Some Convergence Results on Stable Infinite Moving Average Processes and Stable Self-Similar Processes

by Sami Umut Can

SOME CONVERGENCE RESULTS ON STABLE INFINITE MOVING AVERAGE PROCESSES AND STABLE SELF-SIMILAR PROCESSES

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Non-Gaussian stable stochastic models have attracted growing interest in recent years, due to their connections to limit theorems and due to empirical evidence pointing to heavier-than-Gaussian probability tails in many natural situations. We study the structure of two broad classes of stable stochastic processes through some convergence results.

In the first half of the thesis, we study the integrated periodogram for discrete-time infinite moving average processes with i.i.d. stable noise. We show that for such processes, a collection of weighted integrals of the periodogram, considered as a function-indexed stochastic process, converges weakly to a limit which can be represented as an infinite Fourier series with i.i.d. stable coefficients. The convergence works under certain assumptions on the Fourier coefficients of the index functions. We also extend the weak convergence results to stochastic volatility processes with stable noise, which are of interest in financial time series analysis.

In the second half, we describe a family of continuous-time stable processes with stationary increments that are asymptotically or exactly self-similar. We show that they arise naturally as a large time scale limit in a situation where many users perform independent random walks and collect heavy-tailed random rewards depending on their position on the integer line. We study various properties of the limiting process. This work generalizes an earlier construction by Cohen and Samorodnitsky (2006).

BIOGRAPHICAL SKETCH

Sami Umut Can was born on October 17, 1980 in Istanbul, Turkey. In 1999 he graduated from the Austrian Sankt Georgs-Kolleg in Istanbul and joined Cornell University in Ithaca, NY on a foreign student scholarship provided by the University. He received his B.A. degree in Mathematics in May 2003.

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Upon completion of his Ph.D. he will join EURANDOM, the European Institute for Statistics, Probability, Stochastic Operations Research and its Applications, in Eindhoven, the Netherlands as a postdoctoral fellow.

To my family.

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CHAPTER 1

INTRODUCTION

1.1 Stable Distributions

Stable distributions are those whose shapes are preserved under convolutions. That is, a random variable X is said to be *stable* if for any integer $n \geq 2$, there is a positive number c_n and a real number d_n such that

$$X_1 + X_2 + \ldots + X_n \stackrel{d}{=} c_n X + d_n \,, \tag{1.1}$$

where $X_1, X_2, \ldots, X_n$ are independent copies of X, and $\stackrel{d}{=}$ denotes equality in distribution. On this simple assumption rests a rich mathematical structure that has been increasingly studied and used for modeling over the last 80 years.

It is apparent from (1.1) that Gaussian distributions are special cases of stable distributions. Non-Gaussian stable laws have much more slowly decaying probability tails: for any non-Gaussian stable random variable X, there is a constant $0 < \alpha < 2$, called the *tail index* of X, such that

$$P(|X| > x) \sim cx^{-\alpha} \text{ as } x \to \infty \tag{1.2}$$

for some $c > 0$. Consequently, all non-Gaussian stable laws have infinite variance, and some have infinite absolute expectation as well. The lack of moments, as well as the lack of density formulas in all but a few cases, have historically made non-Gaussian stable distributions somewhat forbidding for many practitioners. Nevertheless, there are two compelling reasons to consider them in applications. The first reason is the so-called Generalized Central Limit Theorem (see, for example, §33 of Gnedenko and Kolmogorov (1954)), which states that stable distributions are the only distributions that can be obtained as limits of normalized sums of i.i.d. random variables. Since many natural quantities, such as the price of a stock or the noise in a communication system, can be thought of as the sum of many small terms, a stable model should be appropriate to describe such systems. The second reason to consider stable distributions in applications is that there is solid empirical and theoretical evidence pointing to heavier-than-Gaussian tails in many situations. Although stable distributions are by no means the only ones possessing heavy tails, in view of the Generalized Central Limit Theorem just mentioned, they are a natural choice for modeling heavy-tailed random phenomena. Examples in finance and economics are given in Mandelbrot (1963), Fama (1965), Samuelson (1967), Embrechts et al. (1997), Rachev and Mittnik (2000) and Sun et al. (2008). Examples in communication systems are given in Stuck and Kleiner (1974), Nikias and Shao (1995), Crovella and Bestavros (1996) and Willinger et al. (1997). The monographs by Zolotarev (1986), Uchaikin and Zolotarev (1999) and Nolan (2010) list a number of other fields, such as physics, geology, computer science, biology, and medicine, where stable models have been used to describe a large variety of naturally occurring systems.

As a historical note, stable laws were first characterized and studied by Paul Lévy and Aleksandr Khinchine in the 1920s and 1930s; see for example Lévy (1924), Lévy (1925) and Lévy and Khinchine (1936). Classical references on the subject are the monographs by Gnedenko and Kolmogorov (1954) and Feller (1971). More recent and oft-cited treatments include Zolotarev (1986) and Samorodnitsky and Taqqu (1994). Stable laws are special cases of infinitely divisible distributions, which are covered in detail in Sato (1999).

The general univariate stable distribution is characterized by four parameters:

an index of stability $\alpha \in (0, 2]$, which coincides with the tail index α in (1.2) for $\alpha < 2$, a scale parameter $\sigma > 0$, a skewness parameter $\beta \in [-1, 1]$, and a shift parameter $\mu \in \mathbb{R}$. The customary notation for a generic stable distribution is $S_\alpha(\sigma, \beta, \mu)$. The characteristic function of a $S_\alpha(\sigma, \beta, \mu)$ random variable X is given by

$$E(e^{i\theta X}) = \begin{cases} \exp\left\{i\mu\theta - \sigma^\alpha |\theta|^\alpha \left(1 - i\beta \operatorname{sgn}(\theta) \tan \frac{\pi\alpha}{2}\right)\right\} & \text{if } \alpha \neq 1, \\ \exp\left\{i\mu\theta - \sigma|\theta| \left(1 + i\beta \frac{2}{\pi} \operatorname{sgn}(\theta) \log |\theta|\right)\right\} & \text{if } \alpha = 1, \end{cases} \tag{1.3}$$

where sgn denotes the sign function. In this dissertation, we will restrict ourselves to *symmetric* stable distributions, for which $\beta = \mu = 0$. In that case, the characteristic function takes the particularly simple form

$$E(e^{i\theta X}) = e^{-\sigma^\alpha |\theta|^\alpha}, \tag{1.4}$$

which reduces to a centered Gaussian distribution when $\alpha = 2$. A symmetric stable random variable with index of stability α is usually called *symmetric α-stable*, or SαS for short. As we have just observed, S2S is the same as centered Gaussian. A stochastic process $(X(t), t \in T)$ with an arbitrary index set T is called SαS if it has jointly SαS finite dimensional distributions, which is equivalent to the condition that all linear combinations

$$\sum_{j=1}^{k} a_j X(t_j), \quad t_1, \ldots, t_k \in T, \; a_1, \ldots, a_k \in \mathbb{R}$$

are SαS. (Note that, in general, a random vector is *not* necessarily stable even if all linear combinations of its components are univariate stable. However, a random vector is *symmetric* stable if and only if all linear combinations of its components are symmetric stable. See Chapter 2 of Samorodnitsky and Taqqu (1994) for more information.)

In this dissertation, we investigate the structure of two broad classes of stable processes, both of great theoretical and practical importance. One is the class of discrete-time *linear processes with stable innovations*, and the other is that of continuous-time *stable self-similar processes with stationary increments*. In the following two sections, we review some important facts about these classes that are relevant for our discussion.

1.2 Linear Processes with Stable Innovations and the Integrated Periodogram

Discrete-time linear processes of the form

$$X_t = \sum_{j=-\infty}^{\infty} \psi_j \varepsilon_{t-j}, \quad t \in \mathbb{Z}, \tag{1.5}$$

are frequently used for modeling empirical time series. Here, $(\varepsilon_j, j \in \mathbb{Z})$ are i.i.d. random variables called *innovations* or *noise*, and $(\psi_j, j \in \mathbb{Z})$ are constant coefficients called a *linear filter*. Processes of this type are also called (doubly) infinite moving average process. In practical situations, one often considers so-called *causal* representations in (1.5), i.e. $\psi_j = 0$ for $j < 0$, so that the value of X_t does not depend on $(\varepsilon_j, j > t)$. We impose no such restriction. Note that, since the noise terms $(\varepsilon_j, j \in \mathbb{Z})$ are assumed to be i.i.d., the linear process $(X_t, t \in \mathbb{Z})$ is a stationary process, i.e. its finite dimensional distributions are invariant under shifts of the time index.

Naturally, the linear filter $(\psi_j, j \in \mathbb{Z})$ has to satisfy certain conditions, depending on the noise distribution, for the series in (1.5) to converge and the linear process to be well defined. If the noise terms $(\varepsilon_j, j \in \mathbb{Z})$ are assumed to have zero mean and finite variance, as is usually the case in the classical time series literature, a sufficient and necessary condition for well-definedness (in the sense of almost sure convergence in (1.5) for any fixed t) is

$$\sum_{j=-\infty}^{\infty} \psi_j^2 < \infty, \tag{1.6}$$

by virtue of the three-series theorem (Theorem 22.8 in Billingsley (1995)).

In this dissertation, we will consider linear processes with SαS noise terms, which are better suited to describe empirical data that exhibit heavy tails. For such processes, a necessary and sufficient condition for almost sure convergence in (1.5) is

$$\sum_{j=-\infty}^{\infty} |\psi_j|^\alpha < \infty, \tag{1.7}$$

again by the three-series theorem. For an overview of linear processes with infinite variance noise terms, we refer to §13.3 of Brockwell and Davis (1991), §7.12 of Samorodnitsky and Taqqu (1994) and Chapter 7 of Embrechts et al. (1997). For a partial list of applications in economics and engineering, see Davis and Resnick (1986).

Classical (i.e. finite variance) time series analysis often deals with the second (or higher) moment structure of a stationary sequence through the study of its autocovariance and autocorrelation functions in the time domain, and its spectral distribution function in the frequency domain. As natural estimators of these deterministic quantities, the sample autocovariance, the sample autocorrelation and the periodogram (more about it below) have been intensely studied in the classical time series literature, and many efforts have been made to describe their asymptotic behavior as the number of observations increases, with statistical applications in mind. The asymptotic theory of these estimators and their various modifications in the finite variance case can be found in any standard reference on the subject; see, for example, Priestley (1981), Grenander and Rosenblatt (1984) or Brockwell and Davis (1991).

When the marginal distributions of a stationary time series have infinite variance, as is the case with linear processes with SαS noise, the notions of autocovariance, autocorrelation and spectral distribution are not applicable anymore. Nevertheless, one can still study the asymptotic behavior of the corresponding sample statistics, which are perfectly well-defined random objects, in the hope of gaining some insight into the statistical structure of the underlying process and constructing useful statistical tests. Various studies over the last 20 years have shown that the analysis of linear processes with heavy-tailed innovations is very similar to the classical time series analysis in this respect, and by now an asymptotic theory exists for the heavy-tailed case that parallels the classical theory. In contrast to the latter theory, the limits in the heavy-tailed case involve infinite variance stable distributions and processes rather than Gaussian ones. Results on the asymptotic theory for sample autocovariances and sample autocorrelations in the heavy-tailed situation can be found in Davis and Resnick (1985a,b, 1986). Helpful summaries of these and related results can be found in §13.3 of Brockwell and Davis (1991) and Chapter 7 of Embrechts et al. (1997). Spectral estimates in the heavy-tailed case are studied in Klüppelberg and Mikosch (1996a,b) and Mikosch (1998), and it is to spectral estimates, in particular the periodogram, that we now turn our attention.

One of the main goals of classical time series analysis is the study of the spectral properties of the underlying series under the assumption of finite variance of the marginal distributions. In this context, the *periodogram* mentioned above plays a prominent role as an estimator of spectral density. It is defined as

$$I_{n,X}(\lambda) = \left| \frac{1}{\sqrt{n}} \sum_{t=1}^{n} e^{-i\lambda t} X_t \right|^2, \quad \lambda \in [0, \pi]. \tag{1.8}$$

Numerous estimation and test procedures are based on this statistic; see Chapter

4 of Priestley (1981) and Chapter 10 of Brockwell and Davis (1991). In particular, integrated versions of the periodogram of the form

$$J_{n,X}(f) = \int_0^\pi I_{n,X}(\lambda) f(\lambda) d\lambda \tag{1.9}$$

for appropriate classes of real-valued functions $f \in \mathcal{F}$ on $[0,\pi]$ are used for a multitude of applications. We mention a few of them.

We start with the class of the indicator functions

$$\mathcal{F}_I = \left\{ \mathbf{1}_{[0,x]} : x \in [0,\pi] \right\}.$$

In this case, we consider the integrated periodogram

$$J_{n,X}(\mathbf{1}_{[0,x]}) = \int_0^x I_{n,X}(\lambda) d\lambda, \quad x \in [0,\pi],$$

which is a process indexed by $x \in [0,\pi]$. Under the assumption of finite fourth moments for the i.i.d. noise terms and a summability condition slightly stronger than (1.6) for the linear filter, this type of process converges uniformly with probability 1 to the function

$$\sigma_\varepsilon^2 \int_0^x \left|\psi(e^{-i\lambda})\right|^2 d\lambda, \quad x \in [0,\pi],$$

where σ_ε^2 is the variance of the noise terms,

$$\psi(e^{-i\lambda}) = \sum_{j=-\infty}^{\infty} \psi_j e^{-i\lambda j}, \quad \lambda \in [0,\pi], \tag{1.10}$$

is the *transfer function* of the linear filter $(\psi_j, j \in \mathbb{Z})$, and $\left|\psi(e^{-i\lambda})\right|^2$ is the corresponding *power transfer function*; see Mikosch and Norvaiša (1997). The transfer function is one of the essential building blocks of the *spectral density* of the stationary process $(X_t, t \in \mathbb{Z})$:

$$f_X(\lambda) = \frac{\sigma_\varepsilon^2}{2\pi} \left|\psi(e^{-i\lambda})\right|^2 = \frac{1}{2\pi} \sum_{h=-\infty}^{\infty} e^{-ih\lambda} \gamma_X(h), \quad \lambda \in [0,\pi].$$

In words, the spectral density is the Fourier series based on the *autocovariance function*

$$\gamma_X(h) = \mathrm{Cov}(X_0, X_h) = \sigma_\varepsilon^2 \sum_{j=-\infty}^{\infty} \psi_j \, \psi_{j+|h|}, \quad h \in \mathbb{Z}.$$

Since $J_{n,X}(\mathbf{1}_{[0,\cdot]})$ estimates the spectral distribution function of the stationary process $(X_t, t \in \mathbb{Z})$, it has been used for a long time as the *empirical spectral distribution function*, both as an estimator and as a basic tool for constructing goodness-of-fit tests for the underlying spectral distribution function. The theory is presented in detail in Grenander and Rosenblatt (1984); see also Brockwell and Davis (1991) and Priestley (1981).

Since the limit process of the properly centered and normalized process $J_{n,X}(\mathbf{1}_{[0,\cdot]})$ depends on the (in general unknown) spectral density f_X, Bartlett (1954) proposed to consider $(J_{n,X}(f), f \in \mathcal{F}_B)$, where

$$\mathcal{F}_B = \{\mathbf{1}_{[0,x]}/f_X : x \in [0, \pi]\},$$

i.e., he considered the process

$$J_{n,X}(\mathbf{1}_{[0,x]}/f_X) = \int_0^x \frac{I_{n,X}(\lambda)}{f_X(\lambda)} \, d\lambda, \quad x \in [0, \pi] \, .$$

Under the assumption of finite fourth moments for the noise and suitable summability conditions for the linear filter, this process converges uniformly with probability 1 to the function $f(x) \equiv x$. More generally, weighted integrated periodograms of the form

$$J_{n,X}(\mathbf{1}_{[0,x]} \, g) = \int_0^x I_{n,X}(\lambda) g(\lambda) d\lambda, \quad x \in [0, \pi]$$

are used to estimate the spectral density or to perform various tests about the spectrum of the underlying stationary sequence. A general reference on the integrated periodogram and its weighted versions as well as on statistical applications is Chapter 6 of Priestley (1981).

The weighted integrated periodogram is also the basis for one of the classical estimators for fitting ARMA and fractional ARIMA models. This method goes back to early work by Whittle (1951). In this context one considers the functional

$$J_{n,X}(1/f_X(\cdot;\theta)) = \int_0^\pi \frac{I_{n,X}(\lambda)}{f_X(\lambda;\theta)} d\lambda, \quad f_X(\cdot;\theta) \in \mathcal{F}_W,$$

where $\mathcal{F}_W$ is a class of spectral densities indexed by a parameter $\theta \in \Theta \subset \mathbb{R}^d$. The *Whittle estimator* $\widehat{\theta}_n$ of the true parameter $\theta_0 \subset \Theta$ is the minimizer of $J_{n,X}(1/f_X(\cdot;\theta))$ over the parameter set Θ, or over a compact subset of it. This kind of estimation technique is one of the backbones of quasi-maximum likelihood estimation in parametric time series modeling. The so-defined estimator is known to be asymptotically equivalent to the corresponding least squares and Gaussian quasi-maximum likelihood estimators. Equivalence means that the estimator is consistent and asymptotically normal with the same $\sqrt{n}$-rate and asymptotic variance as in the other two cases. A general reference on parameter estimation in ARMA models is Chapter 8 in Brockwell and Davis (1991). When proving the asymptotic normality and consistency of $\widehat{\theta}_n$, one has to study the properties of the sequence $(J_{n,X}(1/f_X(\cdot;\widehat{\theta}_n)))$ which can be considered as weighted integrated periodogram indexed by a class of functions.

The above examples have in common that one always considers a function-indexed stochastic process $(J_{n,X}(f), f \in \mathcal{F})$ for some class $\mathcal{F}$ of functions. In all cases one is interested in the asymptotic behavior of the process $J_{n,X}$, uniformly over the class $\mathcal{F}$. This is analogous to the case of the empirical distribution function indexed by classes of functions. General references in this context are the monographs Pollard (1984) and van der Vaart and Wellner (1996). Early on, this analogy was discovered by Dahlhaus (1988) who gave some uniform convergence theory for $J_{n,X}$ under entropy and exponential moment conditions. The almost sure and weak convergence theory under entropy and power moment conditions was given in Mikosch and Norvaiša (1997). A recent survey of non-parametric statistical methods related to the empirical spectral distribution indexed by classes of functions is Dahlhaus and Polonik (2002).

In Chapter 2 of this dissertation, we aim to give an analogous uniform convergence theory for linear processes $(X_t, t \in \mathbb{Z})$ with i.i.d. SαS innovations. The hope is that this theory can then be used to construct useful statistical estimators or tests about various spectral characteristics of the underlying process, similar to the examples cited above. Although it seems feasible that our theory can be extended to the more general class of linear processes whose noise variables have regularly varying probability tails, we do not attempt to achieve this goal. The price would be more technicalities, the gain would be incremental. We will show how the classical (finite variance) tools and methods have to be modified in the infinite variance stable situation, which can be considered as a boundary case of the classical one when some of the innovations assume extremely large values.

We will also extend our results to *stochastic volatility processes* $(X_t,\ t \in \mathbb{Z})$ of the form

$$X_t = \sigma_t \varepsilon_t, \quad t \in \mathbb{Z}, \tag{1.11}$$

where the *volatility sequence* $(\sigma_t, t \in \mathbb{Z})$ is a strictly stationary non-negative process independent of the i.i.d. multiplicative noise sequence $(\varepsilon_t, t \in \mathbb{Z})$. For our purposes, the noise will be a sequence of i.i.d. SαS random variables, and the logarithm of the volatility sequence will be a linear Gaussian process, as is common in the literature. That is, we will assume that

$$\log \sigma_t = \sum_{j=-\infty}^{\infty} c_j \, \eta_{t-j} \,, \quad t \in \mathbb{Z} \,,$$

where $(c_j, j \in \mathbb{Z})$ is a sequence of real numbers satisfying $\sum_j c_j^2 < \infty$ and $(\eta_j, j \in \mathbb{Z})$ is an i.i.d. standard normal sequence. Stochastic volatility models are standard in financial time series analysis; see, for example, Shephard (2005) and Andersen et al. (2009).

1.3 Stable Self-Similar Processes with Stationary Increments

Self-similar processes are stochastic processes that are invariant in finite-dimensional distributions under suitable scaling of time and space. More precisely, a real-valued stochastic process $(X(t), t \in T)$, where T is either $\mathbb{R}$ or $\mathbb{R}_+ = [0, \infty)$, is called self-similar if for any $c > 0$,

$$(X(ct), t \in T) \stackrel{d}{=} (c^H X(t), t \in T) \tag{1.12}$$

for some constant $H > 0$. Here, $\stackrel{d}{=}$ denotes equality in finite-dimensional distributions. Lamperti (1962) showed that c^H is the only possible form for the scaling factor on the right-hand side of (1.12), assuming $(X(t), t \in T)$ is a non-trivial process that is stochastically continuous at 0. H is called the *index of self-similarity* of the process $(X(t), t \in T)$. A self-similar process with index H is called H-self-similar, or H-ss for short.

The study of self-similar processes is motivated by empirical and theoretical considerations. Aspects of self-similarity appear in fields as diverse as hydrology (Mandelbrot and Wallis (1968)), geophysics (Mandelbrot and Wallis (1969)), turbulence (Mandelbrot (1974)), finance (Cont (2005)), risk theory (Michna (1998)) and communication networks (Leland et al. (1994)), among others. The main theoretical justification for approximate self-similarity in natural situations is provided by the limit theorem due to Lamperti (1962): self-similar processes are the only possible limits that can arise in limiting procedures of the form

$$\lim_{c \to \infty} \left(\frac{1}{f(c)} X(ct), t \in T \right), \tag{1.13}$$

where the limit is understood to be in finite-dimensional distributions, $(X(t), t \in T)$ is a stochastic process and f is a real-valued function satisfying $\lim_{c\to\infty} f(c) = \infty$. We refer to Embrechts and Maejima (2002) for an excellent introduction to the general theory of self-similar processes, and to Taqqu (1986) and Willinger et al. (1996) for comprehensive bibliographical guides to many applications.

In practice, self-similar processes are often used as continuous-time models for deviations from the mean of a cumulative input system in steady state, hence self-similar processes with stationary increments have attracted particular interest. Recall that a real-valued process $(X(t), t \in T)$ has *stationary increments* if

$$(X(t+h) - X(h), t \in T) \stackrel{d}{=} (X(t) - X(0), t \in T), \text{ for all } h \in T.$$

An H-self-similar process with stationary increments is usually abbreviated as H-sssi. Fractional Brownian motions, first introduced in Kolmogorov (1940) and considered in many applications ever since, are perhaps the best known examples of such processes. They are Gaussian H-sssi with $0 < H \leq 1$, the case $H = 1/2$ corresponding to the usual Brownian motion and the case $H = 1$ corresponding to the straight line process with a random (Gaussian) slope. It turns out that fractional Brownian motions are the *only* Gaussian sssi processes, up to multiplicative constants (see, for example, Corollary 7.2.3 of Samorodnitsky and Taqqu (1994)).

In this dissertation, we will consider SαS sssi processes, which are commonly used as models for phenomena exhibiting both self-similarity and heavy tails. Chapter 7 in Samorodnitsky and Taqqu (1994) provides a good exposition on the subject; we refer to the bibliographical guides cited earlier for examples of applications. Maejima (1986) has shown that for a non-trivial SαS H-sssi process with $0 < \alpha \leq 2$, the range of possible values for the exponent of self-similarity is restricted to $0 < H \leq \max(1, 1/\alpha)$. In a significant departure from the Gaussian case, where the exponent of self-similarity determines the law of the sssi process (up to a multiplicative constant), there are generally many different SαS sssi processes for any given feasible pair (α, H) with $0 < \alpha < 2$. The only exception is the case $0 < \alpha < 1, H = 1/\alpha$, which corresponds to a single process, namely the SαS Lévy motion; see Samorodnitsky and Taqqu (1990).

SαS Lévy motions are the heavy-tailed equivalents of the Brownian motion: they are self-similar processes with stationary and independent increments. In light of the Generalized Central Limit Theorem mentioned earlier, it is not surprising that such processes arise as weak limits of normalized partial sums of i.i.d. random variables; see, for example, Corollary 7.1 of Resnick (2007). This makes them ideal approximating models for a number of natural situations; see Barndorff-Nielsen et al. (2001) for examples. For greater flexibility in modeling, efforts have been made over the last few decades to construct SαS sssi processes that do not possess independent increments, and to discover limit theorems that show how such processes could arise naturally as limits of stationary sequences of random variables under scaling and normalizing. The most widely known processes in this context are the *linear fractional stable motion* introduced in Taqqu and Wolpert (1983), Maejima (1983) and Kasahara and Maejima (1988), and the *real harmonizable fractional stable motion* introduced in Cambanis and Maejima (1989). Both processes are defined for $0 < \alpha \leq 2, 0 < H < 1$, and both reduce to the fractional Brownian motion in the case $\alpha = 2$.

An important difference between linear and real harmonizable fractional motions (in the case $0 < \alpha < 2$) is that the increments of the first process form a short-memory sequence, in the sense that they are generated by a *dissipative flow*, while the increments of the latter process form an infinite-memory sequence, in the sense that they are generated by a *positive flow*; see Rosiński (1995). The connection between memory properties of stationary SαS sequences (as observed in the asymptotic behavior of the sequence of partial maxima) and the ergodic theory of nonsingular flows is explained in Samorodnitsky (2004, 2005).

In Cohen and Samorodnitsky (2006), the authors constructed a new class of continuous-time SαS sssi processes for which the increment process is generated by a *conservative null flow* and hence can be regarded as having a finite but long memory. The construction is based on the local time process of a fractional Brownian motion with index of self-similarity H, so the authors called their model the *FBM-H-local time fractional stable motion*. They also showed that, in the case $H = 1/2$, this model arises naturally as a limiting process in a situation where many "users" perform independent symmetric random walks on distinct copies of the integer line and collect i.i.d. heavy-tailed random "rewards" associated with the integers that they visit. As the number of users increases, the properly normalized and time-scaled total reward process of all users converges weakly to the FBM-1/2-local time fractional stable motion (which can also be called the BM-local time fractional stable motion). The Brownian local time appearing in the limiting model can be regarded heuristically as a replacement for the local times of the random walks.

In Chapter 3 of this dissertation, we extend the construction of Cohen and Samorodnitsky (2006) for the case $H = 1/2$, by considering a general *continuous additive functional* of Brownian motion instead of the Brownian local time. Following the authors' terminology, this model can be called the *BM-CAF fractional stable motion*, where CAF stands for continuous additive functional. CAFs of Brownian motion can be thought of as generalizations of the local time concept, since they include the local time as a special case. In fact, every Brownian CAF is a unique mixture of local times at different levels along $\mathbb{R}$, in a sense that will be made precise. This suggests that the BM-CAF fractional stable motion will be similar in structure to the BM-local time fractional motion, and in particular, it will be a natural approximating model for a generalized version of the random rewards scheme described in Cohen and Samorodnitsky (2006). Our aim is to show that this is indeed the case. We will formally introduce the BM-CAF fractional stable motion, explore its similarities and differences with the BM-local time stable motion, and prove that it is a limiting model in a situation where many independent users collect *moving averages* of i.i.d. heavy-tailed random rewards associated with the nodes around them.

CHAPTER 2

WEAK CONVERGENCE OF THE INTEGRATED PERIODOGRAM FOR INFINITE VARIANCE PROCESSES

2.1 Introduction

This chapter discusses the weak convergence of the function-indexed integrated periodogram (1.9) for linear and stochastic volatility processes with SαS noise. Section 2.2 briefly reviews some preliminaries on the periodogram. Section 2.3 proves a weak convergence result for the integrated periodogram of an i.i.d. sequence of SαS random variables, under different assumptions for the cases $\alpha \in (0,1)$ and $\alpha \in [1,2)$. The results of Section 2.3 are extended to linear process with SαS innovations in Section 2.4, and to stochastic volatility processes with SαS innovations in Section 2.5. Finally, Section 2.6 presents two technical lemmas that are used in the proofs of the earlier sections.

2.2 Preliminaries on the Periodogram

Recall the definition (1.8) of the periodogram $I_{n,X}(\lambda)$, $\lambda \in [0,\pi]$. Note that

$$
\begin{aligned}
I_{n,X}(\lambda) &= \frac{1}{n}\left|\sum_{t=1}^{n}\cos(\lambda t)X_t - i\sum_{t=1}^{n}\sin(\lambda t)X_t\right|^2 \\
&= \frac{1}{n}\left(\sum_{t=1}^{n}\cos(\lambda t)X_t\right)^2 + \frac{1}{n}\left(\sum_{t=1}^{n}\sin(\lambda t)X_t\right)^2,
\end{aligned}
$$

which yields the following fundamental decomposition:

$$
I_{n,X}(\lambda) = \gamma_{n,X}(0) + 2\sum_{h=1}^{n-1}\cos(\lambda h)\gamma_{n,X}(h), \tag{2.1}
$$

where

$$\gamma_{n,X}(h) = \frac{1}{n} \sum_{t=1}^{n-|h|} X_t X_{t+h}, \quad |h| \le n-1,$$

denotes the *sample autocovariance function* of the sample $X_1, \ldots, X_n$.

In what follows, we will frequently make use of the *self-normalized periodogram*

$$\widetilde{I}_{n,X}(\lambda) = \frac{I_{n,X}(\lambda)}{\gamma_{n,X}(0)} = \rho_{n,X}(0) + 2 \sum_{h=1}^{n-1} \cos(\lambda h) \rho_{n,X}(h),$$

where

$$\rho_{n,X}(h) = \frac{\gamma_{n,X}(h)}{\gamma_{n,X}(0)}, \quad |h| \le n-1,$$

denotes the *sample autocorrelation function* of $X_1, \ldots, X_n$.

