

# THREE PROBLEMS IN INTERACTING PARTICLE SYSTEMS

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Daniel Ilan Remenik

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## THREE PROBLEMS IN INTERACTING PARTICLE SYSTEMS

Daniel Ilan Remenik, Ph.D.

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This work consists on the study of three problems in the theory of interacting particle systems. The first and third problems are inspired by problems in ecology, while the second one is inspired by problems in finance and economics.

In the first one we study the contact process running on a dynamic random environment on  $\mathbb{Z}^d$ . This can be thought of as model for the spread of an infection on an environment where each sites alternates randomly between being susceptible or blocked for the infection. Our main results are versions of the classical results in the theory of contact processes adapted to our setting. We give a partial description of the set of parameter values for which the process survives or dies out. We also extend the classical block conditions for survival of the process and use them to show that the critical process dies out and that the complete convergence theorem holds in the supercritical case.

In the second problem we study limit theorems for a class of individual-based models which are suitable for studying various problems in economics and finance. We prove a law of large numbers and a central limit theorem for the empirical distribution of the process. Interestingly, both the model and the techniques used in the proofs draw inspiration from some work done in ecology and mathematical physics.

The third problem is inspired by gypsy moth populations, which present the following interesting feature: they grow until they become sufficiently dense, at which point a large epidemic reduces them to a low level. We model this

phenomenon in a discrete time particle system, and we show that the density of occupied sites converges to a dynamical system that presents chaotic behavior for certain parameter values.

## **BIOGRAPHICAL SKETCH**

Daniel Remenik was born in Santiago, Chile, on June 13 1980. In March 1999 he enrolled in the Engineering School of Universidad de Chile, where he received his B.Sc. in Engineering with a Mention in Mathematical Engineering in December 2002 and his degree in Mathematical Engineering in November 2004. In August 2005 he moved to Ithaca, New York to pursue a Ph.D. in Applied Mathematics from Cornell University.

A Gaby

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The best thing about my years as a graduate student in Ithaca is that I spent them with my wife Gaby. I am lucky to have her unconditional support and encouragement and I am lucky to have someone like her who makes me happy and with whom I can laugh no matter what. This was since the beginning a project of us two, and the fact that she is now pursuing graduate studies at Cornell is proof of that, but I am forever grateful to her for her willingness to come to Ithaca with me without really knowing how things would turn out for her.

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## TABLE OF CONTENTS

Biographical Sketch . . . . .	iii
Dedication . . . . .	iv
Acknowledgements . . . . .	v
Table of Contents . . . . .	vii
List of Figures . . . . .	ix
<b>1 Introduction . . . . .</b>	<b>1</b>
1.1 The contact process in a dynamic random environment and the evolution of dispersal distance . . . . .	2
1.2 Limit theorems for individual-based models in economics and finance . . . . .	6
1.3 Chaos in a spatial epidemic model and gypsy moth epidemics . . . . .	8
References . . . . .	11
<b>2 The Contact Process in a Dynamic Random Environment . . . . .</b>	<b>13</b>
2.1 Introduction . . . . .	13
2.2 Preliminaries . . . . .	20
2.2.1 Graphical representation and monotonicity . . . . .	20
2.2.2 Duality . . . . .	24
2.2.3 Positive correlations . . . . .	26
2.3 Survival and extinction . . . . .	27
2.4 Block construction . . . . .	34
2.5 Complete convergence . . . . .	43
References . . . . .	56
<b>3 Limit Theorems for Individual-Based Models in Economics and Finance . . . . .</b>	<b>58</b>
3.1 Introduction . . . . .	58
3.2 Description of the Model . . . . .	62
3.2.1 Introductory example . . . . .	62
3.2.2 Description of the General Model . . . . .	64
3.3 Law of large numbers for $\nu_t^N$ . . . . .	66
3.4 Central limit theorem for $\nu_t^N$ . . . . .	71
3.4.1 General setting . . . . .	72
3.4.2 Statement of the theorem . . . . .	78
3.4.3 Application to concrete type spaces . . . . .	82
3.4.3.1 Finite $W$ . . . . .	82
3.4.3.2 $W = \mathbb{Z}^d$ . . . . .	85
3.4.3.3 $W = \Omega$ , a compact, sufficiently smooth subset of $\mathbb{R}^d$ . . . . .	88
3.4.3.4 $W = \mathbb{R}^d$ . . . . .	89
3.5 Proofs of the Results . . . . .	93

3.5.1	Preliminary computations and proof of Theorem 3.3.1 . . .	93
3.5.2	Proof of Theorem 3.4.1 . . . . .	96
3.5.2.1	Moment estimates for $\nu_t^N$ and $\nu_t$ . . . . .	97
3.5.2.2	Extension of $\langle \nu_t^N, \cdot \rangle$ and $\langle \nu_t, \cdot \rangle$ to $\mathcal{H}_4'$ . . . . .	100
3.5.2.3	The drift term . . . . .	102
3.5.2.4	Uniform estimate for the martingale term in $\mathcal{H}_4'$ . . . . .	103
3.5.2.5	Evolution equation for $\sigma_t^N$ in $\mathcal{H}_3'$ . . . . .	105
3.5.2.6	Uniform estimate for $\sigma_t^N$ on $\mathcal{C}_2'$ . . . . .	107
3.5.2.7	Proof of the theorem . . . . .	107
3.5.3	Proof of Theorems 3.4.2 and 3a-3d . . . . .	113
	References . . . . .	117
<b>4</b>	<b>Chaos in a Spatial Epidemic Model</b>	<b>120</b>
4.1	Introduction . . . . .	120
4.1.1	Mean-field growth on a random 3-regular graph . . . . .	122
4.1.2	Local growth on the $d$ -dimensional torus . . . . .	127
4.1.3	Local growth on $\mathbb{Z}^d$ . . . . .	131
4.2	Proof of Theorem 4.1.1 . . . . .	133
4.3	Proof of Theorem 4.1.2 . . . . .	140
4.4	Proof of Theorem 4.1.3 . . . . .	147
4.5	Proof of Theorem 4.1.4 . . . . .	148
	References . . . . .	160

## LIST OF FIGURES

4.1	Graph of $h_T$ with $\beta = 2 \log 3$ . The point $c = h_T^{-1}(a_0)$ will play a role in the proof of Theorem 4.1.1. . . . .	125
4.2	Orbits of the system $(h_T^k(p))_{k \geq 0}$ started at $p = 0.1$ . The $x$ -axis has the values of $\beta$ used in the simulations, while the $y$ -axis has $h_T^k(p)$ for $k = 501, \dots, 550$ . . . . .	127
4.3	Density process $(\rho_k^N(p))_{k \geq 0}$ running on the 2-dimensional torus with mean-field growth, depicted for $k = 501, \dots, 550$ for different values of the parameter $\beta$ (similar to Figure 4.2). Here $N = 500$ .	131
4.4	Graph of $\rho_k^N$ against $\rho_{k+1}^N$ on the 2-dimensional torus with $N = 750$ and $r_N = 50$ . The graph clearly does not correspond to a function. . . . .	132
4.5	Sequence of densities $\rho_k^N$ of the process running on the 2-dimensional torus with local interactions in the epidemic step for $N = 500$ and $N = 1500$ , both with $r_N = 5$ . As this graph suggests, the fluctuations of the density process get small as $N$ grows if the range $r_N$ is held fixed. . . . .	133
4.6	State of the process at time 200 on a torus of size $450 \times 450$ (black dots are occupied). In this simulation, $\beta = 2.25$ , $r_N = 5$ , and the infection probability at each site is $5 \cdot 10^{-6}$ . This picture corresponds to an intermediate state of the process, after an epidemic event wiped out a big cluster but the process has had time to grow back. . . . .	134
4.7	Infimum of $ (h_T^3)'(p) $ on the relevant interval for $\beta \in (2 \log 2, 2.6)$ . The computation was done for each $\beta$ on a grid of width $2 \cdot 10^{-6}$ on this interval, as explained within the proof of Theorem 4.1.1. The infimum lies above 1.02 for $\beta \in (2 \log 2, 2.48]$ . . . . .	138

# CHAPTER 1

## INTRODUCTION

Interacting particle systems was born as a field in the late 1960's, mainly from the work of Frank Spitzer at Cornell University and Roland Dobrushin at the Institute for Problems of Information Transmission in Moscow. It has since become a major and diverse theme of research in probability theory and statistical physics.

To describe what an interacting particle system is, let us paraphrase the introduction of Tom Liggett's 1985 book on the subject. A typical interacting particle system consists on finitely or infinitely many particles which, in the absence of interaction, would evolve as a system of independent Markov chains. Superimposed on this system there is some stochastic rule governing the interaction between the particles. The system as a whole is still Markovian, but the evolution of each particle no longer is. As a large and complex Markov process, interacting particle systems usually present difficulties which are not present in the study of more classical Markov processes such as random walks or Brownian motion. Therefore the study of these systems has required the development of some specific tools while at the same time it has relied heavily on other tools which had earlier played a minor role in probability theory.

The original motivation for this field was provided by some problems in statistical physics. The main idea was to study stochastic models which describe the temporal evolution of physical systems whose equilibrium measures are the classical Gibbs states and, in particular, the study of phase transitions. But as time passed, in Liggett's words, it became clear that models with a very similar mathematical structure could be naturally formulated in other contexts. Among

these stand out problems in mathematical biology, since interacting particle systems provide a natural way to model spatial phenomena such as the spread of an infection or the competition between different species. More recently, particles systems have proved useful to model problems in economics and finance, in particular by allowing to study problems in which the market is seen as a large and complex system composed by many interacting agents.

It is these last two subjects, mathematical biology and economics and finance, which provided the motivation for the three problems that we consider in this work. During the rest of this introduction we will explain briefly the motivation for these problems and the results we obtained.

## **1.1 The contact process in a dynamic random environment and the evolution of dispersal distance**

In Chapter 2 we study a version of the contact process running on a dynamic environment in  $\mathbb{Z}^d$ . Before describing the process, let us explain our original motivation, even if the problem we ended up studying is somewhat far from it.

The problem we were interested in is that of the evolution of dispersal distances in an ecological system. To fix ideas, consider the evolutionary problem facing a plant that produces seeds. Smaller seeds are lighter and disperse further but have a smaller probability of successfully germinating. Conversely, bigger seeds are better at germinating, but have a smaller dispersal distance. There is thus a trade-off for the plant in its choice of seed size.

This problem has been studied extensively in the mathematical biology lit-

erature, see for instance Ezoë (1998), Bolker and Pacala (1999) and Levin and Muller-Landau (2000). Our goal was to study this problem using a multi-type contact process in  $\mathbb{Z}^2$  where each species has a different birth rate and dispersal distance. To be more precise, consider a continuous time Markov process with state space  $\{0, 1, 2\}^{\mathbb{Z}^2}$ . Sites at state 0 are vacant, while sites at state  $i$ , for  $i = 1, 2$ , are occupied by an individual of species  $i$ . The dynamics is as follows. Each individual of species  $i$  dies at rate 1 and gives birth at rate  $\beta_i$ . Offspring is then sent to a site chosen uniformly at random from a ball of radius  $r_i$  around the parent particle, and the birth is suppressed if the chosen site is not empty. The question is the following: for which parameters  $\beta_i$  and  $r_i$  can the two species coexist? It is not hard to see that if one species has both a bigger birth rate and a bigger dispersal distance, then that species will drive the other one to extinction, so we should think of the case  $\beta_1 \geq \beta_2$  and  $r_1 \leq r_2$  (or vice versa).

A related problem was studied by Neuhauser (1992), who considered equal dispersal distances and different birth and death rates. She conjectured that if the death rates are fixed, then coexistence is possible only for a one-dimensional set of parameter values for the birth rates, and she proved this in the case where the death rates are the same. The difficulty in the case where the death rates are different is that one of the basic tools in the study of contact processes, duality, breaks down. We have the same difficulty in our setting, and thus we were not able to make a lot of progress in the problem as described above. Durrett and Schinazi (1993) studied a version of the process where 2's are allowed to give birth on top of 1's (but not vice versa), while 1's have a long range of dispersal. This simplification allowed them to use duality.

One way to try to get a grasp on this system is to develop a mean-field de-

terministic equation when  $r_2 \rightarrow \infty$  and  $r_1$  is fixed. Heuristically, the limiting differential equation should be

$$\begin{aligned}\frac{du_1(t)}{dt} &= u_1(t) + \beta_1 u_{01}(t) \\ \frac{du_2(t)}{dt} &= u_2(t) + \beta_2 u_2(t) u_0(t),\end{aligned}$$

where  $u_{01}$  is a “local equilibrium distribution”. Unfortunately, it is not clear what two-dimensional distribution should be used to evaluate  $u_{01}$ .

In an attempt to get an idea about what this probability measure must look like, we considered the following simplification of the model. Now there is only one species, the 1’s, which die at rate 1, give birth at rate  $\beta$  and have dispersal distance equal to 1. On the other hand, each site becomes uninhabitable (that is, no births are allowed onto the site) at rate  $\alpha$  and goes back to be habitable at rate  $\delta$ . When a 1 is at a site which becomes uninhabitable it dies. Sites flip between being or not being habitable independently of each other, and this is what we call the dynamic random environment. The uninhabitable sites here play the role of the 2’s above.

We were able to analyze this last process in some detail. The process, and the results we got, are interesting in its own right, since they extend most of the classical theory of the contact process in  $\mathbb{Z}^d$  to this varying environment case. But they unfortunately did not prove useful (at least for now) to attack the original problem. Before briefly discussing the results we obtained, let us remark that we are currently working in an alternative model for the evolution of the dispersal distance which is more tractable, and for which we already have results (Durrett and Remenik, 2009), though they will not be included in this thesis. In this alternative formulation, we replace the multi-type contact process by a multi-type voter model. Now there are no empty sites, so each site is occupied

by either a 1 or a 2, while each species is allowed to give birth on top of the other. In this new formulation we are able to rescale time and space to obtain deterministic limit for the evolution of the system in the form of a reaction diffusion equation (this is similar to what has been done for particle systems with rapid stirring, see Durrett and Neuhauser (1994), and builds upon ideas introduced in Cox, Durrett, and Perkins (2009)). By analyzing this limiting equation we are able to predict what pairs of birth rate and dispersal range are evolutionary stable in the sense that a population with these parameters cannot be driven to extinction by a small population arising from a small mutation, and thus gives a solution to the problem of the evolution of dispersal distance from the perspective of adaptive dynamics.

Coming back to the contact process on a dynamic random environment, we gave a partial description of the phase diagram of the process. We proved three results which together give bounds on the regions on the  $(\beta, \alpha, \delta)$  parameter space where the 1's survive or die out. One of these three results follows from an easy coupling with the classical contact process, the second one uses ideas from the theory of continuum percolation and the last one uses an interesting coupling result for Poisson processes which we borrow from Broman (2007). Here by survival we mean the fact that if the process starts with a single 1 at the origin and all other sites uninhabitable, then there is a positive probability that we will see 1's at all times. Our results imply in particular that, depending on the flip rates for the environment, the process may survive for large enough  $\beta$  or may die out for all  $\beta$ .

