta_{Q_1,3} f_4 * \eta_{Q_2,4}\right] * \eta_{P_3,7} dx,
\end{aligned}
$$

which can then be discretized in the standard way to yield a sum over rapidly decaying terms of averages of generic forms of type Λ_3 given by

$$
\begin{aligned}
& \sum_{\vec{P}\in\mathbb{P}} \frac{1}{|I_{\vec{P}}|^{1/2}} \left\langle \sum_{\vec{R}\in\mathbb{R}:\widetilde{\omega_{R_1}}\supset\supset\omega_{P_2}} \frac{1}{|I_{\vec{R}}|^{1/2}} \langle f_1, \Phi_{R_1,1}\rangle\langle f_5, \Phi_{R_2,5}\rangle \Phi^{n-l}_{R_3,0}, \Phi_{P_1,6} \right\rangle \langle f_2, \Phi_{P_2,2}\rangle \\
\times & \left\langle \int_0^1 BHT^{\alpha,\mathbb{Q}}_{\omega_{P_3}}(f_3, f_4) d\alpha, \Phi_{P_3,7} \right\rangle.
\end{aligned}
$$

While $\mathbb{Q}$ is rank-1 collection of ti-tiles, both $\mathbb{R}$ and $\mathbb{P}$ are not. We now show generalized restricted type estimates for type Λ_3 models. Similar to the arguments presented in proof of Proposition 9, we may reduce the proof of Theorem 22 to the proof of its discretized version:

Theorem 25. *Let $\sigma, \sigma', \tilde{\sigma} \in \{0, \frac{1}{3}, \frac{2}{3}\}^3$ be shifts, and let $\mathbb{P}, \mathbb{Q}, \mathbb{R}$ be finite collections of tri-tiles with shifts $\sigma, \sigma', \tilde{\sigma}$ respectively so that $\mathbb{Q}$ is rank-1. Define the form $\Lambda_{\mathbb{P},\mathbb{Q},\mathbb{R}}$ by*

$$\begin{aligned}
&\Lambda_{\mathbb{P},\mathbb{Q},\mathbb{R}}(f_1,f_2,f_3,f_4,f_5)\\
=\ &\sum_{\vec{P}\in\mathbb{P}}\frac{1}{|I_{\vec{P}}|^{1/2}}\left\langle\sum_{\vec{R}\in\mathbb{R}:\widetilde{\omega_{R_1}}\supset\supset\omega_{P_2}}\frac{\langle f_1,\Phi_{R_1,1}\rangle\langle f_5,\Phi_{R_2,5}\rangle}{|I_{\vec{R}}|^{1/2}}\Phi^{n-l}_{R_3,0},\Phi_{P_1,6}\right\rangle\langle f_2,\Phi_{P_2,2}\rangle\\
\times\ &\left\langle\int_0^1 BHT^{\alpha,\mathbb{Q}}_{\omega_{P_3}}(f_3,f_4)d\alpha,\Phi_{P_3,7}\right\rangle,
\end{aligned}$$

where the $\mathbb{P}$-sum is over all tri-tiles of the form $\vec{P}=(P_1,P_2,P_3),\Phi_{P_1,1}$ is a wave packet on $I_{\vec{P}}\times\omega^{lac}_{|\vec{P}|}:=I_{\vec{P}}\times[c_1|I_{\vec{P}}|^{-1},c_2|I_{\vec{P}}|^{-1}]$ for some absolute constants $c_1<<c_2$, $\Phi_{P_2,2}$ is a wave packet on $I_{\vec{P}}\times\omega_{P_2}$, $\Phi_{P_3,7}$ is a wave packet on $I_{\vec{P}}\times\omega_{P_3}$, and

$$BHT^{\alpha}_{\omega_{P_3}}(f_2,f_3)(x):=\sum_{\vec{Q}\in\mathbb{Q}:|\omega_Q|<<|\omega_P|}\frac{1}{|I_{\vec{Q}}|^{1/2}}\langle f_2,\Phi^{\alpha}_{Q_1,2}\rangle\langle f_3,\Phi^{\alpha}_{Q_2,3}\rangle\Phi^{\alpha}_{Q_3,5}(x),$$

