SPECIES AND HYPERPLANE ARRANGEMENTS

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This dissertation has two leading characters: Hopf monoids in the category of species and the Tits algebra of a real hyperplane arrangement. The relation between these two comes from the work of Aguiar and Mahajan (2013), who showed that a (co)commutative Hopf monoid gives rise to a family of (left)right-modules over the Tits algebra of the braid arrangement in all dimensions. One goal of this thesis is to explore the representation theory of the Tits algebra of arbitrary affine arrangements to extend what is known in the case of linear arrangements and to give an insight into some unanswered questions in the field of Hopf monoids.

In the first part, we extend the study of characteristic elements of a hyperplane arrangement from the linear to the affine case. We present the basic properties of these elements and apply them to derive numerous results about the characteristic polynomial of an arrangement, from Zaslavsky's formulas to more recent results of Kung and of Klivans and Swartz. We construct several examples of characteristic elements, including one in terms of intrinsic volumes of faces of the arrangement.

In the second part, we study deformations $\mathcal{A}$ of a linear arrangement $\mathcal{A}_0$ and endow the Tits algebra of $\mathcal{A}$ with a bimodule structure over the algebra of $\mathcal{A}_0$. The left module structure sheds some light on the study of exponential sequences of arrangements, in the sense of Stanley. In particular, we construct the Hopf monoid of faces associated with such a sequence and use characteristic elements to deduce formulas for certain bivariate polynomial invariants of these arrangements.

In the third part, we endow the polytope subalgebra of deformations of a zonotope with the structure of a module over the Tits algebra of the corresponding hyperplane arrangement. We study algebraic invariants of this module and find relations between statistics on (signed) permutations and the module structure in the case of (type B) generalized permutahedra. In type B, the module structure surprisingly reveals that any family of generators (via signed Minkowski sums) for generalized permutahedra of type B will contain at least 2^{d-1} full-dimensional polytopes. We find a generating family of simplices attaining this minimum. Finally, we prove that the relations defining the polytope algebra are compatible with the Hopf monoid structure of generalized permutahedra, and explain the relationship between the antipode formula of this Hopf monoid and inversion in the polytope algebra.

In the last chapter, we introduce a novel definition of type B Hopf monoids. Unlike standard Hopf monoids and the Hopf monoids in $\mathcal{H}$-species of Bergeron and Choquette (2009), our notion involves a pair of species (one of type A with an involution and one of type B) and a (co)module structure of one over the other. This closer represents the algebraic structure that arises from the Tits algebra of the type B Coxeter arrangement. We study some general constructions like the substitution product of type B objects and the free (commutative) monoid over a positive type B object. We conclude by endowing the type B object of generalized permutahedra and the type B object of symplectic matroids with the structure of a type B Hopf monoid.

BIOGRAPHICAL SKETCH

Jose Dario Bastidas Olaya was born to Jose Luis Bastidas and Martha Irene Olaya in Pasto, Nariño, Colombia on July, 1991. He attended school at Instituto Champagnat in his hometown before moving to Bogotá to study at Universidad de Los Andes. There, he completed a Bachelor of Science and a Master of Science in Mathematics under the supervision of Professor Mauricio Velasco. He began graduate school at Cornell University in 2015, and in 2019 he obtained a Special Masters in Computer Science. He completed his dissertation in 2021 under the supervision of Professor Marcelo Aguiar.

Para mis viejos y abuelos.

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I also want to thank Mauricio Velasco for guiding me during the earlier stages of my career and encouraging me to pursue a graduate program. Mauricio has an incredible ability to spread his enthusiasm and appreciation for all areas of mathematics. I hope one day I can do the same with my students. My gratitude also goes to Federico Ardila, Karola Mészáros, and Vic Reiner for the valuable conversations and advice during these years.

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Last but not least, I want to thank my parents Martha and Jose, my grandparents Segundo, Stella, Ana, and Roberto, and all my extended family for their caring, love, and patience. Your encouragement made this dissertation possible. Gracias.

CONTENTS

LIST OF FIGURES

CHAPTER 1

INTRODUCTION

Species and Hopf monoids

Combinatorial species were originally introduced by Joyal [49] as a tool for studying generating power series from a combinatorial perspective. They provide a unified framework to study families of combinatorial objects. When a family of combinatorial objects has natural operations to *merge* and *break* structures, the corresponding species promotes to a Hopf monoid. The theory of Hopf monoids in the category of species was developed by Aguiar and Mahajan [6, 7] and has received significant attention in recent years, see [5, 15, 17, 22, 34, 66, 70, 74, 75]. Notably, the Hopf monoid of generalized permutahedra of Aguiar and Ardila [1] encompasses several Hopf monoids that had previously been studied on a case-by-case analysis.

Roughly speaking, a species h assigns to each finite set I a collection $\mathsf{h}[I]$ of h-structures on I (think of *linear orders* of I or *set partitions* of I). A Hopf monoid is a species h together with product and coproduct maps

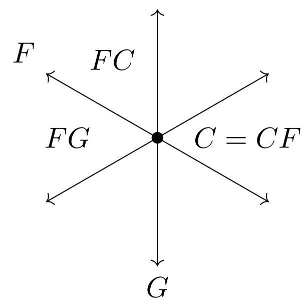
$$\begin{aligned} \mu_{S,T} : \mathsf{h}[S] \times \mathsf{h}[T] &\to \mathsf{h}[I] \\ (x, y) &\mapsto x \cdot y \end{aligned} \qquad \text{and} \qquad \begin{aligned} \Delta_{S,T} : \mathsf{h}[I] &\to \mathsf{h}[S] \times \mathsf{h}[T] \\ z &\mapsto (z|_S, z/_S) \end{aligned}$$

for each finite set I and decomposition $I = S \sqcup T$. These maps need to satisfy certain (co)unitality, (co)associativity and compatibility axioms.

Hyperplane arrangements and the Tits algebra

In his work on Coxeter complexes Σ, Tits [72, 73] defined the *projection* of a simplex $G \in \Sigma$ into another simplex $F \in \Sigma$. This operation can be extended to a product on the set $\Sigma[\mathcal{A}]$ of faces of an arbitrary real hyperplane arrangement $\mathcal{A}$. Bidigare [26] studied the induced semigroup structure and its applications to some well-studied Markov chains; see also [25, 32]. The Tits algebra of a hyperplane arrangement is the linearization $\Bbbk\Sigma[\mathcal{A}]$ of the face semigroup. Saliola [65] studied this algebraic structure and, in particular, constructed a complete system of primitive orthogonal idempotents for the Tits algebra.

Fix a hyperplane arrangement $\mathcal{A}$ in a real vector space. The product of two faces F and G is the first face you encounter after moving a small positive distance from an interior point of F to an interior point of G, as illustrated below



Rendezvous: the braid arrangement

The braid arrangement $\mathcal{A}_d$ in $\mathbb{R}^d$ consists of hyperplanes $\boldsymbol{x}_i = \boldsymbol{x}_j$ for $1 \leq i < j \leq d$. Faces of the braid arrangement are in correspondence with set compositions (i.e. ordered set partitions) $(S_1, \dots, S_k)$ of $[d] := \{1, 2, \dots, d\}$. More generally, we let $\mathcal{A}_I$ denote the braid arrangement in $\mathbb{R}^I$ for any finite set I.

Take a Hopf monoid h. Then, every face $F = (S_1, S_2, \dots, S_k)$ of the braid arrangement $\mathcal{A}_I$ induces a map

$$\mathsf{h}[I] \xrightarrow{\Delta_F} \mathsf{h}[S_1] \times \mathsf{h}[S_2] \times \cdots \times \mathsf{h}[S_k] \xrightarrow{\mu_F} \mathsf{h}[I]$$

obtained by iterating coproducts and products in any meaningful way. A natural question arises:

$$\text{Does } \Sigma[\mathcal{A}_I] \ni F \longmapsto \mu_F \circ \Delta_F \in \mathrm{End}(\mathsf{h}[I]) \text{ induce a module structure?}$$

Aguiar and Mahajan [7] show that the answer is yes whenever h is commutative ($x \cdot y = y \cdot x$) or cocommutative ($z|_S = z/_T$ and $z/_S = z|_T$).

The relation between Hopf monoids and modules over the Tits algebra of the braid arrangement is a driving idea for the work contained in this dissertation. Notably, in Chapters 4 and 5, we first study a particular module over the Tits algebra of an arbitrary linear arrangement, and subsequently construct a Hopf monoid compatible with the module structure in the case of the braid arrangement. Furthermore, the algebraic structures surrounding the Tits algebra of Coxeter arrangement of type B inspired the definition of type B Hopf monoid of Chapter 6.

Chapter 3

Simple representations of the Tits algebra $\Bbbk\Sigma[\mathcal{A}]$ are one-dimensional and indexed by flats X of the arrangement. Let χ_{X} denote the character of the simple module associated to the flat X. An element $w \in \Bbbk\Sigma[\mathcal{A}]$ of the Tits algebra is *characteristic* of parameter $t \in \Bbbk$ if $\chi_{\mathrm{X}}(w) = t^{\dim(\mathrm{X})}$ for all flats X of $\mathcal{A}$.

We extend the theory of characteristic elements for real hyperplane arrangements started in [8, Chapter 12] from the linear to the affine case. We present the basic properties of these elements and apply them to derive numerous results about the characteristic polynomial of an arrangement, from Zaslavsky's formulas to more recent results of Kung.

A main contribution of this chapter is the construction of a characteristic element canonically associated to each arrangement in terms of intrinsic volumes. As a consequence, we recover a result of Klivans and Swartz [50] relating the characteristic polynomial of an arrangement with the intrinsic volumes of its *chambers* (maximal faces).

Chapter 4

An affine arrangement $\mathcal{A}$ is a *deformation* of a linear arrangement $\mathcal{A}_0$ if the hyperplanes of $\mathcal{A}$ are parallel to those of $\mathcal{A}_0$. We endow the Tits algebra $\Bbbk\Sigma[\mathcal{A}]$ of a deformation $\mathcal{A}$ of $\mathcal{A}_0$ with the structure of a bimonoid over $\Bbbk\Sigma[\mathcal{A}_0]$, and subsequently deduce that the bimodule structure arises from an inclusion of algebras $\Bbbk\Sigma[\mathcal{A}_0] \hookrightarrow \Bbbk\Sigma[\mathcal{A}]$.

We then specialize to the case of *exponential sequences of arrangements* in the sense of Stanley [69]. A sequence of arrangements $\mathscr{A} = (\mathcal{A}_1, \mathcal{A}_2, \dots)$ is *exponential* if $\mathcal{A}_d$ is a deformation of the braid arrangement in $\mathbb{R}^d$ and for all $S \subseteq [d]$, the subarrangement of $\mathcal{A}_d$ consisting of hyperplanes parallel to $\boldsymbol{x}_i = \boldsymbol{x}_j$ for $i, j \in S$ is isomorphic to $\mathcal{A}_{|S|}$. The sequences of braid, Shi, Linial, or Catalan arrangements are examples of exponential sequences.

Given an exponential sequence $\mathscr{A}$, we construct the *Hopf monoid of faces of the sequence* $\Sigma_{\mathscr{A}}$. Elements of $\Sigma_{\mathscr{A}}[I]$ are the faces of the arrangement $\mathcal{A}_I \cong \mathcal{A}_{|I|}$ in $\mathbb{R}^I$. The (cocommutative) Hopf monoid structure is given by a (left) action of $\Bbbk\Sigma[(\mathcal{A}_n)_0]$ on $\Bbbk\Sigma[\mathcal{A}_n]$. We use characteristic elements and tools from the theory of Hopf monoids to study bivariate polynomial invariants of the arrangements in $\mathscr{A}$, thus generalizing a result of Stanley.

Chapter 5

Generalized permutahedra were first introduced by [38] in the context of optimization of submodular functions, and have been of central interest to combinatorialists because they serve as polyhedral models for several families of combinatorial structures. They have been extensively studied by Postnikov [61], Postnikov, Reiner and Williams [60], and many others.

Aguiar and Ardila [1] endowed generalized permutahedra with the structure of a Hopf monoid, and obtained an elegant and simple formula for its *antipode*. They observe the similarity between their formula and the formula for inversion in the polytope algebra of McMullen [55]. One of the contributions of this chapter is to give an explanation for this similarity.

We first take a more general approach and for every linear hyperplane arrangement $\mathcal{A}$ with ambient space V, we define the module of generalized zonotopes of $\mathcal{A}$. It is at the same time a subalgebra of the polytope algebra $\Pi(V)$, and a right $\Bbbk\Sigma[\mathcal{A}]$-module. We study numerical invariants arising from this module structure and later specialize to the case of the braid arrangement and the type B Coxeter arrangement. As a surprising consequence, we show that any collection of type B generalized permutahedra in $\mathbb{R}^d$ that affinely spans the corresponding deformation cone (also called type cone) must contain at least 2^{d-1} full dimensional polytopes. In particular, we give two answers to a question of Ardila, Castillo, Eur and Postnikov [14] that asks for a generating family. One attains the theoretical minimum of 2^{d-1} full dimensional polytopes, and the other is invariant under the action of the corresponding Coxeter group.

Chapter 6

The past two decades have witnessed different attempts to extend the theory of species and Hopf monoids to families of objects with groups of symmetries other than $\mathfrak{S}_n$. See [24, 35, 48] for type B (cubical/hyperoctahedral) species, and [64] for Coxeter species.

We give a novel definition of **type B Hopf monoids**. In the same spirit as work of Bergeron and Choquette, the role of finite sets in species is replaced by finite sets with a fixed-point free involution. However, our theory substantially differs from that of previous authors in a central idea. Instead of attempting to define a monoidal structure on the category of type B species, our construction involves an action of the monoidal category of (standard) species on the category of Type B species.

We study some general constructions, such as the *free (commutative) monoid* over a positive type B object, the *substitution product* of type B objects, *convolution monoids*, and a notion of *type B antipode*. We endow (type B) Boolean functions, (bi)submodular functions, and (type B) generalized permutahedra with the structure of a type B Hopf monoid.

Joint work

The contents of Chapters 3 and 4 are joint work with M. Aguiar and S. Mahajan and Chapter 6 is joint work with M. Aguiar.

CHAPTER 2

PRELIMINARIES

2.1 Polyhedral geometry

We review normal cones, normal fans, tangent cones, and recession cones of polyhedra. For more details, see [77].

2.1.1 Polytopes

Let V be a real vector space of dimension d endowed with an inner product $\langle \cdot, \cdot \rangle$, and let $\mathbf{0} \in V$ denote its zero vector. For a polytope $P \subseteq V$ and a vector $\boldsymbol{v} \in V$, let $P_{\boldsymbol{v}}$ denote the face of P *maximized in the direction* $\boldsymbol{v}$. That is,

$$P_{\boldsymbol{v}} := \{\boldsymbol{p} \in P \,:\, \langle \boldsymbol{p}, \boldsymbol{v} \rangle \geq \langle \boldsymbol{q}, \boldsymbol{v} \rangle \text{ for all } \boldsymbol{q} \in P\}.$$

The (outer) **normal cone** of a face F of P is the polyhedral cone

$$N_F P := \{\boldsymbol{v} \in V \,:\, F \leq P_{\boldsymbol{v}}\} = \overline{\{\boldsymbol{v} \in V \,:\, F = P_{\boldsymbol{v}}\}},$$

and the **normal fan** of P is the collection $\Sigma_P = \{N_F P \,:\, F \leq P\}$ of all normal cones of faces of P. There is a natural order-reversing correspondence between faces of P and cones in Σ_P. For a cone $C \in \Sigma_P$, we let $P_C \leq P$ denote the face whose normal cone is C. That is $P_C = P_{\boldsymbol{v}}$ for any $\boldsymbol{v} \in \operatorname{relint}(C)$.

Recall that a fan Σ **refines** Σ' if every cone in Σ' is a union of cones in Σ. We say that a polytope Q is a **deformation** of P if Σ_P refines Σ_Q. The **Minkowski sum** of two polytopes $P, Q \subseteq V$ is the polytope $P + Q := \{\boldsymbol{p} + \boldsymbol{q} \,:\, \boldsymbol{p} \in P,\, \boldsymbol{q} \in Q\}$.

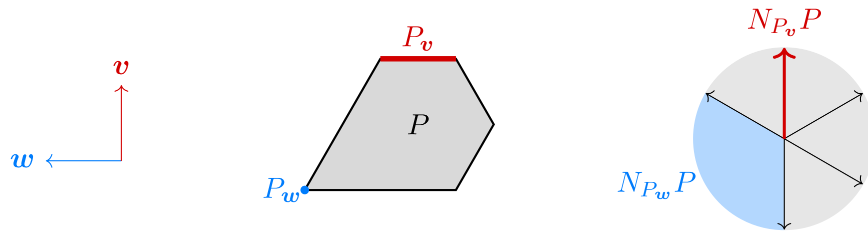


Figure 2.1.1: A 2-dimensional polytope P and two of its faces $P_{\boldsymbol{v}}, P_{\boldsymbol{w}}$ maximized in directions $\boldsymbol{v}, \boldsymbol{w}$, respectively. On the right, the normal fan Σ_P and the normal cones corresponding to the faces $P_{\boldsymbol{v}}, P_{\boldsymbol{w}}$ of P.

We say that a polytope Q is a **Minkowski summand** of P if $P = Q + Q'$ for some polytope Q'. The normal fan of $P + Q$ is the common refinement of Σ_P and Σ_Q, see Figure 2.1.2 for an example involving unbounded polyhedra. Hence, Σ_P refines the normal fan of any of its Minkowski summands, and the Minkowski sum of deformations of P is again a deformation of P.

The f**-polynomial** of a d-dimensional polytope P is

$$f(P, z) = \sum_{i=0}^{d} f_i(P) z^i,$$

where $f_i(P)$ is the number of i-dimensional faces of P. The h**-polynomial** of P is defined by

$$h(P, z) = \sum_{i=0}^{d} h_i(P) z^i = f(P, z-1).$$

The sequences $(f_0(P), \ldots, f_d(P))$ and $(h_0(P), \ldots, h_d(P))$ are the f-vector and h-vector of P, respectively. These polynomials behave nicely with respect to the Cartesian product of polytopes. If $P \subseteq V$ and $Q \subseteq V'$ are polytopes, then

$$f(P \times Q, z) = f(P, z) f(Q, z) \qquad h(P \times Q, z) = h(P, z) h(Q, z),$$

where $P \times Q = \{(\boldsymbol{p}, \boldsymbol{q}) \in V \oplus V' \, : \, \boldsymbol{p} \in P, \, \boldsymbol{q} \in Q\}$.

2.1.2 (Unbounded) Polyhedra

A **polyhedral cone** $C \subseteq V$ is the positive span of a finite collection of vectors in V. In particular, a polyhedral cone always contains the zero vector $\mathbf{0} \in V$. In the present work, cone will always mean polyhedral cone.

A **polyhedron** $P \subseteq V$ is any set that can be written as a Minkowski sum $P = Q + C$, where Q is a polytope and C is a cone. Normal cones and normal fans are defined just like for (bounded) polytopes.

The **recession cone** of a polyhedron P is the cone

$$\text{rec}(P) := \{\boldsymbol{v} \in V \,:\, \boldsymbol{v} + P \subseteq P\}. \tag{2.1.1}$$

If $P = Q + C$, where Q is a polytope and C is a cone, then $C = \text{rec}(P)$. On the other hand, the polytope Q is not completely determined by P (unless P itself is a polytope, in which case $Q = P$ and $C = \text{rec}(P) = \{\mathbf{0}\}$).

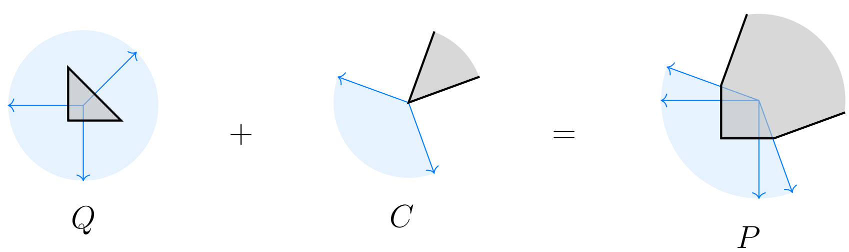


Figure 2.1.2: The normal fan of the polyhedron $P = Q + C$ is the common refinement of the normal fans of Q and of C.

If C is a cone with minimal face O, then $N_O C$ is sometimes called the **polar cone** of C. In this case, the normal fan Σ_C consists precisely of the faces of $N_O C$.

The **tangent cone** of P at a face $F \leq P$, denoted $T_F P$, is the cone

$$T_F P = \{\boldsymbol{v} \in V \,:\, \boldsymbol{x}_0 + t\boldsymbol{v} \in P \text{ for small enough } t > 0\}, \tag{2.1.2}$$

where $\boldsymbol{x}_0$ is any point in the relative interior of F. It is the polar cone to $N_F P$.

2.2 Hyperplane arrangements

We follow [8] for definitions, see the first chapters for more details.

2.2.1 Faces, flats and characteristic polynomial

Let $\mathcal{A}$ be a (real, affine, finite) hyperplane arrangement: a finite collection of affine hyperplanes in a finite-dimensional real vector space V. A subspace of V obtained as intersection of hyperplanes in $\mathcal{A}$ is called a **flat**. The set of flats of $\mathcal{A}$ is denoted by $\mathcal{L}[\mathcal{A}]$, and it forms a graded join-semilattice ordered by inclusion, in particular it is ranked. The ambient space is the top element of $\mathcal{L}[\mathcal{A}]$, we denote it by $\top$. When the arrangement is **central**, that is when all hyperplanes intersect, $\mathcal{L}[\mathcal{A}]$ is a lattice with minimum element equal to the intersection of all hyperplanes in $\mathcal{A}$, we denote it by $\bot$. The **rank** of the arrangement, denoted by $\mathrm{rank}(\mathcal{A})$, is by definition the rank of the poset $\mathcal{L}[\mathcal{A}]$.

The **characteristic polynomial** of $\mathcal{A}$ is

$$\chi(\mathcal{A}, t) := \sum_{\mathrm{Y} \in \mathcal{L}[\mathcal{A}]} \mu(\mathrm{Y}, \top)\, t^{\dim(\mathrm{Y})}, \tag{2.2.1}$$

where μ is the Möbius function of the poset $\mathcal{L}[\mathcal{A}]$. It is a monic polynomial of degree $\dim(V)$.

The **arrangement under a flat** X is the following collection of hyperplanes in ambient space X

$$\mathcal{A}^{\mathrm{X}} = \{\mathrm{H} \cap \mathrm{X} \,:\, \mathrm{H} \in \mathcal{A},\, \mathrm{X} \not\subseteq \mathrm{H},\, \mathrm{H} \cap \mathrm{X} \neq \emptyset\}.$$

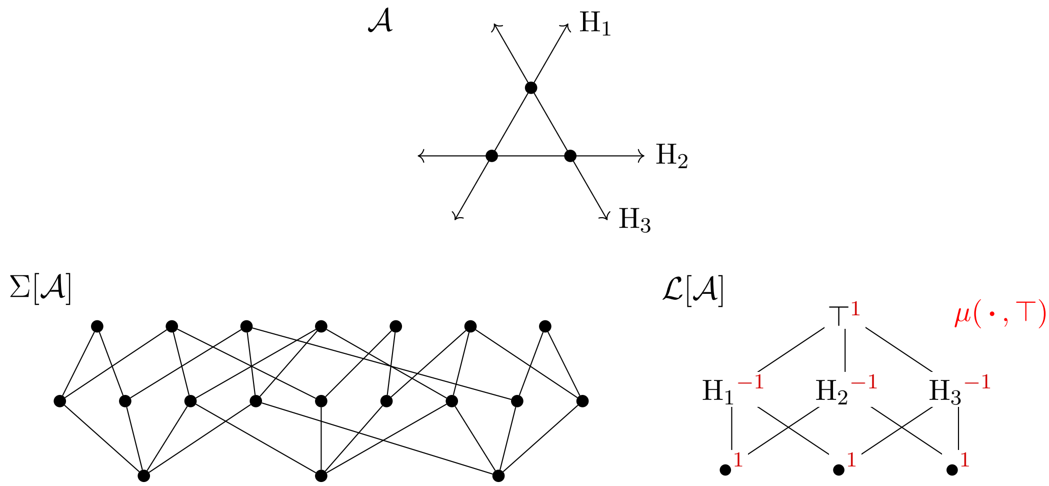


Figure 2.2.1: A 2-dimensional arrangement $\mathcal{A}$ together with its poset of faces (left) and semilattice of flats (right). The Möbius function of the lattice of flats is shown in red. The characteristic polynomial of $\mathcal{A}$ is $\chi(\mathcal{A}, t) = t^2 - 3t + 3$.

This is sometimes called the **restriction** of $\mathcal{A}$ to X. The flats of $\mathcal{A}^{\mathrm{X}}$ are the flats of $\mathcal{A}$ that are contained in X. Hence,

$$\chi(\mathcal{A}^{\mathrm{X}}, t) = \sum_{\mathrm{Y}:\, \mathrm{Y} \leq \mathrm{X}} \mu(\mathrm{Y}, \mathrm{X})\, t^{\dim(\mathrm{Y})}. \tag{2.2.2}$$

Similarly, the **arrangement over a flat** X is

$$\mathcal{A}_{\mathrm{X}} = \{\mathrm{H} \,:\, \mathrm{H} \in \mathcal{A},\ \mathrm{X} \subseteq \mathrm{H}\}.$$

Note that $\mathcal{A}_{\mathrm{X}}$ is a central arrangement in V, since X is precisely the intersection of all hyperplanes in $\mathcal{A}_{\mathrm{X}}$. The flats of $\mathcal{A}_{\mathrm{X}}$ are the flats of $\mathcal{A}$ that contain X. Hence,

$$\chi(\mathcal{A}_{\mathrm{X}}, t) = \sum_{\mathrm{Y}:\, \mathrm{Y} \geq \mathrm{X}} \mu(\mathrm{Y}, \top)\, t^{\dim(\mathrm{Y})}. \tag{2.2.3}$$

The hyperplanes in $\mathcal{A}$ split V into a collection $\Sigma[\mathcal{A}]$ of polyhedra called **faces**. Explicitly, the complement in V of the union of hyperplanes in $\mathcal{A}$ is the disjoint union of open subsets of V; and $\Sigma[\mathcal{A}]$ is the collection of the closures of these regions together with all their faces. The collection $\Sigma[\mathcal{A}]$ is a poset under containment, its maximal elements are called **chambers**. If the arrangement $\mathcal{A}$ is central, then $\Sigma[\mathcal{A}]$ contains a unique minimum element O, which we call the **central face** of the arrangement, it coincides with the minimum flat.

The arrangement $\mathcal{A}$ is **essential** if the minimal flats are points. In general, the minimal flats are pairwise parallel affine subspaces of the same dimension. Intersecting with an orthogonal subspace makes $\mathcal{A}$ essential. Faces of $\mathcal{A}$ are in correspondence with faces of the essentialization. The same applies to flats. A face of $\mathcal{A}$ is **essentially bounded** if the corresponding face of the essentialization is bounded.

The **support** of a face F is the smallest flat $\mathrm{s}(F)$ containing it. Equivalently, it is the affine span of F. The support map

$$\mathrm{s} : \Sigma[\mathcal{A}] \to \mathcal{L}[\mathcal{A}] \tag{2.2.4}$$

is surjective and order preserving.

Given two hyperplane arrangements $\mathcal{A}$ and $\mathcal{A}'$ in vector spaces V and W, respectively, the **product arrangement** $\mathcal{A} \times \mathcal{A}'$ is the hyperplane arrangement in $V \oplus W$ consisting of hyperplanes

$$\mathrm{H} \times W \text{ for each } \mathrm{H} \in \mathcal{A} \quad \text{and} \quad V \times \mathrm{H} \text{ for each } \mathrm{H} \in \mathcal{A}'.$$

There are natural identifications

$$\Sigma[\mathcal{A}] \times \Sigma[\mathcal{A}'] = \Sigma[\mathcal{A} \times \mathcal{A}'] \quad \text{and} \quad \mathcal{L}[\mathcal{A}] \times \mathcal{L}[\mathcal{A}'] = \mathcal{L}[\mathcal{A} \times \mathcal{A}'], \tag{2.2.5}$$

given by

$$(F, G) \longleftrightarrow F \times G \qquad \text{and} \qquad (\mathrm{X}, \mathrm{Y}) \longleftrightarrow \mathrm{X} \times \mathrm{Y},$$

respectively.

2.2.2 The Tits semigroup

The set $\Sigma[\mathcal{A}]$ is a semigroup under the **Tits product**. Informally, the product of two faces F and G is the first face you encounter after moving a small positive distance from an interior point of F to an interior point of G, as illustrated below. Note that F is always a face of FG and $F = FG$ if and only if $\mathrm{s}(G) \leq \mathrm{s}(F)$.

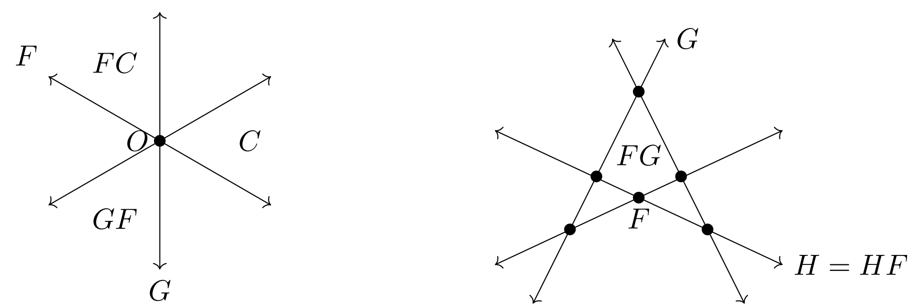


Figure 2.2.2: Product of faces in two arrangements of rank 2.

In order to formalize this product, let us first define the **sign sequence** of a face. There are two half-spaces H^+ and H^- associated to each hyperplane H. The choice of $+$ and $-$ is arbitrary but fixed. For convenience, we also denote $\mathrm{H}^0 = \mathrm{H}$. The sign sequence $(\epsilon_{\mathrm{H}}(F))_{\mathrm{H}\in\mathcal{A}}$ of a face F is defined by

$$\epsilon_{\mathrm{H}}(F) = \begin{cases} 0 & \text{if } F \subseteq \mathrm{H}, \\ + & \text{if } F \subseteq \mathrm{H}^+ \text{ and } F \nsubseteq \mathrm{H}, \\ - & \text{if } F \subseteq \mathrm{H}^- \text{ and } F \nsubseteq \mathrm{H}. \end{cases}$$

Thus, $\epsilon_{\mathrm{H}}(F) = +$ (resp. $\epsilon_{\mathrm{H}}(F) = -$) if and only if $\mathrm{relint}(F)$ is contained in $\mathrm{H}^+ \setminus \mathrm{H}^0$ (resp. $\mathrm{H}^- \setminus \mathrm{H}^0$). Moreover, the sign sequence uniquely determines F, since:

$$F = \bigcap_{\mathrm{H}\in\mathcal{A}} \mathrm{H}^{\epsilon_{\mathrm{H}}(F)}.$$

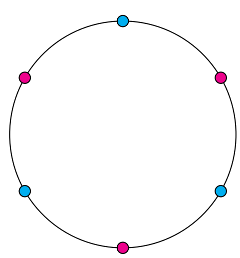
Now we can define the face FG in terms of its sign sequence.

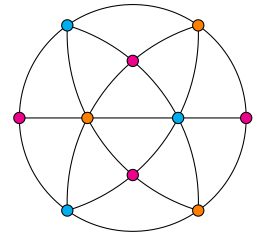
$$\epsilon_{\mathrm{H}}(FG) = \begin{cases} \epsilon_{\mathrm{H}}(F) & \text{if } \epsilon_{\mathrm{H}}(F) \neq 0, \\ \epsilon_{\mathrm{H}}(G) & \text{otherwise.} \end{cases}$$

For linear arrangements, such as the one on the left in Figure 2.2.2, $\Sigma[\mathcal{A}]$ is a monoid and the central face O (obtained by intersecting all hyperplanes in $\mathcal{A}$) is the unit. For affine arrangements, such as the one on the right, this semigroup is nonunital.

2.2.3 The braid arrangement

The **braid arrangement** $\mathcal{A}_d$ in $\mathbb{R}^d$ consists of the diagonal hyperplanes $\boldsymbol{x}_i = \boldsymbol{x}_j$ for $1 \leq i < j \leq d$. Its central face is the line perpendicular to the hyperplane $\boldsymbol{x}_1 + \cdots + \boldsymbol{x}_d = 0$. Intersecting $\mathcal{A}_d$ with this hyperplane and a sphere around the origin we obtain the *Coxeter complex of type* A_{d-1}. The pictures below show the cases $d = 3$ and 4.





Flats and faces of $\mathcal{A}_d$ are in one-to-one correspondence with set partitions and set compositions of $[d] := \{1, 2, \ldots, d\}$, respectively. We proceed to review this correspondence.

A **weak set partition** of a finite set I is a collection $\mathrm{X} = \{S_1, \ldots, S_k\}$ of pairwise disjoint subsets $S_i \subseteq I$ such that $I = S_1 \cup \cdots \cup S_k$. The subsets S_i are the

blocks of X. A **set partition** is a weak set partition with no empty blocks. We write $\mathrm{X} \vdash I$ to denote that X is a set partition of I. Given a partition $\mathrm{X} \vdash [d]$, the corresponding flat of $\mathcal{A}_d$ is the intersection of the hyperplanes $\boldsymbol{x}_i = \boldsymbol{x}_j$ for all i, j that belong to the same block of X, as illustrated in the following example for $d = 8$:

$$\boldsymbol{x}_1 = \boldsymbol{x}_3, \quad \boldsymbol{x}_2 = \boldsymbol{x}_5 = \boldsymbol{x}_6 = \boldsymbol{x}_8 \qquad \longleftrightarrow \qquad \{13, 2568, 4, 7\},$$

where we write 13 to abbreviate the set $\{1, 3\}$, 2568 to abbreviate the set $\{2, 5, 6, 8\}$ and so on. We use X to denote both a flat of $\mathcal{A}_d$ and the corresponding set partition of $[d]$. Observe that $\dim(\mathrm{X})$ is precisely the number of blocks of X as a partition. The partial order relation of $\mathcal{L}[\mathcal{A}_d]$ becomes the ordering by refinement of set partitions. That is, $\mathrm{X} \leq \mathrm{Y}$ if the set partition X is **refined by** Y. Recall that X is refined by Y if every block of X is the union of some blocks in Y. For instance, $\{12345678\} \leq \{13, 2568, 4, 7\} \leq \{1, 28, 3, 4, 56, 7\}$.

If $S \subseteq I$ is a union of blocks of a partition $\mathrm{X} \vdash I$, we let $\mathrm{X}|_S \vdash S$ denote the partition of S formed by the blocks of X contained in S. Let $\mathrm{X} = \{S_1, \dots, S_k\} \vdash [d]$ be a partition. Then, the choice of a flat $\mathrm{Y} \geq \mathrm{X}$ is equivalent to the choice of partitions $\mathrm{Y}|_{S_i} \vdash S_i$ for each block of X. With X and Y as above, the Möbius function of $\mathcal{L}[\mathcal{A}_d]$ is determined by

$$\mu(\perp, \mathrm{X}) = (-1)^{k-1}(k-1)! \quad \text{and} \quad \mu(\mathrm{X}, \mathrm{Y}) = \mu(\perp, \mathrm{Y}|_{S_1}) \dots \mu(\perp, \mathrm{Y}|_{S_k}), \tag{2.2.6}$$

where in each factor, $\perp$ denotes the minimum partition of S_i.

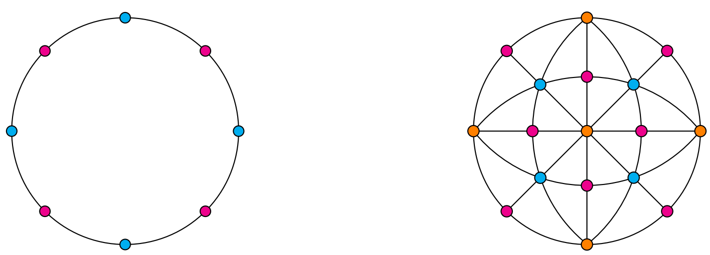
A **set composition** of I is an ordered set partition $F = (S_1, \dots, S_k)$. We write $F \vDash I$ to denote that F is a composition of I, and let $\mathrm{s}(F) \vdash I$ be the underlying (unordered) set partition. Given a set composition $F \vDash [d]$, the corresponding face of $\mathcal{A}_d$ is obtained by intersecting the hyperplanes $\boldsymbol{x}_i = \boldsymbol{x}_j$ whenever i, j are in the same block of F, and the halfspaces $\boldsymbol{x}_i \geq \boldsymbol{x}_j$ whenever the block containing i precedes the block containing j. For example,

$$\boldsymbol{x}_1 = \boldsymbol{x}_3 \geq \boldsymbol{x}_4 \geq \boldsymbol{x}_2 = \boldsymbol{x}_5 = \boldsymbol{x}_6 = \boldsymbol{x}_8 \geq \boldsymbol{x}_7 \qquad \longleftrightarrow \qquad (13, 4, 2568, 7).$$

We sometimes refer to these as *type A* set partitions and *type A* set compositions.

2.2.4 The type B Coxeter arrangement

The **type B Coxeter arrangement** $\mathcal{A}_d^{\pm}$ in $\mathbb{R}^d$ consists of the hyperplanes $\boldsymbol{x}_i = \boldsymbol{x}_j$, $\boldsymbol{x}_i = -\boldsymbol{x}_j$ for $1 \leq i < j \leq d$ and $\boldsymbol{x}_k = 0$ for $1 \leq k \leq d$. Its central face is the trivial cone $\{\mathbf{0}\}$. The *Coxeter complex of type* B_d, obtained by intersecting the arrangement with a sphere around the origin in $\mathbb{R}^d$, is shown below for $d = 2$ and 3.



Flats and faces of $\mathcal{A}_d^{\pm}$ are in correspondence with signed set partitions and signed set compositions of $[\pm d] := \{-d, -d+1, \ldots, -1, 1, \ldots, d-1, d\}$, as originally introduced by Reiner [63].

Let $\boldsymbol{I}$ be a **signed set**: a finite set with an fixed-point free involution $x \mapsto \overline{x}$. For instance, $[\pm d]$ with involution $\overline{x} = -x$. We will denote signed sets with bold letters $\boldsymbol{I}, \boldsymbol{J}, \boldsymbol{S}$.

A subset $S \subseteq \boldsymbol{I}$ is said to be **involution-exclusive** if $S \cap \overline{S} = \emptyset$, where $\overline{S} = \{\overline{x} : x \in S\}$. In contrast, $\boldsymbol{S} \subseteq \boldsymbol{I}$ is said to be **involution-inclusive** if $\boldsymbol{S} = \overline{\boldsymbol{S}}$. In the later case, $\boldsymbol{S}$ is a signed set itself, with involution obtained by restricting that of $\boldsymbol{I}$. Given an involution-exclusive subset $S \subseteq \boldsymbol{I}$, we let $\pm S$ be the involution-inclusive set $S \cup \overline{S}$. Sometimes we refer to involution-exclusive subsets as **admissible**, and to subsets that are not involution-exclusive as **inadmissible**. Note that inadmissible is not the same as involution-inclusive, but any nonempty involution-inclusive is inadmissible. A maximal admissible subset of $\boldsymbol{I}$ is called a **transversal**, its cardinality is necessarily half of the cardinality of $\boldsymbol{I}$. The collection of admissible subsets of $\boldsymbol{I}$ is denoted $\mathcal{P}'(\boldsymbol{I})$. Observe that $\mathcal{P}'(\boldsymbol{I})$ is closed under intersections but not necessarily under unions.

A **signed set partition** of $\boldsymbol{I}$ is a weak set partition of the form $\mathrm{X} = \{\boldsymbol{S}_0, S_1, \overline{S_1}, \dots, S_k, \overline{S_k}\}$, where $\boldsymbol{S}_0$ is involution-inclusive and allowed to be empty, and each S_i for $i \neq 0$ is nonempty and involution-exclusive. We call $\boldsymbol{S}_0$ the **zero block** of X. We write $\mathrm{X} \vdash^B \boldsymbol{I}$ to denote that X is a signed partition of $\boldsymbol{I}$. Given a signed partition $\mathrm{X} \vdash^B [\pm d]$, the corresponding flat of $\mathcal{A}_d^{\pm}$ is the intersection of the hyperplanes $\boldsymbol{x}_i = \boldsymbol{x}_j$ for each i, j in the same block of X, where for $k \in [d]$, we let $\boldsymbol{x}_{\overline{k}}$ denote $-\boldsymbol{x}_k$. In particular, if $k \in [d]$ is in the zero block of X, the corresponding flat lies in the hyperplane $\boldsymbol{x}_k = 0$. For instance, consider the following two examples for $d = 7$:

$$\boldsymbol{x}_1 = \boldsymbol{x}_3, \quad \boldsymbol{x}_2 = -\boldsymbol{x}_4 = \boldsymbol{x}_5, \quad \boldsymbol{x}_6 = \boldsymbol{x}_7 \qquad \longleftrightarrow \qquad \{\emptyset, 13, \bar{1}\bar{3}, 2\bar{4}5, \bar{2}4\bar{5}, 67, \bar{6}\bar{7}\},$$

$$\boldsymbol{x}_1 = \boldsymbol{x}_3 = 0, \quad \boldsymbol{x}_2 = -\boldsymbol{x}_4 = \boldsymbol{x}_5, \quad \boldsymbol{x}_6 = \boldsymbol{x}_7 \quad \longleftrightarrow \qquad \{1\bar{1}3\bar{3}, 2\bar{4}5, \bar{2}4\bar{5}, 67, \bar{6}\bar{7}\}.$$

The zero block in the first partition is empty since the corresponding flat is not contained in any coordinate hyperplane. We use X to denote both a flat of $\mathcal{A}_d^{\pm}$ and the corresponding signed partition of $[\pm d]$. Observe that the number of nonzero blocks of X is $2\dim(\mathrm{X})$. The partial order relation of $\mathcal{L}[\mathcal{A}_d^{\pm}]$ becomes the ordering by refinement of signed partitions. For instance, $\{1\bar{1}3\bar{3}, 2\bar{4}5, \bar{2}4\bar{5}, 67, \bar{6}\bar{7}\} \leq \{\emptyset, 13, \bar{1}\bar{3}, 2\bar{4}5, \bar{2}4\bar{5}, 67, \bar{6}\bar{7}\} \leq \{\emptyset, 13, \bar{1}\bar{3}, 25, \bar{2}\bar{5}, 4, \bar{4}, 67, \bar{6}\bar{7}\}$.

If $\boldsymbol{S}$ is an involution-inclusive union of blocks of $\mathrm{X} \vdash^B \boldsymbol{I}$, we let $\mathrm{X}|_{\boldsymbol{S}} \vdash^B \boldsymbol{S}$ denote the corresponding signed partition. If, on the other hand, S is an involution-exclusive union of blocks of $\mathrm{X} \vdash^B I$, we let $\mathrm{X}|_S \vdash S$ denote the corresponding type A partition.

Let $\mathrm{X} = \{\boldsymbol{S}_0, S_1, \overline{S_1}, \ldots, S_k, \overline{S_k}\}$. A choice of $\mathrm{Y} \geq \mathrm{X}$ is equivalent to the choice of a signed partition $\mathrm{Y}|_{\boldsymbol{S}_0} \vdash^B \boldsymbol{S}_0$ of the zero block and of type A partitions $\mathrm{Y}|_{S_i} \vdash S_i$ for $i = 1, \ldots, k$. Note that in this case, $\mathrm{Y}|_{\overline{S_i}}$ is automatically determined by $\mathrm{Y}|_{S_i}$.

With X and Y as above, the Möbius function of $\mathcal{L}[\mathcal{A}_d^{\pm}]$ is determined by

$$\mu(\perp, \mathrm{X}) = (-1)^k (2k-1)!! \quad \text{and} \quad \mu(\mathrm{X}, \mathrm{Y}) = \mu(\perp, \mathrm{Y}|_{\boldsymbol{S}_0}) \mu(\perp, \mathrm{Y}|_{S_1}) \ldots \mu(\perp, \mathrm{Y}|_{S_k}), \tag{2.2.7}$$

where $(2k-1)!!$ denotes the double factorial $(2k-1)!! := (2k-1)(2k-3)\ldots 1$, and in each factor $\perp$ denotes the minimum (signed) partition of $(\boldsymbol{S}_0)$ S_i.

A **signed composition** is an ordered signed set partition with the property that S_i precedes S_j if and only if $\overline{S_j}$ precedes $\overline{S_i}$. The following examples illustrate the identification between signed compositions of $\pm[d]$ and faces of $\mathcal{A}_d^{\pm}$:

$$\boldsymbol{x}_6 = \boldsymbol{x}_7 > -\boldsymbol{x}_2 = \boldsymbol{x}_4 = -\boldsymbol{x}_5 > \boldsymbol{x}_1 = \boldsymbol{x}_3 > 0 \quad \longleftrightarrow \quad (67, \bar{2}4\bar{5}, 13, \emptyset, \bar{1}\bar{3}, 2\bar{4}5, \bar{6}\bar{7})$$
$$\boldsymbol{x}_6 = \boldsymbol{x}_7 > -\boldsymbol{x}_2 = \boldsymbol{x}_4 = -\boldsymbol{x}_5 > \boldsymbol{x}_1 = \boldsymbol{x}_3 = 0 \quad \longleftrightarrow \quad (67, \bar{2}4\bar{5}, 1\bar{1}3\bar{3}, 2\bar{4}5, \bar{6}\bar{7})$$

Note that we can alternatively describe the first face with the inequalities

$$0 > -\boldsymbol{x}_1 = -\boldsymbol{x}_3 > \boldsymbol{x}_2 = -\boldsymbol{x}_4 = \boldsymbol{x}_5 > -\boldsymbol{x}_6 = -\boldsymbol{x}_7.$$

2.3 Hopf monoids in the category of Species

A comprehensive introduction to the theory of species can be found in the work by Bergeron, Labelle, and Leroux [23]. The category of species possesses more than one monoidal structure. Of central interest for the present work are the *Cauchy* and *Hadamard* product. Aguiar and Mahajan [6, 7] have explored these structures extensively, and have exploited this rich algebraic structure to obtain outstanding combinatorial results.

2.3.1 Hopf monoids in a nutshell

Let $\mathsf{set}^\times$ denote the category of finite sets with bijections as morphisms, and Set the category of sets and arbitrary set functions. The category of set species Sp is the functor category $[\mathsf{set}^\times, \mathsf{Set}]$. Explicitly, a **set species** p consists of the following data:

1. For each finite set I, a set $\mathsf{p}[I]$ of **p-structures**.
2. For each bijection $\sigma : I \to J$, a function $\mathsf{p}[\sigma] : \mathsf{p}[I] \to \mathsf{p}[J]$. These functions satisfy

$$\mathsf{p}[\sigma \circ \tau] = \mathsf{p}[\sigma] \circ \mathsf{p}[\tau] \qquad \text{and} \qquad \mathsf{p}[\mathrm{Id}] = \mathrm{Id}\,.$$

   In particular, they are bijections. We call these maps **relabeling maps**.

A species p is said to be **connected** if $\mathsf{p}[\emptyset]$ consists of exactly one element. Unless otherwise stated, we assume all species to be connected and use ϵ to denote the only p-structure on the empty set. A **morphism** of species $f : \mathsf{p} \to \mathsf{q}$ is a natural transformation; that is, a collection of functions

$$f_I : \mathsf{p}[I] \to \mathsf{q}[I],$$

one for each finite set I, that commute with bijections. Namely, $f_J \circ \mathsf{p}[\sigma] = \mathsf{q}[\sigma] \circ f_I$ for any bijection $\sigma : I \to J$.

The category of species Sp is a *braided monoidal category* under the **Cauchy product** of species, defined by

$$(\mathsf{p} \cdot \mathsf{q})[I] = \coprod_{I=S\sqcup T} \mathsf{p}[S] \times \mathsf{q}[T]. \tag{2.3.1}$$

A **monoid** in $(\mathsf{Sp}, \cdot)$ is a species m together with a morphism $\mu : \mathsf{m} \cdot \mathsf{m} \to \mathsf{m}$ satisfying certain unitality and associativity axioms. Breaking down this definition, a monoid is a species m with a collection of maps

$$\begin{aligned} \mu_{S,T} : \mathsf{m}[S] \times \mathsf{m}[T] &\to \mathsf{m}[I] \\ (x, y) \quad &\mapsto x \cdot y, \end{aligned}$$

one for each finite set I and decomposition $I = S \sqcup T$. These maps satisfy

$$\epsilon \cdot x = x \cdot \epsilon = x \qquad \text{and} \qquad (x \cdot y) \cdot z = x \cdot (y \cdot z),$$

for all decompositions $I = R \sqcup S \sqcup T$, and structures $x \in \mathsf{m}[R]$, $y \in \mathsf{m}[S]$, $z \in \mathsf{m}[T]$. A monoid m is **commutative** if $x \cdot y = y \cdot x$ for all $I = S \sqcup T$, and $x \in \mathsf{m}[S]$, $y \in \mathsf{m}[T]$. A **monoid morphism** $f : \mathsf{m} \to \mathsf{m}'$ is a morphism of species that respects the product: $f_I(x \cdot y) = f_S(x) \cdot f_T(y)$ for all decompositions $I = S \sqcup T$ and structures $x \in \mathsf{m}[S]$, $y \in \mathsf{m}[T]$.

Dually, a **comonoid** is a species c with a collection of maps

$$\begin{aligned} \Delta_{S,T} : \mathsf{c}[I] &\to \mathsf{c}[S] \times \mathsf{c}[T] \\ z \ &\mapsto (z|_S, z/_S), \end{aligned}$$

one for each finite set I and decomposition $I = S \sqcup T$, satisfying axioms dual to those of the product. A comonoid c is **cocommutative** if $z|_S = z/_T$ and $z/_S = z|_T$ for all $I = S \sqcup T$ and $z \in \mathsf{c}[I]$. Here, $z|_S$ is the first component of the coproduct $\Delta_{S,T}(z)$, while $z/_T$ is the second component of the coproduct $\Delta_{T,S}(z)$; both are structures on S.

A **Hopf monoid** is a species h that is both a monoid and a comonoid and such that the product and coproduct are **compatible**, meaning that the coproduct is a morphism of monoids (or, equivalently, that the product is a morphism of comonoids). Suppose $I = S \sqcup T = J \sqcup K$ are two decompositions of the same set I, and we have structures $x \in \mathsf{h}[J]$ and $y \in \mathsf{h}[K]$. Then, the compatibility axiom requires that

$$(x \cdot y)|_S = x|_A \cdot y|_B \qquad \text{and} \qquad (x \cdot y)/_S = x/_A \cdot y/_B, \tag{2.3.2}$$

where $A = S \cap J$ and $B = S \cap K$.

Remark 2.3.1. In the general setting, where h is not required to be connected, the definition above is that of a **bimonoid**. A Hopf monoid is a bimonoid with an additional axiom that is automatically satisfied by connected bimonoids. See [6, Section 2] and Section 6.3.3 for more details.

Example 2.3.2. The **exponential species** E is the species with exactly one structure $*_I$ on each finite set I. It is a bimonoid with the only possible operations: $*_S \cdot *_T = *_I$ and $\Delta_{S,T}(*_I) = (*_S, *_T)$ for all decompositions $I = S \sqcup T$.

Example 2.3.3. We consider the species of **linear orders** L. For any finite set I, $\mathsf{L}[I]$ is the set consisting of all linear orders on I. Given a bijection $\sigma : I \to J$ and a linear order ℓ on I, the linear order $\ell' = \mathsf{L}[\sigma](\ell)$ on J is defined by: $j <_{\ell'} j'$ if and only if $\sigma^{-1}(j) <_\ell \sigma^{-1}(j')$. The species L is a Hopf monoid with the following operations:

$\ell_1 \cdot \ell_2$ is the **concatenation** of ℓ_1 and ℓ_2,

$\ell|_S$ is the **restriction** of ℓ to S,

$\ell/_S = \ell|_T$,

for all $I = S \sqcup T$, $\ell \in \mathsf{L}[I]$, $\ell_1 \in \mathsf{L}[S]$, and $\ell_2 \in \mathsf{L}[T]$. Representing $\ell \in \mathsf{L}[I]$ with the list of elements of I ordered according to ℓ, we see that for example

$$afb \cdot dec = afbdec \qquad \Delta_{\{a,b,c,d\},\{e,f\}}(afbdec) = (abdc, fe).$$

The compatibility axiom is easily verified; for instance, in the previous example we have

$$abdc = ab \cdot dc = afb|_{\{a,b\}} \cdot dec|_{\{c,d\}}.$$

We now shift our attention to the category of **vector species** $\mathsf{Sp}_{\Bbbk}$ which are obtained by replacing Set with the category $\mathsf{Vec}_{\Bbbk}$ (of vector spaces over $\Bbbk$ with linear transformations as morphisms) in the definition of Sp. In the present work, we only consider vector species that are obtained by *linearizing* a set species. Given a set species p, the vector species $\Bbbk\mathsf{p}$ is obtained by setting $\Bbbk\mathsf{p}[I]$ to be the vector space with basis $\mathsf{p}[I]$ and by linearly extending the relabeling maps. The definitions of monoids, comonoids, and bimonoids are obtained by the linearization of the axioms above.

Let $\Bbbk\mathsf{h}$ be a Hopf monoid. The **antipode** of $\Bbbk\mathsf{h}$ is the morphism of species $\mathtt{s} : \Bbbk\mathsf{h} \to \Bbbk\mathsf{h}$ defined by $\mathtt{s}_\emptyset = \mathrm{Id}$, and

$$\mathtt{s}_I(x) = \sum_{I = S_1 \sqcup \cdots \sqcup S_k} (-1)^k \mu_{S_1,\ldots,S_k} \circ \Delta_{S_1,\ldots,S_k}(x), \tag{2.3.3}$$

for $I \neq \emptyset$ and $x \in \Bbbk\mathsf{h}[I]$. The sets S_i in the sum are nonempty, so they form a composition of I. We call (2.3.3) the **Takeuchi formula** for the antipode. The antipode plays the role of *inversion* in a Hopf monoid and it is closely related with reciprocity results for some polynomial invariants that arise from this theory. The number of terms in (2.3.3) grows extremely fast ($\sim n!(\log_2(e))^{n+1}/2$) and usually many cancellations take place. To give a reduced expression for this formula is the content of the *antipode problem* [6, Section 8.4.2]. In general, the antipode is **not** a morphism of monoids nor a morphism of comonoids.

2.3.2 Series and characters

A **series** of a species $\Bbbk\mathsf{p}$ is a morphism $s : \Bbbk\mathsf{E} \to \Bbbk\mathsf{p}$. That is, a series of $\Bbbk\mathsf{p}$ is a collection s of elements $s_I = s_I(*_I) \in \Bbbk\mathsf{p}[I]$, one for each finite set I, such that $\Bbbk\mathsf{p}[\sigma](s_I) = s_J$ for each bijection $\sigma : I \to J$. In particular, the element $s_I \in \Bbbk\mathsf{p}[I]$ needs to be invariant under the relabeling maps induced by permutations $\sigma : I \to I$. The space of series $\mathscr{S}(\Bbbk\mathsf{p})$ is a $\Bbbk$-vector space, with

$$(s+t)_I = s_I + t_I \qquad (c \cdot s)_I = cs_I$$

for all $s, t \in \mathscr{S}(\Bbbk\mathsf{p})$ and $c \in \Bbbk$.

