CONTINUOUS AND DISCRETE SELF-SIMILARITY VIA CLASSIFICATION SCHEMES OF MARKOV PROCESSES, AND THE VAN DANTZIG PROBLEM

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Rohan Sarkar

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CONTINUOUS AND DISCRETE SELF-SIMILARITY VIA CLASSIFICATION SCHEMES OF MARKOV PROCESSES, AND THE VAN DANTZIG PROBLEM

Rohan Sarkar, Ph.D.

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This dissertation has three chapters, the first two of them are focused on the self-similar Markov semigroups, and the third chapter deals with the classical van Dantzig problem, which has some connections with the self-similar Markov processes on the positive real line ($\mathbb{R}_+$). One of the main focus of this thesis is the development of functional analytic theory for continuous and discrete self-similar Markov semigroups, which are studied elaborately in Chapter 2 and Chapter 3 respectively.

Spectral theory of Markov semigroups is a fast growing literature with many applications in the analysis of convergence rate of Markov processes, functional inequalities, analysis of linear PDEs and many more. The Markov semigroups of our interest are self-similar, that is, they have the following scaling property:

$$P_t d_\alpha = d_\alpha P_{\alpha t} \quad \forall \alpha, t > 0$$

where $d_\alpha f(x) = f(\alpha x)$ is the classical dilation operator on $\mathbb{R}_+$. It is important to observe that the notion of self-similarity is dependent on how one defines the dilation operator. More precisely, any multiplicative semigroup $(D_\alpha)_{\alpha>0}$ acting on a certain subset E of $\mathbb{R}$ can be thought as a dilation operator. With this motivation in mind, in Chapter 3 we define *discrete self-similar Markov chains* on $\mathbb{Z}_+$ and study their analytical properties.

Self-similar semigroups, both in continuous and discrete state spaces, are, in general, not self-adjoint (not even normal), which is why their spectral analysis is much more involved. In this dissertation, we exploit the concept of *weak similarity* and *intertwining relationship* whose original motivation comes from group representation theory and functional analysis. In the context of Markov semigroups, we say that two semigroups $P : \mathrm{H}_1 \to \mathrm{H}_1$ and $Q : \mathrm{H}_2 \to \mathrm{H}_2$, where $\mathrm{H}_1, \mathrm{H}_2$

are Hilbert spaces, are weakly similar if there exists a densely defined injective operator Λ with dense range such that $P\Lambda = \Lambda Q$ on a dense set. In particular, when Λ is a Markov kernel, the weak similarity boils down to the intertwining relation. The earliest instance of intertwining relationship in Markov processes goes back to E.B. Dynkin [41], where he considered bijective transformations of Markov processes via the similarity transform of the semigroups. Later, Pitman and Rogers [110] introduced *Markov intertwining* operators to prove that the distribution of the Brownian motion reflected around its running maximum coincides with the Bessel process of dimension 3. In more recent years, Carmona, Petit and Yor [29] proved intertwining relationship among the class of Bessel processes to derive certain distributional identities. Chapter 2 of this thesis is motivated from the aforementioned work, where we prove weak similarity relation among the class of (log) self-similar Markov semigroups on $\mathbf{L}^2(\mathbb{R}, e)$. Using this weak similarity relation and the self-similarity of the self-adjoint Bessel semigroup in $\mathbf{L}^2(\mathbb{R}_+)$, we provide a spectral representation of the non-self-adjoint self-similar Markov semigroup, and a detailed description of the spectrum including the point, continuous and the residual spectrum. We carry out the similar program in Chapter 3 for the *discrete Laguerre chain*, which are obtained by a perturbation of the discrete self-similar Markov chains. We show that both the discrete self-similar and the Laguerre chains have a *gateway* relation with their continuous analogues. This enables us to obtain the spectral representation, hypocoercivity and hypercontractivity of the non-reversible Laguerre chains.

The final chapter of the thesis deals with the classical van Dantzig problem which can be stated very simply as follows: find all analytic characteristic functions (of probability measure) $\mathcal{F}$ such that $V\mathcal{F}$ is also a characteristic function, where $V\mathcal{F}(t) = 1/\mathcal{F}(\mathrm{i}t)$ for all $t \in \mathbb{R}$. In the seminal paper [82], E. Lukacs studied the solution to the above problem within the class of entire functions with real zeros. Let $\mathscr{D}_L$ denote the class of all such functions. After observing the fact that any function solving the van Dantzig problem must be even, by means of analytical methods, Lukacs obtained several closure properties of the class $\mathscr{D}_L$. In this chapter, we first discuss how the class of functions $\mathscr{D}_L$ is related to the Riemann Hypothesis and the Lee-Yang property in statistical mechanics. Then, we provide a new class of solutions to the van Dantzig problem, which may have complex zeros. We point out that these functions are also the eigenfunctions of certain self-similar Markov semigroups on $\mathbb{R}_+$.

All work in this thesis was done in collaboration with Pierre Patie. The work in Chapter 3 was done with the additional collaboration with Laurent Miclo and the work in Chapter 4 was done in collaboration with Takis Konstantopoulos. The contents of Chapter 2 – 4 have been submitted to peer-reviewed journals as follows:

- Pierre Patie and Rohan Sarkar, Weak similarity orbit of (log)-self-similar Markov semigroups on the euclidean space, submitted, 55pp., 2022.
- Laurent Miclo, Pierre Patie and Rohan Sarkar, Discrete self-similar and ergodic Markov chains, *The Annals of Probability*, accepted, 50pp., 2022.
- Takis Konstantopoulos, Pierre Patie, and Rohan Sarkar, A new class of solutions to the van Dantzig problem, the Lee-Yang property, and the Riemann Hypothesis, submitted, 35pp., 2021.

BIOGRAPHICAL SKETCH

Rohan Sarkar was born in Basirhat, his mother's hometown, located in West Bengal, India. Afterwards, his family moved to Sodepur, a suburb area near Kolkata, the state capital of West Bengal, where he grew up with his parents and his elder sister. After finishing his high school education from Sodepur High School, he earned both Bachelor (2015) and Masters (2017) degree in Statistics from Indian Statistical Institute, Kolkata. In Fall 2017, he joined as a Ph.D. student in the department of Operations Research and Information Engineering at Cornell University, where he was advised by Prof. Pierre Patie. Starting from Fall 2022, he will be an Assistant Research Professor in the Mathematics Department at the University of Connecticut.

To my parents and my sister; and, to Eva.

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I owe my deepest gratitude to my parents and my sister for their unconditional love, support and sacrifices they have made, without which I would not have come this far. Lastly, my deep love to Eva, who have been my constant support throughout the last five years, and with whom I have travelled around the world, and spent unforgettable moments in Ames, Iowa.

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CHAPTER 1

INTRODUCTION

A Markov process on $\mathbb{R}_+$ is called self-similar of index 1 if for all $\alpha > 0$, one has the following commutation type relation

$$P_t d_\alpha = d_\alpha P_{\alpha t} \tag{1.1}$$

where $P = (P_t)_{t \geqslant 0}$ is the semigroup associated with the process and $d_\alpha f(x) = f(\alpha x)$ is the dilation operator, that satisfies the semigroup property $d_\alpha d_\beta = d_{\alpha\beta}$ for all $\alpha, \beta > 0$. Self-similar processes are ubiquitous in the theory of Markov processes and they have been studied intensively over the last three decades, both from theoretical and applied perspectives, see e.g. [76, 24, 99]. These interests seem to be attributed to their role played in limit theorems in probability theory. This probably also explains the appearance of self-similar Markov processes in many different studies, such as coalescence-fragmentation [11], random planar maps [15], and also in the study of fractional operators [103, 106] to name but a few. Motivated by limit theorems, Lamperti [76] obtained a complete characterization of these processes. The semigroups $(P_t)_{t \geqslant 0}$ are Fellerian and sub-invariant with respect to the Lebesgue measure on $\mathbb{R}_+$. As a result, they extend to strongly continuous contraction semigroup on $\mathbf{L}^2(\mathbb{R}_+)$. From functional analytic point of view, it is very natural to investigate the spectral properties of P on the Hilbert space $\mathbf{L}^2(\mathbb{R}_+)$. However, this question is very non-trivial because of two reasons: firstly, these semigroups correspond to Markov processes with jumps, so one cannot apply the theory of differential operators in such cases. Secondly, the semigroups are non self-adjoint for which it is very difficult to develop the spectral theory. We circumvent this issue by resorting to the concept of *weak similarity* (WS), whose mathematical interpretation is as follows: an operator P is said to be weakly similar to Q if there exists an injective operator Λ with dense range such that

$$P\Lambda = \Lambda Q$$

on a dense domain. Note that in the case of finite dimension, the identity above reads as the similarity of matrices. Since the injectivity is not sufficient for invertibility in infinite dimension, one cannot expect the similarity of operators from the weak similarity, which also explains the terminology. Nonetheless, in many circumstances, the WS relation transfers the spectrum of Q to that of P. In particular, when Λ is a Markov kernel, the weak similarity relation is termed as the *intertwining relation*. Our general strategy is to prove weak similarity of a non-self adjoint semigroup with a self-adjoint one, whose spectral properties are easier to deal with.

In Chapter 2, we carry on our goal to develop an in-depth and detailed analysis of the non-self-adjoint and non-local pseudo-differential operators (PDOs) denoted by $\mathscr{A}$ in the class of densely defined operators in the weighted Hilbert space $\mathbf{L}^2(\mathbb{R}^d, \mathbf{e}_M)$, where $\mathbf{e}_M(\boldsymbol{x}) = e^{\langle M^{-1}\boldsymbol{x}, \mathbf{1}\rangle}$ with $M \in \mathrm{GL}_d(\mathbb{R})$, the group of linear invertible operators. The spectral analysis of self-adjoint pseudo-differential operators (PDO) in Hilbert space is by now well established and has offered many fascinating and deep ramifications in several branches of mathematics, see for example, the monograph of Shubin [122]. However, although generic, the theory of non-self-adjoint ones is much less understood and unified, something which seems to be attributed to the variety of phenomena, such as the instability of the spectrum under small perturbation that one encounters when studying such operators. We refer to the recent monograph of Sjöstrand [124] for a thorough account on non-self-adjoint differential operators and a detailed study of their spectral asymptotic. The first motivation to investigate this class of (non local) PDOs stems on their intimate connection with the family of self-similar Markov processes on $\mathbb{R}^d_+$. We shall prove that indeed the Dynkin generator of such processes, up to the homeomorphism $\boldsymbol{x} \mapsto e(\boldsymbol{x})$, coincide, on a core that we identify, with a pseudo-differential operator in the class $\mathscr{A}$. Resorting to the PDO representation of the generators, we establish *weak similarity*, see (2.1.2) for definition, between the (log) self-similar semigroups on $\mathbb{R}^d$. As a by product, we obtain several analytical properties of these semigroups which include their spectral theory and representation, existence of point, residual and continuous spectrum, and the integro-differential representation of the generators, which, to the best of our knowledge, have not been explored before.

In Chapter 3, we introduce a new class of *discrete self-similar* Markov chains on $\mathbb{Z}_+$, and explore their connection to the self-similar Markov processes on $\mathbb{R}_+$. More precisely, we construct a large class of continuous time Markov chains (CTMC) on $\mathbb{Z}_+$ that also enjoy a scaling type property like the self-similar Markov processes. Naturally, one cannot expect (1.1) to hold in this setting, because the set of integers is not stable by the dilation operators as defined above. However, in [84], the authors introduced the following signed Binomial kernel defined by

$$\mathbb{D}_\alpha \mathsf{f}(n) = \sum_{k=0}^{n} \binom{n}{k} \alpha^k (1-\alpha)^{n-k} \mathsf{f}(k)$$

which resembles the dilation operator through the multiplicative semigroup property $\mathbb{D}_{\alpha\beta} = \mathbb{D}_\alpha \mathbb{D}_\beta$ for all $\alpha, \beta > 0$. We adapt this notion of dilation to define the discrete self-similarity on $\mathbb{Z}_+$. Like in the case of continuous state space, it is intriguing to characterize the class of all discrete self-similar Markov chains. In the aforementioned paper, the authors showed that a certain class of birth-death chains satisfy discrete self-similarity. The first half of this chapter is devoted to finding a way of constructing the discrete self-similar Markov chains. After proving that these Markov chains must be *upward skip-free*, we establish a "gateway" relationship between the discrete and continuous self-similar Markov processes. Consequently, we obtain an isospectral approximation of continuous self-similar Markov processes by discrete self-similar Markov chains. In the second half of this chapter, we investigate a class of ergodic Markov chains obtained by perturbing the the discrete self-similar CTMCs. This idea was motivated from its continuous analogue, named as the generalized Laguerre semigroups, studied by Patie and Savov [95]. Since the state space is discrete and the Markov chains are non-reversible in our setup, there is no direct way to analyze the rate of convergence to the equilibrium, hypercontractivity via the classical log-Sobolev inequalities. We dealt this case with the help of the *intertweaving relations*, a concept recently introduced by Miclo and Patie [85].

In Chapter 4, we carry out an in-depth analysis of the intriguing van Dantzig problem which consists on characterizing the set $\mathscr{D}$ of analytic characteristic functions $\mathcal{F}$ which remains stable by the action of the mapping $V\mathcal{F}(t) = 1/\mathcal{F}(\mathrm{i}t)$, $t \in \mathbb{R}$. We start by observing that the celebrated Lee-Yang property, appearing in statistical mechanics and quantum field theory, and the Riemann hypothesis can be both rephrased in terms of the van Dantzig problem, and, more specifically, in terms of the set $\mathscr{D}_L \subset \mathscr{D}$ of real-valued characteristic functions that belong to the Laguerre-Pólya class. Motivated by these facts, we proceed by identifying several non-trivial closure properties of the set $\mathscr{D}$ and $\mathscr{D}_L$. This not only revisits but also, by means of probabilistic techniques, deepens the fascinating studies of the set of even characteristic functions in the Laguerre-Pólya class carried out by Pólya [112], de Bruijn [37], Lukacs [82], Newman [90] and more recently by Newman and Wu [91], among others. We continue by providing a new class of entire functions that belong to the set $\mathscr{D}$ but not necessarily to $\mathscr{D}_L$, offering the first examples outside the set $\mathscr{D}_L$. This class, which is derived from some entire functions introduced in [102], is in bijection with a subset of continuous negative-definite functions and includes several notable generalized hypergeometric type functions. Besides identifying the characteristic functions, we also manage to characterize the pair of the corresponding van Dantzig random variables revealing that one of them is infinitely divisible. Finally, we investigate the possibility that the Riemann ξ function belongs to this class.

1.1 Notations

In this section we enlist general notations adapted throughout the thesis.

Functional Spaces and operators

Let $E = \mathbb{R}$ or $\mathbb{R}_+$. For any $k \in \mathbb{N} \cup \{0\}$, we denote by $\mathbf{C}^k(E)$ the class of all functions defined on E that are k-times continuously differentiable. When $k = 0$, we write $\mathbf{C}^0(E)$ simply as $\mathbf{C}(E)$. By $\mathbf{C}_0(E)$, we denote the class of all continuous functions defined on E that vanishes at infinity. We also consider the spaces $\mathbf{C}_b^k(E)$, the class of all k-times continuously differentiable and bounded functions with bounded derivatives. Finally, we denote by $\mathbf{C}_c^\infty(E)$, the class of all compactly supported smooth functions on E. When the functions are assumed to be smooth with all the derivatives vanishing at infinity, we denote them by $\mathbf{C}_0^\infty(E)$. When E is a discrete set, the sets

$\mathbf{C}_b(E), \mathbf{C}_0(E)$ have the same meaning as above, and $\mathbf{C}_c(E)$ denotes the set of all finitely supported functions on E.

For any two Banach spaces $\mathrm{X}_1, \mathrm{X}_2$, we denote by $\mathscr{B}(\mathrm{X}_1, \mathrm{X}_2)$ the space of all bounded linear operators defined from X_1 to X_2. When $\mathrm{X}_1 = \mathrm{X}_2 = \mathrm{X}$, we simply denote the unital algebra by $\mathscr{B}(\mathrm{X})$. For any $T \in \mathscr{B}(\mathrm{H}_1, \mathrm{H}_2)$ where $\mathrm{H}_1, \mathrm{H}_2$ are Hilbert spaces, the adjoint of T is denoted by $\widehat{T}$, which is an element in $\mathscr{B}(\mathrm{H}_2, \mathrm{H}_1)$.

Complex plane, strips and analytic functions

We use $\mathbb{C}$ to denote the complex plane. For any $-\infty \leqslant a < b \leqslant \infty$, we use $\mathbb{C}_{(a,b)}$ to denote the vertical strip $\{z \in \mathbb{C};\ \mathtt{Re}(z) \in (a,b)\}$. For the horizontal strip we use the notation $\mathbb{S}_{(a,b)}$. In the same manner, we use the notations $\mathbb{C}_{[a,b]}$, $\mathbb{S}_{[a,b]}$ to denote the strips including their boundaries. Again, for $-\infty \leqslant a < b \leqslant \infty$, we use $\mathbf{A}_{(a,b)}$ to denote the class of functions that are analytic on the strip $\mathbb{S}_{(a,b)}$. We use $\mathbf{A}_{[a,b]}$ to denote the class of analytic functions on $\mathbb{S}_{(a,b)}$ that extend continuously to the boundary of the strip. We also use the classical notation $\mathrm{i}^2 = -1$.

Boundedness and asymptotics of ratios

For two functions f, g defined on the complex plane, we use the following notations

$$\begin{aligned}
f &\asymp g \text{ means that } \exists c > 0 \text{ such that } c^{-1} \leqslant \frac{f}{g} \leqslant c, \\
f &\overset{a}{\sim} g \text{ means that } \lim_{x \to a} \frac{f(x)}{g(x)} = 1 \text{ for some } a \in [0, \infty], \\
f(x) &\overset{a}{=} \mathrm{O}(g(x)) \text{ means that } \varlimsup_{x \to a} \left| \frac{f(x)}{g(x)} \right| < \infty, \\
f(x) &\overset{a}{=} \mathrm{o}(g(x)) \text{ means that } \lim_{x \to a} \frac{f(x)}{g(x)} = 0.
\end{aligned}$$

Spectrum of operators in Hilbert spaces

For a linear operator A acting on a Hilbert space H, with identity operator I, we write $\mathrm{Spec}(A) = \{\lambda \in \mathbb{C};\ A - \lambda I$ is not invertible$\}$, $\mathrm{Spec}_p(A) = \{\lambda \in \mathrm{Spec}(A);\ A - \lambda I$ is not injective$\}$, $\mathrm{Spec}_c(A) = \{\lambda \in \mathrm{Spec}(A) \setminus \mathrm{Spec}_p(A);\ A - \lambda I$ has a dense range$\}$ and $\mathrm{Spec}_r(A) = \mathrm{Spec}(A) \setminus (\mathrm{Spec}_p(A) \cup \mathrm{Spec}_c(A))$ which corresponds to the spectrum, point spectrum, continuous spectrum and residual spectrum of A respectively, see e.g. the monograph of Dunford and Schwartz [40, XV.8]. Additionally, we denote the approximate point spectrum of A by

$$\mathrm{Spec}_{ap}(A) = \{\lambda \in \mathbb{C}; \exists\, (v_n)_{n\geqslant 0} \subset \mathrm{H} \text{ such that } \|v_n\|_{\mathrm{H}} = 1,\, \|Av_n - \lambda v_n\|_{\mathrm{H}} \to 0\}$$

where $\|.\|_{\mathrm{H}}$ stands for the norm in H.

CHAPTER 2

WEAK SIMILARITY ORBIT OF (LOG) SELF-SIMILAR MARKOV SEMIGROUPS ON THE EUCLIDEAN SPACE

2.1 Introduction

In this chapter, we develop an in-depth and detailed analysis of the non-self-adjoint and non-local PDOs in the class

$$\mathscr{A} = \{\boldsymbol{A}^M_{\texttt{PDO}}[\boldsymbol{\psi}];\ \boldsymbol{\psi} \in \mathbf{N}^d_b(\mathbb{R}), M \in \mathrm{GL}_d(\mathbb{R})\}$$

of densely defined operators in the weighted Hilbert space $\mathbf{L}^2(\mathbb{R}^d, \mathrm{e}_M)$, see (2.19) for definition, and whose symbols take the form

$$\mathrm{a}(\boldsymbol{x}, \boldsymbol{\xi}) = -\langle e(-M^{-1}\boldsymbol{x}), \boldsymbol{\psi}(M^\top \boldsymbol{\xi})\rangle,\ \boldsymbol{x}, \boldsymbol{\xi} \in \mathbb{R}^d \tag{2.1}$$

where $e(\boldsymbol{x}) = (e^{x_1}, \ldots, e^{x_d})$, $\langle ., . \rangle$ is the Euclidean inner product, $\boldsymbol{\psi} = (\psi_1, \ldots, \psi_d) \in \mathbf{N}^d_b(\mathbb{R})$, a vector of continuous definite functions defined in (2.17), and $M \in \mathrm{GL}_d(\mathbb{R})$, the group of $d \times d$ invertible matrices. This includes the study of their domains, a classification scheme by weak similarity orbit, a notion that we introduce in Section 2.1.2, the generation of $\mathcal{C}_0$- positivity-preserving contraction semigroups, the spectral theory and representation of the latter, and representation as integro-differential operators.

Let us now describe our main results and the strategy we have implemented to get them, and, we refer to the diagram 2.1 for a visual description of the path we have followed. Our approach relies on the concept of weak similarity (WS) relation. More specifically, we say that the two linear operators P, Q have a WS relation if

$$P\Lambda = \Lambda Q \ \text{ on } \ \mathbf{D}(\Lambda) \tag{2.2}$$

where $\Lambda \in \mathscr{D}(\mathrm{H})$, i.e. it is a densely defined injective linear operator with a dense range in a Hilbert space H, and we denote its domain by $\mathbf{D}(\Lambda)$, see Section 2.1.2 for a refined definition. In such case, we say that P is in the WS orbit of Q. Since we shall show that, in our context, WS is an equivalence relation, Q may also be seen as a member of the WS orbit of P.

Coming back to our setting, we start by showing, by a combination of Fourier and complex analysis, that, under very mild condition, see Remark 2.1.5 for a discussion, the class of symbols of the form (2.1) is in the WS orbit of the self-adjoint PDO whose symbol is given by

$$\mathrm{a}_0(\boldsymbol{x},\boldsymbol{\xi}) = -\langle e(-\boldsymbol{x}), \boldsymbol{\psi}_0(\boldsymbol{\xi})\rangle, \ \boldsymbol{x},\boldsymbol{\xi} \in \mathbb{R}^d, \tag{2.3}$$

where $\boldsymbol{\psi}_0(\boldsymbol{\xi}) = (\xi_1^2, \ldots, \xi_d^2)$. Let us point out that in [95], in the one-dimensional case, relying on original factorization of some random variables, intertwining relations (this terminology means that Λ in (2.2) is a bounded Markov operator) has been obtained between self-similar semigroups on $\mathbb{R}_+$ under some conditions on the negative definite functions ψ. Although, by the isomorphism aforementioned, these relations could easily be transferred to our framework, the required conditions, which are rather restrictive, could not be relaxed. Moreover, an attempt to obtain weak similarity relations from the representation of the infinitesimal generator as an integro-differential operator or Dynkin characteristic operator given by Lamperti [76] also leads to some restrictive conditions such as, at least, the requirement of analytical extension to some strip of the Lévy-Khintchine exponent.

These are the main reasons that have forced us to consider the study of these generators problem from a different and fresh perspective by starting instead from the class of PDOs with symbols of the form (2.1) to first characterize the WS orbit of (log)-Bessel semigroups for almost all the class of log-Lamperti semigroups. However, this approach gives rise to several issues, such as the existence of a $\mathcal{C}_0$-contraction semigroup associated to this PDO's, the positivity preserving property of the latter as well as their relation to log-Lamperti semigroups. We emphasize that the latter is not a trivial exercise as, to the best of our knowledge, there does not exist an equivalent of Volkinski's formula for PDO's. We proceed by explaining the paths we follow to overcome these difficulties.

First, establishing a link between the PDO with symbol (2.3) and the Laplacian acting on the space of symmetric functions, and, resorting to classical results from the theory of self-adjoint operators, we show that it generates the self-adjoint $\mathcal{C}_0$-contraction semigroup, on the weighted Hilbert space $\mathbf{L}^2(\mathbb{R}^d, \mathbf{e}_M)$, where $\mathbf{e}_M$ is defined in (2.19), of the log-squared Bessel process, which is also positivity preserving. Unfortunately, these classical results do not carry over to the class of symbols of the form (2.1).

To deal with the generation of a, in general non-self-adjoint, $\mathcal{C}_0$-contraction semigroup on the same weighted $\mathbf{L}^2$-space, we first observe that the class of symbols of the form (2.1) is closed under conjugation allowing us to split the analysis of this class of PDO's into two. Then, using, in the one-dimensional case, the WS relation mentioned above we manage to identify a core for one of these two classes as well as the dissipativity property which guarantees the existence of a $\mathcal{C}_0$-contraction semigroup. For the other remaining class, we resort to classical results from the theory of $\mathcal{C}_0$-contraction semigroup. We also manage to lift the WS relations between the PDO's to the level of the $\mathcal{C}_0$-contractions semigroups.

Based on the WS relation and the well-known diagonalisation of the Bessel semigroup, see e.g. [89], we proceed by giving, in Theorem 2.2.4, the spectral representation of the non-self-adjoint $\mathcal{C}_0$-contraction semigroups generated by these PDO's. It is valid, at least, on a domain, defined in terms of objects related to the negative definite function ψ, which is either the full or a dense subset of the Hilbert space $\mathbf{L}^2(\mathbb{R}^d, \mathbf{e}_M)$. The integral representation is expressed in terms of Fourier multiplier operators which are, themselves, defined in terms of the so-called Bernstein-gamma functions associated to the Wiener-Hopf factors of the negative definite functions ψ, see (2.10) for definition. In this direction, we mention that spectral properties of some special instances of self-similar Markov semigroups have already been studied in the literature. For instance, spectral expansion of self-similar semigroups corresponding to fractional derivative operators have been derived in [106], the spectral representation of the transition density of stable processes on the half-line and related objects have been obtained in [68, 70, 71, 88].

Next, to prove the positivity preserving property, thanks again to the characterization of the core by the WS relation, we show that each PDO coincide with the Dynkin operator of a positive self-similar Markov process, see (2.40) for definition. The d-dimensional case is then deduced by means of a tensorization argument combined with a similarity transform.

We proceed by characterizing the nature of the spectrum which depends on the asymptotic behavior of the Bernstein-gamma functions, see (2.10), which we manage to reduce to easy-to-check properties on the Wiener-Hopf factors or, directly, on the negative definite functions. The class spans all parts of the spectrum including the point, residual, approximate or continuous ones, which reveals the flexibility of the concept of WS in this context. It is already worth pointing out that the spectral representation is valid on the full Hilbert space (resp. on a dense domain) when the spectrum is the residual one (resp. point one), something which, to the best of our knowledge, seems to have been unnoticed in the literature.

The remaining part of this chapter is organized as follows. After providing some general notations, we present our main results, including the WS orbit, the nature of the spectrum, the spectral representation in Section 2.2. Before providing the proofs of these results in Section 4.5, we recall, in Section 2.3 some substantial preliminary results and define some tools that will be essential in the course of the proofs. The last Section is devoted to the detailed description of some new examples for which we express the spectral components in terms of known special functions or recently introduced power series generalizing the later. In Figure 2.1, we provide a chart to indicate how the main results are connected to each other.

2.1.1 Notations and preliminaries

Throughout this chapter, we denote by $\mathbf{L}^2(\mathbb{R}^d, \boldsymbol{\mu})$ the class of all square integrable functions $\mathbf{f}$ with respect to the non-negative Borel measure $\boldsymbol{\mu}$ on $\mathbb{R}^d$ endowed with the natural inner-product $\langle \mathbf{f}, \mathbf{g} \rangle_{\boldsymbol{\mu}} = \int_{\mathbb{R}^d} \mathbf{f}(\boldsymbol{x})\overline{\mathbf{g}(\boldsymbol{x})}\boldsymbol{\mu}(d\boldsymbol{x})$. In particular, when $\boldsymbol{\mu}(d\boldsymbol{x}) = e^{\langle \boldsymbol{x}, \mathbf{1} \rangle} d\boldsymbol{x}$ (resp. $\boldsymbol{\mu}(d\boldsymbol{x}) = e(x)dx = e^x dx, x \in \mathbb{R}$) we denote the Hilbert space by $\mathbf{L}^2(\mathbb{R}^d, \mathbf{e})$ (resp. $\mathbf{L}^2(\mathbb{R}, e)$). We use the notation "$\perp$" to indicate the orthogonality relation on Hilbert spaces.

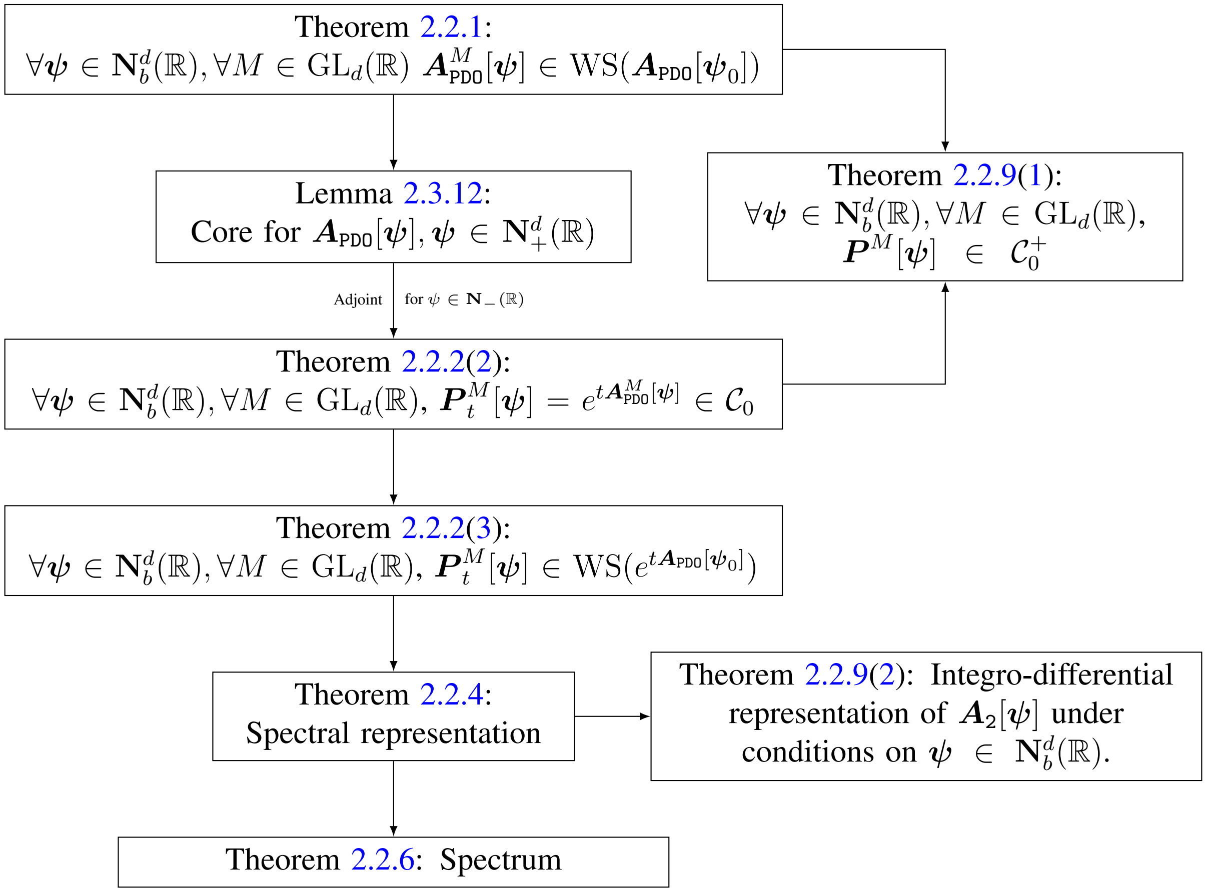

Figure 2.1: User guide on the connection between the main results

For two Banach spaces $\mathrm{X}_1, \mathrm{X}_2$, and an operator $\Lambda : \mathrm{X}_1 \to \mathrm{X}_2$, we denote by $\mathbf{D}(\Lambda)$ the domain of Λ. We also use the set

$$\mathscr{D}_0(\mathrm{X}_1, \mathrm{X}_2) = \{\Lambda : \mathbf{D}(\Lambda) \subset \mathrm{X}_1 \to \mathrm{X}_2 \text{ linear, densely defined, injective with dense range}\}$$

and we write $\mathscr{D}(\mathrm{X}_1, \mathrm{X}_2)$ to denote the set of all densely defined operators from X_1 to X_2. When $\mathrm{X}_1 = \mathrm{X}_2 = \mathrm{X}$, we use the notations $\mathscr{D}_0(\mathrm{X})$ and $\mathscr{D}(\mathrm{X})$ respectively.

Throughout this chapter, we use the notation $\mathcal{C}_0(\mathbf{L}^2(\mathbb{R}^d, \boldsymbol{\mu}))$ (resp. $\mathcal{C}_0^+(\mathbf{L}^2(\mathbb{R}^d, \boldsymbol{\mu}))$) to denote the set of (resp. positivity-preserving) strongly continuous semigroups on the Hilbert space $\mathbf{L}^2(\mathbb{R}^d, \boldsymbol{\mu})$ where $\boldsymbol{\mu}$ is σ-finite measure on $\mathbb{R}^d$. We recall that $(\boldsymbol{P}_t)_{t \geqslant 0} \in \mathcal{C}_0(\mathbf{L}^2(\mathbb{R}^d, \boldsymbol{\mu}))$ is said to be positivity-preserving if for all $t \geqslant 0$, $\boldsymbol{P}_t \mathbf{f} \geqslant 0$ a.s. whenever $\mathbf{f} \geqslant 0$ a.s.

Fourier transform in $\mathbf{L}^2(\mathbb{R}^d, \mathrm{e})$ and multipliers

Let us denote by $\mathcal{F}^{\mathbf{e}}$ the shifted Fourier transform defined for functions $\mathbf{f}$ such that $\sqrt{\mathbf{e}}\mathbf{f}$ is integrable as follows

$$\mathcal{F}^{\mathbf{e}}_{\mathbf{f}}(\boldsymbol{\xi}) = (2\pi)^{-\frac{d}{2}} \int_{\mathbb{R}^d} e^{\mathrm{i}\langle \boldsymbol{\xi}, \boldsymbol{x}\rangle} \sqrt{\mathbf{e}(\boldsymbol{x})}\mathbf{f}(\boldsymbol{x})d\boldsymbol{x} = \mathcal{F}_{\sqrt{\mathbf{e}}\mathbf{f}}(\boldsymbol{\xi}),\ \boldsymbol{\xi} \in \mathbb{R}^d, \tag{2.4}$$

where $\mathcal{F}$ is the usual Fourier transform and we recall that $\mathbf{e}(\boldsymbol{x}) = e^{\langle \boldsymbol{x}, \mathbf{1}\rangle}$, $\boldsymbol{x} \in \mathbb{R}^d, \mathbf{1} = (1, 1, \dots, 1)$. $\mathcal{F}^{\mathbf{e}}$ (resp. $\mathcal{F}$) is an unitary operator from $\mathbf{L}^2(\mathbb{R}^d, \mathbf{e})$ (resp. $\mathbf{L}^2(\mathbb{R}^d)$) into $\mathbf{L}^2(\mathbb{R}^d)$, which explains the notation. Next, for a function $\mathbf{m}^{\mathbf{e}}_{\boldsymbol{\Lambda}} : \mathbb{R}^d \to \mathbb{C}$, we define the *shifted Fourier operator* $\boldsymbol{\Lambda} : \mathbf{L}^2(\mathbb{R}^d, \mathbf{e}) \to \mathbf{L}^2(\mathbb{R}^d, \mathbf{e})$ with multiplier $\mathbf{m}^{\mathbf{e}}_{\boldsymbol{\Lambda}}$ by

$$\mathcal{F}^{\mathbf{e}}_{\boldsymbol{\Lambda}\mathbf{f}}(\boldsymbol{\xi}) = \mathbf{m}^{\mathbf{e}}_{\boldsymbol{\Lambda}}(\boldsymbol{\xi})\, \mathcal{F}^{\mathbf{e}}_{\mathbf{f}}(\boldsymbol{\xi}),\ \boldsymbol{\xi} \in \mathbb{R}^d. \tag{2.5}$$

When the multiplier $\mathbf{m}^{\mathbf{e}}_{\boldsymbol{\Lambda}}$ is analytic in the cylinder $\mathbb{S}^d_{(0,1)}$, we will use the notation $\mathbf{m}_{\boldsymbol{\Lambda}}\left(\boldsymbol{\xi} + \frac{\mathrm{i}}{2}\mathbf{1}\right) = \mathbf{m}^{\mathbf{e}}_{\boldsymbol{\Lambda}}(\boldsymbol{\xi})$, which corresponds to the (shifted) multiplier associated to the classical Fourier transform. $\boldsymbol{\Lambda}$ is a densely defined operator with domain

$$\mathbf{D}(\boldsymbol{\Lambda}) = \left\{\mathbf{f} \in \mathbf{L}^2(\mathbb{R}^d, \mathbf{e});\ \boldsymbol{\xi} \mapsto \mathbf{m}^{\mathbf{e}}_{\boldsymbol{\Lambda}}(\boldsymbol{\xi})\, \mathcal{F}^{\mathbf{e}}_{\mathbf{f}}(\boldsymbol{\xi}) \in \mathbf{L}^2(\mathbb{R}^d)\right\}. \tag{2.6}$$

When $\mathbf{m}^{\mathbf{e}}_{\boldsymbol{\Lambda}}$ is a.e. non-zero, $\boldsymbol{\Lambda} \in \mathscr{D}_0(\mathbf{L}^2(\mathbb{R}^d, \mathbf{e}))$. In Subsection 2.3.4, we will discuss properties of these operators in more details.

Continuous negative definite functions, Wiener-Hopf factorization and Bernstein-gamma functions

Let $\mathbf{N}(\mathbb{R})$ be the class of all continuous negative definite functions defined on $\mathbb{R}$, i.e. for any $\psi \in \mathbf{N}(\mathbb{R})$ there exists a unique quadruple $(\psi(0), \sigma^2, \mathtt{b}, \mu)$ such that

$$\psi(\xi) = \psi(0) - \mathrm{i}\mathtt{b}\xi + \sigma^2\xi^2 + \int_{\mathbb{R}} \left(1 - e^{\mathrm{i}\xi y} + \mathrm{i}y\xi\mathbb{1}_{\{|y|\leqslant 1\}}\right)\mu(dy),\ \xi \in \mathbb{R}, \tag{2.7}$$

where $\psi(0) \geqslant 0, \mathtt{b} \in \mathbb{R}, \sigma^2 \geqslant 0$ and μ is a non-negative Radon measure such that $\int_{\mathbb{R}}(y^2 \wedge 1)\mu(dy) < \infty$, see e.g. [56] and [120, Chap. 4]. It is well-known that the set $\mathbf{N}(\mathbb{R})$ is stable by conjugation, that is $\psi \in \mathbf{N}(\mathbb{R})$ if and only if its conjugate function $\overline{\psi} \in \mathbf{N}(\mathbb{R})$. Moreover, each $\psi \in \mathbf{N}(\mathbb{R})$ admits the following analytic Wiener-Hopf factorization

$$\psi(\xi) = \phi_+(-\mathrm{i}\xi)\phi_-(\mathrm{i}\xi), \;\; \xi \in \mathbb{R}, \tag{2.8}$$

where $\phi_+, \phi_- \in \mathbf{B}$, the set of Bernstein functions, which are functions $\phi : [0, \infty) \mapsto [0, \infty)$ of the form

$$\phi(u) = \phi(0) + \mathtt{d}u + \int_0^\infty (1 - e^{-uy})\nu(dy), \quad u \geqslant 0, \tag{2.9}$$

with $\phi(0) \geqslant 0, \mathtt{d} \geqslant 0$ and ν is a non-negative Radon measure such that $\int_0^\infty (1 \wedge y)\nu(dy) < \infty$. Any $\phi \in \mathbf{B}$ can be extended analytically on $\mathbb{C}_{(0,\infty)}$ that remains continuous on $\mathrm{i}\mathbb{R}$. The Wiener-Hopf factors ϕ_+ and ϕ_- are the Laplace exponents of the so-called ascending and descending ladder height processes corresponding to the Lévy process associated with ψ. Also, existence of jump measures in ϕ_+ (resp. ϕ_-) indicates existence of positive (resp. negative) jumps of the Lévy process determined by (2.38). Note that if ψ is of the form (2.7), it follows immediately that its conjugate admits the factorization

$$\overline{\psi}(\xi) = \phi_-(-\mathrm{i}\xi)\phi_+(\mathrm{i}\xi), \;\; \xi \in \mathbb{R}.$$

For an excellent account on the Wiener-Hopf factorization of Lévy processes, we refer to [12, Chapter VI] and [73, Chapter 6].

Next, for any $\phi \in \mathbf{B}$, let us denote by W_ϕ the so-called Bernstein-gamma function associated to ϕ, that is the unique positive definite function, i.e. the Mellin transform of a positive random variable, satisfying the functional equation

$$W_\phi(z + 1) = \phi(z)W_\phi(z), \;\; z \in \mathbb{C}_{(0,\infty)}, \text{ and } W_\phi(1) = 1. \tag{2.10}$$

We refer to [96] for a thorough account on this function and in particular from [96, Sec. 4], one gets that

$$W_\phi \text{ is analytic and zero-free (at least) on } \mathbb{C}_{[0,\infty)}. \tag{2.11}$$

2.1.2 The notion of weak similarity

The cornerstone of our approach to obtain a detailed spectral theory for the semigroups generated by $\boldsymbol{A}^{M}_{\mathtt{PDO}}[\boldsymbol{\psi}]$ stems on its membership in the weak similarity orbit of the self-adjoint PDO $A_{\mathtt{PDO}}[\boldsymbol{\psi}_0]$ given by (2.3), a concept that we now define. For a Hilbert space H, let us first recall that

$$\mathscr{D}_0(\mathrm{H}) = \{\Lambda : \mathbf{D}(\Lambda) \subseteq \mathrm{H} \mapsto \mathrm{H} \text{ linear, densely defined, closed, injective with dense range}\}$$

and note that it is not, in general, a group of operators.

Definition 2.1.1. For two Hilbert spaces $\mathrm{H}_1, \mathrm{H}_2$ and $A \in \mathscr{D}(\mathrm{H}_1), B \in \mathscr{D}(\mathrm{H}_2)$, we say that A *is weakly similar to* B if there exists $\Lambda \in \mathscr{D}_0(\mathrm{H}_2, \mathrm{H}_1)$ such that

$$A\Lambda = \Lambda B \ \text{ on } \ \mathcal{D} \tag{2.12}$$

with $\mathcal{D}, \Lambda(\mathcal{D})$ being dense in $\mathrm{H}_2, \mathrm{H}_1$ respectively, and

$$\mathcal{D} \subset \mathbf{D}(B) \cap \mathbf{D}(\Lambda), \ \Lambda(\mathcal{D}) \subset \mathbf{D}(A) \ \text{ and } \ B(\mathcal{D}) \subset \mathbf{D}(\Lambda). \tag{2.13}$$

Λ is called a *weak similarity operator*.

From now on we implicitly assume (2.13) whenever we have identities of the type (2.12). In the above definition we note that when A, B are bounded and $\mathcal{D}$ is a core for Λ, the identity (2.12) extends to $\mathbf{D}(\Lambda)$.

Definition 2.1.2. Let $\mathrm{H}_1, \mathrm{H}_2$ be Hilbert spaces and $B \in \mathscr{D}(\mathrm{H}_2)$. We define the *weak similarity orbit* of B in $\mathscr{D} \subset \mathscr{D}(\mathrm{H}_1)$ as the set of all operators $A \in \mathscr{D}$ such that A is weakly similar to B. We write

$$\mathrm{WS}_{\mathscr{D}}(B) = \{A \in \mathscr{D};\ A \text{ is weakly similar to } B\}.$$

When $\mathscr{D} = \mathscr{D}(\mathrm{H}_1)$, we simply denote $\mathrm{WS}(B) = \mathrm{WS}_{\mathscr{D}}(B)$.

2.1.3 Standing assumptions

We now set up the assumptions and notation that will be in force throughout this chapter, and we refer to the remarks below for discussions about their generality. Let

$$\mathbf{N}_+(\mathbb{R}) = \left\{\psi \in \mathbf{N}(\mathbb{R});\ \underline{\lim}_{|\xi|\to\infty} \frac{|\phi_+(\mathrm{i}\xi)|}{|\xi|^{-\kappa}} > 0 \text{ for some } \kappa > 0,\ \frac{W_{\phi_+}(\frac{1}{2}+\mathrm{i}\xi)}{W_{\phi_-}(\frac{1}{2}+\mathrm{i}\xi)} = \mathrm{O}(1)\right\} \tag{2.14}$$

$$\mathbf{N}_-(\mathbb{R}) = \left\{\psi \in \mathbf{N}(\mathbb{R});\ \overline{\psi} \in \mathbf{N}_+(\mathbb{R})\right\} \tag{2.15}$$

and

$$\mathbf{N}_b(\mathbb{R}) = \mathbf{N}_+(\mathbb{R}) \cup \mathbf{N}_-(\mathbb{R}). \tag{2.16}$$

Finally, for any $d \in \mathbb{N}, d \geqslant 2$, we write $\boldsymbol{\psi} = (\psi_1, \ldots, \psi_d) \in \mathbf{N}_b^d(\mathbb{R})$ where

$$\mathbf{N}_b^d(\mathbb{R}) = \mathbf{N}_b(\mathbb{R}) \times \ldots \times \mathbf{N}_b(\mathbb{R}) \quad (d \text{ times}) \tag{2.17}$$

and, for any $k = 1, \ldots, d$, $(\psi_k(0), \sigma_k^2, \mathtt{b}_k, \mu_k)$ is the characteristic quadruplet of ψ_k, see (2.7).

Remark 2.1.3. Since for all $\phi \in \mathbf{B}$, the conjugate of W_ϕ is such that $\overline{W}_\phi(z) = W_\phi(\overline{z})$, $z \in \mathbb{C}_{(0,\infty)}$, we note that if $\psi \in \mathbf{N}_-(\mathbb{R})$ then

$$\underline{\lim}_{|\xi|\to\infty} |\xi|^\kappa |\phi_-(\mathrm{i}\xi)| > 0 \text{ for some } \kappa > 0 \text{ and } \frac{W_{\phi_-}(\frac{1}{2}+\mathrm{i}\xi)}{W_{\phi_+}(\frac{1}{2}+\mathrm{i}\xi)} = \mathrm{O}(1),$$

justifying the notation.

Remark 2.1.4. We point out that, for $\phi \in \mathbf{B}$, the condition $\underline{\lim}_{|\xi|\to\infty} |\xi|^\kappa |\phi(\mathrm{i}\xi)| > 0$ has been termed as the *weak non-lattice* property in [96, (2.29)], and in particular, it implies that

$$\phi \text{ is zero-free on } \mathrm{i}\mathbb{R} \setminus \{0\}. \tag{2.18}$$

Here, the weak non-lattice property is used to ensure the moderate growth of certain functions along the lines $\mathtt{Re}(z) = a$, $0 \leqslant a \leqslant 1$, which plays an important role in the proof of our results. We refer to Section 2.3.6 for more details.

Remark 2.1.5. Although the weak non-lattice property disregards only very specific and even degenerate cases, it seems to be the main restriction in the definition of the set $\mathbf{N}_b(\mathbb{R})$ as we believe that for any $\psi \in \mathbf{N}(\mathbb{R})$, the ratio of its associated Bernstein-gamma functions is either bounded from above or below.

Remark 2.1.6. We also emphasize that, in many instances, the sets $\mathbf{N}_+(\mathbb{R})$ and $\mathbf{N}_-(\mathbb{R})$ can be expressed in terms of the characteristic quadruple of the negative definite function ψ and/or in terms of the parameters defining its Wiener-Hopf factors. For more details, we refer to Proposition 2.2.11 below.

2.2 Main results

2.2.1 The weak similarity orbits $\mathrm{WS}_{\mathscr{A}_M}$

For any $\boldsymbol{\psi} = (\psi_1, \ldots, \psi_d) \in \mathbf{N}_b^d(\mathbb{R})$ and $M \in \mathrm{GL}_d(\mathbb{R})$, the group of $d \times d$ invertible matrices, we consider the pseudo-differential operator

$$\boldsymbol{A}^M_{\mathsf{PDO}}[\boldsymbol{\psi}] : \mathbf{D}(\boldsymbol{A}^M_{\mathsf{PDO}}[\boldsymbol{\psi}]) \subset \mathbf{L}^2(\mathbb{R}^d, \mathbf{e}_M) \to \mathbf{L}^2(\mathbb{R}^d, \mathbf{e}_M)$$

with symbol

$$\mathrm{a}(\boldsymbol{x}, \boldsymbol{\xi}) = -\langle e(-M^{-1}\boldsymbol{x}), \boldsymbol{\psi}(M^\top \boldsymbol{\xi})\rangle,\ \boldsymbol{x}, \boldsymbol{\xi} \in \mathbb{R}^d,$$

where we have set

$$\mathbf{e}_M(\boldsymbol{x}) = \mathbf{e}(M^{-1}\boldsymbol{x}) = e^{\langle M^{-1}\boldsymbol{x}, \mathbf{1}\rangle}, \quad \boldsymbol{x} \in \mathbb{R}^d. \tag{2.19}$$

When $M = \mathrm{Id}$, we simply write $\boldsymbol{A}_{\mathsf{PDO}}[\boldsymbol{\psi}] = \boldsymbol{A}^M_{\mathsf{PDO}}[\boldsymbol{\psi}]$ and $\mathbf{L}^2(\mathbb{R}^d, \mathbf{e}) = \mathbf{L}^2(\mathbb{R}^d, \mathbf{e}_M)$. Let us also define

$$\mathcal{D}(\mathbb{R}) = \{f \in \mathbf{L}^2(\mathbb{R}, e);\ \mathcal{F}_f \text{ is entire and satisfies (2.20)}\}$$

$$\text{For any compact } C \subset \mathbb{C},\ \sup_{z \in C} |\mathcal{F}_f(z)| = \mathrm{O}(e^{-(\frac{\pi}{2}+\epsilon)|\mathrm{Re}(z)|}),\ \epsilon > 0, \tag{2.20}$$

and we write $\boldsymbol{\mathcal{D}}(\mathbb{R}^d) = \otimes_{k=1}^d \mathcal{D}(\mathbb{R})$. We are now ready to state our first main result.

Theorem 2.2.1. *For any $M \in \mathrm{GL}_d(\mathbb{R})$, writing $\mathscr{A}_M = \{\boldsymbol{A}^M_{\mathtt{PDO}}[\boldsymbol{\psi}];\ \boldsymbol{\psi} \in \mathbf{N}^d_b(\mathbb{R})\}$, we have*

$$\mathrm{WS}_{\mathscr{A}_M}(\boldsymbol{A}_{\mathtt{PDO}}[\boldsymbol{\psi}_0]) = \mathscr{A}_M$$

where $\boldsymbol{\psi}_0(\boldsymbol{\xi}) = (\xi_1^2, \ldots, \xi_d^2)$. More specifically, for any $\boldsymbol{\psi} \in \mathbf{N}^d_b(\mathbb{R})$ and $M \in \mathrm{GL}_d(\mathbb{R})$, we have

$$\boldsymbol{A}^M_{\mathtt{PDO}}[\boldsymbol{\psi}]\boldsymbol{\Lambda}_{\boldsymbol{\psi},M} = \boldsymbol{\Lambda}_{\boldsymbol{\psi},M}\boldsymbol{A}_{\mathtt{PDO}}[\boldsymbol{\psi}_0] \ \textit{ on } \ \boldsymbol{\mathcal{D}}(\mathbb{R}^d) \tag{2.21}$$

where $\boldsymbol{\Lambda}_{\boldsymbol{\psi},M} : \mathbf{D}(\boldsymbol{\Lambda}_{\boldsymbol{\psi},M}) \subset \mathbf{L}^2(\mathbb{R}^d, \mathbf{e}) \to \mathbf{L}^2(\mathbb{R}^d, \mathbf{e}_M)$ is such that $\boldsymbol{\Lambda}_{\boldsymbol{\psi},M}\mathbf{f}(\boldsymbol{x}) = \boldsymbol{\Lambda}_{\boldsymbol{\psi}}\mathbf{f}(M\boldsymbol{x})$, and $\boldsymbol{\Lambda}_{\boldsymbol{\psi}}$ is a shifted Fourier multiplier operator associated to

$$\mathbf{m}^{\mathbf{e}}_{\boldsymbol{\Lambda}_{\boldsymbol{\psi}}}(\boldsymbol{\xi}) = \prod_{k=1}^d \frac{W_{\phi_{+,k}}(\frac{1}{2} - \mathrm{i}\xi_k)}{W_{\phi_{-,k}}(\frac{1}{2} + \mathrm{i}\xi_k)} \frac{\Gamma(\frac{1}{2} + \mathrm{i}\xi_k)}{\Gamma(\frac{1}{2} - \mathrm{i}\xi_k)}.$$

Moreover, for any $\boldsymbol{\psi} \in \mathbf{N}^d_+(\mathbb{R})$, $\boldsymbol{\Lambda}_{\boldsymbol{\psi},M} \in \mathscr{B}(\mathbf{L}^2(\mathbb{R}^d, \mathbf{e}), \mathbf{L}^2(\mathbb{R}^d, \mathbf{e}_M))$.

This theorem is proved in Subsection 2.5.3.

2.2.2 Generation of the set $\mathscr{P}$ of $\mathcal{C}_0$-contractions semigroups and $\mathrm{WS}_{\mathscr{P}}$

Let us introduce the linear differential operator

$$\boldsymbol{A}_0\mathbf{f}(\boldsymbol{x}) \ = \ \mathrm{Tr}\left(\Sigma(\boldsymbol{x})\nabla^2\mathbf{f}(\boldsymbol{x})\right), \quad \boldsymbol{x} \in \mathbb{R}^d, \tag{2.22}$$

where Tr and ∇ stand for the trace and the gradient respectively, and, $\Sigma = \Sigma_{\mathbf{1}}$ with, for any $(\sigma_1, \ldots, \sigma_d) \in [0, \infty)^d$ and writing $\boldsymbol{x} = (x_1, \ldots, x_d)$, the diagonal matrix $\Sigma_{\boldsymbol{\sigma}}(\boldsymbol{x})$ is defined as

$$\Sigma_{\boldsymbol{\sigma}}(\boldsymbol{x}) = \mathrm{diag}(\sigma_1 e^{-x_1}, \ldots, \sigma_d e^{-x_d}). \tag{2.23}$$

Theorem 2.2.2. *1. The closure of $(\boldsymbol{A}_{\mathtt{PDO}}[\boldsymbol{\psi}_0], \mathbf{C}^\infty_c(\mathbb{R}^d))$ in $\mathbf{L}^2(\mathbb{R}^d, \mathbf{e})$, denoted by $\boldsymbol{A}_2[\boldsymbol{\psi}_0]$, generates a self-adjoint $\mathcal{C}_0$-semigroup $\boldsymbol{Q} \in \mathcal{C}^+_0(\mathbf{L}^2(\mathbb{R}^d, \mathbf{e}))$. More specifically, $\boldsymbol{Q}$ restricted to $\mathbf{C}_0(\mathbb{R}^d) \cap \mathbf{L}^2(\mathbb{R}^d, \mathbf{e})$ extends to the Feller semigroup of the d-dimensional log-Bessel process of dimension 2 whose generator is the closure of $(\boldsymbol{A}_0, \mathbf{C}^\infty_c(\mathbb{R}^d))$.*

2. *For any $M \in \mathrm{GL}_d(\mathbb{R})$, there exists a one-to-one mapping between the set $\mathscr{A}_M$ and the set*

$$\mathscr{P}_M = \left\{ \boldsymbol{P}^M[\boldsymbol{\psi}];\ \boldsymbol{\psi} \in \mathbf{N}_b^d(\mathbb{R}) \right\} \subset \mathcal{C}_0(\mathbf{L}^2(\mathbb{R}^d, \mathbf{e}_M))$$

such that $\boldsymbol{A}_2^M[\boldsymbol{\psi}] = \boldsymbol{A}_{\mathsf{PDO}}^M[\boldsymbol{\psi}]$ on a dense subset, where $\boldsymbol{A}_2^M[\boldsymbol{\psi}]$ is the $\mathbf{L}^2(\mathbb{R}^d, \mathbf{e}_M)$-generator of $\boldsymbol{P}^M[\boldsymbol{\psi}]$.

More specifically, for any $\psi \in \mathbf{N}_+(\mathbb{R})$, $A_2[\psi]$ is the closure of $(A_{\mathsf{PDO}}[\psi], \Lambda_\psi(\mathbf{C}_c^\infty(\mathbb{R}))$, where $\Lambda_\psi \in \mathscr{B}(\mathbf{L}^2(\mathbb{R}, e))$ is defined in (2.21), and, otherwise, for any $\psi \in \mathbf{N}_b(\mathbb{R}) \setminus \mathbf{N}_+(\mathbb{R})$, $\widehat{P}[\psi] = P[\overline{\psi}]$. The general case is then obtained by tensorization and isomorphism of Hilbert spaces.

3. *For any $M \in \mathrm{GL}_d(\mathbb{R})$, we have*

$$\mathrm{WS}_{\mathscr{P}_M}(\boldsymbol{Q}) = \mathscr{P}_M. \tag{2.24}$$

In fact, the set of weak similarity operators is a multiplicative group which entails that $\mathrm{WS}_{\mathscr{P}}$ forms an equivalence class, where $\mathscr{P} = \bigcup_{M \in \mathrm{GL}_d(\mathbb{R})} \mathscr{P}_M$.

4. *For any $\boldsymbol{\psi} \in \mathbf{N}_b^d(\mathbb{R})$, we have $\widehat{\boldsymbol{P}}^M[\boldsymbol{\psi}] = \boldsymbol{P}^M[\overline{\boldsymbol{\psi}}] \in \mathscr{P}$, where $\widehat{\boldsymbol{P}}^M[\boldsymbol{\psi}]$ stands for the $\mathbf{L}^2(\mathbb{R}^d, \mathbf{e}_M)$-adjoint of $\boldsymbol{P}^M[\boldsymbol{\psi}]$. Thus, $\boldsymbol{P}^M[\boldsymbol{\psi}]$ is self-adjoint in $\mathbf{L}^2(\mathbb{R}^d, \mathbf{e}_M)$ if and only if $\overline{\boldsymbol{\psi}} = \boldsymbol{\psi}$.*

This theorem is proved in Subsection 2.5.4.

Remark 2.2.3. When $d = 1$, $M = \mathrm{Id}$, the identity matrix, writing simply $P[\psi] = \boldsymbol{P}^{\mathrm{Id}}[\boldsymbol{\psi}]$, the identity (2.24) reads as follows

$$\mathrm{WS}_{\mathscr{P}}(Q) = \{P[\psi];\ \psi \in \mathbf{N}_b(\mathbb{R})\} \tag{2.25}$$

where Q is the semigroup on $\mathbf{L}^2(\mathbb{R}, e)$, $e(x) = e^x, x \in \mathbb{R}$, of the log-squared Bessel process of dimension 2. Moreover, $P[\psi]$ is the $\mathbf{L}^2(\mathbb{R}, e)$-extension of the log-self-similar Feller semigroup as reviewed in Section 2.3.3.

2.2.3 Spectral representation of the class $\mathscr{P}$

In this part, we use the weak similarity relation combined with the fact that $\boldsymbol{Q}$ is diagonalisable as it is self-adjoint in $\mathbf{L}^2(\mathbb{R}^d, \mathbf{e})$, to obtain the spectral decomposition of the semigroups in $\mathscr{P}$. To this end, we denote the multiplication semigroup on $\mathbf{L}^2(\mathbb{R}^d, \mathbf{e})$ by $(\boldsymbol{e}_t)_{t\geqslant 0}$, i.e. for any $t \geqslant 0$,

$$\boldsymbol{e}_t \mathbf{f}(\boldsymbol{y}) = e^{-t\langle e(-\boldsymbol{y}),\mathbf{1}\rangle}\mathbf{f}(\boldsymbol{y}), \quad \boldsymbol{y} \in \mathbb{R}^d, \tag{2.26}$$

where we recall that $e(-\boldsymbol{y}) = (e^{-y_1}, \dots, e^{-y_d})$. We are now ready to state the following.

Theorem 2.2.4. *Let $\boldsymbol{P}^M[\boldsymbol{\psi}] \in \mathscr{P}_M$ for some $M \in \mathrm{GL}_d(\mathbb{R})$ and $\boldsymbol{\psi} = (\psi_1, \dots, \psi_d) \in \mathbf{N}_b^d(\mathbb{R})$ such that $\psi_k(\xi) = \phi_{+,k}(-\mathrm{i}\xi)\phi_{-,k}(\mathrm{i}\xi)$ for all $1 \leqslant k \leqslant d$ and $\xi \in \mathbb{R}$. Then, for all $t \geqslant 0$,*

$$\boldsymbol{P}_t^M[\boldsymbol{\psi}] = \boldsymbol{H}_{\boldsymbol{\psi},M}\boldsymbol{e}_t\boldsymbol{H}_{\boldsymbol{\psi},M}^{-1} \ \text{ on } \ \mathrm{D}(\boldsymbol{H}_{\boldsymbol{\psi},M}^{-1}) \tag{2.27}$$

where $\mathrm{D}(\boldsymbol{H}_{\boldsymbol{\psi},M}^{-1}) = \mathrm{Range}(\boldsymbol{H}_{\boldsymbol{\psi},M})$ is dense in $\mathbf{L}^2(\mathbb{R}^d, \mathbf{e})$, $\boldsymbol{H}_{\boldsymbol{\psi},M}\mathbf{f}(\boldsymbol{x}) = \boldsymbol{H}_{\boldsymbol{\psi}}\mathbf{f}(M\boldsymbol{x})$ and $\boldsymbol{H}_{\boldsymbol{\psi}}$ being the shifted Fourier operator with multiplier denoted by $\mathbf{m}_{\boldsymbol{\psi}}^{\mathbf{e}}(\boldsymbol{\xi})$, where

$$\mathbf{m}_{\boldsymbol{\psi}}^{\mathbf{e}}(\boldsymbol{\xi}) = \prod_{k=1}^{d} \frac{W_{\phi_{+,k}}(\frac{1}{2} - \mathrm{i}\xi_k)}{W_{\phi_{-,k}}(\frac{1}{2} + \mathrm{i}\xi_k)}, \quad \boldsymbol{\xi} = (\xi_1, \dots, \xi_d) \in \mathbb{R}^d. \tag{2.28}$$

In fact,

$$\mathrm{D}(\boldsymbol{H}_{\boldsymbol{\psi},M}^{-1}) = \left\{ \mathbf{f} \in \mathbf{L}^2(\mathbb{R}^d, \mathbf{e}_M);\ \boldsymbol{\xi} = (\xi_1, \dots, \xi_d) \mapsto \mathcal{F}_{\mathbf{f}\circ M^{-1}}^{\mathbf{e}}(\boldsymbol{\xi}) \prod_{k=1}^{d} \frac{W_{\phi_{-,k}}(\frac{1}{2} - \mathrm{i}\xi_k)}{W_{\phi_{+,k}}(\frac{1}{2} + \mathrm{i}\xi_k)} \in \mathbf{L}^2(\mathbb{R}^d) \right\}.$$

Under some conditions (see Proposition 2.2.11), $\mathrm{D}(\boldsymbol{H}_{\boldsymbol{\psi},M}^{-1})$ is the full Hilbert space and thus $\boldsymbol{H}_{\boldsymbol{\psi},M}^{-1}$ extends as a bounded operator on the entire Hilbert space. As a result, (2.27) holds on $\mathbf{L}^2(\mathbb{R}^d, \mathbf{e}_M)$.

This theorem is proved in Subsection 2.5.5.

Remark 2.2.5. It can be easily checked that the inverse of $\boldsymbol{H}_{\boldsymbol{\psi},M}$ is $\widehat{\boldsymbol{H}}_{\overline{\boldsymbol{\psi}},M}$. This implies that when $\boldsymbol{\psi} = \overline{\boldsymbol{\psi}}$, $\boldsymbol{H}_{\boldsymbol{\psi},M}$ is an unitary operator, see Subsection 2.3.4. In Theorem 2.2.6(3) below, we show that the boundedness of $\boldsymbol{H}_{\boldsymbol{\psi},M}^{-1}$ along with the condition $\mathbf{m}_{\boldsymbol{\psi}}^{\mathbf{e}} \in \mathbf{L}^2(\mathbb{R}^d, \mathbf{e})$ ensure the existence of residual spectrum of the semigroup $\boldsymbol{P}^M[\boldsymbol{\psi}]$. Hence, Theorem 2.2.4 indicates that existence of residual spectrum, which is empty for self-adjoint linear operators, enables the semigroup to have spectral expansion on the entire Hilbert space.

2.2.4 About the spectra of the $\mathcal{C}_0$-semigroups in $\mathscr{P}$

The next result provides criteria to determine the nature of the spectrum of the family of semigroups $\mathscr{P}$.

Theorem 2.2.6. *Let $\boldsymbol{P}^M[\boldsymbol{\psi}] \in \mathscr{P}$ for some $M \in \mathrm{GL}_d(\mathbb{R})$.*

1. *For all $t \geqslant 0$,*

$$e^{t\mathbb{R}_-} \subseteq \mathrm{Spec}(\boldsymbol{P}_t^M[\boldsymbol{\psi}]).$$

More precisely, if $\mathbf{m}_{\psi}^{\mathbf{e}}$ is bounded then $e^{t\mathbb{R}_-} \subseteq \mathrm{Spec}_{ap}(\boldsymbol{P}_t^M[\boldsymbol{\psi}])$, where $\mathbf{m}_{\psi}^{\mathbf{e}}$ is defined in (2.28). If $1/\mathbf{m}_{\psi}^{\mathbf{e}}$ is bounded, then $e^{t\mathbb{R}_-} \subseteq \mathrm{Spec}_{ap}(\widehat{\boldsymbol{P}}_t^M[\boldsymbol{\psi}])$.

2. *If $\mathbf{m}_{\psi}^{\mathbf{e}} \in \mathbf{L}^2(\mathbb{R}^d)$ then*

$$e^{t\mathbb{R}_-} \subseteq \mathrm{Spec}_p(\boldsymbol{P}_t^M[\boldsymbol{\psi}]) \textit{ and } \mathrm{Spec}_r(\boldsymbol{P}_t^M[\boldsymbol{\psi}]) = \emptyset$$

with, for any $\boldsymbol{q} \in \mathbb{R}_+^d$,

$$\boldsymbol{P}_t^M[\boldsymbol{\psi}]\tau_{\ln \boldsymbol{q}}\mathbf{J}_{\boldsymbol{\psi}}^M(\boldsymbol{x}) = \mathbf{e}(-t\boldsymbol{q})\, \tau_{\ln \boldsymbol{q}}\mathbf{J}_{\boldsymbol{\psi}}^M(\boldsymbol{x})$$

where $\tau_{\boldsymbol{q}}\mathbf{f}(\boldsymbol{x}) = \mathbf{f}(\boldsymbol{x}+\boldsymbol{q})$, $\mathbf{J}_{\boldsymbol{\psi}}^M(\boldsymbol{x}) = \mathbf{J}_{\boldsymbol{\psi}}(M^{-1}\boldsymbol{x})$, and

$$\mathbf{J}_{\psi}(\boldsymbol{x}) = \frac{\mathbf{e}(-\boldsymbol{x}/2)}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} e^{\mathrm{i}\langle \boldsymbol{x},\boldsymbol{\xi}\rangle}\mathbf{m}_{\psi}^{\mathbf{e}}(\boldsymbol{\xi})d\boldsymbol{\xi}$$

where the integral is understood in the $\mathbf{L}^2$-sense.

3. *If $1/\mathbf{m}_{\psi}^{\mathbf{e}} \in \mathbf{L}^2(\mathbb{R}^d)$ then*

$$e^{t\mathbb{R}_-} \subseteq \mathrm{Spec}_r(\boldsymbol{P}_t^M[\boldsymbol{\psi}]) \textit{ and } \mathrm{Spec}_p(\boldsymbol{P}_t^M[\boldsymbol{\psi}]) = \emptyset$$

with, for any $\boldsymbol{q} \in \mathbb{R}_+^d$, $t > 0$ and $\mathbf{f} \in \mathbf{L}^2(\mathbb{R}^d, \mathbf{e}_M)$,

$$\left\langle \boldsymbol{P}_t^M[\boldsymbol{\psi}]\mathbf{f}, \tau_{\ln \boldsymbol{q}}\mathbf{J}_{\frac{M}{\psi}} \right\rangle_{\mathbf{L}^2(\mathbb{R}^d,\mathbf{e}_M)} = \mathbf{e}(-t\boldsymbol{q}) \left\langle \mathbf{f}, \tau_{\ln \boldsymbol{q}}\mathbf{J}_{\frac{M}{\psi}} \right\rangle_{\mathbf{L}^2(\mathbb{R}^d,\mathbf{e}_M)}.$$

4. *If $\mathbf{m}_{\psi}^{\mathbf{e}}$ is both bounded from above and below then*

$$\mathrm{Spec}(\boldsymbol{P}_t^M[\boldsymbol{\psi}]) = \mathrm{Spec}_c(\boldsymbol{P}_t^M[\boldsymbol{\psi}]) = e^{t\mathbb{R}_-}. \tag{2.29}$$

In particular, when $\boldsymbol{P}_t^M[\boldsymbol{\psi}]$ is self-adjoint, i.e. $\boldsymbol{\psi} = \overline{\boldsymbol{\psi}}$, then (2.29) holds.

This theorem is proved in Subsection 2.5.6.

Remark 2.2.7. For the point and residual spectrum, there are plenty of natural examples, and, we detail some of them in Sections 2.6.1 and 2.6.2. More generally, for any $\phi \in \mathbf{B}$, defining

$$\underline{\Theta}_{\phi} = \underline{\lim}_{|\xi|\to\infty} \Theta_{\phi}(|\xi|) \text{ and } \overline{\Theta}_{\phi} = \overline{\lim}_{|\xi|\to\infty} \Theta_{\phi}(|\xi|) \tag{2.30}$$

where $\Theta_{\phi}(|\xi|) = \frac{\int_{1/2}^{\infty} \ln\left(\frac{|\phi(a+\mathrm{i}|\xi|)|}{\phi(a)}\right) da}{|\xi|}$, one easily deduces from [95, Theorem 6.2(1)], that, for any $\xi \in \mathbb{R}$,

$$\left|\frac{W_{\phi_+}\left(\frac{1}{2}+\mathrm{i}\xi\right)}{W_{\phi_-}\left(\frac{1}{2}+\mathrm{i}\xi\right)}\right| \asymp \frac{\sqrt{|\phi_-\left(\frac{1}{2}+\mathrm{i}\xi\right)|}}{\sqrt{|\phi_+\left(\frac{1}{2}+\mathrm{i}\xi\right)|}} e^{-|\xi|(\Theta_{\phi_+}(|\xi|)-\Theta_{\phi_-}(|\xi|))}. \tag{2.31}$$

Thus, if $\underline{\Theta}_{\phi_+} > \overline{\Theta}_{\phi_-}$ (resp. $\underline{\Theta}_{\phi_-} > \overline{\Theta}_{\phi_+}$), then $P[\psi]$ has point spectrum (resp. has residual spectrum). For a proof of the last two statements and also sufficient conditions for both point and residual spectrum in the case $\underline{\Theta}_{\phi_+} = \overline{\Theta}_{\phi_-}$, we refer to Proposition 2.2.11. In Section 2.6.2, we provide examples of ψ's for which $P[\psi]$ is non-self-adjoint and has continuous spectrum. However, in general, though we have always $e^{t\mathbb{R}_-} \subseteq \mathrm{Spec}(\boldsymbol{P}_t^M[\boldsymbol{\psi}])$, the weak similarity relation does not allow to check the reverse inclusion.

Remark 2.2.8. When the function $\mathbf{m}_{\psi}^{\mathbf{e}}$ is polynomially bounded, i.e. there is some positive $p > 0$ such that

$$\left|\mathbf{m}_{\psi}^{\mathbf{e}}(\boldsymbol{\xi})\right| \leqslant C_p(1+\|\boldsymbol{\xi}\|)^p$$

we can still have the Fourier inverse of $\mathbf{m}_{\psi}^{\mathbf{e}}$ in the sense of Schwartz distribution, that could be used to define the functions associated to the residual spectrum. However, the growth of $\mathbf{m}_{\psi}^{\mathbf{e}}$ along the real line, being, in general, exponential, this analysis requires deeper techniques that we plan to develop in a subsequent work.

2.2.5 Positivity-preserving and integro-differential representation of the generators

In this section we provide an integro-differential representation of the $\mathbf{L}^2(\mathbb{R}^d, \mathbf{e}_M)$-generator of the semigroup $\boldsymbol{P}^M[\boldsymbol{\psi}]$. We start by introducing the following subset of $\mathbf{L}^2(\mathbb{R}^d, \mathbf{e})$

$$\boldsymbol{\mathcal{E}} = \bigotimes_{k=1}^{d} \mathcal{E} \text{ with } \mathcal{E} = \mathrm{Span}\{\mathfrak{h}_{\epsilon,\beta};\ \epsilon, \beta > 0\} \tag{2.32}$$

where $\mathfrak{h}_{\epsilon,\beta} : \mathbb{R} \to \mathbb{R}_+$ is defined by $\mathfrak{h}_{\epsilon,\beta}(x) = e^{-(\frac{1}{2}+\epsilon)x} e^{-\beta e^{-x}}$, and $\otimes$ stands for the tensor product of the univariate functions. Next, for any $\boldsymbol{\psi} = (\psi_1, \ldots, \psi_d) \in \mathbf{N}_b^d(\mathbb{R})$, where, for any $k = 1, \ldots d$, $(\psi_k(0), \sigma_k^2, \mathtt{b}_k, \mu_k)$ is the characteristic quadruplet of ψ_k, and $M \in \mathrm{GL}_d(\mathbb{R})$, we consider the linear integro-differential operator acting on smooth and well behaved functions $\mathbf{f} : \mathbb{R}^d \to \mathbb{R}$ as follows

$$\begin{aligned} \boldsymbol{A}^M[\boldsymbol{\psi}]\mathbf{f}(\boldsymbol{x}) &= \mathrm{Tr}\left(M^\top \Sigma_{\boldsymbol{\sigma}}(M^{-1}\boldsymbol{x}) M \nabla^2 \mathbf{f}(\boldsymbol{x})\right) + \left\langle M e_{\mathbf{b}}(-M^{-1}\boldsymbol{x}), \nabla \mathbf{f}(\boldsymbol{x})\right\rangle \\ &\quad + \int_{\mathbb{R}^d} \left(\mathbf{f}(\boldsymbol{x}+\boldsymbol{y}) - \mathbf{f}(\boldsymbol{x}) - \langle \boldsymbol{y}, \nabla \mathbf{f}(\boldsymbol{x})\rangle \mathbb{1}_{\{\|M^{-1}\boldsymbol{y}\| \leqslant 1\}}\right) \langle e(-M^{-1}\boldsymbol{x}), \boldsymbol{\mu}_M(d\boldsymbol{y})\rangle \\ &\quad - \langle e(-\boldsymbol{x}), \boldsymbol{\psi}(0)\rangle\, \mathbf{f}(\boldsymbol{x}) \end{aligned} \tag{2.33}$$

where Tr stands for the trace, $\Sigma_{\boldsymbol{\sigma}}$ is defined in (2.23), ∇ is the gradient, $\mathbf{b} = (\mathtt{b}_1, \ldots, \mathtt{b}_d)$, $e_{\mathbf{b}}(-\boldsymbol{x}) = (\mathtt{b}_1 e^{-x_1}, \ldots, \mathtt{b}_d e^{-x_d})$ with $e_{\mathbf{1}}(-\boldsymbol{x}) = e(-\boldsymbol{x}) = (e^{-x_1}, \ldots, e^{-x_d})$. $\langle ., . \rangle$ and $\| \cdot \|$ are the inner product and the Euclidean norm in $\mathbb{R}^d$ respectively, and, for any $\boldsymbol{B} \in \mathcal{B}(\mathbb{R}^d)$, $\boldsymbol{\mu}_M(\boldsymbol{B}) = \boldsymbol{\mu}(M^{-1}\boldsymbol{B})$ with $\boldsymbol{\mu}(\boldsymbol{B}) = (\mu_1(B_1), \ldots, \mu_d(B_d))$ where for all $1 \leqslant k \leqslant d$, $B_k = \{x \in \mathbb{R} : x\mathfrak{e}_k \in \boldsymbol{B}\}$, $(\mathfrak{e}_1, \ldots, \mathfrak{e}_d)$ being the canonical basis of $\mathbb{R}^d$. Let us now introduce the following condition regarding the Wiener-Hopf factors of $\boldsymbol{\psi}$.

$$\text{For any } 1 \leqslant k \leqslant d, \text{ either } \int_{|y|>1} |y| \mu_k(dy) < \infty \quad \text{or} \quad \phi_{+,k}(0) > 0 \tag{2.34}$$

$$\lim_{|\xi|\to\infty} |\xi|^n e^{-\frac{\pi}{2}|\xi|} \sup_{1 \leqslant k \leqslant d} \left| \frac{W_{\phi_{+,k}}(\frac{1}{2} + \mathrm{i}\xi)}{W_{\phi_{-,k}}(\frac{1}{2} + \mathrm{i}\xi)} \right| = 0 \text{ for all } n \in \mathbb{N} \tag{2.35}$$

When $\phi_{+,k}(0) = 0$, the integrability assumption in (2.34) combined with an application of Taylor's formula imply that the operator $\boldsymbol{A}^M[\boldsymbol{\psi}]$ in (2.33) is well defined on the set

$$\boldsymbol{\mathcal{C}} = \{\mathbf{f} \in \mathbf{C}^2(\mathbb{R}^d);\ \nabla \mathbf{f} \in \mathbf{C}_b^1(\mathbb{R}^d)\}. \tag{2.36}$$

Theorem 2.2.9. *Let $M \in \mathrm{GL}_d(\mathbb{R})$.*

1. *We have*

$$\mathscr{P}_M \subset \mathcal{C}_0^+(\mathbf{L}^2(\mathbb{R}^d, \mathrm{e}_M)).$$

   More specifically, $\boldsymbol{P}^M[\boldsymbol{\psi}]$, when restricted to $\mathbf{C}_0(\mathbb{R}^d) \cap \mathbf{L}^2(\mathbb{R}^d, \mathrm{e}_M)$, extends to the semigroup of the Feller process on $\mathbb{R}^d$ defined, for any $t \geqslant 0$, as $\boldsymbol{Y}_t = M\boldsymbol{X}_t$, where $\boldsymbol{X}_t = (\ln X_t^{(1)}, \ldots, \ln X_t^{(d)})$, the stochastic processes $(X_t^{(k)})_{t \geqslant 0}, k = 1, \ldots, d$, are mutually independent and each of them is a positive self-similar Feller process on $(0, \infty)$, see Section 2.3.3 for more details on these processes.

2. *If $\boldsymbol{\psi}$ satisfies the conditions (2.34) and (2.35), then $\boldsymbol{\mathcal{E}}_{\boldsymbol{\psi}}^M := \boldsymbol{H}_{\boldsymbol{\psi}}^M(\boldsymbol{\mathcal{E}})$ is a core for $\boldsymbol{A}_2[\boldsymbol{\psi}]$, and its restriction on $\boldsymbol{\mathcal{E}}_{\boldsymbol{\psi}}^M$ coincides with $\boldsymbol{A}^M[\boldsymbol{\psi}]$ defined in (2.33).*

This theorem is proved in Section 2.5.7.

Remark 2.2.10. The condition (2.34) is needed to ensure that $\boldsymbol{\mathcal{E}}_{\boldsymbol{\psi}}^M \subset \boldsymbol{\mathcal{C}}$, which entails that the integro-differential operator in (2.33) is well-defined on $\boldsymbol{\mathcal{E}}_{\boldsymbol{\psi}}^M$. In order to weaken this assumption and get that $\boldsymbol{\mathcal{E}}_{\boldsymbol{\psi}}^M \subset \mathbf{C}_b^2(\mathbb{R}^d)$ whenever $\boldsymbol{\psi}$ satisfies (2.35), one would need to develop a refined analysis of the asymptotic behavior of the ratio $\left|\frac{W_{\phi_+}\left(\frac{1}{2}-\mathrm{i}\xi\right)}{W_{\phi_-}\left(\frac{1}{2}+\mathrm{i}\xi\right)}\right|$.

2.2.6 Conditions for point or residual spectrum in terms of the Wiener-Hopf factors

The next proposition provides some sufficient conditions on the characteristic triplet of the Wiener-Hopf factors for the characterization of the presence of point or residual spectrum, which according to Theorem 2.2.6, boils down to identify whether or not $m_\psi = m_{\psi_k} \in \mathbf{L}^2(\mathbb{R})$ for $k = 1, \ldots, d$. For $\phi \in \mathbf{B}$, we recall that $\overline{\Theta}_\phi$ and $\underline{\Theta}_\phi$ are defined in (2.30).

