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**ON THE COMPLEXITY OF  
APPROXIMATING THE MAXIMAL  
INSCRIBED ELLIPSOID FOR  
A POLYTOPE**

By

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**ON THE COMPLEXITY OF  
APPROXIMATING THE MAXIMAL  
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## Abstract

We give a new polynomial bound on the complexity of approximating the maximal inscribed ellipsoid for a polytope.

Key words: maximal inscribed ellipsoid, maximal inscribed paraboloid, path-following Newton's method, computational complexity.

## 1. Introduction

Let  $Q$  be a full-dimensional polytope in  $\mathbb{R}^n$  defined by  $m$  linear inequalities

$$Q = \{x \in \mathbb{R}^n \mid c_i^T x \leq 1, i = 1, \dots, m\}. \quad (1.1)$$

In this paper, we shall study the complexity of the following extremal geometric

### Problem I:

given a polytope (1.1) and a relative accuracy  $\gamma \in (0,1)$  in the volume, find an ellipsoid  $E$ , contained in the polytope, such that

$$\text{vol } E / \text{vol } E^* \geq \gamma,$$

where  $E^*$  is the ellipsoid of maximum volume inscribed in  $Q$ .

One of the motivations for studying the complexity of Problem I is that it appears as a basic subroutine at each iteration of the method of inscribed ellipsoids [11], which achieves relative error  $\epsilon$  of minimization of an arbitrary nondifferentiable convex function  $F$  on  $Q$  in at most

$$6.64 n \ln \frac{1}{\epsilon}$$

iterations. At each iteration of this method it is required to solve Problem I with accuracy

$\gamma = 0.99$  for a current polytope localizing the set of extrema, and to evaluate  $F$  and the subgradient of  $F$  at the center of  $E$ . Note that, similar to the method of volumetric centers by Vaidya [13], the method of inscribed ellipsoids is an optimal method for convex programming in terms of the order of the number of iterations. In particular, good algorithms for solving Problem I may prove useful for developing efficient methods for decomposition and nondifferentiable convex optimization (for a similar motivation see also [5,9,14]).

Another useful application of  $\gamma$ -maximal ellipsoids is related to the fact that they give “well rounding” affine transformations for convex bodies [6]. More precisely, it is known [11] that

$$E \subseteq Q \subseteq n \left( \frac{1 + 3\sqrt{1-\gamma}}{\gamma} \right) E \quad (1.2)$$

for an arbitrary  $n$ -dimensional body  $Q$ , where  $\lambda E$  stands for the ellipsoid obtained by the homothetic dilatation of  $E$  by a factor of  $\lambda$ .

Extremal inscribed and circumscribed ellipsoids are also used for approximating reachability regions for linear control problems, in optimal design [12], and in some other applications. We also find Problem I interesting in itself.

In this paper we show that a  $\gamma$ -maximal ellipsoid for a polytope (1.1) can be computed in at most

$$O\left(m^{3.5} \ln \left[ \frac{mR}{\ln \frac{1}{\gamma}} \right] \ln \left[ \frac{n \ln R}{\ln \frac{1}{\gamma}} \right] \right), \quad (1.3)$$

arithmetical operations, where  $R$  is an *a priori* known ratio of the radii of two Euclidean balls, the first of which is circumscribed about  $Q$  and the second inscribed in  $Q$ . This improves by a factor of  $m$  the best previously known complexity bound for the problem due to Nesterov and Nemirovsky [7]. We also show that the computational cost of the  $m$ th iteration of the method of inscribed ellipsoids can be bounded by  $O(m^{3.5} \ln m \ln \ln n)$  operations. Note that for the method of inscribed ellipsoids one can assume without loss of generality that  $\gamma = 0.99$ ,  $R = 3n$ ,  $m = O(n \ln n)$ .

The paper is organized as follows: In Section 2 we consider four computational problems of finding extremal ellipsoids for convex polytopes and show that these problems can be reduced in linear time to Problem I. In Section 3 Problem I is formulated as a convex program with nonlinear constraints. In Section 4 we describe an algorithm ElliP that reduces the Problem I to a small number

of special convex programming problems  $P$  with linear constraints. The number  $k$  of subproblems  $P$  is indeed very small:  $k \leq 12$  for the method of inscribed ellipsoids with  $n \leq 10^6$  variables. In Section 5 Algorithm ElliP is viewed in geometrical terms. It turns out that each problem  $P$  can be interpreted as the problem of inscribing the maximal paraboloid in a polyhedral cone defined by the pair  $(Q, b)$ , where  $b$  is an interior point of the polytope  $Q$ . In Section 6 we bound the complexity of Problem  $P$  by using the general path-following Newton's method stated in [7]. The number of Newton iterations of the method does not exceed  $O(m^{1/2} \ln [mR/\ln \frac{1}{\gamma}])$  and, though the number of unknowns in the problem grows as  $n(n+1)/2 + n \approx n^2/2$ , the computational cost of one iteration can be bounded by  $O(m^3)$  arithmetical operations. To prove the latter, we develop a system of  $n+m$  linear equations with  $n+m$  unknowns to compute the Newton direction, which is similar to but simpler than the system suggested in [7] for the case of  $n(n+1)/2$  unknowns. In Section 7 we obtain the upper bound (1.3) on the complexity of Problem I. Section 8 of the paper contains some concluding remarks and open questions.

## 2. Extremal ellipsoids

Let  $Q$  be a convex body in  $\mathbb{R}^n$ . It is known that

- among the ellipsoids  $E$ , centered at a given point  $a \in \text{int } Q$  and inscribed in  $Q$ , there exists a unique ellipsoid  $E^*(a)$  of maximum volume;
- there exists a unique maximal ellipsoid

$$E^* = \operatorname{argmax} \{ \operatorname{vol} E \mid E \subseteq Q \}$$

for  $Q$  [3].

Let  $\gamma \in (0,1]$ . An ellipsoid  $E$ , inscribed in  $Q$ , is called  $\gamma$ -maximal for  $Q$ , if  $\operatorname{vol} E \geq \gamma \cdot \operatorname{vol} E^*$ . We say that  $E$  is  $(\gamma, a)$ -maximal for  $Q$ , if  $E$  is centered at  $a$ ,  $E \subseteq Q$ , and  $\operatorname{vol} E \geq \gamma \cdot \operatorname{vol} E^*(a)$ .

Similarly,

- among the ellipsoids, centered at  $a$  and circumscribed about  $Q$ , there exists a unique ellipsoid  $E_*(a)$  of minimum volume;
- there exists a unique minimal ellipsoid

$$E_* = \operatorname{argmin} \{ \operatorname{vol} E \mid Q \subseteq E \}.$$

for an arbitrary convex body  $Q$ . Moreover, the center of  $E_*$  is an interior point of  $Q$  [3].

Again, let  $\gamma \in (0,1]$  be a given relative accuracy in the volume. An ellipsoid  $E$ , containing  $Q$ , is said to be  $\gamma$ -minimal for  $Q$  if  $\gamma \cdot \operatorname{vol} E \leq \operatorname{vol} E_*$ . Next,  $E$  is said to be  $(\gamma, a)$ -minimal for  $Q$  if  $E$  is centered at  $a$ ,  $E \supseteq Q$ , and  $\gamma \cdot \operatorname{vol} E \leq \operatorname{vol} E_*(a)$ .

Suppose without loss of generality that  $Q$  contains the origin  $a = 0$  as an interior point and consider the following four computational problems.

Problem I Given  $\gamma \in (0,1)$  and a full-dimensional polytope  $Q$  in  $\mathbb{R}^n$  defined by  $m$  linear inequalities

$$Q = \{x \in \mathbb{R}^n \mid c_i^T x \leq 1, i = 1, \dots, m\}, \quad (2.1)$$

find a  $\gamma$ -maximal ellipsoid for  $Q$ .

Problem I<sub>0</sub> (“centered” version of I) Find a  $(\gamma, 0)$ -maximal ellipsoid for (2.1).

Problem C Given  $\gamma \in (0,1)$  and a full-dimensional polytope  $Q$  defined as the convex hull of  $m$  points in  $\mathbb{R}^n$

$$Q = \operatorname{conv. hull} \{d_1, \dots, d_m\}, \quad (2.2)$$

find a  $\gamma$ -minimal ellipsoid for  $Q$ .

Problem  $C_0$  (“centered” version of C) Find a  $(\gamma, 0)$ -minimal ellipsoid for (2.2).

In this section we describe five geometric transformations

$$C \rightleftharpoons C_0 \rightleftharpoons I_0 \rightarrow I. \quad (2.3)$$

yielding “linear-time” reductions among the above listed computational problems. We begin with the reduction  $C(n, m, \gamma) \rightarrow C_0(n+1, 2m, \gamma)$ , suggested for the case  $\gamma = 1$  by Titterton [12].