We now let $\Bbbk\mathsf{h}$ be a Hopf monoid. In this case, the space $\mathscr{S}(\Bbbk\mathsf{h})$ is an algebra under the **Cauchy product** of series $s * t$, defined by:

$$(s * t)_I = \sum_{I = S \sqcup T} \mu_{S,T}(s_S \otimes t_T). \tag{2.3.4}$$

The unit of $\mathscr{S}(\Bbbk\mathsf{h})$ is the series

$$u_I = \begin{cases} \epsilon & \text{if } I = \emptyset, \\ 0 & \text{otherwise.} \end{cases}$$

Example 2.3.4. A series of the exponential species $\Bbbk\mathsf{E}$ is a collection s of the form $s_I = a_{|I|} *_I$, where $a_{|I|} \in \Bbbk$ only depends on the cardinality of I. Identifying s with the power series $\sum_{n \geq 0} a_n \frac{x^n}{n!}$, one easily verifies that $\mathscr{S}(\Bbbk\mathsf{E}) \cong \Bbbk[\![x]\!]$ as vector spaces. Moreover, the Cauchy product of series (2.3.4) corresponds to the product of power series, and $\mathscr{S}(\Bbbk\mathsf{E}) \cong \Bbbk[\![x]\!]$ is an isomorphism of algebras.

$$(s * t)_{[n]} = \sum_{[n] = S \sqcup T} a_{|S|} b_{|T|} = \sum_{k=0}^{n} \binom{n}{k} a_k b_{n-k}$$

A series $g \in \mathscr{S}(\Bbbk\mathsf{h})$ is said to be **group-like** if $\Delta_{S,T}(g_I) = g_S \otimes g_T$ and $g_\emptyset = \epsilon$. If a series $g \in \mathscr{S}(\Bbbk\mathsf{h})$ is group-like, then it is invertible under the Cauchy product (See [7, Section 12.3]), with inverse g^{-1} determined by

$$(g^{-1})_I = \mathbb{s}_I(g_I). \tag{2.3.5}$$

A **character** ζ of a monoid $\Bbbk\mathsf{m}$ is a monoid morphism $\zeta : \Bbbk\mathsf{m} \to \Bbbk\mathsf{E}$. That is, a character of $\Bbbk\mathsf{m}$ consists of collection of linear maps $\zeta_I : \Bbbk\mathsf{m}[I] \to \Bbbk$ such that

$$\zeta_\emptyset(\epsilon) = 1 \qquad \zeta_I(x \cdot y) = \zeta_S(x)\zeta_T(y) \qquad \zeta_I(z) = \zeta_J(\mathsf{m}[\sigma](z)),$$

for any $I = S \sqcup T$, $x \in \mathsf{m}[S]$, $y \in \mathsf{m}[T]$, $z \in \mathsf{m}[I]$, and bijection $\sigma : I \to J$.

If $\Bbbk\mathsf{h}$ is a Hopf monoid, a character ζ of $\Bbbk\mathsf{h}$ defines an algebra morphism

$$\Psi_\zeta : \mathscr{S}(\Bbbk\mathsf{h}) \to \mathscr{S}(\Bbbk\mathsf{E}) \cong \Bbbk[\![x]\!]$$

as follows. Given a series s of $\Bbbk\mathsf{h}$, the associated power series is

$$\Psi(s) = \Psi_\zeta(s) = \sum_{n \geq 0} \zeta_{[n]}(s_{[n]}) \frac{x^n}{n!}. \tag{2.3.6}$$

2.3.3 From Hopf monoids to modules

Let h be a Hopf monoid. The associativity and coassociativity axioms imply that for any (weak) set composition $F = (S_1, \ldots, S_k) \vDash I$, there are well defined maps

$$\mu_F : \mathsf{h}[S_1] \otimes \cdots \otimes \mathsf{h}[S_k] \to \mathsf{h}[I] \qquad \text{and} \qquad \Delta_F : \mathsf{h}[I] \to \mathsf{h}[S_1] \otimes \cdots \otimes \mathsf{h}[S_k],$$

obtained by iterating the product and coproduct maps in any meaningful way. In particular, the maps $\mu_{(I)}$ and $\Delta_{(I)}$ are both the identity of $\mathsf{h}[I]$.

For a finite set I, let $\mathcal{A}_I$ denote the braid arrangement in $\mathbb{R}^I$. Recall that $\mathcal{A}_I$ is a central arrangement, and its collection of faces $\Sigma[\mathcal{A}_I]$ forms a monoid under the Tits product.

Theorem 2.3.5 ([7, Theorem 82])**.** *Let* h *be a commutative (resp. cocommutative) Hopf monoid. Then, for every finite set* I*,* $\mathsf{h}[I]$ *is a right (resp. left) module over the monoid* $\Sigma[\mathcal{A}_I]$*. The action of a face* $F \in \Sigma[\mathcal{A}_I]$ *on a structure* $x \in \mathsf{h}[I]$ *is* $\mu_F \circ \Delta_F(x)$*.*

2.3.4 Hopf monoids on linearly ordered sets

Let h be a Hopf monoid in set species. The **groupoid of elements** $\mathsf{el}(\mathsf{h})$ associated to h is the category whose objects are pairs $[I, x]$ where I is a finite set and $x \in \mathsf{h}[I]$, and whose morphisms $[I, x] \to [J, y]$ are bijections $\sigma : I \to J$ such that $\mathsf{h}[\sigma](x) = y$.

An h**-species** is a functor $\mathsf{el}(\mathsf{h}) \to \mathsf{Set}$. The Hopf monoid structure of h endows the category of h-species with two monoidal structures. For our purposes, we will only be interested in the product $\star$ arising from the comonoid structure of h. If p and q are h-species, then $\mathsf{p} \star \mathsf{q}$ is defined by

$$(\mathsf{p} \star \mathsf{q})[I, x] = \coprod_{I = S \sqcup T} \mathsf{p}[S, x|_S] \times \mathsf{q}[T, x/_S]. \tag{2.3.7}$$

Vector species, Hopf monoids, series and characters are defined for h-species in a analogous manner. The role of the exponential species E is played by the exponential h-species E_h, that has precisely one structure on each pair $[I, x]$.

We will pay special attention to L-species in Chapter 4. Observe that the groupoid $\mathsf{el}(\mathsf{L})$ is *thin*: there is at most one morphism from $[I, \ell]$ to $[J, \ell']$. Such a morphism exists precisely when $|I| = |J|$, the corresponding bijection is completely determined by ℓ and ℓ'. In particular, if $\Bbbk\mathsf{h}$ is an L-species, a series $s \in \mathscr{S}(\Bbbk\mathsf{h})$ corresponds to an arbitrary choice of elements $s_n \in \Bbbk\mathsf{h}[[n], \leq]$, where $\leq$ denotes the usual order of $[n]$.

In view of (2.3.7), a monoid in Sp_L is an L-species m with maps

$$\mu^\ell_{S,T} : \mathsf{m}[S, \ell|_S] \times \mathsf{m}[T, \ell|_T] \longrightarrow \mathsf{m}[I, \ell],$$

one for each finite set I, linear order $\ell \in \mathsf{L}[I]$, and decomposition $I = S \sqcup T$, satisfying the usual axioms. A similar observation applies to the collection of maps $\Delta^\ell_{S,T}$ for a comonoid in Sp_L.

CHAPTER 3

CHARACTERISTIC ELEMENTS FOR REAL HYPERPLANE ARRANGEMENTS

The contents of this chapter are joint work with Aguiar and Mahajan, and have been partially published in [3]. We further develop the theory of *characteristic elements* for real hyperplane arrangements started in [8, Chapter 12]. These elements of the Tits algebra determine the characteristic polynomial of the arrangement and also determine the characteristic polynomial of the arrangements under each flat. They are defined by requiring that the simple characters of the Tits algebra evaluate on a characteristic element to powers of a specified parameter.

The Tits algebra is briefly reviewed in Section 3.1. The fact that the characteristic polynomial of an arrangement, which is defined in terms of flats, carries information about the decomposition of space into faces, originates in work of Zaslavsky [76], and is at the root of the combinatorial theory of hyperplane arrangements [45, 69]. The Tits algebra provides a natural setting in which this connection can be further gleaned and developed. Each arrangement possesses many characteristic elements, and the interest is in constructing particular elements from which specific information about the characteristic polynomial can be extracted. This chapter illustrates this fact repeatedly.

We review the notion of characteristic elements in Section 3.2, extending the definitions and main results of [8, Section 12.4] from linear to affine arrangements. The first applications are given in Section 3.3: we derive the fundamental recursion for the characteristic polynomial from a basic functoriality property of characteristic elements, and we employ multiplicativity of characteristic elements to derive an interesting identity due to Kung [52]. Certain characteristic elements of parameters

1 and -1 are discussed in Section 3.4, and employed to derive Zaslavsky's formulas. Section 3.5 builds characteristic elements for the simplest Coxeter arrangements in terms of lattice point counting. A main contribution of this chapter is the construction of a characteristic element canonically associated to each arrangement in terms of intrinsic volumes. This is done in Section 3.7. As an application, we derive a beautiful result of Klivans and Swartz [50] which relates the coefficients of the characteristic polynomial to the intrinsic volumes of the chambers.

3.1 The Tits algebra

Let $\Bbbk$ be a field. The linearization $\Bbbk\Sigma[\mathcal{A}]$ of the Tits semigroup (Section 2.2.2) is the **Tits algebra** of $\mathcal{A}$. See [8, Chapters 1 and 9] for more details. Recall that the semigroup $\Sigma[\mathcal{A}]$ fails to be unital if the arrangement $\mathcal{A}$ is not central. An interesting fact that we review below (Theorem 3.4.1) is that the Tits algebra is always unital. We let $\mathtt{H}_F$ denote the basis element of $\Bbbk\Sigma[\mathcal{A}]$ associated to the face F of $\mathcal{A}$.

We view $\mathcal{L}[\mathcal{A}]$ as a commutative monoid with the join operation $\vee$ for product. This makes the support map (2.2.4) a morphism of semigroups. We let $\mathtt{H}_{\mathrm{X}}$ denote the basis element of $\Bbbk\mathcal{L}[\mathcal{A}]$ associated to the flat X of $\mathcal{A}$, so that $\mathtt{H}_{\mathrm{X}} \cdot \mathtt{H}_{\mathrm{Y}} = \mathtt{H}_{\mathrm{X}\vee\mathrm{Y}}$.

A result of Solomon [67, Theorem 1] shows that the monoid algebra $\Bbbk\mathcal{L}[\mathcal{A}]$ is split-semisimple. This rests on the fact that the unique complete system of orthogonal idempotents for $\Bbbk\mathcal{L}[\mathcal{A}]$ consists of elements $\mathtt{Q}_{\mathrm{X}}$ uniquely determined by

$$\mathtt{H}_{\mathrm{X}} = \sum_{\mathrm{Y}:\, \mathrm{Y}\geq\mathrm{X}} \mathtt{Q}_{\mathrm{Y}} \quad \text{or equivalently} \quad \mathtt{Q}_{\mathrm{X}} = \sum_{\mathrm{Y}:\, \mathrm{Y}\geq\mathrm{X}} \mu(\mathrm{X},\mathrm{Y})\mathtt{H}_{\mathrm{Y}}. \tag{3.1.1}$$

In particular, $\Bbbk\mathcal{L}[\mathcal{A}]$ is the maximal split-semisimple quotient of $\Bbbk\Sigma[\mathcal{A}]$ via the support map and the simple modules of $\Bbbk\Sigma[\mathcal{A}]$ are indexed by flats. The character χ_{X} of the simple module associated with the flat X evaluated on an element

$$w = \sum_{F} w^F \mathtt{H}_F \tag{3.1.2}$$

of $\Bbbk\Sigma[\mathcal{A}]$ yields

$$\chi_{\mathrm{X}}(w) = \sum_{F:\, \mathrm{s}(F) \leq \mathrm{X}} w^F. \tag{3.1.3}$$

3.2 Characteristic elements

The definitions and results in this section extend those of [8, Section 12.4] to the setting of affine arrangements.

3.2.1 Definition and basic properties

Let t be a fixed scalar. An element w of the Tits algebra is **characteristic of parameter** t if for each flat X

$$\chi_{\mathrm{X}}(w) = t^{\dim(\mathrm{X})}, \tag{3.2.1}$$

with $\chi_{\mathrm{X}}(w)$ as in (3.1.3).

Two characteristic elements of the same parameter take the same value on all simple modules, and hence differ by a nilpotent element (an element of the Jacobson radical). The set of characteristic elements of a given parameter is an affine subspace of the Tits algebra of dimension equal to the number of faces minus the number of flats.

One-dimensional characters are multiplicative. We deduce the following result.

Lemma 3.2.1. *If u is a characteristic element of parameter s and v is a characteristic element of parameter t, then uv is a characteristic element of parameter st.*

3.2.2 Relation to the characteristic polynomial

The right-hand sides of (2.2.2) and (3.2.1) are related by Möbius inversion, which implies the following result.

Lemma 3.2.2. *An element w of the Tits algebra is characteristic of parameter t if and only if for every flat* X*,*

$$\sum_{F:\, \mathrm{s}(F)=\mathrm{X}} w^F = \chi(\mathcal{A}^{\mathrm{X}}, t). \tag{3.2.2}$$

In particular, since the chambers are the faces of top support:

$$\sum_{C} w^C = \chi(\mathcal{A}, t), \tag{3.2.3}$$

with the sum over all chambers C of $\mathcal{A}$.

3.2.3 Functoriality

Let $\mathcal{A}'$ be a subarrangement of $\mathcal{A}$: $\mathcal{A}'$ consists of some of the hyperplanes in $\mathcal{A}$. There is a morphism of semigroups

$$f : \Sigma[\mathcal{A}] \to \Sigma[\mathcal{A}'] \tag{3.2.4}$$

which sends a face F of $\mathcal{A}$ to the unique face of $\mathcal{A}'$ whose interior contains the interior of F. This in turn induces a morphism from the Tits algebra of $\mathcal{A}$ to that of $\mathcal{A}'$: if w is as in (3.1.2), then

$$f(w) = \sum_{G \in \Sigma[\mathcal{A}']} f(w)^G \, \mathtt{H}_G, \quad \text{where} \quad f(w)^G = \sum_{F:\, f(F)=G} w^F.$$

This map induces another morphism

$$\bar{f} : \mathcal{L}[\mathcal{A}] \to \mathcal{L}[\mathcal{A}'] \tag{3.2.5}$$

sending a flat X of $\mathcal{A}$ to the minimal flat of $\mathcal{A}'$ containing it. These morphisms satisfy

$$\mathrm{s}(f(F)) = \bar{f}(\mathrm{s}(F)) \tag{3.2.6}$$

for all faces F of $\mathcal{A}$.

Lemma 3.2.3. *Let w be a characteristic element for $\mathcal{A}$ of parameter t, then $f(w)$ is characteristic for $\mathcal{A}'$, of the same parameter.*

Proof. Let w be a characteristic element of parameter t. Take any flat X of $\mathcal{A}'$. Then,

$$\sum_{\mathrm{s}(G) \leq \mathrm{X}} f(w)^G = \sum_{\mathrm{s}(F) \leq \mathrm{X}} w^F = t^{\dim(\mathrm{X})}. \qquad \square$$

Example 3.2.4. Consider the arrangement $\mathcal{A}' = \mathcal{A}_{\mathrm{X}}$ over a flat X. Fix a face F with support X. There is a natural bijection between $\Sigma[\mathcal{A}]_F$, the set of faces of $\mathcal{A}$ containing F, and $\Sigma[\mathcal{A}_{\mathrm{X}}]$ [8, Section 1.7.3]. Under this identification, the map

$$f : \Sigma[\mathcal{A}] \to \Sigma[\mathcal{A}_{\mathrm{X}}] = \Sigma[\mathcal{A}]_F$$

is given by

$$f(G) = FG.$$

Thus, the interior of the face of $\mathcal{A}_{\mathrm{X}}$ corresponding to $H \in \Sigma[\mathcal{A}]_F$ is the union of the interior of all the faces G of $\mathcal{A}$ satisfying $FG = H$. In the linear case, where all faces contain the origin, this is precisely the interior of the tangent cone $T_F H$, which is easily verified from its definition (2.1.2). In the non-linear case, we get the translation of $T_F H$ whose apex (minimal face) contains $\mathrm{s}(F)$.

Moreover, for any flat Y of $\mathcal{A}$, we have that

$$\bar{f}(\mathrm{Y}) = \mathrm{X} \vee \mathrm{Y},$$

since both expressions agree with $\mathrm{s}(FG)$, where G is any face with support Y.

3.3 First applications

3.3.1 The fundamental recursion for the characteristic polynomial

The characteristic polynomial of $\mathcal{A}$ may be calculated recursively by removing one hyperplane at a time. As a first application, we derive a proof of this formula.

Let H be a hyperplane in $\mathcal{A}$ and set $\mathcal{A}' = \mathcal{A} \setminus \{\mathrm{H}\}$. Pick any characteristic element w of parameter t. Applying (3.2.2) to calculate the characteristic polynomial of $\mathcal{A}^{\mathrm{H}}$, and (3.2.3) to calculate that of $\mathcal{A}$, we obtain

$$\chi(\mathcal{A}, t) + \chi(\mathcal{A}^{\mathrm{H}}, t) = \sum w^C + \sum w^F.$$

The first sum is over all chambers C of $\mathcal{A}$ and the second over all faces F of $\mathcal{A}$ with $\mathrm{s}(F) = \mathrm{H}$. By Lemma 3.2.3, we may further employ (3.2.3) to calculate $\chi(\mathcal{A}', t)$ in terms of coefficients of $f(w)$. We obtain

$$\chi(\mathcal{A}', t) = \sum f(w)^D = \sum w^G$$

the first sum being over all chambers D of $\mathcal{A}'$ and the second over all faces G of $\mathcal{A}$ with $f(G) = D$ for some such D. These faces G are either chambers of $\mathcal{A}$ or faces with support H. Comparing the above expressions, we conclude that

$$\chi(\mathcal{A}, t) = \chi(\mathcal{A} \setminus \{\mathrm{H}\}, t) - \chi(\mathcal{A}^{\mathrm{H}}, t). \tag{3.3.1}$$

This derivation of the fundamental recursion is given in [8, Proposition 12.66] (for linear arrangements). The proof in [69, Lemma 2.2], [58, Theorem 2.56] is quite different.

3.3.2 An identity of Crapo

We show how to derive the Crapo identity (1.55) in [8], originally proved in [37], using characteristic elements. We first prove a more general version that was considered by Kung in [51]. The fundamental recursion above is a particular case of the following result.

Proposition 3.3.1. *For any subarrangement $\mathcal{A}'$ of an arrangement $\mathcal{A}$,*

$$\chi(\mathcal{A}', t) = \sum_{\mathrm{X}} \chi(\mathcal{A}^{\mathrm{X}}, t), \tag{3.3.2}$$

where the sum is taken over all flats X *of $\mathcal{A}$ not contained in any hyperplane of $\mathcal{A}'$.*

Proof. First note that a flat X is not contained in any hyperplane of $\mathcal{A}'$ if and only if $\bar{f}(\mathrm{X}) = \top$. Pick any characteristic element w of parameter t for $\mathcal{A}$. By Lemma 3.2.3, the element $f(w)$ is characteristic of parameter t for $\mathcal{A}'$. Applying (3.2.3) to this element, we deduce that

$$\chi(\mathcal{A}', t) = \sum_{C} f(w)^C = \sum_{F:\, \mathrm{s}(f(F))=\top} w^F.$$

On the other hand, applying (3.2.2) to w, we obtain

$$\sum_{\bar{f}(\mathrm{X})=\top} \chi(\mathcal{A}^{\mathrm{X}}, t) = \sum_{\bar{f}(\mathrm{X})=\top} \Big(\sum_{F:\, \mathrm{s}(F)=\mathrm{X}} w^F \Big) = \sum_{F:\, \bar{f}(\mathrm{s}(F))=\top} w^F.$$

The result follows since the two sums above agree by identity (3.2.6). □

Let Z be any flat of $\mathcal{A}$ and $\mathcal{A}' = \mathcal{A}_{\mathrm{Z}}$. From Example 3.2.4 we have that $\bar{f}(\mathrm{X}) = \mathrm{X} \vee \mathrm{Z}$. We deduce the following identity.

Corollary 3.3.2. *For any flat* Z *of an arrangement* $\mathcal{A}$*,*

$$\chi(\mathcal{A}_{\mathrm{Z}}, t) = \sum_{\mathrm{X}\colon \mathrm{X}\vee\mathrm{Z}=\top} \chi(\mathcal{A}^{\mathrm{X}}, t). \tag{3.3.3}$$

3.3.3 The characteristic polynomial on a product. An identity of Kung

For the third application we employ Lemma 3.2.1. Pick characteristic elements u and v of parameters s and t, respectively. Applying (3.2.3) to the characteristic element uv, we obtain

$$\chi(\mathcal{A}, st) = \sum_{C} (uv)^{C} = \sum_{F,G\colon \mathrm{s}(FG)=\top} u^{F} v^{G}.$$

The first sum is over chambers C and the second over faces F and G which multiply to a chamber. This happens precisely when $\mathrm{s}(F) \vee \mathrm{s}(G) = \mathrm{s}(FG) = \top$, since s is a morphism of semigroups. So the previous sum equals

$$\sum_{\mathrm{X},\mathrm{Y}\colon \mathrm{X}\vee\mathrm{Y}=\top} \; \sum_{\substack{F\colon \mathrm{s}(F)=\mathrm{X} \\ G\colon \mathrm{s}(G)=\mathrm{Y}}} u^{F} v^{G}.$$

Combining the preceding with (2.2.2) we obtain

$$\chi(\mathcal{A}, st) = \sum_{\mathrm{X},\mathrm{Y}\colon \mathrm{X}\vee\mathrm{Y}=\top} \chi(\mathcal{A}^{\mathrm{X}}, s)\chi(\mathcal{A}^{\mathrm{Y}}, t).$$

Finally, an application of Corollary 3.3.2 yields

$$\chi(\mathcal{A}, st) = \sum_{\mathrm{X}} \chi(\mathcal{A}^{\mathrm{X}}, s)\chi(\mathcal{A}_{\mathrm{X}}, t). \tag{3.3.4}$$

This identity is due to Kung [52, Theorem 4]. Kung discusses a couple of proofs, all quite different from the one above. Kung's result is for matroids, which covers the case of linear arrangements. The identity above holds for affine arrangements.

3.4 Characteristic elements of parameters ±1

3.4.1 The unit element

When the hyperplanes of $\mathcal{A}$ are in *general position*, the following is [8, Theorem 14.23]. See Section 3.8 for a proof for any affine arrangement.

Theorem 3.4.1. *The Tits algebra of an affine arrangement $\mathcal{A}$ possesses a unit element. The unit is*

$$\upsilon = \sum_F (-1)^{\mathrm{rank}(F)} \mathtt{H}_F, \tag{3.4.1}$$

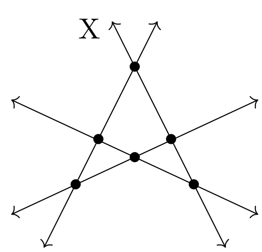
with F running over the set of essentially bounded faces of $\mathcal{A}$.

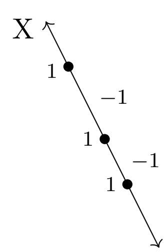
When $\mathcal{A}$ is linear, the only essentially bounded face is the central face O, and $\upsilon = \mathtt{H}_O$.

The unit element acts as the identity on any module, and hence the one-dimensional characters evaluate to 1 on it. This implies the following result.

Proposition 3.4.2. *The unit element υ is characteristic of parameter* 1*.*

Here is a direct proof of the proposition. According to (3.1.3), the character value $\chi_{\mathrm{X}}(\upsilon) = \sum(-1)^{\mathrm{rank}(F)}$ is the Euler characteristic of the complex consisting of the essentially bounded part of X, as illustrated below. The latter is contractible [28, Theorem 4.5.7], and hence the character value is 1.





3.4.2 The Takeuchi element

The **Takeuchi element** is

$$\tau = \sum_F (-1)^{\dim(F)}\, \mathtt{H}_F, \tag{3.4.2}$$

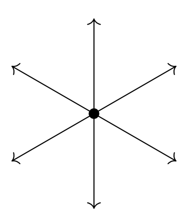
with the sum over all faces F of $\mathcal{A}$.

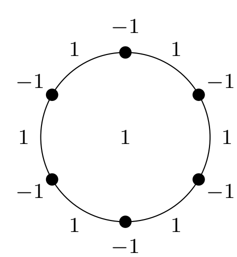
The following extends [8, Corollary 12.57] to the setting of affine arrangements.

Proposition 3.4.3. *The Takeuchi element τ is characteristic of parameter -1.*

This time the proof can be brought down to the calculation of the Euler characteristic of a relative pair of cell complexes $(B, \partial B)$, where B is the complex obtained by dissecting a large ball (containing all the bounded faces) with the hyperplanes in $\mathcal{A}$.

In the central case, one can more simply work with the reduced Euler characteristic of a sphere, as illustrated below.





3.4.3 Application: Zaslavsky's formulas

All chambers of $\mathcal{A}$ are of rank rank$(\mathcal{A})$. Applying (3.2.3) to the unit element υ we obtain that

$$(-1)^{\mathrm{rank}(\mathcal{A})}\chi(\mathcal{A}, 1) = (-1)^{\mathrm{rank}(\mathcal{A})} \sum_{\mathrm{Y}} \mu(\mathrm{Y}, \top)$$

equals the number of essentially bounded chambers in $\mathcal{A}$.

Let d be the dimension of the ambient space V. Employing the Takeuchi element τ instead, we obtain that

$$(-1)^d \chi(\mathcal{A}, -1) = \sum_{\mathrm{Y}} (-1)^{n-\dim(\mathrm{Y})} \mu(\mathrm{Y}, \top)$$

equals the total number of chambers in $\mathcal{A}$. These are Zaslavsky's formulas [76, Theorem A, Theorem C, Corollary 2.2], [53, Proposition 8.1].

Remark 3.4.4. The above proof does not differ substantially from Zaslavsky's. The core topological argument has been shifted to prove the facts that υ and τ are characteristic.

3.5 The Adams elements

3.5.1 Braid arrangement

Let $\mathcal{A}_d$ be the braid arrangement in $\mathbb{R}^d$ (see Section 2.2.3). The **Adams element of type** A_{d-1} (and parameter t) is defined by

$$\alpha_t = \sum_F \binom{t}{\dim(F)} \mathtt{H}_F, \qquad (3.5.1)$$

with the sum over the faces F of $\mathcal{A}_d$. For each positive integer k, the binomial coefficient $\binom{k}{\dim(F)}$ counts the number of points in the relative interior of F with coordinates from $[k] = \{1, \dots, k\}$. On the other hand, given a flat X, the number of points in $\mathrm{X} \cap [k]^d$ is $k^{\dim(\mathrm{X})}$. Since X splits as the disjoint union of the relatively open faces F with $\mathrm{s}(F) \leq \mathrm{X}$, we have that

$$\sum_{F:\, \mathrm{s}(F) \leq \mathrm{X}} \binom{k}{\dim(F)} = k^{\dim(\mathrm{X})}.$$

We have shown the following, for which a different proof is given in [8, Lemma 12.78].

Proposition 3.5.1. *For any nonzero scalar t, the element α_t is characteristic of parameter t.*

There are $d!$ chambers in $\mathcal{A}_d$. As a small application of (3.2.3), we obtain the well-known expression for the characteristic polynomial of the braid arrangement.

$$\chi(\mathcal{A}_d, t) = \sum_C \binom{t}{d} = t(t-1)(t-2)\cdots(t-(d-1)). \tag{3.5.2}$$

It follows from Lemma 3.2.1 that $\alpha_s\alpha_t$ is characteristic of parameter st. In fact, it can be shown that $\alpha_s\alpha_t = \alpha_{st}$, see for instance [8, Lemma 12.80].

3.5.2 Type B Coxeter arrangement

Let $\mathcal{B}_d$ be the Coxeter arrangement of type B in $\mathbb{R}^d$ (see Section 2.2.4). The **Adams element of type** B_d (and odd parameter $2t+1$) is defined by

$$\alpha^{\pm}_{2t+1} = \sum_F \binom{t}{\dim(F)} \mathtt{H}_F. \tag{3.5.3}$$

Proceeding as in Section 3.5.1, but now counting integer points in $[-k,k]^d \cap \mathrm{X}$ for each flat X according to the face in which they lie, one arrives at the following fact, for which a different proof is given in [8, Proposition 12.89].

Proposition 3.5.2. *For any scalar t, the element $\alpha^{\pm}_{2t+1}$ is characteristic of parameter $2t+1$.*

There are $(2n)!!$ chambers in $\mathcal{A}_d^{\pm}$. Employing (3.2.3), one obtains that

$$\chi(\mathcal{A}_d^{\pm}, 2t+1) = (2n)!!\binom{t}{n},$$

which is equivalent to the familiar expression for the characteristic polynomial of the type B Coxeter arrangement:

$$\chi(\mathcal{A}_d^{\pm}, t) = (t-1)(t-3)\cdots(t-(2n-1)). \tag{3.5.4}$$

Related elements $\alpha_{2t}^{\pm}$ are discussed in [8, Section 12.6.3]. These are not characteristic.

3.5.3 Graphic arrangement

Let $\mathcal{A}(G)$ be the graphic arrangement associated to a simple graph G. The ambient space is $\mathbb{R}^I$, where I is the vertex set of G, and $\mathcal{A}(G)$ contains the hyperplane $x_i = x_j$ whenever $\{i, j\}$ is an edge of G. The arrangement is not essential, its rank is

$$\operatorname{rank}(\mathcal{A}(G)) = |I| - c(G),$$

where $c(G)$ is the number of connected components of G.

$\mathcal{A}(G)$ is a subarrangement of the braid arrangement $\mathcal{A}$ in $\mathbb{R}^I$. The **chromatic element** of G of parameter t is defined by

$$\gamma_t = f(\alpha_t),$$

where $f : \Bbbk\Sigma[\mathcal{A}] \to \Bbbk\Sigma[\mathcal{A}(G)]$ is the morphism (3.2.4). Since α_t is characteristic for $\mathcal{A}$, Lemma 3.2.3 implies the following.

Proposition 3.5.3. *The element γ_t is characteristic of parameter t for $\mathcal{A}(G)$.*

Let k be a positive integer. A k-**coloring** of G is a function $\boldsymbol{x} : I \to [k]$, that is, a point in $[k]^I$. The coloring $\boldsymbol{x}$ is **proper** whenever vertices i and j are joined by an edge in G, $\boldsymbol{x}_i \neq \boldsymbol{x}_j$. Let $p(G, t)$ denote the **chromatic polynomial** of G. Thus, $p(G, k)$ is the number of proper k-colorings of G.

Corollary 3.5.4. *The chromatic polynomial of G coincides with the characteristic polynomial of the associated arrangement. That is,*

$$\chi(\mathcal{A}(G), t) = p(G, t).$$

Proof. Applying (3.2.3) to the characteristic element $\gamma_k = f(\alpha_k)$, we obtain

$$\chi(\mathcal{A}(G), k) = \sum_{F:\mathrm{s}(f(F))=\top} \alpha_k^F = \sum_{F:\mathrm{s}(f(F))=\top} |[k]^I \cap \mathrm{relint}(F)|.$$

The sums are over faces of the braid arrangement whose image under f is a chamber. A point $\boldsymbol{x}$ lies in the interior of one such face if and only if it does not lie on any hyperplane of $\mathcal{A}(G)$, and this occurs precisely when the coloring $\boldsymbol{x}$ is proper. □

Example 3.5.5. Let G be the cycle on 4 vertices. Then $\mathcal{A}(G)$ is the smallest nonsimplicial arrangement. It has 3 types of chambers: triangles, small squares, and big squares. The small squares are composed of two triangles from the braid arrangement, and the big squares are composed of four such. The coefficients γ_k^C, for the three types of chambers are, respectively,

$$\binom{k}{4}, \quad 2\binom{k}{4} + \binom{k}{3}, \quad 4\binom{k}{4} + 4\binom{k}{3} + \binom{k}{2}.$$

There are 8 triangles, 4 small squares and 2 large ones. It follows that

$$\chi(\mathcal{A}(G), t) = 24\binom{t}{4} + 12\binom{t}{3} + 2\binom{t}{2}.$$

3.5.4 Coordinate arrangement

The coordinate arrangement $\mathcal{C}_d$ in $\mathbb{R}^d$ consists of the coordinate hyperplanes $\boldsymbol{x}_i = 0$ for $1 \leq i \leq d$. The associated subdivision of the sphere is the *Coxeter complex of type* A_1^d. The *first orthant* is $\bigcap_i \{\boldsymbol{x}_i \geq 0\}$. For each face F of $\mathcal{C}_d$, let

$$\gamma_t^F = \begin{cases} (t-1)^{\operatorname{rank}(F)} & \text{if } F \text{ lies in the first orthant,} \\ 0 & \text{if not.} \end{cases}$$

An argument similar to those in Sections 3.5.1 and 3.5.2, but now counting integer points in $[0, k-1]^d \cap \mathrm{X}$, shows that the element

$$\gamma_t = \sum_F \gamma_t^F \mathtt{H}_F$$

is characteristic of parameter t. In this case, only one chamber appears with nonzero coefficient in γ_t (the first orthant). We obtain

$$\chi(\mathcal{C}_d, t) = (t-1)^d.$$

Remark 3.5.6. The strategy employed in this section to build characteristic elements draws on ideas of Beck and Zaslavsky in [21], and in fact may be further developed to study the polynomials introduced in that work.

3.6 Characteristic elements and valuations

The characteristic elements in Section 3.5 are part of a more general family of examples that arise from a *valuation* defined on a particular collection of subsets of the ambient space V. We explain this construction in this section and use it to define *intrinsic elements*, a family of characteristic elements defined for any hyperplane arrangement, in the next section.

Let V be a set and $\mathcal{C}$ a collection of subsets of V that is closed under finite intersections and unions. A **valuation** v on $\mathcal{C}$ is a function from $\mathcal{C}$ to a commutative ring R satisfying $v(\emptyset) = 0$ and

$$v(C_1) + v(C_2) = v(C_1 \cap C_2) + v(C_1 \cup C_2) \tag{3.6.1}$$

for all $C_1, C_2 \in \mathcal{C}$.

Ehrenborg and Readdy show in [39] that if V is a real vector space and $\mathcal{C}$ is the collection of finite unions of affine subspaces of V, then there is a unique valuation $v : \mathcal{C} \to \mathbb{Z}[t]$ satisfying

$$v(A) = t^{\dim(A)} \quad \text{for all affine subspaces } A \text{ of } V. \tag{3.6.2}$$

Moreover, for any hyperplane arrangement $\mathcal{A} = \{\mathrm{H}_1, \ldots, \mathrm{H}_k\}$ in V,

$$\chi(\mathcal{A}, t) = v\Big(V \setminus \bigcup_{i=1}^{k} \mathrm{H}_i\Big). \tag{3.6.3}$$

The proof reduces to Möbius inversion.

Fix a hyperplane arrangement $\mathcal{A}$. We can apply formula (3.6.3) as long as we have a valuation v defined on a collection $\mathcal{C}$ of subsets of V containing the boolean algebra generated by the flats of $\mathcal{A}$ and satisfying (3.6.2) on flats of $\mathcal{A}$. One such collection is the boolean algebra generated by the relative interior of faces of $\mathcal{A}$. Since the relative interior of different faces of $\mathcal{A}$ are disjoint, there is a one-to-one correspondence between valuations $v : \mathcal{C} \to \Bbbk$ and elements w of the Tits algebra $\Bbbk\Sigma[\mathcal{A}]$. Explicitly,

$$v(C) = \sum_{\mathrm{relint}(F) \subseteq C} w^F \qquad \longleftrightarrow \qquad w^F = v(\mathrm{relint}(F)). \tag{3.6.4}$$

The element w is characteristic of parameter $t \in \Bbbk$ if and only if the corresponding valuation v satisfies (3.6.2) for all flats of $\mathcal{A}$.

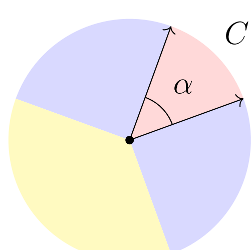
The characteristic elements of Section 3.5 are constructed from a very simple kind of valuation: point enumeration. Let $P \subseteq V$ be a finite subset and $\mathcal{C}$ any collection of subsets of V as above. Then, the function $v : \mathcal{C} \to \mathbb{Z}$ defined by $v(C) = |P \cap C|$ is a valuation.

Example 3.6.1. Let $\mathcal{A}$ be the the braid arrangement in $\mathbb{R}^d$, k a positive integer and v the valuation given by $v(C) = |[k]^d \cap C|$. As observed in Section 3.5.1, for any flat X of $\mathcal{A}$ we have $|[k]^d \cap \mathrm{X}| = k^{\dim(\mathrm{X})}$, so v satisfies (3.6.2). The characteristic element associated to v under the correspondence (3.6.4) is the Adams element α_k.

Remark 3.6.2. As a consequence of the valuation property (3.6.1), the morphism $f : \Sigma[\mathcal{A}] \to \Sigma[\mathcal{A}']$ sends the element associated to a valuation v to the element associated to the valuation $v|_{\mathcal{C}'}$, where $\mathcal{C}' \subseteq \mathcal{C}$ is the boolean algebra generated by the faces of $\mathcal{A}'$.

3.7 Main construction: Intrinsic elements

We employ the notion of **intrinsic volumes** of a convex polyhedral cone C in $\mathbb{R}^d$ [10, Section 2.2]. For each $k = 0, \ldots, d$, let $v_k(C)$ be the proportion of volume of space occupied by points that map to a k-dimensional face of C under the *nearest point* projection.



$$v_2(C) = \alpha/2\pi$$
$$v_1(C) = 1/2$$
$$v_0(C) = 1/2 - \alpha/2\pi$$

Figure 3.7.1: Intrinsic volumes of a 2-dimensional cone in $\mathbb{R}^2$.

$v_k(C)$ is the **k-th dimensional intrinsic volume** of C. We record the following properties of intrinsic volumes. First and foremost, each intrinsic volume v_k is a *valuation* on convex cones: it satisfies (3.6.1) whenever $C_1 \cup C_2$ is convex.

If $k > \dim(C)$, or if k is smaller than the dimension of the minimal face of C, then $v_k(C) = 0$. Also, it is clear by definition that

$$\sum_{k=0}^{d} v_k(C) = 1. \tag{3.7.1}$$

The *Gauss-Bonnet formula* states that if C is not a subspace, then

$$\sum_{k=0}^{n} (-1)^k v_k(C) = 0. \tag{3.7.2}$$

If C is a subspace, then

$$v_k(C) = \begin{cases} 1 & \text{if } \dim(C) = k, \\ 0 & \text{if not.} \end{cases} \tag{3.7.3}$$

A result by Grünbaum [46] states that

$$(-1)^k v_k(C) = \sum_{F \leq C} (-1)^{\dim(F)} v_k(F). \tag{3.7.4}$$

And lastly

$$v_k(C \times C') = \sum_{i=0}^{k} v_i(C) v_{k-i}(C'). \tag{3.7.5}$$

We want to extend this notion to convex polyhedra P, using the same definition. It then turns out that $v_k(P)$ depends only on the recession cone of P (see (2.1.1) for the definition), and each v_k is a valuation on convex polyhedra. To achieve this, let us first recall a more refined definition in the case of polyhedral cones.

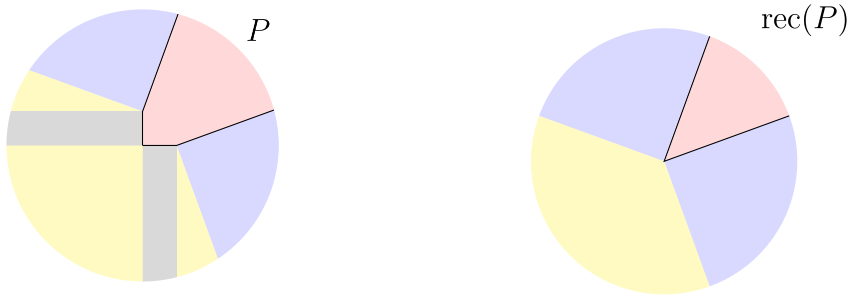


Figure 3.7.2: The k-the dimensional intrinsic volume of P depends only on its recession cone.

The **solid angle** of a cone C is the proportion of the space in the linear span of C that lies inside of C. That is,

$$\alpha(C) = \frac{\mathrm{vol}(C \cap B)}{\mathrm{vol}(\mathrm{span}(C) \cap B)},$$

where B is the unit ball centered at the origin and vol is the Lebesgue measure in $\mathrm{span}(C)$. If C is a linear subspace, we have $\alpha(C) = 1$. We can extend this notion to an arbitrary polyhedron P by setting

$$\alpha(P) = \lim_{r \to \infty} \frac{\mathrm{vol}(P \cap B_r)}{\mathrm{vol}(\mathrm{span}(P) \cap B_r)},$$

where now B_r is the ball of radius r and span denotes affine span. It follows that $\alpha(P) = \alpha(\lambda P)$ for any $\lambda > 0$. Thus,

$$\alpha(P) = \begin{cases} \alpha(\mathrm{rec}(P)) & \text{if } \dim(P) = \dim(\mathrm{rec}(P)), \\ 0 & \text{otherwise.} \end{cases} \tag{3.7.6}$$

Moreover, if P and P' live in orthogonal subspaces of $\mathbb{R}^d$, then

$$\alpha(P + P') = \alpha(P)\alpha(P').$$

Remark 3.7.1. It is worth noting that this notion does not correspond to Beck and Robins' definition of solid angle of a polyhedron P at a point $\boldsymbol{p}$ in [20, Chapter 13]. Their definition coincides with the notion of internal angle β defined below.

Given a polyhedron P and a face $F \leq P$, the **internal and external angles** of P at F are respectively

$$\beta(F, P) = \alpha(T_F P) \quad \text{and} \quad \gamma(F, P) = \alpha(N_F P).$$

Note that if C is a cone with apex O, then $\beta(O, C) = \alpha(C)$.

Let P be a polyhedron and $F \leq P$ one of its faces. The set of points that project to the interior of F under the nearest-point map is precisely $\text{relint}(F) + N_F P$. Define the k-th dimensional intrinsic volume of P as

$$v_k(P) = \sum_{F \leq P} \alpha(F)\alpha(N_F P), \tag{3.7.7}$$

where the sum is over the k-dimensional faces F of P.

Proposition 3.7.2. *For any polyhedron P, $v_k(P) = v_k(\text{rec}(P))$. Moreover, v_k is a valuation on polyhedra.*

Proof. Let $C = \text{rec}(P)$ and write $P = Q + C$ for some polytope Q. We claim that for faces $G \leq C$ and $F \leq P$, $\text{rec}(F) = G$ if and only if $\text{relint}(N_F P) \subseteq \text{relint}(N_G C)$. Indeed, let $\boldsymbol{v} \in \text{relint}(N_F P)$. Since $F = P_{\boldsymbol{v}} = Q_{\boldsymbol{v}} + C_{\boldsymbol{v}}$, we have $C_{\boldsymbol{v}} = \text{rec}(F)$. Thus $\text{rec}(F) = G$ if and only if $C_{\boldsymbol{v}} = G$ if and only if $\boldsymbol{v} \in \text{relint}(N_G C)$.

We will now use the definition (3.7.7) to compute $v_k(P)$. Observe that:

- For a face F in the sum, if $\dim(\text{rec}(F)) \neq k$, then $\alpha(F) = 0$ by (3.7.6).

- For a k-dimensional face $G \leq C$, the only cones in $\mathcal{N}(P)$ that are full-dimensional inside $N_G C$ are of the form $N_F P$ for $F \leq P$ of dimension k with $\text{rec}(F) = G$. Since $\mathcal{N}(P)$ refines $\mathcal{N}(C)$,

$$\text{relint}(N_G C) = \sum_{\substack{\dim(F)=k \\ \text{rec}(F)=G}} \text{relint}(N_F P).$$

Thus, grouping nonzero terms in (3.7.7) according to $\text{rec}(F)$, we get

$$v_k(P) = \sum_{\dim(F)=k} \alpha(F)\alpha(N_F P) = \sum_{G \leq C} \alpha(G)\Big(\sum_{\substack{\dim(F)=k \\ \text{rec}(F)=G}} \text{relint}(N_F P)\Big)$$
$$= \sum_{G \leq C} \alpha(G)\alpha(N_G C) = v_k(C),$$

as we wanted to show. The sums are over k-dimensional faces G of C.

The last statement follows since, whenever $P \cup P'$ is convex,

$$\text{rec}(P \cup P') = \text{rec}(P) \cup \text{rec}(P') \qquad \text{and} \qquad \text{rec}(P \cap P') = \text{rec}(P) \cap \text{rec}(P'). \qquad \square$$

Remark 3.7.3. When $P = C$ is a cone, this definition is equivalent to

$$v_k(C) = \sum_{F \leq C} \beta(O, F)\gamma(F, C)$$

where O is the minimal face of C and the sum is over the k-dimensional faces F of C. In this case, all the terms in the sum are nonzero.

We now show that (3.7.4) continues to hold for arbitrary polyhedra.

Lemma 3.7.4. *For any polyhedron P,*

$$(-1)^k v_k(P) = \sum_{F \leq P} (-1)^{\dim(F)} v_k(F).$$

Proof. Let $C = \mathrm{rec}(P)$. Grouping terms according to $\mathrm{rec}(F)$ and using (3.7.4) and Proposition 3.7.2 we obtain

$$\sum_{F \le P} (-1)^{\dim(F)} \upsilon_k(F) = \sum_{G \le C} \Big(\sum_{\mathrm{rec}(F)=G} (-1)^{\dim(F)} \Big) \upsilon_k(G)$$
$$= \sum_{G \le C} (-1)^{\dim(G)} \upsilon_k(G) = \upsilon_k(C) = \upsilon_k(P).$$

The second equality follows using that, as discussed in the proof of the previous proposition,

$$\mathrm{relint}(N_G C) = \bigsqcup_{\mathrm{rec}(F)=G} \mathrm{relint}(N_F P)$$

and

$$\sum_{\mathrm{rec}(F)=G} (-1)^{\dim(F)} = (-1)^d \sum_{\mathrm{rec}(F)=G} (-1)^{\dim(N_F P)}$$
$$= (-1)^d (-1)^{\dim(N_G C)} = (-1)^{\dim(G)}.$$

□

Let $\mathcal{A}$ be an arrangement in $\mathbb{R}^d$. Each face of $\mathcal{A}$ is a polyhedron. We define the **intrinsic element** of parameter t for $\mathcal{A}$ by

$$\nu_t = \sum_F (-1)^{\dim(F)} \Big(\sum_k (-1)^k \upsilon_k(F) t^k \Big) \mathtt{H}_F. \tag{3.7.8}$$

The following Theorem is a consequence of the discussion in Section 3.6.

Theorem 3.7.5. *The element ν_t is characteristic of parameter t.*

Proof. Given that each υ_k is a valuation, the function υ defined for any (closed) polyhedron P by

$$\upsilon(P) = \sum_k \upsilon_k(P) t^k \tag{3.7.9}$$

extends to a valuation on the Boolean algebra generated by convex polyhedra. Property (3.7.3) shows that υ satisfies condition (3.6.2). Therefore, the element

$w \in \Bbbk\Sigma[\mathcal{A}]$ defined by

$$w^F = v(\text{relint}(F))$$

is characteristic of parameter t. We now verify that $\nu_t = w$. By inclusion-exclusion,

$$\begin{aligned} w^C = v(\text{relint}(C)) &= \sum_{F \leq C} (-1)^{\dim(C)-\dim(F)} v(F) \\ &= (-1)^{\dim(C)} \sum_{F \leq C} \Big(\sum_k (-1)^{\dim(F)} v_k(F) t^k \Big) \\ &= (-1)^{\dim(C)} \sum_k (-1)^k v_k(C) t^k = \nu_t^C. \end{aligned} \tag{3.7.10}$$

In the third step we used Lemma 3.7.4. □

As an immediate consequence of the theorem, we deduce the following result of Klivans and Swartz [50, Theorem 5].

Corollary 3.7.6. *The coefficient of t^k in the characteristic polynomial of $\mathcal{A}$ is*

$$(-1)^{n-k} \sum_C v_k(C), \tag{3.7.11}$$

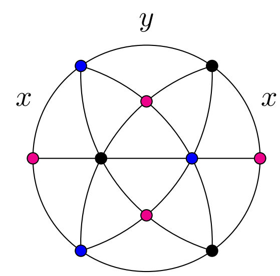
with the sum over all chambers of $\mathcal{A}$.

Example 3.7.7. The following table shows the intrinsic volumes for the braid arrangement of rank 3 in its essentialized realization. The ambient space is the hyperplane $x_1 + \cdots + x_4 = 0$ in $\mathbb{R}^4$. All faces of the same *type* are congruent and hence have the same volumes. The edges of types $(2, 1, 1)$ and $(1, 1, 2)$ have size x and the edges of type $(1, 2, 1)$ have size y, where

$$x = \frac{1}{2\pi} \arccos(\frac{\sqrt{3}}{3}) \quad \text{and} \quad y = \frac{1}{2\pi} \arccos(\frac{1}{3}).$$

The entries above the main diagonal are 0.

face	v_0	v_1	v_2	v_3
center	1			
vertices	1/2	1/2		
short edges	$1/2 - x$	1/2	x	
long edges	$1/2 - y$	1/2	y	
triangles	1/4	11/24	1/4	1/24



Each circle is composed of four edges of size x and two edges of size y, so $4x + 2y = 1$. The arrangement under a circle is combinatorially isomorphic to the braid arrangement of rank 2. Employing (3.7.11) we obtain that its characteristic polynomial is

$$4\big[(\frac{1}{2} - x) - \frac{1}{2}t + xt^2\big] + 2\big[(\frac{1}{2} - y) - \frac{1}{2}t + yt^2\big] = 2 - 3t + t^2.$$

For the characteristic polynomial of the whole arrangement we obtain

$$24\big(-\frac{1}{4} + \frac{11}{24}t - \frac{1}{4}t^2 + \frac{1}{24}t^3\big) = -6 + 11t - 6t^2 + t^3.$$

Both calculations agree with (3.5.2) (up to a factor of t from the essentialization).

We turn to general properties of the intrinsic elements. The recession cone of a face F of $\mathcal{A}$ is a linear subspace if and only if F is essentially bounded. Together with the Gauss-Bonnet formula (3.7.2), this implies that the intrinsic element of parameter 1 of $\mathcal{A}$ is precisely the unit element of the Tits algebra: $\nu_1 = \upsilon$. Formula (3.7.1) implies that the intrinsic element of parameter -1 and the Takeuchi element coincide: $\nu_{-1} = \tau$.

We now prove a multiplicativity result for intrinsic elements. We first review a theorem by McMullen that relates the internal and external angles of cones.

Theorem 3.7.8 ([54, Theorem 3]). *Let C be a cone. Then,*

$$\sum_{F \leq C} (-1)^{\dim(F)} \gamma(O,F)\beta(F,C) = \begin{cases} (-1)^{\dim(C)} & \text{if } C \text{ is a linear subspace,} \\ 0 & \text{otherwise.} \end{cases} \tag{3.7.12}$$

where O is the minimal face of C.

The Gauss-Bonnet formula (3.7.2) can be expressed in a similar manner, permuting the role of β and γ in the theorem. We will need a version of this result that works for general polyhedra.

Corollary 3.7.9. *Let P be a polyhedron and $G \leq P$ a fixed face of P. Then,*

$$\sum_{F:\ G \leq F \leq P} (-1)^{\dim(F)} \alpha(N_G F)\alpha(T_F P) = \begin{cases} (-1)^{\dim(P)} & \text{if } G = P \\ 0 & \text{otherwise.} \end{cases}$$

Proof. Consider the tangent cone $C = T_G P$, whose minimal face is $O = \operatorname{span}(G)$. There is a bijection between the faces F of P containing G and the faces of C; namely, $F \mapsto T_G F$. Moreover, $N_G F = N_O(T_G F)$ and $T_F P = T_{T_G F} C$. The result follows by applying (3.7.12) to C and observing that $T_G P$ is a linear subspace if and only if $G = P$. □

Theorem 3.7.10. *For any parameters s and t, $\nu_s \nu_t = \nu_{st}$.*

Proof. Consider the valuation v as in (3.7.9), so that $\nu_t^F = v(\operatorname{relint}(F))$ for all faces F of $\mathcal{A}$. Following Example 3.2.4, we have that for faces $F \leq P$,

$$\sum_{G:FG=P} \nu_t^G = \sum_{G:FG=P} v(\operatorname{relint}(G)) = v(\operatorname{relint}(T_F P)).$$

Fix any face P of the arrangement $\mathcal{A}$. The coefficient of $\mathtt{H}_P$ in $\nu_s \nu_t$ is

$$\sum_{F \leq P} \Big(\sum_{G:FG=P} \nu_s^F \nu_t^G \Big) = \sum_{F \leq P} \nu_s^F v(\operatorname{relint}(T_F P)).$$

Using the equivalent expressions in (3.7.10), this equals

$$\sum_{F\leq P}\Big(\sum_{k,k'}(-1)^{\dim(F)}\upsilon_k(F)(-s)^k(-1)^{\dim(P)}\upsilon_{k'}(T_FP)(-t)^{k'}\Big)$$

If we fix k and k', the factor of $(-s)^k(-t)^{k'}$ in this expression equals

$$(-1)^{\dim(P)}\sum_{K\leq F\leq K'\leq P}(-1)^{\dim(F)}\alpha(K)\alpha(N_KF)\alpha(T_FK')\alpha(N_{K'}P),$$

where K and K' run over faces of dimension k and k', respectively. Taking the sum over F first, Corollary 3.7.9 implies that the sum is zero unless $K = K'$, in which case $k = k'$ and $(-s)^k(-t)^{k'} = (st)^k$. Thus, the coefficient of $\mathtt{H}_P$ in $\nu_s\nu_t$ is

$$(-1)^{\dim(P)}\sum_{K\leq P}(-1)^{\dim(K)}\alpha(K)\alpha(N_KP)(st)^k$$
$$= (-1)^{\dim(P)}\sum_k(-1)^k\upsilon_k(P)(st)^k = \nu^P_{st},$$

and therefore $\nu_s\nu_t = \nu_{st}$ □

Finally, the following proposition is a consequence of the observation in Remark 3.6.2.

Proposition 3.7.11. *Let $\mathcal{A}'$ be a subarrangement of $\mathcal{A}$ and ν_t be the intrinsic element of parameter t of $\mathcal{A}$. Then, $f(\nu_t)$ is the intrinsic element of parameter t of $\mathcal{A}'$, where f is as in (3.2.4).*

3.7.1 Generalization to Cone angles

Let $\mathcal{C}_d$ be the collection of polyhedral cones in $\mathbb{R}^d$. Backman, Manecke and Sanyal define in [18] a **cone angle** in $\mathbb{R}^d$ as a map $\alpha : \mathcal{C}_d \to \mathbb{R}$ satisfying the following three properties.

1. The valuation property (3.6.1) whenever $C_1 \cup C_2$ is a cone,

2. $\alpha(C) = 0$ if $\dim(C) < d$, and

3. $\alpha(\mathbb{R}^d) = 1$.

Given a cone angle α, the corresponding interior and exterior angles of a polyhedron P at a face $F \leq P$ are defined by

$$\hat{\alpha}(F, P) = \alpha(T_F P + \mathtt{L}(P)^{\perp}) \quad \text{and} \quad \check{\alpha}(F, P) = \alpha(N_F P + \mathtt{L}(F)),$$

respectively, where $\mathtt{L}(P)$ is the linear space parallel to (the affine span of) P. Note that $T_F P + \mathtt{L}(P)^{\perp}$ and $N_F P + \mathtt{L}(F)$ are always full-dimensional.

Let $\mathcal{A}$ be a central arrangement in $\mathbb{R}^d$, [18, Corollary 4.6 (iii)] shows that

$$\sum_{C} \sum_{F \leq C} (-1)^{\dim(F)} \hat{\alpha}(O, F) \check{\alpha}(F, C) t^{\dim(F)} = \chi(\mathcal{A}, t),$$

where the first sum is over the chambers C of $\mathcal{A}$. In fact, this is easily generalized as follows.

Proposition 3.7.12. *Let $\mathcal{A}$ be a central arrangement. For any $t \in \Bbbk$, the element $w \in \Bbbk\Sigma[\mathcal{A}]$ defined by*

$$w^F = (-1)^{\dim(F)} \sum_{G \leq F} (-1)^{\dim(G)} \hat{\alpha}(O, G) \check{\alpha}(-G, -F) t^{\dim(G)}$$

is characteristic of parameter t.

We can extend this result to the case of general hyperplane arrangements by means of the observations in Section 3.6. The result we will prove is the following.

Theorem 3.7.13. *Let $\mathcal{A}$ be an arbitrary arrangement. Given an angle cone α and a parameter $t \in \Bbbk$, the element $w \in \Bbbk\Sigma[\mathcal{A}]$ defined by*

$$w^F = (-1)^{\dim(F)} \sum_{G \leq \operatorname{rec}(F)} (-1)^{\dim(G)} \hat{\alpha}(O, G) \check{\alpha}(-G, -\operatorname{rec}(F)) t^{\dim(G)}$$

is characteristic of parameter t.