Proposition 2.2.11. *Let $\psi \in \mathbf{N}(\mathbb{R})$ where $\psi(\xi) = \phi_+(-\mathrm{i}\xi)\phi_-(\mathrm{i}\xi)$ for all $\xi \in \mathbb{R}$ with $\phi_\pm(z) = \phi_\pm(0) + \mathtt{d}_\pm z + \int_0^\infty (1 - e^{-zy})\nu_\pm(dy)$ for all $z \in \mathbb{C}_{[0,\infty)}$. Then, for all $\xi \in \mathbb{R}$,*

(i) If $\underline{\Theta}_{\phi_+} > \overline{\Theta}_{\phi_-}$ and

$$\sup \left\{ \kappa > 0; \ \varliminf_{|\xi|\to\infty} |\xi|^\kappa |\phi_+(\mathrm{i}\xi)| > 0 \right\} < \infty$$

then $\psi \in \mathbf{N}_+(\mathbb{R})$, $m_\psi \in \mathbf{L}^2(\mathbb{R})$ and $m_{\overline{\psi}} \notin \mathbf{L}^2(\mathbb{R})$. Table 2.1 provides sufficient conditions expressed in terms of the characteristic triplets of ϕ_+ and ϕ_- for $\underline{\Theta}_{\phi_+} > \overline{\Theta}_{\phi_-}$, where $\overline{\nu}_\pm(r) = \nu_\pm(r, \infty)$, and, for any $\alpha > 0$, $\mathbf{RV}(\alpha)$ (resp. q-m) denotes the set of all regularly varying (resp. quasi-monotone) functions of index α, see [18, Chapter 1 and Section 2.7]

$\mathtt{d}_+$	$\mathtt{d}_-$	$\overline{\nu}_+$	$\overline{\nu}_-$	$\left\lvert\frac{W_{\phi_+}\left(\frac{1}{2}+\mathrm{i}\xi\right)}{W_{\phi_-}\left(\frac{1}{2}+\mathrm{i}\xi\right)}\right\rvert$
0	0	$\overline{\nu}_+ \in \mathbf{RV}(\alpha_+)$, $y \mapsto \frac{\overline{\nu}_+(y)}{y^{\alpha_+}}$ is q-m	$\overline{\nu}_- \in \mathbf{RV}(\alpha_-)$, $y \mapsto \frac{\overline{\nu}_-(y)}{y^{\alpha_-}}$ is q-m,	$= \mathrm{O}\left(e^{-(\alpha_+-\alpha_-)\frac{\pi}{2}\lvert\xi\rvert}\right)$ $0 < \alpha_- < \alpha_+ < 1$
> 0	0		$\overline{\nu}_- \in \mathbf{RV}(\alpha)$, $y \mapsto \frac{\overline{\nu}_-(y)}{y^{\alpha}}$ is q-m,	$= \mathrm{O}\left(e^{-(1-\alpha)\frac{\pi}{2}\lvert\xi\rvert}\right)$ $0 < \alpha < 1$

Table 2.1: Conditions for $m_\psi \in \mathbf{L}^2(\mathbb{R})$ when $\underline{\Theta}_{\phi_+} > \overline{\Theta}_{\phi_-}$

(ii) If $\underline{\Theta}_{\phi_+} = \overline{\Theta}_{\phi_-}$, we give in Table 2.2 a set of sufficient conditions on the characteristic triplets of ϕ_+, ϕ_- to ensure that $m_\psi \in \mathbf{L}^2(\mathbb{R})$ and $m_{\overline{\psi}} \notin \mathbf{L}^2(\mathbb{R})$.

$\mathtt{d}_+$	$\mathtt{d}_-$	$\overline{\nu}_+$	$\overline{\nu}_-$	$\left\lvert\frac{W_{\phi_+}\left(\frac{1}{2}+\mathrm{i}\xi\right)}{W_{\phi_-}\left(\frac{1}{2}+\mathrm{i}\xi\right)}\right\rvert$
> 0	0	$\overline{\nu}_+(0) < \infty$		$= \mathrm{O}(\lvert\xi\rvert^{-u})\ \forall u > 0$
> 0	> 0	$\overline{\nu}_+(0) < \infty$	$\overline{\nu}_-(0) = \infty$	$= \mathrm{O}(\lvert\xi\rvert^{-u})\ \forall u > 0$

Table 2.2: Conditions for $m_\psi \in \mathbf{L}^2(\mathbb{R})$ when $\underline{\Theta}_{\phi_+} = \overline{\Theta}_{\phi_-}$

This proposition is proved in Section 2.3.8. We point out that the study of the asymptotic behavior of the ratio $\left|\frac{W_{\phi_+}\left(\frac{1}{2}+\mathrm{i}\xi\right)}{W_{\phi_-}\left(\frac{1}{2}+\mathrm{i}\xi\right)}\right|$ for any pairs of Bernstein functions, in particular any pairs of Lévy measures, is a delicate issue. Our results identify several large classes of Bernstein functions for which such estimate is attainable. We mention that, in the recent paper [87], asymptotic estimates of this ratio when W_{ϕ_+} is the classical gamma function, has been obtained under other type of conditions than ours on the tail of the Lévy measure $\overline{\nu}_-$.

2.3 Preliminaries and auxiliary results

In this section we gather some general facts about several ideas and tools that will be used throughout the remaining part of this chapter, with an emphasis to the theory of shifted Fourier multipliers and the analytical properties of certain ratios of Bernstein-gamma functions that will be important in the sequel.

2.3.1 Fourier transforms, shifted Fourier transforms and some classical results

For any function f in $\mathbf{L}^1(\mathbb{R})$ or $\mathbf{L}^2(\mathbb{R})$, we denote its Fourier transform by $\mathcal{F}_f$, i.e.

$$\mathcal{F}_f(\xi) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-\mathrm{i}x\xi} f(x)\, dx,\ \xi \in \mathbb{R}.$$

As before we define $\mathcal{F}^e$ the shifted Fourier transform, i.e. for any $f \in \mathbf{L}^2(\mathbb{R}, e)$, writing $e(x) = e^x$,

$$\mathcal{F}^e_f(\xi) = \mathcal{F}_{\sqrt{e}f}(\xi) \tag{2.37}$$

in the $\mathbf{L}^2$-sense. Clearly, $\mathcal{F}^e : \mathbf{L}^2(\mathbb{R}, e) \to \mathbf{L}^2(\mathbb{R})$ is an isometry whose inverse is denoted by $\widehat{\mathcal{F}}^e$. Next, we mention a very useful result due to Wiener that will be frequently used.

Theorem 2.3.1 (Wiener's Tauberian Theorem)**.** *Let $f \in \mathbf{L}^2(\mathbb{R})$ be such that $\mathcal{F}_f$ is almost everywhere (a.e.) non-zero. Then, recalling that $\tau_a f(.) = f(. + a)$, the set $\mathrm{Span}\{\tau_a f; a \in \mathbb{R}\}$ is dense in $\mathbf{L}^2(\mathbb{R})$.*

This result is very standard and can be found in [18]. In the following corollary, we mention a variant of Wiener's theorem which will be needed in the subsequent results.

Corollary 2.3.2. Let $f \in \mathbf{L}^2(\mathbb{R}, e)$ be such that $\mathcal{F}_f^e$ is a.e. non-zero. Then, the set $\{\tau_a f; a \in \mathbb{R}\}$ is dense in $\mathbf{L}^2(\mathbb{R}, e)$.

Proof. Note that $f \in \mathbf{L}^2(\mathbb{R}, e)$ implies that $\sqrt{e}f \in \mathbf{L}^2(\mathbb{R})$ where we recall that $\sqrt{e}f = f(x)e^{\frac{x}{2}}, x \in \mathbb{R}$. By means of the Wiener's Tauberian theorem, we have that the set $\{\tau_a \sqrt{e}f; a \in \mathbb{R}\}$ is dense in $\mathbf{L}^2(\mathbb{R})$ which proves the result. We close this part with the following variant of Cauchy contour integration formula that will also be useful later.

Lemma 2.3.3. *Let $f \in \mathbf{A}_{[0,\gamma]}$ for some $\gamma > 0$ and there exists $1 \leqslant p \leqslant \infty$ such that*

$$\lim_{|x|\to\infty} \sup_{b\in[0,\gamma]} \int_{\mathbb{R}} |f(x + \mathrm{i}b)|^p \, dx = 0.$$

Then, one can choose a subsequence $(n_j)_{j\geqslant 1}$ of natural numbers such that for any $0 \leqslant b \leqslant \gamma$,

$$\lim_{j\to\infty} \int_{-n_j}^{n_j} [f(x + \mathrm{i}b) - f(x)]dx = 0.$$

In particular, when $p = 1$, $\int_{\mathbb{R}} f(x)dx = \int_{\mathbb{R}} f(x + \mathrm{i}b)dx$ for all $0 \leqslant b \leqslant \gamma$, and when $p = \infty$, $\lim_{n\to\infty} \int_{-n}^{n} [f(x + \mathrm{i}b) - f(x)]dx = 0$ for all $0 \leqslant b \leqslant \gamma$.

Proof. Let us first assume that $1 \leqslant p < \infty$. From the analyticity of f in the strip $\mathbb{S}_{[0,\gamma]}$ and applying the Cauchy integral formula, for any $x > 0$ and $0 < b \leqslant \gamma$ we have

$$\int_{R_x^b} f(z)dz = 0$$

where R_x^b is the rectangle formed by the points $-x, x, x + \mathrm{i}b, -x + \mathrm{i}b$. Let us estimate the integrals on the vertical lines of R_x^b. By Hölder's inequality, we have

$$\left(\int_0^b |f(\pm x + \mathrm{i}y)|dy\right)^p \leqslant \left(\int_0^\gamma |f(\pm x + \mathrm{i}y)|dy\right)^p \leqslant \gamma^{\frac{p}{q}} \int_0^\gamma |f(\pm x + \mathrm{i}y)|^p dy$$

where $\frac{1}{p} + \frac{1}{q} = 1$ with $q = \infty$ when $p = 1$. Applying Fubini's theorem we get

$$\int_{\mathbb{R}} \left(\int_0^\gamma |f(x + \mathrm{i}y)|dy\right)^p dx \leqslant \gamma^{\frac{p}{q}} \int_0^\gamma \int_{\mathbb{R}} |f(x + \mathrm{i}y)|^p dx\, dy < \infty$$

which implies that there exists a subsequence $(n_j)_{j\geqslant 1}$ of natural numbers such that

$$\lim_{j\to\infty} \int_0^\gamma |f(\pm n_j + \mathrm{i}y)|dy = 0.$$

As a result, for any $0 \leqslant b \leqslant \gamma$, $\lim_{j\to\infty} \left(\int_{-n_j}^{n_j} [f(x + \mathrm{i}b) - f(x)] - \int_{R_{n_j}^b} f(z)dz\right) = 0$, which proves the lemma. When $p = \infty$, for any $0 < b \leqslant \gamma$, the integral on the vertical lines of R_x^b goes to 0 as $x \to \infty$. This completes the proof of the lemma for all $1 \leqslant p \leqslant \infty$.

2.3.2 Negative definite functions on $\mathbb{R}^d$ and pseudo-differential operators

Negative definite functions

A function $\psi : \mathbb{R}^d \to \mathbb{C}$ is called negative definite if for any choice of $p \in \mathbb{N}$ and $(\boldsymbol{\xi}_1, \ldots, \boldsymbol{\xi}_p) \in \mathbb{R}^{d\times p}$, the matrix

$$(\psi(\boldsymbol{\xi}_i) + \overline{\psi}(\boldsymbol{\xi}_j) - \psi(\boldsymbol{\xi}_i - \boldsymbol{\xi}_j))_{1\leqslant i,j\leqslant p}$$

is non-negative Hermitian. It is a well known fact that any continuous negative definite function ψ has the following representation

$$\psi(\boldsymbol{\xi}) = \psi(\mathbf{0}) - \mathrm{i}\langle \mathrm{b}, \boldsymbol{\xi}\rangle + \langle \boldsymbol{\xi}, \Sigma\boldsymbol{\xi}\rangle + \int_{\mathbb{R}^d} \left(1 - e^{\mathrm{i}\langle \boldsymbol{\xi}, \boldsymbol{y}\rangle} + \mathrm{i}\langle \boldsymbol{\xi}, \boldsymbol{y}\rangle \mathbb{1}_{\{\|\boldsymbol{y}\|\leqslant 1\}}\right) \mu(d\boldsymbol{y})$$

where $\psi(\mathbf{0}) \geqslant 0, \mathrm{b} \in \mathbb{R}^d$, Σ is non-negative definite and μ is a positive Radon measure such that $\int_{\mathbb{R}^d} (\|\boldsymbol{y}\|^2 \wedge 1)\mu(d\boldsymbol{y}) < \infty$. We denote the set of all continuous negative definite functions by

$\mathbf{N}(\mathbb{R}^d)$. The class $\mathbf{N}(\mathbb{R}^d)$ comes naturally in the study of Lévy processes. Indeed, for any (possibly killed) Lévy process $Z = (Z_t)_{t\geqslant 0}$ on $\mathbb{R}^d$, there is a unique $\psi \in \mathbf{N}(\mathbb{R}^d)$ such that

$$\mathbb{E}[e^{\mathrm{i}\langle \boldsymbol{\xi}, Z_t\rangle}] = e^{-t\psi(\boldsymbol{\xi})},\ t \geqslant 0,\ \ \boldsymbol{\xi} \in \mathbb{R}^d, \tag{2.38}$$

and the infinitesimal generator of Z is given, for a smooth function $\mathbf{f} \in \mathbf{S}(\mathbb{R}^d)$, by

$$\begin{aligned} L[\psi]\mathbf{f}(\boldsymbol{x}) =& \langle \nabla \mathbf{f}, \Sigma \nabla \mathbf{f}(\boldsymbol{x})\rangle + \langle \mathrm{b}, \nabla \mathbf{f}(\boldsymbol{x})\rangle + \int_{\mathbb{R}^d} (\mathbf{f}(\boldsymbol{x}+\boldsymbol{y}) - \mathbf{f}(\boldsymbol{x}) - \langle \boldsymbol{y}, \nabla \mathbf{f}(\boldsymbol{x})\rangle \mathbb{1}_{\{\|\boldsymbol{y}\|\leqslant 1\}})\mu(d\boldsymbol{y}) \\ & - \psi(\mathbf{0})\mathbf{f}(\boldsymbol{x}). \end{aligned}$$

We refer the monograph of Jacob [56] for a thorough account on this set of functions and its connection to Lévy processes.

Pseudo-differential operators and connection with Markov Processes

Let $\mathbb{a} : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{C}$ be a function which is continuous and, for all $\boldsymbol{x} \in \mathbb{R}^d$, the mapping $\boldsymbol{\xi} \mapsto \mathbb{a}(\boldsymbol{x}, \boldsymbol{\xi})$ is negative definite. From the general theory of *pseudo-differential operators* (PDO in short) for Markov processes, we know that $|\mathbb{a}(\boldsymbol{x}, \boldsymbol{\xi})| \leqslant \gamma(\boldsymbol{x})(1 + \|\boldsymbol{\xi}\|^2)$ for all $\boldsymbol{x}$, γ being a locally finite function. We define the following linear operator

$$-\mathbb{a}(\boldsymbol{x}, D)f(x) = -(2\pi)^{-\frac{d}{2}} \int_{\mathbb{R}^d} e^{\mathrm{i}\langle \boldsymbol{\xi}, \boldsymbol{x}\rangle} \mathcal{F}_{\mathbf{f}}(\boldsymbol{\xi}) \mathbb{a}(\boldsymbol{x}, \boldsymbol{\xi})\, d\boldsymbol{\xi}\ ,\ \ \mathbf{f} \in \mathbf{S}(\mathbb{R}^d),$$

where $\mathbf{S}(\mathbb{R}^d)$ denotes the Schwartz space on $\mathbb{R}^d$. Clearly, the above integral is well defined and we say that $\mathbb{a}(\boldsymbol{x}, D)$ is a PDO with *symbol* $\mathbb{a}(\boldsymbol{x}, \boldsymbol{\xi})$. The class of pseudo-differential operators with negative definite symbols plays an important role in the theory of Markov processes. Courrège [35] showed that if $(\boldsymbol{A}, \mathbf{D}(\boldsymbol{A}))$ is the generator of a Feller semigroup on $\mathbb{R}^d$ such that $\mathbf{C}_c^\infty(\mathbb{R}^d) \subseteq \mathbf{D}(\boldsymbol{A})$, then A is a pseudo-differential operator with symbol $\mathbb{a}(\boldsymbol{x}, \boldsymbol{\xi})$ of the form

$$\mathbb{a}(\boldsymbol{x}, \boldsymbol{\xi}) = \mathbb{a}(\boldsymbol{x}, \mathbf{0}) - \mathrm{i}\langle \boldsymbol{\xi}, \mathbf{b}(\boldsymbol{x})\rangle + \langle \boldsymbol{\xi}, Q(\boldsymbol{x})\boldsymbol{\xi}\rangle + \int_{\mathbb{R}^d\setminus\{0\}} (1 - e^{\mathrm{i}\langle \boldsymbol{\xi}, \boldsymbol{y}\rangle} + \mathrm{i}\langle \boldsymbol{\xi}, \boldsymbol{y}\rangle \mathbb{1}_{\{\|\boldsymbol{y}\|\leqslant 1\}})\mu(\boldsymbol{x}, d\boldsymbol{y})$$

where, for all $\boldsymbol{x}$, $\mathbb{a}(\boldsymbol{x}, \mathbf{0}) \geqslant 0$, $Q(\boldsymbol{x})$ is non-negative definite, $\mathbf{b} : \mathbb{R}^d \to \mathbb{R}^d$ is measurable and $\mu(\boldsymbol{x}, d\boldsymbol{y})$ is a Lévy measure, i.e. $\int_{\mathbb{R}^d} (\|\boldsymbol{y}\|^2 \wedge 1)\mu(\boldsymbol{x}, d\boldsymbol{y}) < \infty$. On the other hand, if we are given a pseudo-differential operator with a negative definite symbol, the question of existence of a unique Feller process corresponding to it, is more subtle. Some results in this direction can be found in [53, Theorem 5.7], but the required conditions on the symbol $\mathtt{a}(\cdot,\cdot)$ are quite stringent. For instance, already in the one-dimensional case, they are not satisfied by the symbol $\mathtt{a}(x,\xi) = e^{-x}\psi(\xi)$ where ψ is a continuous negative definite function. However, in Theorem 2.4.1 below, we present an original approach based on the concept of intertwining to overcome this issue. Indeed, we will show that under some mild conditions on ψ, the PDO with symbol $\mathtt{a}(x,\xi) = e^{-x}\psi(\xi)$ restricted on a certain dense subset of functions, extends uniquely to the generator of a $\mathcal{C}_0$-contraction positivity-preserving semigroup on $\mathbf{L}^2(\mathbb{R}, e)$.

2.3.3 Lamperti mapping, duality, the log-transformation and generators

In his seminal paper, Lamperti [76] established a one-to-one correspondence between the class of all Lévy processes and self-similar Feller processes on the positive real line up to their absorption time at 0. More specifically, for any positive α-self-similar, $\alpha > 0$, Feller process $X = (X_t(x))_{t\geqslant 0}$ issued from $x > 0$, there is a unique Lévy process $Z = (Z_t)_{t\geqslant 0}$ issued from the origin such that, for all $0 \leqslant t < \zeta(x) = \inf\{t \geqslant 0; X_t(x) \leqslant 0\}$,

$$X_t(x) = x\exp(Z_{\varphi(x^{-\alpha}t)})$$

where $\varphi(t) = \inf\left\{s > 0;\ \int_0^s e^{\alpha Z_r}\,dr > t\right\}$. Almost sure finiteness of the first hitting time $\zeta(x)$ depends on the Lévy process Z as well as its lifetime. This is known as the Lamperti mapping and enables to associate to X a unique Lévy-Khintchine exponent and for more details, we refer to [76, Theorem 4.1]. It is not hard to see that for any α-self-similar Feller process $(X_t)_{t\geqslant 0}$ on $(0,\infty)$, the process $(X_t^{1/\alpha})_{t\geqslant 0}$ is a 1-self-similar Feller process on $(0,\infty)$. Since $x \mapsto x^{1/\alpha}$ is a homeomorphism on $(0,\infty)$, the corresponding semigroups are similar to each other. For a 1-self-similar Feller semigroup $P_t^{\mathtt{F}}[\psi]$ corresponding to the Lévy-Khintchine exponent ψ, the adjoint semigroup is also a Feller one and corresponds to the conjugate Lévy-Khintchine exponent $\overline{\psi}$, see

[13, Lemma 2]. That is, for any $f, g \geqslant 0$,

$$\int_{\mathbb{R}_+} P_t^{\mathtt{F}}[\psi] f(x) g(x)\, dx = \int_{\mathbb{R}_+} f(x) P_t^{\mathtt{F}}[\overline{\psi}] g(x)\, dx. \tag{2.39}$$

This ensures that the Lebesgue measure on $\mathbb{R}_+$ is an excessive measure for the semigroup $P^{\mathtt{F}}[\psi]$ and thus, the latter has a natural extension to $\mathbf{L}^2(\mathbb{R}_+)$, which we denote by $P[\psi]$. $P[\psi]$ is a $\mathcal{C}_0$-contraction semigroup on $\mathbf{L}^2(\mathbb{R}_+)$ and its adjoint is $P[\overline{\psi}]$. Since our approach stems on the theory of pseudo-differential operators which are defined on the entire real line $\mathbb{R}$, we consider the logarithm of the 1-self-similar Feller processes. Since the logarithm is a homeomorphism from $(0, \infty)$ to $\mathbb{R}$, the resultant semigroup is still similar to the original one. Due to the $\log$-transformation, the resulting semigroup has $e(x)dx = e^x\, dx, x \in \mathbb{R}$, as an excessive measure and extends to a $\mathcal{C}_0$-contraction semigroup on $\mathbf{L}^2(\mathbb{R}, e)$. Therefore, all the spectral properties remain invariant under this $\log$-transformation. Next, from [76], it is known that the Dynkin characteristic operator of a 1-self-similar process associated to the Lévy-Khintchine exponent ψ is given by

$$A_{\mathtt{D}}[\psi] f(x) = \frac{1}{x} L[\psi](f \circ \exp)(\ln x), \quad x \in \mathbb{R} \tag{2.40}$$

where $L[\psi]$ is the generator of the Lévy process with Lévy-Khintchine exponent ψ and the set $\{f : \mathbb{R}_+ \to \mathbb{R}_+;\ x \mapsto f(x), xf'(x), x^2 f''(x) \in \mathbf{C}_b(\mathbb{R}_+)\} \subseteq \mathbf{D}(A_{\mathtt{D}}[\psi])$. If $A_{\mathtt{D}}[\psi]$ denotes the Dynkin characteristic operator of the log-self-similar process, then

$$A_{\mathtt{D}}[\psi] f(x) = e^{-x} L[\psi] f(x), \quad x \in \mathbb{R}, \tag{2.41}$$

and $\{f : \mathbb{R} \to \mathbb{R};\ f \in \mathbf{C}_b^2(\mathbb{R})\} \subseteq \mathbf{D}(A_{\mathtt{D}}[\psi])$. Also, for any $f \in \mathbf{S}(\mathbb{R})$, from (2.41), we deduce that $A_{\mathtt{D}}[\psi] f = A_{\mathtt{PDO}}[\psi] f$ where

$$A_{\mathtt{PDO}}[\psi] f(x) = -\frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-x} \psi(\xi) \mathcal{F}_f(\xi) e^{\mathrm{i}\xi x}\, d\xi$$

which is a PDO with symbol $\mathtt{a}(x, \xi) = e^{-x}\psi(\xi)$. From now on, we only consider the log-self-similar processes and we denote their semigroups by $P[\psi]$.

2.3.4 Shifted Fourier multiplier operators

We recall that an operator $\Lambda : \mathbf{L}^2(\mathbb{R}, e) \to \mathbf{L}^2(\mathbb{R}, e)$ is called a shifted Fourier multiplier operator if there exists a measurable function $m^e_\Lambda : \mathbb{R} \to \mathbb{C}$ such that, for all $\xi \in \mathbb{R}$,

$$\mathcal{F}^e_{\Lambda f}(\xi) = m^e_\Lambda(\xi)\mathcal{F}^e_f(\xi)$$

for all $f \in \mathbf{L}^2(\mathbb{R}, e)$ with $m^e_\Lambda \mathcal{F}^e_f \in \mathbf{L}^2(\mathbb{R})$. The operator Λ is bounded if and only if m^e is measurable and essentially bounded. In this work, we use a subclass of the shifted Fourier multiplier operators for which the multiplier is analytic on a strip. This class is introduced formally in the following definition.

Definition 2.3.4. Let $\mathscr{M}$ be the collection of all shifted Fourier multiplier operators defined on the weighted Hilbert space $\mathbf{L}^2(\mathbb{R}, e)$ such that for any $\Lambda \in \mathscr{M}$, there exists a function simply denoted by $m^e_\Lambda = m_\Lambda$, the Fourier multiplier, defined on $\mathbb{S}_{(0,1)}$ and that satisfies

(i) m_Λ is analytic in the strip $\mathbb{S}_{(0,1)}$

(ii) m_Λ is zero-free on $\mathbb{S}_{(0,1)}$ and

$$\mathcal{F}^e_{\Lambda f}(\xi) = m_\Lambda\left(\xi + \frac{\mathrm{i}}{2}\right)\mathcal{F}^e_f(\xi), \;\; \xi \in \mathbb{R}, \tag{2.42}$$

with $\mathbf{D}(\Lambda) = \{f \in \mathbf{L}^2(\mathbb{R}, e);\; m_\Lambda\left(\xi + \frac{\mathrm{i}}{2}\right)\mathcal{F}^e_f(\xi) \in \mathbf{L}^2(\mathbb{R})\}$.

If $f \in \mathbf{D}(\Lambda)$ is regular enough and the multiplier m_Λ satisfies some integrability conditions, then on a subspace of $\mathbf{L}^2(\mathbb{R}, e)$, Λ can be expressed as a classical Fourier multiplier operator on $\mathbf{L}^2(\mathbb{R})$.

Proposition 2.3.5. *Suppose that m_Λ extends continuously on $\mathbb{R}$. Let $f \in \mathbf{D}(\Lambda)$ be such that for some $\frac{1}{2} \leqslant \gamma \leqslant 1$,*

(i) $\mathcal{F}_f \in \mathbf{A}_{[0,\gamma]}$ and $\mathcal{F}_f(\cdot + \mathrm{i}b) \in \mathbf{L}^2(\mathbb{R})$ for all $0 \leqslant b \leqslant \gamma$

(ii) $\sup_{b\in[0,\gamma]} \int_\mathbb{R} |m_\Lambda(\xi + \mathrm{i}b)\mathcal{F}_f(\xi + \mathrm{i}b)|^2\, d\xi < \infty$.

Then, $\Lambda f \in \mathbf{L}^2(\mathbb{R}, e^{2bx})$ for any $0 \leqslant b \leqslant \gamma$, and for all $\xi \in \mathbb{R}$,

$$\mathcal{F}_{\Lambda f}(\xi + \mathrm{i}b) = m_\Lambda(\xi + \mathrm{i}b)\mathcal{F}_f(\xi + \mathrm{i}b). \tag{2.43}$$

Proof. We first note that condition (ii) ensures that for any $0 \leqslant b \leqslant \gamma$, there exists $g_b \in \mathbf{L}^2(\mathbb{R})$ such that $\mathcal{F}_{g_b}(\xi) = m_\Lambda(\xi + \mathrm{i}b)\mathcal{F}_f(\xi + \mathrm{i}b)$ for all $\xi \in \mathbb{R}$. Now, from the standard theory of Fourier transform, it is known that for any $0 \leqslant b \leqslant \gamma$,

$$g_b(x) = \lim_{n\to\infty} \frac{1}{\sqrt{2\pi}} \int_{-n}^{n} m_\Lambda(\xi + \mathrm{i}b)\mathcal{F}_f(\xi + \mathrm{i}b)e^{\mathrm{i}\xi x}d\xi$$

where the above convergence holds in $\mathbf{L}^2(\mathbb{R})$. Now, for each $x \in \mathbb{R}$, since the integrand above satisfies the condition of Lemma 2.3.3 with $p = 2$, we can therefore choose a subsequence $(n_j)_{j\geqslant 1}$ such that for all $x \in \mathbb{R}$ and $0 \leqslant b \leqslant \gamma$,

$$\frac{1}{\sqrt{2\pi}} \int_{-n_j}^{n_j} m_\Lambda(\xi)\mathcal{F}_f(\xi)e^{\mathrm{i}\xi x}d\xi - \frac{1}{\sqrt{2\pi}} \int_{-n_j}^{n_j} m_\Lambda(\xi + \mathrm{i}b)\mathcal{F}_f(\xi + \mathrm{i}b)e^{\mathrm{i}\xi x}e^{-bx}d\xi \to 0$$

as $j \to \infty$. Note that the first integrand above converges to $g_0(x)$ and the second integral converges to $e^{-bx}g_b(x)$ in $\mathbf{L}^2(\mathbb{R})$, which implies that $g_b(x) = e^{bx}g_0(x)$ for all $0 \leqslant b \leqslant \gamma$, in particular when $b = \frac{1}{2}$. Now, from the definition of Λf, it is clear that $\Lambda f(x) = e^{-\frac{x}{2}}g_{\frac{1}{2}}(x) = g_0(x)$. The above computation shows that $g_0 \in \mathbf{L}^2(\mathbb{R}, e^{2bx})$ for all $0 \leqslant b \leqslant \gamma$, which proves the first statement of the result. By means of Cauchy-Schwarz inequality, for any $0 < b < \gamma$ we have

$$\left(\int_{-\infty}^{0} |g_0(x)|e^{bx}dx\right)^2 \leqslant \int_{-\infty}^{0} |g_0(x)|^2 dx \int_{-\infty}^{0} e^{2bx}dx < \infty$$
$$\left(\int_{0}^{\infty} |g_0(x)|e^{bx}dx\right)^2 \leqslant \int_{0}^{\infty} |g_0(x)|^2 e^{2\gamma x}dx \int_{0}^{\infty} e^{-2(\gamma-b)x}dx < \infty$$

which shows that $\Lambda f = g_0 \in \mathbf{L}^1(\mathbb{R}, e^{bx})$ for all $0 < b < \gamma$. Hence, for any $0 < b < \gamma$ one has

$$\mathcal{F}_{\Lambda f}(\xi + \mathrm{i}b) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} g_0(x)e^{bx}e^{-\mathrm{i}\xi x}dx = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} g_b(x)e^{-\mathrm{i}\xi x}dx = m_\Lambda(\xi + \mathrm{i}b)\mathcal{F}_f(\xi + \mathrm{i}b).$$

Since $\Lambda f \in \mathbf{L}^2(\mathbb{R}) \cap \mathbf{L}^2(\mathbb{R}, e^{2\gamma x})$ and the right-hand side of the above equation extends continuously to the boundary of $\mathbb{S}_{(0,\gamma)}$, (2.43) follows for all $0 \leqslant b \leqslant \gamma$.

Proposition 2.3.6. *Any $(\Lambda, \mathbf{D}(\Lambda)) \in \mathscr{M}$ is densely defined with dense range in $\mathbf{L}^2(\mathbb{R}, e)$ and $\widehat{\mathcal{F}^e}_{\mathbf{C}_c^\infty(\mathbb{R})}$ is a core.*

Proof. Let $f \in \mathbf{D}(\Lambda)$ be such that $\mathcal{F}_f^e(\xi) \neq 0$ a.e.. Then, for any $a, \xi \in \mathbb{R}$,

$$\mathcal{F}^e_{\tau_a \Lambda f}(\xi) = e^{\mathrm{i}a - \frac{a}{2}} m_\Lambda\left(\xi + \mathrm{i}/2\right) \mathcal{F}^e_f(\xi) = m_\Lambda\left(\xi + \mathrm{i}/2\right) \mathcal{F}^e_{\tau_a f}(\xi) = \mathcal{F}^e_{\Lambda \tau_a f}(\xi),$$

that is $\tau_a \Lambda f \in \mathrm{Range}(\Lambda)$. An application of Wiener's Tauberian theorem yields that the set $\mathrm{Span}\{\tau_a \Lambda f;\ a \in \mathbb{R}\}$ is dense in $\mathbf{L}^2(\mathbb{R}, e)$. Next, we observe that

$$\mathbf{D}(\Lambda) = \left\{ f \in \mathbf{L}^2(\mathbb{R}, e);\ \mathcal{F}^e_f \in \mathbf{L}^2\left(\mathbb{R}, \left(1 + \left|m_\Lambda\left(\xi + \mathrm{i}/2\right)\right|^2\right) d\xi\right) \right\}. \tag{2.44}$$

Since the function $1 + \left|m_\Lambda\left(\cdot + \mathrm{i}/2\right)\right|^2$ is locally integrable, the corresponding weighted L^2-space has $\mathbf{C}_c^\infty(\mathbb{R})$ as a dense subset. Also, the graph norm of Λ, which is given by $\|f\|_\Lambda := \|f\|_{\mathbf{L}^2(\mathbb{R},e)} + \|\Lambda f\|_{\mathbf{L}^2(\mathbb{R},e)}$, is equivalent to the norm of $\mathbf{L}^2\left(\mathbb{R}, \left(1 + \left|m_\Lambda\left(\xi + \mathrm{i}/2\right)\right|^2\right) d\xi\right)$. Thus, $\mathcal{F}^e_{\mathbf{C}_c^\infty(\mathbb{R})}$ is dense in $\mathbf{D}(\Lambda)$ with respect to the graph norm $\|\cdot\|_\Lambda$ and hence is a core of $(\Lambda, \mathbf{D}(\Lambda))$. The class $\mathscr{M}$ also has some nice algebraic properties that are given in the following.

Proposition 2.3.7. *The class $\mathscr{M}$ forms an abelian group with respect to operator multiplication and it is closed under adjoints.*

Proof. We note that any $\Lambda \in \mathscr{M}$ is injective because from (2.42), $\Lambda f \equiv 0$ implies that $\mathcal{F}^e_f = 0$ a.e., as $m_\Lambda(\xi + \mathrm{i}/2) \neq 0$ a.e., which yields $f \equiv 0$. From the definition, it is also clear that any operator in $\mathscr{M}$ is closed. Furthermore, any operator $(\Lambda, \mathbf{D}(\Lambda)) \in \mathscr{M}$ admits an inverse $(\Lambda^{-1}, \mathbf{D}(\Lambda^{-1}))$ which is also closed and $\mathbf{D}(\Lambda^{-1}) = \mathrm{Range}(\Lambda)$ and $\mathrm{Range}(\Lambda^{-1}) = \mathbf{D}(\Lambda)$. It is easy to see that $(\Lambda^{-1}, \mathbf{D}(\Lambda^{-1})) \in \mathscr{M}$ with $m_{\Lambda^{-1}} = \frac{1}{m_\Lambda}$. Also, for any two operators $\{(\Lambda_k, \mathbf{D}(\Lambda_k))\}_{k=1}^2$, $\Lambda_1\Lambda_2$ is densely defined with $m_{\Lambda_1\Lambda_2} = m_{\Lambda_1} m_{\Lambda_2}$. Clearly, $(\Lambda_1\Lambda_2, \mathbf{D}(\Lambda_1\Lambda_2)) \in \mathscr{M}$. For $\Lambda \in \mathscr{M}$, we claim that $\widehat{\Lambda}$, the adjoint of Λ is the Fourier multiplier operator with multiplier $m_{\widehat{\Lambda}} = \overline{m}_\Lambda$. To see this, let $g \in \mathbf{L}^2(\mathbb{R}, e)$ be such that the map

$$f \mapsto \langle \Lambda f, g \rangle_e = \left\langle m_\Lambda\left(\cdot + \mathrm{i}/2\right) \mathcal{F}^e_f, \mathcal{F}^e_g \right\rangle_{\mathbf{L}^2(\mathbb{R})} = \left\langle \mathcal{F}^e_f, \overline{m}_\Lambda\left(\cdot + \mathrm{i}/2\right) \mathcal{F}^e_g \right\rangle_{\mathbf{L}^2(\mathbb{R})}$$

is continuous. This is possible only if $\overline{m}_\Lambda\left(\cdot + \mathrm{i}/2\right) \mathcal{F}^e_g \in \mathbf{L}^2(\mathbb{R})$, which completes the proof of the proposition. Let $\mathscr{M}_e$ be the class of operators in $\mathscr{M}$ with the following additional property

$$\text{For all } (\Lambda, \mathbf{D}(\Lambda)) \in \mathscr{M}_e,\ \left|m_\Lambda\left(\xi + \mathrm{i}/2\right)\right| \leqslant C e^{k|\xi|}, \quad \text{for all } \xi \in \mathbb{R}, \tag{2.45}$$

where C, k are positive constants depending on Λ but does not depend on ξ. Finally, we define $\overline{\mathscr{M}}$ (resp. $\overline{\mathscr{M}}_e$) to be the class of operators Λ in $\mathscr{M}$ (resp. $\mathscr{M}_e$) such that the function m_Λ is analytic on $\mathbb{S}_{(0,1)}$ and extends continuously to $\mathbb{S}_{[0,1]}$. One notes that $(\Lambda, \mathbf{D}(\Lambda)) \in \mathscr{M}$ is bounded if and only if $m_\Lambda(\cdot + \mathrm{i}/2)$ is bounded. We now show that Fourier multiplier operators satisfy the transitivity property when they act as intertwining operators.

Proposition 2.3.8. *Let $A, B \in \mathscr{B}(\mathbf{L}^2(\mathbb{R}, e))$ and $\Lambda \in \mathscr{M}_e$ be such that $A\Lambda f = \Lambda B f$ for all $f \in \mathbf{D}(\Lambda)$. Then $B\Lambda^{-1} f = \Lambda^{-1} A f$ for all $f \in \mathbf{D}(\Lambda^{-1})$. Similarly, if for $A, B, C \in \mathscr{B}(\mathbf{L}^2(\mathbb{R}, e))$, there exist $\Lambda_1, \Lambda_2 \in \mathscr{M}_e$ such that $A\Lambda_1 = \Lambda_1 B$ on $\mathbf{D}(\Lambda_1)$ and $B\Lambda_2 = \Lambda_2 C$ on $\mathbf{D}(\Lambda_2)$, then $A\Lambda_1\Lambda_2 = \Lambda_1\Lambda_2 B$ on $\mathbf{D}(\Lambda_1\Lambda_2)$.*

Proof. If $A\Lambda = \Lambda B f$ on $\mathbf{D}(\Lambda)$, then by injectivity of Λ, we have $B\Lambda^{-1} = \Lambda^{-1} A$ on $\mathrm{Range}(\Lambda)$. We claim that this identity can be extended to

$$\mathbf{D}(\Lambda^{-1}) = \left\{ f \in \mathbf{L}^2(\mathbb{R}, e);\ \xi \mapsto \frac{\mathcal{F}^e_f(\xi)}{m_\Lambda(\xi + \mathrm{i}/2)} \in \mathbf{L}^2(\mathbb{R}) \right\}.$$

By Proposition 2.3.6, the set $\mathcal{F}^e_{\mathbf{C}^\infty_c(\mathbb{R})}$ is a core of Λ^{-1} and, we observe that $\mathcal{F}^e_{\mathbf{C}^\infty_c(\mathbb{R})} \subset \mathrm{Range}(\Lambda)$. Now, for any $f \in \mathbf{D}(\Lambda^{-1})$, let $(f_n)_{n\geqslant 0} \subset \mathcal{F}^e_{\mathbf{C}^\infty_c(\mathbb{R})}$ be such that as $n \to \infty$, $f_n \to f$ and $\Lambda^{-1} f_n \to \Lambda^{-1} f$ in $\mathbf{L}^2(\mathbb{R}, e)$. From the given identity, we have, for all n,

$$B\Lambda^{-1} f_n = \Lambda^{-1} A f_n. \tag{2.46}$$

Now letting $n \to \infty$ and using the fact that $\Lambda^{-1} A$ is closed, as A is bounded, we conclude that $B\Lambda^{-1} f = \Lambda^{-1} A f$ for all $f \in \mathbf{D}(\Lambda^{-1})$. To prove the transitivity property we consider $A, B, C \in \mathscr{B}(\mathbf{L}^2(\mathbb{R}, e))$ such that $A\Lambda_1 = \Lambda_1 B$ on $\mathbf{D}(\Lambda_1)$ and $B\Lambda_2 = \Lambda_2 C$ on $\mathbf{D}(\Lambda_2)$. Then, we already have $A\Lambda_1\Lambda_2 = \Lambda_1\Lambda_2 B$ on $\{f \in \mathbf{L}^2(\mathbb{R}, e);\ f \in \mathbf{D}(\Lambda_2),\ \Lambda_2 f \in \mathbf{D}(\Lambda_1)\}$. To extend the last identity to the full domain

$$\mathbf{D}(\Lambda_1\Lambda_2) = \left\{ f \in \mathbf{L}^2(\mathbb{R}, e);\ \xi \mapsto m_{\Lambda_1}(\xi + \mathrm{i}/2)\, m_{\Lambda_2}(\xi + \mathrm{i}/2)\, \mathcal{F}^e_f(\xi) \in \mathbf{L}^2(\mathbb{R}, e) \right\},$$

we use a similar reasoning as above.

2.3.5 Some basic facts about generators

In the theory of Markov semigroups, there are several notions for defining the generators. It depends on the underlying Banach space and the topology one is interested about. In this article we are concerned about the $\mathbf{L}^2$-generators of the Markov semigroups. The main purpose of this section is to connect the three types of generators, namely, the Feller generator, the Dynkin characteristic operator, and the $\mathbf{L}^2$-generator. We begin with the following result due to Dynkin that relates the characteristic operator and the infinitesimal generator of a Feller process.

Lemma 2.3.9. *[41, Theorem 5.5, Chapter V.3] Let $(A_{\mathtt{F}}, \mathbf{D}(A_{\mathtt{F}}))$ and $(A_{\mathtt{D}}, \mathbf{D}(A_{\mathtt{D}}))$ be respectively the Feller infinitesimal generator and Dynkin characteristic operator of a Feller process with state space E. If for some $f \in \mathbf{D}(A_{\mathtt{D}})$, $A_{\mathtt{D}} f \in \mathbf{C}_0(E)$, then $f \in \mathbf{D}(A_{\mathtt{F}})$ and $A_{\mathtt{F}} f = A_{\mathtt{D}} f$.*

The next result is about the equality of the strong and weak generator of a Markov semigroup on a Hilbert space and the proof can be found in [109].

Lemma 2.3.10. *[109, Theorem 1.3] Let $(P_t)_{t \geqslant 0}$ be a $\mathcal{C}_0$-contraction semigroup of bounded operators on a Banach space $\mathbf{Y}$. Let $(\tilde{A}, \mathbf{D}(\tilde{A}))$ and $(A, \mathbf{D}(A))$ denote the weak and strong generator of this semigroup respectively. Then, $(\tilde{A}, \mathbf{D}(\tilde{A})) = (A, \mathbf{D}(A))$.*

Finally, we have the following result that relates the Feller infinitesimal generator with the $\mathbf{L}^2$-generator of a Markov process.

Lemma 2.3.11. *For a Feller semigroup $(P_t)_{t \geqslant 0}$ with an excessive measure ε, i.e. $\varepsilon P_t \leqslant \varepsilon$, let $(P_t^{(2)})_{t \geqslant 0}$ denote its $\mathbf{L}^2(\mathbb{R}, \varepsilon)$-extension. Let $(A_{\mathtt{F}}, \mathbf{D}(A_{\mathtt{F}}))$ and $(A_2, \mathbf{D}(A_2))$ be the generators of P and $P^{(2)}$ respectively. If $f \in \mathbf{D}(A_{\mathtt{F}}) \cap \mathbf{L}^2(\mathbb{R}, \varepsilon)$ is such that $A_{\mathtt{F}} f \in \mathbf{L}^2(\mathbb{R}, \varepsilon)$, then $f \in \mathbf{D}(A_2)$ and $A_{\mathtt{F}} f = A_2 f$.*

Proof. Consider any $f \in \mathbf{D}(A_{\mathtt{F}}) \cap \mathbf{L}^2(\mathbb{R}, \varepsilon)$ such that $A_{\mathtt{F}} f \in \mathbf{L}^2(\mathbb{R}, \varepsilon)$ and $g \in \mathbf{L}^2(\mathbb{R}, \varepsilon)$. Then, we know that, for all $t \geqslant 0$, $P_t^{(2)} g \in \mathbf{L}^2(\mathbb{R}, \varepsilon)$ and

$$\left\langle (P_t^{(2)} f - f)/t, g \right\rangle_{\mathbf{L}^2(\mathbb{R},\varepsilon)} = \langle (P_t f - f)/t, g \rangle_{\mathbf{L}^2(\mathbb{R},\varepsilon)} . \tag{2.47}$$

Since $A_{\mathtt{F}}$ is the generator of the Feller semigroup $(P_t)_{t \geqslant 0}$, we have

$$\lim_{t \to 0} \|(P_t f - f)/t - A_{\mathtt{F}} f\|_\infty = 0$$

where $\| \cdot \|_\infty$ stands for the supremum norm in $\mathbf{C}_0(\mathbb{R})$. Since $A_{\mathtt{F}} f \in \mathbf{L}^2(\mathbb{R}, \varepsilon)$, we observe that for any $t > 0$,

$$\left\| \frac{P_t f - f}{t} \right\|_{\mathbf{L}^2(\mathbb{R},m)} = \frac{1}{t} \left\| \int_0^t P_s A_{\mathtt{F}} f ds \right\|_{\mathbf{L}^2(\mathbb{R},\varepsilon)} \leqslant \frac{1}{t} \int_0^t \|P_s A_{\mathtt{F}} f\|_{\mathbf{L}^2(\mathbb{R},\varepsilon)} ds \leqslant \|A_{\mathtt{F}} f\|_{\mathbf{L}^2(\mathbb{R},\varepsilon)}.$$

Now, (2.47) implies that if $g \in \mathbf{L}^1(\mathbb{R}, \varepsilon) \cap \mathbf{L}^2(\mathbb{R}, \varepsilon)$, then $\lim_{t \to 0} \left\langle \frac{P_t f - f}{t}, g \right\rangle_{\mathbf{L}^2(\mathbb{R},\varepsilon)} = \langle A_{\mathtt{F}} f, g \rangle_{\mathbf{L}^2(\mathbb{R},\varepsilon)}$. On the other hand, since $\{(P_t f - f)/t;\ t > 0\}$ is bounded in $\mathbf{L}^2(\mathbb{R}, \varepsilon)$, it is therefore weakly compact in $\mathbf{L}^2(\mathbb{R}, \varepsilon)$. Let f^* be a weak subsequential limit of $(P_t f - f)/t$ as $t \to 0$. Then, by (2.47) and the discussion above, we have $\langle f^*, g \rangle_{\mathbf{L}^2(\mathbb{R},\varepsilon)} = \langle f, g \rangle_{\mathbf{L}^2(\mathbb{R},\varepsilon)}$ for all $g \in \mathbf{L}^1(\mathbb{R}, \varepsilon) \cap \mathbf{L}^2(\mathbb{R}, \varepsilon)$. Since $\mathbf{L}^1(\mathbb{R}, \varepsilon) \cap \mathbf{L}^2(\mathbb{R}, \varepsilon)$ is dense in $\mathbf{L}^2(\mathbb{R}, \varepsilon)$, we infer that $\frac{P_t^{(2)} f - f}{t} \to A_{\mathtt{F}} f$ weakly for all $f \in \mathbf{D}(A_{\mathtt{F}}) \cap \mathbf{L}^2(\mathbb{R}, \varepsilon)$, as $t \to 0$. Finally, invoking Lemma 2.3.10, we conclude that $A_{\mathtt{F}} f$ is also the strong limit of $(P_t^{(2)} f - f)/t$, which completes the proof. We close this section with the following lemma that shows that the $\mathcal{C}_0$-contraction semigroup generation property of operators is preserved under the weak similarity relation.

Lemma 2.3.12. *Let the closure of $(B, \mathcal{D})$ be the generator of a $\mathcal{C}_0$-contraction semigroup on a Banach space X. Let $(A, \mathbf{D}(A))$ be a densely defined operator on X such that*

$$A\Lambda = \Lambda B \ \text{ on } \ \mathcal{D} \tag{2.48}$$

where $\Lambda \in \mathscr{B}(\mathrm{X})$ with dense range, and A is dissipative on $\Lambda(\mathcal{D})$. Then, the closure of $(A, \Lambda(\mathcal{D}))$ generates a $\mathcal{C}_0$-contraction semigroup on X.

Proof. We first note that $\mathcal{D}$ is a dense subset of X as it is a core for the generator of a $\mathcal{C}_0$-contraction semigroup on X. Next, by the Hille-Yosida Theorem, $(\alpha I - B)(\mathcal{D})$ is dense in X

for any $\alpha > 0$. Since Λ is bounded and $\mathrm{Range}(\Lambda)$ is dense in X, it follows that both $\Lambda(\mathcal{D})$ and $\Lambda((\alpha I - B)(\mathcal{D}))$ are dense subsets of X. Now, using (2.48), we have $\Lambda((\alpha I - B)(\mathcal{D})) = (\alpha I - A)(\Lambda(\mathcal{D}))$, which shows that the latter subset is dense in X for any $\alpha > 0$. Since A is also dissipative on $\Lambda(\mathcal{D})$, the proof is completed by applying the Hille-Yosida Theorem on the operator $(A, \Lambda(\mathcal{D}))$.

2.3.6 Bernstein-gamma functions and a functional equation

We recall, from (2.9), that $\phi \in \mathbf{B}$ can be written as $\phi(z) = \phi(0) + \mathtt{d}z + \int_0^\infty (1 - e^{-zy})\nu(dy),\ z \in \mathbb{C}_{[0,\infty)}$, where ν is a positive measure such that $\int_0^\infty (y \wedge 1)\nu(dy) < \infty$ and $\phi(0), \mathtt{d} \geqslant 0$. Next, we define

$$W_\phi(z) = \frac{e^{-\gamma_\phi z}}{\phi(z)} \prod_{k=1}^{\infty} \frac{\phi(z)}{\phi(k+z)} e^{\frac{\phi'(k)}{\phi(k)}},\ \ z \in \mathbb{C}_{(0,\infty)},$$

where $\gamma_\phi = \lim_{n\to\infty} \left(\sum_{k=1}^n \frac{\phi'(k)}{\phi(k)} - \ln \phi(n) \right) \in \left[-\ln \phi(1), \frac{\phi'(1)}{\phi(1)} - \ln \phi(1) \right]$. From [96, Section 4] and [95, Theorem 4.2], it is known that W_ϕ is a solution to the functional equation

$$f(z+1) = \phi(z) f(z),\ f(0) = 1, \quad z \in \mathbb{C}_{(0,\infty)}, \tag{2.49}$$

and, satisfies the following important properties.

(W1) W_ϕ is the unique positive definite solution to the above functional equation (2.49), that is it is the unique solution in the set of Mellin transforms of probability measures on $(0, \infty)$

(W2) W_ϕ is zero-free on $\mathbb{C}_{(0,\infty)}$ and $\overline{W}(z) = W(\overline{z})$ for all $z \in \mathbb{C}_{(0,\infty)}$

(W3) W_ϕ is analytic on $\mathbb{C}_{(0,\infty)}$

(W4) For any $z = a + \mathrm{i}\xi$ with $a > 0$

$$|W_\phi(z)| = \frac{\sqrt{\phi(1)}}{\sqrt{\phi(a)\phi(1+a)|\phi(z)|}} e^{G_\phi(a) - A_\phi(z) - E_\phi(z) - R_\phi(a)} \tag{2.50}$$

where $G_\phi(a) = \int_1^{1+a} \ln \phi(u)\, du \leqslant a \ln \phi(1+a)$, $A_\phi(z) = \int_a^\infty \ln\left(\frac{|\phi(u+\mathrm{i}\xi)|}{\phi(u)}\right) du \in [0, \frac{\pi}{2}|\xi|]$ and $\sup_{z\in\mathbb{C}_{(d,\infty)}} |E_\phi(z)| + \sup_{a>d} |R_\phi(a)| < \infty$ for all $d > 0$. We again refer to [96, Theorem 3.1 and Theorem 3.2] for the definition of the quantities $G_\phi, A_\phi, E_\phi, R_\phi$.

W_ϕ is called the Bernstein-gamma function corresponding to ϕ. These functions have been studied extensively in [96]. The next proposition derives the analyticity and growth bounds on the multiplier functions defined in (2.28). We deal with the one-dimensional case here, which is the main ingredient for all the results in this chapter.

Proposition 2.3.13. *Let $\psi \in \mathbf{N}(\mathbb{R})$ be such that $\psi(\xi) = \phi_+(-\mathrm{i}\xi)\phi_-(\mathrm{i}\xi)$ for all $\xi \in \mathbb{R}$ and $\psi_\beta(\xi) = \xi^2 - \mathrm{i}\beta\xi$, $\beta > 0$. Then, the function $W_\psi^{(\beta)} : \mathbb{S}_{(0,1)} \to \mathbb{C}$ defined by*

$$W_\psi^{(\beta)}(z) = \frac{\Gamma(1+\mathrm{i}z+\beta)W_{\phi_+}(-\mathrm{i}z)}{\Gamma(-\mathrm{i}z)W_{\phi_-}(1+\mathrm{i}z)}$$

satisfies the functional equation

$$W_\psi^{(\beta)}(\xi)\psi(\xi) = W_\psi^{(\beta)}(\xi+\mathrm{i})\psi_\beta(\xi), \;\; \xi \in \mathbb{R}, \tag{2.51}$$

and enjoys the following properties.

(1) $W_\psi^{(\beta)}$ is analytic in the strip $\mathbb{S}_{(0,1)}$ and extends continuously on $\mathbb{S}_{[0,1]}$.

(2) Let $\psi \in \mathbf{N}_+(\mathbb{R})$. Then,

$$\sup_{a\in[0,1]} \left| W_\psi^{(\beta)}(\xi+\mathrm{i}a) \right| = \mathrm{O}(|\xi|^u) \textit{ for some } u > \beta. \tag{2.52}$$

Remark 2.3.14. From the proof of Proposition 2.3.13, it is clear that for any $\psi_1, \psi_2 \in \mathbf{N}(\mathbb{R})$, with $\psi_1(\xi) = \phi_+^{(1)}(-\mathrm{i}\xi)\phi_-^{(1)}(\mathrm{i}\xi)$, $\psi_2(\xi) = \phi_+^{(2)}(-\mathrm{i}\xi)\phi_-^{(2)}(\mathrm{i}\xi)$, if we assume that

$$\{z \in \mathrm{i}\mathbb{R};\; \phi_\pm^{(2)}(z) = 0\} \subseteq \{0\} \text{ and } \phi_-^{(1)}(0) > 0,$$

then the solution of the functional equation

$$f(\xi)\psi_2(\xi) = f(\xi+\mathrm{i})\psi_1(\xi) \;\text{ for all } \xi \in \mathbb{R}\setminus\{0\} \tag{2.53}$$

is given by

$$f(\xi) = W[\psi_2;\psi_1](\xi) = \frac{W_{\phi_+^{(2)}}(-\mathrm{i}\xi)}{W_{\phi_+^{(1)}}(-\mathrm{i}\xi)} \frac{W_{\phi_-^{(1)}}(1+\mathrm{i}\xi)}{W_{\phi_-^{(2)}}(1+\mathrm{i}\xi)}.$$

Also, $W[\psi_2;\psi_1] \in \mathbf{A}_{(0,1]}$. In particular, taking $\psi_2 = \psi$, $\psi_1 \equiv 1$, the function $W(\xi) = \frac{W_{\phi_+}(-\mathrm{i}\xi)}{W_{\phi_-}(1+\mathrm{i}\xi)}$ satisfies

$$W(\xi + \mathrm{i}) = \psi(\xi)W(\xi) \quad \text{for all } \xi \in \mathbb{R} \setminus \{0\}, \tag{2.54}$$

and $W \in \mathbf{A}_{(0,1]}$.

2.3.7 Proof of Proposition 2.3.13

Let ψ be such that

$$\psi(\xi) = \phi_+(-\mathrm{i}\xi)\phi_-(\mathrm{i}\xi)$$

for all $\xi \in \mathbb{R}$ with $\phi_+, \phi_- \in \mathbf{B}$. Let us define the following pair of functions

$$\begin{cases} W_+(z) = \frac{W_{\phi_+}(-\mathrm{i}z)}{\Gamma(-\mathrm{i}z)}, & z \in \mathbb{S}_{(0,\infty)} \\ W_-(z) = \frac{\Gamma(1+\mathrm{i}z+\beta)}{W_{\phi_-}(1+\mathrm{i}z)}, & \text{for any } \beta > 0 \text{ and } z \in \mathbb{S}_{(-\infty,1)}. \end{cases}$$

Now, for $z \in \mathbb{S}_{(0,\infty)}$, we have

$$W_+(z+\mathrm{i}) = \frac{W_{\phi_+}(-\mathrm{i}z+1)}{\Gamma(-\mathrm{i}z+1)} = \frac{\phi_+(-\mathrm{i}z)}{-\mathrm{i}z} \frac{W_{\phi_+}(-\mathrm{i}z)}{\Gamma(-\mathrm{i}z)} = \frac{\phi_+(-\mathrm{i}z)}{-\mathrm{i}z} W_+(z)$$

i.e. $W_+(z) = W_+(z+\mathrm{i})\frac{-\mathrm{i}z}{\phi_+(-\mathrm{i}z)}$. Along the same vein, we have, for all $z \in \mathbb{S}_{(-\infty,0)}$,

$$W_-(z+\mathrm{i}) = W_-(z)\frac{\phi_-(\mathrm{i}z)}{\mathrm{i}z+\beta}.$$

Next, using the property (W3) for W_{ϕ_+}, we deduce that W_+ is analytic on $\mathbb{S}_{(0,\infty)}$ and extends continuously to the punctured line $\mathbb{R} \setminus \{0\}$. On the other hand, W_- is also analytic on $\mathbb{S}_{(-\infty,1)}$ and extends continuously on the line $\mathbb{R} + \mathrm{i}$ for any value of $\phi_-(0)$. To deal with the extension of W_+ to the entire real line, we observe that

$$\lim_{\substack{z\to 0\\ \mathtt{Im}(z)>0}} \frac{-\mathrm{i}z}{\phi_+(-\mathrm{i}z)} = \lim_{\substack{z\to 0\\ \mathtt{Re}(z)>0}} \frac{z}{\phi_+(z)} = \begin{cases} 0, & \text{if } \phi_+(0)>0 \\ \frac{1}{\phi_+'(0)} < \infty, & \text{if } \phi_+(0)=0 \end{cases}$$

and, hence W_+ extends continuously to $\mathbb{S}_{[0,\infty)}$. Now, we define $W_\psi^{(\beta)}(z) = W_+(z)W_-(z)$ for all $z \in \mathbb{S}_{(0,1)}$. Clearly, the above results entail that $W_\psi^{(\beta)}$ is analytic on $\mathbb{S}_{(0,1)}$, extends continuously to its boundary and solves (2.51). To show (2.52), by Phragmén-Lindelöf principle, it is enough to prove (2.52) when $a = 0, 1$ respectively. Let us deal with the case $a = 1$. Converting everything in terms of the function $W_\psi^{(\beta)}$, we note that

$$W_\psi^{(\beta)}(\xi + \mathrm{i}) = \frac{W_{\phi_+}(1+\mathrm{i}\xi)\Gamma(\beta - \mathrm{i}\xi)\phi_-(-\mathrm{i}\xi)}{W_{\phi_-}(1-\mathrm{i}\xi)\Gamma(1+\mathrm{i}\xi)}\ .$$

By Stirling asymptotic formula for the gamma function and recalling from [96, Proposition 3.1(4)] that $\phi_-(-\mathrm{i}\xi) = -\mathrm{i}\mathtt{d}_-\xi + \mathrm{o}(|\xi|)$, we get

$$\frac{\Gamma(\beta - \mathrm{i}\xi)\phi_-(-\mathrm{i}\xi)}{\Gamma(1+\mathrm{i}\xi)} = \mathrm{O}(|\xi|^\beta). \tag{2.55}$$

So, it is enough to study the asymptotic behavior of the ratio $\frac{W_{\phi_+}(1+\mathrm{i}\xi)}{W_{\phi_-}(1-\mathrm{i}\xi)}$. For this, we proceed by proving the following lemma.

Lemma 2.3.15. *For any $\phi \in \mathbf{B}$, $a, b \geqslant 0$, and for all $\xi \in \mathbb{R}$ one has*

$$|\phi(a+\mathrm{i}\xi) - \phi(b+\mathrm{i}\xi)| \leqslant \phi(|a-b|) - \phi(0).$$

Thus, for any $a, b > 0$, as $|\xi| \to \infty$, $\phi(a+\mathrm{i}\xi) \asymp \phi(b+\mathrm{i}\xi)$.

Proof. Let ϕ be of the form (2.9). Then, for any $a, b \geqslant 0$ and $\xi \in \mathbb{R}$, we can write

$$\phi(a+\mathrm{i}\xi) - \phi(b+\mathrm{i}\xi) = \mathtt{d}(a-b) + \int_{\mathbb{R}} \left(e^{-(a+\mathrm{i}\xi)y} - e^{-(b+\mathrm{i}\xi)y}\right)\nu(dy)$$

Now, let us assume that $a < b$. Applying triangle inequality to the right-hand side of the above equality, we obtain

$$|\phi(a+\mathrm{i}\xi) - \phi(b+\mathrm{i}\xi)| \leqslant \mathtt{d}(b-a) + \int_{\mathbb{R}} (1 - e^{-(b-a)y})e^{-ay}\nu(dy) \leqslant \phi(b-a) - \phi(0).$$

Since $|\phi(a+\mathrm{i}\xi)| \geqslant \phi(a) > 0$ for any $a>0$, from the above inequality we obtain that

$$\left|\frac{\phi(a+\mathrm{i}\xi)}{\phi(b+\mathrm{i}\xi)}-1\right| \leqslant \frac{\phi(|a-b|)-\phi(0)}{\phi(b)}, \quad \left|\frac{\phi(b+\mathrm{i}\xi)}{\phi(a+\mathrm{i}\xi)}-1\right| \leqslant \frac{\phi(|a-b|)-\phi(0)}{\phi(a)}$$

which completes the proof of the lemma. Now, coming back to the proof of the proposition, from the property (W4) and using the above lemma, we obtain

$$\left|\frac{W_{\phi_+}(1+\mathrm{i}\xi)}{W_{\phi_-}(1-\mathrm{i}\xi)}\frac{W_{\phi_-}(1/2+\mathrm{i}\xi)}{W_{\phi_+}(1/2-\mathrm{i}\xi)}\right| \lesssim e^{A_{\phi_-}(1+\mathrm{i}\xi)-A_{\phi_-}(1/2+\mathrm{i}\xi)+A_{\phi_+}(1/2+\mathrm{i}\xi)-A_{\phi_+}(1+\mathrm{i}\xi)} \tag{2.56}$$

where $f \lesssim g$ means that $\frac{f(\xi)}{g(\xi)}$ is bounded above as $|\xi|\to\infty$. Since $\lim_{|\xi|\to\infty}\frac{\phi_+(u+i\xi)}{u+i\xi} = \mathtt{d}_+$ locally uniformly in $u>0$, from the definition of A_{ϕ_+} in (2.50), we have that

$$\left|A_{\phi_+}(1+\mathrm{i}\xi)-A_{\phi_+}(1/2+\mathrm{i}\xi)\right| = \int_{\frac{1}{2}}^{1}\ln\left(\frac{|\phi_+(u+\mathrm{i}\xi)|}{\phi_+(u)}\right)du \leqslant \int_{\frac{1}{2}}^{1}\ln\left(\frac{|\phi_+(u+\mathrm{i}\xi)|}{\phi_+(\frac{1}{2})}\right)du$$

where the right most integral grows at most at the order of $\ln|\xi|$ as $|\xi|\to\infty$. On the other hand, $A_{\phi_-}(1+\mathrm{i}\xi) \leqslant A_{\phi_-}(1/2+\mathrm{i}\xi)$ for all $\xi\in\mathbb{R}$, see [96, Theorem 3.2(1)]. Therefore, by the boundedness of the ratio $\frac{W_{\phi_+}\left(\frac{1}{2}-\mathrm{i}\xi\right)}{W_{\phi_-}\left(\frac{1}{2}+\mathrm{i}\xi\right)}$ in (2.56) along with the estimate in (2.55), it follows that $W_\psi^{(\beta)}(1+\mathrm{i}\xi) = \mathrm{O}(|\xi|^u)$ for some $u>\beta$. The case $a=0$ can be handled in the similar fashion after observing that $W_{\phi_\pm}(-\mathrm{i}\xi) = W_{\phi_\pm}(1-\mathrm{i}\xi)/\phi_\pm(-\mathrm{i}\xi)$ for any $\xi\in\mathbb{R}$. Since there exists $\kappa>0$ such that $\varliminf_{|\xi|\to\infty}|\xi|^\kappa|\phi_+(-\mathrm{i}\xi)|>0$, using the similar idea as before, one can show that the following ratio

$$\left|\frac{W_{\phi_+}(-\mathrm{i}\xi)}{W_{\phi_-}(1+\mathrm{i}\xi)}\frac{W_{\phi_-}\left(\frac{1}{2}+\mathrm{i}\xi\right)}{W_{\phi_+}\left(\frac{1}{2}-\mathrm{i}\xi\right)}\right|$$

grows at most polynomially as $|\xi|\to\infty$. Hence, the proof of the proposition is completed by applying the Phragmén-Lindelöf principle to the function $z\mapsto(2+z^2)^{-\frac{u}{2}}W_\psi^{(\beta)}(z)$ in the strip $\mathbb{S}_{[0,1]}$. □

2.3.8 Proof of Proposition 2.2.11

From [95, Theorem 6.2(1)], we know that for any $\xi\in\mathbb{R}$,

$$\left|W_\phi\left(\frac{1}{2}+\mathrm{i}\xi\right)\right| \asymp \frac{\sqrt{\phi(\frac{1}{2})}W_\phi(\frac{1}{2})}{\sqrt{|\phi\left(\frac{1}{2}+\mathrm{i}\xi\right)|}}e^{-|\xi|\Theta_\phi(|\xi|)}. \tag{2.57}$$

Clearly, this proves the estimate in (2.31). When condition (i) holds, the ratio

$$\left|\frac{W_{\phi_+}\left(\frac{1}{2}+\mathrm{i}\xi\right)}{W_{\phi_-}\left(\frac{1}{2}+\mathrm{i}\xi\right)}\right|$$

decays exponentially, thanks to (2.31). Moreover, we note that $\lim\limits_{|\xi|\to\infty}|\phi_+(\mathrm{i}\xi)|=\infty$ if $\mathtt{d}_+>0$. Now, writing $\overline{\nu}_\pm(r)=\nu_\pm(r,\infty)$, [96, Theorem 3.3(2)] ensures that if $\mathtt{d}_+>0$, $\mathtt{d}_-=0$ and $\overline{\nu}_-\in\mathbf{RV}(\alpha)$ with $r\mapsto\frac{\overline{\nu}_-(r)}{r^\alpha}$ being quasi-monotone, then $\lim_{|\xi|\to\infty}\Theta_{\phi_+}(|\xi|)=\pi/2$ and $\lim_{|\xi|\to\infty}\Theta_{\phi_-}(|\xi|)=\pi\alpha/2$. When $\mathtt{d}_+=\mathtt{d}_-=0$, $\overline{\nu}_\pm\in\mathbf{RV}(\alpha_\pm)$ and $r\mapsto\frac{\overline{\nu}_\pm(r)}{r^{\alpha_\pm}}$ are quasi-monotone, the previous reference entails that $\lim_{|\xi|\to\infty}\Theta_{\phi_\pm}(|\xi|)=\pi\alpha_\pm/2$. Since $\alpha_+>\alpha_-$, applying (2.31) it follows that $\psi\in\mathbf{N}_+(\mathbb{R})$ and $m_\psi\in\mathbf{L}^2(\mathbb{R})$. This proves (i). To prove (ii), invoking [96, Proposition 6.2], we know that if $\mathtt{d}_-=0$ or $\overline{\nu}_-(0)=\infty$, then for any $u>0$,

$$\lim_{|\xi|\to\infty}|\xi|^u\left|\frac{\Gamma\left(\frac{1}{2}+\mathrm{i}\xi\right)}{W_{\phi_-}\left(\frac{1}{2}+\mathrm{i}\xi\right)}\right|=0.$$

On the other hand, if $\mathtt{d}_+>0$ and $\overline{\nu}_+(0)<\infty$, then

$$\lim_{|\xi|\to\infty}|\xi|^u\left|\frac{\Gamma\left(\frac{1}{2}+\mathrm{i}\xi\right)}{W_{\phi_+}\left(\frac{1}{2}+\mathrm{i}\xi\right)}\right|=\infty$$

whenever $u>\frac{\phi_+(0)+\overline{\nu}_+(0)}{\mathtt{d}_+}$. This proves (ii).

In both of the above two cases, we note that m_ψ and $m_{\overline{\psi}}$ cannot be in $\mathbf{L}^2(\mathbb{R})$ simultaneously as $m_{\overline{\psi}}(\xi)=1/\overline{m}_\psi(\xi)$ for all $\xi\in\mathbb{R}$.

2.4 Proof of Theorem 2.2.1: the one dimensional case

Since the proof of this theorem is rather long, we split it into two parts and start by sketching the main ideas. We first prove it in the one dimensional case, that is $d=1$, and thus $M=\mathrm{Id}$. The general case will follow by tensorization and similarity transform techniques and it is postponed to Section 2.5. In Theorem 2.4.1, we prove the one dimensional case where we first derive the weak similarity relations at the level of pseudo-differential operators. Then, using the results from

Subsection 2.3.5, we show that those pseudo-differential operators generate Markov semigroups and lift the weak similarity relations at the level of semigroups. Subsequently, using Dynkin's theorem (see Lemma 2.3.9) and uniqueness of semigroups generated by operators defined on a core, we show that these semigroups are indeed the ones corresponding to the log-self-similar Markov processes. As a by-product of the weak similarity identity (2.60), we obtain the core and integro-differential representation of the $\mathbf{L}^2(\mathbb{R},e)$-generators of these semigroups, see Proposition 2.4.2, by using some approximation techniques motivated from [96].

Recall from Subsection 2.3.2 that for $\psi \in \mathbf{N}_b(\mathbb{R})$, $P[\psi]$ stands for the $\mathcal{C}_0$-contraction semigroup on $\mathbf{L}^2(\mathbb{R},e)$, which coincides with the log-self-similar Feller semigroup corresponding to the Lévy-Khintchine exponent ψ, when restricted on $\mathbf{C}_0(\mathbb{R})$. We also denote the $\mathbf{L}^2(\mathbb{R},e)$-generator of $P[\psi]$ by $A_2[\psi]$.

Theorem 2.4.1. *(1) Let $\psi_0(\xi)=\xi^2$, $\xi\in\mathbb{R}$, then the closure of the operator $(A_{\mathsf{PDO}}[\psi_0],\mathbf{C}_c^\infty(\mathbb{R}))$ in $\mathbf{L}^2(\mathbb{R},e)$ generates the log-squared Bessel semigroup Q on $\mathbf{L}^2(\mathbb{R},e)$.*

(2) For any $\psi\in\mathbf{N}_b(\mathbb{R})$, the shifted Fourier multiplier operator $\Lambda_\psi\in\mathscr{M}_e$, where $m_{\Lambda_\psi}(z)=\frac{W_{\phi_+}(-iz)}{W_{\phi_-}(1+iz)}\frac{\Gamma(1+iz)}{\Gamma(-iz)}$ for all $z\in\mathbb{S}_{(0,1)}$. In particular, when $\psi\in\mathbf{N}_+(\mathbb{R})$, $\Lambda_\psi\in\mathscr{B}(\mathbf{L}^2(\mathbb{R},e))$.

(3) For any $\psi\in\mathbf{N}_b(\mathbb{R})$, we have $A_{\mathsf{PDO}}[\psi]\in\mathrm{WS}(A_{\mathsf{PDO}}[\psi_0])$. More specifically, recalling that $\mathcal{D}(\mathbb{R})=\{f\in\mathbf{L}^2(\mathbb{R},e);\ \mathcal{F}_f$ is entire and satisfies (2.20)$\}$, we have $\mathcal{D}(\mathbb{R})$ is dense in $\mathbf{L}^2(\mathbb{R},e)$, and,

$$A_{\mathsf{PDO}}[\psi]\Lambda_\psi=\Lambda_\psi A_{\mathsf{PDO}}[\psi_0]\ \ \text{on}\ \ \mathcal{D}(\mathbb{R}). \tag{2.58}$$

In particular, if $\psi\in\mathbf{N}_+(\mathbb{R})$ then (2.58) holds on $\mathbf{C}_c^\infty(\mathbb{R})$.

(4) If $\psi\in\mathbf{N}_+(\mathbb{R})$ then the closure of the operator $(A_{\mathsf{PDO}}[\psi],\Lambda_\psi(\mathbf{C}_c^\infty(\mathbb{R})))$ in $\mathbf{L}^2(\mathbb{R},e)$ generates the semigroup $P[\psi]$ on $\mathbf{L}^2(\mathbb{R},e)$. Therefore, in this case, $\Lambda_\psi(\mathbf{C}_c^\infty(\mathbb{R}))$ is a core for $A_2[\psi]$. In general, for any $\psi\in\mathbf{N}_b(\mathbb{R})$, we have

$$A_2[\psi]=A_{\mathsf{PDO}}[\psi]\ \ \text{on}\ \ \Lambda_\psi(\mathcal{D}(\mathbb{R})) \tag{2.59}$$

and $\Lambda_\psi(\mathcal{D}(\mathbb{R}))$ is dense in $\mathbf{L}^2(\mathbb{R},e)$.

(5) For any $\psi \in \mathbf{N}_b(\mathbb{R})$ and $t \geqslant 0$, we have

$$P_t[\psi]\Lambda_\psi = \Lambda_\psi Q_t \ \text{ on } \ \mathbf{D}(\Lambda_\psi). \tag{2.60}$$

If H_ψ is the shifted Fourier multiplier operator with $m_\psi(z) := m_{H_\psi}(z) = \frac{W_{\phi_+}(-\mathrm{i}z)}{W_{\phi_-}(1+\mathrm{i}z)}$, then $(H_\psi, \mathbf{D}(H_\psi)) \in \mathscr{M}_e$, and, for all $t \geqslant 0$, we have

$$P_t[\psi] = H_\psi e_t H_\psi^{-1} \ \text{ on } \ \mathbf{D}(H_\psi^{-1}) \tag{2.61}$$

where $(e_t)_{t\geqslant 0}$ denotes the multiplication semigroup on $\mathbf{L}^2(\mathbb{R}, e)$, i.e. $e_t f(x) = e^{-te^{-x}} f(x)$. Note that $H_\psi^{-1} = \widehat{H}_{\overline{\psi}}$ as they correspond to the same multiplier.

This theorem is proved in Section 2.4.1.

The item (4) in the above theorem provides us with a core for $A_2[\psi]$ when $\psi \in \mathbf{N}_+(\mathbb{R})$. The general case is more involved as $\mathbf{C}_c^\infty(\mathbb{R})$ may not be included in the domain of Λ_ψ when $\psi \in \mathbf{N}_-(\mathbb{R})$, as $m_\Lambda(\cdot + \mathrm{i}/2)$ can grow exponentially. Although the set $\Lambda_\psi(\mathcal{D}(\mathbb{R}))$ mentioned in item (4) is dense in $\mathbf{L}^2(\mathbb{R}, e)$, it is not a core for $A_2[\psi]$ in general. To deal with this issue, we find a core for the generator of the semigroup $(e_t)_{t\geqslant 0}$, which consists of smooth functions with Fourier transforms decaying exponentially fast, and also invariant under the semigroup $(e_t)_{t\geqslant 0}$. Then, by imposing a mild condition on the ratio $\frac{W_{\phi_+}\left(\frac{1}{2}-\mathrm{i}\xi\right)}{W_{\phi_-}\left(\frac{1}{2}+\mathrm{i}\xi\right)}$, we show that those smooth functions are in the domain of H_ψ defined in item (5), and their image under H_ψ forms a core for $A_2[\psi]$. Additionally, we obtain the integro-differential representation of $A_2[\psi]$. These results are formally stated in the next proposition.

Proposition 2.4.2. *Let $\psi \in \mathbf{N}_b(\mathbb{R})$ and denote by $A_2[\psi]$ the $\mathbf{L}^2(\mathbb{R}, e)$-generator of $P[\psi]$.*

(1) If $\psi(\xi) = -\mathrm{i}\mathtt{d}\xi$ for some $\mathtt{d} > 0$, then $\mathbf{C}_c^\infty(\mathbb{R})$ is a core for $A_2[\psi]$. Otherwise, let us assume that

$$\eta_\psi := \sup\left\{\eta \in \mathbb{R};\ \xi \mapsto |\xi|^{\eta+\frac{1}{2}} \frac{W_{\phi_+}\left(\frac{1}{2}-\mathrm{i}\xi\right)}{W_{\phi_-}\left(\frac{1}{2}+\mathrm{i}\xi\right)} e^{-\frac{\pi}{2}|\xi|} \in \mathbf{L}^2(\mathbb{R})\right\} \in (0, \infty]. \tag{2.62}$$

Writing, for $\epsilon > 0$, $\mathcal{E}(\epsilon) = \mathrm{Span}\{\mathfrak{h}_{\epsilon,\beta}; \beta > 0\}$ *where* $\mathfrak{h}_{\epsilon,\beta}(x) = e^{-(\frac{1}{2}+\epsilon)x}e^{-\beta e^{-x}}, x \in \mathbb{R}$, *the following holds.*

(a) For all $\epsilon > 0$, $\mathcal{E}(\epsilon)$ *is a dense subset of* $\mathbf{L}^2(\mathbb{R}, e)$.

(b) For all $0 < \epsilon < \eta_\psi$, $\mathcal{E}(\epsilon) \subset \mathbf{D}(H_\psi)$ *and* $\mathcal{E}_\psi(\epsilon) := H_\psi(\mathcal{E}(\epsilon))$ *is a core for* $A_2[\psi]$.

(2) Let $\psi \in \mathbf{N}_b(\mathbb{R})$ *be such that, for all* $n \in \mathbb{N}$,

$$\xi \mapsto |\xi|^n \frac{W_{\phi_+}\left(\frac{1}{2} - \mathrm{i}\xi\right)}{W_{\phi_-}\left(\frac{1}{2} + \mathrm{i}\xi\right)} e^{-\frac{\pi}{2}|\xi|} \in \mathbf{L}^1(\mathbb{R}). \tag{2.63}$$

Moreover, if ψ *satisfies one of the following conditions*

(i) $\phi_+(0) > 0$

(ii) $\int_{|y|>1} |y|\mu(dy) < \infty$

then $\mathcal{E}_\psi = \bigcup_{0<\epsilon<\infty} \mathcal{E}_\psi(\epsilon)$ *is core for* $A_2[\psi]$. *Moreover, on* $\mathcal{E}_\psi$, $A_2[\psi]$ *admits the representation as the following integro-differential operator*

$$\begin{aligned} A_2[\psi]f(x) &= e^{-x}\left(\sigma^2 f''(x) + \mathtt{b}f'(x) - \psi(0)f(x)\right. \\ &+ \left.\int_{\mathbb{R}}(f(x+y) - f(x) - y\mathbb{1}_{\{|y|\leqslant 1\}}f'(x))\mu(dy)\right) \end{aligned} \tag{2.64}$$

where $(\psi(0), \mathtt{b}, \sigma^2, \mu)$ *is the quadruplet determining* ψ.

This proposition is proved in Subsection 2.4.2.

Remark 2.4.3. The condition (2.62) is needed only to show that $\mathcal{E}_\psi(\epsilon)$ is in the domain of the $\mathbf{L}^2(\mathbb{R}, e)$-generator of $P[\psi]$. It is always satisfied when $\psi \in \mathbf{N}_+(\mathbb{R})$. The inclusion $\mathcal{E}_\psi(\epsilon) \subset \mathbf{D}(H_\psi)$ and the density of $H_\psi(\mathcal{E}(\epsilon))$ in $\mathbf{L}^2(\mathbb{R}, e)$ is true whenever $\epsilon > 0$ is small enough, depending on ψ, see (2.80) and (2.81).

Remark 2.4.4. We note that the condition (2.63) is always satisfied when $\psi \in \mathbf{N}_+(\mathbb{R})$. More generally, (2.63) holds whenever $\psi(\xi) \neq -\mathrm{i}\mathtt{d}\xi$ and either $\mathtt{d}_- = 0$, or $\mathtt{d}_+ > 0$, or $\int_{\mathbb{R}} \mu(dy) = \infty$, see [96, Theorem 2.3(1)], where $\mathtt{d}_\pm$ is defined in Proposition 2.2.11, and μ is defined in (2.7). However, when $\psi \in \mathbf{N}_+(\mathbb{R})$, we will consider $\Lambda_\psi(\mathbf{C}_c^\infty(\mathbb{R}))$ as core for $A_2[\psi]$, since we have a PDO representation of $A_2[\psi]$ on this set, see (2.67), and it does not require the conditions (2i)-(2ii).

2.4.1 Proof of Theorem 2.4.1

We begin by proving the following lemma.

Lemma 2.4.5. *For any $\psi \in \mathbf{N}(\mathbb{R})$, the operator $A_{\mathtt{PDO}}[\psi]$ is dissipative on the domain $\mathbf{D}_\psi = \{f \in \mathbf{S}(\mathbb{R}) \cap \mathbf{L}^2(\mathbb{R}, e);\ A_{\mathtt{PDO}}[\psi]f \in \mathbf{L}^2(\mathbb{R}, e)\}$.*

Proof. For any $f \in \mathbf{D}_\psi \subset \mathbf{S}(\mathbb{R})$, we have

$$A_{\mathtt{PDO}}[\psi]f(x) = -\frac{e^{-x}}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{\mathrm{i}\xi x} \psi(\xi) \mathcal{F}_f(\xi)\, d\xi = e^{-x} L[\psi]f(x) \tag{2.65}$$

where $L[\psi]$ is the $\mathbf{L}^2(\mathbb{R})$-generator as well as the Feller-generator of the Lévy process with Lévy-Khintchine exponent ψ, see [4, Section 3.4.1 and Theorem 3.3.3]. It is known that an operator $(L, \mathbf{D}(L))$ on a Hilbert space H is dissipative if and only if $\mathtt{Re}\langle Lf, f\rangle \leqslant 0$ for all $f \in \mathbf{D}(L)$. To check this condition for $A_{\mathtt{PDO}}[\psi]$ on $\mathbf{D}_\psi \subset \mathbf{L}^2(\mathbb{R}, e)$, we observe that, for all $f \in \mathbf{D}(L)$,

$$\mathtt{Re}\langle A_{\mathtt{PDO}}[\psi]f, f\rangle_{\mathbf{L}^2(\mathbb{R},e)} = \mathtt{Re}\langle L[\psi]f, f\rangle_{\mathbf{L}^2(\mathbb{R})} \leqslant 0 \tag{2.66}$$

as $L[\psi]$ is the $\mathbf{L}^2(\mathbb{R})$-generator of the Lévy process, hence dissipative. Thus, $(A_{\mathtt{PDO}}[\psi], \mathbf{D}_\psi)$ is dissipative.

Remark 2.4.6. We point out that although the identity (2.65) relates, on the dense subset $\mathbf{D}_\psi$, $A_{\mathtt{PDO}}$ with the $\mathbf{L}^2(\mathbb{R}, e)$- and Feller generator of a Lévy process, we can not yet conclude that $A_{\mathtt{PDO}}$ coincides on this set with the $\mathbf{L}^2(\mathbb{R}, e)$-generator of the log-Lamperti semigroup, as a representation of such generator is not available. Indeed, Lamperti [76] provides only the representation of the generator as a Dynkin operator due to the delicate application of the Volskonskii's formula in this context. However, under the condition (2.63) on the Wiener-Hopf factors of ψ, we shall show that the Dynkin representation coincides with the Feller (and also the $\mathbf{L}^2(\mathbb{R}, e)$) one. To deal with the remaining cases, we will prove and use the fact that our class of PDO's as well as the log-Lamperti class is stable by taking the adjoint in $\mathbf{L}^2(\mathbb{R}, e)$.

