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REFORMULATIONS OF ECONOMIC EQUILIBRIUM PROBLEMS FOR SOLUTION BY QUASI-NEWTON AND SIMPLICIAL ALGORITHMS

By

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Abstract

We discuss various aspects of economic equilibrium problems which suggest certain reformulations before applying solution procedures. Many of the ideas are motivated by quasi-Newton algorithms but they are also useful for piecewise-linear homotopy methods. We also propose a new quasi-Newton update formula for a subclass of equilibrium problems. Limited computational experiments are presented.

1. Introduction

A number of computational procedures can be applied to approximate equilibria in economic models. These methods generate a sequence of price vectors or a sequence of utility weights using either successive linearizations, a sequence of mathematical programming problems, or a homotopy algorithm; see for example the other papers in this volume and those in Scarf and Shoven (1984). In general the homotopy algorithms are guaranteed to converge, but on large problems usually require more computation than the other approaches. One particular method that appears to be very efficient is the successive linear complementarity problem technique of Mathiesen (1983)--see also his paper in this volume.

Here we are concerned with reformulations of equilibrium problems that may render them more amenable to solution approaches that generate sequences of price vectors. These ideas were motivated by quasi-Newton algorithms but are also useful for (piecewise-linear) homotopy methods, for which global convergence can be maintained while employing reformulations for efficiency. We also propose a new quasi-Newton update formula for certain equilibrium problems.

In Section 2 we describe the three problems with which we are concerned and obtain an explicit form for the nonlinear functions involved in one case. Section 3 presents the Newton and quasi-Newton philosophies, while in Section 4 we discuss various formulations of economic equilibrium problems. Section 5 describes the new update formula and why it may be of interest. Finally, Section 6 contains the results of some limited computational experimentation.

2. Problems

In all cases there are n goods indexed by i and m consumers indexed by j. Lower-case roman letters generally denote n—vectors, while lower-case greek letters are reserved for scalars. Upper-case roman letters denote matrices, except that diagonal matrices are denoted by upper-case greek letters. We use nonnegative superscripts for indexing, so that possibly nonnegative powers are indicated by enclosing their arguments in parentheses.

A. Partial Equilibrium, Pure Trade

The economy is described by the following for each consumer j: his initial endowment ω^j of money, his consumption set $\mathbb{R}^n_+$ and a utility function $u^j\colon \mathbb{R}^n_+ \to \mathbb{R}$. There is also an initial endowment $w \in \mathbb{R}^n$ of goods--we assume $w > 0$ (i.e., each component of w is positive).

For given prices $p \geq 0$, consumer j maximizes u^j over his or her budget set

$$\{x^j \in \mathbb{R}^n_+\colon\ p^T x^j \leq \omega^j\};$$

we assume that the maximizing vector exists and is a continuous function d^j of the price vector p. The excess demand vector is then

$$g(p) := \Sigma_j\ d^j(p) - w.$$

We seek a price vector p with

$$p \geq 0,\ g(p) \leq 0,\ p^T g(p) = 0. \qquad (1)$$

The fact that the demand functions d^j arise from utility maximization, even with several regularity assumptions on the u^j's, imposes very little structure besides continuity on the aggregate excess demand function g (Debreu (1974)). However, we will be interested in instances of (1) in which $-g$ is strictly monotone, i.e.

$$(p-q)^T(g(p)-g(q)) < 0 \qquad (2)$$

for all distinct $p,q \geq 0$. Such a property can be deduced under certain hypotheses with a continuum of consumers (Hildenbrandt (1983)). If g is continuously differentiable, (2) follows if g' is negative definite everywhere. It may also be true that g' is negative definite in a neighborhood of a solution to (1). This may occur if the income effects are (locally) dominated by the substitution effects.

B. General Equilibrium, Pure Trade

Here the consumers have an income determined by the sale of their initial endowment vectors $w^j \in \mathbb{R}^n_+$, and the total initial endowment is $w = \Sigma_j w^j$, which we assume to be strictly positive. Hence consumer j maximizes u^j over

$$\{x^j \geq 0: p^T x^j \leq p^T w^j\},$$

yielding his demand $d^j(p)$. Again we set $g(p) = \Sigma_j d^j(p) - w$, and assume that g is defined and continuous at all $p \geq 0$, $p \neq 0$, and moreover satisfies Walras' law $p^T g(p) = 0$. We seek a price vector p with

$$p \geq 0,\ p \neq 0, \quad \text{and} \quad g(p) \leq 0. \tag{3}$$

Because of the homogeneity of g we can normalize p, for example by requiring $e^T p = 1$ where e is a vector of ones of appropriate dimension. Then the problem reduces to one on the price simplex $S^{n-1} = \{p \in \mathbb{R}^n: p \geq 0, e^T p = 1\}$ of dimension $n-1$. As we shall see, however, this is not a particularly natural normalization.

