COMPLETE SETS OF F(n, n/s) SQUARES FOR $s = p^k$

By

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ABSTRACT

It is shown that a complete set of F(n, n/s) exists for $n = s^r$ and $s = p^k$, p a prime number. A proof using construction of the complete set of F-squares is presented. The F-squares constructed are combinatorially orthogonal. Also, a method for constructing an orthogonal array from the constructed F-squares is given.

Key words: combinatorial orthogonality; sum-of-squares orthogonality; orthogonal array; prime power.

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INTRODUCTION

Many unsolved problems in discrete mathematics remain. One such problem is to determine the existence of complete sets of combinatorially orthogonal sets of F(n, n/s) squares. Federer (2003) has demonstrated the existence of sum-of-squares orthogonal F-squares for all values of n. However except for prime numbers, they are not combinatorially orthogonal. We show that complete sets of combinatorially orthogonal F(n, n/s) squares and their corresponding orthogonal arrays exist for $n = s^r$ and $s = p^k$, p a prime number. It is shown how to construct the set of F-squares and hence their existence.

COMPLETE SETS

An F-square of order n is an n-row × n-column square with s symbols and is denoted by F(n, n/s). Each of the s symbols occurs n/s times in each row and in each column of the square. Federer (2003) has defined this as a regular (or standard) F-square. A factorial arrangement of f factors with l levels of each of the factors is all combinations of the factor levels to make $N = l^f$ combinations. The N - 1 degrees of freedom for the N combinations are partitioned into degrees of freedom for main effects of the f factors and their interactions. These interactions are denoted as multiplicative interactions. For l a prime number or power of a prime number, each of the interactions may be partitioned into its geometrical interaction components. For example, a two factor multiplicative interaction A × B with $l = 5$ levels, say, is partitioned into AB, AB^2, AB^3, and AB^4. The

$(l - 1)^2 = 16$ degrees of freedom are partitioned into four sets of four degrees of freedom each. The levels of the geometrical interaction components may be used to construct F(16, 4) squares. When l is a prime number, the usual addition and multiplication operations hold and are taken modulo l. When l is a prime power, the calculus becomes somewhat more complicated in that the marks of a field along with their addition and multiplication tables must be used to obtain the combinations associated with each level. For example for $l = 4 = 2^2$, the marks of the field are 0, 1, x, and x + 1. The addition and multiplication tables are:

Addition

	0	1	x	x+1
0	0	1	x	x+1
1	1	0	x+1	x
x	x	x+1	0	1
x+1	x+1	x	1	0

Multiplication

	0	1	x	x+1
0	0	0	0	0
1	0	1	x	x+1
x	0	x	x+1	1
x+1	0	x+1	1	x

The above construction of F-squares produces a set of pairwise orthogonal F-squares which was called combinatorial orthogonality by Federer (2003). He also introduced the idea of sum-of-squares orthogonality. For this latter type of orthogonality, a multiplicative interaction is partitioned into geometrical components and the sum of the sums of squares for the set of geometrical interaction components adds to that for the multiplicative interaction. For l a prime number, the constructed F-squares have both types of orthogonality (see Federer, 2003).

Let $n = s^r$ and $s = p^k$ where p is a prime number. The following theorem holds.

Theorem: A complete set of combinatorially orthogonal $(s^r - 1)^2 / (s - 1)$ F(n, n/s) squares exists for $s = p^k$ symbols, p a prime number and $n = s^r$.

