ITERATED FRACTIONAL INTEGRALS AND APPLICATIONS TO FOURIER INTEGRAL OPERATORS WITH RATIONAL SYMBOL

A Dissertation

Presented to the Faculty of the Graduate School

of Cornell University

in Partial Fulfillment of the Requirements for the Degree of

Doctor of Philosophy

by

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August 2021

ITERATED FRACTIONAL INTEGRALS AND APPLICATIONS TO FOURIER INTEGRAL OPERATORS WITH RATIONAL SYMBOL

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Cornell University 2021

We prove a full range of strong type L^p estimates for a family of trilinear iterated fractional integral operators, including restricted weak type estimates for a select set of endpoints. Afterward, we apply our result to prove mixed norm estimates for certain types of Fourier integral operators with rational symbol which appear naturally in the study of PDEs.

BIOGRAPHICAL SKETCH

Aleksandra Maalaoui (née Niepla) was born in Stryżów Poland. She moved to New Jersey, USA with her parents at the age of two and a half. Before pursuing her PhD in mathematics at Cornell University she attended Rutgers University in New Brunswick, where she majored in mathematics and physics.

To my family.

ACKNOWLEDGEMENTS

I would like to start by thanking my advisor Camil Muscalu for his time, dedication, patience, and wisdom. When we had a goal in mind, our meetings had no time limit. From him, I learned how to tackle difficult problems by breaking them into smaller and more familiar pieces. I appreciated his encouragement to follow the path that I find most interesting regardless of what others may think. I also appreciated his patience during my explorations and his guidance in linking all of our findings to one broad picture. Camil's unique ability to communicate his thought process and intuition behind various topics of harmonic analysis not only inspired me, but also helped me develop a deeper understanding of the subject matter. I am lucky to have worked with such a talented and kind mathematician.

I am grateful to my committee members Robert Strichartz and Laurent Saloff-coste for their support, inspiration, and questions that led me to think more carefully about my work. I want to reiterate my thanks to Robert Strichartz for the effort he made to attend my defense.

Thank you to Mihaela Ifrim and Daniel Tataru for the insightful discussions and presentations during Camil Muscalu's visit to MSRI that led to the initial motivation behind this project.

I am thankful to Gennady Uraltsev for our productive meetings, especially during my preparation for the A-exam. I appreciated his forwardness and precision when discussing mathematics, as well as his sense of humor.

I would like to thank my fellow graduate students Yujia, Itamar, Andres, Anwesh, Ayah, Ivan, Ryan, José, Zaoli, Emily, Abby, Lila, Rakvi, Maru, and Avery. Each one of them motivated me through their talent and determination. A special thanks to Itamar, Andres, and Ivan for all the board games and fun.

Thank you to Kelly Delp for making me a better math educator. I will miss our cheerful chats.

Thank you to Knight Fu who introduced me to the realm of pure mathematics during my participation in the DRP program at Rutgers University, he spent countless hours with me discussing real analysis, topology, and much more. Thank you to Bryce Weaver for advising me throughout my first research project during my REU at Indiana University. These two individuals had a profound impact on my confidence as a mathematician and my perception of mathematics.

I am grateful for having the opportunity to have learned from Robert Wilson, János Komlós, and Michael Saks during my time at Rutgers University. Thank you for helping me build a strong foundation.

The completion of this thesis would not have been possible without the support of my loving parents and husband. They helped me stay focused and motivated when times were hard. To them I owe everything.

Finally, I would like to thank my little Ayla whose bright and inquisitive smile continuously reassured me to keep right on going!

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CHAPTER 0

STRUCTURE

The primary roles of chapters 1 and 2 are motivational. In chapter 1 we introduce operators $\mathcal{R}_P$ which are Fourier integral operators with rational symbol $m(\xi_1, \ldots, \xi_n) = \frac{1}{P(\xi_1, \ldots, \xi_n)}$ where P is a polynomial. Then, we show that certain instances of operators $\mathcal{R}_P$ arise naturally from the study of PDEs, such as the Schrödinger equation with quadratic nonlinearity. We prove estimates for specific cases using Strichartz estimates together with vector-valued paraproduct estimates. In chapter 2 we provide two motivating examples that highlight the interplay between generalized fractional integral operator estimates and mixed norm estimates for $\mathcal{R}_P$.

Chapters 3 – 6 are solely dedicated to stating and proving $L^{p_1} \times L^{p_2} \times L^{p_3} \to L^r$ estimates for a family of trilinear iterated fractional integral operators $T_{\alpha,\beta}$. The proof consists of dualizing the operators $T_{\alpha,\beta}$ through $L^{1/3}$ and carefully discretizing the resulting form. This specific dualization approach is taken in order to accommodate the presence of two dyadic scales in the discretization of the operator $T_{\alpha,\beta}$. Then, we obtain restricted weak-type estimates through two successive stopping-time arguments and strong type estimates follow by standard interpolation techniques.

Lastly, in chapter 7 we apply specific estimates for a bilinear fractional integral operator proved in [12] and [7] to prove mixed norm estimates for the Fourier integral operator $\mathcal{R}_{\xi_{1,1}\xi_{2,2}+(\xi_{3,3}-\xi_{4,3})}$. Then, we apply the estimates for the trilinear fractional integral operators $T_{\alpha,\beta}$ proved in Chapter 3 to prove mixed norm estimates for $\mathcal{R}_{(\xi_{1,1}\xi_{2,2}+\xi_{5,3}-\xi_{6,3})(\xi_{3,3}\xi_{4,3}+\xi_{5,3}-\xi_{7,3})}$.

CHAPTER 1

FOURIER INTEGRAL OPERATORS WITH RATIONAL SYMBOL

Fourier integral operators with rational symbol form a very large class of operators whose systemic study is yet to be initiated. Techniques from various areas of analysis such as fractional integrals, oscillatory integrals, and restriction are expected to play a role in the study of these operators. In this chapter we will introduce the initial motivation for their study, we present estimates for certain introductory cases, and we conclude by identifying some special cases that will be of particular interest in this work.

1.1 Motivation from Partial Differential Equations

1.1.1 Brief Recap of Paraproducts

We begin by stating the definition and relevant boundedness properties of paraproducts. By *paraproduct* we refer to the bilinear multiplier operator given by

$$\Pi_m(f,g)(x) = \int_{\mathbb{R}^{2d}} m(\xi,\eta)\hat{f}(\xi)\hat{g}(\eta)e^{2\pi i(x\cdot(\xi+\eta))}d\xi d\eta, \tag{1.1}$$

Where $f, g \in \mathcal{S}(\mathbb{R}^d), \xi, \eta, x \in \mathbb{R}^d$, and $m(\xi,\eta)$ is a Coifman-Meyer symbol, that is, it satisfies

$$|\partial^\alpha m(\xi,\eta)| \lesssim \frac{1}{|(\xi,\eta)|^\alpha} \tag{1.2}$$

for sufficiently many multi-indices α.

Theorem 1.1.1. *(Coifman, Meyer [5]) Any bilinear multiplier operator Π_m associated to a symbol $m(\xi,\eta)$ satisfying (1.2) maps $L^p(\mathbb{R}^d) \times L^q(\mathbb{R}^d) \to L^r(\mathbb{R}^d)$ provided that $1 < p, q \leqslant \infty$, $0 < r < \infty$, and $\frac{1}{p} + \frac{1}{q} = \frac{1}{r}$.*

As a result of the work in [2], [6], and the references therein, vector-valued paraproduct estimates are well understood.

Theorem 1.1.2. *(Benea, Muscalu [2] and Di Plinio, Li, et al. [6]) Let f_α, g_α be families of functions indexed by a parameter $\alpha \in \mathbb{R}$. Consider*

$$\Pi_{\mathbb{R}^d}(f_\alpha, g_\alpha)(x_1, x_2, \ldots, x_d),$$

then

$$||\Pi_{\mathbb{R}^d}(f_\alpha, g_\alpha)||_{L^{r_1}_{x_1}\cdots L^{r_d}_{x_d} L^{r_{d+1}}_\alpha} \lesssim ||f_\alpha||_{L^{p_1}_{x_1}\cdots L^{p_d}_{x_d} L^{p_{d+1}}_\alpha} ||g_\alpha||_{L^{q_1}_{x_1}\cdots L^{q_d}_{x_d} L^{q_{d+1}}_\alpha}$$

for $\frac{1}{p_i} + \frac{1}{q_i} = \frac{1}{r_i}$ with $i \in \{1, 2, \ldots, d+1\}$ in the ranges $1 < p_i, q_i \leqslant \infty$ and $0 < r_i < \infty$.

1.1.2 Schrödinger Equation with Quadratic Nonlinearity

The operators B_m defined later in this section, together with the motivation for their study, were introduced to us by Mihaela Ifrim and Daniel Tataru during Camil Muscalu's visit to MSRI in 2018. Insightful discussion with them led to our eventual interest in the more general Fourier integral operators with rational symbol $\mathcal{R}_P$.

Consider the Schödinger equation in $\mathbb{R}^d$ with quadratic nonlinearity

$$\begin{cases} (i\partial_t - \Delta)u &= H(u, u) \\ u(x, 0) &= u_0, \end{cases} \tag{1.3}$$

where we wish to choose u_0 in an adequate Banach space that would guarantee local well-posedness. There are several types of quadratic nonlinearities, some examples can be found in chapter 3 of [4]. For the purposes of this exposition we take H to be a paraproduct as defined in (1.1).

As usual the solution to (1.3) is expected to satisfy

$$u(t) = S(t)u_0 + \int_0^t S(t-t')H(u,u)\,dt'. \tag{1.4}$$

Then, solving for such u becomes equivalent to finding a fixed point of the operator

$$T(u) = S(t)u_0 + \int_0^t S(t-t')H(u,u)\,dt', \tag{1.5}$$

this will naturally depend on the space on which T is defined. The choice of the space is dictated by the nonlinearity H and the initial condition u_0.

A classical result is the existence of global solutions under a smallness assumption of the initial condition u_0 because under this restriction one can show that T is a contraction mapping in a neighborhood of zero. For instance, we refer to chapter 6 of [4]. In the setting where H is quadratic, the contraction coefficient of the mapping T depends linearly on the norm of u. Whereas if one considers a cubic nonlinearity the coefficients will have quadratic growth. Therefore it is not surprising that dealing with a cubic nonlinearity gives more flexibility in finding the appropriate smallness assumption needed on the initial data u_0.

To that end, we attempt a change of variables of type

$$v = u + B(u,u), \tag{1.6}$$

such that

$$(i\partial_t - \Delta)v = Q(u,u,u) \tag{1.7}$$

where $Q(u,u,u)$ is a cubic non-linearly. Provided that B has adequate boundedness properties, then maintaining the smallness assumption on u will also guarantee smallness of $u - v$, and hence the principal part of equation (1.7) can be written as

$$(i\partial_t - \Delta)v = Q(v, v, v) \tag{1.8}$$

because the remaining terms are of higher order and can therefore be neglected at this stage. From the preceding discussion we see that one may prefer to work with equation (1.8). In order to gain meaning from these observations one must answer the following questions,

1) Does such a bilinear operator B exist?
2) If one can find such a B, then what type of boundedness properties does it satisfy?

The answer to the first question is affirmative. In particular, we look for a bilinear operator B of the form

$$B(f, g)(x) = \int_{\mathbb{R}^{2d}} \mathcal{B}(\xi, \eta)\widehat{f}(\xi)\widehat{g}(\eta)e^{2\pi i x\cdot(\xi+\eta)}\, d\xi d\eta, \tag{1.9}$$

where $\mathcal{B}$ is a symbol to be determined later. Then, making the change of variables (1.6) and plugging directly into equation (1.8) we have,

$$\begin{aligned}(i\partial_t - \Delta)v &= (i\partial_t - \Delta)u + (i\partial_t - \Delta)(B(u, u)) \\ &= H(u, u) + i\partial_t B(u, u) - \Delta(B(u, u)) \\ &= H(u, u) + B(i\partial_t u, u) + B(u, i\partial_t u) - \Delta(B(u, u)) \\ &= H(u, u) + B(H(u, u), u) + B(\Delta u, u) + B(u, \Delta u) + B(u, H(u, u)) - \Delta(B(u, u)) \\ &= H(u, u) + B(\Delta u, u) + B(u, \Delta u) - \Delta(B(u, u)) + \text{“cubic terms.”}\end{aligned} \tag{1.10}$$

In order to obtain an equation of the form (1.7) one would need to have

$$H(u,u) + B(\Delta u, u) + B(u, \Delta u) - \Delta(B(u,u)) = 0. \tag{1.11}$$

At the level of symbols this implies

$$\frac{1}{4\pi^2} \cdot m_H(\xi,\eta) + \mathcal{B}(\xi,\eta) \cdot |\xi|^2 + \mathcal{B}(\xi,\eta) \cdot |\eta|^2 - \mathcal{B}(\xi,\eta) \cdot |\xi+\eta|^2 = 0. \tag{1.12}$$

In particular, this means that

$$\mathcal{B}(\xi,\eta) = -\frac{\widetilde{m}_H(\xi,\eta)}{|\xi|^2 + |\eta|^2 - |\xi+\eta|^2}, \tag{1.13}$$

where $\widetilde{m}_H(\xi,\eta) = \frac{1}{4\pi^2} m_H(\xi,\eta)$. Moreover, we can replace the Laplacian with a more general differential operator L. Then our choice for $\mathcal{B}$ becomes

$$\mathcal{B}(\xi,\eta) = \frac{\widetilde{m}_H(\xi,\eta)}{\mathcal{L}(\xi) + \mathcal{L}(\eta) - \mathcal{L}(\xi+\eta)}, \tag{1.14}$$

where $\mathcal{L}(\cdot)$ is the symbol of the differential operator L. Then, in order to answer the second question, it becomes desirable to study the boundedness properties of bilinear maps of type

$$B_{m,\mathcal{L}}(f,g)(x) = \int_{\mathbb{R}^{2d}} \frac{m(\xi,\eta)}{\mathcal{L}(\xi) + \mathcal{L}(\eta) - \mathcal{L}(\xi+\eta)} \widehat{f}(\xi)\widehat{g}(\eta) e^{2\pi i x(\xi+\eta)} d\xi d\eta. \tag{1.15}$$

Since $\mathcal{L}(\cdot)$ is the symbol of a differential operator, the term $\mathcal{L}(\xi)+\mathcal{L}(\eta)-\mathcal{L}(\xi+\eta)$ can lead to a vast array of possible polynomials in (ξ,η). Often controlling the support of $\widehat{f}$ and $\widehat{g}$ leads to more variety in the type of polynomials that appear. For example, taking $\widehat{f}$ and $\widehat{g}$ supported in transveral regions of the plane, like two disjoint cones, one notices that the term $\mathcal{L}(\xi+\eta)$ does not make meaningful contributions to the denominator of the symbol.

One of the most straightforward, yet nontrivial, examples occurs in dimension two and takes the form

$$B_m(f,g)(x) = \int_{\mathbb{R}^4} \frac{m(\xi,\eta)}{\xi_1+\eta_2}\widehat{f}(\xi)\widehat{g}(\eta)e^{2\pi i x(\xi+\eta)}d\xi d\eta. \tag{1.16}$$

Notice that the symbol is very singular. In order to make sense of the integral above, we define it in the following limiting sense

$$B_m(f,g)(x) := \sum_{k\in\mathbb{Z}} \int_{\mathbb{R}^4} \psi_k(\xi_1+\eta_2)m(\xi,\eta)\hat{f}(\xi)\hat{g}(\eta)e^{2\pi i x(\xi+\eta)}\, d\xi d\eta, \tag{1.17}$$

where $\{\psi_k\}_k$ is the standard collection of Schwartz functions with the property that $\frac{1}{x} \sim \sum_k \psi_k$. In particular, we have

$$\text{sign}(\xi) = \sum_k \widehat{\psi_k}(\xi) = \sum_k \widehat{\psi}_k^1(\xi) - \sum_k \widehat{\psi}_k^2(\xi), \tag{1.18}$$

where $\{\psi_k^1\}_k$ is a positive family of Schwartz functions coming from a partition of unity on the collection of shifted dyadic intervals contained in $(0,\infty)$ each with length proportional to their distance from the origin and $\{\psi_k^2\}_k$ is similar for $(-\infty,0)$.

1.1.3 L^2 Estimates for B_m

In fact, estimates for B_m can be obtained as a direct consequence of Theorem 1.1.2.

Theorem 1.1.3. *Consider the operator B_m as defined in (1.17). Then, the following twisted mixed norm estimates hold,*

$$||B_m(f,g)||_{L^{r_1}_{x_1}(L^{r_2}_{x_2})} \leqslant ||f||_{L^{r_2}_{x_2}(L^{u}_{x_1})}||g||_{L^{r_1}_{x_1}(L^{u'}_{x_2})}$$

and

$$||B_m(f,g)||_{L^{r_2}_{x_2}(L^{r_1}_{x_1})} \leqslant ||f||_{L^{r_2}_{x_2}(L^{u}_{x_1})}||g||_{L^{r_1}_{x_1}(L^{u'}_{x_2})}$$

for $1 < r_i, u, u' \leqslant \infty$, *with* $\frac{1}{u} + \frac{1}{u'} = 1$. *Moreover, the only possible diagonal estimates are*

$$||B_m(f,g)||_{L^2(\mathbb{R}^2)} \lesssim ||f||_{L^2(\mathbb{R}^2)}||g||_{L^2(\mathbb{R}^2)}.$$

Proof. Using as intuition that

$$\frac{1}{\xi_1+\eta_2} = \text{p.v.}\ \frac{1}{2i\pi}\int_{\mathbb{R}} \text{sign}(\alpha)e^{-2\pi i x\alpha(\xi_1+\eta_2)}d\alpha. \tag{1.19}$$

we use (1.18) to rewrite

$$\begin{aligned}
B_m(f,g)(x_1,x_2) &= \sum_k \int_{\mathbb{R}^4} \psi_k(\xi_1+\eta_2)m(\xi,\eta)\widehat{f}(\xi)\widehat{g}(\eta)e^{2\pi i x\cdot(\xi+\eta)}d\xi d\eta \\
&= \sum_k \int_{\mathbb{R}^5} (\widehat{\psi}_k^1(\alpha)-\widehat{\psi}_k^2(\alpha))m(\xi,\eta)\widehat{f}(\xi)\widehat{g}(\eta)e^{-2\pi i\alpha(\xi_1+\eta_2)}e^{2\pi i x\cdot(\xi+\eta)}\ d\alpha d\xi d\eta \\
&= \sum_k \int_{\mathbb{R}^5} (\widehat{\psi}_k^1(\alpha)-\widehat{\psi}_k^2(\alpha))m(\xi,\eta)\widehat{f}(\xi)\widehat{g}(\eta)e^{-2\pi i\alpha(\xi_1+\eta_2)}e^{2\pi i x\cdot(\xi+\eta)}\ d\xi d\eta d\alpha \\
&= \sum_k \int_{\mathbb{R}} (\widehat{\psi}_k^1(\alpha)-\widehat{\psi}_k^2(\alpha))\Pi(\mathcal{T}_\alpha^1(f),\mathcal{T}_\alpha^2(g))(x_1,x_2)d\alpha. \qquad (1.20)
\end{aligned}$$

Where $\mathcal{T}_\alpha^1(f)(x_1,x_2) := f(x_1-\alpha,x_2)$ and $\mathcal{T}_\alpha^2(f)(x_1,x_2) := f(x_1,x_2-\alpha)$. The purpose of introducing $\{\psi_k\}_k$ into our definition of B_m rather than using (1.19) directly was to make the Fubini step above possible. Furthermore, write

$$B_m(f,g)(x_1,x_2) = \tag{1.21}$$

$$\sum_k \int_{\alpha>0} \widehat{\psi}_k^1(\alpha)\Pi(\mathcal{T}_\alpha^1(f),\mathcal{T}_\alpha^2(g))(x_1,x_2)d\alpha - \sum_k \int_{\alpha<0} \widehat{\psi}_k^2(\alpha)\Pi(\mathcal{T}_\alpha^1(f),\mathcal{T}_\alpha^2(g))(x_1,x_2)d\alpha, \tag{1.22}$$

Then,

$$|B_m(f,g)(x_1,x_2)| \lesssim \int_{\mathbb{R}} |\Pi(\mathcal{T}_\alpha^1(f),\mathcal{T}_\alpha^2(g))(x_1,x_2)|d\alpha. \tag{1.23}$$

By Theorem 1.1.2

$$||B_m(f,g)||_{L^{r_1}_{x_1}(L^{r_2}_{x_2})} \lesssim ||\Pi_m(\mathcal{T}^1_\alpha(f),\mathcal{T}^2_\alpha(g))||_{L^{r_1}_{x_1}(L^{r_2}_{x_2}(L^1_\alpha))} \tag{1.24}$$

$$\lesssim ||\mathcal{T}^1_\alpha(f)||_{L^{p_1}_{x_1}(L^{p_2}_{x_2}(L^u_\alpha))}||\mathcal{T}^2_\alpha(g))||_{L^{q_1}_{x_1}(L^{q_2}_{x_2}(L^{u'}_\alpha))} \tag{1.25}$$

$$= ||f||_{L^{r_2}_{x_2}(L^u_{x_1})}||g||_{L^{r_1}_{x_1}(L^{u'}_{x_2})} \tag{1.26}$$

In (1.26), we are forced to take $p_1 = q_2 = \infty$ which in turn implies that $p_2 = r_2$ and $q_1 = r_1$. The second set of estimates is proved similarly.

