

# Exact Analysis of a Lost Sales Model under Stuttering Poisson Demand

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We investigate the  $(S-1,S)$  inventory policy under stuttering Poisson demand and generally distributed lead time when the excess demand is lost. We correct results presented in Feeney and Sherbrooke's seminal paper (1966). We also prove that the distribution of ordered unit delivery times becomes increasingly concentrated as the variance-to-mean ratio of demand increases.

*Key words*: Lost Sales, Stuttering Poisson Process, Reversible Markov Chain

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## 1. Introduction and Literature Review

### 1.1. Introduction

We investigate an  $(S - 1, S)$  inventory policy employed in a lost sales environment in this paper. In this environment, demand for an item occurs according to a compound Poisson process. We assume the compounding distribution is a geometric distribution. This demand process is known as a stuttering Poisson process. Additionally, the replenishment lead times are assumed to be independent and identically distributed for each order. Since a replenishment order is placed each time a customer order is accepted, all units accepted in a customer order are replenished together as an order. We investigate the special case in detail in which lead times have an exponential distribution.

Our paper addresses the topics contained in the seminal paper by Feeney and Sherbrooke(F-S) published in 1966. One of the many results reported in their paper is a collection of formulas for the stationary distribution of the number of units on order for general compound Poisson demand processes when demand in excess of supply is lost. For the particular case of geometrically-distributed order-sizes, our results lead to formulas that differ from those presented in their paper.

Since their results do not hold for this special case, we conclude that the exact analysis for general compound Poisson demand processes is still an open question. In this paper, we derive the exact results for the lost sales model with exponential lead times under the  $(S - 1, S)$  inventory policy for the environment we have described. Furthermore, we prove that the stationary distribution does not depend on the lead time distribution but only on its mean. We also demonstrate that, at least for the stuttering Poisson case, the F-S formulas are good approximations when used to set optimal stock levels. We also prove an interesting result on the spread of expected order replenishment delivery times as a function of the variance-to-mean ratio of the demand process. We show that the spread of these times increases and becomes more concentrated as the variance-to-mean ratio grows.

Our motivation for reconsidering the lost sales model derives from its use in modeling a type of emergency order system which we have observed in industry. Such a system contains a regional stocking location (RSL), which serves two types of facilities: a set of field service locations (FSL) and an emergency stocking location (ESL). The field service locations support technical service representatives who make visits to customer sites to repair equipment. The demand processes at the FSLs exhibit a high variance-to-mean ratio. These locations operate on an  $(S - 1, S)$  policy. Demand in excess of on-hand inventory at the FSL is redirected to the ESL. Each FSL places replenishment orders with the RSL. The stationary distribution of the number of units in regular resupply at the FSLs from the RSL is the probability distribution derived in this paper for the lost sales model. In a companion paper, we further develop both exact and approximate expressions for the mean and variance of the number of emergency orders outstanding in this system, and the probability that there are no outstanding emergency orders.

Our paper is organized as follows. In section 2, we introduce the lost sales model and notation. Our focus is on the partial fill case in that section. By partial fill, we mean that the demand in excess of inventory on hand is lost but the remainder of the customer's order is filled from on-hand inventory. In section 3, we construct a Markov chain to represent state transitions and show the reversibility of this Markov chain. Using this property in section 4, we derive the stationary

distribution of the number of units on order for the lost sales model. The complete fill case is mentioned in section 4.3. By complete fill, we mean that a customer order is rejected (all units lost) if there is insufficient stock on hand to fill it completely. Both cases, complete and partial fill, can be extended to general lead time distributions provided only that lead times are independent and identically distributed. In section 5, we compare these exact results with Feeney and Sherbrooke's results. In section 6, we prove a property of the expected ordered unit delivery times. Concluding comments are found in section 7, followed by a glossary of notation. Proofs of theorems for general lead time distributions are deferred to an e-companion appendix.

## 1.2. Review of Literature

There are literally thousands of papers on inventory control problems for both the backorder and the lost sales cases. Discussion of many of them can be found in the textbooks by Arrow *et al.* (1958), Zipkin (2000), Porteus (2002), Muckstadt (2005), Nahmias (2005) and Axsäter (2006). These texts also contain extensive bibliographies.

In this brief review, we highlight research on lost sales problems with Poisson and compound Poisson demand processes, in both continuous and periodic review settings. One such important result due to Palm (1938) establishes that the number of units in resupply in steady state of the lost sales model is a truncated Poisson distribution for any resupply time distribution having a finite mean when the resupply times are independent. Hadley and Whitin (1963) derive performance measures for both the backorders case and the lost sales case for any resupply distribution having a finite mean under Poisson demand. Smith (1977) derives procedures for finding the optimal stocking levels for the (S-1,S) models with Poisson demand process and arbitrary resupply times with lost sales. Johansen and Thorstenson (1993,1996) construct algorithms for finding the optimal (r,Q) policy parameters with Poisson demand process and lost sales. Karlin and Scarf (1958) show in general that an (S-1,S) policy for lost sales is not optimal, and, more recently, Hill (1999) illustrates this fact for the case in which demand is Poisson process and the lead time is constant.

For compound Poisson demand processes, Galliher, *et al.* (1959) generalize the demand assumption to the stuttering Poisson for the backorder case but restrict attention to the constant and exponential resupply time distributions. Feeney and Sherbrooke (1966) provide the generalization discussed in the introduction. Mitchell, *et al.* (1983) also provide empirical support for the stuttering Poisson model by examining actual historical data from several U.S. Air Force bases. Cheung (1996) derives expressions of the steady state distribution and the expectation of the number of backordered units for the (S-1,S) inventory system with compound Poisson demands for the special case of i.i.d. resupply times for each unit ordered. Mohebbi and Posner (1998) derive the stationary distribution of an (s,nQ) inventory system with compound Poisson demands when lead times are Erlang and hyperexponentially distributed and demand in excess of supply is lost. Johnson *et al.* (2003) also find empirical evidence to support the assumption of using the geometric distribution as the compounding distribution for the size of orders from customers. Lu and Randovanović (2007) generalize existing models with compound point processes and derive simple asymptotic expressions for blocking probabilities (i.e. the proportion of lost sales to total demand ) in loss networks.

Most of the papers mentioned above are continuous review inventory control problems for lost sales systems. For the periodic review problems, Morton (1969) generalizes the basic results of Karlin and Scarf (1958) to the periodic review lost sales problems with fixed lead times that are integer multiples of the period's length. Subsequently, Morton (1971) proposes and evaluates myopic policies as effective heuristics for these problems. Nahmias (1979) considers more general periodic review lost sales problems. He includes fixed ordering costs, partial backordering and random lead times. He develops myopic policies for these problems using either (s,S) policies or order-up-to-S policies to manage inventories. Kapalka *et al.* (1999) analyze the (s,S) policy for the single location, single item periodic review inventory model with lost sales and service level constraints with a fraction of a period lead time. Downs *et al.* (2001) develop a linear program to determine an optimal (S-1,S) policy for multiple products when budgets are constrained, lead times are constant, and sales in excess of supply are lost. Janakiraman *et al.* (2004, 2006) provide more

analysis about the optimality and cost comparison for periodic review lost sales and backordering models.

Baganha (1985) criticizes the balance equations used in the Feeney and Sherbrooke paper but does not challenge the result. As we have mentioned, we correct a formula found in Feeney and Sherbrooke's analysis. To our knowledge, no paper exists that addresses the problem studied in our paper.

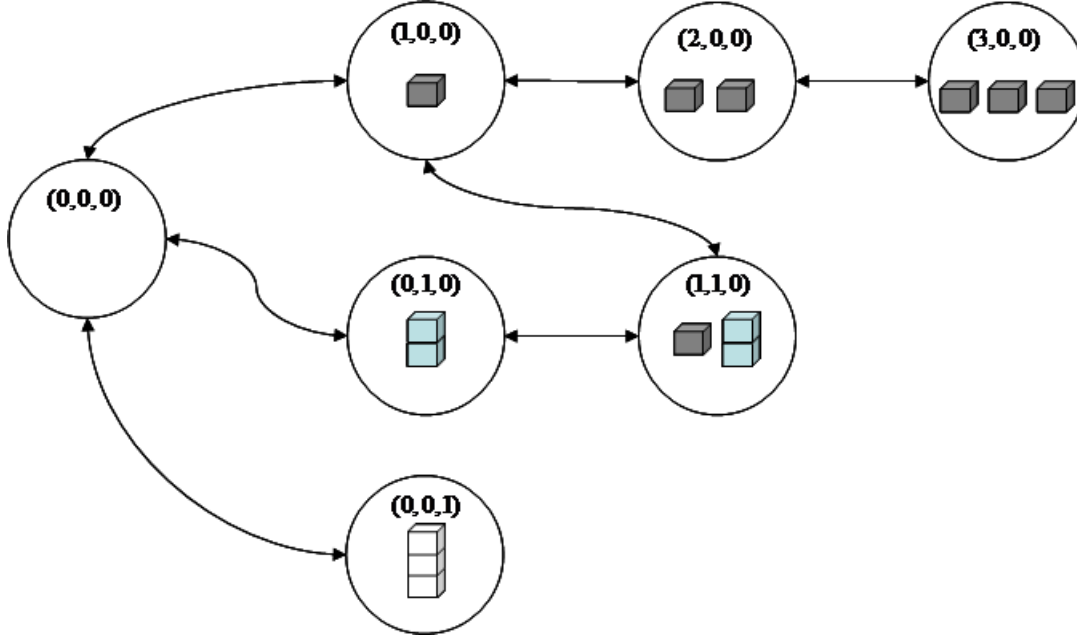
## 2. A Lost Sales Model with Compound Poisson Demand and Exponential Lead Times

We begin our analysis by considering a continuous time model in which demand arrives according to a stationary compound Poisson process. Let  $\lambda$  denote the rate of arrivals of customer orders and let  $X$  denote the order size, which is a positive, integer-valued, random variable. Let  $p_k \equiv P\{X = k\}$  and let  $\bar{P}_k \equiv P\{X > k\}$  for all  $k = 0, 1, 2, \dots$ . We assume at least one unit is ordered for each customer arrival:  $p_0 = 0$  and  $\bar{P}_0 = 1$ , although the results are easily generalized to allow for zero-sized orders. For the special case of the so-called stuttering Poisson process, the order size distribution is geometric. Let  $p$  denote the probability of a unit-sized order under the geometric distribution:  $p_1 = p$ . In this case, for all  $k = 1, 2, \dots$ ,  $p_k = p(1 - p)^{k-1}$  and  $\bar{P}_k = (1 - p)^k$ .

Let  $I_t$  denote the inventory on hand at time  $t$ ,  $t \geq 0$ , a non-negative, integer-valued random variable. We assume that demand in excess of inventory on hand is lost but that a customer's order may be partially filled. That is, fulfilled demand at the time of a customer order is given by  $X_t \wedge I_t$  (defined as  $\min(X_t, I_t)$ ), where  $X_t$  is the size of the customer order, and  $t$  is customer order arrival epoch.

We assume that the system is managed according to an  $(S - 1, S)$  policy. Thus whenever a customer arrives, the accepted demand is  $X \wedge I$ , where  $X$  is the customer order size and  $I$  is inventory on hand at the time of the order, and a replenishment order is placed for  $X \wedge I$  units. The total number of units on order plus on hand is maintained at a constant level,  $S$ .

Finally, we assume that lead times for replenishment orders are independent, exponentially-distributed random variables with rate  $\mu$ . Let  $\tau$  denote the expected replenishment order lead time:



**Figure 1** State Space and Single Order Transitions for  $S = 3$  and Partial Fill Case

$$\tau = 1/\mu.$$

Let  $N_{kt}$  denote the number of replenishment orders of size  $k$  outstanding at time  $t$ , for  $k = 1, 2, \dots, S$ , and let  $N_t = (N_{kt})_{k=1}^S$  denote the vector of outstanding replenishment orders. Given our assumptions of lost sales, partial fills, and an  $(S-1, S)$  policy, it follows that

$$I_t + \sum_{k=1}^S kN_{kt} = S.$$

The stochastic process  $N = \{N_t, t \geq 0\}$  is a finite-state, time-homogeneous Markov process. Let  $V$  index the state space of the underlying Markov chain. That is, we assume the existence of a one-to-one mapping from  $V$  to the set of all possible vectors of outstanding replenishment orders. For each  $i \in V$ , we denote the mapping by  $n(i) = (n_1(i), n_2(i), \dots, n_S(i))$ , where  $n_k(i) \in \{0, 1, \dots, \lfloor S/k \rfloor\}$  for all  $k = 1, \dots, S$ , and  $\sum_{k=1}^S kn_k(i) \leq S$ . Furthermore, the implied number of units on hand is given by

$$n_0(i) \equiv S - \sum_{k=1}^S kn_k(i).$$

The graphic in Figure 1 illustrates the possible states when  $S = 3$  when orders may be partially

filled. For example, the state  $(1, 1, 0)$  corresponds to the situation of three units on order over two orders: one order of size one unit and one order of size two.