Suppose we wish to compute a  $\gamma$ -minimal ellipsoid for a given  $n$ -dimensional polytope (2.2), containing the origin as an interior point. Let us introduce a new “vertical” coordinate  $x_{n+1}$ , and consider the  $(n+1)$ -dimensional polytope

$$Q' = \text{conv. hull} \{ \pm (d_1, 1), \dots, \pm (d_m, 1) \}, \quad (2.4)$$

still containing  $0 \in \mathbb{R}^{n+1}$  as an interior point. Let  $E'$  be a  $(\gamma, 0)$ -minimal ellipsoid for  $Q'$ . Then the intersection of  $E'$  with the hyperplane  $\Pi = \{x \in \mathbb{R}^{n+1} \mid x_{n+1} = 1\}$  gives an  $n$ -dimensional ellipsoid  $E$  which is  $\gamma$ -minimal for  $Q$ . Indeed,  $E' \supseteq Q'$  if and only if  $E' \cap \Pi \supseteq Q$ . Moreover,

$$\text{vol}_{n+1} E' = \text{const}(n) \cdot \text{vol}_n[E' \cap \Pi] \cdot v(h)$$

where  $h > 1$  is the “height” of  $E'$ , and

$$v(h) = h^{n+1}(h^2 - 1)^{-n/2} \geq v(\sqrt{n+1}).$$

In particular, if  $E_*'$  is the  $(1, 0)$ -minimal ellipsoid for  $Q'$ , then the height of  $E_*'$  equals  $\sqrt{n+1}$ , and  $E_*' \cap \Pi$  is the minimal ellipsoid for  $Q$ . Furthermore,

$$\frac{\text{vol}_n(E_*' \cap \Pi)}{\text{vol}_n(E' \cap \Pi)} = \frac{\text{vol}_{n+1} E_*'}{\text{vol}_{n+1} E'} \cdot \frac{v(h)}{v(\sqrt{n+1})} \geq \gamma,$$

i.e.,  $E' \cap H$  is  $\gamma$ -minimal for  $Q$ .

The reverse reduction  $C_0(n, m, \gamma) \rightarrow C(n, 2m, \gamma)$  is simpler. In order to determine a  $(\gamma, 0)$ -minimal ellipsoid for (2.2), it suffices to find a  $\gamma$ -minimal ellipsoid  $E$  for the polytope

$$Q_{\pm} = \text{conv.hull} \{ \pm d_1, \dots, \pm d_m \},$$

and shift  $E$  to the origin. It is easy to see that the shifted ellipsoid  $[E + (-E)]/2$  still contains the centrally symmetric polytope  $Q_{\pm}$ , and by definition this ellipsoid is  $(\gamma, 0)$ -minimal for both  $Q_{\pm}$  and  $Q$ .

Thus, the problem  $C$  is equivalent to its “centered” version.

The equivalence  $I_0(n, m, \gamma) \Leftrightarrow C_0(n, m, \gamma)$  follows by standard polarity arguments: an ellipsoid  $E$  is inscribed in

$$Q = \{x \in \mathbb{R}^n \mid c_i^T x \leq 1, i = 1, \dots, m\} \tag{2.1}$$

if and only if its polar

$$E^{\circ} = \{y \in \mathbb{R}^n \mid x^T y \leq 1 \text{ for all } x \in E\}$$

contains the polar

$$Q^{\circ} = \text{conv. hull} \{c_1, \dots, c_m\} \tag{2.5}$$

of the polytope  $Q$ . If  $E$  is centered at the origin, then  $E^{\circ}$  is centered at the origin as well, and

$$\text{vol } E \cdot \text{vol } E^{\circ} = \mu_n^2,$$

where  $\mu_n$  is the volume of the unit  $n$ -dimensional Euclidean ball. Therefore  $E$  is  $(\gamma, 0)$ -maximal for (2.1) if and only if  $E^O$  is  $(\gamma, 0)$ -minimal for (2.5).

Now for the last reduction  $I_0(n, m, \gamma) \rightarrow I(n, 2m, \gamma)$ : to find a  $(\gamma, 0)$ -maximal ellipsoid  $E$  for (2.1), one can compute a  $\gamma$ -maximal ellipsoid  $E$  for the centrally symmetric polytope

$$Q^\pm = \{x \in \mathbb{R}^n \mid \pm c_i^T x \leq 1, i = 1, \dots, m\},$$

and translate  $E$  to the origin.

We do not know whether there exists a reduction  $I \rightarrow I_0$ , similar to the geometric reductions (2.3); see also Question 2 in Section 8. Henceforth we focus on the computational complexity of Problem I, the most difficult among our four computational problems. In Sections 4 and 5 we will reduce Problem I to a small number of subproblems P, each of which can be interpreted as a problem of the approximate computation of the maximal paraboloid inscribed in a polyhedral cone.

In most of the paper, we need the following technical assumption:  $Q$  contains the unit Euclidean ball  $B_1 = \{x \in \mathbb{R}^n \mid \|x\| \leq 1\}$  and is contained in the Euclidean ball  $B_R = \{x \in \mathbb{R}^n \mid \|x\| \leq R\}$  of a given radius  $R$ :

$$B_1 \subseteq Q \subseteq B_R. \tag{2.6}$$

Note that by means of dilatations we can keep  $R$  constant in the reductions

$C \leftarrow C_0 \rightleftarrows I_0 \rightarrow I$ . In the reduction  $C \rightarrow C_0$  we have  $R' = \sqrt{R^2 + 1} < 2R$  for the polytope  $Q'$ , see (2.4).

### 3. The problem of finding the maximal inscribed ellipsoid as a convex programming problem

Problem I can be reformulated as a convex program [9,10]. This can be done as follows.

An arbitrary ellipsoid  $E$  in  $\mathbb{R}^n$  can be given in the form  $E = \{x \mid x = a + Bz, \|z\| \leq 1\}$  where  $a \in \mathbb{R}^n$  is the center of  $E$  and  $B$  is an  $n \times n$  symmetric positive definite matrix. Thus  $E$  is the image of the Euclidean unit ball  $\{z \mid \|z\| \leq 1\}$  shifted to the point  $a$  after the linear transformation  $B$ . In particular, in the above representation the support function of the ellipsoid has the form

$$\phi_E(c) = \max\{c^T x \mid x \in E\} = c^T a + \|c^T B\|,$$

and its volume is given by  $\text{vol } E = \mu_n \det B$ . Hence, in order to find a  $\gamma$ -maximal ellipsoid  $E$  for (2.1) it suffices to solve the convex program

$$\left. \begin{aligned} f(B) = -\ln \det B \rightarrow \min \\ c_i^T a + \|c_i^T B\| \leq 1, \quad i = 1, \dots, m \end{aligned} \right\} \quad (3.1)$$

with the unknowns  $a \in \mathbb{R}^n$  and  $B \in \mathbb{R}^{n(n+1)/2}$ , to an absolute accuracy of  $\ell_n \frac{1}{\gamma}$  in the functional. Here  $B$  is a symmetric positive definite matrix of order  $n$ . The convexity of  $f(B)$  on the set of positive definite matrices is well known [2].

Letting  $A = B^2$ , (3.1) can be rewritten as

$$\left. \begin{aligned} f(A) = -\ln \det A \rightarrow \min \\ c_i^T A c_i \leq (1 - c_i^T a)^2, \quad i = 1, \dots, m \end{aligned} \right\} \quad (3.2)$$

Problem (3.2) is also formulated in [5]. Note that unlike those of (3.1), the constraints of (3.2) are not convex. However, for a fixed  $a$  (say  $a = 0$ ) these constraints are linear in  $A$ . In particular, Problems  $C$ ,  $C_0$  and  $I_0$  can be reduced to the problem of minimizing  $f(A)$  with  $A$  subject to linear

constraints (see [1] for the case  $I_0$ ).