The above computations actually show that one can interchange the sum and integral in (1.20) to obtain the equality

$$B_m(f,g)(x_1,x_2) = \int_{\mathbb{R}} \text{sign}(\alpha)\Pi(\mathcal{T}^1_\alpha(f),\mathcal{T}^2_\alpha(g))(x_1,x_2)d\alpha. \tag{1.27}$$

Moreover, notice that in particular, the only diagonal estimates resulting from this approach come from taking $r_1 = r_2 = u = u' = 2$. Hence, we obtain

$$||B_m(f,g)||_{L^2(\mathbb{R}^2)} \lesssim ||f||_{L^2(\mathbb{R}^2)}||g||_{L^2(\mathbb{R}^2)}.$$

In fact, by taking the trivial case of $m \equiv 1$ and applying the Brasscamp-Lieb inequality in [3] it follows that these are the only possible diagonal estimates.

$\square$

1.2 Mixed Norm Estimates for Operators of Type B_m

In the spirit of the discussion at the end of Section 1.1.2, we can pose a similar question about more general operators of type B_m, that is

$$B_m^{(n,k)}(f,g)(x) := \int_{\mathbb{R}^4} \frac{m(\xi,\eta)}{\xi_1^n + \eta_2^k} \widehat{f}(\xi)\widehat{g}(\eta) e^{2\pi i x(\xi+\eta)} d\xi d\eta, \tag{1.28}$$

defined in the same limiting sense as in (1.17). A remarkable property of this class of operators is that they can be expressed in terms of convolutions with generalized Airy functions. It is often the case that these convolutions are solutions to well known partial differential equations such as the linear Schrödinger or the KdV equation.

1.2.1 Airy Function and Properties

Before proceeding with the reformulation of operators $B_m^{(n,k)}$ we segue into a brief discussion of the *Airy function* which appears naturally in the study of the linear Korteweg-de Vries (KdV) equation. The contents of this section are discussed in greater detail in chapter 1 of [17], where there is also an analysis of the dispersive properties of a certain generalized nonlinear KdV equation.

To this end, let us consider the linear KdV equation on the real line given by

$$\begin{cases} \partial_t u + \partial_x^3 u = 0, \\ u(0) = g. \end{cases}$$

In this setting the solution can be calculated explicitly. Taking the Fourier transform with respect to the x variable of the first equation, we obtain

$$\partial_t \widehat{u}(t,\xi) = 8\pi^3 i\xi^3 \widehat{u}(t,\xi)$$

while the second equation gives

$$\widehat{u}(0,\xi) = \widehat{g}(\xi).$$

By combining these two we obtain

$$\widehat{u}(t,\xi) = \widehat{g}(\xi)e^{8\pi^3 it\xi^3}.$$

or, equivalently, $u(t,\xi) = (g * K_t)(x)$ where

$$\widehat{K}_t(\xi) = e^{8\pi^3 it\xi^3}.$$

By a scaling argument, it is easy to observe that

$$K_t(x) = \frac{1}{(4\pi^2 t)^{1/3}} Ai\left(\frac{x}{(4\pi^2 t)^{1/3}}\right)$$

where Ai, is the *Airy function* defined by the oscillatory integral

$$Ai(x) := \int_{\mathbb{R}} e^{2\pi i\eta^3} e^{2\pi i\eta x} d\eta. \tag{1.29}$$

In particular, one has $\widehat{Ai} = e^{2\pi i\xi^3}$. The functions K_t are defined a priori as distributions, but they are in fact functions, whose asymptotic behavior can be studied in detail.

Lemma 1.2.1 ([17]). *The Airy function has the following properties:*

(i) *Ai(x) is a bounded function and is* $O(|x|^{-1/4})$ *as* $|x| \to \infty$;

(ii) $D^\alpha(Ai)$ *is bounded for any* $\alpha \in [0, \frac{1}{2}]$.

Note that Lemma 1.2.1 implies that $Ai(x) \in L^r(\mathbb{R})$ for $r > 4$. Moreover, using this lemma one can prove the following dispersive estimates for the solutions of the linear KdV.

Lemma 1.2.2 ([17]). *Let* $g \in L^1$. *Then,*

(i) $||g * K_t|| \lesssim t^{-1/3}||g||_1$

(ii) $||g * K_t||_p \lesssim t^{(-1/3)(1-1/p)}||g||_1$ *for every* $p > 4$.

1.2.2 Reformulation in terms of Generalized Airy Functions

The Airy function discussed in the previous section is helpful in the understanding of $B_m^{(1,3)}(f,g)$, and there is an analogous correlation between $B^{(n,k)}(f,g)$ and more general Airy functions. For $k \geqslant 2$, define the generalized Airy function

$$Ai_k^{\alpha}(x) := \int_{\mathbb{R}} e^{2\pi i \alpha \eta^k} e^{2\pi i \eta x} d\eta = \frac{1}{\alpha^{1/k}} Ai_k^1\left(\frac{x}{\alpha^{1/k}}\right). \tag{1.30}$$

Then, recalling that

$$\frac{1}{\xi_1^n + \eta_2^k} = \text{p.v.}\ \frac{1}{2i\pi} \int_{\mathbb{R}} \text{sign}(\alpha) e^{-2\pi i x \alpha(\xi_1^n + \eta_2^k)} d\alpha, \tag{1.31}$$

and proceeding exactly as in the proof of Theorem 1.1.3, we note that the operators $B_m^{(n,k)}(f,g)$ fall into three main categories:

1. If $n = k = 1$, then

$$B_m^{(1,1)}(f,g)(x_1,x_2) = \int_{\mathbb{R}} \text{sign}(\alpha) \Pi_m(\mathcal{T}_\alpha^1(f), \mathcal{T}_\alpha^2(g))(x_1,x_2) d\alpha$$

2. If $n = 1$, then

$$B_m^{(1,k)}(f,g)(x_1,x_2) = \int_{\mathbb{R}} \text{sign}(\alpha) \Pi_m(\mathcal{T}_\alpha^1(f), g *_2 Ai_k^{\alpha})(x_1,x_2) d\alpha$$

3. Otherwise,

$$B_m^{(n,k)}(f,g)(x_1,x_2) = \int_{\mathbb{R}} \text{sign}(\alpha) \Pi_m(f *_1 Ai_n^{\alpha}, g *_2 Ai_k^{\alpha})(x_1,x_2) d\alpha.$$

Here $f *_1 Ai_n^{\alpha}(x_1,x_2)$ and $g *_2 Ai_k^{\alpha}(x_1,x_2)$ represent convolution in the first and second variable respectively. The first category was the topic of Theorem 1.1.3. Estimates for certain operators in the second category can be proven through the application of Strichartz estimates and the Hardy-Littlewood-Sobolev inequality, which is the topic of sections 1.2.3 and 2.2.1. Little is understood about the operators in the third category. While an approach using Strichartz estimates seems promising at first glance, the restricted range of available estimates falls short for these operators. This category will not be discussed further in this work, and is left as an open question. One fascinating feature of operators in category three is their reminiscence of bilinear restriction, if we take for example $n = k = 2$ and $m(\xi, \eta) = 1$.

1.2.3 Proof via Strichartz Estimates

In this section we combine the well known Strichartz estimates for the linear Schödinger and KdV equations together with Theorem 1.1.2 to obtain mixed norm estimates for $B_m^{1,2}$ and $B_m^{1,3}$ in restricted ranges.

Theorem 1.2.3. *(Strichartz Estimates for Schödinger)*

We have

$$||Ai_2^{\alpha} * f(x)||_{L_\alpha^p L_x^q(\mathbb{R}^{d+1})} \lesssim ||f||_{L^2(\mathbb{R}^d)}$$

with

$$\frac{2}{p} + \frac{d}{q} = \frac{d}{2}$$

and $2 \leqslant p \leqslant \infty, \ \ (p,q) \neq (2,\infty)$.

Theorem 1.2.4. *(Strichartz Estimates for KdV)*

We have

$$||Ai_3^{\alpha} * f(x)||_{L_\alpha^p L_x^q(\mathbb{R}^2)} \lesssim ||f||_{L^2(\mathbb{R})} \tag{1.32}$$

with

$$\frac{3}{p} + \frac{1}{q} = \frac{1}{2}$$

and $6 \leqslant p \leqslant \infty, \ \ 2 \leqslant q \leqslant \infty.$

Theorem 1.2.5. *Consider the family of operators* $B_m^{(n,k)}$ *defined in (1.28). The following twisted mixed norm estimates hold.*

i) $||B_m^{(1,2)}(f,g)||_{L^{r_1}_{x_1}L^{r_2}_{x_2}} \lesssim ||f||_{L^{p_2}_{x_2}L^{p_1}_{x_1}}||g||_{L^{r_1}_{x_1}L^2_{x_2}}$ *for* $1 < p_2 \leqslant \infty, 0 < r_2 < \infty, 6/5 \leqslant p_1 \leqslant 4/3$, $\frac{1}{p_2} + \frac{2}{p_1} = \frac{1}{r_2} + \frac{3}{2}$.

ii) $||B_m^{(1,3)}(f,g)||_{L^{r_1}_{x_1}L^{r_2}_{x_2}} \lesssim ||f||_{L^{p_2}_{x_2}L^{p_1}_{x_1}}||g||_{L^{r_1}_{x_1}L^2_{x_2}}$ *for* $1 < p_2 \leqslant \infty$, $0 < r_2 < \infty$, $8/7 \leqslant p_1 \leqslant 6/5$, *and* $\frac{1}{p_2} + \frac{3}{p_1} = \frac{1}{r_2} + \frac{5}{2}$.

Proof. First, note that for $k \in \{2, 3\}$ one can justify the equality

$$B_m^{1,k}(f,g)(x_1,x_2) = \int_{\mathbb{R}} \operatorname{sign}(\alpha)\Pi_m(\mathcal{T}^1_\alpha f, g *_2 Ai_k^\alpha)(x_1,x_2)\, d\alpha$$

exactly as in the proof of Theorem 1.1.3. Then, by Theorem 1.1.2 we have

$$\begin{aligned} ||B_m^{(1,k)}(f,g)||_{L^{r_1}_{x_1}L^{r_2}_{x_2}} &\lesssim ||\Pi_m(\mathcal{T}^1_\alpha f, g *_2 Ai_k^\alpha)||_{L^{r_1}_{x_1}L^{r_2}_{x_2}L^1_\alpha} \\ &\lesssim ||\mathcal{T}^1_\alpha f||_{L^{\widetilde{p_1}}_{x_1}L^{p_2}_{x_2}L^{p_1}_\alpha}||g *_2 Ai_k^\alpha||_{L^{q_1}_{x_1}L^{q_2}_{x_2}L^{p_1'}_\alpha} &(1.33) \\ &\lesssim ||f||_{L^{p_2}_{x_2}L^{p_1}_{x_1}}||g *_2 Ai_k^\alpha||_{L^{r_1}_{x_1}L^{q_2}_{x_2}L^{p_1'}_\alpha} &(1.34) \end{aligned}$$

for $0 < r_1, r_2 < \infty$, $1 < \widetilde{p_1}, p_2, p_1 q_1, q_2, p_1' \leqslant \infty$ with

$$\frac{1}{r_1} = \frac{1}{\widetilde{p_1}} + \frac{1}{q_1}, \qquad \frac{1}{r_2} = \frac{1}{p_2} + \frac{1}{q_2}, \qquad \frac{1}{p_1} + \frac{1}{p_1'} = 1. \tag{1.35}$$

If in addition we assume that $q_2 \geqslant p_1'$, then we can interchange the order of the norms in order to apply the corresponding Strichartz estimates. Namely, for $k = 2$ we have

$$||g *_2 Ai_2^\alpha||_{L^{r_1}_{x_1}L^{q_2}_{x_2}L^{p_1'}_\alpha} \lesssim ||g *_2 Ai_2^\alpha||_{L^{r_1}_{x_1}L^{p_1'}_\alpha L^{q_2}_{x_2}}$$

$$\lesssim ||g||_{L^{r_1}_{x_1}L^2_{x_2}}, \tag{1.36}$$

with $2 \leqslant p_1' < \infty$ and $\frac{2}{p_1'} + \frac{1}{q_2} = \frac{1}{2}$. This, combined with the requirement that $q_2 \leqslant p_1'$, impose the additional restriction that $4 \leqslant p_1' \leqslant 6$. Finally, combining with (1.35) we obtain the estimates in the desired ranges and exponent relation. The case $k = 3$ is proved similarly, using Strichartz estimates for the KdV equation.

□

Remark 1.2.6. *Notice that due to the range restrictions the above theorem does not imply diagonal estimates for either of the two cases.*

Theorem 1.1.3 and Theorem 1.2.5 summarize our initial observations when beginning the study of operators $B_m^{1,k}$. In fact, these are the only known results regarding operators $B_m^{1,k}$ that allow for insertion of an arbitrary Coifmann-Meyer symbol m. It quickly became evident that even the case of trivial m poses an interesting equation. This leads us to the next section which contains the formal definition of Fourier integral operators with rational symbol.

1.3 General Definition and Specific Cases

We formally define a *Fourier integral operator with rational symbol* as a multilinear operator of the form

$$\begin{aligned}&\mathcal{R}_P(f_1, f_2 \ldots, f_m)(x)\\ &\qquad = \int_{\mathbb{R}^{nm}} \frac{1}{P(\xi_1, \xi_2, \ldots, \xi_m)} \widehat{f_1}(\xi_1)\widehat{f_2}(\xi_2)\ldots\widehat{f_m}(\xi_m) e^{2\pi i(x\cdot(\xi_1+\xi_2+\cdots+\xi_m))} d\xi_1 \ldots d\xi_m,\end{aligned} \tag{1.37}$$

where $\widehat{f_i} \in \mathcal{S}(\mathbb{R}^n)$, $x \in \mathbb{R}^n$, $\xi_i := (\xi_{i,1}, \xi_{i,2}, \ldots, \xi_{i,n}) \in \mathbb{R}^n$, and P is a polynomial in nm variables. We understand the integral in the definition of $\mathcal{R}_P$ in the limiting sense of (1.17). That is,

$$\mathcal{R}_P(f_1, f_2 \ldots, f_m)(x) \\ = \sum_k \int_{\mathbb{R}^{nm}} \psi_k(P(\xi_1, \xi_2, \ldots, \xi_m)) \widehat{f_1}(\xi_1) \widehat{f_2}(\xi_2) \ldots \widehat{f_m}(\xi_m) e^{2\pi i (x \cdot (\xi_1 + \xi_2 + \cdots + \xi_m))} d\xi_1 \ldots d\xi_m, \tag{1.38}$$

where $\{\psi_k\}_k$ is as in (1.18). Note that if $P \equiv 1$, then

$$\mathcal{R}_P(f_1, \ldots, f_m)(x) = f_1 \cdot f_2 \cdots f_m(x).$$

If P has degree one, then $\mathcal{R}_P$ can be written in terms of translations, and estimates can be obtained using Hölder inequalities.

This definition encompasses an extremely wide variety of operators. It is easy to see that in the case $m \equiv 1$ the operators $B_m^{(n,k)}$ form a subset of those defined in (1.37). For certain cases, such as $P((\xi_{1,1}, \xi_{1,2}), (\xi_{2,1}, \xi_{2,2})) = \xi_{1,1} + \xi_{2,2}^3$, imposing the restriction $m \equiv 1$ allows us to take advantage of Lemma 1.2.1 together with the Hardy-Littlewood-Sobolev inequality in a way that is not compatible when an arbitrary paraproduct is present. We prove a full range of L^p estimates for this toy example in Section 2.2.1.

It turns out that the application of fractional integral operators in the study of operators $\mathcal{R}_P$ does not end there. In fact, in section 2.2.2 we discuss a second example of

$$P_1(\xi_{1,1}, \xi_{1,2}, \xi_3) = \xi_{1,1}\xi_{2,2} + \xi_{3,3}$$

where we prove mixed norm estiamtes via a careful concatenation of Hörmander's Oscillatory theorem and the Hardy-Littlewood-Sobolev inequality. As a matter of fact, through the application of more advanced fractional integral operator estimates we are able to prove estimates for more intricate choices of P.

That is, in Chapter 7 we apply the bilinear fractional integral estimates proven in [12] and independently in [7] to prove mixed norm estimates for $\mathcal{R}_P$ with

$$P_2(\xi_1, \xi_2, \xi_3, \xi_4) = \xi_{1,1}\xi_{2,2} + (\xi_{3,3} - \xi_{4,3})$$

for $(\xi_{i,1}, \xi_{i,2}, \xi_{i,3}) \in \mathbb{R}^3$.

In Chapters 3 - 6 we discuss and prove estimates for a family of trilinear iterated fractional integral operators and in Chapter 7 we apply these results to prove mixed norm estimates for operator $\mathcal{R}_P$ with

$$P_3(\xi_1, \xi_2, \ldots, \xi_7) = (\xi_{1,1}\xi_{2,2} + \xi_{5,3} - \xi_{6,3})(\xi_{3,3}\xi_{4,3} + \xi_{5,3} - \xi_{7,3}),$$

where $(\xi_{i,1}, \xi_{i,2}, \xi_{i,3}) \in \mathbb{R}^3$.

CHAPTER 2

GENERALIZED FRACTIONAL INTEGRAL OPERATORS

In this chapter we recall some well known fractional integral operator estimates, and we introduce a trilinear iterated fractional integral operator whose estimates are proven in subsequent chapters.

2.1 An Overview

It appears remarkable that a variety of fractional integral operators wind up being a fundamental tool in understanding certain operators of type $\mathcal{R}_P$. Yet the study of fractional integral operators has been of interest to mathematicians for over a century. A prime example is the well known Hardy-Littlewood Sobolev inequality, which was the key ingredient of Sergei Sobolev's original proof of the Sobolev embedding theorem.

Theorem 2.1.1. *(Hardy-Littlewood-Sobolev Inequality, [10]) Consider the Kernel* $K_\alpha(x) := |x|^{-\alpha}$ *and convolution operator given by* $I_\alpha(f) := f * K_\alpha$*. If* $p > 1$ *and* $\alpha = n\left(1 - \frac{1}{p} + \frac{1}{q}\right)$*, then* $||I_\alpha f||_{L^q(\mathbb{R}^n)} \lesssim ||f||_{L^p(\mathbb{R}^n)}$.

In the early 2000's, estimates for a bilinear fractional integral operator were proven by Kenig and Stein in [12] and independently by Grafakos and Kalton in [7]. We state a truncated version of their result.