Let the pair  $(i, j)$  denote a transition from state  $i \in V$  to state  $j \in V$ . Let

$$\|(i, j)\| \equiv \sum_{k=1}^S |n_k(i) - n_k(j)|, \text{ for all } (i, j) \in V \times V,$$

the number changes in outstanding order levels separating  $i$  from  $j$ . State transitions occur only when either a customer order arrives or a replenishment order arrives. Since the probability that two or more orders arrive simultaneously is infinitesimally small, we focus on single-order transitions, that is, transitions for which  $\|(i, j)\| = 1$ . The arrows in Figure 1 indicate all possible single order transitions for the  $S = 3$  case. For a single-order transition  $(i, j)$ , let  $k_{ij}$  denote the size of the (accepted) customer order or the size of the arriving replenishment order, as appropriate:

$$k_{ij} \equiv \sum_{k=1}^S k |n_k(i) - n_k(j)| \text{ for all } (i, j) \in V \times V \text{ s.t. } \|(i, j)\| = 1.$$

We classify single-order transitions by whether they are customer order arrivals ( $(i, j) \in V_C^2$ ) or replenishment order arrivals ( $(i, j) \in V_R^2$ ):

$$(i, j) \in \begin{cases} V_C^2 & \text{iff } \|(i, j)\| = 1 \text{ and } n_{k_{ij}}(i) < n_{k_{ij}}(j) = n_{k_{ij}}(i) + 1 \\ V_R^2 & \text{iff } \|(i, j)\| = 1 \text{ and } n_{k_{ij}}(i) > n_{k_{ij}}(j) = n_{k_{ij}}(i) - 1 \end{cases}$$

It is easily seen that the infinitesimal generator for this Markov process  $N$  is given by

$$A_{ij} \equiv \begin{cases} n_{k_{ij}}(i)\mu & \text{if } (i, j) \in V_R^2, \\ \lambda p_{k_{ij}} & \text{if } (i, j) \in V_C^2, n_0(j) > 0, \\ \lambda \bar{P}_{k_{ij}-1} & \text{if } (i, j) \in V_C^2, n_0(j) = 0, \\ -(m(i)\mu + \lambda) & \text{if } j = i, \\ 0 & \text{otherwise.} \end{cases}$$

To see this, note that  $(i, j) \in V_R^2$  means a replenishment of size  $k_{ij}$  arrived. In this case, the transition rate is  $n_{k_{ij}}(i)\mu$ . The condition  $(i, j) \in V_C^2, n_0(j) > 0$  means a new customer order of size  $k_{ij}$  arrives and could be satisfied. So the transition rate is  $\lambda p_{k_{ij}}$ . The condition  $(i, j) \in V_C^2, n_0(j) = 0$  means an order arrives with order size greater than or equal to the on-hand inventory level causing  $n_0(j) = 0$

with transition rate  $\lambda \bar{P}_{k_{ij}-1}$ . For any other  $(i, j), j \neq i$ , there is no single step transition between them, so the transition rate is zero. Finally, for  $j = i$ :

$$A_{ii} = - \sum_{j \in V, j \neq i} A_{ij} = - \left( \sum_j n_{k_{ij}}(i) \mu + \lambda \right) = -(m(i) \mu + \lambda).$$

In the case of the stuttering Poisson demand process, this infinitesimal generator simplifies to:

$$A_{ij} = \begin{cases} n_{k_{ij}}(i) \mu & \text{if } (i, j) \in V_R^2, \\ \lambda p^{1\{n_0(j)>0\}} (1-p)^{k_{ij}-1} & \text{if } (i, j) \in V_C^2, \\ -(m(i) \mu + \lambda) & \text{if } j = i, \\ 0 & \text{otherwise,} \end{cases} \quad (1)$$

where  $1\{E\}$  is the indicator function of condition  $E$  ( $1\{E\} = 1$  if  $E$  and  $= 0$  otherwise). Important properties of this infinitesimal generator will be seen to hold only if the demand process is a stuttering Poisson process.

### 3. Reversibility

Our goal is to calculate the stationary distribution of the continuous time Markov process  $N$  defined in the previous section. We find that when the arrival process is a stuttering Poisson process, this Markov process is reversible. As a consequence, we can compute the stationary distribution easily. Now let us first review the definition and properties of a reversible continuous time Markov Chain.

A reversible continuous-time Markov chain is defined and described in Resnick (2005, p433-434).

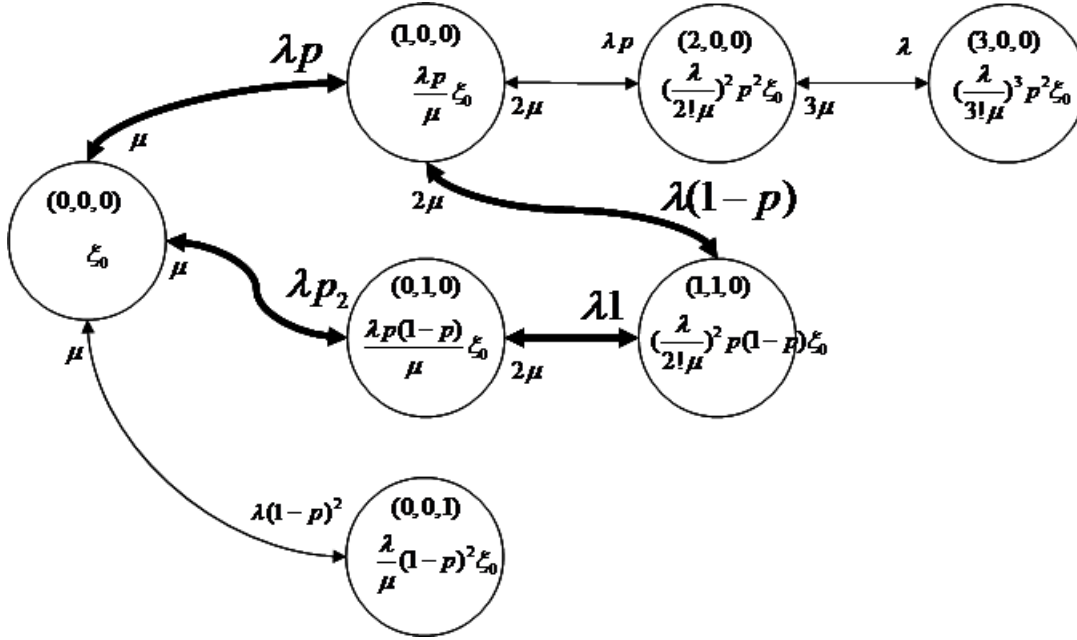
The following proposition characterizes the reversible property.

**PROPOSITION 1.** *A stationary Markov chain  $\{\tilde{X}(t), -\infty < t < \infty\}$  is reversible if and only if when  $\tilde{A}$  is the generator matrix of  $\{\tilde{X}(t)\}$ , the detailed balance equations*

$$\tilde{\xi}_i \tilde{A}_{ij} = \tilde{\xi}_j \tilde{A}_{ji}, \text{ for all } i \neq j, \quad (2)$$

*hold for some  $\tilde{\xi}$ . If a solution  $\tilde{\xi}$  can be found to (2), then  $\tilde{\xi}$  is, in fact, the stationary distribution of  $\{\tilde{X}(t)\}$ .*

For the replenishment order process,  $N$ , we choose as a reference state the state  $i_0$  for which no orders are outstanding ( $n_0(i_0) = S$ ). For any state  $i \in V$ , with at least one outstanding order ( $n_0(i) <$



**Figure 2** Solution to  $S = 3$  Case with Sample Loop

$S$ ), we seek to define a sequence of single-order transitions that will lead from  $i$  to the reference state,  $i_0$ . It is natural to choose each transition to correspond to the delivery of a replenishment order. In this case, the number of transitions required will be given by the total number of outstanding orders in state  $i$ . Let  $m(i) \equiv \sum_{k=1}^S n_k(i)$ , be the total number of outstanding orders in state  $i$ . We form a path of states  $j_0 = i \rightarrow j_1 \rightarrow j_2 \rightarrow \dots \rightarrow j_{m(i)-1} \rightarrow j_{m(i)} = i_0$  in which each transition  $(j_l, j_{l+1})$  corresponds to the delivery of a replenishment order  $((j_l, j_{l+1}) \in V_R^2)$ . Furthermore, for each transition, we choose the size of the arriving replenishment order according to a *largest subscript rule*. That is, let  $j_l$  denote a state on this path,  $l = 0, 1, \dots, m(i) - 1$ . Let  $k_l$  denote the order size of the largest outstanding order:  $k_l = \max \{k \in \{1, 2, \dots, S\} : n_k(j_l) > 0\}$ . We choose as the next state,  $j_{l+1}$ , the state corresponding to the arrival of a replenishment order of size  $k_l$ . That is,  $j_{l+1}$  is the unique state satisfying  $n_k(j_{l+1}) = n_k(j_l) - 1\{k = k_l\}$  for all  $k = 1, \dots, S$ . It should be clear that a path of single-order transitions from the reference state  $i_0$  back to state  $i$  can be found by simply reversing the sequence:  $i_0 \rightarrow j_{m(i)-1} \rightarrow \dots \rightarrow j_1 \rightarrow i$ . Along this reverse path, the transitions all correspond to customer arrivals  $((j_l, j_{l-1}) \in V_C^2)$ .

The graphic in Figure 2 illustrates two possible paths through the state space from state  $(1, 1, 0)$  to the reference state  $(0, 0, 0)$  along one step transitions corresponding to deliveries. One path passes through  $(1, 0, 0)$  and the other through  $(0, 1, 0)$ . Each of these paths has a reverse path from  $(0, 0, 0)$  back to  $(1, 1, 0)$  along one step transitions corresponding to customer arrivals. Arrival transition rates are shown above the transition arc while delivery transition rates are shown below the transition arc.

Suppose that the replenishment order process is reversible and that  $\eta$  is the stationary distribution. Given the largest subscript rule of selecting paths between any state  $i \in V$  and the reference state  $\nu_{i_0}$ , observe that repeated application of (2) yields the following:

$$\xi_i A_{ij_1} A_{j_1 j_2} \dots A_{j_{m(i)-1} i_0} = \xi_{i_0} A_{i_0 j_{m(i)-1}} A_{j_{m(i)-1} j_{m(i)-2}} \dots A_{j_1 i}.$$

This suggests a solution of the form ( $i \in V$ ):

$$\xi_i = \begin{cases} \nu_i \xi_{i_0} & i \neq i_0, \\ \frac{1}{1 + \sum_{j \neq i_0} \nu_j} & i = i_0, \end{cases} \quad (3)$$

where

$$\nu_i \equiv \frac{A_{i_0 j_{m(i)-1}} A_{j_{m(i)-1} j_{m(i)-2}} \dots A_{j_1 i}}{A_{ij_1} A_{j_1 j_2} \dots A_{j_{m(i)-1} i_0}}. \quad (4)$$

In Figure 2, for the  $S = 3$  example, let  $i$  correspond to the state  $(1, 1, 0)$ . In this case, considering the path through state  $(1, 0, 0)$  we have

$$\nu_i = \frac{\lambda p \cdot \lambda(1-p)}{2\mu \cdot \mu} = \frac{\lambda^2 p(1-p)}{2\mu^2}.$$

After normalization (3), we have

$$\xi_i = \frac{\lambda^2 p(1-p)}{2\mu^2} \xi_{i_0}.$$

Observe that since  $p_2 = p(1-p)$  for the stuttering Poisson, we arrive at the same formula for state  $(1, 1, 0)$  whether we consider the path through  $(1, 0, 0)$  or the path through  $(0, 1, 0)$ . The corresponding formulas for the other states in the  $S = 3$  case are shown in this figure.

