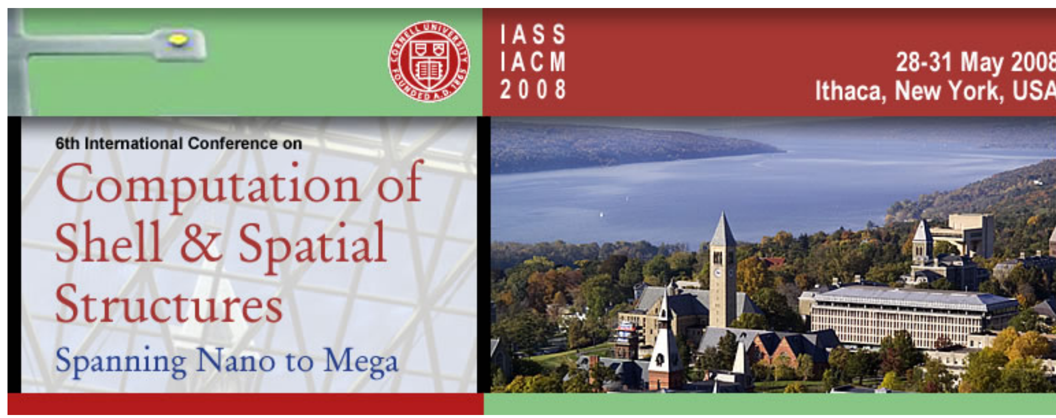
Universität Stuttgart

Fakultät für Bau und-
Umweltingenieurwissenschaften

Baustatik und Baudynamik

Modeling of Shells with Three-dimensional Finite Elements

Manfred Bischoff



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Thomas Cichosz

Michael Gee

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Wolfgang A. Wall

outline

evolution of shell models

solid-like shell or shell-like solid element?

locking and finite element technology

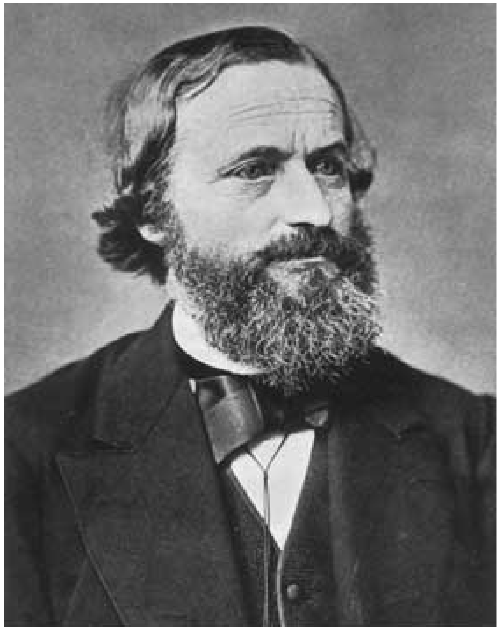
how three-dimensional are
3d-shells / continuum shells / solid shells?

History

early attempts

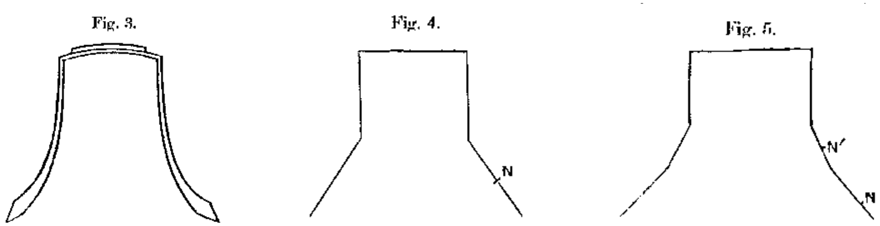
- ring models (Euler 1766)
- lattice models (J. Bernoulli 1789)
- continuous models (Germain, Navier, Kirchhoff, 19th century)

Leonhard Euler 1707 - 1783



Gustav Robert Kirchhoff
1824 - 1887

History



- membrane and bending action
- inextensional deformations

Lord Rayleigh (John W. Strutt)

first shell theory = „Kirchhoff-Love“ theory

"This paper is really an attempt to construct a theory of the vibrations of bells"

August E.H. Love, 1888

All you need is Love?

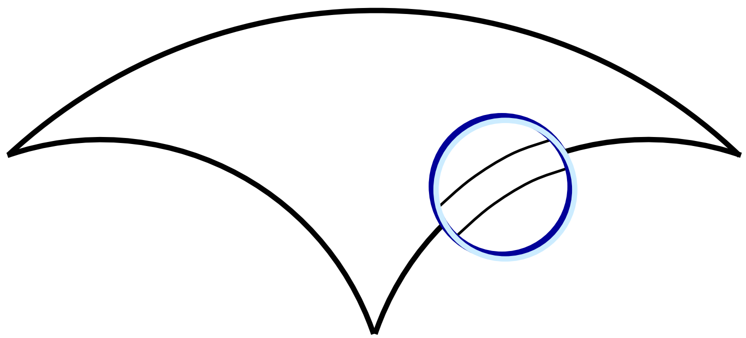
August E.H. Love, 1888

first shell theory = „Kirchhoff-Love“ theory

"This paper is really an attempt to construct a theory of the vibrations of bells"

Evolution of Shell Models

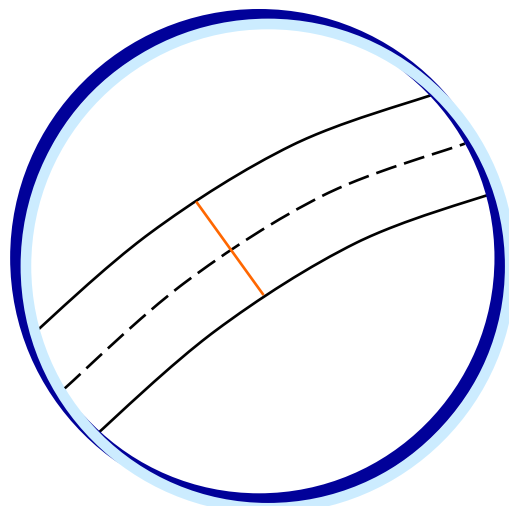
fundamental assumptions



$$\sigma_{zz} = 0, \left(\varepsilon_{zz} = 0\right)$$

contradiction
requires modification
of material law

cross sections remain
- straight
- unstretched
- normal to midsurface

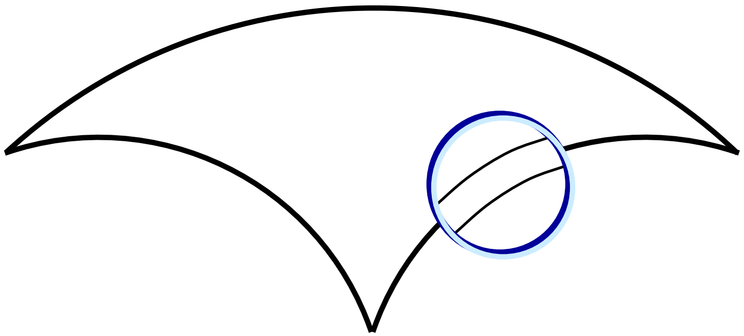


$$\gamma_{xz} = 0$$
$$\gamma_{yz} = 0$$

Kirchhoff-Love

Evolution of Shell Models

fundamental assumptions



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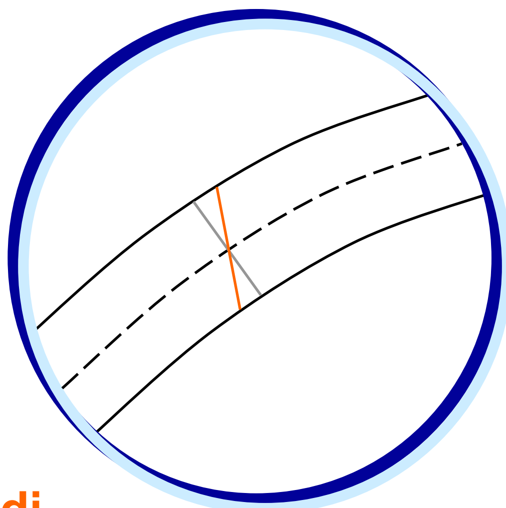


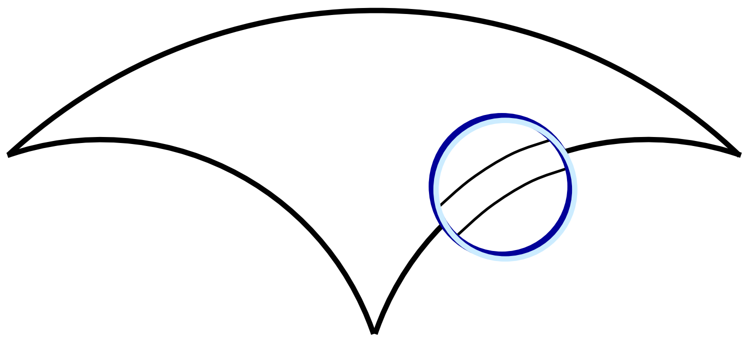
$$\gamma_{xz} \neq 0$$

$$\gamma_{yz} \neq 0$$

Reissner-Mindlin, Naghdi

Evolution of Shell Models

fundamental assumptions



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- straight
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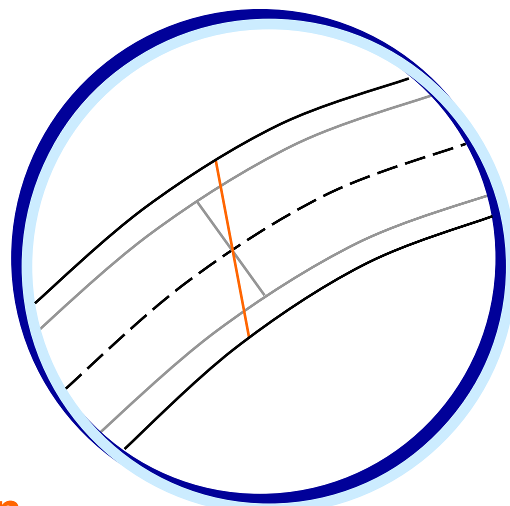
- ~~normal to midsurface~~



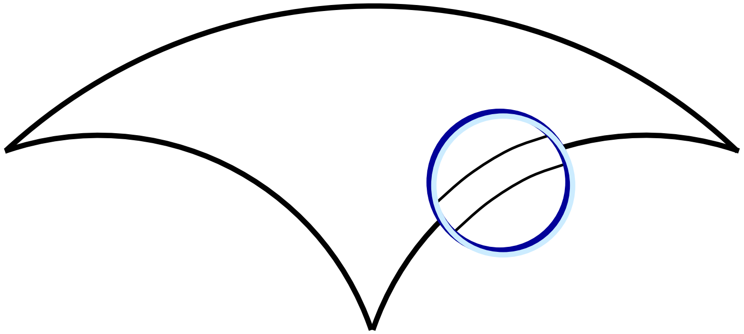
$$\gamma_{xz} \neq 0$$

$$\gamma_{yz} \neq 0$$

7-parameter formulation

Evolution of Shell Models

fundamental assumptions



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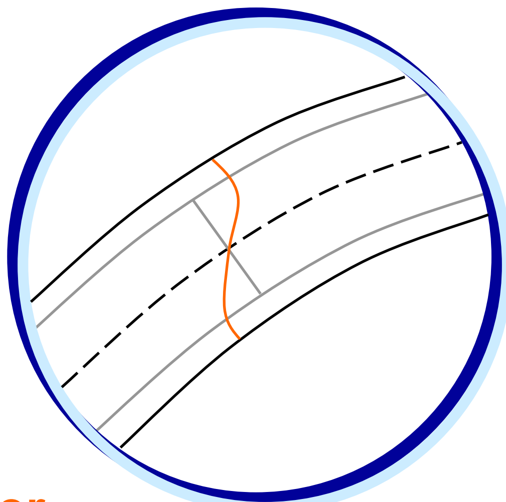


$$\gamma_{xz} \neq 0$$

$$\gamma_{yz} \neq 0$$

multi-layer, multi-director

Evolution of Shell Models

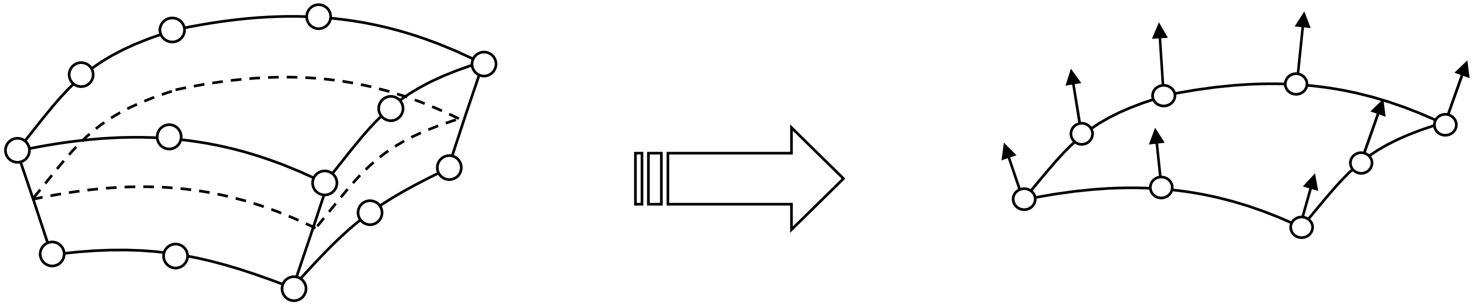
from classical „thin shell“ theories to 3d-shell models

- 1888: Kirchhoff-Love theory
  membrane and bending effects
- middle of 20^{th} century: Reissner/Mindlin/Naghdi
  + transverse shear strains
- 1968: degenerated solid approach (Ahmad, Irons, Zienkiewicz)
  shell theory = semi-discretization of 3d-continuum
- 1990+: 3d-shell finite elements, solid shells,
  surface oriented (“continuum shell”) elements
  Schoop, Simo et al, Büchter and Ramm, Bischoff and Ramm,
  Krätzig, Sansour, Betsch, Gruttmann and Stein, Miehe and Seifert,
  Hauptmann and Schweizerhof, Brank et al., Wriggers and Eberlein,
  Klinkel, Gruttmann and Wagner, and many, many others

since ~40 years parallel development of theories and finite elements

Evolution of Shell Models

the degenerated solid approach
Ahmad, Irons and Zienkiewicz (1968)

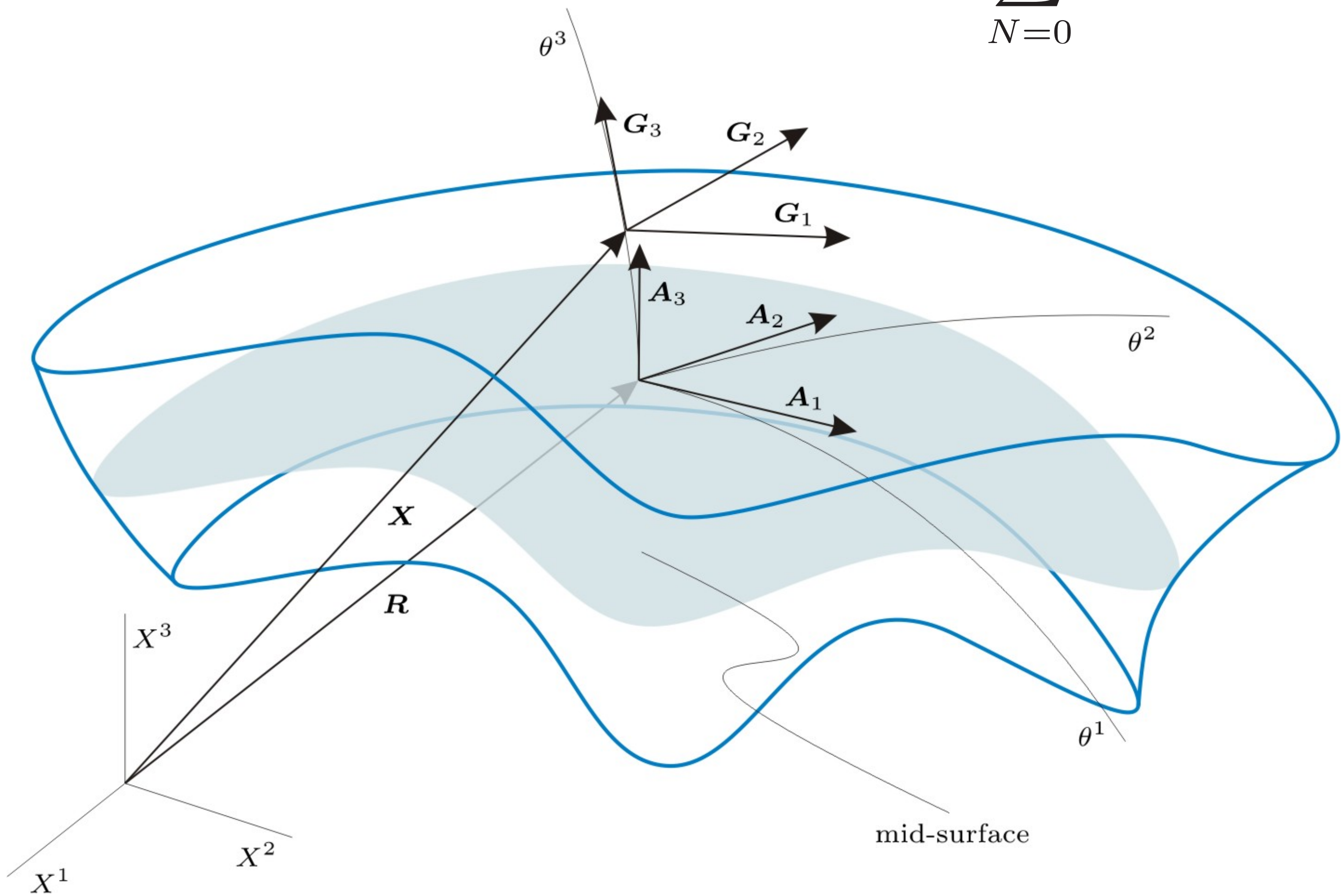


1. take a three-dimensional finite element (brick)
2. assign a mid surface and a thickness direction
3. introduce shell assumptions and
   refer all variables to mid surface quantities
   (displacements, rotations, curvatures, stress resultants)

Derivation from 3d-continuum (Naghdi)

geometry of shell-like body

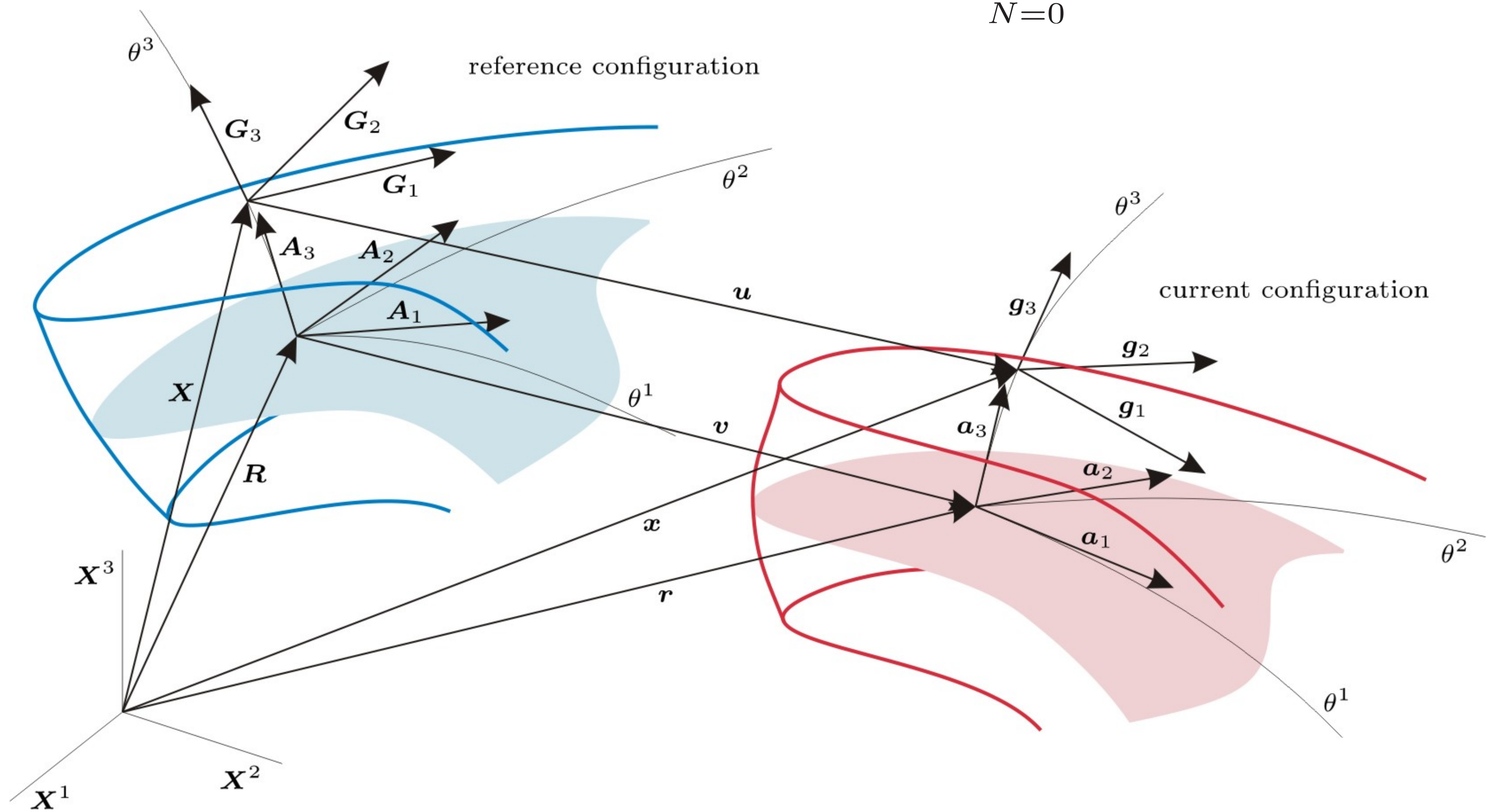
$$\boldsymbol{X}(\theta^1,\theta^2,\theta^3) = \sum_{N=0}^{\infty} (\theta^3)^N \boldsymbol{R}^N(\theta^1,\theta^2)$$



Derivation from 3d-continuum (Naghdi)

deformation of shell-like body

$$\boldsymbol{u}(\theta^1, \theta^2, \theta^3) = \sum_{N=0}^{\infty} (\theta^3)^N \boldsymbol{v}^N(\theta^1, \theta^2)$$



7-parameter Shell Model

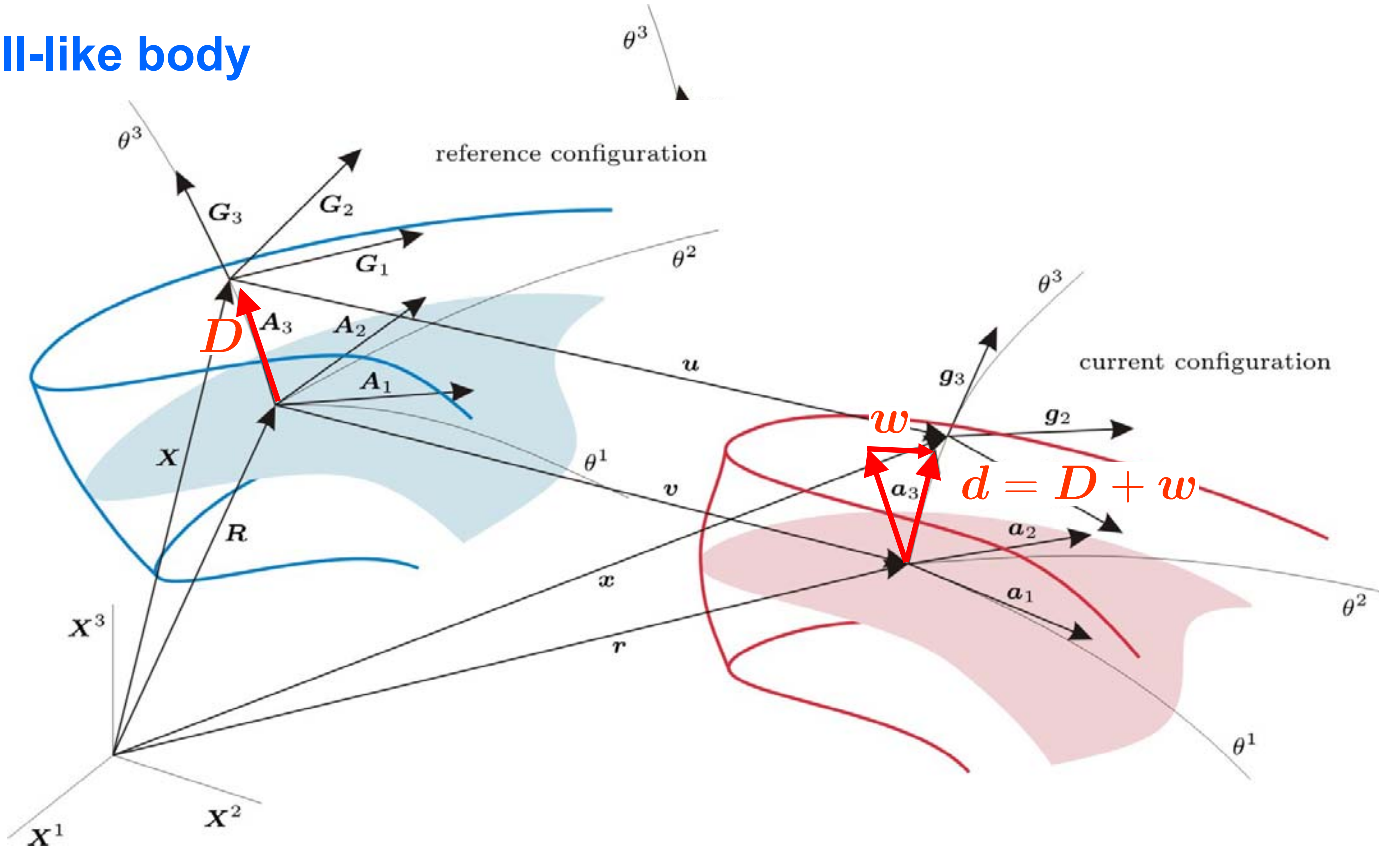
geometry of shell-like body

$$\boldsymbol{X} = \boldsymbol{R} + \theta^3\,\boldsymbol{D}$$

$$X_1 = R_1 + \theta^3\,D_1$$

$$X_2 = R_2 + \theta^3\,D_2$$

$$X_3 = R_3 + \theta^3\,D_3$$



displacements

$$\boldsymbol{u} = \boldsymbol{v} + \theta^3\,\boldsymbol{w}$$

$$u_1 = v_1 + \theta^3\,w_1$$

$$u_2 = v_2 + \theta^3\,w_2$$

$$u_3 = v_3 + \theta^3\,w_3$$

+ 7th parameter for linear transverse normal strain distribution

7-parameter Shell Model

linearized strain tensor in three-dimensional space

$$\boldsymbol{\varepsilon} = \varepsilon_{ij}\, \boldsymbol{G}^i \otimes \boldsymbol{G}^j$$

$$\varepsilon_{ij} = \frac{1}{2}\left(\frac{\partial \boldsymbol{u}}{\partial \theta^i} \cdot \frac{\partial \boldsymbol{X}}{\partial \theta^j} + \frac{\partial \boldsymbol{u}}{\partial \theta^j} \cdot \frac{\partial \boldsymbol{X}}{\partial \theta^i}\right) = \frac{1}{2}\left(\boldsymbol{u}_{,i} \cdot \boldsymbol{G}_j + \boldsymbol{u}_{,j} \cdot \boldsymbol{G}_i\right)$$

approximation (semi-discretization)

$$\boldsymbol{G}_\alpha = \boldsymbol{X}_{,\alpha} = \boldsymbol{R}_{,\alpha} + \theta^3 \boldsymbol{D}_{,\alpha} \qquad \boldsymbol{G}_3 = \boldsymbol{X}_{,3} = \boldsymbol{D}$$

$$\boldsymbol{u}_{,i} = \boldsymbol{v}_{,i} + \theta^3 \boldsymbol{w}_{,i} \qquad \boldsymbol{u}_{,3} = \boldsymbol{w}$$

strain components

$$\varepsilon_{\alpha\beta} = \frac{1}{2}\left[\left(\boldsymbol{v}_{,\alpha} + \theta^3 \boldsymbol{w}_{,\alpha}\right) \cdot \left(\boldsymbol{R}_{,\beta} + \theta^3 \boldsymbol{D}_{,\beta}\right) + \left(\boldsymbol{v}_{,\beta} + \theta^3 \boldsymbol{w}_{,\beta}\right) \cdot \left(\boldsymbol{R}_{,\alpha} + \theta^3 \boldsymbol{D}_{,\alpha}\right)\right]$$

$$\varepsilon_{\alpha 3} = \frac{1}{2}\left[\left(\boldsymbol{v}_{,\alpha} + \theta^3 \boldsymbol{w}_{,\alpha}\right) \cdot \boldsymbol{D} + \boldsymbol{w} \cdot \left(\boldsymbol{R}_{,\alpha} + \theta^3 \boldsymbol{D}_{,\alpha}\right)\right] = \varepsilon_{3\alpha}$$