For a polyhedron P, define

$$v(P) = v(\operatorname{rec}(P)) = \sum_{F \le \operatorname{rec}(P)} \hat{\alpha}(O, F)\check{\alpha}(F, \operatorname{rec}(P))t^{\dim(F)},$$

where $O = \operatorname{rec}(P) \cap (-\operatorname{rec}(P))$ is the minimal face of $\operatorname{rec}(P)$.

Using that $\operatorname{rec}(P \cup Q) = \operatorname{rec}(P) \cup \operatorname{rec}(Q)$ whenever $P \cup Q$ is a polyhedron, and $\operatorname{rec}(P \cap Q) = \operatorname{rec}(P) \cap \operatorname{rec}(Q)$ whenever $P \cap Q \neq \emptyset$, we can verify that v defines a valuation. Moreover, it is clear that $v(A) = t^{\dim(A)}$ for any affine subspace. Thus, the element $w \in \Bbbk\Sigma[\mathcal{A}]$ defined by

$$w^F = v(\operatorname{relint}(F))$$

is characteristic of parameter t. Note that in the central case this is precisely the result of Proposition 3.7.12:

$$\begin{aligned}
v(\operatorname{relint}(F)) = & \sum_{H \le F} (-1)^{\dim(F)-\dim(H)} \sum_{G \le H} \hat{\alpha}(O, G)\check{\alpha}(G, H)t^{\dim(G)} \\
= & (-1)^{\dim(F)} \sum_{G \le F} \hat{\alpha}(O, G)t^{\dim(G)} \sum_{H : G \le H \le F} (-1)^{\dim(H)}\check{\alpha}(G, H) \\
= & (-1)^{\dim(F)} \sum_{G \le F} \hat{\alpha}(O, G)t^{\dim(G)}(-1)^{\dim(G)}\check{\alpha}(-G, -F) \\
= & (-1)^{\dim(F)} \sum_{G \le F} (-1)^{\dim(G)}\hat{\alpha}(O, G)\check{\alpha}(-G, -F)t^{\dim(G)}
\end{aligned}$$

The second equality follows from [18, Lemma 2.4] with $C = N_G F + \mathtt{L}(G)$, so that $-C = N_{-G}(-F) + \mathtt{L}(-G)$.

The affine case requires one additional observation.

$$
\begin{aligned}
v(\operatorname{relint}(F)) &= \sum_{H \le F} (-1)^{\dim(F)-\dim(H)} \sum_{G \le \operatorname{rec}(H)} \hat{\alpha}(O,G)\check{\alpha}(G,\operatorname{rec}(H)) t^{\dim(G)} \\
&= (-1)^{\dim(F)} \sum_{G \le H' \le \operatorname{rec}(F)} \sum_{\operatorname{rec}(H)=H'} (-1)^{\dim(H)} \hat{\alpha}(O,G)\check{\alpha}(G,H') t^{\dim(G)} \\
&= (-1)^{\dim(F)} \sum_{G \le H' \le \operatorname{rec}(F)} (-1)^{\dim(H')} \hat{\alpha}(O,G)\check{\alpha}(G,H') t^{\dim(G)} \\
&= (-1)^{\dim(F)} \sum_{G \le \operatorname{rec}(F)} (-1)^{\dim(G)} \hat{\alpha}(O,G)\check{\alpha}(-G,-\operatorname{rec}(F)) t^{\dim(G)}
\end{aligned}
$$

In the second equality we use that $\sum_{\operatorname{rec}(H)=H'} (-1)^{\dim(H)}$ computes the Euler characteristic of the relative pair of cell complexes $(X, \partial X)$, where $X = \{H : \operatorname{rec}(H) \le H'\}$.

3.8 Proof of Theorem 3.4.1

The definition of v in Theorem 3.4.1 only depends on the rank of the faces, not one their dimension. Thus, in this section we assume without loss of generality that $\mathcal{A}$ is essential.

Let $\Sigma^b[\mathcal{A}]$ denote the set of bounded faces of $\mathcal{A}$. It forms a cell complex that is known to be contractible, see [28, Theorem 4.5.7] [69, Chapter 1, Exercise 7]. We will reduce the proof of Theorem 3.4.1 to the computation of the Euler characteristic of a certain subcomplexes of $\Sigma^b[\mathcal{A}]$.

Lemma 3.8.1. *Let C and D be two different chambers of $\mathcal{A}$. The Euler characteristic of the pair (X, Y), where*

$$X = \{F \in \Sigma^b[\mathcal{A}] : F \le C\} \quad \textit{and} \quad Y = \{F \in X : FD \ne C\},$$

is zero.

Proof. First note that Y is a sub-complex of X: suppose $G \in Y$, $F \leq G$ but $F \notin Y$; then, $GD = GFD = GC = C$, contradicting that $G \in Y$.

Consider a large ball $B \subseteq V$ that contains all the bounded faces of $\mathcal{A}$ in its interior. The intersection of B with the faces of $\mathcal{A}$ forms the following cell complex decomposition of B:

$$\Sigma[B] := \{F \,:\, F \in \Sigma^b[\mathcal{A}]\} \sqcup \{F \cap B, F \cap \partial B \,:\, F \in \Sigma[\mathcal{A}] \setminus \Sigma^b[\mathcal{A}]\}.$$

Note that $\Sigma^b[\mathcal{A}] \subseteq \Sigma[B]$, and for each unbounded face F of $\mathcal{A}$ there are two associated faces $F \cap \partial B \lessdot F \cap B$ of $\Sigma[B]$. Define the following subcomplexes of $\Sigma[B]$:

$$X' = X \cup \{F \cap B, F \cap \partial B \,:\, F \text{ and unbounded face of } C\}$$

and

$$Y' = Y \cup \{F \cap B, F \cap \partial B \,:\, F \text{ and unbounded face of } C \text{ with } FD \neq C\}.$$

Since for each unbounded face F of $\mathcal{A}$ we have $\dim(F \cap B) = \dim(F \cap \partial B) + 1$, the Euler characteristic of (X', Y') and (X, Y) agree.

Now, let x be a point in the interior of $D \cap B$. Y' is precisely the set of faces of $C \cap B$ that are *visible* from x. That is, faces in Y' correspond to points $y \in C \cap B$ such that the segment $\overline{yx}$ intersects $C \cap B$ precisely at y. Consider the map $p_x : C \cap B \to C \cap B$ defined by $p_x(y)$ is the last point inside $C \cap B$ in the segment $\overline{yx}$. p_x is well defined since $C \cap B$ is closed, and it is continuous because $C \cap B$ is convex. By the observation above, $|Y'|$ is the image of this map, and p_x acts as the identity on it. Thus, $p_x : C \cap B \to |Y'|$ is a retraction. Since $|Y'|$ is a retraction of $C \cap B$ and X' is a cell decomposition of $C \cap B$, we conclude that

$$\chi(X, Y) = \chi(X', Y') = 0,$$

as we wanted to show. □

Lemma 3.8.2. *Let H be a face of $\mathcal{A}$ and $G < H$ a proper face. The Euler characteristic of the pair $(T, \partial T)$, where*

$$T = \{F \in \Sigma^b[\mathcal{A}] \,:\, GF \leq H\} \quad \textit{and} \quad \partial T = \{F \in \Sigma^b[\mathcal{A}] \,:\, GF < H\},$$

is zero.

Proof. Define B and $\Sigma[B]$ as in the proof of Lemma 3.8.1. Similarly, let

$$T' = T \cup \{F \cap B, F \cap \partial B \,:\, F \text{ is an unbounded face such that } GF \leq H\}$$

and

$$\partial T' = \partial T \cup \{F \cap B, F \cap \partial B \,:\, F \text{ is an unbounded face such that } GF < H\}.$$

Note that the second complex is non-empty precisely because G is a proper face of H. As before, adding pairs of faces $F \cap B$ and $F \cap \partial B$ does not change the Euler characteristic of the complex, so $\chi(T, \partial T) = \chi(T', \partial T')$.

Finally, note that T' is a cell decomposition of $T_G H \cap B$ and $\partial T'$ is a cell decomposition of $(\partial T_G H) \cap B$. Since both of these spaces are contractible, in fact they are respectively cones over $T_G H \cap \partial B$ and $(\partial T_G H) \cap \partial B$, we have that

$$\chi(T, \partial T) = \chi(T', \partial T') = 0. \qquad \square$$

Proof of Theorem 3.4.1. It is enough show that for any face G of $\mathcal{A}$, $\upsilon \mathtt{H}_G = \mathtt{H}_G$ and $\mathtt{H}_G \upsilon = \mathtt{H}_G$. We do this by comparing the coefficient of $\mathtt{H}_H$ on both sides of the equality, for any face H of $\mathcal{A}$.

First, $\upsilon \mathtt{H}_G = \mathtt{H}_G$ is equivalent to

$$\sum_{\substack{F \,:\, F \in \Sigma^b \\ FG = H}} (-1)^{\operatorname{rank} F} = \begin{cases} 1 & \text{if } H = G, \\ 0 & \text{otherwise.} \end{cases} \tag{3.8.1}$$

If $H = G$, then $FG = G = H$ if and only if F is a face of G. In this case, the sum computes the Euler characteristic of the bounded complex of G, which, by arguments similar to those in the previous lemmas, equals $\chi(G \cap B) = 1$.

If $H \neq G$, the sum is empty unless $\mathrm{s}(G) \leq \mathrm{s}(H)$, so we can assume H is a chamber C. Let D be the chamber containing G that is furthest from C. Namely, choose signs so that $\epsilon_{\mathrm{H}}(C) = +$ for all the hyperplanes of $\mathcal{A}$. D is the chamber containing G such that $\epsilon_{\mathrm{H}}(D) = -$ for all $\mathrm{H} \geq \mathrm{s}(G)$. Then, a face $F \leq C$ satisfies $FG = C$ if and only if $FD = C$. Indeed, if $FD = C$ and $FG \neq C$, then $FG < C$ and for some hyperplane $\epsilon_{\mathrm{H}}(F) = \epsilon_{\mathrm{H}}(G) = 0$; but in that case $\epsilon_{\mathrm{H}}(D) = -$, contradicting that $FD = C$. The result now follows from Lemma 3.8.1.

Now, $\mathtt{H}_G v = \mathtt{H}_G$ is equivalent to

$$\sum_{\substack{F\,:\,F \in \Sigma^b \\ GF = H}} (-1)^{\operatorname{rank} F} = \begin{cases} 1 & \text{if } H = G, \\ 0 & \text{otherwise.} \end{cases} \tag{3.8.2}$$

If $H = G$, then $GF = G = H$ if and only if $\mathrm{s}(F) \leq \mathrm{s}(G)$. Thus, the sum computes the Euler characteristic of $\Sigma^b[\mathcal{A}^{\mathrm{X}}]$.

If $H \neq G$, the sum is trivial unless G is a proper face of H. Fix $G < H$. Note that

$$\sum_{\substack{F\,:\,F \in \Sigma^b \\ GF = H}} (-1)^{\operatorname{rank} F} = \sum_{\substack{F\,:\,F \in \Sigma^b \\ GF \leq H}} (-1)^{\operatorname{rank} F} - \sum_{\substack{F\,:\,F \in \Sigma^b \\ GF < H}} (-1)^{\operatorname{rank} F}$$

computes the Euler characteristic of the pair in Lemma 3.8.2, and is therefore 0. □

CHAPTER 4

DEFORMATIONS OF A LINEAR ARRANGEMENT AND EXPONENTIAL SEQUENCES

The contents of this chapter are joint work with Aguiar and Mahajan, and will appear in an extended version of [3]. An affine arrangement $\mathcal{A}$ is a **deformation** of a linear arrangement $\mathcal{A}_0$ if each hyperplane in $\mathcal{A}$ is parallel to some hyperplane in $\mathcal{A}_0$. Several families of deformations of Coxeter arrangements have been studied by Athanasiadis, Postnikov, Stanley, Suyama and others. See for instance [16, 62, 71].

In this chapter we consider the Tits algebra of a deformation $\mathcal{A}$ as a bimodule over the Tits algebra of the corresponding linear arrangement $\mathcal{A}_0$. We show that the left and right action of any element $w \in \Sigma[\mathcal{A}_0]$ on the unit element $v \in \Bbbk\Sigma[\mathcal{A}]$ agree, thus concluding the existence of a morphism of algebras $i : \Bbbk\Sigma[\mathcal{A}_0] \to \Bbbk\Sigma[\mathcal{A}]$. We use an explicit description of the map i to prove that the image of the intrinsic element of parameter t for $\mathcal{A}_0$ is the intrinsic element of the same parameter for $\mathcal{A}$.

We then study exponential sequences of arrangements in the sense of Stanley [69]. Given an exponential sequence $\mathscr{A}$, we construct a Hopf monoid of faces $\Sigma_{\mathscr{A}}$ and a Hopf monoid of flats $\Sigma_{\mathscr{A}}$ of the sequence. The Hopf monoid $\Sigma_{\mathscr{A}}$ is co-commutative, we show that the induced left module structure over the Tits algebra of the braid arrangement (Section 2.3.4) coincides with the left module structure above. We use techniques from Hopf monoid theory (in particular *series* and *characters*) to obtain results about polynomial invariants associated with these sequences, thus extending results of Stanley.

4.1 The action of a central arrangement on its deformations

Let $\mathcal{A}$ be an affine arrangement. For each hyperplane $\mathrm{H} \in \mathcal{A}$, let H_0 be the linear hyperplane parallel to H. The linearization of $\mathcal{A}$ is the linear arrangement $\mathcal{A}_0 = \{\mathrm{H}_0 : \mathrm{H} \in \mathcal{A}\}$. In this situation we also say that $\mathcal{A}$ is a **deformation** of $\mathcal{A}_0$. Notice that $\mathcal{A}$ contains at least one hyperplane parallel to each of the hyperplanes in $\mathcal{A}_0$.

The set $\Sigma[\mathcal{A}]$ is a bimodule over the monoid $\Sigma[\mathcal{A}_0]$, as explained next.

Pick $F_0 \in \Sigma[\mathcal{A}_0]$ and $G \in \Sigma[\mathcal{A}]$, with F_0 different from the central face of $\mathcal{A}_0$. Then a point in the relative interior of F_0 determines a direction in the ambient space (the vector based at the the origin and with vertex at that point). Define the left and right action of F_0 on G as follows:

- $F_0 \cdot G \in \Sigma[\mathcal{A}]$ is the **last face** one encounters when walking out of (a point in the relative interior of) G in the direction of (a point in the relative interior of) F_0. For the central face $O \in \Sigma[\mathcal{A}_0]$, we set $O \cdot G = G$.
- $G \cdot F_0 \in \Sigma[\mathcal{A}]$ is the **first face** one encounters when walking out of G in the direction of F_0. For the central face $O \in \Sigma[\mathcal{A}_0]$, we set $F \cdot O = F$.

See Figure 4.1.1 for an example. Observe that the face $F_0 \cdot G$ contains a translate of F_0. In particular, if F_0 is not the central face of $\mathcal{A}_0$, the face $F_0 \cdot G$ is (essentially) unbounded.

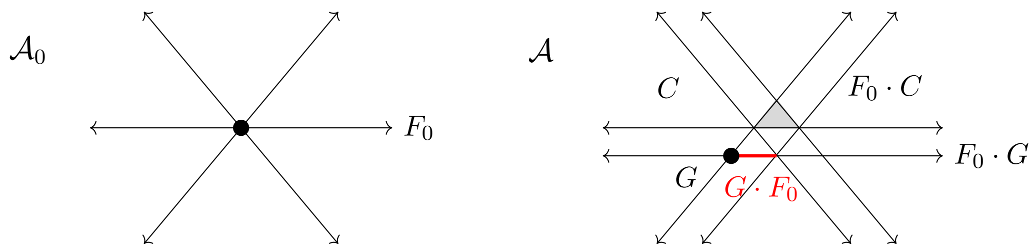


Figure 4.1.1: The arrangement $\mathcal{A}$ is a deformation of $\mathcal{A}_0$. $C = C \cdot F_0$ is a chamber and G is a minimal face of $\mathcal{A}$.

We can define these actions in terms of sign sequences as well. Pick signs so that for every pair of parallel hyperplanes H and H′, the translation of H^+ and $(\mathrm{H}')^+$ to the origin (meaning the origin is in the corresponding hyperplane) agree. Then, for $F_0 \in \Sigma[\mathcal{A}_0]$ and $G \in \Sigma[\mathcal{A}]$,

$$\epsilon_{\mathrm{H}}(F_0 \cdot G) = \begin{cases} \epsilon_{\mathrm{H}_0}(F_0) & \text{if } \epsilon_{\mathrm{H}_0}(F_0) \neq 0 \\ \epsilon_{\mathrm{H}}(G) & \text{otherwise,} \end{cases} \tag{4.1.1}$$

and

$$\epsilon_{\mathrm{H}}(G \cdot F_0) = \begin{cases} \epsilon_{\mathrm{H}}(G) & \text{if } \epsilon_{\mathrm{H}}(G) \neq 0, \\ \epsilon_{\mathrm{H}_0}(F_0) & \text{otherwise.} \end{cases} \tag{4.1.2}$$

With these definitions one easily checks that these products are associative and unital. We also have the following *mixed-associativity* axioms:

$$(F_0 \cdot F) \cdot G_0 = F_0 \cdot (F \cdot G_0) \qquad \text{and} \qquad (F \cdot F_0)G = F(F_0 \cdot G). \tag{4.1.3}$$

The first one is precisely the bimodule axiom. Similarly, one can check the following LRB-type properties:

$$F_0 \cdot G \cdot F_0 = F_0 \cdot G \quad \text{and} \quad (G \cdot F_0)G = G \cdot F_0. \tag{4.1.4}$$

LRB stands for *left regular band*: a semigroup S satisfying $xx = x$ and $xyx = xy$ for all $x, y \in S$.

Remark 4.1.1. The right module structure was described by Aguiar and Petersen in [9, Section 1]. This structure is defined even if $\mathcal{A}$ has infinitely many hyperplanes in each parallelism class ($\mathcal{A}_0$ is still required to be finite). However, in that setting the left action is not well defined (there is no *last face*).

The bimodule structure above descends to an action of $\mathcal{L}[\mathcal{A}_0]$ on $\mathcal{L}[\mathcal{A}]$ satisfying $\mathrm{s}(F_0 \cdot G) = \mathrm{s}(G \cdot F_0) = \mathrm{s}(F_0) \cdot \mathrm{s}(G)$ for all $F_0 \in \Sigma[\mathcal{A}_0]$ and $G \in \Sigma[\mathcal{A}]$. Explicitly, for $\mathrm{X}_0 \in \mathcal{L}[\mathcal{A}_0]$ and $\mathrm{Y} \in \mathcal{L}[\mathcal{A}]$, $\mathrm{X}_0 \cdot \mathrm{Y}$ is the minimum flat of $\mathcal{A}$ containing the affine subspace $\mathrm{X}_0 + \mathrm{Y}$.

Given a flat X_0 of the linear arrangement $\mathcal{A}_0$, we let $\mathcal{A}_{\mathrm{X}_0}$ be the subarrangement of $\mathcal{A}$ consisting of all hyperplanes that contain a translate of X_0. That is,

$$\mathcal{A}_{\mathrm{X}_0} = \{\mathrm{H} \in \mathcal{A} \,:\, \mathrm{X}_0 \leq \mathrm{H}_0\}.$$

Unlike the *arrangement over a flat* (see Section 2.2), this arrangement might not be central.

4.1.1 The map f_0

We define a map

$$f_0 : \Sigma[\mathcal{A}] \to \Sigma[\mathcal{A}_0] \tag{4.1.5}$$

as follows. Given $F \in \Sigma[\mathcal{A}]$, consider the halfspaces and hyperplanes defining F and translate them to the origin. Their intersection defines the face $f_0(F)$ of $\mathcal{A}_0$. Alternatively, $f_0(F)$ is the largest face of $\mathcal{A}_0$ with a translate contained in F. This is precisely the recession cone of F, see [77, Proposition 1.12(i)]. In particular, a face of $\mathcal{A}$ is essentially bounded if and only if f_0 sends it to the central face of $\mathcal{A}_0$.

The map f_0 is order-preserving, but is neither a morphism of semigroups, nor a morphism of $\Sigma[\mathcal{A}_0]$-bimodules. However, its linearization $f_0 : \Bbbk\Sigma[\mathcal{A}] \to \Bbbk\Sigma[\mathcal{A}_0]$ does send the unit element of $\Bbbk\Sigma[\mathcal{A}]$ to the unit element of $\Bbbk\Sigma[\mathcal{A}_0]$.

The sign sequence of $f_0(F)$ cannot be immediately read from the sign sequence of F. However, the following property holds: if $\epsilon_{\mathrm{H}_0}(f_0(F)) = +$ (resp. $-$), then $\epsilon_{\mathrm{H}'} = +$ (resp. $-$) for all hyperplanes $\mathrm{H}' \in \mathcal{A}$ parallel to H_0. The converse is not necessarily true; see the shaded triangle in Figure 4.1.1 for an example. We can sum this up as follows:

$$\epsilon_{\mathrm{H}_0}(f_0(F)) \neq 0 \Rightarrow \epsilon_{\mathrm{H}}(F) = \epsilon_{\mathrm{H}_0}(f_0(F)). \tag{4.1.6}$$

We can similarly define a map

$$\bar{f}_0 : \Pi[\mathcal{A}] \to \Pi[\mathcal{A}_0] \tag{4.1.7}$$

sending a flat to its translation to the origin. It is order preserving. Note that in general $\bar{f}_0(\mathrm{s}(F))$ and $\mathrm{s}(f_0(F))$ do not agree. For example if C is a bounded chamber, then $\mathrm{s}(f_0(C)) = \perp$ and $\bar{f}_0(\mathrm{s}(C)) = \top$.

Property (4.1.6) readily implies that

$$f_0(F) \cdot F = F \quad \text{and} \quad F \cdot f_0(F) = F. \tag{4.1.8}$$

Since $F_0 \cdot G$ contains a translate of F_0, and $f_0(F)$ is the largest face of $\mathcal{A}_0$ with a translate contained in F, we see that

$$F_0 \leq f_0(F_0 \cdot G). \tag{4.1.9}$$

In particular, if $F_0 \cdot G = G$ then $F_0 \leq f_0(G)$. Conversely, suppose that $F_0 \leq f_0(G)$. Then, using (4.1.8),

$$F_0 \cdot G = F_0 \cdot (f_0(G) \cdot G) = F_0 f_0(G) \cdot G = f_0(G) \cdot G = G.$$

We have shown the following.

$$F_0 \cdot G = G \Longleftrightarrow F_0 \leq f_0(G). \tag{4.1.10}$$

Now suppose that $G \cdot F_0 = G$. Taking support on both sides we conclude that $\mathrm{s}(G) + \mathrm{s}(F_0)$ is contained in $\mathrm{s}(G)$. Therefore, a translate of $\mathrm{s}(F_0)$ is contained in $\mathrm{s}(G)$. Equivalently, $\mathrm{s}(F_0) \leq \bar{f}_0(\mathrm{s}(G))$. Conversely, suppose $\mathrm{s}(F_0) \leq \bar{f}_0(\mathrm{s}(G))$. For any $\boldsymbol{x} \in \mathrm{relint}(F_0)$, $\boldsymbol{y} \in \mathrm{relint}(F)$, and $\lambda \in \Bbbk$, we have $\boldsymbol{y}+\lambda\boldsymbol{x} \in \mathrm{s}(G)+\mathrm{s}(F_0) = \mathrm{s}(G)$. Moreover, G is full-dimensional inside $\mathrm{s}(G)$, so $\boldsymbol{y}+\lambda\boldsymbol{x} \in \mathrm{relint}(F)$ for small enough values of $\lambda > 0$. That is, $G \cdot F_0 = G$. We have shown that

$$G \cdot F_0 = G \Longleftrightarrow \mathrm{s}(F_0) \leq \bar{f}_0(\mathrm{s}(G)). \tag{4.1.11}$$

Fix a face F_0 of $\mathcal{A}_0$ and let $\mathrm{X}_0 = \mathrm{s}(F_0)$. Let $f : \Sigma[\mathcal{A}] \to \Sigma[\mathcal{A}_{\mathrm{X}_0}]$ be the morphism (3.2.4): it sends a face G of $\mathcal{A}$ to the minimal face of the subarrangement $\mathcal{A}_{\mathrm{X}_0}$ containing G. That is, $\epsilon_{\mathrm{H}}(f(G)) = \epsilon_{\mathrm{H}}(G)$ for all $\mathrm{H} \in \mathcal{A}_{\mathrm{X}_0}$. Since $\epsilon_{\mathrm{H}_0}(F_0) = 0$ for all $\mathrm{H} \in \mathcal{A}_{\mathrm{X}_0}$ and $\epsilon_{\mathrm{H}_0}(F_0) \neq 0$ for all $\mathrm{H} \in \mathcal{A} \setminus \mathcal{A}_{\mathrm{X}_0}$, it follows that

$$f(G) = f(G') \Longleftrightarrow F_0 \cdot G = F_0 \cdot G'. \tag{4.1.12}$$

Thus, there is a natural bijection

$$\Sigma[\mathcal{A}]_{F_0} := \{F \in \Sigma[\mathcal{A}] \mid f_0(F) \geq F_0\} \longrightarrow \Sigma[\mathcal{A}_{\mathrm{X}_0}]$$

sending a face $F_0 \cdot G$ to $f(G)$. This map is in fact an isomorphism of semigroups. Under this bijection, relatively bounded faces of $\mathcal{A}_{\mathrm{X}_0}$ correspond to faces $G \in \Sigma[\mathcal{A}]_{F_0}$ such that $f_0(G) = F_0$.

4.1.2 The map i

We have a surjective morphism of semigroups $\Sigma[\mathcal{A}] \to \Sigma[\mathcal{A}]_{F_0}$ sending a face G to $F_0 \cdot G$. Its linearization is a morphism of algebras, and sends the unit element $\upsilon \in \Bbbk\Sigma[\mathcal{A}]$ (see (3.4.1)) to the unit element of $\Bbbk\Sigma[\mathcal{A}]_{F_0} \cong \Bbbk\Sigma[\mathcal{A}_{\mathrm{X}_0}]$. For $F \in \Sigma[\mathcal{A}]_{F_0}$ the rank of the corresponding face in $\Sigma[\mathcal{A}_{\mathrm{X}_0}]$ is $\dim(F) - \dim(F_0)$. Therefore,

$$F_0 \cdot \upsilon = \sum_{\substack{F \in \Sigma[\mathcal{A}]: \\ f_0(F) = F_0}} (-1)^{\dim(F) - \dim(F_0)} \mathtt{H}_F. \tag{4.1.13}$$

Using (4.1.3) and (4.1.4), we conclude

$$\upsilon \cdot F_0 = (\upsilon \cdot F_0)\upsilon = \upsilon(F_0 \cdot \upsilon) = F_0 \cdot \upsilon.$$

We define a map

$$i : \Bbbk\Sigma[\mathcal{A}_0] \to \Bbbk\Sigma[\mathcal{A}]$$

by $i(F_0) = F_0 \cdot \upsilon = \upsilon \cdot F_0$. Finally, we can check that bimodule structure is given by the map i. Indeed,

$$F_0 \cdot G = F_0 \cdot (\upsilon G) = (F_0 \cdot \upsilon)G = i(F_0)G$$

and

$$G \cdot F_0 = (G\upsilon) \cdot F_0 = G(\upsilon \cdot F_0) = Gi(F_0).$$

Furthermore, we see that i is a morphism of algebras:

$$i(F_0G_0) = F_0G_0 \cdot \upsilon = F_0 \cdot (G_0 \cdot \upsilon) = i(F_0)(G_0 \cdot \upsilon) = i(F_0)i(G_0),$$

and the bimodule structure is precisely the one induced by this algebra extension.

4.1.3 Characteristic and intrinsic elements

Proposition 4.1.2. *Let $v_0 \in \Bbbk\Sigma[\mathcal{A}_0]$ and $w \in \Bbbk\Sigma[\mathcal{A}]$ be characteristic elements of parameters s and t, respectively. Then, $v_0 \cdot w \in \Bbbk\Sigma[\mathcal{A}]$ is characteristic of parameter st.*

Proof. Take any flat $\mathrm{X} \in \mathcal{L}[\mathcal{A}]$. We have $\mathrm{s}(F_0 \cdot G) \leq \mathrm{X}$ if and only if $\mathrm{s}(F) \leq \bar{f}_0(\mathrm{X})$ and $\mathrm{s}(G) \leq \mathrm{X}$. Then,

$$\sum_{\mathrm{s}(F_0 \cdot G) \leq \mathrm{X}} (v_0 \cdot w)^{F_0 \cdot G} = \sum_{\substack{\mathrm{s}(F_0) \leq \bar{f}_0(\mathrm{X}) \\ \mathrm{s}(G) \leq \mathrm{X}}} v_0^{F_0} w^G = s^{\dim(\bar{f}_0(\mathrm{X}))} t^{\dim(\mathrm{X})} = (st)^{\dim(\mathrm{X})}.$$

Since this occurs for all $\mathrm{X} \in \mathcal{L}[\mathcal{A}]$, the result follows. □

In particular, letting $w = \upsilon$ and using Proposition 3.4.2, we conclude the following.

Corollary 4.1.3. *If $v_0 \in \Bbbk\Sigma[\mathcal{A}_0]$ is characteristic of parameter s, then so is $i(v_0) \in \Bbbk\Sigma[\mathcal{A}]$.*

Proposition 4.1.4. *Let $\nu_t^0 \in \Bbbk\Sigma[\mathcal{A}_0]$ be the intrinsic element for $\mathcal{A}_0$ of parameter t. Then, $i(\nu_t^0) = \nu_t$, the intrinsic element for $\mathcal{A}$ of the same parameter.*

Proof. Using (4.1.13) and the definition of the intrinsic element (3.7.8), we have that the coefficient of $\mathtt{H}_F$ on $i(\nu_t^0)$ is

$$\begin{aligned}(-1)^{\dim(F) - \dim(f_0(F))} (\nu_t^0)^{f_0(F)} &= (-1)^{\dim(F)} \sum_k (-1)^k \upsilon_k(f_0(F)) t^k \\ &= (-1)^{\dim(F)} \sum_k (-1)^k \upsilon_k(F) t^k = (\nu_t)^F.\end{aligned}$$

In the second step we use that the intrinsic volume of F only depends on $f_0(F)$, see Proposition 3.7.2. □

4.2 Exponential sequences of arrangements

Definition 4.2.1 ([69, Section 5.3])**.** A sequence $\mathscr{A} = (\mathcal{A}_1, \mathcal{A}_2, \dots)$ of hyperplane arrangements is called **exponential** if

1. $\mathcal{A}_d$ is a deformation of the braid arrangement in $\mathbb{R}^d$, and
2. for all $S \subseteq [d]$, the arrangement

   $$(\mathcal{A}_d)_S := \{\mathrm{H} \in \mathcal{A}_d \,:\, \mathrm{H} \text{ is parallel to } x_i - x_j = 0 \text{ for some } i, j \in S\}$$

   is combinatorially isomorphic to $\mathcal{A}_k$, where $|S| = k$.

In particular, the number of hyperplanes in any parallelism class is constant (and equal to the number of hyperplanes in $\mathcal{A}_2$).

Well-studied examples of exponential sequences include the Catalan, semiorder, Linial, and Shi arrangements. The definition of the latter two uses the order of the elements of $[d]$. For instance, the Shi arrangement in $\mathbb{R}^d$ consists of hyperplanes

$$x_i - x_j = 0 \quad \text{and} \quad x_i - x_j = 1 \quad \text{for } 1 \le i < j \le d.$$

Thus, to define the Shi arrangement in $\mathbb{R}^I$ for an arbitrary finite set I, we need to first fix a linear order $\ell \in \mathsf{L}[I]$.

Fix a finite set I and a linear order $\ell \in \mathsf{L}[I]$. For $i, j \in I$ with $i <_\ell j$, and $\alpha \in \mathbb{R}$, we let $\mathrm{H}^\alpha_{i,j}$ denote the hyperplane $x_i - x_j = \alpha$ in $\mathbb{R}^I$, and choose positive (and negative) half-spaces by setting

$$(\mathrm{H}^\alpha_{i,j})^+ = \{x \in \mathbb{R}^I \,:\, x_i - x_j \ge \alpha\}.$$

If $i, j \in S \subseteq I$, we use $\mathrm{H}^\alpha_{i,j}$ to denote both a hyperplane in $\mathbb{R}^S$ and a hyperplane in $\mathbb{R}^I$, the ambient space will be clear from context.

The most natural examples of exponential sequences are obtained by fixing constants $\alpha_1 < \alpha_2 < \cdots < \alpha_k$ and letting $\mathcal{A}_d$ consist of hyperplanes

$$\boldsymbol{x}_i - \boldsymbol{x}_j = \alpha_r \quad \text{for } 1 \leq i < j \leq d,\ 1 \leq r \leq k.$$

Henceforth, we assume our exponential sequences are of this form.

4.3 Hopf monoid of flats and faces

Fix an exponential sequence $\mathscr{A} = (\mathcal{A}_1, \mathcal{A}_2, \dots)$ as above. Given a finite set I of cardinality d and a linear order $\ell = i_1 i_2 \dots i_d$ of I, let $\mathcal{A}_\ell$ be the following hyperplane arrangement:

$$\mathcal{A}_\ell = \{\mathrm{H}^\alpha_{i_a,i_b} \subseteq \mathbb{R}^I \,:\, \mathrm{H}^\alpha_{a,b} \in \mathcal{A}_d\}. \tag{4.3.1}$$

It is a deformation of the braid arrangement in $\mathbb{R}^I$.

Definition 4.3.1. The **L-species of faces** $\Sigma_{\mathscr{A}}$ of the exponential sequence $\mathscr{A}$ is defined by:

- The set $\Sigma_{\mathscr{A}}[I, \ell]$ is the collection of faces of the arrangement $\mathcal{A}_\ell$, that is $\Sigma_{\mathscr{A}}[I, \ell] := \Sigma[\mathcal{A}_\ell]$.
- Given $\ell = i_1 i_2 \dots i_d \in \mathsf{L}[I]$ and $\ell' = j_1 j_2 \dots j_d \in \mathsf{L}[J]$, the relabeling map $\Sigma_{\mathscr{A}}[I, \ell] \to \Sigma_{\mathscr{A}}[J, \ell']$ sends a face $F \in \Sigma[\mathcal{A}_\ell]$ to the face $F' \in \Sigma[\mathcal{A}_{\ell'}]$ with sign sequence $\epsilon_{\mathrm{H}^\alpha_{j_a,j_b}}(F') = \epsilon_{\mathrm{H}^\alpha_{i_a,i_b}}(F)$.

We proceed to give $\Sigma_{\mathscr{A}}$ the structure of a Hopf monoid in L-species. Fix a decomposition $I = S \sqcup T$ and a linear order $\ell \in \mathsf{L}[I]$.

- The product $F \cdot G = \mu^{\ell}_{S,T}(F, G)$ of faces $F \in \Sigma_{\mathscr{A}}[S, \ell|_S]$ and $G \in \Sigma_{\mathscr{A}}[T, \ell|_T]$ is defined by the sign sequence

$$\epsilon_{\mathrm{H}^{\alpha}_{i,j}}(F \cdot G) = \begin{cases} \epsilon_{\mathrm{H}^{\alpha}_{i,j}}(F) & \text{if } i, j \in S \\ \epsilon_{\mathrm{H}^{\alpha}_{i,j}}(G) & \text{if } i, j \in T \\ + & \text{if } i \in S,\ j \in T, \\ - & \text{if } i \in T,\ j \in S. \end{cases} \tag{4.3.2}$$

  To verify that this is a valid sign sequence, take any point $\boldsymbol{v} \in \operatorname{relint}(F \times G)$ and consider $\boldsymbol{v} + \lambda \boldsymbol{e}_S$ for $\lambda >> 0$, where $\boldsymbol{e}_S = \sum_{i \in S} \boldsymbol{e}_i$.

- The coproduct $\Delta^{\ell}_{S,T}(F) = (F|_S, F|_T)$ of a face $F \in \Sigma_{\mathscr{A}}[I, \ell]$ is determined by $\epsilon_{\mathrm{H}^{\alpha}_{i,j}}(F|_S) = \epsilon_{\mathrm{H}^{\alpha}_{i,j}}(F)$. Observe that the product is cocommutative.

Observe that the product arrangement $\mathcal{A}_{\ell|_S} \times \mathcal{A}_{\ell|_T}$ is a subarrangement of $\mathcal{A}_{\ell}$. Using the identification $\Sigma[\mathcal{A}_{\ell|_S}] \times \Sigma[\mathcal{A}_{\ell|_T}] = \Sigma[\mathcal{A}_{\ell|_S} \times \mathcal{A}_{\ell|_T}]$ in (2.2.5), the product and coproduct of $\Sigma_{\mathscr{A}}$ have a very concrete description. Let $F \in \Sigma[\mathcal{A}_{\ell|_S} \times \mathcal{A}_{\ell|_T}]$, and take $\boldsymbol{v} \in \operatorname{relint}(F)$. Then, $\mu^{\ell}_{S,T}(F)$ is the only face of $\mathcal{A}_{\ell}$ containing $\boldsymbol{v} + \lambda \boldsymbol{e}_S$ in its interior for $\lambda >> 0$. Conversely, if $F \in \Sigma[\mathcal{A}_{\ell}]$, then $\Delta^{\ell}_{S,T}(F) = f(F)$, where $f : \Sigma[\mathcal{A}_{\ell}] \to \Sigma[\mathcal{A}_{\ell|_S} \times \mathcal{A}_{\ell|_T}]$ is the morphism (3.2.4).

In particular, it follows that

$$\mu^{\ell}_{S,T} \circ \Delta^{\ell}_{S,T}(F) = F_{S,T} \cdot F \tag{4.3.3}$$

where $F_{S,T}$ is the face of the braid arrangement in $\mathbb{R}^I$ corresponding to the composition (S, T) and the action is that of Section 4.1.

We similarly define the **L-species of flats** $\mathcal{L}_{\mathscr{A}}$, by setting $\mathcal{L}_{\mathscr{A}}[I, \ell] := \mathcal{L}[\mathcal{A}_{\ell}]$. With $I = S \sqcup T$ and $\ell \in \mathsf{L}[I]$ as before, the product of flats $\mathrm{X} \in \mathcal{L}_{\mathscr{A}}[S, \ell|_S]$ and

$\mathrm{Y} \in \mathcal{L}_{\mathscr{A}}[T, \ell|_T]$ is

$$\mathrm{X} \cdot \mathrm{Y} = \mathrm{X} \oplus \mathrm{Y}. \tag{4.3.4}$$

Analogously, the coproduct is determined by $\Delta^{\ell}_{S,T}(\mathrm{X}) = (\mathrm{X}|_S, \mathrm{X}|_T)$, where $\mathrm{X}|_S$ is the intersection of the hyperplanes $\mathrm{H}^{\alpha}_{i,j} \in \mathcal{A}_{\ell|_S}$ such that $\mathrm{H}^{\alpha}_{i,j} \in \mathcal{A}_{\ell}$ contains X. The following result easily follows from the definitions.

Proposition 4.3.2. *The collection of support maps* $\mathrm{s}_\ell : \Sigma_{\mathscr{A}}[I, \ell] \to \mathcal{L}_{\mathscr{A}}[I, \ell]$ *forms a morphism of Hopf monoids.*

Given a flat $\mathrm{X} \in \mathcal{L}_{\mathscr{A}}[I, \ell]$, we let $\pi(\mathrm{X}) \vdash I$ be the partition associated to the flat $\bar{f}_0(\mathrm{X})$ of the braid arrangement in $\mathbb{R}^I$, where f_0 is the map (4.1.7). That is, $i, j \in I$ are in the same block of $\pi(\mathrm{X})$ if and only if $\boldsymbol{x}_i - \boldsymbol{x}_j$ is constant on X. It follows by the definition of the product of flats (4.3.4) that

$$\pi(\mathrm{X} \cdot \mathrm{Y}) = \pi(\mathrm{X}) \sqcup \pi(\mathrm{Y}).$$

Observe that for $i, j \in \pi(\mathrm{X})$, the hyperplane $x_i - x_j = \alpha$ containing X might or might not belong to the arrangement $\mathcal{A}_\ell$. Nevertheless, $i, j \in I$ are in the same block of $\pi(\mathrm{X})$ if and only if there are hyperplanes $\mathrm{H}^{\alpha_1}_{i_1,i_2}, \mathrm{H}^{\alpha_2}_{i_2,i_3}, \ldots, \mathrm{H}^{\alpha_{k-1}}_{i_{k-1},i_k} \in \mathcal{A}_\ell$ containing X, with $i = i_1$ and $j = i_k$. Therefore, if $S \subseteq I$ is the union of the blocks in some subpartition $\pi' \subseteq \pi(\mathrm{X})$, then $\pi(\mathrm{X}|_S) = \pi'$.

4.3.1 Antipode

We now consider the linearization of these species, $\Bbbk\Sigma_{\mathscr{A}}$ and $\Bbbk\mathcal{L}_{\mathscr{A}}$. Takeuchi's formula (2.3.3) and (4.3.3) readily imply that for any $w \in \Sigma_{\mathscr{A}}[I, \ell]$

$$\mathbb{s}_\ell(w) = \tau_0 \cdot w,$$

where τ_0 is the Takeuchi element of the braid arrangement in $\mathbb{R}^I$.

Since τ_0 is the intrinsic element of parameter -1, Proposition 4.1.4 implies the following.

Proposition 4.3.3. *The antipode of the Hopf monoid $\Bbbk\Sigma_{\mathscr{A}}$ is given by*

$$\mathrm{s}_\ell(w) = \tau w,$$

where τ is the Takeuchi element of $\mathcal{A}_\ell$.

Observe that the antipode formula in the previous proposition is *internal* to the algebra $\Bbbk\Sigma[\mathcal{A}_\ell]$, and does not involve the Tits algebra of the braid arrangement at all. The multiplicativity of characteristic elements Lemma 3.2.1 and of intrinsic elements Theorem 3.7.10 imply the following.

Corollary 4.3.4. *The antipode of a characteristic element of parameter t is a characteristic element of parameter $-t$. Moreover, the antipode of the intrinsic element of parameter t is the intrinsic element of parameter $-t$.*

4.4 Characteristic elements and power series

In this section we consider the space of series $\mathscr{S}(\Bbbk\Sigma_{\mathscr{A}})$ of the Hopf monoid $\Bbbk\Sigma_{\mathscr{A}}$. A series corresponds to a sequence $(\omega_1, \omega_2, \dots)$, where ω_d is an element of the Tits algebra of $\mathcal{A}_d$.

A series $\omega \in \mathscr{S}(\Bbbk\Sigma)$ is a **characteristic series** of parameter $t \in \Bbbk$ if each element $\omega_\ell \in \Bbbk\Sigma_{\mathscr{A}}[I, \ell]$ is a characteristic element of parameter t.

Proposition 4.4.1. *Let $\omega, \varpi \in \mathscr{S}(\Bbbk\Sigma_{\mathscr{A}})$ be characteristic series of parameters s and t, respectively. Then, $\omega * \varpi$ is a characteristic series of parameter $s + t$.*

Proof. To prove that each element $(\omega * \varpi)_\ell \in \Bbbk\Sigma_{\mathscr{A}}[I,\ell]$ is characteristic of parameter $s+t$, we need to show that for all flats $\mathrm{X} \in \mathcal{L}_{\mathscr{A}}[I,\ell]$,

$$\sum_{\mathrm{s}(G)\leq \mathrm{X}} (\omega * \varpi)^G_\ell = \sum \omega^F_{\ell|_S} \varpi^{F'}_{\ell|_T} \quad \text{equals} \quad (s+t)^{\dim(\mathrm{X})},$$

where the second sum is over all decompositions $I = S \sqcup T$ and faces $F \in \Sigma_{\mathscr{A}}[S, \ell|_S]$, $F' \in \Sigma_{\mathscr{A}}[T, \ell|_T]$ such that $\mathrm{s}(F \cdot F') \leq \mathrm{X}$.

Suppose $I = S \sqcup T$, $F \in \Sigma_{\mathscr{A}}[S, \ell|_S]$, and $F' \in \Sigma_{\mathscr{A}}[T, \ell|_T]$ are such that $\mathrm{s}(F \cdot F') \leq \mathrm{X}$. Then, $\pi(\mathrm{s}(F)) \sqcup \pi(\mathrm{s}(F'))$ is coarser than $\pi(\mathrm{X})$. In particular, S (and T) is the union of the blocks in some subpartition $\pi' \subseteq \pi(\mathrm{X})$. We can then rewrite the second sum above as

$$\sum_{\pi' \subseteq \pi(\mathrm{X})} \Big(\sum_{\mathrm{s}(F)\leq \mathrm{X}|_S} \omega^F_{\ell|_S} \Big) \Big(\sum_{\mathrm{s}(F')\leq \mathrm{X}|_T} \varpi^{F'}_{\ell|_T} \Big) = \sum_{\pi' \subseteq \pi(\mathrm{X})} s^{|\pi'|} t^{|\pi(\mathrm{X})| - |\pi'|} = (s+t)^{|\pi(\mathrm{X})|},$$

where S in the first sum is the union of the blocks of π' and $T = I \setminus S$. Since $|\pi(\mathrm{X})| = \dim(\mathrm{X})$, the result follows. □

The **intrinsic series** of parameter t is the series $\nu \in \mathscr{S}(\Bbbk\Sigma)$ consisting intrinsic elements of the right parameter; that is, such that $\nu_\ell \in \Sigma_{\mathscr{A}}[I,\ell]$ is the in the intrinsic element of parameter t for $\mathcal{A}_\ell$.

In view of Proposition 4.4.1 and of the multiplicativity of intrinsic elements (Theorem 3.7.10), one might wonder if the Cauchy of two intrinsic series ν and ν' of parameters s and t, respectively, is the intrinsic series of parameter $(s+t)$. This is not the case, as it can be seen by computing the coefficient $(\nu * \nu')^F_{[4]}$ where F is a *long edge* as in Example 3.7.7, in the case where each $\mathcal{A}_d$ is the braid arrangement in $\mathbb{R}^d$.

Nevertheless, the intrinsic series satisfies the following interesting property.

Proposition 4.4.2. *The intrinsic series of parameter t is group-like.*

Proof. Using the identification

$$\Bbbk\Sigma[\mathcal{A}_{\ell|_S}] \otimes \Bbbk\Sigma[\mathcal{A}_{\ell|_T}] = \Bbbk\Sigma[\mathcal{A}_{\ell|_S} \times \mathcal{A}_{\ell|_T}],$$

we verify that both $\Delta^{\ell}_{S,T}(\nu_\ell)$ and $\nu_{\ell|_S} \otimes \nu_{\ell|_T}$ are the intrinsic element for $\mathcal{A}_{\ell|_S} \times \mathcal{A}_{\ell|_T}$ of parameter t. The first statement follows from Proposition 3.7.11, since $\Delta^{\ell}_{S,T}(\nu_\ell) = f(\nu_\ell)$, where $f : \Sigma[\mathcal{A}_\ell] \to \Sigma[\mathcal{A}_{\ell|_S} \times \mathcal{A}_{\ell|_T}]$ is the morphism (3.2.4).

On the other hand

$$\begin{aligned}\nu_{\ell|_S} \otimes \nu_{\ell|_T} &= \Big(\sum_{F \in \Sigma[\mathcal{A}_{\ell|_S}]} v(\mathrm{relint}(F))\mathtt{H}_F \Big) \otimes \Big(\sum_{F' \in \Sigma[\mathcal{A}_{\ell|_T}]} v(\mathrm{relint}(F'))\mathtt{H}_{F'} \Big) \\ &= \sum_{(F,F') \in \Sigma[\mathcal{A}_{\ell|_S}] \times \Sigma[\mathcal{A}_{\ell|_T}]} v(\mathrm{relint}(F)) v(\mathrm{relint}(F'))\mathtt{H}_F \otimes \mathtt{H}_{F'} \\ &= \sum_{G \in \Sigma[\mathcal{A}_{\ell|_S} \times \mathcal{A}_{\ell|_T}]} v(\mathrm{relint}(G))\mathtt{H}_G\end{aligned}$$

is also the intrinsic element for $\mathcal{A}_{\ell|_S} \times \mathcal{A}_{\ell|_T}$ of parameter t. In the last step we used Proposition 3.7.2, the fact that $\mathrm{rec}(F \times F') = \mathrm{rec}(F) \times \mathrm{rec}(F')$, and (3.7.5). □

An application of (2.3.5) and Proposition 4.3.3 yields the following.

Corollary 4.4.3. *The intrinsic series of parameter t is invertible, and its inverse is the intrinsic series of parameter $-t$.*

4.5 Characters and polynomial invariants

We consider two bivariate polynomial invariants associated to a hyperplane arrangement. First, the **Whitney polynomial** (or Möbius polynomial) of an arrangement $\mathcal{A}$ is defined by

$$W(\mathcal{A},t,q) = \sum_{\mathrm{X}\leq\mathrm{Y}} \mu(\mathrm{X},\mathrm{Y}) t^{\dim(\mathrm{X})} q^{\operatorname{codim}(\mathrm{Y})}.$$

On the other hand, the **coboundary polynomial** of $\mathcal{A}$ is

$$\overline{\chi}(\mathcal{A},t,q) = \sum_{\text{central } \mathcal{B}\subseteq\mathcal{A}} t^{d-\operatorname{rank}(\mathcal{B})}(q-1)^{|\mathcal{B}|},$$

where d is the dimension of the ambient space of $\mathcal{A}$. Observe that $d - \operatorname{rank}(\mathcal{B})$ is the dimension of the flat of $\mathcal{A}$ obtained by intersecting the hyperplanes in $\mathcal{B}$. A result by Ardila [12, Theorem 3.8] gives a formula for $\overline{\chi}(\mathcal{A},t,q)$ in terms of the Möbius function of $\mathcal{L}[\mathcal{A}]$:

$$\overline{\chi}(\mathcal{A},t,q) = \sum_{\mathrm{X},\mathrm{Y}} \mu(\mathrm{X},\mathrm{Y}) t^{\dim(\mathrm{X})} q^{h(\mathrm{Y})},$$

where $h(\mathrm{Y})$ denotes the number of hyperplanes of $\mathcal{A}$ containing Y.

Fix a parameter $q \in \Bbbk$. We define a character ζ of $\Bbbk\Sigma_{\mathscr{A}}$ by

$$\zeta_\ell(\mathtt{H}_F) = q^{\operatorname{codim}(F)}.$$

Since $\operatorname{codim}(F \cdot G) = \operatorname{codim}(\mathrm{s}(F) \oplus \mathrm{s}(G)) = \operatorname{codim}(\mathrm{s}(F)) + \operatorname{codim}(\mathrm{s}(G))$, ζ is indeed a character. Let $w \in \Bbbk\Sigma_{\mathscr{A}}[d]$ be a characteristic element of parameter t. Grouping faces by their support, we get

$$\zeta_{[d]}(w) = \sum_{F\in\Sigma[\mathcal{A}_d]} w^F q^{\operatorname{codim}(F)} = \sum_{\mathrm{Y}\in\mathcal{L}[\mathcal{A}_d]} \chi(\mathcal{A}_d^{\mathrm{Y}}, t) q^{\operatorname{codim}(\mathrm{Y})} = W(\mathcal{A}_d, t, q). \tag{4.5.1}$$

Theorem 4.5.1. *Let $\mathscr{A} = (\mathcal{A}_1, \mathcal{A}_2, \dots)$ be an exponential sequence of arrangements. Then,*

$$1 + \sum_{d\geq 1} W(\mathcal{A}_d, t, q)\frac{x^d}{d!} = \left(1 + \sum_{d\geq1}\Big(\sum_{k=1}^{d}(-1)^k f_k(\mathcal{A}_d) q^{d-k}\Big)\frac{x^d}{d!}\right)^{-t}, \tag{4.5.2}$$

where $f_k(\mathcal{A}_d)$ denotes the number of k-dimensional faces of $\mathcal{A}_d$.

Proof. Let $\Psi : \mathscr{S}(\Bbbk\Sigma_{\mathscr{A}}) \to \Bbbk[\![x]\!]$ be the algebra morphism associated to the character ζ, as in (2.3.6). If $\omega \in \mathscr{S}(\Bbbk\Sigma_{\mathscr{A}})$ is a characteristic series of parameter t, then (4.5.1) shows that

$$\Psi(\omega) = 1 + \sum_{d\geq 1} W(\mathcal{A}_d, t, q)\frac{x^d}{d!}.$$

In particular, if ω is a characteristic series of parameter -1, Zaslavsky's formulas (see Section 3.4.3) imply

$$\Psi(\omega) = 1+\sum_{d\geq 1}\Big(\sum_{\mathrm{Y}\in\mathcal{L}[\mathcal{A}_d]} \chi(\mathcal{A}_d^{\mathrm{Y}}, -1)q^{\mathrm{codim(Y)}}\Big)\frac{x^d}{d!} = 1+\sum_{d\geq 1}\Big(\sum_{k=0}^{d}(-1)^k f_k(\mathcal{A}_d)q^{d-k}\Big)\frac{x^d}{d!}.$$

The result follows from Proposition 4.4.1 and the fact that Ψ is a morphism of algebras. □

Plugging in $q = 0$ in (4.5.2), we recover the following result by Stanley.

Corollary 4.5.2 ([69, Theorem 5.17])**.** *Let $\mathscr{A} = (\mathcal{A}_1, \mathcal{A}_2, \dots)$ be an exponential sequence of arrangements. Then,*

$$\sum_{d\geq 0} \chi(\mathcal{A}_d, t)\frac{x^d}{d!} = \Big(\sum_{d\geq 0}(-1)^d c^{\mathcal{A}_d}\frac{x^d}{d!}\Big)^{-t},$$

where $c^{\mathcal{A}_d}$ is the number of chambers of the arrangement $\mathcal{A}_d$.

Example 4.5.3. Consider the simplest exponential sequence: for each d, $\mathcal{A}_d$ is the braid arrangement in $\mathbb{R}^d$. Observe that the arrangement $\mathcal{A}_d^{\mathrm{Y}}$ has zero relatively bounded chambers, unless $\mathrm{Y} = \perp$ is the minimum flat. Thus, $W(\mathcal{A}_d, 1, q) = q^{d-1}$ and plugging in $t = 1$ in Theorem 4.5.1 yields

$$1 + \sum_{d\geq 1} q^{d-1}\frac{x^d}{d!} = \bigg(\sum_{d\geq 0}\Big(\sum_{k=0}^{d}(-1)^k f_k(\mathcal{A}_d)q^{d-k}\Big)\frac{x^d}{d!}\bigg)^{-1}.$$

After a reparametrization, we obtain the following generating function for the *ordered Stirling numbers* ($k!s(d, k) = f_k(\mathcal{A}_d)$ is the number of compositions of a set of size d into k parts):

$$\sum_{d,k\geq 0} f_k(\mathcal{A}_d) q^k \frac{x^d}{d!} = \frac{1}{1 - qe^x + q}.$$

We similarly obtain the generating function for the Whitney polynomial of the braid arrangement

$$1 + \sum_{d\geq 1} W(\mathcal{A}_d, t, q)\frac{x^d}{d!} = \Big(1 + \sum_{d\geq 1} q^{d-1}\frac{x^d}{d!}\Big)^t = \big(1 + \tfrac{1}{q}(e^{qx} - 1)\big)^t$$

Let us now consider a different character. Fix a parameter $q \in \Bbbk$, and define the character ϕ by

$$\phi_\ell(\mathtt{H}_F) = q^{h(F)},$$

where $h(F) = h(\mathrm{s}(F))$ is the number of hyperplanes of $\mathcal{A}_\ell$ containing F. Take a decomposition $I = S \sqcup T$, a linear order $\ell \in \mathsf{L}[I]$ and faces $F \in \Sigma_{\mathscr{A}}[S, \ell|_S]$, $G \in \Sigma_{\mathscr{A}}[T, \ell|_T]$. It readily follows from (4.3.2) that a hyperplane of $\mathcal{A}_\ell$ containing $F \cdot G$ corresponds to either a hyperplane of $\mathcal{A}_{\ell|_S}$ containing F or a hyperplane of $\mathcal{A}_{\ell|_T}$ containing G, and thus $h(F \cdot G) = h(F) + h(G)$. Therefore, ϕ is a character of $\Bbbk\Sigma_{\mathscr{A}}$.