The convex programming problem (3.1) can be solved by the ellipsoid method, which yields a polynomial (but poor) bound

$$n^6(n^2 + m) \ln [Rn/\ln \frac{1}{\gamma}]$$

on the number of arithmetical operations sufficient to solve Problems I and C [10,11]. For reasonable  $m$ , this result was substantially improved by Nesterov and Nemirovsky [7]. Using a path-following Newton's method for minimizing the function

$$F_t = - \ln \det A - t \sum_{i=1}^m \ln [(1 - c_i^T a)^2 - c_i^T A c_i]$$

with penalty parameter  $t \downarrow 0$ , see (3.2), they reduced the arithmetical complexity of computing a  $\gamma$ -maximal ellipsoid to the bound

$$O(m^{4.5} \ln [Rm/\ln \frac{1}{\gamma}]). \quad (3.3)$$

Applying the same approach to the centered version of the problem ( $a = 0$ ), Nesterov and Nemirovsky also obtained a better bound

$$O(m^{3.5} \ln [Rm/\ln \frac{1}{\gamma}]) \quad (3.4)$$

on the complexity of finding a  $\gamma$ -minimal ellipsoid for a polytope. In both cases the bound  $O(m^{0.5} \ln [Rm/\ln \frac{1}{\gamma}])$  on the number of Newton steps is "standard" [8]. However, in the case  $a=0$ , where the constraints of (3.2) are linear, the corresponding linear system for computing Newton's direction is simpler. In time  $O(m^3)$  this system can be rewritten as a linear system in  $m$  unknowns, and consequently, it can be solved in  $O(m^3)$  operations. For the general case, where  $a$  is not fixed, Nesterov and Nemirovsky described a more complicated method, which requires  $O(m^4)$  operations per Newton iteration [7, pp. 163-188].

In this paper we reduce (3.2) to a small number of subproblems

$$P(b) \quad \left. \begin{aligned} f(A) &= -\ln \det A \rightarrow \min, \\ c_i^T A c_i &\leq (1 - c_i^T a)(1 - c_i^T b), \quad i = 1, \dots, m \end{aligned} \right\} \quad (3.5)$$

with fixed values of  $b \in \mathbb{R}^n$ . So each  $P(b)$  is a problem of minimizing  $f(A)$  with  $A$  and  $a$  subject to linear constraints. This allows us to bring down the complexity of finding a  $\gamma$ -maximal ellipsoid for a polytope to a bound close to (3.4).

#### 4. Reduction I $\rightarrow$ P

Consider the following algorithm ElliP for computing a  $\gamma$ -maximal ellipsoid for a polytope  $Q$ .

Step 0. Set

$$\delta := \frac{1}{3} \ln \frac{1}{\gamma},$$

$$k := 0$$

$$b_k := b_0 := \text{an arbitrary interior point in } Q.$$

Step 1. Find an approximate solution  $a_k = a(b_k)$  and  $A_k = A(b_k)$  to Problem  $P(b_k)$ , see (3.5), with absolute error  $\delta$  in the functional.

Step 2. Update:

$$b_{k+1} := \frac{1}{2}(b_k + a_k),$$

$$k := k+1.$$

Go to step 1 and start a new iteration.

Let

$$\gamma(b) = \text{vol } E^*(b) / \text{vol } E^* \quad (4.1)$$

be the ratio of the volumes of the  $(1,b)$ -maximal and the maximal ellipsoids for  $Q$ . So  $\gamma(b) \in (0,1)$  for all  $b \in \text{int } Q$ , and  $\gamma(b) = 1$  if and only if  $b = a^*$ , where  $a^*$  is the center of the maximal ellipsoid  $E^*$  for  $Q$ .

Theorem 1. Algorithm ElliP converges in at most  $\kappa+1$  iterations, where

$$\kappa = \lceil \log(\ln \frac{1}{\gamma(b_0)} / \ln \frac{1}{\bar{\gamma}}) \rceil + 1, \quad (4.2)$$

with the ellipsoid

$$E_\kappa = \{x \in \mathbb{R}^n \mid x = b_{\kappa+1} + A_\kappa^{1/2} z, \|z\| \leq 1\}$$

$\gamma$ -maximal for  $Q$ .

*Proof.* Let us first show that  $E_\kappa \subseteq Q$ . By the definition of  $A_\kappa$  (see step 1 in the description of the algorithm) we have

$$c_i^T A_\kappa c_i \leq (1 - c_i^T a_\kappa)(1 - c_i^T b_\kappa), \quad i = 1, \dots, m.$$

Since

$$(1 - c_i^T a)(1 - c_i^T b) \leq (1 - c_i^T \frac{a+b}{2})^2 \quad (4.3)$$

for all  $a, b \in Q$ , we conclude that

$$c_i^T A_\kappa c_i \leq (1 - c_i^T \frac{a_\kappa + b_\kappa}{2})^2 = (1 - c_i^T b_{\kappa+1})^2,$$

see step 2. Therefore

$$\|c_i^T A_\kappa^{1/2}\| \leq 1 - c_i^T b_{\kappa+1}, i = 1, \dots, m.$$

This proves the inclusion  $E_\kappa \subseteq Q$ , see (3.1).

To prove the  $\gamma$ -maximality of  $E_\kappa$ , consider the function  $\phi : Q \times Q \rightarrow \mathbb{R}$  defined as

$$\phi(a,b) = \min \{f(A) \mid c_i^T A c_i \leq (1 - c_i^T a)(1 - c_i^T b), i = 1, \dots, m\}.$$

Here  $f(A) = -\ln \det A$  and  $A$  is a symmetric positive definite matrix of order  $n$ . We need the following property of  $\phi$ : for all  $a, b \in Q$

$$\phi\left(\frac{a+b}{2}, \frac{a+b}{2}\right) \leq \phi(a,b) \leq \frac{\phi(a,a) + \phi(b,b)}{2}. \quad (4.4)$$

The first inequality of (4.4) follows from (4.3). To prove the second inequality suppose that  $A$  and  $B$  are the optimal matrices for  $(a,a)$  and  $(b,b)$ :

$$\left. \begin{aligned} f(A) = \phi(a,a), \quad c_i^T A c_i &\leq (1 - c_i^T a)^2, \\ f(B) = \phi(b,b), \quad c_i^T B c_i &\leq (1 - c_i^T b)^2. \end{aligned} \right\} \quad (4.5)$$

We can assume without loss of generality that  $A$  and  $B$  are diagonal matrices. Multiplying the inequalities (4.5) for each  $i = 1, \dots, m$ , we get

$$c_i^T (AB)^{1/2} c_i \leq [(c_i^T A c_i)(c_i^T B c_i)]^{1/2} \leq (1 - c_i^T a)(1 - c_i^T b).$$

Hence,

$$\phi(a,b) \leq f((AB)^{1/2}) = \frac{1}{2} [\phi(a,a) + \phi(b,b)].$$

Observe that the first inequality of (4.4) implies that the minimum of  $\phi(a,b)$  on  $Q \times Q$  is attained at  $(a^*, a^*)$ :

$$\min \{\phi(a,b) | a, b \in Q\} = \min \{\phi(a,a) | a \in Q\} = \phi(a^*, a^*) = f(A^*) = -2 \ln(\text{vol } E^* / \mu_n).$$

Here  $E^* = \{x \in \mathbb{R}^n | x = a^* + (A^*)^{1/2} z, \|z\| \leq 1\}$  is the maximal ellipsoid for  $Q$ .

Now we can prove the inequality

$$\ln \frac{\text{vol } E^*}{\text{vol } E_\kappa} = \frac{1}{2} [f(A_\kappa) - f(A^*)] \leq \ln \frac{1}{\gamma}, \quad (4.6)$$

equivalent to the  $\gamma$ -maximality of  $E_\kappa$ . From the description of the algorithm we know that

$$f(A_\kappa) \leq \min \{\phi(a, b_\kappa) | a \in Q\} + \delta \leq \phi(b_\kappa, b_\kappa) + \delta,$$

see step 1. Hence

$$\ln \frac{\text{vol } E^*}{\text{vol } E_\kappa} \leq \frac{1}{2} [\phi(b_\kappa, b_\kappa) - \phi(a^*, a^*)] + \frac{\delta}{2}. \quad (4.7)$$

Let

$$\xi_\kappa = \phi(b_\kappa, b_\kappa) - \phi(a^*, a^*) = 2 \ln \frac{1}{\gamma(b_\kappa)},$$

see (4.1). From the description of ElliP and (4.4) we have

$$\begin{aligned} \xi_\kappa &= \phi\left(\frac{a_{\kappa-1} + b_{\kappa-1}}{2}, \frac{a_{\kappa-1} + b_{\kappa-1}}{2}\right) - \phi(a^*, a^*) \\ &\leq \phi(a_{\kappa-1}, b_{\kappa-1}) - \phi(a^*, a^*) \\ &\leq \min \{\phi(a, b_{\kappa-1}) | a \in Q\} + \delta - \phi(a^*, a^*) \\ &\leq \min \left\{ \frac{\phi(a, a) + \phi(b_{\kappa-1}, b_{\kappa-1})}{2} | a \in Q \right\} + \delta - \phi(a^*, a^*) \\ &= \frac{\phi(b_{\kappa-1}, b_{\kappa-1}) - \phi(a^*, a^*)}{2} + \delta \\ &= \frac{1}{2} \xi_{\kappa-1} + \delta. \end{aligned}$$