$\varepsilon_{33} = \boldsymbol{w} \cdot \boldsymbol{D}$ + linear part via 7th parameter

7-parameter Shell Model

in-plane strain components

$$\varepsilon_{\alpha\beta} = \frac{1}{2}\left[\left(\boldsymbol{v}_{,\alpha} + \theta^3 \boldsymbol{w}_{,\alpha}\right) \cdot \left(\boldsymbol{R}_{,\beta} + \theta^3 \boldsymbol{D}_{,\beta}\right) + \left(\boldsymbol{v}_{,\beta} + \theta^3 \boldsymbol{w}_{,\beta}\right) \cdot \left(\boldsymbol{R}_{,\alpha} + \theta^3 \boldsymbol{D}_{,\alpha}\right)\right]$$

$$= \frac{1}{2}\left(\boldsymbol{v}_{,\alpha} \cdot \boldsymbol{R}_{,\beta} + \boldsymbol{v}_{,\beta} \cdot \boldsymbol{R}_{,\alpha}\right)$$

membrane

$$+\frac{1}{2}\theta^3 \left(\boldsymbol{v}_{,\alpha} \cdot \boldsymbol{D}_{,\beta} + \boldsymbol{w}_{,\alpha} \cdot \boldsymbol{R}_{,\beta} \boldsymbol{v}_{,\beta} \cdot \boldsymbol{R}_{,\alpha} + \boldsymbol{w}_{,\beta} \cdot \boldsymbol{R}_{,\alpha}\right)$$

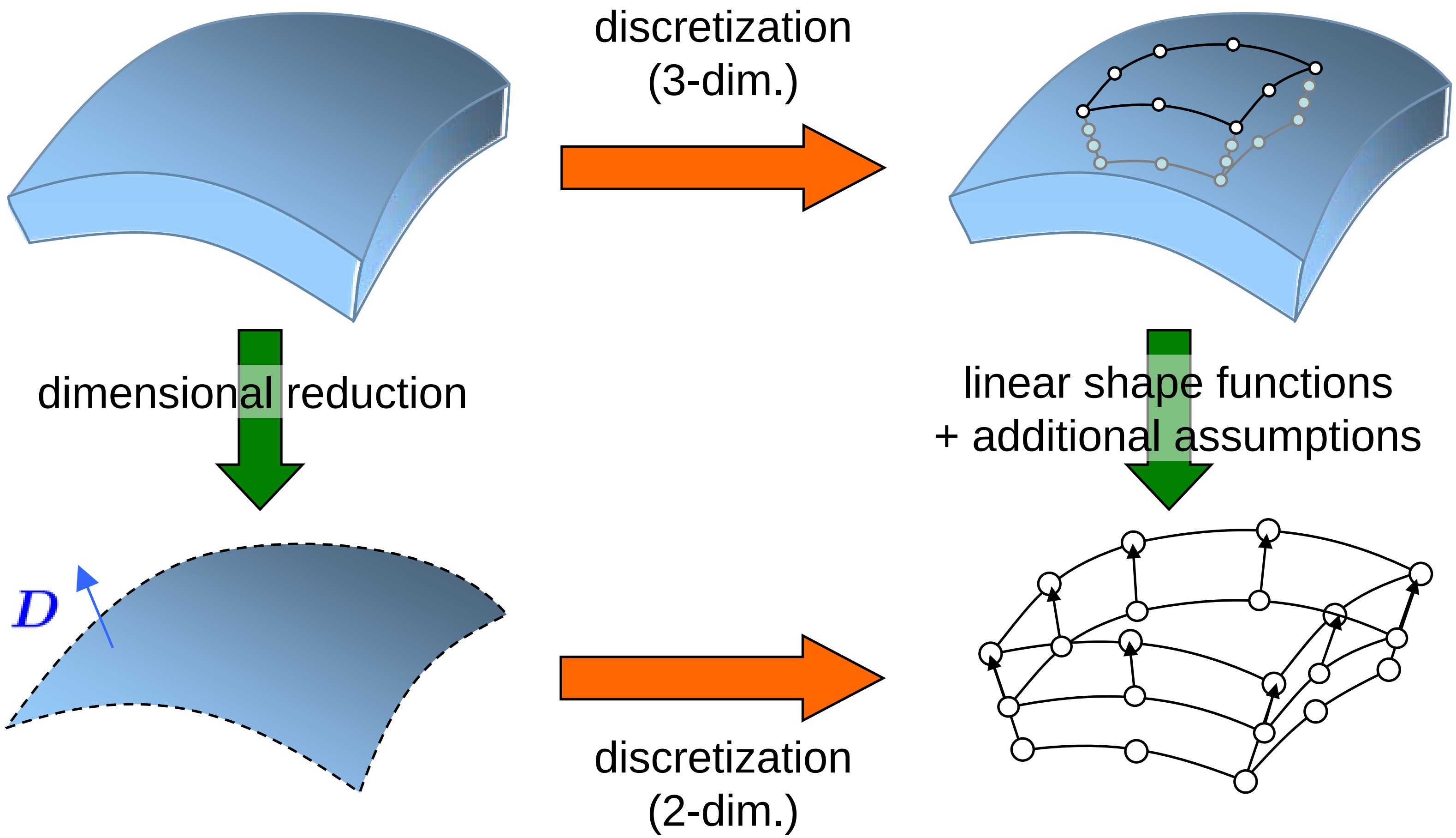
bending

$$+\frac{1}{2}(\theta^3)^2 \cancel{\left(\boldsymbol{w}_{,\alpha} \cdot \boldsymbol{D}_{,\beta} + \boldsymbol{w}_{,\beta} \cdot \boldsymbol{D}_{,\alpha}\right)}$$

higher order effects

Semi-discretization of Shell Continuum

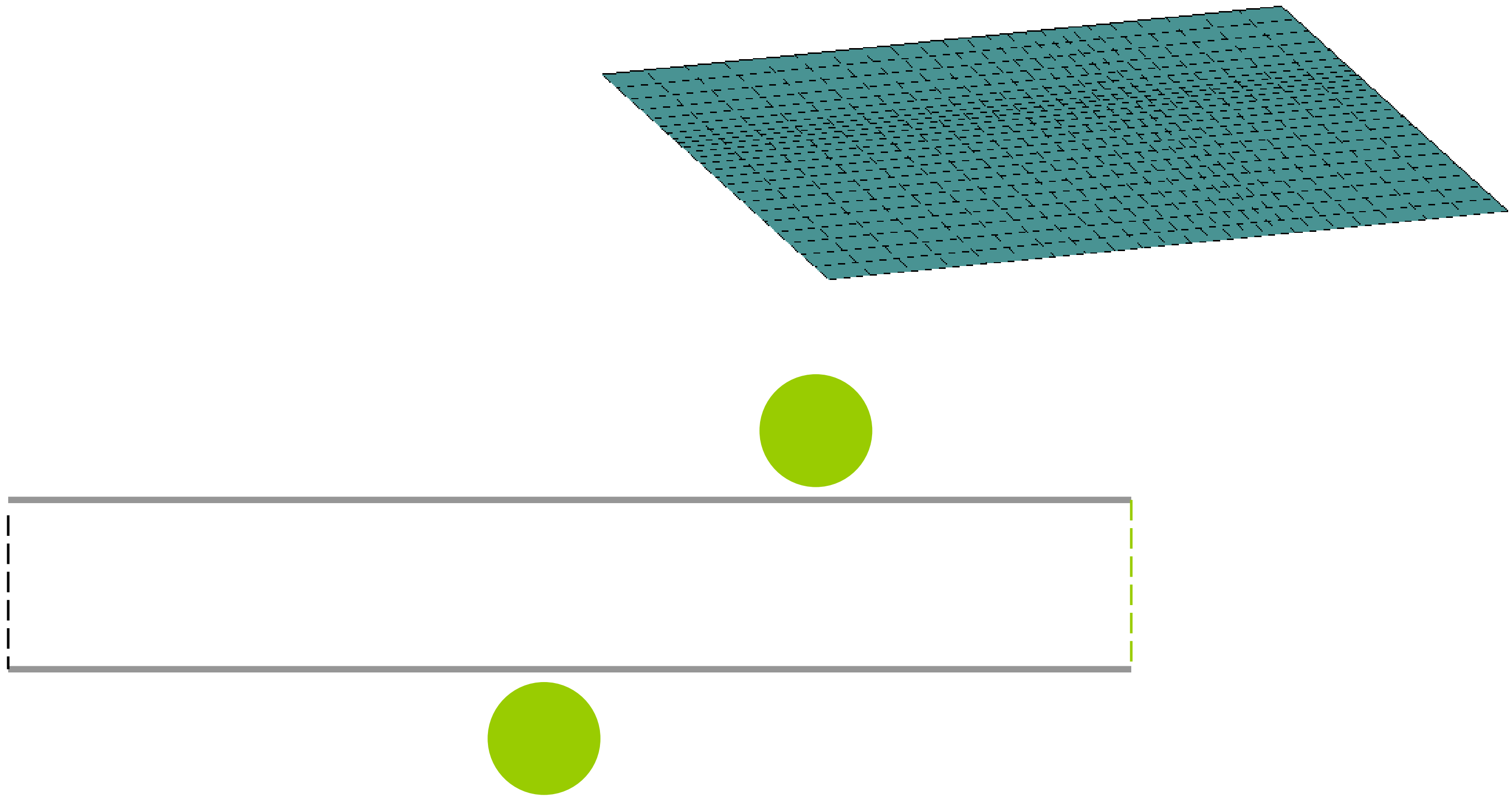
straight cross sections: inherent to *theory* or *discretization*?



equivalence of shell theory and degenerated solid approach, Büchter and Ramm (1992)

Large Strains

metal forming, using 3d-shell elements (7-parameter model)



Large Strains

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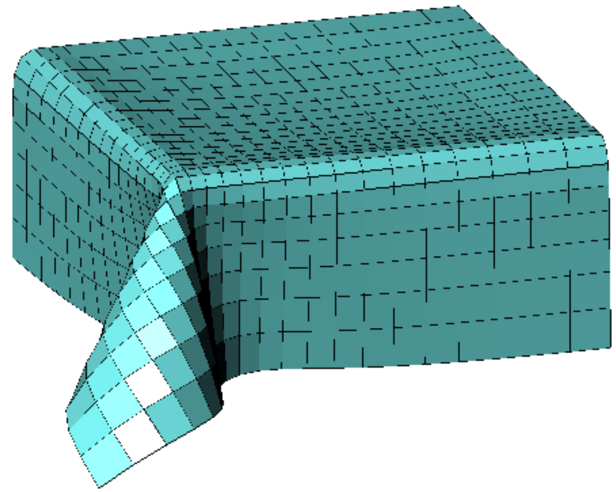
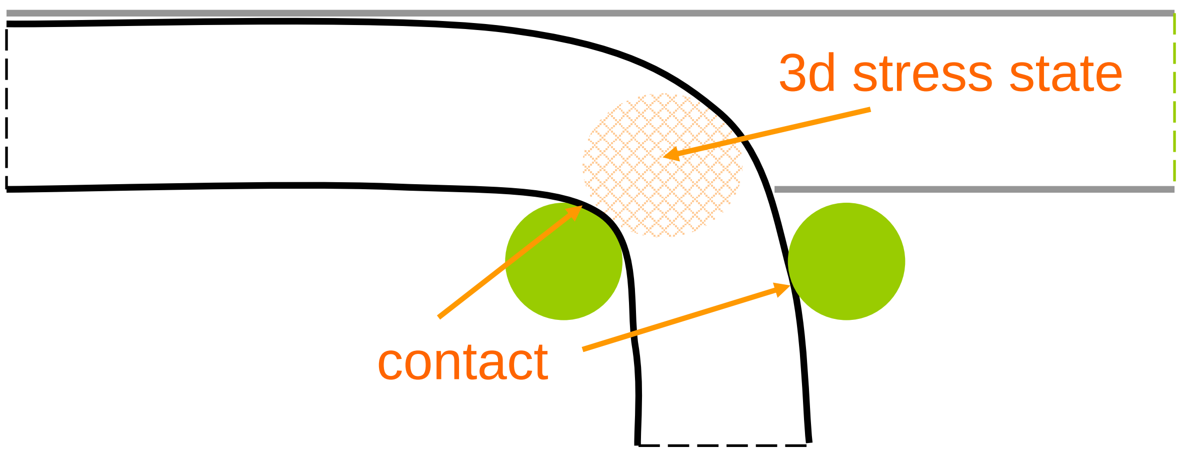
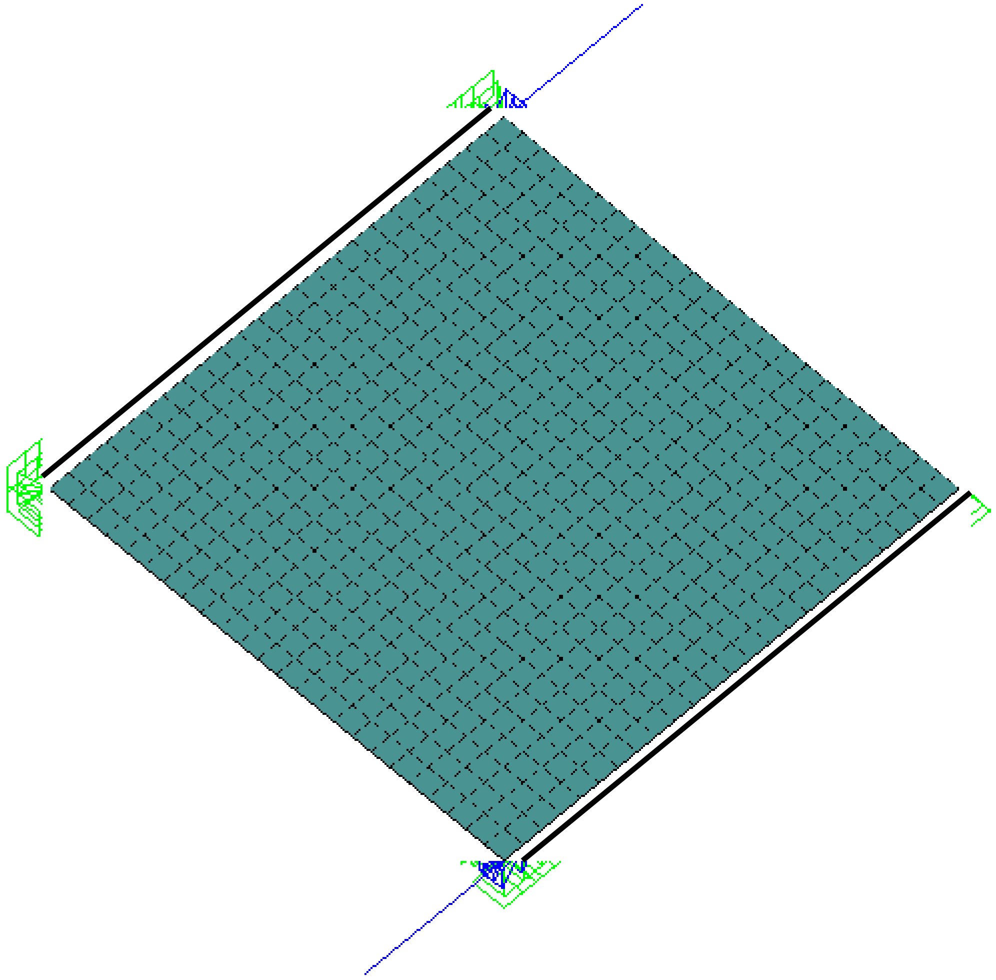




Large Strains

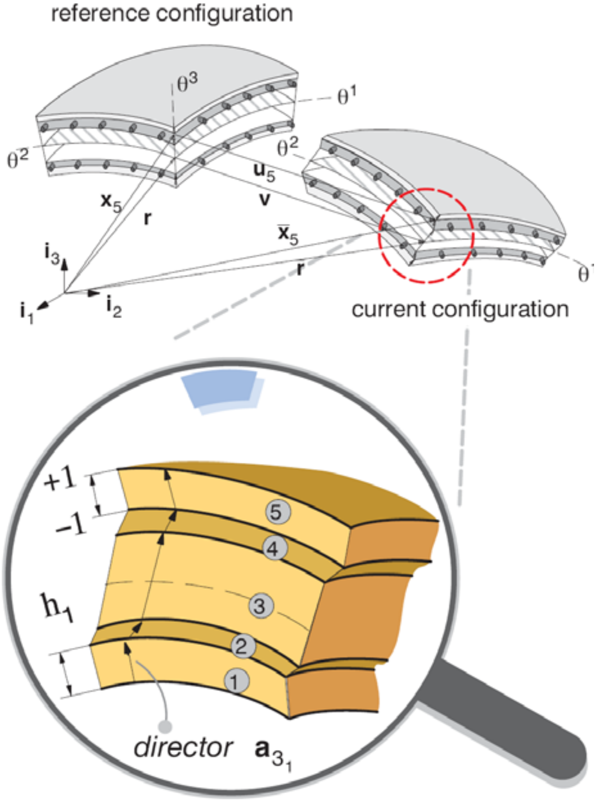
very thin shell (membrane), 3d-shell elements



Motivation

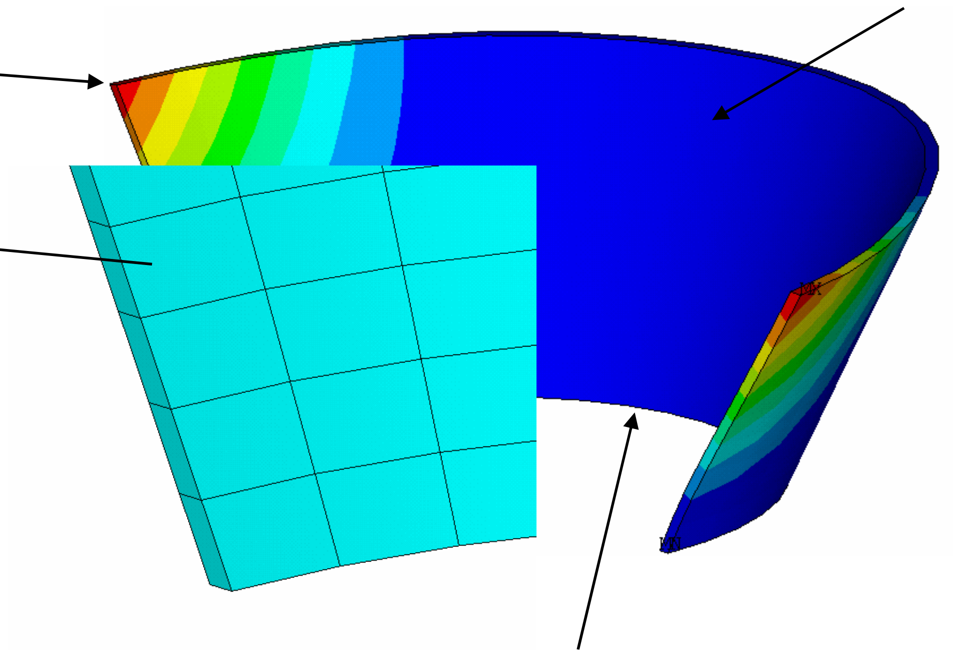
why solid elements instead of 3d-shell elements?

- three-dimensional data from CAD
- complex structures with stiffeners and intersections
- connection of thin and thick regions, layered shells, damage and fracture,…



Shell Analysis with Standard Solid Elements

a naïve approach: take a commercial code and go!

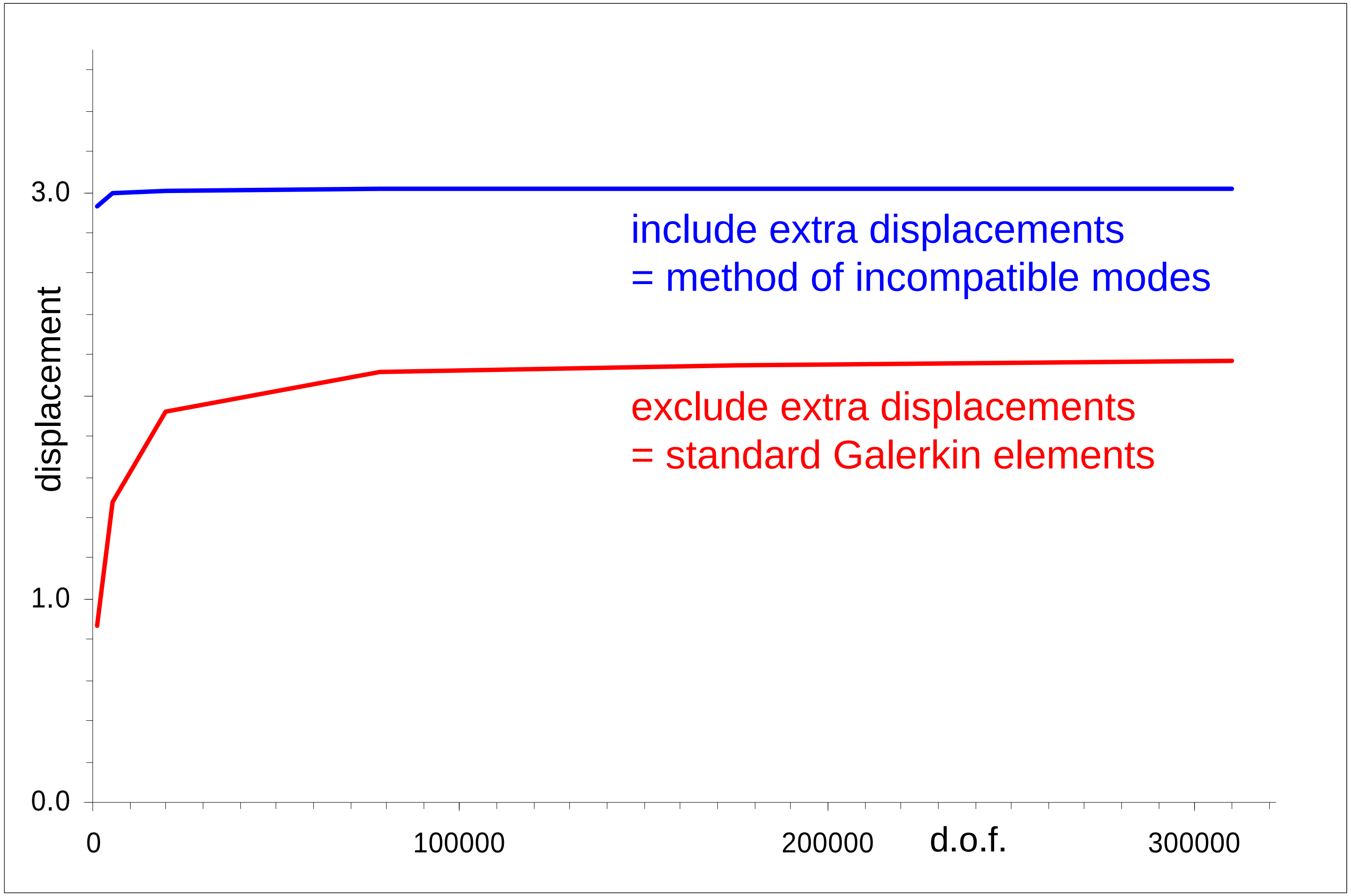


element formulation

- include extra displacements
- exclude extra displacements

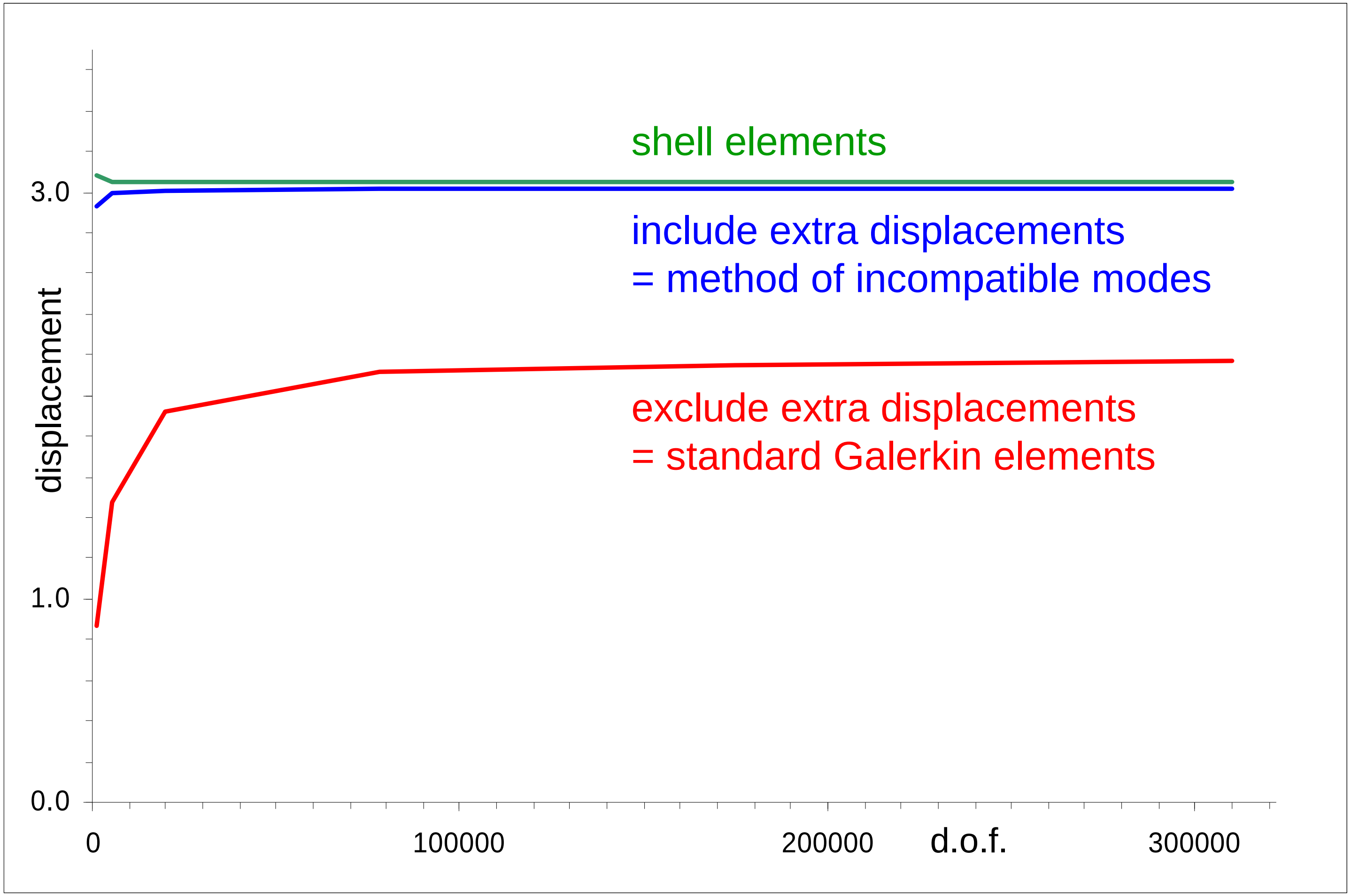
Shell Analysis with Standard Solid Elements

a naïve approach: take a commercial code and go!