If $w \in \Bbbk\Sigma_{\mathscr{A}}[d]$ is a characteristic element of parameter t, then

$$\zeta_{[d]}(w) = \sum_{F\in\Sigma[\mathcal{A}_d]} w^F q^{h(F)} = \sum_{\mathrm{Y}\in\mathcal{L}[\mathcal{A}_d]} \chi(\mathcal{A}_d^{\mathrm{Y}}, t) q^{h(\mathrm{Y})} = \overline{\chi}(\mathcal{A}_d, t, q). \tag{4.5.3}$$

Following the ideas in the proof of Theorem 4.5.1, we deduce the following.

Theorem 4.5.4 ([12, Theorem 5.2])**.** *Let $\mathscr{A} = (\mathcal{A}_1, \mathcal{A}_2, \dots)$ be an exponential sequence of arrangements. Then,*

$$\sum_{d\geq 0} \overline{\chi}(\mathcal{A}_d, t, q)\frac{x^d}{d!} = \Big(\sum_{d\geq 0} \overline{\chi}(\mathcal{A}_d, 1, q)\frac{x^d}{d!}\Big)^t.$$

CHAPTER 5

THE MODULE OF GENERALIZED ZONOTOPES MODULO MCMULLEN RELATIONS

Generalized permutahedra serve as a geometric model for many *classical* (type A) combinatorial objects and have been extensively studied in resent years. Notably, Aguiar and Ardila [1] endowed generalized permutahedra with the structure of a Hopf monoid GP in the category of species and, in so doing, gave a unified framework to study similar algebraic structures over many different families of combinatorial objects. At the same time, McMullen's polytope algebra [55] offers a different algebraic perspective to study generalized permutahedra.

One of the goals of the present chapter is to investigate the compatibility between both structures. To achieve this goal, we take a more general approach and study deformations of an arbitrary zonotope. We then specialize our results to deformations Coxeter permutahedra of type A and type B, revealing remarkable connections with (type B) Eulerian polynomials and statistics over (signed) permutations. The results in type B allow us to find a family of generalized permutahedra that generates all other deformations via signed Minkowski sums, thus solving a question posed by Ardila, Castillo, Eur, and Postnikov [14]. Most of the results presented in this chapter appear in [19].

Let V be a finite-dimensional real vector space. The polytope algebra $\Pi(V)$ is generated by the classes $[P]$ of polytopes $P \subseteq V$. These classes satisfy the following *valuation* and *translation invariance* relations: $[P \cup Q] + [P \cap Q] = [P] + [Q]$ and $[P + \{t\}] = [P]$, whenever $P \cup Q$ is a polytope and for every $t \in V$. The product structure of $\Pi(V)$ is given by the Minkowski sum of polytopes: $[P] \cdot [Q] = [P + Q]$.

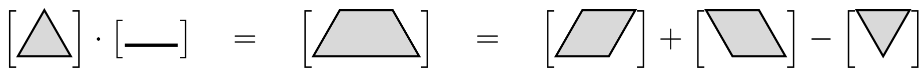


Figure 5.0.1: Different expressions for the class of the trapezoid above in the polytope algebra $\Pi(\mathbb{R}^2)$.

For a fixed polytope $P \subseteq V$, $\Pi(P)$ denotes the subalgebra of $\Pi(V)$ generated by classes of *deformations* of P; see McMullen [56]. We are particularly interested in the case where P is a zonotope corresponding to a linear hyperplane arrangement $\mathcal{A}$. In this case, let $\mathbb{R}\Sigma[\mathcal{A}]$ denote the Tits algebra of $\mathcal{A}$, see Section 5.2.2. It is linearly generated by the elements $\mathtt{H}_F$ as F runs through the faces of the arrangement. The following is a central result of this chapter.

Theorem 5.2.3. *Let P and $\mathcal{A}$ be as above. The algebra $\Pi(P)$ is a right $\mathbb{R}\Sigma[\mathcal{A}]$-module under the following action:*

$$[Q] \cdot \mathtt{H}_F := [Q_{\boldsymbol{v}}],$$

where $\boldsymbol{v} \in \operatorname{relint}(F)$ and $Q_{\boldsymbol{v}}$ denotes the face of Q maximal in the direction *of $\boldsymbol{v}$. Moreover, the action of $\mathtt{H}_F$ is an endomorphism of graded algebras.*

In the particular case of the permutahedron and the braid arrangement, the compatibility between the algebra structure and the action of the Tits algebra is related to the Hopf monoid structure of Aguiar and Ardila. Similar results for the Hopf monoid of extended generalized permutahedra were independently obtained by Ardila and Sanchez in [15].

Theorem 5.5.3. *The species Π defined by $\Pi[I] = \Pi(\pi_I)$, where $\pi_I \subseteq \mathbb{R}^I$ is the standard permutahedron, is a Hopf monoid quotient of* GP.

We embark on further understanding the module structure of $\Pi(Z)$ when Z is a zonotope corresponding to a hyperplane arrangement $\mathcal{A}$. The decomposition of

$\Pi(Z) = \bigoplus_r \Xi_r(Z)$ into its graded components will play an essential role here. McMullen characterized $\Xi_r(Z)$ as the eigenspace of the dilation morphism δ_λ, defined by $\delta_\lambda[P] = [\lambda P]$, with eigenvalue λ^r for any positive $\lambda \neq 1$. Theorem 5.2.3 then implies that each graded component is a $\mathbb{R}\Sigma[\mathcal{A}]$-submodule.

The simple modules over $\mathbb{R}\Sigma[\mathcal{A}]$ are one-dimensional and indexed by the flats of the arrangement [8, Chapter 9]. Given a module M over $\mathbb{R}\Sigma[\mathcal{A}]$, the number of copies of the simple module associated with the flat X that appear as a composition factor of M is $\eta_{\mathrm{X}}(M)$. We propose that studying these algebraic invariants for the modules $\Xi_r(Z)$ yields important geometric and combinatorial information of the deformations of Z. In Sections 5.3 and 5.4, we do this for the algebra of deformations of the permutahedron and the type B permutahedron, respectively. The main results in these sections relate the invariants η_{X} with statistics over (signed) permutations:

Theorem 5.3.1. *For any flat* X *of the braid arrangement in* $\mathbb{R}^d$ *and* $r = 0, 1, \ldots, d-1$,

$$\eta_{\mathrm{X}}(\Xi_r(\pi_d)) = \left|\{\sigma \in \mathfrak{S}_d \,:\, \mathrm{s}(\sigma) = \mathrm{X},\ \mathrm{exc}(\sigma) = r\}\right|.$$

Theorem 5.4.1. *For any flat* X *of type B Coxeter arrangement in* $\mathbb{R}^d$ *and* $r = 0, 1, \ldots, d$,

$$\eta_{\mathrm{X}}(\Xi_r(\pi_d^B)) = \left|\{\sigma \in \mathfrak{B}_d \,:\, \mathrm{s}(\sigma) = \mathrm{X},\ \mathrm{exc}_B(\sigma) = r\}\right|.$$

This surprising relation arises from the remarkable results of McMullen and Brenti that we explain now. McMullen [56] proved that when P is a simple polytope, just like the (type B) permutahedron, the dimension of the graded components $\Xi_r(P)$ are determined by the h-numbers of P. On the other hand, building on top of work by Björner [27], Brenti proved that the h-polynomial of the Coxeter permutahedron associated with any Coxeter group is the corresponding Eulerian polynomial [30, Theorem 2.3].

Work of Postnikov [61], and of Ardila, Benedetti, and Doker [13] show that any generalized permutahedron in $\mathbb{R}^d$ can be written as the *signed Minkowski sum* of the faces of the standard simplex $\Delta_{[d]} = \mathrm{Conv}\{\boldsymbol{e}_j : j \in [d]\}$. As a consequence of Theorem 5.4.1, we show that any such set of generators for type B generalized permutahedra contains at least 2^{d-1} full dimensional polytopes (see Proposition 5.4.8). We manage to obtain a family of generators that attains this minimum.

Theorem 5.4.9. *Every type B generalized permutahedron in* $\mathbb{R}^d$ *can be written uniquely as a signed Minkowski sum of the simplices* Δ_S *and* Δ_S^0 *for* special involution-exclusive *subsets* $S \subseteq [\pm d]$.

Unlike the standard simplex and its faces in the type A case, this collection of generators is not invariant under the action of the corresponding Coxeter group. However, using the set of generators in the previous theorem, we are able to find a different collection of generators that is invariant under the action of $\mathfrak{B}_d$, at the cost of including twice as many full-dimensional polytopes.

Theorem 5.4.12. *Every type B generalized permutahedron in* $\mathbb{R}^d$ *can be written uniquely as a signed Minkowski sum of the simplices* Δ_S^0 *for involution-exclusive subsets* S.

This chapter is organized as follows. We review McMullen's construction in Section 5.1. In Section 5.2.2, we recall the definition of the Tits algebra of a hyperplane arrangement and some of its representation theoretic properties. Characteristic elements and Eulerian idempotents are also reviewed in this section. In Section 5.2, we begin the study of the polytope algebra of deformations of a zonotope as a module over the Tits algebra of the corresponding hyperplane arrangement, which is the central construction for this chapter. We study the polytope algebra of generalized permutahedra in Section 5.3. In particular, we provide a conjectural basis of simultaneous eigenvectors for the action of the *Adams element* and positive dilations on the module $\Pi(\pi_d)$. Section 5.4 contains the analogous results for type B. We also give explicit sets of *signed Minkowski generators* for type B generalized permutahedra; one shows that the lower bound on the number of full-dimensional polytopes in such a collection is tight, and the other is invariant under the action of $\mathfrak{B}_d$. In Section 5.5 we prove that the valuation and translation invariance relations are compatible with the Hopf monoid structure of GP.

5.1 The polytope algebra

We briefly review the construction of McMullen's polytope algebra [55] and its main properties. A hands on introduction to this topic can be found in the survey [33]. The subalgebra relative to a fixed polytope [56] is studied at the end of this section.

5.1.1 Definition and structure theorem

As an abelian group, the **polytope algebra** $\Pi(V)$ is generated by elements $[P]$, one for each polytope $P \subseteq V$. These generators satisfy the relations

$$[P \cup Q] + [P \cap Q] = [P] + [Q], \tag{5.1.1}$$

whenever P, Q and $P \cup Q$ are polytopes; and

$$[P + \{t\}] = [P], \tag{5.1.2}$$

for any polytope P and translation vector $t \in V$. These relations are referred as the **valuation property** and the **translation invariance property**, respectively. The group $\Pi(V)$ is endowed with a commutative product defined on generators by means of the Minkowski sum

$$[P] \cdot [Q] := [P + Q].$$

It readily follows from (5.1.2) that the class of a point $1 := [\{\mathbf{0}\}]$ is the unit of $\Pi(V)$.

For each scalar $\lambda \in \mathbb{R}$, the **dilation morphism** $\delta_\lambda : \Pi(V) \to \Pi(V)$ is defined on generators by $\delta_\lambda[P] = [\lambda P]$. Recall that for any subset $S \subseteq V$ and $\lambda \in \mathbb{R}$, $\lambda S := \{\lambda \boldsymbol{v} : \boldsymbol{v} \in S\}$. One can easily verify that δ_λ preserves the valuation (5.1.1) and translation invariance (5.1.2) relations, and that it defines a morphism of rings. The main structural result for the polytope algebra is the following.

Theorem 5.1.1 ([55, Theorem 1]). *The commutative ring $\Pi(V)$ is almost a graded $\mathbb{R}$-algebra, in the following sense:*

i. as an abelian group, $\Pi(V)$ admits a direct sum decomposition

$$\Pi(V) = \Xi_0(V) \oplus \Xi_1(V) \oplus \cdots \oplus \Xi_d(V);$$

ii. under multiplication, $\Xi_r(V) \cdot \Xi_s(V) = \Xi_{r+s}(V)$, with $\Xi_r(V) = 0$ for $r > d$;

iii. $\Xi_0(V) \cong \mathbb{Z}$, and for $r = 1, \ldots, d$, $\Xi_r(V)$ is a real vector space;

iv. the product of elements in $\bigoplus_{r \geq 1} \Xi_r(V)$ is bilinear.

v. the dilations δ_λ are algebra endomorphisms, and for $r = 0, 1, \ldots, d$, if $x \in \Xi_r(V)$ and $\lambda \geq 0$, then $\delta_\lambda x = \lambda^r x$.

We discuss the definition of the graded components $\Xi_r(V)$ below. The component $\Xi_0(V)$ of degree 0 is simply the subring of $\Pi(V)$ generated by 1, and thus $\Xi_0(V) \cong \mathbb{Z}$. Let Z_1 be the subgroup of $\Pi(V)$ generated by all elements of the form $[P]-1$. Observe that $\delta_0[P] = 1$ for every polytope P, so $Z_1 = \ker \delta_0$ is an ideal. As an abelian group, $\Pi(V)$ has a direct sum decomposition $\Pi(V) = \Xi_0(V) \oplus Z_1$. Moreover, Z_1 is a nil ideal, since for any k-dimensional polytope P,

$$([P]-1)^r = 0 \quad \text{for } r > k.$$

This is [55, Lemma 13]. McMullen also shows that Z_1 has the structure of a vector space (first over $\mathbb{Q}$ and then over $\mathbb{R}$). Therefore, we can define inverse maps

$$1 + Z_1 \underset{\exp}{\overset{\log}{\rightleftarrows}} Z_1$$

by means of their usual power series. In particular, we can define the *log-class* of a k-dimensional polytope P by

$$\log[P] := \log(1 + ([P]-1)) = \sum_{r=1}^{k} \frac{(-1)^{r-1}}{r}([P]-1)^r. \tag{5.1.3}$$

Using the exponential map, we recover $[P]$ from $\log[P]$:

$$[P] = \sum_{r=0}^{k} \frac{1}{r!}(\log[P])^r. \tag{5.1.4}$$

The log and exp maps satisfy the standard properties of logarithms and exponentials. In particular,

$$\log[P+Q] = \log([P] \cdot [Q]) = \log[P] + \log[Q]. \tag{5.1.5}$$

Example 5.1.2. Let $v_1, \dots, v_k \in V$ be nonzero vectors and let $\mathfrak{l}_i$ denote the line segment $\operatorname{Conv}\{\mathbf{0}, v_i\}$. Then, $\log[\mathfrak{l}_i] = [\mathfrak{l}_i] - 1$. We will see that the product $\prod_{i=1}^{k} \log[\mathfrak{l}_i] \in \Xi_k(V)$ is nonzero if and only if the collection $\{v_1, \dots, v_k\}$ is linearly independent. See Section 5.1.1 for an example with $k = 3$.

Consider the polytope $Z = \mathfrak{l}_1 + \mathfrak{l}_1 + \cdots + \mathfrak{l}_k$, a Minkowski sum of segments. Using the logarithm property (5.1.5), we get $(\log[Z])^k = (\sum_{i=1}^k \log[\mathfrak{l}_i])^k = k! \prod_{i=1}^k \log[\mathfrak{l}_i]$. The last equality follows since $k! \prod \log[\mathfrak{l}_i]$ is the only square-free term in the expansion of $(\sum \log[\mathfrak{l}_i])^k$, and $(\log[\mathfrak{l}])^2 = ([\mathfrak{l}] - 1)^2 = 0$ for any line segment $\mathfrak{l}$. Finally, $(\log[Z])^k \neq 0$ if and only if $k \leq \dim(Z)$, and Z being the sum of k line segments has dimension at most k, with equality precisely when the vectors $v_1, \ldots, v_k$ are linearly independent.

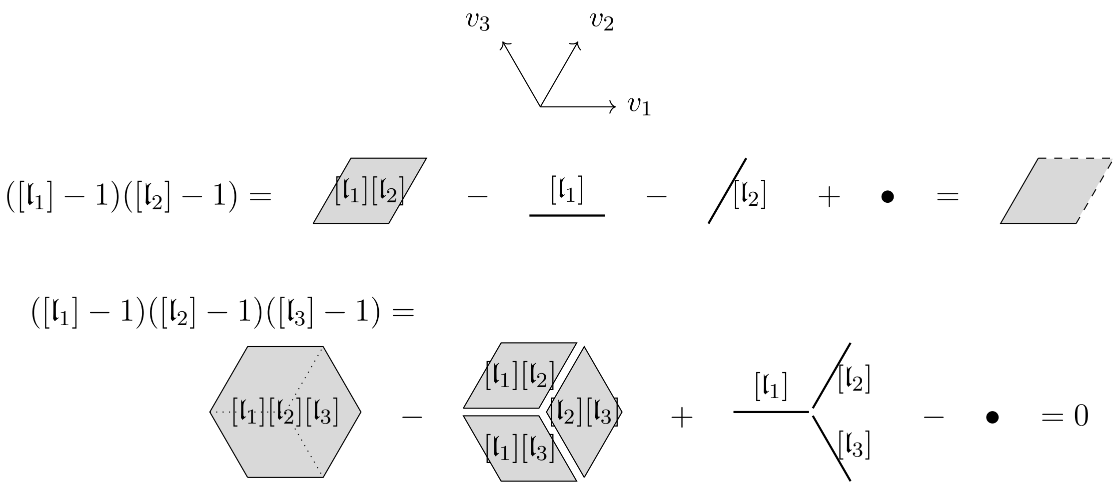


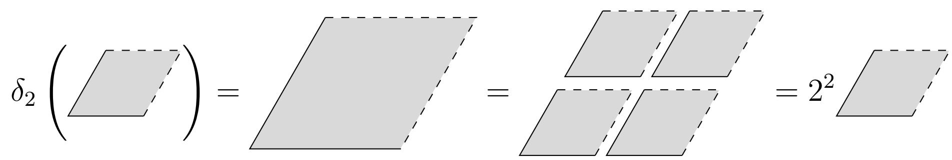
Figure 5.1.1: Vectors v_1, v_2, v_3 lie in the same plane. The vectors v_1, v_2 are linearly independent and $\log[\mathfrak{l}_1] \log[\mathfrak{l}_2]$ represents *the class of a half-open parallelogram*. In contrast, the product $\log[\mathfrak{l}_1] \log[\mathfrak{l}_2] \log[\mathfrak{l}_3]$ is zero.

For $r \geq 1$ let $\Xi_r(V)$ be the subgroup (or subspace) of $\Pi(V)$ generated by elements of the form $(\log[P])^r$. The following result implies that the sum $\Pi(V) = \Xi_0(V) + \Xi_1(V) + \cdots + \Xi_d(V)$ is direct, and characterizes each graded component as the space of eigenvectors for the positive dilations δ_λ.

Lemma 5.1.3 ([55, Lemma 20]). *Let $x \in \Pi(V)$ and $\lambda > 0$, with $\lambda \neq 1$. Then,*

$$x \in \Xi_r(V) \qquad \text{if and only if} \qquad \delta_\lambda x = \lambda^r x. \tag{5.1.6}$$

It is clear by the definition of the graded components $\Xi_r(V)$ that the class of a half-open parallelogram in Section 5.1.1 is in $\Xi_2(V)$. We can also verify this using (5.1.6) with $\lambda = 2$:



Convention 5.1.4. As in later work of McMullen [56, 57], we replace $\Xi_0(V) \cong \mathbb{Z}$ with the tensor product $\Xi_0(V)_\mathbb{R} := \mathbb{R} \otimes \Xi_0(V) \cong \mathbb{R}$ to get a genuine graded $\mathbb{R}$-algebra $\Pi(V)_\mathbb{R} := \Xi_0(V)_\mathbb{R} \oplus Z_1$. To simplify notation, we drop the subscript $\mathbb{R}$ and sometimes wirte Ξ_r and Π instead of $\Xi_r(V)$ and $\Pi(V)_\mathbb{R}$.

For an arbitrary vector $\boldsymbol{v} \in V$, we can define a **maximization operator** $P \mapsto P_{\boldsymbol{v}}$ on the space of all polytopes $P \subseteq V$. The next result shows that it induces a well-defined map on $\Pi(V)$.

Theorem 5.1.5 ([55, Theorem 7])**.** *The map $P \mapsto P_{\boldsymbol{v}}$ induces an endomorphism $x \mapsto x_{\boldsymbol{v}}$ on $\Pi(V)$, defined on generators by*

$$[P] \mapsto [P]_{\boldsymbol{v}} := [P_{\boldsymbol{v}}].$$

This endomorphism commutes with nonnegative dilations.

In particular, the morphism $x \mapsto x_{\boldsymbol{v}}$ restricts to each graded component Ξ_r.

Lastly, the **Euler map** $x \mapsto x^*$ is the linear operator defined on generators by

$$[P]^* = \sum_{Q \leq P} (-1)^{\dim(Q)} [Q]. \tag{5.1.7}$$

The sum runs over all nonempty faces Q of P. Up to a sign, the element $[P]^*$ corresponds to *the class of the interior* of P.

Theorem 5.1.6 ([55, Theorem 2])**.** *The Euler map is an involutory automorphism of* $\Pi(V)$*. Moreover, for* $x \in \Xi_r(V)$ *and* $\lambda < 0$*,*

$$\delta_\lambda x = \lambda^r x^*.$$

5.1.2 Subalgebra relative to a fixed polytope

Fix a polytope $P \subseteq V$. The **subalgebra relative to** P, denoted $\Pi(P)$, is the subalgebra of $\Pi(V)$ generated by classes $[Q]$ of deformations Q of P. It is worth pointing out that if Q, Q' are deformations of P such that $Q \cup Q'$ is a polytope, then both $Q \cup Q'$ and $Q \cap Q'$ are deformations of P. This follows since, in this case, $Q \cup Q' + Q \cap Q' = Q + Q'$ and Minkowski summands of a deformation of P are again deformations of P. Thus, the valuation property (5.1.1) is not introducing classes of new polytopes to $\Pi(P)$.

Remark 5.1.7. McMullen [56] originally defined $\Pi(P)$ as the subalgebra generated by the classes of Minkowski summands of P. The following result of Shephard [47, Section 15.2.7] implies that both definitions are equivalent: a polytope Q is a deformation of P if and only if for small enough $\lambda > 0$, λQ is a Minkowski summand of P.

The relations between $[Q]$ and $\log[Q]$ in (5.1.3) and (5.1.4) show that $\Pi(P)$ is generated by homogeneous elements. Thus, the grading of $\Pi(V)$ induces a grading of $\Pi(P)$. We let $\Xi_r(P) = \Pi(P) \cap \Xi_r(V)$ denote the component of $\Pi(P)$ in degree r. Unlike the full algebra $\Pi(V)$, the subalgebra $\Pi(P)$ has finite dimension. McMullen described the dimension of the graded components of $\Pi(P)$ when P is a simple polytope.

Theorem 5.1.8 ([56, Theorem 6.1])**.** *Let* P *be a* d*-dimensional simple polytope.*

Then,

$$\dim_{\mathbb{R}}(\Xi_r(P)) = h_r(P)$$

for $r = 0, 1, \ldots, d$.

Let F be a face of P and $\boldsymbol{v} \in \operatorname{relint}(N_F P)$. The maximization operator $x \mapsto x_{\boldsymbol{v}}$ defines a morphism of graded algebras

$$\psi_F : \Pi(P) \to \Pi(F) \tag{5.1.8}$$

that only depends on the face F and not on the particular choice of $\boldsymbol{v} \in \operatorname{relint}(N_F P)$.

First observe that this map is well defined; that is, $[Q_{\boldsymbol{v}}] \in \Pi(F)$ for every generator $[Q]$ of $\Pi(P)$. Indeed, if Q is a summand of P, say $P = Q + Q'$, then $F = P_{\boldsymbol{v}} = Q_{\boldsymbol{v}} + Q'_{\boldsymbol{v}}$, so $Q_{\boldsymbol{v}}$ is a Minkowski summand of F. Moreover, since the normal fan of P refines that of Q, then $Q_{\boldsymbol{w}} = Q_{\boldsymbol{v}}$ for any other $\boldsymbol{w} \in \operatorname{relint}(N_F P)$. Therefore the morphism (5.1.8) only depends on F.

Theorem 5.1.9 ([56, Theorem 2.4])**.** *Let P be a simple polytope and F a face of P. Then, the morphism ψ_F is surjective.*

5.2 The polytope algebra as a module

Fix a hyperplane arrangement $\mathcal{A}$ in V. Take a normal vector $\boldsymbol{v}_{\mathrm{H}}$ for each hyperplane $\mathrm{H} \in \mathcal{A}$, and consider the zonotope (Minkowski sum of segments):

$$Z = \sum_{\mathrm{H}} \operatorname{Conv}\{\mathbf{0}, \boldsymbol{v}_{\mathrm{H}}\}. \tag{5.2.1}$$

Its normal fan Σ_Z coincides with the collection of faces $\Sigma[\mathcal{A}]$ of the arrangement $\mathcal{A}$. We say that a polytope Q is a **generalized zonotope** of $\mathcal{A}$ it is a deformation of Z. In this section, we will work with the Tits algebra with **real** coefficients.

5.2.1 The module structure

We now consider the algebra $\Pi(Z)$ introduced in Section 5.1.2. It is generated by the classes of generalized zonotopes of $\mathcal{A}$. It only depends on the arrangement $\mathcal{A}$ and not on the particular choice of normal vectors v_{H}. We start with a simple yet interesting result. Recall the morphism $\psi_F : \Pi(Z) \to \Pi(F)$ defined for every face $F \leq Z$ in (5.1.8), it sends the class $[Q]$ of a generalized zonotope of $\mathcal{A}$ to $[Q_{\boldsymbol{v}}]$ where $\boldsymbol{v} \in V$ is any vector such that $F = Z_{\boldsymbol{v}}$.

Proposition 5.2.1. *Let Z be a zonotope, and F a face of Z. Then, the morphism ψ_F is surjective.*

Proof. Let F be a face of Z, $F \in \Sigma[\mathcal{A}]$ be its normal cone, and $\mathrm{X} = \mathrm{s}(F)$ be the flat orthogonal to F. It follows from (5.2.1) that $F = Z_F$ is a translate of

$$Z_{\mathrm{X}} := \sum_{\mathrm{H}\,:\,\mathrm{H} \supseteq \mathrm{X}} \mathrm{Conv}\{\mathbf{0}, v_{\mathrm{H}}\}.$$

In particular, F is a Minkowski summand of Z. Being a Minkowski summand is a transitive relation. Hence, any generator $[Q]$ of $\Pi(F)$ is also in $\Pi(Z)$. That is, $\Pi(F)$ is a subalgebra of $\Pi(Z)$. Moreover, if $\boldsymbol{v} \in \mathrm{relint}(F)$ and Q is a Minkowski summand of F, then $Q_{\boldsymbol{v}} = Q$. Therefore, the composition $\Pi(F) \hookrightarrow \Pi(Z) \xrightarrow{\psi_F} \Pi(F)$ is the identity map. Consequently, the morphism ψ_F is surjective. □

Remark 5.2.2. Compare with Theorem 5.1.9 and note that we do not assume the zonotope Z to be simple. For an arbitrary polytope P and a face F of P, there is no natural morphism $\Pi(F) \to \Pi(P)$, unlike in the previous case. This is a particular property of zonotopes. Indeed, a polytope P is a zonotope if and only if every face $F \leq P$ is a Minkowski summand of P, see [28, Proposition 2.2.14] for a proof.

Let F be a face of $\mathcal{A}$ and $F = Z_F$ the corresponding face of Z. We define right multiplication by the basis element $\mathtt{H}_F \in \mathbb{R}\Sigma[\mathcal{A}]$ on $\Pi(Z)$ as the projection $\psi_F : \Pi(Z) \to \Pi(F) \subseteq \Pi(Z)$.

Theorem 5.2.3. *The algebra $\Pi(Z)$ is a right $\mathbb{R}\Sigma[\mathcal{A}]$-module under the action above. Explicitly, for a generator $[Q]$ of $\Pi(Z)$ and a basis element $\mathtt{H}_F$ of $\mathbb{R}\Sigma[\mathcal{A}]$,*

$$[Q] \cdot \mathtt{H}_F := [Q_{\boldsymbol{v}}], \qquad \text{where } \boldsymbol{v} \in \operatorname{relint}(F).$$

Moreover, each graded component $\Xi_r(Z)$ is a $\mathbb{R}\Sigma[\mathcal{A}]$-submodule and the action of basis elements $\{\mathtt{H}_F\}_F$ on $\Pi(Z)$ is by (graded) algebra endomorphisms.

Proof. The zero vector belongs to the central face O, so the action is clearly unital. Associativity follows from the following fact about polytopes [47, Section 3.1.5]. If $Q \subseteq V$ is a polytope and $\boldsymbol{v}, \boldsymbol{w} \in V$, then $(Q_{\boldsymbol{v}})_{\boldsymbol{w}} = Q_{\boldsymbol{v}+\lambda\boldsymbol{w}}$ for any small enough $\lambda > 0$. Similarly, the definition of the Tits product is such that if $\boldsymbol{v} \in \operatorname{relint}(F)$ and $\boldsymbol{w} \in \operatorname{relint}(G)$, then $\boldsymbol{v} + \lambda\boldsymbol{w} \in \operatorname{relint}(FG)$ for any small enough $\lambda > 0$. Hence,

$$([Q] \cdot \mathtt{H}_F) \cdot \mathtt{H}_G = [(Q_{\boldsymbol{v}})_{\boldsymbol{w}}] = [Q_{\boldsymbol{v}+\lambda\boldsymbol{w}}] = [Q] \cdot \mathtt{H}_{FG}.$$

It follows that this product gives $\Pi(Z)$ the structure of a right $\mathbb{R}\Sigma[\mathcal{A}]$-module.

The second statement follows directly from Theorem 5.1.5 and the characterization of the graded components Ξ_r in (5.1.6). Indeed, for any $x \in \Xi_r(Z)$ and

$\lambda > 0$,

$$\delta_\lambda(x \cdot \mathtt{H}_F) = \delta_\lambda(x) \cdot \mathtt{H}_F = \lambda^r x \cdot \mathtt{H}_F = \lambda^r (x \cdot \mathtt{H}_F),$$

thus $x \cdot \mathtt{H}_F \in \Xi_r(Z)$. □

5.2.2 Eulerian idempotents and diagonalization

An **Eulerian family** of $\mathcal{A}$ is a collection of idempotent and mutually orthogonal elements $\{\mathtt{E}_{\mathrm{X}}\}_{\mathrm{X} \in \mathcal{L}[\mathcal{A}]} \subseteq \mathbb{R}\Sigma[\mathcal{A}]$ of the form

$$\mathtt{E}_{\mathrm{X}} = \sum_{F \,:\, \mathrm{s}(F) \geq \mathrm{X}} a^F \mathtt{H}_F, \tag{5.2.2}$$

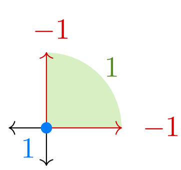
with $a^F \neq 0$ for at least one F with $\mathrm{s}(F) = \mathrm{X}$. It follows that $\{\mathtt{E}_{\mathrm{X}}\}_{\mathrm{X}}$ is a complete system of primitive orthogonal idempotents and that $\mathrm{s}(\mathtt{E}_{\mathrm{X}}) = \mathtt{Q}_{\mathrm{X}}$ [8, Theorem 11.20]. That is,

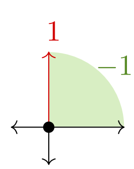
$$\mathtt{E}_{\mathrm{X}}\mathtt{E}_{\mathrm{Y}} = \begin{cases} \mathtt{E}_{\mathrm{X}} & \text{if } \mathrm{X} = \mathrm{Y}, \\ 0 & \text{otherwise}, \end{cases} \qquad \sum_{\mathrm{X}} \mathtt{E}_{\mathrm{X}} = \mathtt{H}_O,$$

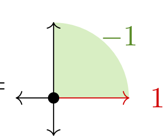
and $\mathtt{E}_{\mathrm{X}}$ cannot be written as the sum of two non-trivial idempotents. The following important property of Eulerian idempotents is [8, Lemma 11.12]. It was first proved by Saliola [65, Lemma 1.4] in for particular families of Eulerian idempotents.

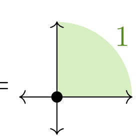
Lemma 5.2.4. *For any face F and flat* X, *if* $\mathrm{s}(F) \not\leq \mathrm{X}$, *then* $\mathtt{H}_F \cdot \mathtt{E}_{\mathrm{X}} = 0$.

Example 5.2.5. Let $\mathcal{C}_2$ be the coordinate arrangement in $\mathbb{R}^2$. The following is an Eulerian family of $\mathcal{C}_2$. Observe that in this example only faces in the first quadrant have non-zero coefficients.

$\mathtt{E}_\perp =$ 

$\mathtt{E}_{x=0} =$ 

$\mathtt{E}_{y=0} =$ 

$\mathtt{E}_\top =$ 

A characteristic element w of *non-critical*[1] parameter t uniquely determines an Eulerian family $\mathtt{E} = \{\mathtt{E}_{\mathrm{X}}\}_{\mathrm{X}}$, which satisfies

$$w = \sum_{\mathrm{X}} t^{\dim(\mathrm{X})} \mathtt{E}_{\mathrm{X}}. \tag{5.2.3}$$

This is a consequence of [8, Propositions 11.9, 12.59]. It follows that the action of such a characteristic elements w on any $\mathbb{R}\Sigma[\mathcal{A}]$-module M is diagonalizable.

Let M be a (right) $\mathbb{R}\Sigma[\mathcal{A}]$-module, $w \in \mathbb{R}\Sigma[\mathcal{A}]$ be a characteristic element of non-critical parameter t and $\{\mathtt{E}_{\mathrm{X}}\}_{\mathrm{X}}$ be the corresponding Eulerian family. Then, we have a decomposition

$$M = \bigoplus_{\mathrm{X}} M \cdot \mathtt{E}_{\mathrm{X}} \tag{5.2.4}$$

of vector spaces. Expression (5.2.3) shows that w acts on $M \cdot \mathtt{E}_{\mathrm{X}}$ by multiplication by $t^{\dim(\mathrm{X})}$. We define

$$\eta_{\mathrm{X}}(M) := \dim_{\mathbb{R}}(M \cdot \mathtt{E}_{\mathrm{X}}).$$

The kernel of the support map s is precisely the radical of $\mathbb{R}\Sigma[\mathcal{A}]$ [8, Proposition 9.22]. Consequently, the character $\chi_M : \mathbb{R}\Sigma[\mathcal{A}] \to \mathbb{R}$ of M factors through $\mathbb{R}\mathcal{L}[\mathcal{A}]$:

$$\begin{array}{ccc} \mathbb{R}\Sigma[\mathcal{A}] & & \\ {\scriptstyle \mathrm{s}}\downarrow & \overset{\chi_M}{\searrow} & \\ \mathbb{R}\mathcal{L}[\mathcal{A}] & \underset{\overline{\chi}_M}{\dashrightarrow} & \mathbb{R} \end{array}$$

Thus, $\eta_{\mathrm{X}}(M) = \dim_{\mathbb{R}}(M \cdot \mathtt{E}_{\mathrm{X}}) = \chi_M(\mathtt{E}_{\mathrm{X}}) = \overline{\chi}_M(\mathtt{Q}_{\mathrm{X}})$ is independent of the characteristic element w. Furthermore, using relations (3.1.1) and the linearity of $\overline{\chi}_M$ we deduce

$$\begin{aligned} \eta_{\mathrm{X}}(M) = \overline{\chi}_M(\mathtt{Q}_{\mathrm{X}}) &= \sum_{\mathrm{Y} \geq \mathrm{X}} \mu(\mathrm{X},\mathrm{Y}) \overline{\chi}_M(\mathtt{H}_{\mathrm{Y}}) \\ &= \sum_{\mathrm{Y} \geq \mathrm{X}} \mu(\mathrm{X},\mathrm{Y}) \chi_M(\mathtt{H}_{F_{\mathrm{Y}}}) = \sum_{\mathrm{Y} \geq \mathrm{X}} \mu(\mathrm{X},\mathrm{Y}) \dim_{\mathbb{R}}(M \cdot \mathtt{H}_{F_{\mathrm{Y}}}) \end{aligned} \tag{5.2.5}$$

[1] $t \in \mathbb{R}$ is non-critical if it is not a root of $\chi(\mathcal{A}^{\mathrm{X}}, t)$ for any flat $\mathrm{X} \in \mathcal{L}[\mathcal{A}]$.

where $F_{\mathrm{Y}} \in \Sigma[\mathcal{A}]$ is such that $\mathrm{s}(F_{\mathrm{Y}}) = \mathrm{Y}$. The last equality follows since $\mathtt{H}_{F_{\mathrm{Y}}}$ is an idempotent element, and thus $\chi_M(\mathtt{H}_{F_{\mathrm{Y}}}) = \dim_{\mathbb{R}}(M \cdot \mathtt{H}_{F_{\mathrm{Y}}})$.

Moreover, the number of composition factors M_{i+1}/M_i isomorphic to the simple module indexed by X in a composition series $0 \subset M_1 \subset M_2 \subset \cdots \subset M_k = M$ of M is precisely $\eta_{\mathrm{X}}(M)$. See [8, Section 9.5 and Theorem D.37] for details.

5.2.3 Simultaneous diagonalization

Let $\lambda > 0$ and let $w \in \mathbb{R}\Sigma[\mathcal{A}]$ be a characteristic element of non-critical parameter t. We know that the dilation morphism δ_λ and the action of w are diagonalizable. Moreover, since δ_λ and the action of w commute, they are simultaneously diagonalizable. A natural question is to determine the eigenvalues of δ_λ and of the action of w in their simultaneous eigenspaces. We completely answer this question in the case of the Coxeter arrangements of type A and B in the next two sections. The following result holds in the general case.

Proposition 5.2.6. *Let $x \in \Pi(Z)$ be a (nonzero) simultaneous eigenvector for δ_λ and w with eigenvalues λ^r and t^k, respectively. Then, $r + k \leq d$.*

Proof. Let $\{\mathtt{E}_{\mathrm{X}}\}_{\mathrm{X}}$ be the Eulerian family associated to w. Using the characterization of the graded components Ξ_r as the eigenspaces of δ_λ, and the decomposition of $\mathbb{R}\Sigma[\mathcal{A}]$-modules in (5.2.4), we deduce that the common eigenspace for δ_λ and w with the given eigenvalues is $\bigoplus_{\mathrm{X}} \Xi_r(Z)\cdot\mathtt{E}_{\mathrm{X}}$, where the sum is over all k-dimensional flats of $\mathcal{A}$. Without loss of generality, we assume that $x \in \Xi_r(Z) \cdot \mathtt{E}_{\mathrm{X}}$ for a single k-dimensional flat X.

Proposition 5.2.1 implies that $\Xi_r(Z) \cdot \mathtt{H}_F = \Xi_r(Z_{\mathrm{Y}})$, where $F \in \Sigma[\mathcal{A}]$ is any face of support Y. Hence, formula (5.2.5) yields

$$\eta_{\mathrm{X}}(\Xi_r(Z)) = \sum_{\mathrm{Y}\geq \mathrm{X}} \mu(\mathrm{X},\mathrm{Y}) \dim_{\mathbb{R}}(\Xi_r(Z_{\mathrm{Y}})). \tag{5.2.6}$$

If $k > d-r$, then $\dim(Z_{\mathrm{Y}}) = d - \dim(\mathrm{Y}) \leq d-k < r$ and $\dim_{\mathbb{R}}(\Xi_r(Z_{\mathrm{Y}})) = 0$ for any flat Y in the sum. So, in this case, $\dim_{\mathbb{R}}(\Xi_r(Z)\cdot \mathtt{E}_{\mathrm{X}}) = \eta_{\mathrm{X}}(\Xi_r(Z)) = 0$. This contradicts that $x \in \Xi_r(Z)\cdot \mathtt{E}_{\mathrm{X}}$ is a nonzero element. Therefore, $r+k \leq d$. $\square$

We have shown that

$$\bigoplus_{\dim(\mathrm{X})=k} \Xi_r(Z)\cdot \mathtt{E}_{\mathrm{X}}$$

is the common eigenspace for δ_λ and w with eigenvalues λ^r and t^k, respectively. It turns out that, if the characteristic element w is **projective** ($w^F = w^{\overline{F}}$), elements in this subspace are also eigenvectors for negative dilations and for the Euler map (5.1.7).

Proposition 5.2.7. *Suppose w is projective. Then, for all* $x \in \bigoplus_{\dim(\mathrm{X})=k} \Xi_r(Z)\cdot \mathtt{E}_{\mathrm{X}}$,

$$x^* = (-1)^{d-k}x \quad \textit{and} \quad \delta_{-1}(x) = (-1)^{d-k-r}x.$$

Proof. In view of Theorem 5.1.6, the two claims are equivalent. We prove the first one.

Since dilations and the Euler map are linear, and $\Xi_r(Z)$ is generated by elements of the form $(\log[P])^r$, we can assume $x = (\log[P])^r \cdot \mathtt{E}_{\mathrm{X}}$ for a fixed generalized zonotope P and a fixed k-dimensional flat X. Moreover, with $\mathtt{E}_{\mathrm{X}}$ as in (5.2.2), we have

$$(\log[P])^r \cdot \mathtt{E}_{\mathrm{X}} = \big((\log[P])^r \cdot \mathtt{E}_{\mathrm{X}}\big)\cdot \mathtt{E}_{\mathrm{X}} = \Big(\sum_{\mathrm{s}(F)\geq \mathrm{X}} a^F (\log[P_F])^r\Big)\cdot \mathtt{E}_{\mathrm{X}}.$$

In the first equality we used that $\mathtt{E}_{\mathrm{X}}$ is idempotent. Consider a term $a^F(\log[P_F])^r$ in the sum. Since $(P_F)_F = P_F$, we have that $N_{P_F}P_F \geq \mathrm{s}(F) \geq \mathrm{X}$. If $N_{P_F}P_F > \mathrm{X}$, take F' maximal inside $N_{P_F}P_F$ and note that

$$(\log[P_F])^r \cdot \mathtt{E}_{\mathrm{X}} = \big((\log[P_F])^r \cdot \mathtt{H}_{F'}\big) \cdot \mathtt{E}_{\mathrm{X}} = 0.$$

The last equality follows from Lemma 5.2.4. Thus, we have shown that $(\log[P])^r \cdot \mathtt{E}_{\mathrm{X}}$ can be written as a linear combination of elements of the form $(\log[Q])^r \cdot \mathtt{E}_{\mathrm{X}}$ with $N_Q Q = \mathrm{X}$. It is then enough to prove the claim for elements of this form, so we now suppose $N_P P = \mathrm{X}$.

$$\begin{aligned}
\big((\log[P])^r \cdot \mathtt{E}_{\mathrm{X}}\big)^* &= (-1)^r \delta_{-1}\big((\log[P])^r \cdot \mathtt{E}_{\mathrm{X}}\big) && \langle\text{Theorem 5.1.6}\rangle \\
&= (-1)^r \delta_{-1}\big((\log[P])^r\big) \cdot \mathtt{E}_{\mathrm{X}} && \langle\mathtt{E}_{\mathrm{X}} \text{ is projective}\rangle \\
&= \big((\log[P])^r\big)^* \cdot \mathtt{E}_{\mathrm{X}} && \langle\text{Theorem 5.1.6}\rangle \\
&= \Big(\sum_{Q \leq P} (-1)^{\dim(Q)} (\log[Q])^r\Big) \cdot \mathtt{E}_{\mathrm{X}} \\
&= (-1)^{\dim(P)} (\log[P])^r \cdot \mathtt{E}_{\mathrm{X}}.
\end{aligned}$$

In the second equality we use that $\delta_{-1}(x \cdot \mathtt{H}_F) = \delta_{-1}(x) \cdot \mathtt{H}_{\overline{F}}$ and that each $\mathtt{E}_{\mathrm{X}}$ is projective if w is. The last equality follows again because for proper faces $Q < P$, $N_Q Q > \mathrm{X}$ and $(\log[Q])^r \cdot \mathtt{E}_{\mathrm{X}} = 0$. The result follows by observing that $\dim(P) = d - k$. □

If in addition $\mathcal{A}$ is a simplicial arrangement, like in the case of reflection arrangements, then Z and each of its faces are simple polytopes. In that case, Theorem 5.1.8 allows us to replace $\dim_{\mathbb{R}}(\Xi_r(Z_{\mathrm{Y}}))$ by $h_r(Z_{\mathrm{Y}})$ in expression (5.2.6). Multiplying by z^r and taking the sum over all values of r, we obtain

$$\sum_r \eta_{\mathrm{X}}(\Xi_r(Z)) z^r = \sum_{\mathrm{Y} \geq \mathrm{X}} \mu(\mathrm{X}, \mathrm{Y}) h(Z_{\mathrm{Y}}, z). \tag{5.2.7}$$

5.2.4 First example: the cube and the coordinate arrangement

Let $\mathcal{C}_d$ be the coordinate arrangement in $\mathbb{R}^d$. We identify the lattice of flats $\mathcal{L}[\mathcal{C}_d]$ with the (opposite) boolean lattice $2^{[d]}$ in the following manner:

$$S \subseteq [d] \longleftrightarrow \mathrm{X}_S := \bigcap_{i \in S} \{\boldsymbol{x} \,:\, \boldsymbol{x}_i = 0\}.$$

Observe that $\mathrm{X}_S \leq \mathrm{X}_T$ if and only if $T \subseteq S$, and in this case $\mu_{\mathcal{L}}(\mathrm{X}_S, \mathrm{X}_T) = \mu_{2^{[d]}}(T, S) = (-1)^{|S \setminus T|}$.

The d-cube $\square_d = [0,1]^d$ a zonotope of $\mathcal{C}_d$. It is the Minkowski sum of the d line segments $\mathfrak{l}_i := \mathrm{Conv}\{\mathbf{0}, \boldsymbol{e}_i\}$ for $i = 1, \dots, d$. It is a simple polytope with h-polynomial $h(\square_d, z) = (1+z)^d$. Furthermore, for any $S \subseteq [d]$ we have

$$(\square_d)_{\mathrm{X}_S} = \sum_{i \in S} \mathfrak{l}_i \cong \square_{|S|}.$$

Let us consider the right $\mathbb{R}\Sigma[\mathcal{C}_d]$-module $\Pi(\square_d)$. For a flat X_S, formula (5.2.7) yields

$$\sum_r \eta_{\mathrm{X}_S}(\Xi_r(\square_d)) z^r = \sum_{T \subseteq S} \mu(\mathrm{X}_S, \mathrm{X}_T) h(\square_{|T|}, z) = \sum_{T \subseteq S} (-1)^{|S \setminus T|} (1+z)^{|T|} = z^{|S|}.$$

Hence,

$$\eta_{\mathrm{X}_S}(\Xi_r(\square_d)) = \begin{cases} 1 & \text{if } |S| = r, \\ 0 & \text{otherwise.} \end{cases}$$

In particular, a series decomposition of $\Xi_r(\square_d)$ contains exactly one copy of the simple module indexed by X_S for every $S \in \binom{[d]}{r}$.

Let us now consider the characteristic element $\gamma_t \in \mathbb{R}\Sigma[\mathcal{C}_d]$ in Section 3.5.4 for

$t \neq 1$. It is defined by

$$\gamma_t = \sum_F \gamma_t^F \mathtt{H}_F, \quad \text{where} \quad \gamma_t^F = \begin{cases} (t-1)^{\dim(F)} & \text{if } F \text{ lies in the first orthant,} \\ 0 & \text{otherwise.} \end{cases}$$

For each $S \subseteq [d]$, let F_S be the intersection of the first orthant with X_S, it is a face of $\mathcal{C}_d$. We have $\mathrm{s}(F_S) = \mathrm{X}_S$ and $T \subseteq S$ if and only if $F_S \leq F_T$. A simple computation shows that the Eulerian family corresponding to the characteristic element γ_t is determined by

$$\mathtt{E}_{\mathrm{X}_S} = \sum_{T \subseteq S} (-1)^{|S \setminus T|} \mathtt{H}_{F_T}.$$

In dimension 2, this is the Eulerian family in Example 5.2.5. For each $S \subseteq [d]$, define

$$y_S = \prod_{i \in S} \log[\mathfrak{l}_i] \in \Pi(\square_d).$$

Example 5.1.2 shows that y_S is a nonzero element of $\Pi(\square_d)$.

We claim that $\{y_S\}_{S \subseteq [d]}$ is a basis of simultaneous eigenvectors of $\Pi(\square_d)$. Explicitly, y_S is an eigenvector for the action of γ_t of eigenvalue $t^{d-|S|}$, and for the action of δ_λ of eigenvalue $\lambda^{|S|}$ ($\lambda > 0$). The second statement is clear, since $\log[\mathfrak{l}_i] \in \Xi_1(\square_d)$. Moreover, using that $\log[\mathfrak{l}_i] = [\mathfrak{l}_i] - 1$, we have

$$y_S = \prod_{i \in S} ([\mathfrak{l}_i] - 1) = \sum_{T \subseteq S} (-1)^{|S \setminus T|} [\square_{\mathrm{X}_T}].$$

On the other hand, observe that

$$[\square_{\mathrm{X}_S}] \cdot \mathtt{E}_{\mathrm{X}_S} = \sum_{T \subseteq S} (-1)^{|S \setminus T|} [\square_{\mathrm{X}_S}] \cdot \mathtt{H}_{F_T} = \sum_{T \subseteq S} (-1)^{|S \setminus T|} [\square_{\mathrm{X}_T}] = y_S.$$

Therefore,

$$y_S \in \Xi_r(\square_d) \cap (\Pi(\square_d) \cdot \mathtt{E}_{\mathrm{X}_S}) = \Xi_r(\square_d) \cdot \mathtt{E}_{\mathrm{X}_S}.$$

The claim follows since $\dim(\mathrm{X}_S) = d - |S|$.

5.2.5 The zonotope module of a product of arrangements

The Cartesian product of two arrangements $\mathcal{A}$ in V and $\mathcal{A}'$ in W is the following collection of hyperplanes in $V \oplus W$:

$$\mathcal{A} \times \mathcal{A}' = \{\mathrm{H} \oplus W \,:\, \mathrm{H} \in \mathcal{A}\} \cup \{V \oplus \mathrm{H} \,:\, \mathrm{H} \in \mathcal{A}'\}.$$

One can easily verify that $\Sigma[\mathcal{A} \times \mathcal{A}'] \cong \Sigma[\mathcal{A}] \times \Sigma[\mathcal{A}']$ as monoids. Hence, $\mathbb{R}\Sigma[\mathcal{A} \times \mathcal{A}'] \cong \mathbb{R}\Sigma[\mathcal{A}] \otimes \mathbb{R}\Sigma[\mathcal{A}']$. In fact, it is also true that

$$\Pi(Z \times Z') \cong \Pi(Z) \otimes \Pi(Z'),$$

where Z and Z' are zonotopes of $\mathcal{A}$ and $\mathcal{A}'$, respectively, and therefore $Z \times Z'$ is a zonotope of $\mathcal{A} \times \mathcal{A}'$. Indeed, every generalized zonotope of $\mathcal{A} \times \mathcal{A}'$ is the Cartesian product of generalized zonotopes of $\mathcal{A}$ and $\mathcal{A}'$. The corresponding isomorphism is induced by

$$\begin{aligned} \Pi(Z) \otimes \Pi(Z') &\to \Pi(Z \times Z') \\ [P] \otimes [Q] \quad &\mapsto \quad [P \times Q] \end{aligned}$$

The fact that this map is well-defined and a morphism of $\mathbb{R}\Sigma[\mathcal{A} \times \mathcal{A}']$-modules follows from the ideas in Section 5.5.1.

5.3 The module of generalized permutahedra

Generalized permutahedra are the deformations of the standard permutahedron $\pi_d \subseteq \mathbb{R}^d$. Edmonds first introduced them under a different name in [38], where he studied their relation to submodular functions and optimization. For a thorough study of the combinatorics of these polytopes, see [1, 60, 61].

In this section, we study the algebra $\Pi(\pi_d)$ of generalized permutahedra and its structure as a module over the Tits algebra of the braid arrangement $\mathcal{A}_d$. We begin with a brief review of the braid arrangement, its relation with the symmetric group, and some statistics on permutations.

5.3.1 The symmetric group and the Eulerian polynomial

The symmetric group $\mathfrak{S}_d$ is the group of *permutations* $\sigma : [d] \to [d]$ under composition. It is the Coxeter group corresponding to the braid arrangement. It acts on $\mathbb{R}^d$ by permuting coordinates:

$$\sigma(\boldsymbol{x}_1, \boldsymbol{x}_2, \dots, \boldsymbol{x}_d) = (\boldsymbol{x}_{\sigma(1)}, \boldsymbol{x}_{\sigma(2)}, \dots, \boldsymbol{x}_{\sigma(d)}).$$

For a permutation $\sigma \in \mathfrak{S}_d$, we let $\mathrm{s}(\sigma)$ denote the **subspace of points fixed by the action** of σ; it is a flat of $\mathcal{A}_d$. In view of the identification between flats of $\mathcal{A}_d$ and partitions of $[d]$, $\mathrm{s}(\sigma)$ can equivalently be defined as the partition of $[d]$ into the disjoint cycles of σ. For example, if in cycle notation $\sigma = (\mathbf{1}3)(\mathbf{2}6\mathbf{5}8)(4)(7)$, then $\mathrm{s}(\sigma) = \{13, 2568, 4, 7\}$.

Recall that $i \in [d-1]$ is a **descent** of $\sigma \in \mathfrak{S}_d$ if $\sigma(i) > \sigma(i+1)$, and $i \in [d-1]$ is an **excedance** of $\sigma \in \mathfrak{S}_d$ if $\sigma(i) > i$. Let $\mathrm{des}(\sigma)$ and $\mathrm{exc}(\sigma)$ denote the number of descents and excedances of σ, respectively. In the example above, $1, 2, 5$ (in bold) are the excedances of σ, and we have $\mathrm{exc}(\sigma) = 3$. We can similarly define descents and excedances for permutations of any set S with a total order $\prec$, we denote the corresponding statistics by $\mathrm{des}_\prec$ and $\mathrm{exc}_\prec$.

It is a classical result that descents and excedances are *equidistributed* in $\mathfrak{S}_d$. That is,

$$A_{d,k} := \big|\{\sigma \in \mathfrak{S}_d \,:\, \mathrm{des}(\sigma) = k\}\big| = \big|\{\sigma \in \mathfrak{S}_d \,:\, \mathrm{exc}(\sigma) = k\}\big|,$$

for all possible values of k. Foata's *fundamental transformation* provides a simple proof of this result. The numbers $A_{d,k}$ are the classical **Eulerian numbers** (OEIS: A008292). The **Eulerian polynomial** $A_d(z)$ is:

$$A_d(z) := \sum_{k=0}^{d-1} A_{d,k} z^k = \sum_{\sigma \in \mathfrak{S}_d} z^{\operatorname{exc}(\sigma)}.$$

The exponential generating function for these polynomials was originally given by Euler himself:

$$A(z,x) = 1 + \sum_{d \geq 1} A_d(z) \frac{x^d}{d!} = \frac{z-1}{z - e^{x(z-1)}}. \tag{5.3.1}$$

See [40, Section 3] for a derivation of this formula.

Let $\mathfrak{C}(S)$ the collection of **cyclic permutations** on a finite set S, and $\mathfrak{C}(d) = \mathfrak{C}([d])$. Given a permutation $\sigma \in \mathfrak{S}_d$ and a block $S \in \mathrm{s}(\sigma)$, the restriction $\sigma|_S$ of σ to S is a cyclic permutation. For example, with σ as before and $S = \{2,5,6\} \in \mathrm{s}(\sigma)$, we have $\sigma|_S = (265) \in \mathfrak{C}(\{2,5,6\})$. A very simple but important observation is that the number of excedances of σ can be computed by adding up the excedances in each cycle in its cycle decomposition. That is, $\operatorname{exc}(\sigma) = \sum_{S \in \mathrm{s}(\sigma)} \operatorname{exc}(\sigma|_S)$. The number of excedances in each cycle $\sigma|_S$ is computed with respect to the natural order in $S \subseteq [d]$.

5.3.2 The module Generalized permutahedra

The permutahedron $\pi_d \subseteq \mathbb{R}^d$ is the convex hull of the $\mathfrak{S}_d$-orbit the point $(1,2,\ldots,d)$. It is a zonotope of the braid arrangement $\mathcal{A}_d$ and has dimension $d-1$. Deformations of π_d are called **generalized permutahedra**. We consider the module $\Pi(\pi_d)$ as in Section 5.2. The main goal of this section will be to prove the following result.