where for each $\alpha\in[0,1]$, the $\mathbb{Q}$-sum is over all tri-tiles of the form $\vec{Q}=(Q_1,Q_2,Q_3),\Phi^{\alpha}_{Q_1,2}$ is a wave packet on $I_{\vec{Q}}\times\omega_{Q_1},\Phi^{\alpha}_{Q_2,3}$ is a wave packet on $I_{\vec{Q}}\times\omega_{Q_2},\Phi^{\alpha}_{Q_3,5}$ is a wave packet on $I_{\vec{Q}}\times\omega_{Q_3}$. Then $\Lambda_{\mathbb{P},\mathbb{Q},\mathbb{R}}$ is restricted type $\vec{\alpha}$ for all admissible tuples in $\vec{\alpha}\in Int[Conv[\mathcal{B}]]$, uniformly in the parameters

$$\sigma,\sigma',\mathbb{P},\mathbb{Q},\left\{\Phi_{P_i,j(i)}\right\},\left\{\Phi^{\alpha}_{Q_i,j(i)}\right\}.$$

For completeness, we record

Proposition 11. *To prove Theorem 22, it suffices to prove Theorem 25.*

Proof. The numerous details required for a full demonstration are a bit tedious to verify and very similar to the proof of Proposition 9. □

Note as before that if $\vec{\alpha}\in Int(Conv[\mathcal{B}])$ has a bad index j, the restricted type estimate will *not* necessarily be uniform in the sense that the major subset E'_j

cannot be chosen independently of the parameters just mentioned. Proposition 11 ensures that taking the expectational set independent of such parameters is not an obstacle in reducing Theorem 22 to Theorem 25. We now prove Theorem 25.

Proof.

3.13 Generalized Restricted Type Estimates near B_1, B_2, B_3

3.13.1 Tile Decomposition

Fix dyadic shifts $\sigma, \sigma', \tilde{\sigma}$ and corresponding tri-tile collections $\mathbb{P}, \mathbb{Q}$, and $\mathbb{R}$ once and for all. By assumption, $\mathbb{Q}$ is rank-1. Moreover, for convenience, we shall subsequently use f_j to denote f_j' for $j = 1, 2, 3, 4$ in Theorem 25 and assume by rescaling that $|E_5| = 1$ and the collections $\mathbb{P}, \mathbb{Q}, \mathbb{R}$ are sparse. Next, set

$$\tilde{\Omega} = \{M1_{E_3} \gtrsim |E_3|\} \bigcup \{M1_{E_4} \gtrsim |E_4|\}.$$

For each $\tilde{d} \geq 0$ set $\mathbb{Q}^{\tilde{d}} := \left\{ \vec{Q} \in \mathbb{Q} : 1 + \frac{dist(I_{\vec{Q}}, \tilde{\Omega}^c)}{|I_{\vec{Q}}|} \simeq 2^{\tilde{d}} \right\}$ and define for $0 < \theta < 1$

$$\begin{aligned}
\Omega_1^{\tilde{d}} &= \left\{ M \left[\int_0^1 BHT^{\alpha, \mathbb{Q}^{\tilde{d}}}(f_3, f_4) d\alpha \right] \gtrsim_\theta 2^{\tilde{d}} |E_3|^{1/2} |E_4|^{1/2} \right\} \\
\Omega_2^{\tilde{d}} &= \left\{ M \left(\left[\int_0^1 \sum_{\vec{Q} \in \mathbb{Q}^{\tilde{d}}} \frac{|\langle f_3, \Phi^\alpha_{Q_{1,3}} \rangle \langle f_4, \Phi^\alpha_{Q_{2,4}} \rangle|}{|I_{\vec{Q}}|} \tilde{1}_{I_{\vec{Q}}} d\alpha \right]^2 \right) \gtrsim 2^{2\tilde{d}} |E_3||E_4| \right\}.
\end{aligned}$$

Lastly, construct

$$\Omega = \bigcup_{\tilde{d} \geq 0} \Omega_1^{\tilde{d}} \bigcup_{\tilde{d} \gtrsim 1} \Omega_2^{\tilde{d}} \bigcup \{M1_{E_1} \gtrsim |E_1|\} \bigcup \{M1_{E_2} \gtrsim |E_2|\} \bigcup \tilde{\Omega}.$$