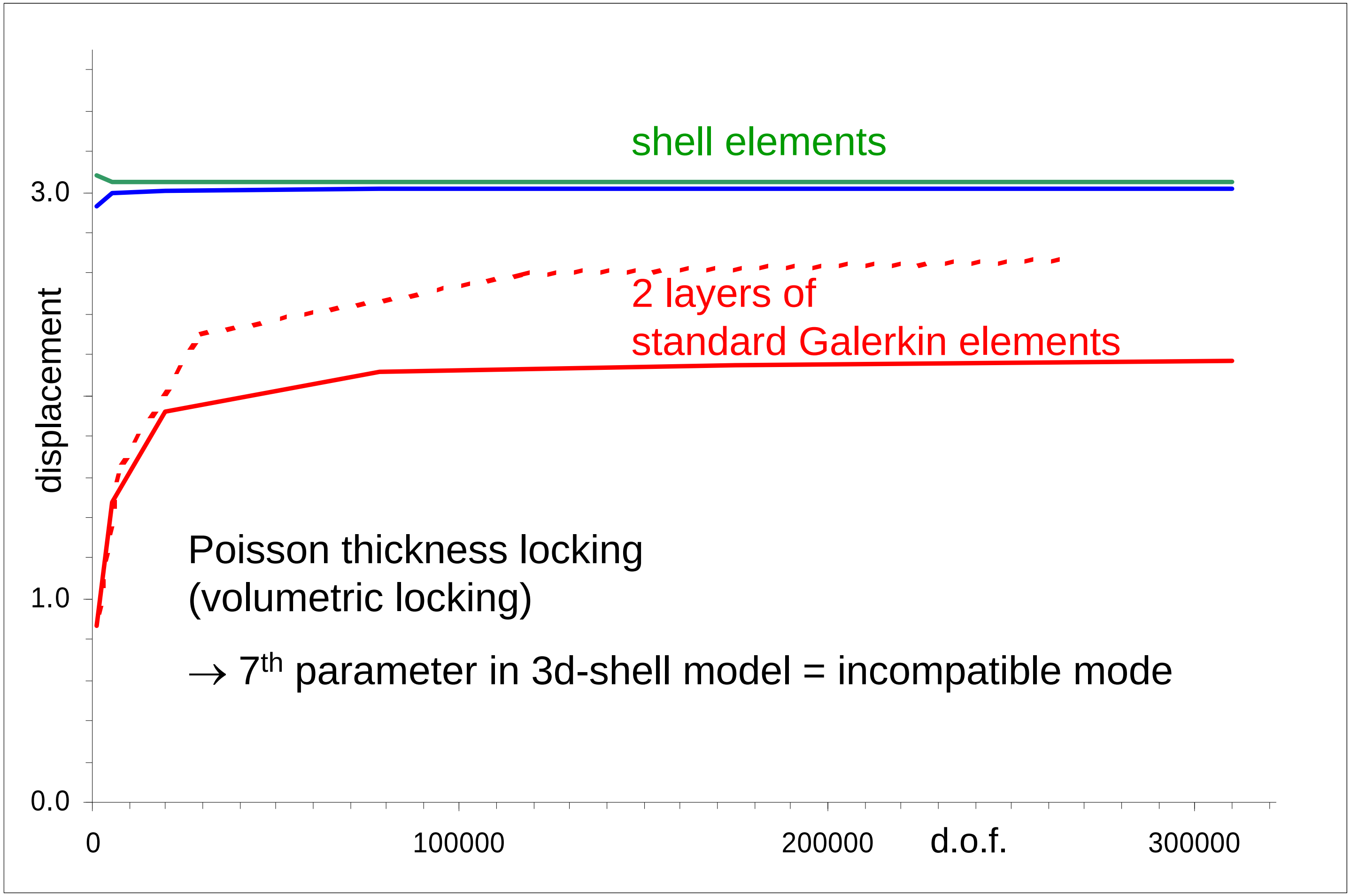
Shell Analysis with Standard Solid Elements

one layer of standard Galerkin elements yields wrong results



Shell Analysis with Standard Solid Elements

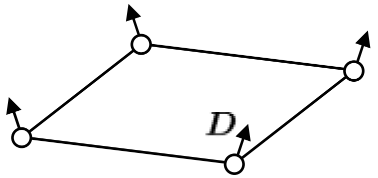
refinement in transverse direction helps (but is too expensive!)



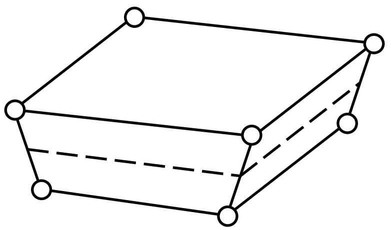
Three-dimensional Analysis of Shells

there are (at least) three different strategies

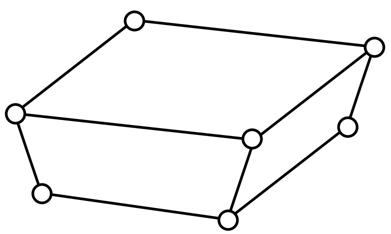
- **3d-shell (e.g. 7-parameter formulation)**
  two-dimensional mesh
  director + difference vector
  6 (+1) d.o.f. per node
  stress resultants



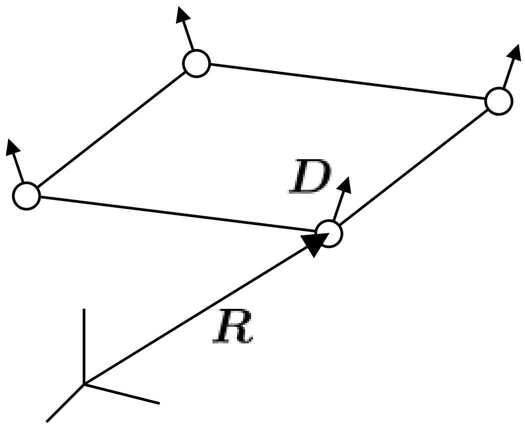
- **continuum shell (solid shell)**
  three-dimensional mesh
  3 d.o.f. per node (+ internal d.o.f.)
  stress resultants

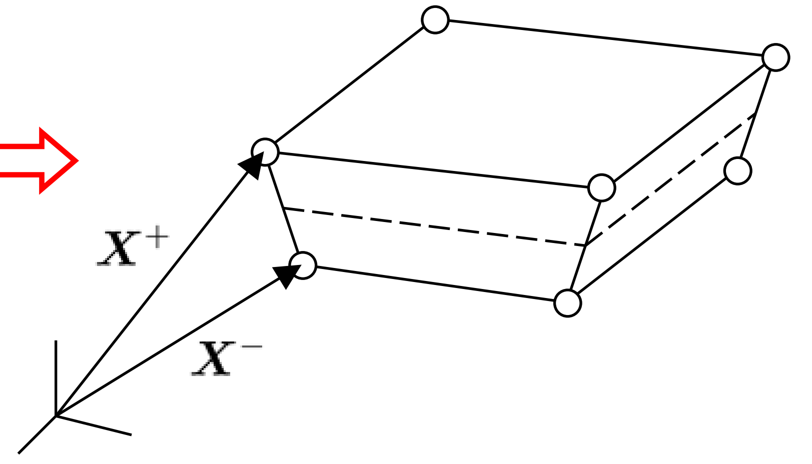


- **3d-solid (brick)**
  three-dimensional mesh
  3 d.o.f. per node (+ internal d.o.f.)
  3d stresses



Surface Oriented Formulation





nodes on upper and lower shell surface

$$\boldsymbol{X} = \boldsymbol{R} + \theta^3\,\boldsymbol{D} \quad \Rightarrow \quad \boldsymbol{X} = \frac{1}{2}\left(1-\theta^3\right)\boldsymbol{X}^- + \frac{1}{2}\left(1+\theta^3\right)\boldsymbol{X}^+$$

nodal displacements instead of difference vector

$$\boldsymbol{u} = \boldsymbol{v} + \theta^3\,\boldsymbol{w} \quad \Rightarrow \quad \boldsymbol{u} = \frac{1}{2}\left(1-\theta^3\right)\boldsymbol{u}^- + \frac{1}{2}\left(1+\theta^3\right)\boldsymbol{u}^+$$

+ 7th parameter for linear transverse normal strain distribution

Surface Oriented Formulation

membrane and bending strains

$$
\begin{aligned}
\varepsilon_{\alpha\beta} = \frac{1}{4}\Big[&\left(\boldsymbol{u}^-_{,\alpha} + \boldsymbol{u}^+_{,\alpha}\right)\cdot\left(\boldsymbol{X}^-_{,\beta} + \boldsymbol{X}^+_{,\beta}\right) + \left(\boldsymbol{u}^-_{,\beta} + \boldsymbol{u}^+_{,\beta}\right)\cdot\left(\boldsymbol{X}^-_{,\alpha} + \boldsymbol{X}^+_{,\alpha}\right)\Big] && \text{membrane}\\
&\left.\begin{aligned}
+\frac{1}{4}\theta^3\Big[&\left(\boldsymbol{u}^+_{,\alpha} - \boldsymbol{u}^-_{,\alpha}\right)\cdot\left(\boldsymbol{X}^-_{,\beta} + \boldsymbol{X}^+_{,\beta}\right) + \left(\boldsymbol{u}^+_{,\beta} - \boldsymbol{u}^-_{,\beta}\right)\cdot\left(\boldsymbol{X}^-_{,\alpha} + \boldsymbol{X}^+_{,\alpha}\right)\\
&+\left(\boldsymbol{u}^-_{,\alpha} + \boldsymbol{u}^+_{,\alpha}\right)\cdot\left(\boldsymbol{X}^+_{,\beta} - \boldsymbol{X}^-_{,\beta}\right) + \left(\boldsymbol{u}^-_{,\beta} + \boldsymbol{u}^+_{,\beta}\right)\cdot\left(\boldsymbol{X}^+_{,\alpha} - \boldsymbol{X}^-_{,\alpha}\right)\Big]
\end{aligned}\right\} && \text{bending}\\
&\xcancel{+\frac{1}{4}(\theta^3)^2\Big[\left(\boldsymbol{u}^+_{,\alpha} - \boldsymbol{u}^-_{,\alpha}\right)\cdot\left(\boldsymbol{X}^+_{,\beta} - \boldsymbol{X}^-_{,\beta}\right) + \left(\boldsymbol{u}^+_{,\beta} - \boldsymbol{u}^-_{,\beta}\right)\cdot\left(\boldsymbol{X}^+_{,\alpha} - \boldsymbol{X}^-_{,\alpha}\right)\Big]} && \text{higher order effects}
\end{aligned}
$$

$\rightarrow$ **“continuum shell” formulation**

Requirements

what we expect from finite elements for 3d-modeling of shells

- asymptotically correct (thickness $\rightarrow$ 0)
- numerically efficient for thin shells (locking-free)
- consistent (patch test)
- competitive to „usual“ 3d-elements for 3d-problems

required for both 3d-shell elements and solid elements for shells

A Hierarchy of Models

thin shell theory (Kirchhoff-Love, Koiter)
3-parameter model

$$\begin{bmatrix} \sigma_0^{\alpha\beta} \\ \sigma_0^{\alpha 3} \\ \sigma_0^{33} \\ \sigma_1^{\alpha\beta} \\ \sigma_1^{\alpha 3} \\ \sigma_1^{33} \end{bmatrix} = \begin{bmatrix} D_0^{\alpha\beta\gamma\delta} & D_0^{\alpha\beta\gamma 3} & D_0^{\alpha\beta 33} & D_1^{\alpha 3\gamma\delta} & D_1^{\alpha\beta\gamma 3} & D_1^{\alpha\beta 33} \\ D_0^{\alpha 3\gamma\delta} & D_0^{\alpha 3\gamma 3} & D_0^{\alpha 333} & D_1^{\alpha 3\gamma\delta} & D_1^{\alpha 3\gamma 3} & D_1^{\alpha 333} \\ D_0^{33\gamma\delta} & D_0^{33\gamma 3} & D_0^{3333} & D_1^{33\gamma\delta} & D_1^{33\gamma 3} & D_1^{3333} \\ D_1^{\alpha\beta\gamma\delta} & D_1^{\alpha\beta\gamma 3} & D_1^{\alpha\beta 33} & D_2^{\alpha\beta\gamma\delta} & D_2^{\alpha\beta\gamma 3} & D_2^{\alpha\beta 33} \\ D_1^{\alpha 3\gamma\delta} & D_1^{\alpha 3\gamma 3} & D_1^{\alpha 333} & D_2^{\alpha 3\gamma\delta} & D_2^{\alpha 3\gamma 3} & D_2^{\alpha 333} \\ D_1^{33\gamma\delta} & D_1^{33\gamma 3} & D_1^{3333} & D_2^{33\gamma\delta} & D_2^{33\gamma 3} & D_2^{3333} \end{bmatrix} \cdot \begin{bmatrix} \varepsilon_{\gamma\delta}^0 \\ \cancel{\varepsilon_{\gamma 3}^0} \\ \boxed{0} \\ \varepsilon_{\gamma\delta}^1 \\ \boxed{0} \\ \boxed{0} \end{bmatrix}$$

modification of material law required

A Hierarchy of Models

first order shear deformation theory (Reissner/Mindlin, Naghdi)
5-parameter model

$$\begin{bmatrix} \sigma_0^{\alpha\beta} \\ \sigma_0^{\alpha 3} \\ \sigma_0^{33} \\ \sigma_1^{\alpha\beta} \\ \sigma_1^{\alpha 3} \\ \sigma_1^{33} \end{bmatrix} = \begin{bmatrix} D_0^{\alpha\beta\gamma\delta} & D_0^{\alpha\beta\gamma 3} & D_0^{\alpha\beta 33} & D_1^{\alpha 3\gamma\delta} & D_1^{\alpha\beta\gamma 3} & D_1^{\alpha\beta 33} \\ D_0^{\alpha 3\gamma\delta} & D_0^{\alpha 3\gamma 3} & D_0^{\alpha 333} & D_1^{\alpha 3\gamma\delta} & D_1^{\alpha 3\gamma 3} & D_1^{\alpha 333} \\ D_0^{33\gamma\delta} & D_0^{33\gamma 3} & D_0^{3333} & D_1^{33\gamma\delta} & D_1^{33\gamma 3} & D_1^{3333} \\ D_1^{\alpha\beta\gamma\delta} & D_1^{\alpha\beta\gamma 3} & D_1^{\alpha\beta 33} & D_2^{\alpha\beta\gamma\delta} & D_2^{\alpha\beta\gamma 3} & D_2^{\alpha\beta 33} \\ D_1^{\alpha 3\gamma\delta} & D_1^{\alpha 3\gamma 3} & D_1^{\alpha 333} & D_2^{\alpha 3\gamma\delta} & D_2^{\alpha 3\gamma 3} & D_2^{\alpha 333} \\ D_1^{33\gamma\delta} & D_1^{33\gamma 3} & D_1^{3333} & D_2^{33\gamma\delta} & D_2^{33\gamma 3} & D_2^{3333} \end{bmatrix} \cdot \begin{bmatrix} \varepsilon_{\gamma\delta}^0 \\ \varepsilon_{\gamma 3}^0 \\ \boxed{0} \\ \varepsilon_{\gamma\delta}^1 \\ \boxed{0} \\ \boxed{0} \end{bmatrix}$$

modification of material law required

A Hierarchy of Models

shear deformable shell + thickness change
6-parameter model

$$
\begin{bmatrix} \sigma_0^{\alpha\beta} \\ \sigma_0^{\alpha 3} \\ \sigma_0^{33} \\ \sigma_1^{\alpha\beta} \\ \sigma_1^{\alpha 3} \\ \sigma_1^{33} \end{bmatrix}
=
\begin{bmatrix}
D_0^{\alpha\beta\gamma\delta} & D_0^{\alpha\beta\gamma 3} & D_0^{\alpha\beta 33} & D_1^{\alpha 3\gamma\delta} & D_1^{\alpha\beta\gamma 3} & D_1^{\alpha\beta 33} \\
D_0^{\alpha 3\gamma\delta} & D_0^{\alpha 3\gamma 3} & D_0^{\alpha 333} & D_1^{\alpha 3\gamma\delta} & D_1^{\alpha 3\gamma 3} & D_1^{\alpha 333} \\
D_0^{33\gamma\delta} & D_0^{33\gamma 3} & D_0^{3333} & D_1^{33\gamma\delta} & D_1^{33\gamma 3} & D_1^{3333} \\
D_1^{\alpha\beta\gamma\delta} & D_1^{\alpha\beta\gamma 3} & D_1^{\alpha\beta 33} & D_2^{\alpha\beta\gamma\delta} & D_2^{\alpha\beta\gamma 3} & D_2^{\alpha\beta 33} \\
D_1^{\alpha 3\gamma\delta} & D_1^{\alpha 3\gamma 3} & D_1^{\alpha 333} & D_2^{\alpha 3\gamma\delta} & D_2^{\alpha 3\gamma 3} & D_2^{\alpha 333} \\
D_1^{33\gamma\delta} & D_1^{33\gamma 3} & D_1^{3333} & D_2^{33\gamma\delta} & D_2^{33\gamma 3} & D_2^{3333}
\end{bmatrix}
\cdot
\begin{bmatrix} \varepsilon_{\gamma\delta}^0 \\ \varepsilon_{\gamma 3}^0 \\ \varepsilon_{33}^0 \\ \varepsilon_{\gamma\delta}^1 \\ \varepsilon_{\gamma 3}^1 \\ \boxed{0} \end{bmatrix}
$$

asymptotically correct for membrane state

A Hierarchy of Models

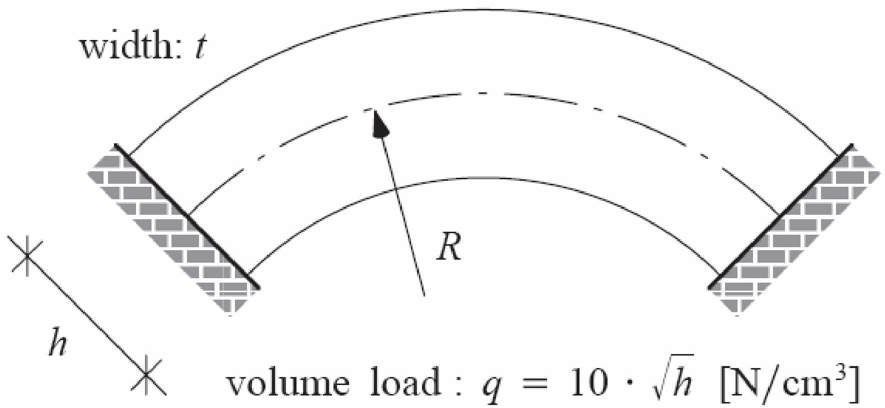
shear deformable shell + linear thickness change
7-parameter model

$$
\begin{bmatrix}
\sigma_0^{\alpha\beta} \\ \sigma_0^{\alpha 3} \\ \sigma_0^{33} \\ \sigma_1^{\alpha\beta} \\ \sigma_1^{\alpha 3} \\ \sigma_1^{33} \\ \sigma_2^{\alpha\beta} \\ \sigma_2^{\alpha 3} \\ \sigma_2^{33}
\end{bmatrix}
=
\begin{bmatrix}
D_0^{\alpha\beta\gamma\delta} & D_0^{\alpha\beta\gamma 3} & D_0^{\alpha\beta 33} & D_1^{\alpha 3\gamma\delta} & D_1^{\alpha\beta\gamma 3} & D_1^{\alpha\beta 33} & D_2^{\alpha 3\gamma\delta} & D_2^{\alpha\beta\gamma 3} & D_2^{\alpha\beta 33} \\
D_0^{\alpha 3\gamma\delta} & D_0^{\alpha 3\gamma 3} & D_0^{\alpha 333} & D_1^{\alpha 3\gamma\delta} & D_1^{\alpha 3\gamma 3} & D_1^{\alpha 333} & D_2^{\alpha 3\gamma\delta} & D_2^{\alpha 3\gamma 3} & D_2^{\alpha 333} \\
D_0^{33\gamma\delta} & D_0^{33\gamma 3} & D_0^{3333} & D_1^{33\gamma\delta} & D_1^{33\gamma 3} & D_1^{3333} & D_2^{33\gamma\delta} & D_2^{33\gamma 3} & D_2^{3333} \\
D_1^{\alpha\beta\gamma\delta} & D_1^{\alpha\beta\gamma 3} & D_1^{\alpha\beta 33} & D_2^{\alpha\beta\gamma\delta} & D_2^{\alpha\beta\gamma 3} & D_2^{\alpha\beta 33} & D_3^{\alpha\beta\gamma\delta} & D_3^{\alpha\beta\gamma 3} & D_3^{\alpha\beta 33} \\
D_1^{\alpha 3\gamma\delta} & D_1^{\alpha 3\gamma 3} & D_1^{\alpha 333} & D_2^{\alpha 3\gamma\delta} & D_2^{\alpha 3\gamma 3} & D_2^{\alpha 333} & D_3^{\alpha 3\gamma\delta} & D_3^{\alpha 3\gamma 3} & D_3^{\alpha 333} \\
D_1^{33\gamma\delta} & D_1^{33\gamma 3} & D_1^{3333} & D_2^{33\gamma\delta} & D_2^{33\gamma 3} & D_2^{3333} & D_3^{33\gamma\delta} & D_3^{33\gamma 3} & D_3^{3333} \\
D_2^{\alpha\beta\gamma\delta} & D_2^{\alpha\beta\gamma 3} & D_2^{\alpha\beta 33} & D_3^{\alpha\beta\gamma\delta} & D_3^{\alpha\beta\gamma 3} & D_3^{\alpha\beta 33} & D_4^{\alpha\beta\gamma\delta} & D_4^{\alpha\beta\gamma 3} & D_4^{\alpha\beta 33} \\
D_2^{\alpha 3\gamma\delta} & D_2^{\alpha 3\gamma 3} & D_2^{\alpha 333} & D_3^{\alpha 3\gamma\delta} & D_3^{\alpha 3\gamma 3} & D_3^{\alpha 333} & D_4^{\alpha 3\gamma\delta} & D_4^{\alpha 3\gamma 3} & D_4^{\alpha 333} \\
D_2^{33\gamma\delta} & D_2^{33\gamma 3} & D_2^{3333} & D_3^{33\gamma\delta} & D_3^{33\gamma 3} & D_3^{3333} & D_4^{33\gamma\delta} & D_4^{33\gamma 3} & D_4^{3333}
\end{bmatrix}
\cdot
\begin{bmatrix}
\varepsilon_{\gamma\delta}^0 \\ \varepsilon_{\gamma 3}^0 \\ \varepsilon_{33}^0 \\ \varepsilon_{\gamma\delta}^1 \\ \varepsilon_{\gamma 3}^1 \\ \varepsilon_{33}^1 \\ 0 \\ \varepsilon_{\gamma 3}^2 \\ 0
\end{bmatrix}
$$

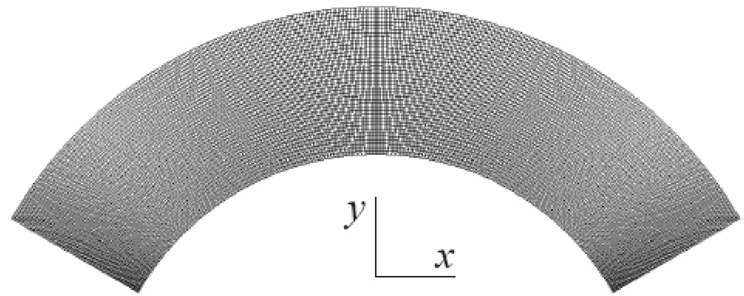
asymptotically correct for membrane +bending

Numerical Experiment (Two-dimensional)

a two-dimensional example: discretization of a beam with 2d-solids



volume load : $q = 10 \cdot \sqrt{h}$ [N/cm^3]



mesh for reference solution

geometry:

$R = 50$ cm

$t = 1$ cm

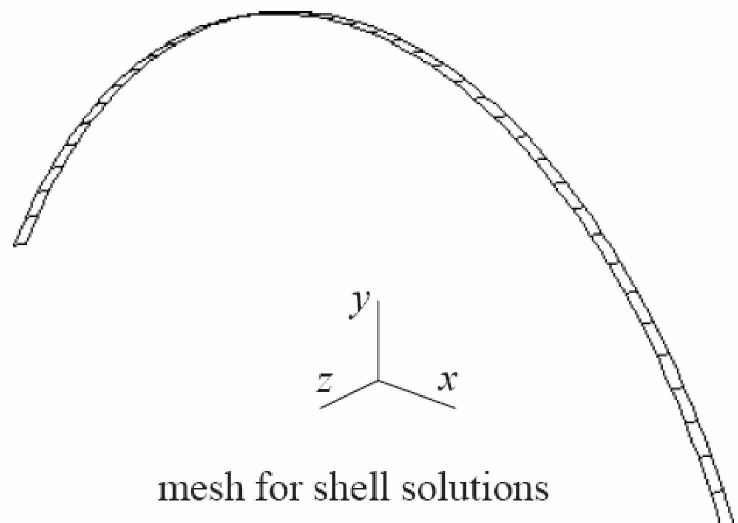
$h = 50 \ldots 0.5$ cm

material:

$E = 21000$ kN/cm^2

$\nu = 0.3$

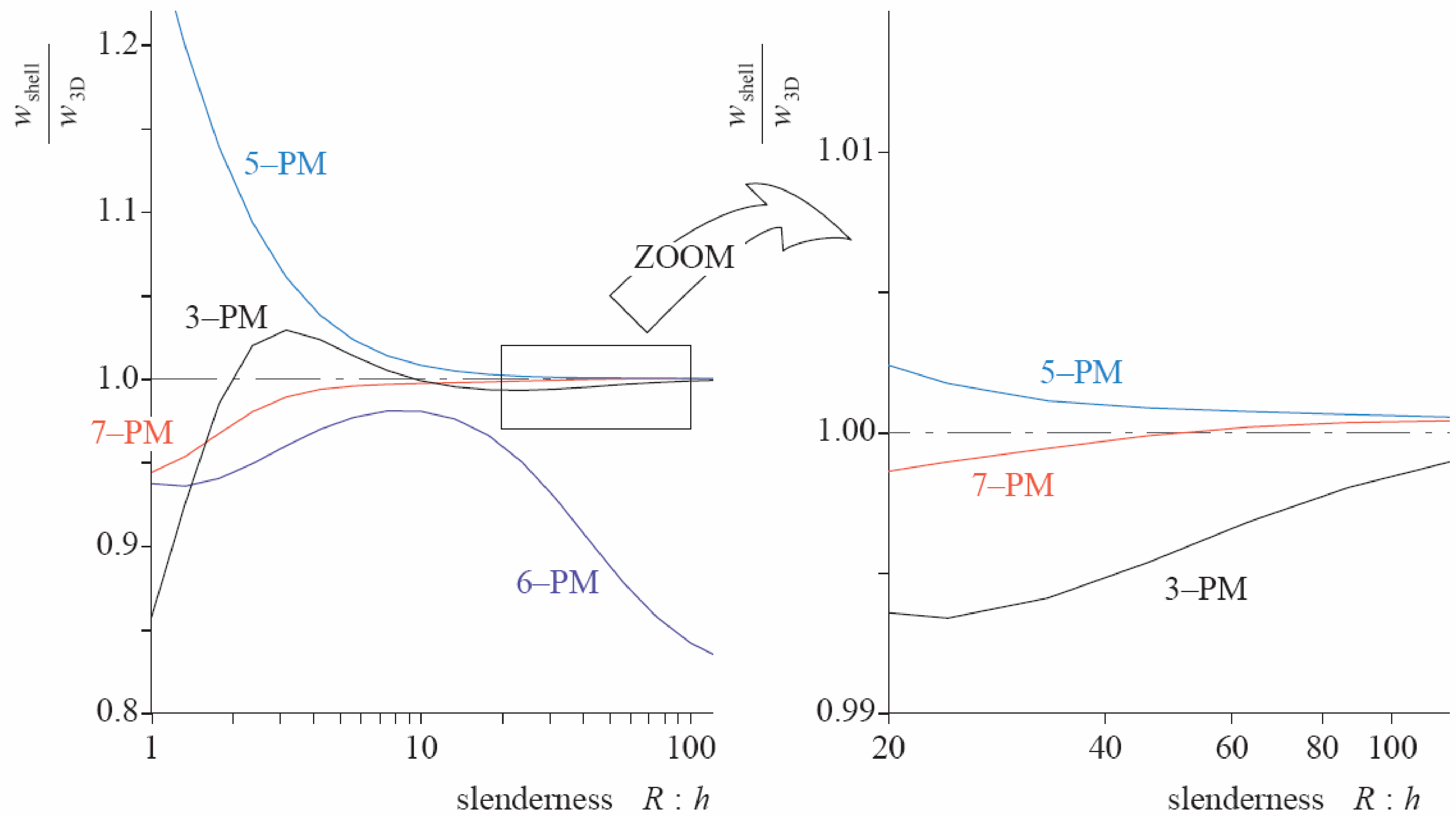
plane strain conditions in z–direction



mesh for shell solutions

Numerical Experiment (Two-dimensional)

a two-dimensional example: discretization of a beam with 2d-solids



Requirements

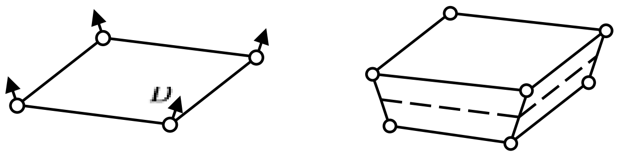
what we expect from finite elements for 3d-modeling of shells