Example: Suppose the utility functions are of constant elasticity of substitution (C.E.S.) so that for appropriate $a_{ij} \geq 0$ and $\sigma^j > 0$ we have

$$u^j(x^j) := \begin{cases} \Sigma_i\ (a_{ij})^{\sigma^j} (x_{ij})^{1-\sigma^j} & \sigma^j \neq 1 \\ \Sigma_i\ a_{ij} \log x_{ij} & \sigma^j = 1. \end{cases}$$

Then the economy is described by the $n \times m$ matrices $A = (a_{ij}) \geq 0$ and $W = (w^1, \ldots, w^m) \geq 0$ and the vector $(\sigma^1, \ldots, \sigma^m) > 0$. Here $We = w > 0$; we also assume $e^T A > 0$, so that each consumer desires something, and by rescaling the utility functions we can then suppose that $e^T A = e^T$. Let us write

$$\Omega := \operatorname{diag}(w) \quad \text{and} \quad \Pi := \operatorname{diag}(p); \tag{4}$$

then for $p > 0$ we find

$$g(p) = \Pi^{-1}(\bar{A}(p)W^T - \Omega)p \tag{5}$$

where

$$\begin{aligned} \bar{A}(p) &:= [\bar{a}^1(p),\ldots,\bar{a}^m(p)] \text{ and} \\ \bar{a}^j(p) &:= (\Pi)^{(1-1/\sigma^j)}a^j/e^T(\Pi)^{(1-1/\sigma^j)}a^j. \end{aligned} \tag{6}$$

and where a^j denotes the jth column of the matrix A. Note that if all σ^j's equal one (i.e., we have Cobb-Douglas utility functions), then $\bar{A}(p) = A$ is constant. For further results on Cobb-Douglas economies see Eaves (1985). Let us make a simple observation that follows immediately from (5): for any Cobb-Douglas economy given by m, A, W there is another given by m', A', W' with $w = We = w' = W'e$, the same excess demand function g, and $m' = n$. This follows by taking $W' = \Omega$ and $A' = (AW^T)\Omega^{-1}$. This is a concrete illustration of the fact that any continuous function g on S^{n-1} satisfying Walras' law is essentially the excess demand function of an economy with n consumers--see Debreu (1974).

Most utility functions used in economic modelling are of C.E.S. type (sometimes nested) and thus the form of the excess demand function in (5) can be a useful guide for devising algorithms to compute equilibria.

C. General Equilibrium, Activity Analysis Production

The consumption side of this economy is described as in case B, but we add a production side. There are ℓ producers indexed by k; the kth has a production set $\{b^k\zeta^k: \zeta^k \geq 0\}$ for some vector b^k of inputs and outputs. We let

$$B := [b^1,\ldots,b^\ell]$$

and assume that free disposal is allowed, so that $B = [-I,B']$. We seek a price vector p and a vector of activity levels z with

$$p \neq 0,\ g(p) = Bz,\ B^T p \leq 0 \quad \text{and} \quad z \geq 0. \qquad (7)$$

Note that $B^T p \leq 0$ implies $p \geq 0$, and again we can renormalize by requiring $p \in S^{n-1}$.

Units

Observe that the price p_i of the ith good is measured in a unit that is the reciprocal of the unit of measurement of good i; $g_i(p)$ and w_i are measured in this unit; and $p_i g_i(p)$ and $p_i w_i$ are dimensionless. (We suppose "money" is dimensionless.) In particular, this implies that normalizations such as $e^T p = \Sigma_i\ p_i = 1$ are somewhat meaningless. It may be useful to view p and $g(p)$ as lying in dual abstract vector spaces; a choice of particular units for the goods specifies particular dual bases for these spaces, and then p_i denotes the component of p corresponding to the ith basis element.

3. Newton and Quasi-Newton Philosophies

In this section we discuss the basic concepts of Newton and quasi-Newton approaches to the solution of hard nonlinear problems. These concepts are used in the next section to motivate certain "natural" reformulations of equilibrium problems.