Proof of Theorem 2.4.1(1).

To prove this result, we change back to $\mathbb{R}_+$ via the homeomorphism $x \mapsto e^x$, which will transform the operator $(A_{\texttt{PDO}}[\psi_0], \mathbf{C}_c^\infty(\mathbb{R}))$ to

$$\widetilde{A}f(r) = rf''(r) + f'(r),\ f \in \mathbf{C}_c^\infty(\mathbb{R}_+).$$

It is enough to show that the operator above is essentially self-adjoint with respect to $\mathbf{C}_c^\infty(\mathbb{R})$. Note that writing $F(x, y) = f(x^2 + y^2)$, one has $Af(x^2 + y^2) = \frac{1}{2}\Delta F(x, y)$, where Δ is the Laplacian on $\mathbb{R}^2$. Therefore, $(\widetilde{A}, \mathbf{C}_c^\infty(\mathbb{R}_+))$ coincides with the operator $(\Delta, \mathbf{C}_c^\infty(\mathbb{R}^2) \cap \mathcal{S})$, where $\mathcal{S}$ is the set of all radially symmetric functions on $\mathbb{R}^2$. Since the latter is essentially self-adjoint in $\mathbf{L}^2(\mathbb{R}^2) \cap \mathcal{S}$, we conclude that $(\widetilde{A}, \mathbf{C}_c^\infty(\mathbb{R}_+))$ is essentially self-adjoint in $\mathbf{L}^2(\mathbb{R}_+)$. Thus, $(A_{\texttt{PDO}}[\psi_0], \mathbf{C}_c^\infty(\mathbb{R}))$ is closable and the closure $(\overline{A}_{\texttt{PDO}}[\psi_0], \mathbf{D}(\overline{A}_{\texttt{PDO}}[\psi_0]))$ is also self-adjoint. Using Lemma 2.4.5, we get that $A_{\texttt{PDO}}[\psi_0]$ is dissipative, so is its closure. By the theory of self-adjoint dissipative operators, we infer that $\overline{A}_{\texttt{PDO}}[\psi_0]$ generates a $\mathcal{C}_0$-contraction semigroup on $\mathbf{L}^2(\mathbb{R}, e)$. On the other hand, from Lemma 2.3.11 and Lemma 2.3.9, we observe that

$$A_2[\psi_0] = A_{\texttt{PDO}}[\psi_0] \text{ on } \mathbf{C}_c^\infty(\mathbb{R})$$

where $A_2[\psi_0]$ is the $\mathbf{L}^2(\mathbb{R}, e)$-generator of the log-squared Bessel semigroup. As $\mathbf{C}_c^\infty(\mathbb{R})$ is dense in $\mathbf{L}^2(\mathbb{R}, e)$, by uniqueness, we conclude that $(\overline{A}_{\texttt{PDO}}[\psi_0], \mathbf{D}(\overline{A}_{\texttt{PDO}}[\psi_0]))$ generates the log-squared Bessel semigroup $(Q_t)_{t\geqslant 0}$ on $\mathbf{L}^2(\mathbb{R}, e)$.

Proof of Theorem 2.4.1(2).

For any $\phi_+, \phi_- \in \mathbf{B}$, we claim that $\frac{W_{\phi_+}\left(\frac{1}{2}-\mathrm{i}\xi\right)}{W_{\phi_-}\left(\frac{1}{2}+\mathrm{i}\xi\right)}\Gamma(\frac{1}{2} + \mathrm{i}\xi)$ is always bounded with respect to ξ. Indeed, we first note that $\xi \mapsto W_{\phi_+}\left(\frac{1}{2} - \mathrm{i}\xi\right)$ is bounded since it is the Mellin transform of a random variable, see [95, Theorem 6.1(3)], and the ratio $\frac{\Gamma(\frac{1}{2}+\mathrm{i}\xi)}{W_{\phi_-}\left(\frac{1}{2}+\mathrm{i}\xi\right)}$ is also the Mellin transform of the exponential functional associated to the subordinator with Laplace exponent ϕ_-, see [51, Theorem 2.2]. Now, recalling that $|\Gamma(\frac{1}{2} + \mathrm{i}\xi)| \sim e^{-\frac{\pi}{2}|\xi|}$ as $|\xi| \to \infty$, the above argument entails that for all $\xi \in \mathbb{R}$,

$$\left|\frac{W_{\phi_+}\left(\frac{1}{2}-\mathrm{i}\xi\right)}{W_{\phi_-}\left(\frac{1}{2}+\mathrm{i}\xi\right)}\right| \leqslant Ce^{\frac{\pi}{2}|\xi|}$$

which shows that $\Lambda_\psi \in \mathscr{M}_e$. In particular, when $\psi \in \mathbf{N}_+(\mathbb{R})$, from (2.14) and the definition of m_{Λ_ψ}, the mapping $\xi \mapsto m_{\Lambda_\psi}(\xi + \mathrm{i}/2)$ is bounded, since $|\Gamma(\frac{1}{2}+\mathrm{i}\xi)| = |\Gamma(\frac{1}{2}-\mathrm{i}\xi)|$ for all $\xi \in \mathbb{R}$. Therefore, $\Lambda_\psi \in \mathscr{B}(\mathbf{L}^2(\mathbb{R}, e))$ if $\psi \in \mathbf{N}_+(\mathbb{R})$.

Proof of Theorem 2.4.1(3).

We first prove the second statement of this item.

Proposition 2.4.7. *For any $\psi \in \mathbf{N}_+(\mathbb{R})$ we have*

$$A_{\texttt{PDO}}[\psi]\Lambda_\psi = \Lambda_\psi A_{\texttt{PDO}}[\psi_0] \ \textit{ on } \ \mathbf{C}_c^\infty(\mathbb{R}). \tag{2.67}$$

Proof. Since the proof is quite technical, we start by outlining the main ideas of its proof. If m_{Λ_ψ} extends continuously on $\mathbb{S}_{[0,1]}$ and the conditions of Proposition 2.3.5 are met for $f \in \mathbf{C}_c^\infty(\mathbb{R})$ with $\gamma = 1$, then we get

$$\mathcal{F}_{A_{\texttt{PDO}}[\psi]\Lambda_\psi f}(\xi + \mathrm{i}) = \mathcal{F}_{L[\psi]\Lambda_\psi f}(\xi) = m_{\Lambda_\psi}(\xi)\psi(\xi)\mathcal{F}_f(\xi)$$

where $L[\psi]$ is the generator of the Lévy process associated to ψ. On the other hand, if one takes $\beta = 0$ in Proposition 2.3.13, it is expected that m_{Λ_ψ} should satisfy the same functional equation in (2.51) with $\beta = 0$, given that $m_{\Lambda_\psi}(\xi+\mathrm{i})$ is well defined for all $\xi \in \mathbb{R}$. As a result, a straightforward application of Proposition 2.3.5 would imply that

$$\mathcal{F}_{\Lambda_\psi A_{\texttt{PDO}}[\psi_0]f}(\xi + \mathrm{i}) = m_{\Lambda_\psi}(\xi + \mathrm{i})\mathcal{F}_{A_{\texttt{PDO}}[\psi_0]f}(\xi + \mathrm{i}) = m_{\Lambda_\psi}(\xi + \mathrm{i})\psi_0(\xi)\mathcal{F}_f(\xi)$$

which would establish that $\mathcal{F}_{A_{\texttt{PDO}}[\psi]\Lambda_\psi f} = \mathcal{F}_{\Lambda_\psi A_{\texttt{PDO}}[\psi_0]f}$. However, the continuous extension of m_{Λ_ψ} on the line $\mathbb{R}+\mathrm{i}$ is not always guaranteed and this is why we perturb the operator Λ_ψ by a parameter

$\beta > 0$, and resort to Proposition 2.3.13 to prove (2.67). To this end, we recall that $\psi_\beta(\xi) = \xi^2 - \mathrm{i}\beta\xi$ for $\beta > 0$ and $\xi \in \mathbb{R}$. Also, let us define

$$m_{\Lambda^{(\beta)}}(z) = \frac{\Gamma(1+\mathrm{i}z)}{\Gamma(1+\beta+\mathrm{i}z)}, \ z \in \mathbb{S}_{(-\infty,1)}.$$

Clearly, $m_{\Lambda^{(\beta)}}$ is analytic and zero-free in $\mathbb{S}_{[0,1)}$. Let $\Lambda^{(\beta)}$ be the shifted Fourier multiplier associated to $m_{\Lambda^{(\beta)}}$. Since $m_{\Lambda^{(\beta)}}$ is bounded on the line $\mathbb{R} + \mathrm{i}/2$, $\Lambda^{(\beta)} \in \mathscr{M} \cap \mathscr{B}(\mathbf{L}^2(\mathbb{R}, e))$. We note that for any $f \in \mathbf{C}_c^\infty(\mathbb{R})$, $\mathcal{F}_f$ is an entire function. Moreover, $m_{\Lambda^{(\beta)}} \in \mathbf{A}_{(-\infty,1)}$ and by the Stirling asymptotic formula of Gamma functions, $m_{\Lambda^{(\beta)}}$ is uniformly bounded on the intervals $(-\infty, a]$ for any $a < 2$. Also, $\mathcal{F}_f(\cdot + \frac{\mathrm{i}a}{2}) \in \mathbf{L}^2(\mathbb{R})$ for all $a < 2$, and therefore by Proposition 2.3.5, it follows that $\Lambda^{(\beta)} f \in \mathbf{L}^2(\mathbb{R}, e^{ax}dx)$ for any $a < 2$.

Lemma 2.4.8. *For any $\beta > 0$ we have*

$$A_{\texttt{PDO}}[\psi_\beta]\Lambda^{(\beta)} = \Lambda^{(\beta)} A_{\texttt{PDO}}[\psi_0] \ \textit{ on } \ \mathbf{C}_c^\infty(\mathbb{R})$$

and $\Lambda^{(\beta)}(\mathbf{C}_c^\infty(\mathbb{R}))$ is dense in $\mathbf{L}^2(\mathbb{R}, e)$.

Proof. By the definition of the operator $\Lambda^{(\beta)}$, the fact that $A_{\texttt{PDO}}[\psi_0](\mathbf{C}_c^\infty(\mathbb{R})) \subset \mathbf{C}_c^\infty(\mathbb{R})$ and Proposition 2.3.5, we have, for any $f \in \mathbf{C}_c^\infty(\mathbb{R})$ and writing $d\tilde{\xi}_x = -e^{\mathrm{i}\xi x}\frac{d\xi}{\sqrt{2\pi}}$,

$$\begin{aligned}\Lambda^{(\beta)} A_{\texttt{PDO}}[\psi_0] f(x) &= -\int_{\mathbb{R}} e^{\mathrm{i}\xi x} m_{\Lambda^{(\beta)}}(\xi)(\xi - \mathrm{i})^2 \mathcal{F}_f(\xi - \mathrm{i})\, \frac{d\xi}{\sqrt{2\pi}} \\ &= \int_{\mathbb{R}} \frac{\Gamma(1+\mathrm{i}\xi)(\xi-\mathrm{i})^2}{\Gamma(1+\beta+\mathrm{i}\xi)} \mathcal{F}_f(\xi - \mathrm{i})\, d\tilde{\xi}_x = \int_{\mathbb{R}} \frac{\mathrm{i}\Gamma(2+\mathrm{i}\xi)(\xi-\mathrm{i})}{\Gamma(1+\beta+\mathrm{i}\xi)} \mathcal{F}_f(\xi - \mathrm{i})\, d\tilde{\xi}_x \end{aligned} \tag{2.68}$$

$$= e^{-x} \int_{\mathbb{R}} \frac{\mathrm{i}\Gamma(1+\mathrm{i}\xi)}{\Gamma(\beta+\mathrm{i}\xi)} \xi \mathcal{F}_f(\xi)\, d\tilde{\xi}_x = e^{-x} \int_{\mathbb{R}} \frac{\Gamma(1+\mathrm{i}\xi)}{\Gamma(1+\beta+\mathrm{i}\xi)} \psi_\beta(\xi) \mathcal{F}_f(\xi)\, d\tilde{\xi}_x \tag{2.69}$$

$$= A_{\texttt{PDO}}[\psi_\beta]\Lambda^{(\beta)} f(x) \tag{2.70}$$

where (2.69) follows by applying Lemma 2.3.3. Now, since the adjoint operator $\widehat{\Lambda}^{(\beta)}$ is also a shifted Fourier multiplier operator with $m_{\widehat{\Lambda}^{(\beta)}}(\xi + \mathrm{i}/2) = m_{\Lambda^{(\beta)}}(-\xi + \mathrm{i}/2) \neq 0$ for a.e. ξ, it is bounded and injective, which implies that the image of any dense set under $\Lambda^{(\beta)}$ is dense in $\mathbf{L}^2(\mathbb{R},e)$. This proves the lemma. Now, let $g \in \Lambda^{(\beta)}(\mathbf{C}_c^\infty(\mathbb{R}))$, that is $g = \Lambda^{(\beta)} f$, $f \in \mathbf{C}_c^\infty(\mathbb{R})$, and,

$$m_{\Lambda_\psi^{(\beta)}}(z) = W_\psi^{(\beta)}(z) = \frac{W_{\phi_+}(-\mathrm{i}z)\Gamma(1+\mathrm{i}z+\beta)}{\Gamma(-\mathrm{i}z)W_{\phi_-}(1+\mathrm{i}z)} \text{ for } z \in \mathbb{S}_{[0,1)}. \tag{2.71}$$

From Proposition 2.3.13(1), it follows that $m_{\Lambda_\psi^{(\beta)}} \in \mathbf{A}_{[0,1]}$. Let $\Lambda_\psi^{(\beta)}$ be Fourier operator associated to $m_{\Lambda_\psi^{(\beta)}}$. Then, $\Lambda_\psi^{(\beta)} \in \mathscr{M}$. Moreover, from our assumption that $\psi \in \mathbf{N}_+(\mathbb{R})$ and the Stirling asymptotic of the gamma function, it follows that $\left|m_{\Lambda_\psi^{(\beta)}}(\xi+\mathrm{i}/2)\right| = \mathrm{O}(|\xi|^\beta)$. From Proposition 2.3.13(2), one gets that $\xi \mapsto m_{\Lambda_\psi^{(\beta)}}(\xi)$ and $\xi \mapsto m_{\Lambda_\psi^{(\beta)}}(\xi+\mathrm{i})$ have at most polynomial growth as $|\xi| \to \infty$. On the other hand, from (2.68) and (2.70), one can show that

$$A_{\texttt{PDO}}[\psi_\beta]g \in \mathbf{L}^2(\mathbb{R},e) \text{ and } \mathcal{F}^e_{A_{\texttt{PDO}}[\psi_\beta]g}(\xi) = \mathrm{O}(|\xi|^{-n}) \text{ for all } n \in \mathbb{N}.$$

Hence, the mapping $\xi \mapsto m_{\Lambda_\psi^{(\beta)}}(\xi+\mathrm{i}/2)\,\mathcal{F}^e_{A_{\texttt{PDO}}[\psi_\beta]g}(\xi) \in \mathbf{L}^2(\mathbb{R}) \cap \mathbf{L}^1(\mathbb{R})$, which ensures that $A_{\texttt{PDO}}[\psi_\beta]g \in \mathbf{D}(\Lambda_\psi^{(\beta)})$. Therefore, we have,

$$\begin{aligned}
\Lambda_\psi^{(\beta)} A_{\texttt{PDO}}[\psi_\beta]g(x) &= \frac{e^{-\frac{x}{2}}}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{\mathrm{i}\xi x} \mathcal{F}^e_{\Lambda_\psi^{(\beta)} A_{\texttt{PDO}}[\psi_\beta]g}(\xi)\, d\xi && (2.72)\\
&= -\frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}+\frac{\mathrm{i}}{2}} e^{\mathrm{i}zx} m_{\Lambda_\psi^{(\beta)}}(z)\psi_\beta(z-\mathrm{i})\mathcal{F}_g(z-\mathrm{i})\, dz && (2.73)\\
&= -\frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{\mathrm{i}\xi x} m_{\Lambda_\psi^{(\beta)}}(\xi)\psi_\beta(\xi-\mathrm{i})\mathcal{F}_g(\xi-\mathrm{i})\, d\xi && (2.74)\\
&= -\frac{e^{-x}}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{\mathrm{i}\xi x} m_{\Lambda_\psi^{(\beta)}}(\xi+\mathrm{i})\psi_\beta(\xi)\mathcal{F}_g(\xi) d\xi\\
&= -\frac{e^{-x}}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{\mathrm{i}\xi x} m_{\Lambda_\psi^{(\beta)}}(\xi)\psi(\xi)\mathcal{F}_g(\xi)\, d\xi\\
&= A_{\texttt{PDO}}[\psi]\Lambda_\psi^{(\beta)} g(x). && (2.75)
\end{aligned}$$

In the computation above, the second identity follows by a change of variable which is valid since ψ_β and $\mathcal{F}_g$ are entire functions, and $\mathcal{F}_{A_{\texttt{PDO}}[\psi_\beta]g}(\xi) = \psi_\beta(\xi-\mathrm{i})\mathcal{F}_g(\xi-\mathrm{i})$ for all $\xi \in \mathbb{R}$. (2.74) follows from (2.73) and Lemma 2.3.3, since $\psi_\beta \mathcal{F}_g \in \mathbf{A}_{(-\infty,1)}$ and it decays faster than any polynomial as $|\xi| \to \infty$, and the identity before (2.75) follows from Proposition 2.3.13(2.51) and Proposition 2.3.5. Now, recalling the definition of the shifted Fourier multiplier operator Λ_ψ, we note that $m_{\Lambda_\psi^{(\beta)}} m_{\Lambda^{(\beta)}} = m_{\Lambda_\psi}$ on $\mathbb{S}_{(0,1)}$, which implies that $\Lambda_\psi = \Lambda_\psi^{(\beta)}\Lambda^{(\beta)}$. Then, plugging in $g = \Lambda^{(\beta)} f$ in the above computation along with Lemma 2.4.8, (2.67) follows. Now, coming back to the proof of item (3), we first show the density of $\mathcal{D}(\mathbb{R})$ in $\mathbf{L}^2(\mathbb{R},e)$. Indeed, we have the following inclusion

$$\mathrm{Span}\{x \mapsto e^{-(x-a)^2};\ a \in \mathbb{R}\} \subset \mathcal{D}(\mathbb{R})$$

and the former set is dense in $\mathbf{L}^2(\mathbb{R},e)$, thanks to Corollary 2.3.2. Next, to prove (2.58), we can mimic the same technique used in the proof of Proposition 2.4.7 after justifying the following fact.

Lemma 2.4.9. *For any $\psi \in \mathbf{N}_b(\mathbb{R})$ and $f \in \mathcal{D}(\mathbb{R})$ we have, for all $u > 0$,*

$$\sup_{b\in[0,1]} \left| m_{\Lambda_\psi^{(\beta)}}(\xi + \mathrm{i}b)\mathcal{F}_f(\xi \pm \mathrm{i}b)\right| = \mathrm{O}(|\xi|^{-u}). \tag{2.76}$$

Proof. From (2.71) and invoking the fact that, for any $b > 0$,

$$\left|\frac{W_{\phi_+}(b+\mathrm{i}\xi)}{W_{\phi_-}(b+\mathrm{i}\xi)}\Gamma(b+\mathrm{i}\xi)\right| = \mathrm{O}(1)$$

we deduce that for $b \in \{0,1\}$, $|m_{\Lambda_\psi^{(\beta)}}(\xi+\mathrm{i}b)| = \mathrm{O}(|\xi|^{\kappa}e^{\pi|\xi|/2})$ for some $\kappa > 0$. Since for any $f \in \mathcal{D}(\mathbb{R})$, $\mathcal{F}_f$ is entire and $\sup_{b\in\{0,1\}} |\mathcal{F}_f(\xi \pm \mathrm{i}b)| = \mathrm{O}(e^{-(\pi/2+\epsilon)|\xi|})$ for some $\epsilon > 0$, the statement of the lemma holds for $b = 0, 1$. Finally, the estimate in (2.76) follows by a straightforward application of the Phragmén-Lindelöf principle as in the proof of Proposition 2.3.13(2). Because of the above lemma, the statement of Lemma 2.4.8 and the computational steps between (2.72) and (2.75) still go through when $\mathbf{C}_c^\infty(\mathbb{R})$ is replaced by $\mathcal{D}(\mathbb{R})$. This completes the proof of item (3).

Proof of Theorem 2.4.1(4)

We show that the closure in $\mathbf{L}^2(\mathbb{R},e)$ of $(A_{\texttt{PDO}}[\psi], \Lambda_\psi(\mathbf{C}_c^\infty(\mathbb{R})))$ generates the semigroup $P[\psi]$. Using the fact that $A_{\texttt{PDO}}[\psi_0]$ is dissipative and $\mathbf{C}_c^\infty(\mathbb{R})$ is its core, thanks to Lemma 2.4.5 and item 1, we get that for any $\alpha > 0$, $(\alpha I - A_{\texttt{PDO}}[\psi_0])(\mathbf{C}_c^\infty(\mathbb{R}))$ is dense in $\mathbf{L}^2(\mathbb{R},e)$. Moreover, since $\xi \mapsto m_{\Lambda_\psi}(\xi + \frac{\mathrm{i}}{2})$ is bounded, and $\overline{m}_{\Lambda_\psi}(\xi + \frac{\mathrm{i}}{2})$ is non-zero on $\mathbb{R}$, it follows that both Λ_ψ and $\widehat{\Lambda}_\psi$ are bounded and injective on $\mathbf{L}^2(\mathbb{R},e)$, and from this we infer that $\Lambda_\psi(\alpha I - A_{\texttt{PDO}}[\psi_0])(\mathbf{C}_c^\infty(\mathbb{R}))$ is dense in $\mathbf{L}^2(\mathbb{R}, e)$. Invoking Lemma 2.3.12, it follows that $(A_{\mathtt{PDO}}[\psi], \Lambda_\psi(\mathbf{C}_c^\infty(\mathbb{R})))$ generates a $\mathcal{C}_0$-contraction semigroup on $\mathbf{L}^2(\mathbb{R}, e)$. Next, to show that this semigroup indeed coincides with $P[\psi]$, let $A_2[\psi]$ denote the $\mathbf{L}^2(\mathbb{R}, e)$-generator of $P[\psi]$. We aim to show that the two operators $A_2[\psi]$ and $A_{\mathtt{PDO}}[\psi]$ are identical when restricted to the set of functions $\Lambda_\psi(\mathbf{C}_c^\infty(\mathbb{R}))$. For that, we first note that $\Lambda_\psi(\mathbf{C}_c^\infty(\mathbb{R})) \subset \mathbf{C}_0^\infty(\mathbb{R})$ for any $\psi \in \mathbf{N}_+(\mathbb{R})$. This follows by observing that for any $\psi \in \mathbf{N}_+(\mathbb{R})$, $f \in \mathbf{C}_c^\infty(\mathbb{R})$, and $u > 0$,

$$\lim_{|\xi|\to\infty} |\xi|^u \sup_{b\in[0,\frac{1}{2}]} \left| m_{\Lambda_\psi}(\xi + ib)\mathcal{F}_f(\xi + ib)\right| = 0.$$

As m_{Λ_ψ} is analytic in the strip $\mathbb{S}_{[0,1)}$, Proposition 2.3.5 yields that $\mathcal{F}_{\Lambda_\psi f} = m_{\Lambda_\psi}\mathcal{F}_f$ and hence, $\Lambda_\psi f \in \mathbf{C}_0^\infty(\mathbb{R})$ by Riemann-Lebesgue lemma. If $A_{\mathtt{D}}[\psi]$ denotes the Dynkin characteristic operator for the log-self-similar Feller process, recalling Lamperti's result, we get

$$A_{\mathtt{D}}[\psi]\Lambda_\psi = A_{\mathtt{PDO}}[\psi]\Lambda_\psi \text{ on } \mathbf{C}_c^\infty(\mathbb{R}).$$

From (2.72), (2.74), (2.75), we observe that for any $f \in \mathbf{C}_c^\infty(\mathbb{R})$, $A_{\mathtt{PDO}}[\psi]\Lambda_\psi f \in \mathbf{C}_0(\mathbb{R}) \cap \mathbf{L}^2(\mathbb{R}, e)$. Therefore, by Lemma 2.3.9, denoting the generator of the Feller semigroup by $A_{\mathtt{F}}[\psi]$, we have $\Lambda_\psi f \in \mathbf{D}(A_{\mathtt{F}}[\psi])$. Finally, using Lemma 2.3.11, we infer that for all $f \in \mathbf{C}_c^\infty(\mathbb{R})$, $\Lambda_\psi f \in \mathbf{D}(A_2[\psi])$ and

$$A_2[\psi]\Lambda_\psi f = A_{\mathtt{F}}[\psi]\Lambda_\psi f = A_{\mathtt{D}}[\psi]\Lambda_\psi f = A_{\mathtt{PDO}}[\psi]\Lambda_\psi f.$$

As $\Lambda_\psi(\mathbf{C}_c^\infty(\mathbb{R}))$ is a core for $A_{\mathtt{PDO}}[\psi]$, the closure of the operators $(A_{\mathtt{PDO}}[\psi], \Lambda_\psi(\mathbf{C}_c^\infty(\mathbb{R})))$ and $(A_2[\psi], \Lambda_\psi(\mathbf{C}_c^\infty(\mathbb{R})))$ must generate the same semigroup $P[\psi]$. To finish the proof of this item, it remains to show (2.59) and the density of $\Lambda_\psi(\mathcal{D}(\mathbb{R}))$. We note that the set $\mathcal{D}(\mathbb{R})$ is closed under translation, that is, if $f \in \mathcal{D}(\mathbb{R})$ then $\tau_a f \in \mathcal{D}(\mathbb{R})$ for all $a \in \mathbb{R}$. Since Λ_ψ is a shifted Fourier multiplier, $\Lambda_\psi(\mathcal{D}(\mathbb{R}))$ is also closed under translation. Therefore, the density of $\Lambda_\psi(\mathcal{D}(\mathbb{R}))$ follows from Corollary 2.3.2. Now, from (2.58) a similar argument involving Lemma 2.3.9 and Lemma 2.3.11 as before will lead to the identity (2.59).

Proof of Theorem 2.4.1(5).

Following the proof of item (4), we can replace $A_{\texttt{PDO}}$ by A_2 and therefore by (2.67), for all $\alpha > 0$ we have,

$$(\alpha I - A_2[\psi])\Lambda_\psi = \Lambda_\psi(\alpha I - A_2[\psi_0]) \text{ on } \mathbf{C}_c^\infty(\mathbb{R}).$$

Recalling that the resolvent operators $R_\alpha[\psi] = (\alpha I - A_2[\psi])^{-1}$ and $R_\alpha[\psi_0] = (\alpha I - A_2[\psi_0])^{-1}$, we get

$$R_\alpha[\psi]\Lambda_\psi = \Lambda_\psi R_\alpha[\psi_0] \text{ on } (\alpha I - A_2[\psi_0])(\mathbf{C}_c^\infty(\mathbb{R})).$$

As $\mathbf{C}_c^\infty(\mathbb{R})$ is a core of $A_2[\psi_0]$, $(\alpha I - A_2[\psi_0])(\mathbf{C}_c^\infty(\mathbb{R}))$ is dense in $\mathbf{L}^2(\mathbb{R}, e)$. Using the boundedness of $\Lambda_\psi, R_\alpha[\psi], R_\alpha[\psi_0]$, we have

$$R_\alpha[\psi]\Lambda_\psi = \Lambda_\psi R_\alpha[\psi_0] \text{ on } \mathbf{L}^2(\mathbb{R}, e).$$

Recalling that $R_\alpha[\psi]$ (resp. $R_\alpha[\psi_0]$) is the Laplace transform of the semigroup $P[\psi]$ (resp. Q), we have

$$\int_0^\infty e^{-\alpha t} P_t[\psi]\Lambda_\psi f dt = \Lambda_\psi \int_0^\infty e^{-\alpha t} Q_t f dt = \int_0^\infty e^{-\alpha t}\Lambda_\psi Q_t f dt$$

where the second equality follows from the boundedness of Λ_ψ. Since $t \mapsto P_t[\psi]\Lambda_\psi f$ and $t \mapsto \Lambda_\psi Q_t f$ are bounded functions, from uniqueness of Laplace transform, we conclude that, for almost every $t \geqslant 0$,

$$P_t[\psi]\Lambda_\psi = \Lambda_\psi Q_t \text{ on } \mathbf{L}^2(\mathbb{R}, e). \tag{2.77}$$

By strong continuity of the semigroups, the above identity extends for all $t \geqslant 0$. This proves (5). If $\psi \in \mathbf{N}_-(\mathbb{R})$, then $\overline{\psi} \in \mathbf{N}_+(\mathbb{R})$ and taking adjoint in (2.60) and using the fact $\widehat{P}[\overline{\psi}] = P[\psi]$, we get

$$Q_t\widehat{\Lambda}_{\overline{\psi}} = \widehat{\Lambda}_{\overline{\psi}}\widehat{P}_t[\overline{\psi}] \iff Q_t\widehat{\Lambda}_{\overline{\psi}} = \widehat{\Lambda}_{\overline{\psi}} P_t[\psi].$$

From the definition of Λ_ψ and $\Lambda_{\overline{\psi}}$, we note that $m_{\Lambda_\psi}^{-1} = \overline{m}_{\Lambda_{\overline{\psi}}}$, which implies that $\widehat{\Lambda}_{\overline{\psi}} = \Lambda_\psi^{-1}$. Hence, using Proposition 2.3.8, we get $P_t[\psi]\Lambda_\psi = \Lambda_\psi Q_t$ on $\mathbf{D}(\Lambda_\psi)$. To prove (2.61), we first observe that the semigroup $(e_t)_{t\geqslant 0}$ corresponds to $\psi \equiv 1$. Thus, from (2.77), we have

$$e_t H = HQ_t \ \text{ on } \ \mathbf{L}^2(\mathbb{R}, e)$$

where $m_H(z) = \frac{\Gamma(1+\mathrm{i}z)}{\Gamma(-\mathrm{i}z)}$ for all $z \in \mathbb{S}_{(0,1)}$. Noting that $m_H(\xi + \mathrm{i}/2) = \frac{\Gamma(1/2+\mathrm{i}\xi)}{\Gamma(1/2-\mathrm{i}\xi)}$ for all $\xi \in \mathbb{R}$, H turns out to be a unitary operator as $|m_H(\xi + \mathrm{i}/2)| = 1$ for all $\xi \in \mathbb{R}$. Hence, Q is unitary similar to the semigroup $(e_t)_{t\geqslant 0}$. Let us define $H_\psi = \Lambda_\psi H$. By Proposition 2.3.7, $H_\psi \in \mathscr{M}_e$ with $m_{H_\psi}(z) = \frac{W_{\phi_+}(-\mathrm{i}z)}{W_{\phi_-}(1+\mathrm{i}z)}$ on $\mathbb{S}_{(0,1)}$ and $\mathbf{D}(H_\psi) = \mathbf{D}(\Lambda_\psi)$. Therefore, for all $f \in \mathbf{D}(H_\psi)$,

$$P_t[\psi]H_\psi f = P_t[\psi]\Lambda_\psi H f = \Lambda_\psi Q_t H f = \Lambda_\psi H e_t f = H_\psi e_t f. \tag{2.78}$$

Also, $m_{H_{\overline{\psi}}} = \frac{1}{\overline{m}_{H_\psi}}$ on $\mathbb{R} + \frac{\mathrm{i}}{2}$. Thus, $H_\psi^{-1} = \widehat{H}_{\overline{\psi}}$, which, from (2.78) yields $P_t[\psi] = H_\psi e_t \widehat{H}_{\overline{\psi}}$ on $\mathbf{D}(\widehat{H}_{\overline{\psi}})$.

2.4.2 Proof of Proposition 2.4.2

Proof of Proposition 2.4.2(1)

When $\psi(\xi) = -\mathrm{i}\mathtt{d}\xi$ for some $\mathtt{d} > 0$, the corresponding semigroup $P_t[\psi]$ is given by

$$P_t[\psi]f(x) = f(\ln(e^x + (1+\mathtt{d})t)).$$

From the above identity, it follows that for any $f \in \mathbf{C}_c^\infty(\mathbb{R})$, $\lim_{t\downarrow 0}(P_t[\psi]f - f)/t$ exists in $\mathbf{L}^2(\mathbb{R}, e)$ and hence, $\mathbf{C}_c^\infty(\mathbb{R}) \subset \mathbf{D}(A_2[\psi])$. Now, trivially, $P_t[\psi](\mathbf{C}_c^\infty(\mathbb{R})) \subset \mathbf{C}_c^\infty(\mathbb{R})$, which shows that $\mathbf{C}_c^\infty(\mathbb{R})$ is a core for the generator of $P[\psi]$.

For the claim (1a), we first note that for any $\epsilon, \beta > 0$, one has for all $\xi \in \mathbb{R}$,

$$\mathcal{F}_{\mathfrak{h}_{\epsilon,\beta}}(\xi) = \frac{\beta^{-\frac{1}{2}-\epsilon-\mathrm{i}\xi}}{\sqrt{2\pi}}\Gamma\left(\frac{1}{2} + \epsilon + \mathrm{i}\xi\right). \tag{2.79}$$

Since $\mathcal{F}_{\mathfrak{h}_{\epsilon,\beta}}$ is non-zero everywhere and $\mathfrak{h}_{\epsilon,\beta} \in \mathbf{L}^2(\mathbb{R}, e)$, writing $\beta = e^a$ for $a \in \mathbb{R}$, the density of $\mathcal{E}(\epsilon)$ follows from Corollary 2.3.2. Next, for (1b), we show that there exists $\delta_\psi > 0$ such that for all $\beta > 0$,

$$\xi \mapsto m_{H_\psi}(\xi + \mathrm{i}/2)\,\mathcal{F}^e_{\mathfrak{h}_{\epsilon,\beta}}(\xi) \in \mathbf{L}^2(\mathbb{R}) \tag{2.80}$$

whenever $0 < \epsilon < \delta_\psi$. Since $\psi(\xi) \neq -\mathrm{i}\mathtt{d}\xi$ for some $\mathtt{d} > 0$, from [96, Theorem 2.3(1)] we know that there exists $\delta_\psi > 0$ such that, for any $\delta_1 < \delta_\psi < \delta_2$,

$$\lim_{|\xi|\to\infty} |\xi|^{\delta_1} \left| \frac{W_{\phi_+}\left(\frac{1}{2} - \mathrm{i}\xi\right)}{W_{\phi_-}\left(\frac{1}{2} + \mathrm{i}\xi\right)} \Gamma\left(\frac{1}{2} + \mathrm{i}\xi\right) \right| = 0, \ \lim_{|\xi|\to\infty} |\xi|^{\delta_2} \left| \frac{W_{\phi_+}\left(\frac{1}{2} - \mathrm{i}\xi\right)}{W_{\phi_-}\left(\frac{1}{2} + \mathrm{i}\xi\right)} \Gamma\left(\frac{1}{2} + \mathrm{i}\xi\right) \right| = \infty. \tag{2.81}$$

Since for all $\xi \in \mathbb{R}$, $\mathcal{F}^e_{\mathfrak{h}_{\epsilon,\beta}}(\xi) = \mathcal{F}_{\mathfrak{h}_{\epsilon,\beta}}(\xi + \mathrm{i}/2) = \frac{\beta^{-\epsilon-\mathrm{i}\xi}}{\sqrt{2\pi}} \Gamma(\epsilon + \mathrm{i}\xi)$, by definition of H_ψ and using the above estimate along with the Stirling approximation, we have for any $\delta_1 < \delta_\psi < \delta_2$,

$$\lim_{|\xi|\to\infty} |\xi|^{\delta_1 + \frac{1}{2} - \epsilon} \left| m_{H_\psi}(\xi + \mathrm{i}/2)\,\mathcal{F}^e_{\mathfrak{h}_{\epsilon,\beta}}(\xi) \right| = 0, \ \lim_{|\xi|\to\infty} |\xi|^{\delta_2 + \frac{1}{2} - \epsilon} \left| m_{H_\psi}(\xi + \mathrm{i}/2)\,\mathcal{F}^e_{\mathfrak{h}_{\epsilon,\beta}}(\xi) \right| = \infty.$$

Clearly, this implies (2.80) whenever $0 < \epsilon < \delta_\psi$, which proves that $\mathcal{E}_\psi(\epsilon) \subset \mathbf{D}(H_\psi)$. To show the density of $\mathcal{E}_\psi(\epsilon)$ in $\mathbf{L}^2(\mathbb{R}, e)$, let $f \in \mathbf{L}^2(\mathbb{R}, e)$ be such that $f \in H_\psi(\mathcal{E}(\epsilon))^\perp$. Then, by the isometry of shifted Fourier transform, we have

$$\mathcal{F}^e_f \perp \left\{ m_{H_\psi}\left(\cdot + \frac{\mathrm{i}}{2}\right) \mathcal{F}^e_{\mathfrak{h}_{\epsilon,\beta}}; \beta > 0 \right\}. \tag{2.82}$$

Recalling that $\mathcal{F}^e_{\mathfrak{h}_{\epsilon,\beta}}(\xi) = \frac{\beta^{-\epsilon-\mathrm{i}\xi}}{\sqrt{2\pi}} \Gamma(\epsilon + \mathrm{i}\xi)$ and choosing $\beta = e^a$, from (2.82), we get

$$\int_{\mathbb{R}} e^{-a\epsilon - \mathrm{i}a\xi} m_{H_\psi}\left(\xi + \frac{\mathrm{i}}{2}\right) \Gamma(\epsilon + \mathrm{i}\xi) d\xi = 0 \quad \text{for all } a \in \mathbb{R}.$$

This implies that $\mathcal{F}^e_f(\xi) = 0$ a.e., which means that $f = 0$ a.e. This completes the proof of the density of $\mathcal{E}_\psi(\epsilon)$.

Now, it remains to show that $H_\psi(\mathcal{E}(\epsilon))$ is a core for $A_2[\psi]$, and, to this end, we need the following two lemmas.

Lemma 2.4.10. *The operator $(I_e, \mathbf{L}^2(\mathbb{R}, e^{|x|}))$, where $I_e f(x) = -e^{-x} f(x)$, is the $\mathbf{L}^2(\mathbb{R}, e)$-generator of $(e_t)_{t \geqslant 0}$, and for any $\epsilon > 0$, $\mathcal{E}(\epsilon) \subset \mathbf{L}^2(\mathbb{R}, e^{|x|})$. Moreover, for any $\epsilon > 0$, $\mathcal{E}(\epsilon)$ is invariant under the semigroup $(e_t)_{t \geqslant 0}$ and hence it is a core of its generator.*

Proof. If $f \in \mathbf{D}(I_e) \cap \mathbf{L}^2(\mathbb{R}, e)$, then $\lim_{t\to 0}(e_t f - f)/t$ must exist in $\mathbf{L}^2(\mathbb{R}, e)$. After observing that $\lim_{t\to 0}(1 - e^{-te^{-x}})/t = e^{-x}$ for all $x \in \mathbb{R}$, by Fatou's lemma, we have

$$\int_{\mathbb{R}} |f(x)|^2 e^{-x} dx \leqslant \varliminf_{t\to 0} \int_{\mathbb{R}} \left(\frac{e_t f(x) - f(x)}{t}\right)^2 e^x dx = \|I_e f\|^2_{\mathbf{L}^2(\mathbb{R},e)} < \infty.$$

This shows that $\mathbf{D}(I_e) \subseteq \mathbf{L}^2(\mathbb{R}, e^{|x|})$. On the other hand, if $f \in \mathbf{L}^2(\mathbb{R}, e^{|x|})$, we first observe that

$$\lim_{t\to 0} \frac{e_t f(x) - f(x)}{t} = I_e f(x) = -e^{-x} f(x)$$

pointwise. Next, recalling the inequality $(1 - e^{-te^{-x}})/t \leqslant e^{-x}$ for all $t > 0$, the dominated convergence theorem yields

$$\lim_{t\to 0} \int_{\mathbb{R}} \left(\frac{e_t f(x) - f(x)}{t} - I_e f(x)\right)^2 e^x dx = 0$$

which shows that $\mathbf{L}^2(\mathbb{R}, e^{|x|}) \subseteq \mathbf{D}(I_e)$, and therefore $\mathbf{D}(I_e) = \mathbf{L}^2(\mathbb{R}, e^{|x|})$. Next, from the definition of $\mathfrak{h}_{\epsilon,\beta}$, it follows that, for any $\epsilon, \beta > 0$, $\mathfrak{h}_{\epsilon,\beta} \in \mathbf{L}^2(\mathbb{R}, e^{|x|})$. Hence, $\mathcal{E}(\epsilon) \subset \mathbf{D}(I_e)$ for any $\epsilon > 0$. Now, for any $\epsilon, \beta > 0$, $e_t \mathfrak{h}_{\epsilon,\beta}(x) = \mathfrak{h}_{\epsilon,\beta+t}$. By linearity of e_t, $\mathcal{E}(\epsilon)$ is indeed invariant. Since $\mathcal{E}(\epsilon)$ is dense in $\mathbf{L}^2(\mathbb{R}, e)$, by [44, Proposition 1.7], it is a core for the generator of $(e_t)_{t\geqslant 0}$.

Coming back to the proof of the claim (1b), we have already shown in Theorem 2.4.1(5) that $P_t[\psi]H_\psi = H_\psi e_t$ on $\mathbf{D}(H_\psi)$ for any $\psi \in \mathbf{N}_b(\mathbb{R})$. Therefore, $P_t[\psi]H_\psi(\mathcal{E}(\epsilon)) = H_\psi e_t(\mathcal{E}(\epsilon))$. By the invariance of $\mathcal{E}(\epsilon)$ under e_t, we get

$$P_t[\psi](H_\psi(\mathcal{E}(\epsilon))) \subseteq H_\psi(\mathcal{E}(\epsilon)).$$

In other words, $\mathcal{E}_\psi(\epsilon) = H_\psi(\mathcal{E}(\epsilon))$ is invariant under the semigroup $P[\psi]$. In the next lemma, we show that $\mathcal{E}_\psi(\epsilon) \subset \mathbf{D}(A_2[\psi])$. This is where the assumption (2.62) is crucial.

Lemma 2.4.11. *If $\psi(\xi) \neq -\mathtt{d}\xi$ for some $\mathtt{d} > 0$ then for any $0 < \epsilon < \eta_\psi$, $H_\psi(\mathcal{E}(\epsilon)) \subseteq \mathbf{D}(A_2[\psi])$ and $A_2[\psi]H_\psi = H_\psi I_e$ on $\mathcal{E}(\epsilon)$.*

Proof. From (2.60) we know, since $\mathcal{E}(\epsilon) \subset \mathbf{D}(H_\psi)$, that

$$P_t[\psi]H_\psi = H_\psi e_t \text{ on } \mathcal{E}(\epsilon). \tag{2.83}$$

For any $f \in \mathcal{E}(\epsilon) \subset \mathbf{D}(I_e)$, we claim that

$$\lim_{t\downarrow 0} H_\psi \frac{e_t f - f}{t} = H_\psi I_e f.$$

It is enough to prove our claim for $f = \mathfrak{h}_{\epsilon,\beta}$ for $\beta > 0$ and $0 < \epsilon < \eta_\psi$. First, we recall that $e_t \mathfrak{h}_{\epsilon,\beta} = \mathfrak{h}_{\epsilon,\beta+t}$ for all $t \geqslant 0$. As H_ψ is a shifted Fourier multiplier operator (hence closed), it suffices to show that the function

$$\xi \mapsto m_{H_\psi}\left(\xi + \frac{\mathrm{i}}{2}\right)\left[\frac{\mathcal{F}^e_{\mathfrak{h}_{\epsilon,\beta+t}}(\xi) - \mathcal{F}^e_{\mathfrak{h}_{\epsilon,\beta}}(\xi)}{t}\right]$$

converges in $\mathbf{L}^2(\mathbb{R})$ as $t \downarrow 0$. From (2.79), using the Stirling asymptotic of the gamma function, we have, as $|\xi| \to \infty$,

$$\sup_{t\in[0,1]}\left|\frac{d}{dt}\mathcal{F}^e_{\mathfrak{h}_{\epsilon,\beta+t}}(\xi)\right| = \frac{1}{\sqrt{2\pi}}\sup_{t\in[0,1]}\left|(\beta+t)^{-\epsilon-\mathrm{i}\xi-1}(\epsilon+\mathrm{i}\xi)\Gamma(\epsilon+\mathrm{i}\xi)\right| \asymp |\xi|^{\epsilon+\frac{1}{2}}e^{-\frac{\pi}{2}|\xi|}.$$

Now, recalling that $m_{H_\psi}(\xi + \mathrm{i}/2) = \frac{W_{\phi_+}\left(\frac{1}{2}-\mathrm{i}\xi\right)}{W_{\phi_-}\left(\frac{1}{2}+\mathrm{i}\xi\right)}$ and invoking the condition (2.62) with the above bound, it follows that for all $0 < \epsilon < \eta_\psi$,

$$\sup_{t\in[0,1]}\int_{\mathbb{R}}\left|m_{H_\psi}\left(\xi+\frac{\mathrm{i}}{2}\right)\right|^2\left|\frac{\mathcal{F}^e_{\mathfrak{h}_{\epsilon,\beta+t}}(\xi) - \mathcal{F}^e_{\mathfrak{h}_{\epsilon,\beta}}(\xi)}{t}\right|^2 d\xi < \infty.$$

Hence, our claim follows from the dominated convergence theorem. Therefore, from (2.83),

$$\lim_{t\downarrow 0}\frac{P_t[\psi]H_\psi f - H_\psi f}{t} = \lim_{t\downarrow 0} H_\psi \frac{e_t f - f}{t} = H_\psi I_e f$$

which implies that $H_\psi f \in \mathbf{D}(A_2[\psi])$ and $A_2[\psi]H_\psi f = H_\psi I_e f$ for all $f \in \mathcal{E}(\epsilon)$, which completes the proof of the lemma. Since the set $\mathcal{E}_\psi(\epsilon)$ is dense in $\mathbf{L}^2(\mathbb{R}, e)$, and invariant under $P[\psi]$, by [44, Proposition 1.7]. it is a core for $A_2[\psi]$.

Proof of Proposition 2.4.2(2)

We proceed by observing that the condition (2.63) implies that $\delta_\psi = \infty$, where δ_ψ is defined in (2.81). This further implies that $\eta_\psi = \infty$, where η_ψ is defined in (2.62). Let us first assume condition (2i), that is $\phi_+(0) > 0$, which ensures that m_{H_ψ} extends continuously on $\mathbb{S}_{[0,1]}$. Now, with the bound in (2.63) and an argument involving the Phragmén-Lindelöf principle as in the proof of Proposition 2.3.13, one can establish that, for any $n \in \mathbb{N}$,

$$\xi \mapsto \sup_{b\in[0,1]} |\xi + \mathrm{i}b|^n |m_{H_\psi}(\xi + \mathrm{i}b)\mathcal{F}_{\mathfrak{h}_{\epsilon,\beta}}(\xi + \mathrm{i}b)| \in \mathbf{L}^1(\mathbb{R}) \cap \mathbf{L}^2(\mathbb{R}).$$

Hence, applying Proposition 2.3.5, we get $H_\psi \mathfrak{h}_{\epsilon,\beta} \in \mathbf{L}^2(\mathbb{R})$ for all $\epsilon, \beta > 0$ and

$$\mathcal{F}_{H_\psi \mathfrak{h}_{\epsilon,\beta}} = m_{H_\psi} \mathcal{F}_{\mathfrak{h}_{\epsilon,\beta}}.$$

Since the function on the right-hand side of the above equation decays faster than any polynomial, we obtain that $H_\psi \mathfrak{h}_{\epsilon,\beta} \in \mathbf{C}_0^\infty(\mathbb{R})$. Also, from Remark 2.3.14, we have $m_{H_\psi}(\xi + \mathrm{i}) = m_{H_\psi}(\xi)\psi(\xi)$ for all $\xi \in \mathbb{R}$. Therefore, from Lemma 2.4.11, for any $\beta, \epsilon > 0$, we get

$$\begin{aligned}
A_2[\psi] H_\psi \mathfrak{h}_{\epsilon,\beta} &= H_\psi I_e \mathfrak{h}_{\epsilon,\beta} \\
&= -\frac{e^{-\frac{x}{2}}}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{\mathrm{i}\xi x} m_{H_\psi}\left(\xi + \mathrm{i}/2\right) \mathcal{F}_{\mathfrak{h}_{\epsilon,\beta}}\left(\xi - \mathrm{i}/2\right) d\xi \\
&= -\frac{e^{-x}}{\sqrt{2\pi}} \int_{\mathbb{R}-\frac{\mathrm{i}}{2}} m_{H_\psi}(z + \mathrm{i}) \mathcal{F}_{\mathfrak{h}_{\epsilon,\beta}}(z) e^{\mathrm{i}zx}\, dz \\
&= -\frac{e^{-x}}{\sqrt{2\pi}} \int_{\mathbb{R}} m_{H_\psi}(\xi + \mathrm{i}) \mathcal{F}_{\mathfrak{h}_{\epsilon,\beta}}(\xi) e^{\mathrm{i}\xi x}\, d\xi \\
&= -\frac{e^{-x}}{\sqrt{2\pi}} \int_{\mathbb{R}} m_{H_\psi}(\xi)\psi(\xi) \mathcal{F}_{\mathfrak{h}_{\epsilon,\beta}}(\xi) e^{\mathrm{i}\xi x}\, d\xi \\
&= A_{\mathtt{PDO}}[\psi] H_\psi \mathfrak{h}_{\epsilon,\beta} = A_{\mathtt{IDO}}[\psi] H_\psi \mathfrak{h}_{\epsilon,\beta}
\end{aligned} \tag{2.84}$$

where we have repeatedly used Lemma 2.3.3 to change the line of integration. Also note that the last equality holds as $\xi \mapsto \xi^2 \mathcal{F}_{H_\psi \mathfrak{h}_{\epsilon,\beta}}(\xi) \in \mathbf{L}^1(\mathbb{R})$. Finally, by linearity of the operators, the last identity extends to $\mathcal{E}_\psi(\epsilon)$, and hence to $\mathcal{E}_\psi$. Next, to get rid of the condition $\phi_+(0) > 0$, we approximate ψ by its small perturbations ψ_q for which we are able to apply the technique discussed above. More formally, we define $\psi_q(\xi) = \psi(\xi) + q$, $q \geqslant 0$. Let $\psi_q(\xi) = \phi_+^{(q)}(-\mathrm{i}\xi)\phi_-^{(q)}(\mathrm{i}\xi)$ be the Wiener-Hopf factorization of ψ_q. From [96, Lemma 7.3, (7.66)], taking $\mathfrak{r} = 1$ with the notation therein, we have, for all $n \in \mathbb{N}$,

$$\varlimsup_{|\xi|\to\infty} |\xi|^n e^{-\frac{\pi}{2}|\xi|} \sup_{0\leqslant q\leqslant 1} \left| m_{H_{\psi_q}}\left(\xi + \mathrm{i}/2\right) - m_{H_\psi}\left(\xi + \mathrm{i}/2\right) \right| = 0. \tag{2.85}$$

Note that with the notations of the aforementioned paper, $\mathtt{N}_\Psi$ coincides with δ_ψ in our case, which equals ∞ due to (2.63). Also, we have replaced $\Gamma(\frac{1}{2}+\mathrm{i}\xi)$ in [96, (7.66)] simply by $e^{-\frac{\pi}{2}|\xi|}$, since they are asymptotically equivalent by the Stirling approximation formula. Therefore, by the dominated convergence theorem and Lemma 2.4.11, we get

$$\begin{aligned}
A_2[\psi]H_\psi\mathfrak{h}_{\epsilon,\beta}(x) &= H_\psi I_e\mathfrak{h}_{\epsilon,\beta}(x) \\
&= -\frac{e^{-\frac{x}{2}}}{\sqrt{2\pi}}\int_{\mathbb{R}} m_{H_\psi}(\xi+\mathrm{i}/2)\,\mathcal{F}_{\mathfrak{h}_{\epsilon,\beta}}(\xi-\mathrm{i}/2)\,e^{\mathrm{i}\xi x}\,d\xi \\
&= -\frac{e^{-\frac{x}{2}}}{\sqrt{2\pi}}\lim_{q\downarrow 0}\int_{\mathbb{R}} m_{H_{\psi_q}}(\xi+\mathrm{i}/2)\,\mathcal{F}_{\mathfrak{h}_{\epsilon,\beta}}(\xi-\mathrm{i}/2)\,e^{\mathrm{i}\xi x}\,d\xi.
\end{aligned} \tag{2.86}$$

Since $\phi_+^{(q)}(0)>0$, (2.84) yields

$$\begin{aligned}
&\lim_{q\downarrow 0}\frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} m_{H_{\psi_q}}(\xi+\mathrm{i}/2)\,\mathcal{F}_{\mathfrak{h}_{\epsilon,\beta}}(\xi-\mathrm{i}/2)\,e^{\mathrm{i}\xi x}\,d\xi \\
&= -e^{-\frac{x}{2}}\lim_{q\downarrow 0}\frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}}\psi_q(\xi)m_{H_{\psi_q}}(\xi)\mathcal{F}_{\mathfrak{h}_{\epsilon,\beta}}(\xi)e^{\mathrm{i}\xi x}\,d\xi \\
&= -e^{\frac{x}{2}}\lim_{q\downarrow 0}A_{\mathtt{IDO}}[\psi_q]H_{\psi_q}\mathfrak{h}_{\epsilon,\beta}(x).
\end{aligned} \tag{2.87}$$

Finally, we need the following lemma to conclude the proof of this theorem.

Lemma 2.4.12. *If ψ satisfies (2ii), then, for any $n\in\mathbb{N}\cup\{0\}$, one has*

$$\lim_{q\downarrow 0}(H_{\psi_q}\mathfrak{h}_{\epsilon,\beta})^{(n)}(x) = (H_\psi\mathfrak{h}_{\epsilon,\beta})^{(n)}(x)$$

$$\lim_{q\downarrow 0}\int_{\mathbb{R}}\mathtt{F}H_{\psi_q}\mathfrak{h}_{\epsilon,\beta}(x,y)\mu(dy) = \int_{\mathbb{R}}\mathtt{F}H_\psi\mathfrak{h}_{\epsilon,\beta}(x,y)\mu(dy)$$

where $\mathtt{F}f(x,y)=f(x+y)-f(x)-y\mathbb{1}_{\{|y|\leqslant 1\}}f'(x)$.

Proof. By definition of H_{ψ_q}, for any $\beta>0$,

$$\lim_{q\downarrow 0}H_{\psi_q}\mathfrak{h}_{\epsilon,\beta}(x) = \frac{1}{\sqrt{2\pi}}e^{-\frac{x}{2}}\lim_{q\downarrow 0}\int_{\mathbb{R}}e^{\mathrm{i}\xi x}m_{H_{\psi_q}}(\xi+\mathrm{i}/2)\,\mathcal{F}^e_{\mathfrak{h}_{\epsilon,\beta}}(\xi)\,d\xi,$$

which, by (2.85) and the dominated convergence theorem, equals

$$\frac{1}{\sqrt{2\pi}}e^{-\frac{x}{2}}\int_{\mathbb{R}}e^{\mathrm{i}\xi x}m_{H_\psi}(\xi+\mathrm{i}/2)\,\mathcal{F}^e_{\mathfrak{h}_{\epsilon,\beta}}(\xi)\,d\xi = H_\psi\mathfrak{h}_{\epsilon,\beta}(x).$$

For the derivatives, we note that, for any $n \in \mathbb{N}$, $x \in \mathbb{R}$,

$$\begin{aligned}(H_\psi \mathfrak{h}_{\epsilon,\beta})^{(n)}(x) &= \frac{1}{\sqrt{2\pi}} e^{-\frac{x}{2}} \int_{\mathbb{R}} (\mathrm{i}\,(\xi + \mathrm{i}/2))^n \, m_{H_\psi}\,(\xi + \mathrm{i}/2)\, \mathcal{F}^e_{\mathfrak{h}_{\epsilon,\beta}}(\xi)\, d\xi \\ &= \lim_{q\downarrow 0} e^{-\frac{x}{2}} \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{\mathrm{i}\xi x} m_{H_{\psi_q}}\,(\xi + \mathrm{i}/2)\,(\mathrm{i}\,(\xi + \mathrm{i}/2))^n \, \mathcal{F}^e_{\mathfrak{h}_{\epsilon,\beta}}(\xi)\, d\xi \\ &= \lim_{q\downarrow 0} (H_{\psi_q} \mathfrak{h}_{\epsilon,\beta})^{(n)}(x).\end{aligned}$$

On the other hand, using the analyticity of the function $z \mapsto (-\mathrm{i}z)^n m_{H_\psi}(z) = \frac{(-\mathrm{i}z)^n}{\phi_+(-\mathrm{i}z)} \frac{W_{\phi_+}(1-\mathrm{i}z)}{W_{\phi_-}(1+\mathrm{i}z)}$ in $\mathbb{S}_{[0,1]}$ for $n \geqslant 1$ and recalling Proposition 2.3.5, we get

$$(H_\psi \mathfrak{h}_{\epsilon,\beta})^{(n)}(x) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{\mathrm{i}\xi x} (\mathrm{i}\xi)^n m_{H_\psi}(\xi) \mathcal{F}_{\mathfrak{h}_{\epsilon,\beta}}(\xi)\, d\xi.$$

Since the modulus of the integrand above decays faster than any polynomial with respect to ξ, we deduce that $(H_\psi \mathfrak{h}_{\epsilon,\beta})^{(n)} \in \mathbf{C}_0^\infty(\mathbb{R})$ for any $n \geqslant 1$, and using (2.85), it is easy to see that, as $q \to 0$, $(H_{\psi_q} \mathfrak{h}_{\epsilon,\beta})^{(n)} \to (H_\psi \mathfrak{h}_{\epsilon,\beta})^{(n)}$ in the uniform topology. Therefore, denoting

$$\mathtt{I} f(x) = \int_{\mathbb{R}} \mathtt{F} f(x,y) \mu(dy) \text{ and } D_q(x) = H_{\psi_q} \mathfrak{h}_{\epsilon,\beta}(x) - H_\psi \mathfrak{h}_{\epsilon,\beta}(x)$$

we have, as $q \to 0$,

$$\begin{aligned}&|\mathtt{I} H_{\psi_q} \mathfrak{h}_{\epsilon,\beta}(x) - \mathtt{I} H_\psi \mathfrak{h}_{\epsilon,\beta}(x)| = |\mathtt{I} D_q(x)| \\ &\leqslant \int_{|y|\leqslant 1} \left| D_q(x+y) - D_q(x) - y D_q'(x) \right| \mu(dy) + \int_{|y|>1} |D_q(x+y) - D_q(x)|\, \mu(dy) \\ &\leqslant \|D_q''\|_\infty \int_{|y|\leqslant 1} y^2 \mu(dy) + \|D_q'\|_\infty \int_{|y|>1} |y| \mu(dy) \to 0\end{aligned}$$

by our assumption on ψ. This proves the lemma. Therefore, using (2.86) and (2.87) and the previous lemma, we conclude that

$$A_2[\psi] H_\psi \mathfrak{h}_{\epsilon,\beta} = A_{\mathtt{IDO}}[\psi] H_\psi \mathfrak{h}_{\epsilon,\beta}.$$

By linearity of the operators, the above identity extends to $\mathcal{E}_\psi$ which completes the proof of (2).

2.5 Proof of Theorem 2.2.1: the multidimensional case

The proof of Theorem 2.2.1 follows from Theorem 2.4.1 by a combination of tensorization and similarity transform techniques that we first describe in the general context of semigroups.

2.5.1 Tensorization and similarity transform of semigroups and their generators

For $d \in \mathbb{N}$, consider the $\mathcal{C}_0$-contraction semigroups $P^{(1)}, P^{(2)}, \ldots, P^{(d)}$ defined on the Hilbert space $\mathbf{L}^2(\mathbb{R}, e)$. For each $t \geqslant 0$, define $\boldsymbol{P}_t$ to be the tensor product of $P_t^{(1)}, \ldots, P_t^{(d)}$. It is plain that $(\boldsymbol{P}_t)_{t \geqslant 0}$ is a $\mathcal{C}_0$-contraction semigroup on $\mathbf{L}^2(\mathbb{R}^d, \mathbf{e})$ where $\mathbf{e}(\boldsymbol{x}) = \otimes_{k=1}^d e(\boldsymbol{x}) = e^{\langle \boldsymbol{x}, \mathbf{1} \rangle}$. If $(A^{(k)}, \mathbf{D}(A^{(k)}))$ denotes the generator of $P^{(k)}$, $k = 1, \ldots, d$, the generator of $(\boldsymbol{P}_t)_{t \geqslant 0}$ is given by

$$\boldsymbol{A} = \sum_{k=1}^{d} I \otimes \ldots \otimes I \otimes A^{(k)} \otimes I \otimes \ldots \otimes I.$$

Since we could not find a proper reference regarding the core for the tensor product of generators, we provide the result in the next lemma.

Lemma 2.5.1. $\otimes_{k=1}^d \mathbf{D}(A^{(k)})$ *forms a core for* $\boldsymbol{A}$. *In particular, for* $k = 1, \ldots, d$, *if* $\mathcal{D}_k \subseteq \mathbf{D}(A^{(k)})$ *are such that* $\mathcal{D}_k$ *is a core for* $P^{(k)}$, *then* $\boldsymbol{\mathcal{D}} = \otimes_{k=1}^d \mathcal{D}_k$ *forms a core for* $\boldsymbol{A}$.

Proof. First note that $\otimes_{k=1}^d \mathbf{D}(A^{(k)}) \subset \mathbf{D}(\boldsymbol{A})$, and it is invariant under the semigroup $\boldsymbol{P}$. Therefore, the first part of the statement follows by [44, Proposition 1.7]. Next, to show that $\boldsymbol{\mathcal{D}} = \otimes_{k=1}^d \mathcal{D}_k$ is a core for $\boldsymbol{A}$, it suffices to show that

(i) $\boldsymbol{\mathcal{D}}$ is dense in $\mathbf{L}^2(\mathbb{R}^d, \mathbf{e})$,

(ii) $(\alpha \boldsymbol{I} - \boldsymbol{A})(\boldsymbol{\mathcal{D}})$ is dense in $\mathbf{L}^2(\mathbb{R}^d, \mathbf{e})$ for any $\alpha > 0$.

Item (i) is true since $\mathcal{D}_k$ is dense in $\mathbf{L}^2(\mathbb{R}, e)$ for each $k = 1, \dots, d$. For item (ii), note that for any $\alpha > 0$, one has

$$(\alpha \boldsymbol{I} - \boldsymbol{A})(\boldsymbol{\mathcal{D}}) = \sum_{k=1}^{d} \mathcal{D}_1 \otimes \cdots \otimes (\alpha I - A^{(k)})(\mathcal{D}_k) \otimes \cdots \otimes \mathcal{D}_d.$$

Since $\mathcal{D}_k$ is a core for $A^{(k)}$, $(\alpha I - A^{(k)})(\mathcal{D}_k)$ is dense in $\mathbf{L}^2(\mathbb{R}, e)$. Therefore, the set on the right hand side of the above equation is also dense in $\mathbf{L}^2(\mathbb{R}^d, \mathbf{e})$. This proves the lemma.

Once a semigroup on $\mathbf{L}^2(\mathbb{R}^d, \mathbf{e})$ is defined, one can construct a class of semigroups via similarity transform of the coordinates of $\mathbb{R}^d$. The resulting semigroup becomes similar to the original one. More specifically, let $(\boldsymbol{P}_t)_{t\geqslant 0}$ be a semigroup on $\mathbf{L}^2(\mathbb{R}^d, \mathbf{e})$, and for any Borel bijection $\mathfrak{g} : \mathbb{R}^d \to \mathbb{R}^d$ one can define a new semigroup by similarity transform as follows, for any $t \geqslant 0$,

$$\boldsymbol{P}_t^{\mathfrak{g}} = \mathrm{d}_{\mathfrak{g}} \boldsymbol{P}_t \mathrm{d}_{\mathfrak{g}}^{-1}$$

where $\mathrm{d}_{\mathfrak{g}} : \mathbf{L}^2(\mathbb{R}^d, \mathbf{e}) \to \mathbf{L}^2(\mathbb{R}^d, \mathbf{e}_{\mathfrak{g}})$ with $\mathrm{d}_{\mathfrak{g}} \mathbf{f} = \mathbf{f} \circ \mathfrak{g}^{-1}$ is an isometry. The semigroup $(\boldsymbol{P}_t^{\mathfrak{g}})_{t\geqslant 0}$ is defined on a new $\mathbf{L}^2$-space, namely, $\mathbf{L}^2(\mathbb{R}^d, \mathbf{e}_{\mathfrak{g}})$, where $\mathbf{e}_{\mathfrak{g}}$ is the push forward measure induced by $\mathfrak{g}$. If $(\boldsymbol{P}_t)_{t\geqslant 0}$ is a $\mathcal{C}_0$-contraction semigroup, so is $(\boldsymbol{P}_t^{\mathfrak{g}})_{t\geqslant 0}$. Furthermore, denoting by $\boldsymbol{A}$, $\boldsymbol{A}_{\mathfrak{g}}$ the generators of $(\boldsymbol{P}_t)_{t\geqslant 0}$, $(\boldsymbol{P}_t^{\mathfrak{g}})_{t\geqslant 0}$ respectively, we have

$$\boldsymbol{A}_{\mathfrak{g}} = \mathrm{d}_{\mathfrak{g}} \boldsymbol{A} \mathrm{d}_{\mathfrak{g}}^{-1} \text{ on } \mathbf{D}(\boldsymbol{A}_{\mathfrak{g}}) = \left\{ \mathbf{f} \in \mathbf{L}^2(\mathbb{R}^d, \mathbf{e}_{\mathfrak{g}});\ \mathbf{f} \circ \mathfrak{g} \in \mathbf{D}(\boldsymbol{A}) \right\}.$$

2.5.2 Tensorization of Fourier multiplier operators

Since Fourier multiplier operators are closed, it is easy to define their tensor product on multi-dimensional $\mathbf{L}^2$ spaces. For instance, let $\{(\Lambda_k, \mathbf{D}(\Lambda_k))\}_{k=1}^d \subset \mathscr{M}$ and $\boldsymbol{\Lambda}$ be the shifted Fourier multiplier on $\otimes_{k=1}^d \mathbf{D}(\Lambda_k)$ defined by

$$\mathcal{F}^{\mathbf{e}}_{\boldsymbol{\Lambda}\mathbf{f}}(\boldsymbol{\xi}) = \prod_{k=1}^{d} m_{\Lambda_k}\left(\xi_k + \mathrm{i}/2\right) \mathcal{F}^{e}_{f_k}(\xi_k)$$

where $\mathbf{f} = \otimes_{k=1}^{d} f_k$, $f_k \in \mathbf{D}(\Lambda_k)$ for $k = 1, \ldots, d$ and $\boldsymbol{\xi} = (\xi_1, \ldots, \xi_d) \in \mathbb{R}^d$. Note that in this case $\mathcal{F}^{\mathbf{e}}$ stands for the multi-dimensional shifted Fourier transform, i.e.

$$\mathcal{F}^{\mathbf{e}}_{\mathbf{f}}(\boldsymbol{\xi}) = (2\pi)^{-\frac{d}{2}} \int_{\mathbb{R}^d} e^{-\mathrm{i}\langle \boldsymbol{x}, \boldsymbol{\xi} + \frac{\mathrm{i}}{2}\mathbf{1}\rangle} \mathbf{f}(\boldsymbol{x})\, d\boldsymbol{x}$$

where we recall that $\mathbf{1}$ is the vector in $\mathbb{R}^d$ consisting of all 1's. We call $\boldsymbol{\Lambda}$ the tensor product of the Λ_k's and write

$$\boldsymbol{\Lambda} = \otimes_{k=1}^{d} \Lambda_k.$$

The operator $\boldsymbol{\Lambda}$ is closable when restricted on $\otimes_{k=1}^{d} \mathbf{D}(\Lambda_k)$ and its closure is the shifted Fourier multiplier operator on $\mathbf{L}^2(\mathbb{R}^d, \mathbf{e})$ with the multiplier function

$$\mathbf{m}_{\boldsymbol{\Lambda}}\left(\boldsymbol{\xi} + \frac{\mathrm{i}}{2}\mathbf{1}\right) = \prod_{k=1}^{d} m_{\Lambda_k}\left(\xi_k + \frac{\mathrm{i}}{2}\right).$$

When taking the tensor product of Fourier multipliers, we will always mean the shifted Fourier multiplier operator mentioned above. Just like the one-dimensional Fourier multiplier operator, Propositions 2.3.6, 2.3.7 and 2.3.8 hold for tensor products of shifted Fourier multiplier operator and tensor product of bounded operators as well. Next, we prove the following lemma regarding the core for the tensor product of Fourier multiplier operators.

Lemma 2.5.2. *Let $(\Lambda_k)_{k=1}^{d}$ be a sequence of shifted Fourier multiplier operators on $\mathbf{L}^2(\mathbb{R}, e)$ such that $\mathcal{D}_k$ is a core for Λ_k for each $k = 1, \ldots, d$. Then, $\boldsymbol{\mathcal{D}} = \otimes_{k=1}^{d} \mathcal{D}_k$ is a core for $\boldsymbol{\Lambda} = \otimes_{k=1}^{d} \Lambda_k$.*

Proof. We recall, from the proof of Proposition 2.3.6, that $\mathcal{D}_k$ is a core for Λ_k if and only if $\mathcal{F}^{e}_{\mathcal{D}_k}$ is dense in $\mathbf{L}^2(\mathbb{R}, 1 + |m^{e}_{\Lambda_k}|^2)$. Let us write $\mathbf{m} = \prod_{k=1}^{d}(1 + |m^{e}_{\Lambda_k}|^2)$. Then, it follows that

$$\bigotimes_{k=1}^{d} \mathbf{L}^2(\mathbb{R}, 1 + |m^{e}_{\Lambda_k}|^2) = \mathbf{L}^2(\mathbb{R}^d, \mathbf{m})$$

and therefore, $\boldsymbol{\mathcal{D}}$ is dense in $\mathbf{L}^2(\mathbb{R}^d, \mathbf{m})$. Since $\mathbf{m} \geqslant 1 + \prod_{k=1}^{d} |m^{e}_{\Lambda_k}|^2 = 1 + |\mathbf{m}^{\mathbf{e}}_{\boldsymbol{\Lambda}}|^2$, this implies that $\boldsymbol{\mathcal{D}}$ is also dense in $\mathbf{L}^2(\mathbb{R}^d, 1 + |\mathbf{m}^{\mathbf{e}}_{\boldsymbol{\Lambda}}|^2)$. Therefore, $\boldsymbol{\mathcal{D}}$ is a core for $\boldsymbol{\Lambda}$.

Proposition 2.5.3. *Let, for each $k = 1, \dots, d$, $A_k, B_k \in \mathscr{D}(\mathbf{L}^2(\mathbb{R}, e))$ and $(\Lambda_k, \mathbf{D}(\Lambda_k)) \in \mathscr{M}$, and, assume that there exists a dense subset $\mathcal{D}_k$ of $\mathbf{L}^2(\mathbb{R}, e)$ such that $A_k\Lambda_k = \Lambda_k B_k$ on $\mathcal{D}_k$. Then,*

$$\boldsymbol{A}\boldsymbol{\Lambda} = \boldsymbol{\Lambda}\boldsymbol{B} \quad \textit{on} \quad \boldsymbol{\mathcal{D}} \tag{2.88}$$

where $\boldsymbol{\mathcal{D}} = \otimes_{k=1}^d \mathcal{D}_k$, $\boldsymbol{A} = \otimes_{k=1}^d A_k$, $\boldsymbol{B} = \otimes_{k=1}^d B_k$, and $\boldsymbol{\Lambda} = \otimes_{k=1}^d \Lambda_k$. In particular, when the A_k, B_k's are bounded operators and $\mathcal{D}_k$ is a core for Λ_k for each k, the above identity extends to $\mathbf{D}(\boldsymbol{\Lambda})$.

Proof. Since any $\mathbf{f} \in \boldsymbol{\mathcal{D}} = \otimes_{k=1}^d \mathcal{D}_k$ is a linear combination of functions of the form $\mathbf{f}(\boldsymbol{x}) = \prod_{k=1}^d f_k(x_k)$, where $\boldsymbol{x} = (x_1, \dots, x_d)$ and $f_k \in \mathcal{D}_k$, by the definition of the tensor product of operators, we have

$$\boldsymbol{A}\boldsymbol{\Lambda}\mathbf{f} = \boldsymbol{A}(\otimes_{k=1}^d \Lambda_k f_k) = \otimes_{k=1}^d A_k\Lambda_k f_k = \otimes_{k=1}^d \Lambda_k B_k f_k = \boldsymbol{\Lambda}\boldsymbol{B}\mathbf{f}, \tag{2.89}$$

which proves (2.88). When A_k, B_k are bounded operators, so are $\boldsymbol{A}$ and $\boldsymbol{B}$. Moreover, if $\mathcal{D}_k$ is a core for Λ_k for $k = 1, \dots, d$, then by Lemma 2.5.2, $\boldsymbol{\mathcal{D}} = \otimes_{k=1}^d \mathcal{D}_k$ is a core for $\boldsymbol{\Lambda}$. Therefore, boundedness of $\boldsymbol{A}, \boldsymbol{B}$ and closedness of $\boldsymbol{\Lambda}$ ensure that (2.89) extends for all $\mathbf{f} \in \mathbf{D}(\boldsymbol{\Lambda})$.

Proposition 2.5.4. *Let $\boldsymbol{\psi} = (\psi_1, \psi_2, \dots, \psi_d) \in \mathbf{N}_b^d(\mathbb{R})$ such that $\psi_k(\xi) = \phi_{+,k}(-\mathrm{i}\xi)\phi_{-,k}(\mathrm{i}\xi)$ and $\psi_k(\xi) \neq -\mathrm{i}d\xi$ for any $k = 1, 2, \dots, d$. Assume that $M \in \mathrm{GL}_d(\mathbb{R})$ and let $\boldsymbol{H}_{\boldsymbol{\psi}}^M$ be defined as in Theorem 2.2.4. Then, $\boldsymbol{H}_{\boldsymbol{\psi}}^M : \mathbf{D}(\boldsymbol{H}_{\boldsymbol{\psi}}^M) \to \mathbf{L}^2(\mathbb{R}^d, \mathbf{e})$ is a linear operator which is densely defined, closed, injective and has dense range with*

$$\mathbf{D}(\boldsymbol{H}_{\boldsymbol{\psi}}^M) = \left\{ \mathbf{f} \in \mathbf{L}^2(\mathbb{R}^d, \mathbf{e}_M);\ \boldsymbol{\xi} = (\xi_1, \dots, \xi_d) \mapsto \mathcal{F}^{\mathbf{e}}_{\mathbf{f}\circ M^{-1}}(\boldsymbol{\xi}) \prod_{k=1}^d \frac{W_{\phi_{+,k}}(\frac{1}{2} - \mathrm{i}\xi_k)}{W_{\phi_{-,k}}(\frac{1}{2} + \mathrm{i}\xi_k)} \in \mathbf{L}^2(\mathbb{R}^d) \right\}. \tag{2.90}$$

Moreover, for sufficiently small $\epsilon > 0$, $\boldsymbol{\mathcal{E}}(\epsilon) \subset \mathbf{D}(\boldsymbol{H}_{\boldsymbol{\psi}}^M)$ and $\boldsymbol{\mathcal{E}}_{\boldsymbol{\psi}}^M(\epsilon) := \boldsymbol{H}_{\boldsymbol{\psi}}^M(\boldsymbol{\mathcal{E}}(\epsilon))$ is dense in $\mathbf{L}^2(\mathbb{R}^d, \mathbf{e}_M)$ where

$$\boldsymbol{\mathcal{E}}(\epsilon) = \bigotimes_{k=1}^d \mathcal{E}(\epsilon) \textit{ with } \mathcal{E}(\epsilon) = \mathrm{Span}\{\mathfrak{h}_{\epsilon,\beta};\ \beta > 0\}$$

and $\mathfrak{h}_{\epsilon,\beta}(x) = e^{-(\frac{1}{2}+\epsilon)x}e^{-\beta e^{-x}}$, $x \in \mathbb{R}$.

Proof. It is enough to prove the above proposition for $d = 1$. The general case will follow by tensorization and similarity transform technique. When $d = 1$, (2.90) follows from Section 2.3.4 and the fact that $\mathcal{E}_\psi(\epsilon)$ is a core for $A_2[\psi]$ for sufficiently small $\epsilon > 0$, follows from Proposition 2.4.2(1b). We are now ready to prove Theorem 2.2.1.

2.5.3 Proof of Theorem 2.2.1

Since the weak similarity operators are Fourier multiplier operators, and weak similarity is stable by the similarity transform, it suffices to prove the theorem when $M = \mathrm{Id}$. From Theorem 2.4.1(3), it follows that for each $k = 1, \ldots, d$, $A_{\text{PDO}}[\psi_k]\Lambda_{\psi_k} = \Lambda_{\psi_k} A_{\text{PDO}}[\psi_0]$ on $\mathcal{D}(\mathbb{R})$. Since $\boldsymbol{A}_{\text{PDO}}[\boldsymbol{\psi}] = \otimes_{k=1}^d A_{\text{PDO}}[\psi_k]$ and $\boldsymbol{\Lambda}_{\boldsymbol{\psi}} = \otimes_{k=1}^d \Lambda_{\psi_k}$, Proposition 2.5.3 yields that

$$\boldsymbol{A}_{\text{PDO}}[\boldsymbol{\psi}]\boldsymbol{\Lambda}_{\boldsymbol{\psi}} = \boldsymbol{\Lambda}_{\boldsymbol{\psi}} \boldsymbol{A}_{\text{PDO}}[\boldsymbol{\psi}_0] \text{ on } \boldsymbol{\mathcal{D}}(\mathbb{R}^d)$$

where $\boldsymbol{\mathcal{D}}(\mathbb{R}^d) = \otimes_{k=1}^d \mathcal{D}(\mathbb{R})$.

When $\boldsymbol{\psi} \in \mathbf{N}_+^d(\mathbb{R})$, it means that $\psi_k \in \mathbf{N}_+(\mathbb{R})$ for each $k = 1, \ldots, d$. Therefore, $\Lambda_{\psi_k} \in \mathscr{B}(\mathbf{L}^2(\mathbb{R}, e))$ for each $k = 1, \ldots, d$. Hence, $\boldsymbol{\Lambda}_{\boldsymbol{\psi}} \in \mathscr{B}(\mathbf{L}^2(\mathbb{R}^d, \mathbf{e}))$. This completes the proof of the theorem.

2.5.4 Proof of Theorem 2.2.2

Proof of Theorem 2.2.2(1)

We first observe that $\boldsymbol{A}_{\text{PDO}}[\boldsymbol{\psi}] = \otimes_{k=1}^d A_{\text{PDO}}[\psi_k]$ for any $\boldsymbol{\psi} = (\psi_1, \ldots, \psi_k) \in \mathbf{N}_b^d(\mathbb{R})$. Next, from Theorem 2.4.1(1), we know that the closure (in $\mathbf{L}^2(\mathbb{R}, e)$) of $(A_{\text{PDO}}[\psi_0], \mathbf{C}_c^\infty(\mathbb{R}))$ generates the log-squared-Bessel semigroup Q on $\mathbf{L}^2(\mathbb{R}, e)$, and its restriction on $\mathbf{C}_0(\mathbb{R}) \cap \mathbf{L}^2(\mathbb{R}, e)$ extends to a Feller semigroup on $\mathbf{C}_0(\mathbb{R})$, whose generator is the closure (in $\mathbf{C}_0(\mathbb{R})$) of $(A_0, \mathbf{C}_c^\infty(\mathbb{R}))$. Note that the tensor product $\boldsymbol{Q} = \otimes_{k=1}^{d} Q$ is generated by the closure of $\boldsymbol{A}_{\texttt{PDO}}[\boldsymbol{\psi}_0]$ restricted on $\otimes_{k=1}^{d}\mathbf{C}_c^{\infty}(\mathbb{R})$, thanks to Lemma 2.5.1. Since $\otimes_{k=1}^{d}\mathbf{C}_c^{\infty}(\mathbb{R}) \subset \mathbf{C}_c^{\infty}(\mathbb{R}^d)$ and both of these two sets are dense in $\mathbf{L}^2(\mathbb{R}^d, \mathbf{e})$, the closure of $(\boldsymbol{A}_{\texttt{PDO}}[\boldsymbol{\psi}_0], \mathbf{C}_c^{\infty}(\mathbb{R}^d))$ also generates $\boldsymbol{Q}$. Moreover, due to tensorization, $\boldsymbol{Q} \in \mathcal{C}_0^{+}(\mathbf{L}^2(\mathbb{R}^d, \mathbf{e}))$. This completes the proof of item (1).

Proof of Theorem 2.2.2(2)

For any $\boldsymbol{\psi} = (\psi_1, \psi_2, \ldots, \psi_d) \in \mathbf{N}_b^d(\mathbb{R})$, we define

$$\boldsymbol{P}[\boldsymbol{\psi}] = \otimes_{k=1}^{d} P[\psi_i].$$

Clearly, $\boldsymbol{P}[\boldsymbol{\psi}] \in \mathcal{C}_0(\mathbf{L}^2(\mathbb{R}^d, \mathbf{e}))$. Also, from Theorem 2.4.1(2.59), we infer that

$$\boldsymbol{A}_2[\boldsymbol{\psi}] = \boldsymbol{A}_{\texttt{PDO}}[\boldsymbol{\psi}] \text{ on } \boldsymbol{\Lambda}_{\boldsymbol{\psi}}(\boldsymbol{\mathcal{D}}(\mathbb{R}^d)). \tag{2.91}$$

Since $\boldsymbol{\Lambda}_{\boldsymbol{\psi}}(\boldsymbol{\mathcal{D}}(\mathbb{R}^d)) = \otimes_{k=1}^{d}\Lambda_{\psi_k}(\mathcal{D}(\mathbb{R}))$ is the tensor product of dense subsets of $\mathbf{L}^2(\mathbb{R}, e)$, it is therefore dense in $\mathbf{L}^2(\mathbb{R}^d, \mathbf{e})$. Next, for $M \in \mathrm{GL}_d(\mathbb{R})$ and $\mathbf{f} \in \mathbf{L}^2(\mathbb{R}^d, \mathbf{e}_M)$, let us define

$$\boldsymbol{P}^M[\boldsymbol{\psi}]\mathbf{f} = \boldsymbol{P}[\boldsymbol{\psi}](\mathbf{f} \circ M) \circ M^{-1}.$$

From the discussion in Subsection 2.5.1, it follows that $\boldsymbol{P}^M[\boldsymbol{\psi}] \in \mathcal{C}_0(\mathbf{L}^2(\mathbb{R}^d, \mathbf{e}_M))$ and $\boldsymbol{A}_2^M[\boldsymbol{\psi}] = \mathrm{d}_M \boldsymbol{A}_2[\boldsymbol{\psi}]\mathrm{d}_{M^{-1}}$ is its $\mathbf{L}^2(\mathbb{R}^d, \mathbf{e}_M)$-generator. Hence, (2.91) yields that

$$\boldsymbol{A}_2^M[\boldsymbol{\psi}] = \boldsymbol{A}_{\texttt{PDO}}^M[\boldsymbol{\psi}] \text{ on } \mathrm{d}_M\boldsymbol{\Lambda}_{\boldsymbol{\psi}}(\boldsymbol{\mathcal{D}}(\mathbb{R}^d))$$

and $\mathrm{d}_M\boldsymbol{\Lambda}_{\boldsymbol{\psi}}(\boldsymbol{\mathcal{D}}(\mathbb{R}^d))$ is also dense in $\mathbf{L}^2(\mathbb{R}^d, \mathbf{e}_M)$ as d_M is an isometry between $\mathbf{L}^2(\mathbb{R}^d, \mathbf{e}_M)$ and $\mathbf{L}^2(\mathbb{R}^d, \mathbf{e})$. The second part of this item follows from the proof of Theorem 2.4.1(5).