PROPOSITION 2. *For the geometric order size distribution, the suggested solution (4) is given by ( $i \in V$ ):*

$$\nu_i \equiv \frac{\left(\frac{\lambda p}{\mu(1-p)}\right)^{m(i)} (1-p)^{S-n_0(i)}}{\prod_{k=1}^S (n_k(i)! p^{1\{n_0(i)=0\}}}. \quad (5)$$

**Proof:** For any path chosen according to the largest subscript rule and for the generators (1),

$$\begin{aligned} A_{i_0 j_{m(i)-1}} A_{j_{m(i)-1} j_{m(i)-2}} \cdots A_{j_1 i} &= \prod_{l=m(i), m(i)-1, \dots, 1} (\lambda p^{1\{n_0(j_l) > 0\}} (1-p)^{(k_{j_l, j_{l-1}} - 1)}) \\ &= \left(\frac{\lambda p}{(1-p)}\right)^{m(i)} (1-p)^{\sum_{l=1}^{m(i)} k_{j_l, j_{l-1}}} p^{-1\{n_0(i)=0\}} \\ &= \left(\frac{\lambda p}{(1-p)}\right)^{m(i)} (1-p)^{S-n_0(i)} p^{-1\{n_0(i)=0\}}. \end{aligned}$$

Considering the path from  $i$  to  $i_0$  and noting that if  $n_k(i) = 0$ ,  $n_k(i)! = 1$ , we get

$$A_{i j_1} A_{j_1 j_2} \cdots A_{j_{m(i)-1} i_0} = \prod_{l=0, 1, \dots, m(i)-1} \mu n_{k_{l, l+1}}(j_l) = \mu^{m(i)} \prod_{k=1}^S (n_k(i)!).$$

Therefore, from (4)

$$\nu_i \equiv \frac{A_{i_0 j_{m(i)-1}} A_{j_{m(i)-1} j_{m(i)-2}} \cdots A_{j_1 i}}{A_{i j_1} A_{j_1 j_2} \cdots A_{j_{m(i)-1} i_0}} = \frac{\left(\frac{\lambda p}{\mu(1-p)}\right)^{m(i)} (1-p)^{S-n_0(i)}}{\prod_{k=1}^S (n_k(i)! p^{1\{n_0(i)=0\}}}.$$

■

THEOREM 1. *For the geometric order size distribution, the replenishment order process,  $N$ , is a reversible stochastic process whose stationary distribution is given by (3) and (5).*

**Proof:** Consider any two distinct states  $i, i' \in V$ , with  $n(i) = (n_1(i), n_2(i), \dots, n_S(i))$  and  $n(i') = (n_1(i'), n_2(i'), \dots, n_S(i'))$ ,  $i \neq i'$ .

1. Since  $i \neq i'$ ,  $\|(i, i')\| \neq 0$ . Suppose  $A_{ii'} = 0$ , then by (1)  $A_{i'i} = 0$ , too. Hence, if  $A_{ii'} = 0$ , we have  $\nu_i A_{ii'} = \nu_{i'} A_{i'i} \equiv 0$ .

2. When  $A_{ii'} \neq 0$  and  $i \neq i'$ , then, by (1),  $\|(i, i')\| = 1$ , and either  $(i, i') \in V_C^2$  or  $(i, i') \in V_R^2$ . Without loss of generality, we assume  $(i, i') \in V_C^2$  and  $(i', i) \in V_R^2$ .

There are two subcases:

•  $n_0(i') > 0$ : In this case, a demand of size  $k_{ii'}$  arrives which is strictly less than  $n_0(i)$ . Then  $A_{ii'} = \lambda p(1-p)^{(k_{ii'}-1)}$ ,  $A_{i'i} = n_{k_{ii'}}(i')\mu = (n_{k_{ii'}}(i) + 1)\mu$  and  $m(i') = m(i) + 1$ . Hence

$$\begin{aligned}
\nu_i A_{ii'} &= \frac{\frac{\lambda p}{\mu(1-p)} \cdot \frac{m(i)}{\prod_{k=1}^S (n_k(i)!)}}{\frac{(1-p)^{S-n_0(i)}}{p^{1\{n_0(i)=0\}}}} \cdot (\lambda p(1-p)^{(k_{ii'}-1)}) \\
&= \frac{\left(\frac{1}{\mu(1-p)}\right)^{m(i)}}{\prod_{k=1}^S (n_k(i)!)}} \cdot (1-p)^{S-n_0(i)+k_{ii'}-1} (\lambda p)^{1+m(i)} \\
&= \frac{\mu \left(\frac{1}{\mu(1-p)}\right)^{m(i')} \frac{n_{k_{ii'}}(i')!}{n_{k_{ii'}}(i)!}}{\prod_{k=1}^S (n_k(i')!) \frac{n_{k_{ii'}}(i)!}{n_{k_{ii'}}(i)!}} \cdot (1-p)^{S-n_0(i')} (\lambda p)^{m(i')} \\
&= \frac{\left(\frac{\lambda p}{\mu(1-p)}\right)^{m(i')}}{\prod_{k=1}^S (n_k(i')!)}} \cdot (1-p)^{S-n_0(i')} \cdot \left(\mu \frac{n_{k_{ii'}}(i')!}{n_{k_{ii'}}(i)!}\right) \\
&= \nu_{i'} \cdot (n_{k_{ii'}}(i')\mu) \\
&= \nu_{i'} A_{i'i}.
\end{aligned}$$

•  $n_0(i') = 0$ : In this case, a demand arrives and the demand size is equal to or greater than  $k_{ii'} = n_0(i)$ , so  $A_{ii'} = \lambda(1-p)^{(k_{ii'}-1)}$ , and  $A_{i'i} = n_{k_{ii'}}(i')\mu$ . But

$$\nu_{i'} = \frac{\left(\frac{\lambda p}{\mu(1-p)}\right)^{m(i)}}{\prod_{k=1}^S (n_k(i')!)}} \frac{(1-p)^{S-n_0(i')}}{p}.$$

Similarly,

$$\begin{aligned}
\nu_i (A_{ii'} p) &= \frac{\frac{\lambda p}{\mu(1-p)} \cdot \frac{m(i)}{\prod_{k=1}^S (n_k(i)!)}}{\frac{(1-p)^{S-n_0(i)}}{p^{1\{n_0(i)=0\}}}} \cdot (\lambda(1-p)^{(k_{ii'}-1)}) p \\
&= \frac{\left(\frac{1}{\mu(1-p)}\right)^{m(i)}}{\prod_{k=1}^S (n_k(i)!)}} \cdot (1-p)^{S-n_0(i)+k_{ii'}-1} (\lambda p)^{1+m(i)} \\
&= \frac{\mu \left(\frac{1}{\mu(1-p)}\right)^{m(i')} \frac{n_{k_{ii'}}(i')!}{n_{k_{ii'}}(i)!}}{\prod_{k=1}^S (n_k(i')!) \frac{n_{k_{ii'}}(i)!}{n_{k_{ii'}}(i)!}} \cdot (1-p)^{S-n_0(i')} (\lambda p)^{m(i')} \\
&= \frac{\left(\frac{\lambda p}{\mu(1-p)}\right)^{m(i')}}{\prod_{k=1}^S (n_k(i')!)}} \cdot (1-p)^{S-n_0(i')} \cdot \left(\mu \frac{n_{k_{ii'}}(i')!}{n_{k_{ii'}}(i)!}\right) \\
&= \nu_{i'} p \cdot (n_{k_{ii'}}(i')\mu) \\
&= (\nu_{i'} p) A_{i'i}.
\end{aligned}$$

Hence,  $\nu_i A_{ii'} = \nu_{i'} A_{i'i}$ .

Therefore, for any  $i, i' \in V$ , we have  $\nu_i A_{ii'} = \nu_{i'} A_{i'i}$  and, after normalization,  $\eta_i A_{ii'} = \eta_{i'} A_{i'i}$ . By Proposition 3.1,  $N$  is a reversible stochastic process whose stationary distribution is given by (3) and (5). ■

**THEOREM 2.** *If the order size distribution satisfies  $p_k > 0$  for all  $k = 1, 2, \dots$ , then the replenishment order process,  $N$ , is a reversible stochastic process for all positive values of  $S$  if and only if the order size distribution is geometric.*

**Proof:** Let  $x$  and  $y$  be positive integers such that  $S = x + y$  and  $P(X = x) > 0$  and  $P(X = y) > 0$ .

Consider the special states

$$n(i_0) = (0, 0, \dots, 0)$$

and

$$\{n(i) : n_x(i) = 1, n_y(i) = 1, n_k(i) = 0, \text{ for } k \neq x, y\}.$$

Now pick the cyclic sequence:  $i_0 \rightarrow i_1 \rightarrow i_2 \rightarrow i'_1 \rightarrow i_0$ , where

$$\{n(i_1) : n_x(i_1) = 1, n_k(i_1) = 0, \text{ for } k \neq x\},$$

and

$$\{n(i'_1) : n_y(i'_1) = 1, n_k(i'_1) = 0, \text{ for } k \neq y\}.$$

If this is a reversible Markov process,

$$\nu_{i_0} A_{i_0, i_1} A_{i_1, i_2} A_{i_2, i'_1} A_{i'_1, i_0} = \nu_{i_0} \lambda^2 p_x P(X \geq y) \mu^2$$

must equal

$$\nu_{i_0} A_{i_0, i'_1} A_{i'_1, i_2} A_{i_2, i_1} A_{i_1, i_0} = \nu_{i_0} \lambda^2 p_y P(X \geq x) \mu^2.$$

i.e.

$$p_x P(X \geq y) = p_y P(X \geq x).$$

Now, if  $\{p_k > 0, \text{ for } k = 1, 2, \dots\}$ , an inductive proof easily establishes  $p_k = p_1(1 - p_1)^{k-1}$  by letting  $x \equiv 1$ . So  $X$  must be geometrically distributed with parameter  $p = p_1$ . Combined with Theorem (3.3), this is a sufficient and necessary condition for reversibility. ■

In summary, we have shown that the steady state distribution of the replenishment order process can be found using the property of reversibility but only for the case of stuttering Poisson demand (when the order size distribution has support on all natural numbers).

## 4. The Stationary Distribution of the Number of Units on Order

In this section we derive an explicit formula for the stationary distribution of the number of units on order in the lost sales model with stuttering Poisson demand. Two variants of the formula are derived: one for the partial fill case and the other for the complete fill case. In both cases, we extend the results to general lead time distributions provided lead times are identically independently distributed.

### 4.1. The Partial Fill Case

Let  $s$  index the number of units on order in the lost sales model,  $s = 0, 1, \dots, S$ . Let  $\pi = (\pi_s)$  denote the stationary distribution of the number of units on order.

We first derive an intermediate quantity. Let  $\eta_{m,s}$  denote the stationary probability of the system having  $m$  orders outstanding and  $s$  units on order:

$$\eta_{m,s} = \sum_{\substack{i \in V \\ S - n_0(i) = s \\ m(i) = m}} \xi_i. \quad (6)$$

Letting  $\bar{\nu} = \frac{1}{1 + \sum_{j \neq i_0} \nu_j}$ , substitution from (3) and (5) yields

$$\begin{aligned} \eta_{m,s} &= \bar{\nu} \frac{(1-p)^s}{p^{1\{s=S\}}} \left( \frac{\lambda p}{\mu(1-p)} \right)^m \sum_{\substack{i \in V \\ S - n_0(i) = s \\ m(i) = m}} \frac{1}{\prod_{k=1}^S (n_k(i)!) } \\ &= \bar{\nu} \frac{\left( \frac{\lambda}{\mu} \right)^m}{p^{1\{s=S\}}} p^m (1-p)^{s-m} \sum_{\substack{i \in V \\ S - n_0(i) = s \\ m(i) = m}} \frac{1}{\prod_{k=1}^S (n_k(i)!) }. \end{aligned} \quad (7)$$

Let  $f_{NB}(\cdot; m, p)$  denote the negative binomial probability distribution with parameters  $m$  and  $p$ :

$$f_{NB}(x; m, p) \equiv \binom{m+x-1}{x} p^m (1-p)^x \text{ for } x = 0, 1, 2, \dots$$

PROPOSITION 3. *For the lost sales model with stuttering Poisson demand and partial fills, the stationary probability of the system having  $m$  orders outstanding and  $s$  units on order is given by*

$$\eta_{m,s} = \bar{\nu} \frac{\left(\frac{\lambda}{\mu}\right)^m}{p^{1\{s=S\}}} \frac{f_{NB}(s-m; m, p)}{m!}.$$

**Proof:** First, let us show that

$$\sum_{\substack{i \in V \\ S-n_0(i)=s \\ m(i)=m}} \frac{m!}{\prod_{k=1}^S (n_k(i)!)} = \binom{s-1}{m-1}. \quad (8)$$

To better understand the combinatorial expressions, we recast the language from orders and order sizes into boxes and balls. We are considering placing  $s$  balls (i.e. units on order) into  $m$  boxes (i.e. orders). Suppose we have placed the  $s$  balls and have used exactly  $m$  boxes. Let  $n_k \in \{0, 1, 2, \dots, s\}$  denote the number of boxes that contain exactly  $k$  balls,  $k = 1, 2, \dots, s$ . We refer to  $n_k$  as the box size count for box size (equivalently, for ball count)  $k$ . Of the  $m!$  permutations of boxes, we are interested only in sequences that are unique with respect to the number of balls in each box. Thus, for example, if  $k_j$  is the number of balls in box  $j$ ,  $j = 1, \dots, m$ , the sequence  $(k_1, k_2, k_3) = (0, 1, 1)$  corresponds to two equivalent permutations of the boxes since boxes numbered 2 and 3 can be reversed in sequence without changing the vector  $(k_1, k_2, k_3)$ . For a given vector of box size counts,  $n \equiv (n_1, n_2, \dots, n_s)$ , the number of permutations of boxes that are unique with respect to box size (i.e. ball count), is given by:

$$\frac{m!}{\prod_{\substack{k \in \{1, \dots, s\} \\ n_k > 0}} n_k!} = \frac{m!}{\prod_{k=1}^s n_k!},$$

where equality comes from the convention that  $0! = 1$ . From this, it follows that the number of ways of assigning  $s$  balls to exactly  $m$  boxes and sequencing the boxes so that the sequence is unique by ball count is given by

$$\sum_{\substack{n=(n_1, n_2, \dots, n_s) \\ \sum_{k=1}^s k n_k = s \\ \sum_{k=1}^s n_k = m}} \frac{m!}{\prod_{k=1}^s n_k!} = \sum_{\substack{i \in V \\ S-n_0(i)=s \\ m(i)=m}} \frac{m!}{\prod_{k=1}^S (n_k(i)!)}.$$

This is the left hand side of (8). Now, we consider the same combinatorial problem from a different perspective. If we take any sequence of balls and place dividers between some of them, we could then assign the balls between dividers to boxes in sequence. The placement of dividers would uniquely define a sequence of ball counts per box. To ensure that exactly  $m$  boxes were used (with positive ball counts in each) we would have to place exactly  $m - 1$  dividers into different positions between the  $s$  balls. (Placing two dividers between the same two balls would imply an empty box, which is not allowed.) Note that only  $s - 1$  positions are available in this partitioning process; therefore, it follows that the number of ways to place these dividers is given by

$$\binom{s-1}{m-1}.$$

From this we get (8).