The general form of a finite-dimensional nonlinear problem is: find $x \in R^n$ such that $P(x)$ holds, where P is some property involving nonlinear functions. The main idea of a Newton-like approach is the following algorithm schema:

Iteration k: Given a current trial point $x^k \in \mathbb{R}^n$:

Test x^k for convergence;

Approximate the functions appearing in P by simpler (usually linear) functions to get P^k;

Solve the model problem: find x^+ such that $P^k(x^+)$ holds;

Update: Use x^+ to get the new trial point x^{k+1} (usually $x^{k+1} := x^+$) and proceed to iteration k+1.

The difference between Newton and quasi-Newton approaches is that the former approximate functions by first- or second-order Taylor approximations, while the latter employ information obtained about the nonlinear functions by evaluations at x^k and x^{k+1} to update the approximations used at iteration k to those for iteration $k+1$.

This general schema includes Newton's method for nonlinear equations (using linear approximations) and for unconstrained minimization (using quadratic approximations), and also embraces naturally the approach to solving nonlinear complementarity problems (e.g., (1)) by solving successive linear complementarity problems. (Eaves (1983) has shown that the latter technique enjoys some of the desirable properties of Newton's method for nonlinear equations.)

Let us illustrate two applications of the quasi-Newton approach.

Nonlinear equations. Find x such that $f(x) = 0$, where $f: \mathbb{R}^n \to \mathbb{R}^n$. Thus $P(x)$ corresponds to $f(x) = 0$ and we let $P^k(x)$ correspond to $f^k(x) = 0$, where

$$f^k(x) := f(x^k) + J^k(x-x^k). \tag{8}$$

We let $x^{k+1} = x^+$ (this is the simplest choice) and

$$J^{k+1} := J^k + \frac{(y-J^k s)s^T}{s^T s}, \qquad (9)$$

where $s = x^{k+1} - x^k$, $y = f(x^{k+1}) - f(x^k)$. Note that J^{k+1} satisfies the secant condition $J^{k+1}s = y$, thus incorporating new information about the function f. This is the algorithm of Broyden (1965); in particular, (9) is known as Broyden's (first or good) update. Given that we have either a factorization of J^k or its inverse explicitly, x^+ is easy to obtain from (8); and since J^{k+1} in (9) is a rank-one update of J^k, a factorization of J^{k+1} or its explicit inverse is easy to derive from one for J^k.

Unconstrained minimization. Find a local minimizer of a C^2 function $\phi: \mathbb{R}^n \to \mathbb{R}$. Then $P(x)$ corresponds to x being a local minimizer of ϕ and $P^k(x)$ to x being a local minimizer of ϕ^k, with

$$\phi^k(x) := \phi(x^k) + f(x^k)^T(x-x^k) + \frac{1}{2}(x-x^k)^T H^k(x-x^k)$$

where $f \equiv \nabla\phi$ (we assume that the gradient of ϕ can be evaluated) and $H^k \approx \nabla^2\phi(x^k)$. If we insist that H^k be symmetric and positive definite then ϕ^k has a unique global minimizer x^+, and the direction x^+-x^k is a descent direction for ϕ. Thus we choose $x^{k+1} = x^k + \lambda(x^+-x^k)$, with $\lambda \geq 0$ selected to give "sufficient decrease" in ϕ, and to assure that $y^T s > 0$ with y and s as above. Finally we set

$$H^{k+1} := H^k - \frac{H^k s s^T H^k}{s^T H^k s} + \frac{y y^T}{y^T s} ; \qquad (10)$$

this update is independently due to Broyden, Fletcher, Goldfarb and Shanno (see Dennis and Schnabel (1983)) and is guaranteed to be symmetric and positive definite if H^k is and $y^T s > 0$.

These two examples are to illustrate that (i) there is no need to take $x^{k+1} = x^+$; and (ii) there is no need to make linear approximations, only to obtain tractable subproblems (i.e. model problems that can be solved easily).

4. Natural Formulations of Economic Equilibrium Problems

In the partial equilibrium situation, case A, we are trying to solve the nonlinear complementarity problem (1). According to our discussion in the previous section, it is natural to solve a sequence of linear complementarity problems of the form:

$$\text{find} \quad p \geq 0 \quad \text{with} \quad g^k(p) \leq 0 \quad \text{and} \quad p^T g^k(p) = 0 \qquad (11)$$

where

$$g^k(p) := g(p^k) + J^k(p-p^k). \qquad (12)$$

This approach has been suggested by Eaves (1983), Hogan (1975) and Mathiesen (1983). However, it is worth noting that under several reasonable assumptions we know that an equilibrium price vector will have all components positive; in this case we are equivalently searching for a (positive) solution to $g(p) = 0$, and then nonlinear scalings can be applied to g, as we shall see later.