The latter recurrence implies

$$\xi_\kappa \leq 2^{-\kappa} \xi_0 + \delta(1 + 2^{-1} + \dots + 2^{-\kappa+1}) < 2^{-\kappa} \xi_0 + 2\delta.$$

Now from (4.7) and (4.1) it follows that

$$\ln \frac{\text{vol } E^*}{\text{vol } E_\kappa} \leq 2^{-\kappa} \ln \frac{1}{\gamma(b_0)} + \frac{3}{2} \delta.$$

Since  $\delta = \frac{1}{3} \ln \frac{1}{\gamma}$ , we obtain (4.6) from (4.2). This completes the proof of the theorem.  $\square$

Selecting  $b_0 = 0$  as the starting point for ElliP, we obtain from (2.6) and (4.2)

$$\kappa \leq \left\lceil \log \left[ \frac{2n \ln R}{\ln \frac{1}{\gamma}} \right] \right\rceil. \quad (4.8)$$

This already low upper bound on the number of iterations of the algorithm ElliP can be still lowered in the case where the algorithm is applied as a subroutine in the method of inscribed ellipsoids [8]. At the  $s$ -th step of this method we have a  $\gamma$ -maximal ellipsoid  $E^s$  for a polytope  $Q^s$ . Next we pass a halfspace  $\pi^s = \{x \in \mathbb{R}^n \mid g_s^T(x - a^s) \geq 0\}$  through the center  $a^s$  of  $E^s$ , and compute a new  $\gamma$ -maximal ellipsoid  $E^{s+1}$  for the polytope  $Q^{s+1} = Q^s \cap \pi^s$ . Since  $Q^{s+1}$  contains the half ellipsoid  $E^s \cap \pi^s$ , we can start ElliP for  $Q^{s+1}$  at the center  $b_0^s$  of the maximal ellipsoid inscribed in the half ellipsoid  $E^s \cap \pi^s$ . Clearly, the point  $b_0^s$  can be computed in  $O(n^2)$  operations, and

$$\gamma(b_0^s) \geq 0.5 \mu_n^{-1/2}$$

as one can inscribe an ellipsoid  $E$  of volume  $0.5 \mu_n^{-1/2}$  in the halfball

$$\{x \in \mathbb{R}^n \mid \|x\| \leq 1, x_1 \geq 0\}$$

(place the center of  $E$  at  $(n^{-1/2}, 0, \dots, 0)$ ). In the method of inscribed ellipsoids we also fix  $\gamma = 0.99$ , see [11]. This yields the following bound

$$\kappa+1 \leq 9 + \log(1 + \ell n \sqrt{n}) \quad (4.9)$$

on the number of subproblems  $P(b_0), \dots, P(b_\kappa)$  that are to be solved at each step of the method inscribed ellipsoids. In particular,  $\kappa+1 \leq 12$  for  $n \leq 10^6$ .

We show in Sections 6 and 7 that the arithmetical complexity of each problem  $P(b)$  does not exceed (3.4). Before that, however, we shall describe a geometric interpretation of these problems.

## 5. Geometric interpretation

### 1<sup>o</sup>. Paraboloids

Let  $\mathcal{P}$  be the set of vertical paraboloids in  $\mathbb{R}^{n+1}$  tangent to the hyperplane

$$\Pi = \{(x, x_{n+1}) \in \mathbb{R}^{n+1} \mid x_{n+1} = 1\}.$$

An arbitrary paraboloid  $P \in \mathcal{P}$  can be represented in the form

$$P = \{(x, x_{n+1}) \in \mathbb{R}^{n+1} \mid x_{n+1} \geq \frac{1}{4} (x - a)^T A^{-1} (x - a) + 1\}, \quad (5.1)$$

where  $A$  is an  $n \times n$  symmetric positive definite matrix and  $a \in \mathbb{R}^n$  is the “center” of the paraboloid. We call the quantity  $V(P) = \mu_n(\det A)^{1/2}$  the “volume” of  $P$ . Geometrically,  $V(P)$  is the  $n$ -volume of the ellipsoid obtained by intersecting  $P$  with the hyperplane  $x_{n+1} = 5/4$ .

Let  $c \in \mathbb{R}^n$  and  $d \in \mathbb{R}$ . Consider the halfspace  $\pi$  in  $\mathbb{R}^{n+1}$  of the form

$$\pi = \{(x, x_{n+1}) \in \mathbb{R}^{n+1} \mid x_{n+1} \geq c^T x + d\}.$$

Clearly,

$$P \subseteq \pi \text{ if and only if } c^T A c \leq 1 - d - c^T a. \quad (5.2)$$

Note that the constraints (5.2) are linear in  $A$  and  $a$ .

## 2<sup>o</sup>. Shadows of paraboloids

For a point  $b \in \mathbb{R}^n$  denote by  $S = S(b)$  the projection of the paraboloid  $P$  from the point  $(b, 0)$  onto the hyperplane  $\pi$ :

$$S = \text{conv.hull}\{P \cup (b, 0)\} \cap \pi.$$

We call  $S$  the “b-shadow” of  $P$ , see Fig. 1. It is easy to see that

$$S \text{ is a } n\text{-dimensional ellipsoid centered at the point } \frac{1}{2}(a + b), \quad (5.3)$$

$$\text{vol } S = \mu_n \left( \left[ 1 + \frac{(a - b)^T A^{-1} (a - b)}{4} \right] \det A \right)^{1/2}. \quad (5.4)$$

Indeed,  $x \in S$  if and only if the ray

$$(b, 0) + t \cdot (x - b, 1), \quad t \in [0, \infty) \quad (5.5)$$

meets  $P$ , or equivalently the quadratic inequality

$$t \geq \frac{1}{4} [(x - b)t - (a - b)]^T A^{-1} [(x - b)t - (a - b)] + 1$$

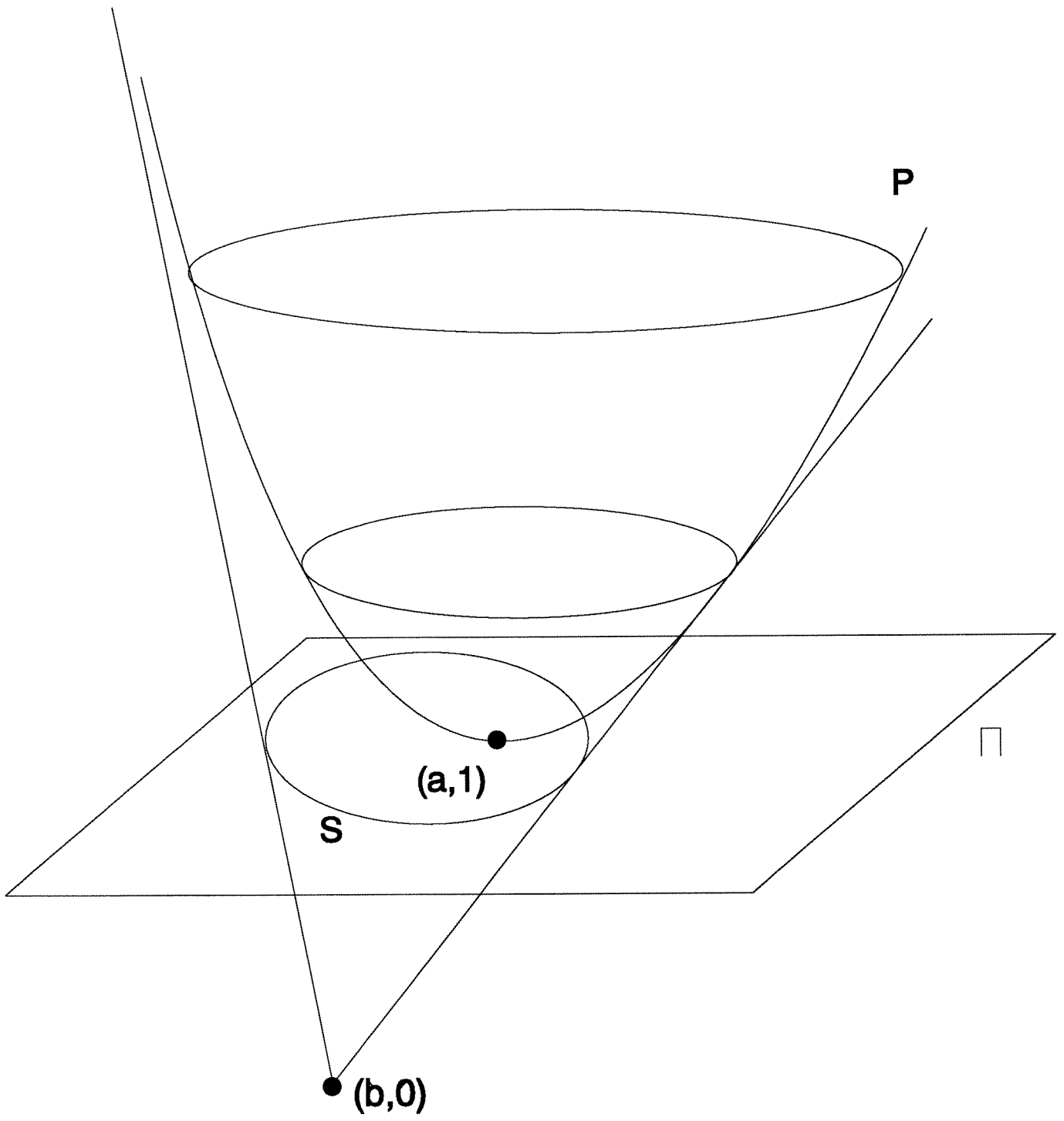


Figure 1.