Proof of Theorem 2.2.2(3)

From Theorem 2.4.1(2.60), we know that for any $\psi \in \mathbf{N}_b(\mathbb{R})$, $P[\psi]\Lambda_\psi = \Lambda_\psi Q$ on $\mathbf{D}(\Lambda_\psi)$, where Q is the log-squared-Bessel semigroup. Let $M = \mathrm{Id}$. Then, $\boldsymbol{P}^M[\boldsymbol{\psi}] = \otimes_{k=1}^{d} P[\psi_k]$ and therefore, using Proposition 2.5.3, we conclude that

$$\boldsymbol{P}[\boldsymbol{\psi}]\boldsymbol{\Lambda}_{\boldsymbol{\psi}} = \boldsymbol{\Lambda}_{\boldsymbol{\psi}}\boldsymbol{Q} \text{ on } \mathbf{D}(\boldsymbol{\Lambda}_{\boldsymbol{\psi}})$$

where $\boldsymbol{\Lambda}_{\boldsymbol{\psi}} = \otimes_{k=1}^d \Lambda_{\psi_k}$ and $\boldsymbol{Q} = \otimes_{k=1}^d Q$. This proves (2.24) when $M = \mathrm{Id}$. The general case will follow from the similarity transform argument.

Proof of Theorem 2.2.2(4)

Let us first assume that $M = \mathrm{Id}$. Then, by tensorization, the adjoint of $\boldsymbol{P}[\boldsymbol{\psi}] = \otimes_{k=1}^d P[\psi_k]$ in $\mathbf{L}^2(\mathbb{R}^d, \mathbf{e})$ equals

$$\widehat{\boldsymbol{P}}[\boldsymbol{\psi}] = \otimes_{k=1}^d \widehat{P}[\psi_k] = \otimes_{k=1}^d P[\overline{\psi}_k] = \boldsymbol{P}[\overline{\boldsymbol{\psi}}].$$

Indeed, $\widehat{\boldsymbol{P}}[\boldsymbol{\psi}] = \boldsymbol{P}[\boldsymbol{\psi}]$ if and only if $\overline{\psi}_k = \psi_k$ for each $k = 1, \ldots, d$. Next, we show that for any $M \in \mathrm{GL}_d(\mathbb{R})$, the adjoint of $\boldsymbol{P}^M[\boldsymbol{\psi}]$ in $\mathbf{L}^2(\mathbb{R}^d, \mathbf{e}_M)$ equals $\boldsymbol{P}^M[\overline{\boldsymbol{\psi}}]$. We recall that the operator $\mathrm{d}_M : \mathbf{L}^2(\mathbb{R}^d, \mathbf{e}_M) \to \mathbf{L}^2(\mathbb{R}^d, \mathbf{e})$ defined by $\mathrm{d}_M \mathbf{f} = \mathbf{f} \circ M^{-1}$ is an isometry and $\widehat{\mathrm{d}}_M = \mathrm{d}_{M^{-1}}$. Now, for any $\mathbf{f}, \mathbf{g} \in \mathbf{L}^2(\mathbb{R}^d, \mathbf{e}_M)$ and $t \geqslant 0$, we have

$$\begin{aligned}
\int_{\mathbb{R}^d} \boldsymbol{P}_t^M[\boldsymbol{\psi}]\mathbf{f}(\boldsymbol{x})\mathbf{g}(\boldsymbol{x})\mathbf{e}_M(\boldsymbol{x})\,d\boldsymbol{x} &= \int_{\mathbb{R}^d} \boldsymbol{P}_t[\boldsymbol{\psi}](\mathbf{f} \circ M)(M^{-1}\boldsymbol{x})\mathbf{g}(\boldsymbol{x})\mathbf{e}(M^{-1}\boldsymbol{x})\,d\boldsymbol{x} \\
&= \det(M) \int_{\mathbb{R}^d} \boldsymbol{P}_t[\boldsymbol{\psi}](\mathbf{f} \circ M)(\boldsymbol{y})\mathbf{g}(M\boldsymbol{y})\mathbf{e}(\boldsymbol{y})\,d\boldsymbol{y} \\
&= \det(M) \int_{\mathbb{R}^d} \boldsymbol{P}_t[\overline{\boldsymbol{\psi}}](\mathbf{g} \circ M)(\boldsymbol{y})\mathbf{f}(M\boldsymbol{y})\mathbf{e}(\boldsymbol{y})\,d\boldsymbol{y} \\
&= \int_{\mathbb{R}^d} \boldsymbol{P}_t^M[\overline{\boldsymbol{\psi}}]\mathbf{g}(\boldsymbol{y})\mathbf{f}(\boldsymbol{y})\mathbf{e}_M(\boldsymbol{y})\,d\boldsymbol{y}.
\end{aligned}$$

Therefore, $\widehat{\boldsymbol{P}}^M[\boldsymbol{\psi}] = \boldsymbol{P}^M[\overline{\boldsymbol{\psi}}]$ and $\boldsymbol{P}^M[\boldsymbol{\psi}]$ is self-adjoint in $\mathbf{L}^2(\mathbb{R}^d, \mathbf{e}_M)$ if and only if $\boldsymbol{\psi} = \overline{\boldsymbol{\psi}}$.

2.5.5 Proof of Theorem 2.2.4

Again, we assume that $M = \mathrm{Id}$ as the general case will follow by similarity transform. We recall from Theorem 2.4.1(2.61) that for any $k = 1, \ldots, d$, we have $P_t[\psi_k]H_{\psi_k} = H_{\psi_k}e_t$ on $\mathbf{D}(H_{\psi_k})$

where $H_{\psi_k} \in \mathscr{M}$ associated to

$$m^e_{\psi_k}(\xi) = m_{H_{\psi_k}}(\xi + \mathrm{i}/2) = \frac{W_{\phi_+}(\frac{1}{2} - \mathrm{i}\xi)}{W_{\phi_-}(\frac{1}{2} + \mathrm{i}\xi)}.$$

Since $\boldsymbol{e}_t = \otimes_{k=1}^d e_t$, applying Proposition 2.5.3 with $\boldsymbol{H}_{\boldsymbol{\psi}} = \otimes_{k=1}^d H_{\psi_i}$, the theorem follows.

2.5.6 Proof of Theorem 2.2.6

Let us prove this theorem for $M = \mathrm{Id}$ as the general case follows by the similarity transform. We first prove items (2) and (3) followed by item (1). If $m_{H_\psi}(\cdot + \mathrm{i}/2) \in \mathbf{L}^2(\mathbb{R})$, there exists a unique function $J_\psi \in \mathbf{L}^2(\mathbb{R}, e)$ such that $\mathcal{F}^e_{J_\psi} = m_{H_\psi}(\cdot + \mathrm{i}/2)$. Thus, for any $f \in \mathbf{D}(H_\psi)$, we have

$$H_\psi f = J_\psi * f$$

where $*$ stands for additive convolution. Also, $m_{H_\psi}\left(\cdot + \frac{\mathrm{i}}{2}\right) \in \mathbf{L}^2(\mathbb{R})$ implies that $\mathbf{C}_c^\infty(\mathbb{R}) \subset \mathbf{D}(H_\psi)$. Now, for any $f \in \mathbf{C}_c^\infty(\mathbb{R})$,

$$\begin{aligned}\int_{\mathbb{R}} (|J_\psi| * |f|)^2(x) e^x dx &= \int_{\mathbb{R}} \left(\int_{\mathbb{R}} |J_\psi(x-y)||f(y)|dy \right)^2 e^x dx \\ &\leqslant \int_{\mathbb{R}} \left(\int_{\mathbb{R}} |J_\psi(x-y)|^2 e^x dx \right)^{\frac{1}{2}} |f(y)|dy \\ &= \|J_\psi\|_{\mathbf{L}^2(\mathbb{R},e)} \int_{\mathbb{R}} e^{\frac{y}{2}} |f(y)|dy < \infty \end{aligned} \tag{2.92}$$

where the first inequality follows from Minkowski's integral inequality. Also, for all $f \in \mathbf{C}_c^\infty(\mathbb{R})$ and $g \in \mathbf{L}^2(\mathbb{R}, e)$,

$$\langle P_t[\psi] H_\psi f, g\rangle_{\mathbf{L}^2(\mathbb{R},e)} = \langle H_\psi f, \widehat{P}_t[\psi] g\rangle_{\mathbf{L}^2(\mathbb{R},e)} = \int_{\mathbb{R}} \left(\int_{\mathbb{R}} J_\psi(x-y) f(y)\, dy \right) \widehat{P}_t[\psi] g(x) e^x\, dx. \tag{2.93}$$

Since (2.92) yields that $|J_\psi| * |f| \in \mathbf{L}^2(\mathbb{R}, e)$, by Cauchy-Schwarz inequality, we have

$$\int_{\mathbb{R}} \int_{\mathbb{R}} |J_\psi(x-y) f(y) \widehat{P}_t[\psi] g(x)| e^x dx\, dy < \infty.$$

Applying Fubini Theorem on the right-hand side of (2.93), for all $f \in \mathbf{C}_c^\infty(\mathbb{R})$ and $g \in \mathbf{L}^2(\mathbb{R}, e)$, we have

$$\langle P_t[\psi]H_\psi f, g\rangle_{\mathbf{L}^2(\mathbb{R},e)} = \int_{\mathbb{R}} \left(\int_{\mathbb{R}} J_\psi(x-y)\widehat{P}_t[\psi]g(x)e^x\,dx\right) f(y)\,dy. \tag{2.94}$$

On the other hand,

$$\begin{aligned}\langle P_t[\psi]H_\psi f, g\rangle_{\mathbf{L}^2(\mathbb{R},e)} &= \langle H_\psi e_t f, g\rangle_{\mathbf{L}^2(\mathbb{R},e)} \\ &= \int_{\mathbb{R}} H_\psi e_t f(x) g(x) e^x\,dx = \int_{\mathbb{R}}\left(\int_{\mathbb{R}} J_\psi(x-y)e_t f(y)\,dy\right) g(x)e^x\,dx \\ &= \int_{\mathbb{R}}\left(\int_{\mathbb{R}} J_\psi(x-y) f(y) e^{-te^{-y}}\,dy\right) g(x)e^x\,dx \\ &= \int_{\mathbb{R}}\left(\int_{\mathbb{R}} J_\psi(x-y) g(x)e^x\,dx\right) f(y)e^{-te^{-y}}\,dy \end{aligned} \tag{2.95}$$

where for the last identity we have invoked again Fubini Theorem. From (2.94), (2.95) and using the density of $\mathbf{C}_c^\infty(\mathbb{R})$ in $\mathbf{L}^2(\mathbb{R}, e)$, we obtain

$$\int_{\mathbb{R}} J_\psi(x-y)\widehat{P}_t[\psi]g(x)e^x\,dx = e^{-te^{-y}} \int_{\mathbb{R}} J_\psi(x-y)g(x)e^x\,dx$$

which entails that, for all $g \in \mathbf{L}^2(\mathbb{R}, e)$,

$$\int_{\mathbb{R}} P_t[\psi]\tau_{-y}J_\psi(x)g(x)e^x\,dx = e^{-te^{-y}}\int_{\mathbb{R}} \tau_{-y}J_\psi(x)g(x)e^x\,dx.$$

Therefore, for all $y \in \mathbb{R}$,

$$P_t[\psi]\tau_{-y}J_\psi(x) = e^{-te^{-y}}\tau_{-y}J_\psi(x).$$

When $d > 1$, $\mathbf{m}^{\mathbf{e}}_{\boldsymbol{\psi}} \in \mathbf{L}^2(\mathbb{R}^d)$ if and only if $m_{H_{\psi_i}} \in \mathbf{L}^2(\mathbb{R})$ for all $k = 1, \ldots, d$. This implies

$$\widehat{\mathcal{F}}^{\mathbf{e}}_{\mathbf{m}^{\mathbf{e}}_{\boldsymbol{\psi}}} = \otimes_{k=1}^d \widehat{\mathcal{F}}^{e}_{m_{H_{\psi_i}}}.$$

Therefore, if $\mathbf{m}^{\mathbf{e}}_{\boldsymbol{\psi}} \in \mathbf{L}^2(\mathbb{R}^d)$ then, for any $t > 0$, $\boldsymbol{P}_t[\boldsymbol{\psi}]\tau_{-\boldsymbol{y}}\mathbf{J}_{\boldsymbol{\psi}} = e^{-t\langle e(-\boldsymbol{y}),\mathbf{1}\rangle}\mathbf{J}_{\boldsymbol{\psi}}$ for all $y \in \mathbb{R}$, where $\mathbf{J}_{\boldsymbol{\psi}} = \otimes_{k=1}^d J_{\psi_i}$. Since $\mathbf{m}^{\mathbf{e}}_{\boldsymbol{\psi}}$ is bounded, $\boldsymbol{H}_{\boldsymbol{\psi}}$ is a bounded operator. Taking the adjoint of the intertwining relation $\boldsymbol{P}_t[\boldsymbol{\psi}]\boldsymbol{H}_{\boldsymbol{\psi}} = \boldsymbol{H}_{\boldsymbol{\psi}}\boldsymbol{e}_t$, we get

$$\boldsymbol{e}_t\widehat{\boldsymbol{H}}_{\boldsymbol{\psi}} = \widehat{\boldsymbol{H}}_{\boldsymbol{\psi}}\widehat{\boldsymbol{P}}_t[\boldsymbol{\psi}].$$

Since $\mathrm{Spec}_r(\boldsymbol{P}_t[\boldsymbol{\psi}]) \subseteq \mathrm{Spec}_p(\widehat{\boldsymbol{P}}_t[\boldsymbol{\psi}])$ and $\mathrm{Spec}_p(\boldsymbol{e}_t) = \emptyset$, we must have $\mathrm{Spec}_p(\widehat{\boldsymbol{P}}_t[\boldsymbol{\psi}]) = \emptyset$. Therefore, $\mathrm{Spec}_r(\boldsymbol{P}_t[\boldsymbol{\psi}]) = \emptyset$, which concludes the proof of item (2).

Item (3) follows from item (2) by replacing $\boldsymbol{P}_t[\boldsymbol{\psi}]$ by $\boldsymbol{P}_t[\overline{\boldsymbol{\psi}}]$. Now coming back to item (1), if one of the conditions in items (2) and (3) holds, then item (1) is trivially true. We again prove this result for $d = 1$ and the general case would be the routine application of tensorization and similarity transform. Let us assume that $m_{H_\psi}(\cdot + \mathrm{i}/2)$ is bounded and not in $\mathbf{L}^2(\mathbb{R})$. We consider $\varphi \in \mathbf{C}_c^\infty(\mathbb{R}), \varphi \geqslant 0$ with $\mathrm{Supp}\,\varphi \subset (-1, 1)$ and for any $y \in \mathbb{R}$, $\varphi_n^y(x) = \sqrt{n}\varphi(n(x-y))$. We first show that (φ_n^y) (when normalized to have unit norm) are approximate eigenfunctions of e_t corresponding to the approximate eigenvalue $e^{-te^{-y}}$. We note that, for any $y \in \mathbb{R}$,

$$\begin{aligned}\|e_t\varphi_n^y - e^{-te^{-y}}\varphi_n^y\|^2_{\mathbf{L}^2(\mathbb{R},e)} &= \int_{\mathbb{R}} ne^{-2te^{-y}}\left[e^{-te^y(e^{-(x-y)}-1)} - 1\right]^2 \varphi(n(x-y))^2 e^x dx \\ &= e^{-2te^{-y}+y}\int_{\mathbb{R}}\left[e^{-te^y(e^{-\frac{x}{n}}-1)} - 1\right]^2 \varphi(x)^2 e^{\frac{x}{n}} dx. \end{aligned} \tag{2.96}$$

Using dominated convergence theorem, it follows that the expression in (2.96) converges to 0 as n tends to infinity. Also, $\|\varphi_n^y\|_{\mathbf{L}^2(\mathbb{R},e)} = \|\varphi\|_{\mathbf{L}^2(\mathbb{R},e)}$ for all $y \in \mathbb{R}$ and $n \geqslant 1$, which implies that $(\varphi_n^y)_{n\geqslant 1}$ is bounded below in $\mathbf{L}^2(\mathbb{R}, e)$. As a result, these functions, when normalized, form approximate eigenfunctions of e_t corresponding to the approximate eigenvalue $e^{-te^{-y}}$. Now, we claim that the functions $H_\psi\varphi_n^y$, when normalized, are approximate eigenfunctions of $P_t[\psi]$ corresponding to $e^{-te^{-y}}$. To prove this, it suffices to show that $\lim_{n\to\infty} P_t[\psi]H_\psi\varphi_n^y = e^{-te^{-y}}H_\psi\varphi_n^y$ in $\mathbf{L}^2(\mathbb{R}, e)$ and $\{\|H_\psi\varphi_n^y\|_{\mathbf{L}^2(\mathbb{R},e)}\}_{n\geqslant 1}$ is bounded below. We first claim that $\|H_\psi\varphi_n^y\|_{\mathbf{L}^2(\mathbb{R},e)} \to \infty$ as $n \to \infty$. To prove our claim, we observe that for all $n \geqslant 1$,

$$\mathcal{F}^e_{\varphi_n^y}(\xi) = e^{-\mathrm{i}y+\frac{y}{2}}\mathcal{F}_\varphi\left(\frac{\xi}{n} + \frac{\mathrm{i}}{2n}\right).$$

Therefore,

$$\mathcal{F}^e_{H_\psi\varphi_n^y}(\xi) = e^{-\mathrm{i}y+\frac{y}{2}} m_{H_\psi}\left(\xi + \frac{\mathrm{i}}{2}\right)\mathcal{F}_\varphi\left(\frac{\xi}{n} + \frac{\mathrm{i}}{2n}\right).$$

Using the isometry of the shifted Fourier transform followed by Fatou's lemma, we obtain

$$\begin{aligned}\varliminf_{n\to\infty} \|H_\psi \varphi_n^y\|^2_{\mathbf{L}^2(\mathbb{R},e)} &= \varliminf_{n\to\infty} \|\mathcal{F}^e_{H_\psi\varphi_n^y}\|^2_{\mathbf{L}^2(\mathbb{R})} \\ &= e^y \varliminf_{n\to\infty} \int_{\mathbb{R}} \left| m_{H_\psi}\left(\xi + \frac{\mathrm{i}}{2}\right)\right|^2 \left|\mathcal{F}_\varphi\left(\frac{\xi}{n} + \frac{\mathrm{i}}{2n}\right)\right|^2 d\xi \\ &\geqslant e^y \int_{\mathbb{R}} \left| m_{H_\psi}\left(\xi + \frac{\mathrm{i}}{2}\right)\right|^2 \varliminf_{n\to\infty} \left|\mathcal{F}_\varphi\left(\frac{\xi}{n} + \frac{\mathrm{i}}{2n}\right)\right|^2 d\xi \\ &= \|\varphi\|^2_{\mathbf{L}^1(\mathbb{R})} \int_{\mathbb{R}} \left| m_{H_\psi}\left(\xi + \frac{\mathrm{i}}{2}\right)\right|^2 d\xi = \infty. \end{aligned} \tag{2.97}$$

With the aid of the intertwining relationship $P_t[\psi]H_\psi = H_\psi e_t$, it is evident that

$$\lim_{n\to\infty} \|P_t[\psi]H_\psi\varphi_n^y - e^{-te^{-y}} H_\psi \varphi_n^y\|_{\mathbf{L}^2(\mathbb{R},e)} = 0.$$

Therefore, (2.97) ensures that the sequence $\{H_\psi\varphi_n^y\}$, when normalized, are approximate eigenfunctions of $P_t[\psi]$ corresponding to the approximate eigenvalue $e^{-te^{-y}}$. When $m_{H_\psi}(\cdot + \frac{\mathrm{i}}{2})^{-1}$ is bounded, we have $\widehat{P}_t[\psi]H_{\overline{\psi}} = H_{\overline{\psi}}e_t$. Therefore, following the same argument as before, we have $e^{t\mathbb{R}_-} \subseteq \mathrm{Spec}_{ap}(\widehat{P}_t[\psi])$. Since $\overline{\mathrm{Spec}(P_t[\psi])} = \mathrm{Spec}(\widehat{P}_t[\psi])$, we conclude that for all $t \geqslant 0$, $e^{t\mathbb{R}_-} \subseteq P_t[\psi]$.

For item (4), let $\boldsymbol{P}_t[\boldsymbol{\psi}]$ be self-adjoint, i.e. $\boldsymbol{\psi} = \overline{\boldsymbol{\psi}}$. Then, for all $k = 1, 2\ldots, d$, $m_{H_{\psi_k}} = (\overline{m}_{H_{\psi_k}})^{-1}$, which means that H_{ψ_k} is unitary. Hence, $\boldsymbol{H}_{\boldsymbol{\psi}}$ is unitary as well and therefore, $\mathrm{Spec}(\boldsymbol{P}[\boldsymbol{\psi}]) = \mathrm{Spec}_c(\boldsymbol{e}_t) = e^{t\mathbb{R}_-}$ for all $t \geqslant 0$.

2.5.7 Proof of Theorem 2.2.9

In the proof of Theorem 2.4.1(4) we have seen that for any $\psi \in \mathbf{N}_+(\mathbb{R})$, the closure of $(A_{\mathtt{PDO}}[\psi], \Lambda_\psi(\mathbf{C}_c^\infty(\mathbb{R}))$ generates $P[\psi]$, which is the $\mathbf{L}^2(\mathbb{R}, e)$-extension of the log self-similar Feller semigroup associated to ψ. By duality, $\widehat{P}[\psi] = P[\overline{\psi}]$ also corresponds to the log self-similar Feller semigroup associated with $\overline{\psi}$. Hence, for any $\psi \in \mathbf{N}_b(\mathbb{R})$ and $f \in \mathbf{C}_0(\mathbb{R}) \cap \mathbf{L}^2(\mathbb{R}, e)$ we have for all $t \geqslant 0$ and $x \in \mathbb{R}$,

$$P_t[\psi]f(x) = \mathbb{E}_x[f(X_t)]$$

where $(X_t)_{t\geqslant 0}$ is the self-similar Feller process associated to ψ. Therefore, item (1) is a direct consequence of tensorization and similarity transform.

Item (2) again follows from Proposition 2.4.2(2) followed by tensorization, similarity transform, and Proposition 2.5.4.

2.6 Examples

2.6.1 Spectrally negative self-similar processes

Let $\psi \in \mathbf{N}(\mathbb{R})$ be the Lévy-Khintchine exponent of a conservative spectrally negative Lévy process with a non-negative mean, that is, $\psi(0) = 0, \psi'(0) \geqslant 0$ and $\mu(0,\infty) = 0$ in (2.7). Note that this entails that the underlying Lévy process, and, hence the associated self-similar process, do not have positive jumps. In this case, the Wiener-Hopf factorization of ψ is

$$\psi(\xi) = -\mathrm{i}\xi\phi(\mathrm{i}\xi),\ \xi \in \mathbb{R},$$

where $\phi \in \mathbf{B}$ with a Lévy measure which is absolutely continuous with a non-increasing density. Let us write $m_{\Lambda_\psi}(z) = \frac{\Gamma(1+\mathrm{i}z)}{W_\phi(1+\mathrm{i}z)}$. From the properties of Bernstein-gamma functions recalled in Section 2.3.6, it is plain that m_{Λ_ψ} is analytic in the strip $\mathbb{S}_{[0,1)}$ and for $z \in \mathbb{S}_{(0,\infty)}$, $m_{\Lambda_\psi}(-\mathrm{i}z)$ is the Mellin transform of a positive random variable, denoted by I_ψ, which has finite moments of all order strictly bigger than -1, see e.g. [98] and references therein. Therefore, $|m_{\Lambda_\psi}(\cdot + \mathrm{i}/2)|$ is bounded on $\mathbb{R}$, and the shifted Fourier multiplier operator Λ_ψ with m_{Λ_ψ} as the multiplier is a bounded operator, and belongs to $\mathscr{M}$. Moreover, by Theorem 2.4.1(5), $P_t[\psi]\Lambda_\psi = \Lambda_\psi Q_t$ on $\mathbf{L}^2(\mathbb{R}, e)$, Q being the log-squared Bessel semigroup. In the next lemma, we show that Λ_ψ is a convolution kernel with respect to a probability measure, which is the additive convolution analogue of [95, Theorem 7.1].

Proposition 2.6.1. *If $\psi(\xi) = -\mathrm{i}\xi\phi(\mathrm{i}\xi)$, $\xi \in \mathbb{R}$, then, the operator $\Lambda^b_\psi \in \mathscr{B}(\mathbf{C}_b(\mathbb{R}))$ where*

$$\Lambda^b_\psi f(x) = \mathbb{E}[f(x - I_\psi)] \tag{2.98}$$

is bounded and

$$\Lambda^b_{\psi\,|\mathbf{C}_b(\mathbb{R})\cap\mathbf{L}^2(\mathbb{R},e)} = \Lambda_{\psi\,|\mathbf{C}_b(\mathbb{R})\cap\mathbf{L}^2(\mathbb{R},e)}.$$

In particular, Λ^b_ψ maps $\mathbf{C}_0([-\infty,\infty))$ to itself.

Proof. We define Λ^b_ψ on $\mathbf{C}_b(\mathbb{R})$ as in (2.98), where I_ψ is such that $\frac{1}{\sqrt{2\pi}}\mathbb{E}[e^{-\mathrm{i}\xi I_\psi}] = m_{\Lambda_\psi}(\xi)$ for all $\xi \in \mathbb{R}$. It easy to see that Λ^b_ψ is bounded (with respect to the supremum topology). Now, for any $f \in \mathbf{C}^\infty_c(\mathbb{R})$,

$$\Lambda^b_\psi f(x) = \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} e^{\mathrm{i}\xi x} m_{\Lambda_\psi}(\xi)\mathcal{F}_f(\xi)\, d\xi = \frac{e^{-\frac{x}{2}}}{\sqrt{2\pi}}\int_{-\infty+\frac{\mathrm{i}}{2}}^{\infty+\frac{\mathrm{i}}{2}} m_{\Lambda_\psi}\left(z + \frac{\mathrm{i}}{2}\right)\mathcal{F}^e_f(z)e^{\mathrm{i}zx}dz.$$

This implies that

$$\Lambda^b_{\psi\,|\mathbf{C}^\infty_c(\mathbb{R})} = \Lambda_{\psi\,|\mathbf{C}^\infty_c(\mathbb{R})}.$$

By density of $\mathbf{C}^\infty_c(\mathbb{R})$ in $\mathbf{L}^2(\mathbb{R},e)$, the above identity ensures that

$$\Lambda^b_{\psi\,|\mathbf{C}_b(\mathbb{R})\cap\mathbf{L}^2(\mathbb{R},e)} = \Lambda_{\psi\,|\mathbf{C}_b(\mathbb{R})\cap\mathbf{L}^2(\mathbb{R},e)}.$$

Proposition 2.6.2. *Let $\psi(\xi) = -\mathrm{i}\xi\phi(\mathrm{i}\xi)$, $\xi \in \mathbb{R}$, and define*

$$J_\psi(x) = \sum_{n=0}^{\infty}(-1)^n \frac{1}{W_\phi(n+1)}\frac{e^{nx}}{n!}. \tag{2.99}$$

Then, J_ψ is an entire function and $J_\psi \in \mathbf{C}_0([-\infty,\infty))$. Moreover, if

$$\xi \mapsto m_{H_\psi}\left(\xi + \frac{\mathrm{i}}{2}\right) = \frac{\Gamma(\frac{1}{2} - \mathrm{i}\xi)}{W_\phi(\frac{1}{2} + \mathrm{i}\xi)} \in \mathbf{L}^2(\mathbb{R})$$

then $J_\psi \in \mathbf{L}^2(\mathbb{R},e)$ and $\tau_{-y}J_\psi$ is an eigenfunction of $P_t[\psi]$ corresponding to the eigenvalue $e^{-te^{-y}}$.

Remark 2.6.3. (i) We mention that the analytic power series $\mathcal{I}_\psi$, defined as

$$J_\psi(z) = \mathcal{I}_\psi(e^{\mathrm{i}\pi}e^z), z \in \mathbb{C},$$

was introduced in [102] and a recent thorough study of this class of entire functions has been carried out in [8]. In the latter reference, several instances of this class of power series are provided in connection with well-known special functions. In particular, when $\psi(\xi) = \xi^2$, then J_ψ boils down to the Bessel function of index 0.

(ii) From [96, Proposition 6.2], $\xi \mapsto \left|m_{H_\psi}(\xi + \mathrm{i}/2)\right| = \left|\frac{\Gamma(1/2-\mathrm{i}\xi)}{W_\phi(1/2+\mathrm{i}\xi)}\right| \in \mathbf{L}^2(\mathbb{R})$ whenever $\phi \in \mathbf{B}_{\mathtt{d}}^c$ or $\phi_+ \in \mathbf{B}_{\mathtt{d}}$ and $\frac{1}{\mathtt{d}}(\phi(0) + \overline{\mu}(0)) > \frac{1}{2}$. In particular, this is always satisfied when $\mathtt{d} > 0$ and $\overline{\mu}(0) = \infty$. However, in the proof, we do not assume any condition on ϕ except that $\xi \mapsto \frac{\Gamma(1/2-\mathrm{i}\xi)}{W_\phi(1/2+\mathrm{i}\xi)} \in \mathbf{L}^2(\mathbb{R})$.

Proof. Since ϕ is increasing, we have $W_\phi(n+1) \geqslant \phi(1)^n$ for all n and therefore, the series defining J_ψ converges absolutely for all $z \in \mathbb{C}$. Now, consider the function

$$J(x) = \sum_{n=0}^{\infty}(-1)^n \frac{e^{nx}}{(n!)^2} = J_0\left(2e^{\frac{x}{2}}\right)$$

where J_0 is the Bessel function of index 0. It is well-known that $J_0 \in \mathbf{C}_0([0,\infty))$ which implies that $J \in \mathbf{C}_0([-\infty,\infty))$. Also,

$$\Lambda_\psi^b J(x) = \mathbb{E}[J(x - I_\psi)] = \mathbb{E}\left[\sum_{n=0}^{\infty}(-1)^n \frac{1}{(n!)^2} e^{n(x-I_\psi)}\right] = \sum_{n=0}^{\infty}(-1)^n \frac{1}{(n!)^2} e^{nx}\mathbb{E}[e^{-nI_\psi}],$$

where the interchange of the sum and expectation is justified as the series in the right-hand side is absolutely convergent. Recalling that $\mathbb{E}[e^{-nI_\psi}] = \frac{n!}{W_\phi(n+1)}$, we obtain $J_\psi = \Lambda_\psi^b J$. From Proposition 2.6.1, we get that $J_\psi \in \mathbf{C}_0([-\infty,\infty))$. Now, from Theorem 2.2.6, we know that $P_t[\psi]$ has point spectrum when $m_{H_\psi}(\cdot + \frac{\mathrm{i}}{2}) \in \mathbf{L}^2(\mathbb{R})$ and the eigenfunction corresponding to the eigenvalue $e^{-te^{-y}}$ is given by $\tau_{-y}K_\psi$, where $K_\psi = \widehat{\mathcal{F}^e}(m_{H_\psi}(\cdot + \frac{\mathrm{i}}{2}))$. The rest of the proof is devoted to show that $K_\psi = J_\psi$. For each $\kappa > 1$, let $\mathfrak{f}_\kappa(x) = \kappa e^{-\kappa x} e^{-e^{-\kappa x}}$. Then, $\int_{\mathbb{R}} \mathfrak{f}_\kappa(x)\, dx = 1$. From the proof of

Theorem 2.2.6, we know that $H_\psi \mathfrak{f}_\kappa = K_\psi * \mathfrak{f}_\kappa$. On the other hand,

$$\begin{aligned}
H_\psi \mathfrak{f}_\kappa(x) &= e^{-\frac{x}{2}} \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \mathcal{F}^e_{H_\psi \mathfrak{f}_\kappa}(\xi) e^{\mathrm{i}\xi x}\, d\xi = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{\mathrm{i}\xi x} m_{H_\psi}\left(\xi + \frac{\mathrm{i}}{2}\right) \mathcal{F}_{\mathfrak{f}_\kappa}\left(\xi + \frac{\mathrm{i}}{2}\right) d\xi \\
&= \frac{1}{\sqrt{2\pi}} e^{-\frac{x}{2}} \int_{\mathbb{R}} e^{\mathrm{i}\xi x} \frac{\Gamma(\frac{1}{2} - \mathrm{i}\xi)}{W_\phi(\frac{1}{2} + \mathrm{i}\xi)} \mathcal{F}_{\mathfrak{f}_\kappa}\left(\xi + \frac{\mathrm{i}}{2}\right) d\xi \\
&= \frac{1}{\sqrt{2\pi}} \int_{-\infty + \frac{\mathrm{i}}{2}}^{\infty + \frac{\mathrm{i}}{2}} e^{\mathrm{i}zx} \frac{\Gamma(-\mathrm{i}z)}{W_\phi(1 + \mathrm{i}z)} \Gamma\left(1 + \frac{\mathrm{i}z}{\kappa}\right) dz \\
&= \frac{1}{\sqrt{2\pi}} \int_{-\infty + \frac{\mathrm{i}}{2}}^{\infty + \frac{\mathrm{i}}{2}} e^{\mathrm{i}zx} \frac{\pi \Gamma(1 + \frac{\mathrm{i}z}{\kappa})}{W_\phi(1 + \mathrm{i}z) \sin(-i\pi z) \Gamma(1 + \mathrm{i}z)}\, dz \\
&=: \frac{1}{\sqrt{2\pi}} \int_{-\infty + \frac{\mathrm{i}}{2}}^{\infty + \frac{\mathrm{i}}{2}} e^{\mathrm{i}zx} G_\kappa(z)\, dz.
\end{aligned}$$

To compute the above integral, we fix $B > 0$, a positive odd integer N and consider the rectangular contour formed by the points $-B + \frac{\mathrm{i}}{2}, B + \frac{\mathrm{i}}{2}, B - \mathrm{i}\frac{N}{2}, -B - \mathrm{i}\frac{N}{2}$ with clockwise orientation. The poles of G_κ in this rectangle are $\{0, -\mathrm{i}, -2\mathrm{i}, \dots, -\lfloor \frac{N}{2} \rfloor \mathrm{i}\}$ with residues $(-1)^n \frac{\Gamma(1 + \frac{n}{\kappa})}{n! W_\phi(n+1)}$, $n = 0, 1, 2, \dots, \lfloor \frac{N}{2} \rfloor$. We split the contour integral into the following four parts

$$\begin{aligned}
I^{(1)}_B &= \int_{-B + \frac{\mathrm{i}}{2}}^{B + \frac{\mathrm{i}}{2}} e^{\mathrm{i}zx} G_\kappa(z)\, dz, \ I^{(2)}_{B,N} = \int_{B - \mathrm{i}\frac{N}{2}}^{B + \frac{\mathrm{i}}{2}} e^{\mathrm{i}zx} G_\kappa(z)\, dz \\
I^{(3)}_{B,N} &= \int_{-B - \mathrm{i}\frac{N}{2}}^{-B + \frac{\mathrm{i}}{2}} e^{\mathrm{i}zx} G_\kappa(z)\, dz, \ I^{(4)}_{B,N} = \int_{-B - \mathrm{i}\frac{N}{2}}^{B - \mathrm{i}\frac{N}{2}} e^{\mathrm{i}zx} G_\kappa(z)\, dz.
\end{aligned}$$

Since the integrals $I^{(2)}_{B,N}$ and $I^{(3)}_{B,N}$ are similar, we deal with only one of them, say $I^{(2)}_{B,N}$. For any fixed N, we observe that, for large values of B,

$$\sup_{t \in [-\frac{N}{2}, \frac{1}{2}]} \left| \frac{\Gamma(1 - \frac{t}{\kappa} + \mathrm{i}\frac{B}{\kappa})}{W_\phi(1 - t + \mathrm{i}B) \sin(\pi(t - \mathrm{i}B)) \Gamma(1 - t + \mathrm{i}\xi)} \right| \leqslant C_N e^{-\frac{\pi}{2\kappa}|B|}.$$

Therefore, for fixed N, $I^{(2)}_{B,N} \to 0, I^{(3)}_{B,N} \to 0$ as $B \to \infty$. For $I^{(4)}_{B,N}$ we proceed as follows. First, we observe that

$$\left| W_\phi\left(1 + \frac{N}{2} + \mathrm{i}\xi\right) \right| \geqslant \left| W_\phi\left(\frac{1}{2} + \mathrm{i}\xi\right) \right| \phi(1)^{\frac{N}{2}}.$$

Then, for any odd natural number N,

$$|I^{(4)}_{B,N}| \leqslant \int_{-B}^{B} \frac{e^{\frac{Nx}{2}}\left|\Gamma(1+\frac{N}{2\kappa}+\frac{\mathrm{i}\xi}{\kappa})\right|}{|W_\phi(\frac{N}{2}+1+\mathrm{i}\xi)||\sin(-\frac{N\pi}{2}-\mathrm{i}\pi\xi)||\Gamma(1+\frac{N}{2}+\mathrm{i}\xi)|}\,d\xi$$

$$\leqslant e^{\frac{Nx}{2}}\phi(1)^{-\frac{N}{2}}\int_{-\infty}^{\infty}\left|\frac{\Gamma(1+\frac{N}{2\kappa}+\frac{\mathrm{i}\xi}{\kappa})}{W_\phi(\frac{1}{2}+\mathrm{i}\xi)\cosh(\pi\xi)\Gamma(1+\frac{N}{2}+\mathrm{i}\xi)}\right|d\xi \quad (2.100)$$

$$\leqslant Ce^{\frac{Nx}{2}}\phi(1)^{-\frac{N}{2}}\left(\frac{N}{2}\right)^{\frac{N}{2}(\frac{1}{\kappa}-1)}\int_{-\infty}^{\infty}\left|\frac{1}{W_\phi(\frac{1}{2}+\mathrm{i}\xi)\cosh(\pi\xi)}\right|d\xi \to 0 \quad (2.101)$$

as $N \to \infty$ since $\kappa > 1$ and (2.101) follows from (3.54) by Stirling approximation. Therefore, $I^{(4)}_{B,N} \to 0$ uniformly in B as $N \to \infty$. Finally, using the residue theorem for meromorphic functions, we conclude that, for all $x \in \mathbb{R}$,

$$H_\psi \mathfrak{f}_\kappa(x) = \sum_{n=0}^{\infty}(-1)^n \frac{\Gamma(1+\frac{n}{\kappa})}{n!W_\phi(n+1)}e^{nx}.$$

On the other hand, it is easy to see that, for all $x \in \mathbb{R}$ and $\kappa > 1$,

$$J_\psi * \mathfrak{f}_\kappa(x) = \sum_{n=0}^{\infty}(-1)^n \frac{\Gamma(1+\frac{n}{\kappa})}{W_\phi(n+1)n!}e^{nx}.$$

Therefore, $K_\psi * \mathfrak{f}_\kappa(x) = J_\psi * \mathfrak{f}_\kappa(x)$ for all $x \in \mathbb{R}$ and $\kappa > 1$. Finally, the proof of the proposition will be completed by means of the following lemma.

Lemma 2.6.4. *If $g \in \mathbf{L}^2(\mathbb{R}, e)$ and $h \in \mathbf{C}_b(\mathbb{R})$ such that $g * \mathfrak{f}_\kappa = h * \mathfrak{f}_\kappa$ for all $\kappa > 1$, then $g = h$ a.e.*

Proof. We first recall the fact that, for any $g \in \mathbf{L}^2(\mathbb{R}, e)$, $\lim_{t\to 0}\|\tau_t g - g\|_e = 0$ and $\|\tau_t g\|_{\mathbf{L}^2(\mathbb{R},e)} = e^{-\frac{t}{2}}\|g\|_{\mathbf{L}^2(\mathbb{R},e^x dx)}$ for all $t \in \mathbb{R}$. Now, for any $\kappa > 1$, $g * \mathfrak{f}_\kappa \in \mathbf{L}^2(\mathbb{R}, e)$ and

$$\|g * \mathfrak{f}_\kappa - g\|_e^2 = \int_{\mathbb{R}} e^x\left(\int_{\mathbb{R}}(\tau_{-y}g(x) - g(x))\mathfrak{f}_\kappa(y)\,dy\right)^2 dx$$

$$\leqslant \int_{\mathbb{R}} e^x\left(\int_{\mathbb{R}}(\tau_{-y}g(x) - g(x))^2\mathfrak{f}_\kappa(y)\,dy\right) dx$$

$$= \int_{\mathbb{R}}\|\tau_{-y}g - g\|_e^2\mathfrak{f}_\kappa(y)\,dy = \int_{\mathbb{R}}\|\tau_{-\frac{y}{\kappa}}g - g\|_e^2\mathfrak{f}_1(y)\,dy. \quad (2.102)$$

Next,

$$\int_{\mathbb{R}}\|\tau_{-\frac{y}{\kappa}}g - g\|_e^2\,\mathfrak{f}_1(y)\,dy \leqslant \int_{\mathbb{R}}(1+e^{\frac{y}{\kappa}})\|g\|_e^2\,\mathfrak{f}_1(y)\,dy = \|g\|_e^2\left(1+\Gamma\left(1-\frac{1}{\kappa}\right)\right)$$

and the right-hand side is bounded with respect to κ. Hence, by applying the dominated convergence theorem, one gets, as $\kappa \to \infty$,

$$\int_{\mathbb{R}} \|\tau_{-\frac{y}{\kappa}} g - g\|^2_{\mathbf{L}^2(\mathbb{R},e)} \mathfrak{f}_1(y)dy \to 0.$$

From (2.102), it follows that $\|g * \mathfrak{f}_\kappa - g\|_{\mathbf{L}^2(\mathbb{R},e)} \to 0$ as $\kappa \to \infty$. Hence, there exists a sequence (κ_n) of positive real numbers such that as $n \to \infty$, $\kappa_n \to \infty$ such that $g * \mathfrak{f}_{\kappa_n} \to g$ a.e. On the other hand, for $h \in \mathbf{C}_b(\mathbb{R})$, $h * \mathfrak{f}_\kappa(x) = \int_{\mathbb{R}} h(x - \frac{y}{\kappa})\mathfrak{f}_1(y)\, dy \to h(x)$ as $\kappa \to \infty$ by continuity of h and the bounded convergence theorem. Therefore, letting $\kappa \to \infty$ on the identity $g * \mathfrak{f}_\kappa = h * \mathfrak{f}_\kappa$, the proof of the lemma follows. Using the above lemma, we conclude that $K_\psi = J_\psi$, which proves the proposition.

2.6.2 Self-similar processes with two sided jumps

Example of a non-self-adjoint semigroup with real continuous spectrum

Consider $\psi \in \mathbf{N}(\mathbb{R})$ such that $\psi(\xi) = \phi_+(-\mathrm{i}\xi)\phi_-(\mathrm{i}\xi)$ for all $\xi \in \mathbb{R}$ and

$$\phi_\pm(z) = m_\pm + \mathtt{d}_\pm z + \int_0^\infty (1 - e^{-zy})\mu_\pm(dy), \quad z \in \mathbb{C}_{[0,\infty)},$$

where $m_\pm \geqslant 0, \mathtt{d}_\pm > 0$ and $\int_0^\infty (y \wedge 1)\mu_\pm(dy) < \infty$. Define $\mathfrak{m}_\pm = m_\pm + \overline{\mu}_\pm(0+)$. Then, we have the following result.

Proposition 2.6.5. *Let* $\mathfrak{m}_\pm, \mathtt{d}_\pm$ *be as above. Let* $\overline{\nu}_\pm$ *denote the tail of the measure* $e^{-ay}\mu_\pm(dy)$*. If* $\mathfrak{m}_\pm < \infty$, $\frac{\mathfrak{m}_+}{\mathtt{d}_+} = \frac{\mathfrak{m}_-}{\mathtt{d}_-}$ *and* $w \mapsto \int_0^\infty \cos(wy)\overline{\nu}_\pm(y)\, dy$ *is in* $\mathbf{L}^1(\mathbb{R})$*, then,* $P_t[\psi]\Lambda_\psi = \Lambda_\psi Q_t$ *with* Λ_ψ *being invertible. Therefore,* $\mathrm{Spec}(P_t[\psi]) = \mathrm{Spec}_c(P_t[\psi]) = e^{t\mathbb{R}_-}$.

Remark 2.6.6. From Vigon's theory of philanthropy [128], if we assume that $\mu_\pm$ has non-increasing density with respect to Lebesgue measure, ϕ_+ and ϕ_- are always Wiener-Hopf factors of some Lévy-Khintchine exponent.

To prove Proposition 2.6.5, we start by showing the following lemma.

Lemma 2.6.7. *Let $\phi \in \mathbf{B}$ be such that $\mathfrak{m} = m + \overline{\mu}(0+) < \infty$, then*

$$|W_\phi(a+\mathrm{i}\xi)| \asymp \sqrt{\phi(a)}W_\phi(a)e^{-\frac{\pi}{2}|\xi|}|\xi|^{a+\frac{\mathfrak{m}}{\mathtt{d}}-\frac{1}{2}}, \ \ \forall a > 0, \tag{2.103}$$

if and only if $w \mapsto \int_0^\infty \cos(wy)\overline{\nu}(y)\,dy \in \mathbf{L}^1(\mathbb{R})$, where $\overline{\nu}$ is the tail of the measure $e^{-ay}\mu(dy)$.

Proof. First note that, from [95, Theorem 6.2(1)], we have, for all $a > 0$,

$$|W_\phi(a+\mathrm{i}\xi)| \asymp \frac{\sqrt{\phi(a)}W_\phi(a)}{\sqrt{\phi(a+\mathrm{i}\xi)}}e^{-|\xi|\Theta_\phi(a,|\xi|)} \quad \text{as } |\xi| \to \infty$$

where

$$|\xi|\Theta_\phi(a,|\xi|) = \int_a^\infty \ln\left(\frac{|\phi(y+\mathrm{i}|\xi|)|}{\phi(y)}\right) dy$$

which, from the proof of [96, Theorem 3.2], turns out to be

$$\Theta_\phi(a,|\xi|) = \int_0^{|\xi|} \arg(\phi(a+\mathrm{i}w))\,dw. \tag{2.104}$$

As $\mathtt{d} > 0$, $|\phi(a+\mathrm{i}\xi)| \sim \mathtt{d}|\xi|$ as $|\xi| \to \infty$. Thus, it is enough to show that $e^{-|\xi|\Theta_\phi(a,|\xi|)} \asymp |\xi|^{\frac{\mathfrak{m}}{\mathtt{d}}+a}e^{-\frac{\pi}{2}|\xi|}$. If $z = a + \mathrm{i}w$, then

$$\phi(z) = z\left(\frac{\mathfrak{m}}{z} + \mathtt{d} - \frac{1}{z}\int_0^\infty e^{-zy}\mu(dy)\right) = zg(z).$$

Now, $\arg(g(z)) = \arctan\left(\frac{\mathtt{Im}(g(z))}{\mathtt{Re}(g(z))}\right)$. We observe that

$$\mathtt{Re}(g(a+\mathrm{i}w)) = \mathtt{d} + \mathrm{O}\left(\frac{1}{w}\right) \tag{2.105}$$

$$\mathtt{Im}(g(a+\mathrm{i}w)) = -\frac{\mathfrak{m}}{w^2+a^2} + \frac{w}{w^2+a^2}\int_0^\infty \sin(wy)e^{-ay}\mu(dy) + \mathrm{O}\left(\frac{1}{w^2}\right). \tag{2.106}$$

From (2.105) and (2.106), we get

$$\frac{\mathtt{Im}(g(a+\mathrm{i}w))}{\mathtt{Re}(g(a+\mathrm{i}w))} = -\frac{\mathfrak{m}}{\mathtt{d}w} + \frac{\frac{w}{w^2+a^2}\int_0^\infty \sin(wy)e^{-ay}\mu(dy)}{\mathtt{Re}(g(a+\mathrm{i}w))} + \mathrm{O}\left(\frac{1}{w^2}\right). \tag{2.107}$$

Now, let us analyze the second term in the above expression. Using integration by parts (or, equivalently, Fubini's theorem), we have

$$\frac{\frac{w}{w^2+a^2}\int_0^\infty \sin(wy)e^{-ay}\mu(dy)}{\mathtt{Re}(g(a+\mathrm{i}w))} = \frac{\frac{w^2}{w^2+a^2}\int_0^\infty \cos(wy)\overline{\nu}(y)\,dy}{\mathtt{Re}(g(a+\mathrm{i}w))}. \tag{2.108}$$

As $\mathtt{Re}(g(a+\mathrm{i}w)) = \mathtt{d} + \mathrm{O}\left(\frac{1}{w}\right)$, the right-hand side of (2.108) is integrable with respect to w if and only if the function $w \mapsto \int_0^\infty \cos(wy)\overline{\nu}(y)\,dy$ is integrable with respect to w. Also, we observe that $\frac{\mathtt{Im}(g(a+\mathrm{i}w))}{\mathtt{Re}(g(a+\mathrm{i}w))} \to 0$ as $w \to \infty$ implies that

$$\arctan\left(\frac{\mathtt{Im}(g(a+\mathrm{i}w))}{\mathtt{Re}(g(a+\mathrm{i}w))}\right) - \frac{\mathtt{Im}(g(a+\mathrm{i}w))}{\mathtt{Re}(g(a+\mathrm{i}w))} = \mathrm{O}\left(\left|\frac{\mathtt{Im}(g(a+\mathrm{i}w))}{\mathtt{Re}(g(a+\mathrm{i}w))}\right|^2\right). \tag{2.109}$$

From (2.107), it is not hard to see that $\int_0^\infty \left|\frac{\mathtt{Im}(g(a+\mathrm{i}w))}{\mathtt{Re}(g(a+\mathrm{i}w))}\right|^2 dw < \infty$. Thus, (2.108) and (2.109) yield that

$$\int_0^{|\xi|} \arctan(g(a+\mathrm{i}w))\,dw = -\frac{\mathfrak{m}}{\mathtt{d}}\ln|\xi| + \mathrm{O}(1).$$

Finally, $\lim_{w\to\infty}\mathtt{Re}(g(a+\mathrm{i}w)) = \mathtt{d} = \lim_{w\to\infty} g(a+\mathrm{i}w) > 0$ implies that for large values of w, $\arg(\phi(a+\mathrm{i}w)) = \arg(a+\mathrm{i}w) + \arg(g(a+\mathrm{i}w))$. Therefore,

$$\begin{aligned}
|\xi|\Theta_\phi(a,|\xi|) &= \int_0^{|\xi|} \arg(\phi(a+\mathrm{i}w))\,dw \\
&= \int_0^{|\xi|} \arg(a+\mathrm{i}w)\,dw + \int_0^{|\xi|} \arg(g(a+\mathrm{i}w))\,dw + \mathrm{O}(1) \\
&= b\arctan\left(\frac{b}{a}\right) - \frac{a}{2}\ln\left(1+\frac{b^2}{a^2}\right) - \frac{\mathfrak{m}}{\mathtt{d}}\ln|\xi| + \mathrm{O}(1) \\
&= \frac{\pi}{2}|\xi| - a\ln|\xi| - \frac{\mathfrak{m}}{\mathtt{d}}\ln|\xi| + \mathrm{O}(1).
\end{aligned}$$

Then, we conclude that

$$e^{-|\xi|\Theta_\phi(a,|\xi|)} \asymp |\xi|^{a+\frac{\mathfrak{m}}{\mathtt{d}}} e^{-\frac{\pi}{2}|\xi|}$$

if and only if $w \mapsto \int_0^\infty \cos(wy)\overline{\nu}(y)\,dy$ is in $\mathbf{L}^1(\mathbb{R})$. This concludes the proof of the lemma. Now, coming back to the proof of Proposition 2.6.5 if both ϕ_+ and ϕ_- satisfy the conditions of the proposition, it is easy to see, from the estimate (2.103), that $\xi \mapsto \left|\frac{W_{\phi_+}(\frac{1}{2}-\mathrm{i}\xi)}{W_{\phi_-}(\frac{1}{2}+\mathrm{i}\xi)}\right|$ is bounded above and below. Therefore, both Λ_ψ and Λ_ψ^{-1} are invertible, which concludes the proof.

Another example with two-sided jumps

Consider ϕ_+, ϕ_- such that

$$\phi_+(z) = \frac{\Gamma(\tilde{\alpha}(1+z))}{\Gamma(\tilde{\alpha}z)} \text{ and } \phi_-(z) = \frac{\Gamma(\rho+\alpha+\alpha z)}{\Gamma(\rho+\alpha z)}, \quad z \in \mathbb{C}_{(0,\infty)},$$

where $\tilde{\alpha}, \alpha \in (0,1),\ \rho > 0$. From [69], it is known that $\phi_+, \phi_- \in \mathbf{B}$ and the Lévy measures of ϕ_+ and ϕ_- are absolutely continuous with non-increasing densities, namely, for $y > 0$,

$$\mu_+(dy) = \frac{1}{\Gamma(1-\tilde{\alpha})} \frac{e^{\frac{y}{\tilde{\alpha}}}}{(e^{\frac{y}{\tilde{\alpha}}}-1)^{1+\tilde{\alpha}}}\, dy,$$
$$\mu_-(dy) = \frac{1}{\Gamma(1-\alpha)} \frac{e^{\frac{(1-\rho)y}{\alpha}}}{(e^{\frac{y}{\alpha}}-1)^{1+\alpha}}\, dy.$$

Thus, from Vigon's theory of philanthropy [128], there is a Lévy process whose Lévy-Khintchine exponent is given by

$$\psi(\xi) = \phi_+(-\mathrm{i}\xi)\phi_-(\mathrm{i}\xi), \quad \xi \in \mathbb{R}.$$

It is immediate that the Bernstein-gamma functions corresponding to ϕ_- and ϕ_+ are given by

$$W_{\phi_+}(z) = \frac{\Gamma(\tilde{\alpha}z)}{\Gamma(\tilde{\alpha})} \text{ and } W_{\phi_-}(z) = \frac{\Gamma(\rho+\alpha z)}{\Gamma(\alpha+\rho)} \quad z \in \mathbb{C}_{(0,\infty)}.$$

Using Stirling formula for the gamma function, we know that for all $a > 0$ and large values of $|\xi|$,

$$|W_{\phi_+}(a+\mathrm{i}\xi)| \asymp |\xi|^{a\tilde{\alpha}-\frac{1}{2}} e^{-\frac{\tilde{\alpha}\pi|\xi|}{2}} \text{ and } |W_{\phi_-}(a+\mathrm{i}\xi)| \asymp |\xi|^{a\alpha+\rho-\frac{1}{2}} e^{-\frac{\alpha\pi|\xi|}{2}}.$$

Also, from the above estimates and the definitions of ϕ_+, ϕ_-, we infer that $\psi \in \mathbf{N}_b(\mathbb{R})$ for any $\alpha, \tilde{\alpha} \in (0,1), \rho > 0$. From Theorem 2.2.6, the multiplier of the Fourier operator H_ψ is given by

$$m_{H_\psi}\left(\xi + \frac{\mathrm{i}}{2}\right) = \frac{W_{\phi_+}(\frac{1}{2}-\mathrm{i}\xi)}{W_{\phi_-}(\frac{1}{2}+\mathrm{i}\xi)},$$

and therefore, $\xi \mapsto m_{H_\psi}\left(\xi+\frac{\mathrm{i}}{2}\right) \in \mathbf{L}^2(\mathbb{R})$ in the following two cases: when $\tilde{\alpha} > \alpha$ or when $\alpha = \tilde{\alpha},\ \rho > \frac{1}{2}$. Similarly, the reciprocal function

$$\xi \mapsto \frac{1}{m_{H_\psi}\left(\xi+\frac{\mathrm{i}}{2}\right)} \in \mathbf{L}^2(\mathbb{R})$$

when (i') $\tilde{\alpha} < \alpha$. Based on these observation, we get the following result.

Proposition 2.6.8. *(i) If $\tilde{\alpha} > \alpha$, or, $\alpha = \tilde{\alpha}$ and $\rho > \frac{1}{2}$, then $e^{t\mathbb{R}_-} \subseteq \mathrm{Spec}_p(P_t[\psi])$ and the eigenfunction corresponding to the eigenvalue $e^{-te^{-y}}$ is given by $\tau_{-y}J_\psi$ where*

$$J_\psi(x) = \mathtt{W}\left(\frac{\alpha}{\tilde{\alpha}}, \alpha+\rho; -e^{\frac{x}{\tilde{\alpha}}}\right)$$

and $\mathtt{W}(\gamma,\beta;z) = \sum_{n=0}^{\infty} \frac{1}{\Gamma(\gamma n+\beta)}\frac{z^n}{n!}$ is the Wright hypergeometric function, which defines an entire function if $\gamma > -1$.

(ii) If $\tilde{\alpha} < \alpha$, then $e^{t\mathbb{R}_-} \subseteq \mathrm{Spec}_r(P_t[\psi])$ and the co-eigenfunction corresponding to $e^{-te^{-y}}$ is given by $\tau_{-y}J_{\overline{\psi}}$ where

$$J_{\overline{\psi}}(x) = e^{\frac{\rho x}{\alpha}}\mathtt{W}\left(\frac{\tilde{\alpha}}{\alpha}, \tilde{\alpha} + \frac{\tilde{\alpha}}{\alpha}\rho; -e^{\frac{x}{\alpha}}\right). \tag{2.110}$$

Proof. The proof will be again based on the integration of the multiplier function on a suitable contour. Assuming the condition (i) or (ii), from the proof of Theorem 2.2.6, we know that

$$J_\psi(x) = e^{-\frac{x}{2}}\widehat{\mathcal{F}}\left(m_{H_\psi}\left(\cdot + \frac{\mathrm{i}}{2}\right)\right)(x).$$

Let us assume that $\tilde{\alpha} > \alpha$. Then, $m_{H_\psi}\left(\cdot + \frac{\mathrm{i}}{2}\right) \in \mathbf{L}^1(\mathbb{R})$. Therefore, by Fourier inversion we get

$$J_\psi(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty+\frac{\mathrm{i}}{2}}^{\infty+\frac{\mathrm{i}}{2}} \frac{\Gamma(-\mathrm{i}\tilde{\alpha}z)}{\Gamma(\rho+\alpha(1+\mathrm{i}z))}e^{\mathrm{i}zx}\,dz := \frac{1}{\sqrt{2\pi}}\int_{-\infty+\frac{\mathrm{i}}{2}}^{\infty+\frac{\mathrm{i}}{2}} G(z)e^{\mathrm{i}zx}\,dz.$$

Choosing the rectangular contour with vertices $-R, R, R-\mathrm{i}\frac{N}{2\tilde{\alpha}}, -R-\mathrm{i}\frac{N}{2\tilde{\alpha}}$, where N is an odd natural number, we define the integrals on the four segments as

$$I_R^{(1)} = \frac{1}{\sqrt{2\pi}}\int_{-R}^{R} G(z)e^{\mathrm{i}zx}\,dz, \quad I_{R,N}^{(2)} = \frac{1}{\sqrt{2\pi}}\int_{-R}^{-R-\mathrm{i}\frac{N}{2\tilde{\alpha}}} G(z)e^{\mathrm{i}zx}\,dz$$
$$I_{R,N}^{(3)} = \frac{1}{\sqrt{2\pi}}\int_{R}^{R-\mathrm{i}\frac{N}{2\tilde{\alpha}}} G(z)e^{\mathrm{i}zx}\,dz, \quad I_{R,N}^{(4)} = \frac{1}{\sqrt{2\pi}}\int_{-R-\mathrm{i}\frac{N}{2\tilde{\alpha}}}^{R-\mathrm{i}\frac{N}{2\tilde{\alpha}}} G(z)e^{\mathrm{i}zx}\,dz.$$

We note that the poles of the function G in the rectangle are $\left\{0, -\frac{\mathrm{i}}{\tilde{\alpha}}, \ldots, -\mathrm{i}\frac{\lfloor\frac{N}{2}\rfloor}{\tilde{\alpha}}\right\}$ with residues $\frac{1}{\tilde{\alpha}n!\Gamma(\frac{n\alpha}{\tilde{\alpha}}+\alpha+\rho)}$. Arguing as in the proof of Proposition 2.6.2, for any fixed N, one can show $I_{R,N}^{(2)}, I_{R,N}^{(3)} \to 0$ as $R \to \infty$. On the other hand, after a change of variable,

$$I_{R,N}^{(4)} = \frac{1}{\sqrt{2\pi}}e^{\frac{Nx}{2\alpha}}\int_{-R}^{R}\frac{\Gamma(-\frac{N}{2}-\mathrm{i}\tilde{\alpha}\xi)}{\Gamma(\rho+\alpha+\frac{N}{2}+\mathrm{i}\alpha\xi)}e^{\mathrm{i}\xi x}\,d\xi$$
$$= \frac{e^{\frac{Nx}{2\alpha}}}{\sqrt{2\pi}}\int_{-R}^{R}\frac{\pi e^{\mathrm{i}\xi x}}{\cosh(\pi\tilde{\alpha}\xi)\Gamma(1+\frac{N}{2}+\mathrm{i}\tilde{\alpha}\xi)\Gamma(\rho+\alpha+\frac{N\alpha}{2\tilde{\alpha}}+\mathrm{i}\alpha\xi)}\,d\xi.$$

Using Stirling formula, we have for all $\xi \in \mathbb{R}$,

$$\left|\Gamma\left(1+\frac{N}{2}+\mathrm{i}\tilde{\alpha}\xi\right)\right| \geqslant \frac{\Gamma(1+\frac{N}{2})}{\cosh^{\frac{1}{2}}(\tilde{\alpha}\pi\xi)}$$
$$\left|\Gamma\left(\rho+\alpha+\frac{N\alpha}{2\tilde{\alpha}}+\mathrm{i}\alpha\xi\right)\right| \geqslant \frac{\Gamma(1+\frac{N\alpha}{2\tilde{\alpha}}+\alpha+\rho)}{\cosh^{\frac{1}{2}}(\alpha\pi\xi)}$$

and, then

$$|I_{R,N}^{(4)}| \leqslant \frac{e^{\frac{Nx}{2\alpha}}}{\sqrt{2\pi}\Gamma(1+\frac{N}{2})\Gamma(\rho+\alpha+\frac{N\alpha}{2\tilde{\alpha}})}\int_{-\infty}^{\infty}\frac{\cosh^{\frac{1}{2}}(\tilde{\alpha}\pi\xi)\cosh^{\frac{1}{2}}(\alpha\pi\xi)}{\cosh(\tilde{\alpha}\pi\xi)}\,d\xi.$$

As $\tilde{\alpha} > \alpha$, the integral on the right-hand side is finite and hence, $I_{R,N}^{(4)} \to 0$ as $N \to \infty$, uniformly in R. Thus, invoking Cauchy integral formula, we have

$$J_\psi(x) = \sum_{n=0}^{\infty}\frac{(-1)^n}{\Gamma(\frac{n\alpha}{\tilde{\alpha}}+\alpha+\rho)}\frac{e^{\frac{nx}{\alpha}}}{n!}.$$

When $\tilde{\alpha} = \alpha$ and $\rho > \frac{1}{2}$, the function $m_{H_\psi}\left(\cdot+\frac{\mathrm{i}}{2}\right) \notin \mathbf{L}^1(\mathbb{R})$ for $\rho \in (\frac{1}{2}, 1]$. In this case, we can use the same idea as in the proof of Proposition 2.6.2, where we have used the convolution of the integrand with respect to a class of kernel functions $\{\mathfrak{f}_\kappa, \kappa > 1\}$ and recalling the fact (see (2.99) with $\phi_+(s) = s+\alpha+\rho$) that

$$x \mapsto \sum_{n=0}^{\infty}\frac{(-1)^n}{\Gamma(n+\alpha+\rho)}\frac{e^{\frac{nx}{\alpha}}}{n!} \in \mathbf{C}_0(\mathbb{R}),$$

we can conclude that

$$J_\psi(x) = \sum_{n=0}^{\infty}\frac{(-1)^n}{\Gamma(n+\alpha+\rho)}\frac{e^{\frac{nx}{\alpha}}}{n!}.$$

When $\tilde{\alpha} < \alpha$, $\frac{1}{m_{H_\psi}}\left(\cdot+\frac{\mathrm{i}}{2}\right) \in \mathbf{L}^2(\mathbb{R})$ and, from the proof of Theorem 2.2.6, one gets

$$J_{\overline{\psi}}(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty-\frac{\mathrm{i}}{2}}^{\infty+\frac{\mathrm{i}}{2}}\frac{\Gamma(\rho-i\alpha z)}{\Gamma(\tilde{\alpha}(1+\mathrm{i}z))}e^{\mathrm{i}zx}\,dz.$$

The poles of the function $z \mapsto \frac{\Gamma(\rho-\mathrm{i}\alpha z)}{\Gamma(\tilde{\alpha}(1+\mathrm{i}z))}$ are $\{-\frac{n+\rho}{\alpha}\mathrm{i},\ n \in \mathbb{N}\cup\{0\}\}$ with residues $\frac{1}{\alpha n!\Gamma(\tilde{\alpha}(1+\frac{n+\rho}{\alpha}))}$. Using the same argument as in the case $\tilde{\alpha} > \alpha$ and Cauchy's theorem of residues, (2.110) follows.

CHAPTER 3

DISCRETE SELF-SIMILAR AND ERGODIC MARKOV CHAINS

3.1 Introduction

In this chapter, we first introduce continuous-time Markov processes with state space the set of all nonnegative integers that also enjoy a scaling type property. Naturally, one cannot expect (1.1) to hold in this setting, because the set of integers is not stable by the dilation operators as defined above. However, in [84], the authors introduced the following signed Binomial kernel defined by

$$\mathbb{D}_\alpha \mathsf{f}(n) = \sum_{k=0}^{n} \binom{n}{k} \alpha^k (1-\alpha)^{n-k} \mathsf{f}(k)$$

which resembles the dilation operator through the multiplicative semigroup property $\mathbb{D}_{\alpha\beta} = \mathbb{D}_\alpha \mathbb{D}_\beta$ for all $\alpha, \beta > 0$, which will be proved in Proposition 3.4.1 below. Furthermore, they showed that the linear birth-death Markov chain, see Remark 3.2.3 below for definition, satisfies the following commutation type relation

$$\mathbb{Q}_t \mathbb{D}_\alpha = \mathbb{D}_\alpha \mathbb{Q}_{\alpha t}$$

where $\mathbb{Q}$ is the associated semigroup. Motivated from this result, we introduce a class of continuous-time Markov chains on $\mathbb{Z}_+$ that satisfy the scaling property as above and are upward skip-free, that is, at any instant the Markov chains do not jump more than one step above and name them *discrete self-similar Markov chains*, see Definition 3.2.1. This class of Markov chains, to the best of our knowledge, have not been identified before. Moreover, we want to understand their connections with self-similar Markov processes. To this end, we resort to intertwining relationship between Markov processes. More specifically, for two Markov semigroups P and Q, we say that they are intertwined if, for all $t \geqslant 0$,

$$P_t \Lambda = \Lambda Q_t$$

for some linear operator Λ. Note that when the underlying processes have different state spaces, one lattice and the other one continuous, we use the terminology gateway relation, coined in [84], to emphasize the unexpected two-sided connection between the two worlds. The term duality is also used in a fast growing and fascinating literature on this topic related to differential operators arising in statistical mechanics, see e.g. [5, 25, 49, 115, 58] and references therein. More generally, the concept of intertwining relation goes back to Dynkin [41] who used it to construct new Markov semigroups from a reference one. These ideas were extended by Rogers and Pitman in [110], leading to the characterization of Markov functions; that is, measurable maps that preserve the Markov property. With the help of the intertwining relationship, we prove the Feller property of the discrete self-similar Markov chains, see Theorem 3.2.6, and obtain the spectrally negative self-similar Markov processes as the scaling limit of these Markov chains, see Theorem 3.2.6(2). The use of intertwining relations to prove limit theorems is not new and, in fact, a general framework was built up by Borodin and Olshanski [22], where they apply it to construct a class of Markov chains on the Thoma cone. Unfortunately, their strategy is not applicable in our situation because their conditions are too stringent for us, namely the set of finitely supported functions are not invariant with respect to the discrete self-similar Markov semigroups. Nonetheless, still resorting to the intertwining relation, we are able to derive explicit formulas for the moments of these Markov chains and we identify their scaling limits by the method of moments. We emphasize that there are many instances of the appearance of positive self-similar Markov processes as the scaling limits of models, such as coalescence-fragmentation processes, see Bertoin [11], random planar maps, see Le Gall and Miermont [77]. We also mention the recent paper by Bertoin and Kortchemski [15] where the authors introduce a class of discrete-time Markov chains whose appropriate scaling limits are positive self-similar Markov processes. It appears that our work offers another class of Markov chains in the domain of attraction of such self-similar Markov processes, with the additional surprising feature that the connection between the two objects goes, thanks to the gateway relation, in both directions.

We proceed by introducing another class of ergodic Markov chains which are obtained by a linear first order perturbation of the generators of the discrete self-similar Markov chains. We name them *skip-free Laguerre* chains. The motivation behind this comes from the fact that their continuous analogue are the generalized Laguerre processes, studied in [95], which are also constructed by perturbation of the generator of self-similar processes by a linear convection term, that is a first order differential operator with a linear coefficient. We show that they generate a class of Feller semigroups of ergodic Markov chains which intertwine with the class of the generalized Laguerre semigroups. Using this connection, we develop the spectral theory, including the spectrum and the eigenvalues expansions, in the Hilbert space ℓ^2 of nonnegative integers weighted with the invariant distributions $\mathbf{n}_\phi$ of the semigroups of these non-reversible chains. As by-product, and under some mild conditions, we prove compactness and also obtain a hypercoercivity estimate for the $\ell^2(\mathbf{n}_\phi)$ convergence to equilibrium, which is given explicitly as a perturbed spectral gap inequality. This part involves a deep theory of non-self-adjoint operators as developed in [95], see Section 3.4.11 for more details.

We continue our analysis of these skip-free Laguerre semigroups by investigating the entropy decay to equilibrium as well as the hypercontractivity property. For self-adjoint Markov semigroups, these two phenomena are equivalent to the (modified) log-Sobolev inequalities. Unfortunately, in our context, this relation fails due to the non-self-adjointness of the semigroups. However, resorting to the idea of interweaving relation, introduced recently in [85], we relate the skip-free Laguerre semigroups with the self-adjoint diffusion Laguerre semigroups and deduce, up to some universal random time, both the entropy decay and the hypercontractivity. Finally, showing that this random time is infinitely divisible, we develop a thorough analysis of the skip-free Laguerre semigroups subordinated with the associated subordinator, which generate a class of ergodic Markov chains with two-sided jumps, for which all the results described above are obtained explicitly.

The remaining part of the chapter is organized as follows. Most of the frequently used notations are defined in Section 3.1.1 while Section 3.2 contains all the main results. We provide some examples in Section 3.3 and Section 3.4 is devoted to the proofs of the main results. Some aspects of spectral theory for non-self-adjoint operators have been reviewed in Subsection 3.4.11 and the results related to interweaving relations have been proved in Subsection 3.4.14.

3.1.1 Notations

For any nonnegative sigma-finite measure μ on $\mathbb{R}_+$ and $p \in [1, \infty]$, $\mathbf{L}^p(\mu)$ denotes the L^p space with weight μ. When $p = 2$, the corresponding Hilbert space is endowed with the inner product denoted by $\langle f, g\rangle_\mu = \int_{\mathbb{R}_+} f(x)\overline{g(x)}\mu(dx)$. When μ is the Lebesgue measure, we simply write $\mathbf{L}^2(\mu) = \mathbf{L}^2(\mathbb{R}_+)$ associated with the inner product $\langle\cdot,\cdot\rangle$. If the underlying space is the set of all integers $\mathbb{Z}_+$, then for any nonnegative discrete measure $\mathbf{m}$ on $\mathbb{Z}_+$, we write $\ell^p(\mathbf{m})$ to denote the weighted ℓ^p space on $\mathbb{Z}_+$ and for $p = 2$, the inner product is written as $\langle \mathsf{f}, \mathsf{g}\rangle_\mathbf{m} = \sum_{n\in\mathbb{Z}_+} \mathsf{f}(n)\mathsf{g}(n)\mathbf{m}(n)$. When $\mathbf{m}$ is the counting measure, we use the notation $\ell^2(\mathbb{Z}_+) = \ell^2(\mathbf{m})$. For any measurable function $f \geqslant 0$ or $f \in \mathbf{L}^1(E, \mu)$, we write $\mu f = \int_E f d\mu$.

3.2 Main Results

3.2.1 Discrete dilation and discrete self-similar Markov chains

We start by introducing a transformation on $\mathbf{C}(\mathbb{Z}_+)$, which we name the *discrete dilation operator*. For any $\alpha > 0$ and $\mathsf{f} \in \mathbf{C}(\mathbb{Z}_+)$, we define

$$\mathbb{D}_\alpha \mathsf{f}(n) = \sum_{r=0}^{n} \binom{n}{r} \alpha^r (1-\alpha)^{n-r} \mathsf{f}(r). \tag{3.1}$$

It should be noted that $\mathbb{D}_\alpha$ is well defined on $\mathbf{C}(\mathbb{Z}_+)$ for all $\alpha \geqslant 0$ and it is a Markov kernel when $\alpha \in [0, 1]$. When $\alpha > 1$, $\mathbb{D}_\alpha \mathsf{f}$ may not be bounded even if f is bounded. For instance, taking $\mathsf{f}(n) = (-1)^n$, for any $n \in \mathbb{Z}_+$, we have $|\mathbb{D}_\alpha \mathsf{f}(n)| = (2\alpha - 1)^n$, which grows exponentially with respect to n. The operator $\mathbb{D}$ shares the multiplicative semigroup property with the dilation operator, that is, for all $\alpha, \beta > 0$, we have $\mathbb{D}_{\alpha\beta} = \mathbb{D}_\alpha \mathbb{D}_\beta$, see Proposition 3.4.1 below. Next, we introduce the discrete self-similar Markov chains which are defined in terms of the operator $\mathbb{D}_\alpha$.

Definition 3.2.1. We say that the semigroup $\mathbb{Q} = (\mathbb{Q}_t)_{t\geqslant 0}$ of a continuous-time Markov chain $\mathbb{X}$

with state space $\mathbb{Z}_+$ is *discrete self-similar* if for all $t \geqslant 0, \alpha \in [0,1]$, the following identity

$$\mathbb{Q}_t \mathbb{D}_\alpha = \mathbb{D}_\alpha \mathbb{Q}_{\alpha t} \tag{3.2}$$

holds on $\mathbf{C}_b(\mathbb{Z}_+)$.

In terms of the law of the Markov chain $\mathbb{X} = (\mathbb{X}(t,n), n \in \mathbb{Z}_+)_{t\geqslant 0}$, where $\mathbb{X}(t,n)$ means that it is issued from n, the discrete self-similarity can be interpreted by the following identity in distribution, for any $\alpha \in [0,1]$, $t \geqslant 0$ and $n \in \mathbb{Z}_+$,

$$\mathtt{B}(\mathbb{X}(t,n),\alpha) \stackrel{(d)}{=} \mathbb{X}(\alpha t, \mathtt{B}(n,\alpha)) \tag{3.3}$$

where $\mathtt{B}(n,\alpha)$ is a Binomial random variable with parameter n and α, and $\mathbb{X}(t,\mathtt{B}(n,\alpha)))$ is the chain at time t with initial law the one of $\mathtt{B}(n,\alpha)$.

Next, we consider the class of triplets (m,σ^2,Π) such that $m,\sigma^2 \geqslant 0$ and Π is a non-negative measure on $\mathbb{R}_+$ that satisfies

$$\int_0^\infty (y \wedge y^2)\Pi(dy) < \infty, \tag{3.4}$$

that is Π is a Lévy measure with a finite first moment away from 0. To each of these triplets, we associate the so-called Bernstein function defined as

$$\phi(u) = m + \sigma^2 u + \int_0^\infty (1-e^{-uy})\overline{\Pi}(y)dy \tag{3.5}$$

where $\overline{\Pi}(y) = \Pi(y,\infty)$ is the tail of the measure Π. Let $\mathbf{B}$ denote the class of all functions of the form (3.5).

We are now ready to introduce a class of discrete operators on $\mathbb{Z}_+$. For any $\phi \in \mathbf{B}$ associated with the triplet (m,σ^2,Π) and $\mathsf{f} \in \mathbf{C}_c(\mathbb{Z}_+)$, we define

$$\mathbb{G}_\phi \mathsf{f}(n) = \sigma^2 n(\partial_+ + \partial_-)\mathsf{f}(n) + (m+\sigma^2)\partial_+\mathsf{f}(n) + \mathbb{G}_\Pi \mathsf{f}(n) \tag{3.6}$$

where $\partial_\pm \mathsf{f}(n) = \mathsf{f}(n\pm 1) - \mathsf{f}(n)$ for all $n \in \mathbb{Z}_+$ and

$$\mathbb{G}_\Pi \mathsf{f}(n) = \frac{1}{n+1}\int_0^\infty \left[\mathbb{D}_{e^{-y}}\mathsf{f}(n+1) - \mathsf{f}(n+1) + y(n+1)\partial_+\mathsf{f}(n)\right]\Pi(dy). \tag{3.7}$$

We are now ready to state our first main result.

Theorem 3.2.2. *The operator* $(\mathbb{G}_\phi, \mathbf{C}_c(\mathbb{Z}_+))$ *generates a Feller Markov chain on* $\mathbb{Z}_+$*, denoted by* $\mathbb{X}_\phi = (\mathbb{X}_\phi(t,n), n \in \mathbb{Z}_+)_{t\geqslant 0}$ *which is self-similar, and* $\mathbf{C}_c(\mathbb{Z}_+)$ *serves as a core for* $\mathbb{G}_\phi$*.*

This theorem is proved in Section 3.4.1.

Remark 3.2.3. When $\Pi \equiv 0$, $\mathbb{X}_\phi$ is the reversible linear birth-death chain with invariant measure

$$\frac{\Gamma(n+m+1)}{\Gamma(n+1)}, \ n \in \mathbb{Z}_+.$$

For a detailed account on such Markov chains, we refer to [84].

Remark 3.2.4. In (3.2), we restrict $\alpha \in [0,1]$ as $\mathbb{D}_\alpha$ is, in this case, a Markov kernel. However, since $\mathbb{D}_\alpha \mathsf{p}_k(n) = \alpha^k \mathsf{p}_k(n)$, for all $\alpha > 0$ and $k, n \in \mathbb{Z}_+$, where p_k is defined in (3.53) below, Theorem 3.4.8, also below, yields that for all $\phi \in \mathbf{B}$ and $t \geqslant 0$, $\mathbb{Q}_t^\phi \mathbb{D}_\alpha \mathsf{p}_k(n) = \mathbb{D}_\alpha \mathbb{Q}_{\alpha t}^\phi \mathsf{p}_k(n)$, where $\mathbb{Q}^\phi$ is the discrete self-similar semigroup generated by $\mathbb{G}_\phi$. This reveals that the discrete self-similarity property also holds in a more general framework than the one given in (3.2).

A continuous-time Markov chain is called upward skip-free if it does not jump more than one step above at any instant, that is, for any $n \in \mathbb{Z}_+$ and $l \geqslant n+2$, $\mathbb{G}(n,l) = 0$ where $\mathbb{G}$ is the generator of the Markov chain. It can be easily shown that the discrete self-similar Markov chain $\mathbb{X}_\phi$ with generator $\mathbb{G}_\phi$ is upward skip-free, see (3.34) below. In the next theorem we show the converse claim that is any discrete self-similar Markov chains must be upward skip-free.

Theorem 3.2.5. *Let* $\mathbb{X}$ *be any continuous-time discrete self-similar Markov chain on* $\mathbb{Z}_+$*. Then* $\mathbb{X}$ *is upward skip-free.*

This theorem is proved in Section 3.4.2.

3.2.2 Connections with self-similar Markov processes: gateway relation and scaling limit

Self-similar Markov processes on the positive real line are well studied as they appear as the weak limits of various Markov processes, see Lamperti [75]. When these processes are spectrally negative, that is, they do not have any positive jumps, and with 0 as an entrance-non-exit boundary, Lamperti [76] showed that they are in bijection with the subset of Bernstein functions $\mathbf{B}$ defined in (3.5) and moreover, the generator of these processes are of the form

$$G_\phi f(x) = \sigma^2 x f''(x) + (m+\sigma^2) f'(x) \tag{3.8}$$

$$+ \frac{1}{x}\int_0^\infty [d_{e^{-y}} f(x) - f(x) + yxf'(x)]\Pi(dy) \tag{3.9}$$

where ϕ is defined in terms of the triplet (m, σ^2, Π), see (3.5) and $f \in \mathbf{C}_c^\infty(\mathbb{R}_+)$. The careful reader will have noticed that the operator $\mathbb{G}_\phi$ in (3.6) is the discrete analogue of the operator G_ϕ, revealing that the former is a natural approximation of the latter. However, we provide below a deeper connection between these class of Markov processes (operators) by establishing a gateway relation between their semigroups, a concept introduced in [85], meaning that the connection goes in both directions. As a by-product, we show that discrete self-similar Markov chains, after scaling appropriately, converge to the self-similar Markov processes in the Skorohod's J_1-topology.