In the general equilibrium pure trade case B, we might similarly consider linearizing g. However here the problem is just (n-1)-dimensional; p can be restricted to S^{n-1} and g(p) satisfies $p^T g(p) = 0$. We can convert the problem to one in $\mathbb{R}^{n-1}$ by choosing an n×(n-1) matrix Z whose columns are orthonormal and orthogonal to $e \in \mathbb{R}^n$ (see e.g. Section 6 of Awoniyi and Todd (1983)). Also let $p^0 = e/n \in S^{n-1}$. Then the problem is equivalently:

$$\text{find} \quad x \in \mathbb{R}^{n-1} \quad \text{with} \quad p^0+Zx \geq 0, \quad g(p^0+Zx) \leq 0,$$

or, if we know that all prices will be positive, to

$$\text{find} \quad x \in \mathbb{R}^{n-1} \quad \text{with} \quad \bar{g}(x) = Z^T g(p^0+Zx) = 0 \qquad (13)$$
$$(\text{and} \quad p^0+Zx \geq 0).$$

Note that $-\bar{g}$ is strictly monotone if $-g$ is.

Finally, in the general equilibrium with production case C, we similarly arrive at an equivalent problem:

$$\begin{aligned} &-\bar{g}(x) + Z^T Bz = 0; \\ &B^T Zx + s = -B^T p^0; \qquad (14) \\ &s \geq 0, \; z \geq 0, \; s^T z = 0; \end{aligned}$$

with $\bar{g}$ as above. Thus in these two cases, we obtain a system of linear equations or a linear complementarity problem by replacing $\bar{g}$ by a linear approximation

$$\bar{g}^k(x) := \bar{g}(x^k) + J^k(x-x^k). \qquad (15)$$

Natural questions that arise concern existence and uniqueness of solutions to the model problems, and how J^k should be updated--in particular is the Broyden update (9) appropriate? We address this last point first.

Note that the Broyden update includes the term s^Ts in the denominator. This implies that the euclidean norm is taken to be a reasonable measure of the distance between two price vectors (or the vectors in $\mathbb{R}^{n-1}$ corresponding to the price vectors in the general equilibrium cases). However, our discussion of the arbitrariness of units suggests that the euclidean norm is not a reasonable measure. The presence of the s^Ts term shows that this arbitrariness affects the algorithm; more precisely, Broyden's update is not invariant under scalings of the domain. To fix this problem, we use a natural scaling of the problem: let

$$q = \Omega p = (w_i p_i)$$

be the new independent variable. This reformulation is equivalent to scaling the units of the goods so that the total amount available of each good is one, and is possible if $w > 0$, as in case A and B. In case C, with production, it can happen that there is no endowment of some goods. In this case, we can scale so that the maximum possible amount of each good that can be produced from the initial endowment is one, by solving the linear programming problems

$$w_i' = \text{maximize} \quad (Bz)_i + w_i$$
$$Bz \leq -w$$
$$z \geq 0$$

for each i. Standard assumptions imply that all these problems have optimal solutions, and the optimal values provide scaling factors to be used in place of the components of w.

The discussion above suggests that we perform a similar rescaling in the range; that is, replace g by $\Omega^{-1}g$. Strictly speaking, this is unnecessary, since Broyden's update is invariant under arbitrary (linear) scalings of the range, if the initial matrix J^0 is appropriately changed. We prefer to simply set J^0 to be a multiple of -I, and thus to rescale both domain and range corresponding to the same change of units. Thus in (1) we may substitute

$$\tilde{g}(q) := \Omega^{-1}g(\Omega^{-1}q) \tag{16}$$

for g, and in (13) and (14)

$$\hat{g}(x) := Z^T\Omega^{-1}g(\Omega^{-1}(q^0+Zx)) \tag{17}$$

for $\bar{g}$ where $q^0 = e/n$; the notation used is to stress that $q = q^0 + Zx$ is dimensionless while the argument of g is $p = \Omega^{-1}q$. If production is allowed, $\Omega' = \text{diag}(w')$ can be used instead of Ω and $(\Omega')^{-1}B$ should replace B. Thus $q_i = w_i p_i$ represents the (dimensionless) value of the total endowment of the ith good; similarly x in (17) represents an orthogonal transformation of these dimensionless

values. In either case, the euclidean norm is reasonable to measure changes in q or x. We might therefore prefer to apply Broyden's method (or a successive linear complementarity problem approach) to the reformulated problem.