Then for large enough implicit constants depending on ϵ, $|\Omega(\epsilon)| \leq 1/2$ and $\tilde{E}_5 := E_5 \cap \Omega(\epsilon)^c$ is a major subset of E_5 since $|E_5| = 1$. Now let $\mathbb{P}^d := \left\{\vec{P} \in \mathbb{P} : 1 + \frac{dist(I_{\vec{P}}, \Omega(\epsilon)^c)}{|I_{\vec{P}}|} \simeq 2^d\right\}$. Assuming $|f_1| \leq 1_{E_1}, |f_2| \leq 1_{E_2}, |f_3| \leq 1_{E_3}, |f_4| \leq 1_{E_4}, |f_5| \leq 1_{E_5 \cap \Omega^c}$, recall that our task in this section is to obtain the estimate $|\Lambda_{\mathbb{P},\mathbb{Q},\mathbb{R}}(f_1, f_2, f_3, f_4, f_5)| \lesssim |E_1|^{\alpha_1}|E_2|^{\alpha_2}|E_3|^{\alpha_3}|E_4|^{\alpha_4}$ for $(\alpha_1, \alpha_2, \alpha_3, \alpha_4)$ in a small neighborhood near an extremal point $\vec{\beta} \in \{B_1, B_2, B_3\}$.

$$
\begin{aligned}
& \Lambda_{\mathbb{P},\mathbb{Q},\mathbb{R}}(f_1, f_2, f_3, f_4, f_5) \\
= & \sum_{\vec{P} \in \mathbb{P}} \frac{1}{|I_{\vec{P}}|^{1/2}} \left\langle \sum_{\vec{R} \in \mathbb{R}(T)} \frac{1}{|I_{\vec{R}}|^{1/2}} \langle f_1, \Phi_{R_1,1}\rangle \langle f_5, \Phi_{R_2,5}\rangle \Phi^{n-l}_{R_3,0}, \Phi_{P_1,6} \right\rangle \langle f_2, \Phi_{P_2,2}\rangle \\
\times & \left\langle \int_0^1 BHT^{\alpha,\mathbb{Q}}_{\omega_{P_3}}(f_3, f_4)\, d\alpha, \Phi_{P_3,7} \right\rangle,
\end{aligned}
$$

where $\mathbb{P}, \mathbb{Q}$, and $\mathbb{R}$ are three tri-tile collections, with the additional assumption that $\mathbb{Q}$ is rank-1. For any subcollection of tri-tiles $\tilde{\mathbb{P}} \subset \mathbb{P}$, let

$$
\begin{aligned}
Size_2\left(f_2, \tilde{\mathbb{P}}\right) &:= \sup_{T \subset \tilde{\mathbb{P}}} \frac{1}{|I_T|^{1/2}} \left(\sum_{\vec{P} \in T} |\langle f_2, \Phi_{P_2,2}\rangle|^2 \right)^{1/2} \\
Size_7^{\tilde{d}}(f_3, f_4, \tilde{\mathbb{P}}) &:= \sup_{T \subset \tilde{\mathbb{P}}} \frac{1}{|I_T|^{1/2}} \left(\sum_{\vec{P} \in T} \left| \left\langle \int_0^1 BHT^{\alpha,\mathbb{Q}^{\tilde{d}}}_{\omega_{P_2}}(f_3, f_4)\, d\alpha, \Phi_{P_3,7} \right\rangle \right|^2 \right)^{1/2},
\end{aligned}
$$

