CHAPTER XI.

NON-PLANE MOTION.

§ 62.—THE SCREW.

In the preceding chapters we have limited ourselves almost entirely to the consideration of mechanisms in which only plane motions occur. These form by far the largest and most important class with which the engineer has practically to deal. We have now to notice some of the principal *non-plane* motions utilised in machinery, and shall in the first instance examine those conditioned by the use of the screw and nut, Fig. 261.[1]

In § 2 we have already noticed the characteristics of screw motion, or twist; and in § 10 we have seen that this motion could be completely constrained by the ordinary screw and nut, a pair of elements which we classed among the *lower* pairs because of its surface contact. Familiar and important as this pair is, there is hardly an instance in which it is used for the sake of its own characteristic helical motion. With scarcely an exception the screw motion is resolved into its two components, rotation and

[1] A more general investigation of screw motion in mechanisms, of which this is the simplest (and a very special) case, will be found in §§ 68 to 70.

translation, and these two motions are employed separately on different links of the chain containing the screw. Fig. 262 shows the most familiar illustration of this. The screw forms

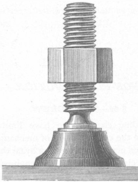


Fig. 262.

part of the link *a* of a three-link chain. The link carries also a turning element or pin, which is paired with *c*, and *c*, in turn, forms a sliding pair with the outside of the nut *b*.

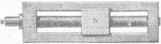


Fig. 263.

The motion of *a* relatively to *c* is a rotation,[1] that of *b* re

[1] It is presupposed that suitable collars prevent any endlong motion of *a* in *c*.

latively to *c* a translation. The motion of *a* relatively to *b* is twist, but this is the one of the three motions of which no use is made under ordinary circumstances.

If the mean radius, or pitch radius, of the screw be called *r*, and the pitch *p*, then any point of the screw at a radius *r* will move through a distance $2\pi r$ relatively to the link *c*, while *b* only moves through a distance *p* relatively to the same link. Any such point will move $\frac{2\pi r}{p}$ times as fast (relatively to *c*) as *b*, and any force applied at it, in its direction of motion, will balance a resistance $\frac{2\pi r}{p}$ times greater than itself to the motion of *b* along *c*. As such a force can be readily caused to act at some radius *R* very greatly larger than *r*, without any alteration in the value of *p*, the screw presents the possibility of attaining in a small compass a very large "mechanical advantage," $\frac{2\pi R}{p}$. We shall see in the next chapter how very seriously this apparent advantage is reduced by unavoidable frictional resistances.

Fig. 263 shows a **screw press**, which is in reality exactly the same chain as the last figure with the link *b* fixed. The relative motions of the links remain exactly as before, but the twist of the link *a* becomes more obvious, as it occurs relatively to the fixed link. In such a case the driving effort upon *a* cannot remain in the same plane (as in the last case), but must change its position axially as the screw goes bodily down or up. Unless, however, the screw be moved by hand, in which case such a change of position does not require to be considered, means are taken to keep the driving effort always in one plane, so that again the actual screw motion does not come into practical considera-tion. Thus the arrangement of Fig. 264 is often used, in which *c* is the fixed link, but *b* carries the screw instead of the nut, and *a* the nut instead of the screw. The screwed part of *b* merely slides in *c*, and the link *a*, which is prevented by shoulders from having an endlong motion, takes externally the form of a belt-pulley or a spur-wheel, which can be driven in the usual way. It is hardly necessary to point out that the interchange of the forms of screw and nut—external and internal screws—make no more change in the mechanism than the interchange of eye and pin in a turning pair, or slot and block in a sliding pair.

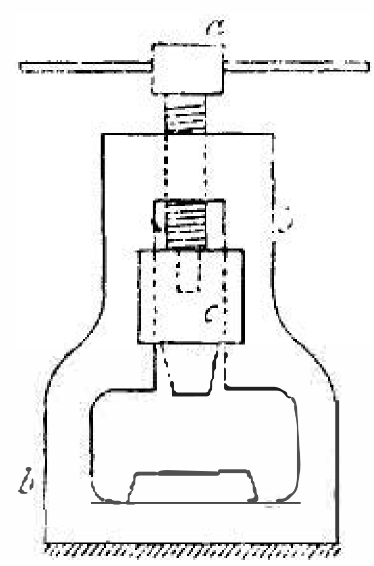


FIG. 263.

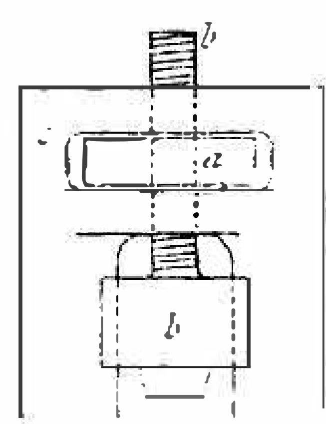


FIG. 264.

In the mechanisms formerly considered we had only to deal with forces acting in, or parallel to, one plane. All other forces or force components were, by hypothesis, balanced as they appeared by stresses in the links (p. 7). Here, however, it happens almost invariably that effort and resistance act in different planes. The effort in most cases (as in the last figure for instance) acts in a plane normal to the axis of the screw, while the resistance acts in a plane passing through that axis, very often, indeed, acting directly in its line. In any case we have still, exactly as before, the condition that all force components tending to cause motions which are incompatible with those permitted by the connection between the links, are entirely balanced by stresses in those links.

In the plane mechanisms hitherto studied we have assumed tacitly that the smallest force acting on any link, and acting in such a direction as to move that link, would move the whole mechanism. Apart from friction, this is strictly true, and under the same conditions it is true with screw mechanisms also. But here, as we shall see later on, the effect of friction is much more serious than where there are only pin-joints or even ordinary slides. With a screw of ordinary proportions, and working with ordinary lubrication, no effort, however great, acting on the nut in the direction

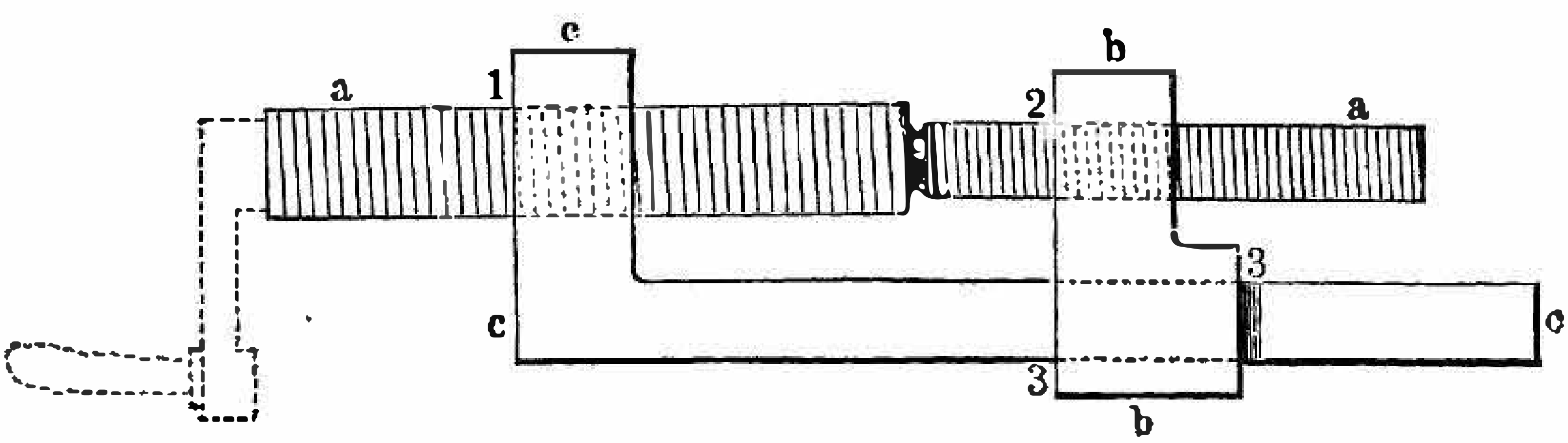


Fig. 265.

of the axis of the screw, could cause rotation of the screw. We therefore obtain here a condition, sometimes of great practical convenience, differing essentially from any with which we have had hitherto to do, namely, that as regards effort and resistance the mechanism is non-reversible. But as this condition depends entirely on frictional resistances its further examination must be deferred.[1]

We shall now notice briefly a few of the principal mechanisms in which screws are used.

The three-link chain shown in Fig. 265 finds several applications; it is commonly known as a **differential**

[1] See § 74.

screw chain.. The link a consists of two screw elements, of different pitches; b carries the nut for one of these elements, and c for the other, and b and c are themselves connected by a slide. Let us write p_c for the pitch of a in c, and p_b for its pitch in b. Then if c be fixed, as is usually the case, each complete turn of a will cause it to move through a distance p_c relatively to c, and (simultaneously) through a distance p_b relatively to b. The corresponding motion of b relatively to c will be the difference or the sum of p_c and p_b, according to whether the two screws are of the same or of opposite hands. In the figure they are of the same hand, and the motion of b relatively to c for one turn of

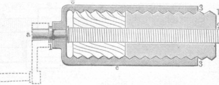


Fig. [illegible]

the screw is equal to $p_c - p_b$. The mechanism is therefore equivalent to that of Fig. 262, with a screw whose pitch was $p_c - p_b$. But this quantity may be excessively small, much smaller than it would be practicable to make the pitch of an ordinary screw, and in this way the "differential" arrangement may enable us to get a mechanical advantage much greater than could conveniently be otherwise obtained by a screw, the advantage being, of course, $\frac{2\pi r}{p_c - p_b}$.

In the chain of Fig. 266 the link b carries two screw elements of different pitches, with which a and c are paired. The connection between a and c is a turning pair. If here c be fixed, the motion of b relatively to it must be twist, but its rotation must be as much slower than that of a as p_a is less than p_b. Thus in one turn of a the link b receives an axial motion of p_a, while at the same time its whole motion (relatively to c) is a twist whose pitch is p_b. It must therefore have made only the fraction $\frac{p_a}{p_b}$ of a complete turn. In a similar way the relative motions of the other links can be examined, their static relations following most easily, as before, from their relative velocities.

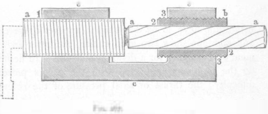


Fig. 267

Fig. 267 shows another three-link chain, but in this case *all* the pairs are screw pairs. If any one (as c) be fixed, the motions of both the others relatively to it, as well as to each other, are all helical. All the three mechanisms to be obtained from this chain are kinematically the same. The chain, which is given by Reuleaux, does not seem to have found any applications as yet in practical work, for the reason, no doubt, that so few uses exist in machinery for helical motions.

There are a number of mechanisms of a more practical kind in which a single screw-pair is used with a number of other links, very often with the special intention of *fixing* the mechanism, *i.e.* of preventing any motion of its links except by the action of those forces which cause the screw to rotate. This fixing, as has been already mentioned, is a consequence of frictional resistances, whose nature and

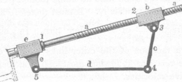


FIG. 268.

magnitude have to be discussed in another chapter. Figs. 268 and 269 show two such chains, the latter an arrangement of Nasmyth's for a dividing machine. In neither case do the problems connected with the chains present any difficulty; the thrust or axial pressure of the screw being

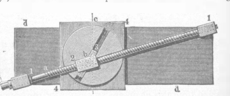


FIG. 269.

once found, it can be taken as a known force, acting on a link of a plane chain, and so dealt with by the methods already discussed.

The steering gear shown in Fig. 270 differs from the other screw mechanisms examined in having a double thread cut on the screw, so that it can work with two half nuts, one right and one left-handed, moving one forwards, while it simultaneously draws the other backwards. It is, in fact, only a modified form of the mechanism of Fig. 271, where the two screws and nuts are complete and separate.

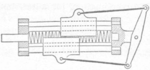


Fig. 270.

The worm and worm-wheel, Fig. 272, form together one of the most familiar combinations containing a screw. We shall see in § 69 that the mechanism shown in the figure is essentially a very special case of screw-wheel gearing, the worm being really a screw-wheel of one, two, &c., teeth, according as it is a single- or double-, &c., threaded. Looking

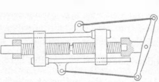


Fig. 271.

at the mechanism at present, however, merely as we have been looking at the other screw-trains mentioned, we see that the pitch circle of the wheel receives from the screw simply the axial component of its motion, exactly as does the link *b* in Fig. 262. The pitch of the teeth of the wheel is the same as the pitch of the screw helix, if it be single-threaded, or half that pitch if it be double-threaded, and so on. Each complete revolution of the screw therefore carries the wheel round through the angle corresponding to one tooth if the screw be single-threaded, two teeth if it be double-threaded, &c., exactly as if the worm were (as it essentially is) a wheel of one, two, &c., teeth. The mechanical advantage of the combination, for a resistance at a radius

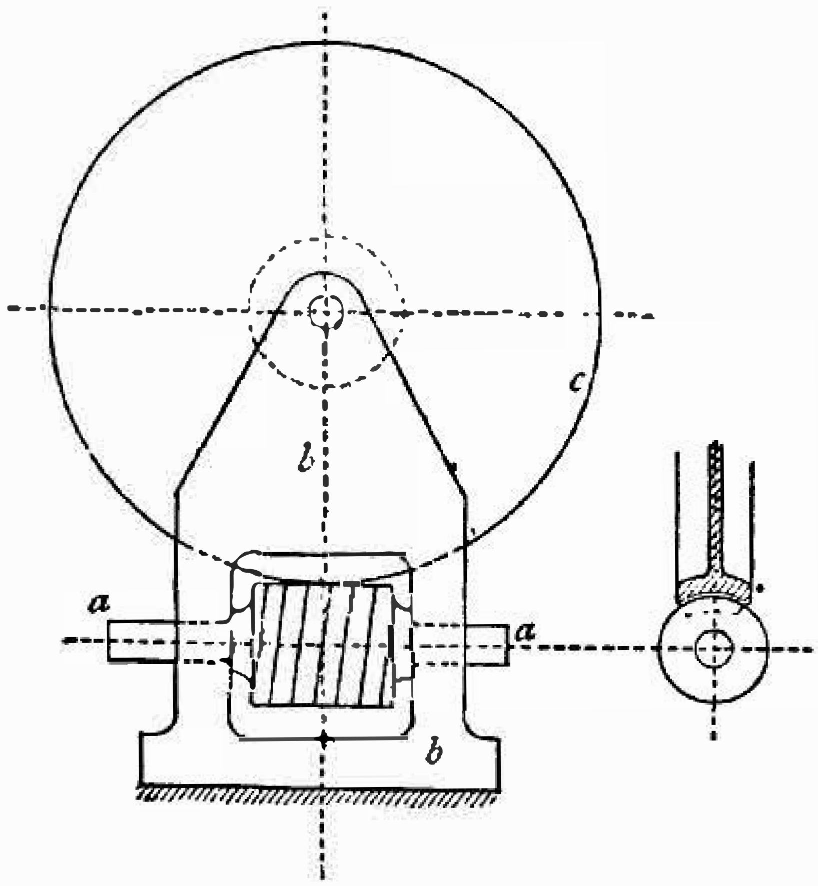


FIG. 272.

equal to that of the wheel, is exactly equal to that of the screw press noticed above, or $\frac{2\pi r}{p}$. But here not only can r be increased, but also the radius at which the resistance acts can often be conveniently made very much less than that of the wheel, as shown, for instance, by the dotted circle. By means of these three links, therefore, a very large mechanical advantage can be gained with the use of very few links and in a very small space.[1] In order that the

[1] As to the relative diameter of the worm and wheel, and of the twist axodes to which they correspond, see § 69.

worm-wheel and worm may gear properly together, it is often assumed that the teethi of the former must be themselves portions of helices having a tangent, in the middle plane of the wheel, coincident with the pitch tangent of the worm-thread. The curvature of such helices being exceedingly small in such a small fraction of their pitch as is represented by the breadth of the worm-wheel, the teeth are usually made straight, and simply inclined at an angle equal to the pitch-angle (or angle whose tangent is $\frac{p}{2\pi r}$) to the position they would occupy in the spur-wheel. If pressure were transmitted from the worm to the wheel always in the middle plane of the latter, this approximation would be reasonably accurate. The point of contact,[1] however, traverses the wheel-tooth from side to side, and (especially if the teeth are hollowed out as sketched to the right of Fig. 272), this causes irregularities of a kind similar to those mentioned on p. 121, in the motion transmitted. This matter has been examined by Professor W. C. Unwin, who has given, in his *Elements of Machine Design*,[2] the only satisfactory investigation of it which we have seen published.

If it is not essential that the axes of the worm and wheel should be at right angles, the arrangement adopted by Mr. Sellers, of Philadelphia, offers many constructive conveniences (Fig. 273). Here the angle between the axes is made less than 90° by an amount exactly equal to the pitch angle of the screw. This turns the worm-wheel into a spur-wheel (a screw-wheel of infinite pitch), its teeth being parallel to its axis. This combination, with reasonably well-formed teeth, runs with exceedingly little friction.

[1] In screw gearing contact occurs at a point on each tooth, not along a line (see § 69).

[2] P. 296, &c., in the fifth edition.

The twist-axis, which takes the place, in such combinations as these, of the virtual axis of rotation of plane mechanisms, as well as of those non-plane mechanisms which are to be examined in §§ 63 to 66, will be found discussed in § 68.

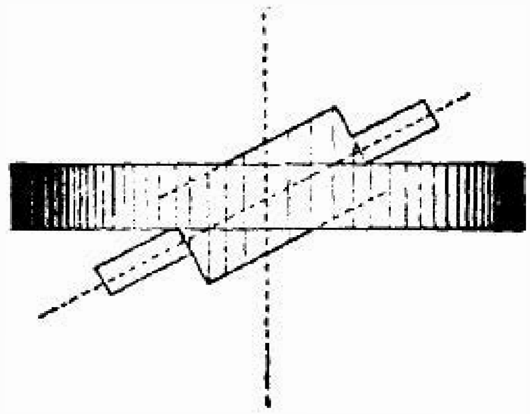


FIG. 273.

§ 63.—CONIC CRANK TRAINS.

IN § 2 we mentioned plane, spheric, and screw motions as the three principal cases with which we had to deal in machinery. Before looking at the more general motions coming under the third head, it will be convenient to examine those of the second. Let PQ, Fig. 274, be a *spheric* section of any body having spheric motion (p. 15), and p and q the paths (on the surface of the sphere) of the points P and Q respectively. We can find a virtual axis for the motion of the body by a method exactly similar to that used for plane motion (p. 40). We can, namely, consider P as moving for the instant along a great circle α touching p in P, and Q along a great circle β, touching q in Q. Drawing great circles α_1 and β_1 normal to α and β, we may say that the instantaneous motion of P is equivalent to a rotation about any diameter of α_1, and that of Q equivalent to a rotation about any diameter of β_1. But α_1 and β_1, being great circles on the same sphere, must have one diameter (here SS_1) in common. The body PQ has therefore for its instantaneous motion a simple rotation about SS_1, which becomes its *virtual axis*. Taking other positions of P and Q we can obtain other virtual axes, and the locus of these axes will again be an *axode* (p. 52). The axode will here, however, consist of a number of lines all passing through the same point O, the centre of the sphere, so that it will be a cone instead of (as for plane motion) a cylinder. In

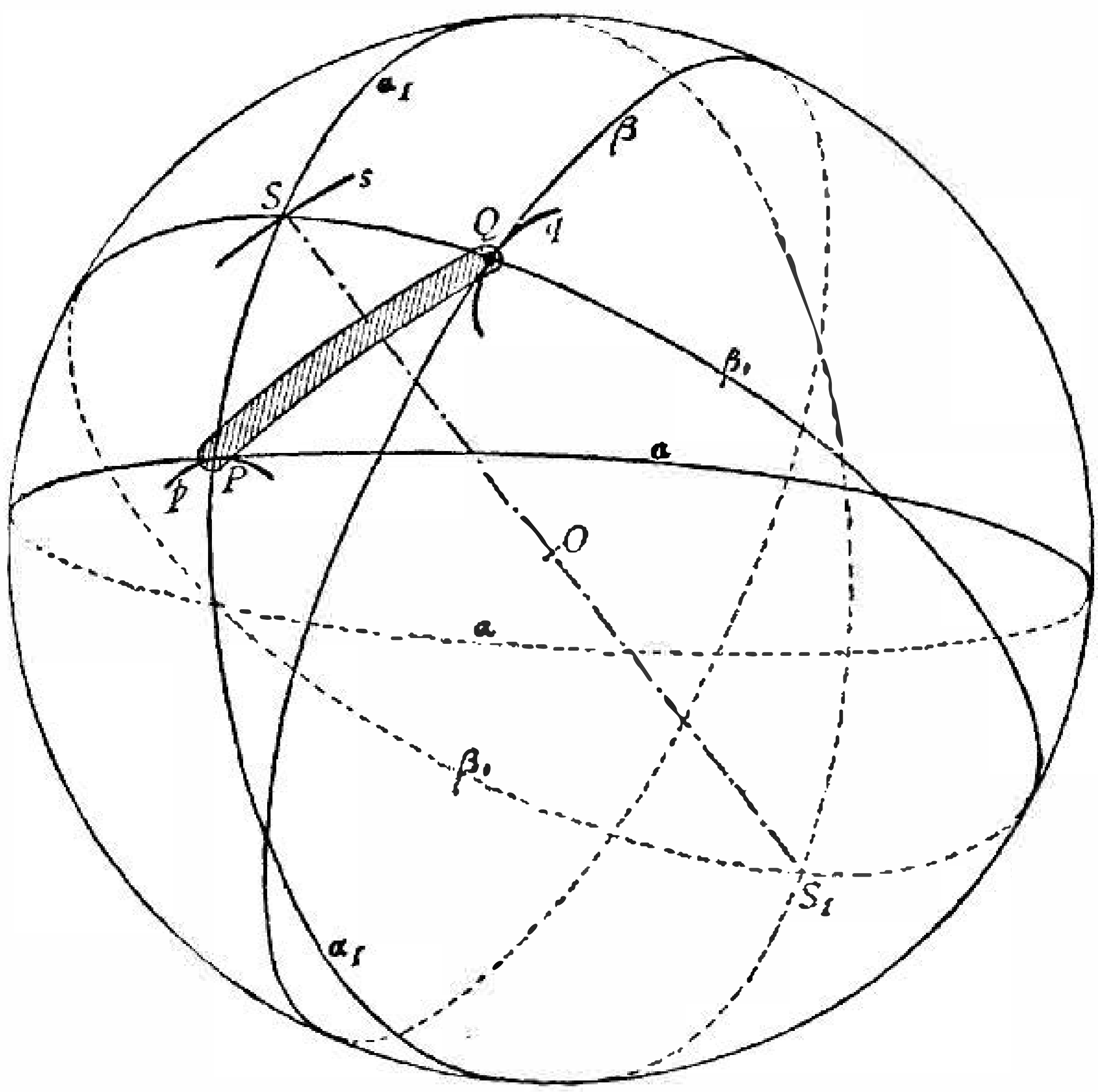


FIG. 274.

general, of course, the cone will be non-circular, just as the cylindrical axode was non-circular, except in the special cases where the centrodes were circular, as on pages 119 and 146. The curvers, which is the locus of the points S, the intersections of the virtual axes with the spheric surface, has some of the properties of the centrode. For the relative (spheric) motions of any two bodies, for instance, measured on the same sphere, there are two such curves, which touch each other always in one point, and that point is the point *S*, which, along with the centre *O*, determines the position of the virtual axis.[1] The two curves *roll* upon one another, as the bodies to which they correspond move, exactly as do the centrodes in plane motion. But we cannot speak of such a point as *S* as a virtual centre, for the different points in *PQ* are not points in a plane passing through *S*, and their virtual motions are not rotations about any one such point, but about different points in the line *SO*. The motion, therefore, when reduced to its lowest terms, is a rotation about an axis, for which axis a point can *not* be substituted, as formerly in § 7.