In view of (2.1) we can rewrite the integrated periodogram $J_{n,X}(f)$ in (1.9) as

$$J_{n,X}(f) = \gamma_{n,X}(0) a_0(f) + 2 \sum_{h=1}^{n-1} a_h(f) \gamma_{n,X}(h), \tag{2.2}$$

where

$$a_h(f) = \int_0^\pi \cos(\lambda h) f(\lambda)\, d\lambda, \quad h \in \mathbb{Z}, \tag{2.3}$$

are the *Fourier coefficients* of f. We also introduce the self-normalized version of $J_{n,X}$:

$$\widetilde{J}_{n,X}(f) = \rho_{n,X}(0) a_0(f) + 2 \sum_{h=1}^{n-1} a_h(f) \rho_{n,X}(h). \tag{2.4}$$

2.3 The i.i.d. Case

In this section we study the limit behavior of the integrated periodogram $J_{n,\varepsilon}$ indexed by classes of functions for an i.i.d. $S_\alpha(1,0,0)$ sequence $(\varepsilon_t, t \in \mathbb{Z})$ with $\alpha \in (0,2)$. In Section 2.3.1 we consider the convergence of the finite-dimensional distributions. In Sections 2.3.2 and 2.3.3 we prove the tightness of the processes in the cases $\alpha \in (0,1)$ and $\alpha \in [1,2)$, respectively, which allows us to conclude weak convergence. In the case $\alpha \in (0,1)$ we solve a more general weak convergence problem for random quadratic forms in the i.i.d. sequence $(\varepsilon_t, t \in \mathbb{Z})$; the convergence of the integrated periodogram indexed by classes of functions is only a special case. The case $\alpha \in [1,2)$ is more involved. Among others, entropy conditions will be needed, and we only prove results on the weak convergence of the integrated periodogram, i.e., we focus on random quadratic forms with Töplitz coefficient matrices given by the Fourier coefficients $a_h(f)$ defined in (2.3).

2.3.1 Convergence of the Finite-Dimensional Distributions

A glance at decomposition (2.2) convinces one that the convergence of the finite-dimensional distributions of $J_{n,\varepsilon}$ is essentially determined by the weak limit behavior of the sample autocovariances $\gamma_{n,\varepsilon}(h)$. For this reason we recall a well known result due to Davis and Resnick (1986); see also §13.3 of Brockwell and Davis (1991).

Lemma 2.3.1. *For every $m \geq 1$,*

$$\left(\frac{n\,\gamma_{n,\varepsilon}(0)}{n^{2/\alpha}}, \frac{n\gamma_{n,\varepsilon}(h)}{(n \log n)^{1/\alpha}}, h = 1, \ldots, m \right) \Longrightarrow (Y_0, Y_1, \ldots, Y_m)\,, \qquad (2.5)$$

where $\Longrightarrow$ denotes weak convergence, the Y_h's are independent, Y_0 is $S_{\alpha/2}(\sigma_1, 1, 0)$ and $(Y_h, h = 1, \ldots, m)$ are i.i.d. $S_\alpha(\sigma_2, 0, 0)$ for some $\sigma_i = \sigma_i(\alpha)$, $i = 1, 2$. In particular,

$$\left((n/\log n)^{1/\alpha} \rho_{n,\varepsilon}(h), h = 1, \ldots, m\right) \Longrightarrow (Y_h/Y_0, h = 1, \ldots, m)\ . \qquad (2.6)$$

The latter result is an immediate consequence of (2.5) and the continuous mapping theorem. Lemma 2.3.1 yields the weak convergence for any finite linear combination of the sample autocovariances and autocorrelations. It also suggests that the weak limit of the standardized process $J_{n,\varepsilon}(f)$ will be determined by the infinite series $\sum_{h=1}^{\infty} a_h(f) Y_h$. But this also means that we need to require additional assumptions on the sequence $(a_h(f), h = 1, 2, \ldots)$.

We will treat this problem in a more general context. Consider a sequence of real numbers

$$\mathbf{a} \in \ell^\alpha := \left\{ (a_1, a_2, \ldots) : \sum_{k=1}^{\infty} |a_k|^\alpha < \infty \right\}.$$

For such an $\mathbf{a}$ we define the sequences of processes

$$\begin{aligned} X_n(\mathbf{a}) &= (n \log n)^{-1/\alpha} \sum_{k=1}^{n-1} a_k \, n\gamma_{n,\varepsilon}(k), \quad Y(\mathbf{a}) = \sum_{k=1}^{\infty} a_k \, Y_k \,, \\ \widetilde{X}_n(\mathbf{a}) &= (n / \log n)^{1/\alpha} \sum_{k=1}^{n-1} a_k \, \rho_{n,\varepsilon}(k), \quad \widetilde{Y}(\mathbf{a}) = Y(\mathbf{a}) / Y_0 \,. \end{aligned} \tag{2.7}$$

Here $Y_0, Y_1, Y_2, \ldots$ are independent stable random variables as described in Lemma 2.3.1. The three-series theorem (Theorem 22.8 in Billingsley (1995)) implies that $\mathbf{a} \in \ell^\alpha$ is equivalent to the a.s. convergence of the infinite series $Y(\mathbf{a})$ in (2.7). However, for the weak convergence of $X_n(\mathbf{a})$ and $\widetilde{X}_n(\mathbf{a})$ we need a slightly stronger assumption:

$$\mathbf{a} \in \ell^\alpha \log \ell := \left\{ (a_1, a_2, \ldots) \in \ell^\alpha : \sum_{k=1}^{\infty} |a_k|^\alpha \log^+ \frac{1}{|a_k|} < \infty \right\},$$

where $\log^+(\cdot) = \max\{0, \log(\cdot)\}$. This assumption ensures convergence in finite-dimensional distributions of the random quadratic forms in (2.7); see Theorem 2.3.2 below. Assumptions of this type frequently occur in the literature on infinite variance quadratic forms; see, for example, Kwapień and Woyczyński (1992). They appear in a natural way in tail estimates for quadratic forms in i.i.d. stable random variables; see Section 2.6.

Now we can formulate our result about the convergence of the finite-dimensional distributions:

Theorem 2.3.2. *For any $\alpha \in (0,2)$,*

$$(X_n(\mathbf{a}), \mathbf{a} \in \ell^\alpha \log \ell) \xrightarrow{f.d.} (Y(\mathbf{a}), \mathbf{a} \in \ell^\alpha \log \ell),$$
$$(\widetilde{X}_n(\mathbf{a}), \mathbf{a} \in \ell^\alpha \log \ell) \xrightarrow{f.d.} (\widetilde{Y}(\mathbf{a}), \mathbf{a} \in \ell^\alpha \log \ell),$$

where $\xrightarrow{f.d.}$ denotes convergence in finite-dimensional distributions.

Proof. Using a Cramér-Wold argument (see §29 of Billingsley (1995)), it will suffice to prove the convergence of one-dimensional distributions. So let $\mathbf{a} = (a_1, a_2, \ldots) \in \ell^\alpha \log \ell$. From (2.5) and the continuous mapping theorem it immediately follows that for every $m \geq 1$,

$$(n \log n)^{-1/\alpha} \sum_{k=1}^{m} a_k \, n\gamma_{n,\varepsilon}(k) \Longrightarrow Y_m(\mathbf{a}) := \sum_{k=1}^{m} a_k \, Y_k \,, \tag{2.8}$$

where $\Longrightarrow$ denotes weak convergence. Also, since $\mathbf{a} \in \ell^\alpha$,

$$Y_m(\mathbf{a}) \Longrightarrow Y(\mathbf{a}) \text{ as } m \longrightarrow \infty,$$

by the three-series theorem. According to Theorem 4.2 in Billingsley (1968), it remains to show that

$$\lim_{m\to\infty} \limsup_{n\to\infty} P\left((n \log n)^{-1/\alpha} \left| \sum_{k=m+1}^{n-1} a_k \, n\gamma_{n,\varepsilon}(k) \right| > \epsilon \right) = 0 \tag{2.9}$$

for every $\epsilon > 0$. We write $p_{n,m}(\mathbf{a};\epsilon)$ for the above probabilities. Note that

$$p_{n,m}(\mathbf{a};\epsilon) = P\left(\left| \sum_{k=m+1}^{n-1} a_k \sum_{j=1}^{n-k} \varepsilon_j \varepsilon_{j+k} \right| > \epsilon (n \log n)^{1/\alpha} \right).$$

Applying Lemma 2.6.1 and the elementary inequality

$$1 + \log^+(ab) \leq (1 + \log^+ a)(1 + \log^+ b), \quad a, b > 0, \tag{2.10}$$

we conclude that

$$\begin{aligned} p_{n,m}(\mathbf{a};\epsilon) &\leq const \, \frac{1 + \log^+ \epsilon}{\epsilon^\alpha} \, \frac{1 + \log n}{n \log n} \sum_{k=m+1}^{n-1} \sum_{j=1}^{n-k} |a_k|^\alpha \left(1 + \log^+ \frac{1}{|a_k|} \right) \\ &\leq const \sum_{k=m+1}^{\infty} |a_k|^\alpha \left(1 + \log^+ \frac{1}{|a_k|} \right), \end{aligned}$$

where the constant in the last line depends on ϵ. Since $\mathbf{a} \in \ell^\alpha \log \ell$, the last expression vanishes as $m \to \infty$, and (2.9) is established. This proves the theorem for $X_n(\mathbf{a})$; the convergence of $\widetilde{X}_n(\mathbf{a})$ can be shown analogously by utilizing (2.6).

□

As an immediate corollary of Theorem 2.3.2 we obtain the following result which solves the problem of finding the limits of the finite-dimensional distributions for the integrated periodogram $J_{n,\varepsilon}$ in (2.2) and its self-normalized version $\widetilde{J}_{n,\varepsilon}$ in (2.4).

Corollary 2.3.3. *Let* $\alpha \in (0,2)$ *and*

$$\mathcal{F} = \left\{ f \in L^2[0,\pi] : \ \mathbf{a}(f) = (a_1(f), a_2(f), \ldots) \in \ell^\alpha \log \ell \right\} ,$$

where $\mathbf{a}(f)$ *is as specified in* (2.3). *Then*

$$\left(n(n \log n)^{-1/\alpha} \big(J_{n,\varepsilon}(f) - a_0(f)\ \gamma_{n,\varepsilon}(0) \big), f \in \mathcal{F} \right) \xrightarrow{f.d.} \big(2Y(\mathbf{a}(f)), f \in \mathcal{F} \big),$$

$$\left((n/\log n)^{1/\alpha} \big(\widetilde{J}_{n,\varepsilon}(f) - a_0(f) \big), f \in \mathcal{F} \right) \xrightarrow{f.d.} \big(2\widetilde{Y}(\mathbf{a}(f)), f \in \mathcal{F} \big).$$

Remark 2.3.4. The condition $\mathbf{a}(f) \in \ell^\alpha \log \ell$ is in general not easily verified. However, if f represents the spectral density of a stationary process (X_n) with absolutely summable autocovariance function γ_X, then, up to a constant multiple, f is represented by the Fourier series of γ_X, and the rate of decay of $\gamma_X(h) \to 0$ as $h \to \infty$ is well known for numerous time series models. For example, if f is the spectral density of an ARMA process, $\gamma_X(h) \to 0$ at an exponential rate (see, e.g., §3.3 of Brockwell and Davis (1991)) and then $\mathbf{a}(f) \in \ell^\alpha \log \ell$ is satisfied for every $\alpha > 0$.

Moreover, for any $0 < \alpha < \beta$, the condition $\mathbf{a}(f) \in \ell^\alpha$ is sufficient for $\mathbf{a}(f) \in \ell^\beta \log \ell$, since

$$\sum_{k=1}^\infty |a_k(f)|^\beta \log^+ \frac{1}{|a_k(f)|} \le const \sum_{k=1}^\infty |a_k(f)|^\alpha .$$

Conditions ensuring that $\mathbf{a}(f) \in \ell^\alpha$ can be found in the literature on Fourier series, for example in Zygmund (2002). His Theorem 3.10 yields for Lipschitz continuous functions f with exponent $\beta \in (0,1]$ that $\mathbf{a}(f) \in \ell^\alpha$ for $\alpha > 2/(2\beta+1)$, but not necessarily for $\alpha = 2/(2\beta+1)$. This means in particular that Lipschitz continuous functions do not necessarily satisfy $\mathbf{a}(f) \in \ell^\alpha$ for small values $\alpha < 1$. Zygmund's Theorem 3.13 states that $\mathbf{a}(f) \in \ell^\alpha$ if f is of bounded variation and Lipschitz continuous with exponent $\beta \in (0,1]$ such that $\alpha > 2/(2+\beta)$, but this statement is not necessarily valid for $\alpha = 2/(2+\beta)$.

We also note that $\mathbf{a}(f) \notin \ell^\alpha$ for $f(\cdot) = \mathbf{1}_{[0,x]}(\cdot), x \in (0,\pi]$, and $\alpha < 1$. Indeed, then $a_k(f) = k^{-1}\sin(xk), k = 1,2,\dots$ and $\sum_{k=1}^\infty |a_k(f)|^\alpha = \infty$. The latter condition implies that the series $Y(\mathbf{a}(f))$ diverges a.s. by the three-series theorem. Hence Corollary 2.3.3 does not apply to the important class of indicator functions when $\alpha < 1$.

2.3.2 Weak Convergence in the Case $\alpha \in (0,1)$

In order to derive a full weak convergence counterpart of the convergence in terms of the finite-dimensional distributions in Corollary 2.3.3 it remains to establish tightness of the corresponding family of laws. We start, once again, in the more general context of random fields indexed by sequences in $\ell^\alpha \log \ell$. Since we are dealing with the weak convergence of infinite-dimensional objects we may expect difficulties which are due to the geometric properties of the underlying path spaces. It is also not completely surprising that the case $\alpha \in (0,1)$ is the "better one" in comparison with $\alpha \in [1,2)$; see for example the results on boundedness, continuity and oscillations of α-stable processes in Chapter 10 of Samorodnitsky and Taqqu (1994). Note, however, that the constraint $\mathbf{a}(f) \in \ell^\alpha \log \ell$ is harder to satisfy for smaller α than for larger α; see Remark 2.3.4.

In the present case $\alpha \in (0,1)$ we introduce the function

$$h(x) = \begin{cases} |x|^\alpha \log(b + |x|^{-1}) & x \neq 0, \\ 0 & x = 0, \end{cases}$$

where b is chosen so large that h is concave on $(0, \infty)$. Notice that $\ell^\alpha \log \ell$ can be characterized as follows:

$$\ell^\alpha \log \ell = \left\{ (a_1, a_2, \ldots) \in \ell^\alpha : \sum_{k=1}^{\infty} h(a_k) \ < \infty \right\},$$

and this set is a linear metric space when endowed with the metric

$$d(\mathbf{a}, \mathbf{b}) = \sum_{k=1}^{\infty} h(a_k - b_k).$$

Assume that $\mathcal{A}$ is a compact subset of $\ell^\alpha \log \ell$ with the additional property that

$$\sum_{k=1}^{\infty} \sup_{\mathbf{a} \in \mathcal{A}} h(a_k) < \infty. \tag{2.11}$$

Observe that $\mathcal{A}$ is then also a compact subset of ℓ^α, and the processes $(Y(\mathbf{a}), \mathbf{a} \in \mathcal{A})$ and $(\widetilde{Y}(\mathbf{a}), \mathbf{a} \in \mathcal{A})$ are sample-continuous, i.e. they have versions for which all sample paths lie in $\mathbb{C}(\mathcal{A})$, the space of continuous functions defined on $\mathcal{A}$ equipped with the uniform topology. This follows from Theorem 10.4.2 of Samorodnitsky and Taqqu (1994).

The following is our main result on the weak convergence of the sequences $X_n(\mathbf{a})$ and $\widetilde{X}_n(\mathbf{a})$ of infinite variance random quadratic forms in the case $\alpha \in (0,1)$.

Theorem 2.3.5. *Assume $\alpha \in (0,1)$. For a compact subset $\mathcal{A}$ of $\ell^\alpha \log \ell$ satisfying (2.11) the following weak convergence results hold in $\mathbb{C}(\mathcal{A})$:*

$$(X_n(\mathbf{a}), \mathbf{a} \in \mathcal{A}) \Longrightarrow (Y(\mathbf{a}), \mathbf{a} \in \mathcal{A}) \quad \textit{and} \quad (\widetilde{X}_n(\mathbf{a}), \mathbf{a} \in \mathcal{A}) \Longrightarrow (\widetilde{Y}(\mathbf{a}), \mathbf{a} \in \mathcal{A}),$$

where X_n, $\widetilde{X}_n$, Y and $\widetilde{Y}$ are as defined in (2.7).

Proof. We first show $X_n \Longrightarrow Y$. In view of Theorem 2.3.2 it will suffice to prove the tightness of the processes X_n in $\mathbb{C}(\mathcal{A})$. We use Theorem 8.2 of Billingsley (1968) to prove tightness. Let $d_{\mathcal{A}}$ denote the restriction of d to $\mathcal{A}$, and note that, for positive ϵ and δ,

$$\begin{aligned}
&P\left(\sup_{d_{\mathcal{A}}(\mathbf{a},\mathbf{b})<\delta} |X_n(\mathbf{a}) - X_n(\mathbf{b})| > \epsilon\right) \\
&= P\left(\sup_{d_{\mathcal{A}}(\mathbf{a},\mathbf{b})<\delta} \left|\sum_{k=1}^{n-1}(a_k - b_k)\sum_{j=1}^{n-k}\varepsilon_j\varepsilon_{j+k}\right| > \epsilon\,(n\log n)^{1/\alpha}\right) \\
&\le P\left(\sum_{k=1}^{n-1}\sup_{d_{\mathcal{A}}(\mathbf{a},\mathbf{b})<\delta}|a_k - b_k|\sum_{j=1}^{n-k}|\varepsilon_j\varepsilon_{j+k}| > \epsilon\,(n\log n)^{1/\alpha}\right) \\
&= P\left(\sum_{1\le s<t\le n}\sup_{d_{\mathcal{A}}(\mathbf{a},\mathbf{b})<\delta}|a_{t-s} - b_{t-s}|\,|\varepsilon_s\varepsilon_t| > \epsilon\,(n\log n)^{1/\alpha}\right) := P_n(\epsilon,\delta).
\end{aligned} \tag{2.12}$$

We want to show that $P_n(\epsilon,\delta)$ can be made arbitrarily small for all n provided δ is small. We solve this problem in a modified form: let $(C_0, C_{s,t}, s, t = 1, 2, \ldots)$ be an array of i.i.d. $S_1(1,0,0)$ random variables, independent of $(\varepsilon_t, t \in \mathbb{Z})$. Denoting $w_k(\delta) = \sup_{d_{\mathcal{A}}(\mathbf{a},\mathbf{b})<\delta}|a_k - b_k|$, we see that

$$C_0 \sum_{1\le s<t\le n} w_{t-s}(\delta)\,|\varepsilon_s\varepsilon_t| \overset{d}{=} \sum_{1\le s<t\le n} w_{t-s}(\delta)\,C_{s,t}|\varepsilon_s\varepsilon_t| \overset{d}{=} \sum_{1\le s<t\le n} w_{t-s}(\delta)\,C_{s,t}\,\varepsilon_s\varepsilon_t\,.$$

Therefore,

$$\begin{aligned}
P_n(\epsilon,\delta) &\le \frac{1}{P(C_0>1)}\,P\left(C_0\sum_{1\le s<t\le n} w_{t-s}(\delta)\,|\varepsilon_s\varepsilon_t| > \epsilon\,(n\log n)^{1/\alpha}\right) \\
&= \frac{1}{P(C_0>1)}\,P\left(\sum_{1\le s<t\le n} w_{t-s}(\delta)\,C_{s,t}\,\varepsilon_s\varepsilon_t > \epsilon\,(n\log n)^{1/\alpha}\right) \\
&:= \frac{1}{P(C_0>1)}\,P_n'(\epsilon,\delta)\,.
\end{aligned}$$

Thus it will suffice to show that $P_n'(\epsilon,\delta)$ can be made arbitrarily small for all n provided δ is small. Applying Lemma 2.6.2 to $P_n'(\epsilon,\delta)$ and taking advantage of the inequality (2.10), we obtain the desired result:

$$P_n'(\epsilon,\delta) \le const\,\frac{1+\log^+\epsilon}{\epsilon^\alpha}\,\frac{1+\log n}{n\log n}\sum_{1\le s<t\le n} w_{t-s}(\delta)^\alpha\left(1+\log^+\frac{1}{w_{t-s}(\delta)}\right)$$

$$\leq const \sum_{k=1}^{\infty} h\Big(\sup_{d_{\mathcal{A}}(\mathbf{a},\mathbf{b})<\delta} |a_k - b_k| \Big) \longrightarrow 0 \quad \text{as } \delta \longrightarrow 0,$$

where the constant in the last line depends on ϵ, and the limit relation is a consequence of condition (2.11).

Next, we prove $\widetilde{X}_n \Longrightarrow \widetilde{Y}$. Once again, it will suffice to prove the tightness of the processes $\widetilde{X}_n$ in $\mathbb{C}(\mathcal{A})$. But note that

$$\widetilde{X}_n(\mathbf{a}) = \frac{n^{2/\alpha}}{\sum_{k=1}^{n} \varepsilon_k^2} X_n(\mathbf{a}) := Z_n X_n(\mathbf{a}), \quad \mathbf{a} \in \mathcal{A},$$

where (X_n) is a tight sequence in $\mathbb{C}(\mathcal{A})$ and Z_n converges in distribution to the reciprocal of an $\alpha/2$-stable random variable; see, e.g., Theorem 1.8.1 of Samorodnitsky and Taqqu (1994). Tightness of $(\widetilde{X}_n)$ follows. □

Theorem 2.3.5 provides the limit process for a very general class of random quadratic forms with infinite first moments. The coefficient matrices of these quadratic forms are given by infinite Töplitz matrices, i.e. matrices with real entries $(T_{ij}, i, j = 1, 2, \ldots)$ of the form

$$T_{ij} = \begin{cases} a_{j-i} & \text{if } j > i, \\ 0 & \text{if } j \leq i, \end{cases}$$

for some sequence $(a_1, a_2, \ldots)$. The conditions on the parameter set $\mathcal{A}$ are restrictions on the coefficient matrices. When specified to the particular case of Fourier coefficients as in (2.3), Theorem 2.3.5 yields the following.

Corollary 2.3.6. *Assume $\alpha \in (0, 1)$ and let*

$$\mathcal{F} = \left\{ f \in L^2[0, \pi] : \ \mathbf{a}(f) = (a_1(f), a_2(f), \ldots) \in \mathcal{A} \right\},$$

where $\mathcal{A}$ is a compact subset of $\ell^\alpha \log \ell$ satisfying (2.11) and $\mathbf{a}(f)$ is as specified in

(2.3). *Then the following weak convergence results hold in* $\mathbb{C}(\mathcal{F})$*:*

$$\begin{aligned}\Big(n(n\log n)^{-1/\alpha}\big(J_{n,\varepsilon}(f)-a_0(f)\gamma_{n,\varepsilon}(0)\big), f\in\mathcal{F}\Big) &\Longrightarrow \big(2Y(\mathbf{a}(f)), f\in\mathcal{F}\big),\\ \Big((n/\log n)^{1/\alpha}\big(\widetilde{J}_{n,\varepsilon}(f)-a_0(f)\big), f\in\mathcal{F}\Big) &\Longrightarrow \big(2\widetilde{Y}(\mathbf{a}(f)), f\in\mathcal{F}\big).\end{aligned} \tag{2.13}$$

Proof. Let $T:\mathcal{F}\to\mathcal{A}$ be defined by $Tf=\mathbf{a}(f)$. We claim that $T\mathcal{F}\subset\mathcal{A}$ is closed, hence compact. Indeed, if $(f_n, n\geq 1)\subset\mathcal{F}$ is such that Tf_n converges in $\ell^\alpha\log\ell$ to some point $\mathbf{a}\in\mathcal{A}$, then (as $0<\alpha<1$) the sequence of functions

$$f_n(\lambda)=\frac{1}{\pi}\sum_{j=-\infty}^{\infty}a_{|j|}(f_n)\cos(j\lambda),\quad \lambda\in[0,\pi],\ n=1,2,\dots$$

converges in $L^1[0,\pi]$ to some function f that has to be in $\mathcal{F}$. Therefore, $\mathbf{a}=Tf\in T\mathcal{F}$, and the latter set is compact. The above argument shows that the $L^2[0,\pi]$ convergence in $\mathcal{F}$ is equivalent to the $\ell^\alpha\log\ell$ convergence in $T\mathcal{F}$. Since Theorem 2.3.5 implies weak convergence of the left-hand side of (2.13) to its right-hand side in $\mathbb{C}(\mathcal{A})$ (when each function $f\in\mathcal{F}$ is identified with $Tf\in\mathcal{A}$), we conclude that weak convergence in (2.13) holds also in $\mathbb{C}(\mathcal{F})$. □

This result provides a solution to the problem of finding the weak limits of the specific random quadratic forms $J_{n,\varepsilon}$ in i.i.d. infinite mean SαS random variables ε_t, uniformly over a whole class of functions $f\in\mathcal{F}$ satisfying some mild conditions.

2.3.3 Weak Convergence in the Case $\alpha\in[1,2)$

Establishing full weak convergence in the case $\alpha\in[1,2)$ is more difficult than in the case $\alpha\in(0,1)$. Indeed, for $\alpha\in(0,1)$ we were allowed to switch from the random variables ε_t to their absolute values, due to the specific geometry of the spaces ℓ^α. The spaces ℓ^α, $\alpha\in[1,2)$, have a much more complicated structure, and therefore the particular geometry of these spaces will be present in proving tightness for the random quadratic forms X_n and $\widetilde{X}_n$. The requirements prescribed by the geometry are usually given by entropy conditions; see Ledoux and Talagrand (1991) for a general treatment of random elements with values in Banach spaces. Entropy conditions are typically needed when α-stable processes with $\alpha \in [1,2)$ appear; see the discussion in Chapter 12 of Samorodnitsky and Taqqu (1994).

In this section we only consider vectors $\mathbf{a} \in \ell^\alpha \log \ell$ of the form (2.3), i.e., they are the Fourier coefficients of some functions f. Corollary 2.3.3 determines the structure of the limit process of the quadratic forms $J_{n,\varepsilon}$ via the convergence of their finite-dimensional distributions. Hence it suffices to show the tightness in $\mathbb{C}(\mathcal{F})$ for suitable classes $\mathcal{F}$. Klüppelberg and Mikosch (1996a) considered the special case of the one-dimensional class $\mathcal{F}_I$ of indicator functions on $[0,\pi]$. We extend their approach to more general classes of functions, using an entropy condition.

For $f, g \in \mathcal{F}$, let

$$d_j(f,g) = j\,|a_j(f) - a_j(g)|, \quad j \geq 1.$$

Each d_j defines a pseudo-metric on $\mathcal{F}$. Let

$$\rho_k(f,g) = \max_{2^k \leq j < 2^{k+1}} d_j(f,g), \quad k \geq 0.$$

Recall that the ϵ-covering number $N(\epsilon, \mathcal{F}, \rho_k)$ of $(\mathcal{F}, \rho_k)$ is the minimal integer m for which we can find functions $f_1, \ldots, f_m \in \mathcal{F}$ such that

$$\sup_{f \in \mathcal{F}} \min_{i=1,\ldots,m} \rho_k(f, f_i) < \epsilon.$$

Theorem 2.3.7. *Assume $\alpha \in [1,2)$, define $\mathbf{a}(f)$ as in (2.3) and let $\mathcal{F}$ be a subset of $L^2[0,\pi]$ satisfying*

(i) $\mathbf{a}(f) \in \ell^\alpha \log \ell$ *for all* $f \in \mathcal{F}$,

(ii) $\exists \beta \in (0, \alpha)$ *such that*

$$N(\epsilon, \mathcal{F}, \rho_k) \le const \left[1 + \left(\frac{2^k}{\epsilon}\right)^{\beta}\right], \quad \epsilon > 0,\ k \ge 0\,. \tag{2.14}$$

Then the weak convergence result (2.13) *holds in* $\mathbb{C}(\mathcal{F})$.

Remark 2.3.8. In contrast to the finite variance case (see Dahlhaus (1988), Mikosch and Norvaiša (1997)) the entropy condition (2.14) is a rather strong one. Indeed, in the papers mentioned integrability of $\log N(\epsilon)$ in a neighborhood of the origin suffices. However, conditions such as (2.14) are common in problems of continuity and boundedness for stable processes; see Chapter 10 in Samorodnitsky and Taqqu (1994).