Theorem 2.1.2. *([12], [7]) Assume that* $0 < \alpha < n$*,* $1 \leqslant p_i \leqslant \infty$*,* $0 < r < \infty$*,* $\frac{1}{p_1} + \frac{1}{p_2} > \frac{\alpha}{n}$*, and define*

$$B_\alpha(f,g) := \int_{\mathbb{R}^n} \frac{f(x-t)g(x+t)}{|t|^{n-\alpha}} dt. \tag{2.1}$$

Then, for

$$\frac{1}{p_1} + \frac{1}{p_2} - \frac{\alpha}{n} = \frac{1}{r}, \tag{2.2}$$

we have

i) *If* $1 < p_i$, $i = 1, 2$,

$$||B_\alpha(f,g)||_{L^r(\mathbb{R}^n)} \lesssim ||f||_{L^{p_1}(\mathbb{R}^n)}||g||_{L^{p_2}(\mathbb{R}^n)}.$$

ii) *If* $1 \leqslant p_i$, $i = 1, 2$, *and either* p_1 *or* p_2 *is one,*

$$||B_\alpha(f,g)||_{L^{r,\infty}(\mathbb{R}^n)} \lesssim ||f||_{L^{p_1}(\mathbb{R}^n)}||g||_{L^{p_2}(\mathbb{R}^n)}.$$

Variants of operator (2.1) were also studied in other works such as [13], [14], and [8]. In our work we introduce the interated trilinear fractional integral operator

$$T_{\alpha,\beta}(f,g,h)(x) = \int_{\mathbb{R}^2} \frac{f(x-t)g(x+t+s)h(x-s)}{|t|^\alpha \cdot |s|^\beta}\, dtds, \tag{2.3}$$

where $f, g, h \in \mathcal{S}(\mathbb{R}), x \in \mathbb{R}$, and $\alpha, \beta \in (0,1)$, and for which we prove a full range of strong type L^p estimates.

A worthwhile observation is that if we take $n = 1$ and replace the kernel $|x|^{-\alpha}$ with $p.v.\frac{1}{x}$ in Theorem 2.1.1, then $I(f)$ becomes the classical Hilbert transform. Hence, the Hilbert transform can be seen as the limiting case of the fractional integral operators I_α as α approaches one.

Analogous to this observation, taking $n = 1$ and replacing $|t|^{-\alpha}$ with $\frac{1}{t}$ in Theorem 2.1, we see that in the limiting case as α approaches 1, B_α approaches the bilinear Hilbert transform (BHT) which is consistent with the exponent relation in (2.2). However, for operator B_α the borderline estimates below $r = 2/3$ are reached whereas estimates for $1/2 < r < 2/3$ are not known for the BHT. The condition that $r > 2/3$ in the case of the BHT is a consequence of the method of proof which is highly nontrivial and requires careful time-frequency analysis techniques. More information on the BHT operator itself, can be found but is not limited to the work in [9], [15], [18], [19], and [20].

Surely, as in the two cases of the Hilbert transform and the BHT, the iterated fractional integral operator $T_{\alpha,\beta}$ follows a similar tale. In this case we note that when replacing both $|t|^{-\alpha}$ and $|s|^{-\beta}$ with t^{-1} and s^{-1} one obtains the Biest operator studied in [16]. Hence, the Biest operator can be seen as the limiting case of the fractional Biest operator as α, β approach one. The estimates in our results regarding $T_{\alpha,\beta}$ are precisely stated in Theorem 3.1.1 and are consistent in the limiting case with those obtained in [16]. However our results for $T_{\alpha,\beta}$ hold arbitrarily close to the end point $r = 1/3$ whereas this sharp range is not known yet for the Biest operator.

Due to the observations mentioned above, we may sometimes informally refer to operators B_α and $T_{\alpha,\beta}$ as the fractional BHT and the fractional Biest, respectively.

2.2 Motivating Examples

2.2.1 Wide Range of Estimates for $\mathcal{R}_{\xi_{1,1}+\xi_{2,2}^3}$ via H-L-S Inequality

Now we will consider our first motivational example.

Theorem 2.2.1. *Let* $1 < p_i, q_i, r_i < \infty$, *and*

$$\mathcal{R}_{\xi_{1,1}+\eta_{2,2}^3}(f,g) = \int_{\mathbb{R}^4} \frac{1}{\xi_{1,1}+\xi_{2,2}^3} \hat{f}(\xi_1)\hat{g}(\xi_2) e^{2\pi i (x\cdot(\xi_1+\xi_2))}\, d\xi_1 d\xi_2, \tag{2.4}$$

defined in the limiting sense of (1.17), for $x = (x_1, x_2)$, $\xi_1 = (\xi_{1,1}, \xi_{1,2})$, $\xi_2 = (\xi_{2,1}, \xi_{2,2}) \in \mathbb{R}^2$ *and* $f, g \in \mathcal{S}(\mathbb{R}^2)$. *Then,*

$$||\mathcal{R}_{\xi_{1,1}+\xi_{2,2}^3}(f,g)||_{L^{r_1}_{x_1} L^{r_2}_{x_2}} \lesssim ||f||_{L^{p_1}_{x_1} L^{p_2}_{x_2}} ||g||_{L^{q_1}_{x_1} L^{q_2}_{x_2}} \tag{2.5}$$

hold for

$$3\left(\frac{1}{p_1} + \frac{1}{q_1} - \frac{1}{r_1}\right) + \left(\frac{1}{p_2} + \frac{1}{q_2} - \frac{1}{r_2}\right) = 3$$

with $\left(\frac{1}{p_1} + \frac{1}{q_1} - \frac{1}{r_1}\right) \leqslant 3/4$ *and* $\left(\frac{1}{p_2} + \frac{1}{q_2} - \frac{1}{r_2}\right) \geqslant 3/4$.

Proof. As in (1.20), we have

$$\mathcal{R}_{\xi_{1,1}+\xi_{2,2}^3}(f,g)(x_1,x_2) = \sum_k \int_{\mathbb{R}^5} \psi_k(\alpha)\hat{f}(\xi_1)\hat{g}(\xi_2) e^{2\pi i [\xi_1\cdot(x_1+\alpha, x_2) + \xi_2\cdot x + \alpha\eta_2^3]}\, d\xi_1 d\xi_2 d\alpha.$$
$$= \int_{\alpha>0} \psi_k^1(\alpha)\mathcal{T}_\alpha^1(f)(x_1,x_2) g *_2 Ai_3^\alpha(x_1,x_2) d\alpha - \int_{\alpha<0} \psi_k^2(\alpha)\mathcal{T}_\alpha^1(f)(x_1,x_2) g *_2 Ai_3^\alpha(x_1,x_2) d\alpha. \tag{2.6}$$

So,

$$|\mathcal{R}_{\xi_{1,1}+\xi_{2,2}^3}(f,g)(x_1,x_2)| \lesssim \int_{\mathbb{R}} |\mathcal{T}_\alpha^1(f)(x_1,x_2) g *_2 Ai_3^\alpha(x_1,x_2)|\, d\alpha.$$

Therefore,

$$\begin{aligned}
&||\mathcal{R}_{\xi_1+\eta_2^3}(f,g)||_{L^{r_1}_{x_1}(L^{r_2}_{x_2})} \\
&\quad\lesssim \left(\int_{\mathbb{R}} \left(\int_{\mathbb{R}} \left(\int_{\mathbb{R}} |\mathcal{T}_\alpha^1(f)(x_1,x_2) g *_2 Ai_3^\alpha(x_1,x_2)| d\alpha\right)^{r_2} dx_2\right)^{\frac{r_1}{r_2}} dx_1\right)^{\frac{1}{r_1}} \\
&\quad\leqslant \left(\int_{\mathbb{R}} \left(\int_{\mathbb{R}} \left(\int_{\mathbb{R}} |\mathcal{T}_\alpha^1(f)(x_1,x_2) g *_2 Ai_3^\alpha(x_1,x_2)|^{r_2} dx_2\right)^{\frac{1}{r_2}} d\alpha\right)^{r_1} dx_1\right)^{\frac{1}{r_1}} \\
&\quad\leqslant \left(\int_{\mathbb{R}} \left(\int_{\mathbb{R}} ||\mathcal{T}_\alpha^1(f)(x_1,\cdot)||_{L^{p_2}_{x_2}} ||g *_2 Ai_3^\alpha(x_1,\cdot)||_{L^{\widetilde{p_2}}_{x_2}} d\alpha\right)^{r_1} dx_1\right)^{\frac{1}{r_1}}
\end{aligned} \tag{2.7}$$

where

$$\frac{1}{p_2} + \frac{1}{\widetilde{p}_2} = \frac{1}{r_2}. \tag{2.8}$$

Moreover, by part (i) of Lemma 1.2.1 we observe that

$$\begin{aligned}
||Ai_3^{\alpha}(\cdot) *_{x_2} g(x_1,\cdot)||_{L_{x_2}^{\tilde{p}_2}} &= \frac{1}{\alpha^{1/3}}||Ai_3^1(\cdot/\alpha^{1/3}) *_{x_2} g(x_1,\cdot)||_{L_{x_2}^{\tilde{p}_2}} \\
&\leqslant \frac{1}{\alpha^{1/3(1-1/\tilde{r})}}||Ai_3^1(\cdot)||_{L^{\tilde{r}}}||g(x_1,\cdot)||_{L_{x_2}^{q_2}} \\
&\lesssim \frac{1}{\alpha^{1/3(1-1/\tilde{r})}}||g(x_1,\cdot)||_{L_{x_2}^{q_2}} \qquad (2.9)
\end{aligned}$$

for

$$\frac{1}{q_2} + \frac{1}{\widetilde{r}} = \frac{1}{\widetilde{p}_2} + 1 \tag{2.10}$$

and $\tilde{r} > 4$. Define

$$F(x_1) := ||f(x_1,\cdot)||_{L_{x_2}^{p_2}} \text{ and } G(x_1) := ||g(x_1,\cdot)||_{L_{x_2}^{q_2}},$$

then combining equations (2.7) and (2.9),

$$\begin{aligned}
||\mathcal{R}_{\xi_1+\eta_2^3}(f,g)||_{L_{x_1}^{r_1}(L_{x_2}^{r_2})} &\lesssim \left(\int_{\mathbb{R}}\left(G(x_1)\int_{\mathbb{R}}\mathcal{T}_{\alpha}^1 F(x_1)\frac{1}{\alpha^{1/3(1-1/\tilde{r})}}d\alpha\right)^{r_1}dx_1\right)^{\frac{1}{r_1}} \\
&= ||G\cdot I_{\sigma}(F)||_{L_{x_1}^{r_1}} \\
&\leqslant ||G||_{L_{x_1}^{q_2}}\cdot||I_{\sigma}(F)||_{L_{x_1}^{\tilde{q}_1}} \\
&\lesssim ||G||_{L_{x_1}^{q_2}}||F||_{L_{x_1}^{p_1}} \qquad (2.11) \\
&= ||g||_{L_{x_1}^{q_1}(L_{x_2}^{q_2})}||f||_{L_{x_1}^{p_1}(L_{x_2}^{p_2})} \qquad (2.12)
\end{aligned}$$

where I_{σ} is as in Theorem 2.1.1 with

$$\sigma = \frac{1}{3}\left(1 - \frac{1}{\widetilde{r}}\right), \tag{2.13}$$

$$\frac{1}{q_2} + \frac{1}{\tilde{q}_1} = \frac{1}{r_1}, \tag{2.14}$$

and

$$\frac{1}{p_1} = 1 - \left(\frac{1}{3}\left(1 - \frac{1}{\tilde{r}}\right)\right) + \frac{1}{\tilde{q}_1}. \tag{2.15}$$

Combining equations (2.14) and (2.15) implies

$$3\left(\frac{1}{p_1} + \frac{1}{q_1} - \frac{1}{r_1}\right) - 2 = \frac{1}{\tilde{r}} \tag{2.16}$$

Similarly, combining equations (2.8) and (2.10) implies

$$\frac{1}{p_2} + \frac{1}{q_2} - \frac{1}{r_2} - 1 = -\frac{1}{\tilde{r}} \tag{2.17}$$

Finally, adding equations (2.16) and (2.17) and recalling that $\tilde{r} > 4$ gives us the final constraint

$$3\left(\frac{1}{p_1} + \frac{1}{q_1} - \frac{1}{r_1}\right) + \left(\frac{1}{p_2} + \frac{1}{q_2} - \frac{1}{r_2}\right) = 3$$

with

$$\left(\frac{1}{p_1} + \frac{1}{q_1} - \frac{1}{r_1}\right) < 3/4 \text{ and } \left(\frac{1}{p_2} + \frac{1}{q_2} - \frac{1}{r_2}\right) > 3/4$$

as needed. In order to obtain equality in the above constraints, which is needed for diagonal estimates, we follow a similar approach,

$$||\mathcal{R}_{\xi_1+\eta_2^3}(f,g)||_{L^{r_1}_{x_1}L^{r_2}_{x_2}} \leqslant ||\mathcal{T}_1^{\alpha}(f)\cdot Ai_3^{\alpha} *_2 g||_{L^{r_1}_{x_1}L^{r_2}_{x_2}L^1_\alpha} \tag{2.18}$$

$$\leqslant ||Ai_3^{\alpha} *_2 g||_{L^{q_1}_{x_1}L^{\tilde{q}_2}_{x_2}L^\infty_\alpha} ||\mathcal{T}_1^{\alpha}(f)||_{L^{\tilde{p}_1}_{x_1}L^{p_2}_{x_2}L^1_\alpha} \tag{2.19}$$

$$\leqslant |||\alpha|^{\sigma}Ai_3^{\alpha} *_2 g||_{L^{q_1}_{x_1}L^{\tilde{q}_2}_{x_2}L^\infty_\alpha} \left|\left|\frac{1}{|\alpha|^{\sigma}}\mathcal{T}_1^{\alpha}(f)\right|\right|_{L^{\tilde{p}_1}_{x_1}L^1_\alpha L^{p_2}_{x_2}}, \tag{2.20}$$

for $\frac{1}{\tilde{p}_1}+\frac{1}{q_1}=\frac{1}{r_1}$ and $\frac{1}{q_1}+\frac{1}{\tilde{q}_2}=\frac{1}{r_2}$, and $0<\sigma<1$ to be determined. As before,

$$\left|\left|\frac{1}{|\alpha|^{\sigma}}\mathcal{T}_1^{\alpha}(f)\right|\right|_{L^{\tilde{p}_1}_{x_1}L^1_\alpha L^{p_2}_{x_2}} \lesssim ||f||_{L^{p_1}_{x_1}L^{p_2}_{x_2}}, \tag{2.21}$$

for $\sigma+\frac{1}{p_1}=\frac{1}{\tilde{p}_1}+1$. This time, we use part (i) of Lemma 1.2.1 to obtain a fractional integral of g in the x_2 variable. That is, since $Ai_3^{\alpha}(x)=\frac{1}{\alpha^{1/3}}Ai_3^1(\frac{x}{\alpha^{1/3}})\lesssim\frac{1}{|\alpha|^{1/4}}\cdot\frac{1}{|x|^{1/4}}$, for $\sigma=1/4$, we have

$$|||\alpha|^{1/4}Ai_3^{\alpha} *_2 g||_{L^{q_1}_{x_1}L^{\tilde{q}_2}_{x_2}L^\infty_\alpha} \lesssim ||I^1_{1/4}(g)||_{L^{q_1}_{x_1}L^{\tilde{q}_2}_{x_2}} \lesssim ||g||_{L^{q_1}_{x_1}L^{q_2}_{x_2}}, \tag{2.22}$$

where $I^1_{1/4}(g)$ denotes fractional integration in the first variable of g, and
$\frac{1}{4}+\frac{1}{q_2}=\frac{1}{r_2}+1$. This completes the proof for the final case of $\frac{1}{p_1}+\frac{1}{q_1}-\frac{1}{r_1}=\frac{3}{4}$ and $\frac{1}{p_2}+\frac{1}{q_2}-\frac{1}{r_2}=\frac{3}{4}$. Notice that the proof for this case holds for $0<r_1,r_2<\infty$, and well as the diagonal case of $p_1=p_2, q_1=q_2$, and $r_1=r_2$.

□

In the preceding examples it became evident that Fourier integral operator with rational symbol of the form $\mathcal{R}_{\xi_{1,1}+\xi^k_{2,2}}$ relies heavily on oscillatory integral estimates for operators (1.30). Mixed norm estimates can be obtained by a concatenation of Hölder inequalities and Strichartz estimates for $g *_2 Ai_k^{\alpha}(x_1,x_2)$. If sufficient dispersion is present, as in the case $k=3$, the full range of diagonal estimates can be obtained as seen in Theorem 2.2.1. However, there are a variety of examples of operators $\mathcal{R}_P$ whose estimates cannot be trivially reduced to the three categories presented in Section 1.2.2. In particular, some choices of P require advanced fractional integral estimates.

2.2.2 Mixed Norm Estimates for $\mathcal{R}_{\xi_{1,1}\xi_{2,2}+\xi_{3,3}}$ via Oscillatory Integral Operator Estimates

Next, we present a second motivational example displaying the interplay between fractional integral operators, oscillatory integrals, and $\mathcal{R}_{\xi_{1,1}\xi_{2,2}+\xi_{3,3}}$. Our analyses of operators $\mathcal{R}_P$ always begin with rewriting the operators using the intuition of equation (1.19). Therefore, it is no surprise that oscillatory integrals continue to play a crucial role in our analyses. The specific case of $\mathcal{R}_{\xi_{1,1}\xi_{2,2}+\xi_{3,3}}$ is insightful as it cannot be rewritten as a vector valued paraproduct type operator as was the case in the previously examined choices for P. In the cases that we consider, Hörmander's oscillatory integral theorem proves to be of significant relevance, and hence we state it below.

Theorem 2.2.2. *(Hörmander's Oscillatory Integral Theorem, [11]) Assume $S(x, y)$ is a two variable smooth function defined in a neighborhood of the origin and $\phi(x, y)$ is a smooth cut off function. If*

$$\left|\frac{\partial^2 S}{\partial_x \partial_y}\right| \geqslant 1, \quad for\ all\ \ (x, y) \in supp(\phi),$$

then there is a constant $C > 0$ independent of λ, such that

$$|\Lambda_2(f,g)| = \left|\int\int e^{2\pi i\lambda S(x,y)} f(y)g(x)\phi(x,y)\ dxdy\right| \leqslant C\min(1,|\lambda|^{-1/2})||f||_2||g||_2.$$

In the special case of $S(x,y) = xy$ we can omit the cut off function ϕ and still obtain the following weaker estimate

$$\left|\int\int e^{2\pi i\lambda S(x,y)} f(y)g(x)\ dxdy\right| \leqslant C|\lambda|^{-1/2}||f||_2||g||_2. \tag{2.23}$$

To see this, write $e^{2\pi i\lambda xy} \sim \sum_{(n,m)\in\mathbb{Z}^2} \varphi_n(\sqrt{\lambda}x)e^{2\pi i\sqrt{\lambda}mx}\varphi_m(\sqrt{\lambda}x)e^{2\pi i\sqrt{\lambda}ny}$ where $\varphi_n(x)$ and $\varphi_m(y)$ are L^∞ normalized bump functions with the property that $\mathrm{supp}(\varphi_n) \subseteq [n-1,n+1], \mathrm{supp}(\varphi_m) \subseteq [m-1,m+1]$ and $|\varphi_n^l| \leqslant C_l$, $|\varphi_m^k| \leqslant C_k$ uniformly in n,m. For more details on this decomposition the reader is referred to chapter 1 of [17]. After normalizing φ_n and φ_m in L^2 and applying Cauchy-Schwartz followed by Bessel's inequality (2.23) follows.

Now we are ready to consider our second example.