Theorem 3.2.6. *1.* ***Gateway relation.*** *For any $\phi \in \mathbf{B}$, let Q^ϕ and $\mathbb{Q}^\phi$ denote the Feller semigroups generated by G_ϕ and $\mathbb{G}_\phi$ respectively. Then, for any $\mathsf{f} \in \mathbf{C}_0(\mathbb{Z}_+)$ and $t \geqslant 0$,*

$$Q_t^\phi \Lambda \mathsf{f} = \Lambda \mathbb{Q}_t^\phi \mathsf{f} \tag{3.10}$$

where $\Lambda \mathsf{f}(x) = \mathbb{E}[\mathsf{f}(\mathrm{Pois}(x))]$, $\mathrm{Pois}(x)$ being a Poisson random variable with parameter $x > 0$.

2. ***Scaling limit.*** *For any $\phi \in \mathbf{B}$, let $\mathbb{X}_\phi$ (resp. $X_\phi = (X_\phi(t,x))_{t \geqslant 0}$) be the discrete self-similar Markov chain (resp. the positive self-similar Markov process issued from x), then, for all*

$x > 0$,

$$\left(\frac{1}{n}\mathbb{X}_\phi(nt, \lfloor nx \rfloor)\right)_{t\geqslant 0} \longrightarrow (X_\phi(t,x))_{t\geqslant 0} \tag{3.11}$$

in Skorohod's J_1-topology.

The intertwining relation (3.10) is proved in Proposition 3.4.3(1) and the scaling limit (3.11) is proved in Section 3.4.4.

Remark 3.2.7. As mentioned to us by an anonymous referee, the gateway relationship (3.10) has the following neat probabilistic interpretation, using the notation of item (2) above,

$$\mathbb{X}_\phi(t, \text{Pois}(x)) \overset{(d)}{=} \text{Pois}(X_\phi(t,x)) \tag{3.12}$$

which is valid for any $t, x > 0$. Using the self-similarity property of X_ϕ, this identity yields, for any fixed $t, x > 0$ and large integer n (but not for $\lfloor nx \rfloor$), $\frac{1}{n}\mathbb{X}_\phi(nt, \text{Pois}(nx)) \overset{(d)}{=} \frac{1}{n}\text{Pois}(X_\phi(nt, nx)) \overset{(d)}{=} \frac{1}{n}\text{Pois}(nX_\phi(t,x)) \to X_\phi(t,x)$ in distribution. Moreover, the identity (3.12) boils down when x tends to 0 to

$$\mathbb{X}_\phi(t, 0) \overset{(d)}{=} \text{Pois}(X_\phi(t,0))$$

where $X_\phi(t,0)$ stands for the entrance law of X_ϕ which is known to exist as $m \geqslant 0$, see e.g. [95].

3.2.3 Discrete Laguerre chains from discrete self-similar Markov chains

Let us now consider a perturbation of the discrete self-similar Markov chains, that is, we introduce a new family of discrete operators on $\mathbf{C}_c(\mathbb{Z}_+)$ defined by

$$\mathbb{L}_\phi \mathsf{f}(n) = \mathbb{G}_\phi \mathsf{f}(n) + n\partial_- \mathsf{f}(n) \tag{3.13}$$

where $\phi \in \mathbf{B}$ and $\mathbb{G}_\phi$ is defined in (3.6). Alternatively, the operator $\mathbb{L}_\phi$ can be represented, for any $\mathsf{f} \in \mathbf{C}_c(\mathbb{Z}_+)$, as follows

$$\mathbb{L}_\phi \mathsf{f}(n) = \sum_{l=0}^{n+1} \mathbb{L}_\phi(n,l)\mathsf{f}(l) \tag{3.14}$$

where

$$\mathbb{L}_\phi(n,l) = \begin{cases} \mathbb{G}_\phi(n,l) & \text{if } l \neq n, n-1 \\ \mathbb{G}_\phi(n,n-1) + n & \text{if } l = n-1 \\ \mathbb{G}_\phi(n,n) - n & \text{if } l = n \end{cases} \tag{3.15}$$

with $\mathbb{G}_\phi(n,l) = \mathbb{G}_\phi\delta_l(n)$ and $\delta_l(n) = \mathbb{1}_{\{l=n\}}$.

Theorem 3.2.8. *1. For any $\phi \in \mathbf{B}$, the operator $(\mathbb{L}_\phi, \mathbf{C}_c(\mathbb{Z}_+))$ generates a Feller Markov semigroup on $\mathbf{C}_0(\mathbb{Z}_+)$, which we denote by $\mathbb{K}^\phi$.*

2. We have, for any $\mathsf{f} \in \mathbf{C}_0(\mathbb{Z}_+)$ and $t \geqslant 0$,

$$\mathbb{K}_t^\phi \mathsf{f} = \mathbb{Q}_{e^t-1}^\phi \mathbb{D}_{e^{-t}} \mathsf{f}. \tag{3.16}$$

3. The semigroup $\mathbb{K}^\phi$ has a unique invariant distribution denoted by $\mathbf{n}_\phi$ and $\mathbf{n}_\phi(n) > 0$ for all $n \in \mathbb{Z}_+$. Moreover, $\mathbf{n}_\phi$ has moments of all orders and it is moment determinate.

4. Finally, the semigroup $\mathbb{K}^\phi$ is self-adjoint in $\ell^2(\mathbf{n}_\phi)$ if and only if $\phi(u) = m + \sigma^2 u$ for some $m, \sigma^2 \geqslant 0$.

We have omitted the proof of the item (1) since it can be obtained by following a line of reasoning similar to the proof of Theorem 3.2.2 from the claims given in Proposition 3.4.11. Item (2) is proved after this latter Proposition. The properties of the invariant distribution in item (3) are proved in Proposition 3.4.11(2) and Proposition 3.4.12. Item (4) is proved in Proposition 3.4.11(4).

Remark 3.2.9. In Proposition 3.4.11 and 3.4.12, we provide additional properties, including several representations, of the invariant measure $\mathbf{n}_\phi$.

We name the Markov semigroup $\mathbb{K}^\phi$ (resp. the Markov chain) the *skip-free Laguerre semigroup* (resp. *skip-free Laguerre chain*). This is motivated by the following observation. The operator $\mathbb{L}_\phi$ can be viewed as the discrete analogue of the generalized Laguerre operator on $\mathbb{R}_+$, studied in [95], and defined by

$$\begin{aligned}L_\phi f(x) =& G_\phi f(x) - xf'(x)\\ =&\sigma^2 x f''(x) + \left(m+\sigma^2 - x\right) f'(x) \qquad (3.17)\\ &+\frac{1}{x}\int_0^\infty \left(d_{e^{-y}} f(x) - f(x) + yxf'(x)\right)\Pi(dy) \qquad (3.18)\end{aligned}$$

where G_ϕ is defined in (3.8) and (σ, β, Π) is the characteristic triplet of ϕ.

We now aim to derive the spectral properties, convergence to the equilibrium and hypercontractivity phenomenon of $\mathbb{K}^\phi$.

3.2.4 Spectral expansion and the spectrum of the skip-free Laguerre semigroups

Since the semigroup $\mathbb{K}^\phi$ has invariant distribution $\mathbf{n}_\phi$, we can extend it on the Hilbert space $\ell^2(\mathbf{n}_\phi)$. If ϕ is as in (3.5), let σ_1 be defined as follows

$$\sigma_1 = \begin{cases} \sigma^2 & \text{if } \sigma^2 > 0 \\ 1 & \text{if } \sigma^2 = 0. \end{cases} \qquad (3.19)$$

We now introduce a sequence of discrete (acting on $\mathbb{Z}_+$) polynomials defined, for $k, n \in \mathbb{Z}_+$, by

$$(1+\sigma_1^{-1})^{-\frac{k}{2}} \sum_{r=0}^{k} (-1)^r \binom{k}{r} \frac{\mathsf{p}_r(n)}{W_\phi(r+1)} \qquad (3.20)$$

where $W_\phi(k+1) = \prod_{r=1}^k \phi(r),\ W_\phi(1) = 1$ and $\mathsf{p}_r(n) = \frac{\Gamma(n+1)}{\Gamma(n+1-r)}$. Since the invariant distribution $\mathbf{n}_\phi$ has finite moments of all order, see Theorem 3.2.8(3), it is plain that, for all $k \in \mathbb{Z}_+$, $\mathsf{P}_k^\phi \in \ell^2(\mathbf{n}_\phi)$. Next, for $k, n \in \mathbb{Z}_+$, we define

$$\mathsf{V}_k^\phi(n) = \frac{(1+\sigma_1^{-1})^{\frac{k}{2}}}{\mathbf{n}_\phi(n)} \sum_{r=0}^{k\wedge n} (-1)^r \frac{(k+n-r)!}{(k-r)!(n-r)!r!} \mathbf{n}_\phi(k+n-r). \qquad (3.21)$$

Theorem 3.2.10. 1. ***Spectrum.*** *For any $\phi \in \mathbf{B}, t \geqslant 0$ and $k \in \mathbb{Z}_+$, $\mathsf{V}_k^\phi \in \ell^2(\mathbf{n}_\phi)$, and*

$$\mathbb{K}_t^\phi \mathsf{P}_k^\phi = e^{-kt}\mathsf{P}_k^\phi, \quad \widehat{\mathbb{K}}_t^\phi \mathsf{V}_k^\phi = e^{-kt}\mathsf{V}_k^\phi$$

where $\widehat{\mathbb{K}}_t^\phi$ is the $\ell^2(\mathbf{n}_\phi)$-adjoint of $\mathbb{K}_t^\phi$. Hence, $\{e^{-kt}; k \in \mathbb{Z}_+\} \subseteq \mathrm{Spec}_p(\mathbb{K}_t^\phi) \cap \mathrm{Spec}_p(\widehat{\mathbb{K}}_t^\phi)$, where for an operator T, $\mathrm{Spec}_p(T)$ denotes the point spectrum of T.

2. ***Biorthogonality.*** *$(\mathsf{P}_k^\phi)_{k\geqslant 0}$ and $(\mathsf{V}_k^\phi)_{k\geqslant 0}$ are biorthogonal sequences in $\ell^2(\mathbf{n}_\phi)$, that is, for all $k, l \in \mathbb{Z}_+$,*

$$\left\langle \mathsf{P}_k^\phi, \mathsf{V}_l^\phi \right\rangle_{\mathbf{n}_\phi} = \mathbb{1}_{\{k=l\}}.$$

3. ***Spectral expansion.*** *If $\sigma^2 > 0$, then, for all $f \in \ell^2(\mathbf{n}_\phi)$ and $t > \frac{1}{2}\log(1+\sigma^{-2})$,*

$$\mathbb{K}_t^\phi \mathsf{f} = \sum_{k=0}^{\infty} e^{-kt} \left\langle \mathsf{f}, \mathsf{V}_k^\phi \right\rangle_{\mathbf{n}_\phi} \mathsf{P}_k^\phi. \tag{3.22}$$

4. ***Compactness.*** *If $\sigma^2 > 0$, then, for all $t > \frac{1}{2}\log(1+\sigma^{-2})$, $\mathbb{K}_t^\phi$ is compact and, denoting by $\mathrm{Spec}(\mathbb{K}_t^\phi)$ the spectrum of $\mathbb{K}_t^\phi$, we have*

$$\mathrm{Spec}(\mathbb{K}_t^\phi) \setminus \{0\} = \mathrm{Spec}_p(\mathbb{K}_t^\phi) = \{e^{-kt}; k \in \mathbb{Z}_+\}.$$

5. ***Transition probabilities.*** *If $(\mathbb{K}_t^\phi(\cdot,\cdot))_{t\geqslant 0}$ denotes the transition probabilities of the skip-free Laguerre chain and $\sigma^2 > 0$, then, for all $t > \frac{1}{2}\log(1+\sigma^{-2})$ and $n, l \in \mathbb{Z}_+$, we have*

$$\mathbb{K}_t^\phi(n,l) = \sum_{k=0}^{\infty} e^{-kt}\mathsf{P}_k^\phi(n)\mathsf{V}_k^\phi(l)\mathbf{n}_\phi(l)$$

where the sum on the right-hand side of the above identity converges absolutely.

This theorem is proved in Section 3.4.12.

Remark 3.2.11. It should be noted that (1) in the above theorem is different from the result in the case of generalized Laguerre semigroups on $\mathbb{R}_+$, their continuous analogue. Indeed, from [95, Theorem 1.22(4)(d)], $e^{-kt} \in \mathrm{Spec}_p(\widehat{K}_t^\phi)$ only if $k \in \mathbb{Z}_\phi$ (see (3.88) for the definition of $\mathbb{Z}_\phi$) and $e^{-kt} \in \mathrm{Spec}_r(\widehat{K}_t^\phi)$ if $k \notin \mathbb{Z}_\phi$, where $\mathrm{Spec}_r(\mathbb{K}_t^\phi)$ stands for the residual spectrum of $\mathbb{K}_t^\phi$. However, for the discrete Laguerre semigroup $\mathbb{K}^\phi$, $e^{-kt} \in \mathrm{Spec}_p(\widehat{\mathbb{K}}_t^\phi)$ for all $k \in \mathbb{Z}_+$.

3.2.5 Convergence to equilibrium

In Theorem 3.2.8(3) we have seen that the non-self-adjoint skip-free Laguerre chains have an unique invariant distribution. In this section, we start by studying the rate of convergence to their invariant distributions via spectral gap inequality, which comes as a by-product of the spectral expansion obtained in the previous theorem. We proceed with explicit rate of convergence to equilibrium in the Φ-entropy sense, which is a consequence of a more subtle relation with the self-adjoint birth-death Laguerre chain, namely an interweaving relation discussed in Section 3.4.14. Before stating the result, let us introduce a few additional objects related to the Bernstein functions. For any $\phi \in \mathbf{B}$ let us define

$$\mathrm{d}_\phi = \min\{u \geqslant 0;\ \phi(-u) = -\infty, \phi(-u) = 0\} \in [0, \infty]. \tag{3.23}$$

If (m, σ^2, Π) is the triplet associated to ϕ, let us write

$$\mathrm{m}_\phi = \lim_{u\to\infty} \frac{\phi(u) - \sigma^2 u}{\sigma^2} = \frac{m + \overline{\overline{\Pi}}(0)}{\sigma^2} \tag{3.24}$$

where $\overline{\overline{\Pi}}(0) = \int_0^\infty \Pi(y, \infty)dy$. The quantity m_ϕ is finite whenever $\sigma^2 > 0$ and $\overline{\overline{\Pi}}(0) \in [0, \infty)$.

Next, for an open interval $I \subseteq \mathbb{R}$, we say that a function $\Phi : I \to \mathbb{R}$ is *admissible* if

$$\Phi \in \mathbf{C}^4(I) \text{ with both } \Phi \text{ and } -\frac{1}{\Phi''} \text{ convex.} \tag{3.25}$$

Given an admissible function Φ, and a probability measure μ on $\mathbb{R}$, we write for any $f : \mathbb{R}_+ \to I$ with $f, \Phi(f) \in \mathbf{L}^1(\mu)$

$$\mathrm{Ent}_\mu^\Phi(f) = \mu\Phi(f) - \Phi(\mu f) \tag{3.26}$$

for the so-called Φ-entropy of f. When $\Phi(x) = x^2, I = \mathbb{R}$, (3.26) is equal to $\mathrm{Var}_\mu(f)$ and when $\Phi(x) = x\log x, I = \mathbb{R}_+$, (3.26) yields the Boltzmann entropy of f with respect to μ. From Jensen's inequality it is plain that the Φ-entropy is always nonnegative. We are now ready to state the following.

Theorem 3.2.12. *Let $\phi \in \mathbf{B}$ be associated with the triplet (m, σ^2, Π) such that $\sigma^2, \mathrm{d}_\phi > 0$ and $\overline{\overline{\Pi}}(0) < \infty$. Then, the following holds.*

1. ***Hypocoercive estimate.*** *For all $\mathsf{f} \in \ell^2(\mathbf{n}_\phi)$ and $t \geqslant 0$, we have*

$$\left\| \mathbb{K}_t^\phi \mathsf{f} - \mathbf{n}_\phi \mathsf{f} \right\|_{\ell^2(\mathbf{n}_\phi)} \leqslant \sqrt{\frac{(\mathrm{m}_\phi + 1)(1+\sigma^2)}{\sigma^2(\mathrm{d}_\phi + 1)}} e^{-t} \|\mathsf{f} - \mathbf{n}_\phi \mathsf{f}\|_{\ell^2(\mathbf{n}_\phi)}. \tag{3.27}$$

2. ***Entropy decay.*** *For all $\beta > \mathrm{m}_\phi$, $t \geqslant 0$ and f such that $\mathsf{f}, \Phi(\mathsf{f}) \in \ell^1(\mathbf{n}_\phi)$, we have*

$$\mathrm{Ent}_{\mathbf{n}_\phi}^{\Phi} \left(\mathbb{K}_{t+\tau_\beta}^\phi \mathsf{f} \right) \leqslant e^{-t}\, \mathrm{Ent}_{\mathbf{n}_\phi}^{\Phi}(\mathsf{f}) \tag{3.28}$$

where, we recall that $\mathbb{K}_{t+\tau_\beta}^\phi \mathsf{f}(n) = \mathbb{E}[\mathsf{f}(\mathbb{X}_\phi(t+\tau_\beta, n)]$ and τ_β is an infinitely divisible positive random variable whose Laplace transform is given by

$$\int_0^\infty e^{-us} \mathbb{P}(\tau_\beta \in ds) = e^{-\phi_\beta(u)}, \quad u > 0, \tag{3.29}$$

with $\phi_\beta(u) = u \log(1+\sigma^{-2}) + \log\left(\frac{\Gamma(u+\beta+1)}{\Gamma(1+\beta)\Gamma(u+1)}\right)$.

Item (1) of the above theorem is proved in Section 3.4.13 and item (2) is proved in Section 3.4.16.

Remark 3.2.13. The estimate in (1) gives the hypocoercivity, in the sense of Villani [129], for the skip-free Laguerre semigroups. This notion continues to attract a lot of interests, especially in the area of kinetic Fokker–Planck equations; see e.g. Baudoin [10] and Dolbeault et al. [39] and the references therein. Unlike this literature, we are able to identify the hypocoercive constants, namely the exponential decay rate as the spectral gap and the constant in front of the exponential, which is greater than 1 as with $\sigma^2, \mathrm{d}_\phi > 0$ we have $\mathrm{m}_\phi > \mathrm{d}_\phi$, is a measure of the deviation of the spectral projections from forming an orthogonal basis. Note that in general, the hypocoercive constants may be difficult to identify and may have little to do with the spectrum. Results in the spirit of (1) have already been obtained by Achleitner et al. [1], Patie and Savov [95] as well as in Patie and Vaidyanathan [97] where a general framework based on intertwining relation is developed.

3.2.6 Hypercontractivity

A Markov semigroup defined on the state space E with invariant distribution μ is said to be hypercontractive if there exists $\alpha > 0$ such that

$$\|P_t\|_{\mathbf{L}^2(E,\mu)\to\mathbf{L}^{p(\alpha t)}(E,\mu)} \leqslant 1$$

where $p(t) = 1 + e^t$ and

$$\|P_t\|_{\mathbf{L}^2(E,\mu)\to\mathbf{L}^{p(\alpha t)}(E,\mu)} = \sup_{f:\|f\|_{\mathbf{L}^2(E,\mu)}=1} \|P_t f\|_{\mathbf{L}^{p(\alpha t)}(E,\mu)}.$$

It is readily seen that the hypercontractivity reflects the regularity of the semigroup. For self-adjoint Markov semigroups, hypercontractivity can be interpreted in terms of their (modified) log-Sobolev constants, see [7, Theorem 5.2.3] and references therein. Nonetheless, even for the self-adjoint birth-death Laguerre chain, it is difficult to obtain a precise value of the (modified) log-Sobolev constant. Using the concept of interweaving, see Section 3.4.14, we circumvent this issue, and in fact, we are able to obtain the hypercontractivity estimates for (non self-adjoint) skip-free Laguerre semigroups up to a random warm-up time.

Theorem 3.2.14. *If $\sigma^2 > 0$ and $\overline{\overline{\overline{\Pi}}}(0) < \infty$, then, for all $\beta > \mathrm{m}_\phi = \frac{m+\overline{\overline{\overline{\Pi}}}(0)}{\sigma^2}$ and $t \geqslant 0$,*

$$\left\|\mathbb{K}^{\phi}_{t+\tau_\beta}\right\|_{\ell^2(\mathbf{n}_\phi)\to\ell^{p(t)}(\mathbf{n}_\phi)} \leqslant 1$$

where τ_β is defined in (3.29).

This theorem is proved in Section 3.4.17.

3.2.7 Bochner subordination of skip-free Laguerre chains

In the previous two sections we have seen that Theorem 2 and Theorem 3.2.14 hold for skip-free Laguerre semigroups up to a random warm-up or delay time denoted by τ_β. However, applying a time-change on the skip-free Laguerre chains, we can obtain a new class of skip-free Markov chains for which the above theorems hold with a deterministic warm-up or delay time. In other words, we obtain a new class of Markov chains (with two sided-jumps of arbitrary size) for which the quantity τ_β can be replaced by a deterministic number. Since τ_β is an infinitely divisible random variable with $\phi_\beta \in \mathbf{B}$ as its Lévy-Khintchine exponent, one can consider the subordinator $(\tau_\beta(t), t \geqslant 0)$ such that $\tau_\beta(1) \overset{(d)}{=} \tau_\beta$. With an abuse of notation, we still denote this subordinator by τ_β. Now, let us consider the subordinated Laguerre semigroup defined by

$$\mathbb{K}_t^{\phi,\tau_\beta} = \int_0^\infty \mathbb{K}_s^\phi \, \mathbb{P}(\tau_\beta(t) \in ds). \tag{3.30}$$

Since $\lim_{t\to\infty} \tau_\beta(t) = \infty$ almost surely, the semigroup $\mathbb{K}^{\phi,\tau_\beta}$ has the same invariant measure $\mathbf{n}_\phi$. Below, we provide the spectral expansion, the Φ-entropy convergence and the hypercontractivity property of $\mathbb{K}^{\phi,\tau_\beta}$.

Theorem 3.2.15. *Let $\phi \in \mathbf{B}$ be associated with the triplet (m, σ^2, Π).*

1. ***Spectral Expansion.*** *If $\sigma^2 > 0$ then for all $\beta > 0$, $\mathsf{f} \in \ell^2(\mathbf{n}_\phi)$ and $t > \frac{1}{2}$ we have*

$$\mathbb{K}_t^{\phi,\tau_\beta}\mathsf{f} = \sum_{k=0}^{\infty} e^{-t\phi_\beta(k)} \langle \mathsf{f}, \mathsf{V}_k^\phi \rangle_{\mathbf{n}_\phi} \mathsf{P}_k^\phi.$$

2. *Φ**-entropy decay.** If $\sigma^2 > 0$, $\overline{\overline{\overline{\Pi}}}(0) < \infty$ and $\beta > \mathrm{m}_\phi$, then for all admissible (see (3.25) for definition) function Φ and f such that $\mathsf{f}, \Phi(\mathsf{f}) \in \ell^1(\mathbf{n}_\phi)$, we have, for all $t \geqslant 0$,*

$$\mathrm{Ent}_{\mathbf{n}_\phi}^\Phi \left(\mathbb{K}_t^{\phi,\tau_\beta}\mathsf{f} \right) \leqslant e^{-\phi_\beta(1)(t-1)_+} \, \mathrm{Ent}_{\mathbf{n}_\phi}^\Phi(\mathsf{f})$$

   where $t_+ = \max(t, 0)$.

3. ***Hypercontractivity.*** *If $\sigma^2 > 0$, $\overline{\overline{\overline{\Pi}}}(0) < \infty$ and $\beta > \mathrm{m}_\phi$, then, for all $t \geqslant 0$,*

$$\left\| \mathbb{K}_{t+1}^{\phi,\tau_\beta} \right\|_{\ell^2(\mathbf{n}_\phi) \to \ell^{q(t)}(\mathbf{n}_\phi)} \leqslant 1$$

   where $q(t) = 1 + (1 + \sigma^{-2})^t$.

This theorem is proved in Section 3.4.18.

3.3 Examples

3.3.1 The discrete Laguerre chains and the Meixner polynomials

Let us consider the Bernstein function $\phi(u) = \sigma^2 u + m$ where $\sigma^2 > 0, m \geqslant 0$. If $\mathbb{K}^\phi$ is the skip-free Laguerre semigroup associated with ϕ, then the generator is given by

$$\mathbb{L}_\phi(n,l) = \begin{cases} \sigma^2 n + m + \sigma^2 + 1 & \text{if } l = n+1 \\ (1+\sigma^2)n & \text{if } l = n-1 \\ -(1+2\sigma^2)n - m - \sigma^2 - 1 & \text{if } l = n \\ 0 & \text{otherwise.} \end{cases}$$

From Proposition 3.4.12, the unique invariant distribution of $\mathbb{L}_\phi$ is given by

$$\mathbf{n}_\phi(n) = \frac{\Gamma\left(n + \frac{m}{\sigma^2} + 1\right)}{\Gamma\left(\frac{m}{\sigma^2} + 1\right) n!} 2^{-n-\frac{m}{\sigma^2}-1}, \; n \in \mathbb{Z}_+.$$

The semigroup $\mathbb{K}^\phi$ is self-adjoint in $\ell^2(\mathbf{n}_\phi)$ and it follows from Theorem 3.2.10 that the eigenfunctions P_k^ϕ of $\mathbb{K}_t^\phi$ corresponding to its eigenvalue e^{-kt} form an orthogonal sequence in $\ell^2(\mathbf{n}_\phi)$. More specifically, writing $\beta = \frac{m}{\sigma^2}$, for all $k \in \mathbb{Z}_+$,

$$\begin{aligned} \mathsf{P}_k^\phi(n) &= \left(1+\sigma^{-2}\right)^{-\frac{k}{2}} \Gamma(\beta+1) \sum_{r=0}^{k} (-\sigma)^{-2r} \binom{k}{r} \frac{\mathsf{p}_r(n)}{\Gamma(r+\beta+1)} \\ &= \left(1+\sigma^{-2}\right)^{-\frac{k}{2}} {}_2F_1\left(-n, -k, \beta+1; -\sigma^{-2}\right) \end{aligned}$$

where

$${}_2F_1(a,b;c;x) = \frac{\Gamma(c)}{\Gamma(a)\Gamma(b)} \sum_{r=0}^{\infty} \frac{\Gamma(r+a)\Gamma(r+b)}{\Gamma(r+c)} \frac{x^r}{r!}. \tag{3.31}$$

From [63, Equation (7)], it follows that

$$\left\|\mathsf{P}_k^\phi\right\|_{\ell^2(\mathbf{n}_\phi)}^2 = \mathfrak{c}_k\left(\beta\right)^{-1}$$

where for any $a > 0$, $\mathfrak{c}_k(a) = \frac{\Gamma(a+k+1)}{\Gamma(a+1)\Gamma(k+1)}$. Finally, for all $\mathsf{f} \in \ell^2(\mathbf{n}_\phi)$ and $t > 0$, we have, in $\ell^2(\mathbf{n}_\phi)$,

$$\mathbb{K}_t^\phi \mathsf{f} = \sum_{k=0}^{\infty} \mathfrak{c}_k\left(\frac{m}{\sigma^2}\right) e^{-kt} \left\langle \mathsf{f}, \mathsf{P}_k^\phi \right\rangle_{\mathbf{n}_\phi} \mathsf{P}_k^\phi.$$

3.3.2 The perturbed Laguerre skip-free chain

Consider the Bernstein function defined for $\mathfrak{m} > 1$ by

$$\phi_{\mathfrak{m}}(u) = \frac{(u+\mathfrak{m}+1)(u+\mathfrak{m}-1)}{u+\mathfrak{m}} = \frac{\mathfrak{m}^2-1}{\mathfrak{m}} + u + \int_0^\infty (1-e^{-uy})e^{-\mathfrak{m}y}dy.$$

Let $\mathbb{G}_{\phi_{\mathfrak{m}}}$ be the generator of the discrete self-similar Markov semigroup associated with $\phi_{\mathfrak{m}}$. Then, according to (3.6), $\sigma^2 = 1, m = \frac{\mathfrak{m}^2-1}{\mathfrak{m}}$ and $\Pi(dy) = \mathfrak{m}e^{-\mathfrak{m}y}dy$. So, the infinitesimal generator $\mathbb{G}_{\mathfrak{m}}$ is given by

$$\mathbb{G}_{\phi_{\mathfrak{m}}}(n,l) = \begin{cases} \frac{\mathfrak{m}\Gamma(l+\mathfrak{m})\Gamma(n-l+2)}{(n+1)\Gamma(n+\mathfrak{m}+2)} & \text{if } l \in [\![0, n-2]\!] \\ \frac{2\mathfrak{m}}{(n+1)(n+\mathfrak{m})(n+\mathfrak{m}+1)} + n & \text{if } l = n-1 \\ \mathfrak{m} - \frac{1}{\mathfrak{m}+n+1} & \text{if } l = n+1 \\ \frac{\mathfrak{m}}{(n+\mathfrak{m})(n+\mathfrak{m}+1)} - \frac{1}{\mathfrak{m}} & \text{if } l = n \\ 0 & \text{if } l > n+1. \end{cases}$$

Now, the corresponding skip-free Laguerre chain has the generator $\mathbb{L}_{\phi_{\mathfrak{m}}}$ given by

$$\mathbb{L}_{\phi_{\mathfrak{m}}}(n,l) = \begin{cases} \mathbb{G}_{\mathfrak{m}}(n,l) & \text{if } l \neq n, n-1 \\ \mathbb{G}_{\mathfrak{m}}(n,n-1) + n & \text{if } l = n-1 \\ \mathbb{G}_{\mathfrak{m}}(n,n) - n & \text{if } l = n. \end{cases}$$

From Proposition 3.4.12, the unique invariant distribution of $\mathbb{L}^{\phi_{\mathfrak{m}}}$ is given by

$$\mathbf{n}_{\phi_{\mathfrak{m}}}(n) = \frac{(n+\mathfrak{m}+1)\Gamma(n+\mathfrak{m})}{(\mathfrak{m}+1)\Gamma(\mathfrak{m})n!} 2^{-(n+\mathfrak{m}+1)}, \ n \in \mathbb{Z}_+.$$

Let us compute the eigenfunctions and co-eigenfunctions of the semigroup $\mathbb{K}^{\phi_{\mathfrak{m}}}$ generated by $\mathbb{L}^{\phi_{\mathfrak{m}}}$. Denoting the eigenfunction (resp. the co-eigenfunction) of $\mathbb{K}_t^{\phi_{\mathfrak{m}}}$ corresponding to the eigenvalue (resp. co-eigenvalue) e^{-kt} by $\mathsf{P}_k^{\phi_{\mathtt{m}}}$ (resp. $\mathsf{V}_k^{\phi_{\mathtt{m}}}$), we have

$$\begin{aligned}
\mathsf{P}_k^{\phi_{\mathtt{m}}}(n) &= 2^{-\frac{k}{2}} \sum_{r=0}^{k} (-1)^r \frac{\binom{k}{r}}{W_{\phi_{\mathtt{m}}}(r+1)} \mathsf{p}_r(n) \\
&= 2^{-\frac{k}{2}} \left[(\mathfrak{m}+1)_2F_1\left(-k,-n,\mathfrak{m}+1;-1\right) - {}_2F_1\left(-k,-n,\mathfrak{m}+2;-1\right) \right], \\
\mathsf{V}_k^{\phi_{\mathtt{m}}}(n) &= \frac{2^{-\frac{k}{2}}}{\mathbf{n}_\phi(n)} \frac{\Gamma(n+k+\mathfrak{m})}{k!n!} \left((n+\mathfrak{m}+k)_2F_1(-k,-n,-n-k-\mathfrak{m};2) \right. \\
&\quad \left. + {}_2F_1(-k,-n,-n-k-\mathfrak{m}+1;2) \right)
\end{aligned}$$

where ${}_2F_1$ is the hypergeometric function defined in (3.31).

3.3.3 The Beta skip-free chain

We consider the Bernstein function $\phi_{\mathtt{m}}$ corresponding to a compound Poisson process with exponential jumps which is defined, for $\mathtt{m} > 1$ and $u > 0$, by

$$\phi_{\mathtt{m}}(u) = \frac{u}{\mathtt{m}(u+\mathtt{m})} = \int_0^\infty (1-e^{-uy})e^{-\mathtt{m}y}dy.$$

Therefore, according to (3.5), $\sigma^2 = 0, m = 0, \Pi(dy) = \mathtt{m}e^{-\mathtt{m}y}dy$ and $\phi_{\mathtt{m}}(\infty) = \frac{1}{\mathtt{m}}$. If $L_{\phi_{\mathtt{m}}}$ denotes the generator of the Laguerre semigroup corresponding to $\phi_{\mathtt{m}}$ in continuous state space, we have for all $f \in \mathbf{C}_c^\infty(\mathbb{R}_+)$,

$$L_{\phi_{\mathtt{m}}}f(x) = -xf'(x) + \frac{\mathtt{m}}{x}\int_0^\infty \left(f(e^{-y}x) - f(x) + yxf'(x)\right)e^{-\mathtt{m}y}dy.$$

The Bernstein-gamma function associated with $\phi_{\mathtt{m}}$ is

$$W_{\phi_{\mathtt{m}}}(k+1) = \frac{\Gamma(\mathtt{m}+1)\Gamma(k+1)}{\mathtt{m}^k\Gamma(k+1+\mathtt{m})},\ k \in \mathbb{Z}_+.$$

From [95, Proposition 2.6(1)], the semigroup generated by $L_{\phi_{\mathtt{m}}}$ admits an unique invariant measure $\nu_{\phi_{\mathtt{m}}}$ which is absolutely continuous with moment sequence $(W_{\phi_{\mathtt{m}}}(k+1))_{k\geqslant 0}$ and given by

$$\nu_{\phi_{\mathtt{m}}}(dx) = \mathtt{m}^2(1-\mathtt{m}x)^{\mathtt{m}-1}dx,\ 0 < x < \frac{1}{\mathtt{m}}.$$

Now coming back to the corresponding skip-free Laguerre chain in the discrete state space, (3.70) implies that the unique invariant distribution of its semigroup $\mathbb{K}^{\phi_{\mathtt{m}}}$ is

$$\begin{aligned}\mathbf{n}_{\phi_{\mathtt{m}}}(n) =&\frac{1}{n!}\sum_{r=0}^{\infty} W_{\phi_{\mathtt{m}}}(n+r+1)\frac{(-1)^r}{r!}\\ =&\frac{1}{n!}\sum_{r=0}^{\infty}\frac{\Gamma(\mathtt{m}+1)\Gamma(n+r+1)}{\mathtt{m}^{n+r}\Gamma(n+r+\mathtt{m}+1)}\frac{(-1)^r}{r!}\\ =&\frac{1}{\mathtt{m}^n}\frac{\Gamma(\mathtt{m}+1)}{\Gamma(n+\mathtt{m}+1)}{}_1F_1\left(n, n+\mathtt{m};\frac{1}{\mathtt{m}}\right)\end{aligned}$$

where ${}_1F_1$ is an hypergeometric function. Finally, from Proposition 3.4.17, the eigenfunction of $\mathbb{K}_t^{\phi_{\mathtt{m}}}$ corresponding to e^{-kt} is given by

$$\mathsf{P}_k^{\phi_{\mathtt{m}}}(n) = 2^{-\frac{k}{2}}\sum_{r=0}^{k}(-1)^r\binom{k}{r}\frac{\mathsf{p}_r(n)}{W_{\phi_{\mathtt{m}}}(r+1)} = 2^{\frac{k}{2}}{}_3F_1(-k,-n,\mathtt{m}+1;1;-\mathtt{m})$$

where ${}_3F_1(a,b,c;d;x) = \frac{\Gamma(d)}{\Gamma(a)\Gamma(b)\Gamma(c)}\sum_{r=0}^{\infty}\frac{\Gamma(r+a)\Gamma(r+b)\Gamma(r+c)}{\Gamma(r+d)}\frac{x^r}{r!}$ and $\mathsf{V}_k^{\phi_{\mathtt{m}}}$ is given by (3.21).

3.4 Proof of the Main Results

We begin this section with some useful facts about to the discrete dilation operator.

Proposition 3.4.1. *1. For all $\alpha > 0$ and $\mathsf{f} \in \mathbf{C}_b(\mathbb{Z}_+)$,*

$$d_\alpha \Lambda \mathsf{f} = \Lambda \mathbb{D}_\alpha \mathsf{f} \tag{3.32}$$

where $d_\alpha f(x) = f(\alpha x)$ is the usual dilation operator on $\mathbb{R}_+$ and Λ is as in Theorem 3.2.6.

2. *For all $\mathsf{f} \in \mathbf{C}(\mathbb{Z}_+)$, and $\alpha, \beta > 0$, $\mathbb{D}_{\alpha\beta}\mathsf{f} = \mathbb{D}_\alpha\mathbb{D}_\beta\mathsf{f}$.*

3. *$\mathbb{D}_1 = \mathrm{Id}$ and for all $\alpha > 0$, $\mathbb{D}_\alpha^{-1} = \mathbb{D}_{1/\alpha}$.*

4. *$(\mathbb{D}_{e^{-t}})_{t\geqslant 0}$ (resp. $(d_{e^{-t}})_{t\geqslant 0}$ form a semigroup on $\ell^2(\mathbb{Z}_+)$ (resp. on $\mathbf{L}^2(\mathbb{R}_+)$) with generator $\partial_-^n = n\partial_-$ (resp. $\partial^x = -x\frac{d}{dx}$) with $\partial^x\Lambda = \Lambda\partial_-^n$ on $\mathbf{C}_c(\mathbb{Z}_+)$.*

Remark 3.4.2. There are several analogies between the two dilation operators which make our choice of the discrete one natural. Indeed, as its continuous analogue, the discrete dilation operator is a multiplicative semigroup, and, the generator of its associated additive semigroup is the discrete analogue of the continuous one, see item 4. However, unlike in the continuous case, $\mathbb{D}_\alpha$ does not have bounded inverse when $\alpha \in (0,1)$.

Proof. The first item follows from [84, Proposition 1]. Next, we note that, for any $\mathsf{f} \in \mathbf{C}_b(\mathbb{Z}_+)$ and $\alpha > 0$,

$$|\mathbb{D}_\alpha \mathsf{f}(n)| \leqslant |(2\alpha - 1)|^n \|\mathsf{f}\|_\infty. \tag{3.33}$$

Then, (3.33) implies that both $\Lambda\mathbb{D}_\alpha\mathsf{f}$ and $d_\alpha\Lambda\mathsf{f}$ are well defined. Now, (1) yields

$$\Lambda\mathbb{D}_{\alpha\beta} = d_{\alpha\beta}\Lambda = d_\alpha d_\beta \Lambda = d_\alpha \Lambda \mathbb{D}_\beta = \Lambda\mathbb{D}_\alpha\mathbb{D}_\beta.$$

Since $\Lambda : \mathbf{C}_b(\mathbb{Z}_+) \to \mathbf{C}_b(\mathbb{R}_+)$ is injective, see [84, Lemma 4(4))], item (2) follows. Item (3) is a direct consequence of item (2). For item (4), it is immediate from item (2) that $(\mathbb{D}_{e^{-t}})_{t\geqslant 0}$ is a translation semigroup on $\mathbf{C}_0(\mathbb{Z}_+)$. Moreover, from (3.32) we have that for all $t \geqslant 0$ and $\mathsf{f} \in \mathbf{C}_0(\mathbb{Z}_+)$

$$d_{e^{-t}}\Lambda\mathsf{f} = \Lambda\mathbb{D}_{e^{-t}}\mathsf{f}.$$

Differentiating the above identity with respect to t when $\mathsf{f} \in \mathbf{C}_c(\mathbb{Z}_+)$ and noting that $d_{e^{-t}} = e^{t\partial^x}$ one obtains that

$$\partial^x \Lambda\mathsf{f} = \Lambda \frac{d}{dt}\mathbb{D}_{e^{-t}}\mathsf{f}_{|t=0}$$

where we used that Λ is a bounded operator. However, from [84, Lemma 4.5], after observing that $\Lambda = \nabla^{-1}$, we have $\partial^x\Lambda\mathsf{f} = \Lambda\partial_-^n\mathsf{f}$ whenever $\mathsf{f} \in \mathbf{C}_c(\mathbb{Z}_+)$. Since Λ is injective on $\mathbf{C}_c(\mathbb{Z}_+)$, we conclude that for all $\mathsf{f} \in \mathbf{C}_c(\mathbb{Z}_+)$ one has

$$\frac{d}{dt}\mathbb{D}_{e^{-t}}\mathsf{f}_{|t=0} = \partial_-^n\mathsf{f}$$

which proves item (4).

3.4.1 Proof of Theorem 3.2.2

It is not difficult to see that the operator $\mathbb{G}_\Pi$ can be simplified as follows. We can write $\mathbb{G}_\Pi f(n) = \sum_{l=0}^{n+1} \mathbb{G}(n,l)\mathsf{f}(l)$ where

$$\mathbb{G}_\Pi(n,l) = \begin{cases} \int_0^\infty \frac{1}{n+1}\binom{n+1}{l} e^{-ly}(1-e^{-y})^{n-l+1}\Pi(dy) & \text{if } l \in [\![0, n-1]\!] \\ \int_0^\infty \frac{1}{n+1}\left(e^{-(n+1)y} - 1 + (n+1)y\right)\Pi(dy) & \text{if } l = n+1 \\ 0 & \text{if } l > n+1 \end{cases} \tag{3.34}$$

and $\mathbb{G}_\Pi(n,n) = -\sum_{l\neq n} \mathbb{G}_\Pi(n,l)$.

To show that $\mathbb{G}_\phi$ is a Markov generator, we need to show that $\mathbb{G}_\phi(n,l) \geqslant 0$ for all $n \neq l$. From the expression of $\mathbb{G}_\phi$ in (3.6), it is enough to show that $\mathbb{G}_\Pi(n,l) \geqslant 0$ for all $l \neq n$, a fact which follows readily from (3.34). To get that $\mathbb{G}_\phi$ generates a Feller semigroup on $\mathbf{C}_0(\mathbb{Z}_+)$, we wish to combine Theorem 3.2 with Corollary 3.2 from [45, Chapter 8]. To this end, the following four conditions need to be checked

(i) $\sup_{n\in\mathbb{Z}_+} \frac{|\mathbb{G}_\phi(n,n)|}{n+1} < \infty$

(ii) $\lim_{n\to\infty} \mathbb{G}_\phi(n,l) = 0$ for all $l \in \mathbb{Z}_+$

(iii) $\sup_{n\in\mathbb{Z}_+} \sum_{l\in\mathbb{Z}_+} \frac{n+1}{l+1}\mathbb{G}_\phi(n,l) < \infty$

(iv) $\sup_{n\in\mathbb{Z}_+} \frac{1}{n+1} \sum_{l\in\mathbb{Z}_+} (l-n)\mathbb{G}_\phi(n,l) < \infty$.

First, we note that

$$\mathbb{G}_\phi(n,l) = \begin{cases} \sigma^2(n+1) + m + \mathbb{G}_\Pi(n,n+1) & \text{if } l = n+1 \\ \sigma^2 n + \mathbb{G}_\Pi(n,n-1) & \text{if } l = n-1 \\ -2\sigma^2 n - \sigma^2 - m + \mathbb{G}_\Pi(n,n) & \text{if } l = n \\ \mathbb{G}_\Pi(n,l) & \text{otherwise.} \end{cases}$$

It is plainly sufficient to check all four conditions above for $\mathbb{G}_\Pi$ merely. From the definition of $\mathbb{G}_\Pi(n,n)$, we get that, for all $n \in \mathbb{Z}_+$,

$$\mathbb{G}_\Pi(n,n) = -\int_0^\infty \frac{1}{n+1}\left(1 - \sum_{l=0}^{n-1}\binom{n+1}{l} e^{-ly}(1-e^{-y})^{n-l+1}\right)\Pi(dy) \\ - \int_0^\infty \frac{1}{n+1}(1 - e^{-(n+1)y} + (n+1)y)\Pi(dy).$$

Since for any $y > 0$, $\sum_{l=0}^{n+1}\binom{n+1}{l} e^{-ly}(1-e^{-y})^{n+1-l} = 1$, the above expression reduces to

$$\mathbb{G}_\Pi(n,n) = \int_0^\infty (e^{-ny} - e^{-(n+1)y} - y)\Pi(dy). \tag{3.35}$$

Next, noting that

$$|e^{-ny} - e^{-(n+1)y} - y| = \left|\int_n^{n+1} y(1-e^{-ry})dr\right| \\ \leqslant (2n+1)y^2\mathbb{1}_{\{y\leqslant 1\}} + y\mathbb{1}_{\{y>1\}},$$

the integral in (3.35) is finite due to (3.4) and therefore,

$$\overline{\lim_{n\to\infty}} \frac{|\mathbb{G}_\phi(n,n)|}{n+1} \leqslant 2\sigma^2 + 2\int_0^1 y^2\Pi(dy) < \infty.$$

This verifies condition (i). Then, for any $l \in \mathbb{Z}_+$ and sufficiently large n,

$$\mathbb{G}_\Pi(n,l) = \frac{1}{n+1}\int_0^\infty \binom{n+1}{l} e^{-ly}(1-e^{-y})^{n-l+1}\Pi(dy).$$

When $l = 0$,

$$\mathbb{G}_\Pi(n,0) = \frac{1}{n+1}\int_0^\infty (1-e^{-y})^{n+1}\Pi(dy) \tag{3.36}$$

and clearly $n \mapsto \int_0^\infty (1-e^{-y})^{n+1}\Pi(dy)$ is a decreasing sequence. Thus, $\lim_{n\to\infty}\mathbb{G}_\Pi(n,0) = 0$. When $l \geqslant 1$, let us define, for all $n \in \mathbb{Z}_+$ with $n \geqslant l+1$,

$$a_n = \int_0^1 e^{-ly}(1-e^{-y})^{n-l+1}\Pi(dy),$$

$$b_n = \int_1^\infty e^{-ly}(1-e^{-y})^{n-l+1}\Pi(dy).$$

We note that both a_n, b_n are well defined if $n \geqslant l+1$. Since, for all $n \geqslant l+1$, $a_{n+1} \leqslant (1-e^{-1})a_n$, we have that $a_n \leqslant a_{l+1}(1-e^{-1})^{n-l-1}$, and thus

$$\lim_{n\to\infty} \binom{n+1}{l} a_n = 0.$$

On the other hand, observing that, for any $y > 0$,

$$\lim_{n\to\infty} \binom{n+1}{l} e^{-ly}(1-e^{-y})^{n-l+1} = 0$$
$$\sup_{n\geqslant 1} \binom{n+1}{l} e^{-ly}(1-e^{-y})^{n+1-l} \leqslant 1,$$

a dominated convergence argument entails that

$$\lim_{n\to\infty} \binom{n+1}{l} b_n = 0$$

which verifies condition (ii). For condition (iii), we first observe that for any $y > 0$ and $l, n \in \mathbb{Z}_+$ with $l \leqslant n+1$, the following identity

$$\frac{1}{n+2}\sum_{j=1}^{n+1}(1-e^{-y})^j - \frac{y}{n+2} = \sum_{l=0}^{n-1}\frac{1}{l+1}\binom{n+1}{l}e^{-ly}\left(1-e^{-y}\right)^{n-l+1} + \frac{\left(e^{-(n+1)y}-1+(n+1)y\right)}{n+2} + \left(e^{-ny}-e^{-(n+1)y}-y\right)$$

holds. As a result of the above identity and invoking (3.34) one gets

$$\sum_{l=0}^{n+1}\frac{n+1}{l+1}\mathbb{G}_\Pi(n,l) = \frac{1}{n+2}\int_0^\infty (1-e^{-y}-y) + \frac{1}{n+2}\sum_{j=2}^{n+1}(1-e^{-y})^j\Pi(dy). \tag{3.37}$$

Since for $y > 0$, $|1-e^{-y}-y| \leqslant y \wedge \frac{y^2}{2}$, using (3.4), we get

$$\int_0^\infty |1-e^{-y}-y|\Pi(dy) \leqslant \int_0^1 \frac{y^2}{2}\Pi(dy) + \int_1^\infty y\Pi(dy) < \infty$$

and, for all $2 \leqslant j \leqslant n+1$,

$$\int_0^\infty (1-e^{-y})^j\Pi(dy) \leqslant \int_0^1 y^2\Pi(dy) + \int_1^\infty \Pi(dy) < \infty.$$

Therefore, condition (iii) is satisfied as well. Finally, the last condition follows since, plainly,

$$\lim_{n\to\infty}\frac{1}{n+1}\sum_{l=0}^{n+1}(l-n)\mathbb{G}_\Pi(n,l) = \lim_{n\to\infty}\frac{1}{n+1}\int_0^\infty (e^{-y}-1+y)\Pi(dy) = 0.$$

Therefore, $\mathbb{G}_\phi$ generates a Feller semigroup on $\mathbf{C}_0(\mathbb{Z}_+)$ with $\mathbf{C}_c(\mathbb{Z}_+)$ as its core.

Next, to prove the discrete self-similarity property of the generated semigroup above, we need the following.

Proposition 3.4.3. *1. Let Q^ϕ and $\mathbb{Q}^\phi$ denote the Feller semigroups generated by G_ϕ and $\mathbb{G}_\phi$ respectively. Then, for any $\mathsf{f} \in \mathbf{C}_0(\mathbb{Z}_+)$ and for all $t \geqslant 0$,*

$$Q_t^\phi \Lambda \mathsf{f} = \Lambda \mathbb{Q}_t^\phi \mathsf{f} \tag{3.38}$$

where we recall that $\Lambda \mathsf{f}(x) = \mathbb{E}[\mathsf{f}(\mathrm{Pois}(x))]$ with $\mathrm{Pois}(x)$ a Poisson random variable with parameter $x > 0$.

2. The counting measure on $\mathbb{Z}_+$, denoted by $\mathbb{m}$, is an excessive measure for the semigroup $\mathbb{Q}^\phi$. Hence $\mathbb{Q}^\phi$ can be extended uniquely to a strongly continuous contraction semigroup on $\ell^2(\mathbb{Z}_+)$, which we again denote by $\mathbb{Q}^\phi$.

3. The operator Λ can be extended uniquely to an operator (also denoted by Λ) in $\mathscr{B}(\ell^2(\mathbb{Z}_+), \mathbf{L}^2(\mathbb{R}_+))$. Keeping the same notation for the extension of Q^ϕ on $\mathbf{L}^2(\mathbb{R}_+)$, we have, for all $\mathsf{f} \in \ell^2(\mathbb{Z}_+)$ and $t \geqslant 0$,

$$Q_t^\phi \Lambda \mathsf{f} = \Lambda \mathbb{Q}_t^\phi \mathsf{f}. \tag{3.39}$$

Moreover, Λ is a quasi-affinity, that is, it is bounded, injective and has dense range.

We split its proof into several parts.

Proof of Proposition 3.4.3(1)

First, let us write

$$\begin{aligned} G_\phi &= G_{m,\sigma^2} + G_\Pi = \sigma^2 x \frac{d^2}{dx^2} + (m + \sigma^2)\frac{d}{dx} + G_\Pi \\ \mathbb{G}_\phi &= \mathbb{G}_{m,\sigma^2} + \mathbb{G}_\Pi = (\sigma^2 n + m + \sigma^2)\partial_+ + n\partial_- + \mathbb{G}_\Pi \end{aligned} \tag{3.40}$$

where, for all $f \in \mathbf{C}_b^2(\mathbb{R}_+)$,

$$G_\Pi f(x) = \frac{1}{x}\int_0^\infty (f(xe^{-y}) - f(x) + yxf'(x))\Pi(dy).$$

Let $\mathbf{P}_\mathfrak{e}$ be the vector space of functions defined on $\mathbb{R}_+$ which are of the form $e^{-x}P(x)$, P being a polynomial. We define the linear operator $\nabla : \mathbf{P}_\mathfrak{e} \to \mathbf{C}_c(\mathbb{Z}_+)$ as follows

$$\nabla f(n) = \frac{d^n}{dx^n}(e^x f(x))(0). \tag{3.41}$$

Lemma 3.4.4. *For any $f \in \mathbf{P}_\mathfrak{e}$,*

$$\mathbb{G}_\phi \nabla f = \nabla G_\phi f.$$

Proof. From [84, Lemma 3], it is known that, for all $f \in \mathbf{P}_\mathfrak{e}$,

$$\mathbb{G}_{m,\sigma^2}\nabla f = \nabla G_{m,\sigma^2} f.$$

Thus, it suffices to prove this lemma replacing $G_\phi, \mathbb{G}_\phi$ by $G_\Pi, \mathbb{G}_\Pi$ respectively. For $y > 0$, let δ_y denote the Dirac measure at y. Taking $\Pi = \delta_y$ and writing G_{δ_y} and $\mathbb{G}_{\delta_y}$ simply as G_y, $\mathbb{G}_y$ respectively, we get

$$G_y f(x) = \frac{1}{x}(f(xe^{-y}) - f(x) + yxf'(x)) \tag{3.42}$$

and

$$\mathbb{G}_y(n,l) = \begin{cases} \frac{1}{n+1}\binom{n+1}{l}(1-e^{-ly})(1-e^{-y})^{n-l+1} & \text{if } l \in [\![0, n-1]\!] \\ \frac{1}{n+1}(e^{-(n+1)y} - 1 + (n+1)y) & \text{if } l = n+1 \\ 0 & \text{if } l > n+1 \end{cases} \tag{3.43}$$

with $\mathbb{G}_y(n,n)$ being such that $\sum_{l=0}^{n+1}\mathbb{G}_y(n,l) = 0$. Then, observing that $\mathbb{G}_\Pi(n,l) = \int_0^\infty \mathbb{G}_y(n,l)\Pi(dy)$ as well as $G_\Pi f(x) = \int_0^\infty G_y f(x)dy$ for all $f \in \mathbf{C}_b^2(\mathbb{R}_+)$. We claim that it suffices to show that, for all $y > 0$ and $f \in \mathbf{P}_\mathfrak{e}$,

$$\mathbb{G}_y \nabla f = \nabla G_y f. \tag{3.44}$$

Indeed, when $f \in \mathbf{P}_{\mathfrak{e}}$,

$$\mathbb{G}_{\Pi}\nabla f(n) = \sum_{l=0}^{n+1} \mathbb{G}_{\Pi}(n,l)\nabla f(l) = \sum_{l=0}^{n+1} \nabla f(l) \int_0^\infty \mathbb{G}_y(n,l)\Pi(dy)$$
$$= \int_0^\infty \mathbb{G}_y \nabla f(n)\Pi(dy).$$

On the other hand,

$$\nabla G_{\Pi} f(n) = \frac{d^n}{dx^n}(e^x G_{\Pi} f(x))(0) = \frac{d^n}{dx^n}\left(e^x \int_0^\infty G_y f(x)\Pi(dy)\right)(0).$$

Since $\mathbf{P}_{\mathfrak{e}} \subset \mathbf{C}_b^\infty(\mathbb{R}_+)$, the above integration and differentiation can be interchanged, therefore yielding

$$\nabla G_{\Pi} f(n) = \int_0^\infty \nabla G_y f(n)\Pi(dy).$$

We now proceed to show (3.44). Since $\mathbf{P}_{\mathfrak{e}} = \mathrm{Span}\{x \mapsto e^{-x}x^l; l \in \mathbb{Z}_+\}$, it suffices to prove (3.44) only for $f(x) = \mathrm{h}_l(x) := e^{-x}x^l$. Now, for $l \geqslant 1$,

$$\nabla G_y \mathrm{h}_l(n) = \frac{d^n}{dx^n}\left[e^{x(1-xe^{-y})}x^{l-1}e^{-yl} - x^{l-1} + y(lx^{l-1} - x^l)\right](0). \tag{3.45}$$

When $l \in [\![1, n-1]\!]$, applying Leibniz rule we get

$$\begin{aligned}\nabla G_y \mathrm{h}_l(n) =& e^{-ly}\sum_{m=0}^{n}\binom{n}{m}\frac{d^m}{dx^m}(x^{l-1})(0)\frac{d^{n-m}}{dx^{n-m}}\left(e^{x(1-e^{-y})}\right)(0)\\ =&\binom{n}{l-1}(l-1)!(1-e^{-y})^{n-l+1}\\ =&\frac{n!}{(n-l+1)!}e^{-ly}(1-e^{-y})^{n-l+1} = l!\mathbb{G}_y(n,l).\end{aligned} \tag{3.46}$$

Also, (3.45) entails

$$\begin{aligned}\nabla G_y \mathrm{h}_n(n) =& n!(e^{-ny}(1-e^{-y}) - y) = n!\mathbb{G}_y(n,n)\\ \nabla G_y \mathrm{h}_{n+1}(n) =& n!(e^{-(n+1)y} - n! + (n+1)y) = (n+1)!\mathbb{G}_y(n,n+1).\end{aligned} \tag{3.47}$$

Finally,

$$\nabla G_y \mathrm{h}_0(n) = \frac{d^n}{dx^n}\left(\frac{1}{x}(e^{x(1-e^{-y})} - 1)\right)(0) = \frac{1}{n+1}(1-e^{-y})^{n+1} = \mathbb{G}_y(n,0). \tag{3.48}$$

On the other hand, for all $l \in \mathbb{Z}_+$, $\mathbb{G}_y \nabla \mathrm{h}_l(n) = l! \mathbb{G}_y(n,l)$. Therefore, combining (3.46), (3.47) and (3.48), we conclude that, for all $n, l \geqslant 0$,

$$\mathbb{G}_y \nabla \mathrm{h}_l(n) = \nabla G_y \mathrm{h}_l(n).$$

This completes the proof of the lemma. The next lemma is a variant of [84, Lemma 4].

Lemma 3.4.5. *$\nabla : \mathbf{P}_{\mathfrak{e}} \to \mathbf{C}_c(\mathbb{Z}_+)$ is bijective with inverse Λ such that $\Lambda \mathsf{f}(x) = \mathbb{E}[\mathsf{f}(\mathrm{Pois}(x))]$ for all $\mathsf{f} \in \mathbf{C}_c(\mathbb{Z}_+)$. Moreover, Λ extends to a bounded operator from $\mathbf{C}_0(\mathbb{Z}_+)$ to $\mathbf{C}_0(\mathbb{R}_+)$.*

We also need the following useful result.

Lemma 3.4.6. *For all $\phi \in \mathbf{B}$, $\mathbf{P}_{\mathfrak{e}} \subset \mathbf{D}(G_\phi)$ where $\mathbf{D}(G_\phi)$ is the domain of the generator of the Feller semigroup Q^ϕ.*

Proof. Denoting the Dynkin characteristic operator of the semigroup Q^ϕ by G_ϕ^D, it follows from [76, Proposition 6.1] that $\mathbf{P}_{\mathfrak{e}} \subset \mathbf{D}(G_\phi^D)$. Also, for all $f \in \mathbf{P}_{\mathfrak{e}}$, $G_\phi^D f = G_\phi f$. In light of [41, Theorem 5.5, Chapter V.3], it suffices to show that for all $f \in \mathbf{P}_{\mathfrak{e}}$, $G_\phi f \in \mathbf{C}_0([0,\infty))$. Since any function $f \in \mathbf{P}_{\mathfrak{e}}$ is of the form $f(x) = e^{-x}P(x)$ for some polynomial P, clearly $G_{m,\sigma^2} f \in \mathbf{C}_0([0,\infty))$. Now, for any $f \in \mathbf{P}_{\mathfrak{e}}$, we have

$$\begin{aligned}
G_\Pi f(x) &= \frac{1}{x}\int_0^\infty [f(xe^{-y}) - f(x) + yxf'(x)]\Pi(dy) \\
&= \frac{1}{x}\int_0^\infty [f(xe^{-y}) - f(x) - (e^{-y}-1)xf'(x)]\Pi(dy) \\
&+ f'(x)\int_0^\infty (e^{-y} - 1 + y)\Pi(dy) \\
&= A_1(x) + A_2(x).
\end{aligned}$$

Since $f \in \mathbf{P}_{\mathfrak{e}}$, it implies that $f' \in \mathbf{C}_0([0,\infty))$ and therefore $A_2(x) \to 0$ as $x \to \infty$. To deal with A_1, by means of Taylor expansion of f up to order 2 we obtain that

$$\begin{aligned}
&\frac{1}{x}\int_0^1 \left| f(xe^{-y}) - f(x) - (e^{-y}-1)xf'(x) \right| \Pi(dy) \\
&\leqslant \frac{x}{2}\int_0^1 \sup_{t \in [e^{-y}x, x]} |f''(t)|(1-e^{-y})^2 \Pi(dy). \qquad (3.49)
\end{aligned}$$

We note that any function $f \in \mathbf{P}_{\mathfrak{e}}$, f is either eventually increasing or decreasing, depending on the sign of the leading coefficient of the polynomial associated with the function. Without loss of generality, we assume that f is eventually decreasing. Since for all $y \in [0,1]$, $e^{-y} \geqslant e^{-1}$, for all large values of x, we have $\sup_{t\in[e^{-y}x,x]} |f''(t)| \leqslant |f''(e^{-1}x)|$, as $f \in \mathbf{P}_{\mathfrak{e}}$ implies $f'' \in \mathbf{P}_{\mathfrak{e}}$. Thus, the right-hand side of (3.49) goes to 0 as $x \to \infty$. On the other hand,

$$\frac{1}{x}|f(xe^{-y}) - f(x) - (e^{-y}-1)xf'(x)| \leqslant (1-e^{-y})\|f'\|_\infty \leqslant y\|f'\|_\infty.$$

Using the dominated convergence theorem, we obtain that

$$\lim_{x\to\infty} \frac{1}{x}\int_1^\infty [f(e^{-y}x) - f(x) + (e^{-y}-1)xf'(x)]\Pi(dy) = 0.$$

This shows that $A_2(x) \to 0$ as $x \to \infty$ which completes the proof of the lemma. Let $\mathbf{D}(\mathbb{G}_\phi)$ denote the domain of the Feller generator $\mathbb{G}_\phi$. Now, coming back to the proofs of Proposition 3.4.3, Lemma 3.4.4, Lemma 3.4.5 and Lemma 3.4.6 imply that, for all $\mathsf{f} \in \mathbf{C}_c(\mathbb{Z}_+)$,

$$\Lambda\mathsf{f} \in \mathbf{P}_{\mathfrak{e}} \subset \mathbf{D}(G_\phi) \text{ and } G_\phi\Lambda\mathsf{f} = \Lambda\mathbb{G}_\phi\mathsf{f}. \tag{3.50}$$

Since $\mathbf{C}_c(\mathbb{Z}_+)$ is a core for the generator $\mathbb{G}_\phi$, for any $\mathsf{f} \in \mathbf{D}(\mathbb{G}_\phi)$ there exists a sequence $\{\mathsf{f}_n\} \subset \mathbf{C}_c(\mathbb{Z}_+)$ such that $\|\mathsf{f}_n - \mathsf{f}\|_\infty \to 0$ and $\|\mathbb{G}_\phi\mathsf{f}_n - \mathbb{G}_\phi\mathsf{f}\|_\infty \to 0$. Therefore, thanks to Lemma 3.4.5,

$$\|\Lambda\mathsf{f}_n - \Lambda\mathsf{f}\|_\infty \to 0,\ \|\Lambda\mathbb{G}_\phi\mathsf{f}_n - \Lambda\mathbb{G}_\phi\mathsf{f}\|_\infty \to 0.$$

Thus, (3.50) entails that $G_\phi\Lambda\mathsf{f}_n$ converges in $\mathbf{C}_0(\mathbb{R}_+)$ which implies that $\Lambda\mathsf{f} \in \mathbf{D}(G_\phi)$ as $(G_\phi, \mathbf{D}(G_\phi))$ is a closed operator, and, for all $\mathsf{f} \in \mathbf{D}(\mathbb{G}_\phi)$,

$$G_\phi\Lambda\mathsf{f} = \lim_{n\to\infty} G_\phi\Lambda\mathsf{f}_n = \lim_{n\to\infty} \Lambda\mathbb{G}_\phi\mathsf{f}_n = \Lambda\mathbb{G}_\phi\mathsf{f}. \tag{3.51}$$

Using Kolmogorov's forward and backward equations, we get, for all $\mathsf{f} \in \mathbf{D}(\mathbb{G}_\phi)$, $t > 0$ and $s \in [0,t]$,

$$\begin{aligned} \frac{d}{ds}Q_s^\phi\Lambda\mathbb{Q}_{t-s}^\phi\mathsf{f} &= Q_s^\phi G_\phi\Lambda\mathbb{Q}_{t-s}^\phi - Q_s^\phi\Lambda\mathbb{G}_\phi\mathbb{Q}_{t-s}^\phi\mathsf{f} \\ &= Q_s^\phi[G_\phi\Lambda - \Lambda\mathbb{G}_\phi]\mathbb{Q}_{t-s}^\phi\mathsf{f} \\ &= 0 \end{aligned}$$

which is due to (3.51) together with the fact that $\mathbb{Q}^{\phi}_{t-s}\mathsf{f} \in \mathbf{D}(\mathbb{G}_{\phi})$. Integrating the above identity, we obtain, for all $\mathsf{f} \in \mathbf{D}(\mathbb{G}_{\phi})$,

$$Q^{\phi}_t \Lambda \mathsf{f} = \Lambda \mathbb{Q}^{\phi}_t \mathsf{f}.$$

Finally, using the density of $\mathbf{D}(\mathbb{G}_{\phi})$ and the boundedness of the operators Q^{ϕ}, $\mathbb{Q}^{\phi}$ and Λ, (3.38) follows.

Proof of Proposition 3.4.3(2)

It is plain that, for any $n \in \mathbb{Z}_+$,

$$\int_0^{\infty} e^{-x}\frac{x^n}{n!}dx = 1,$$

which implies that $\mu\Lambda = \mathbb{m}$ where μ is the Lebesgue measure on $\mathbb{R}_+$. Also, Λ being a positive operator, for any $\mathsf{f} \in \mathbf{C}_0(\mathbb{Z}_+)$ with $\mathsf{f} \geqslant 0$, we have

$$\mathbb{m}\mathsf{f} = \mu\Lambda\mathsf{f} \geqslant \mu Q^{\phi}_t \Lambda \mathsf{f} = \mu\Lambda\mathbb{Q}^{\phi}_t\mathsf{f} = \mathbb{m}\mathbb{Q}^{\phi}_t\mathsf{f}$$

where the second inequality in the above line holds as μ is an excessive measure for Q^{ϕ}. This shows that $\mathbb{m}$ is an excessive measure for $\mathbb{Q}^{\phi}$.

Proof of Proposition 3.4.3(3)

For any $\mathsf{f} \in \mathbf{C}_c(\mathbb{Z}_+)$,

$$\begin{aligned}\|\Lambda\mathsf{f}\|^2_{\mathbf{L}^2(\mathbb{R}_+)} = \int_0^{\infty}(\Lambda\mathsf{f}(x))^2dx &= \int_0^{\infty}\left(\sum_{n=0}^{\infty} e^{-x}\frac{x^n}{n!}\mathsf{f}(n)\right)^2 dx \\ &\leqslant \sum_{n=0}^{\infty}\mathsf{f}(n)^2\int_0^{\infty} e^{-x}\frac{x^n}{n!}dx \\ &= \sum_{n=0}^{\infty}\mathsf{f}(n)^2 = \|\mathsf{f}\|^2_{\ell^2(\mathbb{Z}_+)}.\end{aligned}$$

Using the density of $\mathbf{C}_c(\mathbb{Z}_+)$ in $\ell^2(\mathbb{Z}_+)$, Λ extends uniquely to a bounded operator from $\ell^2(\mathbb{Z}_+)$ to $\mathbf{L}^2(\mathbb{R}_+)$. Finally, for any $\mathsf{f} \in \mathbf{C}_0(\mathbb{Z}_+) \cap \ell^2(\mathbb{Z}_+)$, $\Lambda\mathsf{f} \in \mathbf{C}_0(\mathbb{R}_+) \cap \mathbf{L}^2(\mathbb{R}_+)$. Thus, for all $\mathsf{f} \in$

$\mathbf{C}_0(\mathbb{Z}_+) \cap \ell^2(\mathbb{Z}_+)$, item (1) ensures that

$$Q_t^\phi \Lambda \mathsf{f} = \Lambda \mathbb{Q}_t^\phi \mathsf{f}.$$

Again, using the density of $\mathbf{C}_c(\mathbb{Z}_+)$ in $\ell^2(\mathbb{Z}_+)$, (3.39) follows. Now, it remains to show that Λ is a quasi-affinity. Boundedness of Λ follows from item (2), and one easily checks that, for all $\mathsf{f} \in \ell^2(\mathbb{Z}_+)$,

$$\Lambda \mathsf{f}(x) = \sum_{n=0}^{\infty} e^{-x} \frac{x^n}{n!} \mathsf{f}(n) \quad \text{a.e.}$$

Therefore, $\ker(\Lambda) = \{0\}$, which proves the injectivity. The density of $\mathrm{Range}(\Lambda)$ follows by observing that $\widehat{\Lambda} : \mathbf{L}^2(\mathbb{R}_+) \to \ell^2(\mathbb{Z}_+)$, the adjoint of Λ, takes the following form

$$\widehat{\Lambda} f(n) = \frac{1}{n!} \int_0^\infty f(x) e^{-x} x^n dx = \mathbb{E}[f(\text{Gamma}(n+1))]$$

where Gamma$(n+1)$ is a gamma random variable with $n+1$ as the scale parameter and 1 as the rate parameter. Approximating the $\mathbf{L}^2(\mathbb{R}_+)$ functions by compactly supported continuous functions, it can be shown that $\widehat{\Lambda}$ is an injective operator, which proves that $\mathrm{Range}(\Lambda)$ is dense in $\mathbf{L}^2(\mathbb{R}_+)$. Hence, item (3) is proven, which completes the proof of Proposition 3.4.3.

Corollary 3.4.7. For any $\mathsf{f} : \mathbb{Z}_+ \to \mathbb{R}$ with $f \geqslant 0$, we have

$$Q_t^\phi \Lambda \mathsf{f}(x) = \Lambda \mathbb{Q}_t^\phi \mathsf{f}(x)$$

for all $x \geqslant 0$.

Proof. For any nonnegative function f, we can find $\{\mathsf{f}_n\} \subset \mathbf{C}_c(\mathbb{Z}_+)$ such that $\mathsf{f}_n \uparrow \mathsf{f}$ pointwise. Then, Proposition 3.4.3(1) yields, for all $x \geqslant 0$,

$$Q_t^\phi \Lambda \mathsf{f}_n(x) = \Lambda \mathbb{Q}_t^\phi \mathsf{f}_n(x).$$

Since Λ is a Markov kernel, $\Lambda \mathsf{f}_n \uparrow \Lambda \mathsf{f}$ as well. Writing $Q_t^\phi g(x) = \mathbb{E}_x[g(X_\phi(t))]$ and $\mathbb{Q}_t^\phi \mathsf{g}(n) = \mathbb{E}_n[\mathsf{g}(\mathbb{X}_\phi(t))]$ and invoking the monotone convergence theorem, the proof follows.

End of the Proof of Theorem 3.2.2

From the proof of Proposition 3.4.3(1), we already have that

$$Q^{\phi}_{\alpha t}\Lambda \mathsf{f} = \Lambda \mathbb{Q}^{\phi}_{\alpha t}\mathsf{f}$$

for all $\mathsf{f} \in \mathbf{C}_0(\mathbb{Z}_+)$. By a density argument, the above identity extends for all functions in $\mathbf{C}_b(\mathbb{Z}_+)$. Now, for $\alpha \in [0,1]$, multiplying by d_α both sides of the above equation, we obtain that, for all $\mathsf{f} \in \mathbf{C}_b(\mathbb{Z}_+)$,

$$\Lambda \mathbb{Q}^{\phi}_{t}\mathbb{D}_\alpha \mathsf{f} = Q^{\phi}_{t}\Lambda \mathbb{D}_\alpha \mathsf{f} = Q^{\phi}_{t} d_\alpha \Lambda \mathsf{f} = d_\alpha Q^{\phi}_{\alpha t}\Lambda \mathsf{f} = d_\alpha \Lambda \mathbb{Q}^{\phi}_{\alpha t}\mathsf{f} = \Lambda \mathbb{D}_\alpha \mathbb{Q}^{\phi}_{\alpha t}\mathsf{f}$$

where we used the intertwining relationship between d_α and $\mathbb{D}_\alpha$ given in Proposition 3.4.1. By means of the injectivity of Λ on $\mathbf{C}_b(\mathbb{Z}_+)$, we complete the proof. □

3.4.2 Proof of Theorem 3.2.5

Let $\mathbb{G}$ denote the generator of the discrete self-similar Markov chain $\mathbb{X}$. Then, from the definition of the discrete self-similarity, for any $\alpha \in [0,1]$, we have

$$\mathbb{G}\mathbb{D}_\alpha = \alpha \mathbb{D}_\alpha \mathbb{G} \text{ on } \mathbf{D}(\mathbb{G}).$$

Recalling that, for any $m, n \in \mathbb{Z}_+$, $\mathbb{D}_\alpha(m,n) = \mathbb{D}_\alpha \delta_n(m) = \binom{m}{n}\alpha^n(1-\alpha)^{m-n}\mathbb{1}_{\{n\leqslant m\}}$ we have, for all $l, n \in \mathbb{Z}_+$,

$$\sum_{k\geqslant l}\mathbb{G}(n,k)\binom{k}{l}\alpha^l(1-\alpha)^{k-l} = \sum_{j\leqslant n}\alpha^{j+1}(1-\alpha)^{n-j}\mathbb{G}(j,l). \tag{3.52}$$

Taking $n=0$ and $l=1$ in the above equation, we obtain, for all $k \in \mathbb{Z}_+$, that

$$\sum_{k\geqslant 1}\mathbb{G}(0,k)\alpha(1-\alpha)^{k-1} = \alpha \mathbb{G}(0,1).$$

Using the fact that $\mathbb{G}(0,k) \geqslant 0$ for all $k > 0$, we conclude that $\mathbb{G}(0,k) = 0$ for all $k \geqslant 2$. We now use an induction argument to prove that $\mathbb{G}(n,k) = 0$ for all $k \geqslant n+2$. Let us assume that for all

$n < N \in \mathbb{Z}_+$, $\mathbb{G}(n,k) = 0$ for all $k \geqslant n+2$. Now plugging $n = N, l = N+1$ in (3.52) and using the induction hypothesis, we have

$$\sum_{k \geqslant N+1} \mathbb{G}(N,k) \binom{k}{N+1} \alpha^{N+1} (1-\alpha)^{k-N-1} = \alpha^{N+1} \mathbb{G}(N, N+1).$$

Again invoking the nonnegativity of $\mathbb{G}(N,k)$ for $k \neq N$, we conclude that $\mathbb{G}(N,k) = 0$ for all $k \geqslant N+2$. This completes the induction step and therefore the theorem is proved. □

3.4.3 Factorial moments of discrete self-similar Markov chains

Let us recall that the skip-free Markov chain associated to ϕ is denoted by $\mathbb{X}_\phi$. In the spirit of the work of Bertoin and Yor [14, Proposition 1(i)] on the integer moments of the continuous analogues, we provide an explicit formula for the factorial moments of $\mathbb{X}_\phi(t)$. For $z \in \mathbb{C}$, we recall that $\mathsf{p}_z : \mathbb{Z}_+ \to \mathbb{C}$ is the function defined by

$$\mathsf{p}_z(n) = \frac{\Gamma(n+1)}{\Gamma(n+1-z)}. \tag{3.53}$$

It is well-known that, for any $n, k \in \mathbb{Z}_+$,

$$n^k = \sum_{j=0}^{k} \left\{ \begin{matrix} k \\ j \end{matrix} \right\} \mathsf{p}_j(n) \tag{3.54}$$

where $\left\{ \begin{matrix} k \\ j \end{matrix} \right\}$ are the Stirling numbers of second kind, see [125, p. 81].

Theorem 3.4.8. *For any $n, k \in \mathbb{Z}_+$ and $t \geqslant 0$,*

$$\mathbb{E}\left[\mathsf{p}_k(\mathbb{X}_\phi(t,n))\right] = \mathbb{Q}_t^\phi \mathsf{p}_k(n) = \sum_{l=0}^{k} \binom{k}{l} \frac{W_\phi(k+1)}{W_\phi(l+1)} \mathsf{p}_l(n) t^{k-l} \tag{3.55}$$

where, for all $n \in \mathbb{Z}_+$, $W_\phi(n+1) = \prod_{k=1}^{n} \phi(k)$ and $W_\phi(1) = 1$.

Proof. Defining $p_k(x) = x^k$, from [14, Proposition 1(i)], we have, for all $k \in \mathbb{Z}_+$ and $t \geqslant 0$,

$$\begin{aligned} Q_t^\phi p_k(x) = \mathbb{E}[(X_\phi(t,x))^k] =& x^k + \sum_{l=1}^k \binom{k}{l} \phi(k)\phi(k-1)\cdots\phi(k-l+1)x^{k-l}t^l \\ =& \sum_{l=0}^k \binom{k}{l} \frac{W_\phi(k+1)}{W_\phi(l+1)} x^l t^{k-l}. \end{aligned} \tag{3.56}$$

On the other hand, it is easy to see that, for all $x > 0$, $\Lambda \mathsf{p}_k(x) = p_k(x)$. Applying Corollary 3.4.7 with $f = \mathsf{p}_k$, yields, for all $t, x \geqslant 0$,

$$Q_t^\phi p_k(x) = Q_t^\phi \Lambda \mathsf{p}_k(x) = \Lambda \mathbb{Q}_t^\phi \mathsf{p}_k(x).$$

Recalling that $\nabla = \Lambda^{-1}$, see [84, Lemma 4], we get

$$\mathbb{E}[\mathsf{p}_k(\mathbb{X}_\phi(t,n))] = \mathbb{Q}_t^\phi \mathsf{p}_k(n) = \nabla Q_t^\phi p_k(n) = \frac{d^n}{dx^n}\left(e^x Q_t^\phi p_k(x)\right)(0).$$

Finally using the expression in (3.56) together with the Leibniz rule, the result follows.

Remark 3.4.9. Using (3.54) and the above theorem, $\mathbb{E}\left[\mathbb{X}_\phi^k(t,n)\right]$ can be also computed explicitly for all $n, k \in \mathbb{Z}_+$.

3.4.4 Proof of Theorem 3.2.6(2)

For showing the weak convergence, we need to check the following two facts. First, the tightness property of the sequence $((Y_n(t) = \frac{1}{n}\mathbb{X}_\phi(nt, \lfloor nx \rfloor))_{t\geqslant 0})$ and the finite-dimensional convergence of (Y_n) to X_ϕ. For the tightness property, applying [61, Theorem 16.1] together with the strong Markov property of (Y_n), it is enough to show that $Y_n(h_n) \overset{P}{\to} x$ (in probability) whenever $h_n \to 0$. We will in fact show that $\mathbb{E}[(Y_n(h_n) - x)^2] \to 0$ as $n \to \infty$. From Theorem 3.4.8, we obtain for all $t \geqslant 0$,

$$\mathbb{E}[Y_n(t)] = \frac{1}{n}\mathbb{E}[\mathbb{X}_\phi(nt, \lfloor nx \rfloor)] = \frac{1}{n}\mathbb{E}[\mathsf{p}_1(\mathbb{X}_\phi(nt, \lfloor nx \rfloor))] = \frac{1}{n}(\lfloor nx \rfloor + \phi(1)nt)$$

and

$$\begin{aligned}\mathbb{E}[Y_n^2(t)] =& \frac{1}{n^2}\mathbb{E}[\mathsf{p}_2(\mathbb{X}_\phi(nt,\lfloor nx\rfloor)) + \mathsf{p}_1(\mathbb{X}_\phi(nt,\lfloor nx\rfloor))]\\ =& \frac{1}{n^2}(\mathsf{p}_2(\lfloor nx\rfloor) + 2\phi(2)nt\lfloor nx\rfloor + \phi(1)\phi(2)n^2t^2 + \lfloor nx\rfloor + \phi(1)nt).\end{aligned}$$

Since $h_n \to 0$, the last two equations imply $\mathbb{E}[Y_n(h_n)] \to x$ and $\mathbb{E}[Y_n^2(h_n)] \to x^2$ as $n \to \infty$. Therefore, $\mathbb{E}[(Y_n(h_n) - x)^2] \to 0$, which proves the tightness of (Y_n). Next, to get the finite-dimensional convergence, it is enough to prove that, for all $0 \leqslant t_1 < t_2 < \cdots < t_k$ and $(\alpha_1, \alpha_2, \cdots, \alpha_k) \in \mathbb{Z}_+^k$,

$$\lim_{n\to\infty} \mathbb{E}\left[Y_n^{\alpha_1}(t_1)Y_n^{\alpha_2}(t_2)\cdots Y_n^{\alpha_k}(t_k)\right] = \mathbb{E}\left[X_\phi^{\alpha_1}(t_1,x)X_\phi^{\alpha_2}(t_2,x)\cdots X_\phi^{\alpha_k}(t_k,x)\right], \tag{3.57}$$

as the finite-dimensional distributions of X_ϕ are moment determinate. To prove the above assertion, we need the following lemma.