- asymptotically correct (thickness $\rightarrow$ 0)
- numerically efficient for thin shells (locking-free)
- consistent (patch test)
- competitive to „usual“ 3d-elements for 3d-problems

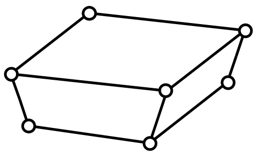
required for both 3d-shell elements and solid elements for shells

Locking Phenomena

3d-shell/continuum shell vs. solid



3d-shell / continuum shell

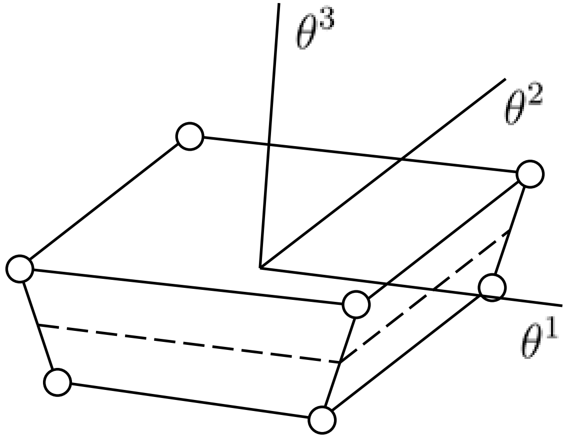


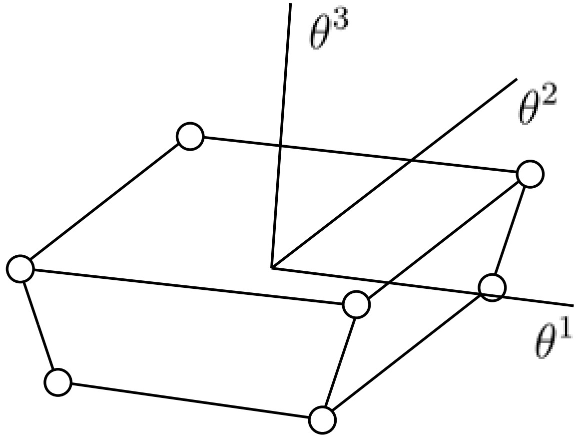
3d-solid

in-plane shear locking	shear locking
transverse shear locking	
membrane locking	(membrane locking)
Poisson thickness locking	volumetric locking
curvature thickness locking	**trapezoidal locking**

Comparison: Continuum Shell vs. 3d-solid

differences with respect to finite element technology and underlying shell theory

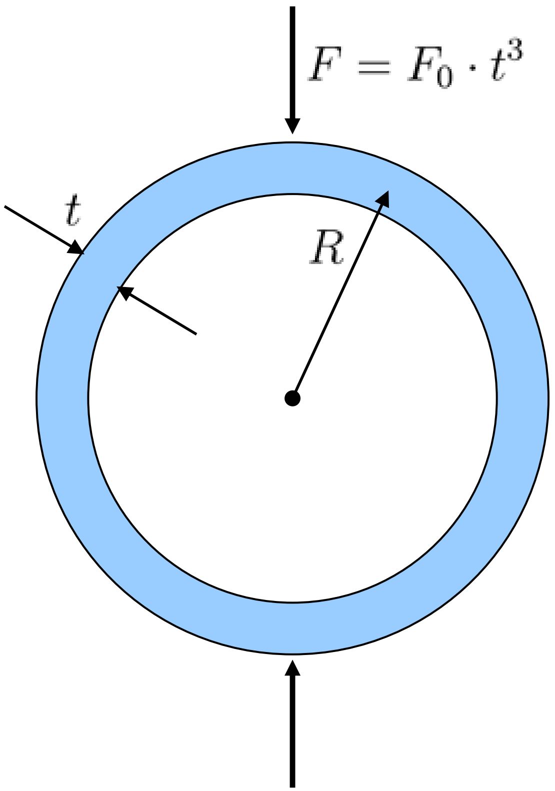


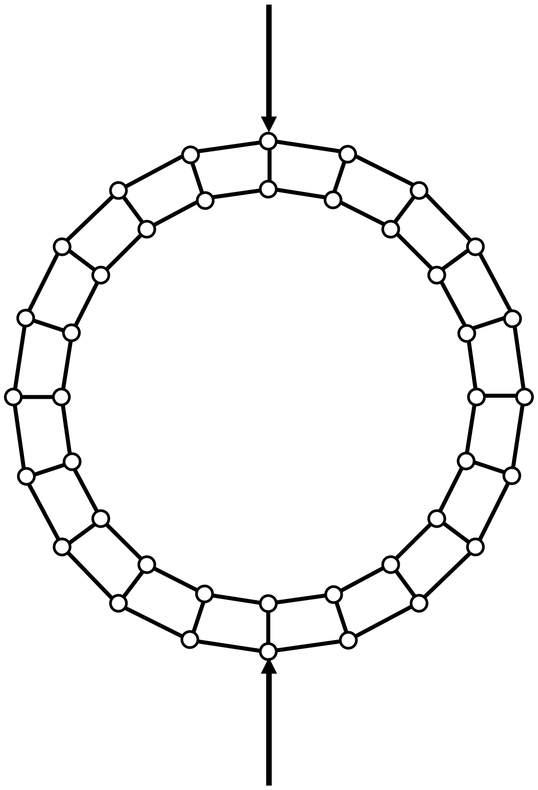


continuum shell	**3d-solid**
stress resultants	stresses
distinct thickness direction	all directions are equal
$\varepsilon_{\alpha\beta}$ linear in θ^3	$\varepsilon_{\alpha\beta}$ quadratic in θ^3

Trapezoidal Locking (Curvature Thickness locking)

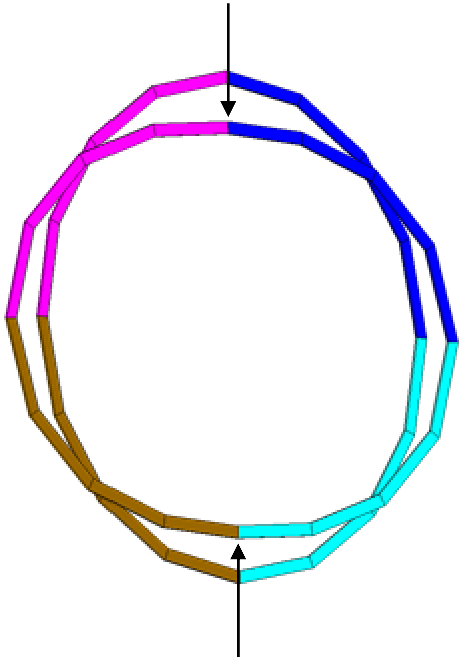
numerical example: pinched ring

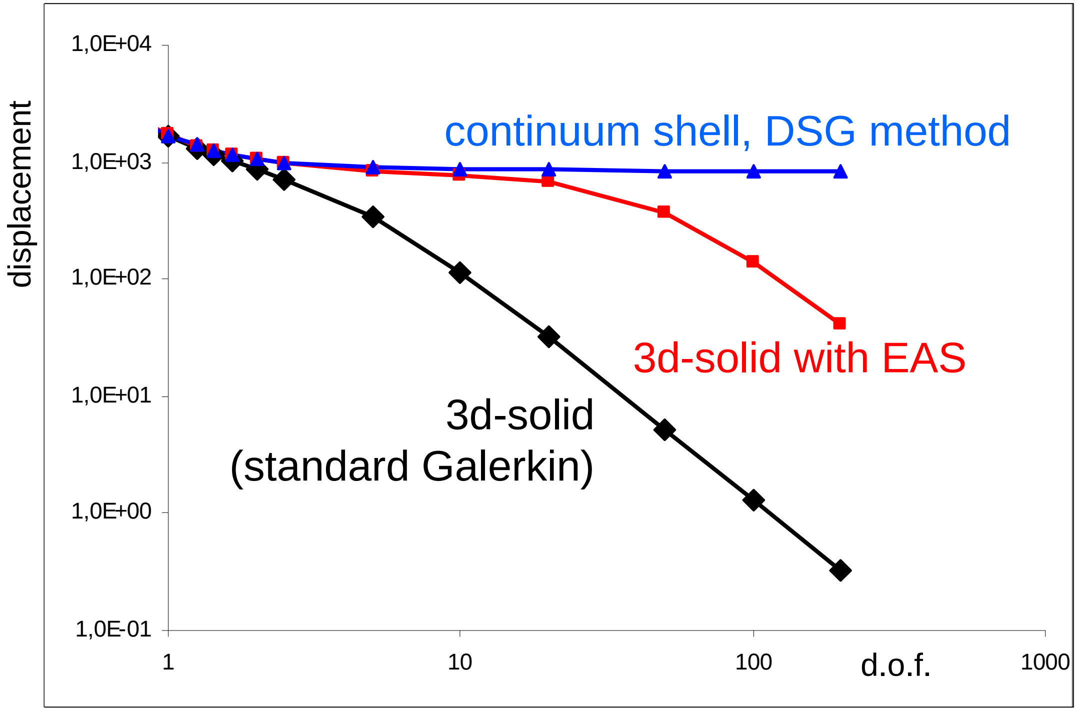




Trapezoidal Locking (Curvature Thickness locking)

numerical example: pinched ring

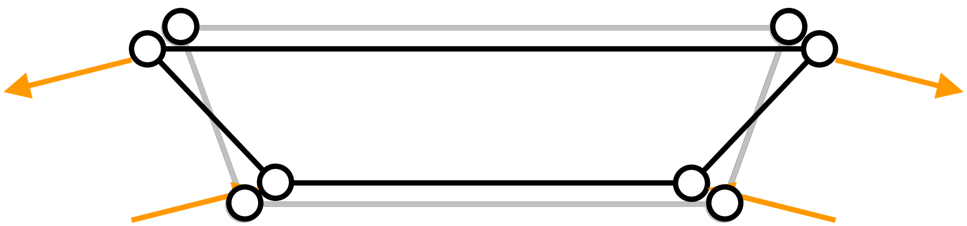


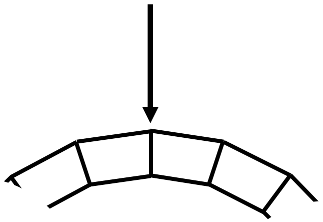


Trapezoidal Locking (Curvature Thickness locking)

origin of locking-phenomenon explained geometrically

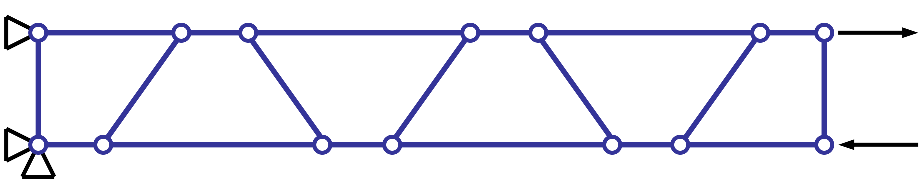
pure „bending“ of an initially curved element





…leads to artificial transverse normal strains and stresses

trapezoidal locking ↔ distortion sensitivity

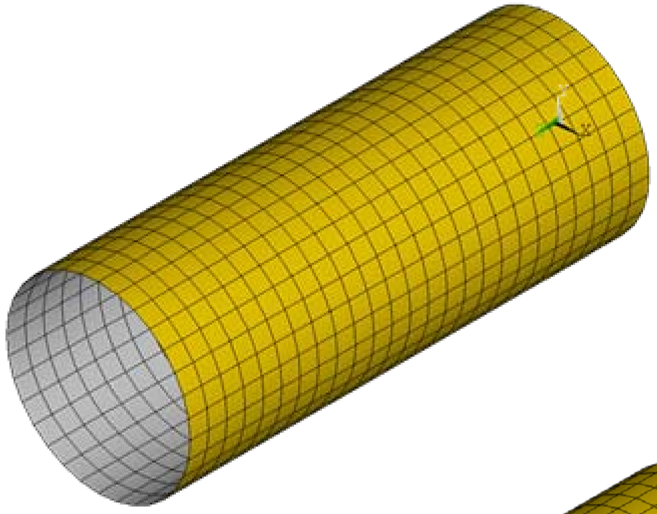


Cylindrical Shell Subject to External Pressure

slenderness $R/t = 100$

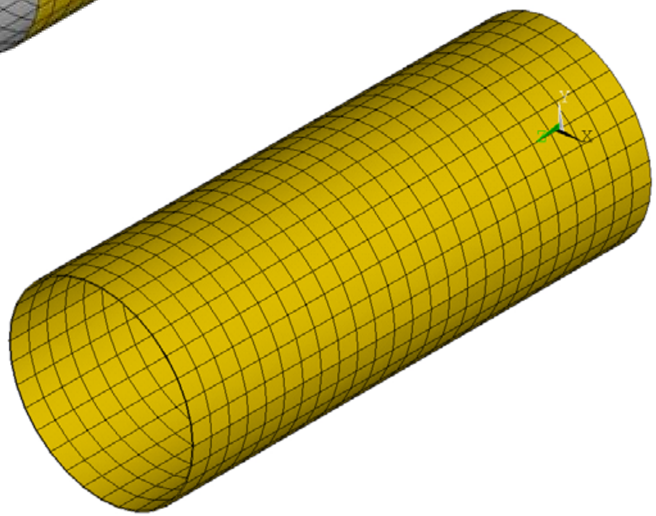
shell elements

coarse mesh, 4608 d.o.f.



3d-solid elements

coarse mesh, 4608 d.o.f.



Cylindrical Shell Subject to External Pressure

slenderness $R/t = 100$

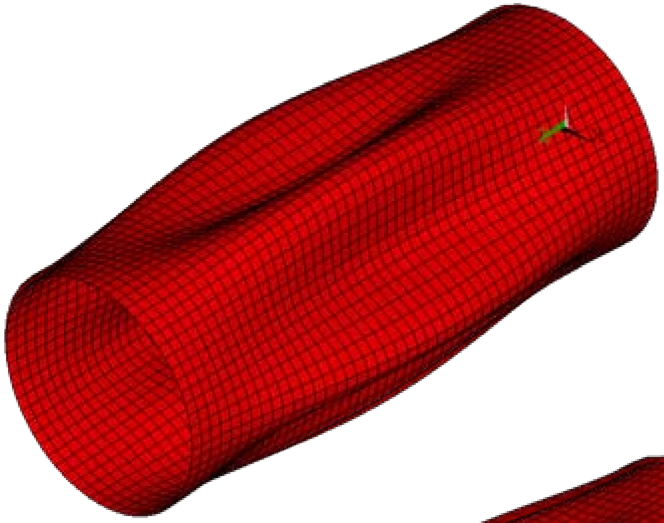
shell elements

coarse mesh, 4608 d.o.f.

$$\lambda_{\text{crit}} = 0.58$$

fine mesh, 18816 d.o.f.

$$\lambda_{\text{crit}} = 0.57$$



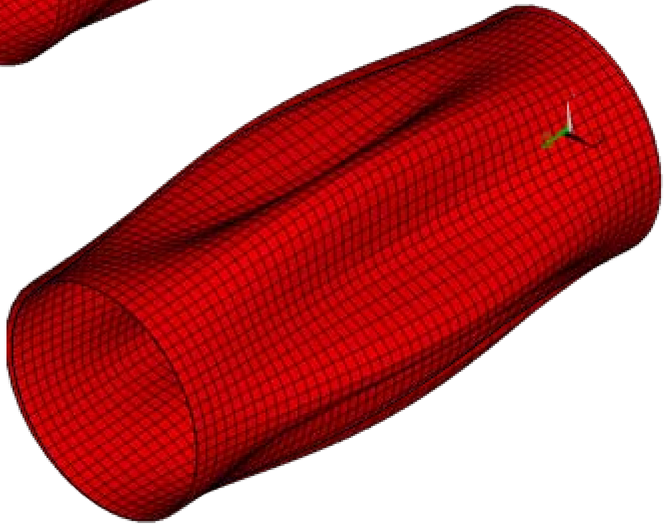
3d-solid elements

coarse mesh, 4608 d.o.f.

$$\lambda_{\text{crit}} = 0.95$$

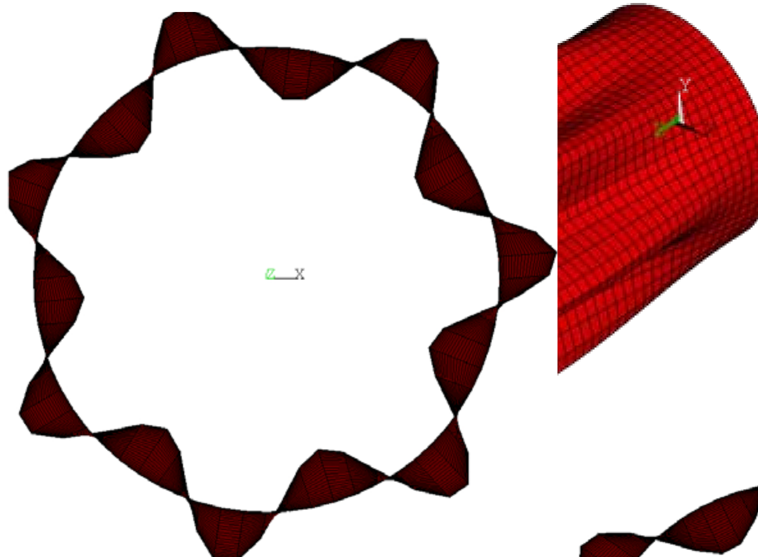
fine mesh, 18816 d.o.f.

$$\lambda_{\text{crit}} = 0.58$$



Cylindrical Shell Subject to External Pressure

slenderness $R/t = 500$



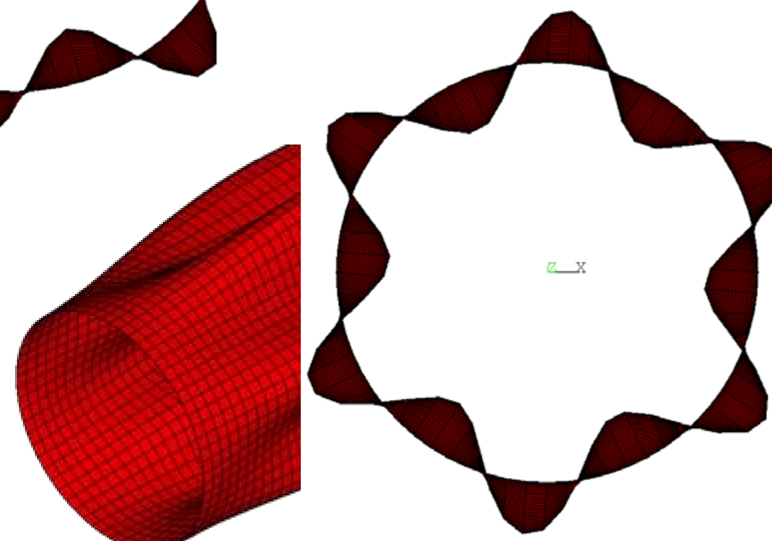
shell elements

coarse mesh, 4608 d.o.f.

$\lambda_{\text{crit}} = 0.01$

fine mesh, 18816 d.o.f.

$\lambda_{\text{crit}} = 0.01$



3d-solid elements

coarse mesh, 4608 d.o.f.

$\lambda_{\text{crit}} = 0.13$ **factor 13!**

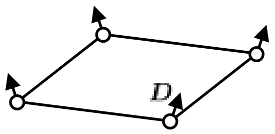
fine mesh, 4608 d.o.f.

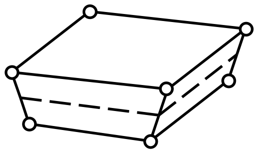
$\lambda_{\text{crit}} = 0.017$ **still 70% error!**

due to trapezoidal locking

Finite Element Technology: Summary

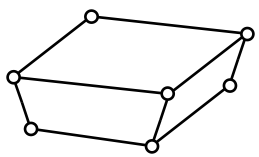
3d-shell, continuum shell, solid shell,…





- stress resultants allow separate treatment of membrane and bending terms
- „anisotropic“ element technology (trapezoidal locking)

3d-solid (brick)



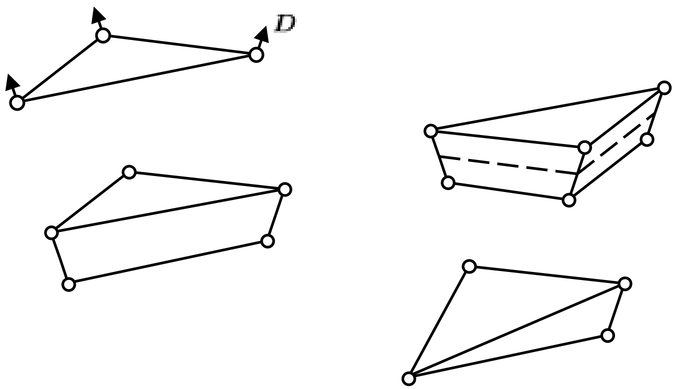
- no “transverse” direction
  no distinction of membrane / bending
- (usually) suffer from trapezoidal locking

general

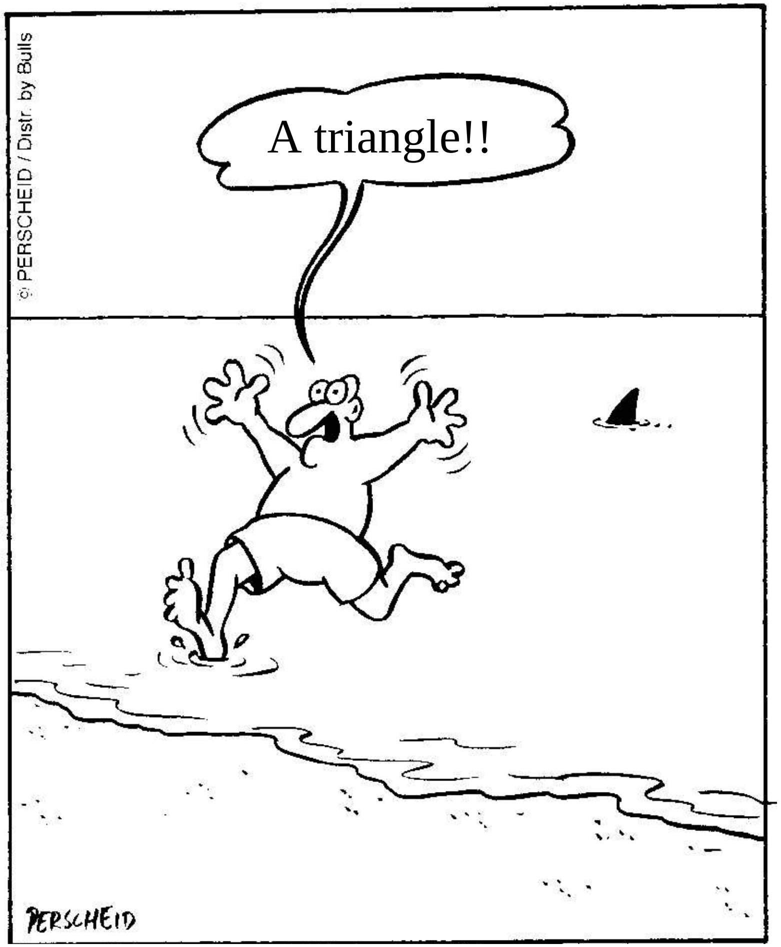
- effective methods for transverse shear locking available
- membrane locking mild when (bi-) linear shape functions are used

Finite Element Technology: Summary

triangles, tetrahedrons and wedges



- tetrahedrons: hopeless
- wedges: may be o.k.
  in transverse direction
- problem: meshing with hexahedrons
  extremely demanding



Requirements

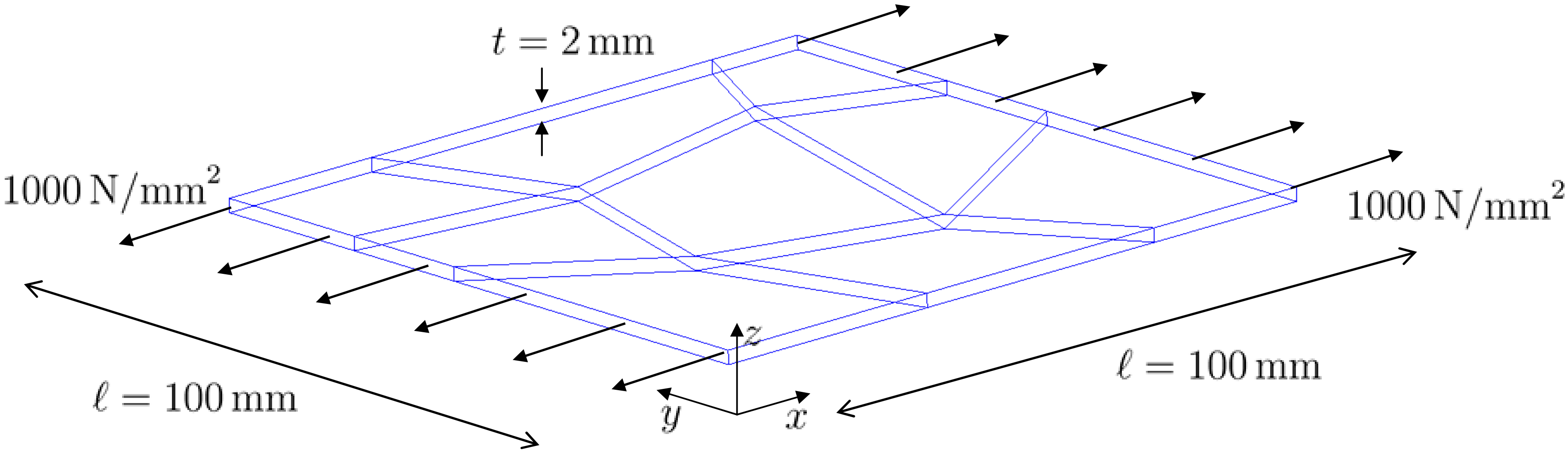
what we expect from finite elements for 3d-modeling of shells

- asymptotically correct (thickness $\rightarrow$ 0)
- numerically efficient for thin shells (locking-free)
- consistent (patch test)
- competitive to „usual“ 3d-elements for 3d-problems

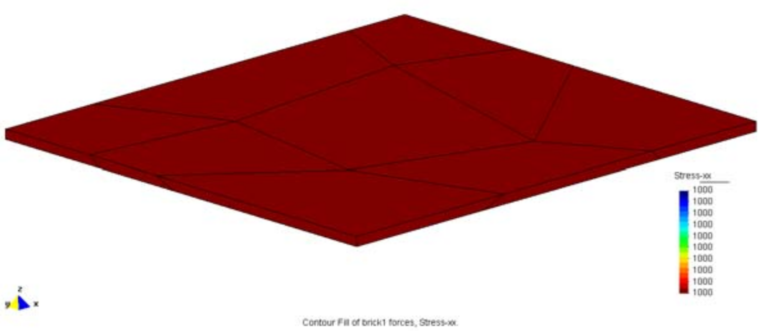
required for both 3d-shell elements and solid elements for shells

Fundamental Requirement: The Patch Test

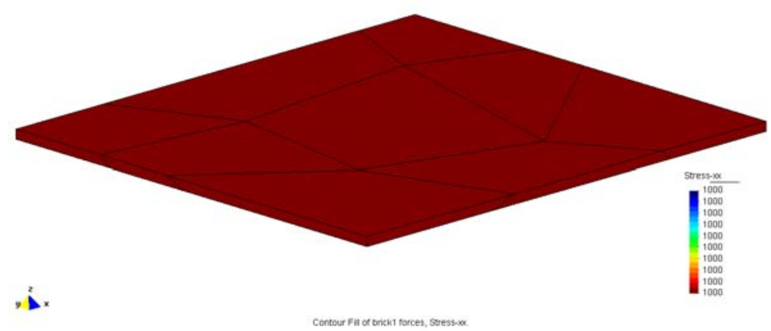
one layer of 3d-elements, σ_x = const.