Theorem 2.2.3. *Let* $0 < r_i \leqslant \infty, 1 \leqslant p_i \leqslant \infty$, *and*

$$\mathcal{R}_{P_1}(f_1,f_2,f_3)(x_1,x_2,x_3) = \int_{\mathbb{R}^9} \frac{\widehat{f_1}(\xi_1)\widehat{f_2}(\xi_2)\widehat{f_3}(\xi_3)}{\xi_{1,1}\xi_{2,2}+\xi_{3,3}} e^{2\pi i(x_1,x_2,x_3)\cdot(\xi_1+\xi_2+\xi_3)}d\xi_1 d\xi_2 d\xi_3, \tag{2.24}$$

defined in the limiting sense of (1.17) for $\xi_i = (\xi_{i,1},\xi_{i,2},\xi_{i,3}) \in \mathbb{R}^3$, *and* $f_i \in \mathcal{S}(\mathbb{R}^3)$. *Then,*

$$||\mathcal{R}_{P_1}(f_1,f_2,f_3)||_{L^{r_1}_{x_1}L^{r_2}_{x_2}L^{r_3}_{x_3}} \lesssim ||f_1||_{L^{p_2}_{x_2}L^{p_3}_{x_3}L^2_{x_1}}||f_2||_{L^{q_1}_{x_1}L^{q_3}_{x_3}L^2_{x_2}}||f_3||_{L^{u_1}_{x_1}L^{u_2}_{x_2}L^{u_3}_{x_3}} \tag{2.25}$$

and

$$||\mathcal{R}_{P_1}(f_1,f_2,f_3)||_{L^{r_2}_{x_2}L^{r_1}_{x_1}L^{r_3}_{x_3}} \lesssim ||f_1||_{L^{p_2}_{x_2}L^{p_3}_{x_3}L^2_{x_1}}||f_2||_{L^{q_1}_{x_1}L^{q_3}_{x_3}L^2_{x_2}}||f_3||_{L^{u_2}_{x_2}L^{u_1}_{x_1}L^{u_3}_{x_3}} \tag{2.26}$$

with $\frac{1}{q_1}+\frac{1}{u_1}=\frac{1}{r_1}$, $\frac{1}{p_2}+\frac{1}{u_2}=\frac{1}{r_2}$, *and* $\frac{1}{p_3}+\frac{1}{q_3}+\frac{1}{u_3}=\frac{1}{r_3}+\frac{1}{2}$, $u_3 > 2$.

Proof. Proceeding again as in (1.20), we have

$$\begin{aligned}
&\mathcal{R}_{P_1}(f_1, f_2, f_3)(x_1, x_2, x_3) \\
&= \sum_k \int_{\mathbb{R}^{10}} \psi_k(\alpha) \hat{f}_1(\xi_1) \hat{f}_2(\xi_2) \hat{f}_3(\xi) e^{2\pi i \alpha(\xi_{1,1}\xi_{2,2}+\xi_{3,3})} e^{2\pi i x\cdot(\xi_1+\xi_2+\xi_3)} d\alpha d\xi_1 d\xi_2 d\xi_3 \\
&= \int_{\mathbb{R}^3} \psi(\alpha) F_1(\xi_{1,1}, x_2, x_3) e^{2\pi i x_1 \xi_{1,1}} F_2(x_1, \xi_{2,2}, x_3) e^{2\pi i x_2 \xi_{2,2}} \\
&\qquad\qquad \times f_3(x_1, x_2, x_3 + \alpha) e^{2\pi i \alpha \xi_{1,1}\xi_{2,2}} d\xi_{1,1} d\xi_{2,2} d\alpha.
\end{aligned} \tag{2.27}$$

Then, considering (1.18) as usual, we obtain

$$\begin{aligned}
|\mathcal{R}_{P_1}(f_1, f_2, f_3)(x_1, x_2, x_3)| \lesssim \int_{\mathbb{R}} \Big| \int_{\mathbb{R}^2} & F_1(\xi_{1,1}, x_2, x_3) e^{2\pi i x_1 \xi_{1,1}} F_2(x_1, \xi_{2,2}, x_3) e^{2\pi i x_2 \xi_{2,2}} \\
& \times f_3(x_1, x_2, x_3 + \alpha) e^{2\pi i \alpha \xi_{1,1}\xi_{2,2}} d\xi_{1,1} d\xi_{2,2} \Big| \, d\alpha.
\end{aligned} \tag{2.28}$$

But (2.23) followed by Plancherel allows us to conclude that

$$\begin{aligned}
&\left| \int_{\mathbb{R}^2} F_1(\xi_{1,1}, x_2, x_3) e^{2\pi i x_1 \xi_{1,1}} F_2(x_1, \xi_{2,2}, x_3) e^{2\pi i x_2 \xi_{2,2}} e^{2\pi i \alpha \xi_{1,1}\xi_{2,2}} d\xi_{1,1} d\xi_{2,2} \right| \\
&\qquad \lesssim \frac{1}{|\alpha|^{1/2}} ||F_1(\cdot, x_2, x_3)||_{L^2_{\xi_{1,1}}} ||F_2(x_1, \cdot, x_3)||_{L^2_{\xi_{2,2}}} \\
&\qquad = \frac{1}{|\alpha|^{1/2}} ||f_1(\cdot, x_2, x_3)||_{L^2_{x_1}} ||f_2(x_1, \cdot, x_3)||_{L^2_{x_2}}.
\end{aligned} \tag{2.29}$$

Combining with (2.28), we have

$$|\mathcal{R}_{P_2}(f_1, f_2, f_3)(x_1, x_2, x_3)| \lesssim I^3_{1/2}(|f|)(x_1, x_2, x_3) ||f_1(\cdot, x_2, x_3)||_{L^2_{x_1}} ||f_2(x_1, \cdot, x_3)||_{L^2_{x_2}}, \tag{2.30}$$

where $I^3_{1/2}$ is convolution with the fractional kernel $\frac{1}{|t|^{1/2}}$ in the third variable only. Hölder's inequality together with Theorem 2.1.1 gives the desired result.

□

The examples above laid the groundwork for us to explore operators $\mathcal{R}_P$ with more complicated choices for P. Having at hand a wide variety of multilinear fractional integration estimates appears to be a necessity for us as we begin to develop a systematic understanding of the operators $\mathcal{R}_P$. In Chapter 7, we delve deeper into these ideas, but beforehand we must state and prove our results regarding iterated fractional integral operators.

CHAPTER 3

MAIN RESULTS: PART I

3.1 L^p Estimates for Iterated Fractional Integral Operators

Theorem 3.1.1. *Let $T_{\alpha,\beta}$ denote the iterated fractional integral operator defined in (2.3). Assume that* $0 < \alpha, \beta < 1$, $1 < p_1, p_2, p_3 < \infty$, $0 < r < \infty$, $\frac{1}{p_1} + \frac{1}{p_2} + \frac{1}{p_3} > 2 - (\alpha + \beta)$, *and*

$$\alpha + \beta + \frac{1}{p_1} + \frac{1}{p_2} + \frac{1}{p_3} = 2 + \frac{1}{r}. \tag{3.1}$$

Then,

$$||T_{\alpha,\beta}(f, g, h)||_{L^r(\mathbb{R})} \lesssim ||f||_{L^{p_1}(\mathbb{R})} ||g||_{L^{p_2}(\mathbb{R})} ||h||_{L^{p_3}(\mathbb{R})}.$$

3.2 Endpoint Estimates

The techniques of our proof allow us to prove restricted weak type estimates in the entire range of theorem 3.1.1. Hence, the end point estimates that we present in this section are stated for informational purposes rather than as a necessity to prove 3.1.1. This differs from the approach used to study the single scale operator (2.1) in [12] and [7] where both authors first prove $L^1 \times L^1 \times L^1 \to L^{\frac{1}{1+\alpha},\infty}$, $L^\infty \times L^p \to L^r$, and $L^p \times L^\infty \to L^r$ end point estimates followed by a careful multilinear interpolation. In fact, our techniques are compatible with the single scale operator (2.1), and can be used to prove all expected strong and restricted weak type estimates. However, due to the presence of two scales in operator (2.3) some weak type estimates, such as $L^1 \times L^1 \times L^1 \to L^{\frac{1}{1+\alpha+\beta},\infty}$ are not covered by our methods, and still remain an open question.

3.2.1 A Restricted Weak Type L^1 Estimate

Theorem 3.2.1. *Let $T_{\alpha,\beta}$ denote the iterated fractional integral operator defined in (2.3). Assume that $0 < \alpha, \beta < 1$, $1 < p_2, p_3 < \infty$, $1 \leqslant p_1 < \infty$, $0 < r < \infty$, $\frac{1}{p_1} + \frac{1}{p_2} + \frac{1}{p_3} > 2 - (\alpha + \beta)$, and*

$$\alpha + \beta + \frac{1}{p_1} + \frac{1}{p_2} + \frac{1}{p_3} = 2 + \frac{1}{r}. \tag{3.2}$$

Then,

$$||T_{\alpha,\beta}(f, g, h)||_{L^{r,\infty}(\mathbb{R})} \lesssim |E_1||E_2|^{1/p_2}|E_3|^{1/p_3},$$

for $|f| \leqslant \chi_{E_1}, |g| \leqslant \chi_{E_2}, |h| \leqslant \chi_{E_3}$ where E_i are sets of finite measure. The analogous statement holds true if instead we take $1 < p_1 < \infty$ and $p_3 = 1$.

Remark 3.2.2. *Restricted weak type estimates in the case that $p_1 = p_3 = 1$ simultaneously or in the case that $p_2 = 1$ are still unknown.*

3.2.2 L^∞ Estimates

The study of the L^∞ estimates of $T_{\alpha,\beta}$ is straightforward, and all possible cases can be reduced to the standard Hardy-Littlewood-Sobolev inequality (Theorem 2.1.1) and to the bilinear fractional integral operator estimates in Theorem 2.1.2.

Theorem 3.2.3. *Assume that $0 < \alpha, \beta < 1, 1 < p_1, p_2, p_3 \leqslant \infty$,*

$\frac{1}{p_1} + \frac{1}{p_2} + \frac{1}{p_3} > 2 - (\alpha + \beta)$, and

$$\alpha + \beta + \frac{1}{p_1} + \frac{1}{p_2} + \frac{1}{p_3} = 2 + \frac{1}{r}. \tag{3.3}$$

(i) If $p_i = \infty$ for exactly one $i \in \{1, 2, 3\}$ then we have

$$||T_{\alpha,\beta}(f, g, h)||_{L^r} \lesssim ||f||_{L^{p_1}} ||g||_{L^{p_2}} ||h||_{L^{p_3}}.$$

(ii) If $p_i = \infty$ for exactly two $i \in \{1, 2, 3\}$ then the only possible bound is:

$$||T_{\alpha,\beta}(f, g, h)||_{L^r} \lesssim ||f||_{L^\infty} ||g||_{L^{p_2}} ||h||_{L^\infty}$$

Proof. Note that due to the restriction $\frac{1}{p_1} + \frac{1}{p_2} + \frac{1}{p_3} > 2 - (\alpha + \beta)$, it is not possible for $p_i = \infty$ for all $i \in \{1, 2, 3\}$. To prove (i) observe that the cases $p_1 = \infty$ or $p_3 = \infty$ are symmetric and yield

$$||T_{\alpha,\beta}(f, g, h)||_{L^r} \leqslant ||f||_{L^\infty} ||B_\beta(I_\alpha(g), h)||_{L^r} \tag{3.4}$$

and

$$||T_{\alpha,\beta}(f, g, h)||_{L^r} \leqslant ||h||_{L^\infty} ||B_\alpha(f, I_\beta(g))||_{L^r} \tag{3.5}$$

respectively. Applying Theorems 2.1.1 and 2.1.2 yields the desired result. In the case that $p_2 = \infty$, we have

$$||T_{\alpha,\beta}(f, g, h)||_{L^r} \leqslant ||g||_{L^\infty} ||I_\alpha(f) \cdot I_\beta(h)||_{L^r}. \tag{3.6}$$

Applying Hölder followed by Theorem 2.1.1 yields the desired result.

To prove (ii), first consider $p_1 = p_3 = \infty$ then,

$$||T_{\alpha,\beta}(f, g, h)||_{L^r} \leqslant ||f||_{L^\infty} ||h||_{L^\infty} ||I_\alpha(I_\beta(g))||_{L^r}. \tag{3.7}$$

Note that in this case, exponent relation (3.3) implies that $r > 1$. We can conclude by applying Theorem 2.1.1 twice. Finally, we look at the cases $p_1 = p_2 = \infty$ and $p_3 = p_2 = \infty$. Consider $g \equiv 1$, together with the observation that in this case $T_{\alpha,\beta}(f, g, h) = I_\alpha(f) \cdot I_\beta(h)$ and the fact that neither I_α nor I_β is not bounded on L^∞. This concludes the proof. □

CHAPTER 4

PRELIMINARIES

4.1 Useful Notation, Definitions, and Theorems

We begin by introducing some notation which will be used throughout the proofs of Theorems 3.1.1 and 3.2.1.

Definition 4.1.1. We denote the set of dyadic intervals at scale $k \in \mathbb{Z}$ as follows:

$$\mathcal{D}_k := \{[2^k n, 2^k(n+1)) : n \in \mathbb{Z}\},$$

moreover, the set of all dyadic intervals is $\mathcal{D} := \bigcup_{k\in\mathbb{Z}} \mathcal{D}_k$.

Definition 4.1.2. Given $I \in \mathcal{D}$, define $c(I)$ to be the interval with the same center as I, but length equal to $c \cdot |I|$ as shown in. That is,

$$c(I) := \left[x_{I_c} - \frac{c}{2}|I|, x_{I_c} + \frac{c}{2}|I|\right)$$

were x_{I_c} is the center of I.

Definition 4.1.3. The L^p average of a measurable function f on the interval I is

$$\text{avg}_{I,p}(f) := \left(\frac{1}{|I|}\int_I f^p(y)dy\right)^{1/p}.$$

The dyadic maximal function together with its boundedness properties will be crucial in the proof of our results.

Definition 4.1.4. Define the dyadic maximal function by

$$M(f)(x) := \sup_{I\in\mathcal{D}:x\in I} \frac{1}{|I|}\int_I |f(y)|dy.$$

Theorem 4.1.5. *The dyadic maximal function satisfies the following boundedness properties:*

i) $||M(f)||_{L^p} \lesssim ||f||_{L^p}$, *for* $1 < p \leqslant \infty$, *and*

ii) $||M(f)||_{L^{1,\infty}} \lesssim ||f||_{L^1}$.

4.2 Dualization through L^r Spaces

Note that in the case $r > 1$, one could attempt a proof of Theorem 3.1.1 by taking advantage of the duality between $L^r \to L^{r'}$ for $\frac{1}{r} + \frac{1}{r'} = 1$. This would reduce the estimates of the trilinear operator $T_{\alpha,\beta}$ to a study of its corresponding $4-$linear form. This technique has proven to be useful in many instances, such as in the analysis of paraproducts in the Banach setting.

However, in the quasi-Banach case, that is for $0 < r < 1$, one has $(L^r)^* = \{0\}$, so this approach fails. One can remedy this by following the approach discussed in chapter 2 of [17]. Namely, rather than proving L^r estimates directly, one can prove $L^{r,\infty}$ estimates followed by a multilinear version of Marcinkiewicz interpolation. Recall that the $L^{r,\infty}$ Lorentz space is defined to be the collection of measurable functions f such that

$$||f||_{r,\infty} := \sup_{\lambda>0} \lambda|\{x : |f(x)| > \lambda\}|^{1/r} < \infty.$$

The benefit in doing so is that even for $0 < r \leqslant 1$, the quasi-norms of these spaces can be dualized. This is exemplified by the *duality lemma* in [17] which states that

$$||f||_{L^{p,\infty}} \sim \sup_{E, 0<|E|<\infty} \inf_{\substack{\tilde{E}\subseteq E \\ major\ subset}} \frac{|\langle f, \chi_{E'}\rangle|}{|E|^{1-\frac{1}{p}}} \tag{4.1}$$

where we say E' is a major subset of E if $E' \subseteq E$ and $|E'| \geqslant |E|/2$.

It turns out that when applying this technique to $T_{\alpha,\beta}$ one falls short in obtaining the full range of estimates in Theorem 3.1.1. Instead, we take advantage of an equivalent duality statement that makes use of L^r norms. This is exactly what was done in [1] where the authors proved a very similar statement to Lemma 2.6 of [17]. The following lemma along with its proof are taken directly from their work.

Lemma 4.2.1. *([1]) The following are equivalent:*

(i) $||f||_{p,\infty} \leqslant A$

(ii) For any set E of finite measure, there exists a subset $\tilde{E} \subseteq E$ with $|\tilde{E}| \simeq |E|$ called the major subset with the property that

$$||f \cdot \chi_{\tilde{E}}||_r \lesssim A \cdot |E|^{\frac{1}{r}-\frac{1}{p}}.$$

This can be reformulated as

$$||f||_{p,\infty} \sim \sup_{0<|E|<\infty} \inf_{\substack{\tilde{E}\subseteq E \\ major\ subset}} \frac{||f \cdot \chi_{\tilde{E}}||_r}{|E|^{\frac{1}{r}-\frac{1}{p}}}.$$

Proof. "(i) $\Rightarrow$ (ii)" Let E be a set of finite measure and consider the set

$$\Omega := \left\{ x : f(x) > C \cdot \frac{A}{|E|^{1/p}} \right\}.$$

Pick $\tilde{E}$ to be $\tilde{E} := E \backslash \Omega$, which is a major subset of E for C large enough. Then we have

$$||f \cdot \chi_{\tilde{E}}||_r \lesssim C \cdot \frac{A}{|E|^{1/p}} \cdot |E|^{1/r} \lesssim A \cdot |E|^{\frac{1}{r}-\frac{1}{p}}.$$

"(ii) $\Rightarrow$ (i)" Let $\lambda > 0$ and set $E := \{x : |f(x)| > \lambda\}$. Then (ii), guarantees the existence of a major subset $\tilde{E}$ of E for which we have

$$\lambda \tilde{E}^{1/r} < ||f \cdot \chi_{\tilde{E}}||_r \lesssim A \cdot |E|^{\frac{1}{r} - \frac{1}{p}}.$$

Since E and $\tilde{E}$ have comparable measure, we get that $\lambda |E|^{1/p} \lesssim A$. □

4.3 Reduction to Restricted Weak Type Estimates via Dualization through $L^{1/3}$

By freezing all but one function and interpolating linearly, or alternatively as a trivial application of the more advanced multilinear interpolation theorem in [7], the proof of Theorem 3.1.1 can be reduced to the following restricted weak type estimates.

Theorem 4.3.1. *Let $T_{\alpha,\beta}$ and p_1, p_2, p_3, r be as in Theorem 3.1.1, and let E_1, E_2, E_3, and E be sets of finite measure. Then, for any $|f| \leqslant \chi_{E_1}, |g| \leqslant \chi_{E_2}, |h| \leqslant \chi_{E_3}$, and $|F| \leqslant \chi_E$ we have*

$$||T_{\alpha,\beta}(f, g, h) \cdot F||_{L^{1/3}} \lesssim |E_1|^{1/p_1} |E_2|^{1/p_2} |E_3|^{1/p_3} |E|^{3-1/r}. \quad (4.2)$$

Note that by Lemma 4.2.1 this is equivalent to the following restricted weak type estimate

$$||T_{\alpha,\beta}(f, g, h)||_{L^{r,\infty}(\mathbb{R})} \lesssim |E_1|^{1/p_1} |E_2|^{1/p_2} |E_3|^{1/p_3}.$$

It is also helpful to apply duality Lemma 4.2.1 in order to reformulate the restricted weak type end point estimates in Theorem 3.2.1 to the equivalent statement below.

Theorem 4.3.2. *Consider the same assumptions as in Theorem 4.3.1, but with* $p_1 = 1$. *Let* E_1, E_2, E_3, *and* E *be sets of finite measure. Then, there exists a set* $\widetilde{E} \subseteq E$ *with* $|\widetilde{E}| > \frac{1}{2}|E|$ *such that for any* $|f| \leqslant \chi_{E_1}, |g| \leqslant \chi_{E_2}, |h| \leqslant \chi_{E_3}$, *and* $|F| \leqslant \chi_{\widetilde{E}}$ *we have*

$$||T_{\alpha,\beta}(f,g,h) \cdot F||_{L^{1/3}} \leqslant |E_1||E_2|^{1/p_2}|E_3|^{1/p_3}|E|^{3-1/r}.$$

The analogous statements holds if we instead take $p_3 = 1$.