where the supremum arising in the definition of the 1-Size is over all 3−trees and the supremum arising in the definition of the 7-Size is over all 2−trees. As before, both sizes generate decompositions of $\mathbb{P}^d$ for each $\tilde{d} \geq 0$, namely $\bigcup_{n_2 \geq N_2(d)} \tilde{\mathbb{P}}^d_{n_2,2}$ and $\bigcup_{\mathfrak{d} \geq N_3(d,\tilde{d})} \mathbb{P}^{d,\tilde{d}}_{\mathfrak{d},3}$ such that $Size_2(f_2, \tilde{\mathbb{P}}^d_{n_2,2}) \lesssim 2^{-n_2}$ and $Size_7^{\tilde{d}}(f_3, f_4, \tilde{\mathbb{P}}^{d,\tilde{d}}_{\mathfrak{d},3}) \lesssim 2^{-\mathfrak{d}}$. Moreover, $\tilde{\mathbb{P}}^d_{n_2,2}$ and $\tilde{\mathbb{P}}^{d,\tilde{d}}_{\mathfrak{d},3}$ can each be written as a union of trees, i.e.

$$\tilde{\mathbb{P}}^d_{n_2,2} = \bigcup_{T\in\mathcal{T}^d_{n_2,2}} \bigcup_{\vec{P}\in T} \vec{P}$$

$$\tilde{\mathbb{P}}^{d,\tilde{d}}_{\mathfrak{d},3} = \bigcup_{T\in\mathcal{T}^{d,\tilde{d}}_{\mathfrak{d},3}} \bigcup_{\vec{P}\in T} \vec{P},$$

such that

$$\sum_{T\in\mathcal{T}^d_{n_2,2}} |I_T| \lesssim 2^{2n_2} \sum_{T\in\mathcal{T}^d_{n_2,2,*}} \sum_{\vec{P}\in T} |\langle f_1, \Phi_{P_2,2}\rangle|^2$$

$$\sum_{T\in\mathcal{T}^{d,\tilde{d}}_{\mathfrak{d},3}} |I_T| \lesssim 2^{2\mathfrak{d}} \sum_{T\in\mathcal{T}^{d,\tilde{d}}_{\mathfrak{d},3,*}} \sum_{\vec{P}\in T} \left|\left\langle \int_0^1 BHT^{\alpha,\mathbb{Q}^{\tilde{d}}}_{\omega_{P_2}}(f_2,f_3)d\alpha, \Phi_{P_2,0}\right\rangle\right|^2,$$

where $\mathcal{T}^d_{n_2,2,*} \subset \mathcal{T}^d_{n_2,2}$ and $\mathcal{T}^{d,\tilde{d}}_{\mathfrak{d},3,*} \subset \mathcal{T}^{d,\tilde{d}}_{\mathfrak{d},3}$, each tree in $\mathcal{T}^d_{n_2,2,*}$ is a $3-$tree and each tree in $\mathcal{T}^{d,\tilde{d}}_{\mathfrak{d},3,*}$ is a $2-$tree, and the collections $\mathcal{T}^d_{n_2,2,*}, \mathcal{T}^{d,\tilde{d}}_{\mathfrak{d},3,*}$ can each be written as the union of a strongly 2-disjoint and $3-$disjoint subcollections respectively. We denote this last property by

$$\mathcal{T}^d_{n_1,1,*} = \mathcal{T}^d_{n_1,1,*,+} \bigcup \mathcal{T}^d_{n_1,1,*,-}$$

$$\mathcal{T}^{d,\tilde{d}}_{\mathfrak{d},2,*} = \mathcal{T}^{d,\tilde{d}}_{\mathfrak{d},2,*,+} \bigcup \mathcal{T}^{d,\tilde{d}}_{\mathfrak{d},2,*,-}.$$

Similar to before, construct $\mathbb{P}^{d,\tilde{d}}_{n_2,\mathfrak{d}} = \tilde{P}^{d,\tilde{d}}_{n_2,2} \cap \tilde{P}^{d,\tilde{d}}_{\mathfrak{d},3}$

3.13.2 Tree Estimate

First, we should recall from [21] the following statement:

Lemma 34. *Let $\tilde{\mathbb{P}} \subset \mathbb{P}$ be any sub collection of tri-tiles. Then, for any $0 < \theta < 1$ and $1-$tree $T \subset \tilde{\mathbb{P}}$,*

$$\left[\frac{1}{|I_T|}\sum_{\vec{P}\in T}\left|\left\langle \int_0^1 \sum_{\vec{Q}\in\mathbb{Q}^{\tilde{d}}:\omega_{Q_3}\supset\supset\omega_{P_3}} \frac{1}{|I_{\vec{Q}}|^{1/2}}\langle f_2, \Phi^{\alpha}_{Q_1,2}\rangle\langle f_3, \Phi^{\alpha}_{Q_2,3}\rangle\Phi^{\alpha}_{Q_3,5}d\alpha, \Phi_{P_3,7}\right\rangle\right|^2\right]^{1/2}$$
$$\lesssim_\theta \quad \left[\sup_{\vec{P}\in\tilde{\mathbb{P}}}\frac{1}{|I_{\vec{P}}|}\int 1_{E_2}\tilde{1}_{I_{\vec{P}}}dx\right]^{\theta}\left[\sup_{\vec{P}\in\tilde{\mathbb{P}}}\frac{1}{|I_{\vec{P}}|}\int 1_{E_3}\tilde{1}_{I_{\vec{P}}}dx\right]^{1-\theta}.$$

Proof. See [21]. □

Now, letting $T \subset \mathbb{P}^{d,\tilde{d}}_{n_2,\mathfrak{d}}$ be a 3-tree, we use the proceeding lemma to conclude

$$\left|\sum_{\vec{P}\in\mathbb{P}}\frac{1}{|I_{\vec{P}}|^{1/2}}\left\langle \sum_{\vec{R}\in\mathbb{R}:\widetilde{\omega_{R_1}}\supset\supset\omega_{P_2}}\frac{\langle f_1,\Phi_{R_1,1}\rangle\langle f_5,\Phi_{R_2,5}\rangle}{|I_{\vec{R}}|^{1/2}}\Phi^{n-l}_{R_3,0},\Phi_{P_1,6}\right\rangle\langle f_2,\Phi_{P_2,2}\rangle\right.$$
$$\times \quad \left.\left\langle\int_0^1 BHT^{\alpha,\mathbb{Q}}_{\omega_{P_3}}(f_3,f_4)d\alpha,\Phi_{P_3,7}\right\rangle\right|$$
$$\leq \quad \frac{\left(\sum_{\vec{P}\in T}\left|\left\langle\sum_{\vec{R}\in\mathbb{R}:\widetilde{\omega_{R_1}}\supset\supset\omega_{P_2}}\frac{\langle f_1,\Phi_{R_1,1}\rangle\langle f_5,\Phi_{R_2,5}\rangle}{|I_{\vec{R}}|^{1/2}}\Phi^{n-l}_{R_3,0},\Phi_{P_1,6}\right\rangle\right|^2\right)^{1/2}}{|I_T|^{1/2}}$$
$$\times \quad \frac{\left(\sum_{\vec{P}\in T}|\langle f_2,\Phi_{P_2,2}\rangle|^2\right)^{1/2}}{|I_T|^{1/2}}\cdot\sup_{\vec{P}\in T}\left[\frac{\left|\left\langle\int_0^1 BHT^{\alpha,\mathbb{Q}^{\tilde{d}}}_{\omega_{P_3}}(f_3,f_4)d\alpha,\tilde{\Phi}^{\infty}_{P_3,7}\right\rangle\right|}{|I_{\vec{Q}}|}\right]|I_T|$$
$$\lesssim_\theta \quad 2^{-\tilde{N}d(1-\theta)}|E_1|^{\theta}2^{-n_2}2^{-\mathfrak{d}}|I_T|.$$