Our second result is a complete convergence theorem, which characterizes all the invariant measures of the process as convex combinations of the lower

invariant measure  $\underline{\nu}$ , which is a product measure with no 1's which is invariant for the environment process, and the upper invariant measure  $\bar{\nu}$ , which is the largest such measure in a precise sense which means roughly that it is the one with most 1's. The theorem also shows that the process started with any initial distribution converges as time goes to infinity to one of these convex combinations. In other words, the process either dies out and its distribution converges to  $\underline{\nu}$ , or it survives and its distribution converges to  $\bar{\nu}$ . The proof relies in extending the classical block construction for the contact process (see Bezuidenhout and Grimmett (1990)), and thus allows us also to show that the process dies out at the critical parameter values.

## **1.2 Limit theorems for individual-based models in economics and finance**

The problem we consider in Chapter 3 was motivated by a suggestion made by Darrell Duffie to Philip Protter. The main goal was to try to give a more sound and general mathematical foundation for a class of models being studied in the mathematical finance literature by Duffie and coauthors among others (see the references section of Chapter 3 for references). The central theme of these models is the attempt to understand the macroeconomic behavior of the markets based on the random local interactions between the individual agents involved. This idea has been around since the mid 1970's (see Föllmer (1974)), and has become popular in the last 5 or 10 years.

The papers of Duffie and coauthors were based in the ideas of Duffie and Sun (2007), who proved what they called an "exact law of large numbers" for

certain systems of interacting agents which each have a certain “type”. In their setting there is a continuum of agents who are repeatedly matched in pairs (all at once) at random. Their exact law of large numbers (the word exact is there presumably because there is no limiting procedure in their statement, since they already start with infinitely many agents), which is proved using non-standard analysis, is a statement about the evolution of the fraction of agents of each type. Although the result is interesting, it falls short of the goal of rigorously justifying their proposed models.

Interestingly, models which were similar in spirit were being developed more or less at the same time to study problems in theoretical population biology (see, e.g., Fournier and Méléard (2004) and Champagnat (2006)). Drawing inspiration from that work, we provided a rigorous justification for the models of Duffie and coauthors, and in fact we developed a much more general framework for this type of models by allowing very general type spaces. A main difference between our approach and that of Duffie and Sun is that we start with a finite number of  $N$  agents, and then obtain a law of large numbers for the empirical measure associated to our processes by taking  $N \rightarrow \infty$ .

An obvious extension to this work was to prove a central limit theorem for these processes. This proved to be a much more delicate problem to attack at the level of generality we were working on because of technical difficulties related with finding the right topology with respect to which to prove convergence. The solution came from generalizing some ideas using Sobolev embeddings which were introduced in Métivier (1987), by seeing our empirical measures as elements in the duals of a chain of abstract Banach and Hilbert spaces. We also gave concrete examples of such chains of spaces (Sobolev spaces among them)

and used them to apply our central limit theorem to some examples related with economics and population genetics.

### **1.3 Chaos in a spatial epidemic model and gypsy moth epidemics**

In the late 1980's the Northeastern United States was in the midst of a gypsy moth infestation. Gypsy moth egg masses are typically laid on branches and trunks of hardwood trees, specially oaks. The larvae feed off the leaves and crown of the trees, and the consequent defoliation may even lead to the death of the trees (depending on the degree of defoliation and on the number of consecutive defoliations which the trees suffer), usually because it leaves them vulnerable to the attack by other insects or organisms. Consequently, people living in the area at that time were not particularly cheerful about the fate of their trees, and spent a good part of their summer taking measures to try to avoid the gypsy moth larvae climbing the trees.

When the next summer came, it was not the trees who had suffered, but instead the gypsy moth larvae, most of whom were either dead or deformed, victims to the nuclear polyhedrosis virus, which is known to spread quickly through the gypsy moth population once it becomes sufficiently dense (see Gould, Elkinton, and Wallner (1990)). The gypsy moth populations are thus driven to a very low density, from where they start to grow again until the density is high enough for the epidemic to attack. This leads to an interesting dynamical behavior for the time evolution of the gypsy moth population density.

In Chapter 4 we study this phenomenon using a discrete time particle system. In the system there are  $N$  sites in some graph, and each one is either vacant or occupied by one individual. The dynamics has two stages. In the first one, no individuals survive but they give birth to a mean  $\beta$  number of offspring which are sent to places chosen at random from the graph according to some given rule. In the second step, each site is attacked by an epidemic with a small probability  $\alpha_N$ , and when an occupied site is attacked the individual, all its neighbors, their neighbors' neighbors and so on die. We assume that this epidemic occurs fast, so all the infected individuals die before the next growing season.

We first consider the system on a random 3-regular graph. We prove that the density of occupied sites converges, as we take  $N \rightarrow \infty$  and  $\alpha_N \rightarrow 0$ , to a discrete time (deterministic) dynamical system, for which we have an explicit formula, which is chaotic for  $\beta > \beta_c = 2 \log 2$ . This chaotic behavior reflects the dynamics of the gypsy moth populations described above: the populations grow until they become sufficiently dense, at which point a large epidemic wipes out the giant component of occupied sites in the graph and the cycle begins again. The critical density and the explicit expression for the dynamical system are related with the percolation probabilities for site percolation on a regular tree of degree 3 and the corresponding critical probability, thanks to the fact that, as we show, a random 3-regular graph looks locally like a tree.

We also consider the process on a  $d$ -dimensional torus. We prove again convergence to a discrete time dynamical system but, since there are no explicit expressions for the percolation probabilities on  $\mathbb{Z}^d$ , we do not have explicit expressions for this system. Nevertheless we prove that in  $d = 2$ , and also in  $3 \leq d < 6$  under a reasonable assumption on the percolation function in these

dimensions, the system presents chaotic behavior for at least an interval of values of  $\beta$  to the right of the critical  $\beta_c$ . We also show that for the system running in all of  $\mathbb{Z}^d$  there are non-trivial invariant measures.

## REFERENCES

- Bezuidenhout, C. and Grimmett, G. (1990). The critical contact process dies out. *Ann. Probab.*, **18**(4), 1462–1482.
- Bolker, B. and Pacala, S. (1999). Optimal dispersal range and seed size in a stable environment. *Amer. Nat.*, **153**, 575–602.
- Broman, E. I. (2007). Stochastic domination for a hidden Markov chain with applications to the contact process in a randomly evolving environment. *Ann. Prob.*, **35**(6), 2263–2293.
- Champagnat, N. (2006). A microscopic interpretation for adaptive dynamics trait substitution sequence models. *Stochastic Process. Appl.*, **116**(8), 1127–1160.
- Cox, J., Durrett, R., and Perkins, E. (2009). Voter model perturbations. In preparation.
- Duffie, D. and Sun, Y. (2007). Existence of independent random matching. *Ann. Appl. Probab.*, **17**(1), 386–419.
- Durrett, R. and Neuhauser, C. (1994). Particle systems and reaction-diffusion equations. *Ann. Probab.*, **22**(1), 289–333.
- Durrett, R. and Remenik, D. (2009). Voter model perturbations II. In preparation.
- Durrett, R. and Schinazi, R. (1993). Asymptotic critical value for a competition model. *Ann. Appl. Probab.*, **3**(4), 1047–1066.
- Ezoe, H. (1998). Optimal dispersal range and seed size in a stable environment. *J. Theor. Biol.*, **190**, 287–293.

- Föllmer, H. (1974). Random economies with many interacting agents. *J. Math. Econom.*, **1**(1), 51–62.
- Fournier, N. and Méléard, S. (2004). A microscopic probabilistic description of a locally regulated population and macroscopic approximations. *Ann. Appl. Probab.*, **14**(4), 1880–1919.
- Gould, J., Elkinton, J., and Wallner, W. (1990). Density-dependent suppression of experimentally created gypsy moth, *Lymantria dispar* (Lepidoptera, Lymantriidae), populations by natural enemies. *J. Anim. Ecol.*, **59**, 213–233.
- Levin, S. and Muller-Landau, H. (2000). The evolution of dispersal and seed size in plant communities. *Evol. Ecol. Res.*, **2**, 409–435.
- Liggett, T. M. (1985). *Interacting particle systems*, volume 276 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, New York.
- Métivier, M. (1987). Weak convergence of measure valued processes using Sobolev-embedding techniques. In *Stochastic partial differential equations and applications (Trento, 1985)*, volume 1236 of *Lecture Notes in Math.*, pages 172–183. Springer, Berlin.
- Neuhauser, C. (1992). Ergodic theorems for the multitype contact process. *Probab. Theory Related Fields*, **91**(3-4), 467–506.

## CHAPTER 2

### THE CONTACT PROCESS IN A DYNAMIC RANDOM ENVIRONMENT \*

We study a contact process running in a random environment in  $\mathbb{Z}^d$  where sites flip, independently of each other, between blocking and non-blocking states, and the contact process is restricted to live in the space given by non-blocked sites. We give a partial description of the phase diagram of the process, showing in particular that, depending on the flip rates of the environment, survival of the contact process may or may not be possible for large values of the birth rate. We prove block conditions for the process that parallel the ones for the ordinary contact process and use these to conclude that the critical process dies out and that the complete convergence theorem holds in the supercritical case.

## 2.1 Introduction

We consider the following version of a contact process running in a dynamic random environment in  $\mathbb{Z}^d$ . The state of the process is represented by some  $\eta \in \mathcal{X} = \{-1, 0, 1\}^{\mathbb{Z}^d}$ , where sites in state 0 are regarded as vacant, sites in state 1 as occupied, and sites in state  $-1$  as blocked (that is, no births of 1's are allowed on that site). The process  $\eta_t$  is defined by the following transition rates:

$$\begin{array}{llll} 0 & \longrightarrow & 1 & \text{at rate } \beta f_1 \\ 1 & \longrightarrow & 0 & \text{at rate } 1 \\ 0, 1 & \longrightarrow & -1 & \text{at rate } \alpha \\ -1 & \longrightarrow & 0 & \text{at rate } \alpha \delta \end{array}$$

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\*Remenik, Daniel (2008), *Annals of Applied Probability*, Vol. 18:2392-2420

where  $f_1$  is the fraction of occupied neighbors at  $L^1$  distance 1.

In words, the  $-1$ 's define a random environment in which each site becomes blocked at rate  $\alpha$  and flips back to being unblocked at rate  $\alpha\delta$ , while the  $1$ 's behave like a nearest neighbor contact process with birth rate  $\beta$  in the space left unblocked by the environment. Observe that when an occupied site becomes blocked, the particle is killed. This version is simpler than the alternative in which only  $0$ 's can turn to  $-1$ 's (mainly because our process satisfies a self-duality relation, see Proposition 2.2.2). However, we feel that our choice is natural: if a site becomes uninhabitable, the particles living there will soon die.

Ever since it was introduced in Harris (1974), the contact process has been object of intensive study, and many extensions and modifications of the process have been considered. In particular, the literature includes several different versions of contact processes in random environments. One class of these processes corresponds to contact processes where the birth and death rates are not homogeneous in space, and they are chosen according to some probability distribution, independently across sites, and remain fixed in time (see, for example, Bramson, Durrett, and Schonmann (1991), Liggett (1992), Andjel (1992), and Klein (1994)). The main question for this class of processes is to determine conditions on the parameters that guarantee or preclude survival.

A different class of models, which are somehow closer to the process we consider, have two species with different parameters or ranges, but one of them behaves independently of the other while the second is restricted to live in the space left by the first. These processes were studied in Durrett and Swindle (1991), Durrett and Møller (1991), and Durrett and Schinazi (1993). The results in these papers (mainly bounds on critical parameters for coexistence and com-

plete convergence theorems) are asymptotic, in the sense that they are proved when the range of one or both types is sufficiently large.

The process we consider differs from both of the types of examples mentioned above: the random environment is dynamic, and it behaves independently across sites. An example of a spin system running in this type of environment was studied in Luo (1992), and corresponds to the Richardson model which would result from ignoring transitions from 1 to 0 in our process. Another example was studied recently in Broman (2007), where the author considers a process in which the environment changes the death rate of the contact process instead of blocking sites. The dynamics of the process  $\Psi_{\delta_0, \delta_1}^{\gamma, p, A}$  introduced there are the same as those of our process if  $\delta_1 = \infty$ . The author considers this case as a tool in the study of his process, but the results of the paper focus on the case  $\delta_1 < \infty$ . We will use one of his results to give a bound on a part of the phase diagram of our process in Theorem 2.1.1.

As mentioned above, the  $-1$ 's evolve independently of the  $1$ 's. They follow an “independent flip process” whose equilibrium is given by the product measure

$$\mu_\rho(\{\eta : \eta(x) = -1\}) = 1 - \mu_\rho(\{\eta : \eta(x) \neq -1\}) = \rho = \frac{1}{1 + \delta} \quad \forall x \in \mathbb{Z}^d.$$

This process is reversible, and its reversible measure is given by  $\mu_\rho$ .

In Section 2.2.1 we will construct our process using the so-called graphical representation. A direct consequence of the construction will be that  $\eta_t$  satisfies some monotonicity properties analogous to those of the contact process. (Here and in the rest of the paper, when we refer to the contact process we mean the “ordinary” nearest-neighbor contact process in  $\mathbb{Z}^d$ ). We consider the following

partial order on configurations:

$$\eta^1 \leq \eta^2 \quad \Leftrightarrow \quad \eta^1(x) \leq \eta^2(x) \quad \forall x \in \mathbb{Z}^d. \quad (2.1)$$

With this order, our process has the following property: given two initial states  $\eta_0^1 \leq \eta_0^2$ , it is possible to couple two copies of the process  $\eta_t^1$  and  $\eta_t^2$  with these initial conditions in such a way that  $\eta_t^1 \leq \eta_t^2$  for all  $t \geq 0$ . We will refer to this property as *attractiveness* by analogy with the case of spin systems (this property is sometimes termed *monotonicity*, see Sections II.2 and III.2 of Liggett (1985) for a discussion of general monotone processes and of attractive spin systems, respectively).

For  $A \subseteq \mathbb{Z}^d$  we define the following probability measure  $\nu_A$  on  $\mathcal{X}$ :  $-1$ 's are chosen first according to their equilibrium measure  $\mu_\rho$  and then  $1$ 's are placed at every site in  $A$  that is not blocked by a  $-1$ . These measures are the initial conditions for  $\eta_t$  that are suitable for duality.

Let  $\underline{\nu} = \nu_\emptyset$ , which corresponds to having the  $-1$ 's at equilibrium and no  $1$ 's. Let also  $\bar{\nu}$  be the limit distribution of the process when starting at the configuration having all sites at state  $1$ , which is obviously the largest configuration in the partial order (2.1). We will show in Proposition 2.2.1 that this limit is well defined and it is stationary, and that  $\underline{\nu}$  and  $\bar{\nu}$  are, respectively, the *lower* and *upper invariant measure* of the process (that is, the smallest and largest stationary distribution of the process).

We will say that the process *survives* if there is an invariant measure  $\nu$  such that

$$\nu(\{\eta: \eta(x) = 1 \text{ for some } x \in \mathbb{Z}^d\}) > 0,$$

or, equivalently, if  $\underline{\nu} \neq \bar{\nu}$  (we remark that, as a consequence of Theorem 2.1.2, every invariant measure for the process is translation invariant, so the above probability is actually 1 whenever it is positive). Otherwise, we will say that the process *dies out*. We will see in Section 2.4 that this definition of survival is equivalent to the following condition: the process started with a single 1 at the origin and everything else at  $-1$  contains 1's at all times with positive probability.