Proof of Theorem 2.3.7. The convergence of the finite-dimensional distributions follows from Theorem 2.3.2, so it remains to prove the tightness in $\mathbb{C}(\mathcal{F})$ of the processes on the left-hand side of (2.13). We first consider the processes in the first line of (2.13). In order to prove that they form a tight sequence, we need to show that

$$\lim_{\delta \to 0} \limsup_{n \to \infty} P\left(\sup_{d(f,g)<\delta} \left| \sum_{j=1}^{n-1} (a_j(f) - a_j(g)) \widehat{\gamma}_{n,\varepsilon}(j) \right| > \epsilon \right) = 0 \tag{2.15}$$

for each $\epsilon > 0$, where d denotes the $L^2[0,\pi]$ metric restricted to $\mathcal{F}$ and

$$\widehat{\gamma}_{n,\varepsilon}(j) = n(n \log n)^{-1/\alpha} \gamma_{n,\varepsilon}(j), \quad |j| \le n-1.$$

Let us denote the probabilities in (2.15) by $P_n(\epsilon, \delta)$. Notice that, for each $1 \le m \le n-1$,

$$\begin{aligned} P_n(\epsilon,\delta) &\le P\left(\sup_{d(f,g)<\delta} \left| \sum_{j=1}^{m} (a_j(f) - a_j(g)) \widehat{\gamma}_{n,\varepsilon}(j) \right| > \epsilon/3 \right) \\ &\quad + 2P\left(\sup_{f \in \mathcal{F}} \left| \sum_{j=m+1}^{n-1} a_j(f) \widehat{\gamma}_{n,\varepsilon}(j) \right| > \epsilon/3 \right) \\ &:= P_{n,m}(\epsilon,\delta) + 2Q_{n,m}(\epsilon). \end{aligned}$$

Furthermore, for each $m \geq 1$,

$$\begin{aligned}
&\lim_{\delta \to 0} \limsup_{n \to \infty} P_{n,m}(\epsilon, \delta) \\
&\qquad \leq \lim_{\delta \to 0} \limsup_{n \to \infty} P\biggl(\sup_{d(f,g)<\delta} \max_{j=1,\ldots,m} |a_j(f) - a_j(g)| \sum_{j=1}^{m} |\widehat{\gamma}_{n,\varepsilon}(j)| > \epsilon \biggr) \\
&\qquad \leq \lim_{\delta \to 0} \limsup_{n \to \infty} P\biggl(\sup_{d(f,g)<\delta} \int_0^\pi |f(\lambda) - g(\lambda)| \, d\lambda \sum_{j=1}^{m} |\widehat{\gamma}_{n,\varepsilon}(j)| > \epsilon \biggr) \\
&\qquad = \lim_{\delta \to 0} P\biggl(\sup_{d(f,g)<\delta} \int_0^\pi |f(\lambda) - g(\lambda)| \, d\lambda \sum_{j=1}^{m} |Y_j| > \epsilon \biggr) = 0,
\end{aligned}$$

where $Y_1, \ldots, Y_m$ are as defined in Lemma 2.3.1. It now follows that (2.15) will be proved once we show that

$$\lim_{m \to \infty} \limsup_{n \to \infty} Q_{n,m}(\epsilon) = 0. \tag{2.16}$$

As in (6.4) on p. 1873 of Klüppelberg and Mikosch (1996a), one can argue that it suffices in (2.16) to consider m and n of some specific form. Let $a < b$ be two positive integers and set

$$m = 2^a - 1 \quad \text{and} \quad n = 2^{b+1} \, .$$

For a large enough we have $\sum_{k=a}^{b} 2^{-k} \leq \epsilon/3$, so that

$$Q_{n,m}(\epsilon) \leq \sum_{k=a}^{b} P\Biggl(\sup_{f \in \mathcal{F}} \Biggl| \sum_{j=2^k}^{2^{k+1}-1} a_j(f) \widehat{\gamma}_{n,\varepsilon}(j) \Biggr| > 2^{-k} \Biggr) := \sum_{k=a}^{b} p_k \, . \tag{2.17}$$

We construct an upper bound for p_k by the following reasoning. Consider an array $(\epsilon_{k,l}, k \geq 0, l \geq 0)$ of positive numbers such that $\epsilon_{k,l} \to 0$ as $l \to \infty$ for each $k \geq 0$. Given integers $k \geq 0$ and $l \geq 0$, one can find $N(\epsilon_{k,l}, \mathcal{F}, \rho_k)$ balls of radius at most $\epsilon_{k,l}$ (in pseudometric ρ_k) covering $\mathcal{F}$. Let $C_{k,l}$ denote the set of the centers of these balls. Also, for each $f \in \mathcal{F}$, let $m_{k,l}(f)$ denote the function in $C_{k,l}$ that minimizes $\rho_k(f, m_{k,l}(f))$. Then we have, for any $k \geq 0$ and $N \geq 1$,

$$\sup_{f \in \mathcal{F}} \Biggl| \sum_{j=2^k}^{2^{k+1}-1} a_j(f) \widehat{\gamma}_{n,\varepsilon}(j) \Biggr|$$

$$\leq \sup_{f\in\mathcal{F}} \left| \sum_{j=2^k}^{2^{k+1}-1} \big(a_j(f) - a_j(m_{k,N}(f))\big)\widehat{\gamma}_{n,\varepsilon}(j) \right|$$
$$+ \sum_{l=1}^{N} \sup_{g_{k,l}\in C_{k,l}} \left| \sum_{j=2^k}^{2^{k+1}-1} \big(a_j(g_{k,l}) - a_j(m_{k,l-1}(g_{k,l}))\big)\widehat{\gamma}_{n,\varepsilon}(j) \right|$$
$$+ \sup_{g_{k,0}\in C_{k,0}} \left| \sum_{j=2^k}^{2^{k+1}-1} a_j(g_{k,0})\widehat{\gamma}_{n,\varepsilon}(j) \right|.$$

Letting $N \to \infty$, we obtain

$$\sup_{f\in\mathcal{F}} \left| \sum_{j=2^k}^{2^{k+1}-1} a_j(f)\widehat{\gamma}_{n,\varepsilon}(j) \right|$$
$$\leq \sup_{g_{k,0}\in C_{k,0}} \left| \sum_{j=2^k}^{2^{k+1}-1} a_j(g_{k,0})\widehat{\gamma}_{n,\varepsilon}(j) \right|$$
$$+ \sum_{l=1}^{\infty} \sup_{g_{k,l}\in C_{k,l}} \left| \sum_{j=2^k}^{2^{k+1}-1} \big(a_j(g_{k,l}) - a_j(m_{k,l-1}(g_{k,l}))\big)\widehat{\gamma}_{n,\varepsilon}(j) \right|,$$

which yields the following bound for the terms p_k in (2.17):

$$p_k \leq P\left(\sup_{g_{k,0}\in C_{k,0}} \left| \sum_{j=2^k}^{2^{k+1}-1} a_j(g_{k,0})\widehat{\gamma}_{n,\varepsilon}(j) \right| > 2^{-k-1} \right)$$
$$+ \sum_{l=1}^{\infty} P\left(\sup_{g_{k,l}\in C_{k,l}} \left| \sum_{j=2^k}^{2^{k+1}-1} \big(a_j(g_{k,l}) - a_j(m_{k,l-1}(g_{k,l}))\big)\widehat{\gamma}_{n,\varepsilon}(j) \right| > 2^{-k-l-1} \right).$$

Further manipulation of the right-hand side yields

$$p_k \leq N(\epsilon_{k,0}, \mathcal{F}, \rho_k)\; p_{k,0} + \sum_{l=1}^{\infty} N(\epsilon_{k,l}, \mathcal{F}, \rho_k)\; p_{k,l}\,, \tag{2.18}$$

with

$$p_{k,0} = \sup_{f\in\mathcal{F}} P\left(\left| \sum_{j=2^k}^{2^{k+1}-1} a_j(f)\; \widehat{\gamma}_{n,\varepsilon}(j) \right| > 2^{-(k+1)} \right),$$
$$p_{k,l} = \sup_{f,g\in\mathcal{F},\, \rho_k(f,g)\leq\epsilon_{k,l-1}} P\left(\left| \sum_{j=2^k}^{2^{k+1}-1} (a_j(f) - a_j(g))\; \widehat{\gamma}_{n,\varepsilon}(j) \right| > 2^{-(k+l+1)} \right).$$

Since $p_{k,0} \le \sum_{l=1}^{\infty} p_{k,l}$ and $N(\epsilon_{k,0}, \mathcal{F}, \rho_k) \le N(\epsilon_{k,l}, \mathcal{F}, \rho_k)$ for all l large enough, (2.18) reduces to

$$p_k \le const \sum_{l=1}^{\infty} N(\epsilon_{k,l}, \mathcal{F}, \rho_k)\, p_{k,l}.$$

Now, by virtue of Lemma 2.6.1, we have for all $f, g \in \mathcal{F}$,

$$P\left(\left| \sum_{j=2^k}^{2^{k+1}-1} (a_j(f) - a_j(g))\, \widehat{\gamma}_{n,\varepsilon}(j) \right| > 2^{-(k+l+1)} \right) \le const\ b_{k,l}\,,$$

where

$$b_{k,l} = 2^{\alpha(k+l)} \sum_{j=2^k}^{2^{k+1}-1} |a_j(f) - a_j(g)|^{\alpha} \left(1 + \log^+(1/|a_j(f) - a_j(g)|)\right)\,.$$

Assuming $\rho_k(f, g) \le \epsilon_{k,l-1}$, we have

$$\begin{aligned} b_{k,l} &\le const\ 2^{\alpha(k+l)}\, \epsilon_{k,l-1}^{\alpha} \sum_{j=2^k}^{2^{k+1}-1} j^{-\alpha} \left(1 + \log j \log^+ \epsilon_{k,l-1}^{-1}\right) \\ &\le const\ 2^{\alpha(k+l)}\, \epsilon_{k,l-1}^{\alpha}\, 2^{-k(\alpha-1)} \left(1 + k \log^+ \epsilon_{k,l-1}^{-1}\right)\,. \end{aligned}$$

Hence we are left to consider

$$\begin{aligned} &\sum_{k=a}^{b} \sum_{l=1}^{\infty} N(\epsilon_{k,l}, \mathcal{F}, \rho_k)\, 2^{k+\alpha l} \epsilon_{k,l-1}^{\alpha} \left(1 + k \log^+ \epsilon_{k,l-1}^{-1}\right) \\ &\quad = \sum_{k=a}^{b} 2^k \sum_{l=1}^{\infty} N(\epsilon_{k,l}, \mathcal{F}, \rho_k)\, \epsilon_{k,l-1}^{\alpha} \left(1 + k \log^+ \epsilon_{k,l-1}^{-1}\right) 2^{\alpha l} \qquad (2.19) \\ &\quad \le const \sum_{k=a}^{b} 2^k \sum_{l=1}^{\infty} \left[1 + \left(\frac{2^k}{\epsilon_{k,l}}\right)^{\beta}\right] \epsilon_{k,l-1}^{\alpha} \left(1 + k \log^+ \epsilon_{k,l-1}^{-1}\right) 2^{\alpha l}\,. \end{aligned}$$

Define the numbers

$$\epsilon_{k,l} = 2^{-\gamma_1 l - \gamma_2 k}\,, \quad k, l \ge 0$$

with $\gamma_1, \gamma_2 > 0$ chosen such that

$$\gamma_1 > \frac{\alpha}{\alpha - \beta} \quad \text{and} \quad \gamma_2 > \frac{1+\beta}{\alpha - \beta}\,.$$

For these parameter choices it is not difficult to see that the last expression in (2.19) converges to zero by first letting $n \to \infty$ (i.e., $b \to \infty$) and then $m \to \infty$

(i.e., $a \to \infty$). This proves (2.16), hence the tightness of the considered processes in $\mathbb{C}(\mathcal{F})$.

We have thus established the weak convergence in the first line of (2.13). The weak convergence in the second line also holds, since we have convergence in finite-dimensional distributions by Theorem 2.3.2, and the processes on the left-hand side form a tight sequence in $\mathbb{C}(\mathcal{F})$. The tightness follows from the identity

$$(n/\log n)^{1/\alpha}\big(\widetilde{J}_{n,\varepsilon}(f) - a_0(f)\big) \\ = \frac{n^{2/\alpha}}{\sum_{t=1}^{n} \varepsilon_t^2}\, n(n\log n)^{-1/\alpha}\big(J_{n,\varepsilon}(f) - a_0(f)\gamma_{n,\varepsilon}(0)\big),$$

since the sequence $n(n\log n)^{-1/\alpha}\big(J_{n,\varepsilon}(\cdot) - a_0(\cdot)\gamma_{n,\varepsilon}(0)\big)$ is tight in $\mathbb{C}(\mathcal{F})$ and the term $n^{2/\alpha}\big/\sum_{t=1}^{n}\varepsilon_t^2$ converges in distribution to the reciprocal of an $\alpha/2$-stable random variable (see, e.g., Theorem 1.8.1 of Samorodnitsky and Taqqu (1994)). □

In what follows, we give examples of function spaces $\mathcal{F}$ satisfying condition (ii) of Theorem 2.3.7.

Example 2.3.9. Consider a space of indexed functions $\mathcal{G}_\Theta = \{g_\theta : \theta \in \Theta\}$ that are defined on $[0,\pi]$. Suppose that each $g_\theta \in \mathcal{G}_\Theta$ is bounded, (Θ,τ) is a compact metric space, and the mapping $\theta \mapsto g_\theta$ is Hölder continuous with exponent $b > 0$ and constant $K > 0$, i.e.

$$\sup_{0\le x\le\pi} |g_{\theta_1}(x) - g_{\theta_2}(x)| \le K\,(\tau(\theta_1,\theta_2))^b \;\text{ for all } \theta_1,\theta_2 \in \Theta\,.$$

Also suppose that the number of balls (in metric τ) of radius at most ϵ necessary to cover Θ is of the order ϵ^{-a} for some $0 < a < b\alpha$. Then, $\mathcal{G}_\Theta$ satisfies

$$N(\epsilon,\mathcal{G}_\Theta,\rho_k) \le const\left[1 + \left(\frac{2^k}{\epsilon}\right)^{a/b}\right], \quad \epsilon > 0, k \ge 0,$$

with $a/b \in (0,\alpha)$. This inequality follows from the following arguments. For any $\epsilon > 0, k \ge 0$, we can find $N \le c_1 + c_2\big((K\pi 2^{k+1})/\epsilon\big)^{a/b}$ balls of radius at most $\left(\epsilon/(K\pi 2^{k+1})\right)^{1/b}$ covering Θ, where $c_1, c_2 > 0$ are constants. Call these balls $B_1, \ldots, B_N$, with centers $\theta_1, \ldots, \theta_N$. Now, given $\theta \in \Theta$, we have $\theta \in B_i$ for some $i \in \{1, \ldots, N\}$ and

$$\begin{aligned}\rho_k\left(g_\theta, g_{\theta_i}\right) &= \max_{2^k \le j < 2^{k+1}} j \left| \int_0^\pi \cos(jx)\left(g_\theta(x) - g_{\theta_i}(x)\right)\, dx \right| \\ &\le 2^{k+1}\pi \sup_{0 \le x \le \pi} |g_\theta(x) - g_{\theta_i}(x)| \\ &\le 2^{k+1}\pi K \tau(\theta, \theta_i)^b \\ &\le 2^{k+1}\pi K \frac{\epsilon}{K\pi 2^{k+1}} = \epsilon\,.\end{aligned}$$

It follows that

$$N(\epsilon, \mathcal{G}_\Theta, \rho_k) \le N \le const\left[1 + \left(\frac{2^k}{\epsilon}\right)^{a/b}\right].$$

Example 2.3.10. Recall the notion of a *Vapnik-Červonenkis (VC) class* that plays an important role in the study of empirical processes. A VC class of sets is defined as follows. Let $\mathcal{C}$ be a collection of subsets of an arbitrary set $\mathcal{X}$. Let $\{x_1, \ldots, x_n\}$ be any finite subset of $\mathcal{X}$, and say that the collection $\mathcal{C}$ *shatters* $\{x_1, \ldots, x_n\}$ if each of the 2^n subsets of the latter can be written as $C \cap \{x_1, \ldots, x_n\}$ for some $C \in \mathcal{C}$. The *VC-index* $V(\mathcal{C})$ of the collection $\mathcal{C}$ is the smallest integer n such that no set of size n is shattered by $\mathcal{C}$. More formally, if we define

$$\Delta_{\mathcal{C}}(x_1, \ldots, x_n) = \#\left\{C \cap \{x_1, \ldots, x_n\} : C \in \mathcal{C}\right\},$$

where $\#$ denotes cardinality, then

$$V(\mathcal{C}) = \min\Big\{n \ge 1 : \max_{x_1, \ldots, x_n \in \mathcal{X}} \Delta_{\mathcal{C}}(x_1, \ldots, x_n) < 2^n\Big\}.$$

Here, the minimum over the empty set is taken to be infinity, so that the index is infinity if and only if $\mathcal{C}$ shatters sets of arbitrarily large size. The collection $\mathcal{C}$ is called a VC class if its VC-index is finite.

To take an example, let $\mathcal{X} = \mathbb{R}$ and let $\mathcal{C} = \{(-\infty, c] : c \in \mathbb{R}\}$, the collection of right-closed half-lines. Then, $\mathcal{C}$ is a VC-class of index 2 because for any two-point set $\{x_1, x_2\}$ with $x_1 < x_2$, the subset $\{x_2\}$ cannot be written as $C \cap \{x_1, x_2\}$ for any $C \in \mathcal{C}$. Similarly, the collection $\mathcal{D} = \{(a, b] : a, b \in \mathbb{R}, a < b\}$ of left-open, right-closed finite intervals is a VC-class of index 3 because for a three-point set $\{x_1, x_2, x_3\}$ with $x_1 < x_2 < x_3$, the subset $\{x_1, x_3\}$ cannot be written as $D \cap \{x_1, x_2, x_3\}$ for any $D \in \mathcal{D}$.

A class of functions mapping a set $\mathcal{X}$ into $\mathbb{R}$ is called a VC-class of index n if the collection of the *subgraphs* of those functions form a VC-class of index n in $\mathcal{X} \times \mathbb{R}$. Recall that the subgraph of a function $f : \mathcal{X} \to \mathbb{R}$ is defined as the set $\{(x, y) \in \mathcal{X} \times \mathbb{R} : y < f(x)\}$ of all points "under the graph" of f. So for $\mathcal{X} = \mathbb{R}$, the collection $\mathcal{C} = \{f \in \mathbb{R}^{\mathbb{R}} : f(\cdot) = c\cdot \text{ for some } c \in \mathbb{R}\}$ of linear functions passing through the origin is a VC-class of index 2, while the collection $\mathcal{D} = \{f \in \mathbb{R}^{\mathbb{R}} : f(\cdot) = c \cdot +d \text{ for some } c, d \in \mathbb{R}\}$ of arbitrary linear functions is a VC-class of index 3.

The relevance of VC-classes for our discussion stems from the following lemma, which is a direct consequence of Theorem 2.6.7 in van der Vaart and Wellner (1996). In the following, $\|f\|_\beta$ denotes the norm $\left(\frac{1}{\pi}\int_0^\pi |f(x)|^\beta dx\right)^{1/\beta}$.

Lemma 2.3.11. *Let $\mathcal{F}$ be a VC-class of functions mapping $[0, \pi]$ into $\mathbb{R}$, with VC-index $V(\mathcal{F}) = 2$. Suppose that there is a function $F : [0, \pi] \to \mathbb{R}$ such that $|f(x)| \leq F(x)$ for all $x \in [0, \pi], f \in \mathcal{F}$, and $\|F\|_\beta < \infty$ for some $\beta \geq 1$. Then, for any $\epsilon > 0$,*

$$N(\epsilon, \mathcal{F}, \|\cdot\|_\beta) \leq const \left(1 + \frac{1}{\epsilon^\beta}\right).$$

Now suppose $\mathcal{F}$ is a class of functions satisfying the hypotheses of Lemma 2.3.11, with $1 \leq \beta < \alpha < 2$. We claim that $\mathcal{F}$ satisfies condition (ii) of Theorem

2.3.7. To see why, let $\epsilon > 0$ and $k \geq 0$. We can find $N \leq const\,(1+(\pi 2^{k+1}/\epsilon)^\beta)$ balls of radius at most $\epsilon/(\pi 2^{k+1})$ covering $\mathcal{F}$ in the norm $\|\cdot\|_\beta$. Call them $B_1, \ldots, B_N$, with centers $f_1, \ldots, f_N$. Now, given $f \in \mathcal{F}$, we have $f \in B_i$ for some $i \in \{1, \ldots, N\}$ and

$$\begin{aligned}\rho_k(f, f_i) &= \max_{2^k \leq j < 2^{k+1}} j \left| \int_0^\pi \cos(jx)(f(x) - f_i(x))\, dx \right| \\ &\leq 2^{k+1} \int_0^\pi |f(x) - f_i(x)|\; dx \\ &\leq 2^{k+1}\pi\, \|f - f_i\|_\beta \\ &\leq 2^{k+1}\pi\, \frac{\epsilon}{\pi 2^{k+1}} = \epsilon\,.\end{aligned}$$

It follows that

$$N(\epsilon, \mathcal{F}, \rho_k) \leq N \leq const \left[1 + \left(\frac{2^k}{\epsilon}\right)^\beta\right].$$

2.4 The Linear Process Case

Recall the definition of the integrated periodogram $J_{n,X}$ indexed by a class of functions $\mathcal{F}$:

$$J_{n,X}(f) = \int_0^\pi I_{n,X}(\lambda) f(\lambda) d\lambda, \quad f \in \mathcal{F}.$$

It is the aim of this section to show that the results for the case of an i.i.d. SαS sequence $(\varepsilon_t, t \in \mathbb{Z})$ translate to the case of a linear process

$$X_t = \sum_{j=-\infty}^{\infty} \psi_j \varepsilon_{t-j}, \quad t \in \mathbb{Z},$$

with i.i.d. SαS noise terms $(\varepsilon_j, j \in \mathbb{Z})$ and a linear filter $(\psi_j, j \in \mathbb{Z})$ satisfying certain summability conditions.

The following decomposition will be crucial:

$$I_{n,X}(\lambda) = I_{n,\varepsilon}(\lambda) \left|\psi(e^{-i\lambda})\right|^2 + R_n(\lambda)\,, \tag{2.20}$$

where $\psi(e^{-i\lambda})$ is the transfer function defined in (1.10) and $R_n(\lambda)$ is some remainder term. This decomposition is analogous to the decomposition of the spectral density f_X of a linear process:

$$f_X(\lambda) = f_\varepsilon(\lambda) \left|\psi(e^{-i\lambda})\right|^2$$

(see Theorem 4.4.1 of Brockwell and Davis (1991)). We will show that the normalized integrated remainder term $\int_0^\pi R_n(\lambda)\, f(\lambda)\, d\lambda$ is negligible uniformly over the class of functions $\mathcal{F}$, in comparison to the normalized main part

$$\int_0^\pi I_{n,\varepsilon}(\lambda)\, |\psi(e^{-i\lambda})|^2\, f(\lambda)\, d\lambda\,, \quad f \in \mathcal{F}\,,$$

which can be treated by the methods of the previous section. Notice that, for a given sequence of coefficients $(\psi_j, j \in \mathbb{Z})$, the functions $|\psi(e^{-i\cdot})|^2 f$ constitute another class of functions on $[0,\pi]$, say $\mathcal{F}_\psi$, and therefore we will study the process

$$J_{n,\varepsilon}(f) = \int_0^\pi I_{n,\varepsilon}(\lambda)\; f(\lambda)\; d\lambda\,, \quad f \in \mathcal{F}_\psi\,,$$

for suitable classes $\mathcal{F}_\psi$.

Lemma 2.4.1. *Let R_n be the remainder term appearing in the decomposition (2.20) of the periodogram $I_{n,X}$. Suppose that the linear filter $(\psi_j, j \in \mathbb{Z})$ of the process X satisfies*

$$\sum_{j=-\infty}^{\infty} |\psi_j|\, |j|^{2/\alpha} \left(1 + \log^+ |j|\right)^{\frac{4-\alpha}{2\alpha}+\tau} < \infty \tag{2.21}$$

for some $\tau > 0$, and $\mathcal{F}$ is a collection of real-valued functions defined on $[0,\pi]$ such that $\sup_{f\in\mathcal{F}} \|f\|_2 < \infty$. Then,

$$\frac{n}{(n\log n)^{1/\alpha}} \sup_{f\in\mathcal{F}} \left| \int_0^\pi f(x) R_n(x)\, dx \right| \xrightarrow{P} 0.$$

Proof. From Proposition 5.1 in Mikosch et al. (1995), substituting $n^{1/2}$ for a_n, we have the following decomposition for R_n:

$$R_n(x) = n^{-1}\Big(\psi(e^{ix}) L_n(x) K_n(-x) + \psi(e^{-ix}) L_n(-x) K_n(x) + |K_n(x)|^2\Big)\,, \tag{2.22}$$

where ψ is the transfer function as defined before, and

$$L_n(x) = \sum_{t=1}^{n} \varepsilon_t e^{-ixt}, \qquad K_n(x) = \sum_{j=-\infty}^{\infty} \psi_j \, e^{-ixj} U_{nj}(x),$$

$$U_{nj}(x) = \left(\sum_{t=1-j}^{n-j} - \sum_{t=1}^{n} \right) \varepsilon_t e^{-ixt}.$$

We first show that

$$\frac{1}{(n \log n)^{1/\alpha}} \sup_{f \in \mathcal{F}} \left| \int_0^{\pi} f(x) \, |K_n(x)|^2 \, dx \right| \xrightarrow{P} 0. \tag{2.23}$$

Note that

$$\begin{aligned}
&\left| \int_0^{\pi} f(x) \, |K_n(x)|^2 \, dx \right| \\
&\quad \le \int_0^{\pi} |f(x)| \left(\sum_{j=-\infty}^{\infty} |\psi_j| \, |U_{nj}(x)| \right)^2 dx \\
&\quad \le const \sum_{j=-\infty}^{\infty} |\psi_j| \int_0^{\pi} |f(x)| \, |U_{nj}(x)|^2 \, dx \\
&\quad = const \left(\sum_{j=-\infty}^{-1} + \sum_{j=1}^{\infty} \right) |\psi_j| \int_0^{\pi} |f(x)| |U_{nj}(x)|^2 \, dx \,.
\end{aligned}$$

The convergence in (2.23) will follow if we can show that the suprema over $f \in \mathcal{F}$ of the two infinite sums in the last expression are bounded in probability as $n \to \infty$. We will prove this for the second sum; the first one can be handled analogously.

We have, by definition of the terms $U_{nj}(x)$, the Cauchy-Schwarz inequality and since, by assumption, $\sup_{f \in \mathcal{F}} \|f\|_2 < \infty$,

$$\begin{aligned}
&\sup_{f \in \mathcal{F}} \sum_{j=1}^{\infty} |\psi_j| \int_0^{\pi} |f(x)| \, |U_{nj}(x)|^2 \, dx \\
&\quad \le \sup_{f \in \mathcal{F}} \sum_{j=1}^{n} |\psi_j| \int_0^{\pi} |f(x)| \left| \sum_{t=1-j}^{0} \varepsilon_t e^{-ixt} - \sum_{t=n-j+1}^{n} \varepsilon_t e^{-ixt} \right|^2 dx \\
&\qquad + \sup_{f \in \mathcal{F}} \sum_{j=n+1}^{\infty} |\psi_j| \int_0^{\pi} |f(x)| \left| \sum_{t=1-j}^{n-j} \varepsilon_t e^{-ixt} - \sum_{t=1}^{n} \varepsilon_t e^{-ixt} \right|^2 dx
\end{aligned}$$

$$\leq const\left(I_1(n)+I_2(n)+I_3(n)+I_4(n)\right),$$

where

$$I_1(n)=\sum_{j=1}^{n}|\psi_j|\left(\int_0^\pi\left|\sum_{t=1-j}^{0}\varepsilon_t e^{-ixt}\right|^4 dx\right)^{1/2},$$
$$I_2(n)=\sum_{j=1}^{n}|\psi_j|\left(\int_0^\pi\left|\sum_{t=n-j+1}^{n}\varepsilon_t e^{-ixt}\right|^4 dx\right)^{1/2},$$
$$I_3(n)=\sum_{j=n+1}^{\infty}|\psi_j|\left(\int_0^\pi\left|\sum_{t=1-j}^{n-j}\varepsilon_t e^{-ixt}\right|^4 dx\right)^{1/2},$$
$$I_4(n)=\sum_{j=n+1}^{\infty}|\psi_j|\left(\int_0^\pi\left|\sum_{t=1}^{n}\varepsilon_t e^{-ixt}\right|^4 dx\right)^{1/2}.$$

It remains to show that each sequence $I_k(n), k=1,2,3,4$, is tight. Now,

$$I_1(n)\stackrel{d}{=}\sum_{j=1}^{n}|\psi_j|\left(\int_0^\pi\left|\sum_{m=1}^{j}\varepsilon_m e^{ixm}\right|^4 dx\right)^{1/2}.$$

Let $\epsilon>0$. Choose $M>0$ so large that the following holds, for $\delta=\frac{2\alpha}{4-\alpha}\tau$:

$$P\left(|\varepsilon_m|>M\,m^{1/\alpha}(1+\log m)^{\frac{1}{\alpha}+\delta}\text{ for some }m\geq 1\right)\leq\epsilon/2\,.$$

Write

$$J_m=\varepsilon_m\mathbf{1}\left\{|\varepsilon_m|\leq Mm^{1/\alpha}(1+\log m)^{\frac{1}{\alpha}+\delta}\right\}.$$

Then, for $k>0$ and δ chosen as above,

$$\begin{aligned}
&P(I_1(n)>k)-\epsilon/2\\
&\leq P\left(\sum_{j=1}^{n}|\psi_j|\left(\int_0^\pi\left|\sum_{m=1}^{j}J_m\,e^{ixm}\right|^4 dx\right)^{1/2}>k\right)\\
&\leq k^{-1}\sum_{j=1}^{n}|\psi_j|\left(\int_0^\pi E\left|\sum_{m=1}^{j}J_m\,e^{ixm}\right|^4 dx\right)^{1/2}\\
&=k^{-1}\sum_{j=1}^{n}|\psi_j|\times
\end{aligned}$$

$$\left(\int_0^\pi E\left(\sum_{m=1}^j J_m^2 + 2\sum_{1\le m_1<m_2\le j} J_{m_1}J_{m_2}\cos((m_1-m_2)x)\right)^2 dx\right)^{1/2}$$

$$\le k^{-1}\sum_{j=1}^n |\psi_j|\left(\int_0^\pi\left(\sum_{m=1}^j E(J_m^4) + 6\sum_{1\le m_1<m_2\le j} E(J_{m_1}^2)E(J_{m_2}^2)\right)dx\right)^{1/2}$$

$$\le const\, k^{-1}\sum_{j=1}^n |\psi_j|\left[\left(\sum_{m=1}^j E(J_m^4)\right)^{1/2} + \sum_{m=1}^j E(J_m^2)\right].$$

Note that for $x \ge 0$,

$$E(\varepsilon_m^4 \mathbf{1}\{|\varepsilon_m|\le x\})$$

$$\le \int_0^\infty P\left(\varepsilon_m^4\mathbf{1}\{|\varepsilon_m|\le x\} > y\right)\, dy \le 2\int_0^{x^4} P\left(\varepsilon_m > y^{1/4}\right)\, dy \le const\; x^{4-\alpha},$$

and

$$E(\varepsilon_m^2 \mathbf{1}\{|\varepsilon_m|\le x\}) \le const\; x^{2-\alpha},$$

by similar reasoning. Therefore, continuing from above,

$$P(I_1(n) > k) - \epsilon/2$$

$$\le const\; k^{-1}\sum_{j=1}^n |\psi_j|\left[\left(\sum_{m=1}^j\left(m^{1/\alpha}(1+\log m)^{\frac{1}{\alpha}+\delta}\right)^{4-\alpha}\right)^{1/2} + \sum_{m=1}^j\left(m^{1/\alpha}(1+\log m)^{\frac{1}{\alpha}+\delta}\right)^{2-\alpha}\right]$$

$$\le const\; k^{-1}\sum_{j=1}^n |\psi_j|\left[j^{2/\alpha}\,(1+\log j)^{\frac{1}{2}(4-\alpha)\left(\frac{1}{\alpha}+\delta\right)} + j^{2/\alpha}\,(1+\log j)^{(2-\alpha)\left(\frac{1}{\alpha}+\delta\right)}\right]$$

$$\le const\; k^{-1}\sum_{j=1}^\infty |\psi_j|\, j^{2/\alpha}\,(1+\log j)^{\frac{(4-\alpha)}{2\alpha}+\tau}.$$

By virtue of (2.21), the last expression can be made smaller than $\epsilon/2$ by choosing k large enough, which proves the tightness of $I_1(n)$. Similar arguments show that $I_j(n)$, $j = 2, 3, 4$, are tight sequences as well. The convergence in (2.23) follows.