Let *a*, *b*, and *c*, be any three bodies each having spheric motion relatively to the other. For these motions there will be three virtual axes, which we may call A_{ab}, A_{ac}, and A_{bc} respectively. The theorem of the three virtual centres (p. 73) is here represented by a theorem as to these axes, which may be stated thus: **If any three bodies, *a*, *b*, and *c*, have spheric motion, their three virtual axes, A_{ab}, A_{ac}, and A_{bc}, are three lines in one plane.** We have already seen that these three lines must all pass through one point, *O*. The simplest proof of this theorem is one corresponding to that formerly given, namely, the followingh[2] The line A_{bc} is a line belonging to both the

[1] As the same construction gives us the points *S* and S_1 simultaneously, and as there is no kinematic difference between them, we may indeed say that each curve is a double one, having two similar and equal parts placed oppositely on the sphere. But as these two parts are precisely similar and equal, and as either one of them by itself, along with the given centre *O* of the sphere, is sufficient to determine the axode, it is unnecessary to trouble ourselves to consider more than the one curve *s*, or point *S*, which happens in any construction to be the more convenient.

[2] If a figure would make it easier for the student to follow this statement, Fig. 30 can be used, taking the paper as a projection of a spherical

bodies b and c. As a line in the former it is turning, relatively to a, about A_{ab}. It must therefore be moving in a plane at right angles to the plane containing the lines A_{ab} and A_{bc}. As a line in c, it is turning, relatively to a, about A_{ac}. It must therefore be moving in a plane at right angles to the plane containing the lines A_{ac} and A_{bc}. It can be moving only in one direction at one time, so that two planes which both pass through it, and are both normal to that direction, must coincide. The planes A_{ab} A_{bc}, and

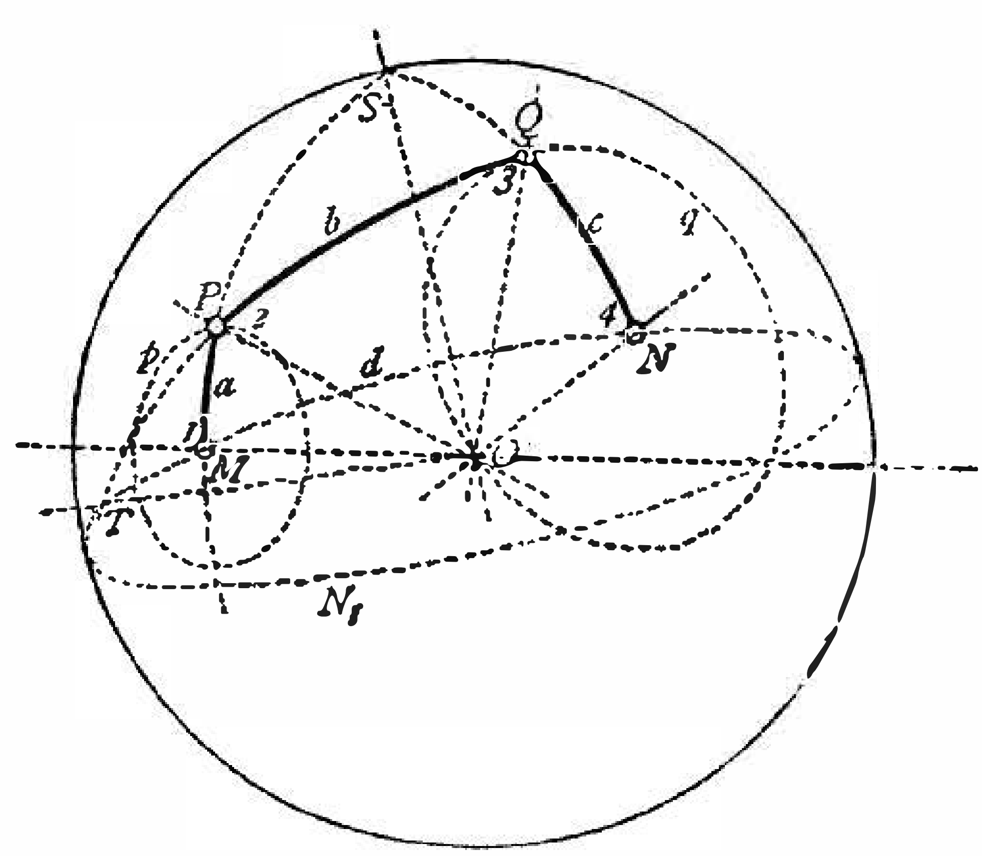


FIG. 275.

A_{ac} A_{bc}, therefore coincide, and the three lines A_{ab}, A_{ac}, and A_{bc} are three lines in one plane. We shall find considerable use for this theorem later on.

Referring again to Fig. 274, it will be noticed that the paths p and q were assumed quite arbitrarily. They may be, for instance, circles on the surface of the sphere, as sketched in Fig. 275. In that case they might be constrained by the use of links MP and NQ, pivoted at M and N

surface, and the points O_{ab}, &c., as the traces on that surface of the axes A_{ab}, &c.

respectively on axes passing through O. If the sphere carrying these axes be supposed fixed, the three links MP, PQ, and QN form along with it a four link mechanism in which the motions are completely constrained. But the *shape* of the links is immaterial (p. 66), so that we may omit the sphere itself, connect M and N by a bar as in the other cases, and we get the four-link chain shown in Fig. 276. Here the essential matter is that the four axes of the pairs of elements should all pass through the same point O. In the case of plane motion the axes were parallel, the point

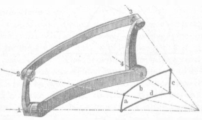


FIG. 276.

O was infinitely distant. In the case of spheric motion, then, we may consider that we have simply brought the point of intersection of the axes nearer, without changing anything else. Comparing the chain here shown with that of Fig. 25, p. 61, it will be seen that it is altered in no other respect. It still contains four links, connected by four turning pairs, and the relative lengths of the links are nearly the same.[1] In both cases, if d be fixed, c will

[1] In this and following figures the links are shown as curved bars. It is hardly necessary to explain that this is done only to make the figure

swing and a rotate, so that the new mechanism may be looked at as simply the old one bent round, so as to bring the point of intersection of its four axes to some near position. For this reason we can still call such a mechanism a lever-crank, but we shall call it a **conic lever-crank**, to distinguish it from the former plane one.

Some characteristic points about these conic mechanisms require notice before we proceed to examine them. In the first place the relative *lengths* of the links are no longer matters of simple linear measurement (for the actual constructive links are not lines on the surface of a sphere), but depend on the *angles* respectively subtended by them. Moreover, no link can subtend an angle greater than a right angle. For if MON, Fig. 275, had been greater than 90°, we could have used N_1 instead of N, and MON_1 would have been less than 90°. We have the same possibility for every link, for we have already seen that to every point, P, Q, &c., there corresponds another on the opposite side of the sphere, having an exactly similar path (p. 490). Hence every link may be said to have either of two angular lengths, as a and $180°—a$, one of which is the supplement of the other. The motions are not affected by which of these lengths are used, but to avoid confusion in speaking of them it is generally convenient to state the length which is less than 90°. The constructive appearance of the mechanism, on the other hand, is so much changed by such alterations as often to place considerable initial difficulties in the way of identifying or understanding it. Thus it is at first sight difficult to recognise Fig. 277, and still more Fig. 278, as being mechanisms not merely similar to,

more intelligible. The links may, just as before, be of any convenient shape or dimensions, straight or curved, so long as only the axes of the elements occupy their proper positions.

but absolutely identical with, that shown in Fig. 275. But examination will show that no change whatever has been made, except the substitution of links subtending the supplementary angles, (in Fig. 277 + 180°) as just mentioned.

In Fig. 279 a conic mechanism is shown, in which two links, *c* and *d*, are made each to subtend a right angle. The constructive form of *d* is made different from that of *c*, that the motions may be realised more easily, but there is no kinematic difference between them. It will be noticed that if *d* be fixed we have a mechanism in which the point

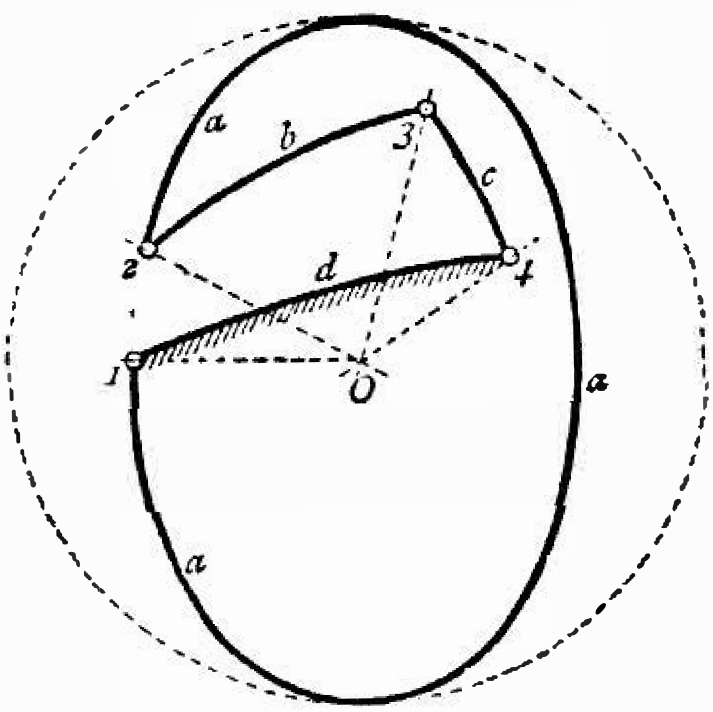


FIG. 277.

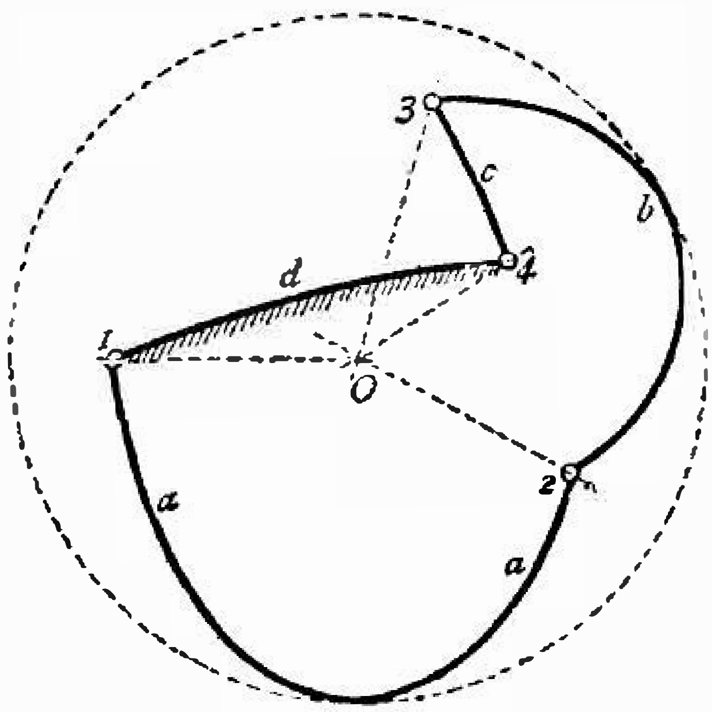


FIG. 278.

3 moves always in a great circle at right angles to one of the axes of *d* (the vertical one in the figure) and in the plane of the other (the horizontal one in the figure). The link *a* rotates as before, and the link *c* still swings or reciprocates, its end-point 3 moving always in the same plane, as we have just seen. The motion of the link *b* corresponds exactly, in consequence, to that of the connecting-rod in the ordinary slider-crank chain (Fig. 26); it might be described accurately enough as a connecting-rod working round a corner!

By expanding[1] the pair 4 in Fig. 279, and bringing it down to the level of the plane 31, the link *b* becomes externally a slider working in a curved slot in *d* (Fig. 280), and the identity of the mechanism with the slider-crank becomes obvious, even on the surface. It may be said to be simply a slider-crank bent round, exactly as in the former case we had a lever-crank bent round. The right-angled links of the conic train correspond to the infinite links of the plane train; motion along a great circle corresponds to motion along a straight line.

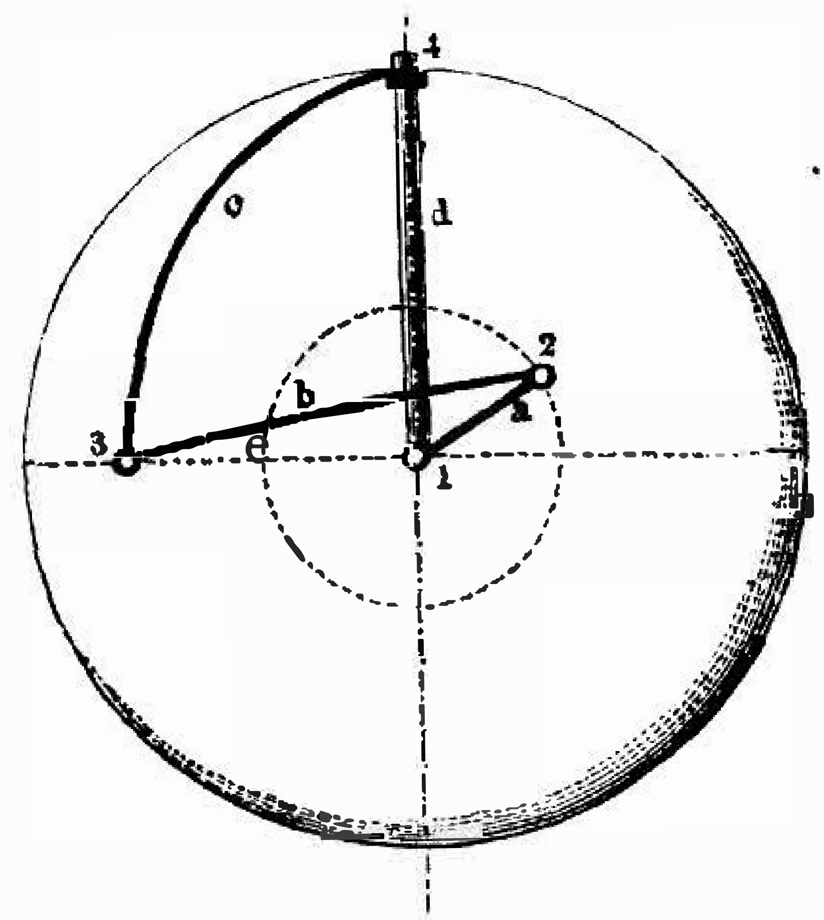


FIG. 279.

The positions of the virtual axes of the links in a conic chain correspond exactly to their positions in the plane chains. For clearly if the points *P* and *Q* (Fig. 274) are constrained in their motion by two rotating or oscillating links, the lines α_1 and β_1, at right angles to the point-paths, must be the centre lines of those links, and the point *S*, which fixes the virtual axis of *PQ*, lies at their join.

[1] See § 52.

Hence, exactly as in the case of plane chains, **the virtual axis**[1] **of either pair of opposite links is the join of the planes of the other two**:[2] **the virtual axis for any pair of adjacent links is the join of their own planes, and is a permanent axis.** Thus in Fig. 281, *SO* is the virtual axis for *b* and *d*, and *TO* for *a* and *c*, and for adjacent links, 1*O* is the virtual axis for *a* and *d*, 2*O* for *b* and *a*, &c. The same lettering is used in

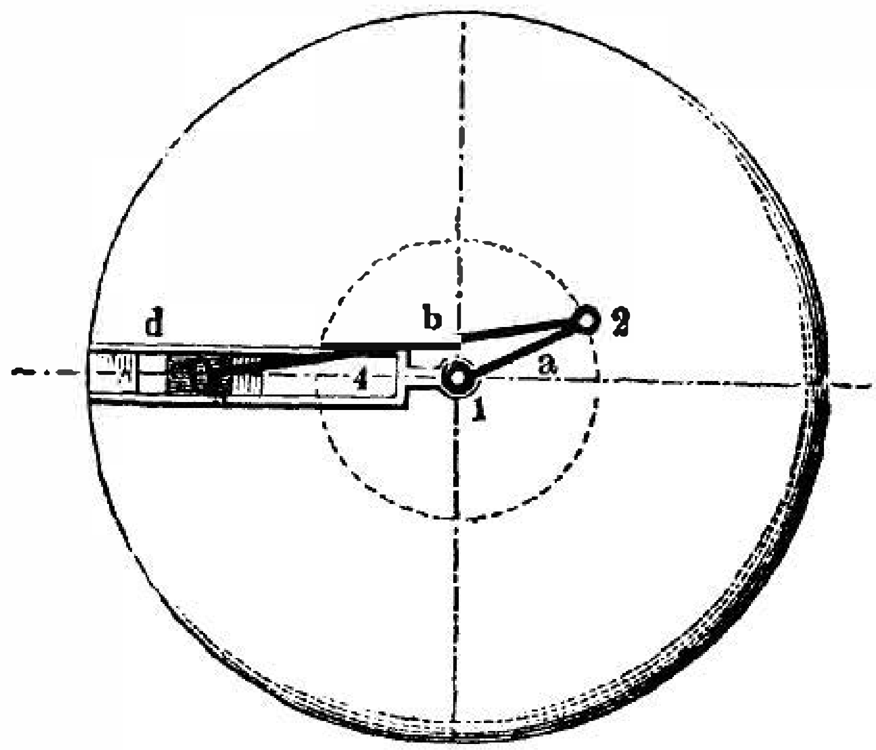


FIG. 280.

Fig. 282, to show how these axes come in the conic slider-crank. It will be there seen how the linear velocities of the points 2 and 3 will be equal when $S2 = S3$, and how *SO* will coincide with 4*O* when the crank is in its mid-position. In this position *S*2 will be at its shortest $(=90^\circ - a)$ and *S*3 at its longest $(=90^\circ)$; therefore for this position of the mechanism (crank in mid-position) there

[1] We have already stated the reasons which compel us here to state the proposition thus instead of with the use of the word *centre*, as formerly.

[2] Compare the theorem on p. 72 and the construction shown in Figs. 29 and 31.

will be the maximum value of the ratio $\frac{\text{vel. } 3}{\text{vel. } 2}$, which corresponds to the ratio $\frac{\text{vel. crosshead}}{\text{vel. crank-pin}}$ in an ordinary engine. There will be two positions, one on each side of the middle, when S_2 and S_3 are equal, and where therefore vel. 2 = vel. 3. If the link *b* subtends a right angle as well as the links *c* and *d*, these two positions will be symmetrical to the mid-position, otherwise they will be unsymmetrical.

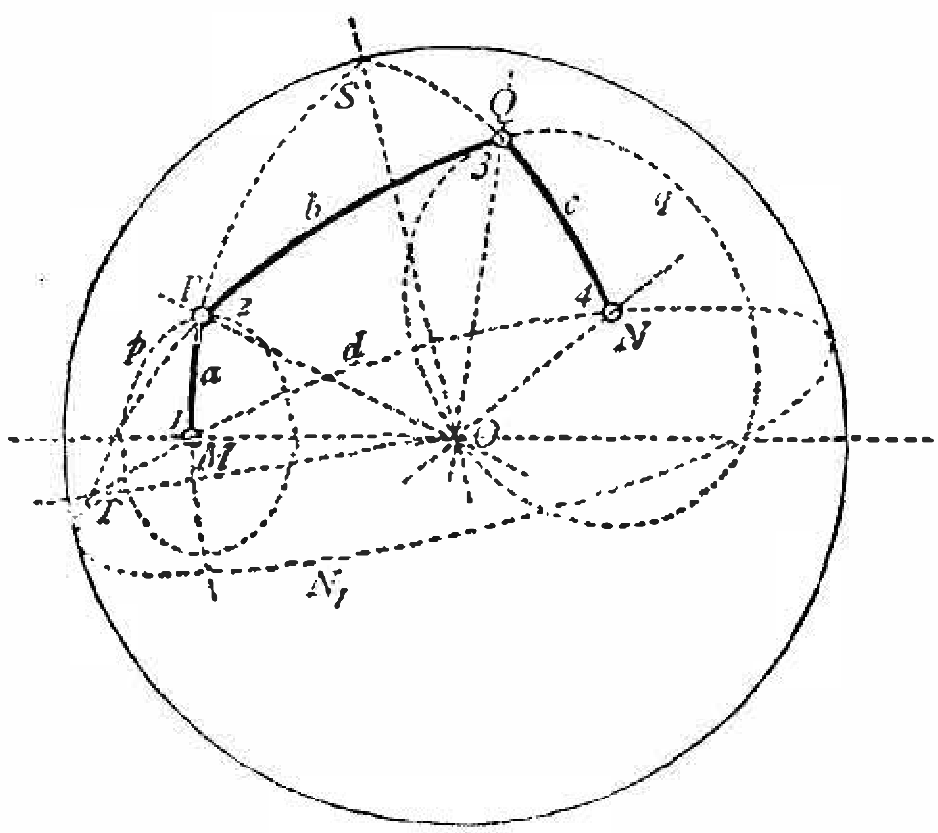


FIG. 281.

As the virtual axes can be found in this way for any conic mechanism, all the kinematic and kinetic problems which we could solve by their help in connection with plane mechanisms can be similarly solved here. Such additional difficulties as there are arise chiefly from the difficulty of recognising the mechanisms under most elaborate constructive disguises, and from the unavoidable necessity of drawing ellipses where formerly straight lines were sufficient. We shall now proceed to examine two or three of the principal practical applications of conic chains, and to work out in connection with them problems similar to those which we have already solved with plane mechanisms. The methods of treatment and solution given in the next two sections will be such as are applicable to any conic trains, although actually applied here only to special ones.