A second monotonicity property that will follow from the construction of  $\eta_t$  is monotonicity with respect to the parameters  $\beta$  and  $\delta$ :

- (i) If  $\alpha$  and  $\delta$  are fixed, and for some  $\beta > 0$  the process survives, then the process also survives for any  $\beta' > \beta$ .
- (ii) If  $\alpha$  and  $\beta$  are fixed, and for some  $\delta > 0$  the process survives, then the process also survives for any  $\delta' > \delta$ .

These properties follow easily from standard coupling arguments. We will denote by  $\beta_c = \beta_c(\alpha, \delta) \in [0, \infty]$  the parameter value such that, fixing these  $\alpha$  and  $\delta$ ,  $\eta_t$  survives for  $\beta > \beta_c$  and dies out for  $\beta < \beta_c$ . We define  $\delta_c = \delta_c(\alpha, \beta)$  analogously.

Our first result provides some bounds on the critical parameters for survival. Let  $\beta_c^{\text{cp}}$  be the critical value of the contact process in  $\mathbb{Z}^d$  (here we are taking the birth rate  $\beta$  to be the total birth rate from each site, so each site sends births to each given neighbor at rate  $\beta/(2d)$ ).

**Theorem 2.1.1.**

- (a) If  $\beta \leq (\alpha + 1)\beta_c^{\text{cp}}$  then the process dies out.

(b) There exists a  $\delta_p > 0$  such that for any  $\delta < \delta_p$  the process dies out (regardless of  $\alpha$  and  $\beta$ ).

(c) Let

$$\bar{\lambda}(\alpha, \beta, \delta) = \frac{1}{2} \left[ \beta + \alpha(1 + \delta) - \sqrt{(\beta - \alpha(1 + \delta))^2 + 4\alpha\beta} \right].$$

If  $\bar{\lambda}(\alpha, \beta, \delta) > (\alpha + 1)\beta_c^{\text{CP}}$  then the process survives.

Part (a) of the theorem is trivial, because the 1's die at rate  $\alpha + 1$ . For part (b), observe that if the complement of the set of sites at state  $-1$  does not space-time percolate, then each 1 in the process is doomed to live in a finite space-time region, and then the process cannot have 1's at all times when started with finitely many occupied sites. We will show by adapting arguments in Meester and Roy (1996) that, with probability 1, no such space-time percolation occurs if  $\delta$  is small enough. For part (c) we will use Broman's result to obtain a suitable coupling with a contact process with birth rate  $\bar{\lambda}(\alpha, \beta, \delta)$  and death rate  $\alpha + 1$ .

In particular, Theorem 2.1.1 implies that if  $\delta$  is large enough then  $\beta_c(\alpha, \delta) < \infty$ , and in fact  $\delta > \frac{\alpha+1}{\alpha}\beta_c^{\text{CP}}$  is enough. To see this, observe that

$$\lim_{\beta \rightarrow \infty} \bar{\lambda}(\alpha, \beta, \delta) = \alpha\delta > (\alpha + 1)\beta_c^{\text{CP}}$$

whenever the above condition on  $\delta$  holds. Then part (c) of the theorem implies that the process survives for these choices of  $\alpha$  and  $\delta$  and large enough  $\beta$ . Another consequence is that  $\delta_p \leq \beta_c^{\text{CP}}$ . Indeed, if  $\delta > \beta_c^{\text{CP}}$ , then  $\delta > \frac{\alpha+1}{\alpha}\beta_c^{\text{CP}}$  for large enough  $\alpha$ , and the previous property implies that the process survives for these choices of  $\alpha$  and  $\delta$  and large enough  $\beta$ .

A significant difficulty in giving a more complete picture of the phase diagram of  $\eta_t$  is that we lack a result about monotonicity with respect to  $\alpha$  analogous to the properties (i) and (ii) (monotonicity with respect to  $\beta$  and  $\delta$ ) men-

tioned above. Observe that the equilibrium density of non-blocked sites is independent of  $\alpha$ , but the environment changes more quickly as  $\alpha$  increases. Simulations suggest that if  $\beta$  and  $\delta$  are given and the process dies out at some parameter value  $\alpha$ , then it also dies out for any parameter value  $\alpha' > \alpha$  (note that part (a) of Theorem 2.1.1 says that the process dies out at least for all  $\alpha$  large enough). But the usual simple arguments based on coupling do not work in this case, since increasing  $\alpha$  increases both the rate at which sites are blocked, which plays against survival, and the rate at which sites are unblocked, which plays in favor of survival, and we have not been able to find an alternative proof.

The second part of our study of  $\eta_t$  investigates the convergence of the process and the structure of its limit distributions. For  $\eta \in \mathcal{X}$  we will write  $\eta = (A, B)$ , where

$$A = \{x \in \mathbb{Z}^d : \eta(x) = 1\}, \quad \text{and} \quad B = \{x \in \mathbb{Z}^d : \eta(x) = -1\}.$$

$\eta_t^\mu = (A_t^\mu, B_t^\mu)$  will denote the process with initial distribution  $\mu$ , and we will refer to  $B_t^\mu$  (or  $B_t$  if no initial distribution is prescribed) as the *environment process*. Observe that the dynamics of the environment process are independent of the 1's in  $\eta_t$ .

**Theorem 2.1.2.** *Denote by  $\tau = \inf\{t \geq 0 : A_t = \emptyset\}$  the extinction time of the process. Then for every initial distribution  $\mu$ ,*

$$\eta_t^\mu \Longrightarrow \mathbb{P}^\mu(\tau < \infty) \underline{\nu} + \mathbb{P}^\mu(\tau = \infty) \bar{\nu},$$

*where the limit is in the topology of weak convergence of probability measures.*

This result, which is usually called a *complete convergence theorem*, implies that all limit distributions are convex combinations of  $\underline{\nu}$  and  $\bar{\nu}$ . Thus, the only interesting non-trivial stationary distribution is  $\bar{\nu}$ .

The proof of Theorem 2.1.2 relies on extending for  $\eta_t$  the classical block construction for the contact process introduced in Bezuidenhout and Grimmett (1990), so that we are able to use the proof of complete convergence for the contact process to prove the corresponding convergence of the contact process part of  $\eta_t$ . As a consequence of this construction we will obtain, just as for the contact process, the fact that the process dies out at the critical parameters  $\beta_c$  and  $\delta_c$  (see Corollary 2.4.4). The arguments involved in this part will depend heavily on a duality relation which will be developed in Section 2.2.2.

The rest of the paper is devoted to the proofs of the two theorems. Section 2.2 describes the construction of  $\eta_t$  and presents some basic preliminary results. Theorem 2.1.1 is proved in Section 2.3. In Section 2.4 we obtain the block conditions for the survival of the process. Finally, in Section 2.5 we use duality and the conditions obtained in Section 2.4 to prove Theorem 2.1.2.

## 2.2 Preliminaries

### 2.2.1 Graphical representation and monotonicity

The graphical representation is one of the basic and most useful tools in the study of the contact process and other interacting particle systems. It will allow us to construct our process from a collection of independent Poisson processes and obtain a single probability space in which copies of the process with arbitrary initial states can be coupled. We will give a rather informal description of this construction, which can be made precise by adapting the arguments of Harris (1972). We refer the reader to Section III.6 of Liggett (1985) for more details

on this construction in the case of an additive spin system.

The construction is done by placing symbols in  $\mathbb{Z}^d \times [0, \infty)$  to represent the different events in the process. For each ordered pair  $x, y \in \mathbb{Z}^d$  at distance 1 let  $N^{x,y}$  be a Poisson process with rate  $\beta/(2d)$ , and take the processes assigned to different pairs to be independent. At each event time  $t$  of  $N^{x,y}$  draw an arrow  $\xrightarrow{1}$  in  $\mathbb{Z}^d \times [0, \infty)$  from  $(x, t)$  to  $(y, t)$  to indicate the birth of a 1 sent from  $x$  to  $y$  (which will only take place if  $x$  is occupied and  $y$  is vacant at time  $t$ ). Similarly, define a family of independent Poisson processes  $(U^{1,x})_{x \in \mathbb{Z}^d}$  with rate 1 and for each event time  $t$  of  $U^{1,x}$  place a symbol  $*_1$  at  $(x, t)$  to indicate that a 1 flips to 0 (i.e. that a particle dies). To represent the environment, consider two families of independent Poisson processes  $(V^x)_{x \in \mathbb{Z}^d}$  and  $(U^{-1,x})_{x \in \mathbb{Z}^d}$  with rates  $\alpha$  and  $\alpha\delta$  respectively. For each event time  $t$  of  $V^x$  place a symbol  $\bullet_{-1}$  at  $(x, t)$  to indicate the birth of a  $-1$  (i.e. the blocking of a site) and for each event time  $t$  of  $U^{-1,x}$  place a symbol  $*_{-1}$  to indicate that a  $-1$  flips to 0 (i.e. the unblocking of a site).

We construct  $\eta_t$  from this percolation structure in the following way. Consider a deterministic initial condition  $\eta_0$  and define the environment process  $B_t$  by setting  $\eta_t(x) = -1$  when  $(x, t)$  lies between symbols  $\bullet_{-1}$  and  $*_{-1}$  (in that order) in the time line  $\{x\} \times [0, \infty)$ , and also if  $\eta_0(x) = -1$  and there is no symbol  $*_{-1}$  in that time line before time  $t$ . Having defined  $B_t$ , we say that there is an *active path* between  $(x, s)$  and  $(y, t)$  if there is a connected oriented path, moving along the time lines in the increasing direction of time and passing along arrows  $\xrightarrow{1}$ , which crosses neither symbols  $*_1$  nor space-time points that were set to  $-1$ . The collection of active paths corresponds to the possible space-time paths along which 1's can move, so we define  $A_t$  by

$$A_t = \{y \in \mathbb{Z}^d : \exists x \in A_0 \text{ with an active path from } (x, 0) \text{ to } (y, t)\}.$$

The arguments of Harris (1972) imply that this construction gives a well defined Markov process with the right transition rates. Moreover, the same realization of this graphical representation can be used for different initial conditions, and this gives the coupling mentioned above (see Section III.6 in Liggett (1985) for more details on this coupling in the case of a spin system). For the rest of the paper we will implicitly use this “canonical” coupling every time we couple copies of  $\eta_t$  with different initial conditions. The attractiveness property mentioned in the Introduction follows directly from this construction, and the monotonicity properties with respect to  $\beta$  and  $\delta$  can be obtained by a simple modification of this coupling (analogous to what is done for the contact process).

Recall the definition of the partial order on configurations given in (2.1). Clearly,

$$\eta^1 \leq \eta^2 \quad \Leftrightarrow \quad A^1 \subseteq A^2 \text{ and } B^1 \supseteq B^2.$$

For probability measures on  $\mathcal{X}$ , which we endow with the product topology, we consider the usual ordering:  $\mu_1 \leq \mu_2$  if and only if  $\int f d\mu_1 \leq \int f d\mu_2$  for every continuous increasing  $f : \mathcal{X} \rightarrow \mathbb{R}$ . We recall that the property  $\mu_1 \leq \mu_2$  is equivalent to the existence of a probability space in which a pair of random variables  $X_1$  and  $X_2$  with distributions  $\mu_1$  and  $\mu_2$  can be coupled in such a way that  $X_1 \leq X_2$  almost surely (see Theorem II.2.4 in Liggett (1985)). We will use this fact repeatedly, and for simplicity we will say that  $X_2$  *dominates*  $X_1$  when this condition holds. We will also use this term to compare two processes, so saying that  $\eta_t^2$  dominates  $\eta_t^1$  will mean that the two processes can be constructed in a single probability space in such a way that  $\eta_t^1 \leq \eta_t^2$  for all  $t \geq 0$ .

The attractiveness property allows us to obtain the lower and upper invariant measure of the process.

**Proposition 2.2.1.** *Let  $\chi_{\mathbb{Z}^d}$  be the probability distribution on  $\mathcal{X}$  assigning mass 1 to the all 1's configuration, and let  $S(t)$  be the semigroup associated to the process. Define*

$$\bar{\nu} = \lim_{t \rightarrow \infty} \chi_{\mathbb{Z}^d} S(t),$$

*where the limit is in the topology of weak convergence of probability measures. Then  $\bar{\nu}$  is the upper invariant measure of the process, that is,  $\bar{\nu}$  is invariant and every other invariant measure is stochastically smaller than  $\bar{\nu}$ . Moreover,*

$$\underline{\nu} = \lim_{t \rightarrow \infty} \nu_{\mathbb{Z}^d} S(t).$$

*Analogously,*

$$\underline{\nu} = \nu_{\emptyset}$$

*is the lower invariant measure of the process.*

*Proof.* Since  $\mu_\rho$  is invariant for the environment and the empty state is a trap for the 1's,  $\underline{\nu}$  is invariant. It is the lower invariant measure because every invariant measure has  $\mu_\rho$  as its projection onto the environment, and  $\nu_{\emptyset}$  is the smallest probability measure on  $\mathcal{X}$  having  $\mu_\rho$  as its marginal on the  $-1$ 's.

For the other part, standard arguments imply that the limit defining  $\bar{\nu}$  exists and is invariant (see, for instance, Sections I.1 and III.2 in Liggett (1985)). Since  $\chi_{\mathbb{Z}^d}$  is larger than any other measure on  $\mathcal{X}$ , it follows by attractiveness that  $\bar{\nu}$  is the largest invariant measure.

Now let  $\nu^* = \lim_{t \rightarrow \infty} \nu_{\mathbb{Z}^d} S(t)$ . As above,  $\nu^*$  is well-defined and invariant, so to prove that  $\nu^* = \bar{\nu}$  it is enough to prove that  $\nu^*$  is larger than any other invariant measure. If  $\nu$  is any invariant measure, its projection onto the  $-1$ 's

must be  $\mu_\rho$ , so for any continuous increasing  $f$ ,

$$\begin{aligned} \int f d\nu &= \mathbb{E}^\nu(f(\eta_0)) = \mathbb{E}^\nu(f(\eta_t)) \\ &\leq \mathbb{E}^{\nu_{\mathbb{Z}^d}}(f(\eta_t)) = \int f d[\nu_{\mathbb{Z}^d} S(t)] \xrightarrow[t \rightarrow \infty]{} \int f d\nu^*. \quad \square \end{aligned}$$

## 2.2.2 Duality

The dual process  $(\hat{\eta}_s^t)_{0 \leq s \leq t} = (\hat{A}_s^t, \hat{B}_s^t)_{0 \leq s \leq t}$  is constructed using the same graphical representation we used for constructing  $\eta_t$ . Our duality relation will require that the process be started with the environment at equilibrium. The dual processes will also be started with measures of the form  $\nu_C$ , for  $C \subseteq \mathbb{Z}^d$ , and the dual process started with this distribution will be denoted by  $(\hat{\eta}_s^{\nu_C, t})_{0 \leq s \leq t}$ .