Therefore,

$$\begin{aligned} \eta_{m,s} &= \bar{\nu} \frac{\left(\frac{\lambda}{\mu}\right)^m}{p^{1\{s=S\}} m!} p^m (1-p)^{s-m} \binom{s-1}{m-1} \\ &= \bar{\nu} \frac{\left(\frac{\lambda}{\mu}\right)^m}{m!} p^{-1\{s=S\}} \binom{s-1}{s-m} p^m (1-p)^{s-m} \\ &= \bar{\nu} \frac{\left(\frac{\lambda}{\mu}\right)^m}{m!} p^{-1\{s=S\}} f_{NB}(s-m; m, p). \end{aligned}$$

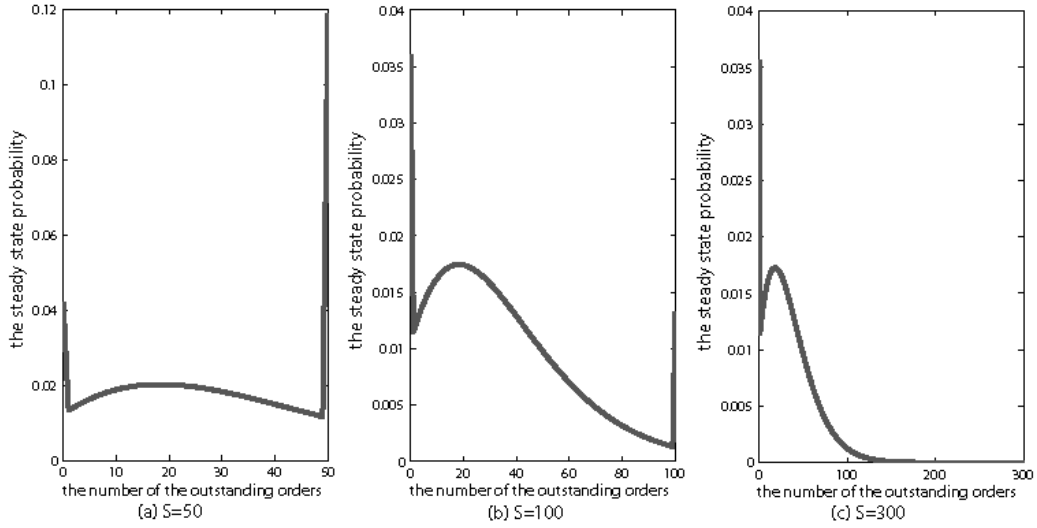
■

**COROLLARY 1.** *For the lost sales model with stuttering Poisson demand and partial fills, the stationary distribution of the number of units on order is given by*

$$\pi_s = \frac{\sum_{m=0}^s \frac{\left(\frac{\lambda}{\mu}\right)^m}{p^{1\{s=S\}} m!} f_{NB}(s-m; m, p)}{G(S)}, \quad (9)$$

where  $G(S) = \sum_{s=0}^S \sum_{m=0}^s \frac{\left(\frac{\lambda}{\mu}\right)^m}{m!} p^{-1\{s=S\}} f_{NB}(s-m; m, p)$ , and  $f_{NB}(s-m; 0, p) = 1\{s=0\}$  when  $m = 0$ .

**THEOREM 3.** *For the lost sales model with stuttering Poisson demand, suppose the replenishment order lead times are independent and identically distributed and have a general distribution with*



**Figure 3** Distribution of Units on Order for the Stuttering Poisson Arrival Process

finite mean  $L = \frac{1}{\mu}$ , with no point mass at zero. For the partial fill case, the stationary distribution of the number of units on order is given by

$$\hat{\pi}_s = \pi_s,$$

where  $\pi_s$  is the stationary distribution of the number of units on order in the lost sales model where lead times are independently identically exponential with mean  $\frac{1}{\mu}$  respectively.

**Proof:** Theorem EC.1 of e-companion appendix shows that the stationary distribution of  $\nu_i$  is unchanged if the lead time has the same mean  $\frac{1}{\mu}$  but has a general distribution where the lead times are independently identically distributed. Therefore the stationary distribution of the number of units on order is still the same as that when lead times are exponentially distributed. ■

The exact stationary distribution of the number of units-on-order (9) is one of the major contributions of this paper.

#### 4.2. The Shape of the Units-on-Order Distribution in the Partial Fill Case

For a lost sales model with Poisson demand, the steady state distribution of the number of units on order is given by

$$\pi_s = \frac{e^{-\frac{\lambda}{\mu}} \frac{(\frac{\lambda}{\mu})^s}{s!}}{\sum_{k=0}^S \frac{e^{-\frac{\lambda}{\mu}} (\frac{\lambda}{\mu})^k}{k!}}, \quad (10)$$

(Muckstadt 2005 p44.). The basic unimodal shape does not change as a function of  $S$ .

Figure 3 (a),(b), and (c) are plots of the steady state distribution of the number of units on order in the stuttering Poisson case (partial fill) for different values of  $S = 50, 100, 300$ , respectively. The mean of the demand per unit time is 5 and the variance per unit time is 100. The distribution is trimodal with additional atoms occurring at 0 and  $S$ . The lead time mean is 7. Observe that, unlike the Poisson-based distribution, the atom at  $S$  becomes more pronounced as  $S$  decreases.

### 4.3. The Complete Fill Case

To this point we have considered only the partial fill case. Another possibility is that a customer order is rejected (all units lost) if there is insufficient stock on hand to fill it completely. We refer to this as the complete fill case. The analysis is very similar to the partial fill case. We have

$$\eta_{m,s} = \eta_{0,0} \left( \frac{\lambda}{\mu} \right)^m \frac{f_{NB}(s-m; m, p)}{m!}, \quad (11)$$

where  $\eta_{0,0}$  is the normalizer. The steady state distribution of units on order is given by the following:

PROPOSITION 4. *For the lost sales model with stuttering Poisson demand and complete fills, the stationary distribution of the number of units on order is given by:*

$$\pi_s = \frac{\sum_{m=0}^s \left( \frac{\lambda}{\mu} \right)^m \frac{f_{NB}(s-m; m, p)}{m!}}{G(S)}, \quad (12)$$

where  $G(S) = \sum_{s=0}^S \sum_{m=0}^s \frac{(\frac{\lambda}{\mu})^m}{m!} f_{NB}(s-m; m, p)$ , and  $f_{NB}(s-m; 0, p) = 1\{s=0\}$  when  $m=0$ . *i.e.* the truncated compound Poisson distribution.

**Proof:** In the case of complete fill the accepted demand is given by  $X1_{X < I}$ , where  $X$  is the customer order size and  $I$  is inventory on hand at the time of the order, as before. The infinitesimal generator in the stuttering Poisson case becomes:

$$A_{ij} \equiv \begin{cases} n_{k_{ij}}(i)\mu & \text{if } (i, j) \in V_R^2, \\ \lambda p_{k_{ij}} & \text{if } (i, j) \in V_C^2, \\ -(m(i)\mu + \lambda(1 - \bar{P}_{n_0(i)})) & \text{if } j = i, \\ 0 & \text{otherwise.} \end{cases} \quad (13)$$

Following the notation and method of section 3, we get

$$\nu_i \equiv \frac{\left(\frac{\lambda p}{\mu(1-p)}\right)^{m(i)}}{\prod_{k=1}^S (n_k(i)!)} (1-p)^{S-n_0(i)} \quad (14)$$

as the complete fill counterpart to (5). Observe that the term  $\frac{1}{p^{1\{n_0(i)=0\}}}$  is needed for the partial fill case (5).

In the analog of Theorem 1 for the complete fill case, simply replace (5) with (14). The proof is identical except that the case  $n_0(i) = 0$  is no different from the  $n_0(i) > 0$  case with complete fills. In the analog of Proposition 2 and Corollary 1, omit the factor  $\frac{1}{p^{1\{n_0(i)=0\}}}$  or  $\frac{1}{p^{1\{s=S\}}}$ . The analog to (9) for the complete fill case becomes (12). ■

This is a bimodal distribution because the mode at  $s = S$  disappears.

**THEOREM 4.** *Suppose in the lost sales model that demand occurs according to stuttering Poisson process and the replenishment order lead times are independent and identically distributed and have general distribution with finite mean  $L = \frac{1}{\mu}$ , where there is no point mass at zero. For the complete fill case, the stationary distribution of the number of units on order is given by*

$$\hat{\pi}_s = \pi_s,$$

where  $\pi_s$ , given by (12), is the stationary distribution of the number of units on order in the lost sales model when lead times are exponentially distributed with mean  $\frac{1}{\mu}$ .

**Proof:** The proof is similar to that of Theorem 3. ■

## 5. Comparison with F-S Results for the Partial Fill Case

Feeney and Sherbrooke (1966) discuss the compound Poisson demand process and give the stationary distribution for lost sales with partial fills allowed. With one exception (Baganha, 1985) this result has been unchallenged for forty years.

Let us restate their formula as follows by substituting for  $\{y, x, s, T\}$  in the original paper with the notation  $\{m, s, S, 1/\mu\}$  in the current paper. Then, using our notation, their formulas become

$$\begin{aligned} h(s) &= \frac{\sum_{m=0}^s \left(\frac{\lambda}{\mu}\right)^m e^{-\frac{\lambda}{\mu}/m!} f^{*m}(s)}{H(S)}, \text{ for } 0 \leq s < S; \\ h(S) &= \frac{\sum_{m=0}^S \left(\frac{\lambda}{\mu}\right)^m e^{-\frac{\lambda}{\mu}/m!} \sum_{i=S}^{\infty} f^{*m}(i)}{H(S)}, \end{aligned} \quad (15)$$

where  $f^{*m}$  is the  $m$ -fold convolution of the order size distribution and  $H(S)$  is the normalizer. These do not agree with (9) when  $f(\cdot)$  is given by the geometric distribution. The difference can be traced to the reduced balance equations (A.7) in their paper(1966). Baganha(1985) noted that (A.7) is inconsistent with the proposed solution (A.8). However, even when corrected (A.7 in Baganha, 1985), these balance equations are built upon an implicit assumption in the F-S derivation that the distribution of order sizes in resupply is the same as the distribution of order sizes in customer arrivals. Since customer orders are filtered by the lost sales process, this assumption means that their analysis is not exact. Their result for the complete fill case does agree with (12), but does not actually solve their steady state equations. As we have shown, exact analysis is possible for the special case of stuttering Poisson demand. However, an exact analysis of the steady state distribution of units on order in the case of lost sales with general compound Poisson demand remains an open question.

In this section, we consider the quality of the F-S result as an approximation.

### 5.1. Analytical Comparison

The following theorem shows that the F-S formula for the stuttering Poisson demand process always overestimates the out-of-stock probability when the targeted inventory level  $S$  is exceeds 1.