Remark 4.3.3. *Notice that the statement of Theorem 4.3.1 does not require the construction of a major set* $\widetilde{E} \subseteq E$, *whereas the more tricky end point case in Theorem 4.3.2 does.*

CHAPTER 5

THE INNER OPERATOR $\widetilde{T}_{\alpha,J}$

We begin a natural discretization procedure of the iterated fractional integral operator $T_{\alpha,\beta}$ in terms for two dyadic scales, one for each singularity. As a result, for a fixed localization of "x" to a dyadic interval J and $|s| \leqslant |J|$ we obtain an "inner operator" which we call $\widetilde{T}_{\alpha,J}$. In this chapter we prove estimates for $\widetilde{T}_{\alpha,J}$ which will be used in Chapter 6 to aid in proving Theorems 3.1.1 and 3.2.1.

5.1 Initial Discretization of $T_{\alpha,\beta}$

Lemma 5.1.1. *For $0 < \gamma < 1$, we can write*

$$\frac{1}{|x|^\gamma} \simeq_\gamma \sum_{k\in\mathbb{Z}} \frac{1}{2^{k\gamma}} \chi_{[-2^k,2^k]}(x) \tag{5.1}$$

where $\chi_{[-2^k,2^k]}$, is the characteristic function on the set $[-2^k, 2^k]$.

Proof. Indeed, for a fixed $x \in \mathbb{R}\backslash\{0\}$ we have:

$$\begin{aligned}
\sum_{k\in\mathbb{Z}} \frac{1}{2^{k\gamma}} \chi_{[-2^k,2^k]}(x) &= \sum_{2^k \geqslant |x|} \frac{1}{2^{k\gamma}} \chi_{[-2^k,2^k]}(x) && (5.2)\\
&= \sum_{2^k \geqslant |x|} \frac{1}{2^{k\gamma}} && (5.3)\\
&= \sum_{k \geqslant \ln_2(|x|)} \frac{1}{2^{k\gamma}} && (5.4)\\
&= \frac{1}{|x|^\gamma} \cdot \frac{1}{1-\frac{1}{2^\gamma}} && (5.5)\\
&\simeq \frac{1}{|x|^\gamma} && (5.6)
\end{aligned}$$

□

We begin by decomposing

$$\frac{1}{|t|^{\alpha}} \simeq \sum_{k\in\mathbb{Z}} \frac{1}{2^{k\alpha}} \chi_{[-2^k,2^k]}(t)$$

and

$$\frac{1}{|s|^{\beta}} \simeq \sum_{j\in\mathbb{Z}} \frac{1}{2^{j\beta}} \chi_{[-2^j,2^j]}(s),$$

We can consider, without loss of generality, the case $k \leqslant j$ since the case $k > j$ can be dealt with equivalently. We have

$$\begin{aligned}
&T_{\alpha,\beta}(f,g,h)(x)\\
&\simeq \sum_{\substack{j,k\in\mathbb{Z},\\ k\leqslant j}} \frac{1}{2^{j\beta}} \cdot \frac{1}{2^{k\alpha}} \int_{|s|\leqslant 2^j} \int_{|t|\leqslant 2^k} f(x-t)g(x+t+s)h(x-s)\, dtds\\
&= \sum_{\substack{j,k\in\mathbb{Z}\\ k\leqslant j}} \sum_{\substack{J\in\mathcal{D}_j,\\ I\in\mathcal{D}_k}} \frac{1}{|J|^{\beta}} \cdot \frac{1}{|I|^{\alpha}} \left(\int_{|s|\leqslant|J|} \int_{|t|\leqslant|I|} f(x-t)g(x+t+s)h(x-s)\, dtds \right) \chi_J(x)\chi_I(x)\\
&= \sum_{J\in\mathcal{D}} \frac{1}{|J|^{\beta}} \left(\sum_{\substack{I\in\mathcal{D}:\\ I\subseteq J}} \frac{1}{|I|^{\alpha}} \left(\int_{|s|\leqslant|J|} \int_{|t|\leqslant|I|} f(x-t)g(x+t+s)h(x-s)\, dtds \right) \chi_I(x) \right) \chi_J(x)\\
&= \sum_{J\in\mathcal{D}} \frac{1}{|J|^{\beta}} \widetilde{T}_{\alpha,J}(f,g,h)(x)\chi_J(x), \qquad (5.7)
\end{aligned}$$

where

$$\widetilde{T}_{\alpha,J} \coloneqq \sum_{\substack{I\in\mathcal{D}:\\ I\subseteq J}} \frac{1}{|I|^{\alpha}} \left(\int_{|s|\leqslant|J|} \int_{|t|\leqslant|I|} f(x-t)g(x+t+s)h(x-s)\, dtds \right) \chi_I(x) \qquad (5.8)$$

Remark 5.1.2. *If we discretize $\frac{1}{|s|^{\beta}}$ only, then it is easy to see that*

$$\widetilde{T}_{\alpha,J} = \left(\int \int_{|s|\leqslant|J|} \frac{f(x-t)g(x+t+s)h(x-s)}{|t|^{\alpha}} dtds \right) \chi_J(x).$$

However, for our purposes the fully discretized version in equation (5.8) is more convenient.

5.2 Estimates for $\widetilde{T}_{\alpha,J}$

Theorem 5.2.1. *Let $1 < p_1, p_2 < \infty$, $0 < r < \infty$, with $\frac{1}{p_1} + \frac{1}{p_2} + \alpha = 1 + \frac{1}{r}$, and E_1, E_2, E_3, E be sets of finite measure. Then,*

$$||\widetilde{T}_{\alpha,J}(f,g,h)\cdot\chi_E||_{L^{1/2}(J)} \lesssim |J|^{1-\frac{1}{p_2}}|E_1\cap J|^{1/p_1}|E_2\cap J|^{1/p_2}|E_3\cap J|^{1/p_2}|E\cap J|^{2-1/r}, \quad (5.9)$$

for $|f| \leqslant \chi_{E_1}, |g| \leqslant \chi_{E_2}, |h| \leqslant \chi_{E_3}$. Moreover, (5.9) implies the strong type estimate

$$||\widetilde{T}_{\alpha,J}(f,g,h)||_{L^r(J)} \lesssim |J|^{1-\frac{1}{p_2}}||f||_{L^{p_1}(J)}||g||_{L^{p_2}(J)}||h||_{L^{p_2}(J)}, \quad (5.10)$$

where in both cases the implicit constant has no additional dependence on $|J|$.

The proof of Theorem 5.2.1 is deferred to Section 5.2.3. In what follows we write $||\widetilde{T}_{\alpha,J}(f,g,h)\cdot\chi_E||_{L^{1/2}}$ in terms of carefully defined averages.

5.2.1 The Dualized Expression of $\widetilde{T}_{\alpha,J}$ in Terms of Averages

For our purposes, we can assume without loss of generality that $f,g,h \geqslant 0$. We have,

$$\begin{aligned}
&||\widetilde{T}_{\alpha,J}(f,g,h)\cdot\chi_E||_{L^{1/2}(J)}^{1/2} \\
&= \left|\left|\sum_{\substack{I\in\mathcal{D}:\\ I\subseteq J}} \frac{1}{|I|^{\alpha}}\left(\int_{|s|\leqslant|J|}\int_{|t|\leqslant|I|} f(x-t)g(x+t+s)h(x-s)\,dtds\right)\chi_I(x)\cdot\chi_E(x)\right|\right|_{L^{1/2}(J)}^{1/2} \\
&\lesssim \sum_{\substack{I\in\mathcal{D}:\\ I\subseteq J}} \frac{1}{|I|^{\frac{\alpha}{2}}}\left|\left|\left(\int_{|s|\leqslant|J|}\int_{|t|\leqslant|I|} f(x-t)g(x+t+s)h(x-s)\,dtds\right)\chi_I(x)\cdot\chi_E(x)\right|\right|_{L^{1/2}(J)}^{1/2} \\
&\lesssim |J|^{\frac{1}{2}}\sum_{\substack{I\in\mathcal{D}:\\ I\subseteq J}} |I|^{\frac{1-\alpha}{2}}\mathrm{avg}_I^{1/2}(\chi_E)\left|\left|\left(\frac{1}{|J|}\int_{|s|\leqslant|J|}\int_{|t|\leqslant|I|} f(x-t)g(x+t+s)h(x-s)\,dtds\right)\chi_I(x)\right|\right|_{L^1}^{1/2}
\end{aligned}$$

$$= |J|^{\frac{1}{2}} \sum_{\substack{I \in \mathcal{D}: \\ I \subseteq J}} |I|^{\frac{1-\alpha}{2}} \mathrm{avg}_I^{1/2}(\chi_E) \cdot \mathcal{I}_{J,I}(f,g,h)^{1/2}, \tag{5.11}$$

where

$$\mathcal{I}_{J,I}(f,g,h) := \frac{1}{|J|} \int_{|s| \leqslant |J|} \int_{|t| \leqslant |I|} \int_I f(x-t) g(x+t+s) h(x-s)\, dxdtds. \tag{5.12}$$

We will write $\mathcal{I}_{J,I}(f,g,h)$ a product of an average of f on the interval $\widetilde{I}$ and an average of $g \otimes h$ on an oblique rectangle indexed by $(I, |J|)$. Begin with the changes of variables $y = x - t$ in the inner most integral, and note that since $x \in I$ and $t \in [-|I|, |I|]$ we have that $y \in \tilde{I}$ where $\widetilde{I} := 3(I)$ is as described in Definition 4.1.2.

So, we have

$$\begin{aligned}
\mathcal{I}_{J,I}(f,g,h) &= \frac{1}{|J|} \int_{\tilde{I}} f(y) \int_{-|J|}^{|J|} \int_{-|I|}^{|I|} g(y+2t+s) h(y+t-s) dtdsdy \\
&= \frac{1}{|J|} \int_{\tilde{I}} f(y) \int_{\tilde{R}} g(y+\tilde{t}) h(y+\tilde{s}) d\tilde{t} d\tilde{s} dy \\
&\leqslant \frac{1}{|J|} \int_{\tilde{I}} f(y) \int_{\tilde{\tilde{R}}} g(y+\tilde{t}) h(y+\tilde{s}) d\tilde{t} d\tilde{s} dy \qquad (5.13) \\
&= \frac{1}{|J|} \int_{\tilde{I}} f(y) \int_{\tilde{\tilde{R}}+(y,y)} g(\tilde{t}) h(\tilde{s}) d\tilde{t} d\tilde{s} dy \\
&\lesssim \left(\int_{\tilde{I}} f(y) dy \right) \cdot \left(\frac{1}{|J|} \int_{\widetilde{R(I,J)}} g(\tilde{t}) h(\tilde{s}) d\tilde{t} d\tilde{s} \right) \qquad (5.14) \\
&= 3|I|^2 \cdot \mathrm{avg}_{\tilde{I}}(f) \cdot \left(\frac{1}{|I||J|} \int_{\widetilde{R(I,J)}} g(\tilde{t}) h(\tilde{s}) d\tilde{t} d\tilde{s} \right) \\
&\simeq |I|^2 \cdot \mathrm{avg}_{\tilde{I}}(f) \cdot \widetilde{\mathrm{avg}}_{I,J}(g,h). \qquad (5.15)
\end{aligned}$$

Where $\widetilde{R}$ is the image of the rectangle $R := [-|I|, |I|] \times [-|J|, |J|]$ under the transformation $A = \left(\begin{smallmatrix} 2 & 1 \\ 1 & -1 \end{smallmatrix}\right)$ which results from the the change of variables $\tilde{t} =$

$2t+s$ and $\tilde{s}=t-s$, and $\tilde{\tilde{R}} := \tilde{A}(R)$, where $\tilde{A} := \left(\begin{smallmatrix} 2 & 2 \\ 2 & -2 \end{smallmatrix}\right)$. In (5.13) we use that $\tilde{R} \subseteq \tilde{\tilde{R}}$, and that g and h are positive functions. Finally, $\widetilde{R(I,J)} := \bigcup_{y\in\tilde{I}} \tilde{\tilde{R}} + (y,y)$. In (5.14) we used that $\tilde{\tilde{R}} + y \subseteq \widetilde{R(I,J)}$ and again that all functions are taken to be positive. Note that $Area(\widetilde{R(I,J)}) = 56|I||J| \simeq |I||J|$. Moreover,

$$\widetilde{\text{avg}}_{I,J}(g,h) := \frac{1}{|I||J|}\int_{\widetilde{R(I,J)}} g(\tilde{t})h(\tilde{s})d\tilde{t}d\tilde{s}, \tag{5.16}$$

is an average of $g \otimes h$ taken on the oblique rectangle $\widetilde{R(I,J)}$. It is also worthwhile to note that for $I \subseteq J$, $\widetilde{R(I,J)} \subseteq \widetilde{R(J,J)}$, and $\widetilde{R(I,J)} \subseteq 10(J)\times 10(J) = \tilde{J}\times\tilde{J}$. Figure 5.1 depicts the various regions of $\mathbb{R}^2$ that are relevant to the understanding the derivation of and properties of $\widetilde{R(I,J)}$.

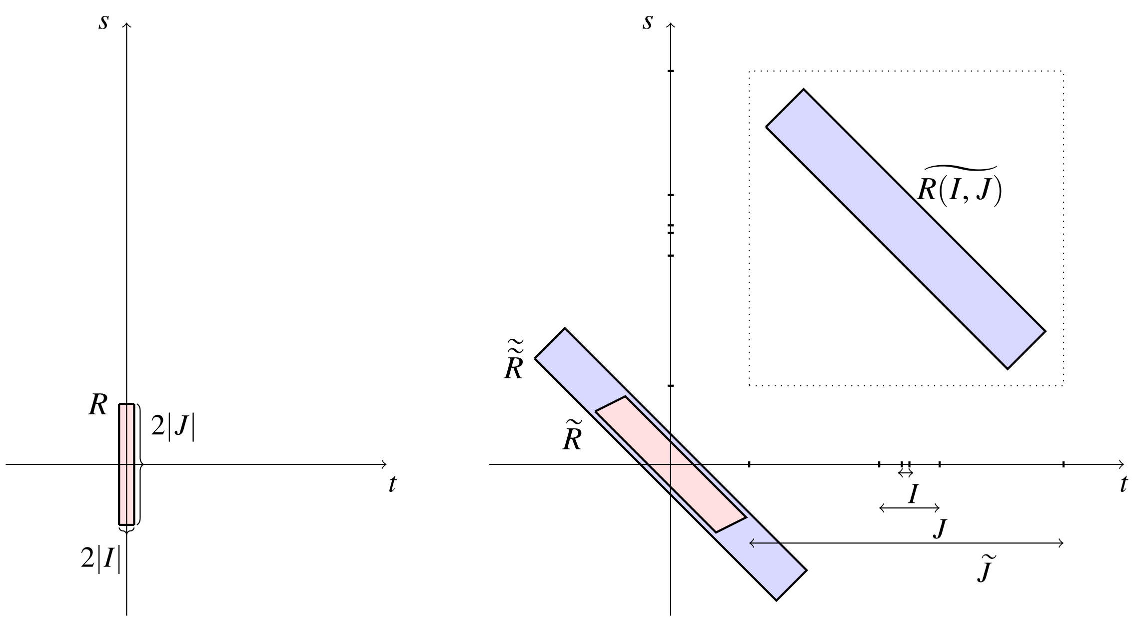

Figure 5.1: Sketch of region $\widetilde{R(I,J)}$.

Finally, combining equations (5.11) and (5.15) we are left with

$$||\widetilde{T}_{\alpha,\beta}(f,g,h)\cdot\chi_E||^{1/2}_{L^{1/2}}(J) \lesssim \sum_{\substack{I\in\mathcal{D}:\\ I\subseteq J}} |I|^{\frac{3-\alpha}{2}} \text{avg}^{1/2}_{\tilde{I}}(f)\cdot\widetilde{\text{avg}}^{1/2}_{I,J}(g,h)\cdot\text{avg}^{1/2}_{I}(\chi_E). \tag{5.17}$$

5.2.2 Properties of Averages

Proposition 5.2.2. *Let I_o be a fixed dyadic interval and $M > 1$, then*

$$\sum_{\substack{I \in \mathcal{D} \\ I \subseteq I_o}} |I|^M \lesssim |I_o|^M.$$

Proof. We have

$$\begin{aligned}
\sum_{\substack{I \in \mathcal{D}: \\ I \subseteq I_o}} |I|^M &= \sum_{\substack{k \in \mathbb{Z}: \\ 2^k \leqslant |I_o|}} \sum_{\substack{I \in \mathcal{D}_k: \\ I \subseteq I_o}} |I|^M \\
&= \sum_{\substack{k \in \mathbb{Z}: \\ 2^k \leqslant |I_o|}} 2^{kM} \cdot |\{I \in \mathcal{D}_k : I \subseteq I_o\}| \\
&= \sum_{\substack{k \in \mathbb{Z}: \\ 2^k \leqslant |I_o|}} 2^{k(M-1)} |I_o| \\
&= |I_o| \sum_{k \leqslant \log_2(|I_o|)} 2^{k(M-1)} \\
&= |I_o| \sum_{-\log_2(|I_o|) \leqslant k} 2^{-k(M-1)} \\
&= |I_o|^M \sum_{k \in \mathbb{Z}_{\geqslant 0}} 2^{k(M-1)} \\
&\lesssim |I_o|^M, \qquad (5.18)
\end{aligned}$$

where convergence of the geometric series is guaranteed because $M > 1$. □

In the following Lemma, $\tilde{I}$ is as in Definition 4.1.2.

Lemma 5.2.3. *Let E be a set of finite measure, $|F| \leqslant \chi_E$, and $M \geqslant 1$. Define*

$$\mathcal{J}_n \coloneqq \{I \in \mathcal{D} : \mathrm{avg}_{I,p}(F) \simeq 2^{-n}\},$$

$$\mathcal{J}'_n \coloneqq \{I \in \mathcal{D} : \mathrm{avg}_{\tilde{I},p}(F) \simeq 2^{-n}\}.$$

Then,

i) $\sum_{\substack{I_o \in \mathcal{J}_n: \\ I_o \text{maximal}}} |I_o|^M \lesssim (2^{np}|E|)^M$,

ii) $\sum_{\substack{I_o \in \mathcal{J}'_n: \\ I_o \text{maximal}}} |I_o|^M \lesssim (2^{np}|E|)^M$.

Proof. (i) Since $|F| \lesssim |\chi_E|$ and $I_o \in \mathcal{J}_n$, we have

$$\text{avg}_{I_o}(F) \simeq 2^{-np} \iff |I_o| \lesssim 2^{np}|E \cap I_o|. \tag{5.19}$$

Then,

$$\sum_{\substack{I_o \in \mathcal{J}_n: \\ I_o \text{ maximal}}} |I_o|^M \lesssim 2^{npM} \sum_{\substack{I_o \in \mathcal{J}_n \\ I_o \text{ maximal}}} |E \cap I_o|^M \tag{5.20}$$

$$\lesssim 2^{npM}|E|^{M-1} \sum_{\substack{I_o \in \mathcal{J}_n \\ I_o \text{ maximal}}} |E \cap I_o| \tag{5.21}$$

$$\lesssim (2^{np}|E|)^M. \tag{5.22}$$

In (5.21) we use that $M \geqslant 1$ and in (5.22) we use that any collection of maximal dyadic intervals is piecewise disjoint.