3d-solid

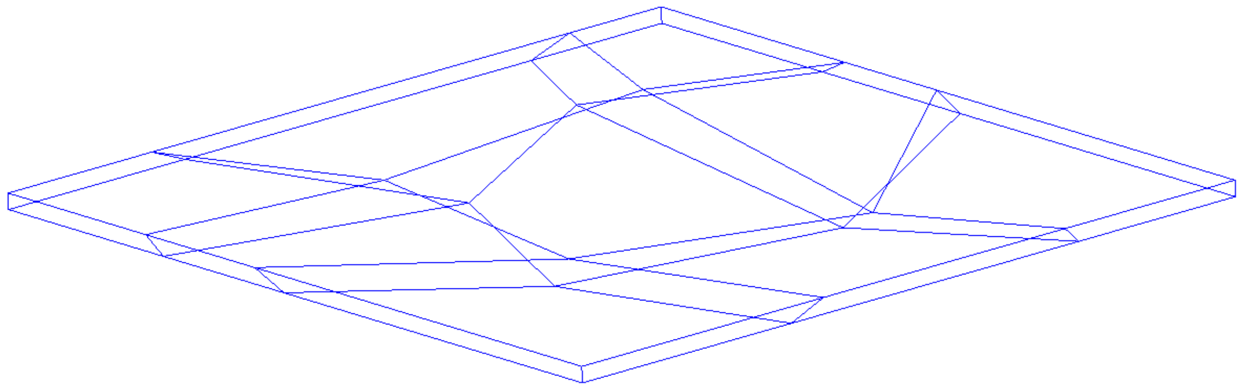


continuum shell, DSG

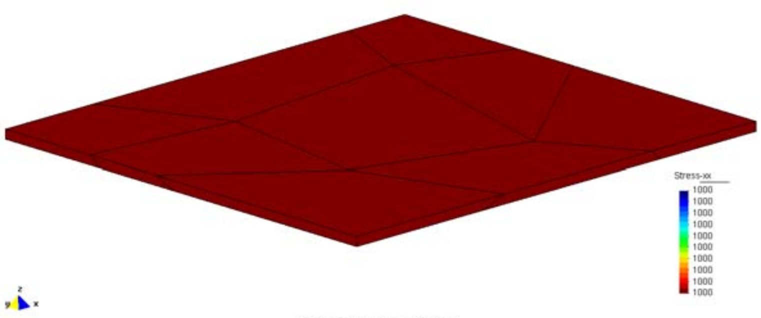


Fundamental Requirement: The Patch Test

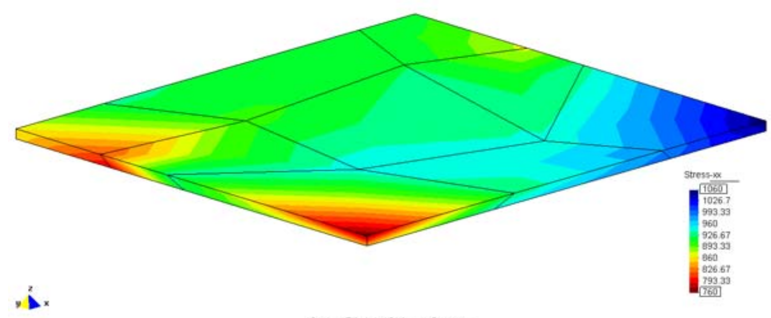
one layer of 3d-elements, σ_x = const., directors skewed



3d-solid



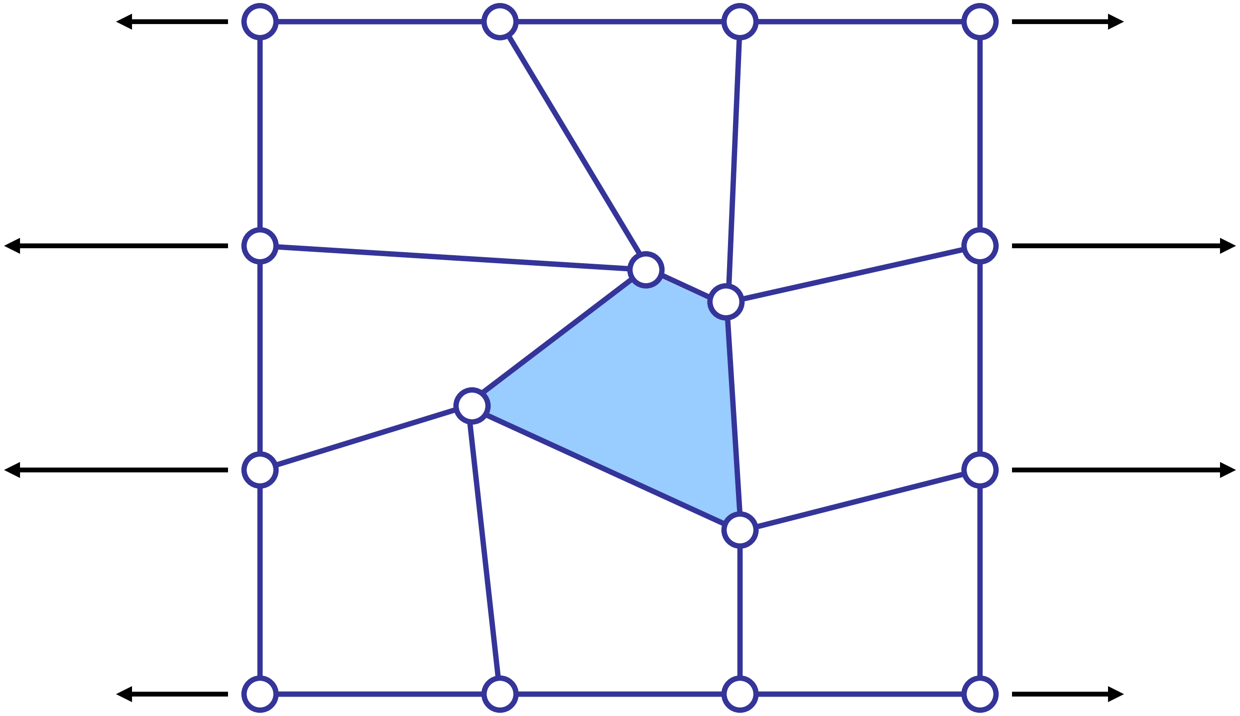
continuum shell, DSG



Two-dimensional Model Problem

the fundamental dilemma of finite element technology

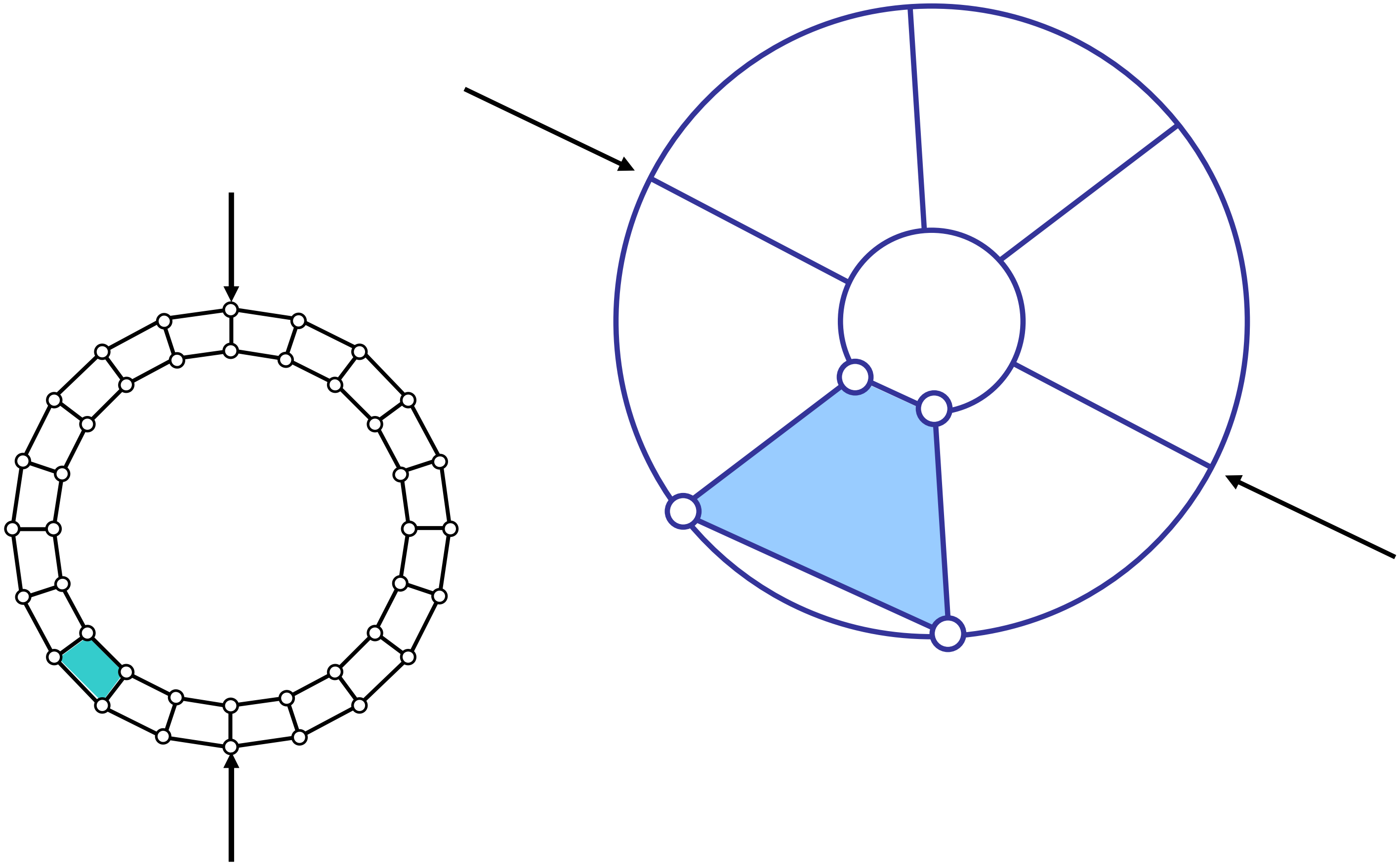
modeling constant stresses



Two-dimensional Model Problem

the fundamental dilemma of finite element technology

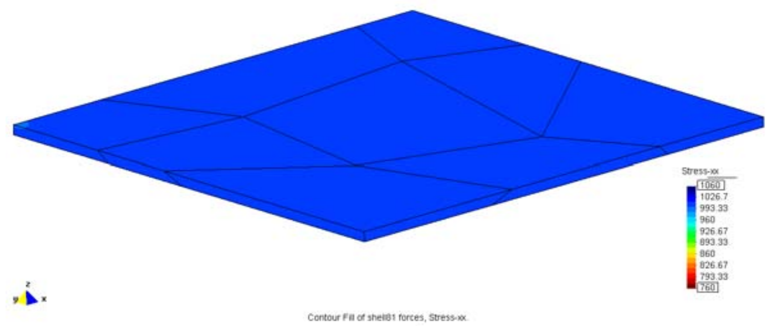
…or pure bending?



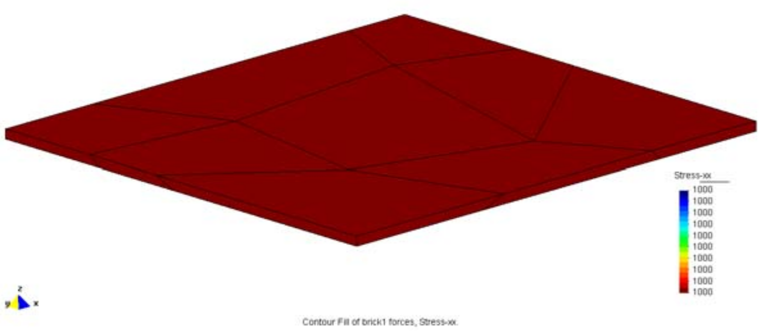
Fundamental Requirement: The Patch Test

one layer of 3d-elements, σ_x = const., directors skewed

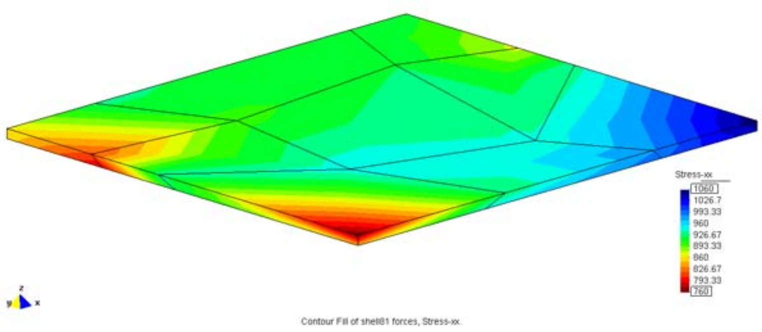
continuum shell, no DSG (trapezoidal locking in bending)



3d-solid



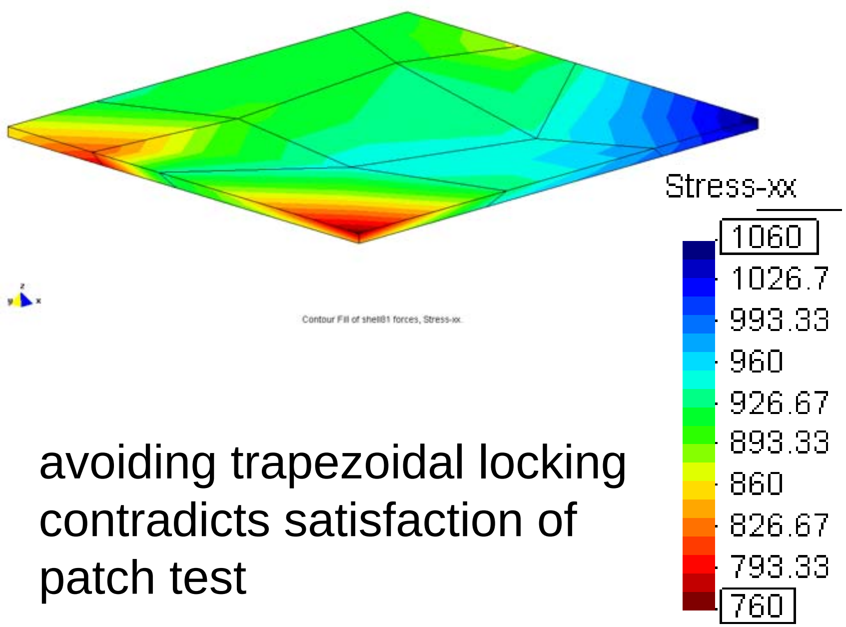
continuum shell, DSG



Fundamental Requirement: The Patch Test

same computational results, different scales for visualization

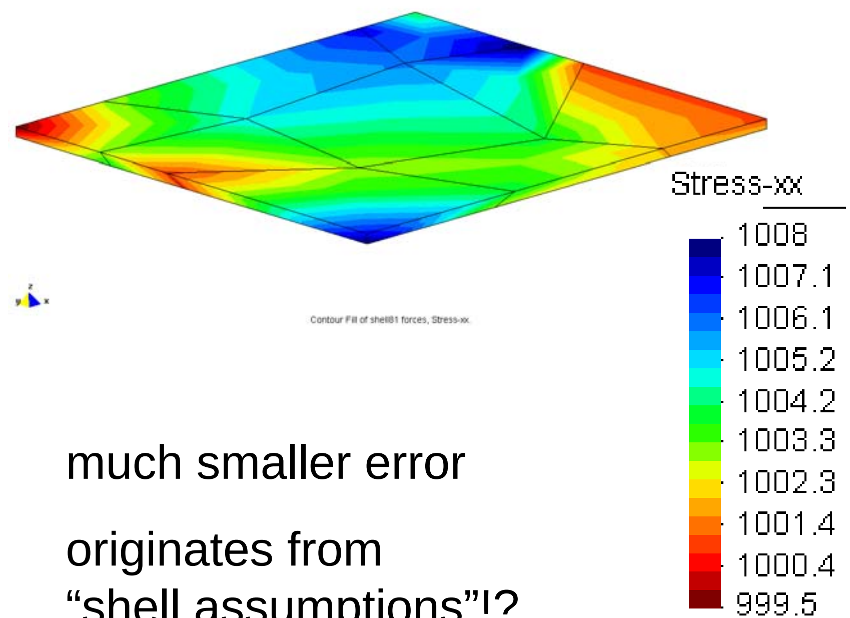
continuum shell, DSG



avoiding trapezoidal locking contradicts satisfaction of patch test

(known since long,
e.g. R. McNeal text book)

continuum shell, no DSG (trapezoidal locking in bending)



much smaller error

originates from
“shell assumptions”!?

Fundamental Requirement: The Patch Test

curvilinear components of strain tensor

$$\boldsymbol{\varepsilon} = \varepsilon \ldots \left(\boldsymbol{e}^l \cdot \boldsymbol{g}_j\right)$$

in-plane strain components

$$\varepsilon_{\alpha\beta} = \frac{1}{2}\left[\left(\boldsymbol{v}_{,\alpha} + \theta^3 \boldsymbol{w}_{,\alpha}\right) \cdot \left(\boldsymbol{R}_{,\beta} + \theta^3 \boldsymbol{D}_{,\beta}\right) + \left(\boldsymbol{v}_{,\beta} + \theta^3 \boldsymbol{w}_{,\beta}\right) \cdot \left(\boldsymbol{R}_{,\alpha} + \theta^3 \boldsymbol{D}_{,\alpha}\right)\right]$$

$$= \frac{1}{2}\left(\boldsymbol{v}_{,\alpha} \cdot \boldsymbol{R}_{,\beta} + \boldsymbol{v}_{,\beta} \cdot \boldsymbol{R}_{,\alpha}\right) \quad \text{membrane}$$

$$+ \frac{1}{2}\theta^3\left(\boldsymbol{v}_{,\alpha} \cdot \boldsymbol{D}_{,\beta} + \boldsymbol{w}_{,\alpha} \cdot \boldsymbol{R}_{,\beta} \boldsymbol{v}_{,\beta} \cdot \boldsymbol{R}_{,\alpha} + \boldsymbol{w}_{,\beta} \cdot \boldsymbol{R}_{,\alpha}\right) \quad \text{bending}$$

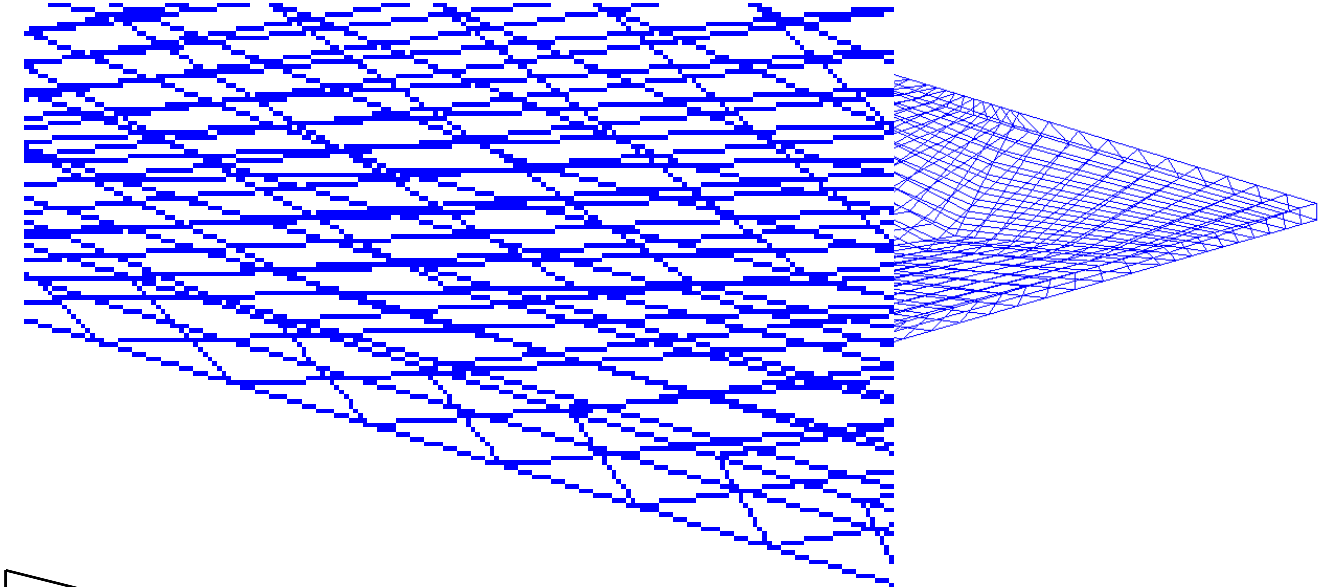
$$\cancel{+ \frac{1}{2}(\theta^3)^2\left(\boldsymbol{w}_{,\alpha} \cdot \boldsymbol{D}_{,\beta} + \boldsymbol{w}_{,\beta} \cdot \boldsymbol{D}_{,\alpha}\right)} \quad \text{higher order effects}$$

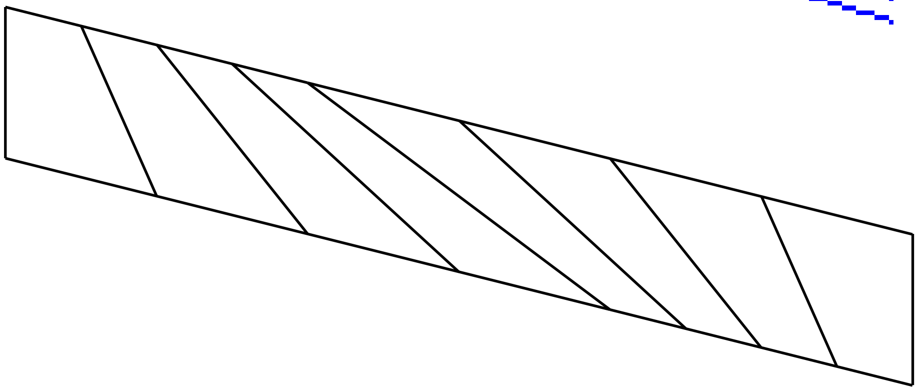
$$\hat{\varepsilon}_{11} = \varepsilon_{kl}\left(\boldsymbol{e}^k \cdot \boldsymbol{g}_1\right)\left(\boldsymbol{e}^l \cdot \boldsymbol{g}_1\right) \qquad \boldsymbol{g}_1 := \frac{\partial \boldsymbol{x}}{\partial \theta^1} = \begin{bmatrix} \frac{1}{4}\left(21 + \theta^3\right) \\ 0 \\ 0 \end{bmatrix}$$

$$\hat{\varepsilon}_{11} = \varepsilon_{11}\left(\frac{21}{4} + \frac{\theta^3}{4}\right)^2 = \varepsilon_{11}\left(\frac{441}{16} + \frac{21}{8}\theta^3 + \frac{1}{6}(\theta^3)^2\right)$$