**THEOREM 5.** *For  $s = 0, 1, \dots, S - 1$ ,*

$$\pi_s G(S) e^{-\frac{\lambda}{\mu}} = h(s) H(S); \quad (16)$$

*For  $s = S = 1$ , then*

$$\pi_S G(S) e^{-\frac{\lambda}{\mu}} = h(S) H(S);$$

*and if  $s = S > 1$ , then*

$$\pi_S G(S) e^{-\frac{\lambda}{\mu}} < h(S) H(S). \quad (17)$$

**Proof:** Recall that  $f$  is the pdf of a geometric distribution with parameter  $p$ . Then

$$f^{*m}(s) = f_{\text{Neg}}(s - m; m, p).$$

Thus, (16) is true for  $s = 0, 1, \dots, S - 1$ .

When  $s = S = 1$ ,

$$\pi_1 G(1) e^{-\frac{\lambda}{\mu}} = \left(\frac{\lambda}{\mu}\right) e^{-\frac{\lambda}{\mu}} \frac{f(1)}{p} = \left(\frac{\lambda}{\mu}\right) e^{-\frac{\lambda}{\mu}} = h(S) H(S).$$

Suppose  $s = S > 1$  and  $i > S$ . When  $m > 1$

$$\frac{f^{*m}(i+1)}{f^{*m}(i)} = \frac{\binom{(i+1)-1}{(i+1)-m} p^m (1-p)^{i+1-m}}{\binom{i-1}{i-m} p^m (1-p)^{i-m}} = \frac{i}{i+1-m} (1-p) > (1-p).$$

This means  $f^{*m}(i+1) > (1-p)f^{*m}(i)$  and

$$f^{*m}(i) > (1-p)^{i-S} f^{*m}(S).$$

Therefore, when  $S > 1$  and  $m > 1$ ,

$$\sum_{i=S}^{\infty} f^{*m}(i) > \sum_{i=S}^{\infty} (1-p)^{i-S} f^{*m}(S) = f^{*m}(S) \sum_{i=0}^{\infty} (1-p)^i = \frac{f^{*m}(S)}{p}.$$

Since  $f^{*0}(i) = 0$  for  $i > 0$  and  $f^{*1}(S)/p = f(S)/p = \sum_{i=S}^{\infty} f(i)$ , we see that for  $S > 1$

$$\pi_S G(S) e^{-\frac{\lambda}{\mu}} = \sum_{m=0}^S \left( \left(\frac{\lambda}{\mu}\right)^m e^{-\frac{\lambda}{\mu}} / m! \right) \frac{f^{*m}(S)}{p} < \sum_{m=0}^S \left( \left(\frac{\lambda}{\mu}\right)^m e^{-\frac{\lambda}{\mu}} / m! \right) \sum_{i=S}^{\infty} f^{*m}(i) = h(S) H(S).$$

After normalization, we have

$$h(S) > \pi_S \text{ and } h(s) < \pi_s, \text{ for } s = 0, 1, \dots, S - 1,$$

when  $S > 1$ . Furthermore,

$$\frac{h(s)}{\pi_s} = \frac{h(s')}{\pi_{s'}},$$

provided  $s, s' < S$ . ■

**Table 1** Relative error of Feeney-Sherbrooke approximation when the ratio  $\frac{\lambda}{\mu}$  changes:

$\frac{\lambda}{\mu}$	$p$	$\frac{c_p}{c_h}$	$S_T^*$	$C_T(S_T^*)$	$S_A^*$	$C_T(S_A^*)$	$\frac{C_T(S_A^*) - C_T(S_T^*)}{C_T(S_T^*)}$
0.7	0.4	10	1	2.5	1	2.5	0
1.4	0.4	10	3	4.3276	3	4.3276	0
2.1	0.4	10	5	5.772	6	5.8255	0.927%
2.8	0.4	10	7	7.0021	9	7.1418	1.99%
3.5	0.4	10	10	8.0842	12	8.3518	3.31 %
4.2	0.4	10	12	9.0512	14	9.2651	2.36 %
4.9	0.4	10	14	9.943	17	10.3204	3.79%
5.6	0.4	10	16	10.774	20	11.3528	5.37%
6.3	0.4	10	18	11.555	22	12.0715	4.46%
7	0.4	10	20	12.294	25	13.0411	6.07 %

## 5.2. Computational Comparison

We now consider the long run cost implications of using the approximate F-S model, (15), to optimize stock levels rather than the exact model (9). The exact model is used to evaluate the solutions.

Let  $c_h$  and  $c_p$  denote the holding cost per unit time and lost sales penalty per unit, respectively. Let  $C(S)$  denote the long run average sum of holding costs and lost sales costs per unit time:

$$\begin{aligned} C(S) &= \sum_{s=0}^S c_h(S-s)\pi_s + \sum_{s=0}^S [\sum_{j=1}^{\infty} c_p \lambda j p (1-p)^{(j-1)+(S-s)}] \pi_s \\ &= \sum_{s=0}^S [c_h(S-s) + c_p \lambda \frac{(1-p)^{S-s}}{p}] \pi_s. \end{aligned}$$

Let  $C_T(S)$  denote the time cost obtained using (9) and let  $C_A(S)$  denote the approximate cost obtained using the F-S approximation (15). Let  $S_T^*$  denote the optimizer of  $C_T(S)$  and  $S_A^*$  the optimizer of  $C_A(S)$ .

We use numerical methods to find  $S_T^*$  and  $S_A^*$ . We also compare  $C_T(S_T^*)$  and  $C_T(S_A^*)$ , which are the true costs under optimized values. In Tables 1 and 2, we fix the mean of the lead time  $\tau = \frac{1}{\mu} = 7$  and vary  $\lambda$ ,  $p$  and  $\frac{c_p}{c_h}$  to study their effects on the difference between the costs obtained using the exact and the F-S models. We observe that the penalty cost of using the F-S model is small, less

**Table 2** Relative error of Feeney-Sherbrooke approximation when the probability  $p$  changes:

$\frac{\lambda}{\mu}$	$p$	$\frac{c_p}{c_h}$	$S_T^*$	$C_T(S_T^*)$	$S_A^*$	$C_T(S_A^*)$	$\frac{C_T(S_A^*) - C_T(S_T^*)}{C_T(S_T^*)}$
3.5	0.1	10	37	34.347	44	34.887	1.57 %
3.5	0.2	10	19	16.851	23	17.2607	2.43 %
3.5	0.3	10	13	11.011	15	11.2027	1.74%
3.5	0.4	10	10	8.0842	12	8.3518	3.31 %
3.5	0.5	10	8	6.3131	9	6.405	1.46 %
3.5	0.6	10	7	5.1412	8	5.2857	2.81 %
3.5	0.7	10	6	4.2819	6	4.2819	0
3.5	0.8	10	5	3.6273	5	3.6273	0
3.5	0.9	10	5	3.1445	5	3.1445	0
3.5	1	10	4	2.7123	4	2.7123	0

**Table 3** Relative error of Feeney-Sherbrooke approximation when the ratio  $\frac{c_p}{c_h}$  changes:

$\frac{\lambda}{\mu}$	$p$	$\frac{c_p}{c_h}$	$S_T^*$	$C_T(S_T^*)$	$S_A^*$	$C_T(S_A^*)$	$\frac{C_T(S_A^*) - C_T(S_T^*)}{C_T(S_T^*)}$
3.5	0.4	1	0	1	0	1	0
3.5	0.4	10	10	8.0842	12	8.3518	3.31 %
3.5	0.4	20	14	11.154	16	11.448	2.64 %
3.5	0.4	30	16	13.002	18	13.2427	1.85%
3.5	0.4	40	18	14.322	20	14.689	2.56%
3.5	0.4	50	19	15.33	21	15.656	2.12%
3.5	0.4	60	20	16.153	22	16.501	2.16%
3.5	0.4	70	21	16.851	22	16.987	0.8%
3.5	0.4	80	21	17.448	23	17.661	1.22%
3.5	0.4	90	22	17.957	24	18.306	1.95%

than 6% in most cases. Consequently it is unlikely that using the F-S formula rather than the exact one is problematic. We further observe that when  $\frac{\lambda}{\mu}$  increases, or  $VTMR = \frac{2-p}{p}$  increases, the difference becomes more significant. However, when  $\frac{c_p}{c_h}$  increases, the difference seems to fluctuate.

We conclude from this analysis that the F-S model provides a reasonably good approximation

for the purpose of stock optimization, at least for the stuttering Poisson case.

## 6. Behavior of the Expected Ordered Unit Delivery Times

In this section we consider the deliveries of units on order in the lost sales model with partial fill and show that for the stuttering Poisson demand process, these deliveries become more concentrated in time as the variance-to-mean ratio of the demand process increases.

This is not a surprising result, as we see in the following example. Consider a lost sales model with an  $(S - 1, S)$  policy where  $S = 10$ . Lead times are exponentially distributed with rate  $\mu$  (the value of  $\mu$  is irrelevant). Compare two demand processes that have identical expected rates of demand: in the first process, demand follows a Poisson process with rate  $\lambda = 1$ ; in the second process, demand follows a compound Poisson process in which orders arrive at rate  $\lambda = 0.01$  but each order is for exactly 100 units. Observe that cumulative expected demand over any constant length of time is the same in both cases. On the other hand, the variance of demand is higher when demand follows the compound Poisson process. If we observe the lost sales system at a random point in time, the probability distribution of the number of units on order of the Poisson system is given by (10). Furthermore, each unit on order corresponds to a unique customer order and each, therefore, belongs to a unique replenishment order. Consequently, conditioned on the number of units on order,  $s$ , the memoryless property of the exponential lead time distribution ensures that the deliveries of these  $s$  units will be spread out in time according to a distribution we will consider in detail in the sequel. For the extreme compound Poisson process, however, it is clear that any arriving customer demand order will always exceed the available stock. The number of units on order will be either 0 or 10 due to the partial fill assumption and the units on order will all belong to a single replenishment order. Thus, the units on order will always arrive together in a single delivery.

Intuitively, this is the limiting distribution of unit deliveries as the variance-to-mean ratio increases: all units arrive in a single delivery after an exponentially distributed lead time. Observe that, under both systems, the expected lead time, for an order, is  $1/\mu$ . It is the spread about

this mean of individual unit deliveries that is of interest. From a managerial perspective, the two systems will behave very differently. In the Poisson system, if you are out of stock you can expect to receive a delivery of at least one unit in one-tenth of a lead time ( $= 1/(S\mu)$ ). In the extreme version of the compound Poisson system, if you are out of stock you can expect to wait a full lead time ( $= 1/\mu$ ) before seeing any units arrive. It is important for service parts planners to understand this phenomenon because, typically, variance-to-mean ratios are higher in the service parts industry than in consumer products environments. As one service parts manager expressed it, “the bad news is worse than I thought: if I am out of stock, I can expect to be out for a long time.”

For the balance of this section, we focus on the stuttering Poisson demand process with parameters  $\lambda$  and  $p$ . Over any fixed length of time,  $T$ , the expected demand is  $\lambda T/p$  and the variance of demand is  $\lambda \left( \frac{1-p}{p^2} \right) T$ . Denote the variance-to-mean ratio by  $VTMR = \frac{1-p}{p}$ . For a constant mean rate of demand,  $\lambda/p = \bar{R}$ , we investigate the impact of increasing the  $VTMR$ . That is, we consider the impact of letting  $p \rightarrow 0$  while keeping  $\lambda = \bar{R}p$ .

We are interested in the spread of delivery times. Let  $O$  denote the number of units on order in steady state. Conditioned on  $O = s$ , let  $M(s)$  denote the number of orders outstanding. Let  $t_m^s$  denote the remaining time until delivery of the  $m^{th}$  order,  $m = 1, \dots, M(s)$ . Under the assumptions of the model, these remaining delivery times are independent, exponentially distributed random variables with mean  $1/\mu$ . Let their order statistics be denoted by  $t_{(h)}^s$ . In particular,  $t_{(1)}^s$  is the remaining time until the delivery of the earliest order and  $t_{(M(s))}^s$  is the remaining time until the delivery of the latest order. Let  $\Delta(s) = \Delta_{\lambda,p,\mu,S}(s) \equiv E \left[ t_{(M(s))}^s - t_{(1)}^s \mid O = s \right]$ , the expected difference in steady state between the earliest and latest order delivery times, conditioned on  $s$  units outstanding. Then  $\Delta(s)$  is a measure of the spread of order delivery times, in steady state, as a function of the parameters of the system. We show that  $\Delta(s) \rightarrow 0$  monotonically as  $p \rightarrow 0$  while keeping  $\lambda = \bar{R}p$ .

There are three steps to obtaining the result. The first is to show that  $E \left[ t_{(m)}^s - t_{(1)}^s \mid M(s) = m, O = s \right]$  is non-decreasing in  $m$ . The second is to show that  $M(s)$  is

stochastically decreasing as  $p \rightarrow 0$  with  $\lambda = \bar{R}p$ . The third step is to show that  $P\{M(s) > 1\}$  converges to 0 as  $p \rightarrow 0$  with  $\lambda = \bar{R}p$ . First, we have the following result.