Proof: The proof is by construction. To illustrate, let $s = 4 = 2^2$ and let $n = 4^2 = 16$. Denote the n rows of the $n \times n$ square as levels of the two factors A and B each at four levels. The row numbers are the 16 combinations of the two factor factorial. Do likewise for the columns of the square for factors C and D. Then set up all the interactions in the row by column interaction. Each of the multiplicative interactions is partitioned into its geometrical components of interaction. This has been done in Table 1. There are $(s^2 - 1)^2 / (s - 1) = 15(15) / 3 = 225 / 3 = 75$ geometrical interaction components. These may be used to construct 75 F(16, 4) squares which are combinatorially and sum-of-squares orthogonal. In addition, $(4^2 - 1) / 3 = 5$ F(16, 4) squares may be constructed from the rows and 5 from the columns. A total of $75 + 5 + 5 = 85 = (4^4 - 1) / 3 = (s^{2r} - 1) / (s - 1)$ F(16, 4) squares are available to construct the orthogonal array OA(256, 4, 85, 2). This leads to the following corollary:

Corollary: An orthogonal array $OA(s^{2r}, s, (s^{2r} - 1)/(s - 1), 2)$ exists for $n = s^r$ and $s = p^k$, p a prime number.

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Obviously the general proof follows directly from the above method of constructing the complete set of F-squares from the multiplicative interactions in a factorial arrangement and their associated geometrical interaction components.

COMMENTS

It has been shown how to construct complete sets of combinatorially orthogonal F(n, n/s) squares for $n = s^r$ and $s = p^k$, p a prime number. Some open questions are as follows. How many combinatorially orthogonal F(4t, t) squares are there when 4t is not equal to s^r, say 12, 20, 24, 28, ….. ? Can one construct a complete set of orthogonal F(12, 3) squares? How about 8 and 16 symbols? These same questions pertain to values of p other than 2.

LITERATURE CITED

Federer, W. T. (2003). Complete sets of F-squares of order n. Utilitas Mathematica (to appear).

Table 1. Partitioning of the interaction degrees of freedom into geometrical components of the interactions for $s = 2^2$.

Source of variation	Degrees of freedom
Total	$256 = n^2 = s^4$
Correction for mean	1
Row	$15 = s^2 - 1$
A	$3 = s - 1$
B	$3 = s - 1$
A × B	$9 = (s - 1)^2$
Column	$15 = s^2 - 1$
C	$3 = s - 1$
D	$3 = s - 1$
C × D	$9 = (s - 1)^2$
Row × Column	$225 = (s^2 - 1)^2$
A × C	9
AC	3
AC^2	3
AC^3	3
A × D	9
AD	3
AD^2	3
AD^3	3
A × C × D	27

ACD		3
ACD^2		3
ACD^3		3
AC^2D		3
AC^2D^2		3
AC^2D^3		3
AC^3D		3
AC^3D^2		3
AC^3D^3		3
B × C	9	
BC		3
BC^2		3
BC^3		3
B × D	9	
BD		3
BD^2		3
BD^3		3
B × C × D	27	
BCD		3
BCD^2		3
BCD^3		3
BC^2D		3
BC^2D^2		3
BC^2D^3		3
BC^3D		3
BC^3D^2		3
BC^3D^3		3
A × B × C	27	
ABC		3
ABC^2		3
ABC^3		3
AB^2C		3
AB^2C^2		3
AB^2C^3		3
AB^3C		3
AB^3C^2		3
AB^3C^3		3
A × B × D	27	
ABD		3
ABD^2		3
ABD^3		3
AB^2D		3
AB^2D^2		3
AB^2D^3		3
AB^3D		3
AB^3D^2		3

AB^3D^3		3
$A \times B \times C \times D$	81	
$ABCD$		3
$ABCD^2$		3
$ABCD^3$		3
ABC^2D		3
ABC^2D^2		3
ABC^2D^3		3
ABC^3D		3
ABC^3D^2		3
ABC^3D^3		3
AB^2CD		3
AB^2CD^2		3
AB^2CD^3		3
AB^2C^2D		3
$AB^2C^2D^2$		3
$AB^2D^2D^3$		3
AB^2C^3D		3
$AB^2C^3D^2$		3
$AB^2C^3D^3$		3
AB^3CD		3
AB^3CD^2		3
AB^3CD^3		3
AB^3C^2D		3
$AB^3C^2D^2$		3
$AB^3C^2D^3$		3
AB^3C^3D		3
$AB^3C^3D^2$		3
$AB^3C^3D^3$		3