However, when it is known that all prices will be strictly positive, a nonlinear scaling of the range may be even better. Recall that Newton-like approaches typically make linear approximations to the nonlinear functions, and note the form of the excess demand function g of (5). If the elasticities σ^j are all equal or close to 1 (so that the utility functions are close to Cobb-Douglas), then Πg is linear or close to linear while g is not. Also zeroes of g and of Πg that are strictly positive coincide. (However, note that Πg may have many other zeroes that are not zeroes of g.) Thus it may be suitable to make linear approximations to

$$\overset{\circ}{g}(p) = \Pi g(p) \tag{18}$$

or to

$$\overset{\circ}{\overline{g}}(x) = Z^T \Pi g(p), \tag{19}$$
$$\text{where} \quad p = p^0 + Zx,$$

instead of to g or $\overline{g}$ themselves. We can also combine this idea with that of scaling to get

$$\overset{\circ}{\sim}(q) = \Pi g(p), \tag{20}$$
$$\text{where} \quad p = \Omega^{-1} q, \quad \text{and}$$

$$\overset{\circ}{\underset{\sim}{g}}(x) = Z^T \Pi g(p), \tag{21}$$
$$\text{where} \quad p = \Omega^{-1}(q^0 + Zx).$$

Another way to view this process is that we are making approximations of the form $\Pi^{-1}(Jp+k)$ to g. When we are seeking zeroes in the model problem, this is tractable; however, when production is present the model problem becomes difficult and we suggest the use of (17). It is also worth remarking that, if $J^Te = 0$, the function $\Pi^{-1}Jp$ is homogeneous of degree zero and satisfies Walras' law. Thus such a function appears a much better model for an excess demand function than any linear function could be.

Approximating g(p) by $\Pi^{-1}Jp$, when $J^Te = 0$ and J has negative diagonal entries and nonnegative off-diagonal entries, corresponds to approximating the economy by one in which all consumers have Cobb-Douglas utility functions. However, the updates we propose for J (or more accurately, for an approximation to the Jacobian of $\hat{g}$ or of $\overset{\circ}{g}$) do not preserve this sign structure, and we do not see how it can be preserved for any reasonable update.

Both Broyden's update (9) and the update (29) to be proposed in the next section are based on least-change principles--see Dennis and Schnabel (1981). Note that the use of a matrix Z with orthonormal columns in (17) and (19) implies that least-change principles in $\mathbb{R}^{n-1}$ and in S^{n-1} lead to the same updates.

To conclude this section we consider the extent to which our reformulations can be combined with piecewise-linear homotopy algorithms, which (at least for cases B and C) guarantee global convergence to a solution. First, it is clear that $\tilde{g}$ and $\hat{g}$ can be used instead of g and $\bar{g}$ in such methods with no penalty, since they merely correspond to a rescaling of the problem. Indeed, these reformulations are likely to

be advantageous, since piecewise-linear methods employ regular triangulations, which are most suited to well-scaled problems. Our computational results in section 6 bear this out. We can use $\hat{g}$ instead of $\bar{g}$ to create a fixed-point problem to which simplicial algorithms can be applied.

More precisely, we consider two functions from S^{n-1} to itself whose fixed points give equilibria. Define

$$f^1(p) := \frac{(p+\mu g(p))_+}{e^T(p+\mu g(p))_+} \tag{22}$$

where μ is positive and for a vector $u = (u_i)$, u_+ denotes the vector with components $(\max\{0,u_i\})$. Alternatively,

$$\begin{aligned} &f^1(p) = p + \lambda(p)h^1(p), \quad \text{where} \\ &\lambda(p) = \mu/e^T(p+\mu g(p))_+ \quad \text{and} \\ &h^1(p) := \max\{g(p),-p/\mu\} - e^T\max\{g(p),-p/\mu\}p, \end{aligned} \tag{23}$$

where $\max\{u,v\}$ denotes the vector with components $\max\{u_i,v_i\}$. The other function we use is

$$\begin{aligned} &f^2(p) = p + \mu h^2(p), \quad \text{where} \\ &h^2(p) := \max\{g(p) + \lambda e, -p/\mu\} \quad \text{and} \\ &\lambda = \lambda(p) \quad \text{is chosen so that} \quad e^T h^2(p) = 0. \end{aligned} \tag{24}$$