LEMMA 1. *If  $\{t_1, t_2, \dots, t_m\}$  are independent and exponentially distributed, each with rate  $\mu$ , then, for  $h = 1, 2, \dots, m$*

$$E[t_{(h)}] = \sum_{k=m-h+1}^m \frac{1}{k\mu},$$

where  $t_{(h)}$  is the  $h^{\text{th}}$  order statistic.

**Proof:** Due to the memoryless property of the exponential distribution random variables, we have that  $t_{(k)} - t_{(k-1)} \sim \text{Exp}((m - k + 1)\mu)$ , for  $k = 1, 2, \dots, m$ , and these differences are independent (Feller 1971 p19, Proposition9). ■

From this lemma, it follows that

$$E[t_{(m)}^s - t_{(1)}^s | M(s) = m, O = s] = \sum_{k=1}^m \frac{1}{k\mu} - \frac{1}{m\mu} = \sum_{k=1}^{m-1} \frac{1}{k\mu}$$

which is non-decreasing in  $m$ .

Let  $\eta_{m,s}(p)$  denote the stationary probability with parameter  $p$  of having  $m$  orders outstanding and  $s$  units on order when  $p$  is the order size parameter. The distribution of  $M(s)$  is given by

$$P\{M(s) = m\} = \eta_{m|s}(p) \equiv \frac{\eta_{m,s}(p)}{\sum_{h=1}^s \eta_{h,s}(p)}.$$

To show that this distribution is stochastically decreasing in  $p$ , we focus on the ratio of the successive probabilities

$$\begin{aligned} r_m^s(p) &\equiv \frac{\eta_{m,s}(p)}{\eta_{m-1,s}(p)} \\ &= \frac{\lambda}{\mu m} \frac{p}{1-p} \frac{(s-m+1)}{(m-1)}, \end{aligned}$$

which is decreasing as  $p \rightarrow 0$  while keeping  $\lambda = \bar{R}p$ .

LEMMA 2. *Suppose we have two random variables,  $M_1$  and  $M_2$ , that take values in  $\{1, 2, \dots, m\}$  with probabilities  $P\{M_h = l\} = f_l^h > 0$  for  $h = 1, 2$  and  $l = 1, 2, \dots, m$ . If*

$$\frac{f_{l+1}^1}{f_l^1} \geq \frac{f_{l+1}^2}{f_l^2},$$

then  $P\{M_1 > l\} \geq P\{M_2 > l\}$  for all  $l = 1, 2, \dots, m$ ; that is,  $M_1$  is stochastically greater than  $M_2$ .

**Proof:** We need to show that  $P(M_1 > k) \geq P(M_2 > k)$ , for  $k \in \{1, 2, \dots, n-1\}$ . Let  $R_k = \frac{f_{k+1}^{(1)}}{f_k^{(1)}}$ ,  $k = 1, 2, \dots, n-1$ . Then

$$f_{k+1}^{(i)} = f_1^{(i)} R_1^{(i)} R_2^{(i)} \cdots R_k^{(i)},$$

and

$$1 = f_1^{(1)}(1 + R_1^{(1)} + \cdots + R_1^{(1)} R_2^{(1)} \cdots R_{n-1}^{(1)}) = f_1^{(2)}(1 + R_1^{(2)} + \cdots + R_1^{(2)} R_2^{(2)} \cdots R_{n-1}^{(2)}).$$

This implies that

$$\frac{f_1^{(2)}}{f_1^{(1)}} = \frac{1 + R_1^{(1)} + \cdots + R_1^{(1)} R_2^{(1)} \cdots R_{n-1}^{(1)}}{1 + R_1^{(2)} + \cdots + R_1^{(2)} R_2^{(2)} \cdots R_{n-1}^{(2)}} \geq 1,$$

since  $R_k^{(1)} \geq R_k^{(2)}$ .

For any value  $k \in \{2, \dots, n-1\}$ , we have

$$\frac{R_1^{(1)} R_2^{(1)} \cdots R_{k-1}^{(1)} + \cdots + R_1^{(1)} R_2^{(1)} \cdots R_{n-1}^{(1)}}{R_1^{(2)} R_2^{(2)} \cdots R_{k-1}^{(2)} + \cdots + R_1^{(2)} R_2^{(2)} \cdots R_{n-1}^{(2)}} \leq \frac{R_1^{(1)} R_2^{(1)} \cdots R_k^{(1)} + \cdots + R_1^{(1)} R_2^{(1)} \cdots R_{n-1}^{(1)}}{R_1^{(2)} R_2^{(2)} \cdots R_k^{(2)} + \cdots + R_1^{(2)} R_2^{(2)} \cdots R_{n-1}^{(2)}}.$$

To obtain this, we need to show that

$$R_1^{(1)} R_2^{(1)} \cdots R_{k-1}^{(1)} (R_1^{(2)} R_2^{(2)} \cdots R_k^{(2)} + \cdots + R_1^{(2)} R_2^{(2)} \cdots R_{n-1}^{(2)}),$$

is less than or equal to

$$R_1^{(2)} R_2^{(2)} \cdots R_{k-1}^{(2)} (R_1^{(1)} R_2^{(1)} \cdots R_k^{(1)} + \cdots + R_1^{(1)} R_2^{(1)} \cdots R_{n-1}^{(1)}).$$

But this follows immediately since  $R_j^{(1)} \geq R_j^{(2)}$  for any  $j$ . Therefore,

$$\frac{1 + R_1^{(1)} + \cdots + R_1^{(1)} R_2^{(1)} \cdots R_{n-1}^{(1)}}{1 + R_1^{(2)} + \cdots + R_1^{(2)} R_2^{(2)} \cdots R_{n-1}^{(2)}} \leq \frac{R_1^{(1)} + \cdots + R_1^{(1)} R_2^{(1)} \cdots R_{n-1}^{(1)}}{R_1^{(2)} + \cdots + R_1^{(2)} R_2^{(2)} \cdots R_{n-1}^{(2)}} \leq \cdots \leq \frac{R_1^{(1)} R_2^{(1)} \cdots R_{n-1}^{(2)}}{R_1^{(2)} R_2^{(2)} \cdots R_{n-1}^{(2)}}.$$

By multiplying  $\frac{f_1^{(1)}}{f_1^{(2)}}$  we have

$$1 = \frac{f_1^{(1)}}{f_1^{(2)}} \frac{1 + R_1^{(1)} + \cdots + R_1^{(1)} R_2^{(1)} \cdots R_{n-1}^{(1)}}{1 + R_1^{(2)} + \cdots + R_1^{(2)} R_2^{(2)} \cdots R_{n-1}^{(2)}} \leq \cdots \leq \frac{f_1^{(1)}}{f_1^{(2)}} \frac{R_1^{(1)} R_2^{(1)} \cdots R_{n-1}^{(2)}}{R_1^{(2)} R_2^{(2)} \cdots R_{n-1}^{(2)}}.$$

Restated,

$$1 = \frac{P(M_1 > 0)}{P(M_2 > 0)} \leq \frac{P(M_1 > 1)}{P(M_2 > 1)} \leq \cdots \leq \frac{P(M_1 > n-1)}{P(M_2 > n-1)}.$$

Hence, for any real value  $k$ ,

$$P(M_1 > k) \leq P(M_2 > k).$$

■

We now establish the following results.

**PROPOSITION 5.** *For the lost sales model with stuttering Poisson demand and exponentially distributed lead times,  $M(s)$ , the number of outstanding orders in steady state, conditioned on  $s$  units on order, is stochastically decreasing as  $p \rightarrow 0$  while keeping  $\lambda = \bar{R}p$ . Furthermore,  $P\{M(s) > 1\}$  converges to 0 as  $p \rightarrow 0$  with  $\lambda = \bar{R}p$ .*

**Proof:** From lemma 2 we see that  $M(s)$  is stochastically decreasing as  $p \rightarrow 0$  when  $\lambda = \bar{R}p$  and that  $r_m^s(p)$  is decreasing in  $p$  for fixed  $\lambda/p$ .

For each  $s$  fixed,  $r_m^s(p) = o(p)$  (that is,  $\lim_{p \rightarrow 0} \frac{r_m^s(p)}{p} = 0$ ). Then we can write

$$\eta_{m,s}(p) = o(p)\eta_{1,s}(p), \text{ for } m = 2, 3, \dots, s.$$

Therefore, the conditional distribution

$$\eta_{m|s}(p) = \frac{\eta_{m,s}}{\sum_{m=1}^s \eta_{m,s}} = o(p), \text{ for } m = 2, 3, \dots, s,$$

and  $\eta_{1|s}(p) \rightarrow 1$ , as  $p \rightarrow 0$ . That is,  $P\{M(s) > 1\}$  converges to 0 as  $p \rightarrow 0$  with  $\lambda = \bar{R}p$ . ■

The main result of this section is as follows.

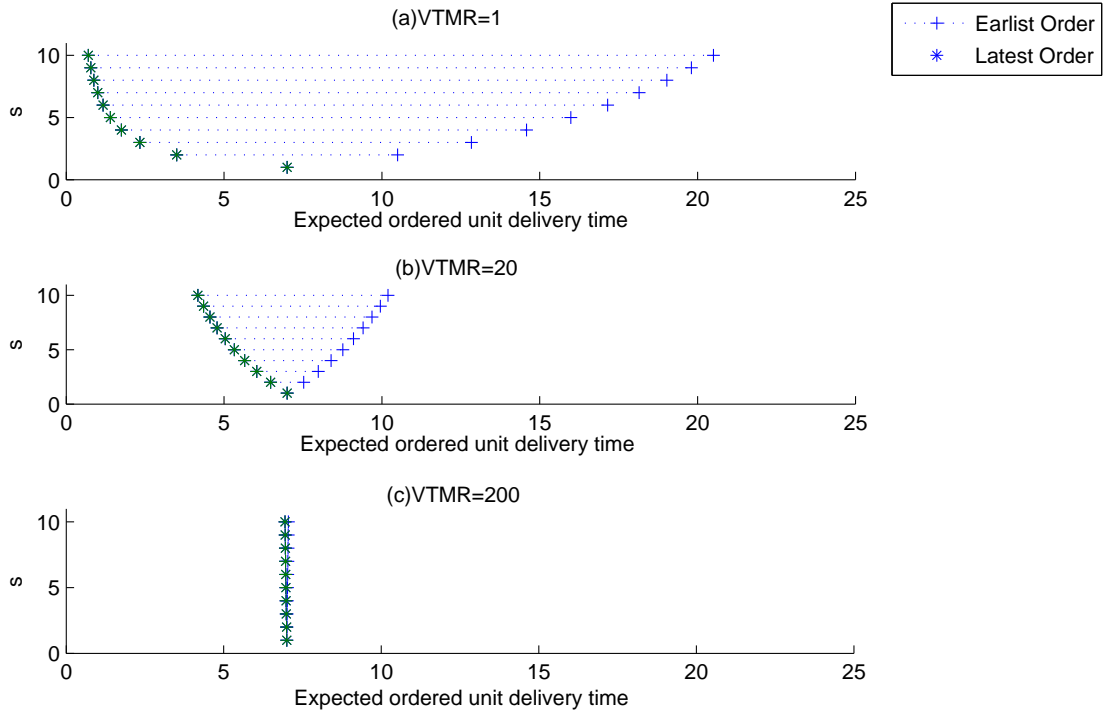
**THEOREM 6.** *For any given  $s = 1, 2, \dots, S$ , the expected spread of deliveries in steady state approaches 0 monotonically as  $p \rightarrow 0$ , when  $\lambda = \bar{R}p$ .*

**Proof:** Since

$$\Delta(s) = E(E[t_{(m)}^s - t_{(1)}^s | M(s) = m, O = s]) = \sum_{m=1}^s \eta_{m|s}(p) E[t_{(m)}^s - t_{(1)}^s | M(s) = m, O = s].$$

Combined with Proposition 5 and the stochastic ordering result (Puterman 2005),  $\Delta(s)$  is decreasing as  $p$  converges to 0 while keeping  $\lambda/p$  constant. ■

**COROLLARY 2.** *As  $p \rightarrow 0$ , the expected ordered unit delivery times, under the condition that  $s$  units are on-order, will converge to  $E(t_{(1)}^1) = \frac{1}{\mu}$ , while keeping  $\lambda = \bar{R}p$ .*



**Figure 4** The Expected Spread of Deliveries

**Proof:** By Proposition(5) and Theorem (6), the expected ordered unit delivery times, under the condition that  $s$  units are on-order, should be the same and equal to

$$E(t_{(1)}^s | O = s) = E(E(t_{(1)}^m | M(s) = m, O = s)) = \frac{1}{\mu} \eta_{1|s}(p) + \sum_{m=2}^s E(t_{(m)}^m - t_{(1)}^m) \eta_{m|s}(p) \rightarrow \frac{1}{\mu}$$

as  $p \rightarrow 0$ . ■

Figure 4 shows the convergence of the expected spread of deliveries as the VTMR goes from 1 to 200 for the case of  $\mu = 5$ ,  $L = 7$  and  $S = 10$ . The points on the left side of each graph are the expected times of the first delivered order with  $s$  units outstanding,  $E(t_{(1)}^s | O = s)$ , and those on the right side are the expected times of the last delivered order with  $s$  units outstanding,  $E(E(t_{(m)}^s | O = s, M(s) = m))$ .