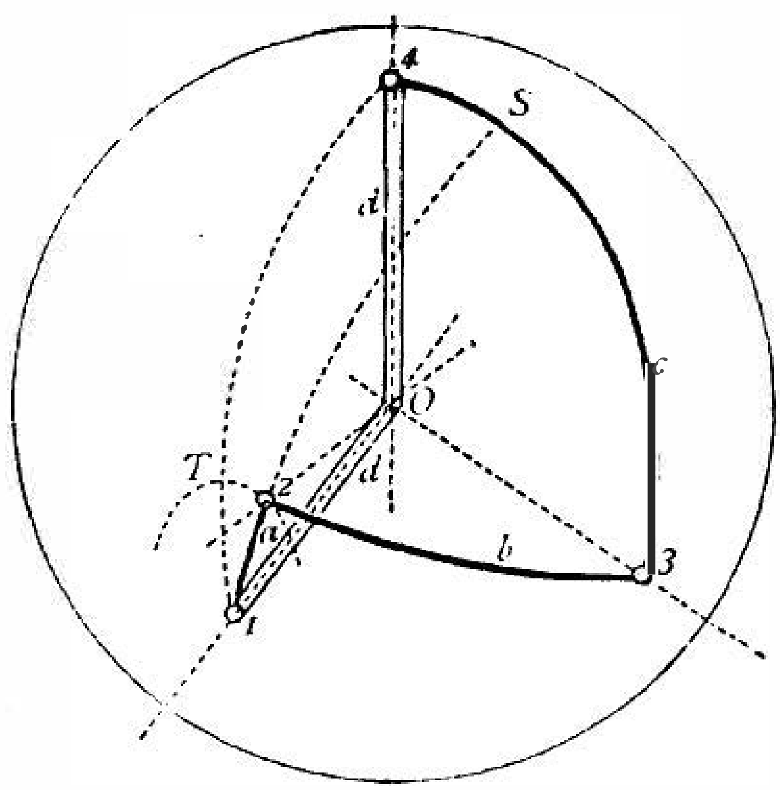


FIG. 282.

§ 64.—THE "UNIVERSAL JOINT."

PROBABLY the most familiar example of a conic crank train occurring in practical work is the Hooke's coupling or **Universal joint**, shown in Figs. 283 to 285, of which the first shows the joint in a form more or less resembling that commonly used in construction, while the two others show it in the schematic form adopted in the last section, and show the link a alternatively made to subtend an acute angle and its supplement. Corresponding links and pairs are noted by the same letters or numbers in the two figures. The chain consists of four links connected by turning pairs, whose axes all meet in one point O. Three of the links—

b, *c*, and *d*—subtend each a right angle, the fourth, *a*, the fixed link, subtends some much larger or (p. 493) smaller angle.

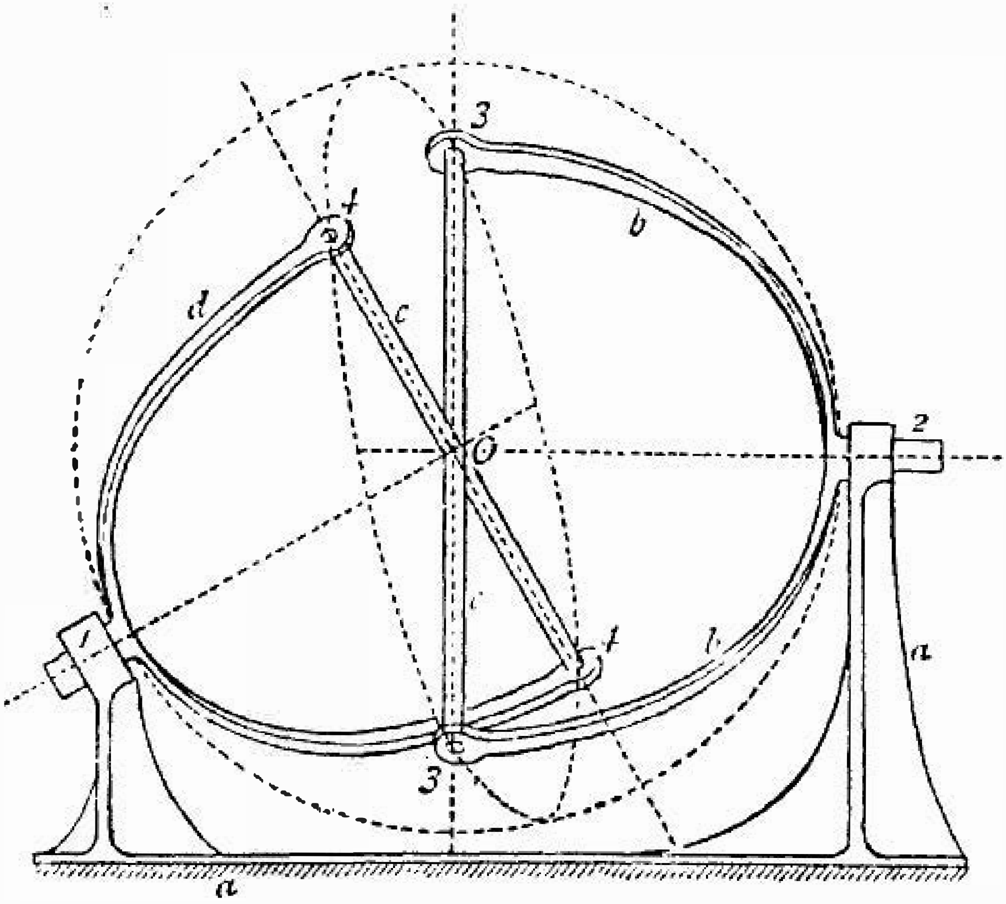


FIG. 283.

In Fig. 283, the pairs 1 and 2 of *a* are placed, as they are usually in construction, on opposite sides of the mechanism.

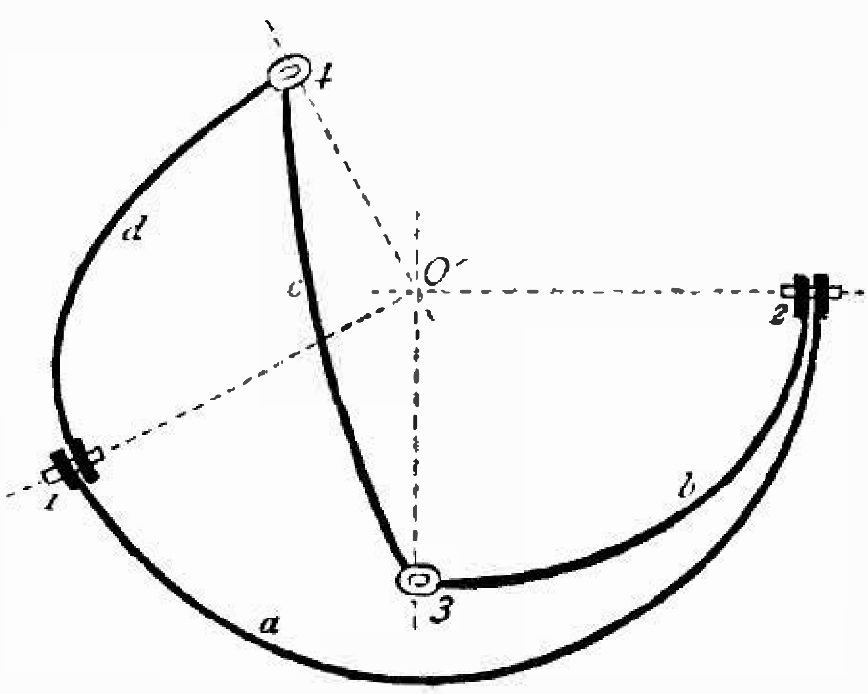


FIG. 284.

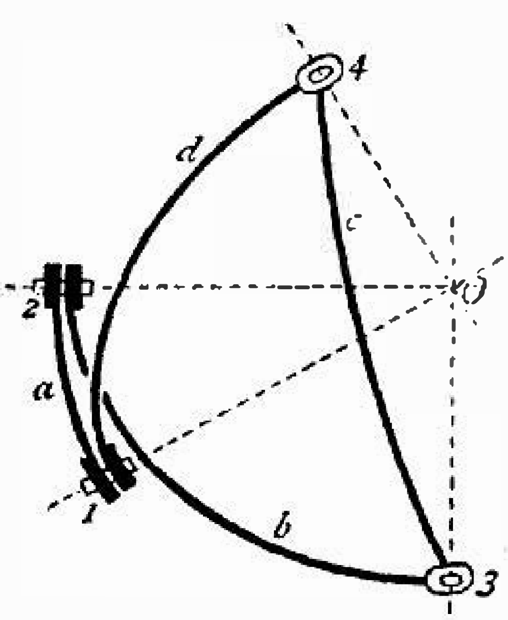


FIG. 285.

In Fig. 285 they are shown on the same side, and it will be seen from this sketch that the mechanism is shown in a position corresponding to those on the lower side of Fig. 55 (p. 111). We have already seen that the change from an angle to its supplement is without influence on the motions occurring in the mechanism, and does not, therefore, constitute any real difference between them. The links *b* and *d* are little altered in appearance, but in Fig. 283, for the sake of securing additional steadiness in the mechanism, *both* ends of 3 and 4 are paired to *c*, and not only one, as in Figs. 284 and 285. Lastly, the link *c* is transformed (like *d* in Fig. 282) into the shape of a cross, paired at 3 and 4 with the double-sided forks of *b* and *d*. Kinematically it is equally a link carrying two elements of turning pairs (pins or eyes), with their axes at right angles to each other, whether it have the form shown in Fig. 284 or be made as the cross of Fig. 283.

If we take an ordinary lever-crank, and fix the short link *a* (as in Fig. 55, p. 111, for instance), we obtain a mechanism commonly known as a "drag-link coupling." The links *b* and *d* revolve on parallel axes, the one driving the other through the "drag-link" *c*. If the mechanism is a parallelogram (Fig. 43), so that $c=a$ and $b=d$, the two revolving links are turning always with the same angular velocity, but the mechanism has two change-points (p. 147). With the ordinary proportions of links, *b* and *d* turn with constantly varying velocity ratio, as we have seen in connection with Fig. 55. When the lever-crank is turned into a slider-crank, and the link *a* again fixed, we get the "quick-return" mechanism of Fig. 126. Here again *b* and *d* revolve, and the one drives the other through the link *c* with constantly varying angular velocity ratio. When, lastly, the lever-crank is turned into a conic chain, and the link *a* fixed, we obtain the mechanism before us, which moves in precisely the same fashion, and is similarly used for the transmission of rotation from one shaft to another. The links *b* and *d* rotate, and the one drives the other through the intervention of the link *c*. Here, again, both driving and driven shafts have their bearings in *a*, the fixed link, but they are now angled to each other instead of being parallel. The angular velocity ratio transmitted is a constantly varying one. The Hooke's joint is in effect a drag-link coupling between shafts whose axes are not parallel, but meet in a point at a finite distance.

It follows from the construction of the mechanism that the planes of *b* and *d* are at right angles to each other four times in each revolution (this can be seen at once from the figures following), and at these instants the two shafts are revolving with the same velocity. The links *b* and *d* thus make quarter-revolutions in the same time. Between these positions the angular velocity ratio varies very much, and varies the more the greater the angle between the shafts. This variation is so great as to make the mechanism practically unusable unless the angle subtended by the link *a* (*i.e.* the angle between the shafts) be small; while if *a* be made $=90°$, the mechanism ceases to be moveable at all.

The first problem in connection with the universal joint is the finding of the position of *d* for any given position of *b* (we may suppose *b* the driving and *d* the driven shaft), the next problem is to find their relative velocities in any given position, and the last to find the corresponding relations between effort and resistance.

Before going on with these problems it is necessary to find where the virtual axes of the mechanism lie. They are shown in Fig. 286, in which four planes are drawn instead of the four bars, in order to make the intersections more clearly visible. The planes are placed exactly in the position of the links of the three last figures, the actual position of the links of Fig. 285 being duplicated by dark lines on the edges of the planes. The plane of the link a, the vertical circle, is taken as coinciding with the plane of the paper. The three axes A_{bc}, A_{cd}, and A_{bd} (see p. 490) are therefore three lines in the plane of the paper. A_{ab} and A_{ad} are the axes of the shafts, and are therefore known. A_{bd} is the join of the planes of the cross (the link c) and the shafts (the link a). The axes A_{bc}, A_{cd}, and A_{bd} are again all in one plane. The two first-named are the arms of the cross, the last we have just found. The plane containing

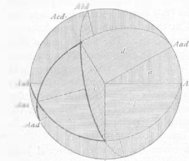


FIG. 286.

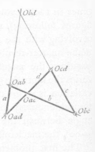


FIG. 287.

them is, of course, the plane of the cross. The axis of the cross c relatively to the fixed link a, A_{ac}, is the only one the drawing of which offers some difficulty. Its position is, as we know, the join of the planes of b and d, the two forks. A_{ac} lies along with A_{ab} and A_{bc} in the plane of the fork b, and along with A_{ad} and A_{cd} in the plane of the fork d. For the sake of showing how completely the same the whole matter is with that which we examined in connection with plane motion, the corresponding plane mechanism is drawn in Fig. 287, in corresponding position to Fig. 286, and its six virtual centres marked.

A most obvious difference between the plane and the spheric mechanisms is felt as soon as any attempts is made to handle them on paper. Any number of positions of the plane mechanism could be drawn at once, without the slightest difficulty, as soon as ever the lengths of the links were given. With the spheric mechanism, on the other hand, although it is easy to draw the chain in the position of Fig. 288 and in a position 90° from it, to draw it in any other position is not so simple a matter, and, indeed, cannot be done without projection in two planes. This does not indicate in any way that these mechanisms belong to a higher order (§ 59), but follows simply from the fact that the non-plane motions cannot be fully represented on paper without projection in two planes.

We shall now proceed with our first problem: the finding of the position of the whole mechanism when that of any one link is given. The link *b*, which we may consider as a driving shaft, is generally the one whose position is given. We shall first see how to find corresponding positions of the links *d* and *c*, the former being the more important.

In Fig. 288 is shown a Universal joint drawn, as before, so that the plane of the paper coincides with that of the two shafts, that is, of the link *a*. The fork *b* starts from a vertical position, the plane of *d* is at right angles to the vertical plane. In the side view, Fig. 289, the two axes of the link *c* appear vertical and horizontal, as *OM* and *ON*. The ellipse *n*, which is the projection (in Fig. 289) of the path of the point *N* (or 4), must be drawn first. Let the link *b* move through any angle, *β*, so that *OM* takes up the position OM_1. The arm *ON* must always *appear*, in Fig. 289, to be at right angles to *OM*, and the projected path of N has already been drawn, so that the point N_1 can be found at once. N_1ON (Fig. 289) is the projection only, on the plane of the paper, of the real angle moved through by ON (*i.e.* by the fork *d*) in a plane inclined at an angle θ to the plane of the paper. The real angle is therefore to be found by turning back the one plane upon the other, which is done if we turn back the ellipse M_3N_3N until it coincides with the circle M_3MN. The point N_1 then comes to N_2, so that N_2ON, or α, is the real angle moved through by *d*, while *b* is turning through the angle β.

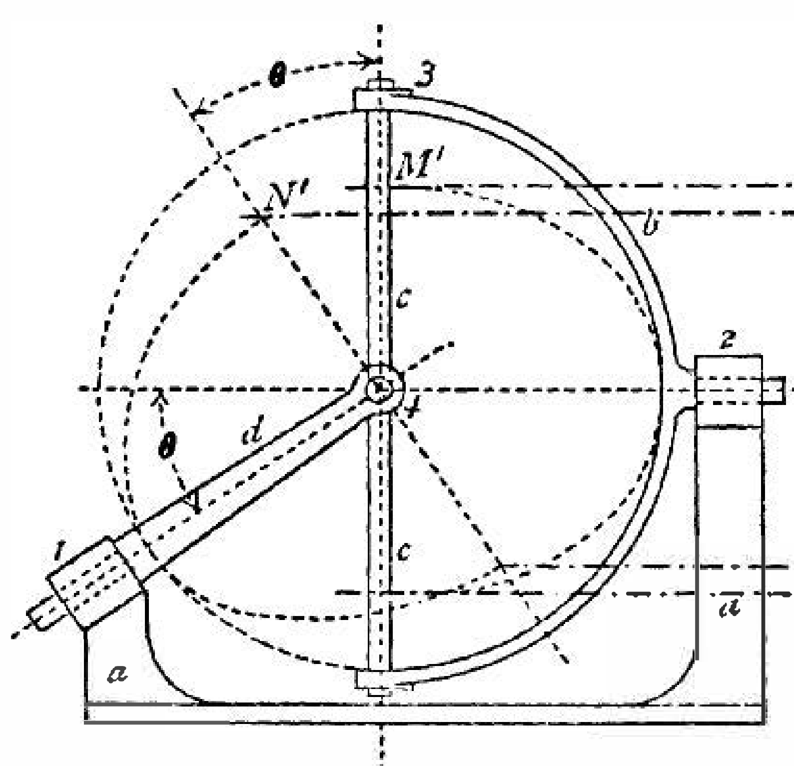


FIG. 288.

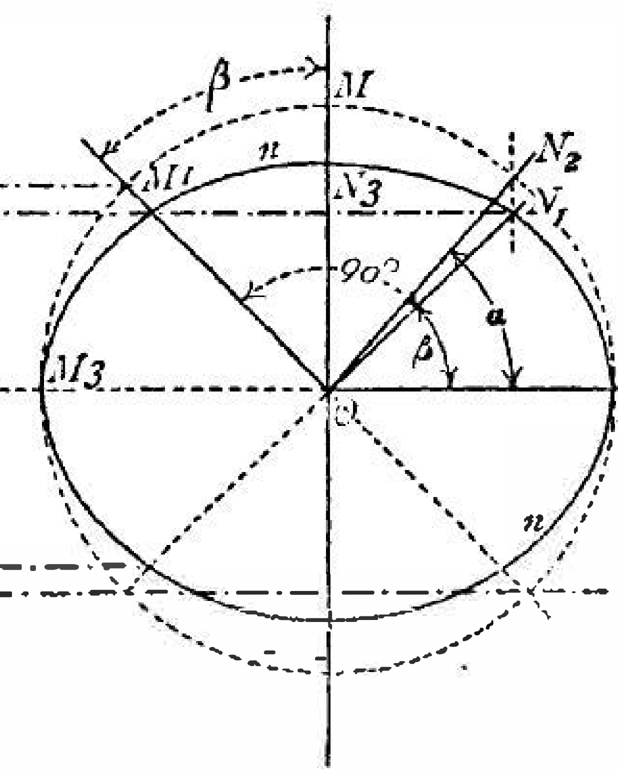


FIG. 289.

The point N_2 is always the point of the circle beyond N_1, and the angle NON_1 is always equal to the angle β. We have thus solved the first part of our problem, the finding of the position of one shaft when that of the other is given. It can easily be seen from the construction that (as has been already mentioned) the two shafts make quarter turns in the same time, although their velocities vary very much within each quadrant. Thus, when OM reaches OM_3, the projection of the position of ON will be ON_3, and the real angle, as well as the projected angle, turned through by ON will be a right angle.

Algebraically, using the symbols of the last two figures,

$$\sin \alpha = \sin \beta \frac{ON_1}{ON_2 \cos \theta} = \sin \beta \frac{ON_1}{ON \cos \theta}.$$

If we wish not only to find the position of d, but also to draw the mechanism in its new position, we may proceed as follows (Figs. 290 and 291):—we know that in the projection, Fig. 290, the one arm of the cross must always appear to lie along the line m and the other along the line n, the one

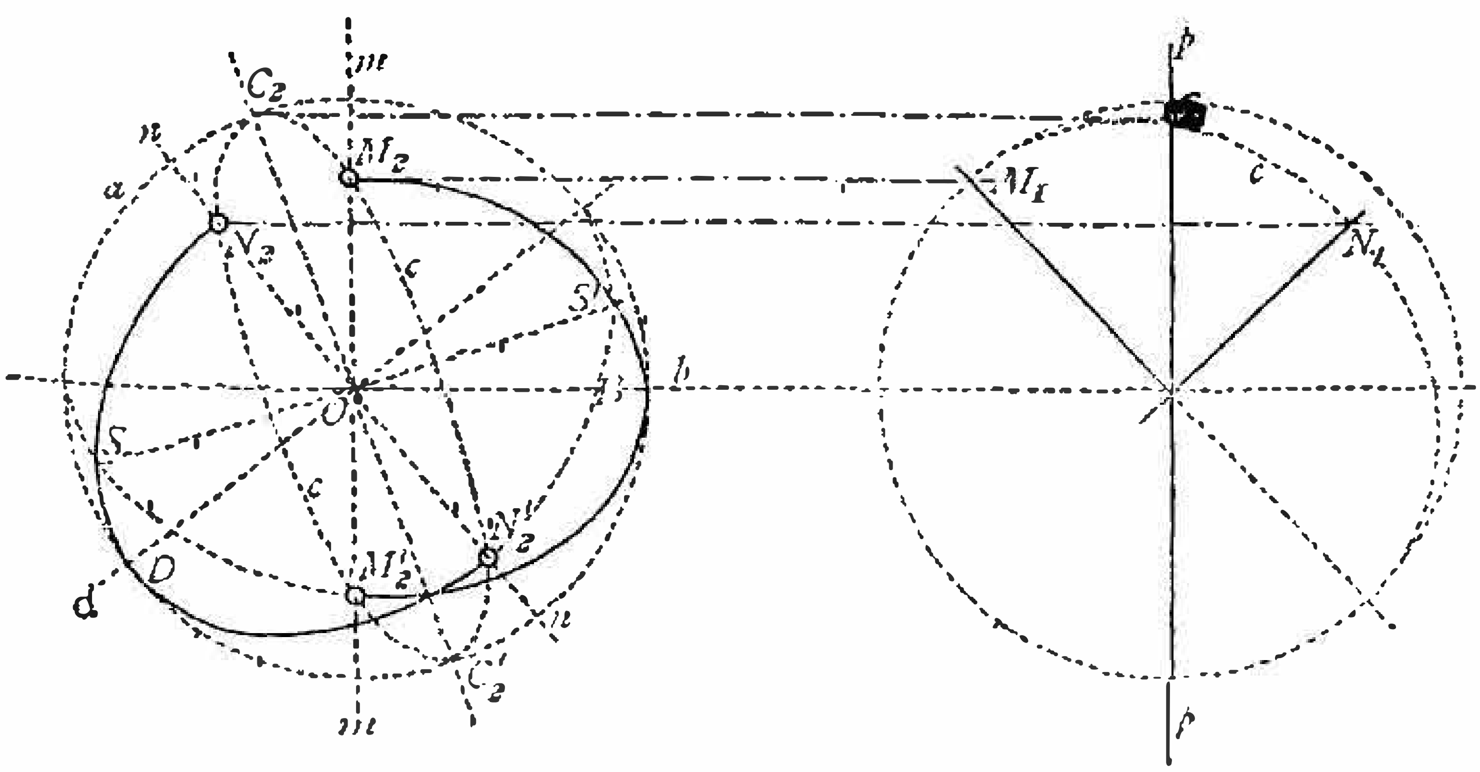


FIG. 290. FIG. 291.

at right angles to the axis of b and the other at right angles to the axis of d. We can therefore project the points M_1 and N_1 from Fig. 291 at once on to these lines, at M_2 and N_2. (The other ends of the cross arms can of course be marked at once, at points equidistant from O, at M'_2 and N'_2). The fork of b will then appear as part of an ellipse, whose semi-axes are OM_2 and OB, and the fork of d as an ellipse whose semi-axes are ON_2 and OD; they are so shown in the figure.