Theorem 5.3.1. *For any flat* $\mathrm{X} \in \mathcal{L}[\mathcal{A}_d]$ *and* $r = 0, 1, \ldots, d-1$,

$$\eta_{\mathrm{X}}(\Xi_r(\pi_d)) = \left|\{\sigma \in \mathfrak{S}_d \,:\, \mathrm{s}(\sigma) = \mathrm{X},\ \mathrm{exc}(\sigma) = r\}\right|.$$

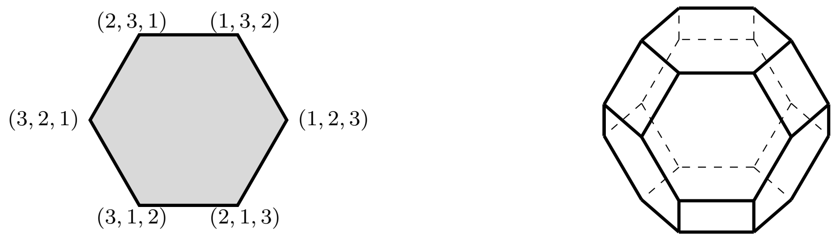


Figure 5.3.1: The permutahedron in $\mathbb{R}^3$ and $\mathbb{R}^4$.

The relation between $\Pi(\pi_d)$ and statistics on $\mathfrak{S}_d$ is via the h-polynomial of π_d. Brenti [30, Theorem 2.3] showed that $h(\pi_d, z) = A_d(z)$. Moreover, for a flat/partition $\mathrm{X} = \{S_1, \ldots, S_k\}$ of $\mathcal{A}_d$, the face $(\pi_d)_{\mathrm{X}}$ is a translate of $\pi_{|S_1|} \times \cdots \times \pi_{|S_k|}$, a product of lower-dimensional permutahedra. Thus,

$$h((\pi_d)_{\mathrm{X}}, z) = A_{|S_1|}(z) \cdot \ldots \cdot A_{|S_k|}(z). \tag{5.3.2}$$

The next lemma is an essential ingredient in the proof of Theorem 5.3.1.

Lemma 5.3.2. *For every* $d \geq 1$,

$$\sum_{\mathrm{X}=\{S_1,\ldots,S_k\}\vdash[d]} \mu(\bot, \mathrm{X}) A_{|S_1|}(z) \cdot \ldots \cdot A_{|S_k|}(z) = \sum_{\sigma \in \mathfrak{C}(d)} z^{\mathrm{exc}(\sigma)}. \tag{5.3.3}$$

Proof. We will show that the exponential generating function of both sides of (5.3.3) are equal to $\log(A(z,x))$, where $A(z,x)$ is the generating function for the Eulerian polynomials in (5.3.1).

First, recall that if $\mathrm{X} = \{S_1, \ldots, S_k\}$, then $\mu(\bot, \mathrm{X}) = (-1)^{k-1}(k-1)!$. Thus, a direct application of the Compositional Formula [68, Theorem 5.1.4] shows that the exponential generating function of the LHS of (5.3.3) is the composition of

$$\sum_{d\geq 1}(-1)^{d-1}(d-1)!\frac{x^d}{d!} = \log(1+x) \qquad \text{with} \qquad \sum_{d\geq 1} A_d(z)\frac{x^d}{d!} = A(z,x)-1,$$

which is precisely $\log(A(z,x))$.

On the other hand, grouping permutations with the same underlying partition $\mathrm{s}(\sigma)$, we obtain

$$A(z,x) = 1+\sum_{d\geq 1}\Big(\sum_{\sigma\in\mathfrak{S}_d} z^{\mathrm{exc}(\sigma)}\Big)\frac{x^d}{d!} = 1+\sum_{d\geq 1}\Big(\sum_{\mathrm{X}\vdash[d]}\Big(\sum_{\substack{\sigma\in\mathfrak{S}_d\\ \mathrm{s}(\sigma)=\mathrm{X}}} z^{\mathrm{exc}(\sigma)}\Big)\Big)\frac{x^d}{d!}.$$

Since a permutation σ with $\mathrm{s}(\sigma)=\mathrm{X}$ is the product of cyclic permutations $\sigma_S\in\mathfrak{C}(S)$ for each block $S\in\mathrm{X}$, and in this case $\mathrm{exc}(\sigma)=\sum_{S\in\mathrm{X}}\mathrm{exc}(\sigma_S)$,

$$\sum_{\substack{\sigma\in\mathfrak{S}_d\\ \mathrm{s}(\sigma)=\mathrm{X}}} z^{\mathrm{exc}(\sigma)} = \prod_{S\in\mathrm{X}}\Big(\sum_{\sigma_S\in\mathfrak{C}(S)} z^{\mathrm{exc}(\sigma_S)}\Big). \tag{5.3.4}$$

Thus, the Exponential Formula [68, Corollary 5.1.6] implies that

$$A(z,x) = \exp\Big(\sum_{d\geq 1}\Big(\sum_{\sigma\in\mathfrak{C}(d)} z^{\mathrm{exc}(\sigma)}\Big)\frac{x^d}{d!}\Big).$$

Taking logarithms on both sides yields the result. □

A small modification in the proof of the previous Lemma immediately gives the following result, which was first discovered by Brenti.

Corollary 5.3.3 ([31, Proposition 7.3])**.** *The following identity holds*

$$1+\sum_{d\geq 1}\Big(\sum_{\sigma\in\mathfrak{S}_d} t^{|\mathrm{s}(\sigma)|}z^{\mathrm{exc}(\sigma)}\Big)\frac{x^n}{n!} = \exp(t\log(A(z,x))) = \left(\frac{z-1}{z-e^{x(z-1)}}\right)^t.$$

An analogous formula for the type B Coxeter group is described in Proposition 5.4.7. We are now ready to prove the main result of this section.

Proof of Theorem 5.3.1. We will compute the values $\eta_{\mathrm{X}}(\Xi_r(\pi_d))$ using formula (5.2.7), which in this case reads

$$\sum_r \eta_{\mathrm{X}}(\Xi_r(\pi_d)) z^r = \sum_{\mathrm{Y}\,:\,\mathrm{Y}\geq\mathrm{X}} \mu(\mathrm{X},\mathrm{Y}) h((\pi_d)_{\mathrm{Y}}, z).$$

Using (2.2.6) and (5.3.2), we can rewrite the expression above as

$$\sum_r \eta_{\mathrm{X}}(\Xi_r(\pi_d)) z^r = \prod_{S\in\mathrm{X}} \Big(\sum_{\mathrm{Y}=\{T_1,\dots,T_\ell\}\vdash S} \mu(\perp,\mathrm{Y}) A_{|T_1|}(z)\cdot\ldots\cdot A_{|T_\ell|}(z)\Big).$$

Now, an application of Lemma 5.3.2 and relation (5.3.4) gives

$$\sum_r \eta_{\mathrm{X}}(\Xi_r(\pi_d)) z^r = \prod_{S\in\mathrm{X}} \Big(\sum_{\sigma\in\mathfrak{C}(S)} z^{\mathrm{exc}(\sigma)}\Big) = \sum_{\substack{\sigma\in\mathfrak{S}_d\\ \mathrm{s}(\sigma)=\mathrm{X}}} z^{\mathrm{exc}(\sigma)}.$$

Finally, taking the coefficient of z^r on both sides of the last equality yields the result. □

Adding over all flats with the same dimension in Theorem 5.3.1, we conclude the following.

Corollary 5.3.4. *Let $w \in \mathbb{R}\Sigma[\mathcal{A}_d]$ be a characteristic element of non-critical parameter t and $\lambda > 0$. The dimension of the simultaneous eigenspace for w and δ_λ with eigenvalues t^k and λ^r is*

$$\big|\{\sigma\in\mathfrak{S}_d \,:\, |\mathrm{s}(\sigma)| = k,\ \mathrm{exc}(\sigma) = r\}\big|.$$

5.3.3 Simultaneous-eigenbasis for the Adams element

Let $\alpha_t \in \mathbb{R}\Sigma[\mathcal{A}_d]$ be the Adams element of parameter t as in Section 3.5.1. It is invariant with respect to the action of $\mathfrak{S}_d$, and its action on $\mathbb{R}\Sigma[\mathcal{A}_d]$-modules is closely related with the *convolution powers of the identity map* of a Hopf monoid, see [7, Section 14.4] and 6.3.3. The corresponding Eulerian idempotents are [8, Theorem 12.75]

$$\mathtt{E}_{\mathrm{X}} = \frac{1}{\dim(\mathrm{X})!} \sum_{\mathrm{s}(F)=\mathrm{X}} \sum_{G \geq F} \frac{(-1)^{\dim(G/F)}}{\deg(G/F)} \mathtt{H}_G, \tag{5.3.5}$$

where $\dim(G/F) = \dim(G) - \dim(F)$ and, $\deg(G/F) = \prod_{S \in F} \left|G|_S\right|$.

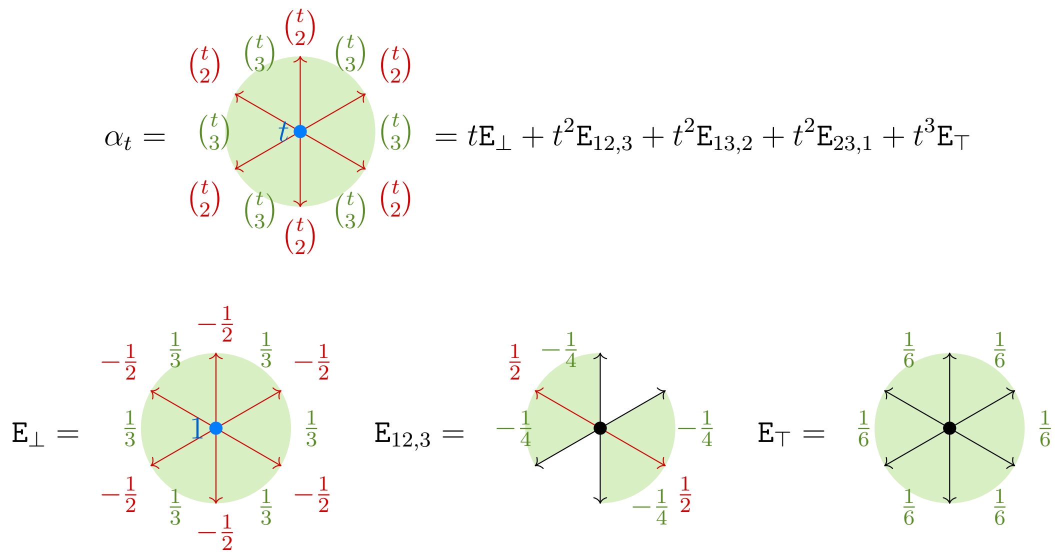


Figure 5.3.2: The Adams element α_t and some of the associated Eulerian idempotents of the braid arrangement in $\mathbb{R}^3$.

Theorem 5.3.1 suggest the existence of a natural basis for $\Xi_r(\pi_d) \cdot \mathtt{E}_{\mathrm{X}}$ indexed by permutations σ with r excedances and $\mathrm{s}(\sigma) = \mathrm{X}$. In this section we will construct a candidate for such basis.

The standard simplex $\Delta_{[d]} \subseteq \mathbb{R}^d$ is the convex hull of the standard basis $\{\boldsymbol{e}_1, \ldots, \boldsymbol{e}_d\}$ of $\mathbb{R}^d$. Similarly, for any nonempty subset $S \subseteq [d]$, we let $\Delta_S = \mathrm{Conv}\{\boldsymbol{e}_i : i \in S\}$. Ardila, Benedetti and Doker showed [13, Proposition 2.4] that every generalized permutahedron P can be written uniquely as a *signed Minkowski sum* of simplices $P = \sum y_S \Delta_S$. This means that we have the following identity:

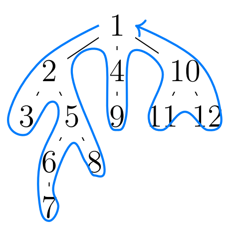
$$P + \sum_{y_S < 0} |y_S| \Delta_S = \sum_{y_S > 0} y_S \Delta_S.$$

Up to translation, this is equivalent to the following identity in $\Xi_1(\pi_d)$:

$$\log[P] = \sum_{S \subseteq [d]} y_S \log[\Delta_S].$$

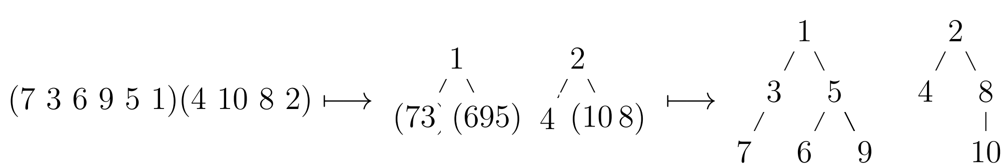
Given that $\log[\Delta_S] = 0$ whenever S is a singleton, and that $[P] = [Q]$ if and only if P is a translate of Q, we conclude that $\{\log[\Delta_S] : S \subseteq [d], \left|S\right| \geq 2\}$ is a linear basis for $\Xi_1(\pi_d)$ and therefore generate $\Pi(\pi_d)$ as an algebra. This agrees with $\dim_{\mathbb{R}}(\Xi_1(\pi_d)) = h_1(\pi_d) = 2^d - d - 1$.

We will use a bijection between *increasing rooted forests* on $[d]$ and permutations in $\mathfrak{S}_d$. An increasing rooted forest is a disjoint union of planar rooted trees where each child is larger than its parent and the children are in increasing order from left to right. Given a rooted forest t, the corresponding permutation $\sigma(t)$ is read as follows. Each connected component of t corresponds to a cycle of $\sigma(t)$. To form a cycle, traverse the corresponding tree counterclockwise and record a node the *second* time you pass by it[2].



$$\longmapsto \qquad (3\ 7\ 6\ 8\ 5\ 2\ 9\ 4\ 11\ 12\ 10\ 1)$$

The inverse can be described inductively by writing each cycle with its minimum element in the last position, and using right to left minima. We omit the details, but provide an example $\sigma \mapsto t(\sigma)$ to illustrate the idea.



(5.3.6)

[2]A similar bijection is described by Peter Luschny in this OEIS entry.

This bijection is such that the connected components of the forest $t(\sigma)$ are the blocks of $\mathrm{s}(\sigma)$. Moreover, the number of leaves of $t(\sigma)$ in $S \in \mathrm{s}(\sigma)$ is $\mathrm{exc}(\sigma|_S)$ (a tree consisting only of its root has zero leaves). Consequently, the total number of leaves of $t(\sigma)$ is $\mathrm{exc}(\sigma)$.

Let $\sigma \in \mathfrak{S}_d$ be a permutation with r excedances and let $\mathrm{X} = \mathrm{s}(\sigma)$. For $1 \leq i \leq r$, let J_i be the elements on the path from the i^{th} leaf of $t(\sigma)$ to the root of the corresponding tree. Define the element

$$x_\sigma = \Big(\prod_{i=1}^{r} \log[\Delta_{J_i}] \Big) \cdot \mathtt{E}_{\mathrm{X}}. \tag{5.3.7}$$

For instance, if σ is the permutation in (5.3.6), then

$$x_\sigma = \Big(\log[\Delta_{\{7,3,1\}}] \log[\Delta_{\{6,5,1\}}] \log[\Delta_{\{9,5,1\}}] \log[\Delta_{\{4,2\}}] \log[\Delta_{\{10,8,2\}}] \Big) \cdot \mathtt{E}_{\mathrm{X}},$$

where $\mathrm{X} = \{1,3,5,6,7,9\}, \{2,4,8,10\}$.

Conjecture 5.3.5. *For fixed* $\mathrm{X} \vdash [d]$ *and* $r \leq d - \big|\mathrm{X}\big|$*, the collection*

$$\{x_\sigma \ : \ \mathrm{s}(\sigma) = \mathrm{X}, \ \mathrm{exc}(\sigma) = r\}$$

is a linear basis of $\Xi_r(\pi_d) \cdot \mathtt{E}_{\mathrm{X}}$.

It follows from the definition (5.3.7) that $x_\sigma \in \Xi_r(\pi_d) \cdot \mathtt{E}_{\mathrm{X}}$. The content of the conjecture is that these elements are linearly independent. Explicit computations show that this is the case for $d = 2, 3, 4$. Proposition 5.3.6 Proposition 5.3.7 below prove the extremal cases $r = 1$ and $r = d - |\mathrm{X}|$ of this conjecture, respectively.

Proposition 5.3.6. *For a subset* $J \subseteq [d]$ *of cardinality at least* 2*, let* $\mathrm{X}_J \vdash [d]$ *be the partition whose only non-singleton block is* J*. Then,*

$$\log[\Delta_J] \cdot \mathtt{E}_{\mathrm{X}_J}$$

is a nonzero element. Furthermore, $\{\log[\Delta_J] \cdot \mathtt{E}_{\mathrm{X}_J} \ : \ J \subseteq [d], \, |J| \geq 2\}$ *is a basis of simultaneous eigenvectors for* $\Xi_1(\pi_d)$.

Proof. First, observe that any cyclic permutation on a set with more than one element has at least one excedance, and only one cyclic permutation attains this minimum. Namely, the only cyclic permutation in $\mathfrak{S}_d$ having one excedance is $(d\, d-1\, \ldots\, 2\, 1)$. Hence, a permutation $\sigma \in \mathfrak{S}_d$ has at least as many excedances as non-singleton blocks in $\mathrm{s}(\sigma)$. It then follows from Theorem 5.3.1 that

$$\dim_{\mathbb{R}}(\Xi_1(\pi_d) \cdot \mathtt{E}_{\mathrm{X}}) = \begin{cases} 1 & \text{if } \mathrm{X} \vdash [d] \text{ has exactly one non-singleton block,} \\ 0 & \text{otherwise.} \end{cases}$$

Thus, the second statement follows from the first.

Since $\{\log[\Delta_J] : J \subseteq [d], \left|J\right| \geq 2\}$ is a linear basis for $\Xi_1(\pi_d)$, it is enough to write $\log[\Delta_J] \cdot \mathtt{E}_{\mathrm{X}_J}$ as a non-trivial linear combination of these basis elements. Observe that if $F = (S_1, S_2, \ldots, S_k)$, then $[\Delta_J] \cdot \mathtt{H}_F = [\Delta_{J \cap S_i}]$ where i is the first index for which the intersection $J \cap S_i$ is nonempty. Thus,

$$[\Delta_J] \cdot \mathtt{H}_F = \begin{cases} [\Delta_J] & \text{if } \mathrm{s}(F) \leq \mathrm{X}_J, \\ [\text{a proper face of } \Delta_J] & \text{otherwise.} \end{cases}$$

Using that the action of $\mathtt{H}_F$ is an algebra morphism, we have $\log[\Delta_J] \cdot \mathtt{H}_F = \log([\Delta_J] \cdot \mathtt{H}_F)$. Hence, the coefficient of $\log[\Delta_J]$ in $\log[\Delta_J] \cdot \mathtt{E}_{\mathrm{X}_J}$ is

$$\frac{1}{\dim(\mathrm{X}_J)!} \sum_{\mathrm{s}(F) = \mathrm{X}_J} 1 = 1.$$

The equality follows since for any flat X of the braid arrangement, $\mathcal{A}_d^{\mathrm{X}}$ has $\dim(\mathrm{X})!$ chambers. □

Note that the element $\log[\Delta_J] \cdot \mathtt{E}_{\mathrm{X}_J}$ in the proposition is precisely the element x_σ for the unique permutation σ with $\mathrm{s}(\sigma) = \mathrm{X}_J$ and $\mathrm{exc}(\sigma) = 1$. Indeed, $t(\sigma)$ consists of a increasing path whose nodes are the elements in J and isolated roots indexed by the elements in $[d] \setminus J$.

Proposition 5.3.7. *For any* $\mathrm{X} = \{S_1, \ldots, S_k\} \vdash [d]$, *the space* $\Xi_{d-k}(\pi_d) \cdot \mathtt{E}_{\mathrm{X}}$ *is* 1-*dimensional. Moreover,*

$$x_{\mathrm{X}} = \prod_{i=1}^{k} \Big(\prod_{j \neq \min(S_i)} \log[\Delta_{\{\min(S_i), j\}}] \Big) \tag{5.3.8}$$

is a nonzero element in $\Xi_{d-k}(\pi_d) \cdot \mathtt{E}_{\mathrm{X}}$.

Proof. Observe that any cyclic permutation on a set with s elements has at most $s-1$ excedances, and only one cyclic permutation attains this maximum. Namely, the only cyclic permutation in $\mathfrak{S}_d$ having $d-1$ excedances is $(1\,2\,\ldots\,d-1\,d)$. Hence, for any $\mathrm{X} = \{S_1, \ldots, S_k\} \vdash [d]$ there is exactly one permutation with $\mathrm{s}(\sigma) = \mathrm{X}$ and $d-k$ excedances. Theorem 5.3.1 then implies that $\dim_{\mathbb{R}}(\Xi_{d-k}(\pi_d) \cdot \mathtt{E}_{\mathrm{X}}) = 1$.

It follows from Example 5.1.2 that the element x_{X} is nonzero, and counting the number of factors in (5.3.8) shows that $x_{\mathrm{X}} \in \Xi_{d-k}(\pi_d)$. Thus, we are only left to prove that $x_{\mathrm{X}} \in \Xi_{d-k}(\pi_d) \cdot \mathtt{E}_{\mathrm{X}}$. That is, that $x_{\mathrm{X}} \cdot \mathtt{E}_{\mathrm{X}} = x_{\mathrm{X}}$

Let $G \in \Sigma[\mathcal{A}_d]$ with $\mathrm{s}(G) > \mathrm{X}$. Then, for some block $S_i \in \mathrm{X}$ and some $a \in S_i$, a and $\min(S_i)$ are not in the same block of $\mathrm{s}(G)$. Hence $[\Delta_{\{\min(S_i), a\}}] \cdot \mathtt{H}_G$ is the class of a point, and $\log[\Delta_{\{\min(S_i), a\}}] \cdot \mathtt{H}_G = \log[\{\mathbf{0}\}] = 0$. Since the action of $\mathtt{H}_G$ is an algebra morphism, we get that $x_{\mathrm{X}} \cdot \mathtt{H}_G = 0$. Therefore,

$$x_{\mathrm{X}} \cdot \mathtt{E}_{\mathrm{X}} = x_{\mathrm{X}} \cdot \Big(\frac{1}{\dim(\mathrm{X})!} \sum_{\mathrm{s}(F) = \mathrm{X}} \mathtt{H}_F \Big) = \frac{1}{\dim(\mathrm{X})!} \sum_{\mathrm{s}(F) = \mathrm{X}} x_{\mathrm{X}} \cdot \mathtt{H}_F = x_{\mathrm{X}}. \qquad \square$$

The element x_{X} in the previous result is x_σ for the only permutation σ with $\mathrm{s}(\sigma) = \mathrm{X}$ and $\mathrm{exc}(\sigma) = d - |\mathrm{X}|$. In this case, the forest $t(\sigma)$ consists of a k trees with node sets $S_1, \ldots, S_k$, respectively. Each tree has $\min(S_i)$ as a root and every other element in S_i as a child of the root.

5.4 The module of type B generalized permutahedra

In this section, we study the algebra $\Pi(\pi_d^B)$ of *type B generalized permutahedra* and its structure as a module over the Tits algebra of the Coxeter arrangement of type B.

5.4.1 The hyperoctahedral group and the Type B Eulerian polynomial

The hyperoctahedral group $\mathfrak{B}_d$ is the group of bijections $\sigma : [\pm d] \to [\pm d]$ satisfying $\sigma(\bar{i}) = \overline{\sigma(i)}$ for all $i \in [\pm d]$ under composition. Elements in $\mathfrak{B}_d$ are called **signed permutations**. The group $\mathfrak{B}_d$ acts on $\mathbb{R}^d$ by permutation and sign changes of coordinates:

$$\sigma(\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots, \boldsymbol{x}_d) = (\boldsymbol{x}_{\sigma(1)}, \boldsymbol{x}_{\sigma(2)}, \ldots, \boldsymbol{x}_{\sigma(d)}).$$

Recall that, for instance, $\boldsymbol{x}_{\bar{1}} = -\boldsymbol{x}_1$. For a signed permutation $\sigma \in \mathfrak{B}_d$, we let $\mathrm{s}(\sigma)$ denote the subspace of points fixed by the action of σ; it is a flat of $\mathcal{A}_d^{\pm}$. Under the identification above, $\mathrm{s}(\sigma)$ is the signed partition of $[\pm d]$ obtained from the underlying the cycle decomposition of σ by merging all the blocks that contain an element i and its negative $\bar{i}$. For example, if in cycle notation $\sigma = (1)(\bar{1})(2\bar{2})(34\bar{3}\bar{4})(5\bar{6})(\bar{5}6)$, then $\mathrm{s}(\sigma) = \{2\bar{2}3\bar{3}4\bar{4}, 1, \bar{1}, 5\bar{6}, \bar{5}6\}$.

Let $\sigma \in \mathfrak{B}_n$. The restriction $\sigma|_{S_0}$ to the zero block $S_0 \in \mathrm{s}(\sigma)$ is a signed permutation of S_0. Its action on $\mathbb{R}^{|S_0|/2}$ does not fix any nonzero vector, so $\mathrm{s}(\sigma|_{S_0}) = \perp$. For a nonzero block $S \in \mathrm{s}(\sigma)$, $\sigma|_S \in \mathfrak{C}(S)$ is a cyclic permutation of the elements in S. The restriction $\sigma|_{\pm S}$ is again a signed permutation, and it is completely determined by either $\sigma|_S$ or $\sigma|_{\overline{S}}$.

We present some statistics on signed permutations. For $\sigma \in \mathfrak{B}_d$, let

$$\begin{aligned}
\mathrm{Des}(\sigma) &= \{i \in [d-1] \cup \{0\} \,:\, \sigma(i) > \sigma(i+1)\} & \mathrm{des}(\sigma) &= \big|\mathrm{Des}(\sigma)\big| \\
\mathrm{Exc}(\sigma) &= \{i \in [d-1] \,:\, \sigma(i) > i\} & \mathrm{exc}(\sigma) &= \big|\mathrm{Exc}(\sigma)\big| \\
\mathrm{Neg}(\sigma) &= \{i \in [d] \,:\, \sigma(i) < 0\} & \mathrm{neg}(\sigma) &= \big|\mathrm{Neg}(\sigma)\big| \\
& & \mathrm{fexc}(\sigma) &= 2\,\mathrm{exc}(\sigma) + \mathrm{neg}(\sigma),
\end{aligned}$$

where we set $\sigma(0) = 0$. Elements in the sets above are **descents**, **excedances** and **negations** of σ, respectively. The last statistic is called the **flag-excedance** of a signed permutation. We define one last statistic, the B-excedance of σ:

$$\mathrm{exc}_B(\sigma) = \lfloor \tfrac{\mathrm{fexc}(\sigma)+1}{2} \rfloor = \mathrm{exc}(\sigma) + \lfloor \tfrac{\mathrm{neg}(\sigma)+1}{2} \rfloor. \tag{5.4.1}$$

Foata and Han [41, Section 9] show that descents and B-excedances are equidistributed. That is,

$$B_{d,k} := \big|\{\sigma \in \mathfrak{B}_d \,:\, \mathrm{des}(\sigma) = k\}\big| = \big|\{\sigma \in \mathfrak{B}_d \,:\, \mathrm{exc}_B(\sigma) = k\}\big|,$$

for all possible values of k. The numbers $B_{d,k}$ are the **Eulerian numbers of type B** (OEIS: A060187). The **type B Eulerian polynomial** $B_d(z)$ is:

$$B_d(z) = \sum_{k=0}^{d} B_{d,k} z^k = \sum_{\sigma \in \mathfrak{B}_d} z^{\mathrm{exc}_B(\sigma)}.$$

The exponential generating function of these polynomials is first due to Brenti [30, Theorem 3.4]. We will be interested in the **type B exponential generating function** of these polynomials:

$$B(z,x) = 1 + \sum_{d \geq 1} B_d(z) \frac{x^d}{(2d)!!} = \frac{(1-z)e^{x(1-z)/2}}{1 - ze^{x(1-z)}}, \tag{5.4.2}$$

where $(2d)!!$ is the double factorial $(2d)!! = (2d)(2d-2)\ldots 2 = 2^d d!$. Substituting x by $2x$ one recovers Brenti's original formula.

5.4.2 The module of type B Generalized permutahedra

The type B permutahedron $\pi_d^B \subseteq \mathbb{R}^d$ is the convex hull of the $\mathfrak{B}_d$-orbit of the point $(1, 2, \ldots, d)$. It is full-dimensional and a zonotope of $\mathcal{A}_d^{\pm}$. We now consider the module $\Pi(\pi_d^B)$. The main result of this section is the following.

Theorem 5.4.1. *For any flat* $\mathrm{X} \in \mathcal{L}[\mathcal{A}_d^{\pm}]$ *and* $r = 0, 1, \ldots, d$*,*

$$\eta_{\mathrm{X}}(\Xi_r(\pi_d^B)) = \big|\{\sigma \in \mathfrak{B}_d \,:\, \mathrm{s}(\sigma) = \mathrm{X},\ \mathrm{exc}_B(\sigma) = r\}\big|.$$

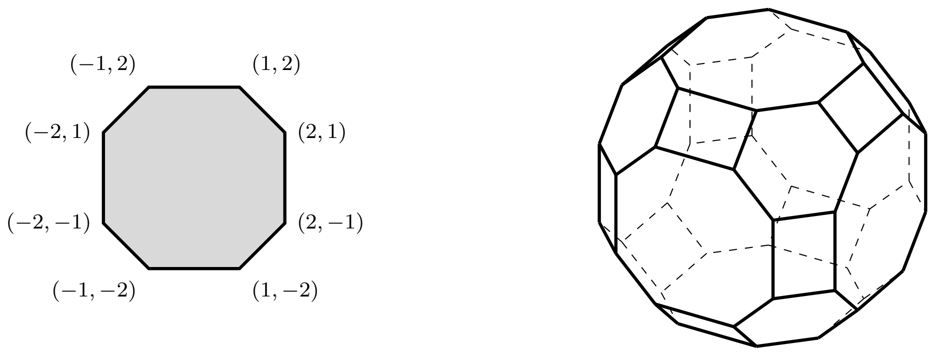


Figure 5.4.1: Type B permutahedron in $\mathbb{R}^2$ and $\mathbb{R}^3$.

As in type A, the relation between $\Pi(\pi_d^B)$ and statistics on $\mathfrak{B}_d$ is due to Brenti's result showing that $h(\pi_d^B, z) = B_d(z)$. For a flat $\mathrm{X} = \{S_0, S_1, \overline{S_1}, \ldots, S_k, \overline{S_k}\}$ of $\mathcal{A}_d^{\pm}$, the face $(\pi_d^B)_{\mathrm{X}}$ is a translate of $\pi^B_{|S_0|/2} \times \pi_{|S_1|} \times \cdots \times \pi_{|S_k|}$, a product of lower-dimensional permutahedra of type A and B, where exactly one factor is of type B. Thus,

$$h((\pi_d^B)_{\mathrm{X}}, z) = B_{|S_0|/2}(z) \cdot A_{|S_1|}(z) \cdot \ldots \cdot A_{|S_k|}(z). \tag{5.4.3}$$

The following result is the analogous in type B of Lemma 5.3.2.

Lemma 5.4.2. *For every* $d \geq 1$*,*

$$\sum_{\mathrm{X}=\{S_0,\ldots,S_k,\overline{S_k}\}\vdash^B[\pm d]} \mu(\bot, \mathrm{X}) B_{|S_0|/2}(z) \cdot A_{|S_1|}(z) \cdot \ldots \cdot A_{|S_k|}(z) = \sum_{\substack{\sigma \in \mathfrak{B}_d \\ \mathrm{s}(\sigma)=\bot}} z^{\mathrm{exc}_B(\sigma)}. \tag{5.4.4}$$

In the same spirit as the proof of Lemma 5.3.2, we will establish (5.4.4) by comparing the type B exponential generating function of both sides of the equality. An important tool in this proof is the following analog of the compositional formula for type B generating functions. For a proof, see Proposition 6.6.4.

Proposition 5.4.3 (Type B Compositional Formula - power series)**.** *Let*

$$f(x) = 1 + \sum_{d\geq 1} f_d \frac{x^d}{(2d)!!} \qquad g(x) = 1 + \sum_{d\geq 1} g_d \frac{x^d}{(2d)!!} \qquad a(x) = \sum_{d\geq 1} a_d \frac{x^d}{d!}.$$

If

$$h(x) = 1 + \sum_{d\geq 1} h_d \frac{x^d}{(2d)!!} \quad \textit{where} \quad h_d = \sum_{\{S_0, S_1, \overline{S_1}, \ldots, S_k, \overline{S_k}\} \vdash^B [\pm d]} f_{|S_0|/2} g_k a_{|S_1|} \ldots a_{|S_k|},$$

then

$$h(x) = f(x) g(a(x)).$$

Taking $g_d = 1$ in the Type B Compositional Formula we deduce the following.

Corollary 5.4.4 (Type B Exponential Formula)**.** *Let $f(x)$ and $a(x)$ be as before. If*

$$h(x) = 1 + \sum_{d\geq 1} h_d \frac{x^d}{(2d)!!} \qquad \textit{where} \qquad h_d = \sum_{\mathrm{X} \vdash^B [\pm d]} f_{|S_0|/2} a_{|S_1|} \ldots a_{|S_k|},$$

then

$$h(x) = f(x) \exp(a(x)/2).$$

In the proof of Theorem 5.3.1, we used that for (type A) permutations $\sigma \in \mathfrak{S}_d$, $\mathrm{exc}(\sigma)$ equals the sum of $\mathrm{exc}(\sigma|_S)$ as S runs through the blocks of $\mathrm{s}(\sigma)$. For signed permutations, one easily checks this also holds for the statistics exc and neg. However, it is not obvious at all that the same is true for exc_B, since its definition uses the floor function.

Consider the order $\prec$ of the elements of any involution-exclusive subset $S \subseteq [\pm d]$ defined by:

$$i \prec j \iff \begin{cases} 0 < i < j, \text{ or} \\ i < 0 < j, \text{ or} \\ j < i < 0. \end{cases} \tag{5.4.5}$$

Proposition 5.4.5. *Let $\sigma \in \mathfrak{B}_d$ and $\mathrm{s}(\sigma) = \{S_0, S_1, \overline{S_1}, \ldots, S_k, \overline{S_k}\}$. Then,*

$$\mathrm{exc}_B(\sigma) = \mathrm{exc}_B(\sigma|_{S_0}) + \mathrm{exc}_\prec(\sigma|_{S_1}) + \cdots + \mathrm{exc}_\prec(\sigma|_{S_k}),$$

where $\mathrm{exc}_\prec(\sigma|_{S_i})$ is the number of usual (type A) excedances of $\sigma|_{S_i}$ with respect to the order $\prec$.

Proof. For $i \geq 1$, write $\sigma|_{S_i} = (j_1 j_2 \ldots j_\ell)$ in cycle notation. Since $\sigma|_{\overline{S_i}} = (\overline{j_1}\,\overline{j_2} \ldots \overline{j_\ell})$, negations of $\sigma|_{\pm S_i}$ are in correspondence with changes of sign in the sequence

$$j_1 \mapsto j_2 \mapsto \cdots \mapsto j_\ell \mapsto j_1.$$

It follows that

$$\mathrm{neg}(\sigma|_{\pm S_i}) = 2 \cdot \left|\{j \in S_i \,:\, j < 0 < \sigma(j)\}\right|$$

is an even number.

Observe that according to the three cases of definition (5.4.5), a $\prec$-excedance of $\sigma|_{S_i}$ corresponds to either

$$\begin{cases} \text{an excedance of } \sigma|_{\pm S_i} \text{ occurring in } S_i, \text{ or} \\ \text{a negation of } \sigma|_{\pm S_i} \text{ occurring in } \overline{S_i}, \text{ or} \\ \text{an excedance of } \sigma|_{\pm S_i} \text{ occurring in } \overline{S_i}, \end{cases}$$

respectively. Since exactly half of the negations of $\sigma|_{\pm S_i}$ occur in $\overline{S_i}$, we deduce that

$$\mathrm{exc}_{\prec}(\sigma|_{S_i}) = \mathrm{exc}(\sigma|_{\pm S_i}) + \tfrac{\mathrm{neg}(\sigma|_{\pm S_i})}{2}.$$

Thus, in view of (5.4.1) and using that $\tfrac{\mathrm{neg}(\sigma|_{\pm S_i})}{2}$ is always an integer,

$$\begin{aligned}
\mathrm{exc}_B(\sigma) &= \mathrm{exc}(\sigma) + \lfloor \tfrac{\mathrm{neg}(\sigma)+1}{2} \rfloor \\
&= \mathrm{exc}(\sigma|_{S_0}) + \sum_i \mathrm{exc}(\sigma|_{\pm S_i}) + \lfloor \tfrac{\mathrm{neg}(\sigma|_{S_0}) + \sum_i \mathrm{neg}(\sigma|_{\pm S_i}) + 1}{2} \rfloor \\
&= \mathrm{exc}(\sigma|_{S_0}) + \sum_i \mathrm{exc}(\sigma|_{\pm S_i}) + \lfloor \tfrac{\mathrm{neg}(\sigma|_{S_0})+1}{2} \rfloor + \sum_i \tfrac{\mathrm{neg}(\sigma|_{\pm S_i})}{2} \\
&= \mathrm{exc}_B(\sigma|_{S_0}) + \mathrm{exc}_{\prec}(\sigma|_{S_1}) + \cdots + \mathrm{exc}_{\prec}(\sigma|_{S_k}).
\end{aligned}$$

□

Proof of Lemma 5.4.2. Recall that $\mu(\perp, \mathrm{X}) = (-1)^k (2k-1)!!$, where $|\mathrm{X}| = 2k+1$. Observe that

$$1 + \sum_{d \geq 1} (-1)^d (2d-1)!! \frac{x^d}{(2d)!!} = \sum_{d \geq 0} \binom{-1/2}{d} x^d = (1+x)^{-1/2}.$$

Using the Type B Compositional formula, we conclude that the type B exponential generating function of the LHS of (5.4.4) is

$$B(z,x)(1 + (A(z,x) - 1))^{-1/2} - 1 = \frac{B(z,x)}{\sqrt{A(z,x)}} - 1,$$

where $A(z,x)$ and $B(z,x)$ are the generating functions in (5.3.1) and (5.4.2), respectively. On the other hand, Proposition 5.4.5 shows that for each partition $\mathrm{X} = \{S_0, S_1, \overline{S_1}, \ldots, S_k, \overline{S_k}\} \vdash^B [\pm d]$,

$$\sum_{\substack{\sigma \in \mathfrak{B}_d \\ \mathrm{s}(\sigma) = \mathrm{X}}} z^{\mathrm{exc}_B(\sigma)} = \Big(\sum_{\substack{\sigma_0 \in \mathfrak{B}(S_0) \\ \mathrm{s}(\sigma_0) = \perp}} z^{\mathrm{exc}_B(\sigma_0)} \Big) \prod_{i=1}^{k} \Big(\sum_{\sigma \in \mathfrak{C}(|S_i|)} z^{\mathrm{exc}(\sigma)} \Big). \tag{5.4.6}$$

In the proof of Lemma 5.3.2, we showed that the (usual) exponential generating function of the terms in the product is $\log(A(z,x))$. An application of the type B

Exponential Formula and (5.4.6) yields

$$B(z,x) = \left(1 + \sum_{d\geq 1}\Big(\sum_{\substack{\sigma\in\mathfrak{B}_d\\ \mathrm{s}(\sigma)=\perp}} z^{\mathrm{exc}_B(\sigma)}\Big)\frac{x^d}{(2d)!!}\right)\exp\left(\frac{\log(A(z,x))}{2}\right)$$

Dividing both sides by $\exp\left(\frac{\log(A(z,x))}{2}\right) = \sqrt{A(z,x)}$ and subtracting 1 yields the result. □

We are now ready to complete the proof of the main theorem in this section. The steps of the proof mirror those of the type A result.

Proof of Theorem 5.4.1. We will again use formula (5.2.7) to compute the values $\eta_{\mathrm{X}}(\Xi_r(\pi_d^B))$. Recall that the formula reads

$$\sum_r \eta_{\mathrm{X}}(\Xi_r(\pi_d^B))z^r \sum_{\mathrm{Y}\geq\mathrm{X}} \mu(\mathrm{X},\mathrm{Y})h((\pi_d^B)_{\mathrm{Y}}, z).$$

Using formulas (2.2.7) and (5.4.3), we see that for a flat $\mathrm{X} = \{S_0,\ldots,S_k,\overline{S_k}\}$ this expression equals

$$\left(\sum_{\substack{\mathrm{Y}\vdash^B S_0\\ \mathrm{Y}=\{T_0,\ldots,T_\ell,\overline{T_\ell}\}}} \mu(\perp,\mathrm{Y})B_{|T_0|/2}A_{|T_1|}\ldots A_{|T_\ell|}\right)$$
$$\cdot\prod_{i=1}^k\left(\sum_{\substack{\mathrm{Y}_i\vdash S_i\\ \mathrm{Y}_i=\{T_1^i,\ldots,T_\ell^i\}}} \mu(\perp,\mathrm{Y}_i)A_{|T_1^i|}(z)\cdot\ldots\cdot A_{|T_\ell^i|}(z)\right).$$

Using Lemmas 5.3.2 and 5.4.2 in each factor, we deduce

$$\sum_r \eta_{\mathrm{X}}(\Xi_r(\pi_d^B))z^r = \Big(\sum_{\substack{\sigma\in\mathfrak{B}(S_0)\\ \mathrm{s}(\sigma)=\perp}} z^{\mathrm{exc}_B(\sigma)}\Big)\prod_{i=1}^k\Big(\sum_{\sigma\in\mathfrak{C}(|S_i|)} z^{\mathrm{exc}(\sigma)}\Big) = \sum_{\substack{\sigma\in\mathfrak{B}_d\\ \mathrm{s}(\sigma)=\mathrm{X}}} z^{\mathrm{exc}_B(\sigma)},$$

where the last equality is (5.4.6). Finally, taking the coefficient of z^r on both sides yields the result. □

Adding over all flats with the same dimension in Theorem 5.4.1, we conclude the following.

Corollary 5.4.6. *Let $w \in \mathbb{R}\Sigma[\mathcal{A}_d^{\pm}]$ be a characteristic element of non-critical parameter t and $\lambda > 0$. The dimension of the simultaneous eigenspace for w and δ_λ with eigenvalues t^k and λ^r is*

$$\left|\{\sigma \in \mathfrak{B}_d \,:\, \dim(\mathrm{s}(\sigma)) = k,\ \mathrm{exc}_B(\sigma) = r\}\right|.$$

As in the type A case, we can modify the proof of the previous Lemma to obtain the generating function for the bivariate polynomials $\sum_{\sigma \in \mathfrak{B}_d} t^{\dim(\mathrm{s}(\sigma))} z^{\mathrm{exc}_B(\sigma)}$. To the best of our knowledge, this is a new result.

Proposition 5.4.7. *The following identity holds*

$$1 + \sum_{d \geq 1} \Big(\sum_{\sigma \in \mathfrak{B}_d} t^{\dim(\mathrm{s}(\sigma))} z^{\mathrm{exc}_B(\sigma)} \Big) \frac{x^n}{(2n)!!} = \frac{(1-z)e^{x(1-z)/2}}{1 - ze^{x(1-z)}} \left(\frac{z-1}{z - e^{x(z-1)}} \right)^{\frac{t-1}{2}}.$$

Proof. We can slightly modify (5.4.6) to obtain

$$\sum_{\substack{\sigma \in \mathfrak{B}_d \\ \mathrm{s}(\sigma) = \mathrm{X}}} t^{\dim(\mathrm{X})} z^{\mathrm{exc}_B(\sigma)} = \Big(\sum_{\substack{\sigma \in \mathfrak{B}(S_0) \\ \mathrm{s}(\sigma) = \perp}} z^{\mathrm{exc}_B(\sigma_0)} \Big) \prod_{i=1}^{k} \Big(t \sum_{\sigma \in \mathfrak{C}(|S_i|)} z^{\mathrm{exc}(\sigma)} \Big).$$

Using the (type B) generating functions of the factors deduced in the proofs of Lemmas 5.3.2 and 5.4.2, and the type B compositional formula, we deduce

$$1 + \sum_{d \geq 1} \Big(\sum_{\sigma \in \mathfrak{B}_d} t^{\dim(\mathrm{s}(\sigma))} z^{\mathrm{exc}_B(\sigma)} \Big) \frac{x^n}{(2n)!!} = \frac{B(z,x)}{\sqrt{A(z,x)}} \exp\Big(\frac{t \log(A(z,x))}{2} \Big),$$

which equals $B(z,x) A(z,x)^{\frac{t-1}{2}}$. Substituting the expressions for $A(z,x)$ and $B(z,x)$ in (5.3.1) and (5.4.2) yields the result. □

Specializing $t := 0$ and $x := 2x$ gives an alternative expression for the exponential generating function of the OEIS sequence A156919. In our context, these coefficients count the number of signed permutations whose action on $\mathbb{R}^d$ has no nonzero fixed point weighted by the statistic exc_B.

5.4.3 Two bases for type B generalized permutahedra

Faces of the standard simplex $\Delta_{[d]}$ correspond to linearly independent rays of the cone of generalized permutahedra in $\mathbb{R}^d$. In the language of the present work, this means that $\{\log[\Delta_S] : S \subseteq [d], |S| \geq 2\}$ forms a linear basis of $\Xi_1(\pi_d)$ and that $\log[\Delta_S]$ is not the sum of the log-classes of other generalized permutahedra (other than trivial dilations of itself). The goal of this section is to establish an analogous result for the type B case.

The standard simplex coincides with the *weight polytope* $P_{\mathfrak{S}_d}(\lambda_1)$ in the sense of Ardila, Castillo, Eur, and Postnikov [14], where λ_1 is the fundamental weight $(1, 0, \ldots, 0)$ of $\mathfrak{S}_d$. In this manner, the cross-polytope $P_{\mathfrak{B}_d}(\lambda_1) = \mathrm{Conv}\{\pm \boldsymbol{e}_i : i \in [d]\}$ is the type B analog of the standard simplex. However, in the same paper the authors point out that the faces of the cross-polytope span a space of roughly half the desired dimension. This is intuitively clear once we notice that the collection of faces of $P_{\mathfrak{B}_d}(\lambda_1)$ entirely contained in one orthant span the same space in $\Xi_1(\pi_d^B)$ as the faces contained in the opposite orthant. The following result shows that it is not possible to find a single type B generalized permutahedron whose faces generate $\Xi_1(\pi_d^B)$.

Proposition 5.4.8. *Let $\mathcal{P} = \{P_\alpha\}_\alpha$ be a collection of type B generalized permutahedra such that $\{\log[P_\alpha]\}_\alpha$ spans $\Xi_1(\pi_d^B)$. Then, $\mathcal{P}$ contains at least 2^{d-1} full dimensional polytopes.*

Proof. Let $\{\mathtt{E}_{\mathrm{X}}\}_{\mathrm{X}} \subseteq \mathbb{R}\Sigma[\mathcal{A}_d^{\pm}]$ be an Eulerian family of $\mathcal{A}_d^{\pm}$. The result follows from the following two facts, which we justify below.

1. The projection of $\log[P]$ to the subspace $\Xi_1(\pi_d^B) \cdot \mathtt{E}_\perp$ is zero unless P is full-dimensional.
2. $\dim_{\mathbb{R}}(\Xi_1(\pi_d^B) \cdot \mathtt{E}_\perp) = 2^{d-1}$.

1. Let P be a type B generalized permutahedron that is not full-dimensional. Then we can choose a face $F \neq O$ of $\mathcal{A}_d^{\pm}$ such that $P_F = P$, for instance any maximal face in $N_P P$. It follows from Lemma 5.2.4 that $\mathtt{H}_F \cdot \mathtt{E}_\perp = 0$. Therefore,

$$\log[P] \cdot \mathtt{E}_\perp = \log([P] \cdot \mathtt{H}_F) \cdot \mathtt{E}_\perp = \log[P] \cdot (\mathtt{H}_F \cdot \mathtt{E}_\perp) = 0.$$

2. Recall that by definition, $\dim_{\mathbb{R}}(\Xi_1(\pi_d^B) \cdot \mathtt{E}_\perp) = \eta_\perp(\Xi_1(\pi_d^B))$. Using Theorem 5.4.1, this is the number of signed permutations $\sigma \in \mathfrak{B}_d$ with $\mathrm{s}(\sigma) = \perp$ and $\mathrm{exc}_B(\sigma) = 1$. A signed permutation $\sigma \in \mathfrak{B}_d$ with $\mathrm{s}(\sigma) = \perp$ is the product of cycles on involution-inclusive subsets $S \subseteq [\pm d]$. Moreover, each such cycle adds at least 1 to the number of negations of σ. Thus, a signed permutation with $\mathrm{s}(\sigma) = \perp$ and $\mathrm{exc}_B(\sigma) = 1$ must be the product of either 1 or 2 cycles and have no excedances. Such cycles are necessarily of the form $(d\, d-1 \ldots 1 \overline{d}\, \overline{d-1} \ldots \overline{1})$. Thus, the permutations counted by $\eta_\perp(\Xi_1(\pi_d^B))$ are in correspondence with *unordered* pairs $\{S, [\pm d] \setminus S\}$ of involution-inclusive subsets of $[\pm d]$, and there are precisely 2^{d-1} many of them.

Alternatively, one can manipulate the generating function in Proposition 5.4.7 (differentiate with respect to z and specialize $t = z = 0$) to deduce that the coefficient of $t^0 z^1 x^d$ is 2^{d-1}. □

In [59], Padrol, Pilaud, and Ritter construct a family of type B generalized permutahedra called *shard polytopes*. They show that any type B generalized permutahedron can be written uniquely as a signed Minkowski sum of these polytopes (up to translation). Already in $\mathbb{R}^3$, there are 14 full-dimensional shard polytopes. We proceed to construct a family of generators that achieves the minimum imposed by the previous proposition.

For a nonempty involution-exclusive subset $S \subseteq [\pm d]$, define the simplices

$$\Delta_S = \text{Conv}\{\boldsymbol{e}_i \mid i \in S\} \qquad \Delta_S^0 = \text{Conv}(\{\boldsymbol{0}\} \cup \{\boldsymbol{e}_i \ : \ i \in S\}),$$

where for $i \in [d]$, $\boldsymbol{e}_{\bar{i}} = -\boldsymbol{e}_i$. Observe that the only full-dimensional simplices in this collection are Δ_S^0 with $|S| = d$. We say that an involution-exclusive subset $S \subseteq [\pm d]$ is **special** if in addition $\min\{|i| \ : \ i \in S\} \in S$. Observe that for any nonempty involution-exclusive subset $S \subseteq [\pm d]$, exactly one of $\{S, \overline{S}\}$ is *special*.

Theorem 5.4.9. *Every type B generalized permutahedron can be written uniquely as a signed Minkowski sum of the simplices Δ_S and Δ_S^0 with S special.*

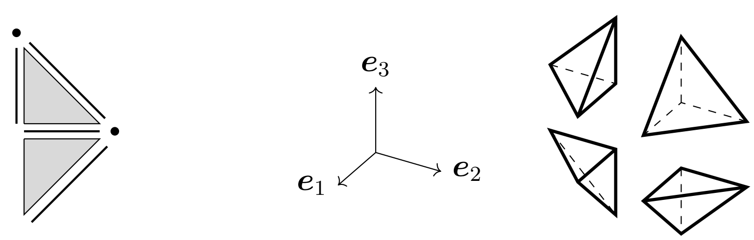


Figure 5.4.2: The type B *generators* of Theorem 5.4.9 in $\mathbb{R}^2$. In $\mathbb{R}^3$, we only show the $4 = 2^{3-1}$ full-dimensional generators.

Remark 5.4.10. Observe that the generating collection $\{\Delta_S \ : \ S \subseteq [d]\}$ for generalized permutahedra is invariant under the action of $\mathfrak{S}_d$. In contrast, the collection of generators for type B generalized permutahedra presented in the previous theorem fails to be invariant under the action of $\mathfrak{B}_d$. This is not by accident. Already in $\mathbb{R}^2$, we see that any collection of type B generalized permutahedra that contains a triangle P (full-dimensional simplex) and that is invariant under the action of $\mathfrak{B}_d$, will contain the rotations of P by 90°, 180°, and 270°, all of which are necessarily different. Thus, such a collection will not attain the minimum number of full-dimensional polytopes required by Proposition 5.4.8.

Before proving the previous theorem, we exhibit a different generating collection of type B generalized permutahedra that is invariant under the action of $\mathfrak{B}_d$. Then, we will show that the collection if the previous theorem generates the same collection of polytopes.

Given a collection of real numbers $\{z_S\}_S \subseteq \mathbb{R}$, one for each nonempty involution-exclusive subset $S \subseteq [\pm d]$, consider the polytope

$$P(\{z_S\}) = \Big\{ \boldsymbol{x} \in \mathbb{R}^d \ : \ \sum_{i \in S} \boldsymbol{x}_i \leq z_S \text{ for all } S \Big\}. \tag{5.4.7}$$

Assume that all the inequalities above are tight. Then, Arcila [11] showed that $P(\{z_S\})$ is a type B generalized permutahedron if and only if the values z_S form a *bisubmodular function* in the sense of Fujishige [42]. That is,

$$z_S + z_J \geq z_{S \cap J} + z_{S \uplus J},$$

where $S \uplus J = (S \cup J) \setminus (\overline{S} \cup \overline{J})$ and we set $z_\emptyset = 0$. Moreover, every type B generalized permutahedron can be written uniquely in this manner. See also [14].

Since $z_S = \max\big\{ \langle \boldsymbol{e}_S, \boldsymbol{p} \rangle \ : \ \boldsymbol{p} \in P(\{z_S\}) \big\}$, it follows that

$$P(\{z_S\}) + \lambda P(\{z'_S\}) = P(\{z_S + \lambda z'_S\})$$

for any $\lambda \in \mathbb{R}$ such that the (signed) Minkowski sum $P(\{z_S\}) + \lambda P(\{z'_S\})$ is defined.

Proposition 5.4.11. *For any $\{y_S\} \subseteq \mathbb{R}$ such that the signed Minkowski sum $\sum_S y_S \Delta^0_S$ is defined,*

$$\sum_S y_S \Delta^0_S = P(\{z_S\}),$$

where $z_S = \displaystyle\sum_{J\,:\,J\cap S\neq\emptyset} y_J$.

Proof. The result immediately follows from the observation preceding the proposition and the following simple computation:

$$\max\big\{\langle \boldsymbol{e}_S, \boldsymbol{p}\rangle \,:\, \boldsymbol{p}\in\Delta^0_J\big\} = \begin{cases} 1 & \text{if } S\cap J\neq\emptyset,\\ 0 & \text{otherwise.}\end{cases} \qquad \square$$

Theorem 5.4.12. *Every type B generalized permutahedron can be written uniquely as a signed Minkowski sum of the simplices* $\{\Delta^0_S \,:\, S\subseteq[\pm d] \text{ involution-exclusive}\}$. *Explicitly,*

$$P(\{z_S\}) = \sum_S y_S\Delta^0_S, \tag{5.4.8}$$

where $y_S = (-1)^{d-|S|-1}\displaystyle\sum_{J\,:\,J\cap\overline{S}=\emptyset}(-1)^{|J|}z_J$.

Proof. To prove (5.4.8), it is enough to show that for all involution-exclusive $S\subseteq[\pm d]$,

$$\max\Big\{\langle \boldsymbol{e}_S, \boldsymbol{p}\rangle \,:\, \boldsymbol{p}\in\sum_K y_K\Delta^0_K\Big\} = z_S.$$

That is, that for all S

$$\sum_{K\,:\,K\cap S\neq\emptyset}(-1)^{d-|K|-1}\Big(\sum_{J\,:\,J\cap\overline{K}=\emptyset}(-1)^{|J|}z_J\Big) = z_S.$$

Regrouping terms, the previous sum is

$$\sum_J(-1)^{d-|J|-1}\Big(\sum_{K\,:\,\substack{K\cap\overline{J}=\emptyset\\ K\cap S\neq\emptyset}}(-1)^{|K|}\Big)z_J$$

So we are only left to show that the internal sum is zero whenever $J\neq S$, and is $(-1)^{d-|S|-1}$ for $J=S$. We consider the different cases below. Recall that all sets we consider are nonempty and involution-exclusive.