(ii) The proof of this portion is more delicate because the averages are taken on dilated dyadic intervals and so the disjointness property does not carry over to the collection $\{\widetilde{I_o}\}_{I_o \in \mathcal{J}'_n : I_o \text{ maximal}}$. So, instead, we will make use of the dyadic maximal function. We have that $I_o \in \mathcal{J}'_n$ implies

$$\text{avg}_{\widetilde{I_o}}(F^p) \simeq 2^{-np} \iff 3|I_o| = |\widetilde{I_o}| \lesssim 2^{np}|E \cap \widetilde{I_o}|.$$

So,

$$\sum_{\substack{I_o \in \mathcal{J}'_n: \\ I_o \text{ maximal}}} |I_o|^M \lesssim 2^{np(M-1)}|E|^{M-1} \sum_{\substack{I_o \in \mathcal{J}' \\ I_o \text{ maximal}}} |I_o| \tag{5.23}$$

$$\simeq 2^{npM}|E|^{M-1} \sum_{\substack{I_o \in \mathcal{J}' \\ I_o \text{ maximal}}} \text{avg}_{\widetilde{I_o}}(F^p)|I_o| \tag{5.24}$$

$$= 2^{npM}|E|^{M-1} \left\| \sum_{\substack{I_o \in \mathcal{J}' \\ I_o \text{ maximal}}} \operatorname{avg}_{\tilde{I_o}}(F^p)\chi_{I_o}(\cdot) \right\|_{L^1} \tag{5.25}$$

$$= 2^{npM}|E|^{M-1} \left\| \sum_{\substack{I_o \in \mathcal{J}' \\ I_o \text{ maximal}}} \operatorname{avg}_{\tilde{I_o}}(F^p)\chi_{I_o}(\cdot) \right\|_{L^{1,\infty}} \tag{5.26}$$

$$\lesssim 2^{npM}|E|^{M-1} \, ||\mathcal{M}(F^p)||_{L^{1,\infty}} \tag{5.27}$$

$$\lesssim 2^{npM}|E|^{M-1} \, ||F^p||_{L^1} \tag{5.28}$$

$$\lesssim (2^{pn}|E|)^M. \tag{5.29}$$

Inequalities (5.25) and (5.27) are a result of the fact that the maximal intervals of $\mathcal{J}_n'$ are disjoint. In (5.26) we use that the L^1 and $L^{1,\infty}$ norms are equal for sums of characteristic functions of disjoints sets. In (5.28) we use that the maximal function maps L^1 into $L^{1,\infty}$. Finally, the last line follows from the assumption that $|F| \lesssim \chi_E$. □

Lemma 5.2.4. *Let E_1, E_2 be sets of finite measure, $|F| \leqslant \chi_{E_1}, |G| \leqslant \chi_{E_2}$, and $M \geqslant 1$. Fix $J \in \mathcal{D}$ and define*

$$\mathcal{J}_{n,J} := \{I \in \mathcal{D} : I \subseteq J, \ \widetilde{avg}_{I,J}(F,G) \simeq 2^{-n}\},$$

where $\widetilde{\operatorname{avg}}_{I,J}(F,G)$ is as in (5.16). Then, the following two properties hold:

i) $\widetilde{avg}_{I,J}(F,G) \lesssim 1$

ii) $\sum_{\substack{I_o \in \mathcal{J}_{n,J}: \\ I_o \text{ maximal}}} |I_o|^M \lesssim \left(\frac{2^n|E_2 \cap \tilde{J}||E_3 \cap \tilde{J}|}{|J|}\right)^M,$

where $\tilde{J} := 10(J)$.

Proof. (i) We have,

$$\widetilde{\operatorname{avg}}_{I,J}(F,G) = \frac{1}{|J||I|} \int_{\widetilde{R(I,J)}} F(\tilde{t})G(\tilde{s}) d\tilde{t} d\tilde{s}$$

$$
\begin{aligned}
&\lesssim \frac{1}{|J||I|}\int_{\widetilde{R(I,J)}} d\tilde{t}d\tilde{s}\\
&\lesssim \frac{\text{area}(\widetilde{R(I,J)})}{|J||I|}\\
&\simeq 1.
\end{aligned}
$$

(ii) Note that $I_o \in \mathcal{J}_{n,J}$ implies

$$
\begin{aligned}
|I_o| &\lesssim \frac{2^n}{|J|}\int_{\widetilde{R(I_o,J)}} |F|(\tilde{t})|G|(\tilde{s})d\tilde{t}d\tilde{s}\\
&\lesssim \frac{2^n}{|J|}\int_{\widetilde{J}\times\widetilde{J}} |F|(\tilde{t})|G|(\tilde{s})d\tilde{t}d\tilde{s}\\
&\lesssim \frac{2^n|E_1\cap\widetilde{J}||E_2\cap\widetilde{J}|}{|J|}.
\end{aligned} \tag{5.30}
$$

Take $B := \begin{pmatrix} 1/2 & 1/2 \\ 1/2 & -1/2 \end{pmatrix}$, and recall that $\widetilde{R(I,J)} \subseteq \widetilde{R(J,J)} \subseteq \widetilde{J}\times\widetilde{J}$. Then,

$$
\widetilde{\text{avg}}(F,G) = \frac{1}{2|I||J|}\int_{B(\widetilde{R(I,J)})} F\otimes G\circ B^{-1}(u,v)\cdot \chi_{B(\widetilde{R(J,J)})}(u,v)\, dudv.
$$

In fact, $B(\widetilde{R(I,J)})$ results in a rectangle with edges parallel to the coordinate axes. More precisely, $(u,v)\in B(\widetilde{R(I,J)})$ if and only if $u = 2x+\tilde{x}$, and $v = 2y$ for $(x,y)\in[-|I|,|I|]\times[-|J|,|J|]$ and $\tilde{x}\in\widetilde{I} = 3(I)$. Hence $B(\widetilde{R(I,J)}) = \bar{I}\times\bar{J}$ with $\bar{I} = \left[x_{I_c} - \frac{1}{2}|I|, x_{I_c} + \frac{7}{2}|I|\right)$ and $\bar{J} = [-2|J|, 2|J|]$, where x_{I_c} is the center of I. A key observation and in fact the motivation for the choice of B is that $I\subseteq\bar{I}$ and $|I|\simeq|\bar{I}|$, see Figure 5.2. Continuing with this notation, we actually have

$$
\widetilde{\text{avg}}_{I,J}(F,G) = \frac{1}{2}\text{avg}_{\bar{I}}(\text{avg}_{\bar{J}}(F\otimes G\circ B^{-1}\cdot\chi_{B(\widetilde{R(J,J)})})),
$$

where we first take the average over $v\in\bar{J}$ and then over $u\in\bar{I}$ of the two variable function $F\otimes G\circ B^{-1}\cdot\chi_{B(\widetilde{R(J,J)})}(u,v)$. In what follows, we will argue similarly as in the proof of part (ii) of Lemma 5.2.3, in particular we will utilize a maximal function estimate applied to $\text{avg}_{\bar{J}}(F\otimes G\circ B^{-1}\cdot\chi_{B(\widetilde{R(J,J)})})(u)$.

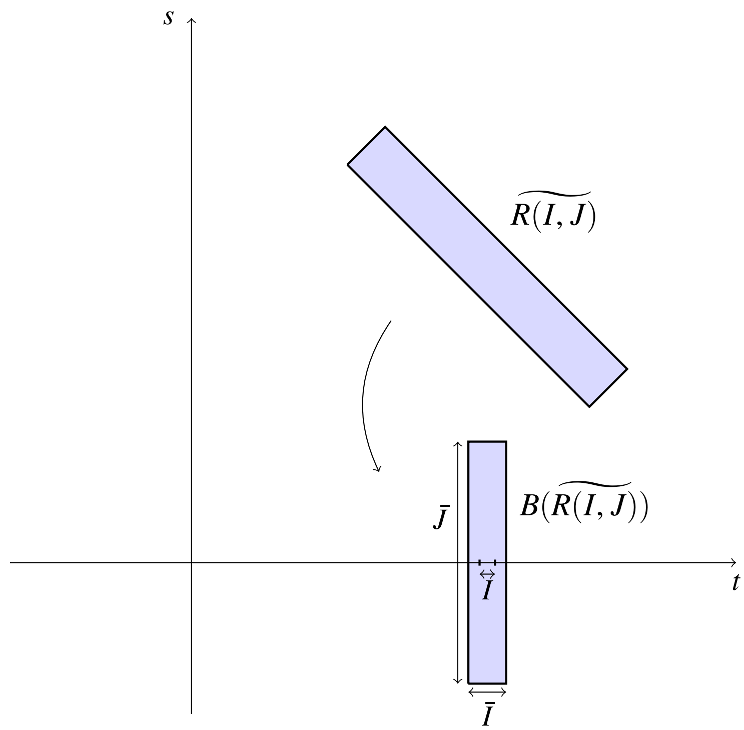

Figure 5.2: Obtaining $\bar{I}$ from $\widetilde{R(I,J)}$.

Now we conclude the proof,

$$\sum_{\substack{I_o \in \mathcal{J}_{n,J}: \\ I_o \text{ maximal}}} |I_o|^M \lesssim \sum_{\substack{I_o \in \mathcal{J}_{n,J} \\ I_o \text{ maximal}}} |I_o|^{M-1} |I_o| \tag{5.31}$$

$$\lesssim \left(\frac{2^n |E_1 \cap \tilde{J}| |E_2 \cap \tilde{J}|}{|J|} \right)^{M-1} \sum_{\substack{I_o \in \mathcal{J}_{n,J} \\ I_o \text{ maximal}}} |I_o| \tag{5.32}$$

$$\lesssim \left(\frac{2^n |E_1 \cap \tilde{J}| |E_2 \cap \tilde{J}|}{|J|} \right)^{M-1} \left\| \sum_{\substack{I_o \in \mathcal{J}_{n,J} \\ I_o \text{ maximal}}} \chi_{I_o}(\cdot) \right\|_{L^1(\mathbb{R})} \tag{5.33}$$

$$= \left(\frac{2^n |E_1 \cap \tilde{J}| |E_2 \cap \tilde{J}|}{|J|} \right)^{M-1} \left\| \sum_{\substack{I_o \in \mathcal{J}_{n,J} \\ I_o \text{ maximal}}} \chi_{I_o}(\cdot) \right\|_{L^{1,\infty}(\mathbb{R})} \tag{5.34}$$

$$\simeq 2^{nM} \left(\frac{|E_1 \cap \tilde{J}| |E_2 \cap \tilde{J}|}{|J|} \right)^{M-1} \left\| \sum_{\substack{I_o \in \mathcal{J}_{n,J} \\ I_o \text{ maximal}}} \widetilde{\operatorname{avg}}_{I,J}(F,G) \chi_{I_o}(\cdot) \right\|_{L^{1,\infty}(\mathbb{R})} \tag{5.35}$$

$$\simeq 2^{nM}\left(\frac{|E_1\cap\tilde{J}||E_2\cap\tilde{J}|}{|J|}\right)^{M-1}\left\|\sum_{\substack{I_o\in\mathcal{J}_{n,J}\\ I_o\text{ maximal}}}\operatorname{avg}_{\bar{I}}(\operatorname{avg}_{\bar{J}}(F\otimes G\circ B^{-1}\cdot\chi_{B(\widetilde{R(J,J)})}))\chi_{I_o(\cdot)}\right\|_{L^{1,\infty}(\mathbb{R})} \tag{5.36}$$

$$\lesssim 2^{nM}\left(\frac{|E_1\cap\tilde{J}||E_2\cap\tilde{J}|}{|J|}\right)^{M-1}\left\|\mathcal{M}(\operatorname{avg}_{\bar{J}}(F\otimes G\circ B^{-1}\cdot\chi_{B(\widetilde{R(J,J)})}))\right\|_{L^{1,\infty}(\mathbb{R})} \tag{5.37}$$

$$\lesssim 2^{nM}\left(\frac{|E_1\cap\tilde{J}||E_2\cap\tilde{J}|}{|J|}\right)^{M-1}\left\|\operatorname{avg}_{\bar{J}}(F\otimes G\circ B^{-1}\cdot\chi_{B(\widetilde{R(J,J)})})\right\|_{L^{1}(\mathbb{R})} \tag{5.38}$$

$$\simeq \frac{2^{nM}}{|J|^M}(|E_1\cap\tilde{J}||E_2\cap\tilde{J}|)^{M-1}\left\|F\otimes G\circ B^{-1}\cdot\chi_{B(\widetilde{R(J,J)})}\right\|_{L^{1}(\mathbb{R}^2)} \tag{5.39}$$

$$\lesssim \frac{2^{nM}}{|J|^M}(|E_1\cap\tilde{J}||E_2\cap\tilde{J}|)^{M-1}\,||F\otimes G||_{L^{1}(\tilde{J}\times\tilde{J})} \tag{5.40}$$

$$\lesssim \left(\frac{2^n|E_1\cap\tilde{J}||E_2\cap\tilde{J}|}{|J|}\right)^{M}. \tag{5.41}$$

In (5.33) we use that all maximal elements of $\mathcal{J}_{n,J}$ are disjoint. Inequality (5.37) follows because $I_o\subseteq\bar{I}_o, |I_o|\simeq|\bar{I}_o|$, and that piecewise disjointness of the maximal intervals allows us to bound the summation pointwise by the one dimensional maximal function. □

5.2.3 First Stopping-Time Argument

Now we begin the proof of Theorem 5.2.1.

Proof. Fix $J\in\mathcal{D}$ and define

$$\mathcal{J}_{n_1} := \{I\subseteq J : \operatorname{avg}_{\tilde{I}}(f)^{1/2}\simeq 2^{-n_1}\}, \tag{5.42}$$

$$\mathcal{J}_{n_2} := \{I\subseteq J : \widetilde{\operatorname{avg}}_{I,J}(g,h)^{1/2}\simeq 2^{-n_2}\}, \tag{5.43}$$

$$\mathcal{J}_{n_3} := \{I\subseteq J : \operatorname{avg}(\chi_E)_I^{1/2}\simeq 2^{-n_3}\}, \tag{5.44}$$

and $\mathcal{J}_{n_1,n_2,n_3} := \mathcal{J}_{n_1} \cap \mathcal{J}_{n_2} \cap \mathcal{J}_{n_3}$. Note that since each of the functions are bounded by characteristic functions, each average can be at most one. By (i) of Lemma 5.2.4 the same hold true for $\widetilde{\text{avg}}_{I,J}(g,h)^{1/2}$. Therefore we only need to consider $n_1, n_2, n_3 \geqslant 0$. Then, continuing from equation (5.11),

$$\frac{1}{|J|^{\frac{1}{2}}}||\widetilde{T}_{\alpha,J}(f,g,h)\cdot\chi_E||^{1/2}_{L^{1/2}(J)} \lesssim \sum_{\substack{I\in\mathcal{D}:\\ I\subseteq J}} |I|^{\frac{3-\alpha}{2}} \text{avg}^{1/2}_{\tilde{I}}(f)\cdot\widetilde{\text{avg}}^{1/2}_{I,J}(g,h)\cdot\text{avg}^{1/2}_{I}(\chi_E)$$

$$= \sum_{n_1,n_2,n_3\in\mathbb{Z}_{\geqslant 0}} 2^{-n_1}2^{-n_2}2^{-n_3} \sum_{I\in\mathcal{J}_{n_1,n_2,n_3}} |I|^{\frac{3-\alpha}{2}} \tag{5.45}$$

$$= \sum_{n_1,n_2,n_3\in\mathbb{Z}_{\geqslant 0}} 2^{-n_1}2^{-n_2}2^{-n_3} \sum_{\substack{I_o\in\mathcal{J}_{n_1,n_2,n_3}:\\ I_o \text{ maximal}}} \sum_{I\in I_o} |I|^{\frac{3-\alpha}{2}} \tag{5.46}$$

$$\lesssim \sum_{n_1,n_2,n_3\in\mathbb{Z}_{\geqslant 0}} 2^{-n_1}2^{-n_2}2^{-n_3} \sum_{\substack{I_o\in\mathcal{J}_{n_1,n_2,n_3}:\\ I_o \text{ maximal}}} |I_o|^{\frac{3-\alpha}{2}}. \tag{5.47}$$

Where in (5.47) we apply Lemma 5.2.2 with $M = \frac{3-\alpha}{2} > 1$. Next, by Lemma 5.2.3 and part (ii) of lemma 5.2.4 we obtain three distinct bounds on the inner most summation.

$$\sum_{\substack{I_o\in\mathcal{J}_{n_1,n_2,n_3}:\\ I_o \text{ maximal}}} |I_o|^{\frac{3-\alpha}{2}} \lesssim \begin{cases} 2^{n_1(3-\alpha)}|E_1\cap J|^{\frac{3-\alpha}{2}} \\ 2^{n_2(3-\alpha)}\left(\frac{|E_2\cap J||E_3\cap J|}{|J|}\right)^{\frac{3-\alpha}{2}} \\ 2^{n_3(3-\alpha)}|E\cap J|^{\frac{3-\alpha}{2}}. \end{cases} \tag{5.48}$$

Finally, interpolating between these three estimates one obtains,

$$\sum_{I\in\mathcal{J}_{n_1,n_2,n_3}} |I|^{\frac{3-\alpha}{2}} \lesssim \left(\sum_{I\in\mathcal{J}_{n_1,n_2,n_3}} |I|^{\frac{3-\alpha}{2}}\right)^{\theta_1}\left(\sum_{I\in\mathcal{J}_{n_1,n_2,n_3}} |I|^{\frac{3-\alpha}{2}}\right)^{\theta_2}\left(\sum_{I\in\mathcal{J}_{n_1,n_2,n_3}} |I|^{\frac{3-\alpha}{2}}\right)^{\theta_3}$$
$$\lesssim 2^{n_1(3-\alpha)\theta_1}|E_1\cap J|^{\frac{(3-\alpha)\theta_1}{2}} 2^{n_2(3-\alpha)\theta_2}\left(\frac{|E_2\cap J||E_3\cap J|}{|J|}\right)^{\frac{(3-\alpha)\theta_2}{2}} 2^{n_3(3-\alpha)\theta_3}|E\cap J|^{\frac{(3-\alpha)\theta_3}{2}}. \tag{5.49}$$

For every $0 \leqslant \theta_1, \theta_2, \theta_3 < 1$ such that $\theta_1 + \theta_2 + \theta_3 = 1$. So,

$$
\begin{aligned}
(5.45) \lesssim |E_1 \cap J|^{\frac{(3-\alpha)\theta_1}{2}} \left(\frac{|E_2 \cap J||E_3 \cap J|}{|J|} \right)^{\frac{(3-\alpha)\theta_2}{2}} |E \cap J|^{\frac{(3-\alpha)\theta_3}{2}} \\
\times \sum_{n_1, n_2, n_3 \in \mathbb{Z}_{\geqslant 0}} 2^{-n_1(1-(3-\alpha)\theta_1)} 2^{-n_2(1-(3-\alpha))\theta_2} 2^{-n_3(1-(3-\alpha))\theta_3} \\
\lesssim |E_1 \cap J|^{\frac{(3-\alpha)\theta_1}{2}} \left(\frac{|E_2 \cap J||E_3 \cap J|}{|J|} \right)^{\frac{(3-\alpha)\theta_2}{2}} |E \cap J|^{\frac{(3-\alpha)\theta_3}{2}} \qquad (5.50)
\end{aligned}
$$

as long as $\theta_i < \frac{1}{3-\alpha}$ for $i = 1, 2, 3$. Finally, we obtain the desired estimate

$$
||\widetilde{T}_{\alpha,J}(f, g, h) \cdot \chi_E||_{L^{1/2}} \lesssim |J|^{1-\frac{1}{p_2}} |E_1 \cap J|^{\frac{1}{p_1}} |E_2 \cap J|^{\frac{1}{p_2}} |E_3 \cap J|^{\frac{1}{p_2}} |E \cap J|^{2-\frac{1}{r}}, \qquad (5.51)
$$

where we take $\frac{1}{p_1} = (3-\alpha)\theta_1$, $\frac{1}{p_2} = (3-\alpha)\theta_2$, and $2-\frac{1}{r} = (3-\alpha)\theta_3$. In particular, the restriction on θ_i means that $p_1, p_2 > 1$, yet we can take $p = p_1 = p_2$ greater than but arbitrarily close to 1 and r greater than but arbitrarily close to $\frac{1}{1+\alpha}$. $\square$

CHAPTER 6

PROOF OF ESTIMATES FOR $T_{\alpha,\beta}$

6.1 Second Stopping-Time Argument: Proof of Theorem 4.3.1

Now we proceed to the proof of Theorem 4.3.1.