has real roots  $t$ . Therefore  $x \in S$  is equivalent to

$$y^T A^{-1} y (1 + \xi^T A^{-1} \xi) \leq (1 + \xi^T A^{-1} y)^2$$

with  $y = x - b$  and  $\xi = (a - b)/2$ . The latter inequality can be written as

$$y^T [A^{-1}(1 + \xi^T A^{-1} \xi) - A^{-1} \xi \xi^T A^{-1}] y \leq 1 + 2 \xi^T A^{-1} y.$$

Since

$$A^{-1} - \frac{A^{-1} \xi \xi^T A^{-1}}{1 + \xi^T A^{-1} \xi} = (A + \xi \xi^T)^{-1},$$

we see that  $x \in S$  if and only if

$$\left(x - \frac{a+b}{2}\right)^T \left[A + \frac{(a-b)(a-b)^T}{2}\right]^{-1} \left(x - \frac{a+b}{2}\right) \leq 1.$$

This proves (5.3); (5.4) follows from

$$\det(A + \xi \xi^T) \equiv (1 + \xi^T A^{-1} \xi) \cdot \det A.$$

Note that (5.4) shows that the volume of any  $b$ -shadow of  $P$  exceeds the “volume”  $\mu_n(\det A)^{1/2}$  of  $P$ , and that the latter quantity is the linear approximation to  $\text{vol } S$  for a close to  $b$ .

3<sup>o</sup>. Geometric interpretation of the problem  $P(b)$ 

Let  $Q$  be a given polytope in  $\mathbb{R}^n$ . Suppose that  $b \in \text{int } Q$ , and consider the polyhedral cone  $K(b) \subset \mathbb{R}^{n+1}$  such that it has the vertex  $(b,0)$  and  $K(b) \cap \Pi = Q$ . In other words,  $K(b)$  is the union of all rays (5.5) which intersect a copy of  $Q$  placed in the hyperplane  $\Pi$ . If  $Q$  is given by (1.1), then  $K(b)$  is defined by the following system of linear inequalities

$$x_{n+1} \geq \frac{c_i^T(x-b)}{1-c_i^T b}, \quad i = 1, \dots, m. \quad (5.6)$$

Now from (5.2) it follows that a paraboloid  $P$  of the form (5.1) is contained in the cone (5.6) if and only if

$$c_i^T A c_i \leq (1-c_i^T a)(1-c_i^T b), \quad i = 1, \dots, m.$$

Therefore Problem  $P(b)$ , see (2.5), can be interpreted as the problem of finding the maximal paraboloid  $P \in \mathcal{P}$  inscribed in the cone  $K(b)$ .

Remark. It can be shown that for an arbitrary  $b \in \text{int } Q$  such a maximal paraboloid is unique.

Note also that  $P \subseteq K(b)$  if and only if the  $b$ -shadow of  $P$ , the ellipsoid  $S$ , is inscribed in the polytope  $Q$ .

4<sup>o</sup>. Geometric interpretation of the reduction  $I \rightarrow P$ 

The iterative procedure ElliP can be interpreted as follows. We select an interior point  $b_0$  in  $Q$  and inscribe a  $\gamma^{1/3}$ -maximal paraboloid  $P_0$  in the cone  $K(b_0)$ . Maximizing the “volume” of  $P \subseteq K(b_0)$ , we maximize the linear approximation (5.4) to the volume of its  $b_0$ -shadow in  $Q$ . Next we

move to the center  $b_1 = (b_0 + a_0)/2$  of the  $b_0$ -shadow of  $P_0$  (see (5.3) and step 2 of ElliP), and start the procedure anew. “Looking at  $Q$  from  $b_1$ ,” we inscribe a  $\gamma^{1/3}$ -maximal paraboloid  $P_1$  in the cone  $K(b_1)$  and so on. As we know from Theorem 1, the shadows  $S(b_0), S(b_1), \dots$  of the paraboloids  $P_0, P_1, \dots$  converge in a small number of iterations to a  $\gamma$ -maximal ellipsoid for  $Q$ . [In fact, instead of the shadows

$$S(b_k) = \{x \in \mathbb{R}^n \mid (x - b_{k+1})^T [A_k + \frac{(a_k - b_k)(a_k - b_k)^T}{2}]^{-1} (x - b_{k+1}) \leq 1\},$$

we used in the proof of Theorem 1 the smaller ellipsoids

$$E_k = \{x \in \mathbb{R}^n \mid (x - b_{k+1})^T A_k^{-1} (x - b_{k+1}) \leq 1\} \subset S(b_k),$$

which also converge to a  $\gamma$ -maximal ellipsoid for  $Q$ . This observation can be used to improve the convergence of ElliP.]

## 6. The complexity of finding a $\gamma$ -maximal paraboloid for a polyhedral cone.

The change of variables

$$\begin{aligned} A &\rightarrow A, \\ a &\rightarrow a - b, \\ c_i &\rightarrow \frac{c_i}{1 - c_i^T b}, \quad i = 1, \dots, m \end{aligned} \tag{6.1}$$

transforms Problem  $P(b)$ , see (3.5), into the following standard problem  $P = P(0)$ :

$$-\ln \det A \rightarrow \min, \quad (6.2)$$

$$c_i^T A c_i + c_i^T a \leq 1, \quad i = 1, \dots, m$$

The latter problem can be solved by the barrier method, stated in Section 3 of [7] for the general convex programming problem

$$f(x) \rightarrow \min, \quad x \in G \subset \mathbb{R}^N \quad (6.3)$$

with a thrice differentiable convex objective function  $f$ . The method is a special Newton procedure that follows the central path of the minimizers  $x(t)$  of the function

$$F_t(x) = f(x) + tg(x), \quad t \downarrow 0.$$

Here  $g$  is a barrier function for the convex feasible region  $G$ .

For the applicability of the method it suffices to check the following three conditions:

1<sup>o</sup>.  $g$  is strongly self-concordant on  $\text{int } G \neq \emptyset$ .

By definition this means that

1.1<sup>o</sup>  $g: \text{int } G \rightarrow \mathbb{R}$  is convex and thrice differentiable;

1.2<sup>o</sup> the level sets  $\{x \in \text{int } G \mid g(x) \leq \ell\}$  of  $g$  are closed in  $\mathbb{R}^N$  for each  $\ell \in \mathbb{R}$ ;

1.3<sup>o</sup> for any  $x \in \text{int } G$  and  $h \in \mathbb{R}^N$ ,

$$|D^3 g(x)[h, h, h]| \leq 2(D^2 g(x)[h, h])^{3/2}. \quad (6.4)$$

2<sup>o</sup>.  $g$  is a  $\sigma$ -self-concordant barrier for  $G$  with some  $\sigma \geq 1$ ,

i.e.,

$$\lambda(x) \leq \sigma^{1/2} \quad (6.5)$$

for all  $x \in \text{int } G$ , where by definition

$$\lambda(x) = \min\{\lambda \geq 0 \mid \forall h \in \mathbb{R}^N: |Dg(x)[h]| \leq \lambda(D^2g(x)[h,h])^{1/2}\}. \quad (6.6)$$

3<sup>o</sup>.  $f$  is  $\beta$ -compatible with  $g$  for some  $\beta \geq 0$ .