Fix  $t > 0$ , and start by choosing  $B_0$  according to  $\mu_\rho$ . Then run the environment process forward in time until  $t$ , using the graphical representation. This defines  $(B_s)_{0 \leq s \leq t}$ . The dual environment is given by  $\hat{B}_s^t = B_{t-s}$ . Now place a 1 at time  $t$  at every site  $x \in C \setminus \hat{B}_0^t$ , that is, every site in  $C$  which is not blocked by the environment at time  $t$ . This defines  $\hat{A}_0^{\nu_C, t}$ , and by the stationarity of the environment process we get an initial condition  $(\hat{A}_0^{\nu_C, t}, \hat{B}_0^t)$  for the dual chosen according to  $\nu_C$ . Having defined  $\hat{A}_0^{\nu_C, t}$  and  $(\hat{B}_s^t)_{0 \leq s \leq t}$ , we define the 1-dual by

$$\hat{A}_s^{\nu_C, t} = \{y \in \mathbb{Z}^d : \exists x \in \hat{A}_0^{\nu_C, t} \text{ with an active path from } (y, t-s) \text{ to } (x, t)\}.$$

That is, the 1-dual is defined by running the contact process for the 1's backwards in time and with the direction of the arrows reversed. An active path in  $\eta_t$  from  $(y, t-s)$  to  $(x, t)$  will be called a *dual active path* from  $(x, t)$  to  $(y, t-s)$  in the dual process.

We could have defined the dual by simply choosing a random configuration

at time  $t$  according to  $\nu_C$  and then running the whole process backwards. The idea of the preceding construction is to allow coupling the process and its dual in the same graphical representation in such a way that the initial state of the environment for  $\eta_s$  is the same as the final state of the environment for  $\hat{\eta}_s^t$  (that is,  $B_0 = \hat{B}_t^t$ ). This allows us to obtain the following duality result:

**Proposition 2.2.2.** *For any  $A, C, D \subseteq \mathbb{Z}^d$ ,*

$$\mathbb{P}^{\nu_A} \left( A_t \cap C \neq \emptyset, B_t \cap D \neq \emptyset \right) = \mathbb{P}^{\nu_C} \left( \hat{A}_t^t \cap A \neq \emptyset, \hat{B}_0^t \cap D \neq \emptyset \right). \quad (2.2)$$

Moreover,  $\eta_t$  satisfies the following self-duality relation: if  $A$  or  $C$  is finite, then

$$\mathbb{P}^{\nu_A} (A_t \cap C \neq \emptyset, B_t \cap D \neq \emptyset) = \mathbb{P}^{\nu_C} (A_t \cap A \neq \emptyset, B_0 \cap D \neq \emptyset). \quad (2.3)$$

*Proof.* The first equality follows directly from coupling the process and its dual using the same realization of the graphical representation. Indeed, if we use this coupling then, by definition,

$$\mathbb{P} \left( \hat{B}_s^{\nu_C, t} = B_{t-s}^{\nu_A} \text{ for every } 0 \leq s \leq t \right) = 1.$$

Calling  $\mathcal{E}$  the  $\sigma$ -algebra generated by the environment process, observe that our construction implies that

$$\mathbb{P}^{\nu_A} (A_t \cap C \neq \emptyset | \mathcal{E}) = \mathbb{P}^{\nu_C} (\hat{A}_t^t \cap A \neq \emptyset | \mathcal{E}).$$

Therefore,

$$\begin{aligned} \mathbb{P}^{\nu_A} \left( A_t \cap C \neq \emptyset, B_t \cap D \neq \emptyset \right) &= \mathbb{E}^{\nu_A} \left( \mathbb{P} (A_t \cap C \neq \emptyset | \mathcal{E}), B_t \cap D \neq \emptyset \right) \\ &= \mathbb{E}^{\nu_C} \left( \mathbb{P} (\hat{A}_t^t \cap A \neq \emptyset | \mathcal{E}), \hat{B}_0^t \cap D \neq \emptyset \right) \\ &= \mathbb{P}^{\nu_C} \left( \hat{A}_t^t \cap A \neq \emptyset, \hat{B}_0^t \cap D \neq \emptyset \right). \end{aligned}$$

(2.3) is obtained from (2.2), the self-duality of the contact process, and the reversibility of the environment.  $\square$

Taking  $A$  finite and  $C = D = \mathbb{Z}^d$  in (2.3) and using the monotonicity of the event  $\{A_t \neq \emptyset\}$  in  $t$  we obtain the following:

$$\mathbb{P}^{\nu_A}(A_t \neq \emptyset \forall t \geq 0) = \bar{\nu}(\{(E, F): E \cap A \neq \emptyset\}).$$

Since  $\bar{\nu}$  is translation invariant, the right side of this equality is positive if and only if  $A \neq \emptyset$  and  $\eta_t$  survives, that is,  $\bar{\nu} \neq \underline{\nu}$ . As a consequence we deduce that the following condition is equivalent to the survival of the process:

For any (or, equivalently, some) finite  $A \subseteq \mathbb{Z}^d$  with  $A \neq \emptyset$ , the process started at  $\nu_A$  contains 1's for every  $t \geq 0$  with positive probability. (S1)

### 2.2.3 Positive correlations

A second property that is central to the study of the contact process is *positive correlations*. Recall that a probability measure  $\mu$  has positive correlations if for every  $f, g$  increasing,

$$\int fg \, d\mu \geq \int f \, d\mu \int g \, d\mu. \tag{2.4}$$

In the following lemma we prove a version of positive correlations for  $\eta_t^{\nu_A}$  with respect to cylinder functions.

**Lemma 2.2.3.** *Let  $f, g$  be increasing real-valued functions on  $\mathcal{X}$  depending on finitely many coordinates. Then if  $\mu_t$  denotes the distribution of  $\eta_t^{\nu_A}$ , (2.4) holds with  $\mu = \mu_t$ , that is,*

$$\mathbb{E}^{\nu_A}(f(\eta_t)g(\eta_t)) \geq \mathbb{E}^{\nu_A}(f(\eta_t)) \mathbb{E}^{\nu_A}(g(\eta_t)).$$

*The same inequality holds if  $\nu_A$  is replaced by any deterministic initial condition.*

*Proof.* Since  $f$  and  $g$  depend on finitely many coordinates and every jump in our process is between states which are comparable in the partial order (2.1), a result of Harris (see Theorem II.2.14 in Liggett (1985)) and attractiveness imply that it is enough to show that the initial distribution of the process has positive correlations in the sense of the lemma. The result with  $\nu_A$  replaced by a deterministic initial condition readily follows.

To show that  $\nu_A$  is positively correlated, consider the process  $\varsigma_t$  defined in  $\mathcal{X}$  by  $\varsigma_0 \equiv 1$  and independent transitions at each site given by

$$\left. \begin{array}{l} 0 \longrightarrow -1 \text{ at rate } \rho \\ -1 \longrightarrow 0 \text{ at rate } 1 - \rho \end{array} \right\} \text{for } x \notin A,$$

$$\left. \begin{array}{l} 1 \longrightarrow -1 \text{ at rate } \rho \\ -1 \longrightarrow 1 \text{ at rate } 1 - \rho \end{array} \right\} \text{for } x \in A.$$

It is clear that  $\varsigma_t$  converges weakly to the measure  $\nu_A$ . Since the initial distribution of  $\varsigma_t$  has positive correlations (because it is deterministic), (2.4) holds for its limit  $\nu_A$ , using again Harris' result.  $\square$

### 2.3 Survival and extinction

In this section we prove Theorem 2.1.1. Throughout the proof we will implicitly use (S1) to characterize survival. We start with the easy part.

*Proof of Theorem 2.1.1, part (a).* Consider the process  $\tilde{\eta}_t$  defined by the following transition rates:

$$\begin{array}{llll}
0, -1 & \longrightarrow & 1 & \text{at rate } \beta f_1 \\
1 & \longrightarrow & 0 & \text{at rate } 1 \\
0, 1 & \longrightarrow & -1 & \text{at rate } \alpha \\
-1 & \longrightarrow & 0 & \text{at rate } \alpha\delta
\end{array}$$

This process corresponds to modifying  $\eta_t$  by ignoring the effect of blocked sites on births. It is easy to couple  $\tilde{\eta}_t$  and  $\eta_t$  using the graphical representation in such a way that if the initial states are the same,  $\eta_t \leq \tilde{\eta}_t$  for all  $t \geq 0$ . Therefore, it is enough to show that  $\tilde{\eta}_t$  dies out, and this follows directly from the hypothesis because the 1's in  $\tilde{\eta}_t$  behave just like a contact process with birth rate  $\beta$  and death rate  $\alpha + 1$ .  $\square$

The proof of part (b) is more involved, and it is based on adapting the techniques of Boolean models in continuum percolation (see Meester and Roy (1996)).

*Proof of Theorem 2.1.1, part (b).* The idea is to show that when  $\delta$  is small, the set of unblocked sites in the environment process  $B_t$  does not “space-time percolate” with probability 1. By this we mean that there is no infinite path in  $\mathbb{Z}^d \times [0, \infty)$  moving between nearest-neighbor sites in  $\mathbb{Z}^d$  and along time lines in the increasing direction of time that uses only non-blocked sites. The conclusion follows directly from this fact, since in that case every 1 will live in a finite space-time box, so it will not be able to contribute to the survival of the process.

By a simple time change, we can consider the environment process as having transitions given by

$$\begin{array}{lll}
-1 & \longrightarrow & 0 & \text{at rate } q \\
0 & \longrightarrow & -1 & \text{at rate } 1 - q,
\end{array}$$

where  $q = \delta/(1 + \delta) \longrightarrow 0$  as  $\delta \longrightarrow 0$ . We still consider this process as defined by the graphical representation, though now the symbols  $\bullet_{-1}$  and  $*_{-1}$  appear at rate  $1 - q$  and  $q$  respectively.

Take the percolation structure given by the graphical representation and draw for every symbol  $*_{-1}$  at a space-time point  $(x, t)$  a box of base  $x + [-2/3, 2/3]^d$  spanning the interval in the time coordinate from  $t$  until the time corresponding to the next symbol  $\bullet_{-1}$  (i.e., these boxes span intervals where the sites are not blocked). Then, since the environment process is translation invariant, the 0's will almost surely not space-time percolate if and only if

$$\mathbb{P}(|\mathcal{W}| = \infty) = 0, \tag{2.5}$$

where  $\mathcal{W}$  denotes the connected component of the union of the boxes that contains the origin at time 0, and  $|\mathcal{W}|$  denotes the number of boxes that form this cluster.

To prove (2.5) we compare this continuum percolation structure with a multitype branching process  $X = (X_{n,i})_{n,i \in \mathbb{N}}$ . The first step in the comparison is to stretch all the boxes so that their heights are all integer-valued. It is enough to show that (2.5) holds after this modification, since increasing the heights of the boxes increases the probability of space-time percolation of the unblocked sites. Assume that the origin is not blocked at time 0, and call  $i_0 \in \mathbb{N}$  the (random) height of its associated box. For simplicity, assume further that all the neighbors of the origin are blocked at time 0, the extension to the general case being straightforward. We start defining  $X$  by saying that the 0-th generation has only

one member, and it is of type  $i_0$  (that is,  $X_{0,j} = \mathbf{1}_{\{j=i_0\}}$ ). The box containing the origin at time 0 is possibly intersected by boxes placed at the  $2d$  neighbors of the origin, and these boxes will constitute the children of the initial member: we let  $X_{1,j}$  be the number of boxes of height  $j$  that intersect the original box. We define the subsequent generations of  $X$  inductively:  $X_{n+1,j}$  is the number of boxes of height  $j$  that intersect boxes of the  $n$ -th generation and which have not been counted up to generation  $n - 1$ . Now let

$$X^\infty = \sum_{n=0}^{\infty} \sum_{i=1}^{\infty} X_{n,i},$$

and observe that every box in  $\mathcal{W}$  is counted in  $X^\infty$ , so

$$|\mathcal{W}| \leq X^\infty \tag{2.6}$$

(recall that  $X$  is constructed from the stretched boxes).

Our goal is to show that  $\mathbb{E}(X^\infty) < \infty$ . To achieve this we will couple  $X$  with another multitype branching process  $Y = (Y_{n,i})_{n,i \in \mathbb{N}}$ , which we define below. The details of this part can be adapted easily from the proof of Theorem 3.2 in Meester and Roy (1996), so we will only sketch the main ideas. Consider a box of height  $i$  based at  $[x - 2/3, x + 2/3]^d \times \{t\}$ , which we will denote by  $B(x, t, i)$ . The boxes of height  $j$  that intersect this box must all have bases of the form  $[y - 2/3, y + 2/3] \times \{s\}$  for some  $y$  at distance 1 of  $x$  and some  $s \in (0 \vee (t - j), t + i]$ . The number of symbols  $*_{-1}$  appearing in the piece  $\{y\} \times (0 \vee (t - j), t + i]$  of the graphical representation above a given neighbor  $y$  of  $x$  is a Poisson random variable with mean  $q[t + i - 0 \vee (t - j)] \leq q[i + j]$ , and each of these symbols corresponds to a box that intersects  $B(x, t, i)$ . Since the probability that any one of these (stretched) boxes is of height  $j$  is  $p_j = \mathbb{P}(Z \in (j - 1, j])$ , where  $Z$  is an exponential random variable with rate  $1 - q$ , we deduce that the number of children of  $B(x, t, i)$  of height  $j$  is a Poisson random variable with mean bounded

by

$$2dp_jq[i + j] \leq 4dqijp_j, \quad (2.7)$$

where we used the fact that  $i + j \leq 2ij$  for positive integers  $i$  and  $j$ . Now let  $Y$  be a multitype branching process where the number of children of type  $j$  of each individual of type  $i$  is a Poisson random variable with mean  $4dqijp_j$  ( $Y_{n,i}$  is the number of individuals of type  $i$  in generation  $n$ ). Then a coupling argument and (2.7) imply that if  $X_{0,i} = Y_{0,i}$  for all  $i \geq 1$  then  $X_{n,i}$  is dominated by  $Y_{n,i}$  for each  $n \geq 0$  and  $i \geq 1$ , and thus

$$\mathbb{E}\left(X^\infty \mid X_{0,k} = \mathbf{1}_{\{k=i_0\}}\right) \leq \mathbb{E}\left(\sum_{n=0}^{\infty} \sum_{j=1}^{\infty} Y_{n,j} \mid Y_{0,k} = \mathbf{1}_{\{k=i_0\}}\right). \quad (2.8)$$

To bound this last sum we recall a standard result in branching processes theory (see, for example, Chapter V in Athreya and Ney (1972)): the expected number of individuals of type  $j$  in the  $n$ -th generation of  $Y$  when starting with one individual of type  $i_0$  is given by

$$\mathbb{E}\left(Y_{n,j} \mid Y_{0,k} = \mathbf{1}_{\{k=i_0\}}\right) = (M^n)_{i_0,j}, \quad (2.9)$$

where  $M$  is the infinite matrix indexed by  $\mathbb{N}$  with  $M_{i,j}$  being the expected number of children of type  $j$  of an individual of type  $i$ . By definition of  $Y$ ,  $M_{i,j} = 4dqijp_j$ , and from this we get inductively a bound for  $(M^n)_{i_0,j}$ :

$$(M^n)_{i_0,j} \leq (4dq)^n i_0 \mathbb{E}(H^2)^{n-1} \mathbb{P}(H = j)j$$

for all  $n \geq 1$ , where  $H$  is a random variable with positive integer values and distribution given by  $\mathbb{P}(H = j) = p_j$ . Using this together with (2.8) and (2.9) gives

$$\begin{aligned} \mathbb{E}\left(X^\infty \mid X_{0,k} = \mathbf{1}_{\{k=i_0\}}\right) &\leq 1 + i_0 \sum_{n=1}^{\infty} \left( (4dq)^n \mathbb{E}(H^2)^{n-1} \sum_{j=1}^{\infty} p_j j \right) \\ &= 1 + 4dq i_0 \mathbb{E}(H) \sum_{n=0}^{\infty} (4dq \mathbb{E}(H^2))^n. \end{aligned} \quad (2.10)$$

Observe that  $H$  is dominated by  $Z + 1$ , so  $\mathbb{E}(H^2) \leq \frac{2(2-q)}{(1-q)^2} + 1$ . Hence,

$$4dq\mathbb{E}(H^2) \leq 4d \left( \frac{2q(2-q)}{(1-q)^2} + q \right) < 1 \quad (2.11)$$

for sufficiently small  $q$ , and then the last sum in (2.10) converges for such  $q$ . This implies by (2.6) that  $\mathbb{E}(|\mathcal{W}|) < \infty$ , so  $\mathbb{P}(|\mathcal{W}| = \infty) = 0$ .  $\square$

Using (2.11) we can get explicit lower bounds for  $\delta_p$ , but these turn out to be rather small (around 0.02 for  $d = 2$  and 0.01 for  $d = 3$ ).