If $T \subset \mathbb{P}^{d,\tilde{d}}_{n_2,\mathfrak{d}}$ be a $2-$tree, then

$$
\begin{aligned}
& \left| \sum_{\vec{P}\in\mathbb{P}} \frac{1}{|I_{\vec{P}}|^{1/2}} \left\langle \sum_{\vec{R}\in\mathbb{R}:\widetilde{\omega_{R_1}}\supset\supset\omega_{P_2}} \frac{\langle f_1, \Phi_{R_1,1}\rangle\langle f_5, \Phi_{R_2,5}\rangle}{|I_{\vec{R}}|^{1/2}} \Phi^{n-l}_{R_3,0}, \Phi_{P_1,6} \right\rangle \langle f_2, \Phi_{P_2,2}\rangle \right. \\
\times & \left. \left\langle \int_0^1 BHT^{\alpha,\mathbb{Q}}_{\omega_{P_3}}(f_3, f_4)d\alpha, \Phi_{P_3,7} \right\rangle \right| \\
\lesssim & \left[\sup_{\vec{P}\in T} \frac{\left| \left\langle \sum_{\vec{R}\in\mathbb{R}:\widetilde{\omega_{R_1}}\supset\supset\omega_{P_2}} \frac{\langle f_1,\Phi_{R_1,1}\rangle\langle f_5,\Phi_{R_2,5}\rangle}{|I_{\vec{R}}|^{1/2}} \Phi^{n-l}_{R_3,0}, \Phi_{P_1,6} \right\rangle \right|}{|I_{\vec{P}}|^{1/2}} \right] \left(\sum_{\vec{P}\in T} \frac{|\langle f_4, \Phi^{lac}_{\vec{P},4}\rangle|^2}{|I_T|} \right)^{1/2} \\
\times & \left(\sum_{\vec{P}\in T} \frac{\left| \left\langle \int_0^1 BHT^{\alpha,\mathbb{Q}^{\tilde{d}}}_{\omega_{P_2}}(f_3, f_4)d\alpha, \Phi_{P_3,7} \right\rangle \right|^2}{|I_T|} \right)^{1/2} |I_T| \\
\lesssim_\theta & \; 2^{-\tilde{N}d(1-\theta)}|E_1|^\theta 2^{-n_2} 2^{-\mathfrak{d}} |I_T|.
\end{aligned}
$$

3.13.3 Size Restrictions

Lemma 35. *Fix $d, \tilde{d}, n_2, \mathfrak{d}$ such that $\mathbb{P}^{d,\tilde{d}}_{n_2,\mathfrak{d}}$ is nonempty. Then*

$$
\begin{aligned}
2^{-n_2} &\lesssim 2^d |E_2| \\
2^{-\mathfrak{d}} &\lesssim 2^{-\tilde{\tilde{N}}(\tilde{d}-d)} |E_3|^{1/2} |E_4|^{1/2}.
\end{aligned}
$$

Proof. The proof is identical to the previous size restriction argument and therefore omitted. □

3.13.4 Synthesis

The l^2 and l^1 energy estimates for $\Lambda_{\mathbb{P},\mathbb{Q},\mathbb{R}}(f_1, f_2, f_3, f_4, f_5)$ are essentially identical to the $\Lambda_{\mathbb{P},\mathbb{Q}}(f_1, f_2, f_3, f_4)$ case, it suffices to assemble all the pieces. Using Theorem

23 as a guide, the reader may check that for all $0 \leq \theta_1, \theta_2, \theta_3 \leq 1$

$$
\begin{aligned}
& |\Lambda_{\mathbb{P},\mathbb{Q},\mathbb{R}}(f_1, f_2, f_3, f_4, f_5)| \\
\lesssim_{\theta,\tilde{\epsilon}} \quad & \sum_{d,\tilde{d}\geq 0} \sum_{n_2 \geq N_2(d)} \sum_{\mathfrak{d} \geq N_3(d,\tilde{d})} 2^{-\tilde{N}d(1-\theta)} 2^{d\theta} |E_1|^{\theta} 2^{-n_2} 2^{-\mathfrak{d}} \\
\times \quad & \min\left\{ 2^{2n_2}|E_2|, 2^{2\mathfrak{d}} 2^{2\tilde{d}} |E_3|^{1+\tilde{\theta}} |E_4|^{2-\tilde{\theta}}, 2^{\frac{\mathfrak{d}}{1-\tilde{\epsilon}}} |E_3|^{1/2} |E_4|^{1/2} \right\} \\
\lesssim \quad & \sum_{d,\tilde{d}\geq 0} \sum_{n_2 \geq N_2(d)} \sum_{\mathfrak{d} \geq N_3(d,\tilde{d})} 2^{-\tilde{N}d(1-\theta)} 2^{d\theta} 2^{-n_2} 2^{-\mathfrak{d}} 2^{2n_2\theta_2} 2^{2\mathfrak{d}\theta_2} 2^{2\tilde{d}\theta_2} \\
\times \quad & |E_1|^{\theta} |E_2|^{\theta_1} |E_3|^{(1+\tilde{\theta})\theta_2} |E_4|^{(2-\tilde{\theta})\theta_2} 2^{\frac{\mathfrak{d}\theta_3}{1-\tilde{\epsilon}}} |E_3|^{\theta_3/2} |E_4|^{\theta_3/2} \\
\leq \quad & \sum_{d,\tilde{d}\geq 0} \sum_{n_2 \geq N_2(d)} \sum_{\mathfrak{d} \geq N_3(d,\tilde{d})} 2^{-n_2(1-2\theta_1)} 2^{-\tilde{N}d(1-\theta)/2} 2^{-\mathfrak{d}\left[1-2\theta_2-\frac{\theta_3}{1-\tilde{\epsilon}}\right]} \\
\times \quad & |E_1|^{\theta} |E_2|^{\theta_1} |E_3|^{(1+\tilde{\theta})\theta_2+\frac{\theta_3}{2}} |E_4|^{(2-\tilde{\theta})\theta_2+\frac{\theta_3}{2}}.
\end{aligned}
$$