Convergence

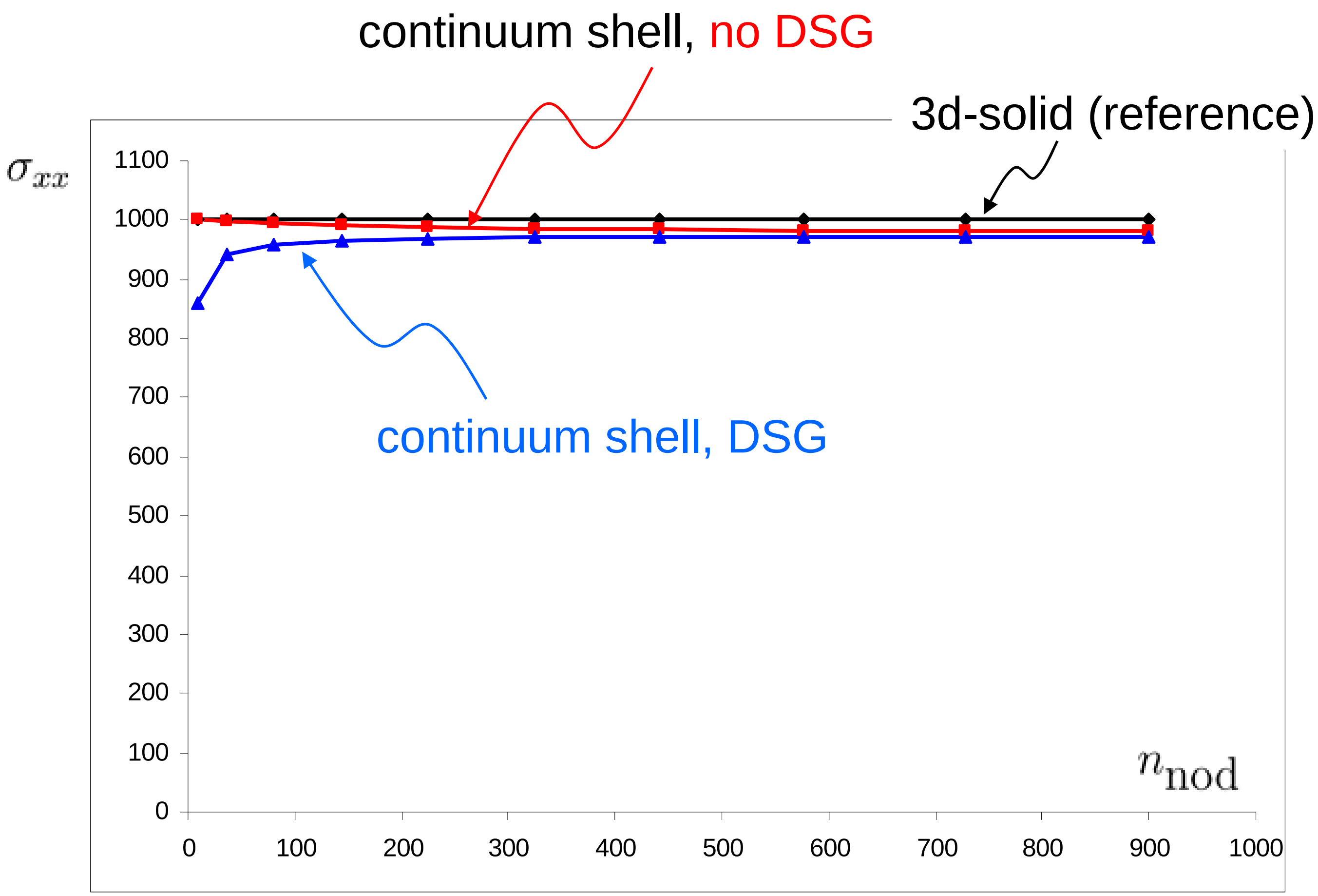
mesh refinement by subdivision





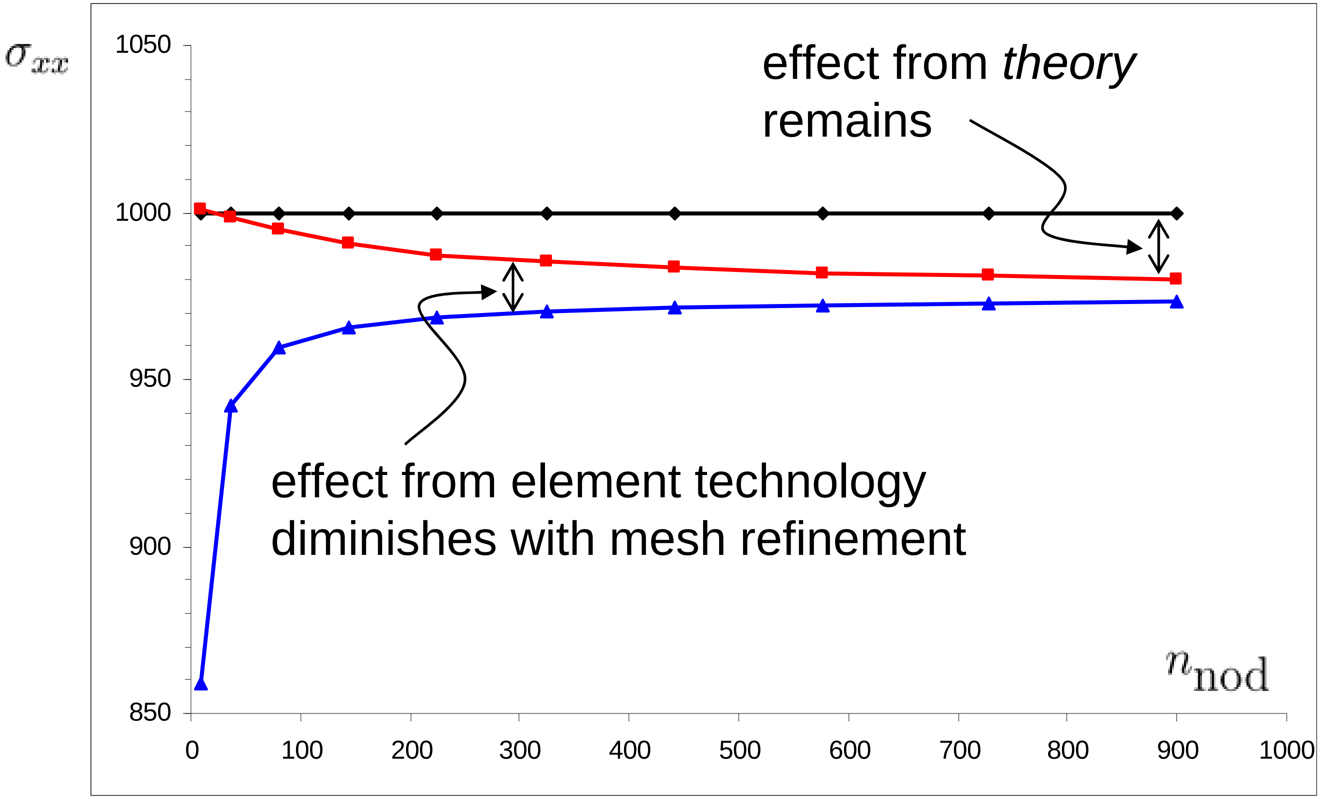
Convergence

mesh refinement



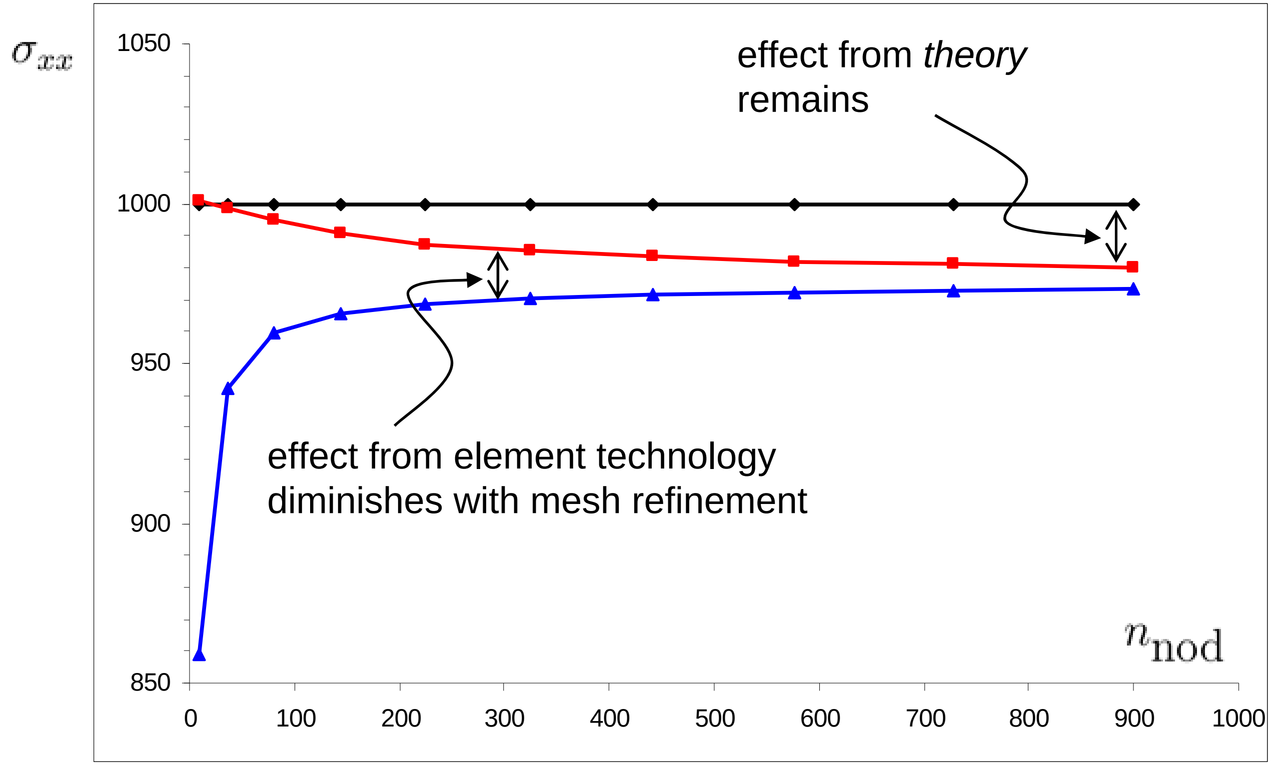
Convergence

mesh refinement



Convergence

mesh refinement



- quadratic terms ought to be included unless directors are normal
- non-satisfaction of patch test harmless when subdivision is used

Requirements

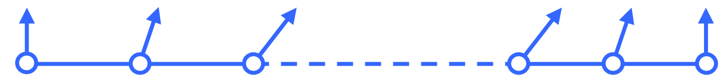
what we expect from finite elements for 3d-modeling of shells

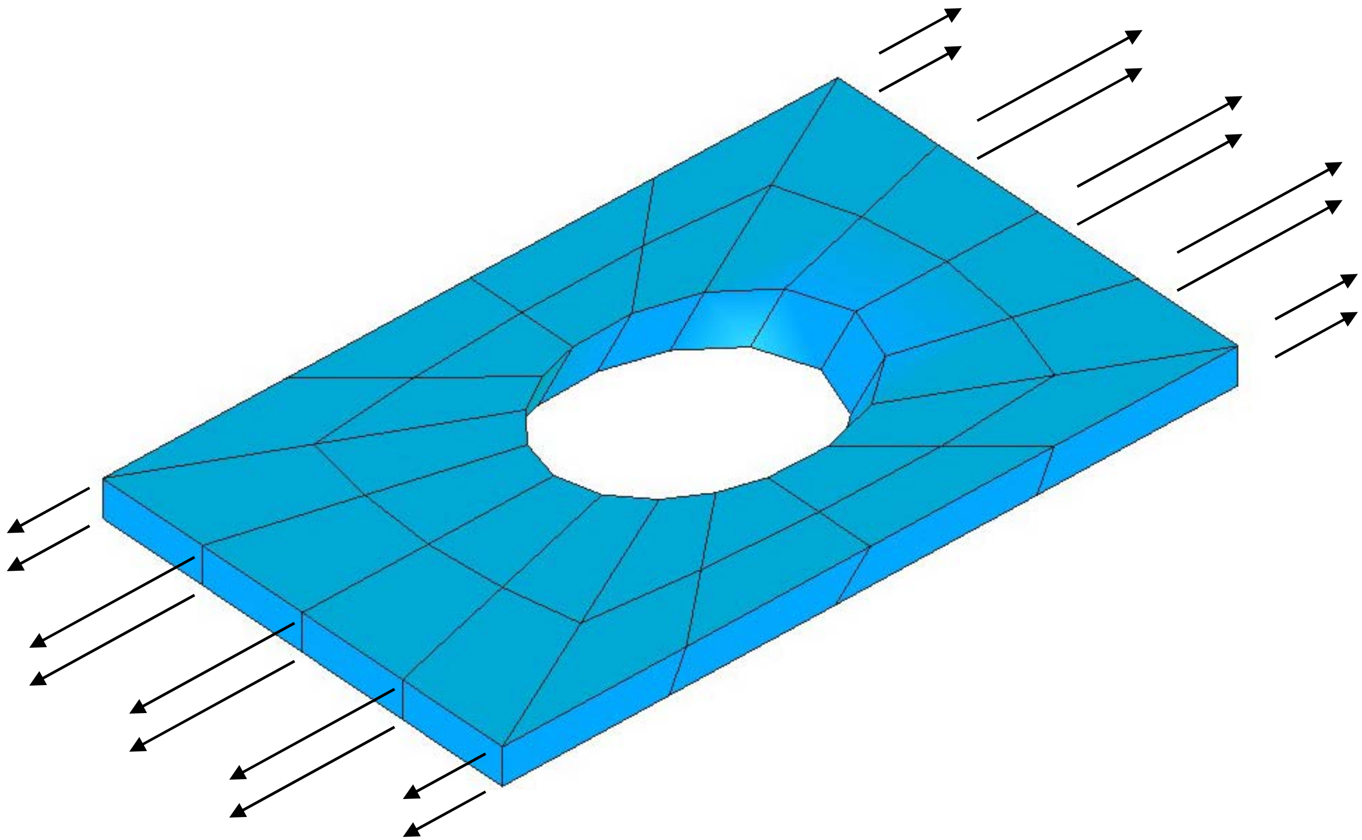
- asymptotically correct (thickness $\rightarrow$ 0)
- numerically efficient for thin shells (locking-free)
- consistent (patch test)
- competitive to „usual“ 3d-elements for 3d-problems

required for both 3d-shell elements and solid elements for shells

Panel with Skew Hole

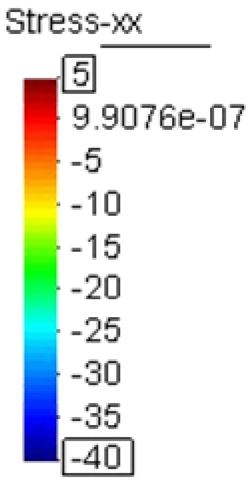
distorted elements, skew directors



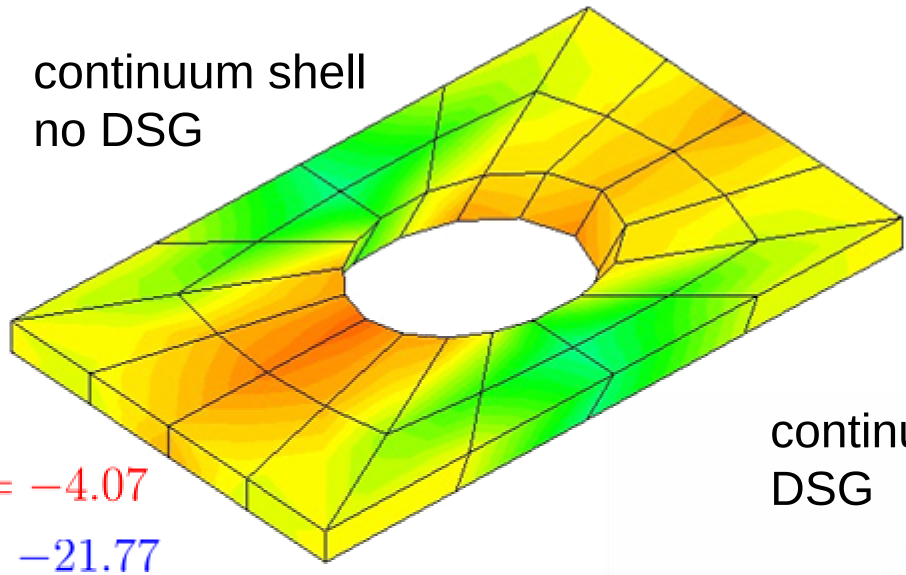


Panel with Skew Hole

continuum shell elements



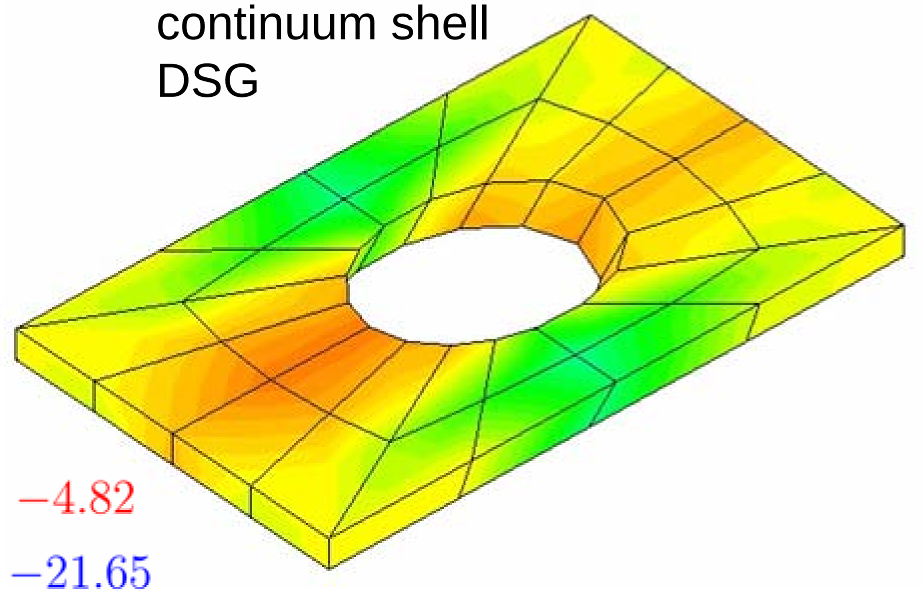
continuum shell
no DSG



$\sigma_{\max} = -4.07$

$\sigma_{\min} = -21.77$

continuum shell
DSG

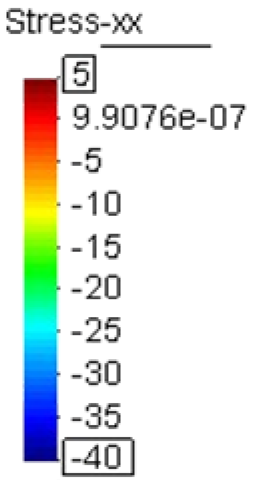


$\sigma_{\max} = -4.82$

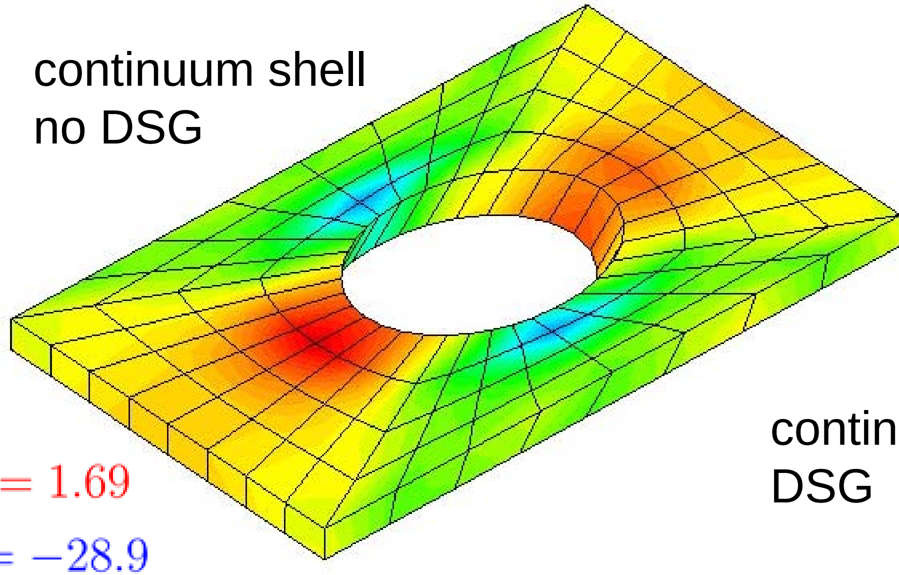
$\sigma_{\min} = -21.65$

Panel with Skew Hole

continuum shell elements



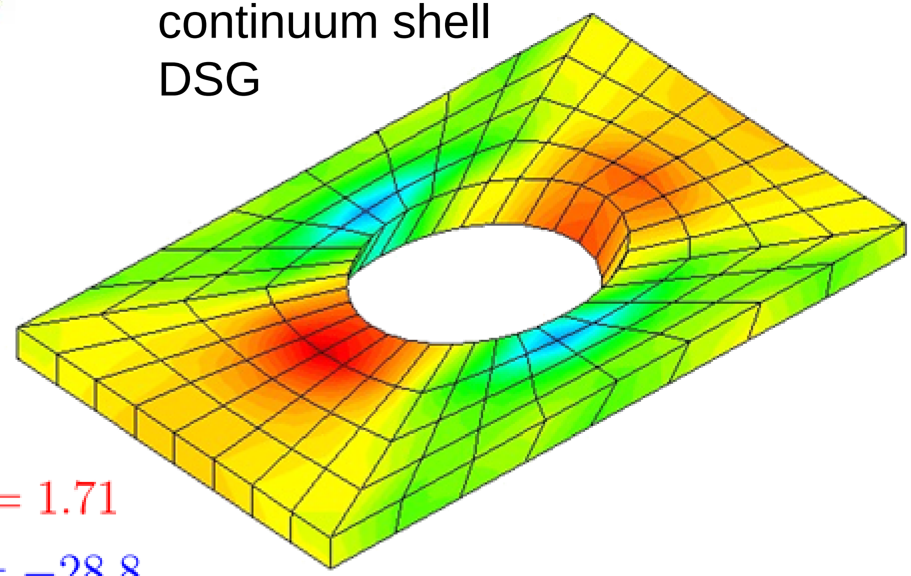
continuum shell
no DSG



$\sigma_{\max} = 1.69$

$\sigma_{\min} = -28.9$

continuum shell
DSG

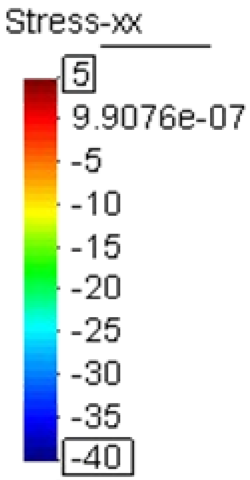


$\sigma_{\max} = 1.71$

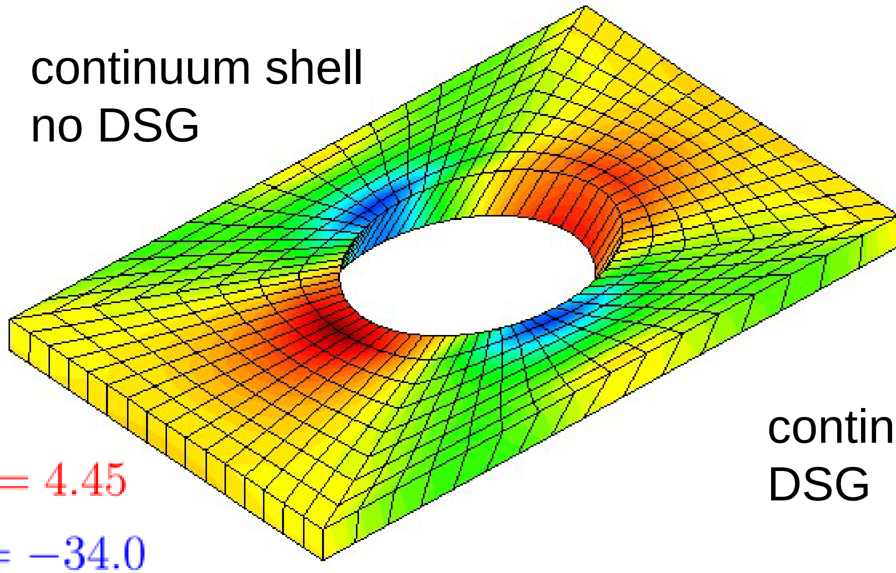
$\sigma_{\min} = -28.8$

Panel with Skew Hole

continuum shell elements



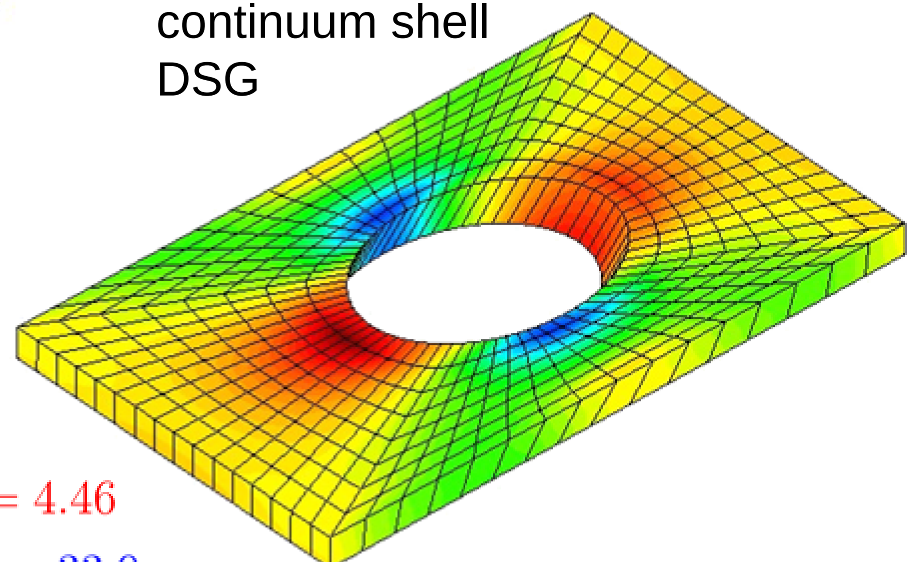
continuum shell
no DSG



$\sigma_{\max} = 4.45$

$\sigma_{\min} = -34.0$

continuum shell
DSG

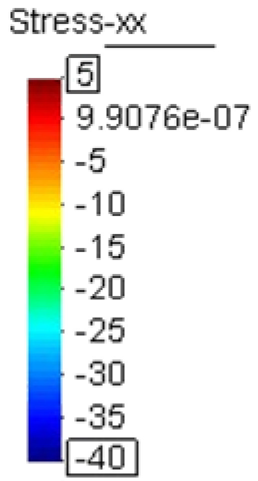


$\sigma_{\max} = 4.46$

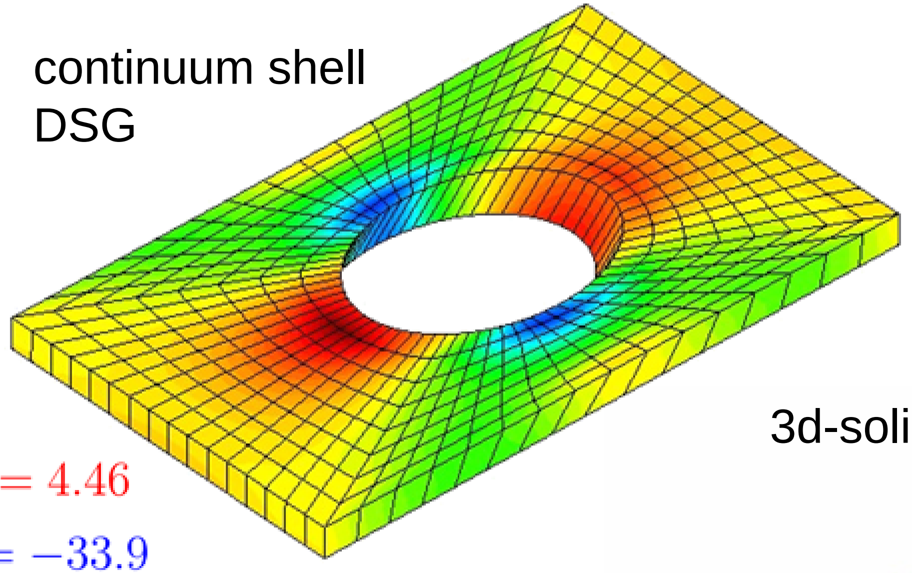
$\sigma_{\min} = -33.9$

Panel with Skew Hole

comparison to brick elements



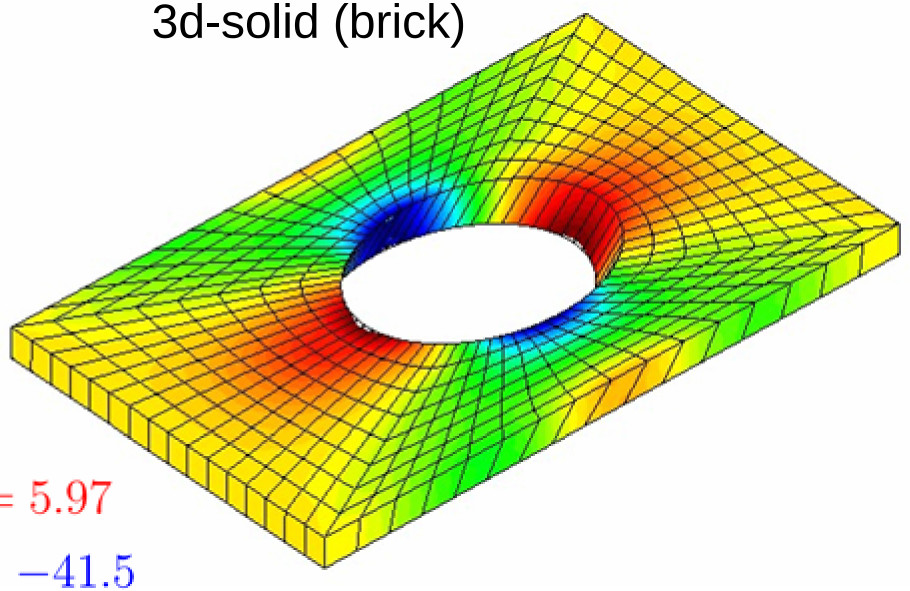
continuum shell
DSG



$\sigma_{\max} = 4.46$

$\sigma_{\min} = -33.9$

3d-solid (brick)