Before proving the last part of Theorem 2.1.1 we need to introduce a result from Broman (2007). Let  $(J_t, X_t)$  be the process with state space  $\{0, 1\} \times \mathbb{N}$  defined as follows.  $J_0$  is a Bernoulli random variable with  $\mathbb{P}(J_0 = 1) = 1 - \mathbb{P}(J_0 = 0) = p$ , and  $X_0 = 0$ . The evolution of the process is given by the following transition rates:

$$\begin{aligned} \text{for } J_t: & \begin{cases} 0 \longrightarrow 1 & \text{at rate } \gamma p \\ 1 \longrightarrow 0 & \text{at rate } \gamma(1-p) \end{cases} \\ \text{for } X_t: & k \longrightarrow k+1 \quad \text{at rate } \sigma_0(1-J_t) + \sigma_1 J_t \end{aligned}$$

where  $\gamma, \sigma_1 > 0$  and  $0 \leq \sigma_0 \leq \sigma_1$ . In words,  $J_t$  acts as the environment, starting at equilibrium and then flipping between states 0 and 1 independently of  $X_t$ , while  $X_t$  is a sort of Poisson process where the rate depends on  $J_t$ . The next lemma recovers the part of Theorem 1.4 in Broman (2007) that is relevant for our purposes. We observe that the original theorem is stated for  $\sigma_0 > 0$ , but the same proof works if  $\sigma_0 = 0$ .

**Lemma 2.3.1.** *Let*

$$\bar{\sigma} = \frac{1}{2} \left[ \sigma_0 + \sigma_1 + \gamma - \sqrt{(\sigma_1 - \sigma_0 - \gamma)^2 + 4\gamma(1-p)(\sigma_1 - \sigma_0)} \right]. \quad (2.12)$$

Then a Poisson process  $N_t(\bar{\sigma})$  with rate  $\bar{\sigma}$  can be coupled with  $(J_t, X_t)$  in such a way that if  $N_t(\bar{\sigma})$  has an arrival at time  $T$ , then so does  $X_t$ . Moreover,  $\bar{\sigma}$  is the largest rate such that this coupling is possible.

Recall that we denote

$$\bar{\lambda}(\alpha, \beta, \delta) = \frac{1}{2} \left[ \beta + \alpha(1 + \delta) - \sqrt{(\beta - \alpha(1 + \delta))^2 + 4\alpha\beta} \right].$$

The following result gives the coupling that we need to prove part (c) of Theorem 2.1.1. Its proof is very similar to that of Theorem 1.7 in Broman (2007), we include here a version based in the graphical representation.

**Proposition 2.3.2.** *Let  $\xi_t$  denote the set of occupied sites of a contact process with birth rate  $\bar{\lambda} = \bar{\lambda}(\alpha, \beta, \delta)$  and death rate  $\alpha + 1$ . Then the processes  $\eta_t$  and  $\xi_t$  can be coupled in such a way that if  $\xi_0 \subseteq A_0$ , then  $\xi_t \subseteq A_t$  for all  $t > 0$ .*

*Proof.* Consider the graphical representation used to construct  $\eta_t$ . Each time line defines an independent copy of the process  $J_t$  introduced above by identifying symbols  $\bullet_{-1}$  and  $*_{-1}$  with  $J_t$  flipping to 0 and 1 respectively, and setting  $\gamma = \alpha(1 + \delta)$  and  $p = \delta/(1 + \delta)$ . Now consider the collection of arrows emanating from that time line ignoring arrows born at times where the site is blocked. By construction, this collection of arrows defines the arrival times of the process  $X_t$  associated to  $J_t$ , with  $\sigma_0 = 0$  and  $\sigma_1 = \beta$ . By Lemma 2.3.1, we can construct a Poisson process  $N_t(\bar{\lambda})$  (where  $\bar{\lambda}$  comes from plugging in our parameters in (2.12)) such that if this process has an arrival at time  $T$ , then there is an arrow at that time for  $\eta_t$ .

We repeat this construction at each time line, getting an i.i.d. collection of Poisson processes  $(N_t^x(\bar{\lambda}))_{x \in \mathbb{Z}^d}$ , and use this collection of processes and the

graphical representation of  $\eta_t$  to construct the graphical representation of  $\xi_t$ : for each arrival time of  $N_t^x(\bar{\lambda})$  put an arrow at that time from  $x$  to the site pointed by the corresponding arrow in the graphical representation of  $\eta_t$ , and for each symbol  $*_1$  and each symbol  $\bullet_{-1}$  for  $\eta_t$  put a death symbol for  $\xi_t$ . It is easy to see that this construction gives a graphical representation for the desired contact process  $\xi_t$ . Moreover, since only the arrows at non-blocked sites can carry births of 1's for  $\eta_t$ , the construction gives a coupling that satisfies the desired monotonicity property. These facts can be checked exactly as in the proof of Theorem 1.7 of Broman (2007) (there the processes  $Y_t$  and  $Y'_t$  correspond to  $A_t$  and  $\xi_t$ ).  $\square$

The proof of the remaining part of Theorem 2.1.1 is now straightforward.

*Proof of Theorem 2.1.1, part (c).* Since  $\frac{\bar{\lambda}(\alpha, \beta, \delta)}{\alpha + 1} > \lambda_c$  implies that the contact process  $\xi_t$  with birth rate  $\bar{\lambda}(\alpha, \beta, \delta)$  and death rate  $\alpha + 1$  survives, the coupling achieved in Proposition 2.3.2 gives the survival of  $\eta_t$ .  $\square$

## 2.4 Block construction

The aim of this section is to establish “block conditions” concerning the process in a finite space-time box that guarantee survival. This was first done in Bezuidenhout and Grimmett (1990). Here we will follow closely Section I.2 of Liggett (1999), together with the corrections to the book that can be found in the author’s website.

Before getting started with the block construction we need to obtain the equivalent condition for survival mentioned in the Introduction, which says

that  $\eta_t$  survives if and only if the following condition holds:

The process started with a single 1 at the origin and everything else at  $-1$  contains 1's at all times with positive probability. (S2)

The sufficiency of this condition is a consequence of (S1) and attractiveness. The necessity will be a consequence of the following stronger result, which is precisely what we will need in the proof of Lemma 2.4.2 below. Let  $\chi_A$  denote the probability measure on  $\mathcal{X}$  that assigns mass 1 to the configuration  $\eta$  with  $\eta|_A \equiv 1, \eta|_{A^c} \equiv -1$ .

**Lemma 2.4.1.** *Suppose that the process survives. Then for any  $\sigma > 0$  there is a positive integer  $n$  such that*

$$\mathbb{P}^{\chi_{[-n,n]^d}}(A_t \neq \emptyset \forall t \geq 0) > 1 - \sigma^2.$$

To obtain (S2) from this result observe that the process started with a single 1 at the origin has  $[-n, n]^d$  fully occupied by time 1 with some positive probability, so we can use the strong Markov property and attractiveness to restart the process at time 1 starting from  $\chi_{[-n,n]^d}$  and obtain  $\mathbb{P}^{\chi_{\{0\}}}(A_t \neq \emptyset \forall t \geq 0) > 0$ . Observe that the lemma is a simple consequence of duality when the initial condition for  $\eta_t$  is  $\nu_{[-n,n]^d}$  instead of  $\chi_{[-n,n]^d}$ . Indeed, using (2.3) with  $D = \mathbb{Z}^d$  gives

$$\begin{aligned} & \lim_{n \rightarrow \infty} \mathbb{P}^{\nu_{[-n,n]^d}}(A_t \neq \emptyset \forall t \geq 0) \\ &= \lim_{n \rightarrow \infty} \lim_{t \rightarrow \infty} \mathbb{P}^{\nu_{[-n,n]^d}}(A_t \neq \emptyset) = \lim_{n \rightarrow \infty} \lim_{t \rightarrow \infty} \mathbb{P}^{\nu_{\mathbb{Z}^d}}(A_t \cap [-n, n]^d \neq \emptyset) \\ &= \lim_{n \rightarrow \infty} \bar{\nu}(\{(E, F) : E \cap [-n, n]^d \neq \emptyset\}) = \bar{\nu}(\{(E, F) : E \neq \emptyset\}). \end{aligned}$$

This last probability is 1 when  $\eta_t$  survives, so in this case given any  $\varepsilon > 0$  we can choose a positive integer  $m$  such that

$$\mathbb{P}^{\nu_{[-m,m]^d}}(A_t \neq \emptyset \forall t \geq 0) > 1 - \varepsilon. \tag{2.13}$$

Recall that in Proposition 2.2.1 we showed that the limit distributions of the processes started at  $\chi_{\mathbb{Z}^d}$  and at  $\nu_{\mathbb{Z}^d}$  are the same. It is then reasonable to expect that the asymptotic behavior as  $t \rightarrow \infty$  of the process started at  $\chi_{[-n,n]^d}$  is similar to that of the process started at  $\nu_{[-n,n]^d}$ , at least for large enough  $n$ . This idea will allow us to derive the lemma from (2.13).

*Proof of Lemma 2.4.1.* Let  $\varepsilon > 0$  and choose  $m$  to be the positive integer obtained in (2.13). To extend this inequality to the process started at  $\chi_{[-n,n]^d}$  we will consider two copies of the process  $\eta_t^1$  and  $\eta_t^2$  coupled using the graphical representation, with  $\eta_t^1$  started at  $\nu_{[-m,m]^d}$  and  $\eta_t^2$  at  $\chi_{[-n,n]^d}$  for some large  $n > m$ . For simplicity we will write  $Q(k) = [-k, k]^d$ .

We want to obtain a space-time cone growing linearly in time such that  $\cup_{t \geq 0} \{t\} \times A_t^{\nu_{Q(m)}}$  is contained in that cone with high probability. To achieve this we compare  $A_t^{\nu_{Q(m)}}$  with a branching random walk  $Z_t$  with branching rate  $\beta/(2d)$  and no deaths (that is, each particle in  $Z_t$  gives birth to a new particle at each neighbor at rate  $\beta/(2d)$ , and multiple particles per site are allowed). Let  $\{p_t(x, y)\}_{x, y \in \mathbb{Z}^d}$  be the transition probabilities of a simple random walk in  $\mathbb{Z}^d$  that moves to each neighbor at rate  $\beta/(2d)$  and let  $C_t$  be the set-valued process given by

$$C_t = \{x \in \mathbb{Z}^d : Z_t(x) > 0\}.$$

For  $D \subseteq \mathbb{Z}^d$ ,  $Z_t^D$  and  $C_t^D$  will denote the processes started with all sites in  $D$  occupied by one particle and no particles outside  $D$ . It is not hard to see that for any  $t > 0$  and any  $x \in \mathbb{Z}^d$ ,

$$\mathbb{E}\left(Z_t^{\{0\}}(x)\right) = e^{\beta t} p_t(0, x)$$

(see, for instance, the proof of Proposition I.1.21 in Liggett (1999)). Therefore,

for any  $D \subseteq \mathbb{Z}^d$ ,

$$\mathbb{E}\left(\left|C_t^{\{0\}} \cap D^c\right|\right) \leq \sum_{x \notin D} \mathbb{E}\left(Z_t^{\{0\}}(x)\right) = e^{\beta t} \sum_{x \notin D} p_t(0, x).$$

From this we get that if  $k > m$  and  $c > 0$  then

$$\mathbb{E}\left(\left|C_t^{Q(m)} \cap Q(k+ct)^c\right|\right) \leq (2m+1)^d e^{\beta t} \sum_{\|x\|_\infty > k-m+ct} p_t(0, x). \quad (2.14)$$

Now if  $X_t$  is the one dimensional random walk starting at 0 and moving to each neighbor at rate  $\beta/(2d)$ , Chebyshev's inequality gives

$$\begin{aligned} \mathbb{P}(|X_t| > k - m + ct) &= 2\mathbb{P}(X_t - k + m - ct > 0) \\ &\leq 2\mathbb{E}(e^{X_t - k + m - ct}) = 2e^{-(k-m)} e^{-ct + \frac{\beta}{2d}(e+e^{-1}-2)t}. \end{aligned}$$

The last equality can be obtained by seeing  $X_t$  as the difference between two independent Poisson random variables, each with mean  $(\beta t)/(2d)$ , and using the fact that the moment generating function of a Poisson random variable  $Y$  with mean  $\lambda$  is  $\mathbb{E}(e^{sY}) = e^{\lambda(e^s-1)}$ . Applying this bound to each coordinate of the  $d$ -dimensional walk we get that

$$\sum_{\|x\|_\infty > k-m+ct} p_t(0, x) \leq d \mathbb{P}(|X_t| > k - m + ct) \leq 2de^{-(k-m)} e^{-ct + \frac{\beta}{2d}(e+e^{-1}-2)t},$$

and then using (2.14) we deduce that  $c$  can be taken large enough so that

$$\mathbb{E}\left(\left|C_t^{Q(m)} \cap Q(k+ct)^c\right|\right) \leq 2d(2m+1)^d e^{-(k-m)} e^{-t}.$$

Observe that, by the definition of  $Z_t$ , the process  $A_t^{\nu_{Q(m)}}$  is dominated by  $C_t^{Q(m)}$ , so the last bound implies that

$$\begin{aligned} \mathbb{E}\left(\int_0^\infty \left|A_t^{\nu_{Q(m)}} \cap Q(k+ct)^c\right| dt\right) \\ \leq \int_0^\infty \mathbb{E}\left(\left|C_t^{Q(m)} \cap Q(k+ct)^c\right|\right) dt \leq 2d(2m+1)^d e^{-(k-m)}. \quad (2.15) \end{aligned}$$

We can use this inequality to estimate the probability that  $A_t \subseteq Q(k+1+ct)$  for all  $t \geq 0$ . Observe that if  $x \in A_t \cap Q(k+1+ct)^c$ , the particle at  $x$  survives at least until time  $t+2/c$  with probability  $e^{-2\alpha(1+\delta)/c}$ , and thus  $x \in A_s \cap Q(k+cs)^c$  for all  $s \in [t+1/c, t+2/c]$  with at least that probability. We deduce that

$$\begin{aligned} \mathbb{E}^{\nu_{Q(m)}} \left( \int_0^\infty |A_t \cap Q(k+ct)|^c dt \right) \\ \geq \mathbb{P}^{\nu_{Q(m)}}(A_t \cap Q(k+1+ct)^c \neq \emptyset \text{ for some } t \geq 0) e^{-2\alpha(1+\delta)/c} \frac{1}{c}. \end{aligned}$$

Therefore, if we let

$$G_1 = \{A_t^1 \subseteq Q(k+1+ct) \forall t \geq 0\},$$

(where  $A_t^1$  denotes the set of 1's in the process  $\eta_t^1$  started at  $\nu_{Q(m)}$ ), we can use this bound together with (2.15) to get

$$\mathbb{P}(G_1^c) \leq 2cd(2m+1)^d e^{2\alpha(1+\delta)/c} e^{-(k-m)}.$$

Choosing now  $k$  large enough yields

$$\mathbb{P}(G_1) > 1 - \varepsilon.$$

Now take  $n > k$ ,  $T > 0$ , let  $(t-T)^+ = (t-T) \vee 0$ , and call  $G_2$  the event that on the space-time region  $\cup_{t \geq 0} \{t\} \times Q(n+c(t-T)^+)$  the environment for  $\eta_t^2$  dominates the environment for  $\eta_t^1$  (with respect to the order (2.1)):

$$G_2 = \{B_t^2 \subseteq B_t^1 \text{ on } Q(n+c(t-T)^+) \forall t \geq 0\}.$$

We want this space-time region to contain the region defining  $G_1$ , so we let  $T = (n-k-1)/c$ .