## 7. Conclusion

In this paper, we conducted an exact analysis of the lost sales model with stuttering Poisson demand and exponentially distributed lead times under the  $(S - 1, S)$  inventory policy. We derived

formulas to calculate the exact stationary distribution of the number of outstanding orders. This result was used to correct the long-standing more general result of Feeney and Sherbrooke for the stuttering Poisson case. We then demonstrated empirically that, at least for the stuttering Poisson case, the Feeney and Sherbrooke formulas are a good approximation (for partial fills) or exact (for complete fills) when used to set optimal stock levels. We also proved an interesting result that the spread of expected order replenishment delivery times becomes more concentrated as the *VTMR* increases. The spread converges to zero around a single point, the mean of the lead time.

In a companion paper we use this lost sales model as the basis for modeling emergency order systems. We develop exact expressions for the first and second moments of the number of outstanding emergency orders and use them to estimate the mean and variance of the number of emergency units on order at the ESL. We also estimate the probability that there are zero emergency order outstanding in steady state.

## Glossary

$\lambda$  rate of customer arrivals

$X$  customer order size

$k = 0, 1, \dots$  order size

$p_k \equiv P\{X = k\}$

$\bar{P}_k \equiv P\{X > k\}$

$I_t$  inventory on hand at time  $t$

$X_t$  size of customer order at time  $t$

$S$  order up to level

$\mu$  delivery rate

$\tau = \frac{1}{\mu}$  expected lead time

$N_{kt}$  the number of replenishment orders size  $k$  at time  $t$

$N_t = (N_{kt})_{k=1}^S$

$N = \{N_t, t \geq 0\}$

$V$  state space of replenishment orders

$n_k(i)$  number of orders size  $k$ ,  $i \in V$

$n(i) = (n_1(i), n_2(i), \dots, n_S(i))$

$n_0(i) \equiv S - \sum_{k=1}^S kn_k(i)$

$m(i)$  the number of orders outstanding

$(i, j)$  transition

$\|(i, j)\| \equiv \sum_{k=1}^S |n_k(i) - n_k(j)|$

$k_{ij} \equiv \sum_{k=1}^S k |n_k(i) - n_k(j)|$

$V_C^2$  customer order arrivals class

$V_R^2$  replenishment order arrivals class

$A_{ij}$  infinitesimal generator for lost sale Markov process

$\tilde{X}(t)$  generic continuous Markov chain

$\tilde{V}$  generic state space

$\tilde{\xi}_i$  generic stationary distribution

$\tilde{A}$  generic generator for  $\tilde{X}$

$i_0$  reference state

$\nu_i$  reversibility rates

$\pi = (\pi_s)$  stationary distribution of number of units on order

$\eta_{m,s} = \sum_{\substack{i \in V \\ S - n_0(i) = s \\ m(i) = m}} \xi_i$  stationary probability of  $m$  orders and  $s$  units on order

$\bar{\nu} = \frac{1}{1 + \sum_{j \neq i_0} \nu_j}$  normalizing constant

$f_{NB}(\cdot; m, p)$  negative binomial probability distribution

$G(S)$  normalizing constant for  $(\pi_s)$  distribution

$c_h$  holding cost per unit

$c_p$  lost sales penalty per unit

$C_{HL}$  long run average cost of holding and lost sales costs

$O$  number of units on order in steady state

$M(s)$  number of orders outstanding conditioned on  $s$

$t_m^s$  time until delivery of  $m$ th order,  $m = 1, \dots, M(s)$

$t_{(h)}^s$   $h$ th order statistic of  $\{t_m^s\}$

$\Delta(s) \equiv E(t_{M(s)}^s - t_{(1)}^s | O) = s$

$\bar{R} = \frac{\lambda}{p}$  constant

$M(s)$  the number of orders outstanding, conditioned on  $O = s$

$\eta_{m,s}(p)$  stationary distribution when order size parameter is  $p$

$\eta_{m|s}(p) = P\{M(s) = m\}$

$r_m^s(p)$  ratio of successive probabilities  $\frac{\eta_{m,s}(p)}{\eta_{m-1,s}(p)}$

$M_1, M_2$  random variables in  $\{1, 2, \dots, m\}$

$(f_i^h)$  distribution of  $M_h$ ,  $h = 1, 2$

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## General Distribution Lead Time

### EC.1. A Lost Sales Model with Compound Poisson Demand and General Lead Times

#### EC.1.1. General Lead Times

We always assume that the lead times are independently identically distributed. The original process  $X(t)$ , with exponentially distributed lead times is a Markov process. When it is extended to the case of general time distributions, it becomes a generalized semi-Markov process (GSMP). By extending the state space, we can obtain a Markov process and derive the stationary distribution of the extended state space. Finally, we could prove the marginal stationary distribution of the number of units on order does not depend on the lead time distribution but only on its mean.

Let  $F(\cdot)$  denote the general cumulative distribution function(CDF) of order lead times with no point mass at zero. Expand the underlying state space from  $V$  to  $V \times \mathfrak{R}_{S,S}^+$ . Here  $U = (u_{s,r}) \in \mathfrak{R}_{S,S}^+$  is an  $S$  by  $S$  matrix with non-negative elements. We construct the lost sales model with generally distributed lead times as a stochastic process  $Z(t)$ , with state space  $V \times \mathfrak{R}_{S,S}^+$ :

$$Z(t) = (i, U) = (n_1(i), n_2(i), \dots, n_S(i); U) = \begin{pmatrix} n_1(i) & n_2(i) & \dots & \dots & n_S(i) \\ u_1^{(1)} & u_2^{(1)} & \dots & \dots & u_S^{(1)} \\ u_1^{(2)} & u_2^{(2)} & \dots & \dots & u_S^{(2)} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ u_1^{(S)} & u_2^{(S)} & \dots & \dots & u_S^{(S)} \end{pmatrix}.$$

Here,  $n_s(i)$  is the number of outstanding orders with size  $s$  and  $u_s^{(1)} \geq u_s^{(2)} \geq \dots \geq u_s^{(S)} \geq 0$  stand for the ordered replenishment ages for orders with size  $s$ . That is  $u_s^{(r)}$  is the age of the  $r$ th oldest replenishment order of size  $s$ . The new process  $Z(t)$  is a Markov process on  $V \times \mathfrak{R}_{S,S}^+$ .

Define  $R(i) = \{(s, r) : r \leq n_s(i)\}$  as the replenishment order index set. So we have  $u_s^{(r)} = 0$  if  $(s, r) \notin R(i)$ . Define

$$\mathfrak{R}_{S,S}(i) \equiv \{U \in \mathfrak{R}_{S,S}^+ : u_s^{(1)} \geq u_s^{(2)} \geq \dots \geq u_s^{(S)} \geq 0, \text{ and } u_s^{(r)} \equiv 0 \text{ if } (s, r) \notin R(i)\}.$$

Each state  $(i, U)$  in this system satisfies the condition  $U \in \mathfrak{R}_{S,S}(i)$  and therefore we have  $(i, U) \in V \times \mathfrak{R}_{S,S}(i) \subseteq V \times \mathfrak{R}_{S,S}^+$ .

Our intent is to show that the stationary distribution of  $Z(t)$  is insensitive to the lead time distribution for a given mean,  $\frac{1}{\mu}$ , under the partial fill case. The proof for the complete fill case is nearly the same.

LEMMA EC.1. *Given state  $i$  ( $(n_1(i), n_2(i), \dots, n_S(i))$ ),*

$$\int_{U \in \mathfrak{R}_{S,S}(i)} \prod_{s,r} [1 - F(u_s^{(r)})] du_1^{(1)} \dots u_S^{(S)} = \frac{1}{\prod_{s=1}^S n_s(i)!} \left(\frac{1}{\mu}\right)^{m(i)},$$

where  $m(i) = \sum_{s=1}^S n_s(i)$ .

**Proof:** Since  $\prod_{s,r} [1 - F(u_s^{(r)})]$  does not depend on the order of  $u_s^{(r)}$ , we could integrate it on the whole space and divide the results by  $n_s(i)!$  for each  $s$  fixed. Therefore,

$$\begin{aligned} \int_{U \in \mathfrak{R}_{S,S}(i)} \prod_{s,r} [1 - F(u_s^{(r)})] du_1^{(1)} \dots u_S^{(S)} &= \prod_{s=1}^S \int_{u_s^{(1)} \geq u_s^{(2)} \geq \dots \geq u_s^{(S)} \geq 0} [1 - F(u_s^{(r)})] du_s^{(1)} \dots u_s^{(S)} \\ &= \prod_{s=1}^S \left[ \int_{t_{s,1}=0}^{\infty} \dots \int_{t_{s,S}=0}^{\infty} [1 - F(t_{s,r})] \frac{1}{n_s(i)!} dt_{s,1} \dots dt_{s,S} \right] \\ &= \prod_{s=1}^S \left\{ \frac{1}{n_s(i)!} \left[ \int_{t=0}^{\infty} [1 - F(t)] dt \right]^{n_s(i)} \right\} \\ &= \prod_{s=1}^S \left\{ \frac{1}{n_s(i)!} \left[ \frac{1}{\mu} \right]^{n_s(i)} \right\} \\ &= \frac{1}{\prod_{s=1}^S n_s(i)!} \left(\frac{1}{\mu}\right)^{m(i)}. \end{aligned}$$

■

The proof of uniqueness and ergodicity of the stationary distribution of this Markov process  $Z(t)$  is a consequence of Theorem 1 in Sevastyanov (1957). The proof just follows the routine of proving the results for a telephone system with refusals (Sevastyanov, 1957, section 3). The stationary distribution of  $Z(t)$  is given by the following theorem. The marginal distribution of  $X(t)$  is seen to be invariant to the form of the lead time distribution.

THEOREM EC.1. *The steady state distribution of this Markov process  $Z(t)$ ,  $\zeta_{(i,U)}$  is*

$$\zeta_{(i,U)} = C \left( \frac{\lambda p}{1-p} \right)^{m(i)} \frac{(1-p)^{S-n_0(i)}}{p^{1\{n_0(i)=0\}}} \prod_{(s,r) \in R(i)} [1 - F(u_s^{(r)})], \quad (\text{EC.1})$$

where  $C = \frac{1}{G(S)}$  is the same normalizer as in Corollary 1. Therefore, the steady state distribution of the original GSMP,

$$\tilde{\xi}_i = \int_{U \in \mathfrak{R}_{S,S}^+} \zeta_{(i,U)} dU = C \frac{\left( \frac{\lambda p}{\mu(1-p)} \right)^{m(i)}}{\prod_{r=1}^S (n_r(i)!)} \frac{(1-p)^{S-n_0(i)}}{p^{1\{n_0(i)=0\}}},$$

which is the same stationary distribution that is obtained when the lead times are exponentially distributed with mean  $\frac{1}{\mu}$ .

**Proof:** Notice that  $\prod_{(s,r)}[1 - F(u_s^{(r)})] = \prod_{(s,r) \in R(i)}[1 - F(u_s^{(r)})]$  since  $1 - F(u_s^{(r)}) = 1$  for  $(s, r)$  outside of  $R(i)$ . Integrating  $\zeta_{(i,U)}$  with respect to  $U \in \mathfrak{R}_{S,S}(i)$  and use LemmaEC.1, we have

$$\begin{aligned} \int_{U \in \mathfrak{R}_{S,S}(i)} \zeta_{(i,U)} dU &= C \left( \frac{\lambda p}{1-p} \right)^{m(i)} \frac{(1-p)^{S-n_0(i)}}{p^{1\{n_0(i)=0\}}} \int_{U \in \mathfrak{R}_{S,S}(i)} \prod_{s,r} [1 - F(u_s^{(r)})] dU \\ &= C \frac{\left( \frac{\lambda p}{\mu(1-p)} \right)^{m(i)}}{\prod_{r=1}^S (n_r(i)!)} \frac{(1-p)^{S-n_0(i)}}{p^{1\{n_0(i)=0\}}} \end{aligned}$$

Let  $U + \Delta t$  (or  $U - \Delta t$ ) denote adding (or subtracting) small  $\Delta t$  (or  $\min(\Delta t, u_s^{(r)})$ ) to  $U$ 's each entry  $u_s^{(r)}$  if  $(s, r) \in R(i)$ .