This latter condition means that:

3.1<sup>o</sup>.  $f$  is lower semicontinuous and convex on  $G$ , and finite and thrice differentiable on  $\text{int } G$ ;

3.2<sup>o</sup>. for all  $x \in \text{int } G$  and  $h \in \mathbb{R}^N$ ,

$$|D^3f(x)[h,h,h]| \leq \beta(3D^2f(x)[h,h])(3D^2g(x)[h,h])^{1/2}. \quad (6.7)$$

Suppose that the conditions 1<sup>o</sup>-3<sup>o</sup> are satisfied. Then, given an initial point  $x_0 \in \text{int } G$  and an absolute error  $\delta > 0$ , the barrier method can produce an approximate solution

$$x^\delta \in \text{int } G, \quad f(x^\delta) \leq \min \{f(x) \mid x \in G\} + \delta$$

to the problem (6.3) in at most

$$O\left(\sigma^{1/2} \ln \left[ \frac{\sigma V_g(f)}{\delta(1 - \pi_{x_g}(x_0))} \right]\right) \quad (6.8)$$

Newton iterations applied to convex combinations of  $f$  and  $g$  (or of  $g$  and some linear form). Here

- (i)  $O(\cdot)$  depends on  $\beta$ ;
- (ii)  $x_g \in \text{int } G$  is the (unique) minimizer of  $g$ ;
- (iii) 
$$V_g(f) = \sup\{f(x) \mid x \in W_{1/2}(x_g)\} - \inf\{f(x) \mid x \in W_{1/2}(x_g)\} \quad (6.9)$$

is the variation of  $f$  on the ellipsoid  $W_{1/2}(x_g)$ , where

$$W_r(x) = \{y \in \mathbb{R}^n \mid D^2g(x)[y - x, y - x] < r^2\}; \quad (6.10)$$

and

$$(iv) \quad \pi_x(y) = \inf\{t \geq 0 \mid x + t^{-1}(y - x) \in G\} \quad (6.11)$$

is the Minkowski function of  $G$  with the pole at  $x$ .

Moreover, it is shown in [7] that under the assumption 1<sup>o</sup>

$$W_1(x) \subset G \quad (6.12)$$

for all  $x \in \text{int } G$ .

To apply these results to our problem (6.2) we set

$$\begin{aligned} x &= (A, a), \\ G &= \{(A, a) \mid (A, a) \text{ satisfies the constraints (6.2) and } A \\ &\quad \text{is positive definite and symmetric}\}, \\ f &= -\ln \det A, \\ g &= f - \sum_{i=1}^m \ln(1 - c_i^T a - c_i^T A c_i). \end{aligned} \quad (6.13)$$

Let us first check the conditions 1<sup>o</sup> - 3<sup>o</sup>.

1<sup>o</sup>) Clearly, the conditions 1.1<sup>o</sup> and 1.2<sup>o</sup> are satisfied. The Taylor expansion of  $g$  has the form

$$\begin{aligned}
g(\mathbf{A} + \mathbf{H}, \mathbf{a} + \mathbf{h}) &= g(\mathbf{A}, \mathbf{a}) - \text{tr}\{\mathbf{A}^{-1}\mathbf{H}\} + \sum_{i=1}^m \left[ \frac{\mathbf{c}_i^T \mathbf{h} + \mathbf{c}_i^T \mathbf{H} \mathbf{c}_i}{1 - \mathbf{c}_i^T \mathbf{a} - \mathbf{c}_i^T \mathbf{A} \mathbf{c}_i} \right] \\
&+ \frac{1}{2} \text{tr}\{(\mathbf{A}^{-1}\mathbf{H})^2\} + \frac{1}{2} \sum_{i=1}^m \left[ \frac{\mathbf{c}_i^T \mathbf{h} + \mathbf{c}_i^T \mathbf{H} \mathbf{c}_i}{1 - \mathbf{c}_i^T \mathbf{a} - \mathbf{c}_i^T \mathbf{A} \mathbf{c}_i} \right]^2 \\
&- \frac{1}{3} \text{tr}\{(\mathbf{A}^{-1}\mathbf{H})^3\} + \frac{1}{3} \sum_{i=1}^m \left[ \frac{\mathbf{c}_i^T \mathbf{h} + \mathbf{c}_i^T \mathbf{H} \mathbf{c}_i}{1 - \mathbf{c}_i^T \mathbf{a} - \mathbf{c}_i^T \mathbf{A} \mathbf{c}_i} \right]^3 + \dots
\end{aligned} \tag{6.14}$$

Hence we get (6.4) and 1.3<sup>o</sup>:

$$\begin{aligned}
|D^3 g(\mathbf{A}, \mathbf{a})[(\mathbf{H}, \mathbf{h}), (\mathbf{H}, \mathbf{h}), (\mathbf{H}, \mathbf{h})]| &= \left| -2 \text{tr}\{(\mathbf{A}^{-1}\mathbf{H})^3\} + 2 \sum_{i=1}^m \left[ \frac{\mathbf{c}_i^T \mathbf{h} + \mathbf{c}_i^T \mathbf{H} \mathbf{c}_i}{1 - \mathbf{c}_i^T \mathbf{a} - \mathbf{c}_i^T \mathbf{A} \mathbf{c}_i} \right]^3 \right| \\
&= |-2 \text{tr}\{\mathbf{X}^3\} + 2 \sum_{i=1}^m \psi_i^3| \\
&\leq 2|\text{tr}\{\mathbf{X}^3\}| + 2 \sum_{i=1}^m |\psi_i|^3 \\
&\leq 2(\text{tr}\{\mathbf{X}^2\})^{3/2} + 2 \left( \sum_{i=1}^m \psi_i^2 \right)^{3/2} \\
&\leq 2(\text{tr}\{\mathbf{X}^2\} + \sum_{i=1}^m \psi_i^2)^{3/2} = 2(\|\mathbf{X}\|^2 + \|\boldsymbol{\psi}\|^2)^{3/2} \\
&= 2(D^2 g(\mathbf{A}, \mathbf{a})[(\mathbf{H}, \mathbf{h}), (\mathbf{H}, \mathbf{h})])^{3/2}.
\end{aligned}$$

Here

$$\mathbf{X} = \mathbf{A}^{-1/2} \mathbf{H} \mathbf{A}^{-1/2}$$

$$\psi_i = \frac{\mathbf{c}_i^T \mathbf{h} + \mathbf{c}_i^T \mathbf{H} \mathbf{c}_i}{1 - \mathbf{c}_i^T \mathbf{a} - \mathbf{c}_i^T \mathbf{A} \mathbf{c}_i}, \quad i = 1, \dots, m,$$

$$\boldsymbol{\psi} = (\psi_1, \dots, \psi_m),$$

and  $\|\cdot\|$  stands for the  $\ell_2$ -norm.

2<sup>o</sup>) Let us prove (6.5) for  $\sigma = 2m$ . By the definition (6.6) of the Newton decrement  $\lambda(A, a)$  of  $g$  at  $(A, a)$  we have

$$\begin{aligned}\lambda(A, a) &= \max_{H, h} \frac{|Dg(A, a)[(H, h)]|}{(D^2g(A, a)[(H, h), (H, h)])^{1/2}} \\ &\leq \max_{X, \psi} \frac{|-\text{tr}\{X\} + \psi_1 + \dots + \psi_m|}{(\|X\|^2 + \|\psi\|^2)^{1/2}} \\ &\leq (n + m)^{1/2} < (2m)^{1/2}.\end{aligned}$$

The last inequality follows from the fact that the polytope (2.1) is bounded and  $n + 1 \leq m$ .

3<sup>o</sup>)  $f$  is  $2 \cdot 3^{-3/2}$ -compatible with  $g$ :

$$\begin{aligned}|D^3f(A, a)[(H, h), (H, h), (H, h)]| &= 2|\text{tr}\{(A^{-1}H)^3\}| \\ &= 2|\text{tr}\{X^3\}| \leq 2 \cdot 3^{-3/2}(3\|X\|^2)(3\|X\|^2)^{1/2} \\ &< 2 \cdot 3^{-3/2}(3\|X\|^2)(3\|X\|^2 + 3\|\psi\|^2)^{1/2} \\ &= 2 \cdot 3^{-3/2}(3D^2f(A, a)[(H, h), (H, h)])(3D^2g(A, a)[(H, h), (H, h)])^{1/2}.\end{aligned}$$

To use the bound (6.8) on the number of Newton iterations we have to obtain upper bounds on the quantities  $V_g(f)$  and  $\pi_{x_g}(x_0)$ . We first prove the following two lemmas.

Lemma 6.1 Let  $x_g = (A_g, a_g)$  be the minimizer of the function  $g$  defined in (6.13). Then

$$A_g^{-1} = \sum_{i=1}^m c_i c_i^T / (1 - c_i^T a_g - c_i^T A_g c_i), \quad (6.15)$$

$$0 = \sum_{i=1}^m c_i / (1 - c_i^T a_g - c_i^T A_g c_i), \quad (6.16)$$

$$n + m = \sum_{i=1}^m 1 / (1 - c_i^T a_g - c_i^T A_g c_i), \quad (6.17)$$

$$\omega_*(A_g) > (2m \cdot \max_{i=1, \dots, m} \|c_i\|^2)^{-1}, \quad (6.18)$$

where  $\omega_*(\cdot)$  is the minimal eigenvalue of  $(\cdot)$ .