Observe that, since we are coupling the processes using the canonical coupling given by the graphical representation, once the environment is equal for

both process at a given site, it stays equal at that site from that time on. In particular,  $B_t^2$  dominates  $B_t^1$  on  $Q(n)$  for all  $t \geq 0$ . For any other site, any symbol  $\bullet_{-1}$  or  $*_{-1}$  leaves the environment equal for both process. Therefore,

$$\begin{aligned} \mathbb{P}(G_2^c) &\leq \sum_{x \notin Q(n)} \mathbb{P}\left(\text{no } \bullet_{-1} \text{ or } *_{-1} \text{ at } x \text{ by time } T + (\|x\|_\infty - n)/c\right) \\ &= \sum_{j>n} |Q(j) \setminus Q(j-1)| e^{-\alpha(1+\delta)(T+(j-n)/c)} \\ &\leq e^{\alpha(1+\delta)(k+1)/c} \sum_{j>n} (2j+1)^d e^{-\alpha(1+\delta)j/c}. \end{aligned}$$

By taking  $n$  large enough we obtain

$$\mathbb{P}(G_2) > 1 - \varepsilon. \quad (2.16)$$

Finally, let

$$G_3 = \{A_t^1 \neq \emptyset \forall t \geq 0\}.$$

By (2.13),  $\mathbb{P}(G_3) > 1 - \varepsilon$ . Observe that on the event  $G_1 \cap G_2 \cap G_3$ ,  $\eta_t^2$  contains 1's at all times with probability 1. Therefore

$$\begin{aligned} \mathbb{P}^{\chi_{[-n,n]^d}}(A_t \neq \emptyset \forall t \geq 0) &\geq \mathbb{P}(G_1 \cap G_2 \cap G_3) \\ &\geq 1 - \mathbb{P}(G_1^c) - \mathbb{P}(G_2^c) - \mathbb{P}(G_3^c) \\ &> 1 - 3\varepsilon, \end{aligned}$$

and choosing  $\varepsilon$  small enough we get the result.  $\square$

In the following lemma we combine and extend for our process the results in Liggett (1999) that lead to the block conditions. Consider the process  ${}_L\eta_t$ , for  $L > 0$ , where no births are allowed outside of  $(-L, L)^d$ . Define  $N_+(L, T)$  to be the maximal number of space-time points in

$$S_+(L, T) = \{(x, s) \in (\{L\} \times [0, L)^{d-1} \times [0, T]: x \in {}_L A_s\}$$

such that each pair of these points having the same spatial coordinate have their time coordinates at distance at least 1.

**Lemma 2.4.2.** *Suppose that the process survives. Then for any  $\sigma > 0$  there is a positive integer  $n$  satisfying the following: for any given pair of positive integers  $N$  and  $M$ , there are choices of a positive integer  $L$  and a positive real number  $T$  such that*

$$\mathbb{P}^{\mathcal{X}_{[-n,n]^d}}(|_L A_T \cap [0, L]^d| > N) \geq 1 - \sigma^{2^{-d}} \quad (2.17a)$$

and

$$\mathbb{P}^{\mathcal{X}_{[-n,n]^d}}(N_+(L, T) > M) \geq 1 - \sigma^{2^{-d}/d}. \quad (2.17b)$$

*Proof.* By Lemma 2.4.1 we can choose a large enough integer  $n$  such that

$$\mathbb{P}^{\mathcal{X}_{[-n,n]^d}}(A_t \neq \emptyset \forall t \geq 0) > 1 - \sigma^2. \quad (2.18)$$

Having this, the proof of the lemma is a simple adaptation of the corresponding proofs for the contact process. To avoid repetition of published results, we will explain the main ideas involved and why the original proofs still work with the random environment, but refer the reader to Section I.2 of Liggett (1999) for the details.

We claim the following: for any finite  $A \subseteq \mathbb{Z}^d$  and any  $N \geq 1$ ,

$$\lim_{t \rightarrow \infty} \lim_{L \rightarrow \infty} \mathbb{P}^{\mathcal{X}^A}(|_L A_t| \geq N) = \mathbb{P}^{\mathcal{X}^A}(A_t \neq \emptyset \forall t \geq 0). \quad (2.19)$$

To see that this is true, we observe that

$$\lim_{L \rightarrow \infty} \mathbb{P}^{\mathcal{X}^A}(|_L A_t| \geq N) = \mathbb{P}^{\mathcal{X}^A}(|A_t| \geq N)$$

and then argue that, conditioned on survival,  $|A_t| \rightarrow \infty$  as  $t \rightarrow \infty$  with probability 1. This follows from the easy fact that there is an  $\varepsilon_N > 0$  such that if

$|A| \leq N$  then the process started with 1's at  $A$  becomes extinct with probability at least  $\varepsilon_N$ , so

$$\mathbb{P}^{X^A}(0 < |A_t| \leq N) \varepsilon_N \leq \mathbb{P}^{X^A}(t < \tau < \infty) \xrightarrow[t \rightarrow \infty]{} 0.$$

The next step is to use positive correlations to localize estimates on the cardinality of  ${}_L A_t$  to a specific orthant of  $\mathbb{Z}^d$ : for every  $N \geq 1$  and  $L \geq n$ ,

$$\mathbb{P}^{X_{[-n, n]^d}}(|{}_L A_t \cap [0, L]^d| \leq N) \leq [\mathbb{P}^{X_{[-n, n]^d}}(|{}_L A_t| \leq 2^d N)]^{2^{-d}}. \quad (2.20)$$

This relation follows easily from the positive correlations result in Lemma 2.2.3, and its proof is identical the proof of Proposition I.2.6 in Liggett (1999).

Observe that (2.18), (2.19), and (2.20) together suffice to obtain (2.17a). The preceding arguments can be modified to obtain similar estimates for  $N_+(L, T)$ , which in turn give (2.17b). The only detail remaining is getting the same  $L$  and  $T$  to work for both inequalities. This is done by obtaining sequences  $L_j \nearrow \infty$  and  $T_j \nearrow \infty$  such that (2.17a) holds with  $L = L_j$  and  $T = T_j$  for every  $j \geq 1$ , and then adapting the arguments above to show that (2.17b) must hold for some pair  $(L_j, T_j)$ . We refer the reader to the proof of Theorem I.2.12 in Liggett's book for the details on how this is achieved, and remark that the argument depends only on properties such as positive correlations and the Feller property which are available both for  $\eta_t$  and the contact process.  $\square$

We state now the block conditions that are equivalent to the survival of the process.

**Theorem 2.4.3.** *The process survives if and only if for any given  $\varepsilon > 0$  there are positive integers  $n$  and  $L$  and a positive real number  $T$  such that the following conditions (BC)*

are satisfied:

$$\mathbb{P}^{\chi_{[-n,n]^d}}(L+2nA_{T+1} \supseteq x + [-n, n]^d \text{ for some } x \in [0, L]^d) > 1 - \varepsilon \quad (\text{BC1})$$

and

$$\mathbb{P}^{\chi_{[-n,n]^d}}(L+2nA_{t+1} \supseteq x + [-n, n]^d \text{ for some } 0 \leq t \leq T \\ \text{and some } x \in \{L+n\} \times [0, L]^{d-1}) > 1 - \varepsilon. \quad (\text{BC2})$$

Observe that these conditions correspond exactly to the conditions in Theorem I.2.12 of Liggett (1999). This will allow us to borrow the arguments from Liggett's book to prove that (BC) implies survival for  $\eta_t$ . The reason why we need the conditions (BC) starting  $\eta_t$  from  $\chi_{[-n,n]^d}$  is because the proof of their sufficiency for survival (as well as their use in the proof of Theorem 2.1.2) demands obtaining repeatedly cubes fully occupied by 1's and, at each step, restarting the process at the lowest possible configuration having those cubes fully occupied.

*Proof of Theorem 2.4.3.* The proof uses the exact same arguments as those in the proofs of Theorems I.2.12 and I.2.23 in Liggett (1999). As before, we will only make some remarks and refer the reader to Liggett's book for the details.

The necessity of (BC) follows from Lemma 2.4.2, by choosing the quantities  $N$  and  $M$  to be large enough to produce the desired boxes filled with 1's.

For the sufficiency of (BC), attractiveness and (S2) imply that it is enough to show that for some  $n > 0$  the process started at  $\chi_{[-n,n]^d}$  contains 1's at all times (by using, as above, the fact that for any given  $n > 0$  the process started at  $\chi_{\{0\}}$  has  $[-n, n]^d$  fully occupied by time 1 with some positive probability). The proof of this fact relies on starting with a large enough cube fully occupied by

1's and then moving its center in an appropriate way. This is used to compare the process with supercritical oriented site percolation, and conclude that such boxes exist for all times with positive probability.  $\square$

The following consequence of Theorem 2.4.3 is obtained in the same way as for the contact process, see Theorem I.2.25 in Liggett (1999) for the details.

**Corollary 2.4.4.** *If  $\beta = \beta_c(\alpha, \delta)$  or  $\delta = \delta_c(\alpha, \beta)$ , then the process dies out.*

## 2.5 Complete convergence

We are ready now to use the block construction of Section 2.4 to prove Theorem 2.1.2. The key step in the proof will be to obtain the result in the special case where the initial distribution  $\mu$  is a probability measure of the form  $\nu_A$ , in which case we can use duality.

**Proposition 2.5.1.** *For every  $A \subseteq \mathbb{Z}^d$ ,*

$$\eta_t^{\nu_A} \Longrightarrow \mathbb{P}^{\nu_A}(\tau < \infty) \underline{\nu} + \mathbb{P}^{\nu_A}(\tau = \infty) \bar{\nu}.$$

To prove the proposition we need a preliminary lemma. Both the proof of the proposition and this lemma are inspired by the proof Theorem 2 in Durrett and Møller (1991).

We will denote by  $\mathbb{P}^{\nu_A, \nu_C}$  the probability measure associated to starting the process at  $\nu_A$  and its dual at  $\nu_C$ , using the same realization of the graphical representation, as explained in Section 2.2.2.

**Lemma 2.5.2.** *For every finite  $C \subseteq \mathbb{Z}^d$  and every  $\varepsilon > 0$ , if  $r$  is a positive real number and  $s$  is large enough, then*

$$\left| \mathbb{P}^{\nu_A, \nu_C} \left( \tau > \frac{s}{2}, \hat{A}_r^{r+s} \neq \emptyset, \hat{B}_0^{r+s} \cap D \neq \emptyset \right) - \mathbb{P}^{\nu_A} \left( \tau > \frac{s}{2} \right) \mathbb{P}^{\nu_C} \left( \hat{A}_r^r \neq \emptyset, \hat{B}_0^r \cap D \neq \emptyset \right) \right| < \varepsilon.$$

Observe that for the (ordinary) contact process, the forward process and the dual are independent when they run on nonoverlapping time intervals, so this fact is trivial and holds with  $s/2$  replaced by  $s$ .

*Proof of Lemma 2.5.2.* Given  $r$  and  $\varepsilon$ , there is a  $q = q(|C|)$  such that every dual active path in  $(\hat{\eta}_u^{\nu_C, r})_{0 \leq u \leq r}$  stays inside  $C + [-q, q]^d$  with probability at least  $1 - \varepsilon$ . To see this, observe that the number of particles in all such dual active paths is dominated by  $X_r$ , where  $(X_r)_{r \geq 0}$  is a branching process starting with  $|C|$  particles and with birth rate  $\beta$  and death rate 0 (we are ignoring deaths and coalescence of paths). By Markov's inequality,  $\mathbb{P}(X_r > q) \leq \mathbb{E}(X_r)/q \leq \varepsilon$  for large enough  $q$ . Since any dual active path in  $\hat{\eta}_t^{\nu_C, r}$  starts inside  $C$ ,  $X_r \leq q$  implies that all dual active paths are contained inside  $C + [-q, q]^d$  up to time  $r$ .

Now denote by  $\eta_t^{(\mu_\rho, s/2)}$  and  $\hat{\eta}_t^{(\mu_\rho, s/2), r}$  modifications of the process and its dual, constructed on the same graphical representation as the original ones, where the environment is reset at time  $s/2$  to its equilibrium  $\mu_\rho$ , independently of its state before  $s/2$  (that is, at time  $s/2$  we replace every  $-1$  by a  $0$  and then flip every site to  $-1$  with probability  $\rho$ , regardless of it being at state  $0$  or  $1$ ). Then for given  $r$  and  $q$ , if  $s$  is large enough we have that

$$\begin{aligned} \mathbb{P}^{\nu_A, \nu_C} (B_u = B_u^{(\mu_\rho, s/2)} \text{ on } C + [-q, q]^d \quad \forall u \in [s, s+r]) \\ \geq (1 - e^{-\alpha(1+\delta)s/2})^{|C+[-q, q]^d|} > 1 - \varepsilon. \end{aligned} \quad (2.21)$$

Indeed, for any given  $x \in C + [-q, q]^d$  the probability that  $B_u$  and  $B_u^{(\mu_\rho, s/2)}$  are equal at  $x$  for every  $u \in [s, s+r]$  is bounded below by the probability that an exponential random variable with parameter  $\alpha(1+\delta)$  is smaller than  $s/2$  (because any symbol  $\bullet_{-1}$  or  $*_{-1}$  above  $(x, s/2)$  leaves the environment at that site equal for both processes from that time on).