It is easy to see that fixed points of f^1 and f^2 (or zeroes of h^1 and h^2) are equilibria, and that $\lambda(p)$ is unique and continuous. The function f^1 appears in Arrow and Hahn (1971) while f^2 was motivated by a suggestion of Eaves. We can then apply a piecewise-linear homotopy algorithm to find a zero of $\bar{k}^j\colon \mathbb{R}^{n-1} \to \mathbb{R}^{n-1}$, with

$$\bar{k}^j(x) := Z^T h^j(p^0+Zx) \tag{25}$$

Note that, when $g(p) + \lambda e \geq -p/\mu$ (we choose $\mu = 10^{-5}$ in our computational tests to encourage this), then $\bar{k}^2(x) = \bar{g}(x)$. However, $\bar{k}^j$ is related to a fixed-point problem so that convergence is assured. We similarly define

$$\hat{k}^j(x) := Z^T \hat{h}^j(q^0+Zx), \tag{26}$$

where $\hat{h}^j$ uses $\Omega^{-1}g(\Omega^{-1}q)$ in place of $g(p)$, in analogy with (17).

Next we ask whether the nonlinear scaling of (18)-(21) can be used while maintaining convergence. For this, let

$$p' = p'(p) := \max\{p, \nu e\} \tag{27}$$

for some $\nu > 0$. Then, with $\Pi' := \mathrm{diag}(p')$, set

$$\tilde{f}^1(p) := \frac{(p+\mu\Pi' g(p))_+}{e^T(p+\mu\Pi' g(p))_+} .$$

There is an alternative representation:

$$\tilde{f}^1(p) = p + \lambda(p)\tilde{h}^1(p), \quad \text{where}$$

$$\lambda(p) = \mu/e^T(p+\mu\Pi' g(p))_+ \quad \text{and} \tag{28}$$

$$\tilde{h}^1(p) := \max\{\Pi' g(p), -p/\mu\} - e^T \max\{\Pi' g(p), -p/\mu\} p.$$

Note that, if $\nu = 1$, then $p' = e$ and $\tilde{f}^1$ reduces to f^1. However, for small ν and μ (we choose $\nu = 10^{-3}$ in our computational tests), if $p \geq \nu e$ and $p \geq -\mu\Pi g(p)$ (which follows if $\mu \leq 1/w_i$, all i), then $\tilde{h}^1(p) = \Pi g(p)$. Thus, except close to the boundary of S^{n-1}, $\tilde{h}^1$ coincides with $\overset{\circ}{g}$. Further, $\tilde{f}^1$ clearly maps S^{n-1} continuously into itself and the proof of Theorem 2.2 in Arrow and Hahn (1971) shows that its fixed points are equilibria. Of course, we can also rescale the problem by replacing p by q and $g(p)$ by $\Omega^{-1} g(\Omega^{-1} q)$ everywhere; then $\tilde{h}^1$ will coincide with $\overset{\circ}{\tilde{g}}$ except close to the boundary of S^{n-1}.

In our computational tests in Section 6, we use slightly modified mappings, since in our examples g is not defined for price vectors with nonpositive components. Following Arrow and Hahn (1971), we replace $g(p)$ in (22)-(23) and $\Pi' g(p)$ in (27)-(28) by

$$(1-\alpha) g(p) + \alpha\rho e \quad \text{and}$$

$$(1-\alpha)\Pi' g(p) + \alpha\rho e$$

respectively, and by ρe if $g(p)$ is undefined, where $\alpha = \alpha(e^T g(p))$ and $\alpha(\lambda) = \max\{0, \min\{1, (\lambda-\rho)/\rho\}\}$ increases from 0 for $\lambda \leq \rho$ to 1 for $\lambda \geq 2\rho$. We choose $\rho = 100n$ in our tests. The resulting functions

are somewhat different than those suggested by Arrow and Hahn, but are smooth in larger neighborhoods of equilibria. It is straightforward to show that the functions obtained are continuous (mapping $\{p: e^T p = 1\}$ into S^{n-1}) and that their fixed points are equilibria. However, this is no longer true for the corresponding modification of f^2 in (24), since any price vector in the boundary of S^{n-1} is then a fixed point. Hence we replace $f^2(p)$ by $f^2(p'')$, where $p'' = \max\{p,\mu e\}/e^T\max\{p,\mu e\}$ for $\mu = 10^{-5}$. Again, rescaling can be performed with these modifications (and makes sense, since α depends on $e^T g(p)$, which is only natural when scaling renders $g(p)$ dimensionless).