Lemma 3.4.10. *For any $t \geqslant 0$ and $k \in \mathbb{Z}_+$,*

$$\lim_{n\to\infty} \mathbb{E}\left[(Y_n(t) - X_\phi(t,x))^k\right] = 0. \tag{3.58}$$

Proof. From Theorem 3.4.8, it is clear that for any $t \geqslant 0$ and $k \in \mathbb{Z}_+$, the sequence $(Y_n^k(t))_{n\geqslant 0}$ is uniformly integrable and, for each $k \in \mathbb{Z}_+$,

$$\begin{aligned}\mathbb{E}[Y_n^k(t)] =& \frac{1}{n^k}\mathbb{E}\left[\mathbb{X}_\phi^k(nt,\lfloor nx\rfloor)\right]\\ =& \frac{1}{n^k}\left(\mathbb{E}[\mathsf{p}_k(\mathbb{X}_\phi(nt,\lfloor nx\rfloor))] + o(n^k)\right).\end{aligned}$$

Therefore,

$$\begin{aligned}\lim_{n\to\infty}\mathbb{E}[Y_n^k(t)] =& \lim_{n\to\infty}\frac{1}{n^k}\mathbb{E}[\mathsf{p}_k(\mathbb{X}_\phi(nt,\lfloor nx\rfloor))]\\ =& \lim_{n\to\infty}\frac{1}{n^k}\left(\mathsf{p}_k(\lfloor nx\rfloor) + \sum_{l=1}^k\binom{k}{l}\frac{W_\phi(k+1)}{W_\phi(k-l+1)}\mathsf{p}_{k-l}(\lfloor nx\rfloor)n^lt^l\right)\\ =& x^k + \sum_{l=1}^k\binom{k}{l}\frac{W_\phi(k+1)}{W_\phi(k-l+1)}x^{k-l}t^l = \mathbb{E}[X_\phi^k(t,x)].\end{aligned}$$

Since the random variable $X_\phi(t,x)$ is moment determinate, the above identity indicates that for each $t \geqslant 0$, as $n \to \infty$,

$$Y_n(t) \xrightarrow{d} X_\phi(t,x).$$

This together with the uniform integrability mentioned above proves the lemma. Now, coming back to the proof of the main theorem, we prove (3.57) by induction. Indeed, (3.57) is satisfied for $k = 1$, thanks to Lemma 3.4.10. Moreover, if for some $k \in \mathbb{Z}_+$ and $(\alpha_1, \alpha_2, \cdots, \alpha_k) \in \mathbb{Z}_+^k$ (3.57) holds, then, by an uniform integrability argument as in the proof of the lemma, one can show that

$$\lim_{n\to\infty} \mathbb{E}\left[\left(Y_n^{\alpha_1}(t_1)\cdots Y_n^{\alpha_k}(t_k) - X_\phi^{\alpha_1}(t_1,x)\cdots X_\phi^{\alpha_k}(t_k,x)\right)^2\right] = 0. \tag{3.59}$$

Now, writing

$$M_{n,k} = Y_n^{\alpha_1}(t_1)\cdots Y_n^{\alpha_k}(t_k),\;\; M_k = X_\phi^{\alpha_1}(t_1,x)\cdots X_\phi^{\alpha_k}(t_k,x),$$

for any $(\alpha_1, \alpha_2, \cdots, \alpha_{k+1}) \in \mathbb{Z}_+^{k+1}$, we have

$$\begin{aligned}
&\left|\mathbb{E}\left[\prod_{i=1}^{k+1} Y_n^{\alpha_i}(t_i) - \prod_{i=1}^{k+1} X_\phi^{\alpha_i}(t_i,x)\right]\right| \\
=&\left|\mathbb{E}\left[M_{n,k}Y^{\alpha_{k+1}}(t_{k+1}) - M_k X_\phi^{\alpha_{k+1}}(t_{k+1},x)\right]\right| \\
=&\left|\mathbb{E}\left[M_{n,k}Y^{\alpha_{k+1}} - M_k Y^{\alpha_{k+1}}(t_{k+1}) + M_k Y^{\alpha_{k+1}}(t_{k+1}) - M_k X_\phi^{\alpha_{k+1}}(t_{k+1},x)\right]\right| \\
\leqslant&\sqrt{\mathbb{E}[(M_{n,k}-M_k)^2]\mathbb{E}[Y^{2\alpha_{k+1}}(t_{k+1})]} \\
&+\sqrt{\mathbb{E}[M_k^2]\mathbb{E}[(Y^{\alpha_{k+1}}(t_{k+1}) - X_\phi^{\alpha_{k+1}}(t_{k+1},x))^2]}.
\end{aligned} \tag{3.60}$$

In view of Lemma 3.4.10, (3.57) and (3.59), the expression on the right-hand side of (3.60) tends to 0 as $n \to \infty$. This completes the induction step of our hypothesis, therefore proving (3.57) for $k+1$. This completes the proof of the finite-dimensional convergence of the process $(Y_n)_{n\geqslant 0}$, which concludes the proof of the theorem. □

3.4.5 Intertwining of the skip-free Laguerre and generalized Laguerre semigroups

In this section we establish the connection between the generalized Laguerre semigroups as defined in [95] and the skip-free Laguerre semigroups introduced therein. From Theorem 1.6(2) in the aforementioned reference, it is known that, for any $\phi \in \mathbf{B}$, the generalized Laguerre semigroup K^{ϕ} on $\mathbb{R}_+$ has a unique invariant distribution ν_ϕ that is absolutely continuous and moment determinate. In the next result we show that the intertwining relationship in (3.38) is retained for the Laguerre semigroups as well. In the next proposition, we use the fact that the semigroup $\mathbb{K}^{\phi}$ has a unique invariant distribution denoted by $\mathbf{n}_\phi$, which is proved in Proposition 3.4.12 below.

Proposition 3.4.11. *1. Let $\phi \in \mathbf{B}$, then we have, for all $t \geqslant 0$ and $\mathsf{f} \in \mathbf{C}_0(\mathbb{Z}_+)$,*

$$K_t^{\phi}\Lambda \mathsf{f} = \Lambda \mathbb{K}_t^{\phi}\mathsf{f} \tag{3.61}$$

where we recall that $\Lambda \mathsf{f}(x) = \mathbb{E}[\mathsf{f}(\text{Pois}(x))]$.

2. $\mathbf{n}_\phi = \nu_\phi \Lambda$ is an invariant distribution of $\mathbb{K}^{\phi}$, and, for all $n \in \mathbb{Z}_+$,

$$\mathbf{n}_\phi(n) = \frac{1}{n!}\int_0^{\phi(\infty)} e^{-x}x^n \nu_\phi(x)dx.$$

3. The Feller semigroup $\mathbb{K}^{\phi}$ extends uniquely to a strongly continuous Markov semigroup on $\ell^2(\mathbf{n}_\phi)$, which is again denoted by $\mathbb{K}^{\phi}$. Furthermore, the operator

$$\Lambda : \mathbf{C}_0(\mathbb{Z}_+) \to \mathbf{C}_0(\mathbb{R}_+)$$

has a unique extension in $\mathscr{B}(\ell^2(\mathbf{n}_\phi), \mathbf{L}^2(\nu_\phi))$, and, for all $t \geqslant 0$ and $\mathsf{f} \in \ell^2(\mathbf{n}_\phi)$,

$$K_t^{\phi}\Lambda \mathsf{f} = \Lambda \mathbb{K}_t^{\phi}\mathsf{f}. \tag{3.62}$$

Moreover, taking the adjoint in the above identity, one gets, for all $t \geqslant 0$ and $f \in \mathbf{L}^2(\nu_\phi)$,

$$\widehat{\mathbb{K}}_t^{\phi}\widehat{\Lambda}_\phi f = \widehat{\Lambda}_\phi \widehat{K}_t^{\phi} f \tag{3.63}$$

where $\widehat{\Lambda}_\phi : \mathbf{L}^2(\nu_\phi) \to \ell^2(\mathbf{n}_\phi)$ *is the adjoint of* Λ*, and, for all* $f \in \mathbf{L}^2(\nu_\phi)$*,*

$$\widehat{\Lambda}_\phi f(n) = \frac{1}{n!\mathbf{n}_\phi(n)} \int_0^\infty e^{-x} x^n \nu_\phi(x) f(x) dx. \tag{3.64}$$

4. $\mathbb{K}^\phi$ *is self-adjoint in* $\ell^2(\mathbf{n}_\phi)$ *if and only if* $\phi(u) = \sigma^2 u + m$ *for some* $\sigma^2, m \geqslant 0$.

Proof. Since $\mathbb{L}^\phi = \mathbb{G}_\phi + n\partial_-$ and $L^\phi = G_\phi - x\frac{d}{dx}$, it suffices to show that, for all $\mathsf{f} \in \mathbf{C}_c(\mathbb{Z}_+)$,

$$-x\frac{d}{dx}\Lambda\mathsf{f}(x) = \Lambda(n\partial_-)f(x). \tag{3.65}$$

From [84, Lemma 23], (3.65) readily follows by considering the reverse intertwining relationship (i.e., taking the inverse of Δ in (45) of the aforementioned reference). Thus, we conclude that, for all $\mathsf{f} \in \mathbf{C}_c(\mathbb{Z}_+)$,

$$L^\phi \Lambda \mathsf{f} = \Lambda \mathbb{L}^\phi \mathsf{f}.$$

The rest of the proof follows similarly as in the proof of Theorem 3.4.3(1).

Next, from the intertwining relation in (1), we deduce that, for all $\mathsf{f} \in \mathbf{C}_0(\mathbb{Z}_+)$,

$$\nu_\phi \Lambda \mathbb{K}_t^\phi \mathsf{f} = \nu_\phi K_t^\phi \Lambda \mathsf{f} = \nu_\phi \Lambda \mathsf{f}$$

implying that $\mathbf{n}_\phi = \nu_\phi\Lambda$ is an invariant finite measure for $\mathbb{K}^\phi$. Now, for any $n \in \mathbb{Z}_+$,

$$\mathbf{n}_\phi(n) = \nu_\phi \Lambda \delta_n = \frac{1}{n!} \int_0^\infty e^{-x} x^n \nu_\phi(x) dx > 0 \tag{3.66}$$

and

$$\sum_{n=0}^\infty \mathbf{n}_\phi(n) = \int_0^\infty \nu_\phi(x) dx = 1.$$

Hence, $\mathbf{n}_\phi$ is the invariant distribution of $\mathbb{K}^\phi$. The uniqueness of the invariant distribution will be proved in Proposition 3.4.12(1). To prove (3), we note that since $\mathbf{C}_0(\mathbb{Z}_+)$ is dense in $\ell^2(\mathbf{n}_\phi)$ and $\Lambda : \mathbf{C}_0(\mathbb{Z}_+) \to \mathbf{C}_0(\mathbb{Z}_+)$ is a Markov kernel, Λ can be uniquely extended to a bounded operator in $\mathscr{B}(\ell^2(\mathbf{n}_\phi), \mathbf{L}^2(\nu_\phi))$. Also, using the density of $\mathbf{C}_0(\mathbb{Z}_+)$ in $\ell^2(\mathbf{n}_\phi)$ and item (1), the identity (3.62) follows. Now, to compute the adjoint $\widehat{\Lambda}_\phi$ of Λ, let us first show that the right-hand side of (3.64) as a function of n belongs to $\ell^2(\mathbf{n}_\phi)$. Using Young's inequality and item (2) one has

$$\sum_{n=0}^{\infty}\frac{1}{\mathbf{n}_\phi(n)^2}\left(\int_0^\infty e^{-x}\frac{x^n}{n!}\nu_\phi(x)f(x)dx\right)^2\mathbf{n}_\phi(n) \leqslant \sum_{n=0}^{\infty}\int_0^\infty e^{-x}\frac{x^n}{n!}f^2(x)\nu_\phi(x)dx$$
$$= \|f\|^2_{\mathbf{L}^2(\nu_\phi)}.$$

Now, writing $\mathsf{f}(n)=\sum_{n=0}^{\infty}\frac{1}{n!\mathbf{n}_\phi(n)}\int_0^\infty e^{-x}x^n f(x)\nu_\phi(x)dx$, we have, for all $\mathsf{g}\in\ell^2(\mathbf{n}_\phi)$,

$$\begin{aligned}\langle \mathsf{g},\mathsf{f}\rangle_{\mathbf{n}_\phi} &= \sum_{n=0}^{\infty}\mathsf{f}(n)\mathsf{g}(n)\mathbf{n}_\phi(n) = \sum_{n=0}^{\infty}\frac{1}{n!}\mathsf{g}(n)\int_0^\infty e^{-x}x^n f(x)\nu_\phi(x)dx \\ &= \int_0^\infty \Lambda\mathsf{g}(x)f(x)\nu_\phi(x)dx\end{aligned}$$

where the third equality is justified by Fubini theorem. This shows that $\widehat{\Lambda}_\phi f=\mathsf{f}$ which proves (3.64). Finally, to justify (4), we note that, for a $\phi\in\mathbf{B}$, $\mathbb{K}^\phi$ is self-adjoint in $\ell^2(\mathbf{n}_\phi)$ if and only if, for all $l,n\in\mathbb{Z}_+$,

$$\mathbb{L}^\phi(n,l)\mathbf{n}_\phi(n)=\mathbb{L}^\phi(l,n)\mathbf{n}_\phi(l). \tag{3.67}$$

Since $\mathbb{L}^\phi(n,l)=0$ whenever $l\geqslant n+2$ and $\mathbf{n}_\phi(n)>0$ for all $n\in\mathbb{Z}_+$ (see the proof of item (2)), the above identity holds only if $\mathbb{L}^\phi(n,l)=0$ for all $l\neq n,n-1,n+1$. This happens only if $\phi(u)=\sigma^2u+m$ for some $\sigma^2,m\geqslant 0$.

3.4.6 Proof of Theorem 3.2.8(2)

First, we recall that, for all $t\geqslant 0$ and $f\in\mathbf{C}_0(\mathbb{R}_+)$, $K_t^\phi f=Q^\phi_{e^t-1}d_{e^{-t}}f$, see e.g. [95]. Then, from (3.61), we have for all $t\geqslant 0$ and $\mathsf{f}\in\mathbf{C}_0(\mathbb{Z}_+)$,

$$\Lambda\mathbb{K}_t^\phi\mathsf{f} = K_t^\phi\Lambda\mathsf{f}=Q^\phi_{e^t-1}d_{e^{-t}}\Lambda\mathsf{f}=Q^\phi_{e^t-1}\Lambda\mathbb{D}_{e^{-t}}\mathsf{f}=\Lambda\mathbb{Q}^\phi_{e^t-1}\mathbb{D}_{e^{-t}}\mathsf{f}$$

where we used, from the third identity onwards, successively (3.38) and (3.32). We conclude the proof by invoking the Feller property of the semigroups as well as the injectivity of Λ on $\mathbf{C}_0(\mathbb{Z}_+)$, see [84, Lemma 4(4))].

3.4.7 The invariant distribution of the skip-free Laguerre semigroup

We now show that the invariant distribution $\mathbf{n}_\phi$ in Proposition 3.4.11(3) is unique and provide several useful representations. We recall that, for any $\phi \in \mathbf{B}$, W_ϕ is the so-called Bernstein-gamma function which is defined as a solution to the following functional equation

$$W_\phi(z+1) = \phi(z)W_\phi(z) \ \forall z \in \mathbb{C}_+, \ W_\phi(1) = 1.$$

The above functional equation has a unique solution in the class of Mellin transforms of probability measures on $\mathbb{R}_+$. For a detailed account of these functions, we refer to [96].

Proposition 3.4.12. *1. For all $\phi \in \mathbf{B}$, the invariant distribution $\mathbf{n}_\phi$ of $\mathbb{K}^\phi$ is unique and is determined by its factorial moments*

$$\mathbf{n}_\phi \mathsf{p}_k = W_\phi(k+1) \tag{3.68}$$

where p_k is defined in (3.53).

2. *For any $n \in \mathbb{Z}_+$ and $0 < c < n+1+\mathrm{d}_\phi$,*

$$\mathbf{n}_\phi(n) = \frac{1}{n!}\frac{1}{2\pi \mathrm{i}}\int_{c-\mathrm{i}\infty}^{c+\mathrm{i}\infty} \Gamma(z)W_\phi(n-z+1)dz \tag{3.69}$$

where $\mathrm{d}_\phi = \min\{u \geqslant 0;\ \phi(-u) = -\infty, \phi(-u) = 0\} \in [0,\infty]$.

3. *If $0 \leqslant \sigma^2 < 1$, then, for any $n \in \mathbb{Z}_+$,*

$$\mathbf{n}_\phi(n) = \frac{1}{n!}\sum_{r=0}^{\infty}(-1)^r \frac{W_\phi(n+r+1)}{r!}. \tag{3.70}$$

Let us first derive the factorial moments of the skip-free Laguerre chains.

Lemma 3.4.13. *For any $t \geqslant 0$ and $k \in \mathbb{Z}_+$,*

$$\mathbb{K}_t^\phi \mathsf{p}_k(n) = \sum_{l=0}^{k} \binom{k}{l} \frac{W_\phi(k+1)}{W_\phi(l+1)} \mathsf{p}_l(n) e^{-tl}\left(1-e^{-t}\right)^{k-l}. \tag{3.71}$$

Proof. Let us recall that Q^ϕ denote the spectrally negative self-similar semigroup associated to ϕ, and, for any $t \geqslant 0$, $x > 0$ and $f \geqslant 0$,

$$K_t^\phi f(x) = Q^\phi_{1-e^{-t}} d_{e^{-t}} f(x),$$

and, writing $p_k(x) = x^k$, $x > 0$, we have, from [14], that, for all $k \in \mathbb{Z}_+$,

$$Q_t^\phi p_k(x) = \sum_{l=0}^k \binom{k}{l} \frac{W_\phi(k+1)}{W_\phi(l+1)} x^l t^{k-l}.$$

Therefore,

$$K_t^\phi p_k(x) = \sum_{l=0}^k \binom{k}{l} \frac{W_\phi(k+1)}{W_\phi(l+1)} e^{-tl} \left(1 - e^{-t}\right)^{k-l} p_l(x).$$

Recalling that $\nabla = \Lambda^{-1}$ and $\nabla p_l = \mathsf{p}_l$ for all $l \in \mathbb{Z}_+$, it follows that

$$\begin{aligned}\mathbb{K}_t^\phi \mathsf{p}_k(n) = \mathbb{K}_t^\phi \nabla p_k(n) =& \nabla K_t^\phi p_k(n) \\ =& \sum_{l=0}^k \binom{k}{l} \frac{W_\phi(k+1)}{W_\phi(l+1)} e^{-tl} \left(1 - e^{-t}\right)^{k-l} \nabla p_l(n) \\ =& \sum_{l=0}^k \binom{k}{l} \frac{W_\phi(k+1)}{W_\phi(l+1)} e^{-tl} \left(1 - e^{-t}\right)^{k-l} \mathsf{p}_l(n)\end{aligned}$$

which completes the proof.

3.4.8 Proof of Proposition 3.4.12

From Lemma 3.4.13, we observe that, for all $k, n \in \mathbb{Z}_+$,

$$\lim_{t\to\infty} \mathbb{K}_t^\phi \mathsf{p}_k(n) = W_\phi(k+1).$$

On the other hand, recalling that $\Lambda \mathsf{p}_k = p_k$ where $p_k(x) = x^k$ and $\nu_\phi p_k = W_\phi(k+1)$, see [95, Proposition 2.6(1)], we get

$$\mathbf{n}_\phi \mathsf{p}_k = \nu_\phi \Lambda \mathsf{p}_k = \nu_\phi p_k = W_\phi(k+1).$$

Now it remains to show that $\mathbf{n}_\phi$ is determined by its moments. Let us write $\mathfrak{e}_a(n) = e^{an}$. Then, applying Tonelli theorem we get

$$\mathfrak{e}_a\mathbf{n}_\phi = \sum_{n=0}^{\infty} e^{an}\mathbf{n}_\phi(n) = \int_0^\infty e^{-x}\sum_{n=0}^{\infty}\frac{(e^a x)^n}{n!}\nu_\phi(x)dx = \int_0^\infty e^{(e^a-1)x}\nu_\phi(x)dx.$$

Next, we have

$$\int_0^\infty e^{(e^a-1)x}\nu_\phi(x)dx = \sum_{r=0}^{\infty} W_\phi(r+1)\frac{(e^a-1)^r}{r!}$$

where we used the fact that $\int_0^{\phi(\infty)} x^n\nu_\phi(x)dx = W_\phi(n+1)$, see [95, Proposition 2.6(1)], and thus $\mathfrak{e}_a\mathbf{n}_\phi < \infty$ as soon as $(e^a-1) < \sigma^{-2}$, that is for at least any $0 < a < \log(1+\sigma^{-2})$. This provides the moment determinacy of $\mathbf{n}_\phi$. Next, to prove (2) and (3), we observe, from Proposition 3.4.14(2), that for all $n \in \mathbb{Z}_+$,

$$\mathbf{n}_\phi(n) = \int_0^{\phi(\infty)} e^{-x}x^n\nu_\phi(x)dx.$$

Expanding the exponential function in the identity above and using a classical Fubini argument, see e.g. [127, Section 1.77], combined with the expression (3.68) of the moment of ν_ϕ, we get

$$\mathbf{n}_\phi(n) = \frac{1}{n!}\int_0^\infty e^{-x}x^n\nu_\phi(x)dx = \frac{1}{n!}\sum_{r=0}^{\infty} W_\phi(n+r+1)\frac{(-1)^r}{r!} \tag{3.72}$$

where the series is absolutely convergent as soon as

$$\lim_{k\to\infty}\frac{\phi(k+n)}{k} = \lim_{k\to\infty}\frac{\phi(k)}{k} = \sigma^2 < 1.$$

To justify the contour integral representation in (3.69), we consider two cases. Assume first that $\sigma^2 > 0$ and we recall that for large $|\mathtt{Im}(z)|$,

$$|\Gamma(z)| \sim C_{\mathtt{Re}(z)}\,|\mathtt{Im}(z)|^{\mathtt{Re}(z)-\frac{1}{2}}e^{-\frac{\pi}{2}|\mathtt{Im}(z)|},\ \mathtt{Re}(z) > 0, \tag{3.73}$$

$$|W_\phi(n-z+1)| \leqslant C_{n-\mathtt{Re}(z)}e^{-\frac{\pi}{2}|\mathtt{Im}(z)|},\ \mathtt{Re}(z) < n+1+\mathrm{d}_\phi,$$

where here and below $C_{\mathtt{Re}(z)} > 0$ is a constant depending only on $\mathtt{Re}(z) > 0$. Note that the first estimate is the classical Stirling formula, see e.g. [94, (2.1.8)], whereas the second bound follows from [95, Theorem 6.2(2b)]. Therefore, the mappings $z \mapsto \Gamma(z)$ and $z \mapsto W_\phi(n-z+1)$ are both in $\mathbf{L}^2(\mathbb{R})$ and holomorphic in the strip $0 < \mathtt{Re}(z) < n+1+\mathrm{d}_\phi$, see [95, Theorem 6.1(2)].

Moreover $z \mapsto W_\phi(n+z)$ and $z \mapsto \Gamma(z)$ are the Mellin transform of $x \mapsto x^n \nu_\phi(x)$, see [95], and $x \mapsto e^{-x}$, respectively. Consequently, both of these functions are in $\mathbf{L}^2(\mathbb{R}_+)$. An application of Parseval identity for the Mellin transform yields

$$\frac{1}{2\pi\mathrm{i}} \int_{c-\mathrm{i}\infty}^{c+\mathrm{i}\infty} \Gamma(z) W_\phi(n-z+1) dz = \int_0^\infty e^{-x} x^n \nu_\phi(x) dx,$$

from where we easily derive the expression (3.69) for $\sigma^2 > 0$. Next, we consider the other case, that is $\sigma^2 = 0$, which ensures that the series representation (3.72) of $\mathbf{n}_\phi(n)$ is valid for all $n \in \mathbb{Z}_+$. Then, using the facts that the mappings $z \mapsto \Gamma(z)$ and $z \mapsto W_\phi(n-z+1)$ are both holomorphic in the strip $0 < \mathtt{Re}(z) < n+1+\mathrm{d}_\phi$ and within this strip, (3.73) still holds and

$$|W_\phi(n-z+1)| \leqslant C_{(n-\mathtt{Re}(z))}.$$

This implies that for all $n \in \mathbb{N}$, the integral (3.69) is absolutely convergent and an application of Cauchy theorem, see [94] for the detailed computation, yields that the contour integral can be expanded as follows

$$\frac{1}{2\pi\mathrm{i}} \int_{c-\mathrm{i}\infty}^{c+\mathrm{i}\infty} \Gamma(z) W_\phi(n-z+1) dz = \sum_{r=0}^{\infty} W_\phi(n+r+1) \frac{(-1)^r}{r!},$$

which completes the proof of (3.70). □

3.4.9 Intertwining between skip-free Laguerre semigroups

It has been shown in [95, Theorem 2.1] that for any $\phi \in \mathbf{B}$, the generalized (non self-adjoint) Laguerre semigroup K^ϕ is intertwined with the diffusive (self-adjoint) Laguerre semigroup K (when $\phi(u) = u$) and the intertwining operator is a multiplicative Markov kernel corresponding to the exponential functional of the subordinator associated with ϕ. An analogous result holds for the skip-free Laguerre semigroups as well (see Theorem 3.4.14 below), although, we prove it under the assumption that σ^2 in (3.5) is positive. The following proposition describes the intertwining operator $\mathbb{I}_\phi$ that links the semigroups corresponding to skip-free (non-reversible) and the reversible

Laguerre chains respectively. Let $\mathcal{P} = \mathrm{Span}\{\mathsf{p}_k,\ k \in \mathbb{Z}_+\}$ and for any $\phi \in \mathbf{B}$ associated with the triplet (m, σ^2, Π), let $\mathbb{I}_\phi : \mathcal{P} \mapsto \mathcal{P}$ be defined by

$$\mathbb{I}_\phi \mathsf{f}(n) = \mathbb{E}\left[\mathsf{f}(\mathsf{B}(I_{\sigma_1}, n))\right] = \sum_{r=0}^{n} \mathsf{f}(r) \binom{n}{r} \mathbb{E}\left[I_{\sigma_1}^r (1 - I_{\sigma_1})^{n-r}\right] \tag{3.74}$$

with $\mathbb{E}\left[I_{\sigma_1}^k\right] = \frac{\sigma_1^k k!}{W_\phi(k+1)}$ for all $k \in \mathbb{Z}_+$, where σ_1 is defined in (3.19) and we recall that $W_\phi(k+1) = \prod_{r=1}^{k} \phi(r),\ W_\phi(1) = 1$.

Theorem 3.4.14. *1. For any $\phi \in \mathbf{B}$, we have the intertwining relation on $\mathcal{P}$*

$$\mathbb{K}_t^\phi \mathbb{I}_\phi = \mathbb{I}_\phi \mathbb{K}_t^{\sigma_1} \tag{3.75}$$

where $\mathbb{K}^{\sigma_1} = \mathbb{K}^\phi$ with $\phi(u) = \sigma_1 u$.

2. Moreover, if $\sigma^2 > 0$, then $\mathbb{I}_\phi : \ell^2(\mathbf{n}_{\sigma^2}) \mapsto \ell^2(\mathbf{n}_\phi)$ is a linear operator that is bounded, injective with a dense range and for all $\mathsf{f} \in \ell^2(\mathbf{n}_{\sigma^2})$,

$$\|\mathbb{I}_\phi \mathsf{f}\|_{\ell^2(\mathbf{n}_\phi)} \leqslant \|\mathsf{f}\|_{\ell^2(\mathbf{n}_{\sigma^2})} \tag{3.76}$$

where $\mathbf{n}_{\sigma^2}$ is the unique invariant distribution of $\mathbb{K}^{\sigma^2}$, and, for all $t \geqslant 0$, $\mathsf{f} \in \ell^2(\mathbf{n}_{\sigma^2})$,

$$\mathbb{K}_t^\phi \mathbb{I}_\phi \mathsf{f} = \mathbb{I}_\phi \mathbb{K}_t^{\sigma^2} \mathsf{f}. \tag{3.77}$$

As a consequence of the above theorem, we obtain the intertwining relationship among the class of discrete self-similar Markov semigroups.

Corollary 3.4.15. For $\phi \in \mathbf{B}$ with $\sigma^2 > 0$, we have

$$\mathbb{Q}_t^\phi \mathbb{I}_\phi = \mathbb{I}_\phi \mathbb{Q}_{\sigma^2 t}$$

both on $\mathbf{C}_0(\mathbb{Z}_+)$ and $\ell^2(\mathbb{Z}_+)$, where we recall that $\mathbb{Q}^\phi$ (resp. $\mathbb{Q}$) is the discrete self-similar semigroup corresponding to the Bernstein function ϕ (resp. $\phi(u) = u$).

We need the following lemma to prove the above theorem.

Lemma 3.4.16. *Recall the definition of* p_z *in* (3.53)*. Then, for all* $k, n \in \mathbb{Z}_+$*, we have*

$$\mathbb{I}_\phi \mathsf{p}_k(n) \quad = \quad \frac{\sigma_1^k k!}{W_\phi(k+1)} \mathsf{p}_k(n)$$

where σ_1 *is defined in* (3.19)*.*

Proof. Recalling the definition of $\mathbb{I}_\phi$ in (3.74), we have, for all $k, n \in \mathbb{Z}_+$,

$$\mathbb{I}_\phi \mathsf{p}_k(r) = \sum_{r=0}^{n} \mathsf{p}_k(r) \binom{n}{r} \mathbb{E}\left[I_{\sigma_1}^r (1 - I_{\sigma_1})^{n-r}\right] = \mathbb{E}\left[\mathsf{p}_k(\mathtt{B}(n, I_{\sigma_1}))\right] \tag{3.78}$$

where $\mathtt{B}$ denotes the Binomial random variable with the parameters written in the parentheses and the moments of I_{σ_1} are given in (3.74). Also, invoking the definition of the discrete dilation operator $\mathbb{D}$, we can write the above quantity as

$$\mathbb{E}\left[\mathsf{p}_k(\mathtt{B}(n, I_{\sigma_1}))\right] = \mathbb{E}[\mathbb{D}_{I_{\sigma_1}} \mathsf{p}_k(n)] = \mathsf{p}_k(n)\mathbb{E}[I_{\sigma_1}^k] = \frac{\sigma_1^k k!}{W_\phi(k+1)} \mathsf{p}_k(n)$$

where the last equality follows from (3.74). This proves the lemma.

3.4.10 Proof of Theorem 3.4.14

From Lemma 3.4.13 and Lemma 3.4.16, we have, for all $t \geqslant 0$ and $k \in \mathbb{N}$,

$$\begin{aligned}
\mathbb{K}_t^\phi \mathbb{I}_\phi \mathsf{p}_k =& \frac{\sigma_1^k k!}{W_\phi(k+1)} \mathbb{K}_t^\phi \mathsf{p}_k \\
=& \frac{\sigma_1^k k!}{W_\phi(k+1)} \sum_{l=0}^{k} e^{-tl}(1 - e^{-t})^{k-l} \binom{k}{l} \frac{W_\phi(k+1)}{W_\phi(l+1)} \mathsf{p}_l \\
=& \sigma_1^k k! \sum_{l=0}^{k} e^{-tl}(1 - e^{-t})^{k-l} \binom{k}{l} \frac{1}{W_\phi(l+1)} \mathsf{p}_l(n).
\end{aligned}$$

On the other hand, recalling that $\mathbb{K}^{\sigma_1} = \mathbb{K}^\phi$ with $\phi(u) = \sigma_1 u$, we have

$$\begin{aligned}
\mathbb{I}_\phi \mathbb{K}_t^{\sigma_1} \mathsf{p}_k =& \sum_{l=0}^{k} e^{-tl}(1 - e^{-t})^{k-l} \sigma_1^{k-l} \binom{k}{l} \frac{k!}{l!} \mathbb{I}_\phi \mathsf{p}_l \\
=& k! \sum_{l=0}^{k} e^{-tl}(1 - e^{-t})^{k-l} \sigma_1^{k-l} \binom{k}{l} \frac{\sigma_1^l}{W_\phi(l+1)},
\end{aligned}$$

which shows that for all $k \in \mathbb{N}$,

$$\mathbb{K}_t^{\phi} \mathbb{I}_\phi \mathsf{p}_k = \mathbb{I}_\phi \mathbb{K}_t^{\sigma_1} \mathsf{p}_k$$

and therefore, on $\mathcal{P}$,

$$\mathbb{K}_t^{\phi} \mathbb{I}_\phi = \mathbb{I}_\phi \mathbb{K}_t^{\sigma_1}.$$

To prove now (2), it is plain, from Lemma 3.4.16, that $\mathbb{I}_\phi(\mathcal{P}) = \mathcal{P}$. Then, under the condition $\sigma^2 > 0$, we have that $\mathcal{P}(I_{\sigma^2} \in [0,1]) = 1$, see [95, Proposition 6.7] (note that $I_{\sigma^2} = \sigma^2 I_\phi$, where I_ϕ is the exponential functional defined in the aforementioned paper) and thus $\mathbb{I}_\phi$ is a Markov operator. By means of Hölder's inequality, one obtains, for any $\mathsf{f} \in \ell^2(\mathbf{n}_{\sigma^2})$,

$$\|\mathbb{I}_\phi \mathsf{f}\|_{\ell^2(\mathbf{n}_\phi)} \leqslant \|\mathbb{I}_\phi \mathsf{f}^2\|_{\ell^1(\mathbf{n}_\phi)} = \mathbf{n}_\phi \mathbb{I}_\phi \mathsf{f}^2. \tag{3.79}$$

Now, for all $k \in \mathbb{Z}_+$, using Lemma 3.4.16 and Proposition 3.4.12(1), we obtain

$$\mathbf{n}_\phi \mathbb{I}_\phi \mathsf{p}_k = \sum_{n \in \mathbb{Z}_+} \mathbf{n}_\phi(n) \mathbb{I}_\phi \mathsf{p}_k(n) = \sum_{n \in \mathbb{Z}_+} \mathbf{n}_\phi(n) \mathsf{p}_k(n) \frac{\sigma^{2k} k!}{W_\phi(k+1)} = \sigma^{2k} k! = \mathsf{p}_k \mathbf{n}_{\sigma^2}$$

which shows that $\mathbf{n}_\phi \mathbb{I}_\phi = \mathbf{n}_{\sigma^2}$ as $\mathbf{n}_\phi, \mathbf{n}_{\sigma^2}$ are moment determinate. Therefore, (3.79) entails that $\mathbb{I}_\phi$ is a bounded operator from $\ell^2(\mathbf{n}_{\sigma^2})$ to $\ell^2(\mathbf{n}_\phi)$ when $\sigma^2 > 0$. Hence, by the density of $\mathcal{P}$ in $\ell^2(\mathbf{n}_{\sigma^2})$, the intertwining relation given by (3.75) extends to $\ell^2(\mathbf{n}_{\sigma^2})$. This completes the proof of the proposition. □

3.4.11 Hilbert sequences and spectral expansion

In this section, we introduce a few notions from non classical harmonic analysis which have been shown recently to be central in the understanding of the spectral expansions of non self-adjoint operators in Hilbert spaces, see e.g. [95]. Two sequences $(\mathsf{P}_k)_{k \geqslant 0}$ and $(\mathsf{V}_k)_{k \geqslant 0}$ are said to be biorthogonal in the Hilbert space $\ell^2(\mathbf{m})$ if for any $k, l \in \mathbb{Z}_+$,

$$\langle \mathsf{P}_k, \mathsf{V}_l \rangle_{\mathbf{m}} = \mathbb{1}_{\{k=l\}}. \tag{3.80}$$

Moreover, a sequence that admits a biorthogonal sequence will be called *minimal* and a sequence that is both minimal and complete, in the sense that its linear span is dense in $\ell^2(\mathbf{m})$, will be called *exact*. It is easy to show that a sequence $(\mathsf{P}_k)_{k\geqslant 0}$ is minimal if and only if none of its elements can be approximated by linear combinations of the others. If this is the case, then a biorthogonal sequence will be uniquely determined if and only if $(\mathsf{P}_k)_{k\geqslant 0}$ is complete. We proceed with some basic notions related to the concept of frames in Hilbert spaces. A recent and thorough account on these Hilbert space sequences can be found in the book of Christensen [32]. A sequence $(\mathsf{P}_k)_{k\geqslant 0}$ in $\ell^2(\mathbf{m})$ is a frame if there exist $A, B > 0$ such that the frame inequalities

$$A\|f\|^2_{\ell^2(\mathbf{m})} \leqslant \sum_{k=0}^{\infty} |\langle f, \mathsf{P}_k\rangle_{\mathbf{m}}|^2 \leqslant B\|f\|^2_{\ell^2(\mathbf{m})} \tag{3.81}$$

hold, for all $f \in \ell^2(\mathbf{m})$. If only the upper bound exists, $(\mathsf{P}_k)_{k\geqslant 0}$ is called a Bessel sequence. A frame sequence is always complete in the Hilbert space and when it is minimal, it is called a Riesz sequence. The latter are very useful objects as they share substantial properties with orthonormal sequences. Indeed, a Riesz sequence always admits a unique biorthogonal sequence $(\mathsf{V}_k)_{k\geqslant 0}$ which is also a Riesz sequence and both together form the so-called Riesz basis. Moreover, the expansion in terms of the Riesz basis of any element of the Hilbert space is unique and convergent in the topology of the norm. When $(\mathsf{P}_k)_{k\geqslant 0}$ is merely a Bessel sequence, that is only the upper frame condition in (3.81) is satisfied, then the so-called synthesis operator, that is the linear operator $\mathcal{S} : \ell^2(\mathbb{Z}_+) \to \ell^2(\mathbf{m})$ defined by

$$\mathcal{S} : \underline{c} = (c_k)_{k\geqslant 0} \mapsto \mathcal{S}(\underline{c}) = \sum_{k=0}^{\infty} c_k \mathsf{P}_k \tag{3.82}$$

is a bounded operator with (operator) norm $\|\mathcal{S}\|_{\mathbf{m}} \leqslant \sqrt{B}$, that is, the series is norm convergent for any sequence in $\ell^2(\mathbb{Z}_+)$. However, $\mathcal{S}$ is not in principle onto as the $(\mathsf{P}_k)_{k\geqslant 0}$ does not form in general a basis of the Hilbert space.

Proposition 3.4.17. *Let* $\phi \in \mathbf{B}$.

1. *For any* $k \in \mathbb{Z}_+$, $\mathsf{P}^{\phi}_k \in \ell^2(\mathbf{n}_{\phi})$, *and, for any* $t > 0$,

$$\mathbb{K}^{\phi}_t \mathsf{P}^{\phi}_k = e^{-kt}\mathsf{P}^{\phi}_k.$$

Moreover, Span$\{\mathsf{P}^{\phi}_k, k > 0\} = \mathcal{P}$ *which is dense in* $\ell^2(\mathbf{n}_\phi)$.

2. *Assume that* $\sigma^2 > 0$. *Then,* $(\mathsf{P}^{\phi}_k)_{k\geqslant 0}$ *is an exact Bessel sequence in* $\ell^2(\mathbf{n}_\phi)$ *with bound* 1 *and for any* $k \in \mathbb{Z}_+$,

$$\left\|\mathsf{P}^{\phi}_k\right\|_{\ell^2(\mathbf{n}_\phi)} \leqslant 1. \tag{3.83}$$

3. *If* $\sigma^2 > 0$ *and* $\mathrm{d}_\phi > 0$. *Then,* $\left(\sqrt{\mathfrak{c}_k(\mathrm{d}_\phi)}\mathsf{P}^{\phi}_k\right)_{k\geqslant 0}$ *is a Bessel sequence with, for all* $k \in \mathbb{Z}_+$,

$$\left\|\mathsf{P}^{\phi}_k\right\|_{\ell^2(\mathbf{n}_\phi)} \leqslant \frac{1}{\sqrt{\mathfrak{c}_k(\mathrm{d}_\phi)}} \tag{3.84}$$

where $\mathfrak{c}_k(\mathrm{d}_\phi) = \frac{\Gamma(k+\mathrm{d}_\phi+1)}{\Gamma(k+1)\Gamma(\mathrm{d}_\phi+1)}$.

Proof. Let $k \in \mathbb{Z}_+$, then it is plain, from Proposition 1, that, as a polynomial, $\mathsf{P}^{\phi}_k \in \ell^2(\mathbf{n}_\phi)$. Then, we recall, from [95, Theorem 7.3] (after multiplying both sides of the next identity by $(1 + \sigma_1^{-1})^{-\frac{k}{2}}$) that, for any $t > 0$ and $k \in \mathbb{Z}_+$,

$$K^{\phi}_t \mathcal{P}^{\phi}_k = e^{-kt}\mathcal{P}^{\phi}_k$$

where we have set

$$\mathcal{P}^{\phi}_k(x) = (1+\sigma_1^{-1})^{-\frac{k}{2}} \sum_{r=0}^{k} (-1)^r \binom{k}{r} \frac{x^r}{W_\phi(r+1)} \in \mathbf{L}^2(\nu_\phi).$$

Thus, since Λ is injective on $\mathcal{P} \subset \ell^2(\mathbf{n}_\phi)$, the algebra of polynomials, we have $\Lambda^{-1}p_k(n) = \mathsf{p}_k(n)$ and thus, by linearity

$$\Lambda^{-1}\mathcal{P}^{\phi}_k(n) = (1+\sigma_1^{-1})^{-\frac{k}{2}} \sum_{r=0}^{k} (-1)^r \frac{\binom{k}{r}}{W_\phi(r+1)} \Lambda^{-1} p_r(n) = \mathsf{P}^{\phi}_k(n). \tag{3.85}$$

Finally, we observe from the gateway relationship (3.62) and the linearity of the operators, that

$$\Lambda \mathbb{K}^{\phi}_t \mathsf{P}^{\phi}_k = K^{\phi}_t \Lambda\Lambda^{-1}\mathcal{P}^{\phi}_k = K^{\phi}_t \mathcal{P}^{\phi}_k = e^{-kt}\mathcal{P}^{\phi}_k = \Lambda e^{-kt}\mathsf{P}^{\phi}_k.$$

The injectivity of Λ on $\mathcal{P}$ yields the eigenfunction property. To complete the proof of (1), we recall the moment determinacy of $\mathbf{n}_\phi$, stated in Proposition 3.4.12(1), which entails, from classical results on the moment problem, the density property of the algebra of polynomials in the weighted Hilbert space, see [2]. Next, when $\sigma^2 > 0$ and $\phi(u) = \sigma^2 u$, we recall, from Example 3.3.1, that $(\mathsf{P}_k^{\sigma^2})_{k\geqslant 0}$ is an orthonormal sequence of eigenfunctions of $\mathbb{K}_t^{\sigma^2}$ associated to the eigenvalues $\{e^{-kt}\}_{k\geqslant 0}$. Now, from Lemma 3.4.16, it is easily seen, from the definition of P_k^{ϕ}, that, for all $k \geqslant 0$,

$$\mathbb{I}_\phi \mathsf{P}_k^{\sigma^2} = \mathsf{P}_k^{\phi}.$$

Since $\mathbb{I}_\phi \in \mathscr{B}\left(\ell^2(\mathbf{n}_{\sigma^2}), \ell^2(\mathbf{n}_\phi)\right)$ whenever $\sigma^2 > 0$, see (3.76), it follows that, for all $k \geqslant 0$, one has

$$\|\mathsf{P}_k^{\phi}\|_{\ell^2(\mathbf{n}_\phi)} \leqslant \|\mathsf{P}_k^{\sigma^2}\|_{\ell^2(\mathbf{n}_{\sigma^2})} \leqslant 1. \tag{3.86}$$

After recalling that $(\mathsf{P}_k^{\sigma^2})_{k\geqslant 0}$ is a complete orthonormal sequence in $\ell^2(\mathbf{n}_{\sigma^2})$, we observe that, for any $\mathsf{f} \in \ell^2(\mathbf{n}_\phi)$,

$$\sum_{k=0}^{\infty} \left\langle \mathsf{f}, \mathsf{P}_k^{\phi} \right\rangle_{\mathbf{n}_\phi} = \sum_{k=0}^{\infty} \left\langle \widehat{\mathbb{I}}_\phi \mathsf{f}, \mathsf{P}_k^{\sigma^2} \right\rangle_{\mathbf{n}_{\sigma^2}} = \|\widehat{\mathbb{I}}_\phi \mathsf{f}\|^2_{\ell^2(\mathbf{n}_{\sigma^2})} \leqslant \|\mathsf{f}\|^2_{\ell^2(\mathbf{n}_\phi)}.$$

This shows that $(\mathsf{P}_k^{\phi})_{k\geqslant 0}$ is a Bessel sequence in $\ell^2(\mathbf{n}_\phi)$. Combining item (1) with the existence of a biorthogonal sequence, see (3.91) below, we get that $(\mathsf{P}_k^{\phi})_{k\geqslant 0}$ is exact, which proves (2). Finally, to prove (3), let $d_\epsilon = \mathrm{d}_\phi - \epsilon$ for some $0 < \epsilon < \mathrm{d}_\phi$ and define $\phi_{d_\epsilon}(u) = \frac{u\phi(u)}{u+d_\epsilon}$. From [95, Lemma 10.3], it follows that $\phi_{d_\epsilon} \in \mathbf{B}$ and

$$\lim_{u\to\infty} \frac{\phi_{d_\epsilon}(u)}{u} = \sigma^2.$$

Now, we need the following whose proof can be carried out by following a line of reasoning similar to the one of Theorem 3.4.14.

Lemma 3.4.18. *For all $t \geqslant 0$,*

$$\mathbb{K}_t^{\phi} \mathbb{I}_{\phi_{d_\epsilon}} = \mathbb{I}_{\phi_{d_\epsilon}} \mathbb{K}_t^{(d_\epsilon,\sigma^2)} \text{ on } \ell^2(\mathbf{n}_{d_\epsilon,\sigma^2}) \tag{3.87}$$

where $\mathbb{K}^{(d_\epsilon,\sigma^2)}$ is the discrete Laguerre semigroup associated to $\phi(u) = \sigma^2(u + d_\epsilon)$ and $\mathbf{n}_{d_\epsilon,\sigma^2}$ denotes its invariant distribution.

Then, the proof of item (1) ensures that

$$\mathsf{P}_k^{(d_\epsilon,\sigma^2)}(n) = (1+\sigma^{-2})^{-\frac{k}{2}} \Gamma(d_\epsilon + 1) \sum_{r=0}^{k} (-1)^r \binom{k}{r} \frac{\mathsf{p}_r(n)}{\Gamma(r + d_\epsilon + 1)}$$

is an eigenfunction of $\mathbb{K}_t^{(d_\epsilon,\sigma^2)}$ corresponding to the eigenvalue e^{-kt}. Therefore, using Lemma 3.4.18, we have that $\mathbb{I}_{\phi_{d_\epsilon}}\mathsf{P}_k^{(d_\epsilon,\sigma^2)}$ is an eigenfunction of $\mathbb{K}_t^{\phi}$ corresponding to the eigenvalue e^{-kt}, and, in fact,

$$\begin{aligned}\mathbb{I}_{\phi_{d_\epsilon}}\mathsf{P}_k^{(d_\epsilon,\sigma^2)}(n) =&(1+\sigma^{-2})^{-\frac{k}{2}}\Gamma(d_\epsilon+1)\sum_{r=0}^{k}\binom{k}{r}\frac{(-1)^r\mathbb{I}_{\phi_{d_\epsilon}}\mathsf{p}_r(n)}{\Gamma(r+d_\epsilon+1)}\\ =&(1+\sigma^{-2})^{-\frac{k}{2}}\sum_{r=0}^{k}\binom{k}{r}\frac{(-1)^r\mathsf{p}_r(n)}{W_\phi(r+1)} = \mathsf{P}_k^{\phi}(n).\end{aligned}$$

Since the sequence $\left(\sqrt{\mathfrak{c}_k(d_\epsilon)}\mathsf{P}_k^{(d_\epsilon,\sigma^2)}\right)_{k\geqslant 0}$ is an orthonormal sequence in $\ell^2(\mathbf{n}_{d_\epsilon,\sigma^2})$, see [63, equation (7)] or Example 3.3.1, and $\mathbb{I}_{\phi_{d_\epsilon}}$ is bounded, we deduce that $\left(\sqrt{\mathfrak{c}_k(d_\epsilon)}\mathsf{P}_k^{\phi}\right)_{k\geqslant 0}$ is a Bessel sequence in $\ell^2(\mathbf{n}_\phi)$ and $\|\mathsf{P}_k^{\phi}\|_{\mathbf{n}_\phi} \leqslant \frac{1}{\sqrt{\mathfrak{c}_k(d_\epsilon)}}$. Letting $\epsilon \downarrow 0$, the proof of (3) follows.

Proposition 3.4.19. *Let $\phi \in \mathbf{B}$, and, for $k \in \mathbb{Z}_+$, V_k^{ϕ} be defined as in (3.21). Then, the following holds.*

1. *For all $k \in \mathbb{Z}_+$, $\mathsf{V}_k^{\phi} \in \ell^2(\mathbf{n}_\phi)$ and, for all $t \geqslant 0$,*

$$\widehat{\mathbb{K}}_t^{\phi}\mathsf{V}_k^{\phi} = e^{-kt}\mathsf{V}_k^{\phi}.$$

2. *For all $k, l \in \mathbb{Z}_+$,*

$$\left\langle \mathsf{P}_k^{\phi}, \mathsf{V}_k^{\phi}\right\rangle_{\mathbf{n}_\phi} = \mathbb{1}_{\{k=l\}}.$$

3. *If $\sigma^2 > 0$ and $\overline{\overline{\Pi}}(0) < \infty$, then $\left((1+\sigma^{-2})^{-\frac{k}{2}}\frac{\mathsf{V}_k^{\phi}}{\sqrt{\mathfrak{c}_k(\mathrm{m}_\phi)}}\right)_{k\geqslant 0}$ is a Bessel sequence in $\ell^2(\mathbf{n}_\phi)$ where we recall that $\mathrm{m}_\phi = \frac{m+\overline{\overline{\Pi}}(0)}{\sigma^2}$ and $\mathfrak{c}_k(\mathrm{m}_\phi) = \frac{\Gamma(k+\mathrm{m}_\phi+1)}{\Gamma(\mathrm{m}_\phi+1)\Gamma(k+1)}$.*

Proof. Let us recall that for $\phi \in \mathbf{B}$ defined by

$$\phi(u) = m + \sigma^2 u + \int_0^\infty (1-e^{-uy})\overline{\Pi}(y)dy,$$

one has

$$\phi(\infty) = \lim_{u\to\infty}\phi(u) = \infty\mathbb{1}_{\{\sigma^2>0\}} + \left(\overline{\overline{\Pi}}(0) + m\right)\mathbb{1}_{\{\sigma^2=0\}}$$

where $0 \leqslant \overline{\overline{\Pi}}(0) = \int_0^\infty \overline{\Pi}(y)dy$. Finally, let $\mathbf{k}_\phi = \infty \mathbb{1}_{\{\sigma^2>0\}} + \frac{\overline{\overline{\Pi}}(0)}{2\phi(\infty)}$, and define the set

$$\mathbb{Z}_\phi = \begin{cases} \mathbb{Z}_+ & \text{if } \mathbf{k}_\phi = \infty \\ \\ \{k \in \mathbb{Z}_+;\ k < \mathbf{k}_\phi\} & \text{otherwise.} \end{cases} \tag{3.88}$$

When both $\overline{\overline{\Pi}}(0) = \infty$ and $\phi(\infty) = \infty$, we have set $\frac{\overline{\overline{\Pi}}(0)}{\phi(\infty)} = \infty$. Also, the condition (3.4) on Π implies that

$$\int_0^\infty (1 \wedge y)\overline{\Pi}(y)dy < \infty$$

and as a consequence, $\overline{\overline{\Pi}}(0) < \infty$ whenever $\overline{\Pi}(0) < \infty$. Thus, $\mathbf{k}_\phi < \infty$ only when $\sigma^2 = 0$ and $\overline{\Pi}(0) < \infty$. It is shown in [95, Theorem 5.2], that $\nu_\phi \in \mathbf{C}_0^{\lfloor 2\mathbf{k}_\phi \rfloor - 1}(\mathbb{R}_+)$ and in [95, Theorem 1.11] that, for any $k \in \mathbb{Z}_\phi$, $\mathrm{V}_k^\phi \in \mathbf{L}^2(\nu_\phi)$, where

$$\mathrm{V}_k^\phi(x) = \frac{(1+\sigma_1^{-1})^{\frac{k}{2}}}{k!} \frac{\frac{d^k}{dx^k}(x^k \nu_\phi(x))}{\nu_\phi(x)}.$$

We now assume that $k \in \mathbb{Z}_\phi$ and recall from [95, Theorem 8.1], that, for all $t > 0$,

$$\widehat{K}_t^\phi \mathrm{V}_k^\phi = e^{-kt}\mathrm{V}_k^\phi.$$

Now, the intertwining relationship (3.63) entails that, for any $t > 0$,

$$\widehat{\mathbb{K}}_t^\phi \widehat{\Lambda}_\phi \mathrm{V}_k^\phi = \widehat{\Lambda}_\phi \widehat{K}_t^\phi \mathrm{V}_k^\phi = e^{-kt}\widehat{\Lambda}_\phi \mathrm{V}_k^\phi.$$

Let us now characterize the quantity $\widehat{\Lambda}_\phi \mathrm{V}_k^\phi$ when $k \in \mathbb{Z}_\phi$. From (3.64) it can be easily checked that $\widehat{\Lambda}_\phi \mathrm{V}_0^\phi(n) = \widehat{\Lambda}_\phi \mathbf{1}(n) = \mathbf{1}$. Writing $\varrho_1 = \frac{1}{2}\log(1+\sigma_1^{-1})$, for any $n, k \in \mathbb{N}$ we have

$$\begin{aligned} e^{-k\varrho_1}\widehat{\Lambda}_\phi \mathrm{V}_k^\phi(n) &= \frac{1}{n!\mathbf{n}_\phi(n)} \int_0^\infty e^{-x}x^n \frac{\frac{d^k}{dx^k}(x^k\nu_\phi(x))}{k!}dx \\ &= \frac{(-1)^k}{n!\mathbf{n}_\phi(n)k!} \int_0^\infty \frac{d^k}{dx^k}(e^{-x}x^n)x^k\nu_\phi(x)dx \qquad (3.89) \\ &= \frac{(-1)^k}{n!\mathbf{n}_\phi(n)k!} \sum_{j=0}^{k\wedge n} (-1)^{k-j}\binom{k}{j}\frac{n!}{(n-j)!}\int_0^\infty e^{-x}x^{k+n-j}\nu_\phi(x)dx \\ &= \frac{1}{\mathbf{n}_\phi(n)} \sum_{j=0}^{k\wedge n} (-1)^j \frac{(k+n-j)!}{(k-j)!(n-j)!j!}\mathbf{n}_\phi(k+n-j) \\ &= e^{-k\varrho_1}\mathrm{V}_k^\phi(n) \end{aligned} \tag{3.90}$$

where we used, for the second identity, the fact that, for all $j = 1, \ldots, k$,

$$\lim_{x \to 0, \phi(\infty)} \frac{d^{k-j}}{dx^{k-j}} (x^k \nu_\phi(x)) \frac{d^j}{dx^j} (e^{-x} x^n) = 0.$$

Indeed, these asymptotic behaviors are deduced easily from [95, Lemma 5.22], which states that for any $x > 0$, $0 \leqslant j \leqslant k$ and $a < \mathrm{d}_\phi$, $\frac{d^{k-j}}{dx^{k-j}}(x^k \nu_\phi(x)) \leqslant C x^{j+a}$ for some constant $C > 0$. Since $\widehat{\Lambda}_\phi : \mathbf{L}^2(\nu_\phi) \mapsto \ell^2(\mathbf{n}_\phi)$ is a bounded linear operator, see Proposition 3.4.11(3), we have that $\mathsf{V}_k^\phi = \widehat{\Lambda}_\phi \mathrm{V}_k^\phi \in \ell^2(\mathbf{n}_\phi)$ and this concludes the proof of (1) when $k \in \mathbb{Z}_\phi$. Now, let $\sigma^2 > 0$. Then, for any $k, l \in \mathbb{Z}_+$, we have, from Propositions 3.4.17 and the previous computation that both $\mathsf{P}_k^\phi, \mathsf{V}_l^\phi \in \ell^2(\mathbf{n}_\phi)$ and using (3.85) and (3.90), we obtain

$$\left\langle \mathsf{P}_k^\phi, \mathsf{V}_l^\phi \right\rangle_{\mathbf{n}_\phi} = \left\langle \mathsf{P}_k^\phi, \widehat{\Lambda}_\phi \mathrm{V}_l^\phi \right\rangle_{\mathbf{n}_\phi} = \left\langle \mathcal{P}_k^\phi, \mathrm{V}_l^\phi \right\rangle_{\nu_\phi} = \mathbb{1}_{\{k=l\}} \tag{3.91}$$

where we used that $(\mathcal{P}_k^\phi, \mathrm{V}_k^\phi)_{k \geqslant 0}$ is a biorthogonal sequence in $\mathbf{L}^2(\nu_\phi)$, see [95, Theorem 1.22(2)], recall that with the notation of this paper, $\mathcal{P}_k^\phi = (1 + \sigma^{-2})^{-\frac{k}{2}} \mathcal{P}_k$ and $\mathrm{V}_k^\phi = (1 + \sigma^{-2})^{\frac{k}{2}} \mathcal{V}_k$. This proves (2) when $\sigma^2 > 0$.

Next, assume that $k \notin \mathbb{Z}_\phi$ which implies that $\mathbf{k}_\phi < \infty$ and thus $\phi(\infty) < \infty$. This entails that the following two-sided bounds hold for any $n \in \mathbb{Z}_+$

$$e^{-\phi(\infty)} W_\phi(n+1) \leqslant \int_0^{\phi(\infty)} e^{-x} x^n \nu_\phi(x) dx \leqslant W_\phi(n+1) \leqslant \phi(\infty)^n \tag{3.92}$$

where the last inequality follows since ϕ is non-decreasing. Thus, we have

$$e^{-\phi(\infty)} \frac{W_\phi(n+1)}{n!} \leqslant \mathbf{n}_\phi(n) \leqslant \frac{W_\phi(n+1)}{n!} \leqslant \frac{\phi(\infty)^n}{n!}. \tag{3.93}$$

Hence, for any $k \in \mathbb{Z}_+$ fixed, with

$$S(k) = \sum_{n=0}^{k} \frac{1}{\mathbf{n}_\phi(n)} \left(\sum_{j=0}^{n} (-1)^j \frac{(k+n-j)!}{(k-j)!(n-j)!j!} \mathbf{n}_\phi(k+n-j) \right)^2 < \infty,$$

we have

$$
\begin{aligned}
e^{-2k\varrho_1}\|\mathsf{V}_k^{\phi}\|^2_{\ell^2(\mathbf{n}_\phi)} &= S(k) + \sum_{n=k}^{\infty} \frac{1}{\mathbf{n}_\phi(n)} \left(\sum_{j=0}^{k} (-1)^j \frac{(k+n-j)!}{(k-j)!(n-j)!j!} \mathbf{n}_\phi(k+n-j) \right)^2 \\
&\leqslant S(k) + \sum_{n=k}^{\infty} \frac{e^{\phi(\infty)} n!}{W_\phi(n+1)} \left(\sum_{j=0}^{k} \frac{(k+n-j)!}{(k-j)!(n-j)!j!} \frac{\phi(\infty)^{k+n-j}}{(k+n-j)!} \right)^2 \\
&\leqslant S(k) + e^{\phi(\infty)} \sum_{n=k}^{\infty} \frac{n!\phi(\infty)^{2n}}{W_\phi(n+1)((n-k)!)^2} \left(\sum_{j=0}^{k} \frac{\phi(\infty)^{k-j}}{(k-j)!j!} \right)^2 \\
&\leqslant S(k) + e^{\phi(\infty)} \left(\sum_{j=0}^{k} \frac{\phi(\infty)^{k-j}}{(k-j)!j!} \right)^2 \sum_{n=k}^{\infty} \frac{n!\phi(\infty)^{2n}}{W_\phi(n+1)((n-k)!)^2} < \infty \qquad (3.94)
\end{aligned}
$$

where the last inequality follows after observing that,

$$
\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{(n+1)\phi(\infty)^2}{\phi(n+1)(n+1-k)^2} = 0
$$

where the a_n's are the coefficient of the last series. When $\phi \in \mathbf{B}$ is such that $\phi(u) = m + \int_0^\infty (1 - e^{-uy})\overline{\Pi}(y)dy$, let us define $\phi_\epsilon \in \mathbf{B}$ as $\phi_\epsilon(u) = \epsilon u + \phi(u)$ with $\epsilon > 0$. Then, from Proposition 3.4.12, it follows that for small values of ϵ and for all $n \in \mathbb{Z}_+$,

$$
\begin{aligned}
\mathbf{n}_{\phi_\epsilon}(n) &= \frac{1}{n!} \sum_{r=0}^{\infty} (-1)^r \frac{W_{\phi_\epsilon}(n+r+1)}{r!}, \\
\mathbf{n}_{\phi}(n) &= \frac{1}{n!} \sum_{r=0}^{\infty} (-1)^r \frac{W_{\phi}(n+r+1)}{r!}.
\end{aligned}
$$

As $\phi_\epsilon(u) \geqslant \phi(u)$ for all ϵ and $u \geqslant 0$, it follows that $W_{\phi_\epsilon}(n) \geqslant W_\phi(n)$ for all $n \in \mathbb{N}$. Also, $W_{\phi_\epsilon} \downarrow W_\phi$ pointwise as $\epsilon \to 0$. Since, for small values of ϵ (e.g. $0 \leqslant \epsilon < 1$),

$$
\sum_{r=0}^{\infty} \frac{W_{\phi_\epsilon}(r+n+1)}{r!} < \infty,
$$

the dominated convergence theorem yields the following pointwise convergence as $\epsilon \to 0$,

$$
\mathbf{n}_{\phi_\epsilon} \to \mathbf{n}_\phi. \qquad (3.95)
$$

Hence, for any $j, k \in \mathbb{Z}_+$,

$$\lim_{\epsilon\to 0}\langle \mathsf{V}_k^{\phi_\epsilon}, \mathsf{p}_j\rangle_{\mathbf{n}_{\phi_\epsilon}} = \lim_{\epsilon\to 0}\sum_{n=0}^{\infty}\mathsf{p}_j(n)\mathsf{V}_k^{\phi_\epsilon}(n)\mathbf{n}_{\phi_\epsilon}(n) \tag{3.96}$$

$$= \lim_{\epsilon\to 0}\sum_{n=0}^{\infty}\mathsf{p}_j(n)\sum_{j=0}^{k\wedge n}(-1)^{k-j}\frac{(k+n-j)!}{(k-j)!(n-j)!j!}\mathbf{n}_{\phi_\epsilon}(k+n-j)$$

$$= \lim_{\epsilon\to 0}\sum_{n=0}^{k}\mathsf{p}_j(n)\sum_{j=0}^{n}(-1)^{k-j}\frac{(k+n-j)!}{(k-j)!(n-j)!j!}\mathbf{n}_{\phi_\epsilon}(k+n-j) \tag{3.97}$$

$$+ \lim_{\epsilon\to 0}\sum_{n=k}^{\infty}\mathsf{p}_j(n)\sum_{j=0}^{k}(-1)^{k-j}\frac{(k+n-j)!}{(k-j)!(n-j)!j!}\mathbf{n}_{\phi_\epsilon}(k+n-j).$$

In (3.97), the first term is a finite sum and therefore

$$\lim_{\epsilon\to 0}\sum_{n=0}^{k}\mathsf{p}_j(n)\sum_{j=0}^{n}(-1)^{k-j}\frac{(k+n-j)!}{(k-j)!(n-j)!j!}\mathbf{n}_{\phi_\epsilon}(k+n-j)$$

$$= \sum_{n=0}^{k}\mathsf{p}_j(n)\sum_{j=0}^{n}(-1)^{k-j}\frac{(k+n-j)!}{(k-j)!(n-j)!j!}\mathbf{n}_{\phi}(k+n-j).$$

For the second term in (3.97), we have

$$\sum_{n=k}^{\infty}\mathsf{p}_j(n)\sum_{j=0}^{k}(-1)^{k-j}\frac{(k+n-j)!}{(k-j)!(n-j)!j!}\mathbf{n}_{\phi_\epsilon}(k+n-j)$$

$$= \sum_{j=0}^{k}(-1)^{k-j}\frac{1}{j!(k-j)!}\sum_{n=k}^{\infty}\mathsf{p}_j(n)\frac{(k+n-j)!}{(n-j)!}\mathbf{n}_{\phi_\epsilon}(k+n-j).$$

Since $\mathbf{n}_{\phi_\epsilon} \to \mathbf{n}_\phi$ pointwise as $\epsilon \to 0$, the distribution $\mathbf{n}_{\phi_\epsilon}$ converges to $\mathbf{n}_\phi$ weakly. Also, for any $k \in \mathbb{Z}_+$, as $\epsilon \to 0$,

$$\sum_{n=0}^{\infty}\mathsf{p}_k(n)\mathbf{n}_{\phi_\epsilon}(n) = W_{\phi_\epsilon}(k+1) \to W_\phi(k+1) = \sum_{n=0}^{\infty}\mathsf{p}_k(n)\mathbf{n}_\phi(n).$$

Applying (3.54) on the previous identity we obtain that, for all $k \in \mathbb{Z}_+$, as $\epsilon \to 0$,

$$\sum_{n=0}^{\infty} n^k \mathbf{n}_{\phi_\epsilon}(n) \to \sum_{n=0}^{\infty} n^k \mathbf{n}_\phi(n). \tag{3.98}$$

Since, for any $k \in \mathbb{Z}_+$ and $j \leqslant k$,

$$\frac{(k+n-j)!}{(n-j)!} = \mathrm{O}(n^k)$$

uniformly with respect to j, using a dominated convergence theorem one can show that for each $j \leqslant k$,

$$\lim_{\epsilon \to 0} \sum_{n=k}^{\infty} \mathsf{p}_j(n) \frac{(k+n-j)!}{(n-j)!} \mathbf{n}_{\phi_\epsilon}(k+n-j) = \sum_{n=k}^{\infty} \mathsf{p}_j(n) \frac{(k+n-j)!}{(n-j)!} \mathbf{n}_{\phi}(k+n-j).$$

Thus, (3.97) yields

$$\lim_{\epsilon \to 0} \left\langle \mathsf{V}_k^{\phi_\epsilon}, \mathsf{p}_j \right\rangle_{\mathbf{n}_{\phi_\epsilon}} = \left\langle \mathsf{V}_k^{\phi}, \mathsf{p}_j \right\rangle_{\mathbf{n}_{\phi}}. \tag{3.99}$$

Now, if $\sigma^2 = 0$, since, for any $k \in \mathbb{Z}_+$, $\mathsf{P}_k^\phi \in \mathcal{P} = \mathrm{Span}\{\mathsf{p}_j, j \in \mathbb{Z}_+\}$ and the coefficient of p_j in $\mathsf{P}_k^{\phi_\epsilon}$ converges to the same in P_k^ϕ for all $j \in \mathbb{Z}_+$, as $\epsilon \to 0$, applying (3.99) it follows that, for all $k, l \in \mathbb{Z}_+$,

$$\mathbb{1}_{\{k=l\}} = \lim_{\epsilon \to 0} \left\langle \mathsf{P}_k^{\phi_\epsilon}, \mathsf{V}_l^{\phi_\epsilon} \right\rangle_{\mathbf{n}_{\phi_\epsilon}} = \left\langle \mathsf{P}_k^{\phi}, \mathsf{V}_l^{\phi} \right\rangle_{\mathbf{n}_{\phi}}$$

where $\phi_\epsilon(z) = \epsilon u + \phi(u)$. This proves (2) for all $\sigma^2 \geqslant 0$ hence for all $\phi \in \mathbf{B}$.

To show that V_k^ϕ is a co-eigenfunction of $\mathbb{K}^\phi$ when $\sigma^2 = 0$, we proceed as follows. Proposition 3.4.17(1) and (3.91) yield that, for $l, k \in \mathbb{Z}_+$, and $t > 0$,

$$\left\langle \widehat{\mathbb{K}}_t^\phi \mathsf{V}_k^\phi, \mathsf{P}_l^\phi \right\rangle_{\mathbf{n}_\phi} = \left\langle \mathsf{V}_k^\phi, \mathbb{K}_t^\phi \mathsf{P}_l^\phi \right\rangle_{\mathbf{n}_\phi} = e^{-tk} \left\langle \mathsf{V}_k^\phi, \mathsf{P}_l^\phi \right\rangle_{\mathbf{n}_\phi} = e^{-tk} \mathbb{1}_{\{k=l\}}.$$

Therefore, for all $\phi \in \mathbf{B}$, $t > 0$ and $k, l \in \mathbb{Z}_+$, we get

$$\left\langle \widehat{\mathbb{K}}_t^\phi \mathsf{V}_k^\phi - e^{-tk} \mathsf{V}_k^\phi, \mathsf{P}_l^\phi \right\rangle_{\mathbf{n}_\phi} = 0.$$

Since $(\mathsf{P}_k^\phi)_{k \geqslant 0}$ is dense in $\ell^2(\mathbf{n}_\phi)$, we deduce that, for all $t \geqslant 0$ and $k \in \mathbb{Z}_+$,

$$e^{tk} \widehat{\mathbb{K}}_t^\phi \mathsf{V}_k^\phi = \mathsf{V}_k^\phi, \tag{3.100}$$

which proves (1) for all $\phi \in \mathbf{B}$.

To prove item (3), it is known from [95, Theorem 10.1(1)] (after multiplying by the factor $(1+\sigma^{-2})^{-\frac{k}{2}}$) that, when $\sigma^2 > 0$ and $\overline{\overline{\overline{\Pi}}}(0) < \infty$, $\left((1+\sigma^{-2})^{-\frac{k}{2}} \frac{\mathrm{V}_k^\phi}{\sqrt{\mathfrak{c}_k(\mathbf{m}_\phi)}} \right)_{k \geqslant 0}$ is a Bessel sequence in $\mathbf{L}^2(\nu_\phi)$. Recalling that, for any $k \geqslant 0$, $\widehat{\Lambda}_\phi \mathrm{V}_k^\phi = \mathsf{V}_k^\phi$, see (3.90), and $\widehat{\Lambda}_\phi : \mathbf{L}^2(\nu_\phi) \to \ell^2(\mathbf{n}_\phi)$ is a contraction, we conclude that $\left((1+\sigma^{-2})^{-\frac{k}{2}} \frac{\mathsf{V}_k^\phi}{\sqrt{\mathfrak{c}_k(\mathbf{m}_\phi)}}\right)_{k\geqslant 0}$ is a Bessel sequence in $\ell^2(\mathbf{n}_\phi)$ and for all $k \in \mathbb{Z}_+$,

$$\|\mathsf{V}_k^\phi\|_{\ell^2(\mathbf{n}_\phi)} \leqslant (1+\sigma^{-2})^{\frac{k}{2}} \sqrt{\mathfrak{c}_k(\mathrm{m}_\phi)}.$$

3.4.12 Proof of Theorem 3.2.10

The proof of the item (1) and (2) follows directly from Proposition 3.4.17(1) and Proposition 3.4.19(1),(2).

Finally for the proof of (3), we now assume that $\sigma^2 > 0$. We recall from (3.19) that $\sigma_1 = \sigma^2$ in this case. Then, for all $\mathsf{f} \in \ell^2(\mathbf{n}_{\sigma^2})$ and $t > 0$, the intertwining relation (3.77) yields that

$$\begin{aligned} \mathbb{K}_t^\phi \mathbb{I}_\phi \mathsf{f} &= \mathbb{I}_\phi \mathbb{K}_t^{\sigma^2} \mathsf{f} \\ &= \mathbb{I}_\phi \sum_{k=0}^\infty e^{-kt} \left\langle \mathsf{f}, \mathsf{P}_k^{\sigma^2} \right\rangle_{\mathbf{n}_{\sigma^2}} \mathsf{P}_k^{\sigma^2} \\ &= \sum_{k=0}^\infty e^{-kt} \left\langle \mathsf{f}, \mathsf{P}_k^{\sigma^2} \right\rangle_{\mathbf{n}_{\sigma^2}} \mathsf{P}_k^\phi \end{aligned} \tag{3.101}$$

where the second identity relies on the spectral decomposition of the reversible birth-death chain, see Example 3.3.1 with $\phi(u) = \sigma^2 u$, whereas the last one is justified as follows. First, since $\sigma^2 > 0$, $\mathbb{I}_\phi : \ell^2(\mathbf{n}_{\sigma^2}) \mapsto \ell^2(\mathbf{n}_\phi)$ is a bounded linear operator, and, with the help of Lemma 3.4.16 and the definition of P_k^ϕ in (3.20), it follows that $\mathbb{I}_\phi \mathsf{P}_k^{\sigma^2} = \mathsf{P}_k^\phi$. Moreover, from Proposition 3.4.17, we have that the sequence $(\mathsf{P}_k^\phi)_{k\geqslant 0}$ is a Bessel sequence and thus its associated synthesis operator $\mathcal{S} : \ell^2(\mathbb{Z}_+) \mapsto \ell^2(\mathbf{n}_\phi)$, see (3.82) for definition, is bounded. Since $\left(\mathsf{P}_k^{\sigma^2}\right)_{k\geqslant 0}$ is an orthonormal sequence in $\ell^2(\mathbf{n}_{\sigma^2})$, it implies that for all $t \geqslant 0$,

$$\left(e^{-kt} \left\langle \mathsf{f}, \mathsf{P}_k^{\sigma^2} \right\rangle_{\mathbf{n}_{\sigma^2}}\right)_{k\geqslant 0} \in \ell^2(\mathbb{Z}_+)$$

and hence the series on the right-hand side of (3.101) is in $\ell^2(\mathbf{n}_\phi)$. Next, as noted before, we have that $\mathbb{I}_\phi \mathsf{P}_k^{\sigma^2} = \mathsf{P}_k^\phi$ for all $k \in \mathbb{Z}_+$. Now, recalling that $(\mathsf{P}_k^\phi, \mathsf{V}_k^\phi)_{k\geqslant 0}$ is biorthogonal in $\ell^2(\mathbf{n}_\phi)$, see

Proposition 3.4.19(2), we have for any $l, k \in \mathbb{Z}_+$,

$$\left\langle \mathsf{P}_l^{\sigma^2}, \widehat{\mathbb{I}}_\phi \mathsf{V}_k^\phi \right\rangle_{\mathbf{n}_{\sigma^2}} = \left\langle \mathbb{I}_\phi \mathsf{P}_l^{\sigma^2}, \mathsf{V}_k^\phi \right\rangle_{\mathbf{n}_\phi} = \left\langle \mathsf{P}_l^\phi, \mathsf{V}_k^\phi \right\rangle_{\mathbf{n}_\phi} = \mathbb{1}_{\{k=l\}}$$

As $(\mathsf{P}_k^{\sigma^2})_{k\geqslant 0}$ is orthonormal in $\ell^2(\mathbf{n}_{\sigma^2})$ (hence biorthogonal to iteself), by uniqueness of biorthogonal sequence we conclude that $\widehat{\mathbb{I}}_\phi \mathsf{V}_k^\phi = \mathsf{P}_k^{\sigma^2}$ for all $k \in \mathbb{Z}_+$. Therefore, writing $\mathsf{g} = \mathbb{I}_\phi \mathsf{f} \in \ell^2(\mathbf{n}_\phi)$, we have, for all $k \in \mathbb{Z}_+$,

$$\left\langle \mathsf{g}, \mathsf{V}_k^\phi \right\rangle_{\mathbf{n}_\phi} = \left\langle \mathsf{f}, \widehat{\mathbb{I}}_\phi \mathsf{V}_k^\phi \right\rangle_{\mathbf{n}_{\sigma^2}} = \left\langle \mathsf{f}, \mathsf{P}_k^{\sigma^2} \right\rangle_{\mathbf{n}_{\sigma^2}}$$

Thus, from (3.101), for $\mathsf{g} \in \mathrm{Ran}(\mathbb{I}_\phi)$, the range of $\mathbb{I}_\phi$, one gets

$$\mathbb{K}_t^\phi \mathsf{g} \;=\; \sum_{k=0}^{\infty} e^{-kt} \left\langle \mathsf{g}, \mathsf{V}_k^\phi \right\rangle_{\mathbf{n}_\phi} \mathsf{P}_k^\phi = \mathbb{S}_t \mathsf{g}$$

where the last identity serves as defining the spectral operator. Note that since $\langle \mathbb{K}_t^\phi \mathsf{g}, \mathsf{V}_k^\phi \rangle_{\mathbf{n}_\phi} = \langle \mathsf{g}, \widehat{\mathbb{K}}_t^\phi \mathsf{V}_k^\phi \rangle_{\mathbf{n}_\phi} = e^{-kt} \langle \mathsf{g}, \mathsf{V}_k^\phi \rangle_{\mathbf{n}_\phi}$, we deduce that

$$\mathbb{S}_t \mathsf{g} = \sum_{k=0}^{\infty} \left\langle \mathbb{K}_t^\phi \mathsf{g}, \mathsf{V}_k^\phi \right\rangle_{\mathbf{n}_\phi} \mathsf{P}_k^\phi.$$

Moreover, as the closure (in $\ell^2(\mathbf{n}_\phi)$) of $\mathrm{Ran}(\mathbb{I}_\phi)$ is $\ell^2(\mathbf{n}_\phi)$, by the bounded linear transformation theorem, $\mathbb{K}_t^\phi$ is the unique continuous extension of the continuous operator $\mathbb{S}_t : \mathrm{Ran}(\mathbb{I}_\phi) \mapsto \ell^2(\mathbf{n}_\phi)$. We now extend the domain of $\mathbb{S}_t$ to $\ell^2(\mathbf{n}_\phi)$. First, by means of Cauchy-Schwartz inequality, we have, for any $\mathsf{g} \in \ell^2(\mathbf{n}_\phi)$ and $k \in \mathbb{N}$,

$$\begin{aligned} \left| \langle \mathsf{g}, \mathsf{V}_k^\phi \rangle_{\mathbf{n}_\phi} \right| \leqslant \|\mathsf{g}\|_{\ell^2(\mathbf{n}_\phi)} \left\| \mathsf{V}_k^\phi \right\|_{\ell^2(\mathbf{n}_\phi)} &= \|\mathsf{g}\|_{\ell^2(\mathbf{n}_\phi)} \|\widehat{\Lambda}_\phi \mathcal{V}_k^\phi\|_{\ell^2(\mathbf{n}_\phi)} \\ &\leqslant \|\mathsf{g}\|_{\ell^2(\mathbf{n}_\phi)} \left\| \mathcal{V}_k^\phi \right\|_{\mathbf{L}^2(\nu_\phi)} \end{aligned}$$

where we used Proposition 3.4.19 and the fact that $\widehat{\Lambda}_\phi$ is a bounded operator. Next, since from [95, Theorem 10.1], we have for k large enough and all $\epsilon > 0$, $\|\mathcal{V}_k^\phi\|_{\mathbf{L}^2(\nu_\phi)} \leqslant C_\epsilon (1 + \sigma^{-2})^{\frac{k}{2}} e^{\epsilon k}$, with $C_\epsilon > 0$, this implies that for all $\mathsf{g} \in \ell^2(\mathbf{n}_\phi)$ and $t > \frac{1}{2} \log(1 + \sigma^{-2})$,

$$\left(e^{-kt} \langle \mathsf{g}, \mathcal{V}_k^\phi \rangle_{\mathbf{n}_\phi} \right)_{k\geqslant 0} \in \ell^2(\mathbb{Z}_+).$$

Finally, the Bessel property of the sequence $(\mathsf{P}_k^{\phi})_{k\geqslant 0}$ entails that $\mathbb{S}_t\mathsf{g} \in \ell^2(\mathbf{n}_{\phi})$, which completes the proof.

For the item (4), we recall from [95, Theorem 10.1] and the proof of Theorem 3.2.10(3) that, for all $\epsilon > 0$ and $k \in \mathbb{Z}_+$, there exists $C_\epsilon > 0$ such that

$$\left\|\mathsf{V}_k^{\phi}\right\|_{\ell^2(\mathbf{n}_\phi)} \leqslant \left\|\mathsf{V}_k^{\phi}\right\|_{\mathbf{L}^2(\nu_\phi)} \leqslant C_\epsilon(1+\sigma^{-2})^{\frac{k}{2}}e^{\epsilon k} \tag{3.102}$$

whenever $\sigma^2 > 0$. Moreover, for all $k \in \mathbb{Z}_+$, $\|\mathsf{P}_k^{\phi}\|_{\ell^2(\mathbf{n}_\phi)} \leqslant 1$. Therefore, (3.22) entails that for all $t > \frac{1}{2}\log(1+\sigma^{-2})$, the operator $\mathbb{K}_t^{\phi}$ can be approximated by the sequence of finite dimensional operators

$$\mathsf{f} \mapsto \sum_{k=0}^{N} e^{-kt}\left\langle \mathsf{f}, \mathsf{V}_k^{\phi}\right\rangle_{\mathbf{n}_\phi} \mathsf{P}_k^{\phi},\ N \geqslant 1,$$

which proves the compactness of the semigroup. Finally, for $l \in \mathbb{Z}_+$, let us choose $\mathsf{f} = \delta_l$ in (3.22). Then, for all $\sigma^2 > 0, t > 0$ and $n \in \mathbb{Z}_+$, we have

$$\mathbb{K}_t^{\phi}(n,l) = \mathbb{K}_t^{\phi}\delta_l(n) = \sum_{k=0}^{\infty} e^{-kt}\mathsf{P}_k^{\phi}(n)\mathsf{V}_k^{\phi}(l)\mathbf{n}_\phi(l) \tag{3.103}$$

where the last identity holds in $\ell^2(\mathbf{n}_\phi)$. Now, from Proposition 3.4.17(2), we have that, for all $k, n \in \mathbb{Z}_+$,

$$\mathsf{P}_k^{\phi}(n)^2\mathbf{n}_\phi(n) \leqslant \|\mathsf{P}_k^{\phi}\|^2_{\ell^2(\mathbf{n}_\phi)} \leqslant 1$$

while, from Jensen's inequality and (3.102), we get, for all $k, l \in \mathbb{Z}_+$, that there exists a uniform constant $C_\epsilon > 0$ such that for all $\epsilon > 0$,

$$|\mathsf{V}_k^{\phi}(l)\mathbf{n}_\phi(l)| \leqslant \|\mathsf{V}_k^{\phi}\|_{\ell^1(\mathbf{n}_\phi)} \leqslant \|\mathsf{V}_k^{\phi}\|_{\ell^2(\mathbf{n}_\phi)} \leqslant C_\epsilon(1+\sigma^{-2})^{\frac{k}{2}}e^{\epsilon k}. \tag{3.104}$$

Since $\mathbf{n}_\phi(n) > 0$ for all $n \in \mathbb{Z}_+$, see (3.66), for all $\frac{1}{2}\log(1+\sigma^{-2}) + \epsilon < t$, we have

$$\sum_{k=0}^{\infty} e^{-kt}|\mathsf{P}_k^{\phi}(n)||\mathsf{V}_k^{\phi}(l)|\mathbf{n}_\phi(l) \leqslant C_\epsilon \sum_{k=0}^{\infty} e^{-kt}\frac{e^{\epsilon k}}{\sqrt{\mathbf{n}_\phi(n)}} < \infty. \tag{3.105}$$

As $\epsilon > 0$ is arbitrary, the proof of the item (5) is completed.