By the decomposition (2.22), the proof will be finished if we can also establish that

$$\frac{1}{(n\log n)^{1/\alpha}} \sup_{f\in\mathcal{F}} \left| \int_0^\pi f(x)\psi(e^{ix}) L_n(x) K_n(-x)\, dx \right| \xrightarrow{P} 0 \tag{2.24}$$

and

$$\frac{1}{(n\log n)^{1/\alpha}} \sup_{f\in\mathcal{F}} \left| \int_0^\pi f(x)\psi(e^{-ix}) L_n(-x) K_n(x)\, dx \right| \xrightarrow{P} 0. \tag{2.25}$$

We will prove (2.24); the arguments for (2.25) will be analogous. We have, by the Cauchy-Schwarz inequality and the identity $|L_n(x)|^2 = n\, I_{n,\varepsilon}(x)$,

$$\begin{aligned} &\left| \int_0^\pi f(x)\psi(e^{ix}) L_n(x) K_n(-x)\, dx \right| \\ &\qquad \le const\ \|f\|_2 \left(\int_0^\pi |L_n(x)K_n(-x)|^2\, dx \right)^{1/2} \\ &\qquad \le const\ \|f\|_2\, n^{1/2} \left(\sup_{0\le x\le\pi} I_{n,\varepsilon}(x) \right)^{1/2} \left(\int_0^\pi |K_n(-x)|^2\, dx \right)^{1/2}. \end{aligned}$$

So we see that

$$\begin{aligned} &\frac{1}{(n\log n)^{1/\alpha}} \sup_{f\in\mathcal{F}} \left| \int_0^\pi f(x)\psi(e^{ix}) L_n(x) K_n(-x)\, dx \right| \\ &\qquad \le const\ \frac{1}{n^{\frac{1}{\alpha}-\frac{1}{2}}} \frac{\left(\sup_{0\le x\le\pi} I_{n,\varepsilon}(x)\right)^{1/2}}{(\log n)^{1/\alpha}} \left(\int_0^\pi |K_n(-x)|^2\, dx \right)^{1/2}. \end{aligned}$$

Similar arguments as for (2.23) ensure the tightness of the sequence $\int_0^\pi |K_n(-x)|^2\, dx$. The tightness of the term

$$\frac{\left(\sup_{0\le x\le\pi} I_{n,\varepsilon}(x)\right)^{1/2}}{(\log n)^{1/\alpha}}$$

follows from Mikosch et al. (2000), Theorem 2.1 (for $0 < \alpha < 1$) and Proposition 3.1 (for $1 \le \alpha < 2$). Thus we conclude that (2.24) holds, and Lemma 2.4.1 is proved. □

By (2.20) we may write for each f

$$\begin{aligned} &J_{n,X}(f) - a_0(f|\psi|^2)\gamma_{n,\varepsilon}(0) \\ &\qquad = J_{n,\varepsilon}(f|\psi|^2) - a_0(f|\psi|^2)\gamma_{n,\varepsilon}(0) + \int_0^\pi f(x) R_n(x)\, dx, \end{aligned}$$

where $|\psi|^2$ denotes $|\psi(e^{-i\cdot})|^2$. Combining this decomposition with Lemma 2.4.1, we can now state the following analogs to Corollary 2.3.6 and Theorem 2.3.7.

Corollary 2.4.2. *Assume $\alpha \in (0,1)$ or $\alpha \in [1,2)$ and let $\mathcal{F}$ be as defined in Corollary 2.3.6 or as in Theorem 2.3.7, respectively. Suppose that the set*

$$\mathcal{F}_\psi = \big\{ f : [0,\pi] \to \mathbb{R} : f|\psi|^2 \in \mathcal{F} \big\}$$

satisfies $\sup_{f \in \mathcal{F}_\psi} \|f\|_2 < \infty$ and (2.21) holds for some $\tau > 0$. Then the following weak convergence results hold in $\mathbb{C}(\mathcal{F})$:

$$\Big(n(n \log n)^{-1/\alpha} \big(J_{n,X}(f) - a_0(f|\psi|^2)\gamma_{n,\varepsilon}(0) \big), f \in \mathcal{F}_\psi \Big) \Longrightarrow \big(2Y(\mathbf{a}(f|\psi|^2)), f \in \mathcal{F}_\psi \big),$$
$$\Big((n/\log n)^{1/\alpha} \big(\widetilde{J}_{n,X}(f) - a_0(f|\psi|^2) \big), f \in \mathcal{F}_\psi \Big) \Longrightarrow \big(2\widetilde{Y}(\mathbf{a}(f|\psi|^2)), f \in \mathcal{F}_\psi \big). \tag{2.26}$$

2.5 The Stochastic Volatility Case

Recall the notion of a stochastic volatility process

$$X_t = \sigma_t \varepsilon_t, \quad t \in \mathbb{Z}, \tag{2.27}$$

where $(\varepsilon_t, t \in \mathbb{Z})$ is an i.i.d. SαS sequence for some $\alpha \in (0,2)$ and $(\log \sigma_t, t \in \mathbb{Z})$ is a linear Gaussian process, i.e.

$$\log \sigma_t = \sum_{j=-\infty}^{\infty} c_j \, \eta_{t-j} \,, \quad t \in \mathbb{Z},$$

where $(c_j, j \in \mathbb{Z})$ is a sequence of real numbers satisfying $\sum_j c_j^2 < \infty$ and $(\eta_j, j \in \mathbb{Z})$ is an i.i.d. standard normal sequence. The results for the integrated periodogram for this process are analogous to the i.i.d. case. In what follows, we will give the results and sketch the ideas of the proofs.

Our first observation is that Lemma 2.3.1 holds in a modified form for the sample autocovariance function $\gamma_{n,X}$ and autocorrelation function $\rho_{n,X}$ of the stochastic volatility process $(X_t, t \in \mathbb{Z})$; see Davis and Mikosch (2001), Theorem 4.1 and Remark 4.2.

Lemma 2.5.1. *For every* $m \geq 1$,

$$\left(\frac{n\gamma_{n,X}(0)}{n^{2/\alpha}}, \frac{n\gamma_{n,X}(h)}{(n\log n)^{1/\alpha}}, h = 1, \ldots, m\right) \Longrightarrow \left(\|\sigma_0\|_\alpha^2\, Y_0, \|\sigma_0\sigma_1\|_\alpha Y_1, \ldots, \|\sigma_0\sigma_m\|_\alpha Y_m\right), \tag{2.28}$$

where the Y_h*'s are independent,* Y_0 *is* $S_{\alpha/2}(\tau_1, 1, 0)$ *and* $(Y_h, h = 1, \ldots, m)$ *are i.i.d.* $S_\alpha(\tau_2, 0, 0)$ *for some* $\tau_i = \tau_i(\alpha)$, $i = 1, 2$, *and for any random variable* Y, $\|Y\|_\alpha = (E|Y|^\alpha)^{1/\alpha}$. *In particular,*

$$\left((n/\log n)^{1/\alpha}\rho_{n,\varepsilon}(h), h = 1, \ldots, m\right) \Longrightarrow \left(\frac{\|\sigma_0\,\sigma_h\|_\alpha}{\|\sigma_0\|_\alpha^2}\, Y_h/Y_0, h = 1, \ldots, m\right). \tag{2.29}$$

Next, let $Y_0, Y_1, Y_2, \ldots$ be the limiting variables given in Lemma 2.5.1. For $\mathbf{a} \in \ell^\alpha$ we define the sequences of processes

$$\begin{aligned} X_n(\mathbf{a}) &= (n\log n)^{-1/\alpha}\sum_{k=1}^{n-1} a_k\, n\gamma_{n,X}(k), \quad Y(\mathbf{a}) = \sum_{k=1}^{\infty} a_k\,\|\sigma_0\sigma_k\|_\alpha\, Y_k\,, \\ \widetilde{X}_n(\mathbf{a}) &= (n/\log n)^{1/\alpha}\sum_{k=1}^{n-1} a_k\,\rho_{n,X}(k), \quad \widetilde{Y}(\mathbf{a}) = Y(\mathbf{a})/(\|\sigma_0\|_\alpha^2\, Y_0)\,. \end{aligned} \tag{2.30}$$

Now we can formulate the following analog of Theorem 2.3.5.

Theorem 2.5.2. *Assume* $\alpha \in (0, 1)$. *For a compact subset* $\mathcal{A}$ *of* $\ell^\alpha \log \ell$ *satisfying* (2.11) *the following weak convergence result holds in* $\mathbb{C}(\mathcal{A})$*:*

$$(X_n(\mathbf{a}), \mathbf{a} \in \mathcal{A}) \Longrightarrow (Y(\mathbf{a}), \mathbf{a} \in \mathcal{A}) \quad \text{and} \quad (\widetilde{X}_n(\mathbf{a}), \mathbf{a} \in \mathcal{A}) \Longrightarrow (\widetilde{Y}(\mathbf{a}), \mathbf{a} \in \mathcal{A})$$

where X_n, $\widetilde{X}_n$, Y *and* $\widetilde{Y}$ *are as defined in* (2.30).

Proof. The proof of the convergence of the finite-dimensional distributions is analogous to the proof of Theorem 2.3.2. Using a Cramér–Wold argument, it suffices to prove the convergence of the one-dimensional distributions. So let $\mathbf{a} = (a_1, a_2, \ldots) \in \ell^\alpha \log \ell$. From (2.28) and the continuous mapping theorem it follows that for every $m \geq 1$,

$$(n \log n)^{-1/\alpha} \sum_{k=1}^{m} a_k \, n \, \gamma_{n,X}(k) \Longrightarrow Y_m(\mathbf{a}) := \sum_{k=1}^{m} a_k \, \|\sigma_0 \sigma_k\|_\alpha \, Y_k \, .$$

Also, since $\mathbf{a} \in \ell^\alpha$ and $\|\sigma_0 \sigma_k\|_\alpha \leq \|\sigma_0\|_{2\alpha}^2$,

$$Y_m(\mathbf{a}) \Longrightarrow Y(\mathbf{a}) \text{ as } m \longrightarrow \infty,$$

by the three-series theorem. According to Theorem 4.2 in Billingsley (1968), it remains to show that

$$\lim_{m \to \infty} \limsup_{n \to \infty} P\left((n \log n)^{-1/\alpha} \left| \sum_{k=m+1}^{n-1} a_k \, n \gamma_{n,X}(k) \right| > \epsilon \right) = 0$$

for every $\epsilon > 0$. We write $p_{n,m}(\mathbf{a}; \epsilon)$ for the above probabilities. Note that

$$p_{n,m}(\mathbf{a}; \epsilon) = P\left(\left| \sum_{k=m+1}^{n-1} a_k \sum_{j=1}^{n-k} \sigma_j \sigma_{j+k} \varepsilon_j \varepsilon_{j+k} \right| > \epsilon (n \log n)^{1/\alpha} \right).$$

Applying Lemma 2.6.1 conditional on $(\sigma_t, t \in \mathbb{Z})$ and using the inequality (2.10), we conclude that

$$\begin{aligned} & p_{n,m}(\mathbf{a}; \epsilon) \\ & \leq const \, \frac{1 + \log^+ \epsilon}{\epsilon^\alpha} \, \frac{1 + \log n}{n \log n} \sum_{k=m+1}^{n-1} \sum_{j=1}^{n-k} E|a_k \sigma_j \sigma_{j+k}|^\alpha \left(1 + \log^+ \frac{1}{|a_k \sigma_j \sigma_{j+k}|} \right). \end{aligned} \tag{2.31}$$

The inequality (2.10) immediately yields the following result for any random variable Y with $E(|Y|^\alpha) < \infty$ and any constant $c \in \mathbb{R}$:

$$\begin{aligned} & E\big(|c\, Y|^\alpha (1 + \log^+(1/|c\, Y|)) \big) \\ & \qquad \leq 2\, |c|^\alpha \, (1 + \log^+(1/|c|)) \, E\big(|Y|^\alpha (1 + \log^+(1/|Y|)) \big). \end{aligned} \tag{2.32}$$

Applying (2.32) to the expectation in (2.31), and observing that σ_1 has finite moments of any order, we obtain

$$\begin{aligned} p_{n,m}(\mathbf{a};\epsilon) &\leq const\, \frac{1+\log^+\epsilon}{\epsilon^\alpha}\, \frac{1+\log n}{n\log n} \sum_{k=m+1}^{n-1}\sum_{j=1}^{n-k} |a_k|^\alpha \left(1+\log^+\frac{1}{|a_k|}\right) \\ &\leq const \sum_{k=m+1}^{\infty} |a_k|^\alpha \left(1+\log^+\frac{1}{|a_k|}\right), \end{aligned}$$

where the constant in the last line depends on ϵ. Since $\mathbf{a}\in\ell^\alpha\log\ell$, the last expression vanishes as $m\to\infty$. This proves the convergence of finite-dimensional distributions for (X_n); the convergence of $(\widetilde{X}_n)$ can be shown analogously by utilizing (2.29).

For the proof of the tightness of (X_n) and $(\widetilde{X}_n)$, one can follow the lines of the proof of Theorem 2.3.5. Again, an application of Lemma 2.6.2 conditional on $(\sigma_t, t\in\mathbb{Z})$ and an application of (2.32) yield the same bounds for P_n' (adapted to the stochastic volatility sequence). □

We obtain the following analog to Corollary 2.3.6 for a stochastic volatility process $(X_t, t\in\mathbb{Z})$ as defined in (2.27).

Corollary 2.5.3. *Assume* $\alpha\in(0,1)$ *and let*

$$\mathcal{F} = \left\{ f\in L^2[0,\pi]:\ \mathbf{a}(f) = (a_1(f), a_2(f),\ldots)\in\mathcal{A}\right\},$$

where $\mathcal{A}$ *is a compact subset of* $\ell^\alpha\log\ell$ *satisfying* (2.11) *and* $\mathbf{a}(f)$ *is as specified in* (2.3). *Then the following weak convergence results hold in* $\mathbb{C}(\mathcal{F})$*:*

$$\begin{aligned} \left(n(n\log n)^{-1/\alpha}\big(J_{n,X}(f) - a_0(f)\gamma_{n,X}(0)\big), f\in\mathcal{F}\right) &\Longrightarrow \big(2Y(\mathbf{a}(f)), f\in\mathcal{F}\big), \\ \left((n/\log n)^{1/\alpha}\big(\widetilde{J}_{n,X}(f) - a_0(f)\big), f\in\mathcal{F}\right) &\Longrightarrow \big(2\widetilde{Y}(\mathbf{a}(f)), f\in\mathcal{F}\big). \end{aligned} \tag{2.33}$$

Finally, we also state an analog of Theorem 2.3.7 for stochastic volatility processes.

Theorem 2.5.4. *Assume $\alpha \in [1,2)$, define $\mathbf{a}(f)$ as in (2.3) and let $\mathcal{F}$ be a subset of $L^2[0,\pi]$ satisfying*

(i) $\mathbf{a}(f) \in \ell^\alpha \log \ell$ *for all* $f \in \mathcal{F}$,

(ii) $\exists \beta \in (0,\alpha)$ *such that*

$$N(\epsilon, \mathcal{F}, \rho_k) \le const \left[1 + \left(\frac{2^k}{\epsilon}\right)^\beta\right], \quad \epsilon > 0,\ k \ge 0\,.$$

Then the weak convergence result (2.33) holds in $\mathbb{C}(\mathcal{F})$.

Proof. The proof is analogous to that of Theorem 2.3.7. As before, one needs to apply Lemma 2.6.1 conditional on $(\sigma_t, t \in \mathbb{Z})$, in combination with (2.32). This will yield the same bounds for the probabilities $P_{n,m}$ and $Q_{n,m}$, adapted to the volatility sequence. □

2.6 Lemmas

For an array $\mathbf{b} = (b_{s,t}, s,t = 1,2,\ldots)$ of real numbers and a sequence $(\varepsilon_t, t = 1,2,\ldots)$ of i.i.d. $S_\alpha(1,0,0)$ random variables, define the quadratic forms

$$Q_{n,\varepsilon}(\mathbf{b}) = \sum_{1\le s\ne t\le n} b_{s,t}\ \varepsilon_s \varepsilon_t$$

and constants

$$\Gamma_n(\mathbf{b}) = \sum_{1\le s\ne t\le n} |b_{s,t}|^\alpha \left(1 + \log^+ \frac{1}{|b_{s,t}|}\right),$$

with the convention that the summands are zero when $b_{s,t} = 0$. The following lemma is a consequence of Theorem 3.1 in Rosiński and Woyczyński (1987); see also Kwapień and Woyczyński (1992).

Lemma 2.6.1. *For $\alpha \in (0,2)$, there exists a positive constant D_α such that for all $x > 0$:*

$$P\left(Q_{n,\varepsilon}(\mathbf{b}) > x\right) \le D_\alpha \, \frac{1+\log^+ x}{x^\alpha} \, \Gamma_n(\mathbf{b})\,.$$

Let now $\mathbf{C} = (C_0, C_{s,t}, s,t = 1,2,\ldots)$ be a sequence of i.i.d. $S_1(1,0,0)$ random variables, independent of $(\varepsilon_t, t \ge 1)$, and $\mathbf{b}$ be as above. The following lemma is a consequence of Lemma 2.6.1:

Lemma 2.6.2. *For $\alpha \in (0,1)$, there exists a positive constant D'_α such that for all $x > 0$:*

$$I_n(x) = P\left(\sum_{1\le s<t\le n} b_{s,t}\, C_{s,t}\, \varepsilon_s\varepsilon_t > x\right) \le D'_\alpha \frac{1+\log^+ x}{x^\alpha} \, \Gamma_n(\mathbf{b})\,. \tag{2.34}$$

Proof. Apply Lemma 2.6.1 to $I_n(x)$, conditionally on $\mathbf{C}$:

$$\begin{aligned} I_n(x) &= P\left(\sum_{1\le s<t\le n} b_{s,t}\, C_{s,t}\, \varepsilon_s\varepsilon_t > x\right) \\ &= E\,P\left(\sum_{1\le s<t\le n} b_{s,t}\, C_{s,t}\, \varepsilon_s\varepsilon_t > x \,\middle|\, \mathbf{C}\right) \qquad (2.35) \\ &\le \mathit{const}\, \frac{1+\log^+ x}{x^\alpha} \, \sum_{s=1}^{n-1}\sum_{t=s+1}^{n} E\left(|b_{s,t}C_0|^\alpha \left(1+\log^+ \frac{1}{|b_{s,t}C_0|}\right)\right). \end{aligned}$$

Because $\alpha \in (0,1)$ we have $E(|C_0|^\alpha) < \infty$, so we can apply the inequality (2.32) to the expectation in the last line, which yields

$$I_n(x) \le \mathit{const}\, \frac{1+\log^+ x}{x^\alpha} \, \sum_{s=1}^{n-1}\sum_{t=s+1}^{n} |b_{s,t}|^\alpha \, \left(1+\log^+ \frac{1}{|b_{s,t}|}\right),$$

as desired. □

CHAPTER 3

THE BM-CAF FRACTIONAL STABLE MOTION

3.1 Introduction

In this chapter we introduce the "Brownian motion-continuous additive functional (BM-CAF) fractional SαS motion" that was mentioned in Section 1.3. It is a generalization of the FBM-H-local time fractional SαS motion introduced in Cohen and Samorodnitsky (2006), in the case $H = 1/2$. In Section 3.2, we briefly discuss the construction of the FBM-H-local time fractional SαS motion and describe the random rewards scheme mentioned in Section 1.3 that converges to it. Section 3.3 gives some preliminary information on Brownian continuous additive functionals, including a fundamental representation theorem which states that each Brownian continuous additive functional can be associated with a unique Radon measure on $\mathbb{R}$. The BM-CAF fractional SαS motion is formally defined in Section 3.4 for a large class of associated Radon measures; the conditions on the associated measures are stronger in the case $\alpha \in (0, 1]$ than in the case $\alpha \in (1, 2]$. In Section 3.5 we show that the BM-CAF fractional SαS motion has stationary increments, and in Section 3.6 we study its increment process, under certain assumptions on the associated measure. In Section 3.7, we turn to the question of self-similarity. It turns out that, unlike the FBM-local time fractional SαS motion, the BM-CAF fractional SαS motion is not always self-similar: under certain assumptions on the associated measure, the latter process will lie in the domain of attraction of the first one, in a sense that will be made precise. In Section 3.8 we study the smoothness of sample paths through their Hölder continuity properties. In Section 3.9 we state and prove a random rewards convergence result similar to the one presented in Cohen and Samorodnitsky (2006), once again under certain limitations on the associated measure. Finally, in Section 3.10 we study a special class of BM-CAF fractional SαS motions for which the associated measure takes on a particular form, and we use it to demonstrate that some of the sufficient conditions introduced in earlier results are merely sufficient and not necessary.

3.2 The FBM-H-Local Time Fractional Stable Motion

In this section we review the construction of the FBM-H-fractional stable motion and the "random rewards scheme" mentioned in Section 1.3 that converges weakly to the FBM-1/2-local time fractional stable motion. Let $(B_H(t), t \geq 0)$ be a fractional Brownian motion with index of self-similarity H, defined on a probability space $(\Omega', \mathcal{F}', \mathbf{P}')$, and let $(l(x,t), x \in \mathbb{R}, t \geq 0)$ be its jointly continuous local time process

$$l(x,t) = \lim_{\epsilon \downarrow 0} \frac{1}{2\epsilon} \int_0^t \mathbf{1}\left\{B_H(s) \in (x-\epsilon, x+\epsilon)\right\} ds,$$

which is known to exist as an almost sure limit (see, for example, Berman (1970)). Also let M be an independently scattered SαS random measure on the space $\Omega' \times \mathbb{R}$ with control measure $\mathbf{P}' \times \text{Leb}$, where Leb denotes the Lebesgue measure on $\mathbb{R}$. M is assumed to be defined on another probability space $(\Omega, \mathcal{F}, \mathbf{P})$. (See Chapter 3 of Samorodnitsky and Taqqu (1994) for information on integrals with respect to stable random measures.) The FBM-H-local time fractional stable motion $(\Gamma(t), t \geq 0)$ is then defined as

$$\Gamma(t) = \int_{\Omega' \times \mathbb{R}} l(x,t) M(d\omega', dx), \quad t \geq 0. \tag{3.1}$$

This is a SαS H'-sssi process with index of self-similarity $H' = 1 - H + H/\alpha$. For $H = 1/2$, it arises as a weak limit of the following discrete scheme.

Let $(W_k, k \in \mathbb{Z})$ be a sequence of i.i.d. symmetric random variables satisfying $P(W_0 > x) \sim cx^{-\alpha}$ as $x \to \infty$, for some $c > 0$ and $0 < \alpha < 2$. Also let $(V_1, V_2, \ldots)$ be a sequence of i.i.d. integer valued random variables having zero mean and unit variance, independent of $(W_k, k \in \mathbb{Z})$. Consider the random walk $S_n = V_1 + \ldots + V_n, n \geq 1$. If one views S_n as describing the "position" of a "user" along the integer line at time n, and W_k as a "reward" associated with position k that will be collected whenever k is visited, then the total reward earned by time n will be

$$R_n = \sum_{j=1}^{n} W_{S_j}, \quad n \geq 1. \tag{3.2}$$

Assuming that there are many such users performing independent random walks and earning independent rewards, the properly normalized and time-scaled total reward process of all users will converge weakly to the FBM-1/2-local time fractional stable motion (which can also be called the BM-local time fractional stable motion) as the number of users increases. A heuristic explanation for this result can be obtained by rewriting (3.2) as

$$R_n = \sum_{k=\infty}^{\infty} \varphi(k, n) W_k, \quad n \geq 1, \tag{3.3}$$

where $\varphi(k, n) = \sum_{j=1}^{n} \mathbf{1}\{S_k = j\}$ is the local time of the random walk $(S_n, n \geq 1)$. Comparing (3.1) and (3.3), one observes that the limiting procedure turns the sum into an integral, the local time of the random walk into that of a Brownian motion, and the heavy-tailed random rewards into a SαS random measure.

The construction of the FBM-H-local time fractional stable motion was likely motivated by a very similar process introduced in Kesten and Spitzer (1979), henceforth called the *Kesten-Spitzer process*. It is defined as

$$\Delta(t) = \int_{\mathbb{R}} l(x, t) M(dx), \quad t \geq 0, \tag{3.4}$$

where $(l(x,t), x \in \mathbb{R}, t \geq 0)$ is the local time of a Brownian motion as before, and M is a SαS random measure defined on $\mathbb{R}$ with Lebesgue control measure, assumed to be independent of the Brownian motion. This is a sssi process that can be seen as a mixture of stable processes; it is *not* a stable process. The random rewards scheme of Cohen and Samorodnitsky (2006) yields the Kesten-Spitzer process in the limit if one considers the total reward of a single user rather than that of many users. (Kesten and Spitzer called the random rewards scheme with a single user a "random walk in a random environment.") Once again, heuristic support for this convergence is provided by the similarity of (3.3) and (3.4).

Cohen and Dombry (2009) generalized the convergence result of Cohen and Samorodnitsky (2006) to $H \neq 1/2$, by considering random walks with dependent steps. More precisely, they assumed that each user performs a random walk $S_n = [V_1 + ... + V_n], n \geq 1$, where $[\cdot]$ denotes the usual "floor" function and the sequence of steps $(V_1, V_2, \ldots)$ forms a stationary Gaussian sequence satisfying

$$\sum_{i=1}^{n} \sum_{j=1}^{n} E(V_i V_j) \sim n^{2H} \text{ as } n \to \infty$$

for some $0 < H < 1$. The properly normalized and time-scaled cumulative reward process of all users then converges weakly to the FBM-H-local time fractional stable motion as the number of users increases.

Dombry and Guillotin-Plantard (2009) replaced the fractional Brownian local time $l(x,t)$ in (3.1) by the local time of a β-stable Lévy motion with $\beta \in (1,2]$, while still assuming M to be a SαS random measure, $0 < \alpha < 2$, independent of the Lévy motion. They showed that the resulting process is again α-stable sssi, and that the random rewards scheme of Cohen and Samorodnitsky (2006) yields their process in the limit if one allows the i.i.d. steps $(V_1, V_2, \ldots)$ to be in the domain of attraction of a β-stable law, rather than having unit variance. Following the terminology of Cohen and Samorodnitsky, they named their process the "β-stable Lévy motion local time fractional α-stable motion."

Our aim is to generalize the construction (3.1) in the case $H = 1/2$ by replacing the integrand local time $l(x, t)$ by a general continuous additive functional of Brownian motion, study the resulting process, and in particular construct a modified version of the random rewards scheme of Cohen and Samorodnitsky (2006) that yields the generalized process in the scaling limit. We start by reviewing some preliminaries on Brownian continuous additive functionals.

3.3 Preliminaries on Brownian Continuous Additive Functionals

Let $B = (B(t), t \geq 0)$ be a Brownian motion defined on a probability space $(\Omega, \mathcal{F}, \mathbf{P})$. A *continuous additive functional* of B is a real-valued process $A = (A(t), t \geq 0)$ such that

(i) A is adapted to the natural filtration of B,

(ii) A is a.s. continuous, non-decreasing and vanishing at zero,

(iii) for each pair (s, t), $A(s + t) = A(t) + A(s) \circ \theta_t$ a.s.,

where $\theta_t : \Omega \to \Omega$ is the right-shift operator satisfying $B(s, \theta_t(\omega)) = B(t+s, \omega)$ for each $\omega \in \Omega$.

Clearly, the local time $(l(x, t), t \geq 0)$ at any point $x \in \mathbb{R}$ is a continuous additive functional, and so is the occupation time of any Borel set Γ,

$$A(t) = \int_0^t \mathbf{1}_\Gamma(B(s))ds.$$

For any continuous additive functional A of Brownian motion, there exists a unique Radon measure ν_A on $\mathbb{R}$ (called the *measure associated with* A) such that

$$(A(t), t \geq 0) = \left(\int_{\mathbb{R}} l(y,t)\nu_A(dy), t \geq 0 \right) \tag{3.5}$$

in finite-dimensional distributions. Conversely, any Radon measure ν_A on $\mathbb{R}$ defines a continuous additive functional A through (3.5). In view of this result, we see that the local time at $x \in \mathbb{R}$ is the continuous additive functional with associated measure δ_x, the Dirac point measure of mass 1 concentrated at x. Similarly, the associated measure of the occupation time of a Borel set Γ is the restriction of the Lebesgue measure on Γ. These results and more on Brownian continuous additive functionals can be found in Chapter X of Revuz and Yor (1999).

In order to replace the local time l in (3.1) with a continuous additive functional A, we need to introduce dependence on a space variable x for A. We do that in the "obvious" way, by defining

$$A(x,t) = \int_{\mathbb{R}} l(x+y,t)\nu_A(dy), \quad x \in \mathbb{R}, t \geq 0, \tag{3.6}$$

i.e. we define $A(x,t)$ to be the value of $A(t)$ for the vertically shifted Brownian motion $(B(t) - x, t \geq 0)$.

3.4 The BM-CAF Fractional Stable Motion

Let $(\Omega', \mathcal{F}', \mathbf{P}')$ be a probability space supporting a Brownian motion $(B(t), t \geq 0)$ with local time process $(l(x,t), x \in \mathbb{R}, t \geq 0)$, and let $(A(x,t), t \geq 0)$ be an arbitrary continuous additive functional (CAF) of B with associated measure ν_A. Let M be a SαS random measure on $\Omega' \times \mathbb{R}$ with control measure $\mathbf{P}' \times \text{Leb}$, where Leb denotes the Lebesgue measure. Suppose M itself lives on some other probability space $(\Omega, \mathcal{F}, \mathbf{P})$. We define the BM-CAF fractional stable motion by

$$
\begin{aligned}
Y(t) &= \int_{\Omega' \times \mathbb{R}} A(x,t) M(d\omega', dx) \\
&= \int_{\Omega' \times \mathbb{R}} \int_{\mathbb{R}} l(x+y,t) \nu_A(dy) M(d\omega', dx), \quad t \geq 0.
\end{aligned}
\tag{3.7}
$$

The first issue that needs to be addressed is that of well-definedness. The following two results identify sufficient conditions on the measure ν_A under which the process (3.7) is a well-defined SαS process. The conditions are more restrictive in the case $0 < \alpha \leq 1$ than in the case $1 < \alpha \leq 2$.