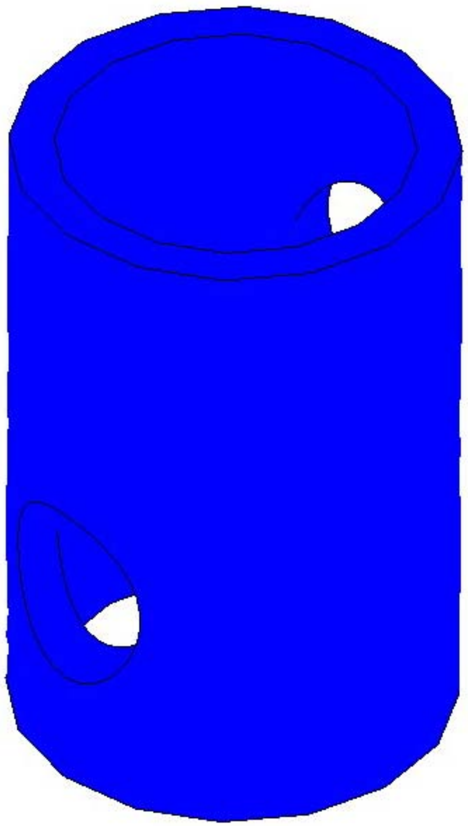


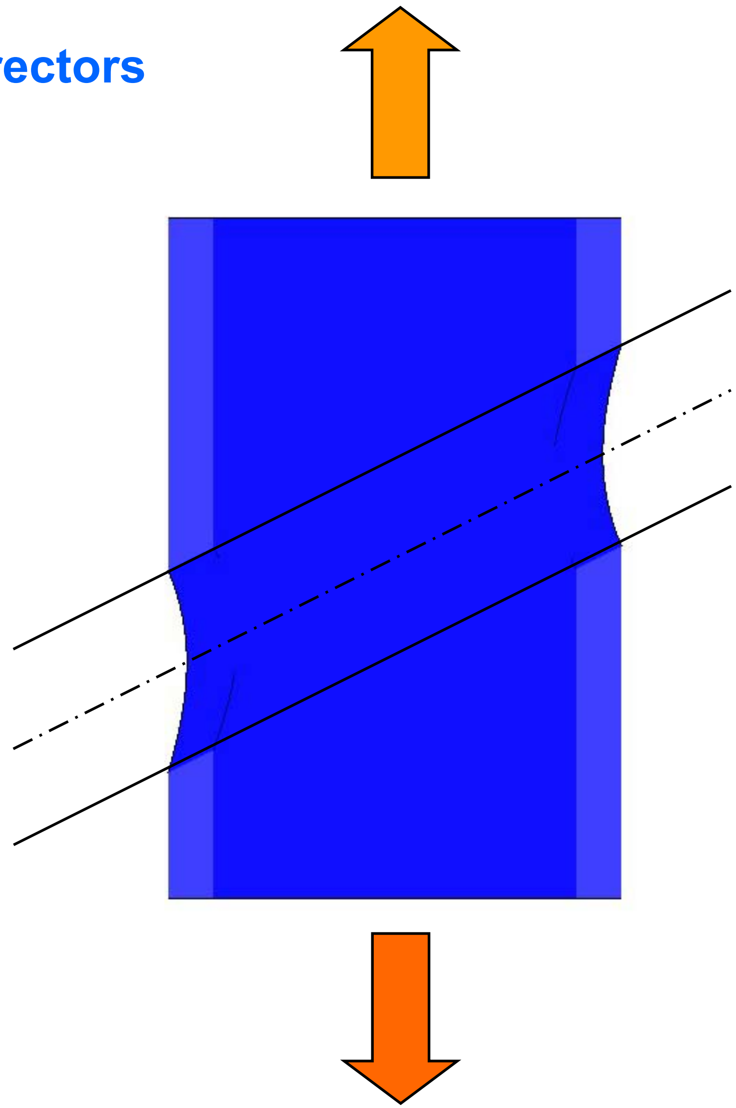
$\sigma_{\max} = 5.97$

$\sigma_{\min} = -41.5$

Cylinder with Skew Hole

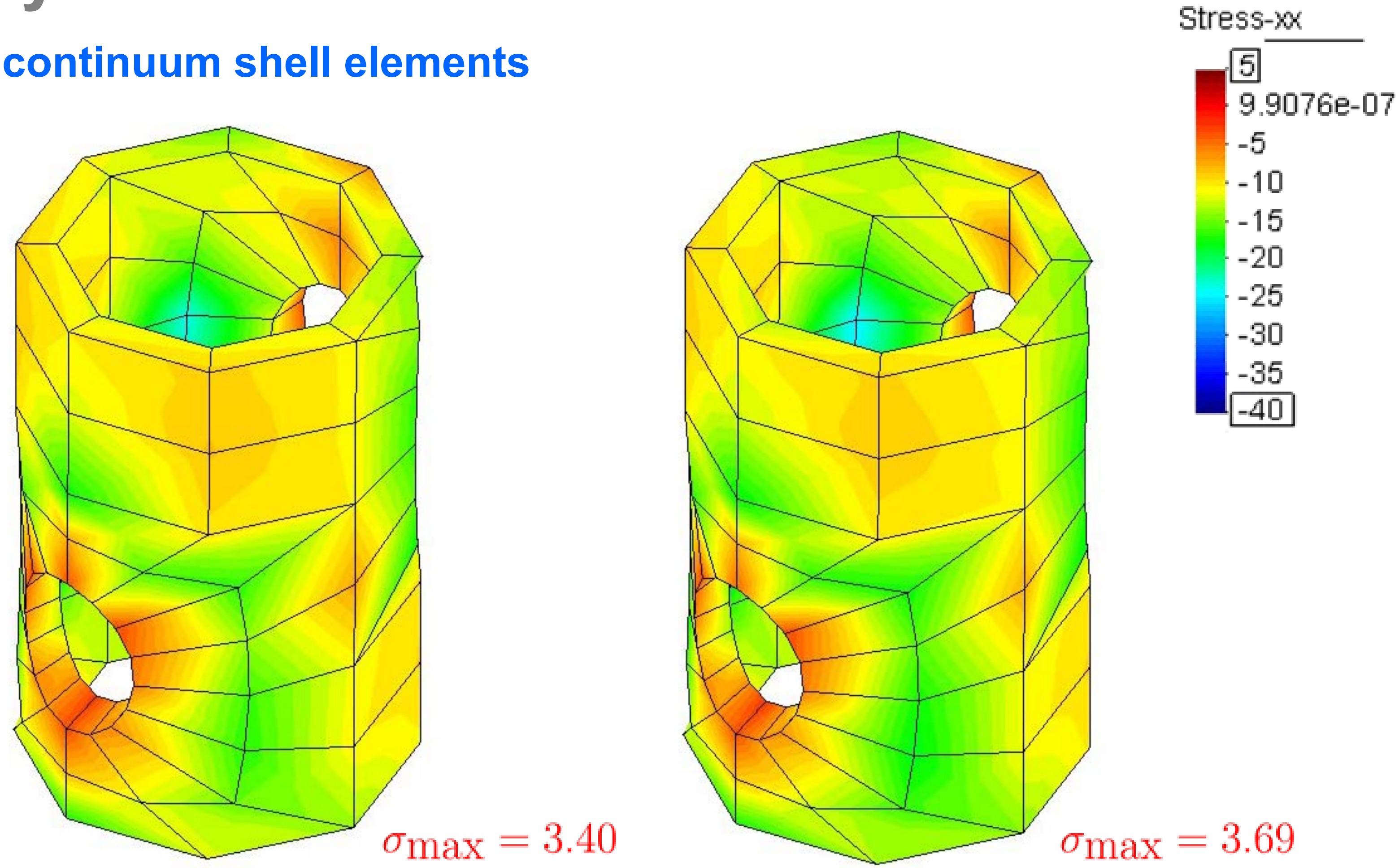
distorted and curved elements, skew directors





Cylinder with Skew Hole

continuum shell elements



$\sigma_{\text{max}} = 3.40$

$\sigma_{\text{min}} = -23.7$

continuum shell
no DSG

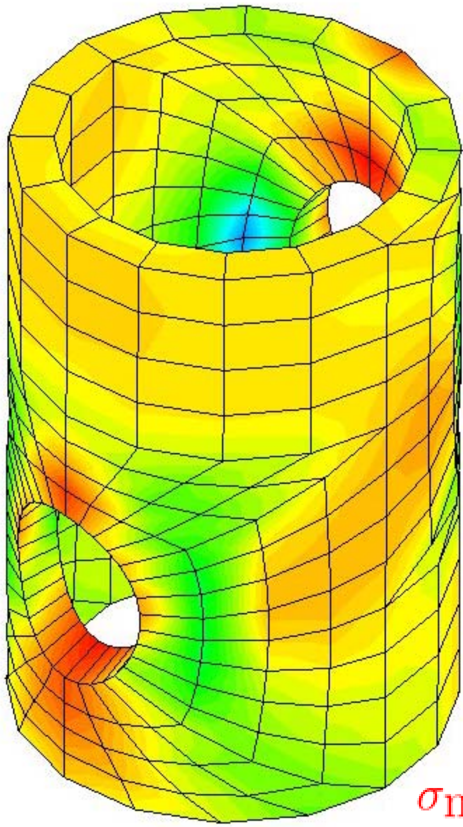
$\sigma_{\text{max}} = 3.69$

$\sigma_{\text{min}} = -24.8$

continuum shell
DSG

Cylinder with Skew Hole

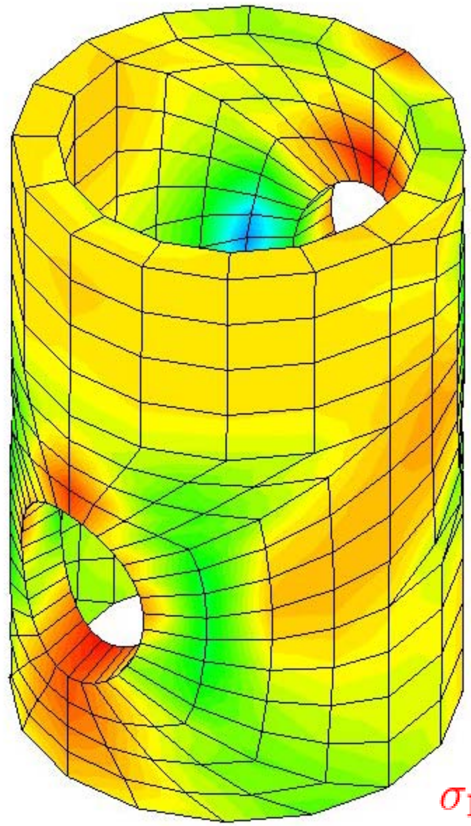
continuum shell elements



$\sigma_{\max} = 5.04$

$\sigma_{\min} = -29.06$

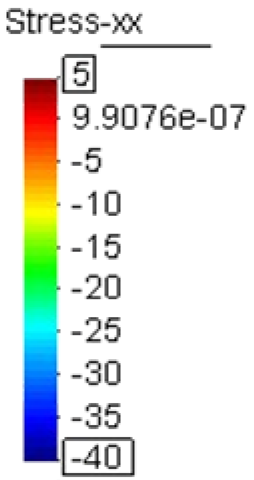
continuum shell
no DSG



$\sigma_{\max} = 5.12$

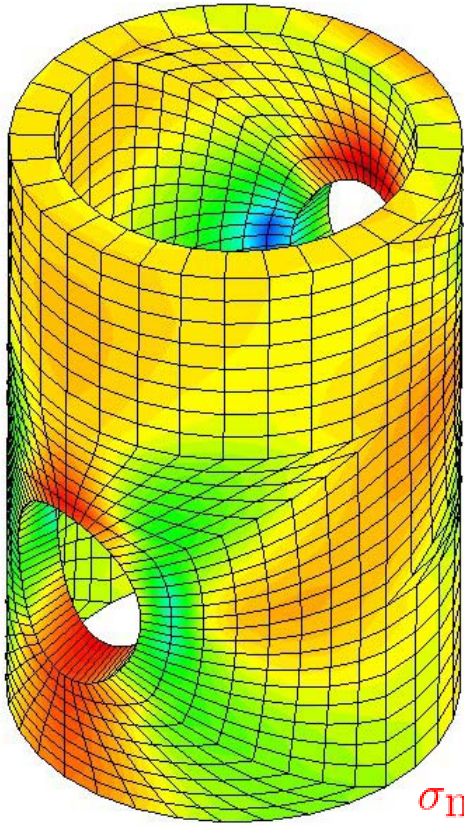
$\sigma_{\min} = -29.24$

continuum shell
DSG



Cylinder with Skew Hole

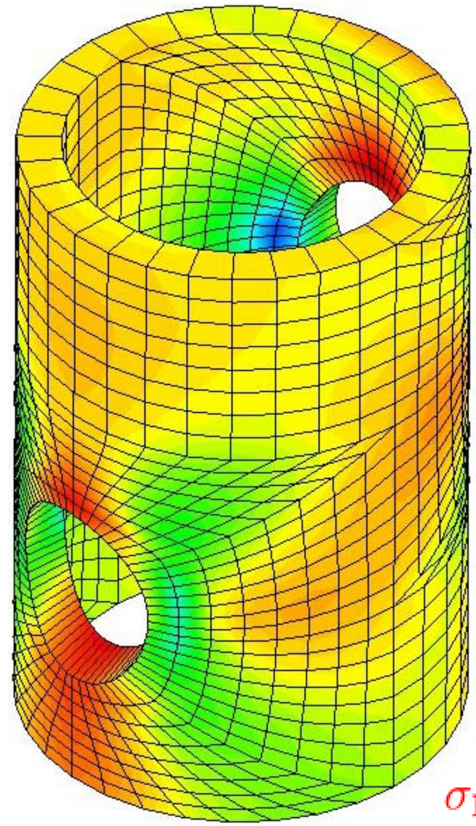
continuum shell elements



$\sigma_{\max} = 3.58$

$\sigma_{\min} = -32.3$

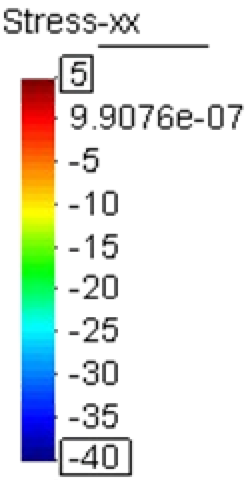
continuum shell
no DSG



$\sigma_{\max} = 3.58$

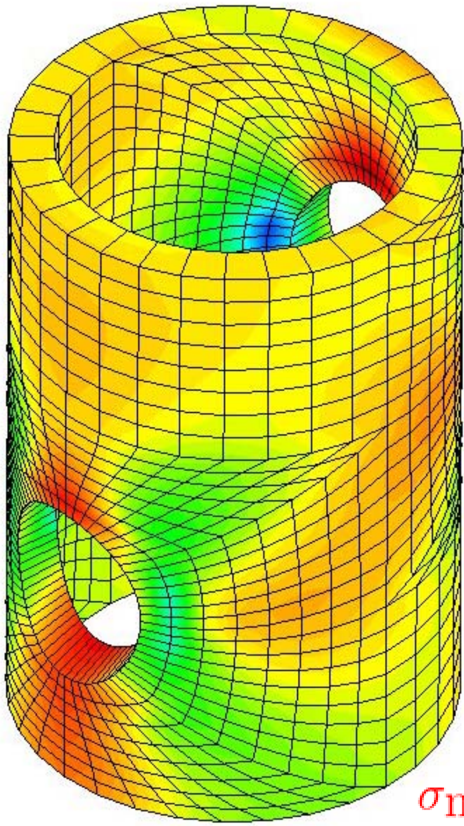
$\sigma_{\min} = -32.3$

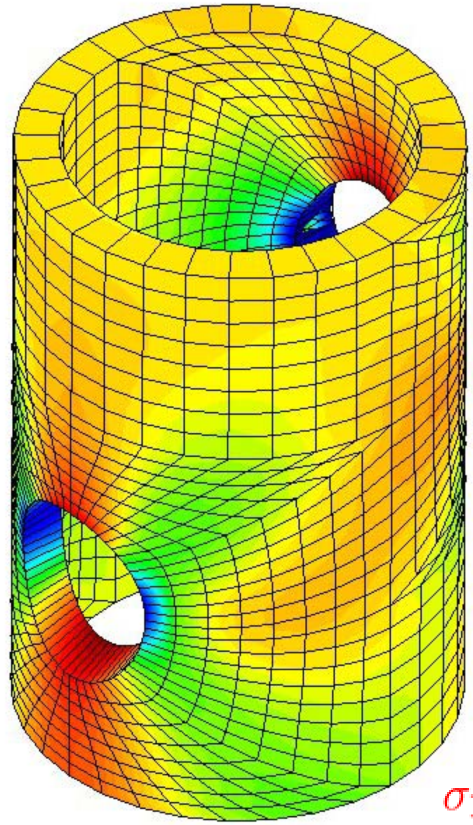
continuum shell
DSG

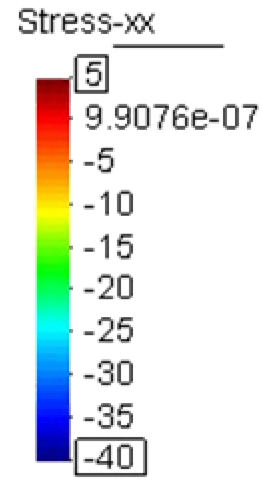


Cylinder with Skew Hole

comparison to brick elements







$\sigma_{\text{max}} = 3.58$

continuum shell
DSG

$\sigma_{\text{min}} = -32.3$

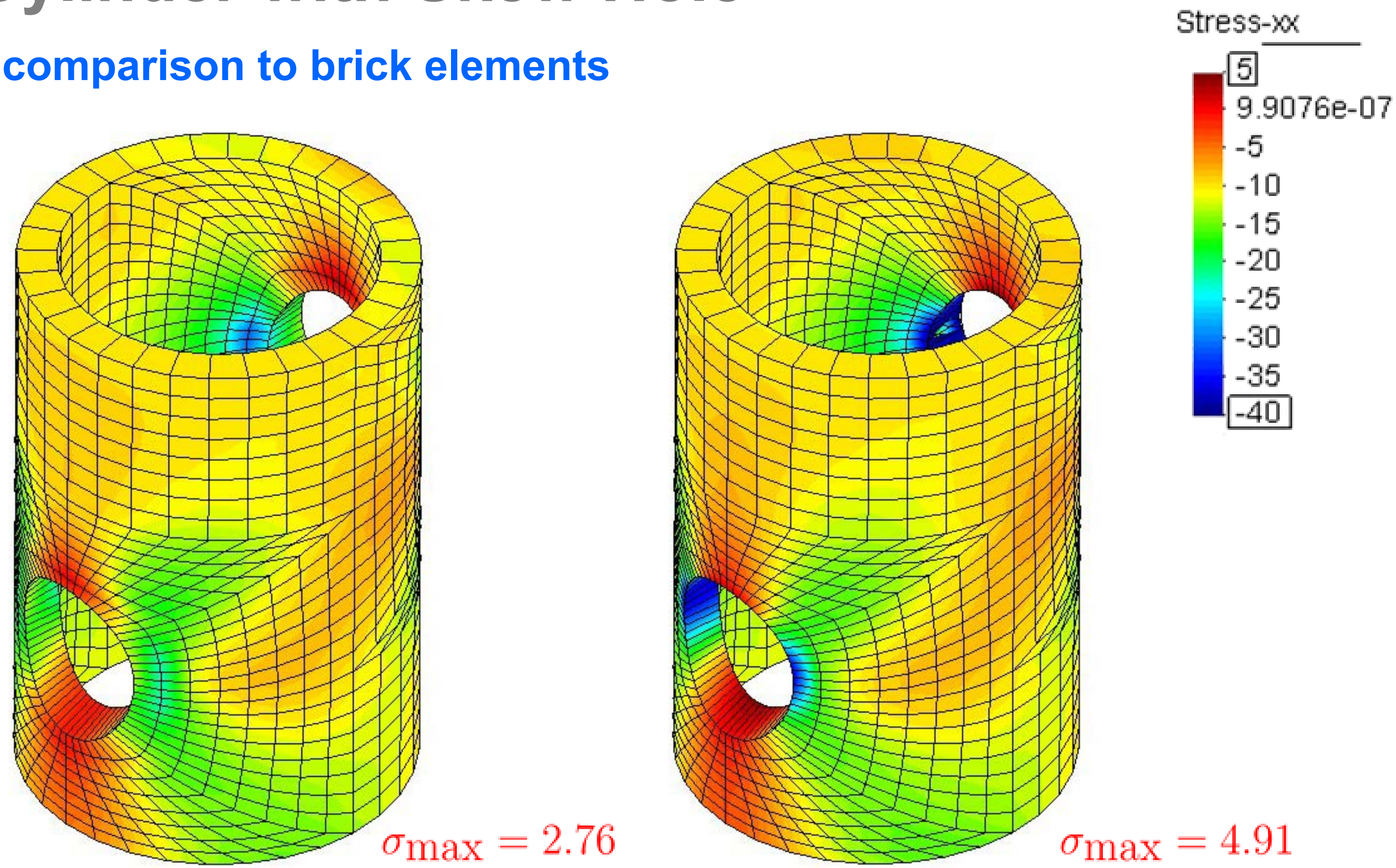
$\sigma_{\text{max}} = 4.91$

3d-solid elements
(bricks)

$\sigma_{\text{min}} = -41.5$

Cylinder with Skew Hole

comparison to brick elements



$\sigma_{\max} = 2.76$

continuum shell $\sigma_{\min} = -30.93$
standard Galerkin

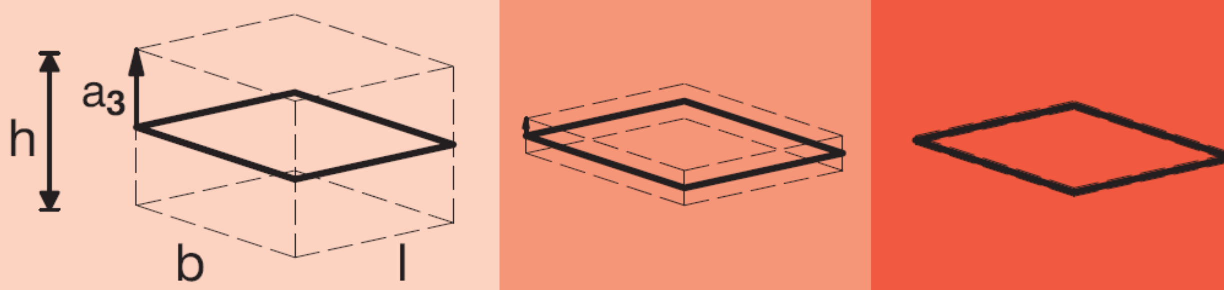
$\sigma_{\max} = 4.91$

3d-solid elements $\sigma_{\min} = -41.5$
(bricks)

The Conditioning Problem

condition numbers for classical shell and 3d-shell elements

	b:l:h = 1:1:1	b:l:h = 1:1:0.1	b:l:h = 1:1:0.01
classical shell:	**5 P:** $\mathbf{c}_{\mathrm{K}} = 2.8 \cdot 10^{2}$	$\mathbf{c}_{\mathrm{K}} = 2.5 \cdot 10^{4}$	$\mathbf{c}_{\mathrm{K}} = 2.8 \cdot 10^{6}$
3d-shell:	**7 P:** $\mathbf{c}_{\mathrm{K}} = 1.0 \cdot 10^{2}$	$\mathbf{c}_{\mathrm{K}} = 1.9 \cdot 10^{5}$	$\mathbf{c}_{\mathrm{K}} = 1.9 \cdot 10^{9}$



Wall, Gee and Ramm (2000)

- spectral condition norm $c_K = \dfrac{\lambda_{\max}}{\lambda_{\min}}$

- thin shells worse than thick shells
- 3d-shell elements ($c_K \propto t^{-3}$) worse than standard shell elements ($c_K \propto t^{-2}$)

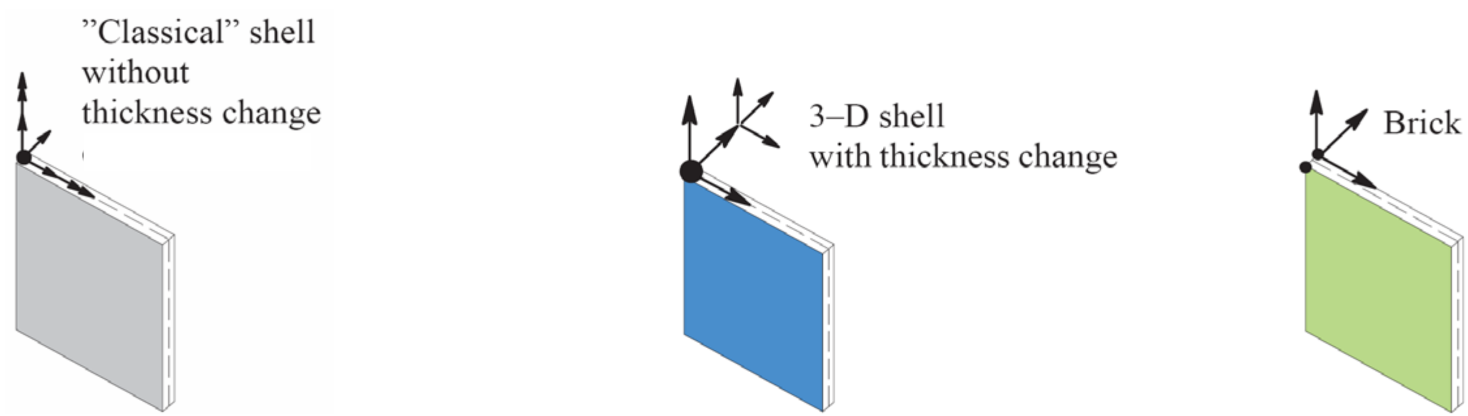
Significance of Condition Number

error evolution in iterative solvers

error of solution vector $\boldsymbol{x}$ after k^{th} iteration $\qquad e^k = \|\boldsymbol{x} - \boldsymbol{x}^k\|$

estimated number of iterations (CG solver) $\qquad k \leq \sqrt{c_K}\,\ln\left(\frac{2}{\varepsilon}\right) + 1$

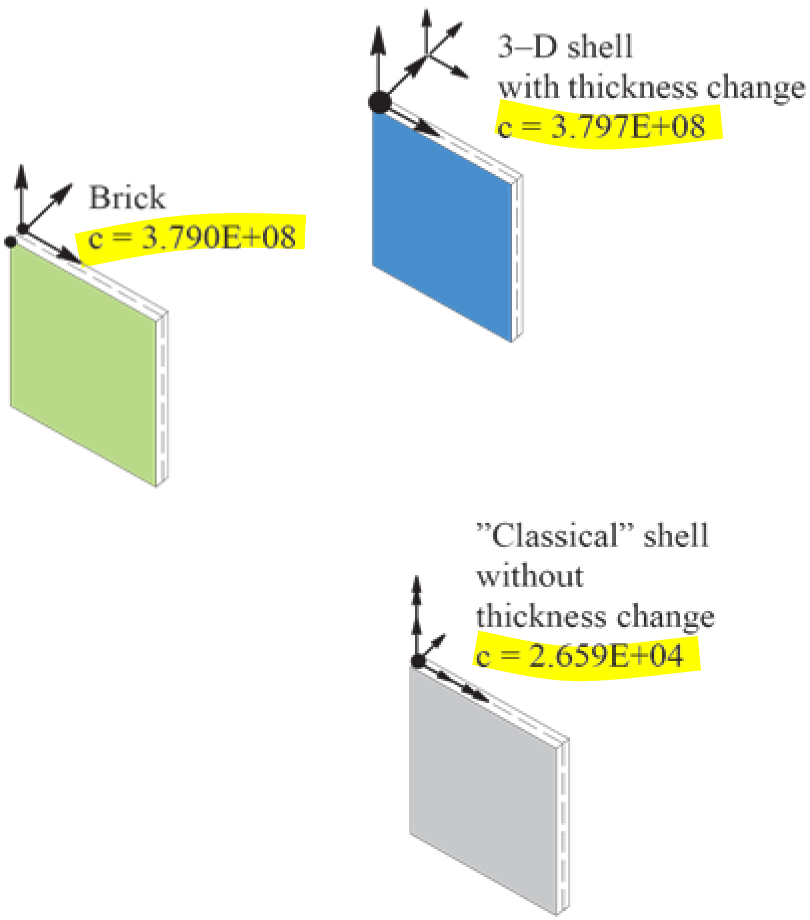
comparison of three different concepts

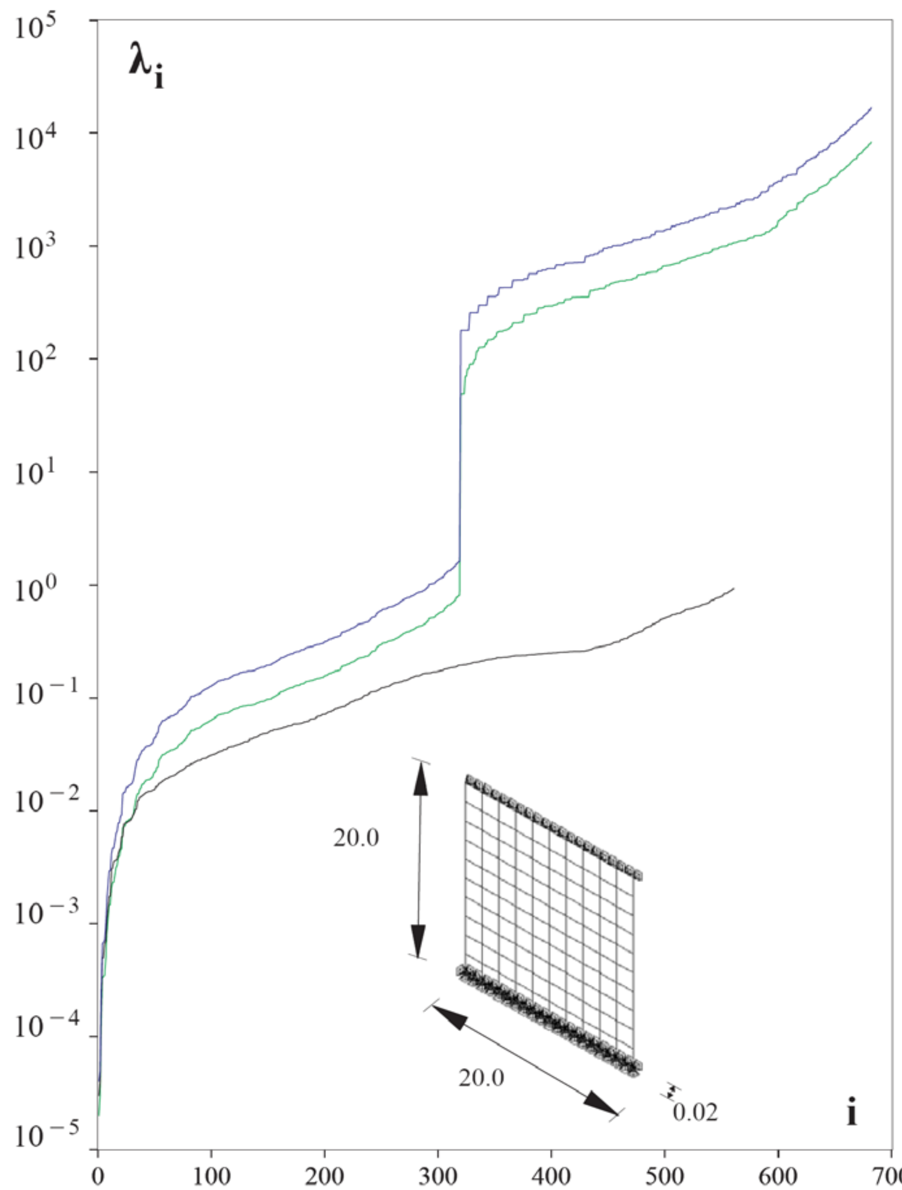


Wall, Gee, Ramm, The challenge of a three-dimensional shell formulation – the conditioning problem, *Proc. IASS-IACM*, Chania, Crete (2000)