3.4.13 Proof of Theorem 3.2.12(1)

From [95, Lemma 10.4], we get that

$$\mathrm{m}_\phi = \lim_{n\to\infty} \frac{\phi(n) - \sigma^2 n}{\sigma^2} = \frac{m + \int_0^\infty \Pi(y,\infty)dy}{\sigma^2} > d_\epsilon = (\mathrm{d}_\phi - \epsilon)\mathbb{1}_{\{d_\phi - \epsilon > 0\}}.$$

Let us write $\varrho = \frac{1}{2}\log(1+\sigma^{-2})$. Then, using (3.22) along with the fact that $\mathsf{P}_0^\phi \equiv 1$, we obtain, for all $\mathsf{f} \in \ell_0^2(\mathbf{n}_\phi) = \{\mathsf{g} \in \ell^2(\mathbf{n}_\phi); \mathbf{n}_\phi \mathsf{g} = 0\}$,

$$\begin{aligned}\mathbb{K}_t^\phi \mathsf{f} &= \sum_{k=1}^\infty e^{-kt}\left\langle \mathsf{f}, \mathsf{V}_k^\phi \right\rangle_{\mathbf{n}_\phi} \mathsf{P}_k^\phi \\ &= \sum_{k=1}^\infty e^{-kt}\sqrt{\frac{\mathfrak{c}_k(\mathrm{m}_\phi)}{\mathfrak{c}_k(\mathrm{d}_\phi)}} e^{k\varrho}\left\langle \mathsf{f}, e^{-k\varrho}\frac{\mathsf{V}_k^\phi}{\sqrt{\mathfrak{c}_k(\mathrm{m}_\phi)}}\right\rangle_{\mathbf{n}_\phi} \sqrt{\mathfrak{c}_k(\mathrm{d}_\phi)}\mathsf{P}_k^\phi. \end{aligned} \tag{3.106}$$

Since $\left(\sqrt{\mathfrak{c}_k(\mathrm{d}_\phi)}\mathsf{P}_k^\phi\right)_{k\geqslant 1}$ is a Bessel sequence with bound 1, we obtain from the boundedness of the synthesis operator, see (3.82), and (4.40) that, writing $\overline{\mathsf{V}_k^\phi} = \frac{\mathsf{V}_k^\phi}{\sqrt{\mathfrak{c}_k(\mathrm{m}_\phi)}}$, for all $\mathsf{f} \in \ell_0^2(\mathbf{n}_\phi)$,

$$\begin{aligned}\|\mathbb{K}_t^\phi \mathsf{f}\|^2_{\ell^2(\mathbf{n}_\phi)} &\leqslant \sum_{k=1}^\infty e^{-2kt}\frac{\mathfrak{c}_k(\mathrm{m}_\phi)}{\mathfrak{c}_k(\mathrm{d}_\phi)} e^{2k\varrho}\left|\left\langle \mathsf{f}, e^{-k\varrho}\overline{\mathsf{V}_k^\phi}\right\rangle_{\mathbf{n}_\phi}\right|^2 \\ &= e^{-2t}\frac{\mathfrak{c}_1(\mathrm{m}_\phi)}{\mathfrak{c}_1(\mathrm{d}_\phi)} e^{2\varrho}\sum_{k=1}^\infty e^{-2(k-1)(t-\varrho)}\frac{\mathfrak{c}_k(\mathrm{m}_\phi)\mathfrak{c}_1(\mathrm{d}_\phi)}{\mathfrak{c}_k(\mathrm{d}_\phi)\mathfrak{c}_1(\mathrm{m}_\phi)}\left|\left\langle \mathsf{f}, e^{-k\varrho}\overline{\mathsf{V}_k^\phi}\right\rangle_{\mathbf{n}_\phi}\right|^2. \end{aligned} \tag{3.107}$$

Now, from the proof of [95, Theorem 1.18(3)], we know that

$$\sup_{k\geqslant 1} e^{-2(k-1)(t-\varrho)}\frac{\mathfrak{c}_k(\mathrm{m}_\phi)\mathfrak{c}_1(\mathrm{d}_\phi)}{\mathfrak{c}_k(\mathrm{d}_\phi)\mathfrak{c}_1(\mathrm{m}_\phi)} \leqslant 1 \iff t > T = \frac{1}{2}\log\left(\frac{\mathrm{m}_\phi + 2}{\mathrm{d}_\phi + 2}\right) + \varrho.$$

Thus, using this bound, the fact that $\left(e^{-k\varrho}\overline{\mathsf{V}_k^\phi}\right)_{k\geqslant 1}$ is also a Bessel sequence (with bound 1) in $\ell^2(\mathbf{n}_\phi)$ and the second inequality in (3.81), we deduce from (3.107) that, for all $\mathsf{f} \in \ell_0^2(\mathbf{n}_\phi)$ and $t > T$,

$$\|\mathbb{K}_t^\phi \mathsf{f}\|^2_{\ell^2(\mathbf{n}_\phi)} \leqslant e^{2\mathrm{t}_{\sigma_1^2}}\frac{\mathfrak{c}_1(\mathrm{m}_\phi)}{\mathfrak{c}_1(\mathrm{d}_\phi)} e^{-2t}\|\mathsf{f}\|^2_{\ell^2(\mathbf{n}_\phi)} = \frac{1+\sigma_1^2}{\sigma_1^2}\frac{\mathrm{m}_\phi + 1}{\mathrm{d}_\phi + 1} e^{-2t}\|\mathsf{f}\|^2_{\ell^2(\mathbf{n}_\phi)}. \tag{3.108}$$

When $t \leqslant T$, $\frac{1+\sigma_1^2}{\sigma_1^2}\frac{\mathrm{m}_\phi+1}{\mathrm{d}_\phi+1}e^{-2t} \geqslant \frac{\mathrm{m}_\phi+1}{\mathrm{d}_\phi+1}\frac{\mathrm{d}_\phi+2}{\mathrm{m}_\phi+2} \geqslant 1$ as $\mathrm{m}_\phi \geqslant \mathrm{d}_\phi$. Therefore, (3.108) holds for all $t > 0$ as $\mathbb{K}^\phi$ is a contraction semigroup. Finally, noting that, for any $\mathsf{f} \in \ell^2(\mathbf{n}_\phi)$, $\mathsf{f} - \mathbf{n}_\phi \mathsf{f} \in \ell_0^2(\mathbf{n}_\phi)$, the proof of the theorem follows.

3.4.14 Interweaving between skip-free and continuous Laguerre semigroups

Following [85], for two Markov semigroups P, P' defined on two Banach spaces B, B' respectively, we say that P has an *interweaving relation* with P' if there exist two Markov kernels $\Lambda : \mathrm{B}' \to \mathrm{B}$ and $\Lambda' : \mathrm{B} \to \mathrm{B}'$, and a non-negative random variable τ such that

$$P\Lambda = \Lambda P' \;\text{ on }\; \mathrm{B}'$$

$$P'\Lambda' = \Lambda' P \;\text{ on }\; \mathrm{B} \text{ and}$$

$$\Lambda\Lambda' = P_\tau = \int_0^\infty P_t \mathbb{P}(\tau \in dt).$$

We call τ the *warm-up time* or the *delay* and we write $P \hookleftarrow P'$ or $P \overset{\tau}{\hookleftarrow} P'$ to emphasize the dependence on τ. Note that when $\tau = \delta_{t_0}$ is the degenerate random variable at $t_0 > 0$, we may simply write, when there is no confusion, $P \overset{t_0}{\hookleftarrow} P'$.

When τ is in addition infinitely divisible we say that P admits an *interweaving relation with an infinitely divisible* warm-up time with P' and we write $P \overset{\tau}{\hookleftarrow} P'$. Finally, when we also have

$$\Lambda\Lambda' = P'_\tau \tag{3.109}$$

we say that there is a *symmetric interweaving relation* between P and P' and we write $P \overset{\tau}{\leftrightsquigarrow} P'$. We refer to [85] for a thorough study and several applications of this concept that refines the one of intertwining relations.

Let now $\mathbb{K}^\phi$ be the skip-free Laguerre semigroup corresponding to the Bernstein function ϕ associated with the triplet (m, σ^2, Π), see (3.5), and K^{σ^2} be the diffusive Laguerre semigroup with generator

$$L^{\sigma^2} = \sigma^2 x \frac{d^2}{dx^2} + (\sigma^2 - x)\frac{d}{dx}. \tag{3.110}$$

Theorem 3.4.20. *If $\sigma^2 > 0$ and $\overline{\overline{\Pi}}(0) = \int_0^\infty \Pi(y,\infty)dy < \infty$, then for all $\beta > \mathrm{m}_\phi = \frac{m + \overline{\overline{\Pi}}(0)}{\sigma^2}$,*

$$\mathbb{K}^\phi \overset{\tau_\beta}{\leftrightsquigarrow} K^{\sigma^2}$$

where τ_β *is an infinite divisible random variable characterized, for any* $u > 0$*, by*

$$\int_0^\infty e^{-ut}\mathbb{P}(\tau_\beta \in dt) = \left(\frac{\sigma^2}{1+\sigma^2}\right)^u \frac{\Gamma(1+\beta)\Gamma(u+1)}{\Gamma(u+\beta+1)} = \left(\frac{\sigma^2}{1+\sigma^2}\right)^u e^{-\phi_\beta(u)}. \tag{3.111}$$

Before proving the theorem, let us show the following lemma.

Lemma 3.4.21. *Let* K^{σ^2} *be the semigroup defined as above and* K *be the semigroup with generator*

$$L = x\frac{d^2}{dx^2} + (1-x)\frac{d}{dx}.$$

Then, for all $t \geqslant 0$*,*

$$K_t^{\sigma^2} d_{\frac{1}{\sigma^2}} = d_{\frac{1}{\sigma^2}} K_t \text{ on } \mathbf{L}^2(\nu)$$

where for $\alpha > 0$*,* $d_\alpha f(x) = f(\alpha x)$ *is the dilation operator on* $\mathbb{R}_+$ *and* $\nu(x)dx = e^{-x}dx, x > 0$*, is the unique invariant distribution of the semigroup* K*.*

Proof. It can be easily checked that if f is a polynomial, then

$$L^{\sigma^2} d_{\frac{1}{\sigma^2}} f = d_{\frac{1}{\sigma^2}} Lf. \tag{3.112}$$

Next, we recall from [95, Theorem 1.6(3)] that the set of all polynomials form a core for L in $\mathbf{L}^2(\nu)$ and $d_{\frac{1}{\sigma^2}}\nu_{\sigma^2} = \nu$ where ν_{σ^2} is the invariant distribution of K^{σ^2}. Since

$$d_{\frac{1}{\sigma^2}} : \mathbf{L}^2(\nu) \to \mathbf{L}^2(\nu_{\sigma^2})$$

is an invertible operator, (3.112) extends at the level of the corresponding semigroups, which proves the lemma.

3.4.15 Proof of Theorem 3.4.20

Let $K^{(\beta)}$ be the self-adjoint Laguerre semigroup with the generator

$$L^{(\beta)} = x\frac{d^2}{dx^2} + (1+\beta-x)\frac{d}{dx}.$$

From [85, Proposition 26], it is known that, for all $\beta > \mathrm{m}_\phi$,

$$K^\phi \overset{\tau^{(\beta)}}{\rightsquigarrow} K^{(\beta)}$$

where $\tau^{(\beta)}$ is an infinitely divisible random variable with Laplace transform given by

$$\int_0^\infty e^{-us}\mathbb{P}(\tau^{(\beta)} \in ds) = \frac{\Gamma(1+\beta)\Gamma(u+1)}{\Gamma(u+\beta+1)}, \quad u > 0.$$

More precisely, for all $t \geqslant 0$,

$$K_t^\phi \mathrm{I}_\phi \widehat{\mathrm{B}}_\beta = \mathrm{I}_\phi \widehat{\mathrm{B}}_\beta K_t^{(\beta)} \quad \text{ on } \mathbf{L}^2(\nu_\beta)$$

$$K_t^{(\beta)} \mathrm{V}_\beta = \mathrm{V}_\beta K_t^\phi \quad \text{ on } \mathbf{L}^2(\nu_\phi)$$

with

$$\mathrm{I}_\phi f(x) = \mathbb{E}[f(xI_\phi)] \tag{3.113}$$

and, for all $k \in \mathbb{Z}_+$, $\mathbb{E}[I_\phi^k] = \frac{k!}{W_\phi(k+1)}$. V_β is another multiplicative Markov kernel associated with the random variable Y_β whose law is determined by its moment sequence given, for all $k \in \mathbb{Z}_+$, by

$$\mathbb{E}[Y_\beta^k] = \Gamma(1+\beta)\frac{W_\phi(k+1)}{\Gamma(k+1+\beta)}.$$

Finally, we have $\widehat{\mathrm{B}}_\beta f(x) = \frac{x^\beta}{\Gamma(\beta)}\int_0^\infty f((1+y)x)y^{\beta-1}e^{-yx}dy, x > 0$, and $\mathrm{I}_\phi \widehat{\mathrm{B}}_\beta \mathrm{V}_\beta = K_{\tau^{(\beta)}}^\phi$. Now, from Proposition 3.4.14, we know that

$$\mathbb{K}_t^\phi \mathbb{I}_\phi = \mathbb{I}_\phi \mathbb{K}_t^{\sigma^2} \text{ on } \ell^2(\mathbf{n}_{\sigma^2}) \tag{3.114}$$

where $\mathbb{K}^{\sigma^2} = \mathbb{K}^\phi$ with $\phi(u) = \sigma^2 u$. On the other hand, from Proposition 3.4.11, it is known that

$$K_t^\phi \Lambda = \Lambda \mathbb{K}_t^\phi \text{ on } \ell^2(\mathbf{n}_\phi) \tag{3.115}$$

and from [84, Proposition 25] along with [85, Proposition 30] and Lemma 3.4.21, we have

$$\mathbb{K}_t^{\sigma^2} \widehat{\Lambda}_{\sigma^2} = \widehat{\Lambda}_{\sigma^2} K_t^{\sigma^2} \quad \text{ on } \mathbf{L}^2(\nu_{\sigma^2})$$

$$K_t^{\sigma^2} d_{\frac{1}{\sigma^2}} \widehat{\mathrm{B}}_\beta = d_{\frac{1}{\sigma^2}} \widehat{\mathrm{B}}_\beta K_t^{(\beta)} \quad \text{ on } \mathbf{L}^2(\nu_\beta)$$

$$K_t^{(\beta)} \mathrm{V}_\beta = \mathrm{V}_\beta K_t^\phi \quad \text{ on } \mathbf{L}^2(\nu_\phi)$$

where ν_{σ^2} (resp. ν_β) equals ν_ϕ (the invariant distribution of the semigroup K^ϕ, see Proposition 3.4.11(2)) with $\phi(u) = \sigma^2 u$ (resp. $\phi(u) = u + \beta$), $d_\alpha f(x) = f(\alpha x)$ is the dilation operator and $\widehat{\Lambda}_{\sigma^2} : \mathbf{L}^2(\nu_{\sigma^2}) \to \ell^2(\mathbf{n}_{\sigma^2})$ is a Markov operator defined by

$$\widehat{\Lambda}_{\sigma^2}(n, dx) = \frac{\sigma^{2(n-1)}}{(1+\sigma^2)^n} \frac{x^n}{n+1} e^{-x\left(1+\sigma^{-2}\right)} dx.$$

By transitivity of the intertwining relation, it follows that

$$\mathbb{K}_t^\phi \mathbb{I}_\phi \widehat{\Lambda}_{\sigma^2} = \mathbb{I}_\phi \widehat{\Lambda}_{\sigma^2} K_t^{\sigma^2} \quad \text{on } \mathbf{L}^2(\nu_{\sigma^2}) \tag{3.116}$$

$$K_t^{\sigma^2} \Upsilon = \Upsilon \mathbb{K}_t^\phi \qquad \text{on } \ell^2(\mathbf{n}_\phi) \tag{3.117}$$

where $\Upsilon = d_{\frac{1}{\sigma^2}} \widehat{\mathrm{B}}_\beta \mathrm{V}_\beta \Lambda$. Now, from (3.116) and (3.117), it remains to show that $\mathbb{I}_\phi \widehat{\Lambda}_{\sigma^2} \Upsilon = \mathbb{K}_{\tau_\beta}^\phi$, where τ_β is defined as in the proposition.

Lemma 3.4.22. *The operator* I_ϕ *in* (3.113) *commutes with the dilation operator* d*. Moreover, if* $\sigma^2 > 0$ *then* $d_{\sigma^2} \mathrm{I}_\phi \Lambda = \mathbb{I}_\phi \Lambda$ *on* $\ell^2(\mathbf{n}_\phi)$.

Proof. Since I_ϕ is a multiplicative Markov kernel, commutation with the dilation operator follows readily. Now, for the intertwining relationship, by density of $\mathcal{P} = \mathrm{Span}\{\mathsf{p}_k; k \in \mathbb{Z}_+\}$ in $\ell^2(\mathbf{n}_\phi)$, it suffices to show that, for all $k \in \mathbb{Z}_+$,

$$d_{\sigma^2} \mathrm{I}_\phi \Lambda \mathsf{p}_k = \Lambda \mathbb{I}_\phi \mathsf{p}_k.$$

However, $\mathbb{I}_\phi \mathsf{p}_k = \frac{\sigma^{2k} k!}{W_\phi(k+1)}$ and $d_{\sigma^2} \mathrm{I}_\phi \Lambda \mathsf{p}_k = d_{\sigma^2} \mathrm{I}_\phi p_k = \frac{\sigma^{2k} k!}{W_\phi(k+1)}$, which proves the lemma. Coming back to the main proof, by an application of Lemma 3.4.21 and Lemma 3.4.22, we obtain

$$\Lambda \mathbb{I}_\phi \widehat{\Lambda}_{\sigma^2} \Upsilon = d_{\sigma^2} \mathrm{I}_\phi \Lambda \widehat{\Lambda}_{\sigma^2} \Upsilon = d_{\sigma^2} \mathrm{I}_\phi \Lambda \widehat{\Lambda}_{\sigma^2} d_{\frac{1}{\sigma^2}} \widehat{\mathrm{B}}_\beta \mathrm{V}_\beta \Lambda. \tag{3.118}$$

Again invoking [85, Proposition 25], we have $\Lambda\widehat{\Lambda}_{\sigma^2} = K^{\sigma^2}_{\log(1+\sigma^{-2})}$. Writing $\varrho = \frac{1}{2}\log(1+\sigma^{-2})$ as before, (3.118) yields

$$\begin{aligned}
\Lambda \mathbb{I}_\phi \Upsilon =& d_{\sigma^2} \mathrm{I}_\phi K^{\sigma^2}_\gamma d_{\frac{1}{\sigma^2}} \widehat{\mathrm{B}}_\beta \mathrm{V}_\beta \Lambda \\
=& d_{\sigma^2} \mathrm{I}_\phi d_{\frac{1}{\sigma^2}} \widehat{\mathrm{B}}_\beta K^{(\beta)}_\gamma \mathrm{V}_\beta \Lambda \\
=& \mathrm{I}_\phi \widehat{\mathrm{B}}_\beta \mathrm{V}_\beta K^\phi_{2\varrho} \Lambda \\
=& K^\phi_{\tau^{(\beta)}} K^\phi_{2\varrho} \Lambda \\
=& \Lambda \mathbb{K}^\phi_{\tau_\beta}
\end{aligned}$$

where in the last line of the above equation, we used the fact that $\tau_\beta = \tau^{(\beta)} + 2\varrho$. By injectivity of Λ, it follows that $\mathbb{I}_\phi \Upsilon = \mathbb{K}^\phi_{\tau_\beta}$. This proves the proposition. □

3.4.16 Proof of Theorem 3.2.12(2)

We recall that L^{σ^2} is the generator of the self-adjoint Laguerre diffusion defined in (3.110) whose invariant distribution is $\nu_{\sigma^2}(x)dx = \frac{1}{\sigma^2}\nu(x/\sigma^2) = \frac{1}{\sigma^2}e^{-x/\sigma^2}dx,\ x > 0$. Let us first prove the Φ-entropy decay for K^{σ^2}, the semigroup generated by L^{σ^2}, that is, for all admissible function Φ and $f \in \mathbf{L}^1(\nu_{\sigma^2})$ with $\Phi(f) \in \mathbf{L}^1(\nu_{\sigma^2})$ one has

$$\mathrm{Ent}^\Phi_{\nu_{\sigma^2}}(K^{\sigma^2}_t f) \leqslant e^{-t}\mathrm{Ent}^\Phi_{\nu_{\sigma^2}}(f). \tag{3.119}$$

In Lemma 3.4.21, we have shown that the semigroups K^{σ^2} and K are equivalent via the similarity transform induced by the dilation operator d_{σ^2}. We first claim that it is enough to prove the exponential entropy decay in (3.119) replacing K^{σ^2} by K. To see why, we note that for any $\sigma^2 > 0$, $f \in \mathbf{L}^1(\nu_{\sigma^2})$ with $\Phi(f) \in \mathbf{L}^1(\nu_{\sigma^2})$, one has by the change of variable along with Lemma 3.4.21 that,

$$\begin{aligned}
\int_0^\infty \Phi(K^{\sigma^2}_t f(x))\nu_{\sigma^2}(x)dx &= \int_0^\infty \frac{1}{\sigma^2}\Phi\left(K_t d_{\sigma^2} f\left(\frac{x}{\sigma^2}\right)\right)\nu\left(\frac{x}{\sigma^2}\right)dx \\
&= \int_0^\infty \Phi(K_t d_{\sigma^2} f(x))\nu(x)dx.
\end{aligned}$$

We also observe by the change of vairable that for any $f \in \mathbf{L}^1(\nu_{\sigma^2})$, one has $\Phi(\nu_{\sigma^2} f) = \Phi(\nu d_{\sigma^2} f)$. As a result, we have

$$\mathrm{Ent}^{\Phi}_{\nu_{\sigma^2}}(K^{\sigma^2}_t f) = \mathrm{Ent}^{\Phi}_{\nu}(K_t d_{\sigma^2} f)$$

which proves our claim. Next, we state the following result regarding the exponential entropy decay of the semigroup K generated by L.

Lemma 3.4.23. *For any Φ as above and $f \in \mathbf{L}^1(\nu)$ with $\Phi(f) \in \mathbf{L}^1(\nu)$, one has*

$$\mathrm{Ent}^{\Phi}_{\nu}(K_t f) \leqslant e^{-t}\mathrm{Ent}^{\Phi}_{\nu}(f).$$

Proof. Since L is a diffusion operator, from [20, Equations (6) and (7)], it suffices to show that for an admissible function Φ and $f \in \mathbf{L}^1(\nu)$ with $\Phi(f) \in \mathbf{L}^1(\nu)$, one has the following Φ-entropy inequality

$$\mathrm{Ent}^{\Phi}_{\mu}(f) \leqslant \mu(\Phi''(f)\Gamma(f)) \tag{3.120}$$

where Γ is the carré-du-champ operator, see [7, Section 1.4.2] associated to L, that is, for smooth funcitons

$$\Gamma(f) = L(f^2) - 2fLf.$$

From [26, Theorem 2.1(2)] it follows that (3.120) is equivalent to the fact that the operator L satisfies the curvature dimension condtion $CD(\frac{1}{2}, \infty)$, which is indeed true from [7, Section 2.7.3]. Hence the lemma is proved. Now coming back to the proof of Theorem 3.2.12(2), due to the interweaving relation in Theorem 3.4.20 and the estimate in (3.119), the proof of this theorem follows directly from [85, Theorem 8]. □

3.4.17 Proof of Theorem 3.2.14

For ergodic self-adjoint diffusion semigroups, we know from [7, Theorem 5.2.3] that the hypercontractivity can be interpreted in terms of the log-Sobolev constants corresponding to the semigroups.

Let us consider the self-adjoint Laguerre semigroup K^{σ^2} defined in Proposition 3.4.20. For this semigroup, the invariant distribution is $\nu_{\sigma^2}(x)dx = \frac{1}{\sigma^2}e^{-x/\sigma^2}dx,\ x > 0$, and the log-Sobolev constant is

$$c_{LS} = \inf_{f\in\mathbf{C}_b^1(\mathbb{R}_+):\|f\|_{\mathbf{L}^2(\nu_{\sigma^2})}=1} \frac{4\int_{\mathbb{R}_+} xf'(x)^2\nu_{\sigma^2}(dx)}{\int_{\mathbb{R}_+} f(x)^2\log(f(x)^2)\nu_{\sigma^2}(dx)}. \tag{3.121}$$

The numerator in the above expression is four times the Dirichlet energy associated to L^{σ^2} defined by

$$\mathcal{E}(f,f) = -\langle L^{\sigma^2}f, f\rangle_\nu = \int_{\mathbb{R}_+} xf'(x)^2\nu_{\sigma_1^2}(dx).$$

It was shown by Bakry [6] that $c_{LS} = 1$. Hence, by applying [7, Theorem 5.2.3], we infer that for all $t \geqslant 0$,

$$\|K_t^{\sigma^2}\|_{\mathbf{L}^2(\nu_{\sigma^2})\to\mathbf{L}^{p(t)}(\nu_{\sigma^2})} \leqslant 1$$

where $p(t) = 1 + e^t$ and $\nu(dx) = e^{-x}dx,\ x > 0$. Having the above hypercontractivity estimate, the rest of the proof follows from [85, Theorem 9] and Theorem 3.4.20. □

3.4.18 Proof of Theorem 3.2.15

First, we note that the semigroup $\mathbb{K}^{\phi,\tau_\beta}$ has the same invariant distribution $\mathbf{n}_\phi$ as $\mathbb{K}^\phi$. Let us recall that for $\sigma^2 > 0$, $\varrho = \frac{1}{2}\log(1+\sigma^{-2})$. If $t > \frac{1}{2}$, Theorem 3.2.10(3) entails that, for all $s > 0$ and $\mathsf{f} \in \ell^2(\mathbf{n}_\phi)$, we have

$$\mathbb{K}_{s+2\varrho t}^\phi \mathsf{f} = \sum_{k=0}^\infty e^{-2k\varrho t}e^{-ks}\langle \mathsf{f}, \mathsf{V}_k^\phi\rangle_{\mathbf{n}_\phi}\mathsf{P}_k^\phi \ \text{ in } \ \ell^2(\mathbf{n}_\phi). \tag{3.122}$$

For $t \geqslant 0$, let us define the random variable $\widetilde{\tau}_\beta(t)$ such that $\tau_\beta(t) = 2\varrho t + \widetilde{\tau}_\beta(t)$. Indeed, from (3.29) it follows that for all $t \geqslant 0$,

$$\log\mathbb{E}\left[e^{-u\widetilde{\tau}_\beta(t)}\right] = -\log\left(\frac{\Gamma(u+\beta+1)}{\Gamma(1+\beta)\Gamma(u+1)}\right).$$

Then, integrating both sides of (3.122) with respect to $\mathbb{P}(\widetilde{\tau}_\beta(t) \in ds)$ with $t > \frac{1}{2}$ we obtain

$$\begin{aligned}
\mathbb{K}_t^{\phi,\tau_\beta}\mathsf{f} = \int_0^\infty \mathbb{K}_s^\phi \mathsf{f}\,\mathbb{P}(\tau_\beta(t) \in ds) = \int_0^\infty \mathbb{K}_{s+2\varrho t}^\phi \mathsf{f}\,\mathbb{P}(\widetilde{\tau}_\beta(t) \in ds)& \\
= \int_0^\infty \left(\sum_{k=0}^\infty e^{-2k\varrho t} e^{-ks} \langle \mathsf{f}, \mathsf{V}_k^\phi\rangle_{\mathbf{n}_\phi} \mathsf{P}_k^\phi\right) \mathbb{P}(\widetilde{\tau}_\beta(t) \in ds)& \\
= \sum_{k=0}^\infty e^{-2k\varrho t} \langle \mathsf{f}, \mathsf{V}_k^\phi\rangle_{\mathbf{n}_\phi} \mathsf{P}_k^\phi \int_0^\infty e^{-ks}\mathbb{P}(\widetilde{\tau}_\beta(t) \in ds)&
\end{aligned}$$

where the last equality follows due to Fubini theorem with the help of the estimates $\|\mathsf{P}_k^\phi\|_{\ell^2(\mathbf{n}_\phi)} \leqslant 1,\ \|\mathsf{V}_k^\phi\|_{\ell^2(\mathbf{n}_\phi)} \leqslant C_\epsilon e^{k(\varrho+\epsilon)}$ for arbitrary $\epsilon > 0$ and $k \in \mathbb{Z}_+$, see Proposition 3.4.17 and the proof of Theorem 3.2.10(3). The proof of this item is concluded by recalling that, for all $k \in \mathbb{Z}_+$,

$$e^{-2\varrho t} \int_0^\infty e^{-ks}\mathbb{P}(\widetilde{\tau}_\beta(t) \in ds) = e^{-t\phi_\beta(k)}.$$

For the next item, applying Jensen's inequality we observe that, for all $\beta, t > 0$,

$$\mathrm{Ent}_{\mathbf{n}_\phi}^\Phi\left(\mathbb{K}_{t+1}^{\phi,\tau_\beta}\mathsf{f}\right) \leqslant \int_0^\infty \mathrm{Ent}_{\mathbf{n}_\phi}^\Phi\left(\mathbb{K}_{s+\tau_\beta}^\phi \mathsf{f}\right) \mathbb{P}(\tau_\beta(t) \in ds).$$

Using Theorem 2, when $\sigma^2 > 0$ and $\beta > \mathrm{m}_\phi$, and with the right-hand side of the above inequality is bounded above by

$$\int_0^\infty e^{-s}\mathrm{Ent}_{\mathbf{n}_\phi}^\Phi(\mathsf{f})\mathbb{P}(\tau_\beta(t) \in ds) = e^{-t\phi_\beta(1)}\mathrm{Ent}_{\mathbf{n}_\phi}^\Phi(\mathsf{f}).$$

This proves (2). Finally, with $\tau_\beta = \tau_\beta(1)$ chosen independent of $(\tau_\beta(t))_{t\geqslant 0}$ which we recall is a subordinator, we have, by the triangle inequality, that, for any $\mathsf{f} \in \ell^2(\mathbf{n}_\phi)$,

$$\begin{aligned}
\left\|\mathbb{K}_{t+1}^{\phi,\tau_\beta}\mathsf{f}\right\|_{\ell^{p(\alpha t)}(\mathbf{n}_\phi)} &= \left\|\int_0^\infty \mathbb{K}_{s+\tau_\beta}^\phi \mathsf{f}\,\mathbb{P}(\tau_\beta(t) \in ds)\right\|_{\ell^{p(\alpha t)}(\mathbf{n}_\phi)} \\
&= \left\|\int_0^\infty \mathbb{K}_{s+2\varrho t+\tau_\beta}^\phi \mathsf{f}\,\mathbb{P}(\widetilde{\tau}_\beta(t) \in ds)\right\|_{\ell^{p(\alpha t)}(\mathbf{n}_\phi)} \\
&\leqslant \int_0^\infty \left\|\mathbb{K}_{s+2\varrho t+\tau_\beta}^\phi \mathsf{f}\right\|_{\ell^{p(2\varrho t)}(\mathbf{n}_\phi)} \mathbb{P}(\widetilde{\tau}_\beta(t) \in ds).
\end{aligned}$$

Invoking Theorem 3.2.14, the right-hand side of the above inequality is bounded above by

$$\|\mathsf{f}\|_{\ell^2(\mathbf{n}_\phi)} \int_0^\infty \mathbb{P}(\widetilde{\tau}_\beta(t) \in ds) = \|\mathsf{f}\|_{\ell^2(\mathbf{n}_\phi)}$$

which completes the proof.

CHAPTER 4

A NEW CLASS OF SOLUTIONS TO THE VAN DANTZIG PROBLEM, THE LEE-YANG PROPERTY, AND THE RIEMANN HYPOTHESIS

4.1 Introduction

In his seminal paper [82], E. Lukacs provided comprehensive and fascinating insights towards the characterization of the class of functions

$$\mathscr{D} = \{\mathcal{F} \in \mathscr{P}_+;\ V\mathcal{F} \in \mathscr{P}_+\} \tag{4.1}$$

where

$$V\mathcal{F}(t) = \frac{1}{\mathcal{F}(\mathrm{i}t)}, t \in \mathbb{R},$$

and $\mathscr{P}_+$ stands for the set of continuous bounded positive-definite functions, normalized to be 1 at the origin. Note that by Bochner's theorem, $\mathscr{P}_+$ is the set of characteristic functions of real-valued random variables, see e.g. [120]. This question was posed, as a prize-winning problem, by David van Dantzig, a Dutch algebraic topologist, in Nieuw Archief voor Wiskunde.

If $\mathcal{F} \in \mathscr{D}$ then the function $V\mathcal{F}$ is called the (van Dantzig) reciprocal of $\mathcal{F}$, and also belongs to $\mathscr{D}$. This last fact entails, see Proposition 4.3.1, that any $\mathcal{F} \in \mathscr{D}$ admits an analytic extension on a cross section of the complex plane including the imaginary line in its interior. We also let

$$\mathscr{D}_2 = \{[\mathcal{F}, V\mathcal{F}] \in \mathscr{P}_+ \times \mathscr{P}_+;\ \mathcal{F}(\mathrm{i}t) \cdot V\mathcal{F}(t) = 1 \text{ for all } t \in \mathbb{R}\}. \tag{4.2}$$

We say that $\mathcal{F}$ is a van Dantzig function and that $[\mathcal{F}, V\mathcal{F}]$ is a van Dantzig pair. Note that the mapping $V : \mathscr{D} \to \mathscr{D}$ is a multiplicative involution, that is, $V\mathcal{F}_1\mathcal{F}_2 = V\mathcal{F}_1 V\mathcal{F}_2$ and $V \circ V\mathcal{F} = \mathcal{F}$, see Proposition 4.3.2 below.

The first historical instances of non-trivial van Dantzig pairs are

$$\left[\cos t, \frac{1}{\cosh t}\right], \left[\frac{\sin t}{t}, \frac{t}{\sinh t}\right] \text{ and } \left[e^{-\frac{t^2}{2}}, e^{-\frac{t^2}{2}}\right]. \tag{4.3}$$

We notice that, in the last example,

$$\mathcal{F}_{\mathrm{N}}(t) = e^{-\frac{t^2}{2}}$$

is the characteristic function of a standard normal random variable for which it is immediate that $V\mathcal{F}_{\mathrm{N}}(t) = 1/\mathcal{F}_{\mathrm{N}}(\mathrm{i}t) = \mathcal{F}_{\mathrm{N}}(t)$ and hence $\mathcal{F}_{\mathrm{N}} \in \mathscr{D}_S$, where

$$\mathscr{D}_S = \{\mathcal{F} \in \mathscr{P}_+;\ \mathcal{F} = V\mathcal{F}\}$$

is the invariant set of V or self-reciprocal elements of $\mathscr{D}$. It is easy to see that, for any $\mathcal{F} \in \mathscr{D}$, the mapping $t \mapsto \mathcal{F}(t)V\mathcal{F}(t)$ is in $\mathscr{D}_S$, meaning that the set $\mathscr{D}_S$ contains many more elements than $\mathcal{F}_{\mathrm{N}}$. However, the following fact, due to Lukacs, offers an original characterization of the characteristic function $\mathcal{F}_{\mathrm{N}}$

$$\big\{[\mathcal{F}_1, \mathcal{F}_2] \in \mathscr{D}_2;\ -\log \mathcal{F}_j \in \mathbb{N}(\mathbb{R}),\ j = 1, 2\big\} = \big\{[\mathcal{F}_{\mathrm{N}}, \mathcal{F}_{\mathrm{N}}]\big\}$$

where $\mathbb{N}(\mathbb{R})$ is the set of negative definite functions on $\mathbb{R}$, see [120, Chapter 4] for more information on this set. In other words, if $\mathcal{F}$ and $V\mathcal{F}$ are infinitely divisible then they are identical and both equal to $\mathcal{F}_{\mathrm{N}}$.

Regarding the first example in (4.3), it is immediate that the mapping $t \mapsto \cos t$ is the characteristic function of a random variable taking values ± 1 with equal probability, whereas $t \mapsto (\cosh t)^{-1}$ is the characteristic function of a random variable with density $(2\cosh(\pi x/2))^{-1}$, $x \in \mathbb{R}$. For the second example, the mapping $t \mapsto \sin t/t$ corresponds to the characteristic function of a uniformly distributed random variable on the interval $[-1, 1]$ and $t \mapsto t(\sinh t)^{-1}$ to the one of an absolutely continuous probability measure whose density is $\frac{\pi}{4}(1 - \tanh(\pi x/2))^2, x \in \mathbb{R}$.

Another classical example, discussed at the end of Lukacs's paper, and presented in a more general form in [59, 50], is expressed in terms of the entire functions

$$\mathcal{J}_\nu(t) = \Gamma(\nu + 1)t^{-\nu} J_\nu\left(t\right), \quad \mathcal{I}_\nu(t) = \mathcal{J}_\nu(\mathrm{i}t), \tag{4.4}$$

where J_ν is the Bessel function of the first kind of index ν, and given by the pair

$$\left[\mathcal{J}_\nu, \frac{1}{\mathcal{I}_\nu}\right] \in \mathscr{D}_2, \quad \nu > -\frac{1}{2}. \tag{4.5}$$

The corresponding pair of random variables are described in details in [59], see also (4.65) and Lemma 4.5.7. This set of examples turns out to be the only canonical solutions to the van Dantzig problem available in the literature. Note that they all belong to the subclass $\mathscr{D}_L \subset \mathscr{D}$, originally identified by Lukacs [82], which stands for the set of even entire characteristic functions in the Laguerre-Pólya class. That is, entire functions which are locally the limit of a series of polynomials whose roots are all real, see Section 4.2 for further discussion on this class. Throughout, we shall provide several ways of generating new instances in $\mathscr{D}_L$ and also present new subclasses of $\mathscr{D}$.

We start by identifying a connection between two fundamental problems in mathematics and the subclass $\mathscr{D}_L \subseteq \mathscr{D}$. First, we explain how the Riemann hypothesis can be equivalently formulated by the membership of the Riemann ξ function to the class $\mathscr{D}_L$, something which, in a different context, was observed by Roynette and Yor [117]. Similarly, the celebrated Lee-Yang property in statistical physics, discovered first by Lee and Yang in connection to the Ising model on a finite lattice, is also equivalent to the requirement that the partition function, viewed as a function on the imaginary line, belongs to $\mathscr{D}_L$, see [55] for a thorough account on this topic. This connection relies on the theory of Pólya frequency functions developed by Schoenberg [121], see Section 4.2.

In view of its importance, we first aim to develop an in-depth analysis of the subclass $\mathscr{D}_L$. On the one hand, we adapt several substantial results that one can find in the number theory and statistical mechanics literature to provide new information about the class $\mathscr{D}_L$. Conversely, by means of probabilistic techniques combined with the theory of entire functions, we also provide original closure properties of the class $\mathscr{D}_L$ which give new insights to the two aforementioned problems. For instance, we find necessary conditions for the product of two independent random variables to belong or to remain in $\mathscr{D}_L$, see Theorem 4.3.13. In the same vein, in Theorem 4.3.14, we revisit, improve and extend to the class $\mathscr{D}$, a recent result due to Newman and Wu [91] regarding the closure property of this class under locally uniform convergence.

Another objective of this chapter is to identify a new subclass, denoted by $\mathscr{D}_P$, of analytic characteristic functions that belong to the class $\mathscr{D}$, that is, are solutions to the van Dantzig problem.

More specifically, to each Laplace exponent Ψ of a (possibly killed) spectrally negative Lévy process which is positive on the interval $[1/2, \infty)$, we associate the function $\mathcal{J}_\Psi \in \mathscr{D}_P$ which is defined by

$$\mathcal{J}_\Psi(t) = 1 + \sum_{n=1}^{\infty} \frac{(-1)^n}{\Psi(1)\cdots\Psi(n)} t^{2n}.$$

Since $\lim_{u\to\infty} \Psi(u) = \infty$, we deduce easily that $\mathcal{J}_\Psi$ defines an entire function. We refer to sections 4.4 and 4.5.2 for more information about the objects introduced here and this class of functions, whose analysis is intimately related to the Wiener-Hopf factorization of the Laplace exponent Ψ. We also point out that the entire function $\mathcal{I}_\Psi(t) = \mathcal{J}_\Psi(\mathrm{i}\sqrt{t})$ was introduced in [102], and has appeared in various mathematical contexts recently, in probability theory [99], in the spectral theory of some non-self-adjoint operators [95, 107], in the study of special functions [133] and a chapter is devoted to this class of function in the monograph [74]. Note that when $\Psi(n) = n(\nu + n)$ is the Laplace exponent of a (scaled) Brownian motion with drift $\nu > -\frac{1}{2}$, then $\mathcal{J}_\Psi$ boils down to the Bessel-Clifford function $\mathcal{J}_\nu$ as defined in (4.4). The class $\mathscr{D}_P$ includes a wide range of other well-known special functions such as several hypergeometric functions, the Mittag-Leffler functions, and the Wright functions, among others, see Section 4.4.2.

Theorem 4.4.1 states that $\mathcal{J}_\Psi \in \mathscr{D}$. Its proof relies on identifying the two random variables whose characteristic functions are $\mathcal{J}_\Psi$ and $1/\mathcal{I}_\Psi, \mathcal{I}_\Psi(t) = \mathcal{J}_\Psi(\mathrm{i}t)$, see Lemmas 4.5.8 and 4.5.7, respectively. This is achieved by introducing a Markov operator, see (4.60), that serves on the one hand to show that $\mathcal{J}_\Psi$ is the characteristic function of the product of two independent random variables (one having as characteristic function the Bessel function $\mathcal{J}_0$). On the other hand, it also turns out to be an intertwining operator between two Markov semigroups, see e.g. [95] for a review of this concept. From this fact, we deduce first that $t \mapsto \mathcal{I}_\Psi(\sqrt{t})$ is an invariant function for one of these semigroups and then its reciprocal is the Laplace transform of a positive random variable associated to the Markov process, from where we conclude by invoking an argument involving the Bochner subordination of a Brownian motion. Moreover, since $\mathcal{J}_\nu \in \mathscr{D}_L \cap \mathscr{D}_P$, it is natural to wonder whether $\mathscr{D}_P \subset \mathscr{D}_L$. This is a very delicate question as it is difficult, in general, to identify the location of zeros of a power series. However, we manage to provide necessary conditions on

Ψ for $\mathcal{J}_\Psi \in \mathscr{D}_L$, that is the entire function has only real zeros. We also identify instances of entire functions in $\mathscr{D}_P$ which have non real zeros, revealing that $\mathscr{D}_L \subsetneq \mathscr{D}$, see Theorem 4.4.4.

The remaining part of the chapter is organized as follows. In Section 4.2, we introduce the Lukacs class $\mathscr{D}_L$ and present its connection with the Riemann hypothesis and the Lee-Yang property. Section 4.3 is devoted to a thorough analysis of the sets $\mathscr{D}$ and $\mathscr{D}_L$, including some closure properties of $\mathscr{D}_L$ under various mappings, and to the identification of original ways to generate new elements in $\mathscr{D}_L$. In Section 4.4 we introduce the new subclass $\mathscr{D}_P$ and provide some interesting properties. Finally, in Section 4.5, we collect the proofs of the results presented in the two previous sections.

4.2 The Lukacs class $\mathscr{D}_L$

We start by recalling that an entire function is in the Laguerre-Pólya class $\mathscr{LP}$ if it is the local uniform limit of a sequence of polynomials with real coefficients and real zeros only. In fact, based on ideas of Laguerre, Pólya and Schur [113] showed that φ is in $\mathscr{LP}$ if and only if

$$\varphi(z) = Kz^m e^{-c^2z^2+az} \prod_{k=1}^{\infty} \left(1 - \frac{z}{z_k}\right) e^{z/z_k}, \ z \in \mathbb{C}, \tag{4.6}$$

for some $m \in \mathbb{Z}_+$, $K, c, a \in \mathbb{R}$, $z_k \in \mathbb{R} \setminus \{0\}$ such that $\sum 1/z_k^2 < \infty$, and, where z_k, $k \in \mathbb{N}$, are the nonzero zeros of the entire function φ, arranged in order of nondecreasing modulus.

Lukacs was interested in even characteristic functions in the class $\mathscr{LP}$ and made the observation that such functions are automatically in $\mathscr{D}$, something that we explain in the sequel. Let now

$$\mathscr{LP}_e = \{\varphi \in \mathscr{LP};\ \varphi \text{ is even and } \varphi(0) = 1\}. \tag{4.7}$$

The representation (4.6) immediately gives that $\varphi \in \mathscr{LP}_e$ must be of the form

$$\varphi(z) = e^{-c^2z^2} \prod_{k=1}^{\infty} \left(1 - \frac{z^2}{z_k^2}\right), \quad c \in \mathbb{R}, \quad z_k > 0, \quad \sum_k 1/z_k^2 < \infty. \tag{4.8}$$

Indeed, the even functions in $\mathscr{LP}$ are the functions of the form (4.6) with $m = 0$, $K = 1$ and zeros that are symmetrically placed around 0 on the real axis, resulting in precisely the form (4.8). Next, we mention that the order ρ and the exponent of convergence ϱ of an entire function φ in the Laguerre-Pólya class are such that

$$0 \leqslant \varrho \leqslant \rho \leqslant 2$$

where we recall that $\rho = \overline{\lim}_{r\to\infty} \frac{\log\log\max_{|z|=r}|\varphi(z)|}{\log r}$ and $\varrho = \inf\left\{\alpha > 0;\ \sum_{k\geqslant 1}|z_k|^{-\alpha} < \infty\right\}$. Here and throughout, we refer to the monograph of Levin [79] for information related to entire functions.

The functions in the examples (4.3) and (4.5) are all of the form (4.8). However, we emphasize that if an entire function φ is of the form (4.8) then it needs not be the case that the mapping $t \mapsto \varphi(t)$ is positive-definite on $\mathbb{R}$. Nevertheless, Schoenberg, following Hadamard, proved that the reciprocal of a normalized Laguerre-Pólya entire function (non-necessarily even) is in $\mathscr{P}_+$. In particular, we have the following.

Theorem 4.2.1. *[121, Theorem 1] If φ is of the form (4.8) then*

$$t \mapsto \frac{1}{\varphi(\mathrm{i}t)} = e^{-ct^2}\prod_{k=1}^{\infty}\left(1+\frac{t^2}{z_k^2}\right)^{-1} \tag{4.9}$$

is the characteristic function of a symmetric Pólya frequency density f_P, namely,

$$\mathcal{F}_{f_P}(t) = \frac{1}{\varphi(\mathrm{i}t)} = \int_{\mathbb{R}} e^{\mathrm{i}tx} f_P(x)dx$$

where f_P is a symmetric probability density function on $\mathbb{R}$ such that, for all $n \in \mathbb{N}$, $x_1 < \cdots < x_n$, $y_1 < \cdots < y_n$, the determinant of the matrix $[f_P(x_j - y_k)]_{j,k=1}^n$ is non-negative.

Moreover, the probability measure with density f_P is infinitely divisible, that is, equivalently, $t \mapsto -\log \mathcal{F}_{f_P}(\mathrm{i}t) \in \mathbb{N}(\mathbb{R})$, see Kwaśnicki [72, Proposition 5.3].

We point out that an easy way to see why (4.9) is a characteristic function is by a probabilistic argument. Indeed, let $\mathrm{N}, \mathrm{Z}_1, \mathrm{Z}_2, \ldots$ be independent random variables where N is a standard normal and each Z_j a standard Laplace random variable, that is, it has density $\frac{1}{2}e^{-|x|}$, $x \in \mathbb{R}$. Then, one observes that the random variable

$$\mathrm{X} = c\sqrt{2}\mathrm{N} + \sum_{j=1}^{\infty} \frac{\mathrm{Z}_j}{z_j} \tag{4.10}$$

has characteristic function (4.9). Then, relying on Theorem 4.2.1, Lukacs observed the very useful fact that if an even characteristic function $\mathcal{F}$ is in the Laguerre-Pólya class then it is necessarily solution to the van Dantzig problem. This leads us to introduce the following class.

Definition 4.2.2. Let

$$\mathscr{D}_L = \mathscr{L}\mathscr{P}_e \cap \mathscr{P}_+ \subset \mathscr{D}$$

be the Lukacs class of solutions to the van Dantzig problem.

The examples in (4.3) and (4.5) are not just in $\mathscr{D}$ but also in $\mathscr{D}_L$. Further elements of $\mathscr{D}_L$ can be generated by means of the mappings described in Theorem 4.3.5 below, and, also from the subclass of $\mathscr{D}$ that we introduce and study in Section 4.4, see Theorem 4.4.4. We already mention that in Section 4.3 we provide some (partial) characterizations of the class $\mathscr{D}_L$.

The first appearance of this class of Laguerre-Pólya characteristic functions traces back to Pólya [113] who, motivated by Riemann hypothesis, was interested in characterizing all (complex valued) functions f on $\mathbb{R}$ such that the analytic extension of

$$t \mapsto \int_{\mathbb{R}} e^{\mathrm{i}tx} f(x)dx \text{ is an entire function with only real zeros.} \tag{4.11}$$

We refer to the recent paper by Newman and Wu [92] for an excellent account on Pólya's approach and to de Bruijn's fascinating contributions to this problem. Theorems 4.3.10, 4.3.11 and 4.3.12 below form the adaptation of these results in the context of the van Dantzig problem. From that time onwards, the Laguerre-Pólya class of characteristic functions has become ubiquitous and plays a central role in various fields of mathematics. In what follows, we describe the connection between the class $\mathscr{D}_L$ and both the Riemann hypothesis, and, the Lee-Yang property that appears in some statistical mechanics models and in Euclidean quantum field theory.

4.2.1 The Lukacs class and the Riemann hypothesis

There is a fascinating literature describing the role played by the Riemann ζ function in probability theory, see the excellent papers [16, 17] and the references therein. In the spirit of the work of Roynette and Yor [117, Théorème V.3.2], we now explain how the Riemann hypothesis can be formulated in terms of the van Dantzig problem. Consider the Riemann ζ function

$$\zeta(z) = \sum_{n=1}^{\infty} n^{-z}, \ \mathtt{Re}(z) > 1,$$

which, when extended meromorphically to the whole complex plane, has a single simple pole at 1 with residue 1 and the following zeros: the trivial ones located at the negative even integers and the nontrivial ones lying in the critical strip $0 < \mathtt{Re}(z) < 1$. The Riemann hypothesis states that all nontrivial zeros are located on the critical line $\mathtt{Re}(z) = 1/2$. It is well-known that the function

$$\eta(z) = \frac{1}{2} z(z-1)\pi^{-z/2}\Gamma(z/2)\zeta(z), \quad z \in \mathbb{C},$$

is an entire function (because $1 - z$ cancels the pole of ζ, whereas the trivial zeros of ζ cancel the poles of the gamma function $\Gamma(z/2)$, located at the same places). Hence the zeros of η are the nontrivial zeros of ζ. Moreover, it satisfies

$$\eta(z) = \eta(1-z), \quad z \in \mathbb{C}.$$

Performing an affine transformation on $\mathbb{C}$ so that the critical line maps onto the real line, we get the Landau function

$$\xi(z) = \eta(\tfrac{1}{2} + \mathrm{i}z), \quad z \in \mathbb{C}.$$

The functional equation above then becomes

$$\xi(z) = \xi(-z), \quad z \in \mathbb{C}.$$

Using the standard integral expression for the gamma function and the definition of the ζ function one can show that

$$\xi(t) = \int_{-\infty}^{\infty} e^{\mathrm{i}tx}\Phi(x)dx, \ t \in \mathbb{R}, \tag{4.12}$$

where

$$\Phi(x) = \sum_{n=1}^{\infty} \left(4\pi^2 n^4 e^{9x/2} - 6\pi n^2 e^{5x/2}\right) e^{-\pi n^2 e^{2x}}.$$

It is plain that $\Phi(x) > 0$ for all $x \geqslant 0$. Since ξ is even, one gets that $\Phi(x) > 0$ for all $x \in \mathbb{R}$. Also, Φ is integrable as $\int_{-\infty}^{\infty} \Phi(x)dx = \xi(0) < \infty$, we have that the function $\Phi/\xi(0)$ is the density of a symmetric real-valued random variable and $\xi/\xi(0)$ is its characteristic function. We have the following.

Theorem 4.2.3. *The function $\xi/\xi(0)$ is in $\mathscr{D}_L$ if and only if the Riemann hypothesis holds.*

Proof. If the function $\xi/\xi(0)$ is in $\mathscr{D}_L$ then it has only real zeros which means that η has all its zeros on the critical line. Hence the Riemann hypothesis holds. If the Riemann hypothesis holds then $\xi/\xi(0)$ is an even entire characteristic function with real zeros only. Using the result of Proposition 4.3.3 below, we have that $\xi/\xi(0)$ is in $\mathscr{D}_L$. The previous result combined with Theorem 4.2.1 yields this reformulation.

Corollary 4.2.4. The function $t \mapsto \xi(0)/\xi(\mathrm{i}t)$ is the characteristic function of a symmetric Pólya frequency function if and only if the Riemann hypothesis holds.

Further connections between the Riemann ξ and the van Dantzig problem will be discussed in Section 4.4.1.

4.2.2 The Lukacs class and the Lee-Yang property

Entire characteristic functions with only real zeros appear naturally in various models of statistical mechanics and quantum field theory. We briefly state the examples taken from the excellent paper [92]. In their pioneering works, Lee and Yang [78, 131] considered the Ising model in the presence of an external magnetic field and discovered that the zeros, in the magnetic field variable, of the partition function are purely imaginary. This is equivalent to the following. Let $\boldsymbol{\mu}$ denote the probability measure on $\{-1,1\}^N$, $N \in \mathbb{N}$, defined by

$$\boldsymbol{\mu}(\boldsymbol{x}) = K e^{\sum_{j,k=1}^N J_{j,k} x_j x_k}, \quad \boldsymbol{x} = (x_1, \ldots, x_N),$$

where, here and below, $J_{j,k} \geqslant 0$ and $J_{j,k} = J_{k,j}$ for all j, k and K is a positive normalizing constant. Then the partition function

$$\mathcal{P}_{\boldsymbol{\lambda},\mu}(z) = \sum_{\boldsymbol{x} \in \{-1,1\}^N} e^{z \boldsymbol{\lambda} \cdot \boldsymbol{x}} \boldsymbol{\mu}(\boldsymbol{x}), \quad \boldsymbol{\lambda} = (\lambda_1, \ldots \lambda_N) \in \mathbb{R}_+^N,$$

is an entire function whose zeros are all purely imaginary. This was discovered by Lee and Yang and refer to it as *the Lee-Yang property*. We emphasize that the location and distribution of the zeros of the partition function are useful to determine substantial properties of the underlying physical system such as phase transitions, the infinite volume limit and existence of a mass gap under an external magnetic field. In relation to the van Dantzig problem, one observes that the mapping $t \mapsto \mathcal{P}_{\boldsymbol{\lambda},\boldsymbol{\mu}}(\mathrm{i}t)$ is the characteristic function of the random variable

$$\boldsymbol{\lambda} \cdot \boldsymbol{x} = \sum_{j=1}^N \lambda_j x_j$$

under the measure $\boldsymbol{\mu}$. By Theorem 4.2.1, the Lee-Yang property is then equivalent to the statement that this characteristic function is in $\mathscr{D}_L$. Generalizing the Lee-Yang result, Simon and Griffiths [123] showed that if μ_0 is a symmetric probability measure[1] on $\mathbb{R}$ such that

$$\int_{\mathbb{R}} e^{bx^2} \mu_0(dx) < \infty \text{ for all } b \in \mathbb{R} \text{ and } \mathcal{F}_{\mu_0}(z) \neq 0 \text{ for all } \mathtt{Im}(z) < 0$$

then, to the probability measure

$$\boldsymbol{\mu}_\beta(d\boldsymbol{x}) = K e^{\beta \sum_{j,k} J_{j,k} x_j x_k} \prod_{k=1}^N \mu_0(dx_k) \quad \boldsymbol{x} = (x_1, \ldots, x_N) \in \mathbb{R}^N,$$

there corresponds the partition function

$$\mathcal{P}_{\boldsymbol{\lambda},\boldsymbol{\mu}_\beta}(z) = \int_{\mathbb{R}^N} e^{z \boldsymbol{\lambda} \cdot \boldsymbol{x}} \boldsymbol{\mu}_\beta(d\boldsymbol{x}), \quad \boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_N) \in \mathbb{R}_+^N,$$

with the property that, for all $\beta \geqslant 0$, the zeros of the entire function $z \mapsto \mathcal{P}_{\boldsymbol{\lambda},\boldsymbol{\mu}_\beta}(z)$ are all purely imaginary. Hence, the mapping $t \mapsto \mathcal{P}_{\boldsymbol{\lambda},\boldsymbol{\mu}_\beta}(\mathrm{i}t)$ is in $\mathscr{D}_L$. We also point out that these Lee-Yang type

[1] In fact, a signed measure is also allowed in [123].

theorems arise in quantum field theory, and refer again to [92] for a more detailed description. We simply point out that an important measure in this context is the measure $\mu_0(dx) = e^{-ax^4-bx^2}dx$, where $a > 0$ and $x, b \in \mathbb{R}$. From Theorem 4.2.1, we deduce the following.

Theorem 4.2.5. *A partition function $\mathcal{P}$ has the Lee-Yang property if and only if $t \mapsto \mathcal{P}(\mathrm{i}t) \in \mathscr{D}_L$.*

4.3 Properties of the van Dantzig and Lukacs classes

We collect here several properties of both the classes $\mathscr{D}$ and $\mathscr{D}_L$. We start by presenting some basic properties that both classes share. Some of them were stated in Lukacs' paper [82] without proof; for sake of conciseness, we indicate the main lines of proof. Then, we move to the properties to the Lukacs class $\mathscr{D}_L$ which are of different types. Some results can be found in the number theory or statistical mechanics literature in some form that we adapt, revisit or extend to identify new properties for the set $\mathscr{D}_L$. Moreover, based on ideas coming from probability theory, we also present original and substantial results about this set, see Theorem 4.3.13, and, we improve, in Theorem 4.3.14, a very interesting closure property due to Newman and Wu. Let us start with the following simple but useful result which is a reformulation of [80, Theorems A.2.1 and A.2.2, page 335].

Proposition 4.3.1. *If $\mathcal{F} \in \mathscr{D}$ then $\mathcal{F}$ is a real and even function, meaning that the associated random variables are symmetric. Moreover, $\mathcal{F}$ admits an analytic extension to some cross $\{z \in \mathbb{C};\ |\mathtt{Im}(z)| < z_1\}, \{z \in \mathbb{C}; |\mathtt{Re}(z)| < z_2\}$ where $z_1 > 0, z_2 > 0$. The same claims hold for $V\mathcal{F}$.*

We also point out that since $\mathcal{F}$ is even, a theorem of Schoenberg on positive-definite radial functions entails that the mapping $t \mapsto \mathcal{F}(\sqrt{t})$ is completely monotone on $\mathbb{R}_+$, that is, it is the Laplace transform of a non-negative Radon measure on $[0, \infty)$, see e.g. [27].

We proceed with the following the simple observation which follows readily from the previous Proposition since $V \circ V\mathcal{F}(t) = 1/V\mathcal{F}(\mathrm{i}t) = \mathcal{F}(-t) = \mathcal{F}(t)$.

Proposition 4.3.2. *The mapping V defined by* (4.2) *is an involution on $\mathscr{D}$.*

In other words, the set $\mathscr{D}_2$ is closed under commutation, that is, if $[\mathcal{F}, V\mathcal{F}] \in \mathscr{D}_2$ then $[V\mathcal{F}, \mathcal{F}] \in \mathscr{D}_2$.

Next, from the Definition 4.2.2 of $\mathscr{D}_L$, a first natural question is to understand whether the Lukacs class $\mathscr{D}_L$ contains all possible entire characteristic functions with only real zeros, that is, whether there exists such an entire function of order $\rho > 2$. Here is the definitive answer to this issue, which is, in fact, a direct consequence of a very nice result due to Gol'dberg and Ostrovs'ki [47], see also [83, Theorem 4.4.1], regarding entire characteristic functions having only real zeros. Note that, in these references, the statement is proven for a more general class of entire functions, namely the ones possessing the so-called ridge property, that is, $|\varphi(z)| \leqslant |\varphi(\mathtt{Im}(z))|$.

Proposition 4.3.3. *Every entire characteristic function with only real zeros belongs to the class $\mathscr{D}_L$.*

We proceed by deriving some closure properties of the sets $\mathscr{D}$ and $\mathscr{D}_L$. First, since multiplication of characteristic functions remain characteristic function as they correspond to addition of independent random variables, we get that $\mathscr{D}$ is stable by multiplication and having the constant function 1 as identity element, we obtain that it is a monoid. Similarly, since any reciprocal of a function in $\mathscr{D}_L$ being the moment generating function, at least on an imaginary strip, of a Pólya frequency function, it remains to identify transforms that preserve the positive definiteness property of a Laguerre-Pólya function. It is the program that we develop in the remaining part of this section. In this spirit, there is this first closure property which follows readily since the product of two characteristic functions in $\mathscr{L}\mathscr{P}$, that is of the form (4.8), remain an even entire characteristic function in $\mathscr{L}\mathscr{P}$, see the item (4.3.4) above for the same property for the set $\mathscr{D}$.

Proposition 4.3.4. *The sets $\mathscr{D}$ and $\mathscr{D}_L$ constitute a monoid under the operation of function multiplication.*

Note that the previous claim could also be interpreted as the set $\mathscr{D}_L$ being invariant by the convolution of probability distributions or equivalently by taking the sum of independent random variables. The following mappings and results were proposed by Lukacs

$$L^{(p)}f(t) = t^{p-2}\frac{f^{(p)}(t)}{f^{(2)}(0)}, \quad t \in \mathbb{R}, \quad p = 1, 2, \tag{4.13}$$

where, for a sufficiently smooth function f, we write $f^{(p)}(t) = \frac{d^p}{dt^p}f(t)$.

Theorem 4.3.5. *[82, Theorem 4] We have, for $p = 1, 2$, $L^{(p)}(\mathscr{LP}) \subset \mathscr{LP}$ and $L^{(p)}(\mathscr{D}_L) \subset \mathscr{D}_L$.*

The proof of this theorem relies on Laguerre's theory, see e.g. Borel [21, Ch. 2], which provides a closure property of the class $\mathscr{LP}$ by differentiation. Then Lukacs showed, by analytical means, that his mappings leave the set $\mathscr{P}_+$ invariant. In Section 4.5 we shall provide an alternative proof of this last fact based on probabilistic arguments.

Remark 4.3.6.
(i) Since $L^{(p)}$, $p = 1, 2$, are differential operators, we easily get that the unique invariant of $L^{(1)}$ (resp. $L^{(2)}$) in $\mathscr{P}_+$, i.e. $L^{(1)}\mathcal{F} = \mathcal{F}$, is $\mathcal{F}(t) = e^{-\sigma^2 t^2/2}$ (resp. $\mathcal{F}(t) = \cos(\sigma t)$, $\sigma \in \mathbb{R}$).

(ii) For any finite sequence $(p_1, \dots, p_n)$ whose elements take values in $\{1, 2\}$ we have that $L^{(p_n)} \circ \dots \circ L^{(p_1)}(\mathbb{D}_L) \subseteq \mathbb{D}_L$. For example, one observes that

$$L^{(1)}\cos(t) = \frac{\sin t}{t} \text{ and } L^{(1)} \circ L^{(1)}\cos(t) = 3\frac{\sin t - t\cos t}{t^3}.$$

More generally, define $\mathcal{F}_{n+1}(t) = L^{(1)}\mathcal{F}_n(t), n \geqslant 0$, with $\mathcal{F}_0(t) = \cos t$. Thus, $\mathcal{F}_n$ is obtained by the n-fold application of the operator $L^{(1)}$ to the cosine function. We obtain the expression

$$\mathcal{F}_{n+1}(t) = \frac{-(2n+1)!!}{t^{2n+1}}\texttt{Re}(\overline{P}_n(-\mathrm{i}t)\mathrm{i}e^{-\mathrm{i}t})$$

where $(2n+1)!! = (2n+1)(2n-1)(2n-3)\cdots 1$ and $\overline{P}_n$ is the so-called reverse Bessel polynomial, expressed in terms of the degree-n Bessel polynomial P_n via $\overline{P}_n(x) = x^n P_n(1/x)$, see [67]. Note that $\mathcal{F}_n \to \mathcal{F}_0$ pointwise as $n \to \infty$ and this is natural from the probabilistic interpretation of the operator $L^{(1)}$, see Lemma 4.5.1.

We proceed with the following two results that give a characterization or a partial characterization of the set $\mathscr{D}_L$. First, we have the following Lévy-Khintchine type representation which follows readily from Theorem 4.2.1 combined with the characterization result due to Kwaśnicki of the Laplace transform of Pólya frequency densities.

Theorem 4.3.7. *[72, Proposition 5.3] If $\mathcal{F} \in \mathscr{D}_L$ then, for all $t \geqslant 0$,*

$$\mathcal{F}(t) = e^{\Psi_L(t)} \tag{4.14}$$

where $\Psi_L(t) = -ct^2 - \int_{-\infty}^{\infty} \left(\frac{1}{t+r} - \frac{1}{r} + \frac{t}{r^2}\right) \rho(r)dr$, $c \geqslant 0$ and $\rho : \mathbb{R} \to \mathbb{Z}$ is an even non-decreasing integer-valued function such that $\int_{-\infty}^{\infty} \frac{|\rho(r)|}{|r|^3}\, dr < \infty$.

The next theorem, proved by Newman [90], who was, as discussed in Section 4.2.2, motivated by problems arising in statistical physics and Euclidean field theory, characterizes a subclass of $\mathscr{D}_L$. For a random variable X and a real number λ such that $Z_\lambda := \int_{-\infty}^{\infty} e^{\lambda x^2} F_{\mathrm{X}}(dx) < \infty$, let X_λ denote a random variable whose distribution F_{X_λ} is given by

$$F_{\mathrm{X}_\lambda}(dx) = \frac{1}{Z_\lambda} e^{\lambda x^2} F_{\mathrm{X}}(dx),\ x \in \mathbb{R}. \tag{4.15}$$

Theorem 4.3.8. *[90, Theorem 1] Let X be a symmetric random variable. Then, $\mathcal{F}_{\mathrm{X}_\lambda} \in \mathscr{D}_L$ for all $\lambda \in D_{\mathrm{X}} = \{\lambda \in \mathbb{R};\ \mathcal{F}_{\mathrm{X}^2}(-\mathrm{i}\lambda) < \infty\} \supseteq (-\infty, 0]$ if and only if either, for some x_0,*

$$F_{\mathrm{X}} = \frac{1}{2}\left(\delta_{x_0} + \delta_{-x_0}\right),$$

or F_{X} is absolutely continuous with respect to the Lebesgue measure with a density f_{X} which takes the form

$$f_{\mathrm{X}}(x) = Kx^{2m} e^{-\alpha x^4 - \beta x^2} \prod_{k=1}^{N} \left(\left(1 + \frac{x^2}{a_k^2}\right) e^{-\frac{x^2}{a_k^2}}\right) \tag{4.16}$$

where $K > 0$ is a normalizing constant, m a nonnegative integer, α, β real numbers, N a nonnegative integer or ∞, with the a_k positive, and either $\alpha = 0$, $\sum a_k^{-4} < \infty$, or $\alpha > 0$, $\beta + \sum a_k^{-2} > 0$ (the case $\sum a_k^{-2} = \infty$ is allowed).

Remark 4.3.9. This theorem does not fully characterize the set $\mathscr{D}_L$ as, for instance, $\mathcal{J}_0$, the Bessel function of order 0, see (4.5), is in $\mathscr{D}_L$, but it is the characteristic function of the arc-sine law whose density, see (4.65), does not have the form (4.16). Another well-known function that does not belong to Newman characterization is the function

$$t \mapsto \mathcal{F}_{\lambda,\Phi}(t) = \frac{1}{\int_{\mathbb{R}} e^{\lambda x^2}\Phi(x)dx}\int_{\mathbb{R}} e^{\mathrm{i}tx}e^{\lambda x^2}\Phi(x)dx,$$

introduced by Pólya, where Φ is the inverse Fourier transform of the Landau function ξ; see (4.12). It is well-known that

$$\mathcal{F}_{\lambda,\Phi} \in \mathscr{D}_L \text{ if and only if } \lambda \geqslant \Lambda_{DN}$$

where Λ_{DN} is the celebrated de Bruijn-Newman constant. Since, by Theorem 4.2.3, the Riemann hypothesis is equivalent to $\Lambda_{DN} \leqslant 0$, this observation has motivated an intensive research activity on the computation of Λ_{DN}. The current state of art is $0 \leqslant \Lambda_{DN} \leqslant 0.22$ and was obtained by Tao and collaborators [126].

The following three claims are adaptation to our setting of some deep results due to Pólya [112] (the first two) and to de Bruijn [37], see also [92, sections 2.2 and 2.3]. We omit their proofs as they follow readily from the aforementioned results combined with Theorem 4.2.1.

Theorem 4.3.10. *[112] Let $f : \mathbb{R} \to \mathbb{R}_+$ be the density of a symmetric random variable such that, for some $A, \alpha > 0$,*

$$f(x) \leqslant Ae^{-x^{2+\alpha}}, \; x \geqslant 0. \tag{4.17}$$

Let $\varphi : \mathbb{R} \to \mathbb{R}_+$ be even, real analytic, and such that $\int_{\mathbb{R}} f(x)\varphi(x)dx = 1$. Assume further that $t \mapsto \mathcal{F}_f(t) = \int_{\mathbb{R}} e^{\mathrm{i}tx}f(x)dx \in \mathscr{D}_L$. Then,

$$t \mapsto \mathcal{F}_{f\cdot\varphi}(t) = \int_{\mathbb{R}} e^{\mathrm{i}tx}f(x)\varphi(x)dx \in \mathscr{D}_L$$

if and only if the analytic extension of φ is such that $t \mapsto \varphi(\mathrm{i}t) \in \mathscr{LP}$.

Note that the function φ in the above theorem is called a universal factor by Pólya. An interesting and simple application of this theorem is for the entire function $\varphi(x) = e^{\lambda x^2}$ where $\lambda > 0$.

Indeed, using the notation of Theorem 4.3.8, if X is a symmetric random variable with an absolutely continuous distribution whose density satisfies the bound (4.17) and for some $\underline{\lambda} \in \mathbb{R}$, $\mathcal{F}_{\mathrm{X}_{\underline{\lambda}}} \in \mathscr{D}_L$, then $\mathcal{F}_{\mathrm{X}_\lambda} \in \mathscr{D}_L$ for all $\lambda \geqslant \underline{\lambda}$ as plainly here $D_{\mathrm{X}} = \mathbb{R}$. For instance, from the expression (4.65), we deduce that, for any $\nu < \frac{1}{2}$ and $\lambda > 0$, the mapping

$$t \mapsto K_\lambda \int_{|x|<2} e^{\mathrm{i}tx} e^{\lambda x^2} (4 - x^2)^{-\nu - \frac{1}{2}} dx \in \mathscr{D}_L \tag{4.18}$$

where $K_\lambda > 0$ is a normalizing constant. However, note that since the density above when $\lambda = 0$ is not of the form (4.16), by Theorem 4.3.8, there exists $\underline{\lambda} < 0$ such that the entire function

$$t \mapsto K_{\underline{\lambda}} \int_{|x|<2} e^{\mathrm{i}tx} e^{\underline{\lambda} x^2} (4 - x^2)^{-\nu - \frac{1}{2}} dx \tag{4.19}$$

has non-real zeros.

Theorem 4.3.11. *[37] Let $f : [0, \infty) \to \mathbb{R}_+$ be the density of a positive random variable, and, for all $y \geqslant 0$,*

$$f(y) \leqslant B e^{-y^{\frac{1}{2}+\beta}}$$

for some $B, \beta > 0$. Suppose that f has an analytic extension in a neighborhood of the origin. Then the function

$$\mathcal{M}_f(t) = \int_0^\infty y^{t-1} f(y) dy$$

has an extension on $\mathbb{C}$ as a meromorphic function. If the function $\mathcal{M}_f$ has only negative zeros and n is a positive integer, then, writing $\frac{1}{C_n} = \int_{\mathbb{R}} f(x^{2n}) dx$,

$$t \mapsto C_n \int_{\mathbb{R}} e^{\mathrm{i}tx} f(x^{2n}) dx \in \mathscr{D}_L.$$

Theorem 4.3.12. *[92] Let f be an entire function such that its derivative $f^{(1)}$ is the limit (uniform in any bounded domain) of a sequence of polynomials, all of whose roots lie on the imaginary axis. Suppose further that f is not a constant, $f(x) = f(-x)$, and $f(x) \geqslant 0$ for $x \in \mathbb{R}$ and $\int_{\mathbb{R}} e^{-f(x)} dx = 1$. Then*

$$t \mapsto \int_{\mathbb{R}} e^{\mathrm{i}tx} e^{-f(x)} dx \in \mathscr{D}_L.$$

As instances illustrating these results, there are the following entire functions that were derived by Pólya

$$t \mapsto \quad K\int_{\mathbb{R}} e^{\mathrm{i}tx}\cosh(ax)e^{-a\cosh x}\,dx, \qquad t \mapsto K\int_{\mathbb{R}} e^{\mathrm{i}tx}e^{-x^{2n}}\,dx \text{ and}$$
$$t \mapsto \quad K\int_{\mathbb{R}} e^{\mathrm{i}tx}e^{-ax^{4n}+bx^{2n}+cx^2}\,dx \in \mathscr{D}_L \tag{4.20}$$

where K is, in each expression, a normalizing constant, and $n \in \mathbb{N}, a > 0$.

We continue Pólya's and de Bruijn's line of research by presenting original additional closure properties of the set $\mathscr{D}_L$. More specifically, we investigate its stability under product of independent variables. We have already mentioned that this property is intimately connected to the concept of intertwining relationship between Markov semigroups, as we will discuss later in this chapter. To state it, we say that a positive linear operator Λ on the space of bounded borelian functions is Markov multiplicative if there exists a random variable I with distribution function F_{I}, such that, for any bounded borelian function f, writing $\Lambda = \Lambda_{\mathrm{I}}$,

$$\Lambda_{\mathrm{I}}f(t) = \int_{\mathbb{R}} f(xt)F_{\mathrm{I}}(dx). \tag{4.21}$$

In the following, we identify a mapping from the set of even entire functions in $\mathscr{P}_+$ into $\mathscr{D}_L$. In other words, we provide a way of creating entire characteristic functions with only real zeros from any even characteristic functions. We also find necessary conditions on a Markov multiplicative operator to leave invariant the set $\mathscr{D}_L$. Let us now denote by $\mathscr{LP}_+$ the class of functions in $\mathscr{LP}$ with only strictly negative zeros and of the form

$$\varphi(z) = Ke^{az}\prod_{k=1}^{\infty}\left(1+\frac{z}{z_k}\right)e^{-z/z_k}$$

with $K, a \in \mathbb{R}$ and $z_k > 0$ for all k with $\sum_{k\geqslant 1} z_k^{-2} < \infty$.

Theorem 4.3.13. *1. Let $\mathcal{F}_{\mathrm{D}} \in \mathscr{P}_+$ be entire and even, and, assume that there exist a random variable I such that for any non-negative integer n*

$$\mathcal{M}_{\mathrm{I}}(2n)\mathcal{M}_{\mathrm{D}}(2n) = a_{\varphi}(n)G(n)\Gamma(2n+1)$$

where $\varphi(z) = \sum_{n=0}^{\infty} a_{\varphi}(n)z^n \in \mathscr{LP}$ and $G \in \mathscr{LP}_+$. Then, $\Lambda_{\mathrm{I}}\mathcal{F}_{\mathrm{D}} \in \mathscr{D}_L$.

2. *Let us now assume that there exist $\varphi \in \mathscr{LP}_+$ and a random variable $\mathrm{I_L}$ such that for any non-negative integer n,*

$$\varphi(n) = \mathcal{M}_{\mathrm{I_L}}(2n).$$

Then, $\Lambda_{\mathrm{I}_L}(\mathscr{D}_L) \subset \mathscr{D}_L$.

Before continuing further, let us first illustrate part (1) of this theorem by an example. Let us take $\varphi(z) = e^z \in \mathscr{LP}$ and then $a_\varphi(n) = \frac{1}{n!}, n \geqslant 0$. For instance, let $\mathrm{D} = \mathrm{J}_0$ where J_0 is the symmetric random variable whose distribution is the arc-sine law and is recalled in (4.65) below. Then, from (4.69), $\mathcal{M}_{\mathrm{J}_0}(2n) = \frac{(2n)!}{n!n!}$ and $\mathcal{F}_{\mathrm{J}_0} = \mathcal{J}_0 \in \mathscr{D}_L$, the Bessel function of order 0, see (4.5) above.

Next fix $b > 0$ and define I_b as the positive random variable with distribution $F_{\mathrm{I_b}}(dx) = \frac{2b}{b+1}e^{-x^2}x\left(\frac{x^2}{b}+1\right)dx, x > 0$. Simple algebra yields that $\mathcal{M}_{\mathrm{I_b}}(2n) = \frac{(n+b)}{b}n!$, and, with the previous choice of φ, the equation $\mathcal{M}_{\mathrm{I_b}}(2n)\mathcal{M}_{\mathrm{J}_0}(2n) = a_\varphi(n)G(n)\Gamma(2n+1)$ gives

$$G(n) = \frac{(2n)!n!(n+b)n!}{bn!n!(2n)!} = \frac{(n+b)}{b}.$$

Since $G(z) = \frac{z+b}{b}$ is in $\mathscr{LP}_+$, the theorem above shows that

$$\Lambda_{\mathrm{I}_b}\mathcal{F}_{\mathrm{J}_0}(t) = \sum_{n=0}^{\infty}(-1)^n\frac{n+b}{n!b}t^{2n} = \frac{b-t^2}{b}e^{-t^2} \in \mathscr{D}_L,$$

for all $b > 0$. Note that this characteristic function already appeared in [80, Example III.8] where the authors showed that it is not decomposable with respect to the additive convolution of probability measures. However, our approach reveals that it is decomposable with respect to the multiplicative one. We point out that Theorem 4.3.13 enables one to generate many new examples of elements in $\mathscr{D}_L$, and, we refer to Theorem 4.5.6 where this idea is exploited and to Section 4.4.2 below, where some additional examples are provided. Another interesting aspect of the set $\mathscr{D}_L$ is the following additional closure property.

Theorem 4.3.14. *The sets $\mathscr{D}$ and $\mathscr{D}_L$ are closed under locally uniform convergence.*

Note that the locally uniform convergence of characteristic functions is equivalent to the pointwise convergence to a continuous function at 0, see [19, Theorem 3.2.1], and, by the Lévy continuity theorem, see [23, Theorem 8.28], this implies the weak convergence of the corresponding sequence of random variables to a random variable. In other words, Theorem 4.2.5 entails that Theorem 4.3.14 is a generalization to [91, Theorem 7] regarding the closure under weak convergence of the set $\mathcal{X}$, which is the set of symmetric probability measures F whose characteristic function is in $\mathscr{L}\mathscr{P}_e$ and such that $\int_{\mathbb{R}} e^{bx^2} F(dx) < \infty$ for some $b > 0$. We do not need this last condition for our closure result. However, the result from Newman and Wu [91, Theorem 7] ensures that the gaussian tail property for elements in $\mathscr{D}_L$ is preserved under weak convergence.

4.4 The new class $\mathscr{D}_P$

To present the main results of this Section, we start by introducing some objects and notation. First, with $\mathbb{M}_+(\mathbb{R}_+)$ denoting the set of non-negative Radon measures on $(0, \infty)$, we define the mapping $\Psi : \mathbb{R}_+ \to \mathbb{R}$ by

$$\Psi(u) = -\kappa + au + \frac{1}{2}\sigma^2 u^2 - \int_0^\infty (1 - \mathrm{e}^{-ur} - ur\mathbb{I}_{\{r<1\}})\mu(\mathrm{d}r) \tag{4.22}$$

where $\kappa \geqslant 0$, $a \in \mathbb{R}$, $\sigma^2 \geqslant 0$ and $\mu \in \mathbb{M}_+(\mathbb{R}_+)$ is such that $\int_0^\infty \inf(1, r^2)\mu(\mathrm{d}r) < \infty$. We exclude the case when $\sigma = 0$, $a + \int_0^1 r\mu(\mathrm{d}r) \leqslant 0$ and $\int_0^\infty \inf(1, r)\mu(\mathrm{d}r) < \infty$, which is seen as degenerate in our context. This function has a nice probabilistic interpretation. Indeed, it is the so-called Laplace exponent of a possibly killed real-valued spectrally negative Lévy process $Y = (Y_t)_{t\geqslant 0}$, i.e. a stochastic process without positive jumps and with stationary and independent increments starting from 0 and, when $\kappa > 0$, it is killed at an independent exponential time of parameter κ. Moreover, we have, for any $u, t \geqslant 0$,

$$\mathcal{F}_{Y_t}(-\mathrm{i}u) = e^{\Psi(u)t}. \tag{4.23}$$

Note that under the three conditions we excluded above, Y is negative-valued and has non-increasing sample paths. We refer to the monograph [74] for a thorough study of these processes. It is a well established (and an easy to check) fact that the Laplace exponent Ψ is strictly convex on $[0,\infty)$ with

$$\lim_{u\to\infty}\Psi(u)=+\infty \text{ and } \Psi \text{ is increasing on } [\theta,\infty) \text{ where } \theta=\sup\{u\geqslant 0;\ \Psi(u)=0\}. \tag{4.24}$$

In fact, by convexity, we have

$$\theta>0 \text{ if and only if (i) } \kappa>0 \text{ or (ii) } \kappa=0 \text{ and } \Psi^{(1)}(0^+)=\lim_{u\downarrow 0}\Psi^{(1)}(u)<0 \tag{4.25}$$

and a monotone convergence argument yields that $\Psi^{(1)}(0^+)=a-\int_1^\infty r\mu(\mathrm{d}r)\in[-\infty,+\infty)$. We are now ready to define the sets

$$\mathbb{N}=\{\Psi:\mathbb{R}_+\to\mathbb{R} \text{ of the form } (4.22)\}^2 \tag{4.26}$$

and

$$\mathbb{N}_{\mathscr{D}}=\left\{\Psi\in\mathbb{N};\ 0\leqslant\theta<\tfrac{1}{2}\right\}. \tag{4.27}$$

Note that (4.25) entails that

$$\Psi\in\mathbb{N}_{\mathscr{D}} \text{ if } \kappa=0 \text{ and } \Psi^{(1)}(0^+)\geqslant 0 \tag{4.28}$$

as, in this case, $\theta=0$. Moreover, from (4.24), one easily gets that a necessary and sufficient condition for a function Ψ to be in $\mathbb{N}_{\mathscr{D}}$ is that $\Psi\in\mathbb{N}$ with $\Psi(\frac{1}{2})>0$. The notation of the set $\mathbb{N}_{\mathscr{D}}$ is motivated by the following facts. On the one hand, due to the infinite divisibility of Y_1, we have, for any $\Psi\in\mathbb{N}$, that $z\mapsto-\Psi(\mathrm{i}z)\in\mathbb{N}(\mathbb{R})$, the set of continuous and negative-definite functions on $\mathbb{R}$. On the other hand, we now define a class of entire functions that are generated by the set $\mathbb{N}_{\mathscr{D}}$, that will be shown to belong to $\mathscr{D}$.