Theorem 3.4.1. *Suppose* $0 < \alpha \leq 1$ *and* ν_A *satisfies*

$$
\sum_{i=0}^{\infty} \beta^{(1-\alpha)i} \nu_A\big([\beta^i, \beta^{i+1})\big)^{\alpha} + \sum_{i=0}^{\infty} \beta^{(1-\alpha)i} \nu_A\big([-\beta^{i+1}, -\beta^i)\big)^{\alpha} < \infty \tag{3.8}
$$

for some constant $\beta > 1$*. Then,* $(Y(t), t \geq 0)$ *in (3.7) is a well-defined SαS process.*

Remark 3.4.2. The value of β in (3.8) does not matter, in the sense that (3.8) holds for all $\beta > 1$ if it holds for one. Indeed, given $\gamma = \beta^c$ for some $c > 0$,

$$
\begin{aligned}
\sum_{i=0}^{\infty} \gamma^{(1-\alpha)i} \nu_A\big([\gamma^i, \gamma^{i+1})\big)^{\alpha} &= \sum_{i=0}^{\infty} \beta^{c(1-\alpha)i} \nu_A\big([\beta^{ci}, \beta^{ci+c})\big)^{\alpha} \\
&\leq \sum_{i=0}^{\infty} \beta^{(1-\alpha)([ci]+1)} \nu_A\big([\beta^{[ci]}, \beta^{[ci]+[c]+2})\big)^{\alpha} \\
&\leq \mathit{const} \sum_{i=0}^{\infty} \beta^{(1-\alpha)([ci]+1)} \sum_{j=0}^{[c]+1} \nu_A\big([\beta^{[ci]+j}, \beta^{[ci]+j+1})\big)^{\alpha} \\
&\leq \mathit{const} \sum_{j=0}^{[c]+1} \sum_{i=0}^{\infty} \beta^{(1-\alpha)([ci]+j)} \nu_A\big([\beta^{[ci]+j}, \beta^{[ci]+j+1})\big)^{\alpha} \\
&\leq \mathit{const} \sum_{i=0}^{\infty} \beta^{(1-\alpha)i} \nu_A\big([\beta^i, \beta^{i+1})\big)^{\alpha}.
\end{aligned}
$$

Proof of Theorem 3.4.1. We need to check that, for any fixed $t \geq 0$,

$$
\mathbf{E}' \left(\int_{\mathbb{R}} \left(\int_{\mathbb{R}} l(x+y,t)\, \nu_A(dy) \right)^{\alpha} dx \right) < \infty \,. \tag{3.9}
$$

It will suffice to prove

$$\mathbf{E}'\left(l_*(t)^\alpha \int_{\mathbb{R}} \left(\int_{\mathbb{R}} \mathbf{1}\{|x+y| \le M(t)\}\, \nu_A(dy)\right)^\alpha dx\right) < \infty\,, \tag{3.10}$$

with

$$l_*(t) = \sup_{x\in\mathbb{R}} l(x,t), \tag{3.11}$$

$$M(t) = \sup_{0\le s\le t} |B(s)|. \tag{3.12}$$

It is known that for any fixed $t \ge 0$, $l_*(t)$ has finite moments of all orders; see, for example, Theorem 1.7 of Borodin (1986). It is also known that $M(t)$ has Gaussian-like probability tails, or more precisely,

$$\mathbf{P}'\left(M(t) \ge x\right) \le const \int_{x/\sqrt{2t}}^\infty e^{-u^2} du \le const\, e^{-x^2/2t}$$

for $x > 0$; see, for example, §10.2 of Ross (2006). In particular, for any fixed $t \ge 0$, $M(t)$ has finite moments of all orders as well. In the following, we will make frequent use of these facts without explicitly mentioning them each time. Let us denote $I(x,y) = \mathbf{1}\{|x+y| \le M(t)\}$ for notational convenience. We first prove

$$\mathbf{E}'\left(l_*(t)^\alpha \int_{\mathbb{R}} \left(\int_1^\infty I(x,y)\, \nu_A(dy)\right)^\alpha dx\right) < \infty\,. \tag{3.13}$$

The left hand side of (3.13) can be decomposed as

$$\begin{aligned} I_1 + I_2 := &\ \mathbf{E}'\left(l_*(t)^\alpha \int_0^\infty \left(\int_1^\infty I(x,y)\nu_A(dy)\right)^\alpha dx\right) \\ &+ \mathbf{E}'\left(l_*(t)^\alpha \int_0^\infty \left(\int_1^\infty I(-x,y)\nu_A(dy)\right)^\alpha dx\right). \end{aligned}$$

Defining $B_i = [\beta^i, \beta^{i+1})$, we have

$$\begin{aligned} I_1 &= \mathbf{E}'\left(l_*(t)^\alpha \int_0^\infty \left(\sum_{i=0}^\infty \int_{B_i} I(x,y)\, \nu_A(dy)\right)^\alpha dx\right) \\ &\le \mathbf{E}'\left(l_*(t)^\alpha \left(\sum_{i=0}^\infty \nu_A(B_i)\right)^\alpha \int_0^\infty \mathbf{1}\{x \le M(t)\}\, dx\right) \end{aligned} \tag{3.14}$$

$$\leq \sum_{i=0}^{\infty} \nu_A(B_i)^\alpha \, \mathbf{E}' \left(l_*(t)^\alpha \, M(t) \right) < \infty \, ,$$

where the finiteness follows from Cauchy-Schwarz inequality and (3.8). The term I_2 can be further decomposed as

$$\begin{aligned} I_2 &= \mathbf{E}' \left(l_*(t)^\alpha \int_0^{\beta^2} \left(\int_1^\infty I(-x,y) \, \nu_A(dy) \right)^\alpha dx \right) \\ &\quad + \mathbf{E}' \left(l_*(t)^\alpha \int_{\beta^2}^\infty \left(\int_1^\infty I(-x,y) \, \nu_A(dy) \right)^\alpha dx \right) \\ &:= I_{21} + I_{22} \, . \end{aligned}$$

Note that

$$\begin{aligned} I_{21} &= \mathbf{E}' \left(l_*(t)^\alpha \int_0^{\beta^2} \left(\sum_{i=0}^{\infty} \int_{B_i} I(-x,y) \, \nu_A(dy) \right)^\alpha dx \right) \\ &\leq \mathbf{E}' \left(l_*(t)^\alpha \int_0^{\beta^2} \left(\sum_{i=0}^{\infty} \nu_A(B_i) \right)^\alpha dx \right) \qquad (3.15) \\ &= \beta^2 \left(\sum_{i=0}^{\infty} \nu_A(B_i)^\alpha \right) \mathbf{E}' \left(l_*(t)^\alpha \right) < \infty \end{aligned}$$

by (3.8), so it remains to prove that $I_{22} < \infty$. We write

$$I_{22} = \sum_{j=2}^{\infty} \mathbf{E}' \left(l_*(t)^\alpha \int_{B_j} \left(\sum_{i=0}^{\infty} \int_{B_i} I(-x,y) \, \nu_A(dy) \right)^\alpha dx \right) := \sum_{j=2}^{\infty} r_j \, .$$

For $j \geq 2$,

$$\begin{aligned} r_j &\leq \mathbf{E}' \left(l_*(t)^\alpha \int_{B_j} \left(\sum_{i=0}^{j-2} \int_{B_i} I(-x,y) \, \nu_A(dy) \right)^\alpha dx \right) \\ &\quad + \mathbf{E}' \left(l_*(t)^\alpha \int_{B_j} \left(\sum_{i=j-1}^{j+1} \int_{B_i} I(-x,y) \, \nu_A(dy) \right)^\alpha dx \right) \\ &\quad + \mathbf{E}' \left(l_*(t)^\alpha \int_{B_j} \left(\sum_{i=j+2}^{\infty} \int_{B_i} I(-x,y) \, \nu_A(dy) \right)^\alpha dx \right) \\ &:= r_j^{(1)} + r_j^{(2)} + r_j^{(3)} \, . \end{aligned}$$

We will show that

$$\sum_{j=2}^{\infty} r_j^{(k)} < \infty \quad \text{for } k = 1, 2, 3. \qquad (3.16)$$

Let $Z_j(t) = \mathbf{1}\{\beta^j - \beta^{j-1} \leq M(t)\}$ and note that

$$
\begin{aligned}
r_j^{(1)} &= \mathbf{E}'\left(Z_j(t)\, l_*(t)^\alpha \int_{B_j} \left(\sum_{i=0}^{j-2} \int_{B_i} I(-x,y)\,\nu_A(dy)\right)^\alpha dx\right) \\
&\leq \int_{B_j} \sum_{i=0}^{j-2} \nu_A(B_i)^\alpha \, dx \; \mathbf{E}'\left(Z_j(t)\, l_*(t)^\alpha\right) \\
&\leq \left(\beta^{j+1} - \beta^j\right) \left(\sum_{i=0}^{\infty} \nu_A(B_i)^\alpha\right) \mathbf{E}'(Z_j(t)\, l_*(t)^\alpha) \\
&\leq c_1 \left(\beta^{j+1} - \beta^j\right) \mathbf{E}'\left(l_*(t)^{2\alpha}\right)^{1/2} \mathbf{P}'\left(\beta^j - \beta^{j-1} \leq M(t)\right)^{1/2} \\
&= c_1 \beta^j\, \mathbf{P}'\left(c_2 \beta^{j-1} \leq M(t)\right)^{1/2} \\
&\leq c_1 \beta^j \exp\left(-c_2 \beta^{2(j-1)}\right),
\end{aligned}
$$

where c_1 and c_2 are positive constants that may change from instance to instance and may depend on t. Since the last expression is summable over j, (3.16) is true for $k = 1$. Next, note that

$$
\begin{aligned}
\sum_{j=2}^{\infty} r_j^{(2)} &\leq \sum_{j=2}^{\infty} \mathbf{E}'\left(l_*(t)^\alpha \int_{B_j} \sum_{i=j-1}^{j+1} \left(\int_{B_i} I(-x,y)\,\nu_A(dy)\right)^\alpha dx\right) \\
&\leq \sum_{j=2}^{\infty} \left(\beta^{j+1} - \beta^j\right)^{1-\alpha} \mathbf{E}'\left(l_*(t)^\alpha \sum_{i=j-1}^{j+1} \left(\int_{B_j} \int_{B_i} I(-x,y)\,\nu_A(dy)\,dx\right)^\alpha\right) \\
&\leq \sum_{j=2}^{\infty} \left(\beta^{j+1} - \beta^j\right)^{1-\alpha} \mathbf{E}'\left(l_*(t)^\alpha \sum_{i=j-1}^{j+1} \left(\int_{B_i} \int_{\mathbb{R}} I(-x,y)\,dx\,\nu_A(dy)\right)^\alpha\right) \\
&\leq \sum_{j=2}^{\infty} \left(\beta^{j+1} - \beta^j\right)^{1-\alpha} \mathbf{E}'\left(2 l_*(t)^\alpha M(t)^\alpha\right) \sum_{i=j-1}^{j+1} \nu_A(B_i)^\alpha \\
&\leq \mathit{const}\; \mathbf{E}'\left(2 l_*(t)^\alpha M(t)^\alpha\right) \sum_{j=0}^{\infty} \beta^{j(1-\alpha)}\, \nu_A(B_j)^\alpha,
\end{aligned}
$$

where the expectation is finite by Cauchy-Schwarz inequality and the sum over j is finite by (3.8). Thus (3.16) is established for $k = 2$ as well. Finally,

$$
\begin{aligned}
r_j^{(3)} &= \mathbf{E}'\left(Z_{j+2}(t)\, l_*(t)^\alpha \int_{B_j} \left(\sum_{i=j+2}^{\infty} \int_{B_i} I(-x,y)\,\nu_A(dy)\right)^\alpha dx\right) \\
&\leq \int_{B_j} \sum_{i=j+2}^{\infty} \nu_A(B_i)^\alpha \, dx \; \mathbf{E}'\left(Z_{j+2}(t)\, l_*(t)^\alpha\right)
\end{aligned}
$$

$$
\begin{aligned}
&\leq \left(\beta^{j+1}-\beta^{j}\right)\left(\sum_{i=0}^{\infty}\nu_A(B_i)^{\alpha}\right)\mathbf{E}'(Z_{j+2}(t)\,l_*(t)^{\alpha})\\
&\leq c_1\left(\beta^{j+1}-\beta^{j}\right)\mathbf{E}'\left(l_*(t)^{2\alpha}\right)^{1/2}\mathbf{P}'\left(\beta^{j+2}-\beta^{j+1}\leq M(t)\right)^{1/2}\\
&= c_1\beta^{j}\,\mathbf{P}'\left(c_2\beta^{j}\leq M(t)\right)^{1/2}\\
&\leq c_1\beta^{j}\exp\left(-c_2\beta^{2j}\right)\ ,
\end{aligned}
$$

where c_1 and c_2 are positive constants that may change from instance to instance and may depend on t. Since the last expression is summable over j, (3.16) is true for $k = 3$. It now follows that $I_{22} < \infty$, and combined with (3.15), this yields $I_2 < \infty$. Thus we have shown (3.13).

Now, it can be shown by analogous arguments that

$$
\mathbf{E}'\left(l_*(t)^{\alpha}\int_{\mathbb{R}}\left(\int_{-\infty}^{-1}I(x,y)\,\nu_A(dy)\right)^{\alpha}dx\right)<\infty\,, \tag{3.17}
$$

and also note that

$$
\begin{aligned}
&\mathbf{E}'\left(l_*(t)^{\alpha}\int_{\mathbb{R}}\left(\int_{-1}^{1}I(x,y)\,\nu_A(dy)\right)^{\alpha}dx\right)\\
&\qquad\leq \mathbf{E}'\left(l_*(t)^{\alpha}\int_{-M(t)-1}^{M(t)+1}\nu_A\left([-1,1)\right)^{\alpha}dx\right)\\
&\qquad= \nu_A\left([-1,1)\right)^{\alpha}\mathbf{E}'\left(2l_*(t)^{\alpha}(M(t)+1)\right)<\infty.
\end{aligned} \tag{3.18}
$$

Combining (3.13), (3.17) and (3.18) yields (3.10), and well-definedness follows. □

Theorem 3.4.3. *Suppose* $1 < \alpha \leq 2$ *and* ν_A *satisfies*

$$
\sum_{i=0}^{\infty}\nu_A\big([\beta^{i},\beta^{i+1})\big)^{\alpha}+\sum_{i=0}^{\infty}\nu_A\big([-\beta^{i+1},-\beta^{i})\big)^{\alpha}<\infty \tag{3.19}
$$

for some constant $\beta > 1$. *Then,* $(Y(t), t \geq 0)$ *in (3.7) is a well-defined* SαS *process.*

Remark 3.4.4. As in condition (3.8) of Theorem 3.4.1, condition (3.19) holds for all $\beta > 1$ if it holds for one. We omit the proof.

Proof of Theorem 3.4.3. The proof is similar to that of Theorem 3.4.1. Using the same notation as in that proof, it will suffice to prove that

$$\mathbf{E}'\left(l_*(t)^\alpha \int_{\mathbb{R}} \left(\int_{\mathbb{R}} I(x,y)\,\nu_A(dy)\right)^\alpha dx\right) < \infty\,, \tag{3.20}$$

and the first step is to show that

$$\mathbf{E}'\left(l_*(t)^\alpha \int_{\mathbb{R}} \left(\int_1^\infty I(x,y)\,\nu_A(dy)\right)^\alpha dx\right) < \infty\,. \tag{3.21}$$

The left hand side of (3.21) can be decomposed as

$$\begin{aligned} I_1 + I_2 := \mathbf{E}'&\left(l_*(t)^\alpha \int_0^\infty \left(\int_1^\infty I(x,y)\nu_A(dy)\right)^\alpha dx\right) \\ &+ \mathbf{E}'\left(l_*(t)^\alpha \int_0^\infty \left(\int_1^\infty I(-x,y)\nu_A(dy)\right)^\alpha dx\right) \end{aligned}$$

Defining $B_i = [\beta^i, \beta^{i+1})$ and $N(t) = \max\{i \geq 0 : \beta^i \leq M(t)\}$, we have

$$\begin{aligned} I_1 &= \mathbf{E}'\left(l_*(t)^\alpha \int_0^\infty \left(\sum_{i=0}^{N(t)} \int_{B_i} I(x,y)\,\nu_A(dy)\right)^\alpha dx\right) \\ &\leq \mathbf{E}'\left(l_*(t)^\alpha \left(\sum_{i=0}^{N(t)} \nu_A(B_i)\right)^\alpha \int_0^\infty \mathbf{1}\{x \leq M(t)\}\,dx\right) \\ &\leq \mathbf{E}'\left(l_*(t)^\alpha\, N(t)^{\alpha-1} \sum_{i=0}^{N(t)} \nu_A(B_i)^\alpha \int_0^\infty \mathbf{1}\{x \leq M(t)\}\,dx\right) \\ &\leq \left(\sum_{i=0}^{\infty} \nu_A(B_i)^\alpha\right) \mathbf{E}'\left(l_*(t)^\alpha\, N(t)^{\alpha-1} M(t)\right) < \infty\,, \end{aligned} \tag{3.22}$$

since the sum over $i \geq 0$ is finite by (3.19) and the random variables $l_*(t), M(t)$ and $N(t)$ have all moments finite. (Finite moments for $N(t)$ are implied by the fact that $N(t) \leq \log M(t)/\log\beta$.)

The term I_2 can be further decomposed as

$$\begin{aligned} I_2 &= \mathbf{E}'\left(l_*(t)^\alpha \int_0^{\beta^2} \left(\int_1^\infty I(-x,y)\,\nu_A(dy)\right)^\alpha dx\right) \\ &\quad + \mathbf{E}'\left(l_*(t)^\alpha \int_{\beta^2}^\infty \left(\int_1^\infty I(-x,y)\,\nu_A(dy)\right)^\alpha dx\right) \\ &:= I_{21} + I_{22}\,. \end{aligned}$$

Defining $\widetilde{N}(t) = \max\{i \geq 0 : \beta^i \leq M(t) + \beta^2\}$, we see that

$$\begin{aligned}
I_{21} &\leq \mathbf{E}'\Bigg(l_*(t)^\alpha \int_0^{\beta^2} \Bigg(\sum_{i=0}^{\widetilde{N}(t)} \int_{B_i} I(-x,y)\, \nu_A(dy) \Bigg)^\alpha dx \Bigg) \\
&\leq \mathbf{E}'\Bigg(l_*(t)^\alpha \Bigg(\sum_{i=0}^{\widetilde{N}(t)} \nu_A(B_i) \Bigg)^\alpha \Bigg) \int_0^{\beta^2} 1\, dx \\
&\leq \beta^2\, \mathbf{E}'\Bigg(l_*(t)^\alpha\, \widetilde{N}(t)^{\alpha-1} \sum_{i=0}^{\widetilde{N}(t)} \nu_A(B_i)^\alpha \Bigg) \\
&\leq \beta^2 \Bigg(\sum_{i=0}^{\infty} \nu_A(B_i)^\alpha \Bigg) \mathbf{E}'\big(l_*(t)^\alpha\, \widetilde{N}(t)^{\alpha-1} \big) < \infty
\end{aligned} \tag{3.23}$$

as before, so it remains to prove that $I_{22} < \infty$. We write

$$I_{22} = \sum_{j=2}^{\infty} \mathbf{E}' \Bigg(l_*(t)^\alpha \int_{B_j} \Bigg(\sum_{i=0}^{\infty} \int_{B_i} I(-x,y)\, \nu_A(dy) \Bigg)^\alpha dx \Bigg) := \sum_{j=2}^{\infty} r_j\, .$$

For $j \geq 2$,

$$\begin{aligned}
r_j \leq 3^{\alpha-1} \Bigg[&\mathbf{E}' \Bigg(l_*(t)^\alpha \int_{B_j} \Bigg(\sum_{i=0}^{j-2} \int_{B_i} I(-x,y)\, \nu_A(dy) \Bigg)^\alpha dx \Bigg) \\
&+ \mathbf{E}' \Bigg(l_*(t)^\alpha \int_{B_j} \Bigg(\sum_{i=j-1}^{j+1} \int_{B_i} I(-x,y)\, \nu_A(dy) \Bigg)^\alpha dx \Bigg) \\
&+ \mathbf{E}' \Bigg(l_*(t)^\alpha \int_{B_j} \Bigg(\sum_{i=j+2}^{\infty} \int_{B_i} I(-x,y)\, \nu_A(dy) \Bigg)^\alpha dx \Bigg) \Bigg] \\
:= 3^{\alpha-1} &\left(r_j^{(1)} + r_j^{(2)} + r_j^{(3)} \right) .
\end{aligned}$$

We will show that

$$\sum_{j=2}^{\infty} r_j^{(k)} < \infty \quad \text{for } k = 1, 2, 3. \tag{3.24}$$

Let $Z_j(t) = \mathbf{1}\{\beta^j - \beta^{j-1} \leq M(t)\}$ and note that

$$\begin{aligned}
r_j^{(1)} &= \mathbf{E}' \Bigg(Z_j(t)\, l_*(t)^\alpha \int_{B_j} \Bigg(\sum_{i=0}^{j-2} \int_{B_i} I(-x,y)\, \nu_A(dy) \Bigg)^\alpha dx \Bigg) \\
&\leq (j-1)^{\alpha-1} \mathbf{E}' \Bigg(Z_j(t)\, l_*(t)^\alpha \int_{B_j} \sum_{i=0}^{j-2} \nu_A(B_i)^{\alpha-1} \int_{B_i} I(-x,y)^\alpha \nu_A(dy)\, dx \Bigg)
\end{aligned}$$

$$
\begin{aligned}
&\leq (j-1)^{\alpha-1}\mathbf{E}'\left(Z_j(t)\, l_*(t)^\alpha\right)\sum_{i=0}^{j-2}\nu_A(B_i)^\alpha\int_{B_j}1\,dx \\
&\leq (j-1)^{\alpha-1}\left(\beta^{j+1}-\beta^j\right)\left(\sum_{i=0}^{\infty}\nu_A(B_i)^\alpha\right)\mathbf{E}'(Z_j(t)\, l_*(t)^\alpha) \\
&\leq c_1(j-1)^{\alpha-1}\left(\beta^{j+1}-\beta^j\right)\mathbf{E}'\left(l_*(t)^{2\alpha}\right)^{1/2}\mathbf{P}'\left(\beta^j-\beta^{j-1}\leq M(t)\right)^{1/2} \\
&= c_1(j-1)^{\alpha-1}\beta^j\,\mathbf{P}'\left(c_2\beta^{j-1}\leq M(t)\right)^{1/2} \\
&\leq c_1(j-1)^{\alpha-1}\beta^j\exp\left(-c_2\beta^{2(j-1)}\right),
\end{aligned}
$$

where c_1 and c_2 are positive constants that may change from instance to instance and may depend on t. Since the last expression is summable over j, (3.24) is true for $k=1$. Next, note that

$$
\begin{aligned}
\sum_{j=2}^{\infty} r_j^{(2)} &\leq 3^{\alpha-1}\sum_{j=2}^{\infty}\mathbf{E}'\left(l_*(t)^\alpha\int_{B_j}\sum_{i=j-1}^{j+1}\left(\int_{B_i}I(-x,y)\,\nu_A(dy)\right)^\alpha dx\right) \\
&\leq 3^{\alpha-1}\sum_{j=2}^{\infty}\mathbf{E}'\left(l_*(t)^\alpha\int_{B_j}\sum_{i=j-1}^{j+1}\left(\nu_A(B_i)^{\alpha-1}\int_{B_i}I(-x,y)\,\nu_A(dy)\right)dx\right) \\
&\leq 3^{\alpha-1}\sum_{j=2}^{\infty}\mathbf{E}'\left(l_*(t)^\alpha\sum_{i=j-1}^{j+1}\left(\nu_A(B_i)^{\alpha-1}\int_{B_i}\int_{\mathbb{R}}I(-x,y)\,dx\,\nu_A(dy)\right)\right) \\
&= 3^{\alpha-1}\sum_{j=2}^{\infty}2\mathbf{E}'(l_*(t)^\alpha M(t))\sum_{i=j-1}^{j+1}\nu_A(B_i)^\alpha \\
&\leq \mathit{const}\;\mathbf{E}'(l_*(t)^\alpha M(t))\sum_{j=0}^{\infty}\nu_A(B_j)^\alpha,
\end{aligned}
$$

where the last expression is finite by Cauchy-Schwarz inequality and by the assumption (3.19), so that (3.24) is true for $k=2$ as well. It remains to prove (3.24) for $k=3$. Letting

$$
K(t)=\min\left\{i\geq 0:\beta^{i+1}-\beta^i>M(t)\right\},
$$

we see that

$$
r_j^{(3)}=\mathbf{E}'\left(Z_{j+2}(t)\, l_*(t)^\alpha\int_{B_j}\left(\sum_{i=j+2}^{\infty}\int_{B_i}I(-x,y)\,\nu_A(dy)\right)^\alpha dx\right)
$$

$$
\begin{aligned}
&\leq \mathbf{E}'\left(Z_{j+2}(t)\, l_*(t)^\alpha\, K(t)^{\alpha-1} \int_{B_j} \sum_{i=j+2}^{\infty} \left(\int_{B_i} I(-x,y)\, \nu_A(dy)\right)^\alpha dx\right) \\
&\leq \mathbf{E}'\left(Z_{j+2}(t)\, l_*(t)^\alpha\, K(t)^{\alpha-1} \int_{B_j} \sum_{i=j+2}^{\infty} \left(\nu_A(B_i)^{\alpha-1} \int_{B_i} I(-x,y)\, \nu_A(dy)\right) dx\right) \\
&\leq \left(\beta^{j+1} - \beta^j\right) \mathbf{E}'\left(Z_{j+2}(t)\, l_*(t)^\alpha\, K(t)^{\alpha-1}\right) \sum_{i=j+2}^{\infty} \nu_A(B_i)^\alpha \\
&\leq const\, \beta^j\, \mathbf{E}'\left(l_*(t)^{2\alpha} K(t)^{2(\alpha-1)}\right)^{1/2} \mathbf{P}'\left(\beta^{j+2} - \beta^{j+1} \leq M(t)\right)^{1/2} \\
&= const\, \beta^j\, \mathbf{P}'\left(\beta^{j+2} - \beta^{j+1} \leq M(t)\right)^{1/2},
\end{aligned}
$$

where we use the fact that $l_*(t)$ and $K(t)$ have all moments finite. The last expression is summable over j as before, so we conclude that (3.24) holds for $k = 3$. It follows that $I_{22} < \infty$, and combined with (3.23), this yields $I_2 < \infty$. Thus we obtain (3.21).

Now, it can be shown by analogous arguments that

$$
\mathbf{E}'\left(l_*(t)^\alpha \int_{\mathbb{R}} \left(\int_{-\infty}^{-1} I(x,y)\, \nu_A(dy)\right)^\alpha dx\right) < \infty\,. \tag{3.25}
$$

Moreover,

$$
\begin{aligned}
&\mathbf{E}'\left(l_*(t)^\alpha \int_{\mathbb{R}} \left(\int_{-1}^{1} I(x,y)\, \nu_A(dy)\right)^\alpha dx\right) \\
&\qquad \leq \mathbf{E}'\left(l_*(t)^\alpha \int_{-M(t)-1}^{M(t)+1} \nu_A\left([-1,1)\right)^\alpha dx\right) \\
&\qquad = \nu_A\left([-1,1)\right)^\alpha \mathbf{E}'\left(2 l_*(t)^\alpha (M(t)+1)\right) < \infty.
\end{aligned} \tag{3.26}
$$

Combining (3.21), (3.25) and (3.26) yields (3.20), and well-definedness follows. □

3.5 Stationary Increments

In this section we prove that the BM-CAF fractional SαS motion has stationary increments. Recall that a stochastic process $(X(t), t \geq 0)$ has stationary increments if

$$(X(t+s)-X(s), t \geq 0) \stackrel{d}{=} (X(t)-X(0), t \geq 0)$$

for any $s > 0$. Here, $\stackrel{d}{=}$ denotes equality in finite-dimensional distributions, as usual. We will need the following lemma, which is proved for $d = 1$ in the Appendix of Samorodnitsky (2010). The proof for $d > 1$ is analogous.

Lemma 3.5.1. *Let $(B(t), t \geq 0)$ be a Brownian motion defined on $(\Omega, \mathcal{F}, P)$, with local time process $(l(x,t), x \in \mathbb{R}, t \geq 0)$. Then for any $y_1, \ldots, y_d \in \mathbb{R}$, the law of*

$$((l(x+y_i, t+s) - l(x+y_i, s), i = 1, \ldots, d), x \in \mathbb{R}, t \geq 0)$$

under $Leb \times P$ does not depend on $s \geq 0$.

The following is the main result of this section.