We have thus been able to draw the links of the mechanism in their right relative positions. Before we can go further we must, as we know, find the six virtual axes which belong to the mechanism. Four of these we have already found in Fig. 290, the four corresponding to the four pairs of adjacent links, the lines *b*, *d*, *m*, and *n* in the figure. The first two lie in the plane of the paper, the last two are inclined to it at angles made determinate by Fig. 289. The two axes of opposite links, A_{bd} and A_{ac}, have still to be found. The first of these is at the join of the planes of *c* and *a*, as shown in Fig. 286. It is therefore a line in the plane of the paper in Fig. 290, and in a plane (as *p*) at right angles to the paper in Fig. 291. It is probably most easily found by taking both *a* and *c* as circular plane figures, and finding their intersection. The projection of *c* in Fig. 291 will be an ellipse whose semi-axes are OM_1 and ON_1. It is only necessary to draw as much of this ellipse as will give us the point C_1, where it cuts the plane *p*. This point can then be projected to the *a* circle in Fig. 290, and so gives us C_2, the line $C_2OC'_2$ being the line we require, the intersection of the planes of *c* and *a* of the cross and the shafts. (This line is also the axis of the *c* ellipse sketched in Fig. 290, and formerly in Fig. 286, which passes through the four points, N_2, M'_2, N'_2, and M_2, and has, like the others, the point *O* for its centre.)

The virtual axis of *a* relatively to *c* is the join of the planes of the two forks *b* and *d*, as shown in Fig. 286. It can here be easily found, without the aid of the second view, by drawing the fork ellipses until they cut each other (as both are projections of great circles on the same sphere they *must* cut somewhere) in the point *S*. The line *SOS'* is the required axis A_{ac}.

Having now the means of drawing the virtual axes as well as the links of the mechanism, we can proceed to problems connected with the relative velocities of its different links. Angular velocities are here of so much greater importance than linear velocities that we may confine ourselves to them. The method to be used for finding the relative angular velocities of two links —say *b* and *d*, the two shafts—is essentially identical with that employed in the similar case with plane motion which was illustrated in Fig. 45, p. 99. We require, first, the three virtual axes, A_{ab}, A_{ad}, and A_{bd} (*a* being the fixed link), which we have just found, and which we know to be three lines in the plane of the paper in Figs. 288 or 290. The axis A_{bd} being a line common to the two bodies *b* and *d*, *any point in it* must have the same linear velocity relatively to *a*, whichever body it may be considered to belong to. But each body is turning, relatively to *a*, about a known fixed axis, A_{ab} for *b*, and A_{ad} for *d*. The angular velocities of the bodies are therefore inversely proportional to the radii, measured from these axes, of any one point in their common line. The similar case for plane motion was stated on pp. 96 and 97, § 15. We then required to find three virtual centres—the fixed point of each link and the common point of the two links—and these three points were in one line. Here we require, similarly, to find three virtual axes—the fixed axis of each link and their common axis—and these three lines are in one plane. Formerly we said that the virtual centre of the two moving bodies was a point which had the same linear velocity in each. Now we can say that the virtual axis of the two moving bodies is a line all of whose points have the same linear velocities in each.[1] The necessary constructions are therefore essentially the same as formerly, differing only because projection on two planes

[1] This also might, of course, have been equally said before, but for reasons which were explained we preferred to express it in the former fashion.

is here required, while formerly everything could be shown upon one plane only.

Having found the three lines A_{ab}, A_{ad}, and A_{bd}, as in Fig. 292, which simply duplicate the axes found in Fig. 290, we have only to measure the distances of any point, as C, in A_{bd} from A_{ab} and A_{ad}; these distances being respectively CB and CD in the figure. They can be measured directly, as all the lines with which we are here concerned are lines in the plane of the paper. The angular velocities of the links

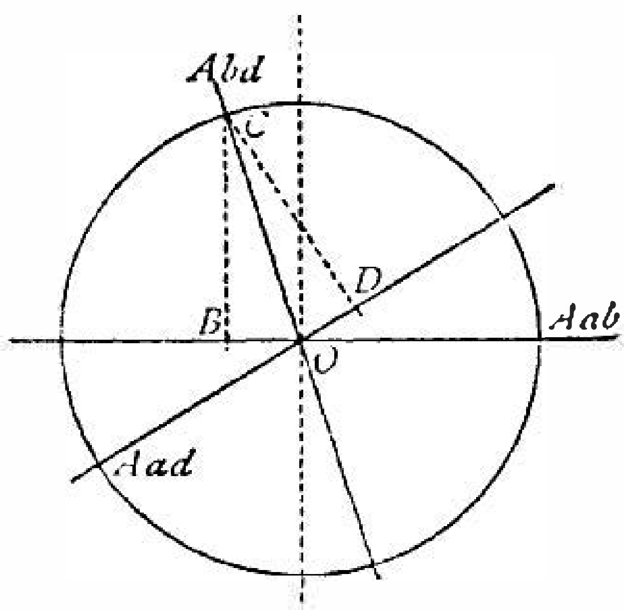


FIG. 292.

b and d are, therefore, inversely proportional to the distances CB and CD, that is,

$$\frac{\text{angular vel. } b}{\text{angular vel. } d} = \frac{CD}{CB},$$

Algebraically this ratio is equal to

$$\frac{\tan\alpha + \cot\alpha}{\tan\beta + \cot\beta},$$

using the nomenclature for the angles employed in Fig. 289 above. The limiting values of this ratio are $\cos\theta$ when OM (Fig. 289) is in the plane of a, and $\frac{1}{\cos\theta}$ when ON is in the plane of a, the angle θ being the angle between the shafts.

At one position within each quadrant the two shafts have for an instant the same velocity,[1] and as we have already seen, they make quarter-revolutions in the same time, so that the *mean* value of their angular velocity ratio is unity.

Were it desired to find the relative angular velocities of *c* and one of the other links, say *b*, a problem which may occasionally occur in some modern machines, it can be found in a precisely similar manner. First, the three axes A_{ab}, A_{ac}, and A_{bc} are found; they all lie (see Fig. 286) in the plane of the fork *b*. Then the distances are measured from any point in A_{bc} to the other two axes, and the calculation made exactly as in the last case. The only additional complication is that the plane of *b* is not, as we have drawn the mechanism, the plane of the paper, and has to be turned down into it, about A_{ab} (exactly as we did on p. 504), before the distances can be measured.[2] A figure drawn like Fig. 286 will generally make the position of the virtual axes quite clear and intelligible, although their position may be a little difficult to realise in looking at the mechanism itself; it may often be worth while, therefore, in working practical problems with these mechanisms, to use such a figure for reference.

The only problems which now remain to be considered respecting the class of mechanisms of which we have been taking the universal joint as representative, are those involving the static or kinetic balance of forces in them. The general problem which was treated for plane motion in § 40, the finding of the force which must act in a given direction at *any* point of any link in order to balance a given force acting at any point in any other link, can be solved here by the same general and direct method which was employed

[1] See the curves of velocities in Fig. 307, § 65.
[2] An example of this calculation occurs in § 65.

before. The application of this method is practically troublesome, however, because of the amount of projection it involves. It is more convenient, and for many purposes sufficient, to suppose all forces to be acting through the two moving virtual axes in the mechanism which are also permanent axes, namely (if a be the fixed link) the axes A_{bc} and A_{cd}. In order to do this, it is only necessary to divide the magnitude of each force by the alteration of radius. This will be clear from Fig. 293, where, for the sake of

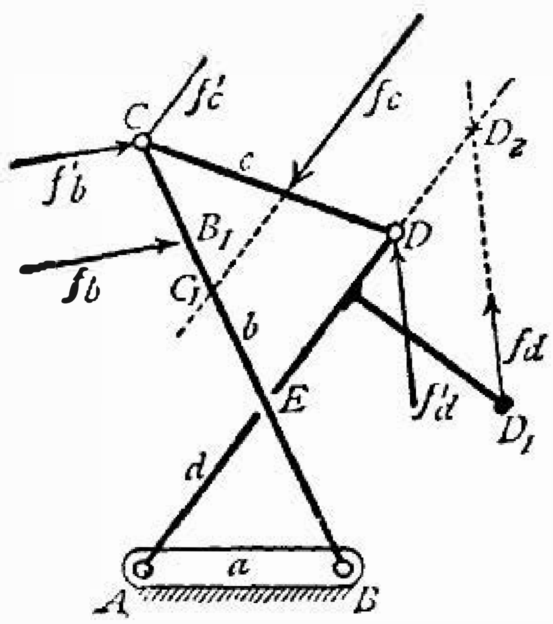


FIG. 293.

clearness, a plane mechanism is drawn. Instead of taking the force f_b at B_1, it is shifted parallel to itself to C (which is here O_{bc}), and is taken as $f'_b = f_b \times \frac{BB_1}{BC}$. Similarly f_d is taken as acting at D ($= O_{cd}$) instead of at D_1, and its magnitude is supposed changed to $f'_d = f_d \times \frac{AD_2}{AD}$. Forces acting on c at any other points than C or D can in the same way be reduced to either of those points; thus f_c may be replaced by $f'_c = f_c \times \frac{C_1E}{CE}$. It is generally most convenient to perform this reduction arithmetically. It is not always quite so easy to see what the distances BB_1, BC, &c., are in the conic mechanisms as in the plane, and Fig. 294 is therefore drawn to correspond to the last one, but for a conic train. Here a force f_d acting on d in any direction at D_1, may be replaced by a force f'_d, acting in a parallel direction at D, and equal to $f_d \times \frac{D_1A}{DO}$. Similarly any force f_b acting on b in any direction at B_1 may be replaced by a force f'_b acting in a parallel direction at C, and equal to $f_b \times \frac{B_1B}{CO}$. The case is not quite so simple here with forces on the link c.

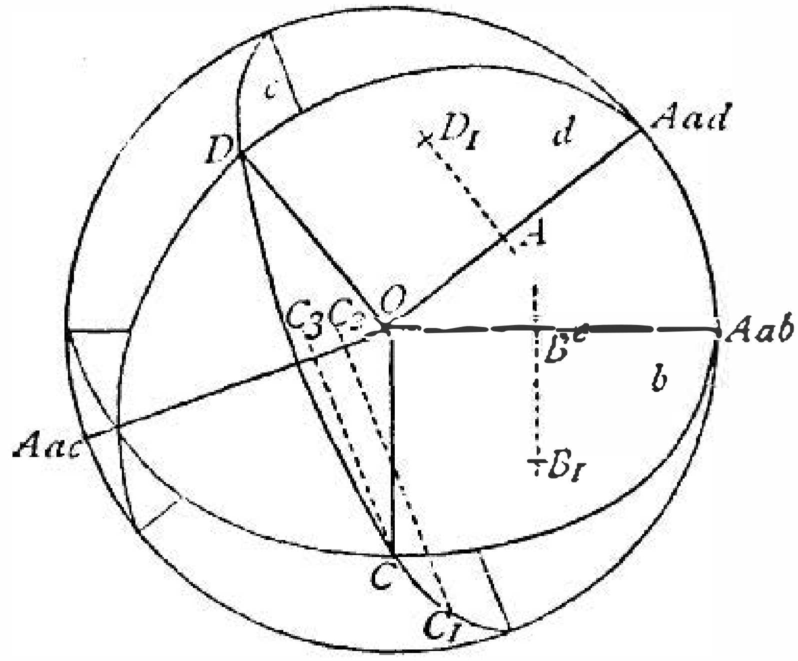


FIG. 294.

Any force f_c at C_1 may be replaced by a force f'_c acting at C, and equal to $f_c \times \frac{C_1C_2}{CC_3}$. But the radii C_1C_2 and CC_3 are not parallel (although their projections are), so that these distances cannot be measured on the paper, their real lengths must be found by projection. Further, the new force f'_c cannot be taken as parallel to f_c, but only as making the same angle to OC that f_c made to OC_1. In this case, as also in cases where such points as B_1 or D_1 do not lie actually in the planes of the forks, it is more convenient to treat the force in the way described in the following paragraphs, before transferring it to another point.

In dealing with plane mechanisms, we assumed that the forces in action were always in the plane of the mechanism. Any components which they had normal to that plane were balanced by the profiles (p. 54) of the elements, which rendered impossible all motions not parallel to the plane. In the plane itself we virtually resolved the force at any point into components parallel and normal to the direction in which the point was moving. The parallel component had to be balanced by other external force or forces, the normal component was balanced by the stresses in the links. Considering the force as acting on a body at some particular point, its tendency to turn the body about its virtual centre, or *moment* about the virtual centre (p. 269), was equal to the product of its component parallel to the direction of motion of the point and the virtual radius of the point. If the question were merely of the moment of the force upon the body, without reference to its action at any particular point in the force line, we saw that it was unnecessary to resolve the force in its own plane. The moment was simply equal to the whole magnitude of the force multiplied by the virtual radius of the force line—that is, its perpendicular distance from the virtual centre.

These simple matters have been repeated here that they may be the more distinctly before us in their applications to spheric motion, with which we have now to deal. The links in a conic mechanism have at each instant just the same sort of motion as those of a plane mechanism—each link, namely, is in rotation about a determinate axis. But these axes, so far as they are non-permanent or instantaneous, are continually shifting their *direction* as well as their position. The direction of the plane normal to the axis—or plane of effective force, as we may call it—is therefore continually changing, and even at any one instant, as the axes of the different links are not parallel, the planes for the different links are not parallel, instead of being, as before, all coincident. This difference introduces some new complexity into the problems; but if it only be borne in mind, the whole of the last paragraph may be taken as applying equally to spheric and to plane mechanisms. Two or three illustrations may be given here; they will serve to show the essential identity of the problems as well as to illustrate their superficial differences.

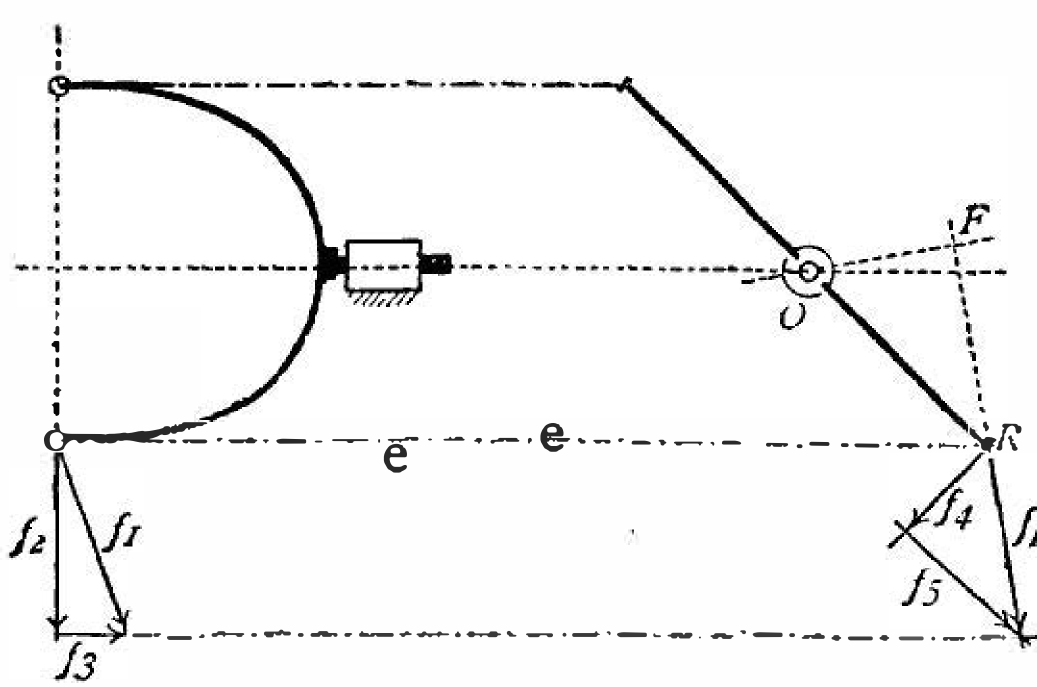


FIG. 295. FIG. 296.

A force f_1, given by two projections in Figs. 295 and 296, acts on one of the forks of a universal joint. Resolving into f_2 and f_3 in Fig. 295, we find at once f_2 as the projection of the nett turning force, and f_3 as the component in the direction of the axis. The latter causes axial pressure or thrust in the bearings, but causes no motion of the fork, and is therefore negligible so far as motions are concerned. The plane of the paper in the second view, Fig. 296, is a plane at right angles to the axis, and is therefore the plane in which effective forces act. The projection of f_1 in this plane, namely f'_1, gives the real length of f_2. If our object is merely to find the turning moment, we have it at once as $f'_1 \times OF$. But if we wish to know more completely what is occurring, we must resolve f'_1 in the direction of motion of the point R, at which it really acts, and normal to that direction, *i.e.*, into f_4 and f_5. The effective turning moment is then $f_4 \times OR$ (which is, of course, $= f'_1 \times OF$), and the component f_5 is the magnitude of the side pressure in the bearing.

We get nothing essentially different from this if the given force act upon the cross (the link *c*, Fig. 283) instead of upon *b* or *d*. We have only to choose the position of the planes of projection so that they bear the same relation to the virtual axis of *c* (A_{ac}, Fig. 286) as the planes of Figs. 295 and 296 bear to the axis of *b*, A_{ab}. One of them, namely, should contain the axis, and the other be at right angles to it.

It will be seen that in neither case have we done more than, or differently from, what we should have done with a simple "turning pair," if only the force in action had not lain in a plane at right angles to its axis.

We may now assume that we have to deal only with forces resolved in the direction of motion of the points on which they act, and, further (as we have always tacitly assumed in connection with plane motion), that the force so resolved may be taken as acting in any plane normal to its virtual axis,[1] so that it may be shifted, if necessary, parallel to the virtual axis to any extent. With these assumptions we can proceed to look at such cases of the

[1] Forces acting on different points of the link may, therefore, first be each resolved in planes at right angles to the virtual axis and normal to such planes, and then the resolved parts in the parallel planes may be all added together or otherwise treated, so far as motion is concerned, as if they were all in one plane.

static balance of forces as we considered in §§ 38 to 41 for plane mechanisms. For the balance of forces on any one link we may either simply balance moments, as on p. 285, or may proceed by resolution through the virtual axis, as on p. 280, whichever happens to be most convenient. It is not necessary to give any example of this. For the balance of forces on different links, whether adjacent or non-adjacent, it is frequently most simple to determine the relative angulari velocities of the links, and to calculate the balanced forces from these, remembering that the turning moments on any links which are in static equilibrium must have magnitudes inversely proportional to the angular velocities of the links. Suppose, for example, that b and d, the two shafts, are the two links concerned, and that their angular velocities for the instant are v_b and v_d respectively, the ratio between those quantities being found as on p. 508. Let f_b be the resolved part (as f_4 in Fig. 296) of the forces acting on b, and tending to cause its rotation, and r_b be the radius of f_b (as OR in Fig. 296), then (using similar lettering for forces, &c., on d) we must have

$$\frac{M_d}{M_b} = \frac{r_d f_d}{r_b f_b} = \frac{v_b}{v_d}$$

and $f_d = f_b \cdot \frac{v_b}{v_d} \cdot \frac{r_b}{r_d}$.

The force f_d thus calculated is the force which, acting in a plane normal to A_{ad}, the virtual axis of d, and at a distance r_d from that axis, will balance the force f_b acting in exactly similar fashion on b. If the real force on d is not acting in the plane mentioned, but at some angle θ to it, the whole magnitude of the force must be $\frac{f_d}{\cos\theta}$, while it will have a component equal to $f_d \tan\theta$ acting along the direction of the axis A_{ad}, and causing axial thrust in the bearings of the mechanism.

For the purpose of finding the variation of balanced resistance and making a diagram such as Fig. 146, p. 321, as we shall presently do, this method is all that is required. But for designing a machine it may be insufficient,—we may require to work through the intermediate link *c* in order to find the stresses to which it is subject in transmitting the driving effort from *b* to *d*. Suppose, for example, that a known force acts at the point 3, Fig. 288, and in the direction of its motion, let it be required to find the force in the line 43 necessary to balance it, which would, of course, be the stress in a straight bar connecting-rod coupling those

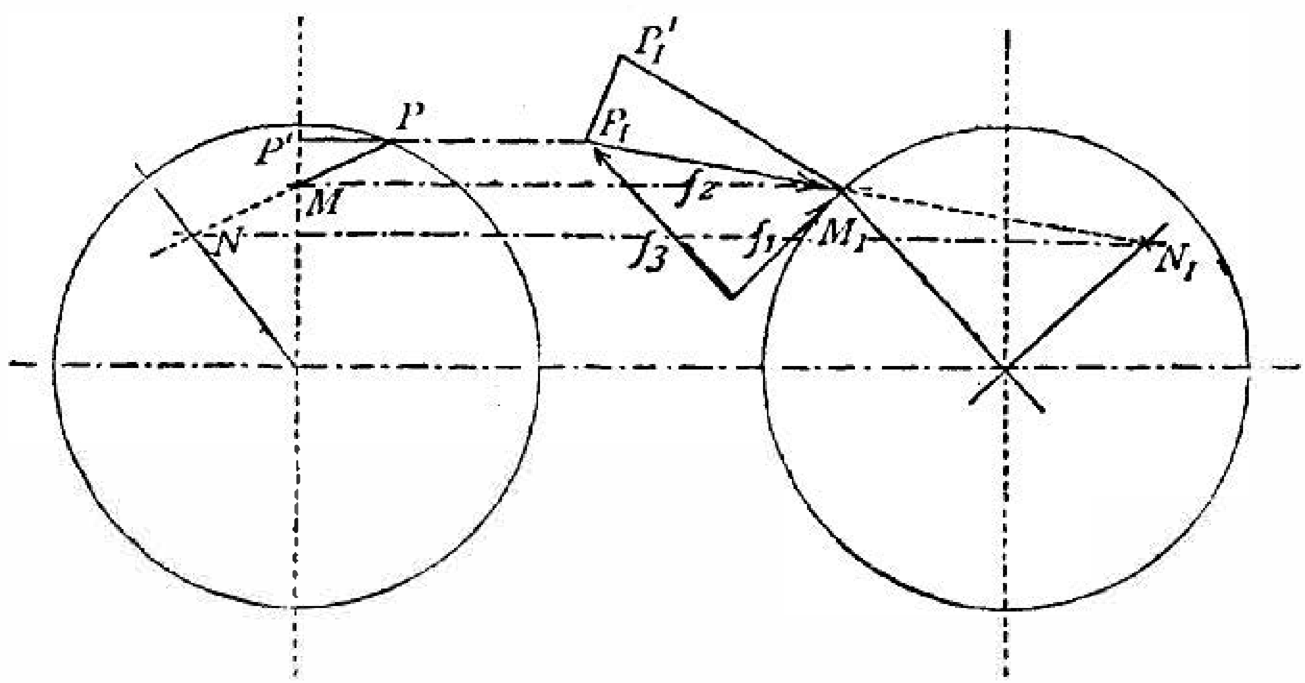


FIG. 297. FIG. 298.

points. In Figs. 297 and 298, the points M and M_1 are the two projections of 3, and N and N_1 the two projections of 4, their positions corresponding to those before determined and sketched in Fig. 288, and the planes of projection being arranged as before. The given force on M (in the plane of Fig. 298) is f_1, and this force can be resolved (in that plane) into f_2 in the given direction N_1M_1, and f_3 through the virtual axis. f_3 is balanced by the pressure of the shaft against its bearings. f_2, or P_1M_1, is the projection, in thes given plane, of the force required to balance f_1. In Fig. 297

PM is the projection of the same force. The real length of this force can be found by turning it down in the usual way, namely, by setting off $P_1P'_1$ equal to PP'. The real value of the pressure acting along the line *NM* is thus found to be $M_1P'_1$.