For any $\Psi\in\mathbb{N}$, we introduce the power series

$$\mathcal{J}_\Psi(t)=\sum_{n=0}^{\infty}\frac{(-1)^n}{W_\Psi(n+1)}t^{2n} \tag{4.29}$$

[2] Note that a similar type of set, denoted by $\mathbf{N}$, was introduced in Chapter 1. $\mathbb{N}$ is the set of Laplace exponents of spectrally negative Lévy processes, while $\mathbf{N}$ denotes the set of all Lévy-Khintchine exponents.

where, here and below, for a function φ defined on $\mathbb{R}_+$, we set

$$W_\varphi(1) = 1, \quad W_\varphi(n+1) = \prod_{k=1}^{n} \varphi(k), \quad n \geqslant 1.$$

Since, from (4.24), $\lim_{n\to\infty} \frac{W_\Psi(n+2)}{W_\Psi(n+1)} = \lim_{n\to\infty} \Psi(n+1) = \infty$, $\mathcal{J}_\Psi$ defines an entire function.

We also write

$$\mathcal{I}_\Psi(t) = \mathcal{J}_\Psi(\mathrm{i}t) = \sum_{n=0}^{\infty} \frac{1}{W_\Psi(n+1)} t^{2n} \tag{4.30}$$

and point out that the entire function $\mathcal{I}_\Psi(\sqrt{t})$ was introduced in [102] where it was shown that it is an invariant function of some self-similar integro-differential operator. Therein, the complete monotonicity property was identified for several of its transformations, and, we also refer to [8] for a more recent and refined studies of this class of functions. We are now ready to introduce the following class of entire functions

$$\mathscr{D}_P = \{\mathcal{J}_\Psi \text{ of the form (4.29) with } \Psi \in \mathbb{N}_{\mathscr{D}}\}. \tag{4.31}$$

Theorem 4.4.1. *We have $\mathscr{D}_P \subseteq \mathscr{D}$, and, for all $\Psi \in \mathbb{N}_{\mathscr{D}}$,*

$$\left[\mathcal{J}_\Psi, \frac{1}{\mathcal{I}_\Psi}\right] \in \mathscr{D}_2 \tag{4.32}$$

with

$$t \mapsto -\log \mathcal{J}_\Psi(t) \notin \mathbb{N}(\mathbb{R}) \text{ but } t \mapsto \log \mathcal{I}_\Psi(t) = \phi_\Psi(t^2) \in \mathbb{N}(\mathbb{R}) \tag{4.33}$$

where the function ϕ_Ψ is a Bernstein function that belongs to the class $\mathbf{B}_J$, that is, it is of the form (4.45) below with the additional property that $r \mapsto r\overline{\mu}(r)$ is non-increasing on $\mathbb{R}_+$. Finally, writing $\mathcal{F}_\Psi(t) = \frac{\mathcal{J}_\Psi(t)}{\mathcal{I}_\Psi(t)}$, $t \in \mathbb{R}$, we have $\mathcal{F}_\Psi \in \mathscr{D}_S$, that is $\mathcal{F}_\Psi(t)\mathcal{F}_\Psi(\mathrm{i}t) = 1$ for all $t \in \mathbb{R}$.

Remark 4.4.2. The random variables whose characteristic functions appear above shall be explicitly described in Section 4.5.2.

Remark 4.4.3. Note that in [101], it is proved that the entire function $\mathcal{J}_\Psi(\sqrt{t})$ has its smallest (in modulus) zero, say z_1, which is simple and located on the positive real line. On the one hand, this shows that $\mathcal{J}_\Psi$ is not the characteristic function of an infinitely divisible variable as their characteristic functions are zero-free, see [119]. On the other hand, $z_1 < 0$ corresponds to a singularity of the Bernstein function ϕ_Ψ. However, from Theorem 4.4.1, we get the identity, for all $t \in \mathbb{R}$,

$$\mathcal{I}_\Psi(t)\mathcal{J}_\Psi(t) = e^{-\phi_\Psi(t^2)}e^{\phi_\Psi(-t^2)} = 1, \tag{4.34}$$

which entails that the Bernstein function ϕ_Ψ admits a meromorphic extension on $\mathbb{C}$, but it is not necessarily a Pick function (see the definition below). This reveals that such an identity cannot be possible for Bernstein functions, especially for those having an essential singularity, e.g. $\phi(u) = u^a, 0 < a < 1$.

The purpose of the next result is to explain how the classes $\mathscr{D}_P$ and $\mathscr{D}_L$ are related, which consists on investigating the difficult issue of locating the zeros of the entire function $\mathcal{J}_\Psi \in \mathscr{D}_P$. To state it, we recall that a Bernstein Pick function [120, p. 56] is a Bernstein function which admits an holomorphic extension which maps the upper half-plane into its closure. We say that an entire function (resp. Pick meromorphic function) has the 1-separation property if its sequence of zeros $(z_k)_{k\geqslant 1}$ (resp. and poles $(\rho_k)_{k\geqslant 1}$) satisfies, for all k, $z_{k+1} < z_k - 1$ (resp. $\rho_k = z_k - 1 > z_{k+1}$). We now introduce the set

$$\mathbf{B}_{P_1} = \{\phi \in \mathbf{B}; \phi \text{ is a Pick function having the } 1 \text{ separation property}\}. \tag{4.35}$$

Theorem 4.4.4. *Let $\phi \in \mathbf{B}_{P_1}$. Then, $u \mapsto \Psi(u) = u\phi(u) \in \mathbb{N}_{\mathscr{D}}$ and $\mathcal{J}_\Psi \in \mathscr{D}_L \cap \mathscr{D}_P$. However, $\mathscr{D}_P \nsubseteq \mathscr{D}_L$ as there are Ψ's $\in \mathbb{N}_{\mathscr{D}}$ such that $\mathcal{J}_\Psi$ has at least a non-real zero.*

Remark 4.4.5. In Section 4.4.2, we provide instances of the two situations presented in this theorem, see e.g. the Bernstein functions that define the Bessel functions and the Fox-Wright functions, see 4.4.2, which both belong to $\mathscr{D}_L$. On the other hand there are the examples involving hypergeometric functions and the Mittag-Leffler functions which have non-real zeros.

We proceed with the following result that shows that the Lukacs mappings, introduced in (4.13), also leave our class $\mathscr{D}_P$ invariant.

Proposition 4.4.6. *Let $L^{(p)}, p = 1, 2,$ be the operators that were defined in (4.13) above. Then, we have $L^{(1)}(\mathscr{D}_P) \subset \mathscr{D}_P$. Moreover, $L^{(2)}(\mathscr{D}_P) \subset \mathscr{D}$ and the same remains true for their iterates.*

4.4.1 The Riemann ξ function and the class $\mathscr{D}_P$

A natural and important question that arises at this stage is to understand whether the Riemann ξ function defined in (4.12) belongs to the class $\mathscr{D}_P$. Indeed, this would yield a power series representation of this function whose coefficients would be expressed in terms of negative definite functions offering new tools to study the location of its zeros, using for instance Theorem 4.4.4. To this end, let us recall that the Riemann ξ function, defined in (4.12), can be expressed in terms of the following power series

$$\Theta(z) = \xi(\sqrt{z}) = \sum_{n=0}^{\infty} \frac{\gamma(n)}{n!} z^n, \ z \in \mathbb{C}, \tag{4.36}$$

where, with $F(n) = \int_1^{\infty} (\log x)^n \, x^{-3/4} \, \theta_0(x) \, dx$ and $\theta_0(x) = \sum_{n=1}^{\infty} e^{-\pi n^2 x}$ the theta series, we have set

$$\gamma(n) = \frac{n!}{(2n)!} \frac{32\binom{2n}{2} F(2n-2) - F(2n)}{2^{2n-1}}, \tag{4.37}$$

see e.g. [48]. The question whether there exists $\Psi \in \mathbb{N}_{\mathscr{D}}$ such that $\mathcal{J}_\Psi = \xi$ boils down to the existence of $\phi \in \mathbf{B}_{\mathscr{D}}$ such that, for all $n \in \mathbb{N}$,

$$\frac{1}{W_\phi(n+1)} = \frac{n!}{(2n)!} \cdot \frac{32\binom{2n}{2} F(2n-2) - F(2n)}{2^{2n-1}}, \tag{4.38}$$

which, after some easy algebra, is equivalent to show that

$$\phi(n+1) = -\frac{G(2n)}{8(n+1)G(2n+2)}, \tag{4.39}$$

where we have set $G(2n) = 64n(2n-1)\frac{F(2n-2)}{F(2n)} - 1$. Since this question does not seem straightforward, we investigate, instead here, whether this possibility could be excluded from the properties that we know about elements of the class $\mathscr{D}_P$ and the Riemann ξ function. Recalling that the latter is an entire function of order 1 with infinite type and it is the characteristic function of the density of a probability measure whose support is $\mathbb{R}$, we have the following.

Proposition 4.4.7. *Let $\Psi \in \mathbb{N}_{\mathscr{D}}$ such that $\Psi(u) = \frac{u^2}{\ell(u)}, u \in \mathbb{R}$, with $\lim_{u\to\infty} \ell(u) = \infty$, ℓ being a slowly varying function at infinity, i.e. for every $u > 0$, $\lim_{t\to\infty} \frac{\ell(ut)}{\ell(t)} = 1$. Then $\mathcal{J}_\Psi$ is an entire function of order 1 and infinite type. This condition holds when $\Psi(u) = (u-\theta)\phi(u), \theta \geqslant 0$, with ϕ a special Bernstein function, i.e. $\phi(u)\widehat{\phi}(u) = u$, such that its conjugate Bernstein function $\widehat{\phi}(u) = \ell(u)$. Moreover, under this condition, $\mathcal{J}_\Psi$ is the characteristic function of a probability density function whose support is $\mathbb{R}$.*

The first part of the Proposition follows readily from [8, Proposition 2.1], see also Proposition 4.5.4 below, where we notice that when the order is 1, with the notation of Proposition 4.5.4, $\underline{\Psi} = 2$ and thus the type $\tau_\Psi \geqslant \left(\overline{\lim_{n\to\infty}} \frac{n}{\phi(n)}\right)^{\frac{1}{2}} = \left(\overline{\lim_{n\to\infty}} \ell(n)\right)^{\frac{1}{2}} = \infty$ under the condition of the Proposition. The last claim is a specific instance of Lemma 4.5.8.

Remark 4.4.8. One instance when $\Psi(u) = \frac{u^2}{\ell(u)}, u \in \mathbb{R}$, with ℓ as in the Proposition is when $\Psi(u) = ue^{\mathrm{W}(u)}$, where W is the Lambert function. Indeed, it is well known that for all $u \geqslant 0$, $\mathrm{W}(u)e^{\mathrm{W}(u)} = u$ and W is a complete Bernstein function that is a Bernstein function whose Lévy measure is absolutely continuous with a completely monotone density, and $\lim_{u\to\infty} \frac{\mathrm{W}(u)}{\ln u} = 1$, and hence it is a special Bernstein function with $e^{W(u)}$ as conjugate, see [93]. Therefore $\Psi(u) = ue^{\mathrm{W}(u)} \sim \frac{u^2}{\ln u}$, for large u.

4.4.2 Some examples in the class $\mathscr{D}_P$

In this section, we give several specific examples of the function $\mathcal{J}_\Psi$ including the modified Bessel functions, the Mittag-Leffler functions and several type of hypergeometric functions, and refer to [46, 64] as classical references on these functions. The interested reader can also consult the monograph [120] for several examples of Bernstein functions from which one can provide additional interesting instances of $\mathcal{J}_\Psi$.

Bessel functions

Let $\Psi(u) = u(u+\nu)$, $\nu > -\frac{1}{2}$. We get that $W_\Psi(n+1) = n!\frac{\Gamma(n+\nu+1)}{\Gamma(\nu+1)}$ and thus

$$\mathcal{J}_\Psi(t) = \sum_{n=0}^{\infty}(-1)^n \frac{\Gamma(\nu+1)}{\Gamma(n+\nu+1)}\frac{t^{2n}}{n!} = \Gamma(\nu+1)t^{-\nu}J_\nu(2t) \tag{4.40}$$

where J_ν stands for the Bessel function of order ν. It is well-known that $\mathcal{J}_\Psi \in \mathscr{D}_L$.

Confluent hypergeometric function

Let $0 < a < 1 < a+b$ and $\Psi(u) = u\frac{u+1-a}{u+b}$. Note that, in this case $\sigma = 0$, and hence the support of the variable $\mathrm{D}_{\phi,\theta}$ is the real line. Moreover, simple algebra yields

$$\Psi(u) \;=\; u\frac{1-a}{b} + u\int_0^\infty \left(1-e^{-ur}\right)(a+b-1)\,e^{-br}dr.$$

We have $W_\Psi(n+1) = n!\frac{\Gamma(b+1)\Gamma(n+2-a)}{\Gamma(n+b+1)\Gamma(2-a)}$ and thus

$$\mathcal{J}_\Psi(t) = \sum_{n=0}^{\infty}(-1)^n \frac{\Gamma(n+b+1)\Gamma(2-a)}{\Gamma(b+1)\Gamma(n+2-a)}\frac{t^{2n}}{n!} = {}_1F_1\left(b+1;2-a;-t^2\right)$$

where ${}_1F_1$ is the confluent hypergeometric function. If $a+b = 2,3,\ldots$, then $\mathcal{J}_\Psi \in \mathscr{D}_L$, see e.g. [62, Theorem 4]. Therein, the authors conjecture, in particular, that $\mathcal{J}_\Psi \in \mathscr{D}_L$ for all $a+b > 1$.

Fox-Wright function

Let $\alpha \in (0,1)$ and $\beta \geqslant \alpha$, then

$$\Psi(u) = u\frac{\Gamma(\alpha u+\beta)}{\Gamma(\alpha(u-1)+\beta)} = u\frac{\Gamma(\beta)}{\Gamma(\beta-\alpha)} + \frac{u}{\Gamma(1-\alpha)}\int_0^\infty \left(1-e^{-ur}\right)\frac{e^{-\frac{\beta}{\alpha}r}}{\left(1-e^{-r/\alpha}\right)^{\alpha+1}}dr$$

yielding $W_\Psi(n+1) = n!\frac{\Gamma(\alpha n+\beta)}{\Gamma(\beta)}$ and thus

$$\mathcal{J}_\Psi(t) = \sum_{n=0}^{\infty}(-1)^n \frac{\Gamma(\beta)}{\Gamma(\alpha n+\beta)}\frac{t^{2n}}{n!} = {}_0\Psi_1^*\left((\beta,\alpha);-t^2\right)$$

where ${}_0\Psi_1^*$ is the normalized Fox-Wright function, which is in $\mathscr{D}_L$ by Laguerre Theorem, see e.g. [118, Theorem 4] as the mapping $z \mapsto \frac{1}{\Gamma(\alpha z+\beta)} \in \mathscr{LP}_+$.

Mittag-Leffler function

Let now $\alpha \in (1,2)$ and $\beta \in (\alpha-1,\alpha)$ and set

$$\Psi(u) = \frac{\Gamma(\alpha u+\beta)}{\Gamma(\alpha u+\beta-\alpha)} = \frac{\alpha\Gamma(\beta)}{\Gamma(\beta-\alpha+1)}u + \frac{(\alpha-1)u}{\Gamma(2-\alpha)}\int_0^\infty \left(1-e^{-ur}\right)\frac{e^{-\frac{\beta}{\alpha}r}}{(1-e^{-r/\alpha})^\alpha}dr.$$

Then, $W_\Psi(n+1) = \frac{\Gamma(\alpha n+\beta)}{\Gamma(\beta)}$ and

$$\mathcal{J}_\Psi(t) = \sum_{n=0}^{\infty}(-1)^n\frac{\Gamma(\beta)}{\Gamma(\alpha n+\beta)}t^{2n} = \Gamma(\beta)E_{\alpha,\beta}(-t^2)$$

where $E_{\alpha,\beta}(t) = \sum_{n=0}^\infty \frac{t^n}{\Gamma(\alpha n+\beta)}$ is the general Mittag-Leffler function. It is well known, see [42, Ch. 3.2], that, for $1<\alpha<2$, $E_{\alpha,\beta}$ have nonreal zeros. For the last 3 examples, we do not know whether $\mathcal{J}_\Psi \in \mathscr{D}_L$ or not.

The Barnes-Hypergeometric function

Let now $\alpha \in (1,2)$, $\rho \in (0,1/\alpha]$ and set

$$\Psi(u) = \frac{\Gamma(\alpha\rho+\alpha u)u}{\Gamma(\alpha u)} = u\int_0^\infty (1-e^{-ur})\frac{\rho}{\Gamma(1-\alpha\rho)}e^{-\rho r}\left(1-e^{-r/\alpha}\right)^{-\alpha\rho-1}dr.$$

Observe that

$$W_\Psi(n+1) = n!\frac{G\left(1,\frac{1}{\alpha}\right)}{G\left(1+\rho,\frac{1}{\alpha}\right)}\frac{G\left(n+1+\rho,\frac{1}{\alpha}\right)}{G\left(n+1,\frac{1}{\alpha}\right)} \tag{4.41}$$

where the Barnes G-function $G(n;\frac{1}{\alpha})$ is defined as the unique log-convex solution to recurrence equation $G(n+1;\frac{1}{\alpha}) = \Gamma(\alpha n)\,G(n;\frac{1}{\alpha})$, and refer to [81] for more details on this example. Then,

$$\mathcal{J}_\Psi(t) = \sum_{n=0}^{\infty}(-1)^n\frac{G\left(n+1,\frac{1}{\alpha}\right)G\left(1+\rho,\frac{1}{\alpha}\right)}{G\left(1,\frac{1}{\alpha}\right)G\left(n+1+\rho,\frac{1}{\alpha}\right)}\frac{t^{2n}}{n!}.$$

Hypergeometric function

Let $\alpha>0$, and, writing $\alpha_1 = \frac{\alpha+\sqrt{\alpha^2+4}}{2}$ and $\alpha_2 = \frac{\alpha-\sqrt{\alpha^2+4}}{2}$, consider

$$\Psi(u) = \frac{u}{u+\alpha}\left(u+\alpha_1\right)\left(u+\alpha_2\right).$$

It follows that $W_\Psi(n+1) = n!\frac{\Gamma(\alpha+1)\Gamma(n+\alpha_1+1)\Gamma(n+\alpha_2+1)}{\Gamma(n+\alpha+1)\Gamma(\alpha_1+1)\Gamma(\alpha_2+1)}$ and thus

$$\mathcal{J}_\Psi(t) = \sum_{n=0}^{\infty}(-1)^n \frac{\Gamma(n+\alpha+1)\Gamma(\alpha_1+1)\Gamma(\alpha_2+1)}{\Gamma(\alpha+1)\Gamma(n+\alpha_1+1)\Gamma(n+\alpha_2+1)}\frac{t^{2n}}{n!} = {}_1F_2\left(\alpha+1;\alpha_1+1,\alpha_2+1;-t^2\right)$$

where ${}_1F_2$ is an hypergeometric function.

Power-gamma function

Let $\alpha \in (0,1)$, $\gamma \geqslant 0$ and consider

$$\Psi(u) = u\,(u+\gamma)^\alpha = \gamma^\alpha u + u\int_0^\infty \left(1-e^{-ur}\right)e^{-\gamma r}\frac{r^{-\alpha-1}}{\Gamma(-\alpha)}dr.$$

It follows that $W_\Psi(n+1) = n!\frac{\Gamma^\alpha(n+\gamma+1)}{\Gamma^\alpha(\gamma+1)}$ and thus

$$\mathcal{J}_\Psi(t) = \sum_{n=0}^{\infty}(-1)^n \frac{\Gamma^\alpha(\gamma+1)}{\Gamma^\alpha(n+\gamma+1)}\frac{t^{2n}}{n!}.$$

4.5 Proofs

4.5.1 Proofs of Section 4.3

Proof of Theorem 4.3.5

First, resorting to the Laguerre theory, see e.g. Borel [21, Ch. 2], on the additional number of zeros obtained by differentiating an entire function of finite genus, we get that if $\mathcal{F} \in \mathscr{LP}$ then $t \mapsto \mathcal{F}^{(1)}(t)/t$ and $t \mapsto \mathcal{F}^{(2)}(t)$ are also in $\mathscr{LP}$. Recall that $\mathscr{D}_L$ contains even and positive definite functions in $\mathscr{LP}$ that take value 1 at 0. Suppose that $\mathcal{F}$ is even with $\mathcal{F}(0) = 1$. it is easy to see that $L^{(p)}\mathcal{F}$, $p = 1, 2$, are also even and take value 1 at 0. Hence, to show that $L^{(p)}(\mathscr{D}_L) \subset \mathscr{D}_L$ it suffices to show that if $\mathcal{F}$ is a characteristic function then so are the $L^{(p)}\mathcal{F}$, $p = 1, 2$. One can check the latter by a probabilistic argument. First, recall if X is a positive random variable with finite expectation $m_{\mathrm{X}}(1)$ and distribution F_{X} then the random variable $\mathrm{X}(1)$ with distribution

$$F_{\mathrm{X}(1)}(dx) = \frac{x}{m_{\mathrm{X}}(1)} F_{\mathrm{X}}(dx),\ x \geqslant 0,$$

is called the size-biased version of X. Size-biasing appears frequently and naturally in probability theory, most notably in the theory of stationary point processes on the real line and in the theory of branching processes and random walks.

Lemma 4.5.1. *Let $\mathcal{F}_X$ be the characteristic function of a real-valued random variable X with finite second moment. Let $\mathrm{X}^2(1)$ be the size-biased version of X^2. Then $L^{(p)}\mathcal{F}_X$ is the characteristic function of the random variable*

$$\mathrm{U}^{(p)} \times \sqrt{\mathrm{X}^2(1)} \tag{4.42}$$

where $\mathrm{U}^{(p)}$ is a uniform random variable on the interval $[-1,1]$ if $p = 1$ or a random variable taking values $+1$ or -1 with probability $1/2$ each if $p = 2$, and, in both cases, is taken independent of $\mathrm{X}^2(1)$.

Proof. We first recall that, for a random variable X, we denote its distribution function by F_X. Then, using the definition of size-biasedness and writing $\mathrm{X}^{(p)} = \mathrm{U}^{(p)} \times \sqrt{\mathrm{X}^2(1)}$ we obtain, since the variables are independent, that

$$F_{\mathrm{X}^{(1)}}(dx) = \frac{1}{m_{\mathrm{X}}(2)} \left(\int_{|y|>|x|} |y| F_{\mathrm{X}(1)}(dy) \right) dx, \quad F_{\mathrm{X}^{(2)}}(dx) = \frac{x^2}{m_{\mathrm{X}}(2)} F_{\mathrm{X}(1)}(dx),$$

where, by assumption, $m_{\mathrm{X}}(2) = \int_{-\infty}^{\infty} x^2 F_{\mathrm{X}}(dx) < \infty$. We can then directly verify that $L^{(p)}f(t) = \int_{-\infty}^{\infty} e^{\mathrm{i}tx} F_{\mathrm{X}^{(p)}}(dx)$, $p = 1, 2$.

Proof of Theorem 4.3.13

First, note that if $\mathcal{F}_{\mathrm{X}} \in \mathscr{P}_+$, then for any random variable I, $\Lambda_{\mathrm{I}}\mathcal{F}_{\mathrm{X}} \in \mathscr{P}_+$ as it is the characteristic function of the random variable XI where the two random variables are considered to be independent. Next, take $\mathcal{F}_{\mathrm{D}} \in \mathscr{P}_+$ even and entire, that is, D is a symmetric real-valued random variable such that, for any $t \in \mathbb{R}$,

$$\mathcal{F}_{\mathrm{D}}(t) = \sum_{n=0}^{\infty}(-1)^n \frac{\mathcal{M}_{\mathrm{D}}(2n)}{(2n)!} t^{2n}.$$

Moreover, under the conditions of the item (1), that is $\varphi \in \mathscr{LP}$ and $G \in \mathscr{LP}_+$, Laguerre's theorem [118, Theorem 4] entails that the function

$$f(z) = \sum_{n=0}^{\infty} G(n) a_\varphi(n) z^n$$

is an entire function with only real zeros. We made the assumption that $G(n)a_\varphi(n) = \mathcal{M}_{\mathrm{D}}(2n)\mathcal{M}_{\mathrm{I}}(2n)/(2n)!$. Since these are nonnegative numbers, f cannot have nonnegative zeros. On the other hand,

$$\begin{aligned}\Lambda_{\mathrm{I}}\mathcal{F}_{\mathrm{D}}(t) = \int_{\mathbb{R}} \mathcal{F}_{\mathrm{D}}(xt) F_{\mathrm{I}}(dx) &= \sum_{n=0}^{\infty}(-1)^n \frac{\mathcal{M}_{\mathrm{D}}(2n)\mathcal{M}_{\mathrm{I}}(2n)}{(2n)!} t^{2n} \\ &= \sum_{n=0}^{\infty}(-1)^n G(n) a_\varphi(n) t^{2n} = f(-t^2)\end{aligned}$$

where the interchange of the integral and sum is justified by a classical Fubini argument as the series defines an entire function. Since f has only real negative zeros, it follows that $\Lambda_{\mathrm{I}}\mathcal{F}_{\mathrm{D}}$ is an even entire function with only real zeros. Since $\Lambda_{\mathrm{I}}\mathcal{F}_{\mathrm{D}}$ is a characteristic function, it is in $\mathscr{P}_+$. Hence $\Lambda_{\mathrm{I}}\mathcal{F}_{\mathrm{D}} \in \mathscr{D}_L$.

For the second one, let $\mathcal{F} \in \mathscr{D}_L$, and thus, one has from (4.11), that $\mathcal{F}$ is of the form (4.8), and thus $t \mapsto \mathcal{F}(\sqrt{t}) \in \mathscr{LP}$. Since $\mathcal{F} \in \mathscr{D}_L$ there exists a symmetric real-valued random variable D such that $\mathcal{F} = \mathcal{F}_{\mathrm{D}}$. Proceeding as above, we get, for any $t \in \mathbb{R}$,

$$\Lambda_{\mathrm{I}}\mathcal{F}_{\mathrm{D}}(t) = \sum_{n=0}^{\infty}(-1)^n \frac{\mathcal{M}_{\mathrm{D}}(2n)\mathcal{M}_{\mathrm{I}}(2n)}{(2n)!} t^{2n}.$$

We apply again Laguerre's Theorem [118, Theorem 4]. On the one hand, by assumption, there is a function $\varphi \in \mathscr{LP}_+$ such that $\mathcal{M}_{\mathrm{I}}(2n) = \varphi(n)$. On the other hand, we observed above that

$$f(z) = \sum_{n=0}^{\infty} \frac{\mathcal{M}_{\mathrm{D}}(2n)}{(2n)!} z^n = \mathcal{F}(\sqrt{z}) \in \mathscr{LP}.$$

Since $\Lambda_{\mathrm{I}}\mathcal{F}_{\mathrm{D}}(t) = f(-t^2)$ we conclude, as above, that $\Lambda_{\mathrm{I}}\mathcal{F}_{\mathrm{D}} \in \mathscr{D}_L$.

Proof of Theorem 4.3.14

Let first $(\mathcal{F}_n)_{n\geqslant 0}$ be a sequence in $\mathscr{D}$ and assume that for all t in a bounded interval, $\lim_{n\to\infty}\mathcal{F}_n = \mathcal{F}$ uniformly. Then, according to the Lévy continuity theorem, see [23, Theorem 8.28], $\mathcal{F} \in \mathscr{P}_+$ and is continuous on $\mathbb{R}$. Let us write, for all $t \in \mathbb{R}, n \geqslant 0$, $G_n(t) = \frac{1}{\mathcal{F}_n(t)}$. Since by assumption, for all $n \geqslant 0$, $t \mapsto G_n(\mathrm{i}t) \in \mathscr{P}_+$, and, $\mathcal{F}_n$ is real, even and non-vanishing around 0, we get that G_n is well defined and even in a neighborhood of 0. Hence, it is the moment generating function of a (unique) symmetric random variable. Moreover, we have, for all $t \in \mathbb{R}$, $\lim_{n\to\infty} G_n(t) = G(t) = \frac{1}{\mathcal{F}(t)}$, and, by continuity of $\mathcal{F}$ and the fact that $\mathcal{F}(0) = 1$, there exits $\alpha > 0$ such that G is finite-valued on $|t| < \alpha$. According to [36, Theorem 3], G is the moment generating function of a random variable, whose law is uniquely determined by G, see [36, Theorem 1]. Finally, its characteristic function is plainly the mapping $t \mapsto G(\mathrm{i}t)$ and is such that, for all $t \in \mathbb{R}$, $\mathcal{F}(t)G(\mathrm{i}t) = 1$, that is $\mathcal{F} \in \mathscr{D}$.

Now assume that the sequence $(\mathcal{F}_n)_{n\geqslant 0}$ is in $\mathscr{D}_L$. We shall show that its locally uniform limit $\mathcal{F} \in \mathscr{D}_L$. Since, for all $n \geqslant 0$, $\mathcal{F}_n \in \mathscr{D}_L$, by Theorem 4.2.1, we have that $\mathcal{G}_n : t \mapsto \frac{1}{\mathcal{F}_n(\mathrm{i}t)}$ is the characteristic function of a symmetric Pólya frequency function. As $\mathscr{D}_L \subset \mathscr{D}$, from the previous proof, we deduce that, for all $t \in \mathbb{R}$, $\lim_{n\to\infty} \mathcal{G}_n(t) = \mathcal{G}(t) = \frac{1}{\mathcal{F}(\mathrm{i}t)}$ with $\mathcal{G} \in \mathscr{P}_+$. However, the set of Pólya frequency functions (probability measures) is closed under weak convergence as it is the closure, under weak convergence, of probability measures whose characteristic functions are the reciprocal of a polynomial having only real roots, see [52, Chap. III, Theorem 4.1]. We obtain, using the fact that $\mathcal{G}_n$ is real and even for all $n \geqslant 0$, that the mapping $t \mapsto \mathcal{F}(t) = \frac{1}{\mathcal{G}(\mathrm{i}t)} \in \mathscr{L}\mathscr{P}_e$ and hence $\mathcal{F} \in \mathscr{P}_+ \cap \mathscr{L}\mathscr{P}_e = \mathscr{D}_L$ which completes the proof.

4.5.2 Proofs of Section 4.4

A Wiener-Hopf type mapping between sets of negative-definite functions

We start the proof by a one-to-one mapping, emanating from the Wiener-Hopf factorization, between the set $\mathbb{N}_{\mathscr{D}}$ and a set of Bernstein functions. This allows us to provide a representation of the coefficients of the function in terms of a Stieltjes moment sequence that will be helpful in identifying one of the random variables solving the van Dantzig problem.

To this end we introduce some notation. First, let us denote by $\mathbf{B}$ the set of Bernstein functions, i.e. functions $\phi : (0, \infty) \to [0, \infty)$ having derivatives of all orders such that $(-1)^{n+1}\phi^{(n)} \geqslant 0$ for all $n \geqslant 1$; see [120, Ch. 3] These functions are in a one-to-one correspondence with functions that admit the so-called Lévy-Khintchine representation

$$\phi(u) = \nu + \frac{\sigma^2}{2}u + \int_0^\infty (1 - e^{-ur})\mu(dr),\ u \geqslant 0, \tag{4.43}$$

where $\nu, \sigma \geqslant 0$ and $\mu \in \mathbb{M}_+(\mathbb{R}_+)$ such that $\int_0^\infty \inf(1, r)\mu(dr) < \infty$. The set $\mathbf{B}$ is invariant under several transformations. In particular, it is a convex cone, and, it is also stable by the action of the semigroup of translations. Indeed, for any $\beta \geqslant 0$, easy algebra yields

$$\phi(u + \beta) = \phi(\beta) + \frac{\sigma^2}{2}u + \int_0^\infty (1 - e^{-ur})e^{-\beta r}\mu(dr),\ u \geqslant 0, \tag{4.44}$$

which is plainly a Bernstein function as $\phi(\beta) \geqslant 0$ and $e^{-\beta r}\mu(dr)$ is a Lévy measure as defined above. We shall also need the subset $\mathbf{B}_J \subset \mathbf{B}$ which is the convex cone of Bernstein functions which take the form

$$\phi(u) = \nu + \frac{\sigma^2}{2}u + \int_0^\infty (1 - e^{-ur})\overline{\mu}(r)dr \tag{4.45}$$

where $\nu, \sigma \geqslant 0$ and $\overline{\mu}$ is a non-negative and non-increasing function on $\mathbb{R}_+$ such that $\int_0^\infty \inf(1, r)\overline{\mu}(r)dr < \infty$. We refer to the monograph [120] for an excellent account on all these sets of functions.

Next, for a Radon measure μ on $(0,\infty)$ and $\theta \geqslant 0$, let

$$\overline{\mu}_\theta(r) := \int_r^\infty e^{\theta(r-s)}\mu(ds), \quad r > 0.$$

Letting $\mathcal{M}_\theta$ be the set of all functions $\overline{\mu}_\theta$, where μ ranges over Radon measures, such that $\int_0^\infty \min(1,r)\overline{\mu}_\theta(r)dr < \infty$, we define, in relation to the van Dantzig problem, the following set of Bernstein functions:

$$\mathbf{B}_{\mathscr{D}} = \left\{\phi \in \mathbf{B}; \mu(dr) = \overline{\mu}_\theta(r)dr, \text{ with } \overline{\mu}_\theta(r) \in \mathcal{M}_\theta, 0 \leqslant \theta < \frac{1}{2}\right\}. \tag{4.46}$$

Note first that when $\theta > 0$ the function $\overline{\mu}_\theta$ may fail to be monotone, for a suitable choice of μ. Note also that when $\theta = 0$ we have $\overline{\mu}_0(r) = \int_r^\infty \mu(ds)$ is simply a non-negative and non-increasing function that satisfies the integrability condition above,meaning that $\mathbf{B}_J$ is a strict subset of $\mathbf{B}_{\mathscr{D}}$.

We have the following.

Proposition 4.5.2. *1. There exists a one-to-one mapping between the sets $\mathbb{N}_{\mathscr{D}}$ and $\mathbf{B}_{\mathscr{D}}$. More specifically, for any $\Psi \in \mathbb{N}_{\mathscr{D}}$ of the form (4.22), we have, for any $u \geqslant 0$,*

$$\Psi(u) = (u-\theta)\phi(u) \tag{4.47}$$

where, with $\nu_\theta = \frac{\kappa}{\theta}\mathbb{I}_{\{\theta>0\}} + \Psi^{(1)}(0^+)\mathbb{I}_{\{\theta=0\}}$, we have set $\phi(u) = \nu_\theta + \frac{1}{2}\sigma^2 u + \int_0^\infty (1 - \mathrm{e}^{-ur})\overline{\mu}_\theta(r)dr \in \mathbf{B}_{\mathscr{D}}$. Moreover, with such a notation, we have, for all $z \in \mathbb{C}$,

$$\mathcal{J}_\Psi(z) = \mathcal{J}_{\phi,\theta}(z) = \sum_{n=0}^\infty \frac{(-1)^n\Gamma(1-\theta)}{\Gamma(n+1-\theta)W_\phi(n+1)} z^{2n}. \tag{4.48}$$

2. *Let us define the mapping $\mathcal{T}_\beta f(u) = \frac{u}{u+\beta} f(u+\beta)$. Then, $(\mathcal{T}_\beta)_{\beta \geqslant 0}$ form a semigroup, i.e., $\mathcal{T}_\beta \circ \mathcal{T}_\gamma = \mathcal{T}_{\beta+\gamma}$, and, for any $\beta \geqslant 0$, $\mathcal{T}_\beta(\mathbb{N}) \subset \mathbb{N}$ and $\mathcal{T}_\beta(\mathbb{N}_{\mathscr{D}}) \subset \mathbb{N}_{\mathscr{D}}$.*
3. *Let us write $\overline{\mathcal{T}}_\beta f(u) = \frac{u-\beta}{u+\beta} f(u+\beta)$. Then, for any $\beta \geqslant \theta$, $\overline{\mathcal{T}}_\beta(\mathbb{N}) \subset \mathbb{N}$ and for any $\theta \leqslant \beta < \frac{1}{2}$, $\overline{\mathcal{T}}_\beta(\mathbb{N}_{\mathscr{D}}) \subset \mathbb{N}_{\mathscr{D}}$.*

Proof. Let $\Psi \in \mathbb{N}_{\mathbb{D}}$. Then, by means of the Wiener-Hopf factorization of Lévy-Khintchine exponents of spectrally negative Lévy process, see e.g. [74], there exists $\phi \in \mathbf{B}$ such that, for all

$u \geqslant 0$,

$$\Psi(u) = (u-\theta)\phi(u) \tag{4.49}$$

where we recall that $\theta = \sup\{u \geqslant 0; \Psi(u) = 0\}$, see (4.25). In order to characterize ϕ, we write $\overline{\nu}_\theta = a + \sigma^2\theta + \int_0^1 (1 - \mathrm{e}^{-\theta r}) r\mu(\mathrm{d}r) - \int_1^\infty e^{-\theta r}\overline{\mu}_\theta(r)dr$ and, observe first that

$$\begin{aligned}\Psi(u+\theta) &= (\overline{\nu}_\theta + \int_1^\infty e^{-\theta r}\overline{\mu}_\theta(r)dr)u + \frac{1}{2}\sigma^2 u^2 - \int_0^\infty (1 - \mathrm{e}^{-ur} - ur\mathbb{I}_{\{r<1\}})e^{-\theta r}\mu(\mathrm{d}r) \\ &= u\left(\overline{\nu}_\theta + \frac{1}{2}\sigma^2 u + \int_0^\infty (1-\mathrm{e}^{-ur})e^{-\theta r}\overline{\mu}_\theta(r)dr\right)\end{aligned}$$

and, thus

$$\Psi(u) = (u-\theta)\left(\nu_\theta + \frac{1}{2}\sigma^2 u + \int_0^\infty (1-\mathrm{e}^{-ur})\overline{\mu}_\theta(r)dr\right)$$

which, after recalling that $-\theta\phi(0) = \Psi(0) = -\kappa$ and $\Psi^{(1)}(0^+) = \phi(0)$ if $\theta = 0$, completes the proof of the first item as the rest follows at once. Next, the claim of item (2) in the case $\beta = 0$ is obvious and thus we assume that $\beta > 0$. In [28], see also [108, Proposition 2.1], it is shown that $(\mathcal{T}_\beta)_{\beta\geqslant 0}$ is a semigroup and the set $\mathbb{N}$ is invariant under the action of the mapping $\mathcal{T}_\beta, \beta \geqslant 0$, that is $\mathcal{T}_\beta(\mathbb{N}) \subset \mathbb{N}$. Thus, it simply remains to show that $\mathcal{T}_\beta(\mathbb{N}_\mathscr{D}) \subset \mathbb{N}_\mathscr{D}$ for all $\beta \geqslant 0$. Let $\Psi \in \mathbb{N}_\mathscr{D}$. Then, note that $\mathcal{T}_\beta\Psi(0) = 0$, i.e. $\kappa = 0$ in (4.22), and, there exists $0 \leqslant \theta < \frac{1}{2}$ such that $\Psi(\theta) = 0$. Suppose, first, that $\beta \geqslant \theta$, then recalling that Ψ is non-decreasing on $[\theta, \infty)$ with $\Psi(\theta) \geqslant 0$, we get the claim, from (4.28), by observing that $(\mathcal{T}_\beta\Psi)^{(1)}(0) = \frac{\Psi(\beta)}{\beta} \geqslant 0$. Next, assuming that $0 < \beta < \theta$, one easily sees that $\mathcal{T}_\beta\Psi(\theta-\beta) = \frac{\theta-\beta}{\theta}\Psi(\theta) = 0$ and since $0 < \theta - \beta < \frac{1}{2}$, we deduce that also $\mathcal{T}_\beta\Psi \in \mathbb{N}_\mathbb{D}$, completing the proof of this item as the semigroup property is obvious. For the last claim, first observe that, $\overline{\mathcal{T}}_\beta\Psi(u) = \frac{u+\beta-2\beta}{u+\beta}\Psi(u+\beta) + \Psi(\beta) - \Psi(\beta) = \mathcal{T}_{2\beta,\beta}\Psi(u) - \Psi(\beta)$ (the last identity serves to fix a notation), and then we know from [28, Proposition 2.2], that $\mathcal{T}_{2\beta,\beta}(\mathbb{N}) \subset \mathbb{N}$ and we obtain the statement since $\Psi(\beta) \geqslant 0$ as we choose $\beta \geqslant \theta$. Finally, if $\Psi \in \mathbb{N}_\mathbb{D}$, then, from (4.24), Ψ is positive on $[\frac{1}{2}, \infty)$ and, from the preceding discussion, $\overline{\mathcal{T}}_\beta\Psi \in \mathbb{N}$ with $\overline{\mathcal{T}}_\beta\Psi(\frac{1}{2}) = \frac{\frac{1}{2}-\beta}{\frac{1}{2}+\beta}\Psi(\frac{1}{2}+\beta) > 0$.

Lemma 4.5.3. *Let $\Psi \in \mathbb{N}_{\mathscr{D}}$, and, for $p = 1, 2$, we set $L_1^{(p)} = L^{(p)}$, where we recall that $L^{(p)}$ was defined in (4.13), and, for $k = 1, 2, \ldots$, we define*

$$L_{k+1}^{(p)} = L^{(p)} \circ L_k^{(p)}.$$

1. *For any $k = 1, 2, \ldots$, we have*

$$L_k^{(1)} \mathcal{J}_\Psi = \mathcal{J}_{\mathcal{T}_k \Psi} \tag{4.50}$$

where we recall that the mapping $\mathcal{T}_k$ was defined in Proposition 4.5.2(2).

2. *Moreover, for any $k = 1, 2, \ldots$, we have*

$$L_k^{(2)} \mathcal{J}_\Psi = \mathcal{J}_{\overline{\mathcal{T}}_{\frac{1}{2}}^{2k} \Psi} \tag{4.51}$$

where, $\overline{\mathcal{T}}_\beta^1 = \overline{\mathcal{T}}_\beta$ and $\overline{\mathcal{T}}_\beta^k = \overline{\mathcal{T}}_\beta \circ \overline{\mathcal{T}}_\beta^{k-1}$, and, the mapping $\overline{\mathcal{T}}_\beta$ was defined in Proposition 4.5.2(3).

Proof. To prove the identity (4.50), we first let $k = 1$ and observe that

$$\mathcal{J}_\Psi^{(1)}(t) = \sum_{n=1}^{\infty} \frac{2n(-1)^n t^{2n-1}}{W_\Psi(n+1)} = \frac{-2t}{\Psi(1)} \sum_{n=0}^{\infty} \frac{(-1)^n t^{2n}}{\prod_{k=1}^{n} \frac{k}{k+1} \Psi(k+1)} = \frac{-2t}{W_\Psi(2)} \mathcal{J}_{\mathcal{T}_1 \Psi}(t). \tag{4.52}$$

Moreover, differentiating the right-hand side one more time yields

$$\mathcal{J}_\Psi^{(2)}(0) = \lim_{t \to 0} \frac{-2}{W_\Psi(2)} \mathcal{J}_{\mathcal{T}_1 \Psi}(t) = \frac{-2}{W_\Psi(2)}$$

where we used that $\mathcal{J}_{\mathcal{T}_1 \Psi}^{(1)}(0) = 0$, which, itself, follows by applying (4.52) to $\mathcal{J}_{\mathcal{T}_1 \Psi}$. Hence $L^{(1)} \mathcal{J}_\Psi = \mathcal{J}_{\mathcal{T}_1 \Psi}$. To complete the proof for $p = 1$, we resort to an induction argument combined with the semigroup property of the mapping $\mathcal{T}_k$. Indeed, for any $k = 1, \ldots$, one has

$$L_{k+1}^{(1)} \mathcal{J}_\Psi = L^{(1)} \circ L_k^{(1)} \mathcal{J}_\Psi = L^{(1)} \mathcal{J}_{\mathcal{T}_k \Psi} = \mathcal{J}_{\mathcal{T}_{k+1} \Psi}.$$

On the other hand, we note that

$$\begin{aligned}\mathcal{J}^{(2)}_{\Psi}(t) &= \frac{-2}{W_{\Psi}(2)}\frac{d}{dt}\sum_{n=0}^{\infty}\frac{(-1)^n}{W_{\mathcal{T}_1\Psi}(n+1)}t^{2n+1} = \frac{-2}{W_{\Psi}(2)}\sum_{n=0}^{\infty}\frac{(-1)^n(2n+1)}{W_{\mathcal{T}_1\Psi}(n+1)}t^{2n} \\ &= \frac{-2}{W_{\Psi}(2)}\sum_{n=0}^{\infty}\frac{(-1)^n}{\prod_{k=1}^{n}\frac{k-\frac{1}{2}}{k+\frac{1}{2}}\frac{k}{k+1}\Psi(k+1)}t^{2n} \\ &= \frac{-2}{W_{\Psi}(2)}\mathcal{J}_{\overline{\mathcal{T}}^{2}_{\frac{1}{2}}\Psi}(t)\end{aligned}$$

which provides the proof of the last claim for $k=1$. As above invoking an induction argument yields

$$L^{(2)}_{k+1}\mathcal{J}_{\Psi} = L^{(2)}\circ L^{(2)}_{k}\mathcal{J}_{\Psi} = L^{(2)}\mathcal{J}_{\overline{\mathcal{T}}^{2k}_{\frac{1}{2}}\Psi} = \mathcal{J}_{\overline{\mathcal{T}}^{2k+2}_{\frac{1}{2}}\Psi}$$

which completes the proof.

We proceed with the following result regarding the order and the type of the entire function $\mathcal{J}_{\Psi}$. It is a slight refinement of a result obtained recently by Bartholmé and Patie [8] in the case $\theta = 0$, and, for the entire function $\mathcal{I}_{\Psi}(\sqrt{t}) = \mathcal{J}_{\Psi}(\mathrm{i}\sqrt{t})$, this later transformation affects simply the order by a factor of 2. To deal with the general case $\theta > 0$, it is not difficult to see that both the order and type remain the same when replacing the term $n!$ by $C\Gamma(n+1-\theta), C>0$, in the coefficients of a power series. We left the details of the easy modification to the reader and recall that a measurable function ℓ on $\mathbb{R}_+$ is said to be slowly varying at infinity if, for every $u>0$, $\lim_{t\to\infty}\frac{\ell(ut)}{\ell(t)} = 1$.

Proposition 4.5.4. *[8, Proposition 2.1] Let $\Psi \in \mathbb{N}_{\mathscr{D}}$. Then, $\mathcal{J}_{\Psi}$ is an entire function of order*

$$\rho_{\Psi} = \frac{2}{\underline{\Psi}} \in [1,2] \tag{4.53}$$

where $\underline{\Psi} = \sup\{a>0;\ \lim_{u\to\infty}u^{-a}\Psi(u) = \infty\} = \varliminf_{u\to\infty}\frac{\ln\Psi(u)}{\ln u} \in [1,2]$, is its so-called Blumenthal-Getoor lower index. Moreover, its type τ_{Ψ} is given by

$$\tau_{\Psi} = \underline{\Psi}e^{-\frac{\underline{\Psi}-1}{\underline{\Psi}}}\varlimsup_{n\to\infty}e^{-\frac{1}{\underline{\Psi}}\left(\frac{\int_1^n\ln\phi(u)du}{n}-(\underline{\Psi}-1)\ln n\right)} \geqslant \left(\varlimsup_{n\to\infty}\frac{n^{\underline{\Psi}-1}}{\phi(n)}\right)^{\frac{1}{\underline{\Psi}}} \tag{4.54}$$

where we recall that $\Psi(u) = (u-\theta)\phi(u)$. In particular, $\rho_{\Psi} = 1$ if $\Psi(u) = \frac{u^2}{\ell(u)}$, and, otherwise, $\rho_{\Psi} = \frac{2}{\mathfrak{a}}, \mathfrak{a}\in[1,2)$, if $\Psi(u) = u^{\mathfrak{a}}\ell(u)$ where, in both cases, ℓ is a slowly varying function, and, $\rho_{\Psi} = 2$ also if $\int_0^{\infty} r\mu(dr) < \infty$.

Useful bounds for the Bernstein-gamma functions

For a Bernstein function $\phi \in \mathbf{B}$, we write $z_\phi = \inf\{u > 0; \phi(-u) = 0\} \in [0,\infty]$ and $e_\phi = \sup\{u > 0; \int_1^\infty e^{ur}\mu(dr) < \infty\} \in [0,\infty]$ and set

$$a_\phi = \min(z_\phi, e_\phi).$$

We also denote by W_ϕ the so-called Bernstein-gamma function which admits the following generalized Weierstrass product representation

$$W_\phi(z) = \frac{e^{-\gamma_\phi z}}{\phi(z)} \prod_{k=1}^{\infty} \frac{\phi(k)}{\phi(k+z)} e^{\frac{\phi'(k)}{\phi(k)}z}$$

where

$$\gamma_\phi = \lim_{n\to\infty} \left(\sum_{k=1}^{n} \frac{\phi'(k)}{\phi(k)} - \log \phi(n) \right).$$

From [96, Theorem 4.1], we know that W_ϕ defines a zero-free and analytic function on the right-half plane $\mathtt{Re}(z) > -e_\phi$, which admits a meromorphic extension to the right-half plane $\mathtt{Re}(z) > -a_\phi$. We mention that when $\phi(z) = z$, then W_ϕ boils down to the gamma function as the infinite product above corresponds to its Weierstrass representation and γ_ϕ is the Euler-Mascheroni constant. This class of functions has been thoroughly studied in the papers by Patie and Savov [95, 96]. Thereout, it has been shown that $W_\phi \in \mathscr{P}_+$ and it is the unique element in $\mathscr{P}_+$ solution to the functional equation $W_\phi(z+1) = \phi(z)W_\phi(z), W_\phi(1) = 1$. We also find in [96] the following result which will be useful in the sequel.

Proposition 4.5.5. *[96, Proposition 6.2] Let $\phi \in \mathbf{B}$ and write $\overline{\mu}(r) = \int_r^\infty \mu(ds), r > 0$. Then, the following estimates holds.*

1. *If $\sigma^2 > 0$ we have, for any $\epsilon, a > 0$ such that $\int_0^\infty e^{-ar}\overline{\mu}(r)dr < 1$, as $|b| \to \infty$,*
$$\left| \frac{\Gamma(a+ib)}{W_\phi(a+ib)} \right| \leqslant C|b|^{-\frac{2}{\sigma^2}(\overline{\mu}(1/|b|)+\phi(0))+\epsilon} \tag{4.55}$$
where $C > 0$.

2. *If $\sigma^2 = 0$ then, for any $u \geqslant 0$ and $a > 0$ fixed,*
$$\lim_{|b|\to\infty} |b|^u \left| \frac{\Gamma(a+ib)}{W_\phi(a+ib)} \right| = 0. \tag{4.56}$$

Characterization and properties of the van Dantzig pair of random variables

To define the van Dantzig pair of random variables that appear in Theorem 4.4.1, we need to introduce some notations. Let $\Psi \in \mathbb{N}$ and recall that it is the Laplace exponent of a spectrally negative Lévy process Y, see (4.23). Then, according to Lamperti [76], there exists a 1-self-similar Markov process on $(0,\infty)$ denoted by $X = (X_t)_{t\geqslant 0}$ such that

$$X_t = e^{Y_{A_t}}, \quad 0 \leqslant t < \xi = \inf\{t > 0; X_t = 0\}, \tag{4.57}$$

where $A_t = \inf\left\{s > 0;\ \int_0^s e^{Y_u} du > t\right\}$. Moreover, if $\theta = 0$, i.e. the conditions (4.25) are not fulfilled, then $\xi = \infty$ almost surely and X is in fact a Feller process on $[0,\infty)$. On the other hand, if $0 < \theta < 1$, then ξ is finite almost surely but there exists an unique 1-self-similar extension of the process X which is also a Feller process on $[0,\infty)$ and with the path property to leave the recurrent boundary point 0 continuously, see [116]. From now on, if $0 \leqslant \theta < 1$, we denote by $X(\Psi) = (X_t(\Psi))_{t\geqslant 0}$ the realization of the 1-self-similar Feller semigroup $(P_t^{\Psi})_{t\geqslant 0}$ on $[0,\infty)$ (that is the recurrent extension if $\theta > 0$) associated to Y and let T_Ψ be a random variable whose distribution is that of

$$\inf\{t > 0;\ X_t(\Psi) \geqslant 1\},$$

conditional on $X(\Psi)$ starting from 0. We also need to introduce the exponential functional of a subordinator, a positive random variable which has been intensively studied over the last two decades and refer to [108] for a nice historical account and a thorough study. Let $\phi \in \mathbf{B}$ and let

$$I_\phi = \int_0^\infty e^{-Z_t} dt \tag{4.58}$$

where $(Z_t)_{t\geqslant 0}$ is a subordinator such that $\mathcal{F}_{Z_t}(\mathrm{i}u) = e^{-\phi(u)t}, u, t \geqslant 0$. Next, we recall, from [30], the following expression for the integer moments of I_ϕ,

$$\mathcal{M}_{I_\phi}(n) = \frac{n!}{W_\phi(n+1)}, \quad n = 0, 1, \ldots, \tag{4.59}$$

and the notation of the associated Markov operator, see (4.21),

$$\Lambda_{I_\phi} f(t) = \int_0^\infty f(xt) F_{I_\phi}(dx). \tag{4.60}$$

We are now ready to state the following result which can be found in [102]. To emphasize the role played by the concept of intertwining in the van Dantzig problem, we provide another original proof. Intertwining, in our context, refers to the relation (4.63) below. To state the theorem, recall the notation

$$\mathcal{J}_{-\theta}(x) = \Gamma(1-\theta)t^{\theta} J_{-\theta}(2x), \quad \mathcal{I}_{-\theta}(x) = \Gamma(1-\theta)t^{\theta} I_{-\theta}(2x),$$

appearing in (4.40), where $J_{-\theta}$ (respectively $I_{-\theta}$) is the ordinary (respectively modified) Bessel function of the first kind of order $-\theta$.

Theorem 4.5.6. *[100, Theorem 2.1] Let $\Psi \in \mathbb{N}$ with $0 \leqslant \theta < 1$. Then, writing $\widetilde{\mathcal{I}}_{\Psi}(x) = \mathcal{I}_{\Psi}(\sqrt{x})$ and $\widetilde{\mathcal{I}}_{-\theta}(x) = \mathcal{I}_{-\theta}(\sqrt{x})$, we have*

$$\Lambda_{\mathrm{I}_{\phi}} \widetilde{\mathcal{I}}_{-\theta}(x) = \widetilde{\mathcal{I}}_{\Psi}(x), \quad x > 0, \tag{4.61}$$

where $\phi(u) = \frac{\Psi(u)}{u-\theta} \in \mathbf{B}$. Moreover, T_{Ψ} is a positive self-decomposable random variable, and, for any $u > 0$, we have

$$\mathcal{F}_{\mathrm{T}_{\Psi}}(\mathrm{i}u) \;=\; \frac{1}{\mathcal{I}_{\Psi}(\sqrt{u})} = e^{-\phi_{\Psi}(u)} \tag{4.62}$$

where ϕ_{Ψ} is the Bernstein function defined in Theorem 4.4.1.

Proof. Let $\Psi \in \mathbb{N}$ with $0 \leqslant \theta < 1$. We recall that in [95] for the case $\theta = 0$, and in [105] for the case $0 < \theta < 1$, the following intertwining relation has been identified

$$P_t^{\Psi} \Lambda_{\mathrm{I}_{\phi}} = \Lambda_{\mathrm{I}_{\phi}} P_t^{\Psi_{\theta}} \tag{4.63}$$

where $\Psi_{\theta}(u) = u(u-\theta) \in \mathbb{N}$. Then, using successively Tonelli theorem and the identity (4.59), we obtain that, for any $x > 0$,

$$\Lambda_{\mathrm{I}_{\phi}} \widetilde{\mathcal{I}}_{-\theta}(x) \;=\; \sum_{n=0}^{\infty} \frac{\mathcal{M}_{\mathrm{I}_{\phi}}(n)}{n!\Gamma(n+1-\theta)} x^n = \sum_{n=0}^{\infty} \frac{1}{W_{\phi}(n+1)\Gamma(n+1-\theta)} x^n = \widetilde{\mathcal{I}}_{\Psi}(x).$$

It is well-known that the mapping $x \mapsto d_q \widetilde{\mathcal{I}}_{-\theta}(x) = \widetilde{\mathcal{I}}_{-\theta}(qx), q > 0$, is a q-invariant for the squared Bessel semigroup $P^{\Psi_{\theta}}$, that is, for all $t \geqslant 0$, $e^{-qt} P_t^{\Psi_{\theta}} d_q \widetilde{\mathcal{I}}_{-\theta}(x) = d_q \widetilde{\mathcal{I}}_{-\theta}(x)$. Now observing that

$d_q\Lambda_{\mathrm{I}_\phi} = \Lambda_{\mathrm{I}_\phi} d_q$, we deduce from the intertwining relation above that

$$\begin{aligned} P_t^\Psi d_q \widetilde{\mathcal{I}}_\Psi(x) &= P_t^\Psi d_q \Lambda_{\mathrm{I}_\phi} \widetilde{\mathcal{I}}_{-\theta}(x) = P_t^\Psi \Lambda_{\mathrm{I}_\phi} d_q \widetilde{\mathcal{I}}_{-\theta}(x) \\ &= \Lambda_{\mathrm{I}_\phi} P_t^{\Psi_\theta} d_q \widetilde{\mathcal{I}}_{-\theta}(x) = e^{qt} \Lambda_{\mathrm{I}_\phi} d_q \widetilde{\mathcal{I}}_{-\theta}(x) \\ &= e^{qt} d_q \Lambda_{\mathrm{I}_\phi} \widetilde{\mathcal{I}}_{-\theta}(x) = e^{qt} d_q \widetilde{\mathcal{I}}_\Psi(x), \end{aligned}$$

which shows, since it is positive, that $d_q\widetilde{\mathcal{I}}_\Psi, q > 0$, is a q-invariant function for the semigroup P^Ψ. Then, applying Dynkin's formula to the bounded stopping $\mathrm{T}_\Psi(t) = \inf(\mathrm{T}_\Psi, t)$, we get, for all $t \geqslant 0$,

$$P^\Psi_{\mathrm{T}_\Psi(t)} d_q \widetilde{\mathcal{I}}_\Psi(x) = 1$$

where $P^\Psi_{\mathrm{T}_\Psi(t)}$, $t \geqslant 0$, is the semigroup corresponding to the process $(\mathrm{X}_{\mathrm{T}_\Psi(\mathrm{t})})_{\mathrm{t}\geqslant 0}$. Differentiating term by term the series $\widetilde{\mathcal{I}}_\Psi$, we observe that $x \mapsto d_q\widetilde{\mathcal{I}}_\Psi(x)$ is, for all $q > 0$, non-decreasing on $\mathbb{R}_+$. This allows us to invoke a dominated convergence argument, while combined with the absence of positive jumps of $\mathrm{X}(\Psi)$, which entails that $\mathrm{X}_{\mathrm{T}_\Psi}(\Psi) = 1$ almost surely, gives the first identity in (4.62). The rest of the statement, that is, T_Ψ is a positive self-decomposable variable is justified in [102]. We are ready to state the following result that identifies the first random variable of our pair of solutions to the van Dantzig problem.

Lemma 4.5.7. *Let $\Psi \in \mathbb{N}$ with $0 \leqslant \theta < 1$ and $\mathrm{B} = (\mathrm{B}_t)_{t\geqslant 0}$ be a standard Brownian motion independent of $\mathrm{X}(\Psi)$. Then, the random variable $\overline{\mathrm{D}}_\Psi = \sqrt{2}\mathrm{B}_{\mathrm{T}_\Psi}$ is real-valued, symmetric, and infinitely divisible. Moreover, for any $t \in \mathbb{R}$,*

$$\mathcal{F}_{\overline{\mathrm{D}}_\Psi}(t) = \frac{1}{\mathcal{I}_\Psi(t)}. \tag{4.64}$$

Proof. According to Theorem 4.5.6, T_Ψ is self-decomposable and hence infinitely divisible. Since B is a symmetric Lévy process, we have, by Bochner subordination, see [119, Theorem 30.1], that $\overline{\mathrm{D}}_\Psi$ is real-valued, symmetric, and infinitely divisible. Next, using the independence of B and $\mathrm{X}(\Psi)$ and hence of T_Ψ, one gets, for any $t \in \mathbb{R}$,

$$\mathcal{F}_{\overline{\mathrm{D}}_\Psi}(t) = \mathcal{F}_{\mathrm{T}_\Psi}\left(\mathrm{i}t^2\right) = \frac{1}{\mathcal{I}_\Psi(t)}$$

where the last line follows from (4.62). To characterize the second random variable, we need to introduce the random variable J_ν, $\nu \in (-\infty, \frac{1}{2})$, whose law is absolutely continuous with a density f_{J_ν} given by

$$f_{\mathrm{J}_\nu}(x) = \frac{2^{2\nu}\Gamma(1-\nu)}{\Gamma(\frac{1}{2})\Gamma(\frac{1}{2}-\nu)}\,(4-x^2)^{-\nu-\frac{1}{2}}\mathbb{I}_{\{|x|\leqslant 2\}}. \tag{4.65}$$

We note that the characteristic function corresponding to the random variable J_ν is the function $\mathcal{J}_{-\nu}(t) = \Gamma(1-\nu)t^\nu J_{-\nu}(2t)$; see [9, p. 38]. We are now ready to state the following which defines a new class of random variables indexed by the whole set of Bernstein functions, and, when restricted to the subet $\mathbf{B}_{\mathscr{D}}$ gives the other set of van Dantzig variables.

Lemma 4.5.8. *Let $\phi \in \mathbf{B}$ and define, for any $\nu < \frac{1}{2}$, the random variable*

$$\mathrm{D}_{\phi,\nu} = \sqrt{\mathrm{I}_\phi} \times \mathrm{J}_\nu$$

where J_ν is chosen independent of I_ϕ. Then, $\mathrm{D}_{\phi,\nu}$ is a symmetric random variable taking values in the possibly infinite interval $\left(-\frac{2\sqrt{2}}{\sigma}, \frac{2\sqrt{2}}{\sigma}\right)$ (we use the convention $\frac{1}{0} = \infty$). Moreover, all its even moments exist and are given by

$$\mathcal{M}_{\mathrm{D}_{\phi,\nu}}(2n) = \frac{\Gamma(2n+1)\Gamma(1-\nu)}{W_\phi(n+1)\Gamma(n+1-\nu)}, \quad n = 0, 1, \ldots, \tag{4.66}$$

and, for any $t \in \mathbb{R}$, we have

$$\mathcal{F}_{\mathrm{D}_{\phi,\nu}}(t) = \mathcal{J}_{\phi,\nu}(t) \tag{4.67}$$

where the entire function $\mathcal{J}_{\phi,\nu}$ is defined as in (4.48). Moreover, the law of $\mathrm{D}_{\phi,\nu}$ is absolutely continuous with a density $f_{\mathrm{D}_{\phi,\nu}}$ which is continuous on $\mathbb{R}$ and $f_{\mathrm{D}_{\phi,\nu}} \in \mathbf{C}_0^\infty(\mathbb{R}\setminus\{0\})$ (resp. $f_{\mathrm{D}_{\phi,\nu}} \in \mathbf{C}_0^p(\mathbb{R}\setminus\{0\})$, where $p = \lceil \frac{2}{\sigma^2}\left(\overline{\mu}(0^+)+\phi(0)\right) - \nu - \frac{1}{2}\rceil$) if $\sigma^2 = 0$ or $\overline{\mu}(0^+) = \infty$ (resp. otherwise), such that $f_{\mathrm{D}_{\phi,\nu}}(x) = f_{\mathrm{D}_{\phi,\nu}}(-x)$, and, for any $n \in \mathbb{N}$ (resp. $n = 0, \ldots, p$), $x > 0$ and $a > \frac{1}{2}+n$,

$$f^{(n)}_{\mathrm{D}_{\phi,\nu}}(x) = (-1)^n\frac{\Gamma(1-\nu)}{2\pi i}\int_{a-i\infty}^{a+i\infty} x^{1-2z}\frac{\Gamma(z)}{\Gamma(z-n)}\frac{\Gamma(2z-1-2n)}{W_\phi(z-n)\Gamma(z-\nu-n)}dz. \tag{4.68}$$

Proof. First, note that the symmetry property of J_ν entails the one of $\mathrm{D}_{\phi,\nu}$. Since the support of J_ν is $[-2, 2]$ and the one of I_ϕ is $\left[0, \frac{2}{\sigma^2}\right]$, see e.g. [96, Theorem 2.4], we deduce readily the one of

$\mathrm{D}_{\phi,\nu}$. Being symmetric, only the even moments are non-zero and are given, for any $n = 0, 1, \ldots$, by

$$\mathcal{M}_{\mathrm{D}_{\phi,\nu}}(2n) = \mathcal{M}_{\mathrm{J}_\nu}(2n)\mathcal{M}_{\mathrm{I}_\phi}(n)$$

where we used that the random variables are independent. Using the identity (4.59) and, observing, from the duplication formula of the gamma function, that

$$\begin{aligned} \int_{-2}^{2} x^{2n} f_{\mathrm{J}_\nu}(x)dx &= \frac{\Gamma(1-\nu)}{\Gamma(\frac{1}{2})\Gamma(\frac{1}{2}-\nu)} 2^{2n} \int_0^1 y^{n-\frac{1}{2}}(1-y)^{-\nu-\frac{1}{2}}dy \\ &= 2^{2n}\frac{\Gamma(1-\nu)\Gamma(n+\frac{1}{2})}{\Gamma(\frac{1}{2})\Gamma(n+1-\nu)} \\ &= \frac{\Gamma(1-\nu)(2n)!}{\Gamma(n+1-\nu)n!} \end{aligned} \tag{4.69}$$

we derive easily the identity (4.66). Next, since, see e.g. [59, Theorem 1], for any $t \in \mathbb{R}$,

$$\mathcal{F}_{\mathrm{J}_\nu}(t) = \mathcal{J}_{-\nu}(t) = \sum_{n=0}^{\infty} \frac{(-1)^n\Gamma(1-\nu)}{\Gamma(n+1-\nu)n!}t^{2n} \tag{4.70}$$

and recalling, from [96], that the law of I_ϕ is absolutely continuous, the independence of I_ϕ and J_ν yields, for any $t \in \mathbb{R}$, that

$$\begin{aligned} \mathcal{F}_{\mathrm{D}_{\phi,\nu}}(t) &= \int_0^\infty \mathcal{F}_{\mathrm{J}_\nu}(\sqrt{x}t) f_{\mathrm{I}_\phi}(x)dx \\ &= \int_0^\infty \mathcal{J}_\nu(\sqrt{x}t) f_{\mathrm{I}_\phi}(x)dx \\ &= \sum_{n=0}^{\infty} \frac{(-1)^n\Gamma(1-\nu)\mathcal{M}_{\mathrm{I}_\phi}(n)}{\Gamma(n+1-\nu)n!}t^{2n} \\ &= \mathcal{J}_{\phi,\nu}(t) \end{aligned}$$

where to justify the interchange of sums in the third equality, we resort to a classical Fubini's argument whose details are described in [127, Section 1.77], and relies on the fact that the series $\mathcal{J}_{\phi,\nu}$ is absolutely convergent on $\mathbb{R}$. For the last identity we used (4.59). Next, it is not difficult to see that the computation (4.69) extends to any complex z such that $\mathtt{Re}(z) > -\frac{1}{2}$, and, from [96, Theorem 2.4], the expression (4.59) also extends (at least) to $\mathtt{Re}(z) > -1$, to get, for $\mathtt{Re}(z) > -\frac{1}{2}$,

$$\mathcal{M}_{\mathrm{D}^2_{\phi,\nu}}(z) = \frac{\Gamma(2z+1)\Gamma(1-\nu)}{W_\phi(z+1)\Gamma(z+1-\nu)},$$

which proves (4.66). Then, recalling the Stirling formula of the gamma function, for any $a > 0$ and $|b|$ large,

$$|\Gamma(a + \mathrm{i}b)| \sim Ce^{-|b|\frac{\pi}{2}}|b|^{a-\frac{1}{2}} \tag{4.71}$$

where, here and below, $C > 0$ is a generic constant, we deduce, that, for any $a > -\frac{1}{2}$ and $|b|$ large,

$$\left|\frac{\Gamma(2a + 1 + 2\mathrm{i}b)}{\Gamma(a + 1 + \mathrm{i}b)\Gamma(a + 1 - \nu + \mathrm{i}b)}\right| \sim C|b|^{\nu-\frac{1}{2}}.$$

This combines with (4.55) when $\sigma^2 > 0$ and $\overline{\mu}(0^+) < \infty$, gives for any $\epsilon > 0$ and $a > -\frac{1}{2}$,

$$\left|\mathcal{M}_{\mathrm{D}^2_{\phi,\nu}}(a + \mathrm{i}b)\right| \leqslant C|b|^{-\frac{2}{\sigma^2}\left(\overline{\mu}(0^+)+\phi(0)\right)+\nu-\frac{1}{2}+\epsilon}. \tag{4.72}$$

We deduce that the mapping $b \mapsto |b|^p \left|\mathcal{M}_{\mathrm{D}^2_{\phi,\nu}}(a + \mathrm{i}b)\right|$ is integrable on $\mathbb{R}$ whenever $p < \frac{2}{\sigma^2}\left(\overline{\mu}(0^+) + \phi(0)\right) - \nu - \frac{1}{2}$. Then, invoking classical results on Mellin inversion, see e.g. [95, Section 1.7.4], we get the Mellin-Barnes representation of $f_{\mathrm{D}^2_{\phi,\nu}}$ which takes the form for any $x > 0$ and $a > \frac{1}{2}$,

$$f_{\mathrm{D}^2_{\phi,\nu}}(x) = \frac{\Gamma(1-\nu)}{2\pi\mathrm{i}}\int_{a-\mathrm{i}\infty}^{a+\mathrm{i}\infty} x^{-z}\frac{\Gamma(2z-1)}{W_\phi(z)\Gamma(z-\nu)}dz.$$

Next since $f_{\mathrm{D}_{\phi,\nu}}$ is symmetric, we have that $f_{\mathrm{D}_{\phi,\nu}}(x) = xf_{\mathrm{D}^2_{\phi,\nu}}(x^2), x > 0$, from where we deduce the Mellin-Barnes representation of $f_{\mathrm{D}_{\phi,\nu}}$. Next, since the mapping $z \mapsto \frac{\Gamma(2z-1)}{W_\phi(z)\Gamma(z-\nu)}$ is meromorphic on $(a', a' + \frac{1}{2}), 0 < a' < \frac{1}{2}$ with a simple pole at $\frac{1}{2}$, an application of the residues theorem yields

$$f_{\mathrm{D}_{\phi,\nu}}(x) = \frac{\Gamma(1-\nu)}{2W_\phi(\frac{1}{2})\Gamma(\frac{1}{2}-\nu)} + \frac{\Gamma(1-\nu)}{2\pi\mathrm{i}}\int_{a'-\mathrm{i}\infty}^{a'+\mathrm{i}\infty} x^{1-2z}\frac{\Gamma(2z-1)}{W_\phi(z)\Gamma(z-\nu)}dz$$

from where we easily conclude, as $a' < \frac{1}{2}$, that $\lim_{x\downarrow 0} f_{\mathrm{D}_{\phi,\nu}}(x) = \frac{\Gamma(1-\nu)}{2W_\phi(\frac{1}{2})\Gamma(\frac{1}{2}-\nu)}$. Moreover, by Mellin inversion, see again [95, Section 1.7.4], and by symmetry, we get that $f \in \mathbf{C}_0(\mathbb{R})$. From the same reference and by a similar reasoning, we obtain the expression and the smoothness properties for the successive derivatives for all $n \leqslant p$. The cases $\sigma = 0$ or $\overline{\mu}(0^+) = \infty$ follow easily by means of a similar reasoning and using (4.56) in place of (4.55).

We continue our program with the following observation which is the key step in proving Proposition 4.4.6 for the $L^{(2)}$ mapping. To state it, we recall that $X(\gamma)$, $\gamma \in \mathbb{R}$, is the γ-length-biased random variable of a non-negative random variable X, if its γ^{th} moment $m_{\mathrm{X}}(\gamma)$ is finite and

$$F_{\mathrm{X}(\gamma)}(dx) = \frac{x^{\gamma}}{m_{\mathrm{X}}(\gamma)} F_{\mathrm{X}}(dx).$$

This notion was used in the proof of the Lukacs mapping defined in (4.3.5). We are now ready to state the following.

Proposition 4.5.9. *For any $\phi \in \mathbf{B}$, we have $\mathcal{T}_1\phi(u) = \frac{u}{u+1}\phi(u+1) \in \mathbf{B}$. Moreover, with the notation of (4.59), we have, for any $t \in \mathbb{R}$,*

$$\mathcal{F}_{\mathrm{I}_{\mathcal{T}_1\phi}}(t) = \mathcal{F}_{\mathrm{I}_{\phi}(1)}(t)$$

and, for any twice continuously differentiable function $\mathcal{F}_X \in \mathscr{P}_+$,

$$L^{(2)}\Lambda_{\mathrm{I}_\phi}\mathcal{F}_X = \Lambda_{\mathrm{I}_\phi(1)}L^{(2)}\mathcal{F}_X \tag{4.73}$$

where $\Lambda_{\mathrm{I}_\phi} f(t) = \mathbb{E}\left[f(t\sqrt{\mathrm{I}_\phi})\right]$ and I_ϕ is chosen independent of X. In particular, for any $k = 1, 2, \ldots$, and, any $t \in \mathbb{R}$, with $\nu < \frac{1}{2}$,

$$L_k^{(2)}\mathcal{F}_{\mathrm{D}_{\phi,\nu}}(t) = \mathcal{F}_{\mathrm{J}_\nu(2k)\sqrt{\mathrm{I}_\phi(k)}}(t) = \mathcal{F}_{\mathrm{D}_{\phi,\nu}(2k)}(t). \tag{4.74}$$

Proof. The first statement can be found in [28]. Then, observing that $W_{\mathcal{T}_1\phi}(n+1) = \prod_{k=1}^{n} \frac{k}{k+1}\phi(k+1) = \frac{W_\phi(n+2)}{W_\phi(1)(n+1)}$, we deduce, from (4.59), that, for any integer n,

$$\mathcal{M}_{\mathrm{I}_{\mathcal{T}_1\phi}}(n) = \frac{n!}{W_{\mathcal{T}_1\phi}(n+1)} = \frac{(n+1)!W_\phi(1)}{W_\phi(n+2)} = \frac{\mathcal{M}_{\mathrm{I}_\phi}(n+1)}{\mathcal{M}_{\mathrm{I}_\phi}(1)} = \mathcal{M}_{\mathrm{I}_\phi(1)}(n) \tag{4.75}$$

which yields the second claim since these random variables are moment determinate, see [30], and the characteristic function uniquely determines the law of a random variable. Next, as $\mathcal{F}_X$ is twice continuously differentiable, one gets, for any $t \in \mathbb{R}$,

$$\begin{aligned}
L^{(2)}\Lambda_{\mathrm{I}_\phi}\mathcal{F}_X(t) &= L^{(2)}\mathcal{F}_{X\sqrt{\mathrm{I}_\phi}}(t) \\
&= \frac{\mathcal{F}^{(2)}_{X\sqrt{\mathrm{I}_\phi}}(t)}{\mathcal{F}^{(2)}_{X\sqrt{\mathrm{I}_\phi}}(0)} \\
&= \frac{1}{\mathcal{F}^{(2)}_{\sqrt{\mathrm{I}_\phi}}(0)} \int_0^\infty \frac{\mathcal{F}^{(2)}_X(t\sqrt{x})}{\mathcal{F}^{(2)}_X(0)} x f_{\mathrm{I}_\phi}(x)dx \\
&= \Lambda_{\mathrm{I}_\phi(1)}L^{(2)}\mathcal{F}_X(t)
\end{aligned}$$

where for the third equality, we used that X and I_ϕ are independent and for the last one that $\mathcal{F}^{(2)}_{\sqrt{\mathrm{I}_\phi}}(0) = \int_0^\infty x f_{\mathrm{I}_\phi}(x)dx = \frac{1}{\phi(1)}$. To prove the last relation, we recall that $\mathrm{D}_{\phi,\nu} = \sqrt{\mathrm{I}_\phi}\mathrm{J}_\nu$, where J_θ is chosen independent of I_ϕ and thus resorting to the commutation type relation (4.73), one gets, for any $t \in \mathbb{R}$,

$$L^{(2)}\mathcal{F}_{\mathrm{D}_{\phi,\nu}}(t) = L^{(2)}\Lambda_{\mathrm{I}_\phi}\mathcal{F}_{\mathrm{J}_\nu}(t) = \Lambda_{\mathrm{I}_\phi(1)}L^{(2)}\mathcal{F}_{\mathrm{J}_\nu}(t) = \Lambda_{\mathrm{I}_\phi(1)}\mathcal{F}_{\mathrm{J}_\nu(2)}(t) = \mathcal{F}_{\mathrm{J}_\nu(2)\sqrt{\mathrm{I}_{\mathrm{I}_\phi(1)}}}(t)$$

which provides the claim for $k = 1$. Then, an induction argument gives for any k,

$$L^{(2)}_{k+1}\mathcal{F}_{\mathrm{D}_{\phi,\nu}}(t) = L^{(2)} \circ L^{(2)}_k\mathcal{F}_{\mathrm{D}_{\phi,\nu}}(t) = L^{(2)}\mathcal{F}_{\mathrm{J}_\nu(2k)\sqrt{\mathrm{I}_{\mathrm{I}_\phi(k)}}} = \mathcal{F}_{\mathrm{J}_\nu(2k+2)\sqrt{\mathrm{I}_{\mathrm{I}_\phi(k+1)}}}(t)$$

which completes the proof.

End of the proof of Theorem 4.4.1

To complete the proof of Theorem 4.4.1, we take $\Psi \in \mathbb{N}_{\mathscr{D}}$, and, recall from Proposition 4.5.2, that there exists $\phi \in \mathbf{B}_{\mathscr{D}}$ and $0 \leqslant \theta < \frac{1}{2}$ such that $\Psi(u) = (u-\theta)\phi(u)$. Then combining lemmas 4.5.7 and 4.5.8, we obtain, using the notation of the latter lemma, that is $\mathrm{D}_{\phi,\theta} = \sqrt{\mathrm{I}_\phi}\mathrm{J}_\theta$,

$$\mathcal{F}_{\overline{\mathrm{D}}_\Psi}(\mathrm{i}t)\mathcal{F}_{\mathrm{D}_{\phi,\theta}}(t) = 1, \quad t \in \mathbb{R}.$$

Hence $[\mathcal{F}_{\overline{\mathrm{D}}_\Psi}, \mathcal{F}_{\mathrm{D}_{\phi,\theta}}]$ form a van Dantzig pair, which completes the proof of the theorem after invoking Theorem 4.5.6.

Proof of Theorem 4.4.4

We simply sketch the proof of the first claim as the detailed arguments can be found in [101]. Let $\phi \in \mathbf{B}_{P_1}$. Then, observing that $W_\Psi(n+1) = n!W_\phi(n+1), n \geqslant 0$, we write $F_\phi = \frac{1}{W_\phi}$, that is F_ϕ is a function which is analytic and zero-free on the half-plane $\mathtt{Re}(z) > z_1$, z_1 being the largest zero of ϕ which is simple, and, at least on this former half-plane, F_ϕ is solution to the recurrence equation

$$F_\phi(z+1) = \frac{1}{\phi(z)}F_\phi(z). \tag{4.76}$$

From this recurrence equation, we get that z_1 is also a simple zero of F_ϕ, and F_ϕ admits an analytic extension to $\mathtt{Re}(z) > \rho_1 > z_2$ due to the interlacing property. Then, the 1-separation property entails that $\rho_1 = z_1 - 1$ is a simple pole of ϕ, and, hence, ρ_1 is neither a zero nor a pole of F_ϕ, which, yields, by the recurrence (4.76), that F_ϕ admits an analytic extension to $\mathtt{Re}(z) > z_2 = \rho_2 - 1$. An induction argument gives that F_ϕ is indeed an entire function with the sequence of simple zeros $(z_k)_{k\geqslant 1}$, which completes the proof of the first claim by invoking Laguerre theorem [118, Theorem 4] as $\mathcal{J}_\Psi(z) = \sum_{n=0}^\infty \frac{(-1)^n F_\phi(n+1)}{n!} z^{2n}$. For the last claim, we invoke, for instance, the Mittag-Leffler function presented in example 4.4.2 which has non-real zeros.

Proof of Proposition 4.4.6

The case $p = 1$ is simply the combination of Lemma 4.5.3, Proposition 4.5.2 and Theorem 4.4.1. To prove the case $p = 2$, we take $\Psi \in \mathbb{N}_{\mathscr{D}}$ and recall that $\mathrm{D}_{\phi,\theta} = \sqrt{\mathrm{I}_\phi}\mathrm{J}_\theta$, where $\Psi(u) = (u-\theta)\phi(u)$. Then, using (4.68), Proposition 4.5.2 and the identity (4.74), one gets, for any $k = 1, 2, \dots,$

$$L_k^{(2)} \mathcal{J}_\Psi(t) = L_k^{(2)} \mathcal{F}_{\mathrm{D}_{\phi,\theta}}(t) = \mathcal{F}_{\mathrm{J}_\theta(2k)\sqrt{\mathrm{I}_{\mathrm{I}_\phi(k)}}}(t). \tag{4.77}$$

On the other hand, using (4.51), and recalling the identity $\mathcal{I}_\Psi(t) = \mathcal{J}_\Psi(e^{\mathrm{i}\frac{\pi}{2}}t)$, one has

$$L_k^{(2)} \mathcal{I}_\Psi = \mathcal{I}_{\overline{\mathcal{T}}_{\frac{1}{2}}^{2k}\Psi}. \tag{4.78}$$

However, since $\Psi \in \mathbb{N}_{\mathscr{D}}$, then $\frac{1}{2} > \theta$ and, from Proposition 4.5.2(3), the mapping $u \mapsto \overline{\mathcal{T}}_{\frac{1}{2}}^{2k}\Psi(u) = \overline{\mathcal{T}}_{\frac{1}{2}}^{1} \circ \overline{\mathcal{T}}_{\frac{1}{2}}^{2k-1}\Psi(u) = \frac{u-\frac{1}{2}}{u+\frac{1}{2}}\overline{\mathcal{T}}_{\frac{1}{2}}^{2k-1}\Psi(u+\frac{1}{2}) \in \mathbb{N}$ with $\overline{\mathcal{T}}_{\frac{1}{2}}^{2k}\Psi(\frac{1}{2}) = 0$. Then, $\overline{\mathcal{T}}_{\frac{1}{2}}^{2k}\Psi$ fulfills the requirement of Theorem 4.5.6 and hence

$$\mathcal{F}_{\mathrm{T}_{\overline{\mathcal{T}}_{\frac{1}{2}}^{2k}\Psi}}(\mathrm{i}u) \quad = \quad \frac{1}{\mathcal{I}_{\overline{\mathcal{T}}_{\frac{1}{2}}^{2k}\Psi}(\sqrt{u})}$$

which concludes the proof after invoking Lemma 4.5.7.

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