The property discussed at the first paragraph of the proof together with (2.21) imply that

$$\left| \mathbb{P}^{\nu_A, \nu_C} \left( \tau > \frac{s}{2}, \hat{A}_r^{r+s} \neq \emptyset, \hat{B}_0^{r+s} \cap D \neq \emptyset \right) - \mathbb{P}^{\nu_A, \nu_C} \left( \tau > \frac{s}{2}, \hat{A}_r^{(\mu_\rho, s/2), r+s} \neq \emptyset, \hat{B}_0^{(\mu_\rho, s/2), r+s} \cap D \neq \emptyset \right) \right| < 2\varepsilon.$$

The statement of the lemma follows now from the independence of disjoint parts of the graphical representation and the stationarity of  $B_t$ , since

$$\begin{aligned} & \mathbb{P}^{\nu_A, \nu_C} \left( \tau > \frac{s}{2}, \hat{A}_r^{(\mu_\rho, s/2), r+s} \neq \emptyset, \hat{B}_0^{(\mu_\rho, s/2), r+s} \cap D \neq \emptyset \right) \\ &= \mathbb{P}^{\nu_A} \left( \tau > \frac{s}{2} \right) \mathbb{P}^{\nu_C} \left( \hat{A}_r^{(\mu_\rho, s/2), r+s} \neq \emptyset, \hat{B}_0^{(\mu_\rho, s/2), r+s} \cap D \neq \emptyset \right) \\ &= \mathbb{P}^{\nu_A} \left( \tau > \frac{s}{2} \right) \mathbb{P}^{\nu_C} \left( \hat{A}_r^r \neq \emptyset, \hat{B}_0^r \cap D \neq \emptyset \right). \quad \square \end{aligned}$$

*Proof of Proposition 2.5.1.* The result is straightforward in the subcritical case. If the process survives, and since weak convergence in this setting corresponds to the convergence of the finite-dimensional distributions, it is enough to prove that the following three properties hold for any two finite subsets  $C, D$  of  $\mathbb{Z}^d$ :

$$\mathbb{P}^{\nu_A}(A_t \cap C \neq \emptyset) \xrightarrow[t \rightarrow \infty]{} \mathbb{P}^{\nu_A}(\tau = \infty) \bar{\nu}(\{(E, F): E \cap C \neq \emptyset\}), \quad (\text{c1})$$

$$\begin{aligned} \mathbb{P}^{\nu_A}(B_t \cap D \neq \emptyset) &= \mathbb{P}^{\nu_A}(\tau < \infty) \underline{\nu}(\{(E, F): F \cap D \neq \emptyset\}) \\ &\quad + \mathbb{P}^{\nu_A}(\tau = \infty) \bar{\nu}(\{(E, F): F \cap D \neq \emptyset\}), \end{aligned} \quad (\text{c2})$$

and

$$\begin{aligned} & \mathbb{P}^{\nu^A}(A_t \cap C \neq \emptyset, B_t \cap D \neq \emptyset) \\ & \xrightarrow[t \rightarrow \infty]{} \mathbb{P}^{\nu^A}(\tau = \infty) \bar{\nu}(\{(E, F): E \cap C \neq \emptyset, F \cap D \neq \emptyset\}). \end{aligned} \tag{c3}$$

Indeed, all the finite-dimensional distributions of the process are determined by these probabilities via the inclusion-exclusion formula. Observe that the right side of (c2) is equal to  $\mu_\rho(\{\eta: \eta(x) = -1 \text{ for some } x \in D\})$ .

(c1) follows from the same arguments used in Liggett (1999) for the contact process. Using duality (Proposition 2.2.2), the proof of Theorem I.1.12 in that book applies in the same way to obtain the fact that (c1) holds if and only if for every  $x \in \mathbb{Z}^d$  and every  $A \subseteq \mathbb{Z}^d$ ,

$$\mathbb{P}^{\nu^A}(\tau = \infty) = \mathbb{P}^{\nu^A}(x \in A_t \text{ i.o.}) \tag{2.22a}$$

and

$$\lim_{n \rightarrow \infty} \liminf_{t \rightarrow \infty} \mathbb{P}^{\nu_{[-n, n]^d}}(A_t \cap [-n, n]^d \neq \emptyset) = 1. \tag{2.22b}$$

The analogous conditions are checked for the contact process in the proof of Theorem I.2.27 in Liggett (1999). (2.22a) follows from the same proof after some minor modifications, so we will skip the argument. For (2.22b), Theorem 2.4.3 allows us to use Liggett's arguments to get the desired limit when  $\nu_{[-n, n]^d}$  is replaced by  $\chi_{[-n, n]^d}$ , so given any  $\varepsilon > 0$  we can choose a large enough integer  $m$  such that

$$\liminf_{t \rightarrow \infty} \mathbb{P}^{\chi_{[-m, m]^d}}(A_t \cap [-m, m]^d \neq \emptyset) > 1 - \varepsilon. \tag{2.23}$$

Given this  $m$ , we can choose a large enough  $n$  so that the process started at  $\nu_{[-n, n]^d}$  contains at time 0 a fully occupied cube of side  $2m + 1$  (contained in  $[-n, n]^d$ ) with probability at least  $1 - \varepsilon$  (in fact, any translate of  $[-m, m]^d$  contained in  $[-n, n]^d$  is fully occupied by 1's with some probability  $p > 0$ , so we

only need to choose  $n$  so that  $[-n, n]^d$  contains enough disjoint translates of  $[-m, m]^d$ . On this event we can restart the process by putting every site outside that cube at state  $-1$  and use attractiveness, translation invariance, and (2.23) to get

$$\liminf_{t \rightarrow \infty} \mathbb{P}^{\nu_{[-n, n]^d}}(A_t \cap [-n, n]^d \neq \emptyset) > (1 - \varepsilon)^2,$$

whence (2.22b) follows. There is only one detail to consider: in his book, Liggett only proves the condition analogous to (2.22b) in the case  $d \geq 2$ , because it is simpler and the case  $d = 1$  was already done in Liggett (1985). The difficulty in the one-dimensional case arises from the fact that certain block events are not independent. This can be overcome by comparing with  $k$ -dependent oriented site percolation instead of ordinary oriented site percolation (see Theorem B26 in Liggett (1999)). We refer the reader to Section 5 of Durrett and Schonmann (1987), where the authors use a similar block construction to derive a complete convergence theorem for a general class of one-dimensional growth models.

(c2) is trivial due to the stationarity of the environment process. To prove (c3) we start by observing that

$$\begin{aligned} \mathbb{P}^{\nu^A}(A_{r+s} \cap C \neq \emptyset, B_{r+s} \cap D \neq \emptyset) \\ = \mathbb{P}^{\nu^A, \nu^C}(A_s \cap \hat{A}_r^{r+s} \neq \emptyset, \hat{B}_0^{r+s} \cap D \neq \emptyset), \end{aligned} \quad (2.24)$$

which follows from constructing  $(\eta_u^{\nu^A})_{0 \leq u \leq r+s}$  and  $(\hat{\eta}_u^{\nu^C, r+s})_{0 \leq u \leq r+s}$  on the same

copy of the graphical representation. On the other hand,

$$\begin{aligned}
& \left| \mathbb{P}^{\nu_A, \nu_C} \left( A_s \cap \hat{A}_r^{r+s} \neq \emptyset, \hat{B}_0^{r+s} \cap D \neq \emptyset \right) \right. \\
& \quad \left. - \mathbb{P}^{\nu_A, \nu_C} \left( A_s \neq \emptyset, \hat{A}_r^{r+s} \neq \emptyset, \hat{B}_0^{r+s} \cap D \neq \emptyset \right) \right| \\
&= \mathbb{P}^{\nu_A, \nu_C} \left( A_s \neq \emptyset, \hat{A}_r^{r+s} \neq \emptyset, A_s \cap \hat{A}_r^{r+s} = \emptyset, \hat{B}_0^{r+s} \cap D \neq \emptyset \right) \quad (2.25) \\
&\leq \mathbb{P}^{\nu_A, \nu_C} \left( A_s \neq \emptyset, \hat{A}_r^{r+s} \neq \emptyset, A_s \cap \hat{A}_r^{r+s} = \emptyset \right) \\
&= \mathbb{P}^{\nu_A, \nu_C} \left( A_s \neq \emptyset, \hat{A}_r^{r+s} \neq \emptyset \right) - \mathbb{P}^{\nu_A, \nu_C} \left( A_s \cap \hat{A}_r^{r+s} \neq \emptyset \right).
\end{aligned}$$

Observe that

$$\mathbb{P}^{\nu_A} (s/2 < \tau < \infty) \xrightarrow{s \rightarrow \infty} 0.$$

Thus, for any given  $D \subseteq \mathbb{Z}^d$  and  $\varepsilon > 0$ , and for large enough  $s$ , we can write

$$\begin{aligned}
& \left| \mathbb{P}^{\nu_A, \nu_C} \left( A_s \neq \emptyset, \hat{A}_r^{r+s} \neq \emptyset, \hat{B}_0^{r+s} \cap D \neq \emptyset \right) \right. \\
& \quad \left. - \mathbb{P}^{\nu_A, \nu_C} \left( \tau > s/2, \hat{A}_r^{r+s} \neq \emptyset, \hat{B}_0^{r+s} \cap D \neq \emptyset \right) \right| \quad (2.26) \\
&= \mathbb{P}^{\nu_A, \nu_C} \left( s/2 < \tau \leq s, \hat{A}_r^{r+s} \neq \emptyset, \hat{B}_0^{r+s} \cap D \neq \emptyset \right) \\
&\leq \mathbb{P}^{\nu_A} (s/2 < \tau < \infty) < \frac{\varepsilon}{3}.
\end{aligned}$$

Putting the previous observations together we get, for large enough  $s$

$$\begin{aligned}
& \left| \mathbb{P}^{\nu_A} \left( A_{r+s} \cap C \neq \emptyset, B_{r+s} \cap D \neq \emptyset \right) \right. \\
& \quad \left. - \mathbb{P}^{\nu_A, \nu_C} \left( A_s \neq \emptyset, \hat{A}_r^{r+s} \neq \emptyset, \hat{B}_0^{r+s} \cap D \neq \emptyset \right) \right| \\
&= \left| \mathbb{P}^{\nu_A, \nu_C} \left( A_s \cap \hat{A}_r^{r+s} \neq \emptyset, \hat{B}_0^{r+s} \cap D \neq \emptyset \right) \right. \\
& \quad \left. - \mathbb{P}^{\nu_A, \nu_C} \left( A_s \neq \emptyset, \hat{A}_r^{r+s} \neq \emptyset, \hat{B}_0^{r+s} \cap D \neq \emptyset \right) \right| \quad \text{by (2.24)} \\
&\leq \mathbb{P}^{\nu_A, \nu_C} \left( A_s \neq \emptyset, \hat{A}_r^{r+s} \neq \emptyset \right) - \mathbb{P}^{\nu_A, \nu_C} \left( A_s \cap \hat{A}_r^{r+s} \neq \emptyset \right) \quad \text{by (2.25)} \\
&\leq \frac{\varepsilon}{3} + \left| \mathbb{P}^{\nu_A, \nu_C} \left( \tau > s/2, \hat{A}_r^{r+s} \neq \emptyset \right) - \mathbb{P}^{\nu_A, \nu_C} \left( A_s \cap \hat{A}_r^{r+s} \neq \emptyset \right) \right| \quad \text{by (2.26),}
\end{aligned}$$

where we used  $D = \mathbb{Z}^d$  and the fact that  $\mathbb{P}^{\nu_A, \nu_C} (\hat{B}_0^{r+s} \neq \emptyset) = 1$  in the application of (2.26). Using again this fact to apply Lemma 2.5.2 with  $D = \mathbb{Z}^d$ , and then

using duality we get

$$\begin{aligned}
& \left| \mathbb{P}^{\nu_A, \nu_C} \left( \tau > s/2, \hat{A}_r^{r+s} \neq \emptyset \right) - \mathbb{P}^{\nu_A, \nu_C} \left( A_s \cap \hat{A}_r^{r+s} \neq \emptyset \right) \right| \\
& \leq \frac{\varepsilon}{3} + \left| \mathbb{P}^{\nu_A} (\tau > s/2) \mathbb{P}^{\nu_C} \left( \hat{A}_r^r \neq \emptyset \right) - \mathbb{P}^{\nu_A, \nu_C} \left( A_s \cap \hat{A}_r^{r+s} \neq \emptyset \right) \right| \\
& = \frac{\varepsilon}{3} + \left| \mathbb{P}^{\nu_A} (\tau > s/2) \mathbb{P}^{\nu_{\mathbb{Z}^d}} (A_r \cap C \neq \emptyset) - \mathbb{P}^{\nu_A} (A_{r+s} \cap C \neq \emptyset) \right|
\end{aligned}$$

for large enough  $s$ . By (c1), the last difference converges to 0 as  $r, s \rightarrow \infty$ , so we finally get

$$\begin{aligned}
& \left| \mathbb{P}^{\nu_A} \left( A_{r+s} \cap C \neq \emptyset, B_{r+s} \cap D \neq \emptyset \right) \right. \\
& \quad \left. - \mathbb{P}^{\nu_A, \nu_C} \left( A_s \neq \emptyset, \hat{A}_r^{r+s} \neq \emptyset, \hat{B}_0^{r+s} \cap D \neq \emptyset \right) \right| < \varepsilon
\end{aligned}$$

for large enough  $r, s$ .

This calculation implies that in order to prove (c3) it is enough to show that

$$\begin{aligned}
& \mathbb{P}^{\nu_A, \nu_C} \left( A_s \neq \emptyset, \hat{A}_r^{r+s} \neq \emptyset, \hat{B}_0^{r+s} \cap D \neq \emptyset \right) \\
& \quad \xrightarrow[r, s \rightarrow \infty]{} \mathbb{P}^{\nu_A} (\tau = \infty) \bar{\nu} \left( (E, F) : E \cap C \neq \emptyset, F \cap D \neq \emptyset \right).
\end{aligned}$$

Repeating the previous application of (2.26) and Lemma 2.5.2 we get that, for large enough  $s$ ,

$$\begin{aligned}
& \left| \mathbb{P}^{\nu_A, \nu_C} \left( A_s \neq \emptyset, \hat{A}_r^{r+s} \neq \emptyset, \hat{B}_0^{r+s} \cap D \neq \emptyset \right) \right. \\
& \quad \left. - \mathbb{P}^{\nu_A} (\tau > s/2) \mathbb{P}^{\nu_C} \left( \hat{A}_r^r \neq \emptyset, \hat{B}_0^r \cap D \neq \emptyset \right) \right| \\
& \leq \frac{\varepsilon}{2} + \left| \mathbb{P}^{\nu_A, \nu_C} \left( \tau > s/2, \hat{A}_r^{r+s} \neq \emptyset, \hat{B}_0^{r+s} \cap D \neq \emptyset \right) \right. \\
& \quad \left. - \mathbb{P}^{\nu_A} (\tau > s/2) \mathbb{P}^{\nu_C} \left( \hat{A}_r^r \neq \emptyset, \hat{B}_0^r \cap D \neq \emptyset \right) \right| \\
& \leq \varepsilon.
\end{aligned}$$

Therefore, we can finally reduce to proving that

$$\begin{aligned}
& \mathbb{P}^{\nu_A} (\tau > s/2) \mathbb{P}^{\nu_C} \left( \hat{A}_r^r \neq \emptyset, \hat{B}_0^r \cap D \neq \emptyset \right) \\
& \quad \xrightarrow[r, s \rightarrow \infty]{} \mathbb{P}^{\nu_A} (\tau = \infty) \bar{\nu} \left( (E, F) : E \cap C \neq \emptyset, F \cap D \neq \emptyset \right).
\end{aligned}$$

This follows easily from duality, since (2.2) yields

$$\begin{aligned} \mathbb{P}^{\nu^A}(\tau > s/2) \mathbb{P}^{\nu^C}(\hat{A}_r^r \neq \emptyset, \hat{B}_0^r \cap D \neq \emptyset) \\ = \mathbb{P}^{\nu^A}(\tau > s/2) \mathbb{P}^{\nu^{\mathbb{Z}^d}}(A_r \cap C \neq \emptyset, B_r \cap D \neq \emptyset), \end{aligned}$$

and this last term converges to the desired limit as  $r, s \rightarrow \infty$ .  $\square$

We extend now Proposition 2.5.1 to the general case.