Theorem 3.5.2. *Let $(Y(t), t \geq 0)$ be a BM-CAF fractional stable motion as defined in (3.7), with α and ν_A satisfying the hypotheses of Theorem 3.4.1 or Theorem 3.4.3. Then, $(Y(t), t \geq 0)$ has stationary increments.*

Proof. Let $\nu_1, \nu_2, \ldots$ be a sequence of discrete measures defined as follows:

$$\nu_n = \sum_{i=-n^2}^{n^2} \nu_A\left(\left[\frac{i}{n}, \frac{i+1}{n}\right)\right) \delta_{\frac{i}{n}},$$

with δ_x denoting the Dirac point measure of mass 1 concentrated at x. Also let $\theta_1, \ldots, \theta_k \in \mathbb{R}, 0 \leq t_1 < \ldots < t_k$ and $s \geq 0$. We have

$$\begin{aligned}
&\mathbf{E} \exp\left(i \sum_{j=1}^{k} \theta_j \big(Y(t_j+s) - Y(s)\big)\right) \\
&= \exp\left(-\int_{\mathbb{R}} \mathbf{E}' \left|\sum_{j=1}^{k} \theta_j \int_{\mathbb{R}} \big(l(x+y, t_j+s) - l(x+y, s)\big)\, \nu_A(dy)\right|^{\alpha} dx\right) \qquad (3.27) \\
&= \exp\left(-\int_{\mathbb{R}} \mathbf{E}' \lim_{n\to\infty} \left|\sum_{j=1}^{k} \theta_j \int_{\mathbb{R}} \big(l(x+y, t_j+s) - l(x+y, s)\big)\, \nu_n(dy)\right|^{\alpha} dx\right).
\end{aligned}$$

Now note that for each $n \geq 1$,

$$\begin{aligned}
&\left| \sum_{j=1}^{k} \theta_j \int_{\mathbb{R}} \big(l(x+y, t_j+s) - l(x+y,s) \big) \, \nu_n(dy) \right|^{\alpha} \\
&\qquad \leq const \left(\int_{\mathbb{R}} \big(l(x+y, t_k+s) - l(x+y,s) \big) \, \nu_n(dy) \right)^{\alpha} \\
&\qquad \leq const \; l_*(t_k+s)^{\alpha} \left(\int_{\mathbb{R}} \mathbf{1}\{ |x+y| \leq M(t_k+s) \} \, \nu_n(dy) \right)^{\alpha} \\
&\qquad \leq const \; l_*(t_k+s)^{\alpha} \left(\int_{\mathbb{R}} \mathbf{1}\{ |x+y| \leq M(t_k+s) + 1 \} \, \nu_A(dy) \right)^{\alpha},
\end{aligned}$$

where M is as defined in (3.12). The last expression is integrable with respect to Leb $\times$ $\mathbf{P'}$; the proof is analogous to that of (3.10) or (3.20), depending on the value of α. Therefore we can apply the dominated convergence theorem to the last expression in (3.27) and conclude that it is the same as

$$\lim_{n \to \infty} \exp \left(- \int_{\mathbb{R}} \mathbf{E'} \left| \sum_{j=1}^{k} \theta_j \int_{\mathbb{R}} \big(l(x+y, t_j+s) - l(x+y,s) \big) \, \nu_n(dy) \right|^{\alpha} dx \right),$$

which is in turn equal to

$$\lim_{n \to \infty} \exp \left(- \int_{\mathbb{R}} \mathbf{E'} \left| \sum_{j=1}^{k} \theta_j \int_{\mathbb{R}} l(x+y, t_j) \, \nu_n(dy) \right|^{\alpha} dx \right), \tag{3.28}$$

by Lemma 3.5.1. Another application of the dominated convergence theorem now allows us to move the limit in (3.28) back under the expectation and conclude that

$$\begin{aligned}
&\mathbf{E} \exp \left(i \sum_{j=1}^{k} \theta_j \big(Y(t_j+s) - Y(s) \big) \right) \\
&\qquad = \exp \left(- \int_{\mathbb{R}} \mathbf{E'} \left| \sum_{j=1}^{k} \theta_j \int_{\mathbb{R}} l(x+y, t_j) \, \nu_A(dy) \right|^{\alpha} dx \right) \\
&\qquad = \mathbf{E} \exp \left(i \sum_{j=1}^{k} \theta_j Y(t_j) \right),
\end{aligned}$$

which proves the theorem. □

3.6 The Increment Process

In this section we investigate the BM-CAF fractional SαS noise

$$Z_n = Y(n+1) - Y(n), \quad n \geq 0,$$

for a BM-CAF fractional SαS process $(Y(t), t \geq 0)$. By Theorem 3.5.2, $(Z_n, n \geq 0)$ is a stationary SαS process. Rosiński (1995) has shown that any stationary SαS sequence $(X_n, n \geq 0)$ can be represented as

$$X_n = \int_E f_n(x)\, M(dx), \quad n \geq 0,$$

where M is a SαS random measure on a measurable space $(E, \mathcal{E})$ with a σ-finite control measure m, and $(f_n(\cdot), n \geq 0)$ are functions in $L^\alpha(m, \mathcal{E})$ of the form

$$f_n(x) = a_n(x) \left(\frac{dm \circ \phi^n}{dm}(x) \right)^{1/\alpha} f \circ \phi^n(x), \quad x \in E,$$

for $n \geq 0$. Here, $\phi : E \to E$ is a measurable nonsingular map (i.e., a one-to-one map with both ϕ and ϕ^{-1} measurable, mapping the control measure m into an equivalent measure),

$$a_n(x) = \prod_{j=0}^{n-1} u \circ \phi^j(x), \quad x \in E,$$

for $n \geq 0$, with $u : E \to \{-1, 1\}$ a measurable function and $f \in L^\alpha(m, \mathcal{E})$. Many properties of the stationary SαS sequence $(X_n, n \geq 0)$ are closely connected with the ergodic-theoretic properties of the flow (group of maps) $(\phi^n, n \geq 0)$.

A basic fact from ergodic theory is the existence of the so-called Hopf decomposition of the set E with respect to the flow $(\phi^n, n \geq 0)$. This means that the set E can be decomposed into a disjoint union $E = C \cup D$, such that C and D are measurable sets that are invariant under the map ϕ, and the flow is *conservative* on C and *dissipative* on D; see, e.g., Krengel (1985) for details. This allows us to write

$$X_n = \int_C f_n(x)\, M(dx) + \int_D f_n(x)\, M(dx) := X_n^C + X_n^D, \tag{3.29}$$

a unique in law decomposition of a stationary SαS process into a sum of two independent SαS processes, one *generated by a conservative flow* and the other *generated by a dissipative flow.* An i.i.d. SαS sequence is generated by a dissipative flow (i.e. the component X^C in (3.29) vanishes). We refer to Rosiński (1995) for more information.

Based on the growth rate of the partial maxima $M_n = \max_{1\le j\le n} |X_j|$, Samorodnitsky (2004) proposed to regard stationary SαS processes generated by dissipative flows as *short memory* processes (with partial maxima growing at the same rate as those of an i.i.d. SαS sequence) and stationary SαS processes generated by conservative flows as *long memory* processes (with partial maxima growing at a strictly slower rate than those of an i.i.d. SαS sequence). An alternative classification of flows, into *null* and *positive* flows, was put forward in Samorodnitsky (2005). While a stationary SαS process generated by a dissipative flow is generated by a null flow, processes generated by conservative flows can be generated by either positive or null flow. Moreover, a stationary SαS process is ergodic if and only if it is generated by a null flow. This suggests that processes generated by positive flows can be viewed as having infinite memory, while those generated by conservative null flows can be viewed as having finite but long memory.

It has already been mentioned in Section 1.3 that the linear fractional stable noise is generated by a dissipative flow (and hence can be labeled a short-memory process), the real harmonizable stable noise is generated by a positive flow (and hence can be labeled an infinite-memory process), and the FBM-local time fractional stable noise is generated by a conservative null flow (and hence can be labeled a finite-but-long memory process). The next result shows that if $\nu_A(\mathbb{R}) < \infty$, the BM-CAF fractional SαS noise is also generated by a conservative null flow.

Theorem 3.6.1. *Let $(Y(t), t \geq 0)$ be a well-defined BM-CAF fractional SαS process with an associated measure ν_A satisfying $\nu_A(\mathbb{R}) < \infty$. Then the BM-CAF fractional SαS noise $Z_n = Y(n+1) - Y(n), n \geq 0$, is generated by a conservative null flow.*

Proof. The proof is a modification of that of Theorem 4.1 in Cohen and Samorodnitsky (2006). Note that the noise $(Z_n, n \geq 0)$ has an integral representation

$$Z_n = \int_{\Omega' \times \mathbb{R}} \left(A(x, n+1)(\omega') - A(x, n)(\omega')\right) M(d\omega', dx), \quad n \geq 0. \tag{3.30}$$

Let C be the space of continuous functions from $[0, \infty)$ to $\mathbb{R}$ and $\mathbf{P}'_1$ the Wiener measure on C, under which the coordinate map $B(\omega', t) := \omega'(t)$ is a Brownian motion. Let m be a σ-finite measure on C defined by $m = (\mathbf{P}'_1 \times \text{Leb}) \circ T^{-1}$, where $T : C \times \mathbb{R} \to C$ is given by $T(\omega', x) = \omega' - x$. Define a measurable function $\mathcal{A} : C \to \mathbb{R}$ by $\mathcal{A}(\omega') = A(0, 1)(\omega')$. An alternative representation of the process in (3.30) is then

$$Z_n = \int_C \mathcal{A} \circ \phi^n(\omega') M_1(d\omega'), \quad n \geq 0, \tag{3.31}$$

where M_1 is a SαS random measure on C with control measure m, and $\phi : C \to C$ is given by $\phi(\omega') = \omega'(\cdot + 1)$. The stationarity of the increments of the Brownian motion implies that the map ϕ preserves the measure m. A conclusion is that the flow $(\phi^n, n \geq 0)$ and the underlying measure space on which $(\phi^n, n \geq 0)$ acts are the same, independently of the value of α. Therefore, it is sufficient to prove the theorem in the case $\alpha = 1$, which we will assume until the end of the proof. (Note that $(Y(t), t \geq 0)$ is well-defined for $\alpha = 1$, since $\nu_A(\mathbb{R}) < \infty$ is equivalent to condition (3.8) with $\alpha = 1$.)

We continue with the representation (3.30). Note that, by the recurrence property of Brownian motion, $l(x,t) \to \infty$ for all $x \in \mathbb{R}$ on a set of probability 1. Combined with the representation (3.6), this yields that $A(x,t) \to \infty$ for all $x \in \mathbb{R}$ on a set of probability 1. Therefore,

$$\sum_{n=0}^{m} \left(A(x,n+1)(\omega') - A(x,n)(\omega')\right) = A(x,m+1)(\omega') \longrightarrow \infty \quad \text{as } m \to \infty$$

outside a subset of $\Omega' \times \mathbb{R}$ of measure 0. By Corollary 4.2 of Rosiński (1995), this implies that the BM-CAF fraction stable noise $(Z_n, n \geq 0)$ is generated by a conservative flow.

In order to prove that $(Z_n, n \geq 0)$ is generated by a null flow, we will apply Corollary 2.2 of Samorodnitsky (2005) to the obvious two-sided extension $(Z_n, n \in \mathbb{Z})$. By symmetry, it will be enough to find a nonincreasing nonnegative sequence $(w_n, n \geq 0)$ such that

$$\sum_{n=0}^{\infty} w_n = \infty \tag{3.32}$$

and

$$\sum_{n=0}^{\infty} w_n \left(A(x,n+1)(\omega') - A(x,n)(\omega')\right) < \infty \tag{3.33}$$

for $\mathbf{P}' \times$ Leb-almost every (ω', x).

Let $w_n = (1+n)^{-\theta}$ for some $1/2 < \theta \leq 1$. Since $\theta \leq 1$, the condition (3.32) is satisfied. To check (3.33), it will be enough to find a strictly positive function g such that

$$\mathbf{E}'\left(\int_{\mathbb{R}} g(x) \sum_{n=0}^{\infty} w_n \left(A(x,n+1)(\omega') - A(x,n)(\omega')\right)\, dx \right) < \infty.$$

Note that

$$
\begin{aligned}
&\mathbf{E}'\Big(\int_{\mathbb{R}} g(x)\sum_{n=0}^{\infty} w_n\left(A(x,n+1)(\omega') - A(x,n)(\omega')\right)dx\Big)\\
&\quad = \mathbf{E}'\Big(\int_{\mathbb{R}} g(x)\sum_{n=0}^{\infty} w_n\int_{\mathbb{R}}\left(l(x+y,n+1) - l(x+y,n)\right)\nu_A(dy)\,dx\Big)\\
&\quad = \sum_{n=0}^{\infty} w_n\int_{\mathbb{R}}\mathbf{E}'\Big(\int_{\mathbb{R}} g(z-y)\left(l(z,n+1)-l(z,n)\right)dz\Big)\nu_A(dy)\\
&\quad = \sum_{n=0}^{\infty} w_n\int_{\mathbb{R}}\int_n^{n+1}\mathbf{E}'\big(g\left(B(t)-y\right)\big)dt\,\nu_A(dy).
\end{aligned}
\tag{3.34}
$$

We choose $g(x) = e^{-x^2/2}$ so that for any $t \geq 0$ and $y \in \mathbb{R}$,

$$
\mathbf{E}'\big(g(B(t)-y)\big) = \sqrt{\frac{1}{1+t}}\exp\left(-\frac{y^2}{2(1+t)}\right).
$$

The last expression of (3.34) is therefore equal to

$$
\begin{aligned}
&\sum_{n=0}^{\infty} w_n\int_{\mathbb{R}}\int_n^{n+1}\sqrt{\frac{1}{1+t}}\exp\left(-\frac{y^2}{2(1+t)}\right)dt\,\nu_A(dy)\\
&\quad \leq \sum_{n=0}^{\infty} w_n\int_{\mathbb{R}}\int_n^{n+1}\sqrt{\frac{1}{1+t}}\,dt\,\nu_A(dy)\\
&\quad \leq \sum_{n=0}^{\infty} w_n\int_{\mathbb{R}}\sqrt{\frac{1}{1+n}}\,\nu_A(dy)\\
&\quad \leq \nu_A(\mathbb{R})\sum_{n=0}^{\infty} w_n\sqrt{\frac{1}{1+n}},
\end{aligned}
$$

and the infinite sum on the last line is finite by the choice of $(w_n, n \geq 0)$. Hence, (3.33) follows. □

3.7 Asymptotic Self-Similarity

A natural question to ask is whether the self-similarity of the BM-local time fractional stable motion carries over to the BM-CAF fractional stable motion. The following result identifies a class of BM-CAF fractional stable motions that are in the domain of attraction of the BM-local time fractional stable motion, in the sense that they converge to it in finite dimensional distributions under proper scaling of time and space. Thus the processes in this class are generally not self-similar, but they can be considered *asymptotically* self-similar.

Theorem 3.7.1. *Suppose* $0 < \alpha \leq 2$ *and* ν_A *satisfies*

$$\sum_{i=0}^{\infty} \beta^i \nu_A\big([\beta^i, \beta^{i+1})\big)^\alpha + \sum_{i=0}^{\infty} \beta^i \nu_A\big([-\beta^{i+1}, -\beta^i)\big)^\alpha < \infty \tag{3.35}$$

for some constant $\beta > 1$*. Then, for* $H = \frac{1}{2} + \frac{1}{2\alpha}$*,*

$$\left(\frac{1}{c^H} Y(ct), t \geq 0\right) \xrightarrow{f.d.} (|\nu_A|\, \Gamma(t), t \geq 0) \quad as \ \ c \to \infty\,,$$

where $\overset{f.d.}{\to}$ *denotes convergence in finite-dimensional distributions,* $|\nu_A| = \nu_A(\mathbb{R})$ *and* $(\Gamma(t), t \geq 0)$ *is the BM-local time fractional SαS motion defined in (3.1).*

Proof. We will take advantage of the following scaling property of the Brownian local time, which follows immediately from the self-similarity of Brownian motion. For any $c > 0$,

$$\big(l(\sqrt{c}x, ct), x \in \mathbb{R}, t \geq 0\big) \overset{d}{=} \big(\sqrt{c}\, l(x,t), x \in \mathbb{R}, t \geq 0\big)\,. \tag{3.36}$$

Now, let $0 \leq t_1 < t_2 < \ldots < t_m$ and $\theta_1, \ldots, \theta_m \in \mathbb{R}$. By (3.36), we have for any $c > 0$

$$\begin{aligned}
&\mathbf{E} \exp\left(i \sum_{j=1}^m \theta_j \frac{1}{c^H} Y(ct_j)\right) \\
&\quad = \exp\left(-\int_{\mathbb{R}} \mathbf{E}' \left|\sum_{j=1}^m \frac{\theta_j}{c^H} \int_{\mathbb{R}} l(x+y, ct_j)\, \nu_A(dy)\right|^\alpha dx\right) \\
&\quad = \exp\left(-c^{\alpha\left(\frac{1}{2}-H\right)} \int_{\mathbb{R}} \mathbf{E}' \left|\sum_{j=1}^m \theta_j \int_{\mathbb{R}} l\left(\frac{x+y}{\sqrt{c}}, t_j\right)\, \nu_A(dy)\right|^\alpha dx\right) \\
&\quad = \exp\left(-\int_{\mathbb{R}} \mathbf{E}' \left|\sum_{j=1}^m \theta_j \int_{\mathbb{R}} l\left(x + \frac{y}{\sqrt{c}}, t_j\right)\, \nu_A(dy)\right|^\alpha dx\right).
\end{aligned} \tag{3.37}$$

We want to take the limit of this expression as $c \to \infty$. We claim that

$$\begin{aligned}
\lim_{c\to\infty} \int_{\mathbb{R}} \mathbf{E}' \left| \sum_{j=1}^{m} \theta_j \int_{\mathbb{R}} l\left(x + \frac{y}{\sqrt{c}}, t_j\right) \, \nu_A(dy) \right|^{\alpha} dx \\
&= \int_{\mathbb{R}} \mathbf{E}' \left| \sum_{j=1}^{m} \theta_j \int_{\mathbb{R}} \lim_{c\to\infty} l\left(x + \frac{y}{\sqrt{c}}, t_j\right) \, \nu_A(dy) \right|^{\alpha} dx \\
&= \int_{\mathbb{R}} \mathbf{E}' \left| \sum_{j=1}^{m} \theta_j \int_{\mathbb{R}} l(x, t_j) \, \nu_A(dy) \right|^{\alpha} dx \\
&= \int_{\mathbb{R}} \mathbf{E}' \left| \sum_{j=1}^{m} \theta_j \, |\nu_A| \, l(x, t_j) \right|^{\alpha} dx,
\end{aligned} \tag{3.38}$$

so that

$$\begin{aligned}
\lim_{c\to\infty} \mathbf{E} \exp\left(i \sum_{j=1}^{m} \theta_j \frac{1}{c^H} Y(ct_j) \right) &= \exp\left(- \int_{\mathbb{R}} \mathbf{E}' \left| \sum_{j=1}^{m} \theta_j \, |\nu_A| \, l(x, t_j) \right|^{\alpha} dx \right) \\
&= \mathbf{E} \exp\left(i \sum_{j=1}^{m} \theta_j \, |\nu_A| \, \Gamma(t_j) \right),
\end{aligned}$$

which proves the theorem. The only step that requires justification is the first equality in (3.38), and we now prove it using Lebesgue's Dominated Convergence Theorem.

For the rest of the proof, we will assume that $0 < \alpha < 1$. The arguments for the case $1 \leq \alpha \leq 2$ will be identical, up to different constants in some bounds. Note that

$$\begin{aligned}
\mathbf{E}' &\left| \sum_{j=1}^{m} \theta_j \int_{\mathbb{R}} l\Big(x + \frac{y}{\sqrt{c}}, t_j\Big) \, \nu_A(dy) \right|^{\alpha} \\
&\leq \mathbf{E}' \Big(\int_{\mathbb{R}} \sum_{j=1}^{m} |\theta_j| l\Big(x + \frac{y}{\sqrt{c}}, t_j\Big) \, \nu_A(dy) \Big)^{\alpha} \\
&\leq \mathbf{E}' \Big(\int_{-\infty}^{-\sqrt{c}} \sum_{j=1}^{m} |\theta_j| l\Big(x + \frac{y}{\sqrt{c}}, t_j\Big) \, \nu_A(dy) \Big)^{\alpha} \\
&\qquad + \mathbf{E}' \Big(\int_{-\sqrt{c}}^{\sqrt{c}} \sum_{j=1}^{m} |\theta_j| l\Big(x + \frac{y}{\sqrt{c}}, t_j\Big) \, \nu_A(dy) \Big)^{\alpha}
\end{aligned}$$

$$
\begin{aligned}
&+ \mathbf{E}'\Big(\int_{\sqrt{c}}^{\infty} \sum_{j=1}^{m} |\theta_j| l\Big(x + \frac{y}{\sqrt{c}}, t_j\Big)\, \nu_A(dy)\Big)^{\alpha} \\
&:= h_1(c,x) + h_2(c,x) + h_3(c,x).
\end{aligned}
$$

We will show that each of these components is bounded uniformly over c by an integrable function of x, which will justify passing the limit through the outer integral in (3.38). Let $M(t) = \sup_{0\le s\le t} |B(s)|$ and $l_*(t) = \sup_{x\in\mathbb{R}} l(x,t)$ as before. Then,

$$
\begin{aligned}
&h_2(c,x) \\
&\le \max\{|\theta_1|,\dots,|\theta_m|\}\, \mathbf{E}'\left(\int_{-\sqrt{c}}^{\sqrt{c}} \sum_{j=1}^{m} l_*(t_j) \mathbf{1}\left\{ \left| x + \frac{y}{\sqrt{c}} \right| \le M(t_j) \right\} \nu_A(dy) \right)^{\alpha} \\
&\le m \max\{|\theta_1|,\dots,|\theta_m|\}\, \mathbf{E}' l_*(t_m)^{\alpha} \left(\int_{-\sqrt{c}}^{\sqrt{c}} \mathbf{1}\left\{ \left| x + \frac{y}{\sqrt{c}} \right| \le M(t_m) \right\} \nu_A(dy) \right)^{\alpha} \\
&\le \mathit{const}\, \mathbf{E}'\big(l_*(t_m)^{\alpha} \mathbf{1}\{|x| \le M(t_m)+1\} \big) \nu_A\left([-\sqrt{c}, \sqrt{c})\right)^{\alpha} \\
&\le \mathit{const}\, \mathbf{E}'\big(l_*(t_m)^{\alpha} \mathbf{1}\{|x| \le M(t_m)+1\} \big),
\end{aligned}
$$

and the last expression is integrable over x since both $M(t_m)$ and $l_*(t_m)$ have all moments finite.

Next, note that

$$
h_3(c,x) = h_3(c,x)\mathbf{1}\{x > -\beta^2\} + h_3(c,x)\mathbf{1}\{x \le -\beta^2\}.
$$

We have

$$
\begin{aligned}
&h_3(c,x)\mathbf{1}\{x > -\beta^2\} \\
&\quad = \mathbf{E}'\Big(\int_{\sqrt{c}}^{\infty} \sum_{j=1}^{m} |\theta_j| l\Big(x + \frac{y}{\sqrt{c}}, t_j\Big)\, \nu_A(dy)\Big)^{\alpha} \mathbf{1}\{-\beta^2 < x < M(t_m)\} \\
&\quad \le \mathit{const}\, \mathbf{E}'\big(l_*(t_m)^{\alpha} \nu_A\left([\sqrt{c}, \infty)\right)^{\alpha} \mathbf{1}\{-\beta^2 < x < M(t_m)\}\big) \\
&\quad \le \mathit{const}\, \mathbf{E}'\big(l_*(t_m)^{\alpha} \mathbf{1}\{-\beta^2 < x < M(t_m)\}\big),
\end{aligned}
$$

where the last expression is integrable as before. Now, let $B_k = [\beta^k, \beta^{k+1})$ and $\sqrt{c}B_k = \left[\sqrt{c}\beta^k, \sqrt{c}\beta^{k+1}\right)$. Then,

$$h_3(c,x)\mathbf{1}\{x \le -\beta^2\} = \sum_{k=2}^{\infty} h_3(c,x)\mathbf{1}\{-x \in B_k\} := \sum_{k=2}^{\infty} g_k(c,x),$$

with

$$\begin{aligned} g_k(c,x) \le \mathbf{E}' &\left(\sum_{i=0}^{k-2} \int_{\sqrt{c}B_i} \sum_{j=1}^{m} |\theta_j| l\left(x + \frac{y}{\sqrt{c}}, t_j\right) \nu_A(dy)\right)^{\alpha} \mathbf{1}\{-x \in B_k\} \\ &+ \mathbf{E}' \left(\sum_{i=k-1}^{k+1} \int_{\sqrt{c}B_i} \sum_{j=1}^{m} |\theta_j| l\left(x + \frac{y}{\sqrt{c}}, t_j\right) \nu_A(dy)\right)^{\alpha} \mathbf{1}\{-x \in B_k\} \\ &+ \mathbf{E}' \left(\sum_{i=k+2}^{\infty} \int_{\sqrt{c}B_i} \sum_{j=1}^{m} |\theta_j| l\left(x + \frac{y}{\sqrt{c}}, t_j\right) \nu_A(dy)\right)^{\alpha} \mathbf{1}\{-x \in B_k\} \\ := g_k^{(1)}(c,x) &+ g_k^{(2)}(c,x) + g_k^{(3)}(c,x). \end{aligned}$$

Letting $Z_k(t) = \mathbf{1}\{\beta^k - \beta^{k-1} \le M(t)\}$, we see that

$$\begin{aligned} \sum_{k=2}^{\infty} g_k^{(1)}(c,x) &\le const \sum_{k=2}^{\infty} \mathbf{E}'\big(Z_k(t_m)\, l_*(t_m)^{\alpha}\big) \left(\sum_{i=0}^{k-2} \int_{\sqrt{c}B_i} \nu_A(dy)\right)^{\alpha} \mathbf{1}\{-x \in B_k\} \\ &\le const \sum_{i=0}^{\infty} \nu_A(\sqrt{c}B_i)^{\alpha} \sum_{k=2}^{\infty} \mathbf{E}'\big(Z_k(t)\, l_*(t_m)^{\alpha}\big)\mathbf{1}\{-x \in B_k\} \qquad (3.39) \\ &\le const \sum_{i=0}^{\infty} \nu_A(B_i)^{\alpha} \sum_{k=2}^{\infty} \mathbf{E}'\big(Z_k(t)\, l_*(t_m)^{\alpha}\big)\mathbf{1}\{-x \in B_k\}, \end{aligned}$$

where the last inequality follows from the fact that, for $\beta^n \le \sqrt{c} < \beta^{n+1}$,

$$\nu_A(\sqrt{c}B_i)^{\alpha} \le \nu_A(B_{n+i})^{\alpha} + \nu_A(B_{n+i+1})^{\alpha}. \qquad (3.40)$$

The last expression in (3.39) is integrable since

$$\begin{aligned} &\int_{\mathbb{R}} \sum_{k=2}^{\infty} \mathbf{E}'\big(Z_k(t_m)\, l_*(t_m)^{\alpha}\big)\mathbf{1}\{-x \in B_k\}\, dx \\ &\quad = \sum_{k=2}^{\infty} \left(\beta^{k+1} - \beta^k\right) \mathbf{E}'\big(Z_k(t_m)\, l_*(t_m)^{\alpha}\big) \\ &\quad \le c_1 \sum_{k=2}^{\infty} \beta^k \mathbf{E}'\left(l_*(t_m)^{2\alpha}\right)^{1/2} \mathbf{P}'\left(\beta^k - \beta^{k-1} \le M(t_m)\right)^{1/2} \end{aligned}$$

$$
\begin{aligned}
&= c_1 \sum_{k=2}^{\infty} \beta^k \, \mathbf{P}' \left(c_2 \beta^{k-1} \le M(t_m) \right)^{1/2} \\
&= c_1 \sum_{k=2}^{\infty} \beta^k \exp\left(-c_2 \beta^{2(k-1)} \right) < \infty,
\end{aligned}
$$

where c_1 and c_2 are positive constants that may change from instance to instance. Also,

$$
\begin{aligned}
\sum_{k=2}^{\infty} g_k^{(2)}(c,x) &\le const \sum_{k=2}^{\infty} \mathbf{E}'\big(l_*(t_m)^\alpha\big) \left(\sum_{i=k-1}^{k+1} \int_{\sqrt{c}B_i} \nu_A(dy) \right)^{\alpha} \mathbf{1}\{-x \in B_k\} \\
&\le const \sum_{k=2}^{\infty} \mathbf{1}\{-x \in B_k\} \sum_{i=k-1}^{\infty} \nu_A(\sqrt{c}B_i)^\alpha \\
&\le const \sum_{k=2}^{\infty} \mathbf{1}\{-x \in B_k\} \sum_{i=k-1}^{\infty} \nu_A(B_i)^\alpha,
\end{aligned}
$$

where the last inequality follows from (3.40) as before. The last expression is integrable because

$$
\begin{aligned}
\int_{\mathbb{R}} \sum_{k=2}^{\infty} \mathbf{1}\{-x \in B_k\} \sum_{i=k-1}^{\infty} \nu_A(B_i)^\alpha \, dx &= \sum_{k=2}^{\infty} \left(\beta^{k+1} - \beta^k \right) \sum_{i=k-1}^{\infty} \nu_A(B_i)^\alpha \\
&= const \sum_{k=1}^{\infty} \beta^k \sum_{i=k}^{\infty} \nu_A(B_i)^\alpha \\
&= const \sum_{i=1}^{\infty} \nu_A(B_i)^\alpha \sum_{k=1}^{i} \beta^k \\
&\le const \sum_{i=1}^{\infty} \beta^i \nu_A(B_i)^\alpha \sum_{k=0}^{\infty} \beta^{-k} \\
&= const \sum_{i=1}^{\infty} \beta^i \nu_A(B_i)^\alpha < \infty,
\end{aligned}
$$

by (3.35). Finally, we also have

$$
\begin{aligned}
\sum_{k=2}^{\infty} g_k^{(3)}(c,x) &\le \sum_{k=2}^{\infty} \mathbf{E}'\big(Z_{k+2}(t_m)\, l_*(t_m)^\alpha\big) \left(\sum_{i=k+2}^{\infty} \int_{\sqrt{c}B_i} \nu_A(dy) \right)^{\alpha} \mathbf{1}\{-x \in B_k\} \\
&\le \sum_{i=0}^{\infty} \nu_A(\sqrt{c}B_i)^\alpha \sum_{k=2}^{\infty} \mathbf{E}'\big(Z_{k+2}(t_m)\, l_*(t_m)^\alpha\big) \mathbf{1}\{-x \in B_k\} \\
&\le 2 \sum_{i=0}^{\infty} \nu_A(B_i)^\alpha \sum_{k=2}^{\infty} \mathbf{E}'\big(Z_{k+2}(t_m)\, l_*(t_m)^\alpha\big) \mathbf{1}\{-x \in B_k\},
\end{aligned}
$$

and the last expression is integrable as before. We have thus shown that $h_3(c,x)$ is bounded uniformly over c by an integrable function of x. It can be shown by analogous arguments that $h_1(c,x)$ is similarly bounded.

We conclude that we are justified in exchanging the limit with the outer integral in (3.38). But once we do that, we can also interchange the limit with the expectation since for any $x \in \mathbb{R}$, $c > 0$, and $\mathbf{P}'$-a.s.,

$$\left| \sum_{j=1}^{m} \theta_j \int_{\mathbb{R}} l\Big(x + \frac{y}{\sqrt{c}}, t_j\Big)\, \nu_A(dy) \right|^{\alpha} \le \left| \sum_{j=1}^{m} \theta_j \right|^{\alpha} \nu_A(\mathbb{R})^{\alpha} l_*(t_m)^{\alpha}\,,$$

where the right-hand side has finite expectation under $\mathbf{P}'$. Finally, the limit also goes through the inner integral since for any $y \in \mathbb{R}$, $x \in \mathbb{R}$, $c > 0$, $j \le m$ and $\mathbf{P}'$-a.s.,

$$l\Big(x + \frac{y}{\sqrt{c}}, t_j\Big) \le l_*(t_m)\,,$$

and the right-hand side is integrable with respect to ν_A. □

Theorem 3.7.1 identifies a class of BM-CAF fractional stable motions that yield the BM-local time fractional stable motion (up to a multiplicative constant) in the large time scale limit, or under "shrinking" of the time scale. It turns out that a subclass of those processes yield the *same* limiting process, up to different multiplicative constants, in the small time scale limit as well. Being attracted to the same limiting process in both large and small time scale limits is an interesting behavior that, to our knowledge, has not been described in literature before.