**Proof.** The equations (6.15) and (6.16) are the first-order optimality conditions  $\partial g / \partial H = 0$  and  $\partial g / \partial h = 0$ , see (6.14). Multiplying (6.15) by  $A_g$  and (6.16) by  $a_g^T$ , we get (6.17):

$$\begin{aligned} n &= \text{tr}(A_g^{-1} A_g) = \text{tr} \left\{ \sum_{i=1}^m \frac{c_i c_i^T A_g}{1 - c_i^T a_g - c_i^T A_g c_i} \right\} = \text{tr} \left\{ \sum_{i=1}^m \frac{c_i a_g^T + c_i c_i^T A_g}{1 - c_i^T a_g - c_i^T A_g c_i} \right\} \\ &= \sum_{i=1}^m \frac{c_i^T a_g + c_i^T A_g c_i}{1 - c_i^T a_g - c_i^T A_g c_i} = -m + \sum_{i=1}^m 1 / (1 - c_i^T a_g - c_i^T A_g c_i). \end{aligned}$$

The last inequality (6.18) can be obtained as follows:

$$\begin{aligned} [\omega_*(A_g)]^{-1} &= \max \{ x^T A_g^{-1} x \mid x \in \mathbb{R}^n, \|x\| = 1 \} \\ &= \max \left\{ \sum_{i=1}^m \frac{(c_i^T x)^2}{1 - c_i^T a_g - c_i^T A_g c_i} \mid x \in \mathbb{R}^n, \|x\| = 1 \right\} \quad (\text{by (6.15)}) \\ &\leq (n + m) \max_i \max \{ (c_i^T x)^2 \mid x \in \mathbb{R}^n, \|x\| = 1 \} \quad (\text{by (6.17)}) \\ &= (n + m) \max_i \|c_i\|^2 < 2m \cdot \max_i \|c_i\|^2. \end{aligned}$$

□

**Lemma 6.2** Suppose that  $Q$  is contained in some Euclidean ball of radius  $R$ . Then

$$\omega^*(A) \leq \frac{25}{16} R^2 \quad (6.19)$$

for all feasible points  $(A, a) \in G$ , where  $\omega^*(\cdot)$  is the maximal eigenvalue of  $(\cdot)$ .

Proof. From the definition of paraboloids (5.1) it follows that for any feasible point  $(A, a) \in G$  the ellipsoid  $E = \{X \in \mathbb{R}^n | (x - a)^T A^{-1} (x - a) \leq 1\}$  is contained in the intersection of the cone  $K = K(0)$  with the hyperplane  $x_{n+1} = 5/4$ . Hence  $E$  can be covered by a copy of the polytope  $5Q/4$ , and consequently by an Euclidean ball of radius  $5R/4$ .  $\square$

Now we can prove that

$$V_g(f) \leq 2n \ln (2m R \max \| c_i \|); \quad (6.20)$$

see (6.9) and (6.10) for the definition of  $V_g(f)$ . Indeed, (6.12) implies that  $\omega_*(A) \geq \frac{1}{2} \omega_*(A_g)$  for all  $(A, a) \in W_{1/2}(A_g, a_g)$ . Since

$$-n \ln \omega^*(A) \leq f = -\ln \det A \leq -n \ln \omega_*(A),$$

(6.20) follows from (6.18) and (6.19).

Let us select the pair

$$\begin{aligned} A_0 &= 0.5 I / \max \| c_i \|^2 \\ a_0 &= 0 \end{aligned}$$

as the starting point  $x_0$  for the barrier method ( $I$  is the identity matrix of order  $n$ ). Since

$$c_i^T A_0 c_i + c_i^T a_0 \leq \frac{1}{2}, \quad i = 1, \dots, m,$$

and

$$\omega_*(A_0) = 0.5 / \max \| c_i \|^2,$$

$$\omega^*(A_g) \leq \frac{25}{16} R^2,$$

we get from the definition (6.11) of the Minkowski function

$$\frac{1}{1 - \pi_{\mathbf{x}_g}(\mathbf{x}_0)} \leq 4 R^2 \max \| \mathbf{c}_i \|^2. \quad (6.21)$$

Letting  $\delta = \ln \frac{1}{\gamma}$  and substituting (6.20) and (6.21) in (6.8), we see that the number of Newton iterations of the barrier method for inscribing a  $\gamma$ -maximal paraboloid in the cone  $K(0)$  does not exceed

$$O(m^{1/2} \ln[m R \max \| \mathbf{c}_i \| / \ln \frac{1}{\gamma}]). \quad (6.22)$$

The arithmetical cost of one Newton iteration can be bounded by  $O(m^3)$  operations, as in the case  $\mathbf{a} \equiv 0, \mathbf{h} \equiv 0$  considered in [9]. Indeed, let  $F$  be a convex combination of  $\mathbf{g}, \mathbf{f}$  and some linear form of the variables, say

$$F = - \ln \det A - \tau \sum_{i=1}^m \ln (1 - \mathbf{c}_i^T \mathbf{a} - \mathbf{c}_i^T A \mathbf{c}_i) + \langle L, A \rangle + \langle \ell, \mathbf{a} \rangle.$$

Here  $L$  is a given symmetric matrix of order  $n$ ,  $\ell$  is a given  $n$ -dimensional vector, and

$$\langle L, A \rangle = \text{tr} (LA),$$

$$\langle \ell, \mathbf{a} \rangle = \ell^T \mathbf{a}$$

stands for the inner product in the Euclidean space  $(A, \mathbf{a})$ . Then

$$\begin{aligned} DF(A, \mathbf{a}) [H, \mathbf{h}] &= - \text{tr}(A^{-1}H) + \tau \sum_{i=1}^m \frac{\mathbf{c}_i^T \mathbf{h} + \mathbf{c}_i^T H \mathbf{c}_i}{1 - \mathbf{c}_i^T \mathbf{a} - \mathbf{c}_i^T A \mathbf{c}_i} + \langle L, H \rangle + \langle \ell, \mathbf{h} \rangle \\ &= \langle -A^{-1} + L + \tau \sum_{i=1}^m C_i \Delta_i^{-1}, H \rangle + \langle \ell + \tau \sum_{i=1}^m \mathbf{c}_i \Delta_i^{-1}, \mathbf{h} \rangle, \end{aligned}$$

where

$$C_i = c_i c_i^T \text{ and } \Delta_i = 1 - c_i^T a - c_i^T A c_i, i = 1, \dots, m.$$

Therefore both the  $A$ -component

$$\nabla_A = -A^{-1} + L + \tau \sum_{i=1}^m \Delta_i^{-1} C_i$$

and the  $a$ -component

$$\nabla_a = \ell + \tau \sum_{i=1}^m \Delta_i^{-1} c_i$$

of the gradient of  $F$  can be computed at any point  $(A, a)$  in  $O(mn^2)$  arithmetical operations.

Furthermore,

$$D^2F(A, a)[(H, h), (H, h)] = \langle A^{-1} H A^{-1}, H \rangle + \tau \sum_{i=1}^m \Delta_i^{-2} (\langle C_i, H \rangle + \langle c_i, h \rangle)^2,$$

and to find the Newton direction  $(H^0, h^0)$  of  $F$  at  $(A, a)$  one has to solve the following system of linear equations

$$\frac{1}{2} \frac{\partial D^2F(A, a)[(H, h), (H, h)]}{\partial H} = A^{-1} H A^{-1} + \tau \sum_{i=1}^m \Delta_i^{-2} \{ \langle C_i, H \rangle + \langle c_i, h \rangle \} C_i = -\nabla_A, \quad (6.23)$$

$$\frac{1}{2} \frac{\partial D^2F(A, a)[(H, h), (H, h)]}{\partial h} = \tau \sum_{i=1}^m \Delta_i^{-2} \{ \langle C_i, H \rangle + \langle c_i, h \rangle \} c_i = -\nabla_a.$$

Let

$$\lambda_i = \tau \Delta_i^{-2} \{ \langle C_i, H \rangle + \langle c_i, h \rangle \}, i = 1, \dots, m. \quad (6.24)$$