In § 16, Figs. 55 and 56, we gave a diagram of relative angular velocities of an ordinary drag-link coupling, and in the same section, p. 113, it was pointed out that the chain of Figs. 32 or 53, with the link *a* fixed, gave also a coupling ("Oldham's") for parallel shafts, but one which transmitted a constant velocity ratio equal to unity, instead of a varying one. That mechanism was one with three infinitely long links (based on the slider-crank with infinitely long connecting-rod, §§ 12 and 52), and was one, therefore, corresponding exactly to the Hooke's joint, with its three right angled links. Turned into a conic train, however, the velocity ratio transmitted is no longer constant, although (as also with the mechanism of Fig. 55) its *mean* value is unity. Its value varies very greatly at different parts of the stroke. A diagram of velocities will be found in the next section in connection with the engines based upon this mechanism, which are there examined.

§ 65.—DISC ENGINES.

It is now some sixty years since some inventive mind, longing for novelty, found out that any such conic chain as we have examined in § 63 could be made available as a steam engine. From the form then given to the link *b* the engine was called a **disc engine**, and this name has been subsequently applied generically to the large number of steam engines which have been founded on conic mechanisms. The real nature of these mechanisms was practically unrecognisable until the kinematic analysis of Reuleaux appeared, and Reuleaux himself was the first to point out its application to them, and their essential identity with certain familiar plane mechanisms—ground over which we have followed him in §§ 63 and 64. In his book[1] he examines and analyses a number of those forms of disc engines which had been proposed previous to its publication, and there is no need that we should go over the same ground. None of the forms which he describes were such as to give any promise of practical success, and probably not a single one of these older forms still survives. Lately however, some other forms of disc engine have been devised, in which the one advantage of such engines, the high speed, has been made the most of, and the several disadvantages have been, by careful design, much reduced. Of these recent disc engines we shall examine in this section the two latest, each of which appears to have possible future practical usefulness, for high speed driving, before it. One is the invention of Mr. Beauchamp Tower. It has been already used to a considerable extent for electric lighting purposes. It will be found described in a paper by Mr. R. H. Heenan, published in the *Proceedings of the Institution of Mechanical Engineers* for 1885. The second is the invention of Mr. John Fielding—it has appeared for the first time at the International Inventions Exhibition of 1885. A description of it has been published in the *Engineer* for June, 1885, and in *Engineering* for July 31st, 1885.

The Tower engine, called by its inventor the "isphericali" engine, is based directly on the mechanism of the universal joint. It consists, that is, of a conic crank train of four links, three of them subtending right angles, the fourth subtending a smaller angle, which in this case is always

[1] *Kinematics of Machinery*, pp. 384-399, and plates xxviii.-xxxi.

made 45°. This short link is the fixed one, and takes the form of a closed spherical chamber carrying two shaft bearings. The links *b* and *d* are alike, each of them has for one element a shaft working in a bearing in *a*. For the rest, each is made externally in the shape of a sector of a sphere subtending (at least in its ideal form) an angle of 90°; that is, a quarter of a solid sphere. The link *c*, the cross of the universal joint, becomes a disc piston, pivotted or hinged to the inner edges of the two sectors by two pins, which are, of course, at right angles to each other. Both the shafts *b* and *d* are caused to rotate by the action of the steam; the rotation of one of them (*b*), which is called the main shaft, is kept as uniform as possible by a fly-wheel, and this shaft is the only one to which the resistance is applied. The other shaft (*d*) is called the dummy shaft; its speed of rotation is allowed to vary continually, as the velocity ratio varies, with different positions of the two shafts; but this is no practical drawback because no work is taken off this shaft directly.

Apart from constructional details, the Tower engine is represented by Figs. 299 to 302, which show it in four different positions, one-eighth of a revolution apart, so that Fig. 302 represents a position three-eighths of a revolution in advance of that of Fig. 299. The starting position, Fig. 299, is identical with that of the universal joint in Fig. 288 of the last section, and the identity of the two mechanisms will be clear without any further explanation. The direction of rotation of the shafts is shown by the arrows. The figures make it easy to understand the working of the mechanism as an engine. The disc *c* divides the sphere always into two equal parts, hemispheres. Half of each of these parts is occupied by the sectors, *b* on the one side, *d* on the other. Let us look at the left-hand half

Fig. 299.

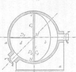


Fig. 300.

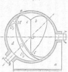
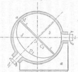


Fig. 301.



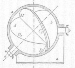
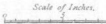
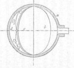


Fig. 302.





alone in the first place, the space in which the *d* sector moves. In Fig. 299 the face D_2 of the sector is close against the lower half, C_2, of the disc and fills the lower half of the hemisphere, the upper half, between C_1 and D_1, being empty. As the position changes to that of Fig. 300, all three links move round, as we know; but, *relatively to each other*, the sector and disc turn about the pivot connecting them, whose axis is, of course, A_{cd}. The sector swings away from the disc, so as to leave a space between C_2 and D_2, and diminish (by an exactly corresponding volume) the space between D_1 and C_1. In Fig. 301 the whole mechanism has made a quarter of a revolution. The pin connecting *c* and *d*, formerly at right angles to the plane of the paper (Fig. 299), now lies in that plane; the disc itself lies in a plane at right angles to the paper. *Relatively to the disc* the sector *d* has turned through 45°, so that it occupies now the centre of its hemisphere, the spaces between C_2 and D_2 and between C_1 and D_1 being equal. The lower half of Fig. 301, (which is a plan projected from the upper half), shows this position clearly. Another eighth of a turn of *b* brings the mechanism to the position of Fig. 302. Here the space between C_2 and D_2 has become nearly as much as that originally between D_1 and C_1, and has come to the upper side, while the originally large space has been continually diminishing. The sum of the two spaces must always be equal to one quarter the whole capacity of the sphere. Another eighth of a turn of *b* would bring the mechanism into a position precisely the same as that of Fig. 299, as far as appearance goes, but with C_1 and D_1 close against each other in the lower part of the casing, and D_2 and C_2 at right angles in the upper part, the whole mechanism having made half a turn. Another half-turn would bring the mechanism back to the position of Fig. 299, the space between D_2 and C_2 now gradually diminishing and that between D_1 and C_1 gradually increasing. If, when the mechanism is in the position of Fig. 299, any fluid under pressure (the fluid in the Tower engine is of course *steam*) be admitted to the space between C_2 and D_2, and if, at the same time, any portion of such fluid as may be already between C_1 and D_1 be allowed free escape, the pressure of the fluid will force the two surfaces apart, and by so doing cause *d* to turn relatively to *c* and compel the whole mechanism, consequently, to move in the manner we have just noticed. If steam be the working fluid it can be "cut off," as in an ordinary engine, at any point before the half revolution is completed, *i.e.* at any point before the space to which it is admitted reaches its maximum volume of one quarter the volume of the sphere. After cut off, the steam can go on expanding, as the volume of its chamber increases, just exactly as the steam in a cylinder goes on expanding after cut off as the piston moves forward. At half a revolution the steam occupies its maximum volume—the piston has, in effect, reached the end of its stroke. During the next half revolution an opening is provided by which the steam can pass away—exactly as during the return stroke of an ordinary steam piston, until at the end of one whole revolution the whole steam is exhausted, the exhaust passage closed and the admission port opened again for another stroke. On the left side of the disc *c* this whole cycle of processes is gone through *twice* in every revolution, once on each side of the sector *d*. At the same time the same changes go on in the same revolution, on the opposite side of the disc, once on each side of the sector *b*. During each revolution, therefore, the steam is alternately allowed to occupy, and expelled from, *four* spaces, *each* equal in volume to one quarter the sphere. If, therefore, there were no expansion—that is, if steam were admitted to each space during half a revolution—the volume of steam used per revolution would be exactly equal to the volume of the sphere, a statement which at first sight is apt to appear paradoxical. This condition is not altered very greatly even when the sectors and disc are made in their practical constructive forms, for although the disc *c* has to be of considerable thickness, to allow for steamtight packing round its edge, the faces of the sectors are reduced by an exactly corresponding amount. The space taken up by the joints (or hinges, as Mr. Tower calls them) is, however, permanently unavailable as steam space, and in an engine intended for working at a very high speed this space is not inconsiderable.[1]

In Mr. Tower's engine the link *a* is made the "cylinder" and the link *c* the "piston," an arrangement probably first adopted, although in a much cruder form, by Lariviere and Braithwaite, about 1868.[2]

Mr. Fielding has adopted, in his disc engine, the plan of making the links *b* and *d* the "cylinders," keeping *c* still as the piston, and in this respect, although hardly in any other, (apart from its kinematic identity), his engine resembles that of Taylor and Davies, patented in 1836.[3] The principle of the Fielding engine, apart altogether from any constructive details, is shown in Fig. 303, which is sketched in such a way as to be readily compared with the Tower engine, as well as the mechanisms sketched in the last section. The angle of 45° between *b* and *d*, which is convenient for the Tower engine, would here be inconvenient, and is made only half as great, namely, 22·5°. The link *d*

[1] In the drawings of a spherical engine of 8 inches diameter these hinges are shown as much as 1¾ inches in diameter.

[2] *Kinematics of Machinery*, p. 393, plate xxx.

[3] *Ibid.*

carries two short cylinders, which, from their shape, we may call "ring-cylinders,"[1] D_1 and D_2. The link *b* is precisely similar in shape, the figure shows one of its cylinders only, which lies directly in front of the other in the position shown. The link *c* carries four corresponding "ring-pistons," C_1 and C_2 on the *d* side and two others on the *b* side. These pistons work steam tight in the cylinders with ordinary packing rings, and no other steam packing of any

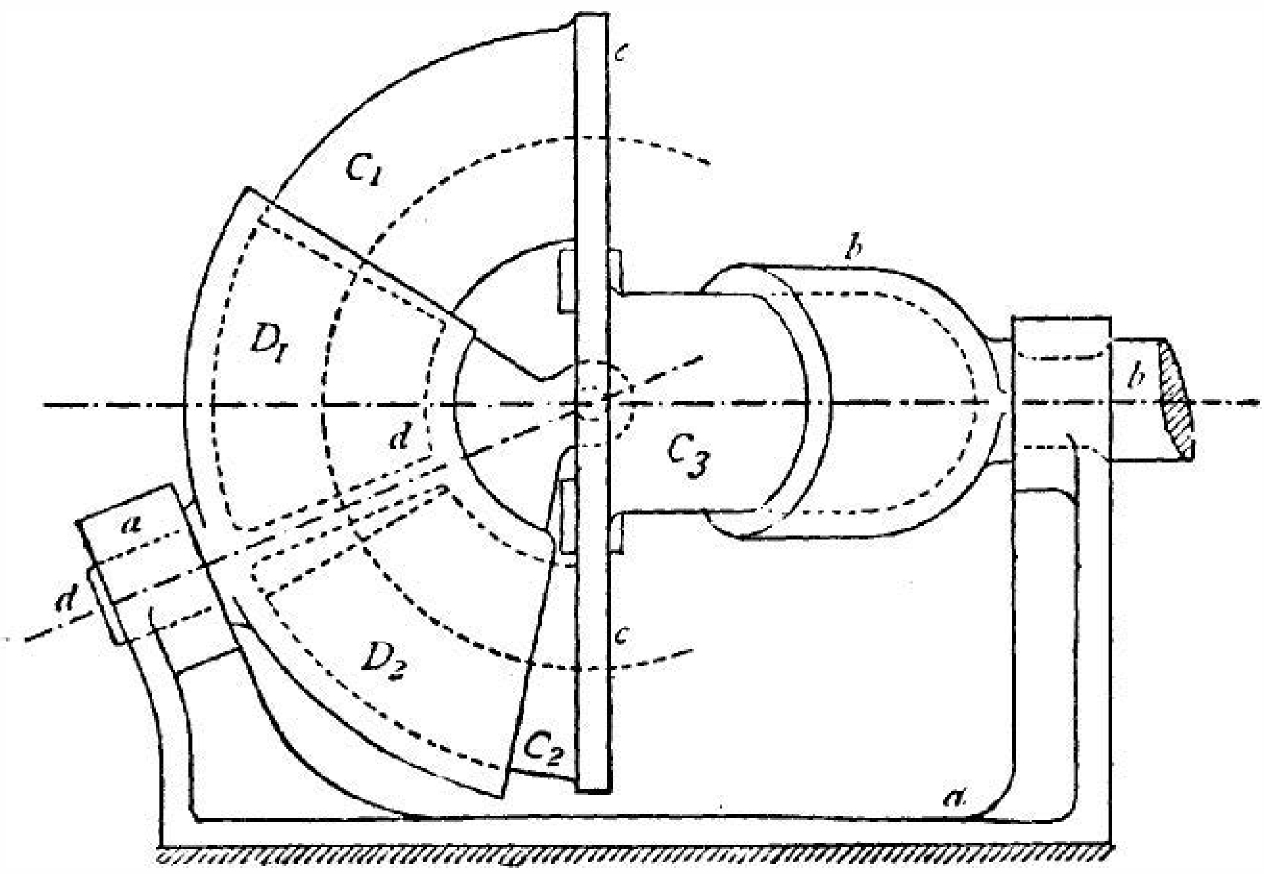


FIG. 303.

complexity is required, a point which seems most advantageous. If the motions of the engine be followed up as we followed those of the Tower engine it will be seen that in each revolution of the engine each piston makes one complete stroke into and out of its cylinder.[2] A very interesting point about this engine is that although the pin

[1] This type of cylinder was first introduced by Mr. Fielding, we believe, in some of Mr. Tweddell's hydraulic riveting machines.

[2] Mr. Fielding has made some of his engines "compound," by making one pair of the cylinders larger than the other, and ingeniously arranging the valves so as to work these as low-pressure cylinders in the usual way.

joints between b and d and the link c are still used, they have become kinematically unnecessary. The pistons C_1 and C_2 are simply an expanded form (§ 52) of the solid of revolution necessary for the turning pair whose axis is A_{cd}, and similarly the cylinders and pistons connecting c and b form really a turning pair, with axis A_{bc}, connecting these two links. It may perhaps be found possible, presently, to dispense with the pin joints altogether, and make the pairing of piston and cylinder accurate and steady enoughi to serve instead.

In § 64 we worked out a number of static problems in connection with the universal joint, which of course apply equally to such modifications of the same mechanism as we are now examining. It is worth while, however, to go further than this, and make a complete examination of the engine kinematically, such as we made of some plane mechanisms in Chapter IX. We shall do this with the Tower engine. Once the method of working is thoroughly understood the student will not find it a difficult matter to make a similar analysis of the working of any other engine of the same or similar type. Our object shall be to find the turning effort (or moment) due to the action of the steam upon the main shaft (the link b) in a sufficient number of different positions to allow us to represent it by a curve. We shall assume (as with an ordinary engine) that the speed of the main shaft is kept practically constant by a fly-wheel or its equivalent, and shall find the resistances due to the accelerations (positive or negative) of the dummy shaft and sector (link d) and the disc or piston (link c). We can then put these together to find the real effective moment on the main shaft (as formerly in § 47), from which the necessary size of the fly-wheel can be determined as before, and the whole working of the engine properly analysed.

We know, in the first place, that apart from friction the work in foot-pounds done on the main shaft per revolution must be numerically equal to the whole volume filled with steam per revolution,[1] in cubic feet, multiplied by the mean pressure of the steam in pounds per square foot. This product, multiplied by the number of revolutions per minute and divided by 33,000, gives, of course, the horse-power. In an eight-inch Tower engine the volume of steam used per revolution is about 244 cubic inches, and if we take the pressure as 100 pounds per square inch, continued throughout the whole stroke (*i.e.* without expansion), we get the work done per revolution as

$$\frac{244}{1728} \times 100 \times 144 = 2030 \text{ foot-pounds},$$

which, at 1000 revolutions per minute, corresponds to over 60 horse-power. We shall use these figures later on as a check on our detailed working. The non-expansive working is only assumed as a simplification. In the Tower engine described in Mr. Heenan's paper the steam was cut off at about half-stroke, and we shall show later on the effect of an early cut-off in the distribution of effort.

The steam pressure which drives the engine acts between the sector and the disc. In any given position of the engine two quadrants have steam pressure in them, as we have seen, one between *c* and *d*, and one between *c* and *b*. In Fig. 299, p. 520, let us assume that steam is just entering between *c* and *d* at the lower side of the engine, and that there is steam also in the space shown to the right of the disc between *c* and *b*. It may save trouble if we call these spaces the *d*-space and the *b*-space respectively. In each case the steam pressure is normal to the

[1] This is ideally, as we have seen, the whole volume of the sphere in the Tower engine.

surface on which it acts, and distributed uniformly over it. By calculation we find that the actual area on which the steam is acting is about 17·7 square inches, and that the radius of the centre of gravity of that area is about 2·2 inches. The total pressure on the piston is therefore 1770 pounds. Instead of taking this pressure as acting at its real radius, it will be convenient to suppose it to act at a radius equal to the radius of the sphere (see p. 511). We may therefore consider that we are dealing with a pressure of $1770 \times \frac{2·2}{4·0}$ or in all 970 pounds, at a radius of 4 inches. Adopting this simplification we may consider the pressure in the *d*-space all concentrated at the point *N*, and the pressure in the *b*-space all concentrated at the point *M*, and amounting in each case to 970 pounds. The point *N* is a point on the axis A_{bc}, and is therefore a point of *b*, the main shaft itself, as well as of the disc *c*. The point *M* is a point on the axis A_{cd}, and is therefore a point of *d*, the dummy shaft, as well as of the disc *c*. It is much more convenient, for geometrical reasons, to consider these points as belonging to *b* and *d* respectively, than as belonging to *c*, and we shall accordingly do so. It will be noticed in such views as Figs. 300 and 302 that the projection of the point *N* lies always upon a line at right angles to the axis of *b*, and that of the point *M* always upon a line at right angles to the axis of *d*. The real directions of the pressures must always be at right angles respectively to the lines joining *N* and *M* to *O*, the centre of the sphere.

The resultant pressure of the steam in the *b*-space acts always, as can readily be seen, in the plane of the main shaft, and therefore has no moment about it directly. It is, therefore, most convenient to determine first its turning effort upon the link *d*, and from this find the moment upon *b* by the use of the (previously calculated) angular velocity ratio of *b* and *d* (see below, Figs. 306-7). The resultant pressure of the steam in the *d*-space acts always, in the same fashion, through the axis of *d*, but it has a moment about the axis of *b* in all positions except those which correspond to the "dead points," that is the beginning and end of the stroke. We shall therefore first determine the action of the steam in the *d* space in turning round *b*; the necessary construction is shown in Figs. 304 and 305.

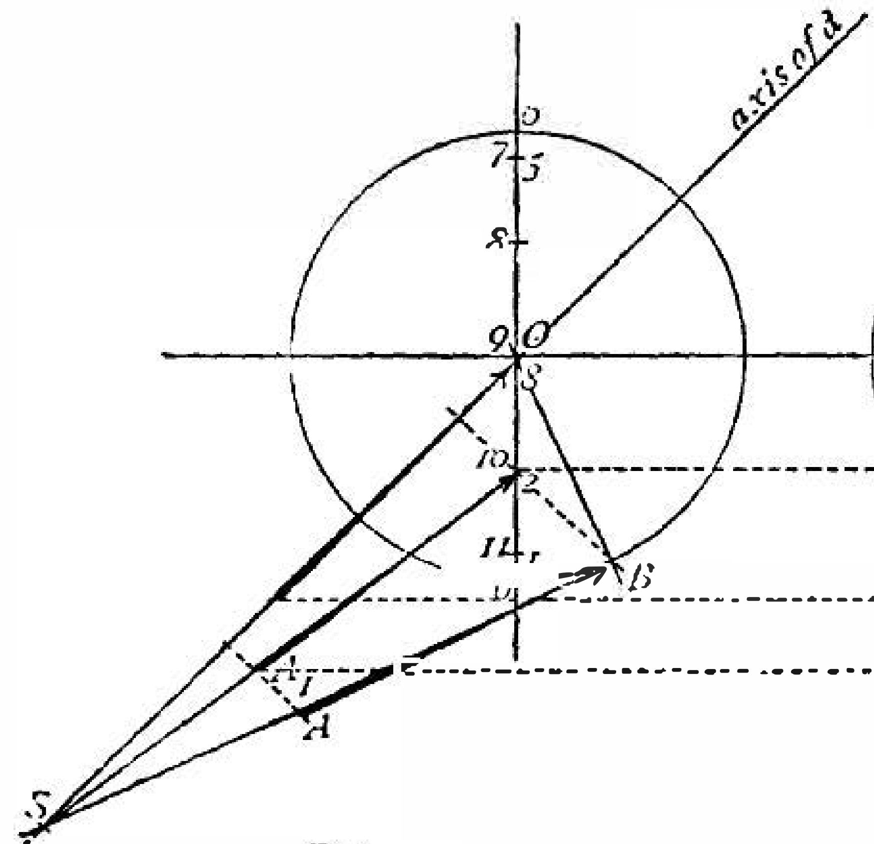


FIG. 304.

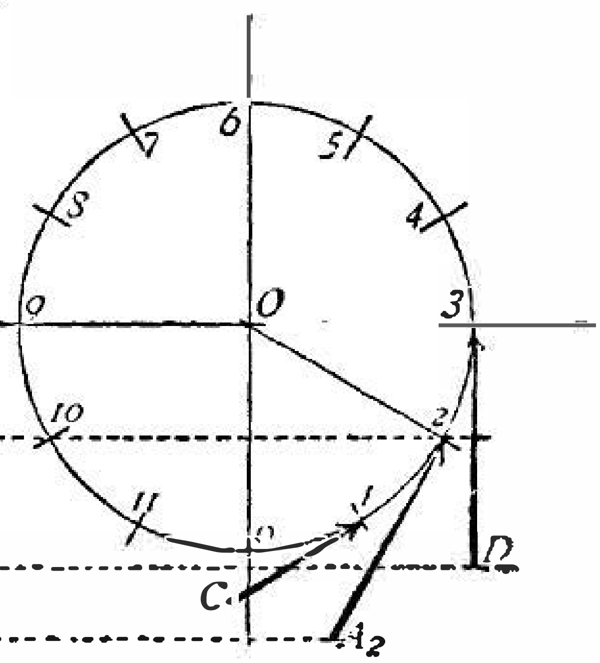


FIG. 305.

Taking (say) twelve equidistant positions of *b* (Fig. 305), we must first determine, by simple projection, the corresponding positions of the point *N*, which are numbered 0 to 11 in Fig. 304. At position 0 we have a "dead point." The pressure at *N* acts in the plane of the paper, and therefore in the plane of the axis of *b*, about which, therefore, it can have no moment. To find the turning effort on *b* at any other position we can have recourse to some such construction as the following, which is worked out for position 2. The position of *N* is known to be that of the point 2 in Fig. 305.