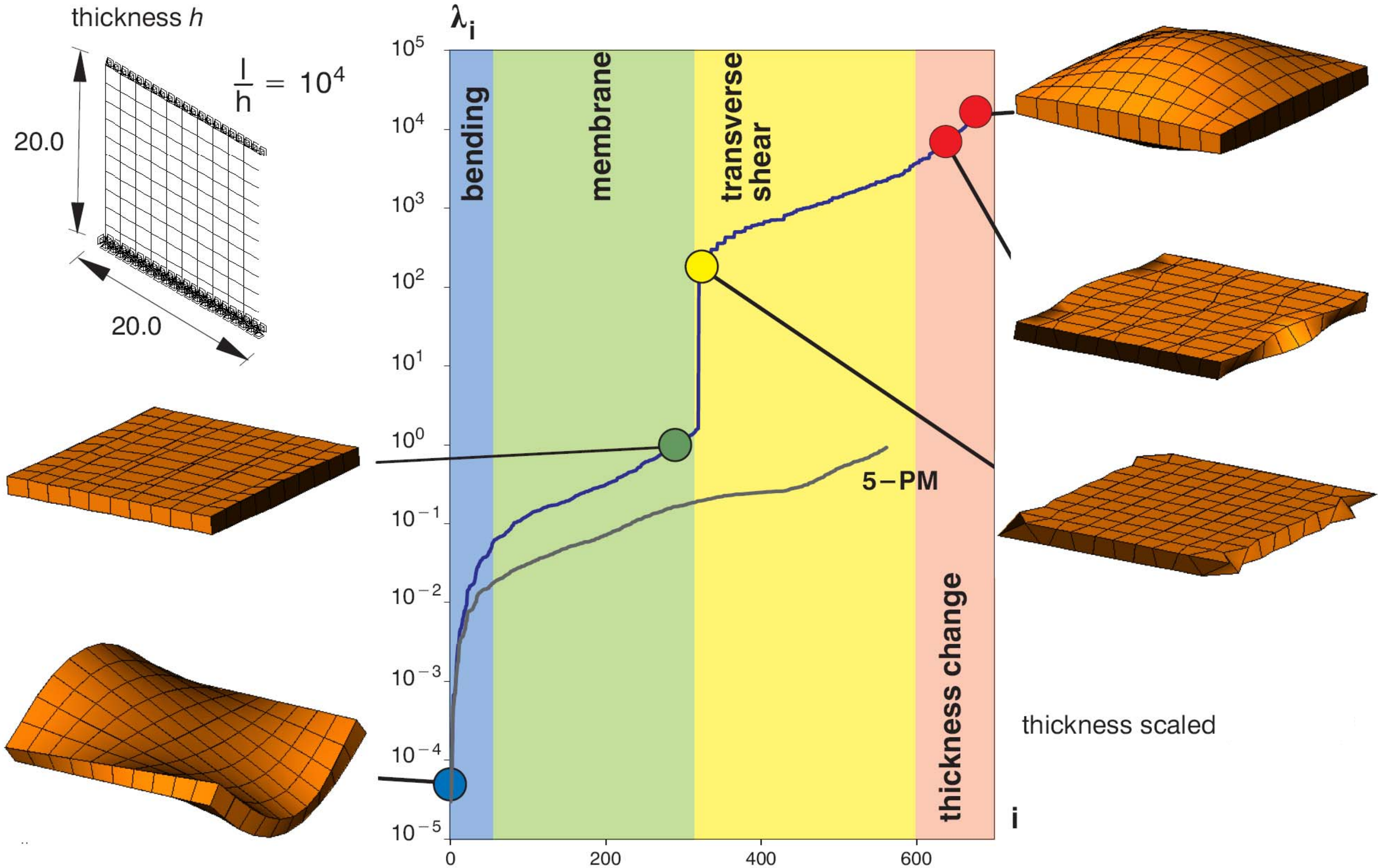
Eigenvalue Spectrum

shell, 3d-shell and brick



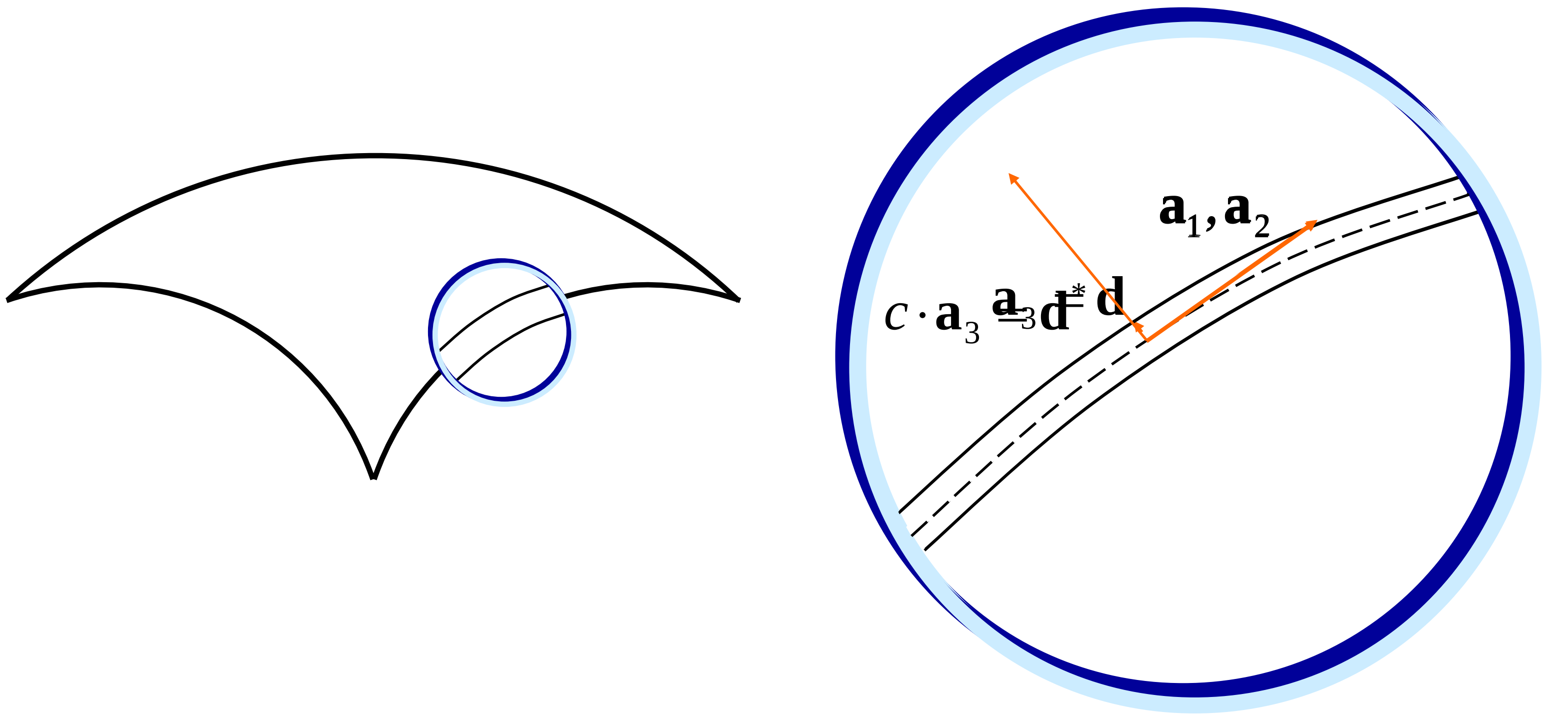


Eigenvectors (Deformation Modes)



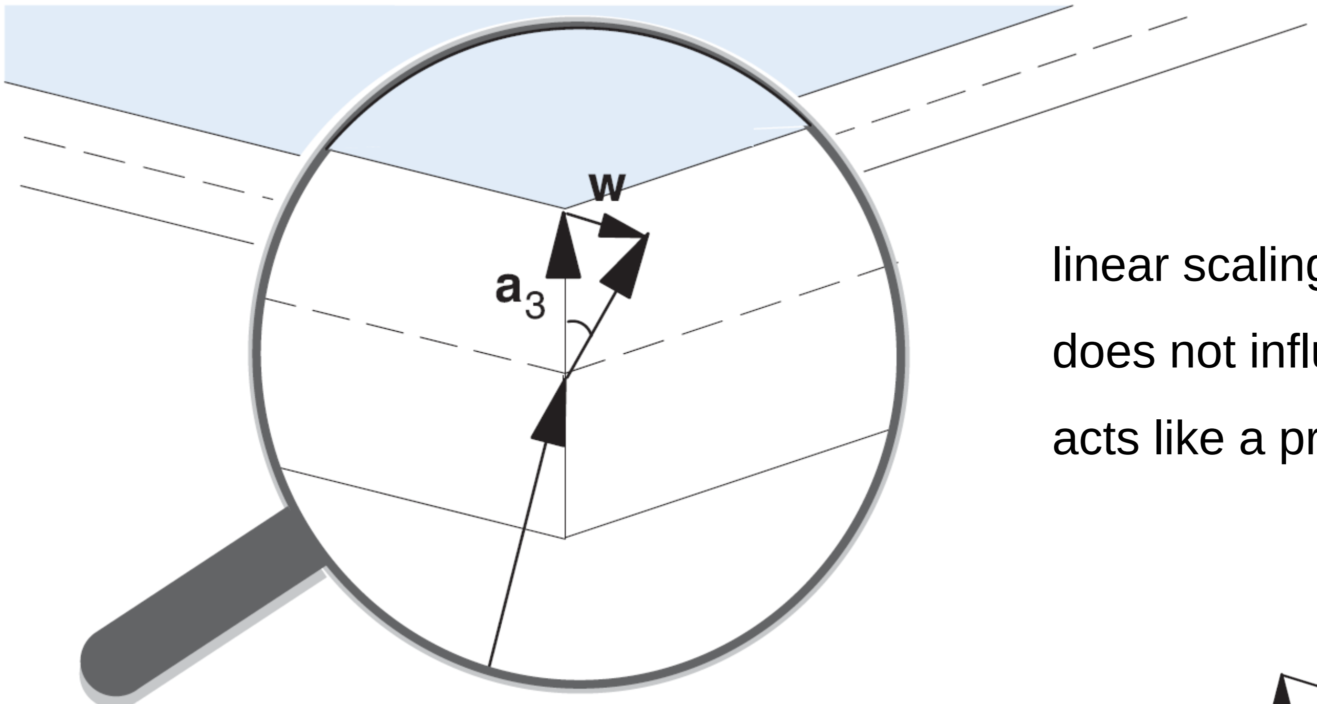
Scaled Director Conditioning

scaling of director



Scaled Director Conditioning

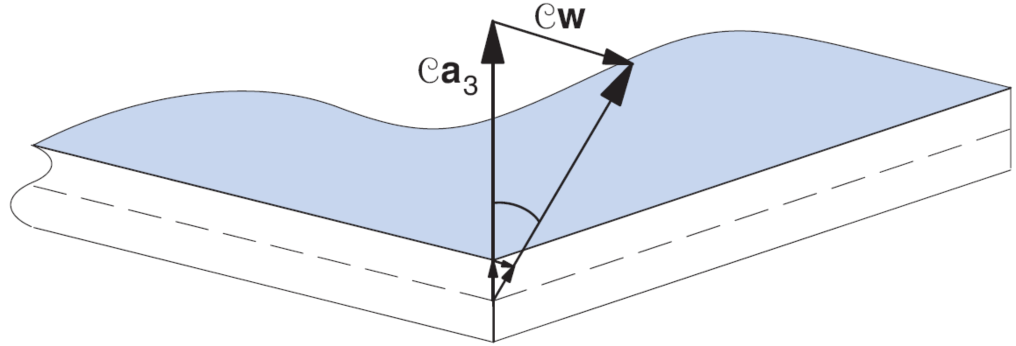
scaling of director



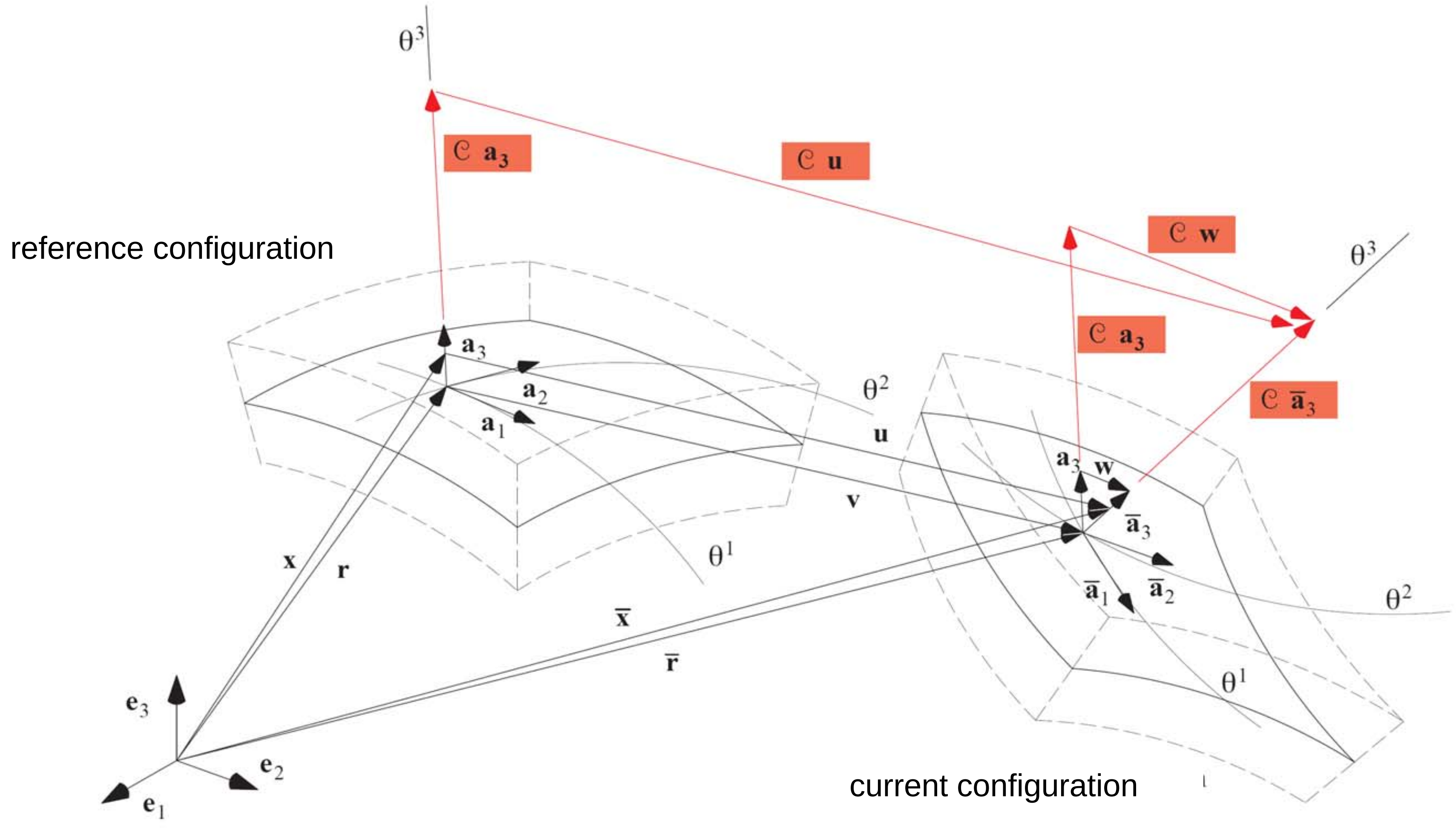
linear scaling of **w**

does not influence results

acts like a preconditioner

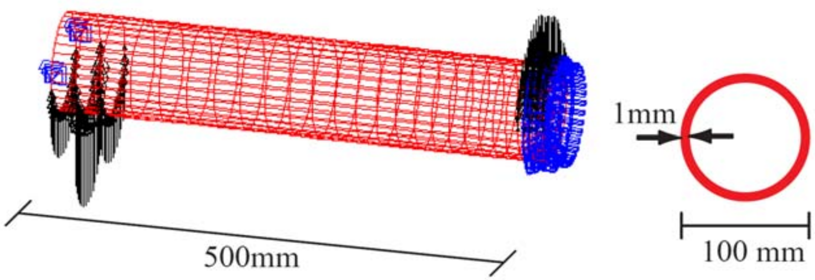


Scaled Director Conditioning

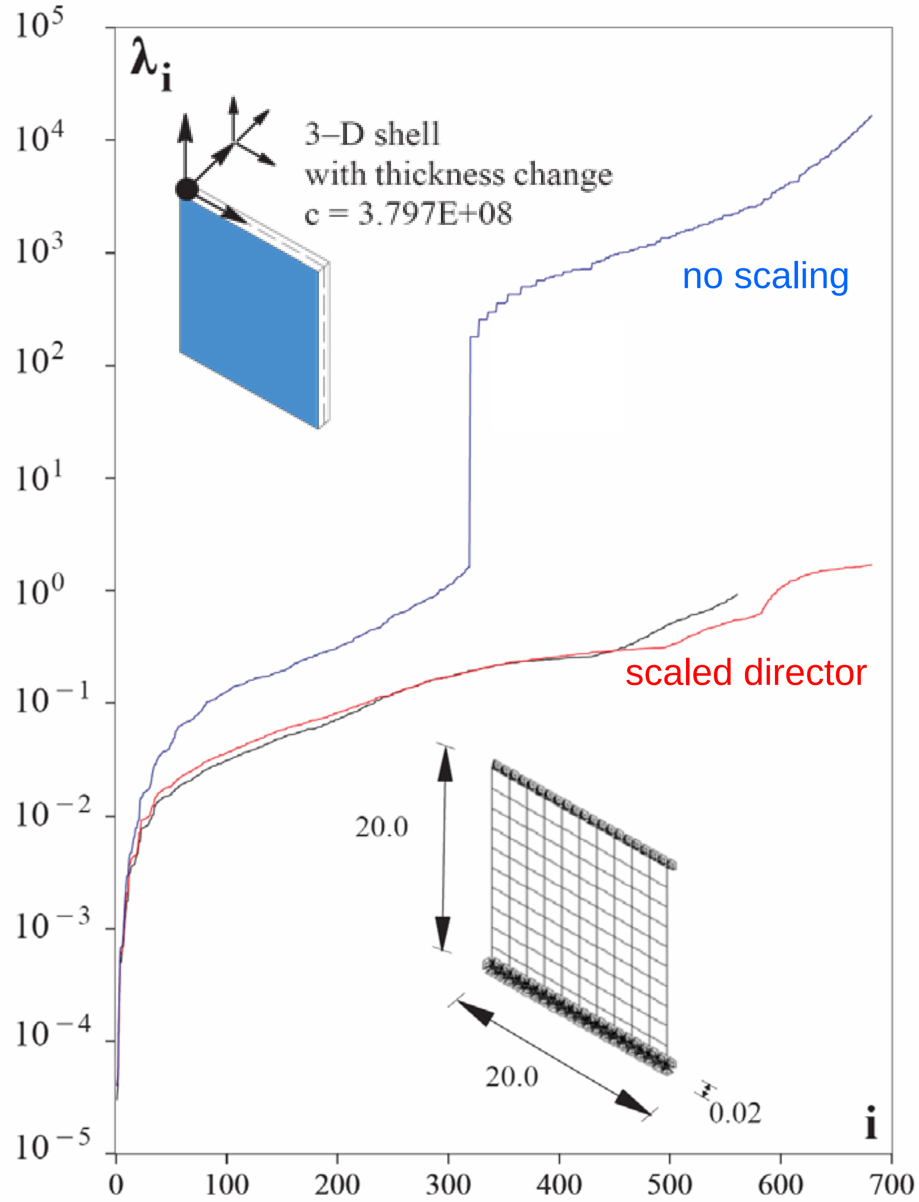


Improved Eigenvalue Spectrum

numerical example

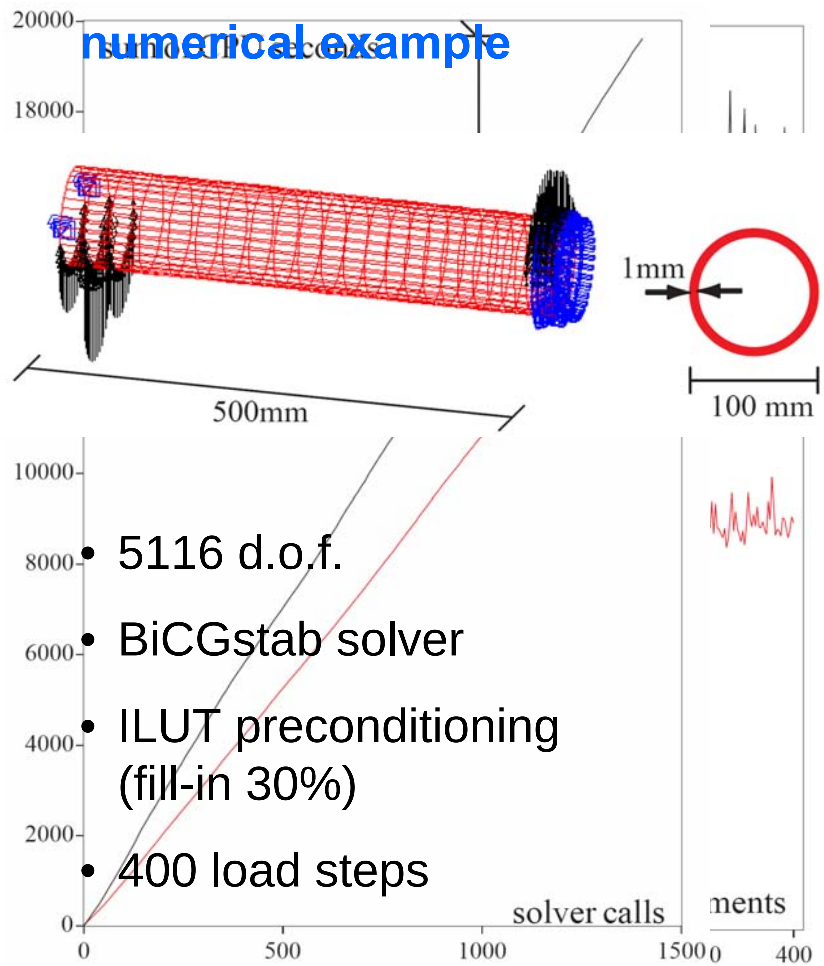


- 5116 d.o.f.
- BiCGstab solver
- ILUT preconditioning (fill-in 30%)
- 400 load steps



Improved Eigenvalue Spectrum

numerical example



- 5116 d.o.f.
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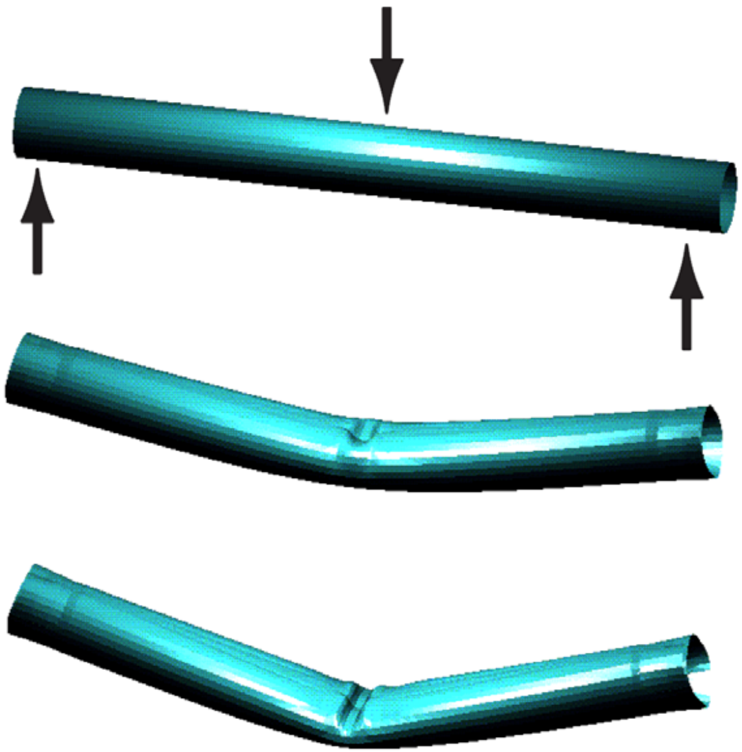
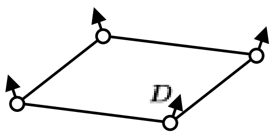
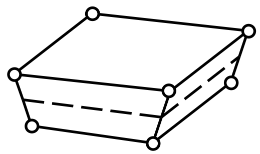
- 400 load steps



Conclusions

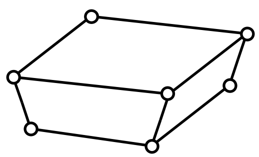
3d-shell and continuum shell (solid shell)





- mechanical ingredients identical
- stress resultants
- flexible and most efficient finite element technology
- neglecting higher order terms bad for 3d-applications
- best for 3d-analysis of “real” shells

3d-solids



- usually suffer from trapezoidal locking (curvature thickness locking)
- pass all patch tests (consistent)
- higher order terms naturally included
- best for thick-thin combinations