From the matrix equation (6.23) it follows that  $H$  can be represented in the form

$$H = -A\nabla_A A - \sum_{j=1}^m \lambda_j AC_j A. \quad (6.25)$$

Treating  $\lambda_1, \dots, \lambda_m$  as unknowns, we can substitute (6.25) in (6.24) and obtain the following system of  $m + n$  linear equations in  $m + n$  unknowns  $\lambda, h$

$$\begin{aligned} \lambda_i = -\tau\Delta_i^{-2} \left[ \sum_{j=1}^m \langle C_i, AC_j A \rangle \lambda_j - \langle c_i, h \rangle + \langle C_i, A\nabla_A A \rangle \right], \quad i = 1, \dots, m, \\ \sum_{j=1}^m c_j \lambda_j = -\nabla_a, \end{aligned} \quad (6.26)$$

equivalent to the system (6.23). Since  $A$  is symmetric,

$$\langle C_i, A\nabla_A A \rangle = \text{tr}(c_i c_i^T A \nabla_A A) = \langle Ac_i, \nabla_A Ac_i \rangle$$

and

$$\langle C_i, AC_j A \rangle = \text{tr}(c_i c_i^T Ac_j c_j^T A) = \langle c_i, Ac_j \rangle^2.$$

Thus, letting

$$f_i = Ac_i, \quad i = 1, \dots, m, \quad (6.27)$$

(6.26) can be rewritten as

$$\begin{aligned} \lambda_i = -\tau\Delta_i^{-2} \left[ \sum_{j=1}^m \langle c_i, f_j \rangle^2 \lambda_j - \langle c_i, h \rangle + \langle f_i, \nabla_A f_i \rangle \right], \quad i = 1, \dots, m, \\ \sum_{j=1}^m c_j \lambda_j = -\nabla_a. \end{aligned} \quad (6.28)$$

Now it is easy to see that the vectors  $f_i, i = 1, \dots, m$  and all the coefficients of the system (6.28) can be computed in  $O(m^3)$  arithmetical operations. As  $n < m$ , the system itself can also be solved in  $O(m^3)$  operations. Having found the solution  $(\lambda^0, h^0)$  to the system (6.28), one can determine the  $A$ -component of the Newton direction by the formula (6.25)

$$H^0 = -A\nabla_A A - \sum_{j=1}^m \lambda_j^0 f_j f_j^T$$

in  $O(m^3)$  operations as well.

Remark For the case  $a \equiv 0, h \equiv 0$ , considered in [7], the system (6.28) turns into

$$\lambda_i = - \frac{\tau}{(1 - \langle c_i, f_i \rangle)^2} \left[ \sum_{j=1}^m \langle c_i, f_j \rangle^2 \lambda_j + \langle f_i, \nabla_A f_i \rangle \right], \quad i = 1, \dots, m,$$

which is simpler than the system derived in [7], see pp.175-177.

We have thus shown that the computational cost of one Newton iteration is  $O(m^3)$  arithmetical operations. Combining this fact with the bound (6.22) on the number of Newton iterations and with (6.1), we obtain the following result.

Theorem 2. Let  $Q = \{x \in \mathbb{R}^n \mid c_i^T x \leq 1, i = 1, \dots, m\}$  be a polytope that can be covered by an Euclidean ball of radius  $R$ , and let  $b \in \text{int } Q$ . To inscribe a  $\gamma$ -maximal paraboloid in the polyhedral cone  $K(b)$  defined by (5.6), it suffices to perform

$$O\left(m^{3.5} \ln\left[\frac{mR}{r(b) \ln \frac{1}{\gamma}}\right]\right) \quad (6.29)$$

arithmetical operations, where

$$\frac{1}{r(b)} = \max_{i=1, \dots, m} \frac{\|c_i\|}{1 - c_i^T b}. \quad (6.30)$$

Proof.  $\square$

Note that  $r(b)$  is the Euclidean distance from  $b$  to the boundary of  $Q$ . Observe also that in Theorem 2 we can replace the adjective “ $\gamma$ -maximal” by “ $\gamma^{1/3}$ -maximal”.

7. Bounding the complexity of determining a  $\gamma$ -maximal ellipsoid for a polytope.

**Theorem 3** Let  $Q = \{x \in \mathbb{R}^n \mid c_i x \leq 1\}$  be a polytope satisfying (2.6). To find a  $\gamma$ -maximal ellipsoid for  $Q$ , it suffices to perform

$$O(m^{3.5} \cdot \ln\left[\frac{mR}{\ln \frac{1}{\gamma}}\right] \cdot \ln\left[\frac{n \ln R}{\ln \frac{1}{\gamma}}\right]) \quad (7.1)$$

arithmetical operations.

**Proof.** Applying ElliP, we reduce the problem of finding a  $\gamma$ -maximal ellipsoid for  $Q$  to  $\kappa$  subproblems  $P(b_0), P(b_1), \dots, P(b_\kappa)$ , each of which is the problem of finding a  $\gamma^{1/3}$ -maximal paraboloid inscribed in the corresponding cone  $K(b_s)$ ,  $s = 0, 1, \dots, \kappa$ . Here

$$b_0 = 0, b_1, \dots, b_\kappa \in \text{int } Q, \quad \kappa \leq \log[2n \ln R / \ln \frac{1}{\gamma}]$$

is the sequence of points generated by ElliP. The condition (1.6) implies  $r(b_0) \geq 1$ , see (6.30), and by (6.29) the complexity of  $P(b_0)$  does not exceed

$$O(m^{3.5} \ln[mR / \ln \frac{1}{\gamma}]) \quad (7.2)$$

arithmetical operations. Now we have to show that the points  $b_1, \dots, b_\kappa$  do not come too close to the boundary of  $Q$ , i.e. to bound the quantities  $r(b_1), \dots, r(b_\kappa)$  from below. Since from the description of ElliP we know that  $b_{s+1} = \frac{1}{2}(b_s + a_s)$ ,  $a_s \in \text{int } Q$ , and the distance function  $r : \text{int } Q \rightarrow \mathbb{R}$  is concave and positive, we have the following trivial recurrence:

$$r(b_0) \geq 1, \quad r(b_{s+1}) > \frac{1}{2} r(b_s).$$

The number  $\kappa+1$  of iterations of ElliP is so small that even this recurrence yields an estimate

$$\min\{r(b_1), \dots, r(b_\kappa)\} > 2^{-\kappa} \geq \frac{\ln \frac{1}{\gamma}}{2n \ln R}, \quad (7.3)$$

sufficient for our needs. Indeed, substituting (7.3) in (6.29), we see that the upper bound (7.2) still applies to each of the problems  $P(b_1), \dots, P(b_\kappa)$ . Multiplying (7.2) by (4.8), we get (7.1).  $\square$

Applying Ellip in the method of inscribed ellipsoids, we also obtain from (4.9) and (1.2) the following

Corollary The complexity of the  $m$ -th iteration of the method of inscribed ellipsoids does not exceed  $O(m^{3.5} \cdot \ln m \cdot \ln \ln n)$  arithmetical operations.

Note that in the latter method one may assume without loss of generality that  $m = O(n \ln n)$ . This yields the bound  $O(n^{3.5 + \epsilon})$  arithmetical operations per iteration of the method of inscribed ellipsoids and improves the bound  $O(n^{4.5 + \epsilon})$  in [7].

## 8. Concluding remarks and questions

1) We believe that the Newton system (6.28) needs further investigation. In particular, is it possible to use “Karmarkar’s speed-up” to reduce the average iteration cost to  $O(m^{2.5})$  arithmetical operations?

2) Consider the following algorithm IC:

Step 0. Set

$$k := 0,$$

$$b_k := b_0 := \text{an arbitrary interior point of } Q.$$

Step 1. Find the minimal covering ellipsoid  $E_k$  for the set of points  $c_i/(1 - c_i^T b_k)$ ,  $i = 1, \dots, m$ .

Step 2. Update:

$b_{k+1} :=$  the center of the polar of  $E_k$ ,

$k := k + 1$ .

Go to 1 and start a new iteration.

Does the sequence of the polars  $E_0^o, E_1^o, \dots, E_k^o \subseteq Q$  converge to the maximal inscribed ellipsoid for  $Q$ ?

3) We conjecture that the following “polar” versions of the problems I and C are NP-hard:

- find a  $\gamma$ -minimal covering ellipsoid for a given polytope (2.1);

- find a  $\gamma$ -maximal inscribed ellipsoid for a given polytope (2.2).

It is known [4] that these problems are NP-hard in the case where we consider balls instead of ellipsoids.

4) Suppose that  $Q = \{x \in \mathbb{R}^n \mid c_i^T x \leq 1, i = 1, \dots, m\}$  is a rational polytope:  $c_1, \dots, c_m \in \mathbb{Z}^n$ . Can one obtain non-trivial bounds on the algebraic degrees of the entries of the maximal inscribed ellipsoid  $E^* = E^*(Q)$  as functions of  $n$ ?

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