5. Maintaining Monotonicity

We have pointed out in Section 2 that under certain conditions the excess supply function $-g$ is globally strictly monotone. This is a strong condition which implies for instance the uniqueness of equilibria; indeed, the axiom of revealed preference holds. For completeness we give a short proof.

Proposition 5.1. If $-g$ ($-\bar{g}$) is strictly monotone, there is at most one solution to (1) ((14)).

Proof. Suppose p and q both solve (1). Then

$$(p-q)^T(g(p)-g(q)) = -q^T g(p) - p^T g(q) \geq 0.$$

Since $-g$ is strictly monotone, $p = q$. Next suppose x and x' solve (14), with associated (z,s) and (z',s'). Then

$$\begin{aligned}(x-x')^T(\bar{g}(x)-\bar{g}(x')) &= (x-x')^T(Z^TB(z-z')) \\ &= (B^TZ(x-x'))^T(z-z') \\ &= (s'-s)^T(z-z') \\ &= (s')^Tz + s^Tz' \geq 0.\end{aligned}$$

Again, strict monotonicity implies that $x = x'$. Note that we cannot claim that $z = z'$, although this will follow under a nondegeneracy assumption.

Even if $-g$ is not globally strictly monotone, it may well be strictly monotone in a neighborhood of an equilibrium. For instance, if substitution effects dominate income effects near p, then $-g'(p)$ should be negative definite (although not necessarily symmetric). In this case there are cogent reasons for approximating $-g$ or $-\bar{g}$ by a strictly monotone linear function during the application of a quasi-Newton approach. As we have seen, this will ensure that the model problems have at most one solution. It can further be shown that a solution exists, so that the algorithm is well-defined. Indeed, for (1) a well-known result states that if $-g$ is linear and strictly monotone then a unique solution exists. For (14), $B^TZx \leq -B^Tp^0$ implies $Zx \geq -p^0$, which has a bounded feasible region, and again a standard result gives existence.

The question then arises how strict monotonicity can be preserved in a quasi-Newton update. Todd (1984) shows that, if $-J$ is strictly monotone and $y^Ts < 0$, then

$$J^+ := J + \frac{(y-Js)(y+J^Ts)^T}{(y+J^Ts)^Ts} \qquad (29)$$

is such that $-J^+$ is strictly monotone. This update can therefore be used in place of Broyden's.

6. Computational Results

Here we describe some very preliminary computer experimentation on pure trade general equilibrium models. Problems E1, E2 and E3 are the three examples described by Scarf (1967) with 5, 8 and 10 goods respectively. All have C.E.S. utility functions for all consumers. E4 is a perturbation of E1; w_{33} is changed to 16 from 15, and (σ^j) to (1,2,.5) from (.9,1.3,.8). E5 and E6 are then rescalings of E4; the unit used to measure good 3 is 4 times smaller (E5) or 16 times larger (E6) than in E4.

We tested 10 quasi-Newton and 8 piecewise-linear homotopy methods on each problem. The methods are denoted QjO (quasi-Newton, using function ℓ^j, and original variables) and QjS (quasi-Newton, using function ℓ^j, and dimensionless scaled variables), $j = 0,1,\ldots,4$ and similarly PjO and PjS, $j = 0,1,\ldots,3$. In all cases we start with the center of the simplex corresponding to p^0 or q^0 equal to e/n and x^0 equal to 0, and the termination criterion is that the maximum component of $\Omega^{-1}g(p)$, denoted $\|\Omega^{-1}g(p)\|_\infty$, not exceed 10^{-10}.

The quasi-Newton methods initialize J^0 to $-\alpha I$ with $\alpha = 1/2n\|\ell^j(x^0)\|_\infty$, so that the first step goes at most half-way to the boundary of the price simplex. Subsequently, we choose

$$x^{k+1} = x^k + \bar{\lambda}(x^+ - x^k), \quad \text{where} \tag{30}$$
$$\bar{\lambda} = \max\{\lambda \leq 1: p_i^{k+1} \geq .05p_i^k \text{ for all } i\};$$

here $p^k = p^0 + Zx^k$. The function whose zero is sought is ℓ^j, where

$$\ell^j(x) = \begin{cases} Z^T\Pi g(p) & \text{(see (19))} \quad \text{if} \quad j = 0 \\ Z^T h^1(p) & \text{(see (23))} \quad \text{if} \quad j = 1 \\ Z^T h^2(p) & \text{(see (24))} \quad \text{if} \quad j = 2 \\ Z^T g(p) & \text{(see (13))} \quad \text{if} \quad j = 3 \text{ or } 4 \end{cases}$$