Theorem 3.7.2. *Suppose* $0 < \alpha \le 2$ *and* ν_A *is of the form*

$$\nu_A = \sum_{i=1}^{n} \mu_i\, \delta_{a_i}\,,$$

where $\mu_1, \ldots, \mu_n > 0$, $a_1, \ldots, a_n \in \mathbb{R}$ *and* δ_{a_i} *is the Dirac point measure of mass 1* *concentrated at a_i. Then, for $H = \frac{1}{2} + \frac{1}{2\alpha}$,*

$$\left(\frac{1}{c^H} Y(ct), t \geq 0\right) \stackrel{f.d.}{\longrightarrow} \left(\left(\sum_{i=1}^{n} \mu_i^\alpha\right)^{1/\alpha} \Gamma(t), t \geq 0\right) \quad as \;\; c \downarrow 0\,, \tag{3.41}$$

where $\stackrel{f.d.}{\to}$ denotes convergence in finite-dimensional distributions and $(\Gamma(t), t \geq 0)$ is the BM-local time fractional $S\alpha S$ motion defined in (3.1).

Proof. Let $0 \leq t_1 < t_2 < \ldots < t_m$ and $\theta_1, \ldots, \theta_m \in \mathbb{R}$. As in (3.37), we have for any $c > 0$

$$\begin{aligned}
&\mathbf{E} \exp\left(i \sum_{j=1}^{m} \theta_j \frac{1}{c^H} Y(ct_j)\right) \\
&\quad = \exp\left(-\int_{\mathbb{R}} \mathbf{E}' \left|\sum_{j=1}^{m} \theta_j \int_{\mathbb{R}} l\left(x + \frac{y}{\sqrt{c}}, t_j\right) \nu_A(dy)\right|^\alpha dx\right) \\
&\quad = \exp\left(-\int_{\mathbb{R}} \mathbf{E}' \left|\sum_{j=1}^{m} \theta_j \sum_{i=1}^{n} \mu_i l\left(x + \frac{a_i}{\sqrt{c}}, t_j\right)\right|^\alpha dx\right) \\
&\quad := \exp\left(-\int_{\mathbb{R}} \mathbf{E}' S(c,x) dx\right).
\end{aligned} \tag{3.42}$$

We want to take the limit of this expression as $c \downarrow 0$. We decompose $S(c,x)$ as

$$\begin{aligned}
S(c,x) = &\sum_{i=1}^{n} \mu_i^\alpha \left|\sum_{j=1}^{m} \theta_j l\left(x + \frac{a_i}{\sqrt{c}}, t_j\right)\right|^\alpha \mathbf{1}\left\{l\left(x + \frac{a_{i'}}{\sqrt{c}}, t_m\right) = 0 \text{ for all } i' \neq i\right\} \\
&+ \sum_{k=1}^{n-1} \left|\sum_{i=1}^{n} \mu_i \sum_{j=1}^{m} \theta_j l\left(x + \frac{a_i}{\sqrt{c}}, t_j\right)\right|^\alpha \mathbf{1}_{G_k(c,x)} \\
:= &\, S_1(c,x) + S_2(c,x)
\end{aligned} \tag{3.43}$$

where $G_k(c,x)$ denotes the event that

$$l\left(x + \frac{a_{k'}}{\sqrt{c}}, t_m\right) = 0 \text{ for all } k' < k,$$
$$l\left(x + \frac{a_k}{\sqrt{c}}, t_m\right) l\left(x + \frac{a_{k''}}{\sqrt{c}}, t_m\right) \neq 0 \text{ for some } k'' > k.$$

We first show that

$$\int_{\mathbb{R}} \mathbf{E}' S_2(c,x) dx \to 0 \;\text{ as } c \downarrow 0. \tag{3.44}$$

Observe that

$$\begin{aligned}
&\int_{\mathbb{R}} \mathbf{E}' S_2(c,x) dx \\
&\quad \leq const \sum_{k=1}^{n-1} \int_{\mathbb{R}} \mathbf{E}' \left(\sum_{i=1}^{n} \mu_i \sum_{j=1}^{m} |\theta_j| \, l \left(x + \frac{a_i}{\sqrt{c}}, t_j \right) \right)^{\alpha} \mathbf{1}_{G_k(c,x)} \, dx \\
&\quad \leq const \sum_{k=1}^{n-1} \int_{\mathbb{R}} \mathbf{E}' \big(l_*(t_m)^{\alpha} \mathbf{1}_{G_k(c,x)} \big) \, dx \\
&\quad \leq const \sum_{k=1}^{n-1} \int_{\mathbb{R}} \left(\mathbf{E}' l_*(t_m)^{2\alpha} \right)^{1/2} \mathbf{P}' (G_k(c,x))^{1/2} dx,
\end{aligned}$$

by Cauchy-Schwarz inequality. Since $l_*(t_m)$ has finite moments of all orders, we see that

$$\begin{aligned}
&\int_{\mathbb{R}} \mathbf{E}' S_2(c,x) dx \\
&\leq const \sum_{k=1}^{n-1} \int_{\mathbb{R}} \mathbf{P}' (G_k(c,x))^{1/2} dx \\
&\leq const \sum_{k=1}^{n-1} \int_{\mathbb{R}} \mathbf{P}' \left\{ l \left(x + \frac{a_k}{\sqrt{c}}, t_m \right) l \left(x + \frac{a_{k''}}{\sqrt{c}}, t_m \right) \neq 0 \text{ for some } k'' > k \right\}^{1/2} dx \\
&\leq const \sum_{k=1}^{n-1} \int_{\mathbb{R}} \mathbf{P}' \left\{ l\left(x, t_m\right) l \left(x + \frac{a_{k''} - a_k}{\sqrt{c}}, t_m \right) \neq 0 \text{ for some } k'' > k \right\}^{1/2} dx \\
&:= const \sum_{k=1}^{n-1} \int_{\mathbb{R}} p_k(c,x)^{1/2} dx. \qquad (3.45)
\end{aligned}$$

Fix $k \in \{1, \ldots, n-1\}$. It is clear that for any $x \in \mathbb{R}, p_k(c,x) \to 0$ as $c \downarrow 0$. Also, for any $c > 0, p_k(c,x) \leq \mathbf{P}' \{ l(x,t_m) \neq 0 \}$, with

$$\int_{\mathbb{R}} \mathbf{P}' \{ l(x,t_m) \neq 0 \}^{1/2} dx \leq \int_{\mathbb{R}} \mathbf{P}' \{ |x| \leq M(t_m) \}^{1/2} dx < \infty,$$

since $M(t_m)$ has Gaussian-like probability tails. It now follows, by the Dominated Convergence Theorem, that the last expression in (3.45) vanishes as $c \downarrow 0$, and (3.44) is established.

Next, note that

$$\begin{aligned}
&\lim_{c\downarrow 0}\int_{\mathbb{R}} \mathbf{E}' S_1(c,x)dx \\
&= \sum_{i=1}^n \mu_i^\alpha \lim_{c\downarrow 0}\int_{\mathbb{R}} \mathbf{E}'\left|\sum_{j=1}^m \theta_j l(x,t_j)\right|^\alpha \mathbf{1}\left\{l\left(x+\frac{a_{i'}-a_i}{\sqrt{c}},t_m\right)=0 \text{ for all } i'\neq i\right\}dx \\
&= \sum_{i=1}^n \mu_i^\alpha \int_{\mathbb{R}} \mathbf{E}'\left|\sum_{j=1}^m \theta_j l(x,t_j)\right|^\alpha \lim_{c\downarrow 0}\mathbf{1}\left\{l\left(x+\frac{a_{i'}-a_i}{\sqrt{c}},t_m\right)=0 \text{ for all } i'\neq i\right\}dx \\
&= \sum_{i=1}^n \mu_i^\alpha \int_{\mathbb{R}} \mathbf{E}'\left|\sum_{j=1}^m \theta_j l(x,t_j)\right|^\alpha dx, \qquad (3.46)
\end{aligned}$$

where once again the Dominated Convergence Theorem provides justification for moving the limit:

$$\left|\sum_{j=1}^m \theta_j l(x,t_j)\right|^\alpha \mathbf{1}\left\{l\left(x+\frac{a_{i'}-a_i}{\sqrt{c}},t_m\right)=0 \text{ for all } i'\neq i\right\} \leq const\, l(x,t_m)^\alpha,$$

and the right-hand side is integrable over $\mathbb{R}\times\Omega'$ with respect to Leb $\times\, \mathbf{P}'$.

By the decomposition (3.43) and the convergences (3.45), (3.46), we conclude that

$$\begin{aligned}
\lim_{c\downarrow 0}\int_{\mathbb{R}} \mathbf{E}' S(c,x)dx &= \sum_{i=1}^n \mu_i^\alpha \int_{\mathbb{R}} \mathbf{E}'\left|\sum_{j=1}^m \theta_j l(x,t_j)\right|^\alpha dx \\
&= \mathbf{E}\exp\left(i\sum_{j=1}^m \theta_j\left(\sum_{i=1}^n \mu_i\right)^{1/\alpha}\Gamma(t_j)\right),
\end{aligned}$$

or, in view of (3.42),

$$\lim_{c\downarrow 0}\mathbf{E}\exp\left(i\sum_{j=1}^m \theta_j\frac{1}{c^H}Y(ct_j)\right) = \mathbf{E}\exp\left(i\sum_{j=1}^m \theta_j\left(\sum_{i=1}^n \mu_i\right)^{1/\alpha}\Gamma(t_j)\right),$$

which completes the proof. □

3.8 Hölder Continuity

Theorem 3.8.1. *Let $(Y(t), t\geq 0)$ be a BM-CAF fractional $S\alpha S$ motion as defined in (3.7), with α and ν_A satisfying the hypotheses of Theorem 3.4.1 or Theorem*

3.4.3. Then, $(Y(t), t \geq 0)$ has a version with continuous sample paths satisfying

$$\sup_{0\leq s<t\leq 1/2} \frac{|Y(t)-Y(s)|}{(t-s)^{1/2}\log\left(\frac{1}{t-s}\right)} < \infty \quad a.s. \tag{3.47}$$

Proof. We use the series representation

$$Y(t) \overset{d}{=} C_\alpha \sum_{j=1}^{\infty} G_j \Gamma_j^{-1/\alpha} e^{X_j^2/2\alpha} \int_{\mathbb{R}} l_j(X_j+y,t)\,dy\,, \tag{3.48}$$

where C_α is a constant determined by α, $(G_j), (\Gamma_j), (X_j), (l_j)$ are independent sequences, $(G_j), (X_j)$ are i.i.d. standard normal random variables, (Γ_j) are arrival times of a unit rate Poisson process, and (l_j) are i.i.d. copies of Brownian local time. We refer to §3.10 of Samorodnitsky and Taqqu (1994) for information on the series representation of stable stochastic integrals.

Assume that (G_j) are defined on some probability space $(\Omega_1, \mathcal{F}_1, \mathbf{P}_1)$, while the other random variables on the right-hand side of (3.48) are defined on some other probability space $(\Omega_2, \mathcal{F}_2, \mathbf{P}_2)$, so that $(Y(t), t \geq 0)$ is defined on the product of these two spaces.

We let

$$K_j = \sup_{\substack{x\in\mathbb{R}\\ 0\leq s<t\leq 1/2}} \frac{l_j(x,t)-l_j(x,s)}{(t-s)^{1/2}\left(\log\left(\frac{1}{t-s}\right)\right)^{1/2}}\,, \quad j=1,2,\ldots$$

As mentioned in Cohen and Samorodnitsky (2006), K_j has finite moments of all orders. Note that, for fixed $\omega_2 \in \Omega_2$, $Y(t)$ is a centered Gaussian process with incremental variance

$$\begin{aligned}
&E_1(Y(t)-Y(s))^2\\
&\quad = C_\alpha^2 \sum_{j=1}^{\infty} \Gamma_j^{-2/\alpha} e^{X_j^2/\alpha} \left(\int_{\mathbb{R}} \left(l_j(X_j+y,t)-l_j(X_j+y,s)\right)\, \nu_A(dy)\right)^2\\
&\quad \leq C_\alpha^2 \sum_{j=1}^{\infty} \Gamma_j^{-2/\alpha} e^{X_j^2/\alpha} K_j^2 (t-s) \log\left(\frac{1}{t-s}\right)
\end{aligned}$$

$$\times \left(\int_{\mathbb{R}} \mathbf{1} \left\{ M_j(t) \geq |X_j + y| \right\} \nu_A(dy) \right)^2$$
$$:= J(\omega_2)(t-s) \log \left(\frac{1}{t-s} \right)$$

for all $0 \leq s < t \leq 1/2$, with $M_j(t) = \sup_{0 \leq r \leq t} |B_j(r)|$. We will prove that J is a $\mathbf{P}_2$-a.s. finite random variable on $(\Omega_2, \mathcal{F}_2, P_2)$. By Theorem 1.4.2 of Samorodnitsky and Taqqu (1994), it will suffice to show that

$$E_2\, e^{X_j^2/2} K_j^{\alpha} \left(\int_{\mathbb{R}} \mathbf{1} \left\{ M_j(t) \geq |X_j + y| \right\} \nu_A(dy) \right)^{\alpha} < \infty\,, \tag{3.49}$$

or equivalently,

$$E_2\, K_j^{\alpha} \int_{\mathbb{R}} \left(\int_{\mathbb{R}} \mathbf{1} \left\{ M_j(t) \geq |x + y| \right\} \nu_A(dy) \right)^{\alpha} dx < \infty\,. \tag{3.50}$$

But the proof of (3.50) is identical to that of (3.10) if $0 < \alpha < 1$, and that of (3.20) if $1 \leq \alpha \leq 2$, provided that one replaces $l_*(t)$ with K_j and $\mathbf{E}'$ with E_2.

We now conclude, by classical results on moduli of continuity of Gaussian processes (see, e.g., Corollary 2.3 of Dudley (1973)), that $(Y(t), t \geq 0)$ has a version with continuous paths satisfying

$$\sup_{\substack{0 \leq s < t \leq 1/2 \\ s,t \in \mathbb{Q}}} \frac{|Y(t) - Y(s)|}{(t-s)^{1/2} \log\left(\frac{1}{t-s}\right)} < \infty \quad \mathbf{P}_1\text{-a.s.}$$

For such a version, we also have, by Fubini's Theorem,

$$\sup_{\substack{0 \leq s < t \leq 1/2 \\ s,t \in \mathbb{Q}}} \frac{|Y(t) - Y(s)|}{(t-s)^{1/2} \log\left(\frac{1}{t-s}\right)} < \infty \quad \mathbf{P}_1 \times \mathbf{P}_2\text{-a.s.}\,,$$

which is equivalent to the statement of the theorem. □

3.9 A Limit Theorem

Our aim in this section is to generalize the "random rewards" scheme presented in Cohen and Samorodnitsky (2006) and outlined in Section 3.2. We start by setting up the notation.

Let $(W_k^{(i)}, k \in \mathbb{Z}, i \geq 1)$ be an array of i.i.d. SαS random variables with scale parameter 1. Further, let $(V_k^{(i)}, k \geq 1, i \geq 1)$ be an array of i.i.d. mean zero and unit variance integer-valued random variables, independent of $(W_k^{(i)}, k \in \mathbb{Z}, i \geq 1)$. Let $S_n^{(i)} = V_1^{(i)} + \ldots + V_n^{(i)}, n \geq 0$ be the i^{th} random walk, $i = 1, 2, \ldots$, and define for $j \in \mathbb{Z}$ and $n \geq 1$

$$\varphi^{(i)}(j, n) = \sum_{k=1}^{n} \mathbf{1}\{S_k^{(i)} = j\},$$

the number of times the i^{th} random walk visits state j by time n. Define $\varphi^{(i)}(j, t)$ for noninteger values of t by linear interpolation, i.e. for $n < t < n + 1$, let

$$\varphi^{(i)}(j, t) = (t - n)\varphi^{(i)}(j, n + 1) + (1 - t + n)\varphi^{(i)}(j, n).$$

We note here that the results presented below can likely be generalized to an array $(W_k^{(i)}, k \in \mathbb{Z}, i \geq 1)$ of i.i.d. infinite-variance random variables that are in the domain of attraction of a SαS distribution, but we do not pursue that goal in order to keep the technicalities at a minimum.

Theorem 3.9.1. *Let $(b_n, n \geq 1)$ be a sequence of positive integers with $b_n \to \infty$, let ν_A be a finite measure on $\mathbb{R}$ whose support is contained in $[-\kappa, \kappa)$ for some positive integer κ, and define, for $n \geq 1$ and $t \geq 0$,*

$$Y_n(t) = \frac{1}{(nb_n^{\alpha+1})^{1/\alpha}} \sum_{i=1}^{n} \sum_{k=-\infty}^{\infty} \varphi^{(i)}(k, b_n^2 t) \sum_{j=-\infty}^{\infty} W_{k-j}^{(i)} \nu_A\left(\left[\frac{j}{b_n}, \frac{j+1}{b_n}\right)\right). \tag{3.51}$$

Then we have, as $n \to \infty$,

$$\big(Y_n(t), t \geq 0\big) \Longrightarrow \big(Y(t), t \geq 0\big)$$

weakly in $\mathbb{C}([0, \infty))$, where Y is the BM-CAF fractional stable motion with associated measure ν_A.

Remark 3.9.2. One way to interpret this result is the following. Suppose many independent "users," indexed by $i \geq 0$, are performing independent random walks $(S_n^{(i)}, n \geq 0)$ on distinct integer lines. The numbers (or "positions") along each integer line are assigned i.i.d. SαS random "rewards" $(W_k^{(i)}, n \geq 0)$. Whenever user i visits position k, she collects a weighted average

$$\sum_{j=-\infty}^{\infty} W_{k-j}^{(i)} \nu_A\left(\left[\frac{j}{b_n}, \frac{j+1}{b_n}\right)\right)$$

of the rewards around k, where the weighting is determined by the measure ν_A and does not depend on k. In other words, the collected amounts form a "moving average" of the i.i.d. rewards. If there are many such users earning rewards independently, their cumulative total reward process can be approximated by the BM-CAF fractional stable motion, up to proper scaling of time and space.

Proof of Theorem 3.9.1. Note that $Y_n(t)$ in (3.51) can also be written as

$$Y_n(t) = \frac{1}{(nb_n^{\alpha+1})^{1/\alpha}} \sum_{i=1}^{n} \sum_{k=-\infty}^{\infty} W_k^{(i)} \sum_{j=-\infty}^{\infty} \varphi^{(i)}(k+j, b_n^2 t)\, \nu_A\left(\left[\frac{j}{b_n}, \frac{j+1}{b_n}\right)\right),$$

since all the sums involved have finitely many non-zero terms. In the following, we will work with this representation.

Let $(B^{(i)}(t), t \geq 0), i = 1, 2, \ldots$ be a sequence of i.i.d. Brownian motions with jointly continuous local time processes $(l^{(i)}(x,t), x \in \mathbb{R}, t \geq 0), i = 1, 2, \ldots,$ such that for every $T > 0$,

$$\sup_{x \in \mathbb{R}, 0 \leq t \leq nT} \left| \varphi^{(i)}([x], t) - \sqrt{n}\, l^{(i)}\left(\frac{x}{\sqrt{n}}, \frac{t}{n}\right)\right| \longrightarrow 0 \tag{3.52}$$

in probability as $n \to \infty, i = 1, 2, \ldots$. (Such a sequence of Brownian motions exists, by Borodin (1982).) Define, for $n \geq 1$ and $t \geq 0$,

$$X_n(t) = \frac{1}{(nb_n)^{1/\alpha}} \sum_{i=1}^{n} \sum_{k=-\infty}^{\infty} W_k^{(i)} \int_{\mathbb{R}} l^{(i)}\left(\frac{k}{b_n} + y, t\right) \nu_A(dy). \tag{3.53}$$

We first show that for any $t \geq 0$,

$$E_n(t) := Y_n(t) - X_n(t) \longrightarrow 0 \text{ in probability} \tag{3.54}$$

as $n \to \infty$. For notational simplicity, we take $t = 1$. We have

$$\begin{aligned}
&E_n(1) \\
&= \frac{1}{(nb_n^{\alpha+1})^{1/\alpha}} \sum_{i=1}^{n} \sum_{k=-\infty}^{\infty} W_k^{(i)} \left(\sum_{j=-\infty}^{\infty} \varphi^{(i)}(k+j, b_n^2)\, \nu_A\left(\left[\frac{j}{b_n}, \frac{j+1}{b_n}\right)\right) \right. \\
&\qquad\qquad \left. - b_n \int_{\mathbb{R}} l^{(i)}\left(\frac{k}{b_n} + y, 1\right) \nu_A(dy) \right) \\
&= \frac{1}{(nb_n^{\alpha+1})^{1/\alpha}} \sum_{i=1}^{n} \sum_{k=-\infty}^{\infty} W_k^{(i)} \int_{\mathbb{R}} \left(\varphi^{(i)}\left([k + yb_n], b_n^2\right) - b_n l^{(i)}\left(\frac{k}{b_n} + y, 1\right)\right) \nu_A(dy) \\
&:= \frac{1}{(nb_n^{\alpha+1})^{1/\alpha}} \sum_{i=1}^{n} \sum_{k=-\infty}^{\infty} W_k^{(i)} D_{k,n}^{(i)}.
\end{aligned}$$

Since the last expression is equal in distribution to

$$\left(\frac{1}{nb_n^{\alpha+1}} \sum_{i=1}^{n} \sum_{k=-\infty}^{\infty} \left| D_{k,n}^{(i)} \right|^{\alpha} \right) W_1^{(1)},$$

the convergence (3.54) will be proven if we can show that

$$\frac{1}{nb_n^{\alpha+1}} \sum_{i=1}^{n} \sum_{k=-\infty}^{\infty} \left| D_{k,n}^{(i)} \right|^{\alpha} \longrightarrow 0 \text{ in probability} \tag{3.55}$$

as $n \to \infty$. The expectation of the left-hand side of (3.55) is

$$\begin{aligned}
&\frac{1}{b_n^{\alpha+1}} E \sum_{k=-\infty}^{\infty} \left| D_{k,n}^{(1)} \right|^{\alpha} \\
&= \frac{1}{b_n^{\alpha+1}} E \sum_{k=-\infty}^{\infty} \left| D_{k,n}^{(1)} \right|^{\alpha} \mathbf{1}\left\{ \left| D_{k,n}^{(1)} \right| \leq 1 \right\} + \frac{1}{b_n^{\alpha+1}} E \sum_{k=-\infty}^{\infty} \left| D_{k,n}^{(1)} \right|^{\alpha} \mathbf{1}\left\{ \left| D_{k,n}^{(1)} \right| > 1 \right\} \\
&:= p_1 + p_2 \,.
\end{aligned}$$

Now, letting

$$M^{(i)}(m) = \max\left\{ \kappa m + \sup_{0 \leq k \leq m^2} \left| S_k^{(i)} \right|, m\left(\sup_{0 \leq t \leq 1} \left| B^{(i)}(t) \right| + \kappa \right) \right\}$$

for positive integers m, we see that

$$p_1 = \frac{1}{b_n^{\alpha+1}} E \sum_{k=-M^{(1)}(b_n)}^{M^{(1)}(b_n)} \left| D_{k,n}^{(1)} \right|^{\alpha} \mathbf{1} \left\{ \left| D_{k,n}^{(1)} \right| \le 1 \right\}$$
$$\le \frac{1}{b_n^{\alpha+1}} E \left(2M^{(1)}(b_n) + 1 \right). \tag{3.56}$$

It is an easy consequence of Doob's martingale inequalities that

$$E\left(M^{(1)}(m)^r \right) \le const\; m^r \tag{3.57}$$

for integers $m \ge 1$ and real numbers $r \ge 1$, where the constant depends on r. Therefore, continuing from (3.56), we obtain

$$p_1 \le const\; \frac{1}{b_n^{\alpha+1}}\, b_n \longrightarrow 0$$

as $n \to \infty$.

Next, we consider p_2. By repeated use of Hölder's inequality,

$$p_2 \le \frac{1}{b_n^{\alpha+1}} E \left(\sum_{k=-\infty}^{\infty} \left| D_{k,n}^{(1)} \right|^2 \right)^{\frac{\alpha}{2}} \left(\sum_{k=-\infty}^{\infty} \mathbf{1} \left\{ \left| D_{k,n}^{(1)} \right| > 1 \right\} \right)^{1-\frac{\alpha}{2}}$$
$$\le \frac{1}{b_n^{\alpha+1}} E \left(\sum_{k=-\infty}^{\infty} \left| D_{k,n}^{(1)} \right|^2 \right)^{\frac{\alpha}{2}} \left(2M^{(1)}(b_n) + 1 \right)^{1-\frac{\alpha}{2}} \mathbf{1} \left\{ \sup_k \left| D_{k,n}^{(1)} \right| > 1 \right\}^{1-\frac{\alpha}{2}}$$
$$\le \frac{1}{b_n^{\alpha+1}} \left(E \sum_{k=-\infty}^{\infty} \left| D_{k,n}^{(1)} \right|^2 \right)^{\frac{\alpha}{2}} \left(E \left(2M^{(1)}(b_n) + 1 \right) \mathbf{1} \left\{ \sup_k \left| D_{k,n}^{(1)} \right| > 1 \right\} \right)^{1-\frac{\alpha}{2}}.$$

But note that

$$E \sum_{k=-\infty}^{\infty} \left| D_{k,n}^{(1)} \right|^2$$
$$\le const \int_{\mathbb{R}} E \sum_{k=-\infty}^{\infty} \left(\varphi^{(1)} \left([k + y b_n], b_n^2 \right) - b_n l^{(1)} \left(\frac{k}{b_n} + y, 1 \right) \right)^2 \nu_A(dy)$$
$$\le const \int_{\mathbb{R}} \left(E \sum_{k=-\infty}^{\infty} \varphi^{(1)} \left(k, b_n^2 \right)^2 + b_n^2\, E \sum_{k=-\infty}^{\infty} l^{(1)} \left(\frac{k}{b_n} + y, 1 \right)^2 \right) \nu_A(dy)$$
$$\le const\; b_n^3,$$

where the last inequality follows from Lemma 1 of Kesten and Spitzer (1979) and from the fact that the largest value of a Brownian local time at time 1 has all moments finite. Furthermore,

$$\sup_k \left| D_{k,n}^{(1)} \right| \leq |\nu_A| \sup_{x\in\mathbb{R}} \left| \varphi^{(1)}([x], b_n^2) - b_n\, l^{(1)} \left(\frac{x}{b_n}, 1 \right) \right| := |\nu_A| \Delta^{(1)}(b_n),$$

where $|\nu_A| = \nu_A(\mathbb{R})$. Thus we obtain

$$\begin{aligned} p_2 &\leq const\, \frac{1}{b_n^{\alpha+1}}\, b_n^{\frac{3\alpha}{2}} \left(EM^{(1)}(b_n)^{3/2} \right)^{\frac{2-\alpha}{3}} \left(P\left(\Delta^{(1)}(b_n) > |\nu_A|^{-1} \right) \right)^{\left(1-\frac{\alpha}{2}\right)/3} \\ &\leq const\, \frac{1}{b_n^{\alpha+1}}\, b_n^{\frac{3\alpha}{2}} \left(b_n^{3/2} \right)^{\frac{2-\alpha}{3}} \left(P\left(\Delta^{(1)}(b_n) > |\nu_A|^{-1} \right) \right)^{\left(1-\frac{\alpha}{2}\right)/3} \\ &= const\, \left(P\left(\Delta^{(1)}(b_n) > |\nu_A|^{-1} \right) \right)^{\left(1-\frac{\alpha}{2}\right)/3} \longrightarrow 0, \end{aligned}$$

by (3.52). Note that in the middle line we take advantage of the inequality (3.57). Thus (3.55) follows, and (3.54) is established.

The next step is to show that the finite-dimensional distributions of the process $(X_n(t), t \geq 0)$ in (3.53) converge to those of $(Y(t), t \geq 0)$. For this, it is enough to show that, for every $m \geq 1$, $0 < t_1 < \ldots < t_m$ and $\theta_1, \ldots, \theta_m \in \mathbb{R}$,

$$\sum_{j=1}^m \theta_j X_n(t_j) \xrightarrow{d} \sum_{j=1}^m \theta_j Y(t_j) \qquad \text{as } n \to \infty\,.$$

We will see that this is true for $m = 1$ and $t_1 = 1$; the general case is similar. So we will show that

$$\frac{1}{(nb_n)^{1/\alpha}} \sum_{i=1}^n \sum_{k=-\infty}^{\infty} W_k^{(i)} \int_{\mathbb{R}} l^{(i)} \left(\frac{k}{b_n} + y, 1 \right) \nu_A(dy) \xrightarrow{d} Y(1). \tag{3.58}$$

Since both sides of (3.58) are conditionally SαS random variables, it will suffice to show the convergence in probability of the scale parameters. That is, it will suffice to show that

$$\begin{aligned} &\frac{1}{nb_n} \sum_{i=1}^n \sum_{k=-\infty}^{\infty} \left(\int_{\mathbb{R}} l^{(i)} \left(\frac{k}{b_n} + y, 1 \right) \nu_A(dy) \right)^\alpha \\ &\qquad\qquad \longrightarrow E \int_{\mathbb{R}} \left(\int_{\mathbb{R}} l\,(x+y,1)\, \nu_A(dy) \right)^\alpha dx \end{aligned} \tag{3.59}$$

in probability. Let us denote the absolute difference

$$\left| \frac{1}{nb_n} \sum_{i=1}^{n} \sum_{k=-\infty}^{\infty} \left(\int_{\mathbb{R}} l^{(i)} \left(\frac{k}{b_n} + y, 1 \right) \nu_A(dy) \right)^\alpha \right.$$
$$\left. - \frac{1}{b_n} E \sum_{k=-\infty}^{\infty} \left(\int_{\mathbb{R}} l \left(\frac{k}{b_n} + y, 1 \right) \nu_A(dy) \right)^\alpha \right|$$

by δ_n. By Chebyshev's inequality,

$$\begin{aligned} P(\delta_n > \epsilon) &\le \frac{1}{\epsilon^2 n b_n^2} E \left(\sum_{k=-\infty}^{\infty} \left(\int_{\mathbb{R}} l^{(1)} \left(\frac{k}{b_n} + y, 1 \right) \nu_A(dy) \right)^\alpha \right)^2 \\ &\le \frac{c}{\epsilon^2 n b_n^2} E \left(2M^{(1)}(b_n) + 1 \right)^2 l_*(1)^{2\alpha} \\ &\le \frac{c}{\epsilon^2 n} \longrightarrow 0. \end{aligned}$$

Moreover,

$$\frac{1}{b_n} E \sum_{k=-\infty}^{\infty} \left(\int_{\mathbb{R}} l \left(\frac{k}{b_n} + y, 1 \right) \nu_A(dy) \right)^\alpha \longrightarrow E \int_{\mathbb{R}} \left(\int_{\mathbb{R}} l\,(x + y, 1)\, \nu_A(dy) \right)^\alpha dx$$

by the Dominated Convergence Theorem. Hence the convergence (3.59) follows, and (3.58) is proven.