Proof. Let E_1, E_2, E_3, and E be sets of finite measure, and f, g, h be measurable functions such that $|f| \leqslant \chi_{E_1}$, $|g| \leqslant \chi_{E_2}$, $|h| \leqslant \chi_{E_3}$, and $|F| \leqslant \chi_E$. Dualizing through $L^{1/3}$ as in the theorem statement, we obtain:

$$
\begin{aligned}
||T_{\alpha,\beta}(f,g,h)\cdot F||_{L^{1/3}}^{1/3} &= \left|\left|\sum_{J\in\mathcal{D}} \frac{1}{|J|^{\beta}} \widetilde{T}_{\alpha,J}(f,g,h)\cdot \chi_J \cdot F\right|\right|_{L^{1/3}}^{1/3} \\
&\lesssim \sum_{J\in\mathcal{D}} \frac{1}{|J|^{\beta/3}} \left|\left|\widetilde{T}_{\alpha,J}(f,g,h)\cdot \chi_J \cdot F\right|\right|_{L^{1/3}}^{1/3} \\
&\lesssim \sum_{J\in\mathcal{D}} \frac{1}{|J|^{\beta/3}} \left|\left|\widetilde{T}_{\alpha,J}(f,g,h)\cdot \chi_E\right|\right|_{L^{1/2}(J)}^{1/3} ||\chi_E||_{L^1(J)}^{1/3} \\
&= \sum_{J\in\mathcal{D}} |J|^{\frac{1-\beta}{3}} \left|\left|\widetilde{T}_{\alpha,J}(f,g,h)\cdot \chi_E\right|\right|_{L^{1/2}(J)}^{1/3} \operatorname{avg}_J^{1/3}(\chi_E). && (6.1) \\
&\lesssim \sum_{J\in\mathcal{D}} |J|^{\frac{2-\beta-\frac{1}{p}}{3}} |E_1\cap J|^{1/3p} |E_2\cap J|^{1/3p} |E_3\cap J|^{1/3p} |E\cap J|^{(1/3\tilde{r})} \operatorname{avg}_J^{1/3}(\chi_E) && (6.2) \\
&= \sum_{J\in\mathcal{D}} |J|^{\frac{5-\alpha-\beta}{3}} \operatorname{avg}_{J,p}^{1/3}(\chi_{E_1}) \operatorname{avg}_{J,p}^{1/3}(\chi_{E_2}) \operatorname{avg}_{J,p}^{1/3}(\chi_{E_3}) \operatorname{avg}_{J,\tilde{r}}^{1/3}(\chi_E) \operatorname{avg}_J^{1/3}(\chi_E). && (6.3) \\
&= \sum_{J\in\mathcal{D}} |J|^{\frac{5-\alpha-\beta}{3}} \operatorname{avg}_{J,p}^{1/3}(\chi_{E_1}) \operatorname{avg}_{J,p}^{1/3}(\chi_{E_2}) \operatorname{avg}_{J,p}^{1/3}(\chi_{E_3}) \operatorname{avg}_{J,\tilde{\tilde{r}}}^{1/3}(\chi_E). && (6.4)
\end{aligned}
$$

Where (6.2) follows from the application of Theorem 5.9 with $p_1 = p_2 = p$. In particular, this imposes the relation

$$\frac{2}{p} + \frac{1}{\tilde{r}} = 3 - \alpha. \tag{6.5}$$

In (6.4) we combined the two averages of χ_E, into one with $\frac{1}{\tilde{\tilde{r}}} = 1 + \frac{1}{\tilde{r}}$. Now, we begin a second stopping-time argument. That is, define:

$$\mathcal{J}_{n_i} := \{J \in \mathcal{D} : \operatorname{avg}_{J,p}(\chi_{E_i})^{1/3} \simeq 2^{-n_i}\}, \tag{6.6}$$

for $i = 1, 2, 3$,

$$\mathcal{J}_{n_4} := \{J \in \mathcal{D} : \operatorname{avg}_{J,\tilde{\tilde{r}}}(\chi_E)^{1/3} \simeq 2^{-n_4}\} \tag{6.7}$$

$$\tag{6.8}$$

and $\mathcal{J} := \bigcap_{i=1}^{4} \mathcal{J}_{n_i}$. Then,

$$\begin{aligned}
(6.3) &= \sum_{n_1,\dots,n_4 \in \mathbb{Z}_{\geqslant 0}} 2^{-n_1} 2^{-n_2} 2^{-n_3} 2^{-n_4} \sum_{J \in \mathcal{J}} |J|^{\frac{5-\alpha-\beta}{3}} \\
&= \sum_{n_1,\dots,n_4 \in \mathbb{Z}_{\geqslant 0}} 2^{-n_1} 2^{-n_2} 2^{-n_3} 2^{-n_4} \sum_{\substack{J_o \in \mathcal{J} \\ J_o \text{ maximal}}} \sum_{J \subseteq J_o} |J|^{\frac{5-\alpha-\beta}{3}} \\
&\lesssim \sum_{n_1,\dots,n_4 \in \mathbb{Z}_{\geqslant 0}} 2^{-n_1} 2^{-n_2} 2^{-n_3} 2^{-n_4} \sum_{\substack{J_o \in \mathcal{J} \\ J_o \text{ maximal}}} |J_o|^{\frac{5-\alpha-\beta}{3}}.
\end{aligned} \tag{6.9}$$

Noting the $\frac{5-\alpha-\beta}{3} > 1$, we applied Lemma 5.2.2 to obtain (6.9), and we apply Lemma 5.2.3 to obtain four distinct bounds on the inner summation:

$$\sum_{J_o \in \mathcal{J}} |J_o|^{\frac{5-\alpha-\beta}{3}} \lesssim \begin{cases} 2^{pn_i(5-\alpha-\beta)} |E_i|^{\frac{5-\alpha-\beta}{3}} & \text{if } i = 1, 2, 3 \\ 2^{\tilde{\tilde{r}} n_4(5-\alpha-\beta)} |E|^{\frac{5-\alpha-\beta}{3}}. \end{cases} \tag{6.10}$$

Interpolating between these estimates, we have

$$\sum_{J_o \in \mathcal{J}} |J_o|^{\frac{5-\alpha-\beta}{3}} \lesssim \prod_{i=1}^{3} 2^{pn_i(5-\alpha-\beta)\theta_i} |E_i|^{\frac{(5-\alpha-\beta)\theta_i}{3}} 2^{\tilde{\tilde{r}}n_4(5-\alpha-\beta)\theta_4} |E|^{\frac{(5-\alpha-\beta)\theta_4}{3}}. \tag{6.11}$$

So,

$$\begin{aligned}
(6.9) \lesssim & |E_1|^{\frac{(5-\alpha-\beta)\theta_1}{3}} |E_2|^{\frac{(5-\alpha-\beta)\theta_2}{3}} |E_3|^{\frac{(5-\alpha-\beta)\theta_3}{3}} |E|^{\frac{(5-\alpha-\beta)\theta_4}{3}} \times \\
& \left(\sum_{n_1,\dots,n_4 \in \mathbb{Z}_{\geqslant 0}} 2^{-n_1(1-p(5-\alpha-\beta)\theta_1)} 2^{-n_2(1-p(5-\alpha-\beta)\theta_2)} 2^{-n_3(1-p(5-\alpha-\beta)\theta_3)} 2^{-n_4(1-\tilde{\tilde{r}}(5-\alpha-\beta))\theta_4} \right) \\
\lesssim & |E_1|^{\frac{(5-\alpha-\beta)\theta_1}{3}} |E_2|^{\frac{(5-\alpha-\beta)\theta_2}{3}} |E_3|^{\frac{(5-\alpha-\beta)\theta_3}{3}} |E|^{\frac{(5-\alpha-\beta)\theta_4}{3}}
\end{aligned} \tag{6.12}$$

as long as $\theta_i < \frac{1}{p(5-\alpha-\beta)}$ (for $i = 1,2,3$), and $\theta_4 < \frac{1}{\tilde{\tilde{r}}(5-\alpha-\beta)}$. This restriction on θ_i is compatible with exponent relation (6.5) since, we can always choose p arbitrarily close to 1, for which

$$1 = \sum_i \theta_i < \frac{1}{5-\alpha-\beta} \cdot \left(\frac{3}{p} + \frac{1}{\tilde{\tilde{r}}} \right) = \frac{4 + 1/p - \alpha}{5-\alpha-\beta}$$

holds true. So, taking $\frac{1}{p_1} = (5-\alpha-\beta)\theta_1$, $\frac{1}{p_2} = (5-\alpha-\beta)\theta_2$, $\frac{1}{p_3} = (5-\alpha-\beta)\theta_3$, and $3 - \frac{1}{r} = (5-\alpha-\beta)\theta_4$ we obtain the estimate

$$||T_{\alpha,\beta}(f,g,h) \cdot F||_{L^{1/3}} \lesssim |E_1|^{\frac{1}{p_1}} |E_2|^{\frac{1}{p_2}} |E_3|^{\frac{1}{p_3}} |E|^{3-\frac{1}{r}}. \tag{6.13}$$

with $\frac{1}{p_1} + \frac{1}{p_2} + \frac{1}{p_3} + \alpha + \beta = \frac{1}{r} + 2$, for $1 < p_1, p_2, p_3 < \infty$ and $0 < r < \infty$ as desired. □

6.2 Proof of Theorem 4.3.2

Next, we begin the proof of Theorem 4.3.2. This time, we will need to construct a major subset $\widetilde{E} \subseteq E$ in order to reach an L^1 end point restricted weak type estimate.

Proof. Let E_1, E_2, E_3, and E be sets of finite measure and $|f| \leqslant \chi_{E_1}$, $|g| \leqslant \chi_{E_2}$, and $|h| \leqslant \chi_{E_2}$. The first step is to notice that it is enough to prove the statement for the case $|E| = 1$. To see this, let us assume that for any E with $|E| = 1$ there exists a subset $\widetilde{E} \subseteq E$ with $|\widetilde{E}| \simeq |E| = 1$ for which the estimate

$$||T_{\alpha,\beta}(f,g,h) \cdot F||_{L^{1/3}} \lesssim |E_1|^{1/p_1}|E_2|^{1/p_2}|E_3|^{1/p_3} \tag{6.14}$$

holds. Then, the general claim follows for arbitrary E by a direct scaling argument. Namely, for any E of finite measure and $\lambda > 0$, we have

$$\begin{aligned} ||T_{\alpha,\beta}(f,g,h) \cdot F||_{L^{1/3}} &= \lambda^3 ||T_{\alpha,\beta}(f,g,h)(\lambda \cdot) \cdot F(\lambda \cdot)||_{L^{1/3}} \\ &= \lambda^{5-\alpha-\beta} ||T_{\alpha,\beta}(f_\lambda, g_\lambda, h_\lambda) \cdot F_\lambda||_{L^{1/3}}, \end{aligned} \tag{6.15}$$

where the subscript λ indicates scaling the respective function by λ. In particular, notice that (6.15) comes from the observation that

$$\begin{aligned} T_{\alpha,\beta}(f,g,h)(\lambda x) &= \int_{\mathbb{R}^2} \frac{f(\lambda x - t) g(\lambda x + t + s) h(\lambda x - s)}{|t|^\alpha |s|^\beta} \, dtdts \\ &= \lambda^{2-\alpha-\beta} \int_{\mathbb{R}^2} \frac{f_\lambda(x-t) g_\lambda(x+t+s) h_\lambda(x-s)}{|t|^\alpha |s|^\beta} \, dtds \\ &= \lambda^{2-\alpha-\beta} T_{\alpha,\beta}(f_\lambda, g_\lambda, h_\lambda)(x). \end{aligned} \tag{6.16}$$

Taking $\lambda = |E|$, ensures that $|F_\lambda| \leqslant \chi_{\frac{1}{|E|} \cdot E}$ and (6.15) together with our assumption implies that

$$||T_{\alpha,\beta}(f,g,h) \cdot F||_{L^{1/3}} \lesssim |E_1|^{1/p_1}|E_2|^{1/p_2}|E_3|^{1/p_3}|E|^{5-\alpha-\beta-\frac{1}{p_1}-\frac{1}{p_2}-\frac{1}{p_3}}, \tag{6.17}$$

which yields the desired estimate provided that $\frac{1}{p_1} + \frac{1}{p_2} + \frac{1}{p_3} + \alpha + \beta = \frac{1}{r} + 2$. So, it is enough to prove (6.14) under the assumption that $|E| = 1$ and $p_1 = 1$. Next, we proceed to the construction of the major subset.

Consider the set $\Omega \coloneqq \{x : \mathcal{M}(f)(x) > C|E_1|\}$. We will take our major subset to be $\widetilde{E} \coloneqq E \backslash \Omega$. Note that since $\mathcal{M}$ maps L^1 into $L^{1,\infty}$, for C large enough we have $|\Omega| < \frac{1}{2}|E|$. So, in fact $|\widetilde{E}| > \frac{1}{2}|E|$ as needed. In order to begin our primary stopping time argument, we define $\mathcal{J}_{n_1}, \mathcal{J}_{n_2}, \mathcal{J}_{n_3}$, and $\mathcal{J}_{n_1,n_2,n_3}$ as in the proof of Theorem 5.2.1, with the exception of replacing E with $\widetilde{E}$ in the definition of $\mathcal{J}_{n_3}$. The key observation here is that if $I \cap \Omega^C = \varnothing$, then $\text{avg}_I^{1/2}(\chi_{\widetilde{E}}) = 0$. So we need only consider the $I \subseteq J$ that intersect Ω^C in our summation. Moreover, for $I \in \mathcal{J}_{n_1}$, we have

$$2^{-2n_1} \simeq \frac{1}{|\widetilde{I}|}\int_{\widetilde{I}} f(x)dx \tag{6.18}$$

$$\leqslant M(f)(\bar{x}) \tag{6.19}$$

$$\lesssim |E_1|, \tag{6.20}$$

where we take $\bar{x} \in I \cap \Omega^C$, and we keep in mind that $I \subseteq \widetilde{I}$. Consequently, following similar steps to those of Theorem 4.2.1, we obtain

$$\frac{1}{|J|^{\frac{1}{2}}} \cdot ||\widetilde{T}_{\alpha,J}(f,g,h) \cdot \chi_{\widetilde{E}}||_{L^{1/2}}^{1/2} \lesssim \tag{6.21}$$

$$|E_1 \cap J|^{\frac{(3-\alpha)\theta_1}{2}} \left(\frac{|E_2 \cap J||E_3 \cap J|}{|J|}\right)^{\frac{(3-\alpha)\theta_2}{2}} |\widetilde{E} \cap J|^{\frac{(3-\alpha)\theta_3}{2}}$$

$$\times \sum_{\substack{n_1,n_2,n_3 \in \mathbb{Z}_{\geqslant 0} \\ 2^{-n_1} \leqslant C|E_1|^{\frac{1}{2}}}} 2^{-n_1(1-(3-\alpha)\theta_1)} 2^{-n_2(1-(3-\alpha))\theta_2} 2^{-n_3(1-(3-\alpha))\theta_3}$$

$$\lesssim |E_1 \cap J|^{\frac{1}{2}} \left(\frac{|E_2 \cap J||E_3 \cap J|}{|J|}\right)^{\frac{(3-\alpha)\theta_2}{2}} |\widetilde{E} \cap J|^{\frac{(3-\alpha)\theta_3}{2}}. \tag{6.22}$$

For every $0 \leqslant \theta_1, \theta_2, \theta_3 < 1$ such that $\theta_1 + \theta_2 + \theta_3 = 1$, as long as $\theta_i < \frac{1}{3-\alpha}$. So,

$$||\widetilde{T}_{\alpha,J}(f,g,h) \cdot \chi_{\widetilde{E}}||_{L^{1/2}} \lesssim |J|^{2+\frac{1}{p}} \text{avg}_{J,1}(\chi_{E_1}) \text{avg}_{J,p}(\chi_{E_2}) \text{avg}_{J,p}(\chi_{E_3}), \tag{6.23}$$

where we take $\frac{1}{p} = (3-\alpha)\theta_2$ for θ_2 less than but arbitrarily close to $\frac{1}{3-\alpha}$ so that p is greater than but arbitrarily close to 1. Then, proceeding as in (6.1) in the proof of Theorem 4.3.1 we obtain

$$||T_{\alpha,\beta}(f,g,h) \cdot F||_{L^{1/3}}^{1/3} \lesssim \sum_{J \in \mathcal{D}} |J|^{\frac{3+\frac{1}{p}-\beta}{3}} \text{avg}_{J,1}^{1/3}(\chi_{E_1}) \text{avg}_{J,p}^{1/3}(\chi_{E_2}) \text{avg}_{J,p}^{1/3}(\chi_{E_3}) \text{avg}_{J,1}^{1/3}(\chi_{\widetilde{E}}). \tag{6.24}$$

Now we conclude the proof with our second and final stopping time iteration. Define $\mathcal{J}_{n_1}, \mathcal{J}_{n_2}, \mathcal{J}_{n_3}$, and $\mathcal{J}_{n_4}$ analogously to those in the primary stopping time iteration. Notice that as in the primary iteration, the presence of $\text{avg}_{J,1}^{1/3}(\chi_{\tilde{E}})$ ensures that we only need to consider $J \in \mathcal{D}$ for which $J \cap \Omega^c \neq \emptyset$ and once again, we have $2^{-3n_1} \lesssim |E_1|$. Moreover, since $1/p$ can be taken arbitrarily close to 1, we can take it in particular to ensure that $\frac{3+\frac{1}{p}-\beta}{3} > 1$ and $\frac{3+\frac{1}{p}-\alpha}{3} > 1$ for the implicit symmetric argument. Then, proceeding as in the proof of Theorem 4.3.1 yields

$$||T_{\alpha,\beta}(f,g,h) \cdot F||_{L^{1/3}} \lesssim |E_1||E_2|^{1/p_2}|E_3|^{1/p_3} \tag{6.25}$$

where $1/p_2$ and $1/p_3$ are greater than, but can be taken arbitrarily close to 1. This concludes the proof of Theorem 4.3.2 for the case $p_1 = 1$. The case $p_2 = 1$ is proved by a symmetric argument where in the derivation of the model we instead group functions f and g together when deriving $\widetilde{\text{avg}}$. □