The resultant pressure acting upon it is at right-angles to the line joining 2 to the centre point O, and is at the same time a line in the plane containing the axis of the shaft d and the line $O2$ or ON (compare Fig. 300, p. 520). We can make this plane coincide with the plane of the paper by simply turning it about the axis of d. The corresponding position of the point 2 is found at once by drawing $2B$ (Fig. 304) at right-angles to the axis of d, the point B being on the circle. The pressure will then be represented by a line AB in the plane of the paper and at right-angles to OB, which is of course the projection of $O2$. If the effect of the acceleration of the different masses is *not* to be considered the line AB may be set off to represent the pressure on the disc (either total or per square inch) on any convenient scale whatever. If, however, the accelerative resistances are to be taken into account it is most convenient to work them out first, and use the same scale for piston-pressure as has been used for them. In this case the line AB will represent the equivalent to the total pressure on the piston at its peripheral radius, or, in this instance, 970 pounds. (The scales used here are all shown on the figures.) We have now to turn back the plane, with AB in it, so that the point B comes to 2, and find where A will come. This is easily done; continue BA until it cuts the axis of d (the line about which the plane was turned) in S, and join S to 2. We know then that the projection of A must lie upon the line $S2$, and we know also that it must lie upon a line through A at right-angles to the axis—it is therefore at once found to be A_1. The line $A_1 2$ is therefore the projection in the plane of the paper in Fig. 304 of the whole steam-pressure with which we are dealing. The plane of the paper is a plane *containing* the axis of b. What we wish to find is the component of AB *at right-angles* to that axis; it therefore only remains to project $A_1 2$ upon a plane at right angles to that axis, and our problem is solved. The plane of Fig. 305 is, as we know, at right-angles to *b*. We have already in that figure the projection of the point 2. We know that the real direction of the pressure lies in a plane at right-angles to the radius *O*2. Its projection in Fig. 305 must therefore be a line at right-angles to that radius, and the position of A_2 can be found at once by projection from A_1. We have, therefore, as the solution of our problem, that the pressure acting between the sector *d* and the piston *c*, when the link *b*, or main shaft, is in the position 2, is equivalent to a force $A_2 2$ acting at a radius *O*2, or (what is the same thing) that the turning moment on *b*, due to the steam-pressure, is $O2 \times A_2 2$. By precisely similar construction *C*1 can be found as the turning effort on *b* for position 1, and if the pressure on the piston is constant (as we have assumed for the present) throughout the stroke, the turning effort at 5 is the same as at 1, and at 4 the same as at 2. For the effort at position 3—as the lines corresponding to *BA* and $2A_1$ are parallel to each other and to *OS*,—it only is necessary to set off the pressure magnitude along *OS*, and project at once to *D*3.

It is hardly necessary to point out that there is no necessity whatever for constructing Fig. 305 separately from Fig. 304, as has been done for clearness' sake. In practice one circle may conveniently be made to serve the purposes of both figures. It may also be noticed that if the pressures at points 4 and 5 differ from those at points 2 and 1, it is still not necessary to make separate constructions for them. The construction shown for position 2, for instance, will serve equally for 4, if instead of *AB* (Fig. 304) there be set off a distance corresponding to the new pressure, and the corresponding point projected into Fig. 305 instead of A_1.

A diagram of turning effort, similar to the diagram of crank-pin effort formerly drawn (Fig. 155), can now be made for this engine by setting off a base-line representing on any scale the length of the path of the point *N*—*i.e.* equal to the circumference of the circle in Fig. 305, dividing it into twelve equal parts, and setting up at each one, as an ordinate, the corresponding pressure $C1$, $A_2 2$, $D3$, &c. This is done in Fig. 306, *I*, where the similar curve from 6 to 12 is put to complete the revolution.

We have seen that we cannot find directly the turning effort on *b*, due to the steam-pressure between *b* and *c*, because that pressure has no direct tendency to turn *b*, and does so only because it turns *d*, and this rotation cannot occur without the simultaneous rotation of *b*. It is unnecessary to make any further construction to find the turning effort on *d* due to this steam-pressure in the *b* space, for it must be precisely the same as the effort on *b* due to the pressure in the *d* space, which we have just found. Remembering only that 3 and 9 are now the dead points instead of 0 and 6 (as can be seen at once from Figs. 299 and 301), we may therefore set out the turning effort on *d* at once, as in Fig. 306, *II*, using the same ordinates as those we have just found for the line *bb* in the same figure. To combine the two diagrams, however, so as to find the total turning effort on *b*, we cannot simply add the two ordinates together (as we should do if they were diagrams for the two cylinders of an ordinary engine), for the angular velocities of *b* and *d* are, as we know, very different, so that a given effort on *d* may be equivalent to an effort of very different magnitude, although in a corresponding position, on *b*. We must, therefore, find first the angular velocity ratio between *b* and *d* for each of the twelve positions of *b*, by one of the methods of the last section (p. 508), and it is convenient to

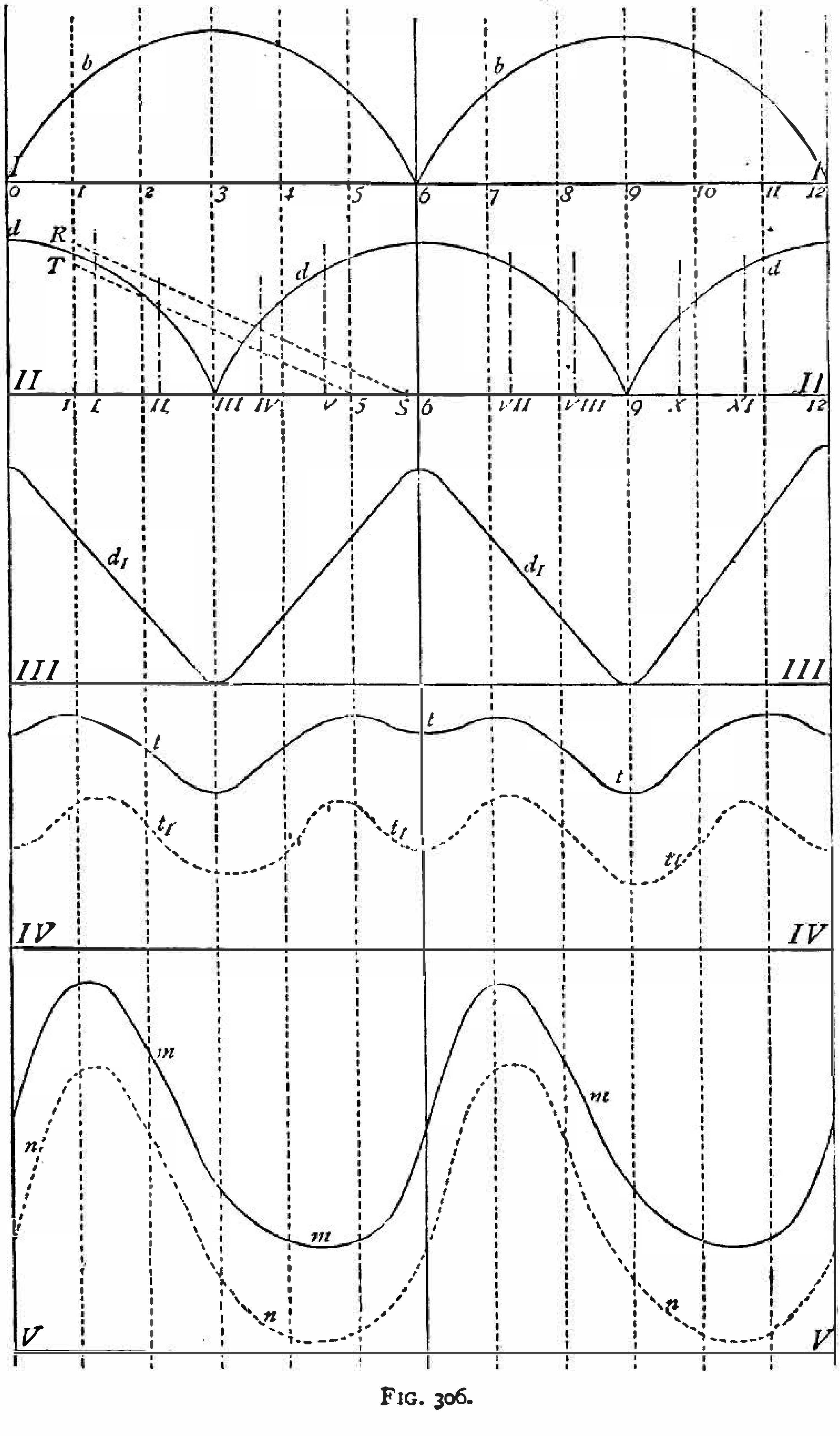

b
b
I
I
0
1
2
3
4
5
6
7
8
9
10
11
12
d
R
T
d
d
II
II
I
II
III
IV
V
5
S
6
VII
VIII
9
X
XI
12
d₁
d₁
III
III
t
t
t
t₁
t₁
t₁
IV
IV
m
m
m
n
n
n
V
V

FIG. 306.

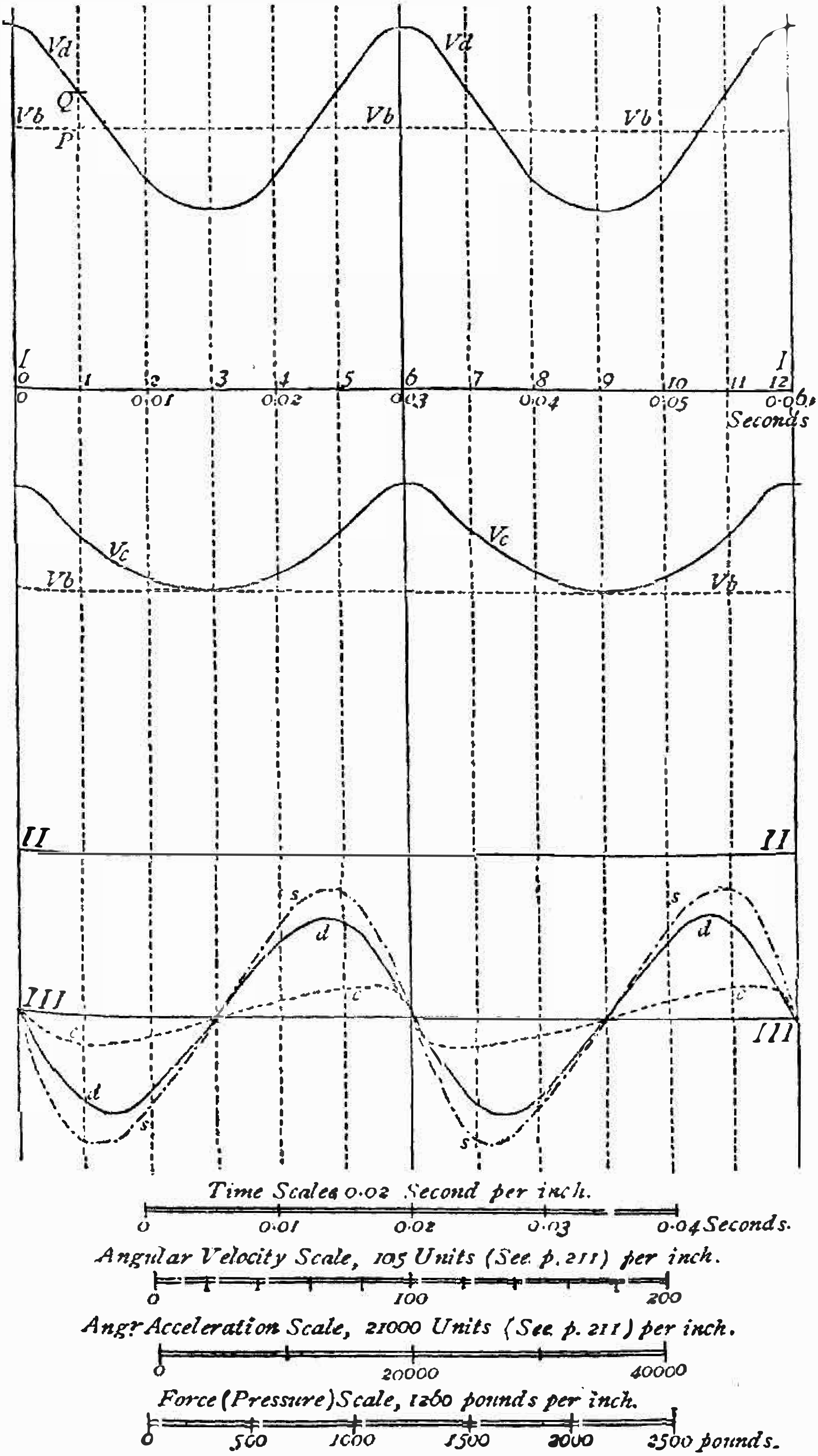

Vd
Q
Vb
P
I
0 1 2 3 4 5 6 7 8 9 10 11 12
0 0·01 0·02 0·03 0·04 0·05 0·06
Seconds
Vc
Vb
II
III
s
d
c
Time Scale 0·02 Second per inch.
0 0·01 0·02 0·03 0·04 Seconds.
Angular Velocity Scale, 105 Units (See p. 211) per inch.
0 100 200
Angr Acceleration Scale, 21000 Units (See p. 211) per inch.
0 20000 40000
Force (Pressure) Scale, 1260 pounds per inch.
0 500 1000 1500 2000 2500 pounds.

FIG. 307.

plot these out in a diagram, as in Fig. 307, *I*. Here any height is taken to represent the (assumed constant) angular velocity of *b*, and the calculated or constructed angular velocities of *d* are set off on the same scale (the lines in the figure are marked v_b and v_d respectively). It will be found convenient for some purposes to represent the angular velocity of *b* by a length equal to that of four of the divisions 01, 12, 23, &c., on the base line. It will be noticed that only four different values of the angular velocity of *d* have really to be found, the rest are all duplicates of these, the velocity at 4, 8, and 10 being the same as at 2, that at 5, 7, and 11 the same as at 1, &c.

It would not be right, however, to take now any ordinate of the *d* curve, and simply multiply it by the ratio $\frac{\angle^r \text{ vel } d}{\angle^r \text{ vel } b}$, and add the ordinate so found to the ordinate of the *b* curve directly above it. For although the ordinate of the *d* curve at 0, for instance, gives the turning effort on *d* which is contemporaneous with the dead point of *b*, the ordinate of the *d* curve at 1 does *not* give the turning effort on *d* contemporaneous with the ordinate of *b* at 1, and the angular velocity ratio between the shafts at 1 in Fig. 307. Contemporaneous points must first be found by the method of the last section (p. 504), and marked on the base line of curve *d*, *I*, *II*, *III*, *IV*, &c. To find the real effort on *b* at position 1, due to pressure transmitted to it from *d* through *c*, it is only necessary to take the ordinate of the *d* curve at *I*, multiply it by the angular velocity ratio at 1, and add the product to the ordinate of the *b* curve at 1. This may perhaps be most rapidly done as follows:—Carry the ordinate at *I* back to 1, as 1*T*. Set off along the base line distances equal to the angular velocities 1*P* and 1*Q* in Fig. 307. (If the dimension for the velocity of *b* has been chosen as recommended above, it will be unnecessary to measure and set off 1*P*, because it will be always equal to the length of four of the equal base line divisions already drawn.) 15 in Fig. 306, *II*, will thus stand for the velocity of *b*, and 1*S* (= 1*Q* in Fig. 307) for the velocity of *d*. Drawing *SR* parallel to 5*T* we obtain 1*R*, the required pressure on *b* in position 1, for by similar triangles $1R = 1T \times \frac{1S}{15}$ = pressure on $d \times \frac{\angle^r \text{ vel } d}{\angle^r \text{ vel } b}$. Carrying out this construction for a sufficient number of points in *d*, we obtain the ordinates which are plotted out as a new curve $d_1 d_1$ in Fig. 306, *III*. Lastly, adding together the ordinates of *b* and d_1, we get the curve *t t* (*IV*), whose ordinates represent the total turning efforts on *b* at a radius equal (in this case) to four inches, the pressure scale being, of course, still the same as that used originally in Fig. 305.

Going on now to the direct consideration of the effect of acceleration in the engine, we notice at once the general resemblance of the problem to that of § 47. We have here again three moving links, one of which rotates with a velocity assumed to be sensibly uniform. Of the other two one (the link *d*) moves about a fixed axis[1] with certain large changes of velocity, which we have now completely determined and diagrammed. The remaining link here, the disc *c*, has motions analogous in certain important points to those of the connecting rod in an ordinary engine, which corresponds to it in being the link which transmits motion to the main shaft of the engine. Both are links which not only undergo varying accelerations, but for which also the accelerations occur about varying

[1] The "reciprocating parts" of an ordinary steam engine are in the same condition, but the axis about which they turn is an infinitely distant one.

axes.[1] The angular velocity of *c* relatively to *b* can be found in precisely the same manner as that in which we have found (p. 508) the angular velocity of *d* relatively to *b*. We find, namely, the position of the line A_{bc}, which is common to *b* and *c*; choosing any convenient point in that line, we find its distances from A_{ac} and A_{ab} respectively; we then say that the angular velocities of the two bodies are inversely as these distances. We thus know, without any construction, that for positions 0 and 6, where A_{ac} coincides with A_{ad}, the link *c* must have the same angular velocity as the link *d*, and that for positions 3 and 9, where A_{ac} coincides with A_{ab}, the link *c* must have the same angular velocity as the link *b*. The former will be the maximum, the latter the minimum, value of the angular velocity ratio of *c* to *b*, and the value of the former we already know (p. 508) to be $\frac{1}{\cos 45^\circ}$, or 1·41, while the value of the latter is unity. The intermediate points in the curve on Fig. 307, *II.* have been found by the following construction, which is analogous to that of Fig. 292, already given. The three virtual axes which we require are A_{ab}, A_{ac}, and A_{bc}. We have already seen (Figs. 286 and 290) how to find the position of each. In Fig. 308 they are respectively represented (in position 2 of *b*) by the lines *OP*, *OQ* and *OR*. These three lines are in one plane (p. 490), and the points *P*, *Q*, and *R* lie all in one circle. The real relative position of the axes is therefore obtained at once by simply turning down the plane about *OP* until it coincides with the plane of the paper, so that *Q* comes to Q_1 and *R* to

[1] The motion of the disc has its most exact representative in plane mechanisms by the motion of the link *c* in the chain of Fig. 32, supposing it converted into a mechanism of which *a* is the fixed link, a mechanism on which innumerable rotary engines have been based.

R_1. We then have at once $\frac{\angle^r \text{ vel } c}{\angle^r \text{ vel } b} = \frac{OR_1}{SR_1} = \frac{1}{\cos POQ}$.
It will be noted that, just as in the upper curve in Fig. 306, the twelve points on the curve of angular velocity of c require only the calculations of *four* different ordinates, and of these we know one (for positions 0 and 6) to be equal to an ordinate of the former curve, and another (for positions 3 and 9) to be equal to unity.[1]

The two curves of velocity which we have now drawn may be considered as curves drawn on a *time base* (p. 194), for the equal abscissæ 01, 12, 23, &c., correspond to equal motions

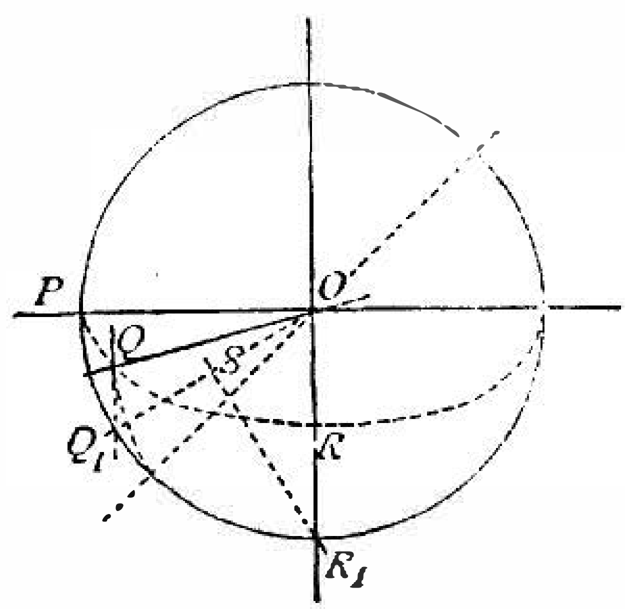


FIG. 308.

of a body (*b*) whose velocity is uniform, and therefore to equal intervals of time. If we take the engine as making 1000 revolutions per minute, one revolution occupies 0·06 second, and each of the twelve divisions of the base line corresponds, therefore, to 0.005 second. Using the method given in § 28, p. 194, for finding the acceleration from a given velocity curve on a time base, we can now draw the curves of Fig. 307, *III.*, of which *dd* represents the acceleration of

[1] For positions 0 and 6, the angle $POQ = \theta$ (the angle between the shafts) $= 45°$, and $\cos POQ = 0·707$; for position 3 and 9 the angle $POQ = 0$, and $\cos POQ = 1$.

the link *d*, *cc* that of the link *c*, and *ss* the sum of the two accelerations. Ordinates above the axis are positive, that is, they correspond to an *increasing* velocity, ordinates below the axis are negative, corresponding to a *decreasing* velocity.

We know that acceleration curves can be read off at once as curves of force or (in this case) pressure (p. 339). We have only, therefore, to determine the scale on which our curves may be so read in order to compound them with the pressure curves of Fig. 306. We saw in § 31 that

$$f\,r\,t = v_a\,I,$$

where f was the force which, applied at radius r for a time t could produce an angular velocity v_a in a body whose moment of inertia about its virtual axis (from which also r was measured) was I. The angular acceleration $a = \frac{v_a}{t}$ so that $f\,r = a\,I$, or $f = \frac{a\,I}{r}$.

In the diagrams as drawn in Figs. 306 and 307 the angular velocity of *b* is represented by a height of one inch. The assumed velocity of 1000 revolutions per minute $= 1000 \times \frac{2\pi}{60} = 105$ angular units (per second), so that the scale of angular velocity is 105 units per inch. Four divisions on the time scale (*i.e.* the distance 04, &c.) are made equal to one inch, so that the time scale is 0·02 second per inch. Each time interval being $\frac{1}{200}$ second, the acceleration scale[1] is $(105 \times 200) = 21{,}000$ units per inch. The value of I for the sector *d* (the units being feet and pounds) is about 0·02, and r, the radius at which we have assumed the pressure to act, is $\frac{1}{3}$ foot. We have, therefore, $f = a \times 3 \times {\cdot}02 = {\cdot}06\,a$, so that the force scale is $21{,}000 \times {\cdot}06 = 1260$ pounds per

[1] See p. 199.

inch.[1] It has been already (p. 529) pointed out that it is convenient to calculate this scale before setting off the piston pressures (Fig. 306), and use it for them. If this has not been done the ordinates *ss* of the total acceleration curve must be reduced to the scale used for Fig. 306 before being further used.