It remains to prove the tightness of the sequence $(Y_n(t), t \ge 0)$ in $\mathbb{C}([0,\infty))$. Given $K > 0$, we write

$$\begin{aligned} Y_n(t) = &\frac{1}{(nb_n^{\alpha+1})^{1/\alpha}} \sum_{i=1}^{n} \sum_{k=-\infty}^{\infty} W_k^{(i)} \mathbf{1}\{|W_k^{(i)}| > K(nb_n)^{1/\alpha}\} \\ &\qquad \times \sum_{j=-\infty}^{\infty} \varphi^{(i)}(k+j, b_n^2 t)\, \nu_A \left(\left[\frac{j}{b_n}, \frac{j+1}{b_n} \right) \right) \\ &+ \frac{1}{(nb_n^{\alpha+1})^{1/\alpha}} \sum_{i=1}^{n} \sum_{k=-\infty}^{\infty} W_k^{(i)} \mathbf{1}\{|W_k^{(i)}| \le K(nb_n)^{1/\alpha}\} \\ &\qquad \times \sum_{j=-\infty}^{\infty} \varphi^{(i)}(k+j, b_n^2 t)\, \nu_A \left(\left[\frac{j}{b_n}, \frac{j+1}{b_n} \right) \right) \\ := &\, Y_{1,n}(t) + Y_{2,n}(t). \end{aligned}$$

Note that

$$P\left(\sup_{0\le t\le 1}\big|Y_{1,n}(t)\big|=0\right)^{1/n} \ge P\left(\text{for all } |k|\le M^{(1)}(b_n), \big|W_k^{(1)}\big|\le K(nb_n)^{1/\alpha}\right)$$
$$= E\,P\left(\big|W_1^{(1)}\big|\le K(nb_n)^{1/\alpha}\right)^{2M^{(1)}(b_n)+1}$$
$$\ge E\left(1-cK^{-\alpha}(nb_n)^{-1}\right)^{2M^{(1)}(b_n)+1}$$
$$\ge 1+E\left(2M^{(1)}(b_n)+1\right)\log\left(1-cK^{-\alpha}(nb_n)^{-1}\right)$$
$$\ge 1+c_1 b_n\log\left(1-c_2K^{-\alpha}(nb_n)^{-1}\right),$$

where c, c_1, c_2 are, as usual, positive constants that may change from instance to instance. It now follows that

$$P\left(\sup_{0\le t\le 1}\big|Y_{1,n}(t)\big|>0\right)\le 1-\left(1+c_1\log\left(1-c_2K^{-\alpha}(nb_n)^{-1}\right)^{b_n}\right)^n.$$

Letting n go to infinity, we obtain

$$\limsup_{n\to\infty} P\left(\sup_{0\le t\le 1}\big|Y_{1,n}(t)\big|>0\right)\le 1-\exp\left(-c_2K^{-\alpha}\right).$$

Since the right-hand side converges to zero as $K\to\infty$, it follows from the decomposition of $Y_n(t)$ above that it will suffice to prove the tightness of the processes $(Y_{2,n}(t), 0\le t\le 1)$ for each fixed K.

Now, for any $0\le s\le t\le 1$, we have

$$E\left(Y_{2,n}(t)-Y_{2,n}(s)\right)^2$$
$$=\frac{1}{n^{\frac{2}{\alpha}-1}b_n^{\frac{2}{\alpha}+2}}E\left(\left(W_1^{(1)}\right)^2\mathbf{1}\{\big|W_1^{(1)}\big|\le K(nb_n)^{1/\alpha}\}\right)$$
$$\times E\sum_{k=-\infty}^{\infty}\left(\sum_{j=-\infty}^{\infty}\left(\varphi^{(i)}(k+j,b_n^2t)-\varphi^{(i)}(k+j,b_n^2s)\right)\nu_A\left(\left[\frac{j}{b_n},\frac{j+1}{b_n}\right)\right)\right)^2.$$

Since, for large x,

$$E\big(W_1^{(1)}\big)^2\mathbf{1}\{\big|W_1^{(1)}\big|\le x\}\le 4\int_0^x yP\big(W_1^{(1)}>y\big)\,dy\le cx^{2-\alpha},$$

we see that, for large n,

$$\begin{aligned}
&E\left(Y_{2,n}(t) - Y_{2,n}(s)\right)^2 \\
&\le c\, b_n^{-3} E \sum_{k=-\infty}^{\infty} \left(\sum_{j=-\infty}^{\infty} \left(\varphi^{(i)}(k+j, b_n^2 t) - \varphi^{(i)}(k+j, b_n^2 s) \right) \nu_A \left(\left[\frac{j}{b_n}, \frac{j+1}{b_n} \right) \right) \right)^2 \\
&= c\, b_n^{-3} E \sum_{k=-\infty}^{\infty} \left(\int_{\mathbb{R}} \left(\varphi^{(i)}\left([k+yb_n], b_n^2 t\right) - \varphi^{(i)}\left([k+yb_n], b_n^2 s\right) \right) \nu_A(dy) \right)^2 \\
&\le c\, b_n^{-3} \int_{\mathbb{R}} E \sum_{k=-\infty}^{\infty} \left(\varphi^{(i)}\left([k+yb_n], b_n^2 t\right) - \varphi^{(i)}\left([k+yb_n], b_n^2 s\right) \right)^2 \nu_A(dy) \\
&\le c\, b_n^{-3} \int_{\mathbb{R}} \left(b_n^2 (t-s) \right)^{3/2} \nu_A(dy) \\
&= c(t-s)^{3/2},
\end{aligned}$$

as in the proof of Lemma 7 in Kesten and Spitzer (1979). We can now appeal to Theorem 12.3 in Billingsley (1968) to conclude the tightness of $(Y_{2,n}(t), 0 \le t \le 1)$ and, hence, complete the proof. □

3.10 A Special Case

In this section we study BM-CAF fractional SαS motions whose associated measures are of the form $\nu_A(dy) = y^{-\lambda} \mathbf{1}_{[0,\infty)}(y)\, dy$ for some $0 < \lambda < 1$. That is, we study processes of the form

$$Y(t) = \int_{\Omega' \times \mathbb{R}} \int_0^\infty l(x+y, t)\, y^{-\lambda}\, dy\, M(d\omega', dx), \quad t \ge 0, \tag{3.60}$$

where $(l(x,t), x \in \mathbb{R}, t \ge 0)$ is the local time of a Brownian motion $(B(t), t \ge 0)$ defined on $(\Omega', \mathcal{F}', \mathbf{P}')$, and M is a SαS random measure on $\Omega' \times \mathbb{R}$ with control measure $\mathbf{P}' \times$ Leb. M lives on a probability space $(\Omega, \mathcal{F}, \mathbf{P})$.

Theorem 3.10.1. *Suppose* $1 < \alpha \le 2$ *and* $1/\alpha < \lambda < 1$. *Then, the process*

$(Y(t), t \geq 0)$ *in (3.60) is a well-defined SαS process. It is self-similar with exponent*

$$H = 1 - \frac{\lambda}{2} + \frac{1}{2\alpha} = 1 - \frac{1}{2}\left(\lambda - \frac{1}{\alpha}\right). \tag{3.61}$$

Remark 3.10.2. Well-definedness does *not* follow from Theorem 3.4.3 in the present case, since condition (3.19) is violated. Indeed, for any $\beta > 1$,

$$\sum_{i=0}^{\infty} \nu_A\left([\beta^i, \beta^{i+1})\right)^\alpha = \sum_{i=0}^{\infty} \left(\int_{\beta^i}^{\beta^{i+1}} y^{-\lambda}\, dy\right)^\alpha = const \sum_{i=0}^{\infty} \beta^{\alpha(1-\lambda)i} = \infty,$$

Hence Theorem 3.10.1 shows that in the case $1 < \alpha \leq 2$, condition (3.19) is not necessary for well-definedness. Also, the self-similarity result proves that the BM-CAF fractional stable motion is not always in the domain of attraction of the BM-local time fractional stable motion.

Proof of Theorem 3.10.1. For well-definedness, we need to check that

$$\mathbf{E}' \int_{\mathbb{R}} \left(\int_0^\infty l(x+y, t)\, y^{-\lambda}\, dy\right)^\alpha dx < \infty. \tag{3.62}$$

It will suffice to prove

$$\mathbf{E}'\, l_*(t)^\alpha \int_{\mathbb{R}} \left(\int_0^\infty \mathbf{1}\{|x+y| \leq M(t)\}\, y^{-\lambda}\, dy\right)^\alpha dx < \infty, \tag{3.63}$$

with $l_*(t)$ and $M(t)$ as defined in (3.11) and (3.12). The left hand side of (3.63) can be decomposed as

$$\mathbf{E}'\, l_*(t)^\alpha \int_{-\infty}^{-M(t)} \left(\int_{-M(t)-x}^{M(t)-x} y^{-\lambda}\, dy\right)^\alpha dx + \mathbf{E}'\, l_*(t)^\alpha \int_{-M(t)}^{M(t)} \left(\int_0^{M(t)-x} y^{-\lambda}\, dy\right)^\alpha dx$$

$$:= I_1 + I_2.$$

Note that

$$\begin{aligned} I_1 &= \mathbf{E}'\, l_*(t)^\alpha \int_{M(t)}^\infty \left(\int_{x-M(t)}^{x+M(t)} y^{-\lambda}\, dy\right)^\alpha dx \\ &= const\, \mathbf{E}'\, l_*(t)^\alpha \int_{M(t)}^\infty \left((x+M(t))^{1-\lambda} - (x-M(t))^{1-\lambda}\right)^\alpha dx \end{aligned}$$

$$= const\, \mathbf{E}'\, l_*(t)^\alpha M(t)^{1+(1-\lambda)\alpha} \int_1^\infty \big((u+1)^{1-\lambda} - (u-1)^{1-\lambda}\big)^\alpha\, du$$
$$= const\, \mathbf{E}'\, l_*(t)^\alpha M(t)^{1+(1-\lambda)\alpha} < \infty\,,$$

since $\big((u+1)^{1-\lambda} - (u-1)^{1-\lambda}\big)^\alpha \sim u^{-\alpha\lambda}$ as $u \to \infty$, and $l_*(t)$ and $M(t)$ have finite moments of all orders. Also,

$$I_2 = const\, \mathbf{E}'\, l_*(t)^\alpha \int_{-M(t)}^{M(t)} (M(t)-x)^{(1-\lambda)\alpha}\, dx$$
$$= const\, \mathbf{E}'\, l_*(t)^\alpha M(t)^{1+(1-\lambda)\alpha} < \infty\,,$$

and (3.63) follows.

For self-similarity, note that for any $c > 0, \theta_1, \ldots, \theta_m \in \mathbb{R}$ and $t_1, \ldots, t_m \geq 0$ we have, using (3.36),

$$\begin{aligned}
&\mathbf{E} \exp\left(i \sum_{j=1}^m \theta_j Y(ct_j) \right) \\
&\quad = \exp\left(- \int_{\mathbb{R}} \mathbf{E}' \left| \sum_{j=1}^m \theta_j \int_0^\infty l(x+y, ct_j)\, y^{-\lambda}\, dy \right|^\alpha dx \right) \\
&\quad = \exp\left(- \int_{\mathbb{R}} \mathbf{E}' \left| \sum_{j=1}^m \theta_j \int_0^\infty \sqrt{c}\, l\left(\frac{x+y}{\sqrt{c}}, t_j \right) y^{-\lambda}\, dy \right|^\alpha dx \right) \\
&\quad = \exp\left(- \int_{\mathbb{R}} \mathbf{E}' \left| \sum_{j=1}^m \theta_j c^{1-\frac{\lambda}{2}+\frac{1}{2\alpha}} \int_0^\infty l(u+v, t_j)\, v^{-\lambda}\, dv \right|^\alpha du \right) \\
&\quad = \mathbf{E} \exp\left(i \sum_{j=1}^m \theta_j c^{1-\frac{\lambda}{2}+\frac{1}{2\alpha}} Y(t_j) \right).
\end{aligned}$$

Therefore, $(Y(t), t \geq 0)$ is H-self-similar, with H as defined in (3.61). □

Next, we prove a finite-dimensional analogue of Theorem 3.9.1 for the process $(Y(t), t \geq 0)$ defined in (3.60). Note that Theorem 3.9.1 does not apply in this case, since the measure $\nu_A(dy) = y^{-\lambda} \mathbf{1}_{[0,\infty)}(y)\, dy$ clearly does not satisfy its hypotheses.

As in Section 3.9.1, let $(W_k^{(i)}, k \in \mathbb{Z}, i \geq 1)$ be an array of i.i.d. SαS random variables with scale parameter 1. Further, let $(V_k^{(i)}, k \geq 1, i \geq 1)$ be an array of i.i.d. mean zero and unit variance integer-valued random variables, independent of $(W_k^{(i)}, k \in \mathbb{Z}, i \geq 1)$. Let $S_n^{(i)} = V_1^{(i)} + \ldots + V_n^{(i)}, n \geq 0$ be the i^{th} random walk, $i = 1, 2, \ldots$, and define for $j \in \mathbb{Z}$ and $n \geq 1$

$$\varphi^{(i)}(j,n) = \sum_{k=1}^{n} \mathbf{1}\{S_k^{(i)} = j\},$$

the number of times the i^{th} random walk visits state j by time n. Define $\varphi^{(i)}(j,t)$ for noninteger values of t by linear interpolation, i.e. for $n < t < n+1$, let

$$\varphi^{(i)}(j,t) = (t-n)\varphi^{(i)}(j,n+1) + (1-t+n)\varphi^{(i)}(j,n).$$

Theorem 3.10.3. *Let $(b_n, n \geq 1)$ be a sequence of positive integers with $b_n \to \infty$, $1 < \alpha \leq 2$ and $1/\alpha < \lambda < 1$. Define, for $n \geq 1$ and $t \geq 0$,*

$$Y_n(t) = \frac{1}{n^{1/\alpha} b_n^{2-\lambda+1/\alpha}} \sum_{i=1}^{n} \sum_{k=-\infty}^{\infty} \varphi^{(i)}(k, b_n^2 t) \sum_{j=0}^{\infty} W_{k-j}^{(i)} \left((j+1)^{1-\lambda} - j^{1-\lambda}\right). \quad (3.64)$$

Then we have, as $n \to \infty$,

$$\big(Y_n(t), t \geq 0\big) \xrightarrow{f.d.} \big(Y(t), t \geq 0\big),$$

where $\xrightarrow{f.d.}$ denotes convergence in finite-dimensional distributions and $(Y(t), t \geq 0)$ is the process defined in (3.60).

Proof. The outline of the proof is the same as in Theorem 3.9.1. We work with the representation

$$Y_n(t) = \frac{1}{n^{1/\alpha} b_n^{2-\lambda+1/\alpha}} \sum_{i=1}^{n} \sum_{k=-\infty}^{\infty} W_k^{(i)} \sum_{j=0}^{\infty} \varphi^{(i)}(k+j, b_n^2 t) \left((j+1)^{1-\lambda} - j^{1-\lambda}\right).$$

Let $(B^{(i)}(t), t \geq 0), i = 1, 2, \ldots$ be a sequence of i.i.d. Brownian motions with jointly continuous local time processes $(l^{(i)}(x,t), x \in \mathbb{R}, t \geq 0), i = 1, 2, \ldots$, such that for every $T > 0$,

$$\sup_{x \in \mathbb{R}, 0 \leq t \leq nT} \left| \varphi^{(i)}([x], t) - \sqrt{n}\, l^{(i)}\left(\frac{x}{\sqrt{n}}, \frac{t}{n}\right) \right| \xrightarrow{L^2} 0 \quad (3.65)$$

as $n \to \infty, i = 1, 2, \ldots$. Such a sequence of Brownian motions exists, by Kang and Wee (1997). Define, for $n \geq 1$ and $t \geq 0$,

$$X_n(t) = \frac{1}{(nb_n)^{1/\alpha}} \sum_{i=1}^{n} \sum_{k=-\infty}^{\infty} W_k^{(i)} \int_0^\infty l^{(i)} \left(\frac{k}{b_n} + y, t \right) y^{-\lambda} \, dy. \tag{3.66}$$

We first show that for any $t \geq 0$,

$$E_n(t) := Y_n(t) - X_n(t) \longrightarrow 0 \text{ in probability} \tag{3.67}$$

as $n \to \infty$. For notational simplicity, we take $t = 1$. We have

$$\begin{aligned} &E_n(1) \\ &= \frac{1}{(nb_n^{\alpha+1})^{1/\alpha}} \sum_{i=1}^{n} \sum_{k=-\infty}^{\infty} W_k^{(i)} \left(\sum_{j=0}^{\infty} \varphi^{(i)}(k+j, b_n^2)\, b_n^{\lambda-1} \left((j+1)^{1-\lambda} - j^{1-\lambda} \right) \right. \\ &\qquad\qquad \left. - b_n \int_0^\infty l^{(i)} \left(\frac{k}{b_n} + y, 1 \right) y^{-\lambda} \, dy \right) \\ &= \frac{1}{(nb_n^{\alpha+1})^{1/\alpha}} \sum_{i=1}^{n} \sum_{k=-\infty}^{\infty} W_k^{(i)} \int_0^\infty \left(\varphi^{(i)} \left([k + yb_n], b_n^2 \right) - b_n l^{(i)} \left(\frac{k}{b_n} + y, 1 \right) \right) y^{-\lambda} \, dy \\ &= \frac{1}{n^{1/\alpha} b_n^{2-\lambda+1/\alpha}} \sum_{i=1}^{n} \sum_{k=-\infty}^{\infty} W_k^{(i)} \int_k^\infty \left(\varphi^{(i)} \left([u], b_n^2 \right) - b_n l^{(i)} \left(\frac{u}{b_n}, 1 \right) \right) (u-k)^{-\lambda} \, du \\ &:= \frac{1}{n^{1/\alpha} b_n^{2-\lambda+1/\alpha}} \sum_{i=1}^{n} \sum_{k=-\infty}^{\infty} W_k^{(i)} \int_k^\infty D^{(i)}(u, b_n)(u-k)^{-\lambda} \, du. \end{aligned}$$

Thus, in order to prove (3.67), it will suffice to show that

$$\frac{1}{nb_n^{(2-\lambda)\alpha+1}} \sum_{i=1}^{n} \sum_{k=-\infty}^{\infty} \left| \int_k^\infty D^{(i)}(u, b_n)(u-k)^{-\lambda} \, du \right|^\alpha \longrightarrow 0 \text{ in probability} \tag{3.68}$$

as $n \to \infty$. For integers $m \geq 1$, we define

$$K^{(i)}(m) = \max \left\{ 1 + \sup_{0 \leq k \leq m^2} \left| S_k^{(i)} \right|, m \Big(\sup_{0 \leq t \leq 1} \left| B^{(i)}(t) \right| \Big) \right\}.$$

Then, the expectation of the left-hand side of (3.68) is

$$\begin{aligned}
&\frac{1}{b_n^{(2-\lambda)\alpha+1}}\, E\left(\sum_{k=-\infty}^{\infty}\left|\int_k^{\infty} D^{(1)}(u,b_n)(u-k)^{-\lambda}\,du\right|^{\alpha}\right)\\
&= \frac{1}{b_n^{(2-\lambda)\alpha+1}}\, E\Bigg(\sum_{k=-\infty}^{-K^{(1)}(b_n)-1}\left|\int_{-K^{(1)}(b_n)}^{K^{(1)}(b_n)} D^{(1)}(u,b_n)(u-k)^{-\lambda}\,du\right|^{\alpha}\Bigg)\\
&\quad + \frac{1}{b_n^{(2-\lambda)\alpha+1}}\, E\Bigg(\sum_{k=-K^{(1)}(b_n)}^{K^{(1)}(b_n)}\left|\int_{k}^{K^{(1)}(b_n)} D^{(1)}(u,b_n)(u-k)^{-\lambda}\,du\right|^{\alpha}\Bigg)\\
&:= p_1+p_2\,.
\end{aligned} \tag{3.69}$$

Defining $D_*^{(i)}(m)=\sup_{u\in\mathbb{R}}\left|D^{(i)}(u,m)\right|$ and omitting the superscript "(1)" for notational convenience, we see that

$$p_1 \le \frac{1}{b_n^{(2-\lambda)\alpha+1}}\, E\left(D_*(b_n)^{\alpha}\sum_{k=-\infty}^{-K(b_n)-1}\left(\int_{-K(b_n)}^{K(b_n)}(u-k)^{-\lambda}\,du\right)^{\alpha}\right),$$

with

$$\begin{aligned}
&\sum_{k=-\infty}^{-K(b_n)-1}\left(\int_{-K(b_n)}^{K(b_n)}(u-k)^{-\lambda}\,du\right)^{\alpha}\\
&= \sum_{k=K(b_n)+1}^{\infty}\left((k+K(b_n))^{1-\lambda}-(k-K(b_n))^{1-\lambda}\right)^{\alpha}\\
&= K(b_n)^{(1-\lambda)\alpha+1}\frac{1}{K(b_n)}\sum_{k=K(b_n)+1}^{\infty}\left(\left(\frac{k}{K(b_n)}+1\right)^{1-\lambda}-\left(\frac{k}{K(b_n)}-1\right)^{1-\lambda}\right)^{\alpha}\\
&\le K(b_n)^{(1-\lambda)\alpha+1}\int_1^{\infty}\left((u+1)^{1-\lambda}-(u-1)^{1-\lambda}\right)^{\alpha}du\\
&= const\; K(b_n)^{(1-\lambda)\alpha+1},
\end{aligned}$$

since $(u+1)^{1-\lambda}-(u-1)^{1-\lambda}\sim u^{-\lambda}$ as $u\to\infty$. Thus we obtain

$$\begin{aligned}
p_1 &\le const\,\frac{1}{b_n^{(2-\lambda)\alpha+1}}\, E\left(D_*(b_n)^{\alpha}K(b_n)^{(1-\lambda)\alpha+1}\right)\\
&\le const\,\frac{1}{b_n^{(2-\lambda)\alpha+1}}\,\left(E(D_*(b_n)^2)\right)^{\alpha/2}\left(E\left(K(b_n)^{\frac{2(1-\lambda)\alpha+2}{2-\alpha}}\right)\right)^{1-\alpha/2}\\
&\le const\,\frac{b_n^{(1-\lambda)\alpha+1}}{b_n^{(2-\lambda)\alpha+1}}\,\left(E(D_*(b_n)^2)\right)^{\alpha/2}\longrightarrow 0 \quad \text{as } n\to\infty\,,
\end{aligned} \tag{3.70}$$

by (3.65).

The term p_2 can be bounded similarly:

$$p_2 \le \frac{1}{b_n^{(2-\lambda)\alpha+1}} E\Bigg(D_*(b_n)^\alpha \sum_{k=-K(b_n)}^{K(b_n)} \left(\int_k^{K(b_n)} (u-k)^{-\lambda} \, du \right)^\alpha \Bigg),$$

with

$$\begin{aligned} \sum_{k=-K(b_n)}^{K(b_n)} \left(\int_k^{K(b_n)} (u-k)^{-\lambda} \, du \right)^\alpha &\le \sum_{k=-K(b_n)}^{K(b_n)} \left(\int_0^{2K(b_n)} u^{-\lambda} \, du \right)^\alpha \\ &= const \sum_{k=-K(b_n)}^{K(b_n)} K(b_n)^{(1-\lambda)\alpha} \\ &= const \; K(b_n)^{(1-\lambda)\alpha+1}. \end{aligned}$$

It follows that

$$p_2 \le const \, \frac{1}{b_n^{(2-\lambda)\alpha+1}} E\left(D_*(b_n)^\alpha K(b_n)^{(1-\lambda)\alpha+1} \right) \longrightarrow 0 \quad \text{as } n \to \infty \,,$$

as in (3.70). Thus we have established (3.68).

The next step is to show that the finite-dimensional distributions of the process $(X_n(t), t \ge 0)$ in (3.66) converge to those of $(Y(t), t \ge 0)$. For this, it is enough to show that, for every $m \ge 1$, $0 < t_1 < \ldots < t_m$ and $\theta_1, \ldots, \theta_m \in \mathbb{R}$,

$$\sum_{j=1}^m \theta_j X_n(t_j) \stackrel{d}{\longrightarrow} \sum_{j=1}^m \theta_j Y(t_j) \qquad \text{as } n \to \infty \,.$$

We will see that this is true for $m = 1$ and $t_1 = 1$; the general case is similar. So we will show that

$$\frac{1}{(nb_n)^{1/\alpha}} \sum_{i=1}^n \sum_{k=-\infty}^\infty W_k^{(i)} \int_0^\infty l^{(i)} \left(\frac{k}{b_n} + y, 1 \right) y^{-\lambda} \, dy \stackrel{d}{\longrightarrow} Y(1). \tag{3.71}$$

Since both sides of (3.71) are conditionally SαS random variables, it will suffice to show the convergence in probability of the scale parameters. That is, it will suffice to show that

$$\frac{1}{nb_n}\sum_{i=1}^{n}\sum_{k=-\infty}^{\infty}\left(\int_0^\infty l^{(i)}\left(\frac{k}{b_n}+y,1\right)y^{-\lambda}\,dy\right)^\alpha \longrightarrow E\int_{\mathbb{R}}\left(\int_0^\infty l\,(x+y,1)\,y^{-\lambda}\,dy\right)^\alpha dx \tag{3.72}$$

in probability. Let us denote the absolute difference

$$\left|\frac{1}{nb_n}\sum_{i=1}^{n}\sum_{k=-\infty}^{\infty}\left(\int_0^\infty l^{(i)}\left(\frac{k}{b_n}+y,1\right)y^{-\lambda}\,dy\right)^\alpha - \frac{1}{b_n}\,E\sum_{k=-\infty}^{\infty}\left(\int_0^\infty l\left(\frac{k}{b_n}+y,1\right)y^{-\lambda}\,dy\right)^\alpha\right|$$

by δ_n. Now, by Chebyshev's inequality,

$$\begin{aligned}
P(\delta_n>\epsilon) &\le \frac{1}{\epsilon^2 n b_n^2}\,E\left(\sum_{k=-\infty}^{\infty}\left(\int_0^\infty l\left(\frac{k}{b_n}+y,1\right)y^{-\lambda}\,dy\right)^\alpha\right)^2\\
&\le \frac{1}{\epsilon^2 n b_n^{2+2(1-\lambda)\alpha}}\,E\left(\sum_{k=-\infty}^{\infty}\left(\int_0^\infty l\left(\frac{u}{b_n},1\right)(u-k)^{-\lambda}\,du\right)^\alpha\right)^2\\
&\le \frac{const}{\epsilon^2 n b_n^{2+2(1-\lambda)\alpha}}\,E\left(\sum_{k=-\infty}^{-K(b_n)-1}\left(\int_{-K(b_n)}^{K(b_n)} l\left(\frac{u}{b_n},1\right)(u-k)^{-\lambda}\,du\right)^\alpha\right)^2\\
&\quad+\frac{const}{\epsilon^2 n b_n^{2+2(1-\lambda)\alpha}}\,E\left(\sum_{k=-K(b_n)}^{K(b_n)}\left(\int_{k}^{K(b_n)} l\left(\frac{u}{b_n},1\right)(u-k)^{-\lambda}\,du\right)^\alpha\right)^2\\
&:= p_1'+p_2'\,.
\end{aligned}$$

Note the similarity of p_1', p_2' to p_1, p_2 in (3.69). By arguments analogous to the ones used for p_1 and p_2, one can show that

$$p_1'+p_2' \le const\,\frac{b_n^{2+2(1-\lambda)\alpha}}{\epsilon^2 n b_n^{2+2(1-\lambda)\alpha}} \longrightarrow 0$$

as $n\to\infty$, hence $\delta_n\to 0$ in probability. Moreover, we have

$$\frac{1}{b_n}\,E\sum_{k=-\infty}^{\infty}\left(\int_{\mathbb{R}} l\left(\frac{k}{b_n}+y,1\right)y^{-\lambda}\,dy\right)^\alpha \longrightarrow E\int_{\mathbb{R}}\left(\int_{\mathbb{R}} l\,(x+y,1)\,y^{-\lambda}\,dy\right)^\alpha dx$$

by the Dominated Convergence Theorem. The convergence (3.72) follows, hence (3.71) is proven, and so is the theorem. □

Corollary 3.10.4. *Let $(b_n, n \geq 1)$ be a sequence of positive integers with $b_n \to \infty$, $1 < \alpha \leq 2$ and $1/\alpha < \lambda < 1$. Define, for $n \geq 1$ and $t \geq 0$,*

$$Y_n(t) = \frac{1-\lambda}{n^{1/\alpha} b_n^{2-\lambda+1/\alpha}} \sum_{i=1}^{n} \sum_{k=-\infty}^{\infty} \varphi^{(i)}(k, b_n^2 t) \sum_{j=0}^{\infty} j^{-\lambda} W_{k-j}^{(i)} \,. \tag{3.73}$$

Then we have, as $n \to \infty$,

$$\big(Y_n(t), t \geq 0\big) \xrightarrow{f.d.} \big(Y(t), t \geq 0\big),$$

where $\xrightarrow{f.d.}$ denotes convergence in finite-dimensional distributions and $(Y(t), t \geq 0)$ is the process defined in (3.60).

Proof. The proof is analogous to that of Theorem 3.10.3. □

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