CHAPTER 7

MAIN RESULTS: PART II

7.1 Mixed Norm Estimates for $\mathcal{R}_{\xi_{1,1}\xi_{2,2}+(\xi_{3,3}-\xi_{4,3})}$

Theorem 7.1.1. *Let* $r = (r_1, r_2, r_3)$ *and* $p_i = (p_{i,1}, p_{i,2}, p_{i,3})$ *for* $i \in \{1, 2, 3\}$ *with* $0 < r_i \leqslant \infty, 1 \leqslant p_{i,1}, p_{i,2}, p_{i,3} \leqslant \infty$ *with the additional restriction that* $p_{3,3}, p_{4,3} > 1$ *and at most one of them equals infinity. Consider*

$$\mathcal{R}_{P_2}(f_1, f_2, f_3, f_4)(x) := \int_{\mathbb{R}^{12}} \frac{\widehat{f_1}(\xi_1)\widehat{f_2}(\xi_2)\widehat{f_3}(\xi_3)\widehat{f_4}(\xi_4)}{\xi_{1,1}\xi_{2,2} + (\xi_{3,3} - \xi_{4,3})} e^{2\pi i x\cdot(\xi_1+\xi_2+\xi_3+\xi_4)} d\xi_1 d\xi_2 d\xi_3 d\xi_4,$$

defined in the limiting sense of (1.17), for $x = (x_1, x_2, x_3), \xi_i = (\xi_{i,1}, \xi_{i,2}, \xi_{i,3}) \in \mathbb{R}^3$ *and* $f_i \in \mathcal{S}(\mathbb{R}^3)$*. Then,*

i) $|\mathcal{R}_{P_2}(f_1, f_2, f_3, f_4)(x)| \lesssim ||f_1(\cdot, x_2, x_3)||_{L^2_{x_1}} ||f_2(x_1, \cdot, x_3)||_{L^2_{x_2}} B^3_{1/2}(|f_3|, |f_4|)(x_1, x_2, x_3)$, *where* $B^3_{1/2}$ *is the multilinear fractional integral operators in (2.1) applied to the* x_3 *variable only.*

ii) $||\mathcal{R}_{P_2}(f_1, f_2, f_3)||_{L^{r_1}_{x_1} L^{r_2}_{x_2} L^{r_3}_{x_3}} \lesssim ||f_1||_{L^{p_{1,2}}_{x_2} L^{p_{1,3}}_{x_3} L^2_{x_1}} ||f_2||_{L^{p_{2,1}}_{x_1} L^{p_{2,3}}_{x_3} L^2_{x_2}} ||f_3||_{L^{p_{3,1}}_{x_1} L^{p_{3,2}}_{x_2} L^{p_{3,3}}_{x_3}} ||f_4||_{L^{p_{4,1}}_{x_1} L^{p_{4,2}}_{x_2} L^{p_{4,3}}_{x_3}}$ *and*

$$||\mathcal{R}_{P_2}(f_1, f_2, f_3)||_{L^{r_2}_{x_2} L^{r_1}_{x_1} L^{r_3}_{x_3}} \lesssim ||f_1||_{L^{p_{1,2}}_{x_2} L^{p_{1,3}}_{x_3} L^2_{x_1}} ||f_2||_{L^{p_{2,1}}_{x_1} L^{p_{2,3}}_{x_3} L^2_{x_2}} ||f_3||_{L^{p_{3,2}}_{x_2} L^{p_{3,1}}_{x_1} L^{p_{3,3}}_{x_3}} ||f_4||_{L^{p_{4,2}}_{x_2} L^{p_{4,1}}_{x_1} L^{p_{4,3}}_{x_3}},$$

with $\sum_{i=1}^4 \frac{1}{p_{i,3}} = \frac{1}{r_3} + \frac{1}{2}, \frac{1}{p_{1,2}} + \frac{1}{p_{3,2}} + \frac{1}{p_{4,2}} = \frac{1}{r_2}$*, and* $\frac{1}{p_{2,1}} + \frac{1}{p_{3,1}} + \frac{1}{p_{4,1}} = \frac{1}{r_1}$*.*

Proof. As in (1.20), we write

$$\mathcal{R}(f_1, f_2, f_3, f_4)(x) = \Big(\sum_k \int_{\mathbb{R}^{13}} \psi_k(\alpha)\hat{f_1}(\xi_1)\hat{f_2}(\xi_2)\hat{f_3}(\xi_3)\hat{f_4}(\xi_4) e^{2\pi i\alpha(\xi_{1,1}\xi_{2,2}+(\xi_{3,3}-\xi_{4,3}))} \times e^{2\pi i x\cdot(\xi_1+\xi_2+\xi_3+\xi_4)} d\alpha d\xi_1 d\xi_2 d\xi_3 d\xi_4\Big) \quad (7.1)$$

$$= \Big(\sum_k \int_{\mathbb{R}^7} \psi_k(\alpha)\hat{f}_1(\xi_1)\hat{f}_2(\xi_2)f_3(x_1,x_2,x_3+\alpha)f_4(x_1,x_2,x_3-\alpha)$$

$$\times e^{2\pi i\alpha\xi_{1,1}\xi_{2,2}}e^{2\pi i x\cdot(\xi_1+\xi_2)}d\xi_1 d\xi_2 d\alpha\Big) \qquad (7.2)$$

$$= \Big(\sum_k \int_{\mathbb{R}^3} \psi(\alpha)F_1(\xi_{1,1},x_2,x_3)e^{2\pi i x_1\xi_{1,1}}F_2(x_1,\xi_{2,2},x_3)e^{2\pi i x_2\xi_{2,2}}e^{2\pi i\alpha\xi_{1,1}\xi_{2,2}}$$

$$\times f_3(x_1,x_2,x_3+\alpha)f_4(x_1,x_2,x_3-\alpha)d\xi_{1,1}d\xi_{2,2}d\alpha\Big) \qquad (7.3)$$

Then, using (1.18), we have

$$|\mathcal{R}(f_1,f_2,f_3,f_4)(x)| \lesssim \int_{\mathbb{R}}\Big|\int_{\mathbb{R}^2} F_1(\xi_{1,1},x_2,x_3)e^{2\pi i x_1\xi_{1,1}}F_2(x_1,\xi_{2,2},x_3)e^{2\pi i x_2\xi_{2,2}}e^{2\pi i\alpha\xi_{1,1}\xi_{2,2}}$$

$$\times f_3(x_1,x_2,x_3+\alpha)f_4(x_1,x_2,x_3-\alpha)d\xi_{1,1}d\xi_{2,2}\Big|\,d\alpha$$

$$\lesssim ||f_1(\cdot,x_2,x_3)||_{L^2_{x_1}}||f_2(x_1,\cdot,x_3)||_{L^2_{x_2}}B^3_{1/2}(|f_3|,|f_4|)(x_1,x_2,x_3). \qquad (7.4)$$

Where F_i is the inverse Fourier transform of $\hat{f}_i$ in all but the i^{th} variable, and in the last inequality we apply (2.23) followed by Plancherel. This completes the proof of (i). Then, (ii) follows by applying Hölder together with Theorem 2.1.2.

□

Remark 7.1.2. *In particular, Theorem 7.1.1 implies the diagonal estimates*

$$||\mathcal{R}_{P_2}(f_1,f_2,f_{3,f_4})||_{L^r(\mathbb{R}^3)} \lesssim ||f_1||_{L^2(\mathbb{R}^3)}||f_2||_{L^2(\mathbb{R}^3)}||f_3||_{L^{p_3}(\mathbb{R}^3)}||f_4||_{L^{p_4}(\mathbb{R}^3)},$$

with $\frac{1}{p_3}+\frac{1}{p_4}+\frac{1}{2}=\frac{1}{r}$.

7.2 Mixed Norm Estimates for $\mathcal{R}_{(\xi_{1,1}\xi_{2,2}+\xi_{5,3}-\xi_{6,3})(\xi_{3,3}\xi_{4,3}+\xi_{5,3}-\xi_{7,3})}$

Consider the operator

$$\mathcal{R}_{P_4}(f_1,\ldots,f_7)(x) = \int_{\mathbb{R}^{21}} \frac{\widehat{f_1}(\xi_1)\widehat{f_2}(\xi_2)\widehat{f_3}(\xi_3)\ldots\widehat{f_7}(\xi_y)}{(\xi_{1,1}\xi_{2,2}+\xi_{5,3}-\xi_{6,3})(\xi_{3,3}\xi_{4,3}+\xi_{5,3}-\xi_{7,3})} e^{2\pi i x\cdot(\xi_1+\cdots+\xi_7)} d\xi_1\ldots d\xi_7 \tag{7.5}$$

defined in the limiting sense:

$$\mathcal{R}_{P_4}(f_1,\ldots,f_7)(x) \coloneqq \sum_{(k,j)\in\mathbb{Z}^2} \Big(\int_{\mathbb{R}^{21}} \psi_k(\xi_{1,1}\xi_{2,2}+\xi_{5,3}-\xi_{6,3})\psi_j(\xi_{3,3}\xi_{4,3}+\xi_{5,3}-\xi_{7,3}) \times \widehat{f_1}(\xi_1)\widehat{f_2}(\xi_2)\widehat{f_3}(\xi_3)\ldots\widehat{f_7}(\xi_y) e^{2\pi i x\cdot(\xi_1+\cdots+\xi_7)} d\xi_1\ldots d\xi_7 \Big) \tag{7.6}$$

where ψ_k and ψ_j are as in (1.18).

Theorem 7.2.1. *Let $r = (r_1, r_2, r_3)$ and $p_i = (p_{i,1}, p_{i,2}, p_{i,3})$ for $i \in \{1, 2, \ldots, 7\}$ with $0 < r_j \leqslant \infty$, $1 \leqslant p_{i,1}, p_{i,2}, p_{i,3} \leqslant \infty$. Then,*

i) $|\mathcal{R}_{P_3}(f_1, f_2, \ldots, f_7)(x)| \lesssim ||f_1(\cdot, x_2, x_3)||_{L^2_{x_1}} ||f_2(x_1, \cdot, x_3)||_{L^2_{x_2}} ||f_3(x_1, x_2, \cdot)||_{L^2_{x_3}} \times ||f_4(x_1, x_2, \cdot)||_{L^2_{x_3}} T^3_{1/2,1/2}(|f_5|, |f_6|, |f_7|)(x_1, x_2, x_3)$

where $T^3_{1/2,1/2}$ is the operator in (2.3) applied in only the x_3 variable.

ii) $||\mathcal{R}_{P_3}(f_1, \ldots, f_7)||_{L^{r_1}_{x_1} L^{r_2}_{x_2} L^{r_3}_{x_3}} \lesssim ||f_1||_{L^{p_{1,2}}_{x_2} L^{p_{1,3}}_{x_3} L^2_{x_1}} ||f_2||_{L^{p_{2,1}}_{x_1} L^{p_{2,3}}_{x_3} L^2_{x_2}} ||f_3||_{L^{p_{3,1}}_{x_1} L^{p_{3,2}}_{x_2} L^2_{x_3}} ||f_4||_{L^{p_{4,1}}_{x_1} L^{p_{4,2}}_{x_2} L^2_{x_3}}$

$||f_5||_{L^{p_{5,1}}_{x_1} L^{p_{5,2}}_{x_2} L^{p_{5,3}}_{x_3}} ||f_6||_{L^{p_{6,1}}_{x_1} L^{p_{6,2}}_{x_2} L^{p_{6,3}}_{x_3}} ||f_7||_{L^{p_{7,1}}_{x_1} L^{p_{7,2}}_{x_2} L^{p_{7,3}}_{x_3}}$

and

$$||\mathcal{R}_{P_3}(f_1,\ldots,f_7)||_{L^{r_2}_{x_2}L^{r_1}_{x_1}L^{r_3}_{x_3}} \lesssim ||f_1||_{L^{p_{1,2}}_{x_2}L^{p_{1,3}}_{x_3}L^2_{x_1}} ||f_2||_{L^{p_{2,1}}_{x_1}L^{p_{2,3}}_{x_3}L^2_{x_2}} ||f_3||_{L^{p_{3,2}}_{x_2}L^{p_{3,1}}_{x_1}L^2_{x_3}} ||f_4||_{L^{p_{4,2}}_{x_2}L^{p_{4,1}}_{x_1}L^2_{x_3}}$$

$$||f_5||_{L^{p_{5,2}}_{x_2}L^{p_{5,1}}_{x_1}L^{p_{5,3}}_{x_3}} ||f_6||_{L^{p_{6,2}}_{x_2}L^{p_{6,1}}_{x_1}L^{p_{6,3}}_{x_3}} ||f_7||_{L^{p_{7,2}}_{x_2}L^{p_{7,1}}_{x_1}L^{p_{7,3}}_{x_3}}$$

with $\frac{1}{p_{1,2}}+\frac{1}{p_{2,3}}+\frac{1}{p_{5,3}}+\frac{1}{p_{6,3}}+\frac{1}{p_{7,2}}=\frac{1}{r_3}+1$, $\frac{1}{p_{1,2}}+\sum_{i=3}^{7}\frac{1}{p_{i,2}}=\frac{1}{r_2}$, $\sum_{i=2}^{7}\frac{1}{p_{i,1}}=\frac{1}{r_1}$, *and the additional restriction that* $p_{5,3}, p_{6,3}, p_{7,3} > 1$ *and are subject to the restrictions in Theorem 3.2.3 for the* L^∞ *endpoint case.*

Proof. Iterating (1.20) twice,

$$\begin{aligned}
\mathcal{R}_{P_3}(f_1,f_2,\ldots,f_7)(x) &= \Big(\sum_{(k,j)\in\mathbb{Z}^2}\int_{\mathbb{R}^{23}}\psi_k(t)\psi_j(s)\widehat{f_1}(\xi_1)\widehat{f_2}(\xi_2)\ldots\widehat{f_7}(\xi_7)e^{2\pi i s(\xi_{1,1}\xi_{2,2}+\xi_{5,3}-\xi_{6,3})}\\
&\qquad\times e^{2\pi i t(\xi_{3,3}\xi_{4,3}+\xi_{5,3}-\xi_{7,3})}e^{2\pi i x\cdot(\xi_1+\cdots+\xi_7)}\,dsdtd\xi_1\ldots d\xi_7\Big)\\
&= \Big(\sum_{(k,j)\in\mathbb{Z}^2}\int_{\mathbb{R}^{14}}\psi_k(t)\psi_j(s)\widehat{f_1}(\xi_1)e^{2\pi i x_1\xi_{1,1}}\widehat{f_2}(\xi_2)e^{2\pi i x_2\xi_{2,2}}\widehat{f_3}(\xi_3)e^{2\pi i x_3\xi_{3,3}}\widehat{f_4}(\xi_4)e^{2\pi i x_4\xi_{4,3}}\\
&\qquad\times f_5(x_1,x_2,x_3+s+t)f_6(x_1,x_2,x_3-s)f_7(x_1,x_2,x_3-t)e^{2\pi i s\xi_{1,1}\xi_{2,2}}e^{2\pi i t\xi_{3,3}\xi_{4,3}}\\
&\qquad\times e^{2\pi i x_1(\xi_{2,1}+\xi_{3,1})}e^{2\pi i x_2(\xi_{1,2}+\xi_{3,2})}e^{2\pi i x_3(\xi_{1,3}+\xi_{2,3})}d\xi_1d\xi_2d\xi_3d\xi_4dsdt\Big)\\
&= \Big(\sum_{(k,j)\in\mathbb{Z}^2}\int_{\mathbb{R}^{6}}\psi_k(t)\psi_j(s)F_1(\xi_{1,1},x_2,x_3)e^{2\pi i x_1\xi_{1,1}}F_2(x_2,\xi_{2,2},x_3)e^{2\pi i x_2\xi_{2,2}}F_3(x_1,x_2,\xi_{3,3})e^{2\pi i x_3\xi_{3,3}}\\
&\qquad\times F_4(x_1,x_2,\xi_{4,3})e^{2\pi i x_3\xi_{4,3}}e^{2\pi i s\xi_{1,1}\xi_{2,2}}e^{2\pi i t\xi_{3,3}\xi_{4,3}}f_5(x_1,x_2,x_3+s+t)\\
&\qquad\times f_6(x_1,x_2,x_3-s)f_7(x_1,x_2,x_3-t)\;d\xi_{1,1}d\xi_{2,2}d\xi_{3,3}d\xi_{4,3}dsdt\Big)
\end{aligned} \tag{7.7}$$

Moreover, by our choice of ψ_k and ψ_j the computation above implies

$$|\mathcal{R}_{P_3}(f_1, f_2, \ldots, f_7)(x)| \lesssim$$

$$\int_{\mathbb{R}^2} \Big| \int_{\mathbb{R}^4} F_1(\xi_{1,1}, x_2, x_3) e^{2\pi i x_1 \xi_{1,1}} F_2(x_2, \xi_{2,2}, x_3) e^{2\pi i x_2 \xi_{2,2}} F_3(x_1, x_2, \xi_{3,3}) e^{2\pi i x_3 \xi_{3,3}} F_4(x_1, x_2, \xi_{4,3})$$

$$\times e^{2\pi i x_3 \xi_{4,3}} e^{2\pi i s \xi_{1,1}\xi_{2,2}} e^{2\pi i t \xi_{3,3}\xi_{4,3}} f_5(x_1, x_2, x_3 + s + t) f_6(x_1, x_2, x_3 - s)$$

$$\times f_7(x_1, x_2, x_3 - t)\, d\xi_{1,1} d\xi_{2,2} d\xi_{3,3} d\xi_{4,3} ds dt \Big|\, ds dt$$

$$\lesssim ||f_1(\cdot, x_2, x_3)||_{L^2_{x_1}} ||f_2(x_1, \cdot, x_3)||_{L^2_{x_2}} ||f_3(x_1, x_2, \cdot)||_{L^2_{x_3}}$$

$$\times ||f_4(x_1, x_2, \cdot)||_{L^2_{x_3}} T^3_{1/2,1/2}(|f_5|, |f_6|, |f_7|)(x_1, x_2, x_3).$$

where as before F_i denotes the inverse Fourier transform of $\widehat{f_i}$ taken in all but the i^{th} variable, and the last inequality follows from Hörmander's oscillatory integral theorem. Then, as before (ii) follows from Hölder and Theorem 3.1.1.

□

BIBLIOGRAPHY

[1] C. Benea, C. Muscalu. *Quasi-Banach valued inequalities via the helicoidal method*. J. Funct. Anal. 273 (2017), no. 4, 1295–1353.

[2] C. Benea, C. Muscalu. *Mixed-Norm Estimates via the Helicoidal Method.* Preprint, ArXiv:2007.01080.

[3] H. J. Brascamp, E. H. Lieb. *Best constants in Young's inequality, its converse, and its generalization to more than three functions.* Adv. Math. 20 (1976), 151–173.

[4] T. Cazenave. *Semilinear Schrödinger Equations, Courant Lect. Notes vol. 10.* New York University, Courant Institute of Mathematical Sciences. American Mathematical Society, 2003.

[5] R. Coifman, Y. Meyer. *Wavelets, Calderón-Zygmund Operators and Multilinear Operators.* Cambridge University Press, 1997.

[6] F. Di Plinio, K. Li, H. Martikainen, E. Vuorinen. *Multilinear singular integrals on non-commutative L^p spaces.* Math. Ann. 378 (2020), no. 3-4, 1371–1414.

[7] L. Grafakos, N. Kalton. *Some remarks on multilinear maps and interpolation.* Math. Ann. 319 (2001), 151–180.

[8] L. Grafakos. *On multilinear fractional integrals.* Studia Math. 102 (1992), No. 1, 49–56.

[9] L. Grafakos, L. Xiaochun. *Uniform bounds for the bilinear Hilbert transforms, I.* Ann. of Math. 159 (2004), 889-933.

[10] L.I. Hedberg. *On Certain Convolution Inequalities.* Proceedings of the American Mathematical Society, 36.2 (1972), 505–510.

[11] L. Hörmander. *Oscillatory Integrals and Multipliers on FL^p.* Ark. Mat. 11 (1973), 1-11.

[12] C. Kenig, E. Stein. *Multilinear estimates and fractional integration.* Math. Res. Lett. 6 (1999), no. 1, 1-15.

[13] V. Kokilashvili, M. Mastyło, A. Meskhi. *On the Boundedness of Multilinear Fractional Integral Operators.* J. Geom. Anal. 30 (2020), 667–679.

[14] S. Kuk, S. Lee. *Endpoint bounds for multilinear fractional integrals.* Math. Res. Lett. 19 (2012), no. 5, 1145-1154.

[15] M. Lacey, C. Thiele. *On Calderón's conjecture.* Ann. of Math. 149 (1999), (2), 475–496.

[16] C. Muscalu, T. Tao, C. Thiele. *L^p estimates for the Biest II. The Fourier Case.* Math. Ann. 329 (2004), (3), 427-461.

[17] C. Muscalu and W. Schlag. *Classical and Multilinear Harmonic Analysis Volume II.* Cambridge University Press, 2013.

[18] R. Oberlin, C. Thiele. *New uniform bounds for a Walsh model of the bilinear Hilbert transform.* Indiana Univ. Math. J. 60 (2011), (5), 1693-1712.

[19] C. Thiele. *A uniform estimate.* Ann. of Math. 156 (2) (2002), (2), 519-563.

[20] L. Xiaochun. *Uniform bounds for the bilinear Hilbert transforms, II.* Rev. Math. Iberoamericana, 22 (2006), no. 3, 1069-1126.