If there is $j \in J \setminus S$, we can pair sets K appearing in the sum in following manner: $K \leftrightarrow K\Delta\{j\}$, where Δ denotes the symmetric difference. Indeed, since $j \notin \overline{J}$ and $j \notin S$, a set K appears in the sum if and only if $K\Delta\{j\}$ does (the condition $K \cap S \neq \emptyset$ guarantees $K \setminus \{j\}$ is nonempty). Since $(-1)^{|K|} + (-1)^{|K\Delta\{j\}|} = 0$, the overall sum is zero. Thus, from this point we assume $J \subseteq S$

Suppose there is $s \in S \setminus J$, and pick $j \in J$. We divide the sets K appearing in the sum in two groups: those containing s and those not containing s. We pair the sets in the first group via $K \leftrightarrow K\Delta\{j\}$. Since $K \cap \overline{J} = \emptyset$, no set contains $\overline{j}$, thus $K\Delta\{j\}$ is involution exclusive and appears in the sum if and only if K does. Now, we pair the sets in the second group (those not containing s) via $K \leftrightarrow K\Delta\{\overline{s}\}$. Since $s \notin J$ and $\overline{s} \notin S$, $K\Delta\{\overline{s}\} \cap \overline{J} \neq \emptyset$ appears in the sum if and only if $K \cap \overline{J} = \emptyset$ does.

Finally, suppose $J = S$. Write any K appearing in the sum as $A \cup B$, where $A = K \cap S$ and $B = K \setminus S$. Then, the internal sum above becomes

$$\Big(\sum_{\emptyset \neq A \subseteq S} (-1)^{|A|} \Big) \Big(\sum_{B \subseteq [\pm d] \setminus \pm S} (-1)^{|B|} \Big).$$

By the binomial theorem, the first factor is $(1-1)^{|S|} - (-1)^0 = -1$. For the second factor, choosing B in the sum corresponds to chose whether $i \in B$, $\overline{i} \in B$, or $i, \overline{i} \notin B$ for each $i \in [d] \setminus \pm S$. Hence, the second factor equals

$$\sum_{\boldsymbol{v} \in \{-1,0,1\}^{d-|S|}} (-1)^{|\boldsymbol{v}|} = (-1+1-1)^{d-|S|} = (-1)^{d-|S|}$$

where $|\boldsymbol{v}|$ denotes the number of nonzero entries of $\boldsymbol{v}$. Thus, identity (5.4.8) is proved. Uniqueness follows from the number of elements in the generating set. $\square$

Proof of Theorem 5.4.9. Observe that there are exactly d zero-dimensional polytopes in this collection: Δ_S with $|S| = 1$. Since they minimally generate all translations in $\mathbb{R}^d$ and $h_1(\pi_d^B) = 3^d - d - 1$, the statement is equivalent to showing that the collection

$$\{\log[\Delta_S] : S \text{ special}, |S| \geq 2\} \cup \{\log[\Delta_S^0] : S \text{ special}\} \tag{5.4.9}$$

spans $\Xi_1(\pi_d^B)$. By Theorem 5.4.12, the collection $\{\log[\Delta_S^0] : S$ involution-exclusive$\}$ generates $\Xi_1(\pi_d^B)$, so it suffices to write each $\log[\Delta_S^0]$ as a linear combination of the elements in (5.4.9). We do this inductively on $|S|$.

Since $\Delta_{\bar{i}}^0$ is a translate of Δ_i^0, we have that $\log[\Delta_{\bar{i}}^0] = \log[\Delta_i^0]$ is in (5.4.9) for all $i \in [\pm d]$ (either i or $\bar{i}$ is in $[d]$).

We use $\lambda = -1$ and $r = 1$ in Theorem 5.1.6 to conclude that for any polytope P:

$$\log[-P] = -\sum_{Q \leq P} (-1)^{\dim(Q)} \log[Q].$$

Applying this identity to $P = \Delta_S^0 = -\Delta_{\bar{S}}^0$ yields

$$\log[\Delta_{\bar{S}}^0] = -\sum_{J \subsetneq S} (-1)^{|J|} \log[\Delta_J^0] + \sum_{J \subseteq S} (-1)^J \log[\Delta_J].$$

The result now follows by induction. □

5.5 Hopf monoid structure

Aguiar and Ardila introduced the Hopf monoid of generalized permutahedra GP in [1]. It contains many other interesting combinatorial Hopf monoids as submonoids. In this section we show that the valuation (5.1.1) and translation invariance (5.1.2) properties define a Hopf monoid quotient of GP.

5.5.1 The McMullen (co)ideal

As a species, $\mathsf{GP}[I]$ is the vector space with basis

$$\mathrm{GP}[I] = \{P \subseteq \mathbb{R}^I \,:\, P \text{ is a generalized permutahedron}\}.$$

The product $\mu_{S,T}$ is defined by

$$\mu_{S,T}(P \otimes Q) = P \times Q,$$

for all permutahedra $P \in \mathrm{GP}[S]$ and $Q \in \mathrm{GP}[T]$. In particular, GP is a commutative monoid. Let F be the face of the braid arrangement in $\mathbb{R}^I$ corresponding to the composition (S, T), and let $\boldsymbol{v} \in \mathrm{relint}(F)$. Then, for any $P \in \mathrm{GP}[I]$, the face $P_{\boldsymbol{v}}$ decomposes as a product of generalized permutahedra $P|_S \times P/_S$, with $P|_S \in \mathrm{GP}[S]$ and $P/_S \in \mathrm{GP}[T]$, see [1, Proposition 5.2]. The coproduct is defined by

$$\Delta_{S,T}(P) = P|_S \otimes P/_S.$$

Aguiar and Ardila also give the following grouping-free and cancellation-free formula for its antipode. For a generalized permutahedron $P \in \mathrm{GP}[I]$,

$$\mathrm{s}_I(P) = (-1)^{|I|} \sum_{Q \leq P} (-1)^{\dim(Q)} Q. \tag{5.5.1}$$

We now introduce the subspecies Mc of GP. The space $\mathsf{Mc}[I] \subseteq \mathsf{GP}[I]$ is the subspace spanned by elements

$$P \cup Q + P \cap Q - P - Q \qquad \text{for} \quad P, Q \in \mathrm{GP}[I] \text{ such that } P \cup Q \text{ is convex}, \tag{5.5.2}$$

and

$$P_{+\boldsymbol{t}} - P \qquad \text{for} \quad P \in \mathrm{GP}[I] \text{ and } \boldsymbol{t} \in \mathbb{R}^I, \tag{5.5.3}$$

where $P_{+\boldsymbol{t}}$ denotes the Minkowski sum $P + \{\boldsymbol{t}\}$. The sums and differences in (5.5.2) and (5.5.3) correspond to the vector space structure of $\mathsf{GP}[I]$, not to Minkowski sum or difference.

Remark 5.5.1. Recall from Section 5.1.2 that if $P \cup Q$ is a polytope, then $P \cup Q$ and $P \cap Q$ are necessarily generalized permutahedra. Thus, the elements (5.5.2) are indeed in $\mathsf{GP}[I]$.

The following result shows that Mc define relations compatible with the Hopf monoid structure of GP. Ardila and Sanchez [15] prove a similar result for extended generalized permutahedra by realizing Mc as the kernel of a Hopf monoid morphism.

Theorem 5.5.2. *The subspecies* Mc *is an ideal and a coideal of* GP*. That is,*

$$\mu_{S,T}\big(\mathsf{Mc}[S]\otimes\mathsf{GP}[T]\big) \subseteq \mathsf{Mc}[I] \quad \textit{and} \quad \Delta_{S,T}\big(\mathsf{Mc}[I]\big) \subseteq \mathsf{Mc}[S]\otimes\mathsf{GP}[T]+\mathsf{GP}[S]\otimes\mathsf{Mc}[T],$$

for any $I = S \sqcup T$*. Therefore, the quotient species* $\widetilde{\Pi}$ *defined by*

$$\widetilde{\Pi}[I] = \mathsf{GP}[I]/\mathsf{Mc}[I]$$

inherits the Hopf monoid structure of GP*.*

Proof. For generators of Mc of the form (5.5.3), the result follows from the following two observations. If $P \in \mathrm{GP}[S]$, $Q \in \mathrm{GP}[T]$ and $\boldsymbol{t} \in \mathbb{R}^S$, then

$$P_{+\boldsymbol{t}} \times Q = (P \times Q)_{+(\boldsymbol{t},\boldsymbol{0})}.$$

If $P \in \mathrm{GP}[I]$ and $\boldsymbol{t} \in \mathbb{R}^T$, then

$$\Delta_{S,T}(P_{+\boldsymbol{t}}) = (P|_S)_{+\boldsymbol{t}_S} \otimes (P|_S)_{+\boldsymbol{t}_T},$$

where $\boldsymbol{t}_S$ and $\boldsymbol{t}_T$ denote the projections of $\boldsymbol{t}$ to $\mathbb{R}^S$ and $\mathbb{R}^T$, respectively.

We will now focus on generators of Mc of the form (5.5.2). Fix an arbitrary finite set I and a nontrivial decomposition $I = S \sqcup T$. Let $\boldsymbol{v} \in \mathbb{R}^I$ be any vector in the interior of the corresponding face of the braid arrangement.

Suppose $P, P', P \cup P' \in \mathsf{GP}[S]$ and $Q \in \mathsf{GP}[T]$. Then,

$$(P \cup P') \times Q = (P \times Q) \cup (P' \times Q), \qquad (P \cap P') \times Q = (P \times Q) \cap (P' \times Q),$$

and $(P \cup P') \times Q = (P \times Q) \cup (P' \times Q)$ is a polytope if and only if $P \cup P'$ is. It follows that

$$\mu_{S,T}\big((P \cup P' + P \cap P' - P - P') \otimes Q\big)$$
$$= (P \times Q) \cup (P' \times Q) + (P \times Q) \cap (P' \times Q) - P \times Q - P' \times Q \in \mathsf{Mc}[I].$$

Since GP is commutative, this proves that Mc is an ideal.

Now, let $P, Q, P \cup Q \in \mathsf{GP}[I]$. There are two possibilities:

i. The face $(P \cup Q)_{\boldsymbol{v}}$ of $P \cup Q$ is completely contained in P or in Q. Without loss of generality, suppose the former. Then $(P \cup Q)_{\boldsymbol{v}} = P_{\boldsymbol{v}}$ and, necessarily, $(P \cap Q)_{\boldsymbol{v}} = Q_{\boldsymbol{v}}$. Hence,

$$\Delta_{S,T}\big(P \cup Q + P \cap Q - P - Q\big) = \Delta_{S,T}(P) + \Delta_{S,T}(Q) - \Delta_{S,T}(P) - \Delta_{S,T}(Q) = 0$$

ii. The face $(P \cup Q)_{\boldsymbol{v}}$ is not contained in P nor in Q. Hence, $(P \cup Q)_{\boldsymbol{v}} = P_{\boldsymbol{v}} \cup Q_{\boldsymbol{v}}$ and $(P \cap Q)_{\boldsymbol{v}} = P_{\boldsymbol{v}} \cap Q_{\boldsymbol{v}}$. Expanding the first equality we have

$$(P \cup Q)|_S \times (P \cup Q)/_S = (P|_S \times P/_S) \cup (Q|_S \times Q/_S).$$

The union of two Cartesian products $A \times B$ and $C \times D$ is again a Cartesian product if and only if one contains the other or either $A = C$ or $B = D$. By assumption, the is no containment between $P_{\boldsymbol{v}}$ and $Q_{\boldsymbol{v}}$. We can therefore assume without loss of generality that

$$P|_S = Q|_S. \tag{5.5.4}$$

Projecting to $\mathbb{R}^S$ and $\mathbb{R}^T$, we further see that

$$(P \cup Q)|_S = P|_S \cup Q|_S = P|_S \qquad \text{and} \qquad (P \cup Q)/_S = P/_S \cup Q/_S. \tag{5.5.5}$$

In particular, $P/_S \cup Q/_S$ is a generalized permutahedron. On the other hand, expanding $(P \cap Q)_{\boldsymbol{v}} = P_{\boldsymbol{v}} \cap Q_{\boldsymbol{v}}$, we have

$$(P \cap Q)|_S \times (P \cap Q)/_S = (P|_S \times P/_S) \cap (P|_S \times Q/_S) = P|_S \times (P/_S \cap Q/_S).$$

Comparing factors, we deduce

$$(P \cap Q)|_S = P|_S \qquad \text{and} \qquad (P \cap Q)/_S = P/_S \cap Q/_S. \tag{5.5.6}$$

Putting together (5.5.4), (5.5.5) and (5.5.6), we conclude

$$\begin{aligned} \Delta_{S,T}\big(P\cup Q + P\cap Q - P - Q\big) &= \Delta_{S,T}(P\cup Q) + \Delta_{S,T}(Q\cap Q) - \Delta_{S,T}(P) - \Delta_{S,T}(Q) \\ &= P|_S \otimes (P/_S \cup Q/_S) + P|_S \otimes (P/_S \cap Q/_S) - P|_S \otimes P/_S - P|_S \otimes Q/_S \\ &= P|_S \otimes \big(P/_S \cup Q/_S + P/_S \cap Q/_S - P/_S - Q/_S\big) \in \mathsf{GP}[S] \otimes \mathsf{Mc}[T]. \end{aligned}$$

Thus, in either case we get $\Delta_{S,T}\big(P \cup Q + P \cap Q - P - Q\big) \in \mathsf{Mc}[S] \otimes \mathsf{GP}[T] + \mathsf{GP}[S] \otimes \mathsf{Mc}[T]$. That is, Mc is a coideal of GP. $\square$

Comparing the generators of the (co)ideal Mc with the relations defining McMullen's polytope algebra, it is natural to ask if $\widetilde{\Pi}[I]$ agrees with $\Pi(\pi_I)$, where $\pi_I \subseteq \mathbb{R}^I$ is the standard permutahedron. The answer is no. For instance, in the polytope algebra, the structure of $\mathbb{R}$-vector space is defined so that

$$\alpha\big([\overset{1}{\underline{\quad\quad}}] - 1\big) = [\overset{\alpha}{\underline{\quad\quad\quad}}] - 1,$$

where the numbers over the segments denote their length. However, if α is an irrational number, then

$$\big(\alpha \overset{1}{\underline{\quad\quad}} - \alpha \bullet\big) - \big(\overset{\alpha}{\underline{\quad\quad\quad}} - \bullet\big) \notin \mathsf{Mc}[I].$$

Nevertheless, the proof of the previous theorem works verbatim to show that the Hopf monoid operations are well-defined in $\Pi(\pi_I)$.

Theorem 5.5.3. *The species* Π *defined by* $\Pi[I] = \Pi(\pi_I)$ *is a Hopf monoid, with product and coproduct defined for any decomposition* $I = S \sqcup T$ *by*

$$\mu_{S,T}([Q_1] \otimes [Q_2]) = [Q_1 \times Q_2] \qquad \textit{and} \qquad \Delta_{S,T}([P]) = [P|_S] \otimes [P/_S]$$

for all classes of generalized permutahedra $[Q_1] \in \Pi[S]$, $[Q_2] \in \Pi[T]$, *and* $[P] \in \Pi[I]$. *Moreover,* Π *is a Hopf monoid quotient of* GP *via the morphism* $P \mapsto [P]$.

It immediately follows from the definitions above that, if $F \in \Sigma[\mathcal{A}_d]$ is the face corresponding to a composition (S, T) of $[d]$, then

$$[P] \cdot \mathtt{H}_F = \mu_{S,T} \circ \Delta_{S,T}([P])$$

for all generalized permutahedra $P \subseteq \mathbb{R}^d$. Thus, in the case of generalized permutahedra, the module structure of Section 5.3 is precisely the one induced from the Hopf monoid structure above.

The antipode

The antipode formula of GP (5.5.1) descends to the quotient Π, but it is no longer grouping-free. The Euler map (5.1.7) allows us to write the antipode formula of Π in a very compact form:

$$\mathrm{s}_I([P]) = (-1)^{|I|}[P]^* = (-1)^{|I|}[-P]^{-1},$$

where $[-P]^{-1}$ is the *multiplicative* inverse of the class $[-P]$ in the polytope algebra. The last equality is [55, Theorem 12]. This surprising relation between two different notions of inversion, one in McMullen's polytope algebra and the other in Aguiar and Ardila's Hopf monoid, can be explained through the module structure of this chapter.

Let $\tau \in \mathbb{R}\Sigma[\mathcal{A}_I]$ be the Takeuchi element of the braid arrangement in $\mathbb{R}^I$. It is the Adams element of parameter -1. In particular, it is projective and the corresponding Eulerian family $\{\mathtt{E}_{\mathrm{X}}\}$ is (5.3.5). For a generalized permutahedron P, write

$$[P] = \sum_{r,k} P_{r,k} \quad \text{where} \quad P_{r,k} \in \bigoplus_{\dim(\mathrm{X})=k} \Xi_r(\pi_I) \cdot \mathtt{E}_{\mathrm{X}}.$$

Then,

$$\$_I([P]) = [P] \cdot \tau = \sum_{r,k} P_{r,k} \cdot \tau = \sum_{r,k} (-1)^k P_{r,k}.$$

On the other hand, using Proposition 5.2.7 we have

$$([P])^* = \sum_{r,k} (P_{r,k})^* = \sum_{r,k} (-1)^{|I|-k} P_{r,k} = (-1)^{|I|} \$_I([P]).$$

5.5.2 Higher monoidal structures

We have just proved that Π is a Hopf monoid in the symmetric monoidal category $(\mathsf{Sp}, \cdot)$. The algebra str ucture of each space $\Pi[I]$ defined by McMullen can also be defined for GP. In both cases, this endows the species with the structure of a monoid in the symmetric monoidal category $(\mathsf{Sp}, \times)$ of species with the *Hadamard product*. The Hadamard product of two species p and q is defined by

$$(\mathsf{p} \times \mathsf{q})[I] = \mathsf{p}[I] \otimes \mathsf{q}[I].$$

Hence, a monoid in $(\mathsf{Sp}, \times)$ consists of a species p with an algebra structure on each space $\mathsf{p}[I]$. For generalized permutahedra, these structures are compatible in a very special way.

Theorem 5.5.4. *The species of generalized permutahedra* GP *and its quotient* Π *are* $(2,1)$*-monoids in the 3-monoidal category* $(\mathsf{Sp}, \cdot, \times, \cdot)$.

See [6, Chapter 7] for the definition of higher monoidal categories and of monoids in such categories. The notation $(2, 1)$ indicates that GP is a monoid with respect to the first two monoidal structures (Cartesian product and Minkowski sum, respectively) and a comonoid with respect to the last (coproduct maps $\Delta_{S,T}$).

Proof. We only discuss the remaining compatibility axioms: the compatibility between Cartesian product and Minkowski sum, and the compatibility between Minkowski sum and the coproduct.

The compatibility between Cartesian product and Minkowski sum boils down to the identity

$$(P_1 + P_2) \times (Q_1 + Q_2) = (P_1 \times Q_1) + (P_2 \times Q_2)$$

for $P_1, P_2 \in \mathrm{GP}[S]$ and $Q_1, Q_2 \in \mathrm{GP}[T]$, which one easily verifies for arbitrary sets $P_1, P_2 \subseteq \mathbb{R}^S$ and $Q_1, Q_2 \subseteq \mathbb{R}^T$.

On the other hand, the compatibility between Minkowski sum and the coproduct is equivalent to the following identity for generalized permutahedra $P, Q \in \mathrm{GP}[I]$:

$$(P + Q)|_S \otimes (P + Q)/_S = (P|_S + Q|_S) \otimes (P/_S + Q/_S).$$

This follows by projecting the identity $(P + Q)_{\boldsymbol{v}} = P_{\boldsymbol{v}} + Q_{\boldsymbol{v}}$ to $\mathbb{R}^S$ and $\mathbb{R}^T$, respectively, where $\boldsymbol{v}$ is any vector in the interior of the face of the braid arrangement corresponding to the composition (S, T). □

The compatibility between Minkowski sum and the Hopf monoid operations refines the last statement in Theorem 5.2.3; which, in the language of this section, states that the maps $\mu_{S,T} \circ \Delta_{S,T}$ are compatible with the Minkowski sum operation.

CHAPTER 6

TYPE B HOPF MONOIDS

The contents of this chapter are joint work with Aguiar [2]. The notions related to sets with a fixed-point free involutions are essential to this chapter, and can be reviewed in Section 2.2.4. We set one additional piece of notation.

Let $\boldsymbol{I}$ be a point with a fixed-point free involution, and $\mathrm{X} \vdash^{B} \boldsymbol{I}$ be a type B partition of $\boldsymbol{I}$. Then, both X_0 (the zero block of X) and $\mathrm{X}_{\pm} := \mathrm{X} \setminus \{\mathrm{X}_0\}$ (the collection of non-zero blocks of X) are sets with a fixed-point free involution, the one induced by the involution of $\boldsymbol{I}$. The following is an example of a type B partition of $[\pm 4]$, where we omit the brackets of each individual block for simplicity

$$\mathrm{X} = \{2\,4\,\overline{2}\,\overline{4}\,,\,1\,\overline{3}\,,\,\overline{1}\,3\},$$

Thus, $\mathrm{X}_0 = \{2, 4, \overline{2}, \overline{4}\}$ is a set of four elements, and $\mathrm{X}_{\pm} = \{1\,\overline{3}\,,\,\overline{1}\,3\}$ is a set of two blocks, each block containing two elements.

6.1 Type B species

A **type B set species** is a functor m from the category of finite sets with a fixed-point free involution with involution-preserving bijections, to the category of sets. Explicitly, a type B species m consists of:

1. For each finite set with a fixed-point free involution $\boldsymbol{I}$, a set $\mathsf{m}[\boldsymbol{I}]$ of **m-structures**.
2. For each bijection $\sigma : \boldsymbol{I} \to \boldsymbol{J}$ that preserves involutions (i.e. $\sigma(\bar{i}) = \overline{\sigma(i)}$), a

function $\mathsf{m}[\sigma] : \mathsf{m}[\boldsymbol{I}] \to \mathsf{m}[\boldsymbol{J}]$. These functions satisfy

$$\mathsf{m}[\sigma \circ \tau] = \mathsf{m}[\sigma] \circ \mathsf{m}[\tau] \qquad \text{and} \qquad \mathsf{m}[\mathrm{Id}] = \mathrm{Id}\,. \tag{6.1.1}$$

In particular, they are bijections.

The involution of $\boldsymbol{I}$ is a (involution-preserving) bijection, and thus determines a map $\mathsf{m}[\boldsymbol{I}] \to \mathsf{m}[\boldsymbol{I}]$. Conditions (6.1.1) imply that this map is an involution on $\mathsf{m}[\boldsymbol{I}]$, which might or might not have fixed points. We denote the image of an element $x \in \mathsf{m}[\boldsymbol{I}]$ under this map by $\overline{x}$. It follows that for all $x \in \mathsf{m}[\boldsymbol{I}]$ and involution-preserving bijection $\sigma : \boldsymbol{I} \to \boldsymbol{J}$,

$$\mathsf{m}[\sigma](\overline{x}) = \overline{\mathsf{m}[\sigma](x)}. \tag{6.1.2}$$

A **morphism** $f : \mathsf{m} \to \mathsf{n}$ of type B species is a natural transformation of functors. That is, f consists of a collection of maps $f_{\boldsymbol{I}} : \mathsf{m}[\boldsymbol{I}] \to \mathsf{n}[\boldsymbol{I}]$ such that $\mathsf{n}[\sigma] \circ f_{\boldsymbol{I}} = f_{\boldsymbol{J}} \circ \mathsf{m}[\sigma]$ for all bijections $\sigma : \boldsymbol{I} \to \boldsymbol{J}$ that preserve involutions. It follows from (6.1.2) that the involutions $\mathsf{m}[\boldsymbol{I}] \to \mathsf{m}[\boldsymbol{I}]$ in the previous paragraph form an involution of type B species $\mathsf{m} \to \mathsf{m}$.

We define type B analogs to some important species.

Example 6.1.1. The **type B exponential species** E^B is the species with exactly one structure $*_{\boldsymbol{I}}$ on each finite set with a fixed-point free involution $\boldsymbol{I}$.

Example 6.1.2. The species $\mathsf{1}$ and X are defined by

$$\mathsf{1}[I] = \begin{cases} \{*_\emptyset\} & \text{if } I = \emptyset, \\ \emptyset & \text{otherwise,} \end{cases} \qquad \text{and} \qquad \mathsf{X}[I] = \begin{cases} I & \text{if } |I| = 1, \\ \emptyset & \text{otherwise.} \end{cases}$$

They are the unit for the Cauchy product and the substitution product, respectively. See [6, Section 8.1.2]. Their type B analogs are

$$\mathsf{1}^B[\boldsymbol{I}] = \begin{cases} \{*_\emptyset\} & \text{if } \boldsymbol{I} = \emptyset, \\ \emptyset & \text{otherwise,} \end{cases} \qquad \text{and} \qquad \mathsf{X}^B[\boldsymbol{I}] = \begin{cases} \boldsymbol{I} & \text{if } |\boldsymbol{I}| = 2, \\ \emptyset & \text{otherwise.} \end{cases}$$

Example 6.1.3. A type B linear order on $\boldsymbol{I}$ is a total order $\boldsymbol{\ell}$ such that $i \leq_{\boldsymbol{\ell}} j$ if and only if $\bar{j} \leq_{\boldsymbol{\ell}} \bar{i}$. The species of **type B linear orders** is denoted by L^B. Note that the condition on $\boldsymbol{\ell} \in \mathsf{L}^B[\boldsymbol{I}]$ above implies that $\overline{\boldsymbol{\ell}} = \mathrm{rev}(\boldsymbol{\ell})$, the reversal of the linear order.

6.1.1 Type B generating functions

Let m be a type B species. The type B generating function $\mathsf{m}(x)$ of m is

$$\mathsf{m}(x) = \sum_{d \geq 0} m_d \frac{x^d}{(2d)!!},$$

where $\mathsf{m}_d = |\mathsf{m}[\pm d]|$. This is analogous to the definition of exponential generating function for a type A species, but with using the double factorial $(2d)!! = 2^d\, d!$ instead of $d!$.

Example 6.1.4. The power series of the species $\mathsf{1}^B$, X^B and E^B are respectively

$$1 \qquad 2\frac{x^1}{2!!} = x \qquad \sum_{d \geq 0} \frac{x^d}{(2d)!!} = e^{x/2}.$$

6.2 Type B bimonoids

6.2.1 Type A species with an involution

Let $\mathbf{Sp}^A$ denote the category of pairs (p, θ) where p is a species and $\theta : \mathsf{p} \to \mathsf{p}$ is an involution in the category of species $\mathbf{Sp}$. The morphisms in $\mathbf{Sp}^A$ are the same as in $\mathbf{Sp}$. In particular, they **do not** necessarily preserve involutions.

A pair (p, θ) in $\mathbf{Sp}^A$ is an **involutive monoid** if p is a usual monoid in $\mathbf{Sp}$ and the following diagram commutes.

$$\begin{array}{ccccc} \mathsf{p} \cdot \mathsf{p} & & \xrightarrow{\quad\mu\quad} & & \mathsf{p} \\ {\scriptstyle \beta_{\mathsf{p},\mathsf{p}}}\downarrow & & & & \downarrow{\scriptstyle \theta} \\ \mathsf{p} \cdot \mathsf{p} & \xrightarrow{\theta \cdot \theta} & \mathsf{p} \cdot \mathsf{p} & \xrightarrow{\mu} & \mathsf{p} \end{array}$$

The morphism $\beta_{\mathsf{p},\mathsf{p}}$ is the usual *braiding* of the category $\mathbf{Sp}$, see [6, Chapter 8]. Namely, $\beta_{\mathsf{p},\mathsf{p}}(x, y) = (y, x)$. Thus, a pair (p, θ) is an involutive monoid if p is a monoid and

$$\theta_I(x \cdot y) = \theta_T(y) \cdot \theta_S(x) \tag{6.2.1}$$

for all decompositions $I = S \sqcup T$, and structures $x \in \mathsf{p}[S]$ and $y \in \mathsf{p}[T]$.

Observe that if θ is the trivial involution $\theta = \mathrm{id}$, then $(\mathsf{p}, \mathrm{id})$ is an involutive monoid if and only if p is a commutative monoid. In more generality, observe that if p is commutative, then (p, θ) is an involutive monoid if and only if θ is a morphism of monoids.

Involutive comonoids are defined dually. Explicitly, an involutive comonoid (p, θ) satisfies

$$\theta_I(z)|_S = \theta_S(z/_T) \qquad \text{and} \qquad \theta_I(z)/_S = \theta_T(z|_T), \tag{6.2.2}$$

for all decompositions $I = S \sqcup T$, and structures $z \in \mathsf{p}[I]$. An **involutive bimonoid** (resp. **involutive Hopf monoid**) is an involutive monoid and comonoid (p, θ) such that p is a bimonoid (resp. Hopf monoid) in the usual sense.

Example 6.2.1. As observed above, any commutative monoid p yields an involutive monoid $(\mathsf{p}, \mathrm{id})$. Similarly, any cocommutative comonoid is an involutive comonoid. In particular, the exponential species E and the species of partitions $\mathsf{\Pi}$ give rise to involutive Hopf monoids $(\mathsf{E}, \mathrm{id})$ and $(\mathsf{\Pi}, \mathrm{id})$.

Example 6.2.2. Consider the species $(\mathsf{L}, \mathrm{rev})$ where rev is the *reversal morphism*: $i <_{\mathrm{rev}(\ell)} j$ if and only if $j <_\ell j$. Then, $(\mathsf{L}, \mathrm{rev})$ is in an involutive Hopf monoid with the usual operations of concatenation and restriction. Indeed, we clearly have

$$\mathrm{rev}(\ell \cdot \ell') = \mathrm{rev}(\ell') \cdot \mathrm{rev}(\ell) \qquad \text{and} \qquad \mathrm{rev}(\ell)|_S = \mathrm{rev}(\ell|_S) = \mathrm{rev}(\ell/_T).$$

Similarly, the species of compositions $\mathsf{\Sigma}$ can be endowed with the involution rev, which reverses the order of the blocks of a composition, to obtain an involutive Hopf monoid.

To simplify the notation, we will sometimes drop the subscript I from the map $\theta_I : \mathsf{p}[I] \to \mathsf{p}[I]$, and will write θ_{p} for both the morphism of species $\mathsf{p} \to \mathsf{p}$ and each of its components $\mathsf{p}[I] \to \mathsf{p}[I]$.

6.2.2 The action of Sp on $\mathbf{Sp}^B$

We introduce an action of the category of species $\mathbf{Sp}$ on the category of type B species $\mathbf{Sp}^B$. Given a species p and a type B species m, we define a type B species $\mathsf{p} \bullet \mathsf{m}$ on sets by

$$(\mathsf{p} \bullet \mathsf{m})[\boldsymbol{I}] = \coprod_{\boldsymbol{I} = S \sqcup \boldsymbol{T} \sqcup \overline{S}} \mathsf{p}[S] \times \mathsf{m}[\boldsymbol{T}]. \tag{6.2.3}$$

We decompose $\boldsymbol{I}$ into an involution-inclusive subset $\boldsymbol{T}$ and its complement, which we further decompose into an involution-exclusive subset S and its image under the involution. There is one such decomposition for each $S \in \mathcal{P}'(\boldsymbol{I})$, in particular S and $\boldsymbol{T}$ are allowed to be empty. Thus, a $(\mathsf{p} \cdot \mathsf{m})$-structure on $\boldsymbol{I}$ is a tuple (x, y) where x is a p-structure on an involution-exclusive subset $S \in \mathcal{P}'(\boldsymbol{I})$ and y is an m-structure on $\boldsymbol{I} \setminus \pm S$.

For a involution preserving bijection $\sigma : \boldsymbol{I} \to \boldsymbol{J}$, the map $(\mathsf{p} \cdot \mathsf{m})[\sigma]$ sends the component $\mathsf{p}[S] \times \mathsf{m}[\boldsymbol{T}]$ of $(\mathsf{p} \cdot \mathsf{m})[\boldsymbol{I}]$ to the component $\mathsf{p}[\sigma(S)] \times \mathsf{m}[\sigma(\boldsymbol{T})]$ of $(\mathsf{p} \cdot \mathsf{m})[\boldsymbol{J}]$ via the product map $\mathsf{p}[\sigma|_S] \times \mathsf{m}[\sigma|_{\boldsymbol{T}}]$.

It is not hard to verify that there are natural isomorphisms

$$\mathsf{1} \cdot \mathsf{m} = \mathsf{m} \quad \text{and} \quad (\mathsf{p} \cdot \mathsf{q}) \cdot \mathsf{m} = \mathsf{p} \cdot (\mathsf{q} \cdot \mathsf{m}).$$

They identify $(*_\emptyset, x) \in (\mathsf{1} \cdot \mathsf{m})[\boldsymbol{I}]$ with $x \in \mathsf{m}[\boldsymbol{I}]$, and $((x, y), z) \in ((\mathsf{p} \cdot \mathsf{q}) \cdot \mathsf{m})[\boldsymbol{I}]$ with $(x, (y, z)) \in (\mathsf{p} \cdot (\mathsf{q} \cdot \mathsf{m}))[\boldsymbol{I}]$, respectively.

Proposition 6.2.3. *The generating function of* $\mathsf{p} \cdot \mathsf{m}$ *is* $\mathsf{p}(x)\mathsf{m}(x)$*.*

Proof. Summing over $k = |S|$ in (6.2.3), we have

$$(\mathsf{p} \cdot \mathsf{m})_d = \sum_{k=0}^{d} \binom{d}{k} 2^k \mathsf{p}_k \mathsf{m}_{d-k} = 2^d d! \sum_{k=0}^{d} \frac{\mathsf{p}_k}{k!} \frac{\mathsf{m}_{d-k}}{2^{d-k}(d-k)!}.$$

This is precisely the coefficient of $\frac{x^d}{(2d)!!}$ in

$$\left(\sum_{k \geq 0} \mathsf{p}_k \frac{x^k}{k!} \right) \left(\sum_{k \geq 0} \mathsf{m}_k \frac{x^k}{(2k)!!} \right) = \mathsf{p}(x)\mathsf{m}(x). \qquad \square$$

Example 6.2.4. Let T^B be the species of transversals. That is, for every set with a fixed-point free involution $\boldsymbol{I}$,

$$\mathsf{T}^B[\boldsymbol{I}] = \{ S \in \mathcal{P}'(\boldsymbol{I}) \, : \, 2|S| = |\boldsymbol{I}| \}.$$

Identifying $(*_S, *_\emptyset) \in \mathsf{E} \cdot \mathsf{1}^B[I]$ with S, we see that $\mathsf{T}^B = \mathsf{E} \cdot \mathsf{1}^B$. Proposition 6.2.3 then yields $\mathsf{T}^B(x) = e^x$, which recovers the fact that $\boldsymbol{I}$ has $2^{|\boldsymbol{I}|/2}$ transversals.

Example 6.2.5. Let P^B be the species of involution-exclusive subsets. That is, for every set with a fixed-point free involution $\boldsymbol{I}$, $\mathsf{P}^B[\boldsymbol{I}] = \mathcal{P}'(\boldsymbol{I})$. Identifying $(*_S, *_{\boldsymbol{I} \setminus \pm S}) \in \mathsf{E} \cdot \mathsf{E}^B[I]$ with S, we see that $\mathsf{P}^B = \mathsf{E} \cdot \mathsf{E}^B$. Thus, $\mathsf{P}^B(x) = e^{3x/2}$ and we recover that $\boldsymbol{I}$ contains $3^{|\boldsymbol{I}|/2}$ involution-exclusive subsets.

Example 6.2.6. Let OP^B be the species of *ordered* involution-exclusive subsets. That is, $\mathsf{OP}^B[\boldsymbol{I}]$ consists of all linear orders on involution-exclusive subsets of $\boldsymbol{I}$. For example, $\mathsf{OP}^B[\pm 2]$ consists of 13 elements: the empty order, 4 singleton orders (1, $\bar{1}$, 2, $\bar{2}$), and 8 orders on transversals (12, 21, $\bar{1}2$, $2\bar{1}$, $1\bar{2}$, $\bar{2}1$, $\bar{1}\bar{2}$, $\bar{2}\bar{1}$). One similarly verifies that $\mathsf{OP}^B = \mathsf{L} \cdot \mathsf{E}^B$. Thus,

$$\mathsf{OP}^B(x) = \frac{e^{x/2}}{1-x}.$$

The number of OP^B-structures on $[\pm d]$ for the first values of d are $1, 3, 13, 79, 633$. This is the number of *ways to sort a spreadsheet with d columns* (OEIS: A010844).

Example 6.2.7. A type B linear order is completely determined by the order on the first half of its elements. Thus, $\mathsf{L}^B = \mathsf{L} \cdot \mathsf{1}^B$. It follows that $\mathsf{L}^B(x) = \dfrac{1}{1-x}$.

6.2.3 Type B objects, monoids, and comonoids

A **type B object** is an object of the category $\mathbf{Sp}^A \times \mathbf{Sp}^B$. That is, a type B object is a triple $(\mathsf{p}, \theta, \mathsf{m})$ where p is a species, $\theta : \mathsf{p} \to \mathsf{p}$ is an involution of species, and m is a type B species. A **morphism** $(f, g) : (\mathsf{p}, \theta_{\mathsf{p}}, \mathsf{m}) \to (\mathsf{q}, \theta_{\mathsf{q}}, \mathsf{n})$ of type B objects is a pair of morphism $f : \mathsf{p} \to \mathsf{q}$ and $g : \mathsf{m} \to \mathsf{n}$ in $\mathbf{Sp}$ and $\mathbf{Sp}^B$, respectively. The involution θ will play a central role in the definition of type B bimonoids below.

First, a **type B monoid** is a type B object $(\mathsf{p}, \theta, \mathsf{m})$, such that (p, θ) is an involutive monoid with product μ, together with a morphism of type B species $\alpha : \mathsf{p} \cdot \mathsf{m} \to \mathsf{m}$ making the following diagrams commute.

$$\begin{array}{ccc} \mathsf{p}\cdot\mathsf{p}\cdot\mathsf{m} & \xrightarrow{\mathrm{Id}\cdot\alpha} & \mathsf{p}\cdot\mathsf{m} \\ {\scriptstyle \mu\cdot\mathrm{Id}}\downarrow & & \downarrow{\scriptstyle \alpha} \\ \mathsf{p}\cdot\mathsf{m} & \xrightarrow[\alpha]{} & \mathsf{m} \end{array} \qquad\qquad \begin{array}{ccc} \mathsf{p}\cdot\mathsf{m} & \xrightarrow{\quad\alpha\quad} & \mathsf{m} \\ {\scriptstyle \iota\cdot\mathrm{Id}}\nwarrow & & \nearrow{\scriptstyle \cong} \\ & 1\cdot\mathsf{m} & \end{array}$$

Explicitly, the morphism α has components

$$\begin{aligned} \alpha_{S,\boldsymbol{T}} : \mathsf{p}[S] \times \mathsf{m}[\boldsymbol{T}] &\longrightarrow \mathsf{m}[\boldsymbol{I}] \\ (x, y) \quad &\longmapsto x \cdot y \end{aligned}$$

for each decomposition $\boldsymbol{I} = S \sqcup \boldsymbol{T} \sqcup \overline{S}$. The commutative diagrams above correspond to the associativity and unitality axioms:

$$x \cdot (y \cdot z) = (x \cdot y) \cdot z \qquad \text{and} \qquad \epsilon \cdot w = w$$

for all decompositions $I = R \sqcup S \sqcup \boldsymbol{T} \sqcup \overline{S} \sqcup \overline{R}$, structures $x \in \mathsf{p}[R]$, $y \in \mathsf{p}[S]$, $z \in \mathsf{m}[\boldsymbol{T}]$, and $w \in \mathsf{m}[\boldsymbol{I}]$. Here, $\epsilon \in \mathsf{p}[\emptyset]$ denotes the unit of the monoid p as in Section 2.3.1.

A **morphism of type B monoids** $(f, g) : (\mathsf{p}, \theta_{\mathsf{p}}, \mathsf{m}) \to (\mathsf{q}, \theta_{\mathsf{q}}, \mathsf{n})$ is a morphism of type B objects such that $f : \mathsf{p} \to \mathsf{q}$ is a morphism of monoids and the following diagram commutes.

$$\begin{array}{ccc} \mathsf{p}\cdot\mathsf{m} & \xrightarrow{\alpha} & \mathsf{m} \\ {\scriptstyle f\cdot g}\downarrow & & \downarrow{\scriptstyle f} \\ \mathsf{q}\cdot\mathsf{n} & \xrightarrow[\alpha]{} & \mathsf{n} \end{array}$$

That is, $f_{\boldsymbol{I}}(x \cdot y) = f_S(x) \cdot g_{\boldsymbol{T}}(y)$ for all decompositions $\boldsymbol{I} = S \sqcup \boldsymbol{T} \sqcup \overline{S}$ and structures $x \in \mathsf{p}[S]$, $y \in \mathsf{m}[\boldsymbol{T}]$.

Type B comonoids and their morphisms are defined dually. The comodule morphism $\delta : \mathsf{m} \to \mathsf{m} \bullet \mathsf{p}$ of a type B comonoid $(\mathsf{p}, \theta, \mathsf{m})$ has components

$$\begin{aligned} \delta_{S,\boldsymbol{T}} : \mathsf{m}[\boldsymbol{I}] &\longrightarrow \mathsf{p}[S] \times \mathsf{m}[\boldsymbol{T}] \\ z &\longmapsto (z|_S, z/_S) \end{aligned}$$

for each decomposition $\boldsymbol{I} = S \sqcup \boldsymbol{T} \sqcup \overline{S}$. This coaction satisfies the usual coassociativity

$$(z|_{R\sqcup S})|_R = z|_R \qquad (z|_{R\sqcup S})/_R = (z/_R)|_S \qquad (z/_R)/_S = z/_{R\sqcup S}$$

and counitality $z|_\emptyset = \epsilon$, $z/_\emptyset = z$ axioms. A morphism of comonoids (f, g) satisfies $g_{\boldsymbol{I}}(z)|_S = f_S(z|_S)$ and $g_{\boldsymbol{I}}(z)/_S = g_{\boldsymbol{T}}(z/_S)$ for all decompositions $\boldsymbol{I} = S \sqcup \boldsymbol{T} \sqcup \overline{S}$ and structures $z \in \mathsf{m}[\boldsymbol{I}]$.

Example 6.2.8. Consider the type B object of linear orders $(\mathsf{L}, \mathrm{rev}, \mathsf{L}^B)$, with L^B and $(\mathsf{L}, \mathrm{rev})$ as in Examples 6.1.3 and 6.2.2, respectively. We endow it with the structure of a type B monoid and type B comonoid as follows. Given a decomposition $\boldsymbol{I} = S \sqcup \boldsymbol{T} \sqcup \overline{S}$, a linear order $\ell \in \mathsf{L}[S]$, and a type B linear order $\boldsymbol{\ell}' \in \mathsf{L}^B[\boldsymbol{J}]$,

$$\alpha_{S,\boldsymbol{T}}(\ell, \boldsymbol{\ell}') = \ell \cdot \boldsymbol{\ell}' \quad \text{is the concatenation} \quad \ell\, \boldsymbol{\ell}' \ \mathrm{rev}(\overline{\ell})$$

For example, with $S = \{1, \overline{3}\} \subseteq [\pm 4]$ and $\boldsymbol{T} = \{2, \overline{2}, 4, \overline{4}\}$,

$$1\overline{3} \cdot 2\overline{4}4\overline{2} = \alpha_{S,\boldsymbol{T}}(1\overline{3}, 2\overline{4}4\overline{2}) = 1\overline{3}2\overline{4}4\overline{2}3\overline{1}.$$

Now, for $\boldsymbol{\ell} \in \mathsf{L}^B[\boldsymbol{I}]$, define $\delta_{S,\boldsymbol{T}}(\boldsymbol{\ell}) = (\boldsymbol{\ell}|_S, \boldsymbol{\ell}/_S)$ by

$$\begin{aligned} \boldsymbol{\ell}|_S &\quad \text{is the restriction of } \boldsymbol{\ell} \text{ to } S, \\ \boldsymbol{\ell}/_S &\quad \text{is the restriction of } \boldsymbol{\ell} \text{ to } \boldsymbol{T}. \end{aligned}$$

For example,

$$\delta_{\{\overline{1},2,\overline{4}\},\{3,\overline{3}\}}(1\overline{3}2\overline{4}4\overline{2}3\overline{1}) = (2\overline{4}\overline{1}, \overline{3}3)$$

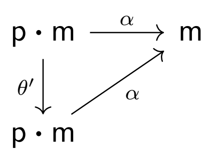
(Co)commutative (co)monoids

Let $(\mathsf{p}, \theta, \mathsf{m})$ be a type B object. The involution θ of p induces an involution $\theta' = \theta'_{\mathsf{p},\mathsf{m}}$ of $\mathsf{p} \cdot \mathsf{m}$ as follows. The component $\theta'_{\boldsymbol{I}}$ maps $\mathsf{p}[S] \times \mathsf{m}[\boldsymbol{J}]$ to $\mathsf{p}[\overline{S}] \times \mathsf{m}[\boldsymbol{J}]$ via

$$\theta'_{\boldsymbol{I}}(x, y) := (\theta_{\overline{S}}(\overline{x}), y) = (\overline{\theta_S(x)}, y),$$

where $\overline{x} = \mathsf{p}[^{-}](x) \in \mathsf{p}[\overline{S}]$. The equality $\theta_{\overline{S}}(\overline{x}) = \overline{\theta_S(x)}$ follows from the naturality of the involution θ. To simplify notation, we will write $\widetilde{x} = \overline{\theta_S(x)}$.

A type B monoid $(\mathsf{p}, \theta, \mathsf{m})$ is **commutative** if p is a commutative monoid and the following diagram commutes.



That is, a monoid $(\mathsf{p}, \theta, \mathsf{m})$ is commutative if p is commutative and for every decomposition $\boldsymbol{I} = S \sqcup \boldsymbol{T} \sqcup \overline{S}$, and structures $x \in \mathsf{p}[S]$, $y \in \mathsf{m}[\boldsymbol{T}]$,

$$\widetilde{x} \cdot y = x \cdot y. \qquad (6.2.4)$$

Cocommutative comonoids are defined dually. Explicitly, a comonoid $(\mathsf{p}, \theta, \mathsf{m})$ is cocommutative if q is cocommutative and for every decomposition $\boldsymbol{I} = S \sqcup \boldsymbol{T} \sqcup \overline{S}$, and structure $z \in \mathsf{m}[\boldsymbol{I}]$,

$$z|_{\overline{S}} = \widetilde{z|_S} \qquad \text{and} \qquad z/_{\overline{S}} = z/_S.$$

Example 6.2.9. The type B (co)monoid of linear orders $(\mathsf{L}, \mathrm{rev}, \mathsf{L}^B)$ is cocommutative but not commutative. Commutativity fails in any nontrivial example, for instance

$$12 \cdot 3\bar{3} = 123\bar{3}\bar{2}\bar{1}, \qquad \text{but} \qquad \widetilde{12} \cdot 3\bar{3} = \bar{2}\bar{1} \cdot 3\bar{3} = \bar{2}\bar{1}3\bar{3}12.$$

Cocommutativity boils down to the fact that $\overline{\boldsymbol{\ell}} = \mathrm{rev}(\boldsymbol{\ell})$. Explicitly,

$$\boldsymbol{\ell}|_{\overline{S}} = \overline{\mathrm{rev}(\boldsymbol{\ell})}|_{\overline{S}} = \overline{\mathrm{rev}(\boldsymbol{\ell})|_{S}} = \overline{\mathrm{rev}(\boldsymbol{\ell}|_{S})} = \widetilde{\boldsymbol{\ell}|_{S}}.$$

The second cocommutativity condition follows trivially since, by definition, both $\boldsymbol{\ell}/_{S}$ and $\boldsymbol{\ell}/_{\overline{S}}$ are the restriction of the order $\boldsymbol{\ell}$ to $\boldsymbol{T}$.

6.2.4 Type B bimonoids

A type B object $(\mathsf{p}, \theta, \mathsf{m})$ is a **type B bimonoid** if it is a type B monoid and comonoid, p is a bimonoid in the usual sense, and the following diagram commutes

$$\begin{array}{ccccc} \mathsf{p}\cdot\mathsf{m} & \xrightarrow{\alpha} & \mathsf{m} & \xrightarrow{\delta} & \mathsf{p}\cdot\mathsf{m} \\ {\scriptstyle \Delta^{(2)}\cdot\delta}\downarrow & & & & \uparrow{\scriptstyle \mu^{(2)}\cdot\alpha} \\ \mathsf{p}\cdot\mathsf{p}\cdot\mathsf{p}\cdot\mathsf{p}\cdot\mathsf{m} & & \xrightarrow[\beta']{} & & \mathsf{p}\cdot\mathsf{p}\cdot\mathsf{p}\cdot\mathsf{p}\cdot\mathsf{m} \end{array}$$

where β' is the following composition of braiding maps and involutions:

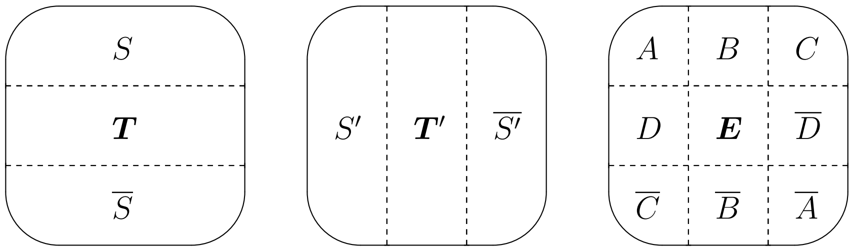
$$\beta' = \big(\mathrm{Id}\cdot(\beta_{\mathsf{p},\mathsf{p}}\cdot\mathrm{Id}\circ\mathrm{Id}\cdot\beta_{\mathsf{p},\mathsf{p}}\circ\beta_{\mathsf{p},\mathsf{p}}\cdot\mathrm{Id})\cdot\mathrm{Id}\big)\circ\mathrm{Id}\cdot\mathrm{Id}\cdot\theta'_{\mathsf{p},\mathsf{p}\cdot\mathsf{m}}$$

The internal parenthesis *interchanges* the second and fourth factors, it corresponds to one of the two possible paths in the usual *hexagon diagram* of a braided monoidal category. That is, $\beta'(x_1, x_2, x_3, x_4, y) = (x_1, x_4, \widetilde{x_3}, x_2, y)$. This is the **type B compatibility axiom**.

Explicitly, the type B compatibility axiom for a bimonoid $(\mathsf{p}, \theta, \mathsf{m})$ requires that for any two decompositions $\boldsymbol{I} = S \sqcup \boldsymbol{T} \sqcup \overline{S} = S' \sqcup \boldsymbol{T}' \sqcup \overline{S'}$, and structures $x \in \mathsf{p}[S]$, $y \in \mathsf{m}[\boldsymbol{T}]$,

$$(x\cdot y)|_{S'} = x|_A \cdot y|_D \cdot \widetilde{x/_{A\sqcup B}} \qquad \text{and} \qquad (x\cdot y)/_{S'} = (x/_A)|_B \cdot y/_D, \qquad (6.2.5)$$

where $A = S \cap S'$, $B = S \cap \boldsymbol{T}'$, and $D = \boldsymbol{T} \cap S'$, as represented in the following diagram.



(6.2.6)

Observe that $\widetilde{x/_{A \sqcup B}} = \widetilde{x}|_{\overline{C}} \in \mathsf{p}[\overline{C}]$ and $y/_D \in \mathsf{m}[\boldsymbol{E}]$.

Example 6.2.10. We continue our example of linear orders by showing that $(\mathsf{L}, \operatorname{rev}, \mathsf{L}^B)$ is a type B bimonoid. Let $S, \boldsymbol{T}, S', \boldsymbol{T}'$ and their intersections be as above. For $\ell \in \mathsf{L}[S]$ and $\boldsymbol{\ell}' \in \mathsf{L}^B[\boldsymbol{T}]$, observe that in $\ell \cdot \boldsymbol{\ell}' = \ell \boldsymbol{\ell}' \operatorname{rev}(\overline{\ell})$

- elements of A (ordered according to ℓ) precede elements of D (ordered according to $\boldsymbol{\ell}'$), which in turn precede elements of $\overline{C}$ (ordered according to $\operatorname{rev}(\overline{\ell})$), and
- elements of B (ordered according to ℓ) precede elements of $\boldsymbol{E}$ (ordered according to $\boldsymbol{\ell}'$).

Since the coaction morphism δ is given by restriction of linear orders, each component of the compatibility axiom (6.2.5) follows from one of the observations above. In particular, note that $\operatorname{rev}(\overline{\ell})|_{\overline{C}} = \operatorname{rev}(\overline{\ell|_C}) = \widetilde{\ell/_{A \sqcup B}}$.

Two immediate consequences of the compatibility axiom are the following.

$$\delta_{S,\boldsymbol{T}} \circ \mu_{S,\boldsymbol{T}}(x, y) = (x, y) \tag{6.2.7}$$

$$\delta_{\overline{S},\boldsymbol{T}} \circ \mu_{S,\boldsymbol{T}}(x, y) = (\widetilde{x}, y) = \theta_{\mathsf{p},\mathsf{m}}(x, y). \tag{6.2.8}$$

In particular, the action maps $\mu_{S,\boldsymbol{T}}$ are injective, while the coaction maps $\delta_{S,\boldsymbol{T}}$ are surjective.

A **morphism of type B bimonoids** $(f, g) : (\mathsf{p}, \theta_{\mathsf{p}}, \mathsf{m}) \to (\mathsf{q}, \theta_{\mathsf{q}}, \mathsf{n})$ is simply a morphism of both type B monoids and type B comonoids. These morphisms satisfy two surprising properties.

First, the component f is completely determined by g, in the following sense. Let $S \in \mathcal{P}'(\boldsymbol{I})$ and $x \in \mathsf{p}[S]$. Then, for any $y \in \mathsf{m}[\boldsymbol{I} \setminus \pm S]$,

$$f_S(x) = f_S((x \cdot y)|_S) = g_{\boldsymbol{I}}(x \cdot y)|_S.$$

The first equality is (6.2.7) and the second is part of the definition of a morphism of comonoids.

Second, the component f respects the involutions. Indeed, with the same setup as above,

$$f_{\overline{S}}(\widetilde{x}) = f_{\overline{S}}((x \cdot y)|_{\overline{S}}) = g_{\boldsymbol{I}}((x \cdot y))|_{\overline{S}} = \big(f_S(x) \cdot g_{\boldsymbol{T}}(y)\big)|_{\overline{S}} = \widetilde{f_S(x)}.$$

The first and last equalities are (6.2.8). Finally, naturality implies

$$f_S(\theta_{\mathsf{p}}(x)) = \overline{f_{\overline{S}}(\widetilde{x})} = \overline{\widetilde{f_S(x)}} = \theta_{\mathsf{q}}(f_S(x)).$$

6.3 Convolution modules and the antipode

In this section we work with (type B) species with values on a vector space. That is, $\mathsf{p}[I]$ and $\mathsf{m}[\boldsymbol{I}]$ are vector spaces over a fixed field $\Bbbk$. The definition of $\mathsf{p} \bullet \mathsf{m}$ for *vector species* is obtained by replacing Cartesian products and disjoint unions with tensor products and direct sums, respectively.

6.3.1 Type A with an involution

Given two objects $(\mathsf{p}, \theta_{\mathsf{p}})$ and $(\mathsf{q}, \theta_{\mathsf{q}})$ of $\mathbf{Sp}^A$, we define a new pair $\big(\operatorname{hom}(\mathsf{p}, \mathsf{q}), \ ^{\theta}\big)$ by

$$\operatorname{hom}(\mathsf{p}, \mathsf{q})[I] = \{f : \mathsf{p}[I] \to \mathsf{q}[I]\}$$

for all finite sets I, and

$$f^{\theta} = \theta_{\mathsf{q}} \circ f \circ \theta_{\mathsf{p}},$$

for all $f \in \operatorname{hom}(\mathsf{p}, \mathsf{q})[I]$. We emphasize that maps in $\operatorname{hom}(\mathsf{p}, \mathsf{q})[I]$ are **not** required to respect the involutions, otherwise we would have $f^{\theta} = f$.

We similarly define an involution on $\operatorname{Hom}_{\mathbf{Sp}^A}\big((\mathsf{p}, \theta_{\mathsf{p}}), (\mathsf{q}, \theta_{\mathsf{q}})\big) = \operatorname{Hom}_{\mathbf{Sp}}(\mathsf{p}, \mathsf{q})$. For $f : \mathsf{p} \to \mathsf{q}$, define $f^{\theta} : \mathsf{p} \to \mathsf{q}$ to have components $(f^{\theta})_I = (f_I)^{\theta}$ for each finite set I. The following lemmas show that the involution $f \mapsto f^{\theta}$ behaves well with respect to the morphisms of monoids and comonoids.