The scale of the accelerations of *c* differs from that of *d* because of the different value of *I*. Approximately the value of the moment of inertia of *c* is half as great as that of *d*, or 0·01, so that the acceleration ordinates derived direct from the *c* curve (Fig. 307, *II*) have to be halved before being set off in Fig. 307, *III*. This has been done before plotting them in the curves shown. It has to be noticed that the value of *I* for the disc *c* is *not* a constant quantity, for the virtual axis (about which *I* is to be measured) does not occupy a constant position in the body. This case is altogether analogous to that of the connecting rod discussed in § 49. The error caused by assuming the value of *I* to be constant is too small to be of any importance to us, and we have therefore neglected it. Practically a very reasonable approximation to the result is obtained by making the same assumption about the link *c* that is commonly made about the connecting rod of a steam-engine, namely, that half its mass shares the motion of the main shaft, and has therefore no acceleration, and that the other half may be taken as part of the mass of the link *d*, and as sharing its accelerations. If this approximation were used in the present case it would give a total acceleration about $1\frac{2}{3}$ as great as that found by our more exact method. The saving of trouble by the

[1] It will be noticed that the pressures we are here dealing with are *total* pressures at an assumed radius, *not* pressures reduced to unit area of piston and mean radius. The reduction can easily be done (see p. 527) if required

omission of the construction of Fig. 308, and the curve Fig. 307, *II*, is of course considerable.

Going back now to Fig. 306, *V*, we have in the curve *mm* the sum of the ordinates of the curve *tt* above it, and of the curve *ss* in Fig 307, *III*, (the two curves being supposed to be drawn on the same pressure scale). The extraordinary result of the accelerative resistances we see at once. Whereas, without them the driving effort was fairly uniform, the ratio of maximum to minimum being about 1·5, when we take into account the accelerations at 1000 revolutions per minute this ratio is increased to about 3·4.h The dotted curve t_1t_1 shows the variation of nominal driving effort, if the steam were cut off at about $\frac{1}{4}$ stroke, and *nn* the real effort under the same conditions at 1000 revolutions per minute, the ratio just mentioned being increased from about two to over twenty. Large as these alterations are, it will be seen, from the table on p. 352, how much larger changes would be produced in any engine of the ordinary type if it were run at anything like the same number of revolutions per minute. The advantage here is no doubt mainly due to the fact that the links which oscillate or swing relatively to each other are both in continued rotation relatively to the fixed link. Thus, although the reciprocating motion is not really done away with, one of its most serious drawbacks is obviated—the reciprocating links do not come to rest, relatively to the fixed link, twice in every revolution, as do the piston and rod, &c., of an ordinary engine.

As a check on the working and diagramming the *mean* effective turning effort ought to be measured from the curve in Fig. 306, *V*. It should be, and in this case is, equal to the known mean steam pressure at 4 inches radius, or here 970 pounds, which corresponds to 2030 ft.-pounds per stroke.

The working out of a Fielding engine may proceed in precisely the same way as that which has been employed for the Tower engine. It is not necessary here to say anything further than that this engine has the advantage, from the point of view of steady running, that the angle between the shafts is much less than in the spherical engine, $22^\circ{\cdot}5$ instead of 45° being used. This makes the ratio of maximum to minimum angular velocity of the dummy shaft $\left(\frac{1}{\cos^2\theta}\right)$, only 1·17 instead of 2. Supposing the masses of the rotating parts to be the same in both engines, the resistances due to acceleration at any given speed will be about $\frac{1}{3}$ of the amount of those just diagrammed. But at the same time an engine of the Fielding type has a net volume for steam less than that of a Tower engine (both being of the same external diameter) in somewhere about the same ratio. The enormous effective volume of the latter engine depends essentially, as we have seen, on the use of a large angle between the shafts, and this unavoidably entails irregularities of driving effort due to the great accelerative resistances, which have to be rendered as unimportant as possible by the use of sufficiently heavy rotating masses (flywheel or its substitute) upon the driving shaft.

§ 66.—BEVEL GEARING.

If we treat the spur wheel chains of Chapter VI. as we have just treated the linkwork mechanisms of the earlier chapters—if, namely, we transform the plane into spheric motion, by bringing the point of intersection of the axes to a finite distance, we obtain the type of wheel gearing known asi **bevel gearing.**[1] It is possible to reproduce, in this form, *all* the spur wheel chains, simple, compound, annular, or epicyclic. Practically, however, very little use is made of any of these changed mechanisms, except the simplest of all, which is shown in Fig. 309, and which is directly derived from the spur wheel train of Fig. 58, p. 117, by inclining its shafts at an angle (in this case 45°) to each other. We have just the same characteristics here as formerly with the spur wheel train; the one shaft drives the other with a constant velocity ratio, but in the opposite sense to that in which it is itself rotating. The portions of the surfaces of the cylindric axodes, which we saw to be

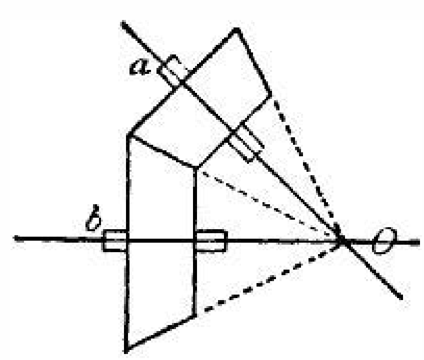


FIG. 309.

formerly the pitch surfaces, are now replaced by portions of the conic axodes. The motion of the toothed bevel wheels corresponds to that of the rolling of the conic pitch surfaces, just as formerly the motion of the spur wheels corresponded to the rolling of the cylindric pitch surfaces. We form teeth on the former for exactly the same reason as we did on the latter; and the actual transmission of motion is accompanied by the sliding on one another of these teeth, exactly as we saw formerly.

Under these circumstances it is not necessary for us to say more than a very few words about this form of non-

[1] If a pair of bevel wheels are of the same size they are often called *mitre wheels*.

plane motion. It will be noticed at once that just as the links in the conic chains had no dimensions which could properly be called their *lengths* (p. 493), so here the wheels have no one special diameter.[1] Wheels of the most various diameters may transmit the same velocity ratio between the same two shafts. Thus let a and b (Fig. 310) be the axes of two shafts intersecting in O, between which it is desired to transmit a known velocity ratio. Let this ratio be such that $\frac{\text{ang. vel. } b}{\text{ang. vel. } a} = \frac{OA}{OB}$. Draw about the point O any circles with radii OA and OB; through A draw a parallel to the axis

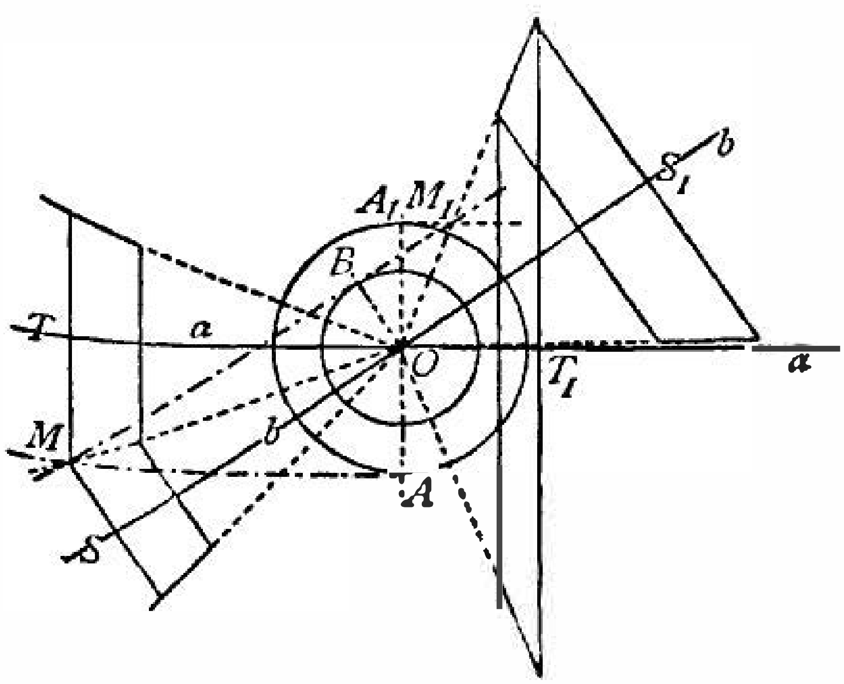


FIG. 310.

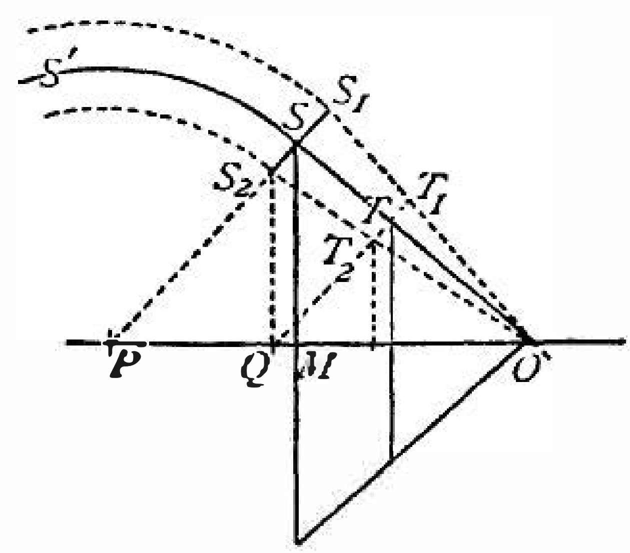


FIG. 311.

a, and through B to the axis b. Call the point where these parallels intersect M, and join MO. Then MO is the line of contact of the two pitch cones, which can, therefore, be at once drawn. Any pair of frustra of these may be used for the wheels, for the ratio $\frac{MT}{MS}$ between their radii must always be the same, and must always be equal to $\frac{OA}{OB}$.

The finding of the shape of the teeth for bevel wheels

[1] Conventionally the largest pitch diameter of a bevel wheel is spoken of as the diameter of the wheel.

involves no difficulty. Let O (Fig. 311) be the vertex for such a wheel, MS its radius, and S_1S_2 the required depth of tooth.[1] We may treat the spheric surface S_2SS_1 as if it were itself a part of a cone with vertex at P ($OSP = 90°$) complementary to the pitch cone. This cone can be developed by drawing circles through S_2, S, and S_1 with P for a centre. If the circle SS' be now used as a pitch circle, and teeth drawn on it (see § 18) with the right depth in the usual way, the profiles of these teeth will be the profiles required. The corresponding profiles for the inner sides of the teeth can be found by developing the cone TQ in exactly the same fashion. It is hardly necessary to point out that all lines along the teeth,

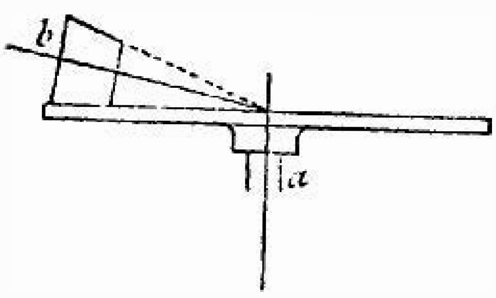


FIG. 312.

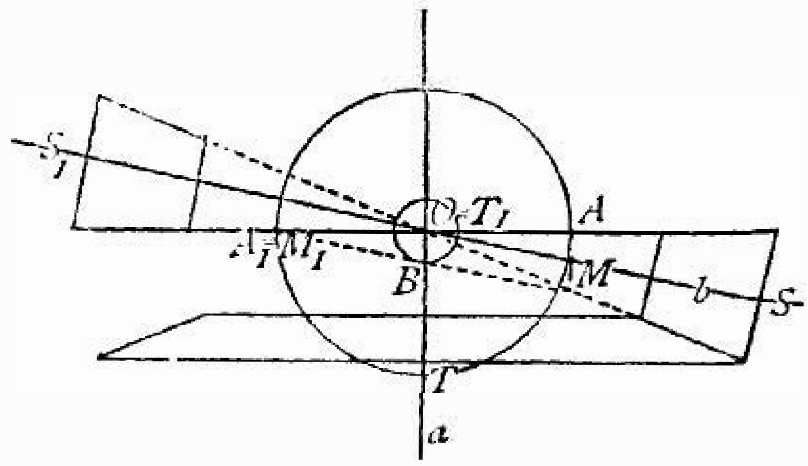


FIG. 313.

such as S_1T_1, S_2T_2, &c., must pass through O as well as the line ST on the pitch surface. Annular bevel wheels, although they are kinematically quite correct, are rarely, if ever, used. They come out at once from the construction of Fig. 310, if only the point A_1, at the opposite end of the diameter, be used instead of A. In the event of the angle M_1OS_1 being equal to the angle A_1OS_1, the annular wheel becomes a disc or flat wheel, as shown in Fig. 312, which may often be a quite convenient arrangement of gearing. Fig. 313, which corresponds to Fig. 310, shows the double

[1] Determined mainly by considerations of strength, &c., such as will be found discussed in Chap. ix. of Professor Unwin's *Elements of Machine Design*, &c.

construction in this case. It will be noticed that, for any given sense of rotation of *b*, the shaft *a* will be driven by the flat wheel in the opposite sense to that in which it would be driven by the cone wheel, which is, of course, the essential characteristic of an annular train.

It is comparatively easy to make correctly shaped *patterns* for the teeth of bevel wheels, and the shape of the bevel tooth, when the wheel is cast, will be fairly near the shape of the pattern. But if it is required that the profiles of the teeth should be really accurate, they must be, as with spur gearing, *machined*, and this operation is one presenting some practical difficulties. The most recently devised machine for this purpose, a very ingenious one, is probably that of Mr. Bilgram, described in *Engineering*, vol. xl. p. 21, the principle of which will repay examination.

§ 67.—THE BALL AND SOCKET JOINT.

THE familiar combination known as the **Ball and Socket Joint** is not, as might be at first sight supposed, a pair of elements. It does not *constrain* any relative motion between the bodies which it connects; it only permits the one to have spheric motion, in any direction, relatively to the other. It cannot, therefore, be used as the sole connection between two bodies in a mechanism or machine unless the relative motions of those bodies be completely constrained by what we have called *chain closure* (p. 410), or its equivalent. In that case it forms, essentially, an example of *reduction* (p. 403). The links connected by the ball and socket joint could not be directly connected by any one lower pair, but might be connected with the use of lower pairs only if one or more links were inserted between them, exactly as in the cases examined in § 53. The ball and socket joint, however, could not be replaced by *plane* links; its motion is spheric, and the links which it virtually replaces would have to form some part of a spheric combination. The ball joint has, of course, *surface* contact, but as it is not a pair of elements this is no contradiction to the statement on page 57, that none but lower pairs of elements had surface contact.

Fig. 314 shows a mechanism which has occasionally found application, and which belongs to a class which will be mentioned in § 70. It is a *simple* chain, each link having

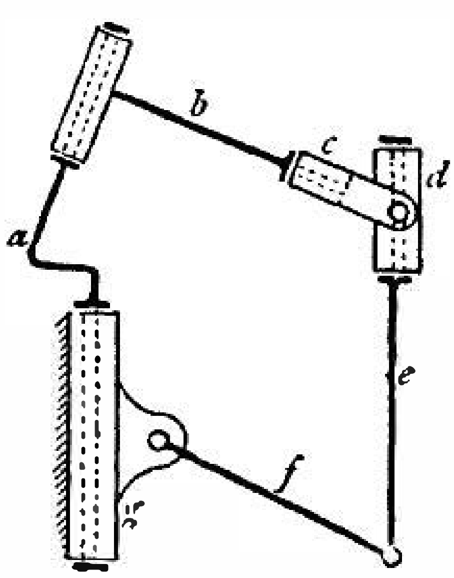


FIG. 314.

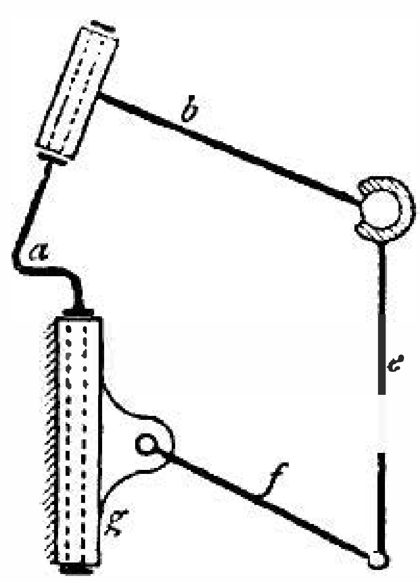


FIG. 315.

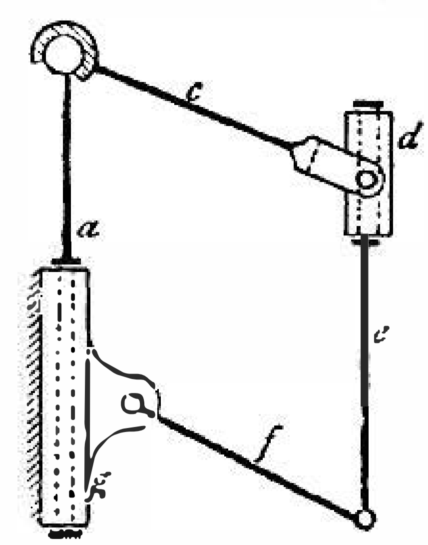


FIG. 316.

only two elements. It contains seven links, each paired to its neighbour by a turning pair. The axes of three pairs, namely, *bc*, *cd*, and *de*, pass through one point, an arrangement which so constrains the relative motion of *b* and *e* that the axes of these two links always intersect, and always intersect *in the same point*. The motion of *b* relatively to *e* is therefore a spheric motion about that point. If we wish to dispense with the links *c* and *d* we may proceed as in § 53, by forming on *b* a suitable element, and finding its envelope on *e*. If we choose for the element a sphere whose centre is at the join of the axes of *b* and *e*, its envelope on the latter link will obviously be simply a corresponding hollow sphere, and the mechanism, so reduced, will take the form of Fig. 315. The motions of the remaining links are here sufficient to constrain the relative motions of *b* and *e*, and the ball joint becomes in this very special case available for use as a higher pair. The motion of *c* relatively to *a* is also such that a certain line on *c* (namely, the axis of the pair *bc*) always passes through the same point on *a*. We might, therefore, omit *b*, and pair *c* to *a* (as in Fig. 316) by a ball joint. By doing this, however, the chain becomes unconstrained, for the link *a* can be rotated without transmitting any motion to the rest of the chain, the ball joint being incapable of transmitting rotation about its own centre. The investigation of the conditions under which a ball joint can be used in a reduced chain without destroying its constrainment does not present any great difficulties, but the case is one which occurs so seldom that we shall not here enter into it.

§ 68.—HYPERBOLOIDAL OR SKEW GEARING.

THERE remains yet to be noticed a class of mechanisms having non-plane motions, but coming under none of the categories hitherto examined in this chapter. These mechanisms may contain only turning pairs (as for example the one illustrated in the last section), or they may contain also screws or cams or higher pairing of any kind. Their characteristic is that some or all of their links have, relatively to some of the other links, a general screw motion, for which reason we may give them the generic name of **general screw mechanisms**. By general screw motion is meant a twist which bears the same relation to simple screw motion that rotation about an instantaneous axis bears to rotation about a permanent axis. A body having such a motion is at each instant twisting relatively to (say) the fixed link. But both pitch and axis of twist may, and often do, vary from instant to instant. There is here neither virtual centre nor virtual axis, neither centrode nor axode, cylindric or conic. The motion cannot be represented as a rotation, or by a *rolling* of two surfaces of any kind. For any two bodies having general screw motion relatively to each other, it is always possible (but sometimes extremely difficult) to find a line that is common to both the bodies for the instant, and about and upon which each is simultaneously *turning* and *sliding*, at the instant, relatively to the other. Such a line may be called the axis of virtual twist, or simply the **twist-axis**, of the two bodies. As an axis, it may be said to be a line common to the two bodies, but it is not a common line in the same sense as the virtual axis of rotation, for as a line in one body it slides along its own direction relatively to the other body. The complete series of twist-axes for the relative motions of two bodies, that is, the loci of these axes, form a pair of ruled surfaces **(twist-axodes)** which may be properly looked upon as the general case of the simpler axodes of plane and spheric motion. As the bodies move, successive lines on these surfaces come into coincidence, and the motion of the one body relatively to the other is always a twisting about the coincident line, exactly as in plane motion there is always a rotation about the coincident line of the two axodes.

A detailed study of these mechanisms—a subject in which comparatively very little work has yet been done—would take up an amount of space altogether out of proportion to their importance, for their applications in practical machinery are comparatively few, and their complexities, from anything like a general point of view, are very great. We shall here not attempt any such complete examination (perhaps we may have some future opportunity of dealing with the subject), but shall merely mention a few of the principal examples which occur in actual work. The simplest of these cases occurs where the twist is the same for each twist-axis, and of these cases the simplest again is no doubt the one

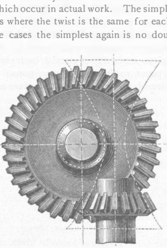


Fig. 317.

(analogous to spur and bevel wheel gearing) where the twist axodes are used directly, altered only by being toothed, for the transmission of rotation between shafts whose axes cross, but do not meet, each other. Gearing of this kind (an example of which is shown in Fig. 317) is known as **skew wheel gearing**, the wheels being often called, from their quasi-conical form, **skew bevel wheels**. The twist-axodes, frustra of which correspond to the pitch surfaces of the skew bevel wheels, are hyperboloids of revolution whose axes are the axes of the shafts and which have always one coincident line or generator in a position corresponding to the coincident line of the pitch surfaces of spur wheels. The position of this line must be such that the distances of every point in it from the two axes must be inversely proportional to the required angular velocities of the wheels, and it can be found in the following manner. Let SA and SB (Fig. 318) be projections of two crossed axes a and b, both parallel to the plane of the paper, so that the common normal to the two axes passes through S and is normal to the plane of the paper.

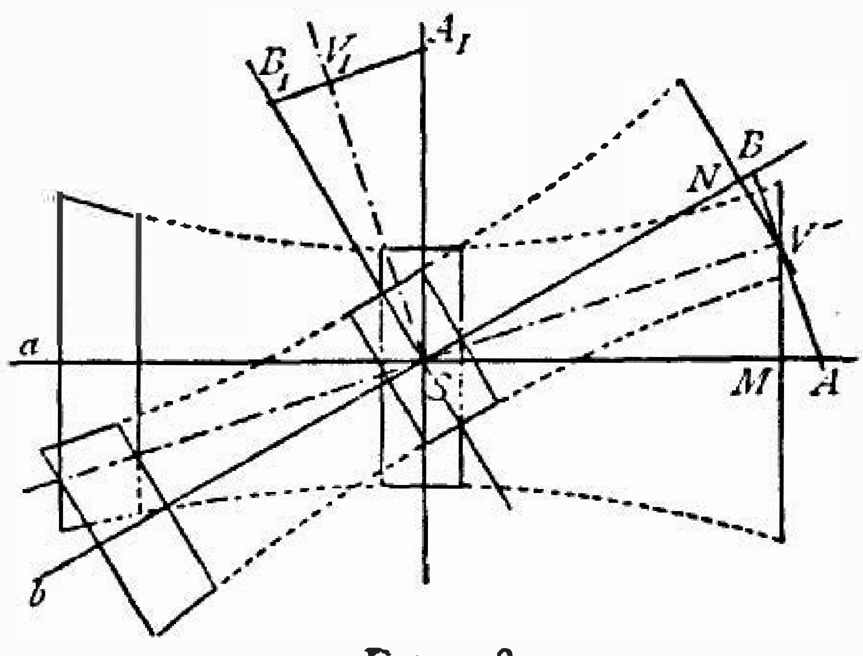


FIG. 318.