We claim that for  $\Delta t$  sufficiently small, there will occur at most one event (customer arrival or order replenishment delivery) in the interval  $(t, t + \Delta t]$  for any  $t$ . This follows because the delivery process is simply a shifted, filtered version of the arrival process. Consequently, the combined process is a filtered version of a Poisson process (refer to Resnick 2005, section 4.4 page 316). So now we choose a  $\Delta t$  sufficiently small so that at most one event happens within the interval  $(t, t + \Delta t]$ .

Define  $Q_{(i,U),(j,U')}(\Delta t)$  as the transition probability from state  $(i, U)$  to state  $(j, U')$  during time  $\Delta t$ . Since  $Z(t)$  is a Markov process,  $Q$  has no dependence on  $t$ . For sufficiently small  $\Delta t$ , we have the following transition probabilities:

*Case 1 :* If no customer arrives,  $n_0(i) = S$ , and  $(j, U') = (i_0, O)$ , where  $O$  is the matrix with zeros entries,

$$Q_{(i_0,O),(i_0,O)}(\Delta t) = 1 - \lambda \Delta t + o(\Delta t).$$

*Case 2 :* If no replenishment order arrives when  $(i, U)$  has  $n_0(i) = 0$  (any arrival is lost), we have

$$Q_{(i,U),(i,U+\Delta t)}(\Delta t) = \prod_{(s,r) \in R(i)} \frac{1 - F(u_s^{(r)} + \Delta t)}{1 - F(u_s^{(r)})}.$$

*Case 3 :* If no customer arrives for general state  $(i, U)$  with  $0 < n_0(i) < S$ ,

- (Case 3a) When no customer arrives, or no replenishment order arrives during time  $\Delta t$  case,

$$Q_{(i,U),(i,U+\Delta t)}(\Delta t) = \prod_{(s,r) \in R(i)} \frac{1 - F(u_s^{(r)} + \Delta t)}{1 - F(u_s^{(r)})} (1 - \lambda \Delta t + o(\Delta t))$$

- (Case 3b) Now suppose no customer arrives but one replenishment order of size  $k_{ij}$  ( $(i,j) \in V_R^2$ ) arrives. Suppose that order is the  $l$ th oldest order,  $1 \leq l \leq n_{k_{ij}}(i)$ . Let  $U_{i,j}^{l-}$  be the same as  $U$  except that the element  $u_{k_{ij}}^{(l)}$  is deleted so that the  $l$ th column changes from

$$(u_{k_{ij}}^{(1)}, \dots, u_{k_{ij}}^{(n_{k_{ij}}(i))}, 0, \dots, 0)',$$

to

$$(u_{k_{ij}}^{(1)}, \dots, u_{k_{ij}}^{(l-1)}, u_{k_{ij}}^{(l+1)}, \dots, u_{k_{ij}}^{(n_{k_{ij}}(i))}, 0, 0, \dots, 0)'$$

Actually,  $U_{i,j}^{l-}$  is  $U$  after recording delivery of  $l$ th oldest order of size  $k_{ij}$ . Thus,

$$Q_{(i,U),(j,U_{i,j}^{l-} + \Delta t)}(\Delta t) = \frac{F(u_{k_{ij}}^{(l)} + \Delta t) - F(u_{k_{ij}}^{(l)})}{1 - F(u_{k_{ij}}^{(l)})} \prod_{(s,r) \in R(i)/(k_{ij},l)} \frac{1 - F(u_s^{(r)} + \Delta t)}{1 - F(u_s^{(r)})} (1 - \lambda \Delta t + o(\Delta t)).$$

*Case 4* : When  $(i,U)$  satisfies  $n_0(i) > 0$ , and one customer arrives with accepted order size  $k_{ij}$  ( $(i,j) \in V_C^2$ ) and has age  $u$  ( $0 < u \leq \Delta t$ ) at the end of the interval and no replenishment order arrives, we have

$$Q_{(i,U),(j,U_{i,j})}(\Delta t) = \prod_{(s,r) \in R(i)} \frac{1 - F(u_s^{(r)} + \Delta t)}{1 - F(u_s^{(r)})} (A_{ij} \Delta t + o(\Delta t)) \left(\frac{1}{\Delta t}\right) (1 - F(u)).$$

Here  $U_{i,j} = U + \Delta t$  except that the new replenishment order caused by the new arrival has age  $u_{k_{ij}}^{(n_{k_{ij}}(j))} = u$ . Notice that  $(\frac{1}{\Delta t})$  is the conditional density of the new replenishment order with  $u$  being the age at the end of the interval  $(0, \Delta t]$ . This is because of the uniformly distributed arrival time of the Poisson process conditioned on one arrival occurring during an interval of length  $\Delta t$ .

A special case when  $(i,U) = (i_0, O), (i_0, j) \in V_C^2$ , we have

$$Q_{(i_0,O),(j,U_{i_0,j})}(\Delta t) = \frac{A_{i_0,j} \lambda \Delta t + o(\Delta t)}{\Delta t} (1 - F(u)),$$

where  $U_{i_0,j} = O$  except  $u_{k_{ij}}^{(1)} = u$ .

Define  $P_{(i,U)}(t) = P[Z(t) = (i, U)]$ . Making use of the Markov property and  $Q_{(i,U)(j,U')}(\Delta t)$ , we obtain:

*Case 1* For  $(i, U) = (i_0, O)$ ,

$$P_{(i_0,O)}(t + \Delta t) = P_{(i_0,O)}(t)(1 - \lambda\Delta t) + \sum_{\{j:(j,i_0) \in V_R^2\}} \int_0^\infty P_{(j,U)}(t) \frac{F(u_{k_{ji}}^{(1)} + \Delta t) - F(u_{k_{ji}}^{(1)})}{1 - F(u_{k_{ji}}^{(1)})} du_{k_{ji}}^{(1)} + o(\Delta t), \quad (\text{EC.2})$$

except  $u_{k_{ji}}^{(1)}$ , the other entries of  $U$  are zeros.

*Case 2* For  $(i, U)$  with  $n_0(i) = 0$ ,

$$P_{(i,U)}(t + \Delta t) = P_{(i,U-\Delta t)}(t) \prod_{(s,r) \in R(i)} \frac{1 - F(u_s^{(r)})}{1 - F(u_s^{(r)} - \Delta t)} + o(\Delta t). \quad (\text{EC.3})$$

*Case 3* For general  $(i, U)$  with  $0 < n_0(i) < S$ , and  $u_s^{(r)} > 0$  for all  $(s, r) \in R(i)$ ,

$$\begin{aligned} & P_{(i,U)}(t + \Delta t) \\ &= P_{(i,U-\Delta t)}(t) \prod_{(s,r) \in R(i)} \frac{1 - F(u_s^{(r)} + \Delta t)}{1 - F(u_s^{(r)})} (1 - \lambda\Delta t) \\ &+ \sum_{\{(j,U'):(j,i) \in V_R^2, U' = (U - \Delta t)_{ji}^+\}} \int_0^\infty P_{(j,U')}(t) \prod_{(s,r) \in R(i)} \frac{1 - F(u_s^{(r)})}{1 - F(u_s^{(r)} - \Delta t)} \frac{F(u) - F(u - \Delta t)}{1 - F(u - \Delta t)} du (1 - \lambda\Delta t) \\ &+ o(\Delta t), \end{aligned} \quad (\text{EC.4})$$

where  $(U - \Delta t)_{ji}^+$  is the  $U - \Delta t$  inserting  $u_{k_{ji}}^{(l)} = u$  for some  $l \leq n_{k_{ji}}(j)$ .

*Case 4* For general  $(i, U)$ , with one  $u_s^{(r)} = u$  with  $0 < u \leq \Delta t$  for  $(s, r) \in R(i)$ ,

$$P_{(i,U)}(t + \Delta t) = P_{(j,U-\Delta t)}(t) \prod_{(s,r) \in R(j)} \frac{1 - F(u_s^{(r)})}{1 - F(u_s^{(r)} - \Delta t)} (A_{ji}\Delta t + o(\Delta t)) \frac{1}{\Delta t} (1 - F(u)), \quad (\text{EC.5})$$

where  $(j, i) \in V_C^2$ .

Define  $P_{(i,U)}^*(t) = \frac{P_{(i,U)}(t)}{\prod_{s,r} [1 - F(u_s^{(r)})]}$ , which is the conditional probability in state  $i$  given the ages of replenishment orders at time  $t$ . Assume the existence of  $\frac{\partial P_{(i,U)}^*(t)}{\partial t}$  and  $\frac{\partial P_{(i,U)}^*(t)}{\partial u_s^{(r)}}$ . Dividing equations (EC.2)-(EC.4) by  $\Delta t$  and letting  $\Delta t \rightarrow 0$  in equation (EC.2)-(EC.5), we obtain the following system of integro-differential equations

*Case 1*

$$\frac{\partial P_{(i_0,O)}^*(t)}{\partial t} + \lambda P_{(i_0,O)}^*(t) = \sum_{\{j:(j,i_0) \in V_R^2\}} \int_0^\infty P_{(j,U)}^*(t) dF(u_{k_{ij}}^{(1)}),$$

$u_{k_{i_0j}}^{(1)}$  is the only positive entry of  $U$ .

*Case 2* For  $(i, U)$  with  $n_0(i) = 0$ ,

$$\frac{\partial P_{(i,U)}^*(t)}{\partial t} + \sum_{(s,r) \in R(i)} \frac{\partial P_{(i,U)}^*(t)}{\partial u_s^{(r)}} = 0.$$

*Case 3* For general  $(i, U)$  with  $i \neq i_0$  and  $0 < n_0(i) < S$ ,

$$\frac{\partial P_{(i,U)}^*(t)}{\partial t} + \sum_{(s,r) \in R(i)} \frac{\partial P_{(i,U)}^*(t)}{\partial u_s^{(r)}} + \lambda P_{(i,U)}^*(t) = \sum_{\{(j,U'):(j,i) \in V_R^2, U'=(U-\Delta t)_{ji}^+\}} \int_0^\infty P_{(j,U')}^*(t) dF(u) \quad (\text{EC.6})$$

where  $(U - \Delta t)_{ji}^+$  is the  $U - \Delta t$  inserting  $u_{k_{ji}}^{(l)} = u$  for some  $l \leq n_{k_{ji}}(j)$ .

*Case 4* for  $(j, i) \in V_C^2$ ,

$$P_{(i,U)}^*(t) = A_{ji} P_{(j,U)}^*(t).$$

If we start with the stationary distribution, then all the derivatives with respect to time  $t$  vanish.

Dropping the dependence on  $t$ , we have

*Case 1*

$$\lambda P_{(i_0,O)}^* = \sum_{\{j:(j,i_0) \in V_R^2\}} \int_0^\infty P_{(j,U)}^* dF(u_{k_{ji_0}}^{(1)}), \quad (\text{EC.7})$$

$u_{k_{ji_0}}^{(1)}$  is the only positive entry of  $U$ .

*Case 2* For  $(i, U)$  with  $n_0(i) = 0$ ,

$$\sum_{(s,r) \in R(i)} \frac{\partial P_{(i,U)}^*}{\partial u_s^{(r)}} = 0. \quad (\text{EC.8})$$

*Case 3* For general  $(i, U)$  with  $i \neq i_0$  and  $0 < n_0(i) < S$ ,

$$\sum_{(s,r) \in R(i)} \frac{\partial P_{(i,U)}^*}{\partial u_s^{(r)}} + \lambda P_{(i,U)}^* = \sum_{\{(j,U'):(j,i) \in V_R^2, U'=(U-\Delta t)_{ji}^+\}} \int_0^\infty P_{(j,U')}^* dF(u) \quad (\text{EC.9})$$

where  $(U - \Delta t)_{ji}^+$  is the  $U - \Delta t$  inserting  $u_{k_{ji}}^{(l)} = u$  for some  $l \leq n_{k_{ji}}(j)$ .

*Case 4* For  $(i, j) \in V_C^2$ ,

$$P_{(i,U)}^*(t) = A_{ji} P_{(j,U)}^*(t). \quad (\text{EC.10})$$

Let  $\zeta_{(i,U)}$  be given by (EC.1). It is straightforward to verify that the substitution  $P_{(i,U)}^*$  by  $\frac{\zeta_{(i,U)}}{\prod_{s,r}[1-F(u_s^{(r)})]}$  satisfies equations (EC.7)-(EC.10). Therefore,  $\zeta_{(i,u)}$  is the stationary distribution of  $Z(t)$ . ■