Take $\tilde{\epsilon} \simeq 0$. To produce generalized restricted type estimates near $B_1 = \left(1, 1, \frac{1}{2}, \frac{1}{2}, -1\right)$, set $\theta \simeq 1, \tilde{\theta} = \frac{1}{2}, \theta_1 \simeq 0, \theta_2 \simeq 0, \theta_3 \simeq 1$. For $B_2 = \left(1, \frac{1}{2}, \frac{1}{2}, 1, -1\right)$, set $\theta \simeq 1, \tilde{\theta} \simeq 0, \theta_1 \simeq \frac{1}{2}, \theta_2 \simeq \frac{1}{2}, \theta_3 \simeq 0$. Lastly, for $B_3 = \left(1, \frac{1}{2}, 1, \frac{1}{2} - 1\right)$, set $\theta \simeq 1, \tilde{\theta} \simeq 1, \theta_2 \simeq \frac{1}{2}, \theta_2 \simeq \frac{1}{2}, \theta_3 \simeq 0$.

3.14 Generalized Restricted Type Estimates near $B_4, B_5, B_6, B_7, B_8, B_9, B_{10}$

The model $\Lambda_{\mathbb{P},\mathbb{R},\mathbb{Q}}$ is symmetric in positions 1 and 5, and so estimate near B_1, B_2, B_3 ensures estimates near B_4, B_5, B_6. Generalized restricted type estimates near B_7, B_8, B_9, B_{10} follow from the following observations:

$$
\begin{aligned}
& |\Lambda_{\mathbb{P},\mathbb{Q},\mathbb{R}}(f_1,f_2,f_3,f_4,f_5)| \\
\lesssim_{\theta,\tilde{\epsilon}} \quad & \sum_{d,\tilde{d}\geq 0}\sum_{n_2\geq N_2(d)}\sum_{\mathfrak{d}\geq N_3(d,\tilde{d})} 2^d 2^{-\mathfrak{d}\left[1-2\theta_2-\frac{\theta_3}{1-\tilde{\epsilon}}\right]} \\
\times \quad & |E_1|^{1-\theta}|E_5|^{\theta}2^{-n_2(1-2\theta_1)}|E_3|^{(1+\tilde{\theta})\theta_2+\frac{\theta_3}{2}}|E_4|^{(2-\tilde{\theta})\theta_2}|E_4|^{\frac{\theta_3}{2}}
\end{aligned}
$$