Lemma 6.3.1. *Let $f : (\mathsf{p}, \theta_{\mathsf{p}}) \to (\mathsf{q}, \theta_{\mathsf{q}})$ be a morphism of (involutive) monoids, then f^{θ} is also a morphism of monoids.*

Proof. This follows from the following simple computation:

$$\begin{aligned} f_I^{\theta}(x \cdot y) = \theta_{\mathsf{q}}\big(f_I(\theta_{\mathsf{p}}(x \cdot y))\big) = \theta_{\mathsf{q}}\big(f_I(\theta_{\mathsf{p}}(y) \cdot \theta_{\mathsf{p}}(x))\big) = \theta_{\mathsf{q}}\big(f_T(\theta_{\mathsf{p}}(y)) \cdot f_S(\theta_{\mathsf{p}}(x))\big) \\ = \theta_{\mathsf{q}}\big(f_S(\theta_{\mathsf{p}}(x))\big) \cdot \theta_{\mathsf{q}}\big(f_T(\theta_{\mathsf{p}}(y))\big) = f_S^{\theta}(x) \cdot f_T^{\theta}(y). \quad \square \end{aligned}$$

Lemma 6.3.2. *Let $f : (\mathsf{p}, \theta_{\mathsf{p}}) \to (\mathsf{q}, \theta_{\mathsf{q}})$ be a morphism of (involutive) comonoids, then f^{θ} is also a morphism of comonoids.*

Proof. We verify the claim directly:

$$\begin{aligned} f_S^{\theta}(z|_S) = \theta_{\mathsf{q}}\big(f_S(\theta_{\mathsf{p}}(z|_S))\big) = \theta_{\mathsf{q}}\big(f_S(\theta_{\mathsf{p}}(z)/_T)\big) \\ = \theta_{\mathsf{q}}\big(f_I(\theta_{\mathsf{p}}(z))/_T\big) = \theta_{\mathsf{q}}\big(f_I(\theta_{\mathsf{p}}(z))\big)|_S = f_I^{\theta}(z)|_S. \end{aligned}$$

A similar computation shows that $f^\theta_T(z/_S) = f^\theta_I(z)/_S$, hence f^θ is indeed a morphism of comonoids. $\square$

If p is a comonoid and q is a monoid, then $\hom(\mathsf{p},\mathsf{q})$ is naturally endowed with a monoid structure as follows. Given $f \in \hom(\mathsf{p},\mathsf{q})[S]$ and $g \in \hom(\mathsf{p},\mathsf{q})[T]$, the product $f \cdot g$ is defined by

$$(f \cdot g)(x) = f(x|_S) \cdot g(x/_S),$$

for all $x \in \mathsf{p}[I]$. See [7, Section 3.2] for details.

Moreover, if $(\mathsf{p}, \theta_\mathsf{p})$ is an involutive comonoid and $(\mathsf{q}, \theta_\mathsf{q})$ is an involutive monoid, then the product above turns $\big(\hom(\mathsf{p},\mathsf{q}),\ {}^\theta\big)$ into an involutive monoid. Indeed, with f, g, and x as above:

$$\begin{aligned}(f \cdot g)^\theta(x) &= \theta_\mathsf{q}\big((f \cdot g)(\theta_\mathsf{p}(x))\big) = \theta_\mathsf{q}\big(f(\theta_\mathsf{p}(x)|_S) \cdot g(\theta_\mathsf{p}(x)/_S)\big)\\ &= \theta_\mathsf{q}\big(f(\theta_\mathsf{p}(x/_T)) \cdot g(\theta_\mathsf{p}(x|_T))\big) = \theta_\mathsf{q}\big(g(\theta_\mathsf{p}(x|_T))\big) \cdot \theta_\mathsf{q}\big(f(\theta_\mathsf{p}(x/_T))\big)\\ &= g^\theta(x|_T) \cdot f^\theta(x/_T) = (g^\theta \cdot f^\theta)(x).\end{aligned} \tag{6.3.1}$$

We turn our attention back to $\mathrm{Hom}_{\mathbf{Sp}^A}\big((\mathsf{p}, \theta_\mathsf{p}), (\mathsf{q}, \theta_\mathsf{q})\big)$. The **convolution** of $f, g \in \mathrm{Hom}_{\mathbf{Sp}^A}\big((\mathsf{p}, \theta_\mathsf{p}), (\mathsf{q}, \theta_\mathsf{q})\big) = \mathrm{Hom}_{\mathbf{Sp}}(\mathsf{p},\mathsf{q})$ is the following composition of morphisms of species:

$$f * g := \mu \circ (f \bullet g) \circ \Delta,$$

see [6, Section 1.2.4]. That is, for all $x \in \mathsf{p}[I]$,

$$(f * g)_I(x) = \sum_{I = S \sqcup T} f_S(x|_S) \cdot g_T(x/_S).$$

The identity element $u \in \mathrm{Hom}_{\mathbf{Sp}^A}\big((\mathsf{p}, \theta_\mathsf{p}), (\mathsf{q}, \theta_\mathsf{q})\big)$ for the convolution product is determined by $u_\emptyset(\epsilon_\mathsf{p}) = \epsilon_\mathsf{q}$ and $u_I(x) = 0$ whenever $I \neq \emptyset$.

A computation similar to (6.3.1) shows that for all $f, g : (\mathsf{p}, \theta_\mathsf{p}) \to (\mathsf{q}, \theta_\mathsf{q})$,

$$(f * g)^\theta = g^\theta * f^\theta. \tag{6.3.2}$$

6.3.2 Type B

Let $(\mathsf{p}, \theta_{\mathsf{p}}, \mathsf{m})$ be a type B comonoid and $(\mathsf{q}, \theta_{\mathsf{q}}, \mathsf{n})$ be a type B monoid. We endow $\mathrm{Hom}_{\mathbf{Sp}^A \times \mathbf{Sp}^B}\big((\mathsf{p}, \theta_{\mathsf{p}}, \mathsf{m}), (\mathsf{q}, \theta_{\mathsf{q}}, \mathsf{n})\big)$ with the structure of a left $\mathrm{Hom}_{\mathbf{Sp}}(\mathsf{p}, \mathsf{q})$-module. Given $h \in \mathrm{Hom}_{\mathbf{Sp}}(\mathsf{p}, \mathsf{q})$ and $(f, g) \in \mathrm{Hom}_{\mathbf{Sp}^A \times \mathbf{Sp}^B}\big((\mathsf{p}, \theta_{\mathsf{p}}, \mathsf{m}), (\mathsf{q}, \theta_{\mathsf{q}}, \mathsf{n})\big)$, define

$$h * (f, g) := (h * f * h^{\theta}, h * g),$$

where the convolutions in the first component occur in $\mathrm{Hom}_{\mathbf{Sp}}(\mathsf{p}, \mathsf{q})$, and

$$h * g := \alpha \circ (h \cdot g) \circ \delta.$$

That is, for all $x \in \mathsf{m}[\boldsymbol{I}]$,

$$(h * g)_{\boldsymbol{I}}(x) = \sum_{\boldsymbol{I} = S \sqcup \boldsymbol{T} \sqcup \overline{S}} h_S(x|_S) \cdot g_{\boldsymbol{T}}(x/_S).$$

The associativity in the first component follows from the usual associativity of $\mathrm{Hom}_{\mathbf{Sp}}(\mathsf{p}, \mathsf{q})$ and (6.3.2). In the second component, the associativity of $*$ is a consequence of the coassociativity of $(\mathsf{p}, \theta_{\mathsf{p}}, \mathsf{m})$ and the associativity of $(\mathsf{q}, \theta_{\mathsf{q}}, \mathsf{n})$.

For regular species, if p is a bimonoid and q is a commutative monoid, then the convolution of two monoid morphisms $f, g : \mathsf{p} \to \mathsf{q}$ is again a morphism of monoids. A dual statement holds for comonoid morphisms, in this case p is required to be cocommutative and q must be a bimonoid. The following are type B analogs of these results.

Proposition 6.3.3. *Suppose* $(\mathsf{p}, \theta_{\mathsf{p}}, \mathsf{m})$ *is a bimonoid and* $(\mathsf{q}, \theta_{\mathsf{q}}, \mathsf{n})$ *is a commutative monoid. If* $h : \mathsf{p} \to \mathsf{q}$ *and* $(f, g) : (\mathsf{p}, \theta_{\mathsf{p}}, \mathsf{m}) \to (\mathsf{q}, \theta_{\mathsf{q}}, \mathsf{n})$ *are monoid morphisms, then so is* $h * (f, g)$.

Proof. The computation bellow uses the compatibility axiom of $(\mathsf{p}, \theta_{\mathsf{p}}, \mathsf{m})$, the monoid morphism properties, and the commutativity of $(\mathsf{q}, \theta_{\mathsf{q}}, \mathsf{n})$, in that order.

Let $\boldsymbol{I} = S \sqcup \boldsymbol{T} \sqcup \overline{S}$, $x \in \mathsf{p}[S]$ and $y \in \mathsf{m}[\boldsymbol{T}]$. Then,

$$\begin{aligned}
(h * g)_{\boldsymbol{I}}(x \cdot y) &= \sum_{S' \in \mathcal{P}'(\boldsymbol{I})} h_{S'}\big((x \cdot y)|_{S'}\big) \cdot g_{\boldsymbol{T}'}\big((x \cdot y)/_{S'}\big) \\
&= \sum_{S' \in \mathcal{P}'(\boldsymbol{I})} h_{S'}\big(x|_A \cdot y|_D \cdot \widetilde{x/_{A \sqcup B}}\big) \cdot g_{\boldsymbol{T}'}\big((x/_A)|_B \cdot y/_D\big) \\
&= \sum_{S' \in \mathcal{P}'(\boldsymbol{I})} h_A(x|_A) \cdot h_D(y|_D) \cdot h_{\overline{C}}(\widetilde{x/_{A \sqcup B}}) \cdot f_B((x/_A)|_B) \cdot g_{\boldsymbol{E}}(y/_D) \\
&= \sum_{S' \in \mathcal{P}'(\boldsymbol{I})} h_A(x|_A) \cdot h_D(y|_D) \cdot h^\theta_C(x/_{A \sqcup B}) \cdot f_B((x/_A)|_B) \cdot g_{\boldsymbol{E}}(y/_D) \\
&= \sum_{\substack{S = A \sqcup B \sqcup C \\ D \in \mathcal{P}'(\boldsymbol{T})}} h_A(x|_A) \cdot f_B((x/_A)|_B) \cdot h^\theta_C(x/_{A \sqcup B}) \cdot h_D(y|_D) \cdot g_{\boldsymbol{E}}(y/_D) \\
&= (h * f * h^\theta)_S(x) \cdot (h * g)_{\boldsymbol{T}}(y).
\end{aligned}$$

The step where we replace $h_{\overline{C}}(\widetilde{x/_{A \sqcup B}})$ by $h^\theta_C(x/_{A \sqcup B})$ is an instance of (6.2.4). □

A similar argument shows the following.

Proposition 6.3.4. *Suppose $(\mathsf{p}, \theta_{\mathsf{p}}, \mathsf{m})$ is a cocommutative comonoid and $(\mathsf{q}, \theta_{\mathsf{q}}, \mathsf{n})$ is a bimonoid. If $h : \mathsf{p} \to \mathsf{q}$ and $(f, g) : (\mathsf{p}, \theta_{\mathsf{p}}, \mathsf{m}) \to (\mathsf{q}, \theta_{\mathsf{q}}, \mathsf{n})$ are morphisms of comonoids, then so is $h * (f, g)$.*

Proof. Let $\boldsymbol{I} = S' \sqcup \boldsymbol{T}' \sqcup \overline{S'}$ and $z \in \mathsf{m}[\boldsymbol{I}]$. Then,

$$\begin{aligned}
(h * g)_{\boldsymbol{I}}(z)|_{S'} &= \sum_{S \in \mathcal{P}'(\boldsymbol{I})} \big(h_S(z|_S) \cdot g_{\boldsymbol{T}}(z/_S)\big)|_{S'} \\
&= \sum_{S \in \mathcal{P}'(\boldsymbol{I})} h_S(z|_S)|_A \cdot g_{\boldsymbol{T}}(z/_S)|_D \cdot \widetilde{h_S(z|_S)/_{A \sqcup B}} \\
&= \sum_{S \in \mathcal{P}'(\boldsymbol{I})} h_A(z|_A) \cdot f_D(z|_D) \cdot \widetilde{h_C(z|_C)} \\
&= \sum_{S \in \mathcal{P}'(\boldsymbol{I})} h_A(z|_A) \cdot f_D(z|_D) \cdot h^\theta_{\overline{C}(z|_{\overline{C}})} \\
&= (h * f * h^\theta)(z|_{S'}).
\end{aligned}$$

And

$$
\begin{aligned}
(h * g)_{\boldsymbol{I}}(z)/_{S'} &= \sum_{S \in \mathcal{P}'(\boldsymbol{I})} \big(h_S(z|_S) \cdot g_{\boldsymbol{T}}(z/_S)\big)/_{S'} \\
&= \sum_{S \in \mathcal{P}'(\boldsymbol{I})} (h_S(z|_S)/_A)|_B \cdot g_{\boldsymbol{T}}(z/_S)/_D \\
&= \sum_{S \in \mathcal{P}'(\boldsymbol{I})} h_B((z/_{S'})|_B) \cdot g_{\boldsymbol{E}}((z/_{S'})/_B) \\
&= (h * g)(z/_{S'}). \qquad \square
\end{aligned}
$$

6.3.3 Type B Hopf monoids and the antipode

Recall that a bimonoid p is a **Hopf monoid** if $\mathrm{Id} \in \mathrm{End}_{\mathbf{Sp}}(\mathsf{p}) := \mathrm{Hom}_{\mathbf{Sp}}(\mathsf{p}, \mathsf{p})$ is invertible with respect to the convolution product. In this case, the inverse of the identity is denoted $\mathbb{s}$ and is called the **antipode** of p.

If (p, θ) is an **involutive Hopf monoid**, then Takeuchi's formula (2.3.3) shows that for all $x \in \mathsf{p}[I]$,

$$
\begin{aligned}
\mathbb{s}_I(\theta(x)) &= \sum_{(S_1,\ldots,S_k) \vDash I} (-1)^k \mu_{S_1,\ldots,S_k} \circ \Delta_{S_1,\ldots,S_k}(\theta(x)) \\
&= \sum_{(S_1,\ldots,S_k) \vDash I} (-1)^k \theta\big(\mu_{S_k,\ldots,S_1} \circ \Delta_{S_k,\ldots,S_1}(x)\big) = \theta(\mathbb{s}_I(x)).
\end{aligned}
$$

That is, the antipode and the involution commute. In particular,

$$
\mathbb{s}^\theta = \mathbb{s} \qquad \text{and} \qquad \mathbb{s} * \mathrm{Id} * \mathbb{s}^\theta = \epsilon * \mathbb{s} = \mathbb{s}.
$$

A type B object $(\mathsf{p}, \theta, \mathsf{m})$ is a **type B Hopf monoid** if it is a type B bimonoid and p is a Hopf monoid. In this case, we define the **type B antipode** of $(\mathsf{p}, \theta, \mathsf{m})$ to be the pair

$$
(\mathbb{s}, \mathbb{s}^{\pm}) := \mathbb{s} * (\mathrm{Id}_{\mathsf{p}}, \mathrm{Id}_{\mathsf{m}}).
$$

Explicitly, for $x \in \mathsf{m}[\boldsymbol{I}]$,

$$\mathtt{s}^{\pm}_{\boldsymbol{I}}(x) = \sum_{\boldsymbol{I}=S\sqcup \boldsymbol{T}\sqcup \overline{S}} \mathtt{s}_S(x|_S) \cdot x/_S.$$

Using Takeuchi's formula (2.3.3) for the antipode of p and the (co)associativity of $(\mathsf{p}, \theta, \mathsf{m})$, we obtain the following type B analog of Takeuchi's formula for $\mathtt{s}^{\pm}$:

$$\mathtt{s}^{\pm}_{\boldsymbol{I}}(x) = \sum_{(S_1,\ldots,S_k,\boldsymbol{T},\overline{S_k}\ldots,\overline{S_1})\models^B \boldsymbol{I}} (-1)^k \alpha_{S_1,\ldots,S_k,\boldsymbol{T}} \circ \delta_{S_1,\ldots,S_k,\boldsymbol{T}}(x). \tag{6.3.3}$$

Observe that $\mathrm{Id}_{\mathsf{p}} * \mathtt{s}^{\pm} = \mathrm{Id}_{\mathsf{p}} * \mathtt{s} * \mathrm{Id}_{\mathsf{m}} = \mathrm{Id}_{\mathsf{m}}$. This yields the following recursive formula to compute $\mathtt{s}^{\pm}$:

$$\mathtt{s}^{\pm}_{\boldsymbol{I}}(x) = x - \sum_{\substack{\boldsymbol{I}=S\sqcup \boldsymbol{T}\sqcup \overline{S} \\ S\neq\emptyset}} x|_S \cdot \mathtt{s}^{\pm}_{\boldsymbol{T}}(x/_S). \tag{6.3.4}$$

We refer to this formula as the type B Milnor-Moore formula.

In general, the antipode $\mathtt{s}$ is not a morphism of Hopf monoids from p to itself. Rather, it can be seen as a Hopf monoid morphism $\mathsf{p} \to \mathsf{p}^{\mathrm{op,cop}}$ (for details, see [6, Section 1.2.9]). In concrete terms, this means that $\mathtt{s}$ *reverses products and coproducts*:

$$\mathtt{s}_I(x \cdot y) = \mathtt{s}_T(y) \cdot \mathtt{s}_S(x) \qquad \text{and} \qquad \Delta_{S,T}(\mathtt{s}_I(z)) = (\mathtt{s}_S \otimes \mathtt{s}_T)(z/_T \otimes z|_T),$$

for all decompositions $I = S \sqcup T$ and structures $x \in \mathsf{p}[S]$, $y \in \mathsf{p}[T]$, $z \in \mathsf{p}[I]$. The following two propositions are the type B analogs of this result.

Proposition 6.3.5. *The antipode* $(\mathtt{s}, \mathtt{s}^{\pm})$ *reverses products. That is, for all decompositions* $\boldsymbol{I} = S \sqcup \boldsymbol{T} \sqcup \overline{S}$, *and structures* $x \in \mathsf{p}[S]$, $y \in \mathsf{m}[\boldsymbol{T}]$,

$$\mathtt{s}^{\pm}_{\boldsymbol{I}}(x \cdot y) = \mathtt{s}_{\overline{S}}(\widetilde{x}) \cdot \mathtt{s}^{\pm}_{\boldsymbol{T}}(y). \tag{6.3.5}$$

Proof. We follow the notation in (6.2.6).

$$\begin{aligned}
\mathfrak{s}^{\pm}_{\boldsymbol{I}}(x\cdot y) &= \sum_{\boldsymbol{I}=S'\sqcup \boldsymbol{T}'\sqcup \overline{S'}} \mathfrak{s}_{S'}((x\cdot y)|_{S'})\cdot (x\cdot y)/_{S'} \\
&= \sum_{\boldsymbol{I}=S'\sqcup \boldsymbol{T}'\sqcup \overline{S'}} \mathfrak{s}_{S'}(x|_A\cdot y|_D\cdot \widetilde{x}|_{\overline{C}})\cdot (x/_A)|_B\cdot y/_D \\
&= \sum_{\boldsymbol{I}=S'\sqcup \boldsymbol{T}'\sqcup \overline{S'}} \mathfrak{s}_{\overline{C}}(\widetilde{x}|_{\overline{C}})\cdot \mathfrak{s}_D(y|_D)\cdot \mathfrak{s}_A(x|_A)\cdot (x/_A)|_B\cdot y/_D
\end{aligned}$$

If we first fix $A\sqcup B = S\setminus (S\cap \overline{S'})$ and first sum over the possible values of $S\cap S' = A\subseteq A\sqcup B$, the third and fourth factor are computing $(\mathfrak{s}*\mathrm{Id}_{\mathsf{p}})_{A\sqcup B}(x|_{A\sqcup B})$. By definition of the antipode, this is 0 unless $A\sqcup B=\emptyset$ (in which case $C=S$). Thus, the sum above equals

$$\sum_{\boldsymbol{T}=D\sqcup \boldsymbol{E}\sqcup \overline{D}} \mathfrak{s}_{\overline{S}}(\widetilde{x})\cdot \mathfrak{s}_D(y|_D)\cdot y/_D = \mathfrak{s}_{\overline{S}}(\widetilde{x})\cdot (\mathfrak{s}*\mathrm{Id}_{\mathsf{m}})_{\boldsymbol{T}}(y) = \mathfrak{s}_{\overline{S}}(\widetilde{x})\cdot \mathfrak{s}^{\pm}_{\boldsymbol{T}}(y). \qquad \square$$

Proposition 6.3.6. *The antipode $(\mathfrak{s},\mathfrak{s}^{\pm})$ reverses coproducts. That is, for all decompositions $\boldsymbol{I}=S'\sqcup \boldsymbol{T}'\sqcup \overline{S'}$, and structures $x\in \mathsf{m}[\boldsymbol{I}]$,*

$$\delta_{S',\boldsymbol{T}'}(\mathfrak{s}^{\pm}_{\boldsymbol{I}}(x)) = \mathfrak{s}_{S'}(\widetilde{x|_{\overline{S'}}})\otimes \mathfrak{s}^{\pm}_{\boldsymbol{T}'}(x/_{\overline{S'}}).$$

Proof. The statement is clear if $S'=\emptyset$, so we assume that $S'\neq\emptyset$ and consequently $\boldsymbol{T}'$ is a proper subset. Applying $\delta_{S',\boldsymbol{T}'}$ to (6.3.4) yields

$$\delta_{S',\boldsymbol{T}'}(\mathfrak{s}^{\pm}_{\boldsymbol{I}}(x)) = x|_{S'}\otimes x/_{S'} - \sum_{\substack{\boldsymbol{I}=S\sqcup \boldsymbol{T}\sqcup \overline{S}\\ S\neq\emptyset}} \delta_{S',\boldsymbol{T}'}(x|_S\cdot \mathfrak{s}^{\pm}_{\boldsymbol{T}}(x/_S)). \tag{6.3.6}$$

By induction on the size of $\boldsymbol{T}$, we have that for any decomposition $\boldsymbol{T}=D\sqcup \boldsymbol{E}\sqcup \overline{D}$

$$\delta_{D,\boldsymbol{E}}(\mathfrak{s}^{\pm}_{\boldsymbol{T}}(x/_S)) = \mathfrak{s}_D(\widetilde{(x/_S)|_{\overline{D}}})\otimes \mathfrak{s}^{\pm}_{\boldsymbol{E}}((x/_S)/_{\overline{D}}),$$

hence, the sum above equals

$$\sum_{\substack{\boldsymbol{I}=S\sqcup \boldsymbol{T}\sqcup \overline{S}\\ S\neq\emptyset}} (x|_S)|_A\cdot \mathfrak{s}_D(\widetilde{(x/_S)|_{\overline{D}}})\cdot \widetilde{(x|_S)/_{A\sqcup B}}\otimes ((x|_S)/_A)|_B\cdot \mathfrak{s}^{\pm}_{\boldsymbol{E}}((x/_S)/_{\overline{D}}), \tag{6.3.7}$$

with $A, B, C, D, \boldsymbol{E}$ as in (6.2.6). We first consider the terms where $A = S'$ (thus, $C = D = \emptyset$, $B \in \mathcal{P}'(T')$ can be chosen arbitrarily, and $S = A \sqcup B$). The sum of these terms is

$$\sum_{\boldsymbol{T}'=B\sqcup\boldsymbol{E}\sqcup\overline{B'}} x|_{S'} \otimes (x/_{S'})|_B \cdot \mathbb{s}^{\pm}_{\boldsymbol{E}}((x/_{S'})/_B) = x|_{S'} \otimes (\mathrm{Id}_{\mathsf{p}} * \mathbb{s}^{\pm})_{\boldsymbol{T}'}(x/'_S) = x|_{S'} \otimes x/_{S'}.$$

We now consider the terms where $A \sqcup B \neq \emptyset$ and $A \subsetneq S'$ (thus, $C \sqcup \overline{D} = \overline{S'} \setminus \overline{A}$ is nonempty). If we first fix A and B satisfying these conditions, then all the factors in the sum are constant except for the later two in the first component of the tensor. Using the that θ_{p} reverses products (6.2.1), that it commutes with $\mathbb{s}$, and the coassociativity of p, we deduce

$$\mathbb{s}_D(\widetilde{(x/_S)|_{\overline{D}}}) \cdot \widetilde{(x|_S)/_{A\sqcup B}} = \overline{\theta_{\mathsf{p}}}((x|_S)/_{A\sqcup B} \cdot \mathbb{s}_{\overline{D}}((x/_S)|_{\overline{D}}) = \overline{\theta_{\mathsf{p}}}(y|_C \cdot \mathbb{s}_{\overline{D}}(y/_C)),$$

where $\overline{\theta_{\mathsf{p}}}(z) = \theta_{\mathsf{p}}(\overline{z})$ and $y = (x/_{A\sqcup B})|_{\overline{S'}\setminus\overline{A}}$. Since

$$\sum_{C\subseteq\overline{S'}\setminus\overline{A}} \overline{\theta_{\mathsf{p}}}(y|_C \cdot \mathbb{s}_{\overline{D}}(y/_C)) = \overline{\theta_{\mathsf{p}}}((\mathrm{Id}_{\mathsf{p}} * \mathbb{s})_{\overline{S'}\setminus\overline{A}}(y)) = 0,$$

the sum of the corresponding terms in (6.3.7) is zero. Finally, we consider the terms where $A = B = \emptyset$ (thus, $C \sqcup \overline{D} = \overline{S'}$, $C = S \neq \emptyset$, and $\boldsymbol{E} = \boldsymbol{T}'$). The sum of these terms is

$$\begin{aligned}
&\sum_{D\subsetneq\overline{S'}} \mathbb{s}_D(\widetilde{(x/_C)|_{\overline{D}}}) \cdot \widetilde{x|_C} \otimes \mathbb{s}^{\pm}_{\boldsymbol{T}'}((x/_C)/_{\overline{D}}) \\
&\qquad = \sum_{D\subsetneq\overline{S'}} \mathbb{s}_D(\widetilde{(x|_{\overline{S'}})}|_D) \cdot \widetilde{(x|_{\overline{S'}})}/_D \otimes \mathbb{s}^{\pm}_{\boldsymbol{T}'}(x/_{\overline{S'}}) = -\mathbb{s}_{\overline{S'}}(\widetilde{(x|_{\overline{S'}})}) \otimes \mathbb{s}^{\pm}_{\boldsymbol{T}'}(x/_{\overline{S'}}).
\end{aligned}$$

We used coassociativity and (6.2.2) to deduce $\widetilde{(x/_C)|_{\overline{D}}} = \widetilde{(x|_{\overline{S'}})/_C} = \widetilde{(x|_{\overline{S'}})}|_D$, and similarly $\widetilde{x|_C} = \widetilde{(x|_{\overline{S'}})}/_D$.

Finally, separating the sum (6.3.6) into the cases above, we obtain

$$\begin{aligned}
\delta_{S',\boldsymbol{T}'}(\mathbb{s}^{\pm}_{\boldsymbol{I}}(x)) &= x|_{S'} \otimes x/_{S'} - \Big(x|_{S'} \otimes x/_{S'} + 0 - \mathbb{s}_{\overline{S'}}(\widetilde{(x|_{\overline{S'}})}) \otimes \mathbb{s}^{\pm}_{\boldsymbol{T}'}(x/_{\overline{S'}})\Big) \\
&= \mathbb{s}_{\overline{S'}}(\widetilde{(x|_{\overline{S'}})}) \otimes \mathbb{s}^{\pm}_{\boldsymbol{T}'}(x/_{\overline{S'}}). \qquad \square
\end{aligned}$$

If p is a Hopf monoid and q is a commutative monoid, then $\mathrm{Hom}_{\mathrm{Mon}}(\mathsf{p}, \mathsf{q})$ is a group under the convolution product; the inverse of f is $f \circ \mathbb{s}$. This statement has two parts. First, it says that if $f : \mathsf{p} \to \mathsf{q}$ is a morphism of monoids, then so is $f \circ \mathbb{s}$. Second, $f * (f \circ \mathbb{s}) = u$, the unit of $\mathrm{Hom}_{\mathrm{Mon}}(\mathsf{p}, \mathsf{q})$. The following is the type B analog of this result.

Proposition 6.3.7. *Let* $(\mathsf{p}, \theta_{\mathsf{p}}, \mathsf{m})$ *be a type B Hopf monoid with antipode* $(\mathbb{s}, \mathbb{s}^{\pm})$ *and* $(\mathsf{q}, \theta_{\mathsf{q}}, \mathsf{n})$ *be a type B commutative monoid. If* $(f, g) : (\mathsf{p}, \theta_{\mathsf{p}}, \mathsf{m}) \to (\mathsf{q}, \theta_{\mathsf{q}}, \mathsf{n})$ *is a morphism of monoids, then so is* $(f^\theta \circ \mathbb{s}, g \circ \mathbb{s}^{\pm})$. *Moreover,* $f * (f^\theta \circ \mathbb{s}, g \circ \mathbb{s}^{\pm}) = (f, g)$.

Proof. We verify the first statement directly. Let $\boldsymbol{I} = S \sqcup \boldsymbol{T} \sqcup \overline{S}$, $x \in \mathsf{p}[S]$ and $y \in \mathsf{m}[\boldsymbol{T}]$. Then,

$$\begin{aligned} f^\theta_S(\mathbb{s}_S(x)) \cdot g_{\boldsymbol{T}}(\mathbb{s}^{\pm}_{\boldsymbol{T}}(y)) &= \widetilde{f_{\overline{S}}(\mathbb{s}_{\overline{S}}(\widetilde{x}))} \cdot g_{\boldsymbol{T}}(\mathbb{s}^{\pm}_{\boldsymbol{T}}(y)) \\ &= f_{\overline{S}}(\mathbb{s}_{\overline{S}}(\widetilde{x})) \cdot g_{\boldsymbol{T}}(\mathbb{s}^{\pm}_{\boldsymbol{T}}(y)) && \langle \mathsf{q} \text{ is commutative} \rangle \\ &= g_{\boldsymbol{I}}(\mathbb{s}_{\overline{S}}(\widetilde{x}) \cdot \mathbb{s}^{\pm}_{\boldsymbol{T}}(y)) && \langle (f,g) \text{ is a morphism of monoids} \rangle \\ &= g_{\boldsymbol{I}}(\mathbb{s}^{\pm}_{\boldsymbol{I}}(x \cdot y)) && \langle (\mathbb{s}, \mathbb{s}^{\pm}) \text{ reverses products (6.3.5)} \rangle \end{aligned}$$

For the second statement, we use that $f^\theta \circ \mathbb{s}$ is the convolution inverse of f^θ and consequently $f^\theta * (f^\theta \circ \mathbb{s}) * (f^\theta)^\theta = f$. Moreover, for all $\boldsymbol{I}$ and $z \in \mathsf{m}[\boldsymbol{I}]$,

$$\begin{aligned} (f * (g \circ \mathbb{s}^{\pm}))_{\boldsymbol{I}}(z) &= \sum_{\boldsymbol{I} = S \sqcup \boldsymbol{T} \sqcup \overline{S}} f_S(z|_S) \cdot g_{\boldsymbol{T}}(\mathbb{s}^{\pm}_{\boldsymbol{T}}(z/_S)) \\ &= \sum_{\boldsymbol{I} = S \sqcup \boldsymbol{T} \sqcup \overline{S}} g_{\boldsymbol{I}}(z|_S \cdot \mathbb{s}^{\pm}_{\boldsymbol{T}}(z/_S)) = g_{\boldsymbol{I}}\Big(\sum_{\boldsymbol{I} = S \sqcup \boldsymbol{T} \sqcup \overline{S}} z|_S \cdot \mathbb{s}^{\pm}_{\boldsymbol{T}}(z/_S) \Big) \\ &= g_{\boldsymbol{I}}\big((\mathrm{Id}_{\mathsf{p}} * \mathbb{s}^{\pm})_{\boldsymbol{I}}(z) \big) = g_{\boldsymbol{I}}(z). \end{aligned}$$

Therefore, $f * (f^\theta \circ \mathbb{s}, g \circ \mathbb{s}^{\pm}) = (f, g)$. ∎

6.4 Characters and polynomial invariants

A **character** of a Hopf monoid p is a morphism of monoids $\zeta : \mathsf{p} \to \mathsf{E}$. Since the monoid E is commutative, the collection $\mathbb{X}(\mathsf{p}) := \mathrm{Hom}_{\mathrm{Mon}}(\mathsf{p}, \mathsf{E})$ forms a group under the convolution product called the **group of characters** of p, see [1, 4].

Fix a character $\zeta \in \mathbb{X}(\mathsf{p})$. For every structure $x \in \mathsf{p}[I]$ we have a function $\chi_I(x) : \mathbb{N} \to \Bbbk$ defined by

$$\chi_I(x)(n) := \zeta_I^{*n}(x). \tag{6.4.1}$$

Expanding this definition, one obtains

$$\begin{aligned} \chi_I(x)(n) &= \sum_{I = S_1 \sqcup \cdots \sqcup S_n} \zeta_I \circ \mu_{S_1,\ldots,S_n} \circ \Delta_{S_1,\ldots,S_n}(x) \\ &= \sum_{k=0}^{|I|} \Bigg(\sum_{\substack{I = S_1 \sqcup \cdots \sqcup S_k \\ S_i \neq \emptyset}} \zeta_I \circ \mu_{S_1,\ldots,S_k} \circ \Delta_{S_1,\ldots,S_k}(x) \Bigg) \binom{n}{k}, \end{aligned}$$

which shows that $\chi_I(x)$ is a polynomial of degree at most $|I|$. Moreover, the naturality of ζ shows that $\chi_I(x)$ is an *invariant* of x; that is, $\chi_I(x) = \chi_J(y)$ whenever $y = \mathsf{p}[\sigma](x)$ for some bijection $\sigma : I \to J$.

Proposition 6.4.1 ([1]). *The polynomial invariants χ satisfy:*

1. *For all decompositions $I = S \sqcup T$ and structures $x \in \mathsf{p}[S]$ and $y \in \mathsf{p}[T]$,*

$$\chi_I(x \cdot y) = \chi_S(x)\chi_T(y).$$

2. *For all structures $x \in \mathsf{p}[I]$ and scalars n, m,*

$$\chi_I(x)(n+m) = \sum_{I = S \sqcup T} \chi_S(x|_S)(n)\, \chi_T(x/_S)(m).$$

3. *For all structures $x \in \mathsf{p}[I]$ and scalars n,*

$$\chi_I(x)(-n) = \chi_I(\mathrm{s}(x))(n).$$

In this section we define characters of a type B Hopf monoid, define two polynomial invariants associated to a character, and prove properties analogous to those of Proposition 6.4.1 for these new invariants.

6.4.1 Type A with an involution

Let $(\mathsf{p}, \theta_{\mathsf{p}})$ be an involutive Hopf monoid. Given a character $\zeta \in \mathbb{X}(\mathsf{p})$, we define the following two new polynomial invariants for each structure $x \in \mathsf{p}[I]$:

$$\psi_I(x)(2n+1) := \big(\zeta^{*(n+1)} * (\zeta^{\theta})^{*n}\big)_I(x), \tag{6.4.2}$$

$$\psi_I^{\pm}(x)(2n) := \big(\zeta^{*n} * (\zeta^{\theta})^{*n}\big)_I(x). \tag{6.4.3}$$

To verify that $\psi_I(x)$ and $\psi_I^{\pm}(x)$ are indeed polynomials of degree at most $|I|$, we express them as a finite sum of polynomial functions of appropriate degrees, as follows.

$$\psi_I(x)(2n+1) = \sum_{I=S\sqcup T} \chi_S(x|_S)(n+1)\chi_T^{\theta}(x/_S)(n), \tag{6.4.4}$$

$$\psi_I^{\pm}(x)(2n) = \sum_{I=S\sqcup T} \chi_S(x|_S)(n)\chi_T^{\theta}(x/_S)(n), \tag{6.4.5}$$

where χ^{θ} denotes the polynomial invariants associated to the character ζ^{θ}.

Proposition 6.4.2. *The polynomial invariants ψ and $\psi^{\pm}$ satisfy:*

1. *For all decompositions $I = S \sqcup T$ and structures $x \in \mathsf{p}[S]$ and $y \in \mathsf{p}[T]$,*

$$\psi_I(x \cdot y) = \psi_S(x)\psi_T(y) \qquad \textit{and} \qquad \psi_I^{\pm}(x \cdot y) = \psi_S^{\pm}(x)\psi_T^{\pm}(y).$$

2. *For all structures $x \in \mathsf{p}[I]$ and scalars t,*

$$\psi_I(x)(-t) = (\psi^{\theta})_I(\mathbb{s}(x))(t) \qquad \textit{and} \qquad \psi_I^{\pm}(x)(-t) = (\psi^{\theta})_I^{\pm}(\mathbb{s}(x))(t).$$

Proof. Both $\zeta^{*(n+1)} * (\zeta^\theta)^{*n}$ and $\zeta^{*n} * (\zeta^\theta)^{*n}$ are characters, hence multiplicative, and the first statement follows.

In order to verify the first claim in *2.*, it suffices to consider the case $t = 2n+1$ where n is a positive integer. We use expression (6.4.4) and Proposition 6.4.1 to deduce

$$\begin{aligned}
\psi_I(x)(-2n-1) &= \sum_{I=S\sqcup T} \chi_S(x|_S)(-n)\, \chi_T^\theta(x/_S)(-n-1) \\
&= \sum_{I=S\sqcup T} \chi_S(\mathbb{s}_S(x|_S))(n)\, \chi_T^\theta(\mathbb{s}_T(x/_S))(n+1) \\
&= \sum_{I=S\sqcup T} \chi_T^\theta(\mathbb{s}_I(x)|_T)(n+1)\, \chi_S(\mathbb{s}_I(x)/_T))(n) \\
&= (\psi^\theta)_I(\mathbb{s}(x))(2n+1)
\end{aligned}$$

as claimed. A similar computation, but using (6.4.5) and $t = 2n$ instead, proves the claim for $\psi^\pm$. □

6.4.2 Type B

Let $(\mathsf{p}, \theta_{\mathsf{p}}, \mathsf{m})$ be a type B Hopf monoid. A **character** of $(\mathsf{p}, \theta_{\mathsf{p}}, \mathsf{m})$ is a morphism of type B monoids $(\zeta, \xi) : (\mathsf{p}, \theta_{\mathsf{p}}, \mathsf{m}) \to (\mathsf{E}, \mathrm{id}, \mathsf{E}^B)$, and $\mathbb{X}(\mathsf{p}, \theta_{\mathsf{p}}, \mathsf{m})$ denotes the collection of characters of $(\mathsf{p}, \theta_{\mathsf{p}}, \mathsf{m})$. Extending the definition of the identity character $u \in \mathbb{X}(\mathsf{p})$, we define $(u, u^B) \in \mathbb{X}(\mathsf{p}, \theta_{\mathsf{p}}, \mathsf{m})$ by

$$u^B_{\boldsymbol{I}}(x) = \begin{cases} 1 & \text{if } x = \epsilon_\emptyset, \\ 0 & \text{if } \boldsymbol{I} \neq \emptyset. \end{cases}$$

Proposition 6.3.3 shows that the group $\mathbb{X}(\mathsf{p})$ acts on $\mathbb{X}(\mathsf{p}, \theta_{\mathsf{p}}, \mathsf{m})$. Given a character (ζ,ξ) of $(\mathsf{p},\theta_{\mathsf{p}},\mathsf{m})$ and a positive integer n, we consider the characters:

$$\zeta^{*n} * (\zeta,\xi) = (\zeta^{*(n+1)} * (\zeta^{\theta})^{*n}, \zeta^{*n} * \xi) \tag{6.4.6}$$

$$\zeta^{*n} * (u, u^{B}) = (\zeta^{*n} * (\zeta^{\theta})^{*n}, \zeta^{*n} * u^{B}), \tag{6.4.7}$$

and use them to define the following new polynomial invariants for each structure $x \in \mathsf{m}[\boldsymbol{I}]$:

$$\chi_{\boldsymbol{I}}(x)(2n+1) := \big(\zeta^{*n} * \xi\big)_{\boldsymbol{I}}(x) \tag{6.4.8}$$

$$\chi^{\pm}_{\boldsymbol{I}}(x)(2n) := \big(\zeta^{*n} * u^{B}\big)_{\boldsymbol{I}}(x). \tag{6.4.9}$$

In order to verify that these invariants are indeed polynomial and of degree at most $|\boldsymbol{I}|/2$, we expand the definition

$$\begin{aligned}\chi_{\boldsymbol{I}}(x)(2n+1) &= \sum_{\boldsymbol{I}=S_1\sqcup\cdots\sqcup S_n\sqcup \boldsymbol{T}\sqcup\overline{S_n}\sqcup\cdots\sqcup\overline{S_1}} \xi_{\boldsymbol{I}}\circ\alpha_{S_1,\dots,S_n,\boldsymbol{T}}\circ\delta_{S_1,\dots,S_n,\boldsymbol{T}}(x)\\ &= \sum_{k=0}^{|\boldsymbol{I}|/2}\Bigg(\sum_{\substack{\boldsymbol{I}=S_1\sqcup\cdots\sqcup S_k\sqcup \boldsymbol{T}\sqcup\overline{S_k}\sqcup\cdots\sqcup\overline{S_1}\\ S_i\neq\emptyset}} \xi_{\boldsymbol{I}}\circ\alpha_{S_1,\dots,S_k,\boldsymbol{T}}\circ\delta_{S_1,\dots,S_k,\boldsymbol{T}}(x)\Bigg)\binom{n}{k}.\end{aligned} \tag{6.4.10}$$

Observe that we have implicitly used that (ζ,ξ) is a character, and

$$\zeta_{S_1}(x_1)\cdot\ldots\cdot\zeta_{S_n}(x_n)\cdot\xi_{\boldsymbol{T}}(y) = \xi_{\boldsymbol{I}}(x_1\cdot\ldots\cdot x_n\cdot y).$$

A similar expression with $\boldsymbol{T}=\emptyset$ shows that $\chi^{\pm}_{\boldsymbol{I}}(x)$ is a polynomial invariant of degree at most $|\boldsymbol{I}|/2$.

Theorem 6.4.3. *The polynomial invariants χ and $\chi^{\pm}$ satisfy:*

1. *For all decompositions $\boldsymbol{I} = S\sqcup\boldsymbol{T}\sqcup\overline{S}$ and structures $x\in\mathsf{p}[S]$ and $y\in\mathsf{m}[\boldsymbol{T}]$,*

$$\chi_{\boldsymbol{I}}(x\cdot y) = \psi_S(x)\chi_{\boldsymbol{T}}(y) \qquad \textit{and} \qquad \chi^{\pm}_{\boldsymbol{I}}(x\cdot y) = \psi^{\pm}_S(x)\chi^{\pm}_{\boldsymbol{T}}(y).$$

2. *For all structures $x\in\mathsf{p}[I]$ and scalars s,t,*

$$\chi_{\boldsymbol{I}}(x)(s+t) = \sum_{\boldsymbol{I}=S\sqcup\boldsymbol{T}\sqcup\overline{S}} \chi_S(x|_S)(\tfrac{s}{2})\,\chi_{\boldsymbol{T}}(x/_S)(t),$$

and

$$\chi^{\pm}_{\boldsymbol{I}}(x)(s+t) = \sum_{\boldsymbol{I}=S\sqcup \boldsymbol{T}\sqcup \overline{S}} \chi_S(x|_S)(\tfrac{s}{2})\, \chi^{\pm}_{\boldsymbol{T}}(x/_S)(t).$$

3. For all structures $x \in \mathsf{p}[I]$,

$$\chi_{\boldsymbol{I}}(x)(-1) = \xi_{\boldsymbol{I}}(\mathbb{s}^{\pm}_{\boldsymbol{I}}(x)).$$

Proof. The claims in *1.* follow from the multiplicativity of the characters (6.4.6) and (6.4.7), and the definition of the polynomial invariants involved.

To prove the first claim in *2.*, it is enough to consider the case $s = 2n$ and $t = 2m+1$ for positive integers n, m. In this case, expanding definition (6.4.8) yields

$$\begin{aligned}\chi_{\boldsymbol{I}}(x)(2(n+m)+1) &= \left(\zeta^{*(n+m)} * \xi\right)_{\boldsymbol{I}}(x)\\ &= \sum_{\boldsymbol{I}=S\sqcup \boldsymbol{T}\sqcup \overline{S}} \zeta^{*n}_S(x|_S)\cdot \left(\zeta^{*m} * \xi\right)_{\boldsymbol{T}}(x/_S)\\ &= \sum_{\boldsymbol{I}=S\sqcup \boldsymbol{T}\sqcup \overline{S}} \chi_S(x|_S)(n)\cdot \chi_{\boldsymbol{T}}(x/_S)(2m+1),\end{aligned}$$

as we wanted to show. A similar analysis using definition (6.4.9) in the case $s = 2n$ and $t = 2m$ with n, m positive integers yields the result for $\chi^{\pm}_{\boldsymbol{I}}(x)$.

For the last statement, we use (6.4.10) with $n = -1$ to deduce

$$\chi_{\boldsymbol{I}}(x)(-1) = \sum_{(S_1,\ldots,S_k,\boldsymbol{T},\overline{S_k}\ldots,\overline{S_1})\vDash^B \boldsymbol{I}} \binom{-1}{k} \xi_{\boldsymbol{I}} \circ \alpha_{S_1,\ldots,S_k,\boldsymbol{T}} \circ \delta_{S_1,\ldots,S_k,\boldsymbol{T}}(x).$$

The result follows by using the linearity of $\xi_{\boldsymbol{I}}$ and the type B analog of Takeuchi's formula (6.3.3). □

6.5 Series

A series of a species p is a morphism $s : \mathsf{E} \to \mathsf{p}$, see Section 2.3.2 for a more explicit definition. Similarly, a **series** of a type B object $(\mathsf{p}, \theta_{\mathsf{p}}, \mathsf{m})$ is a morphism $(s, s^B) : (\mathsf{E}, \mathrm{id}, \mathsf{E}^B) \to (\mathsf{p}, \theta_{\mathsf{p}}, \mathsf{m})$. Explicitly, a series of $(\mathsf{p}, \theta_{\mathsf{p}}, \mathsf{m})$ corresponds to a choice of elements $s_I = s_I(*_I) \in \mathsf{p}[I]$ and $s'_{\boldsymbol{I}} = s^B_{\boldsymbol{I}}(*_{\boldsymbol{I}}) \in \mathsf{m}[\boldsymbol{I}]$ such that

$$\mathsf{p}[\sigma](s_I) = s_J \qquad \text{and} \qquad \mathsf{m}[\tau](s'_{\boldsymbol{I}}) = s'_{\boldsymbol{J}}$$

for any bijection $\sigma : I \to J$ and any involution-preserving bijection $\tau : \boldsymbol{I} \to \boldsymbol{J}$. In particular, the element $s_{[n]}$ is $\mathfrak{S}_n$-invariant and $s_{[\pm n]}$ is $\mathfrak{B}_n$-invariant. We let $\mathscr{S}(\mathsf{p}, \theta_{\mathsf{p}}, \mathsf{m})$ the space of series of $(\mathsf{p}, \theta_{\mathsf{p}}, \mathsf{m})$.

Let $(\mathsf{p}, \theta_{\mathsf{p}}, \mathsf{m})$ be a type B monoid. We consider the action of $\mathrm{Hom}_{\mathbf{Sp}}(\mathsf{E}, \mathsf{p})$ on $\mathrm{Hom}_{\mathbf{Sp}^A \times \mathbf{Sp}^B}\big((\mathsf{E}, \mathrm{id}, \mathsf{E}^B), (\mathsf{p}, \theta_{\mathsf{p}}, \mathsf{m})\big)$ in Section 6.3.2. That is, we endow $\mathscr{S}(\mathsf{p}, \theta_{\mathsf{p}}, \mathsf{m})$ with the structure of a left module over the algebra $\mathscr{S}(\mathsf{p})$. The action of a series $t \in \mathscr{S}(\mathsf{p})$ on a series $(s, s') \in \mathscr{S}(\mathsf{p}, \theta_{\mathsf{p}}, \mathsf{m})$ is determined by

$$t * (s, s') = (t * s * \theta(t), t * s'),$$

where the first component is the Cauchy product of the series $t, s, \theta(t) \in \mathscr{S}(\mathsf{p})$, the series $\theta(t)$ is defines by $\theta(t)_I = \theta_{\mathsf{p}}(t_I)$ for all I, and

$$(t * s')_{\boldsymbol{I}} = \sum_{\boldsymbol{I} = S \sqcup \boldsymbol{T} \sqcup \overline{S}} t_S \cdot s'_{\boldsymbol{T}}.$$

Example 6.5.1. We extend Example 2.3.4 to the type B case. A series (s, s') of $(\mathrm{E}, \mathrm{id}, \mathrm{E}^B)$ is determined by

$$s_{[n]} = a_n *_{[n]} \qquad \text{and} \qquad s'_{[\pm n]} = b_n *_{[\pm n]}$$

for arbitrary scalars $a_n, b_n \in \Bbbk$. We identify (s, s') with the following pair of formal power series in $\Bbbk[\![x]\!]$

$$\sum_{n \geq 0} a_n \frac{x^n}{n!} \qquad \text{and} \qquad \sum_{n \geq 0} b_n \frac{x^n}{(2n)!!}.$$

Then, the action of $\mathscr{S}(\mathsf{E}) \cong \Bbbk[\![x]\!]$ on $(\mathrm{E}, \mathrm{id}, \mathrm{E}^B) \cong \Bbbk[\![x]\!] \times \Bbbk[\![x]\!]$ is given by

$$h(x) * (f(x), g(x)) = (h^2(x)f(x), h(x)g(x)).$$

6.6 Substitution product of type B objects

The substitution product of species is defined in [6, Section 8.7.1]. If p is a species and q is a **positive species** (i.e. $\mathsf{q}[\emptyset] = \emptyset$), the substitution product $\mathsf{p} \circ \mathsf{q}$ is the species

$$(\mathsf{p} \circ \mathsf{q})[I] = \coprod_{\mathrm{X} \vdash I} \mathsf{p}[\mathrm{X}] \times \Big(\prod_{S \in \mathrm{X}} \mathsf{q}[S] \Big).$$

Thus, a $(\mathsf{p} \circ \mathsf{q})$-structure on I is a pair (x, y) where $x \in \mathsf{p}[\mathrm{X}]$ for some partition X of I and y is a tuple of q-structures $y_S \in \mathsf{q}[S]$, one for each block $S \in \mathrm{X}$. The species X is the unit with respect to the substitution product.

We define an analog of the substitution product for type B objects. We say that a type B object $(\mathsf{p}, \theta_{\mathsf{p}}, \mathsf{m})$ is **positive** if p is a positive species. Observe that we do not ask $\mathsf{m}[\emptyset]$ to be empty.

Given two type B objects $(\mathsf{p}, \theta_{\mathsf{p}}, \mathsf{m})$ and $(\mathsf{q}, \theta_{\mathsf{q}}, \mathsf{n})$, the second one positive, define a new type B object

$$(\mathsf{p}, \theta_{\mathsf{p}}, \mathsf{m}) \circ (\mathsf{q}, \theta_{\mathsf{q}}, \mathsf{n}) := (\mathsf{p} \circ \mathsf{q}, \theta_{\mathsf{p}} \circ \theta_{\mathsf{q}}, \mathsf{m} \circ_B (\mathsf{q}, \mathsf{n})),$$

where $\mathsf{p} \circ \mathsf{q}$ and $\theta_{\mathsf{p}} \circ \theta_{\mathsf{q}}$ come from the usual substitution product of species, and

$$(\mathsf{m} \circ_B (\mathsf{q}, \mathsf{n}))[\boldsymbol{I}] := \coprod_{\mathrm{X} \vdash^B \boldsymbol{I}} \mathsf{m}[\mathrm{X}_\pm] \times \Big(\prod_{S \in \mathrm{X}_\pm} \mathsf{q}[S] \Big)^{\sim} \times \mathsf{n}[\mathrm{X}_0].$$

Here, $\big(\prod_{S \in \mathrm{X}_\pm} \mathsf{q}[S] \big)^{\sim}$ denotes the $\sim$-invariant part of $\prod_{S \in \mathrm{X}_\pm} \mathsf{q}[S]$. Namely, it contains tuples $(x_S)_{S \in \mathrm{X}_\pm} \in \prod_{S \in \mathrm{X}_\pm} \mathsf{q}[S]$ such that $x_{\overline{S}} = \widetilde{x_S}$.

Remark 6.6.1. Related substitution operations of a (regular) species with a type B (Hyperoctahedral) species are defined by Choquette in his Ph.D. thesis [35].

Example 6.6.2. Let us consider a $(\mathsf{E}^B \circ_B (\mathsf{E}_+, \mathsf{E}^B))$-structure on $\boldsymbol{I}$. It corresponds to a triple $(*_{\mathrm{X}_\pm}, (*_S)_{S\in \mathrm{X}_\pm}, *_{\mathrm{X}_0})$, where $\{\mathrm{X}_0\} \sqcup \mathrm{X}_\pm$ is a type B partition of $\boldsymbol{I}$. Observe that the triple is completely determined by the partition $\{\mathrm{X}_0\} \sqcup \mathrm{X}_\pm$. Thus, identifying $(*_{\mathrm{X}_\pm}, (*_S)_{S\in \mathrm{X}_\pm}, *_{\mathrm{X}_0})$ with $\{\mathrm{X}_0\} \sqcup \mathrm{X}_\pm$, we deduce $\Pi^B = \mathsf{E}^B \circ_B (\mathsf{E}^B, \mathsf{E}_+)$. Moreover, one easily verifies that

$$(\Pi, \mathrm{id}, \Pi^B) = (\mathsf{E}, \mathrm{id}, \mathsf{E}^B) \circ (\mathsf{E}_+, \mathrm{id}, \mathsf{E}^B).$$

Proposition 6.6.3. *The type B object* $(\mathsf{X}, \mathrm{id}, 1^B)$ *is the unit of the substitution product of type B objects.*

Proof. Since X is the substitution unit in **Sp**, we only need to verify that

$$1^B \circ_B (\mathsf{q}, \mathsf{n}) = \mathsf{n} \qquad \text{and} \qquad \mathsf{m} \circ_B (\mathsf{X}, 1^B) = \mathsf{m}.$$

For the first identity, observe that $1^B[\mathrm{X}_\pm] \neq \emptyset$ if and only if $\mathrm{X}_\pm = \emptyset$, in which case $\left(\prod_{S\in \mathrm{X}_\pm} \mathsf{q}[S]\right)^{\sim}$ consists of the empty tuple and $\mathrm{X}_0 = \boldsymbol{I}$. Identifying $(*_\emptyset, (), x) \in (1^B \circ_B (\mathsf{q}, \mathsf{n}))[\boldsymbol{I}]$ with $x \in \mathsf{n}[\boldsymbol{I}]$, we deduce $1^B \circ_B (\mathsf{q}, \mathsf{n}) = \mathsf{n}$.

The second identity is deduced in a similar manner. Indeed, $1^B[\mathrm{X}_0] \neq \emptyset$ if and only if $\mathrm{X}_0 = \emptyset$ and $\prod_{S\in \mathrm{X}_\pm} \mathsf{X}[S] \neq \emptyset$ if and only if each block $S \in \mathrm{X}_\pm$ is a singleton. The result follows by identifying $(x, (\{i\})_{i\in \boldsymbol{I}}, *_\emptyset) \in (\mathsf{m} \circ_B (\mathsf{X}, 1^B))[\boldsymbol{I}]$ with $x \in \mathsf{m}[\boldsymbol{I}]$. □

Proposition 6.6.4 (Type B compositional formula - species)**.** *Suppose* q *is a positive species. The generating series of* $\mathsf{m} \circ_B (\mathsf{q}, \mathsf{n})$ *is* $\mathsf{m}(\mathsf{q}(x))\mathsf{n}(x)$.