Let the angular velocity ratio of b to a be given, and let it be required to find the generator for the (hyperboloidal) pitch surfaces of skew bevel wheels which will transmit this ratio. The angle ASB must first be divided by SV so that $\frac{VM}{VN} = \frac{\text{ang. vel. } b}{\text{ang. vel. } a}$, which can be done exactly as described for bevel wheels in § 66, Fig. 310. Then SV is the projection of the required generator, which lies in a plane parallel to that of the paper as drawn. Drawing through V a line at right angles to SV, we have at once $\frac{VA}{VB}$ as the constant ratio of the distance of every point in the generator from the axes a and b, and therefore the required ratio between the radii (or diameters) of a and b, which are no longer proportional themselves to the velocity ratio.

It requires now to be proved that skew wheels with this diametral ratio will transmit the required velocity ratio. Draw SA_1, SV_1, and SB_1 at right angles to SA, SV, and SB respectively, and consider the contact between the pitch surfaces upon the normal to the shafts, *i.e.* upon the line through S and normal to the plane of the paper. The direction of the line of contact is SV, and both wheels must have the same velocity normal to that direction.[1] Drawing A_1B_1 parallel to SV, we obtain at once SA_1 and SB_1 as the peripheral velocities of a and b respectively, on any scale on which SV_1 represents their common velocities in its own direction. The angular velocities of the two wheels must be directly as their peripheral velocities and inversely as their radii, or

$$\frac{v_b}{v_a} = \frac{SB_1}{SA_1} \cdot \frac{r_a}{r_b} = \frac{SB}{SA} \cdot \frac{r_a}{r_b} = \frac{VA}{VB} \cdot \frac{SB}{SA}.$$

But $\frac{VA}{VM} = \frac{SA}{SV}$, and $\frac{VB}{VN} = \frac{SB}{SV}$ so that

$$\frac{VA}{VB} = \frac{VM \cdot SA}{VN \cdot SB},$$

and $$\frac{v_b}{v_a} = \frac{VM \cdot SA}{VN \cdot SB} \cdot \frac{SB}{SA} = \frac{VM}{VN},$$

which is the required ratio with which we started.

Any pair of corresponding sections of the hyperboloids may be used for pitch surfaces, two pairs being shown in

[1] Or put otherwise, instead of the two wheels having the same peripheral velocity, as with spur gearing, they have *different* peripheral velocities, but these different velocities must have *equal components along the direction* SV_1.

the figure. The directions of the flanks of the teeth on the pitch surfaces must correspond to the directions of the generator. Frequently frustra of tangential cones are employed in this gearing instead of frustra of the hyperboloids. In that case the shape of the tooth profiles is obtained by designing them in the way described for bevel wheels on p. 544.[1] If sections from the *throats* of the hyperboloids be chosen, the teeth may be made of uniform section right across, like those of spur wheels, but, of course, skewed at the proper angle. The ratio between the numbers of teeth in the wheels must be proportional (inversely) to their intended velocity ratio, and *not* proportional directly to their diameters. The pitch of the teeth on the two wheels, measured circumferentially, is of course different. If SV_1 be the normal pitch (*i.e.* the pitch measured at right angles to the face of the tooth), which is *the same in both wheels*, then SB_1 must be the circumferential pitch of b and SA_1 of a.

If the perpendicular distance between the shafts be t, then the diameters of the two pitch surfaces at the throats, or smallest parts, are respectively,

$$t.\frac{VA}{AB} \text{ for } a, \text{ and } t.\frac{VB}{AB} \text{ for } b.$$

At any other places the diameters can be found from the data in the figure by the ordinary projective constructions.

If the distance SV_1 represent on any scale the common peripheral velocity of the two wheels normal to the direction of the twist axis, then the distances V_1B_1 and V_1A_1 represent on the same scale their velocities of sliding along that axis. The velocity with which each one slides relatively to the other is therefore $B_1V_1 + V_1A_1$, or B_1A_1. This corresponds,

[1] A more exact approximation will be found in *Der Constructeur*, third edition, p. 452, or fourth edition, p. 553.

of course, to the axial component of the twisting motion of the axodes. In this, the simplest case of general screw motion which we have in machinery, the magnitude of the twist (which may be expressed conveniently enough as the ratio $\frac{SV}{BA}$) is the same for each twist axis, as we have already noticed.

We may consider that spur wheels are the special case of skew wheels where SB coincides in direction with SA, and where, therefore, $B_1A_1 = O$. Looking at matters in this way, we see that the teeth of skew wheels must have the same rubbing action in planes normal to their lines of contact (*i.e.* normal to SV) as ordinary spur wheels, and *in addition* a rubbing action in the direction of those lines. As all such action involves the expenditure of work in overcoming the frictional resistances which the surfaces offer to sliding one on the other, the frictional losses in skew wheel gearing are necessarily considerably greater than in spur gearing. More will be said about this matter in the next chapter.

A skew-wheel train may, of course, contain an annular wheel, or it may be made epicyclic by fixing one of the wheels instead of the frame. Practically an annular skew wheel would be so troublesome to make that we are not likely to see one, especially as it is always possible (as we have seen with bevel wheels) to alter the sense of rotation transmitted without the use of an annular wheel. There is no particular difficulty about making an epicyclic skew train, but no occasion seems yet to have occurred for using one.

§ 69.—SCREW WHEELS.

IF we desire to drive two crossed shafts, one from the other, by a pair of wheels whose teeth can always touch each other *along a straight line*, we have no alternative but to use the skew wheels described in the last section. We must, moreover, use for their pitch surfaces hyperboloids generated by the revolution of one particular line (fixed as we fixed *SV* in Fig. 318) for each particular angular position of the shafts

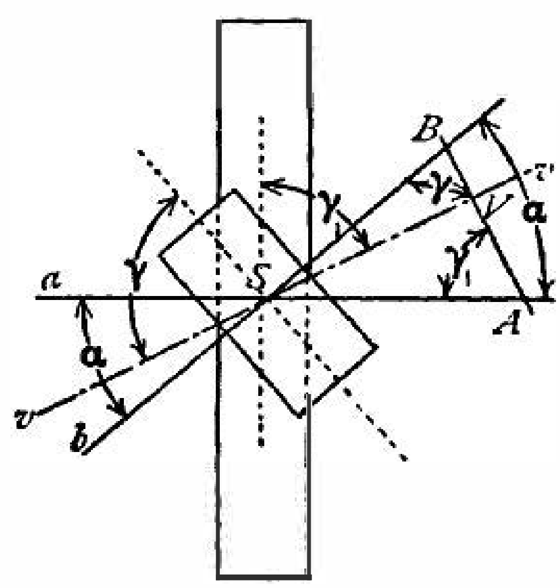


FIG. 319.

and velocity ratio to be transmitted. If, however, we are content to transmit motion through teeth which touch each other on *one point only*[1] at each instant, we have a much larger choice of possibilities. Thus if in Fig. 319, *SA* and *SB* are again projections of the axes of crossed shafts (drawn in the same way as in Fig. 318, § 68), we may take any line *SV* between *SA* and *SB*, or *in coincidence with either of them*, as the common tangent at *S* to screw lines drawn on cylindrical surfaces which have *a* and *b* as their axes, and

[1] In one point (for each pair of teeth), speaking kinematically only. Physically, of course, the point becomes a small but undefined area.

which touch at some point on the normal through S. If on such cylinders, or slices of them, we build up helical teeth corresponding to the assumed tangent, we shall have what are called **screw wheels**, each wheel being, in effect, a portion of a many-threaded screw.

If the tangent SV be taken in the same position as that of the coincident line in skew wheels, the pitch diameters of the screw wheels, for a given velocity ratio, will be the same as those of the throats of the corresponding hyperboloids. In this case the screw wheels will differ very little in appearance from the skew wheels. In the former, however, the faces of the teeth will be helical instead of straight, and in actual working, contact between any pair of teeth will begin on one side, pass through the point S, and end on the other side, instead of taking place simultaneously all across the teeth.

For any given velocity ratio, whether the common tangent in mid-position be taken as mentioned in the last paragraph or not, the hyperboloids having a coincident generator, as found in § 68, will remain the twist-axodes for the motion transmitted by the wheels, no matter what the diameters of the latter may be. These diameters are easy to find in any case. Let V be any point upon the (arbitrarily chosen) common tangent SV, and BA a line through V normal to SV. Then we have already proved (p. 551) that (using the same symbols as before)

$$\frac{r_a}{r_b} = \frac{v_b}{v_a}\cdot\frac{SA}{SB} = \frac{v_b}{v_a}\cdot\frac{\sin\gamma}{\sin\gamma_1}.$$

If, then, SV coincide with SB (Fig. 320), $\gamma = 90°$, and $\gamma_1 = 90° - \alpha$, and the ratio between the diameters of the wheels is greater than the velocity ratio, a condition in itself disadvantageous. The teeth on b are here parallel to its axis, as in a spur wheel. If SV bisect the angle ASB (Fig. 321)

$\gamma = \gamma_1$, and the diametral ratio corresponds, as with spur wheels, to the velocity ratio. This is the condition of minimum friction. If SV coincide with SA (Fig. 322) the teeth on the a wheel are parallel to its axis, the angle $\gamma = 90^\circ - \alpha$ and $\gamma_1 = 90^\circ$, and the diametral ratio is *less* than the velocity ratio. Intermediate positions have, of course, corresponding characteristics. If V be nearer B than A, the diametral ratio is greater than the velocity ratio, and *vice versâ.*

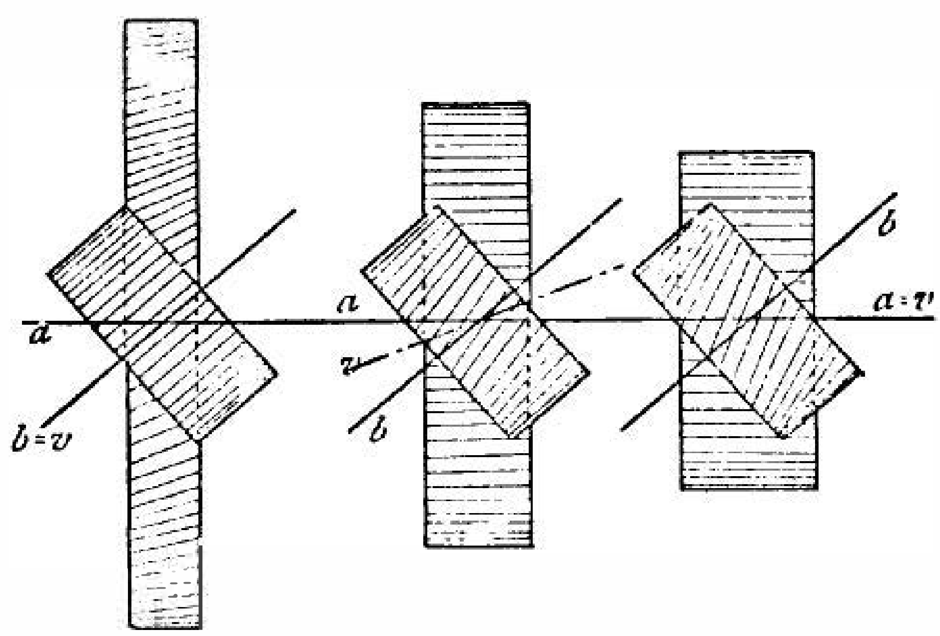


FIG. 320. FIG. 321. FIG. 322.

In the most common case of screw wheels occurring in practice, the angle α, between the shafts, is a right angle, and the velocity ratio transmitted is large. In this case the combination becomes the worm and worm-wheel (Fig. 323), which we have already looked at in § 62 from another point of view. If we were here to make $\gamma = 90^\circ$, the pitch of the helix on a would become $= 0$, and that on $b = \infty$. If we were to make $\gamma_1 = 90^\circ$, the pitch on a would become $= \infty$ and on $b = 0$. In neither case, therefore, would the mechanism work. In practice γ is made some small angle, and (as $\gamma + \gamma_1 = 90^\circ$) γ_1 is a much larger one. A very large velocity ratio can therefore be transmitted by a pair of screw wheels (as the worm and wheel really are) of much smaller diametral ratio. Thus, for example, if $\gamma = 10°$, γ_1 must $= 80°$, the value of $\frac{\sin \gamma}{\sin \gamma_1} = 0{\cdot}175$. In such a case any velocity ratio r can be transmitted by a worm and wheel whose diameters are in the ratio of $(0{\cdot}175\ r)$ to each other. This is, of course, a great practical convenience. If a pair of skew wheels were used under similar conditions, with a value of r of 50, their diametral ratio would have to be 50, instead of $0{\cdot}175 \times 50$, or $8{\cdot}75$.

In both cases equally the number of teeth in the wheels must be proportional to the velocity-ratio transmitted, but with screw wheels the number of teeth means *the number of threads in the screw*, for that is the real number of teeth that would be shown by any section of the wheel normal to its axis. A single-threaded screw, such as is often used for a worm, if cut by a plane at right angles to its axis, would show only *one* tooth and one space—it is in reality a one-toothed wheel. A double-threaded screw, similarly, is equivalent to a wheel of two teeth, and so on.

The *pitch* of screw wheels has to be determined in the same way as that of skew wheels. The pitch *of the teeth*, measured at right angles to the common tangent, must be the same in both wheels, or in wheel and worm, but this quantity must not be confused with the pitch *of the screw lines*. The real pitch of the teeth in screw wheels is the distance represented by the spaces between the lines in Figs. 320 and 322 of this section. This is determined from the normal pitch exactly as on p. 552, and we do not concern ourselves at all with the relation between this (circumferential) quantity and the (axial) pitch of the helices on which the threads are formed, which is in these cases always a very much *larger* distance. In the case of the worm of Fig. 323, the case is, however, reversed. Taken as a single-threaded screw, or one-toothed screw wheel, the pitch of its tooth is equal to its circumference, while the pitch of the helix, or distance from one convolution to the next, measured axially, is a much *smaller* quantity. It is the former, or tooth-pitch, which is given by the calculation on p. 552. In this particular case, however, it is the helical pitch which is the visible thing, and the real nature of the worm as a screw wheel is sometimes obscured by the confusion between the two pitches. In this particular case the circumferential

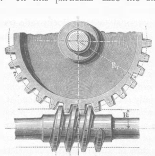


FIG. [illegible]

pitch of the worm wheel is equal to the axial pitch of the worm helix if it be single threaded, to half that pitch if it be double threaded, and so on.

We have already mentioned (p. 487) the Sellers worm gearing, in which the shafts are set at an angle less than 90° by an amount equal to γ, so that $\gamma_1 = \alpha$, SV coinciding with SA. In this case the teeth of the wheel, like those of the wheel in Fig. 322, lie parallel to its axis; the wheel in fact simply becomes, or may become, a spur wheel, and its construction is correspondingly simplified. The form of the teeth of worm wheels has already been mentioned in § 62.

We have stated on p. 554 that the tangent line of screw wheels may be taken anywhere between *SB* and *SA*. Kinematically it may be taken outside these limits also, but in this case the friction due to the sliding of the teeth (see p. 553) becomes excessively great, without counterbalancing advantage of any other kind.

Longitudinal sliding of the teeth on one another produces additional friction in screw wheels (as compared with spur or bevel wheels) exactly as in skew wheels, but not to the

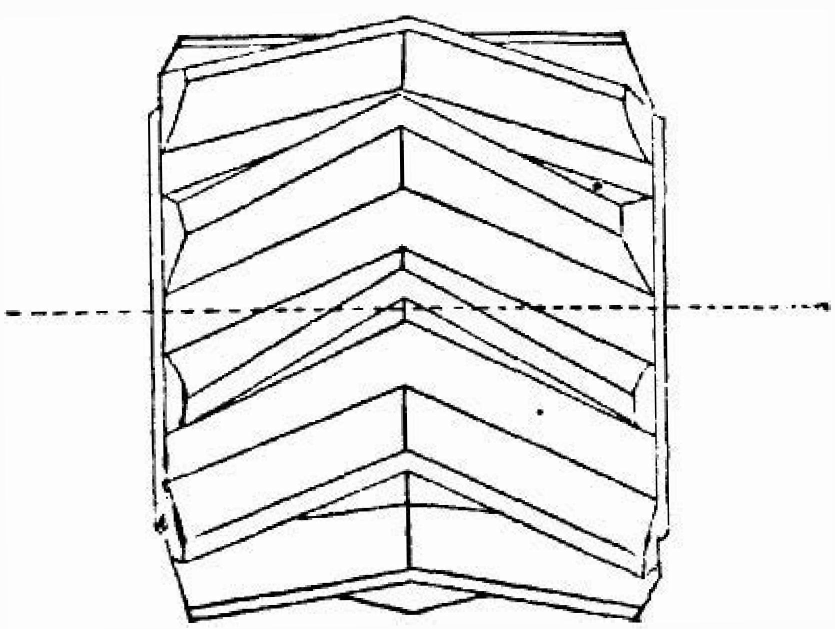


FIG. 324.

same extent, the area of surface in contact being much smaller (see p. 554). In both cases also the obliquity of the pressure causes end thrust in the journals of one or both of the shafts. This is often more serious in screw than in skew gearing, because of the greater obliquity. If the shafts are *parallel*, this difficulty can be got over by the use of the double-helical wheels of Fig. 324, which were mentioned in § 19 (p. 131). These wheels, although they are used with parallel shafts, are real screw wheels; the contact of the teeth is a point contact only, and not a line contact, and there is always contact in at least one pair of points along the pitch line. They present certain practical difficulties in manufacture, but these have been long ago overcome, and very large numbers of them are used on the Continent, and also by some English makers. The small surface of contact makes them work very "sweetly" if the teeth are reasonably well formed. It will be noticed that their relative motion is represented simply by the rolling of cylindrical pitch surfaces, or axodes, as with spur wheels, and not by the twisting together of hyperboloidal surfaces, as in the case of screw wheels with crossed axes or skew bevel wheels. The cylindrical pitch surfaces here show themselves at once as being a special case of the hyperboloidal surfaces.

§ 70.—GENERAL SCREW MECHANISMS.

THERE remain to be mentioned **general screw mechanisms** (see p. 547) of a much more complex kind than the hyperboloidal or screw wheels of the last two sections. Of such mechanisms two have been already illustrated in Figs. 314 and 315 in § 67, a third is shown in Fig. 325. Of these a modification of Fig. 314 has found actual, if not very practical, application in machines. The other two have not, so far as we know, had any practical applications. A general investigation into their conditions of constrainment, or determination of the twist-axodes of the different links, does not appear as yet to have been made. Such a determination is not very difficult in the case of Figs. 314 and 315, where also the mechanism can be drawn in any position, without any difficulty, by the ordinary constructions of orthographic projection. With Fig. 324, however, these constructions alone do not enable the motions of the mechanism to be drawn. The case appears to be analogous, among non-plane mechanisms, to the case of the third order in plane mechanisms which was discussed in § 59 (p. 458). The extremely limited practical importance of these mechanisms makes it unsuitable that anything in the nature of a general discussion of their properties should be attempted here. It is much to be hoped that some competent geometer will presently take them in hand, and classify and analyse them.

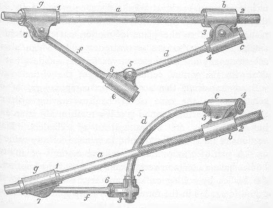


Fig. 305.

We may here summarise briefly the conditions of motion which we have found to exist in the elements of mechanisms and in the mechanisms themselves. The **turning pair**, first of all, gives us a simple rotation about a fixed and permanent (p. 47) axis at a finite distance, and we saw that the motion of the **sliding pair** was merely the special case in which the axis of rotation (equally fixed and permanent) was at infinity. The **screw pair** gives us motion about a fixed and permanent twist-axis, the magnitude of the twist being also constant. It might be considered to include the former pairs as special cases, the one (turning pair) where the pitch of the twist had become zero, and the other (sliding pair) where the rotation of the twist had become zero. Passing from elements to chains or mechanisms, we find that for mechanisms having **plane motion**, the virtual motion of every link is a rotation about a fixed (permanent or instantaneous) axis, at a finite distance or at infinity. In this case, also, all the virtual axes are parallel, so that all planes parallel to the plane of motion cut the axode in similar and equal curves or centrodes, and we can always substitute any one of these curves for the axode (that is, deal with the virtual centre instead of the virtual axis), without impairing the accuracy or completeness of our solutions. In the case of mechanisms having **spheric motion**, the virtual motion is still a rotation about an axis, but the axodes are cones instead of cylinders. Plane sections of these axodes are not, in general, of any value to us. A pair of axodes for the relative motions of any two bodies have a common vertex, and represent the motion of the bodies by rolling on one another,[1] exactly as do the cylindric axodes in the former case. Any sphere which has its centre at the vertex of such a pair of axodes, cuts them both in a pair of spheric sections which roll upon one another as the bodies move, and which touch each other in a point (as S in Fig. 274), which determines, along with the centre of the sphere, the virtual axis. This point of contact is not, however, a virtual centre, as the bodies do not, virtually or otherwise, rotate about it, and these spheric sections of

[1] The proof of the rolling is exactly the same as that given in § 9 for plane centrodes, and does not need to be written out again in full.

the conic axodes are not, therefore, really centrodes. No virtual centres, in the sense in which we have defined these points, exist for spheric motions, and the sets of lines each containing three virtual centres are replaced by sets of planes each containing three virtual axes. With **general screw mechanisms**, lastly, neither virtual axis nor centre exist, the virtual motion is no longer a simple rotation of any kind, but as twist is already reduced to its lowest terms. It may be noticed that in dealing with twist as we did in § 62, looking separately at its two components, rotation and sliding, we virtually resolved it into a pair of rotations, one about the axis of the screw, and the other about an axis at right angles to it and in the same plane, but at an infinite distance. No particular convenience, however, for our purposes, comes from this way of looking at the matter. We have found three different cases of screw motion to occur ino ur work. The regular twist of the screw pair is the first, where the twist-axis is permanent and where the magnitude of the twist is constant. The cases (skew wheels and screw wheels) examined in §§ 68 and 69 come next, and bear the same relation to the screw pair that the motions of a spur-wheel chain (omitting all consideration of the teeth in both cases) bear to those of a turning pair. The twist-axis, namely (for one or more pairs of links), becomes an instantaneous instead of a permanent axis. But the magnitude of the twist is constant, so that the hyperboloids[1] being given, and the value of the twist, the motions are as fully, if not quite as simply, determined as those of rotating bodies whose centrodes are known. This determinateness has nothing to do with the hyperboloidal form of the twist-

[1] It will be remembered that it is these surfaces, the twist-axodes, and not the helical surfaces or their base cylinders, which really determine the motion of screw wheels (p. 555).

axodes, but depends on the constancy of the twist. So long as this condition exists, the relative screw motion of two bodies may be geometrically represented by the loci of their twist-axes as completely as the relative plane motion of two bodies can be by their centrodes. In the third case of general screw mechanisms, however, the case specially dealt with at the commencement of this section, the value of the twist differs with each twist-axis, and varies quite independently of the change of position of the axis. The motions of the mechanism do not seem, in this case, to be determinate by aid of the twist axodes alone, geometrically, but to require also for their determination some expression for the rate of change of the magnitude of the twist itself. It is perhaps fortunate for engineers that problems of this kind have not yet made their appearance in practical work.