Again take $\tilde{\epsilon} \simeq 0$. To produce generalized restricted type estimates near $B_7 = \left(0, -\frac{3}{2}, \frac{1}{2}, 1, 1\right)$, set $\theta \simeq 1, \tilde{\theta} \simeq 0, \theta_1 \simeq 1/2, \theta_2 \simeq 1/2, \theta_3 \simeq 0$. For $B_8 = \left(1, -\frac{3}{2}, \frac{1}{2}, 1, 0\right)$, set $\theta \simeq 0, \tilde{\theta} \simeq 0, \theta_1 \simeq 1/2, \theta_2 \simeq 1/2, \theta_3 \simeq 0$. For $B_9 = \left(0, -\frac{3}{2}, 1, \frac{1}{2}, 1\right)$, set $\theta \simeq 1, \tilde{\theta} \simeq 1, \theta_1 \simeq 1/2, \theta_2 \simeq 1/2, \theta_3 \simeq 0$. For $B_{10} = \left(1, -\frac{3}{2}, 1, \frac{1}{2}, 0\right)$, set $\theta \simeq 0, \tilde{\theta} \simeq 1, \theta_1 \simeq 1/2, \theta_2 \simeq 1/2, \theta_3 \simeq 0$.

3.15 Generalized Restricted Type Estimates near $B_{11}, B_{12}, B_{13}, B_{14}, B_{15}, B_{16}$

By modifying the adjoint tile decomposition for $\Lambda_{\mathbb{P},\mathbb{Q}}$, it is not difficult to observe

$$
\begin{aligned}
& |\Lambda_{\mathbb{P},\mathbb{Q},\mathbb{R}}(f_1,f_2,f_3 1_{\Omega^c},f_4)| \\
\lesssim_{\theta,\tilde{\theta},\tilde{\epsilon}} \quad & \sum_{\tilde{d},n_1,n_4,k\geq 0}\sum_{\mathfrak{d}\geq -k} 2^{-n_1(1-\theta)}2^{-n_5\theta}2^{-n_4}2^{-\mathfrak{d}} \\
\times \quad & \min\left\{2^{2n_2}|E_2|, 2^{2n_2}2^{n_1(1-\theta)}2^{n_5\theta}|E_1|^{1-\theta}|E_5|^{\theta}, 2^{2\mathfrak{d}}2^{-k/2}|E_4|^{1/2}, 2^{2\mathfrak{d}}2^{\tilde{d}}|E_4|^{2-\tilde{\theta}}, 2^{\frac{\mathfrak{d}}{1-\tilde{\epsilon}}}|E_4|^{1/2}\right\}.
\end{aligned}
$$

Take $\tilde{\epsilon} \simeq 0$. Using $0 \leq \theta_1, \theta_2, \theta_3, \theta_4, \theta_5 \leq 1$ to denote the weightings assigned to each term in the above minimum, we may deduce suitable weak type estimates in a neighborhood of $B_{11} = \left(0, \frac{1}{2}, -\frac{1}{2}, 1, 0\right)$ by taking $\theta = 1/2, \tilde{\theta} \simeq 0, \theta_1 \simeq 0, \theta_2 \simeq \frac{1}{2}, \theta_3 \simeq 0, \theta_4 \simeq \frac{1}{2}, \theta_5 \simeq 0$. For estimates near $B_{12} = \left(\frac{1}{2}, 0, -\frac{1}{2}, 1, 0\right)$, take $\theta \simeq 0, \tilde{\theta} \simeq 0, \theta_1 \simeq$

$0, \theta_2 \simeq 1/2, \theta_3 \simeq 0, \theta_4 \simeq \frac{1}{2}, \theta_5 \simeq 0$. For estimates near $B_{13} = \left(0, 0, -\frac{1}{2}, 1, \frac{1}{2}\right)$, take $\theta \simeq 1, \tilde{\theta} \simeq 1, \theta_1 \simeq 0, \theta_2 \simeq 1/2, \theta_3 \simeq 0, \theta_4 \simeq \frac{1}{2}, \theta_5 \simeq 0$. Some care has to be taken to ensure summability over $k \geq 0$. That this is possible is again straightforward, and so details are left to the reader. Generalized restricted type estimates near $B_{14} = \left(0, \frac{1}{2}, 1, -\frac{1}{2}, 0\right), B_{15} = \left(\frac{1}{2}, 0, 1, -\frac{1}{2}, 0\right), B_{16} = \left(0, 0, 1, -\frac{1}{2}, \frac{1}{2}\right)$ are obtained by symmetry with B_{11}, B_{12}, B_{13}. This concludes the proof of Theorem 22.

$\square$

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