and $p = p^0 + Zx$. These formulae are modified in the obvious way if scaling is performed--see (21), (26) and (17) for $j = 0$, $j = 1$ or 2, and $j = 3$ or 4 respectively. So far, there appears to be no difference between Q30 (Q3S) and Q40 (Q4S). The distinction is in the update used; while for $j = 0,1,2$ and 3 the Broyden update is employed, $j = 4$ uses the new update (29). (Although the model does not guarantee that $y^T s < 0$ at each iteration, this was observed in practice in all examples.)

Table 1 gives the number of function evaluations required. The results on this limited test set are remarkably consistent. In all cases rescaling improves the computational performance, especially for the (deliberately) badly-scaled problems E5 and E6. Problems E4-E6, which only differ in scaling, give identical results as expected for the rescaled methods. Finally, using ℓ^0 (basically Πg) yields a substantial improvement over any of the other formulations, especially when the problems are not rescaled. Methods Q2 and Q3 performed identically, which is not too surprising: if $g(p) + \lambda e \geq -p/\mu$ (recall, we chose $\mu = 10^{-5}$), then $Z^T h^2(p) = Z^T g(p)$. Finally, we should comment on the two failures. In our primitive implementation, the choice of step-size in (30) is the only concession to global convergence; the failure of Q10 on E2 appears related to this, as the iterates tried to converge to a point on the boundary of the price simplex until they were repulsed, when they tried to converge to another boundary point. Also we explicitly updated the inverse of J^k; the failure of Q40 on E5 seems to be related to the near-singularity of several J^k matrices. More sophisticated implementations (see Powell (1970)) might eliminate these difficulties.

The piecewise-linear methods all used the PLALGO code (Todd (1980)) with default settings (hence Merrill's method with triangulation J_1) except that the grid size was initially $1/n$ and after each major cycle reduced by a factor .37. The mappings used were those described in Section 4. With the modifications outlined there, method PjO (PjS) used $\bar{k}^j$ ($\hat{k}^j$), given by (25) ((26)) for $j = 1,2$. For $j = 0$, these functions were based on $\tilde{f}^1$ and $\tilde{h}^1$ in (28), and for $j = 3$, on $Z^T g(p)$ as for the quasi-Newton methods.

The results are given in Table 2, where an entry p/q means that p linear programming pivots and q function evaluations were required. Once again the beneficial effects of rescaling are demonstrated convincingly. Similarly, POO and POS perform much better than the other methods, showing the advantage of the nearly linear function $\Pi' g(p)$. Indeed, method POS, with guaranteed convergence, requires a number of function evaluations not much greater than those for the best quasi-Newton method, QOS. The very poor behavior of all "unscaled" methods on the poorly-scaled problem E5 should also be noted.

These results, while based on limited testing, substantiate the more theoretical discussion on section 3 and argue for the use of reformulations before applying numerical methods. Also, the new update (29), while not designed for this class of problems, seems to perform reasonably well and deserves further investigation.

Problem	E1	E2	E3	E4	E5	E6
Dimension	5	8	10	5	5	5
Method						
Q00	20	19	19	22	40	16
Q10	62	*	24	22	79	49
Q20	25	35	23	23	53	24
Q30	25	35	23	23	53	24
Q40	26	25	22	20	*	23
Q0S	10	16	14	10	10	10
Q1S	11	17	15	13	13	13
Q2S	10	17	15	12	12	12
Q3S	10	17	15	12	12	12
Q4S	10	18	15	13	13	13

*Failed to Converge

Table 1

Problem	E1	E2	E3	E4	E5	E6
Dimension	5	8	10	5	5	5
Method						
P00	54/58	41/51	82/93	74/77	215/202	23/32
P10	59/65	99/109	75/86	76/82	417/377	39/49
P20	69/74	89/99	69/82	124/127	184/178	41/54
P30	68/77	83/94	82/94	77/83	468/417	41/52
P0S	5/11	24/32	23/32	6/14	6/14	6/14
P1S	10/21	47/58	35/45	10/21	10/21	10/21
P2S	10/20	49/60	34/46	10/22	10/22	10/22
P3S	10/21	51/62	36/46	11/22	11/22	11/22

Table 2

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