Proof. An application of the usual compositional formula [68, Theorem 5.1.4] shows that the coefficient of $\frac{x^d}{(2d)!!}$ in $\mathsf{m}(\mathsf{q}(x))\mathsf{n}(x)$ is

$$(2d)!! \sum_{k=0}^{d} \Big(\frac{1}{k!} \sum_{\{S_1,\dots,S_r\} \vdash [k]} \frac{\mathsf{m}_r}{2^r} \, \mathsf{q}_{|S_1|} \dots \mathsf{q}_{|S_r|} \Big) \frac{\mathsf{n}_{d-k}}{2^{d-k}\,(d-k)!}$$
$$= \sum_{k=0}^{d} \binom{d}{k} \Big(\sum_{\{S_1,\dots,S_r\} \vdash [k]} 2^{k-r} \, \mathsf{m}_r \, \mathsf{q}_{|S_1|} \dots \mathsf{q}_{|S_r|} \, \mathsf{n}_{d-k} \Big)$$
$$= \sum_{\{\boldsymbol{S}_0, S_1, \dots, S_r, \overline{S_1}, \dots, \overline{S_r}\} \vdash^B [\pm d]} \mathsf{m}_r \, \mathsf{q}_{|S_1|} \dots \mathsf{q}_{|S_r|} \, \mathsf{n}_{|\boldsymbol{S}_0|/2},$$

which is precisely the number of $(\mathsf{m} \circ_B (\mathsf{q}, \mathsf{n}))$-structures on $[\pm d]$. To verify the last equality, note that choosing a type B partition $\{\boldsymbol{S}_0, S_1, \overline{S_1}, \dots, S_k, \overline{S_k}\} \vdash^B [\pm d]$ with $|\boldsymbol{S}_0| = 2(d-k)$ is equivalent to:

1. choosing a subset $S \in \binom{[d]}{k}$ and setting $\boldsymbol{S}_0 = \pm([d] \setminus S)$,
2. choosing a partition $\{K_1, \dots, K_r\}$ of S, and
3. constructing blocks $\{S_i, \overline{S_i}\}$ from K_i as follows: for each $j \in K_i \setminus \{\max K_i\}$, choose whether j and $\max K_i$ will be in the same or in *opposite* blocks.

There are precisely 2^{k-r} possible choices in the last step. □

Example 6.6.5. Continuing Example 6.6.2, we deduce

$$\mathsf{\Pi}^B(x) = \mathsf{E}^B(\mathsf{E}_+(x))\mathsf{E}^B(x) = e^{\frac{e^x-1}{2}} e^{\frac{x}{2}} = e^{\frac{e^x+x-1}{2}}.$$

The number of $\mathsf{\Pi}^B$-structures on $[\pm d]$ for the first values of d are $1, 2, 6, 24, 116$. These are the **type B Bell numbers**, or *Dowling numbers* (OEIS: A007405).

Example 6.6.6. Some type B species can be obtained in more than one way using the action of **Sp** on $\mathbf{Sp}^B$ and the substitution product. For instance, $(\mathsf{\Sigma}, \mathrm{rev}, \mathsf{\Sigma}^B) = (\mathsf{L}, \mathrm{rev}, \mathsf{L}^B) \circ_B (\mathsf{E}_+, \mathrm{id}, \mathsf{E}^B)$. Thus,

$$\mathsf{\Sigma}^B = \mathsf{E}^B \cdot \mathsf{\Sigma} \qquad \text{and} \qquad \mathsf{\Sigma}^B = \mathsf{L}^B \circ_B (\mathsf{E}_+, \mathsf{E}^B).$$

The formulas in Proposition 6.2.3 and Proposition 6.6.4 give two "different" expressions for the same generating function:

$$e^{x/2} \cdot \frac{1}{2 - e^x} = \frac{1}{1 - (e^x - 1)} \cdot e^{x/2}.$$

The number of Σ^B-structures on $[\pm d]$ for the first values of d are $1, 3, 17, 147$. These are the **Type B Fubini numbers** or **Type B ordered Bell numbers** (OEIS: A080253)

Proposition 6.6.7. *The substitution product on positive type B objects is associative.*

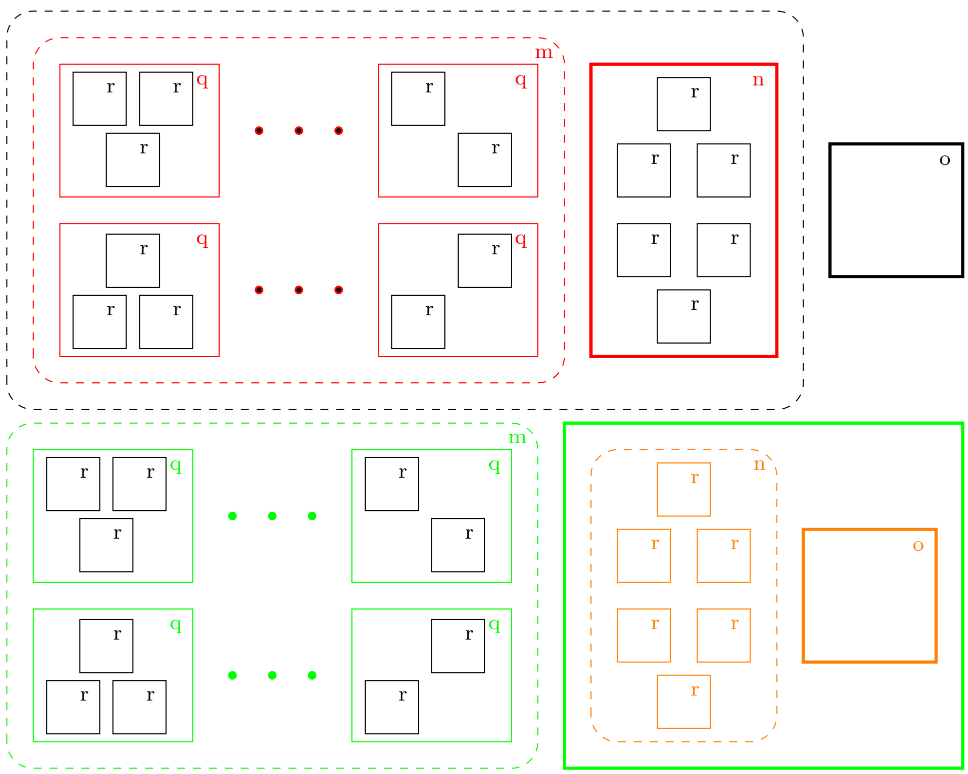


Figure 6.6.1: Associativity of the type B substitution product.

Sketch of proof. The associativity in the first two components is the usual associativity of the substitution product in species. For the type B component, we need to prove that

$$(\mathsf{m} \circ_B (\mathsf{q}, \mathsf{n})) \circ_B (\mathsf{r}, \mathsf{o}) = \mathsf{m} \circ_B (\mathsf{q} \circ \mathsf{r}, \mathsf{n} \circ_B (\mathsf{r}, \mathsf{o})).$$

By definition, the first species on a set $\boldsymbol{I}$ is

$$\coprod_{\mathrm{X} \vdash^B \boldsymbol{I}} \Big(\coprod_{\mathrm{Y} \vdash^B \mathrm{X}_\pm} \mathsf{m}[\mathrm{Y}_\pm] \times \big(\prod_{C \in \mathrm{Y}_\pm} \mathsf{q}[C] \big)^{\sim} \times \mathsf{n}[\mathrm{Y}_0] \Big) \times \big(\prod_{A \in \mathrm{X}_\pm} \mathsf{r}[A] \big)^{\sim} \times \mathsf{o}[\mathrm{X}_0],$$

while the second is

$$\coprod_{\mathrm{W} \vdash^B \boldsymbol{I}} \mathsf{m}[\mathrm{W}_\pm] \times \Big(\prod_{B \in \mathrm{W}_\pm} \big(\coprod_{\mathrm{V} \vdash B} \mathsf{q}[\mathrm{V}] \times \prod_{E \in \mathrm{V}} \mathsf{r}[E] \big) \Big)^{\sim} \times \Big(\coprod_{\mathrm{Z} \vdash^B \mathrm{W}_0} \mathsf{n}[\mathrm{Z}_\pm] \times \big(\prod_{D \in \mathrm{Z}_\pm} \mathsf{r}[D] \big)^{\sim} \times \mathsf{o}[\mathrm{Z}_0] \Big).$$

Figure 6.6.1 represents how to take a structure of the first kind to a structure of the second kind. Different colors represent different (type B) partitions. Solid squares represent the blocks of a (type B) partition, a bold square represents the zero block of a type B partition, and a dashed rounded rectangle represents the collection of nonzero blocks of a type B partition. In the first picture, X is represented in black and Y in red. In the second picture, W is green, the partitions V are black and Z is orange. □

A similar argument shows the following property of the substitution product.

Proposition 6.6.8. *For all species* p, q *and type B species* m, n,

$$(\mathsf{p} \cdot \mathsf{m}) \circ_B (\mathsf{q}, \mathsf{n}) = (\mathsf{p} \circ \mathsf{q}) \cdot (\mathsf{m} \circ_B (\mathsf{q}, \mathsf{n})).$$

6.7 The type B bimonoids of set compositions and set partitions

We have already discussed how (Π, id) and (Σ, rev) yield involutive bimonoids. In this section, we present how $(\Pi, \mathrm{id}, \Pi^B)$ and $(\Sigma, \mathrm{rev}, \Sigma^B)$ can be endowed with the structure of type B bimonoids. The (co)action maps naturally extend the (co)multiplication of their type A component.

Fix a decomposition $\boldsymbol{I} = S \sqcup \boldsymbol{T} \sqcup \overline{S}$. Given partitions $\mathrm{X} \vdash S$, $\mathrm{Y} \vdash^B \boldsymbol{T}$, and compositions $F \vDash I$, $G \vDash^B \boldsymbol{T}$, their products are defined by

$$\begin{aligned} \mathrm{X} \cdot \mathrm{Y} \quad &= \mathrm{X} \sqcup \mathrm{Y} \sqcup \overline{\mathrm{X}}, \\ F \cdot G \quad &\text{is the concatenation } F\, G\, \overline{\mathrm{rev}(F)}. \end{aligned}$$

It immediately follows that $(\Pi, \mathrm{id}, \Pi^B)$ is commutative, while $(\Sigma, \mathrm{rev}, \Sigma^B)$ is not. Observe that the type B partition $\mathrm{X} \cdot \mathrm{Y}$ satisfies

$$(\mathrm{X} \cdot \mathrm{Y})_0 = \mathrm{Y}_0 \qquad \text{and} \qquad (\mathrm{X} \cdot \mathrm{Y})_\pm = \mathrm{X} \sqcup \mathrm{Y}_\pm \sqcup \overline{\mathrm{X}}.$$

Conversely, given a type B partition $\mathrm{X} \vdash^B \boldsymbol{I}$, and a type B composition $F = (S_1, \ldots, S_k, \boldsymbol{S}_0, \overline{S_k}, \ldots, \overline{S_1}) \vDash^B \boldsymbol{I}$, their coproduct is determined by

$$\begin{aligned} \mathrm{X}|_S &= \{S \cap B \,:\, B \in \mathrm{X}\}^{\frown}, \\ \mathrm{X}/_S &= \{\boldsymbol{T} \cap B \,:\, B \in \mathrm{X}\}^{\frown}, \\ F|_S &= (S \cap S_1, \ldots, S \cap S_k, S \cap \boldsymbol{S}_0, S \cap \overline{S_k}, \ldots, S \cap \overline{S_1})^{\frown}, \\ F/_S &= (\boldsymbol{T} \cap S_1, \ldots, \boldsymbol{T} \cap S_k, \boldsymbol{T} \cap \boldsymbol{S}_0, \boldsymbol{T} \cap \overline{S_k}, \ldots, \boldsymbol{T} \cap \overline{S_1})^{\frown}. \end{aligned}$$

where the superscript $^\frown$ denotes that all empty intersections (other than the zero block in $\mathrm{X}/_S \vdash^B \boldsymbol{T}$ and $F/_S \vDash^B \boldsymbol{T}$) have been removed. Since $\overline{\mathrm{X}} = \mathrm{X}$ and $\overline{F} = \mathrm{rev}(F)$, it follows that both $(\Pi, \mathrm{id}, \Pi^B)$ and $(\Sigma, \mathrm{rev}, \Sigma^B)$ are cocommutative. The compatibility axiom is verified following the argument in Example 6.2.10.

6.8 The free (commutative) monoid

The **free monoid** over a positive type B object $(\mathsf{p}, \theta, \mathsf{m})$ is

$$\mathcal{T}(\mathsf{p}, \theta, \mathsf{m}) := (\mathsf{L}, \mathrm{rev}, \mathsf{L}^B) \circ (\mathsf{p}, \theta, \mathsf{m}) = (\mathsf{L} \circ \mathsf{p}, \mathrm{rev}_\theta, \mathsf{L}^B \circ_B (\mathsf{p}, \mathsf{m})).$$

A $(\mathsf{L} \circ \mathsf{p})$-structure on a finite set I is a tuple $(F, p_1, \ldots, p_k)$ where $F = (S_1, \ldots, S_k)$ is a composition of I and $p_i \in \mathsf{p}[S_i]$. The involution rev_θ is determined by

$$\mathrm{rev}_\theta(F, p_1, \ldots, p_k) = (\mathrm{rev}(F), \theta(p_k), \ldots, \theta(p_1)),$$

where $\mathrm{rev}(F) = (S_k, \ldots, S_1)$.

Similarly, a $(\mathsf{L}^B \circ_B (\mathsf{p}, \mathsf{m}))$-structure on a set with an involution $\boldsymbol{I}$ is a tuple $(F, p_1, \ldots, p_k, m)$, where $F = (S_1, \ldots, S_k, \boldsymbol{T}, \overline{S_k}, \ldots, \overline{S_k})$ is a type B composition of $\boldsymbol{I}$, $p_i \in \mathsf{p}[S_i]$, and $m \in \mathsf{m}[\boldsymbol{T}]$.

The type B object $\mathcal{T}(\mathsf{p}, \theta, \mathsf{m})$ is a type B monoid with the following product:

$$(F, p_1, \ldots, p_k) \cdot (G, p'_1, \ldots, p'_r, m) = (F \cdot G, p_1, \ldots, p_k, p'_1, \ldots, p'_r, m). \tag{6.8.1}$$

The morphism $(\iota, \iota^B) : (\mathsf{p}, \theta, \mathsf{m}) \to \mathcal{T}(\mathsf{p}, \theta, \mathsf{m})$ of type B objects is determined by

$$\iota_I(p) = ((I), p) \qquad \text{and} \qquad \iota^B_{\boldsymbol{I}}(m) = ((\boldsymbol{I}), m) \tag{6.8.2}$$

for all $p \in \mathsf{p}[I]$ and $m \in \mathsf{m}[\boldsymbol{I}]$. Observe that the composition $(\boldsymbol{I})$ of $\boldsymbol{I}$ does not have nonzero blocks, so $((\boldsymbol{I}), m)$ is indeed a $(\mathsf{L}^B \circ_B (\mathsf{p}, \mathsf{m}))$-structure on $\boldsymbol{I}$.

Proposition 6.8.1. *Let $(\mathsf{p}, \theta_\mathsf{p}, \mathsf{m})$ be a positive type B object, $(\mathsf{q}, \theta_\mathsf{q}, \mathsf{n})$ be a type B monoid, and $(f, g) : (\mathsf{p}, \theta_\mathsf{p}, \mathsf{m}) \to (\mathsf{q}, \theta_\mathsf{q}, \mathsf{n})$ be a morphism of type B objects. Then there is a unique morphism $\mathcal{T}(f, g) : \mathcal{T}(\mathsf{p}, \theta_\mathsf{p}, \mathsf{m}) \to (\mathsf{q}, \theta_\mathsf{q}, \mathsf{n})$ of monoids that makes the following diagram commute.*

$$\begin{array}{ccc} (\mathsf{p}, \theta_\mathsf{p}, \mathsf{m}) & \xrightarrow{(f,g)} & (\mathsf{q}, \theta_\mathsf{q}, \mathsf{n}) \\ {\scriptstyle (\iota,\iota^B)}\downarrow & \nearrow {\scriptstyle \mathcal{T}(f,g)} & \\ \mathcal{T}(\mathsf{p}, \theta_\mathsf{p}, \mathsf{m}) & & \end{array}$$

Proof. Define the morphism $\mathcal{T}(f) : \mathsf{L} \circ \mathsf{p} \to \mathsf{q}$ as in [6, Section 11.2]. That is,

$$\mathcal{T}(f)_I(F, p_1, \ldots, p_k) = f_{S_1}(p_1) \cdot \cdots \cdot f_{S_k}(p_k). \tag{6.8.3}$$

Observe that the expression on the right is a product of q-structures. Similarly, define a morphism $\mathcal{T}^0(f, g) : \mathsf{L}^B \circ_B (\mathsf{p}, \mathsf{m}) \to \mathsf{n}$ by

$$\mathcal{T}^B(f, g)_{\boldsymbol{I}}(F, p_1, \ldots, p_k, m) = f_{S_1}(p_1) \cdot \cdots \cdot f_{S_k}(p_k) \cdot g_{\boldsymbol{T}}(m) \tag{6.8.4}$$

Define the morphism of type B objects $\mathcal{T}(f, g) := (\mathcal{T}(f), \mathcal{T}^B(f, g))$. It immediately follows from (6.8.1), (6.8.3), and (6.8.4) that $\mathcal{T}(f, g)$ is a morphism monoids, and from (6.8.2) that the diagram above commutes. Uniqueness follows since every $(\mathsf{L}^B \circ_B (\mathsf{p}, \mathsf{m}))$-structure factors uniquely as product of structures of the form (6.8.2):

$$(F, p_1, \ldots, p_k, m) = ((S_1), p_1) \cdot \cdots \cdot ((S_k), p_k) \cdot ((\boldsymbol{T}), m). \qquad \square$$

We can similarly define the **free commutative monoid** over a positive type B object $(\mathsf{p}, \theta, \mathsf{m})$:

$$\mathcal{S}(\mathsf{p}, \theta, \mathsf{m}) := (\mathsf{E}, \mathrm{id}, \mathsf{E}^B) \circ (\mathsf{p}, \theta, \mathsf{m}) = (\mathsf{E} \circ \mathsf{p}, \theta', \mathsf{E}^B \circ_B (\mathsf{p}, \mathsf{m})).$$

A $(\mathsf{E} \circ \mathsf{p})$-structure on a finite set I is a pair $(\mathrm{X}, \{p_S\}_{S \in \mathrm{X}})$, where X is a partition of I and $p_S \in \mathsf{p}[S]$ for all $S \in \mathrm{X}$. The involution θ' is determined by

$$\theta'(\mathrm{X}, \{p_S\}_{S \in \mathrm{X}}) = (\mathrm{X}, \{\theta(p_S)\}_{S \in \mathrm{X}}).$$

Similarly, a $(\mathsf{E}^B \circ_B (\mathsf{p}, \mathsf{m}))$-structure on $\boldsymbol{I}$ is a triple $(\mathrm{X}, \{p_S\}_{S \in \mathrm{X}_\pm}, m)$, where X is a type B partition of $\boldsymbol{I}$, the elements $p_S \in \mathsf{p}[S]$ for all $S \in \mathrm{X}$ are such that $p_{\overline{S}} = \widetilde{p_S}$, and $m \in \mathsf{m}[\mathrm{X}_0]$.

The type B object $\mathcal{S}(\mathsf{p}, \theta, \mathsf{m})$ is a type B (commutative) monoid with the following product:

$$(\mathrm{X}, \{p_S\}_{S \in \mathrm{X}}) \cdot (\mathrm{Y}, \{p'_K\}_{K \in \mathrm{Y}_\pm}, m) = (\mathrm{X} \cdot \mathrm{Y}, \{p_S\}_{S \in \mathrm{X}} \sqcup \{p'_K\}_{K \in \mathrm{Y}_\pm} \sqcup \{\overline{p_{\overline{S}}}\}_{S \in \overline{\mathrm{X}}}, m)$$

The morphism $(\iota, \iota^B) : (\mathsf{p}, \theta, \mathsf{m}) \to \mathcal{S}(\mathsf{p}, \theta, \mathsf{m})$ of type B objects is determined by

$$\iota_I(p) = (\{I\}, \{p\}) \qquad \text{and} \qquad \iota^B_{\boldsymbol{I}}(m) = (\{\boldsymbol{I}\}, \emptyset, m)$$

for all $p \in \mathsf{p}[I]$ and $m \in \mathsf{m}[\boldsymbol{I}]$. A proof similar to that of Proposition 6.8.1 shows the following.

Proposition 6.8.2. *Let $(\mathsf{p}, \theta_{\mathsf{p}}, \mathsf{m})$ be a positive type B object, $(\mathsf{q}, \theta_{\mathsf{q}}, \mathsf{n})$ be a commutative type B monoid, and $(f, g) : (\mathsf{p}, \theta_{\mathsf{p}}, \mathsf{m}) \to (\mathsf{q}, \theta_{\mathsf{q}}, \mathsf{n})$ be a morphism of type B objects. Then there is a unique morphism $\mathcal{S}(f, g) : \mathcal{S}(\mathsf{p}, \theta_{\mathsf{p}}, \mathsf{m}) \to (\mathsf{q}, \theta_{\mathsf{q}}, \mathsf{n})$ of monoids that makes the following diagram commute.*

$$\begin{array}{ccc} (\mathsf{p}, \theta_{\mathsf{p}}, \mathsf{m}) & \xrightarrow{(f,g)} & (\mathsf{q}, \theta_{\mathsf{q}}, \mathsf{n}) \\ {\scriptstyle (\iota,\iota^B)}\downarrow & \nearrow {\scriptstyle \mathcal{S}(f,g)} & \\ \mathcal{S}(\mathsf{p}, \theta_{\mathsf{p}}, \mathsf{m}) & & \end{array}$$

6.9 Type B Boolean functions

A Boolean function on a set I is a function $z : 2^I \to \mathbb{R}$ such that $z(\emptyset) = 0$. Analogously, a **type B Boolean function** on a set with a fixed-point free involution $\boldsymbol{I}$ is a function $z : \mathcal{P}'(\boldsymbol{I}) \longrightarrow \mathbb{R}$ such that $z(\emptyset) = 0$. Let BF^B denote the species of type B Boolean functions and BF the usual species of Boolean functions. We define an involution $\theta : \mathsf{BF} \to \mathsf{BF}$ as follows. Let $z \in \mathsf{BF}[I]$, then for all $A \subseteq I$

$$\theta(z)(A) := z(I \setminus A) - z(I).$$

Clearly $\theta(z)(\emptyset) = 0$, so $\theta(z)$ is again a Boolean function. Moreover, if $z' = \theta(z)$, then

$$\theta(z')(A) = z'(I \setminus A) - z'(I) = \big(z(A) - z(I)\big) - \big(z(\emptyset) - z(I)\big) = z(A),$$

so θ is indeed an involution.

Aguiar and Ardila [1] introduced a Hopf monoid structure on Boolean functions that we review now. Fix a decomposition $I = S \sqcup T$. If $z \in \mathsf{BF}[S]$ and $z' \in \mathsf{BF}[T]$, their product is defined by

$$(z \cdot z')(A) := z(A \cap S) + z'(A \cap T)$$

for all $A \subseteq I$. Since BF is commutative, it follows that (BF, θ) is an involutive monoid. On the other hand, the coproduct $\Delta_{S,T}(z)$ of $z \in \mathsf{BF}[I]$ is determined by

$$z|_S(A) := z(A) \qquad \text{and} \qquad z/_S(B) := z(B \cup S) - z(S),$$

for all $A \subseteq S$ and $B \subseteq T$. We now verify that (BF, θ) is an involutive comonoid. Indeed,

$$\begin{aligned}\theta(z)|_S(A) = \theta(z)(A) &= z(I \setminus A) - z(I) \\ &= \big(z(I \setminus A) - z(T)\big) - \big(z(I) - z(T)\big) \\ &= z/_T(S \setminus A) - z/_T(S) = \theta(z/_T)(A),\end{aligned}$$

for all $A \subseteq S$ and

$$\begin{aligned}\theta(z)/_S(B) &= \theta(z)(B \cup S) - \theta(z)(S) \\ &= \big(z(I \setminus (B \cup S)) - z(I)\big) - \big(z(I \setminus S) - z(I)\big) \\ &= z(T \setminus B) - z(T) = z|_T(T \setminus B) - z|_T(T) = \theta(z|_T)(B),\end{aligned}$$

for all $B \subseteq T$.

We now endow $(\mathsf{BF}, \theta, \mathsf{BF}^B)$ with the structure of a commutative type B bimonoid. For a decomposition $\boldsymbol{I} = S \sqcup \boldsymbol{T} \sqcup \overline{S}$ and Boolean functions $z \in \mathsf{BF}[S]$, $z' \in \mathsf{BF}^B[\boldsymbol{T}]$, let $\alpha_{S,\boldsymbol{T}}(z, z') = z \cdot z'$ be defined by

$$(z \cdot z')(A) := z(A \cap S) + z'(A \cap \boldsymbol{T}) + \widetilde{z}(A \cap \overline{S}) \tag{6.9.1}$$

for all $A \in \mathcal{P}'(\boldsymbol{I})$. Observe that $\widetilde{z} = \theta(\overline{z})$ is a Boolean function on $\overline{S}$ and $z \cdot z' = \widetilde{z} \cdot z'$. Conversely, if $z \in \mathsf{BF}^B[\boldsymbol{I}]$, define $\delta_{S,\boldsymbol{T}}(z) = (z|_S, z/_S)$ by

$$z|_S(A) := z(A) \qquad \text{and} \qquad z/_S(B) := z(B \cup S) - z(S)$$

for all $A \subseteq S$ and $B \in \mathcal{P}'(\boldsymbol{T})$. The (co)associativity of the (co)action can be checked similarly to that of the (co)product of BF; we omit the details. We will, however, verify the compatibility axiom.

Let $\boldsymbol{I} = S \sqcup \boldsymbol{T} \sqcup \overline{S} = S' \sqcup \boldsymbol{T}' \sqcup \overline{S}'$ be two decompositions of $\boldsymbol{I}$, and let their pairwise intersections be as in (6.2.6). We need to verify that, for $z \in \mathsf{BF}[S]$ and $z' \in \mathsf{BF}^B[\boldsymbol{T}]$,

$$(z \cdot z')|_{S'} = z|_A \cdot z|_D \cdot \widetilde{z}|_{\overline{C}} \qquad \text{and} \qquad (z \cdot z')/_{S'} = (z/_A)|_B \cdot z'/_D.$$

The first condition readily follows from (6.9.1) since $A = S' \cap S$, $D = S' \cap \boldsymbol{T}$, and $\overline{C} = S' \cap \overline{S}$. Now, for all $K \subseteq \boldsymbol{T}'$,

$$\begin{aligned}
(z \cdot z')/_{S'}(K) &= (z \cdot z')(K \cup S') - (z \cdot z')(S') \\
&= \big(z((K \cup S') \cap S) + z'((K \cup S') \cap \boldsymbol{T}) + \widetilde{z}((K \cup S') \cap \overline{S})\big) \\
&\qquad - \big(z(S' \cap S) + z'(S' \cap \boldsymbol{T}) + \widetilde{z}(S' \cap \overline{S})\big) \\
&= z((K \cap B) \cup A) - z(A) + z'((K \cap \boldsymbol{E}) \cup D) - z'(D) + \widetilde{z}((K \cap \overline{B}) \cup \overline{C}) - \widetilde{z}(\overline{C}) \\
&= z/_A(K \cap B) + z'/_D(K \cap \boldsymbol{E}) + \widetilde{z}/_{\overline{C}}(K \cap \overline{B}) = ((z/_A)|_B \cdot z'/_D)(K),
\end{aligned}$$

as we wanted to show. The last equality uses that $z/_A(K \cap B) = (z/_A)|_B(K \cap B)$ and $(\widetilde{z}/_{\overline{C}})|_{\overline{B}} = \widetilde{(z|_{A \cup B})/_A} = \widetilde{(z/_A)|_B}$, which are instances of the involutive comonoid axioms and the coassociativity of the coproduct.

6.10 Type B Submodular functions

Let SF denote the subspecies of BF consisting of submodular functions. That is, Boolean functions z on I such that $z(A) + z(B) \geq z(A \cap B) + z(A \cup B)$ for all $A, B \subseteq I$. The species SF is actually a sub Hopf monoid of BF, see [1]. The involution θ of BF restricts to SF. Indeed,

$$\begin{aligned}\theta(z)(A) + \theta(z)(B) &= z(I \setminus A) - z(I) + z(I \setminus B) - z(I) \\ &\geq z((I \setminus A) \cup (I \setminus B)) - z(I) + z((I \setminus A) \cap (I \setminus B)) - z(I) \\ &= z(I \setminus (A \cap B)) - z(I) + z(I \setminus (A \cup B)) - z(I) \\ &= \theta(z)(A \cap B) + \theta(z)(A \cup B),\end{aligned}$$

for all $A, B \subseteq I$. Thus, (SF, θ) is an involutive bimonoid.

Let SF^B be the type B species of bisubmodular functions, in the sense of Fujishige [42]; see Section 5.4.3. Remember that $z \in \mathsf{BF}[\boldsymbol{I}]$ is bisubmodular if

$$z(A) + z(B) \geq z(A \cap B) + z(A \uplus B) \tag{6.10.1}$$

for all $A, B \in \mathcal{P}'(\boldsymbol{I})$, where $A \uplus B = (A \cup B) \setminus (\overline{A} \cup \overline{B})$. The type B object $(\mathsf{SF}, \theta, \mathsf{SF}^B)$ is a type B sub-bimonoid of $(\mathsf{BF}, \theta, \mathsf{BF}^B)$; that is, it is closed under the action α and coaction δ.

Let $\boldsymbol{I} = S \sqcup \boldsymbol{T} \sqcup \overline{S}$, $z \in \mathsf{SF}[S]$, and $z' \in \mathsf{SF}^B[\boldsymbol{T}]$. Observe that for $A, B \in \mathcal{P}'(\boldsymbol{I})$,

$$(A \cap \boldsymbol{T}) \uplus (B \cap \boldsymbol{T}) = (A \uplus B) \cap \boldsymbol{T}.$$

Thus, the bisubmodularity of $z \cdot z'$ follows from applying the submodularity of z and $\widetilde{z}$, and the bisubmodularity of z' in (6.9.1).

Now, let $z \in \mathsf{SF}^B[\boldsymbol{I}]$ and $\boldsymbol{I} = S \sqcup \boldsymbol{T} \sqcup \overline{S}$. The submodularity of $z|_S$ follows directly from (6.10.1), since $A \uplus B = A \cup B$ for all $A, B \subseteq S$. The following simple computation shows that $z/_S$ is bisubmodular. For all $A, B \in \mathcal{P}'(\boldsymbol{T})$,

$$\begin{aligned} z/_S(A) + z/_S(B) &= z(A \cup S) - z(S) + z(B \cup S) - z(S) \\ &\geq z\big((A \cup S) \cap (B \cup S)\big) - z(S) + z\big((A \cup S) \uplus (B \cup S)\big) - z(S) \\ &= z\big((A \cap B) \cup S\big) - z(S) + z\big((A \uplus B) \cup S\big) - z(S) \\ &= z/_S(A \cap B) + z/_S(A \uplus B). \end{aligned}$$

6.11 Type B generalized permutahedra

Given a set with a fixed-point free involution $\boldsymbol{I}$, let $\overline{\mathbb{R}^{\boldsymbol{I}}}$ be the quotient of $\mathbb{R}^{\boldsymbol{I}}$ where we have identified the basis element $\boldsymbol{e}_{\bar{i}}$ with $-\boldsymbol{e}_i$ for all $i \in \boldsymbol{I}$. For any decomposition $\boldsymbol{I} = S \sqcup \boldsymbol{T} \sqcup \overline{S}$, there is a canonical isomorphism $\mathbb{R}^S \times \overline{\mathbb{R}^{\boldsymbol{T}}} \cong \overline{\mathbb{R}^{\boldsymbol{I}}}$ obtained by linearly extending $(\boldsymbol{e}_i, \boldsymbol{0}) \mapsto \boldsymbol{e}_i$ for all $i \in S$, and $(\boldsymbol{0}, \boldsymbol{e}_j) \mapsto \boldsymbol{e}_j$ for all $j \in \boldsymbol{T}$. In particular, we have $\mathbb{R}^I \cong \overline{\mathbb{R}^{\boldsymbol{I}}}$ for any transversal I of $\boldsymbol{I}$. In this manner, we can define the type B Coxeter arrangement $\mathcal{B}_{\boldsymbol{I}}$ and type B generalized permutahedra in $\overline{\mathbb{R}^{\boldsymbol{I}}}$ just as in Section 2.2.4 and Section 5.4, respectively.

We endow $(\mathsf{GP}, \theta, \mathsf{GP}^B)$, where θ is the involution $\theta(P) = -P$, with the structure of a commutative type B Hopf monoid. The species GP is a Hopf monoid, see [1] and Section 5.5. Moreover, the commutativity of GP and the identities $(-P) \times (-Q) = -(P \times Q)$ and $(-P)_{\boldsymbol{v}} = -(P_{-\boldsymbol{v}})$ show that (GP, θ) is an *involutive* Hopf monoid. The action and coaction of GP on GP^B are defined below.

Fix a decomposition $\boldsymbol{I} = S \sqcup \boldsymbol{T} \sqcup \overline{S}$. For $P \in \mathsf{GP}[S]$ and $Q \in \mathsf{GP}^B[\boldsymbol{T}]$, define

$$P \cdot Q := P \times Q \subseteq \mathbb{R}^S \times \overline{\mathbb{R}^{\boldsymbol{T}}} \cong \overline{\mathbb{R}^{\boldsymbol{I}}}.$$

The normal fan of P coarsens (the fan consisting of the faces of) $\mathcal{A}_S$, the braid arrangement in $\mathbb{R}^S$, and the normal fan of Q coarsens $\mathcal{B}_{\boldsymbol{T}}$; thus, the normal fan of

$P \times Q$ coarsens $\mathcal{A}_S \times \mathcal{B}_{\boldsymbol{T}}$, a subarrangement of $\mathcal{B}_{\boldsymbol{I}}$. Hence, the polytope $P \cdot Q$ is a type B generalized permutahedron. This product is clearly associative. Moreover, observe that $\widetilde{P} = \{\boldsymbol{x} \in \mathbb{R}^{\overline{S}} : -\overline{\boldsymbol{x}} \in P\}$, where $\overline{\sum a_i \boldsymbol{e}_i} = \sum a_i \boldsymbol{e}_{\bar{i}}$, is the image of P under the canonical isomorphism $\mathbb{R}^S \cong \overline{\mathbb{R}^{\pm S}} \cong \mathbb{R}^{\overline{S}}$. Thus,

$$P \times Q = \widetilde{P} \times Q,$$

and $(\mathsf{GP}, \theta, \mathsf{GP}^B)$ is commutative.

Let $F \in \Sigma[\mathcal{B}_{\boldsymbol{I}}]$ be the face corresponding to the type B composition $(S, \boldsymbol{T}, \overline{S})$. Then $(\mathcal{B}_{\boldsymbol{I}})_{\mathrm{s}(F)} = \mathcal{A}_S \times \mathcal{B}_{\boldsymbol{T}}$. In particular, if $P \subseteq \overline{\mathbb{R}^{\boldsymbol{I}}}$ is a type B generalized permutahedron, then the normal fan of P_F coarsens $\mathcal{A}_S \times \mathcal{B}_{\boldsymbol{T}}$. Consequently, there are polynomials $P|_S \subseteq \mathbb{R}^S$ and $P/_S \subseteq \overline{\mathbb{R}^{\boldsymbol{T}}}$ such that

$$P_F = P|_S \times P/_S$$

and the normal fan of $P|_S$ (resp. $P/_S$) coarsens $\mathcal{A}_S$ (resp. $\mathcal{B}_{\boldsymbol{T}}$). That is, $P|_S \in \mathsf{GP}[S]$ and $P/_S \in \mathsf{GP}^B[\boldsymbol{T}]$. Define the coaction by

$$\delta_{S,\boldsymbol{T}}(P) = (P|_S, P/_S).$$

We verify the compatibility axiom. Take $P \in \mathsf{GP}[S]$ and $Q \in \mathsf{GP}^B[\boldsymbol{T}]$, and consider a second decomposition $\boldsymbol{I} = S' \sqcup \boldsymbol{T}' \sqcup \overline{S'}$. Then,

$$(P \times Q)_{\boldsymbol{e}_{S'}} = P_{\boldsymbol{e}_{S' \cap S} - \boldsymbol{e}_{\overline{S'} \cap S}} \times Q_{\boldsymbol{e}_{S' \cap \boldsymbol{T}}} = (P|_A \times (P/_A)|_B \times P/_{A \sqcup B}) \times (Q|_D \times Q/_D).$$

In the first equality, we use that the canonical isomorphism $\overline{\mathbb{R}^{\pm S}} \cong \mathbb{R}^S$ identifies $\boldsymbol{e}_{S' \cap \pm S}$ with $\boldsymbol{e}_{S' \cap S} - \boldsymbol{e}_{\overline{S'} \cap S}$. The second equality uses the coassociativity of GP and that $\boldsymbol{e}_{S' \cap S} - \boldsymbol{e}_{\overline{S'} \cap S} = (1)\boldsymbol{e}_A + (0)\boldsymbol{e}_B + (-1)\boldsymbol{e}_C$, with $A, B, C, D, \boldsymbol{E}$ as in (6.2.6). The result follows by taking the factors in $\mathbb{R}^{S'}$ and $\overline{\mathbb{R}^{\boldsymbol{T}'}}$ in the expression above.

6.11.1 Isomorphism with (bi)submodular functions

Recall that there is a one-to-one correspondence between submodular functions on I (resp. bisubmodular functions on $\boldsymbol{I}$) and generalized permutahedra on $\mathbb{R}^I$ (resp. type B generalized permutahedra on $\overline{\mathbb{R}^{\boldsymbol{I}}}$) [38] (resp. [11]). We show that the morphism $(\mathsf{SF}, \theta, \mathsf{SF}^B) \to (\mathsf{GP}, \theta, \mathsf{GP}^B)$ arising from these correspondences is an isomorphism of type B Hopf monoids.

Aguiar and Ardila [1] show that the component $\mathsf{SF} \to \mathsf{GP}$ is an isomorphism of Hopf monoids, and one easily checks that it is in fact an isomorphism of *involutive* Hopf monoids $(\mathsf{SF}, \theta) \to (\mathsf{GP}, \theta)$. Indeed, let $z \in \mathsf{SF}[I]$ be a submodular function on I, and $P(z)$ be the corresponding generalized permutahedron. That is,

$$P(z) = \{\boldsymbol{x} \in \mathbb{R}^I \,:\, \boldsymbol{x}(I) = z(I)\,,\, \boldsymbol{x}(S) \leq z(S) \text{ for all } S \subseteq I\},$$

where $\boldsymbol{x}(S) = \sum_{i \in S} \boldsymbol{x}_i$. Then, $-P(z)$ lies in the hyperplane $\boldsymbol{x}(I) = -z(I) = \theta(z)(I)$, and inside this hyperplane is defined by inequalities $-\boldsymbol{x}(S) \leq z(S)$ for all S. Equivalently,

$$\boldsymbol{x}(S) = \boldsymbol{x}(I) - \boldsymbol{x}(I \setminus S) \leq -z(I) + z(I \setminus S) = \theta(z)(S).$$

That is, $-P(z) = P(\theta(z))$, as claimed.

Take a decomposition $\boldsymbol{I} = S \sqcup \boldsymbol{T} \sqcup \overline{S}$ and (bi)submodular functions $z \in \mathsf{SF}[S]$ and $z' \in \mathsf{SF}^B[\boldsymbol{T}]$. We show that $P(z) \times P(z') = P(z \cdot z')$, where $P(z')$ and $P(z \cdot z')$ are defined as in (5.4.7). Indeed, for any $\boldsymbol{x} \in P(z) \times P(z')$ and $A \in \mathcal{P}'(\boldsymbol{I})$,

$$\begin{aligned}
\boldsymbol{x}(A) &= \boldsymbol{x}(A \cap S) + \boldsymbol{x}(A \cap \boldsymbol{T}) + \boldsymbol{x}(S \setminus (\overline{A} \cap S)) - \boldsymbol{x}(S) \\
&\leq z(A \cap S) + z'(A \cap \boldsymbol{T}) + z(S \setminus (\overline{A} \cap S)) - \boldsymbol{x}(S) \\
&= z(A \cap S) + z'(A \cap \boldsymbol{T}) + \theta(z)(\overline{A} \cap S).
\end{aligned}$$

In the last equality we use that $\boldsymbol{x}(S) = z(S)$. To check that this inequality is tight, consider any point in the face $P(z)_F \times Q(z')_G$ of $P(z) \times Q(z')$, where

$$F = (A \cap S, S \setminus (\pm A \cap S), \overline{A} \cap S) \vDash S$$

and

$$G = (A \cap \boldsymbol{T}, \boldsymbol{T} \setminus (\pm A \cap \boldsymbol{T}), \overline{A} \cap \boldsymbol{T}) \vDash^B \boldsymbol{T}.$$

Hence, $P(z) \times P(z') = P(z \cdot z')$ and $(\mathsf{SF}, \theta, \mathsf{SF}^B) \to (\mathsf{GP}, \theta, \mathsf{GP}^B)$ is a morphism of type B monoids.

Now, take a decomposition $\boldsymbol{I} = S \sqcup \boldsymbol{T} \sqcup \overline{S}$ and a bisubmodular function $z \in \mathsf{SF}^B[\boldsymbol{I}]$. To show that $P(z)|_S = P(z|_S) \subseteq \mathbb{R}^S$ and $P(z)/_S = P(z/_S) \subseteq \overline{\mathbb{R}^{\boldsymbol{T}}}$, we verify that

$$P(z)|_S \times P(z)/_S = P(z) \cap \{\boldsymbol{x} \in \overline{\mathbb{R}^{\boldsymbol{I}}} \, : \, \boldsymbol{x}(S) = z(S)\}$$

and $P(z|_S) \times P(z/_S)$ are equal.

(⊆) Let $\boldsymbol{x} \in P(z) \cap \{\boldsymbol{x} \in \overline{\mathbb{R}^{\boldsymbol{I}}} \, : \, \boldsymbol{x}(S) = z(S)\}$. Since $\boldsymbol{x} \in P(z)$, for all $S \subseteq S$ we have $\boldsymbol{x}(A) \leq z(A) = z|_S(A)$. Moreover, for all $B \in \mathcal{P}'(\boldsymbol{T})$ we have

$$\boldsymbol{x}(B) = \boldsymbol{x}(B \sqcup S) - \boldsymbol{x}(S) = \boldsymbol{x}(B \sqcup S) - z(S) \leq z(B \sqcup S) - z(S) = z/_S(B).$$

Thus, $\boldsymbol{x} \in P(z|_S) \times P(z/_S)$.

(⊇) Let $\boldsymbol{x} \in P(z|_S) \times P(z/_S)$. First observe that $\boldsymbol{x}(S) = z|_S(S) = z(S)$, so we only need to show that $\boldsymbol{x} \in P(z)$. First observe that using bisubmodularity (6.10.1) twice, we obtain

$$\begin{aligned} z(C) &\geq z(C \cap S) + z(C \uplus S) - z(S) && \langle A = C,\, B = S \rangle \\ &\geq z(C \cap S) + z((C \cap \boldsymbol{T}) \sqcup S) + z(S \setminus (\overline{C} \cap S)) - 2z(S) && \langle A = C \uplus S,\, B = S \rangle \\ &= z|_S(C \cap S) + z/_S(C \cap \boldsymbol{T}) + z(S \setminus (\overline{C} \cap S)) - z(S) \end{aligned}$$

for all $C \in \mathcal{P}'(\boldsymbol{I})$. Thus,

$$\begin{aligned}\boldsymbol{x}(C) &= \boldsymbol{x}(C \cap S) + \boldsymbol{x}(C \cap \boldsymbol{T}) + \boldsymbol{x}(C \cap \overline{S}) \\ &= \boldsymbol{x}(C \cap S) + \boldsymbol{x}(C \cap \boldsymbol{T}) + \boldsymbol{x}(S \setminus (\overline{C} \cap S)) - \boldsymbol{x}(S) \\ &\leq z|_S(C \cap S) + z/_S(C \cap \boldsymbol{T}) + z|_S(S \setminus (\overline{C} \cap S)) - \boldsymbol{x}(S) \leq z(C),\end{aligned}$$

implying that $\boldsymbol{x} \in P(z)$. The first inequality uses the definition of $P(z|_S)$ and $P(z/_S)$, and the second uses that $\boldsymbol{x}(S) = z(S)$ and the previous equation.

Therefore, $(\mathsf{SF}, \theta, \mathsf{SF}^B)$ and $(\mathsf{GP}, \theta, \mathsf{GP}^B)$ are isomorphic type B Hopf monoids.

6.12 Symplectic matroids

A **symplectic matroid** on a set with a fixed-point free involution $\boldsymbol{I}$ is a collection of involution-exclusive subsets $M \subseteq \mathcal{P}'(\boldsymbol{I})$ that is closed under inclusions and satisfies the following axiom.

For all $X, Y \in M$ such that $|X| < |Y|$, either

1. there is $y \in Y \setminus X$ such that $X \cup \{y\} \in M$, or
2. $X \cup Y$ is inadmissible and there is $z \notin X \cup Y$ such that $X \cup \{z\} \in M$ and $\{\overline{z}\} \cup X \setminus \overline{Y} \in M$.

The elements of M are the **independent** sets of the matroid. A maximal independent set is a **basis** of M. Symplectic matroids were originally introduced by Gelfand and Serganova [43, 44]. The characterization in terms of independent sets above is originally due to Chow [36]. We let M^B denote the type B species of symplectic matroids.

Let M denote the Hopf monoid of Matroids. A matroid $M \in \mathsf{m}[I]$ is a nonempty collection of subsets of I that is closed under inclusions and satisfies: whenever $X, Y \in M$ with $|X| < |Y|$, there is $y \in Y \setminus X$ such that $X \cup \{y\} \in M$. Sets in M are the **independent sets** of the matroid M. A maximal subset in M is called a **basis** of M. We recall the Hopf monoid structure of M below, see [1] for details.

Given a decomposition $I = S \sqcup T$, the product of matroids $M_1 \in \mathsf{M}[S]$ and $M_2 \in \mathsf{M}[T]$ is

$$M_1 \cdot M_2 = \{X \subseteq I \,:\, X \cap S \in M_1,\ X \cap T \in M_2\}.$$

This is the **direct sum** of M_1 and M_2. Conversely, for a matroid $M \in \mathsf{M}[I]$, the **restriction** of M to S, sometimes called the **deletion** of T from M, is the matroid

$$M|_S = M \setminus T = \{X \subseteq S \,:\, X \in M\} \in \mathsf{M}[S],$$

and the **contraction** of S from M is

$$M/_S = \{X \subseteq T \,:\, X \sqcup B \in M \text{ for any basis } B \text{ of } M|_S\} \in \mathsf{M}[T].$$

The species of matroids M has a natural involution given by duality: $M \mapsto M^*$ for any $M \in \mathrm{M}[I]$. Since M is commutative and

$$(M|_S)^* = (M \setminus T)^* = M^*/_T \qquad \text{and} \qquad (M/_S)^* = M^* \setminus S = M^*|_T,$$

we have that $(\mathsf{M},\ ^*)$ is an involutive Hopf monoid.

We proceed to endow $(\mathsf{M},\ ^*, \mathsf{M}^B)$ with the structure of a commutative type B Hopf monoid. Let $\boldsymbol{I} = S \sqcup \boldsymbol{T} \sqcup \overline{S}$ be a decomposition of $\boldsymbol{I}$ and $M \in \mathsf{M}^B[\boldsymbol{I}]$. The **restriction** of M to S is the matroid

$$M|_S := \{X \subseteq S \,:\, X \in M\}.$$

Since subsets of S are always admissible, Chow's axiom above implies the augmentation property for $M|_S$, so $M|_S$ is indeed a matroid on ground set S. On the other hand, the **contraction** of S form M is the symplectic matroid

$$M/_S := \{X \in \mathcal{P}'(\boldsymbol{T}) \,:\, X \sqcup B \in M \text{ for any basis } B \text{ of } M|_S\}.$$

Lemma 6.12.1. *The collection $M/_S$ above is well-defined and is a symplectic matroid on ground set $\boldsymbol{T}$.*

Proof. We first show that the definition of $M/_S$ is independent of B. Let B_1, B_2 be bases of $M|_S$ and suppose $X \in \mathcal{P}'(\boldsymbol{T})$ is such that $X \sqcup B_1 \in M$ but $X \sqcup B_2 \notin M$. Choose a maximal subset $Y \subseteq X$ such that $Y \sqcup B_2 \in M$. Since $|X \sqcup B_1| > |Y \sqcup B_2|$ and $(X \sqcup B_1) \cup (Y \sqcup B_2) \subseteq X \sqcup K$ is admissible, Chow's axiom implies there exists $z \in (X \sqcup B_1) \setminus (Y \sqcup B_2)$ such that $(Y \sqcup B_2) \cup \{z\} \in \boldsymbol{I}$. Since B_2 is a basis of $M|_S$, the element z cannot be in S, so necessarily $z \in X$. However, this contradicts the maximality of $Y \subseteq X$. Therefore, $M/_S$ is well-defined and independent of the choice of basis B of $M|_S$.

Now, fix a basis $B \in M|_S$. It is clear that $M/_S = \{X \in \mathcal{P}'(\boldsymbol{T}) \,:\, X \sqcup B \in M\}$ is subset-closed, so we are only left to verify Chow's axiom for $M/_S$. Let $X, Y \in M/_S$ with $|X| < |Y|$ and suppose there is no $z \in Y \setminus X$ such that $X \cup \{z\} \in M/_S$. Then $X \sqcup B, Y \sqcup B \in M$ are such that $|X \sqcup B| < |Y \sqcup B|$ and no $z \in (Y \sqcup B) \setminus (X \sqcup B)$ satisfies $(X \sqcup B) \cup \{z\} \in M$. Therefore, Chow's axiom for M implies that $(X \sqcup B) \cup (Y \sqcup B)$ is inadmissible and there is $z \notin (X \sqcup B) \cup (Y \sqcup B)$ such that $(X \sqcup B) \cup \{z\} \in M$ and $\{\overline{z}\} \cup (X \sqcup B) \setminus (\overline{Y} \sqcup \overline{B}) = \{\overline{z}\} \cup (X \setminus \overline{Y}) \sqcup B \in M$. Since B is a basis of $M|_S$, this implies $z, \overline{z} \notin S$. That is, $z \in \boldsymbol{T} \setminus (X \cup Y)$ is such that $X \cup \{z\} \in M/_S$ and $\{\overline{z}\} \cup (X \setminus \overline{Y}) \in M/_S$. Therefore, $M/_S$ is a symplectic matroid. $\square$

We define the coproduct of $(\mathsf{M}, \ ^*, \mathsf{M}^B)$ by $\delta_{S,\boldsymbol{T}}(M) = (M|_S, M/_S)$. The coassociativity axiom can be easily verified from the definitions.

Before introducing the product of $(\mathsf{M}, \ ^*, \mathsf{M}^B)$, we recall the following characterization of symplectic matroids due to Gelfand and Serganova.

Theorem 6.12.2 ([29, Theorem 3.3.3])**.** *Let $\mathcal{B}$ be a collection of admissible subsets of $\boldsymbol{I}$ of the same cardinality. Then, $\mathcal{B}$ is the collection of bases of a symplectic matroid if and only if*

$$P_{\mathcal{B}} := \mathrm{Conv}\{e_B \, : \, B \in \mathcal{B}\}$$

if a type B generalized permutahedron.

Remark 6.12.3. The reader might be familiar with a similar result for usual matroids and generalized permutahedra. Observe that in type A, we might drop the hypothesis that the sets in $\mathcal{B}$ have the same cardinality, since otherwise $P_{\mathcal{B}}$ cannot possibly be a generalized permutahedron. In type B, however, there are collections of admissible subsets of $\boldsymbol{I}$ of *different cardinalities* such that the convex hull of their indicator vectors is a type B generalized permutahedron. These sets, of course, cannot be the collection of basis of a symplectic matroid.

Let $\boldsymbol{I} = S \sqcup \boldsymbol{T} \sqcup \overline{S}$ be a decomposition of $\boldsymbol{I}$, and $M \in \mathsf{M}[S]$ and $M' \in \mathsf{M}^B[\boldsymbol{T}]$. We define the product $M \cdot M'$ to be the symplectic matroid

$$M \cdot M' := \{X \in \mathcal{P}'(\boldsymbol{I}) \, : \, X \cap S \in M, \, X \cap \boldsymbol{T} \in M', \, X \cap \overline{S} \in \widetilde{M}\}$$

Let $\mathcal{B}$ be the collection of maximal elements of $M \cdot M'$. Then,

$$\mathcal{B} = \{B \sqcup B' \sqcup \overline{(S \setminus B)} \, : \, B \text{ is a basis of } M \text{ and } B' \text{ is a basis of } M'\}.$$

Observe that the elements in $\mathcal{B}$ have the same cardinality, and that

$$P_{\mathcal{B}} = (2P_M - \boldsymbol{e}_S) \times P_{M'},$$

where we write $P_M \subseteq \mathbb{R}^S$ and $P_{M'} \subseteq \overline{\mathbb{R}^{\boldsymbol{T}}}$ for the basis polytopes of M and M' respectively. Since P_M is a generalized permutahedron and $P_{M'}$ is a type B generalized permutahedron, it follows from Theorem 6.12.2 that $M \cdot M'$ is indeed a symplectic matroid. The associativity and commutativity of the product are clear from the definitions.

Finally, we verify the compatibility axiom (6.2.5). Take two decompositions $\boldsymbol{I} = S \sqcup \boldsymbol{T} \sqcup \overline{S} = S' \sqcup \boldsymbol{T}' \sqcup \overline{S'}$ of $\boldsymbol{I}$, and let $M \in \mathsf{M}[S]$ and $M' \in \mathsf{M}[\boldsymbol{T}]$. Define sets $A, B, C, D, \boldsymbol{E}$ as in (6.2.6). The compatibility axiom requires us to verify that

$$(M \cdot M')|_{S'} = M|_A \cdot M'|_D \cdot \widetilde{M/_{A \sqcup B}} \qquad \text{and} \qquad (M \cdot M')/_{S'} = (M/_A)|_B \cdot M'/_D.$$

We do so explicitly. For the first identity, observe that

$$\begin{aligned}(M \cdot M')|_{S'} &= \{X \subseteq S' \,:\, X \cap S \in M,\, X \cap \boldsymbol{T} \in M',\, X \cap \overline{S} \in \widetilde{M}\} \\ &= \{X \subseteq S' \,:\, X \cap A \in M,\, X \cap D \in M',\, X \cap \overline{C} \in \widetilde{M}\} \\ &= \{X \subseteq S' \,:\, X \cap A \in M|_A,\, X \cap D \in M'|_D,\, X \cap \overline{C} \in \widetilde{M}|_{\overline{C}}\},\end{aligned}$$

the result follows since $\widetilde{M/_{A \sqcup B}} = \overline{(M/_{A \sqcup B})^*} = \overline{M^*|_C} = \widetilde{M}|_{\overline{C}}$. Similarly, fix a basis B_0 of $(M \cdot M')|_{S'}$, then

$$\begin{aligned}&(M \cdot M')/_{S'} = \{X \in \mathcal{P}'(\boldsymbol{T}') \,:\, X \sqcup B_0 \in M \cdot M'\} \\ &= \{X \in \mathcal{P}'(\boldsymbol{T}') \,:\, (X \sqcup B_0) \cap S \in M,\, (X \sqcup B_0) \cap \boldsymbol{T} \in M',\, (X \sqcup B_0) \cap \overline{S} \in \widetilde{M}\}.\end{aligned}$$

Since $X \subseteq \boldsymbol{T}'$ and $B_0 \subseteq S'$, we have that $(X \sqcup B_0) \cap S = (X \cap B) \sqcup B_A$, where $B_A = B \cap A$ is a basis of $M|_A$. Thus, condition $(X \sqcup B_0) \cap S \in M$ is equivalent to $X \cap B \in (M/_A)|_B$. A similar analysis for the other two conditions shows

$$(M \cdot M')/_{S'} = \{X \in \mathcal{P}'(\boldsymbol{T}') \,:\, X \cap B \in (M/_A)|_B,\, X \cap \boldsymbol{E} \in M'/_D,\, X \cap \overline{B} \in (\widetilde{M}/_{\overline{C}})|_{\overline{B}}\}.$$

The result follows by the definition of the product in $(\mathsf{M},\ ^*, \mathsf{M}^B)$ and because $(\widetilde{M}/_{\overline{C}})|_{\overline{B}} = (\widetilde{M|_{A \sqcup B}})|_{\overline{B}} = \widetilde{(M|_{A \sqcup B})/_A} = \widetilde{(M/_A)|_B}$.

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