*Proof of Theorem 2.1.2.* It is enough to show that

$$\lim_{t \rightarrow \infty} \mathbb{E}^\mu(f(\eta_t)) = \mathbb{P}^\mu(\tau < \infty) \int f d\nu + \mathbb{P}^\mu(\tau = \infty) \int f d\bar{\nu} \quad (2.27)$$

for every  $f$  in the space of continuous increasing functions depending on finitely many coordinates of  $\mathcal{X}$ , which we will denote by  $\mathcal{F}$ . To see this, observe that given any two finite subsets  $C, D$  of  $\mathbb{Z}^d$ , the functions

$$f_1(E, F) = \mathbf{1}_{E \cap C \neq \emptyset}, \quad f_2(E, F) = \mathbf{1}_{F \cap D = \emptyset}, \quad \text{and} \quad f_3(E, F) = \mathbf{1}_{E \cap C \neq \emptyset, F \cap D = \emptyset}$$

are all in  $\mathcal{F}$  and (as in the proof of Proposition 2.5.1) all the finite-dimensional distributions of the process can be obtained from  $\mathbb{E}^\mu(f_1(\eta_t))$ ,  $\mathbb{E}^\mu(f_2(\eta_t))$ , and  $\mathbb{E}^\mu(f_3(\eta_t))$  by the inclusion-exclusion formula.

Let  $f$  be a function in  $\mathcal{F}$  and observe that, in particular,  $f$  is bounded. One inequality in (2.27) is easy: by the Markov property and attractiveness, given  $0 < s < t$  we have that

$$\begin{aligned} \mathbb{E}^\mu(f(\eta_t)) &= \mathbb{E}^\mu(f(\eta_t), \tau < s) + \mathbb{E}^\mu(f(\eta_t), \tau \geq s) \\ &= \mathbb{E}^\mu(\mathbb{E}^{\eta^s}(f(\eta_{t-s}), \tau < s) + \mathbb{E}^\mu(\mathbb{E}^{\eta^s}(f(\eta_{t-s})), \tau \geq s)) \\ &\leq \mathbb{E}(f(\eta_{t-s}^0)) \mathbb{P}^\mu(\tau < s) + \mathbb{E}^{\chi^{\mathbb{Z}^d}}(f(\eta_{t-s})) \mathbb{P}^\mu(\tau \geq s), \end{aligned}$$

where  $\eta_t^0$  denotes the process started at the configuration  $\eta \equiv 0$ . Since  $\eta_t^0 \implies \mu_\rho = \underline{\nu}$  and  $\eta_t^{\chi_{\mathbb{Z}^d}} \implies \bar{\nu}$ , we get

$$\limsup_{t \rightarrow \infty} \mathbb{E}^\mu(f(\eta_t)) \leq \mathbb{P}^\mu(\tau < s) \int f d\underline{\nu} + \mathbb{P}^\mu(\tau \geq s) \int f d\bar{\nu},$$

and now taking  $s \rightarrow \infty$  we deduce that

$$\limsup_{t \rightarrow \infty} \mathbb{E}^\mu(f(\eta_t)) \leq \mathbb{P}^\mu(\tau < \infty) \int f d\underline{\nu} + \mathbb{P}^\mu(\tau = \infty) \int f d\bar{\nu}. \quad (2.28)$$

To obtain the other inequality in (2.27) we will begin by considering the case  $\mu = \chi_{[-n,n]^d}$  and showing that, given any  $\varepsilon > 0$  and any  $x \in \mathbb{Z}^d$ ,

$$\liminf_{t \rightarrow \infty} \mathbb{E}^{\chi_{x+[-n,n]^d}}(f(\eta_t), \tau = \infty) \geq \int f d\bar{\nu} - \varepsilon \quad (2.29)$$

for large enough  $n$ . By the translation invariance of  $\eta_t$  and  $\bar{\nu}$ , it is enough to consider the case  $x = 0$ . To show (2.29) we will use the construction introduced in the proof of Lemma 2.4.1. Using the notation of that proof, recall that we showed that, given any  $\gamma > 0$ , there are positive integers  $n > k > m$  such that

$$\mathbb{P}(G_1 \cap G_2 \cap G_3) > 1 - 3\gamma.$$

This means that the processes  $\eta_t^1$  (started at  $\nu_{[-m,m]^d}$ ) and  $\eta_t^2$  (started at  $\chi_{[-n,n]^d}$ ) can be coupled in such a way that, with probability at least  $1 - 3\gamma$ , for all  $t \geq 0$  we have that  $A_t^1 \neq \emptyset$ ,  $A_t^2 \neq \emptyset$ ,  $A_t^1 \subseteq Q(k+1+ct)$ , and  $B_t^2 \subseteq B_t^1$  inside  $Q(k+1+ct)$ .

Let  $G = G_1 \cap G_2 \cap G_3$  and  $\gamma > 0$  and choose  $n > k > m$  so that  $\mathbb{P}(G) > 1 - 3\gamma$ . We will denote by  $\tau^1$  and  $\tau^2$  the extinction times of the processes  $\eta_t^1$  and  $\eta_t^2$ , respectively. Define

$$\check{\eta}_t = (A_t^1, B_t^2)$$

and observe that, on the event  $G$ ,  $\check{\eta}_t$  defines an  $\mathcal{X}$ -valued process and, moreover,  $\eta_t^2 \geq \check{\eta}_t$  for all  $t \geq 0$ . Therefore, since  $f$  is increasing and  $\{\tau^2 = \infty\} \supseteq G$ ,

$$\mathbb{E}(f(\eta_t^2), \tau^2 = \infty) \geq \mathbb{E}(f(\eta_t^2), G) \geq \mathbb{E}(f(\check{\eta}_t), G) \quad (2.30)$$

for all  $t \geq 0$ . Now observe that, trivially,

$$\mathbb{E}(f(\check{\eta}_t), G) = \mathbb{E}(f(\check{\eta}_t), \tau^2 = \infty) - \mathbb{E}(f(\check{\eta}_t), \tau^2 = \infty, G^c),$$

and

$$\mathbb{E}(f(\check{\eta}_t), \tau^2 = \infty, G^c) \leq \|f\|_\infty \mathbb{P}(G^c) < 3\gamma \|f\|_\infty,$$

so

$$\mathbb{E}(f(\check{\eta}_t), G) > \mathbb{E}(f(\check{\eta}_t), \tau^2 = \infty) - 3\gamma \|f\|_\infty \quad (2.31)$$

for all  $t \geq 0$ . On the other hand,

$$|\mathbb{E}(f(\check{\eta}_t), \tau^2 = \infty) - \mathbb{E}(f(\eta_t^1), \tau^2 = \infty)| \xrightarrow[t \rightarrow \infty]{} 0. \quad (2.32)$$

To see this, observe that since  $f$  depends on finitely many coordinates, then given any  $q > 0$ ,  $f(\check{\eta}_s) = f(\eta_s^1)$  for all  $s \geq t$  with probability at least  $1 - q$  if  $t$  is large enough. Indeed, if  $K \subseteq \mathbb{Z}^d$  is the finite set of coordinates of  $\mathcal{X}$  on which  $f$  depends, then repeating the calculations that led to (2.16) we get that

$$\begin{aligned} & \mathbb{P}\left(B_s^1(x) \neq B_s^2(x) \text{ for some } x \in K \text{ and some } s \geq t\right) \\ & \leq \sum_{x \in K} \mathbb{P}(\text{no } \bullet_{-1} \text{ or } *_{-1} \text{ at } x \text{ by time } t) = |K| e^{-\alpha(1+\delta)t} \xrightarrow[t \rightarrow \infty]{} 0. \end{aligned}$$

Therefore, given any  $q > 0$ ,

$$|\mathbb{E}(f(\check{\eta}_t), \tau^2 = \infty) - \mathbb{E}(f(\eta_t^1), \tau^2 = \infty)| \leq \mathbb{E}(|f(\check{\eta}_t) - f(\eta_t^1)|) \leq 2q \|f\|_\infty$$

for large enough  $t$ , and we get (2.32). Finally, we have that

$$\begin{aligned} \mathbb{E}(f(\eta_t^1), \tau^2 = \infty) &= \mathbb{E}(f(\eta_t^1), \tau^1 = \infty) - (\mathbb{E}(f(\eta_t^1), \tau^1 = \infty) - \mathbb{E}(f(\eta_t^1), G)) \\ &\quad - (\mathbb{E}(f(\eta_t^1), G) - \mathbb{E}(f(\eta_t^1), \tau^2 = \infty)), \end{aligned}$$

and since  $G \subseteq \{\tau^1 = \infty\} \cap \{\tau^2 = \infty\}$ ,

$$|\mathbb{E}(f(\eta_t^1), \tau^i = \infty) - \mathbb{E}(f(\eta_t^1), G)| \leq \|f\|_\infty \mathbb{P}(G^c) < 3\gamma \|f\|_\infty$$

for  $i = 1, 2$ . Thus Proposition 2.5.1 implies that

$$\liminf_{t \rightarrow \infty} \mathbb{E}(f(\eta_t^1), \tau^2 = \infty) > \mathbb{P}(\tau^1 = \infty) \int f d\bar{\nu} - 6\gamma \|f\|_\infty,$$

and since  $\mathbb{P}(\tau^1 = \infty) \geq \mathbb{P}(G) > 1 - 3\gamma$  we obtain

$$\liminf_{t \rightarrow \infty} \mathbb{E}(f(\eta_t^1), \tau^2 = \infty) > \int f d\bar{\nu} - 9\gamma \|f\|_\infty. \quad (2.33)$$

Putting (2.30), (2.31), (2.32), and (2.33) together we deduce that

$$\liminf_{t \rightarrow \infty} \mathbb{E}(f(\eta_t^2), \tau^2 = \infty) \geq \int f d\bar{\nu} - 12\gamma \|f\|_\infty,$$

and choosing  $\gamma$  appropriately we obtain (2.29).

Getting back to the proof of the remaining inequality in (2.27), let  $\varepsilon > 0$  and choose  $n \in \mathbb{N}$  so that (2.29) holds. Define

$$N = \inf \{k \in \mathbb{N} : \eta_k \supseteq x + [-n, n]^d \text{ for some } x \in \mathbb{Z}^d\}$$

and let  $p = \mathbb{P}^{x\{0\}}(A_1 \supseteq x + [-n, n]^d \text{ for some } x \in \mathbb{Z}^d) > 0$ . Observe that for any  $k \geq 0$ , if  $A_k \neq \emptyset$  then  $A_{k+1}$  contains some translate of  $[-n, n]^d$  with probability at least  $p$  (by attractiveness and translation invariance) and therefore, since the Poisson processes used in the graphical representation for disjoint time intervals are independent, we deduce that

$$\{\tau = \infty\} \subseteq \{N < \infty\}. \quad (2.34)$$

When  $N < \infty$  we will denote by  $X$  the center of the corresponding fully occupied box. If there is more than one point  $x$  such that  $x + [-n, n]^d$  is fully occupied by 1's at time  $N$ , we pick  $X$  to be the one minimizing  $\phi(x)$ , where  $\phi$  is any fixed bijection between  $\mathbb{Z}^d$  and  $\mathbb{N}$  (this ensures that the events  $\{X = x\}$  are disjoint for different  $x$ ). Then given  $m \in \mathbb{N}$ , the Markov property and attractiveness imply

that

$$\begin{aligned}
\mathbb{E}^\mu(f(\eta_t), \tau = \infty) &\geq \sum_{k=0}^m \mathbb{E}^\mu(f(\eta_t), \tau = \infty, N = k) \\
&= \sum_{k=0}^m \mathbb{E}^\mu(\mathbb{E}^{\eta_k}(f(\eta_{t-k}), \tau = \infty), N = k) \\
&\geq \sum_{k=0}^m \sum_{x \in \mathbb{Z}^d} \mathbb{E}^\mu(\mathbb{E}^{X_{x+[-n, n]^d}}(f(\eta_{t-k}), \tau = \infty), N = k, X = x)
\end{aligned}$$

for  $t \geq m$ . Since  $f$  is bounded, (2.29) implies that

$$\begin{aligned}
\liminf_{t \rightarrow \infty} \mathbb{E}^\mu(f(\eta_t), \tau = \infty) &\geq \left( \int f d\bar{\nu} - \varepsilon \right) \sum_{k=0}^m \sum_{x \in \mathbb{Z}^d} \mathbb{P}^\mu(N = k, X = x) \\
&= \left( \int f d\bar{\nu} - \varepsilon \right) \mathbb{P}^\mu(N \leq m).
\end{aligned}$$

Taking now  $m \rightarrow \infty$  we get by (2.34) that

$$\liminf_{t \rightarrow \infty} \mathbb{E}^\mu(f(\eta_t), \tau = \infty) \geq \left( \int f d\bar{\nu} - \varepsilon \right) \mathbb{P}^\mu(N < \infty) \geq \left( \int f d\bar{\nu} - \varepsilon \right) \mathbb{P}^\mu(\tau = \infty)$$

if  $\varepsilon < \int f d\bar{\nu}$ , and taking  $\varepsilon \rightarrow 0$  we deduce that

$$\liminf_{t \rightarrow \infty} \mathbb{E}^\mu(f(\eta_t), \tau = \infty) \geq \mathbb{P}^\mu(\tau = \infty) \int f d\bar{\nu}.$$

On the other hand, by arguments similar to those that led to (2.28) (using attractiveness to compare with the process started at  $\chi_\emptyset$ ) we get

$$\liminf_{t \rightarrow \infty} \mathbb{E}^\mu(f(\eta_t), \tau < \infty) \geq \mathbb{P}^\mu(\tau < \infty) \int f d\underline{\nu}.$$

We finally deduce that

$$\liminf_{t \rightarrow \infty} \mathbb{E}^\mu(f(\eta_t)) \geq \mathbb{P}^\mu(\tau < \infty) \int f d\bar{\nu} + \mathbb{P}^\mu(\tau = \infty) \int f d\underline{\nu},$$

and the proof is ready. □

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## REFERENCES

- Andjel, E. D. (1992). Survival of multidimensional contact process in random environments. *Bol. Soc. Brasil. Mat. (N.S.)*, **23**(1-2), 109–119.
- Athreya, K. B. and Ney, P. E. (1972). *Branching processes*. Springer-Verlag, New York. Die Grundlehren der mathematischen Wissenschaften, Band 196.
- Bezuidenhout, C. and Grimmett, G. (1990). The critical contact process dies out. *Ann. Probab.*, **18**(4), 1462–1482.
- Bramson, M., Durrett, R., and Schonmann, R. H. (1991). The contact process in a random environment. *Ann. Probab.*, **19**(3), 960–983.
- Broman, E. I. (2007). Stochastic domination for a hidden Markov chain with applications to the contact process in a randomly evolving environment. *Ann. Prob.*, **35**(6), 2263–2293.
- Durrett, R. and Møller, A. M. (1991). Complete convergence theorem for a competition model. *Probab. Theory Related Fields*, **88**(1), 121–136.
- Durrett, R. and Schinazi, R. (1993). Asymptotic critical value for a competition model. *Ann. Appl. Probab.*, **3**(4), 1047–1066.
- Durrett, R. and Schonmann, R. H. (1987). Stochastic growth models. In *Percolation theory and ergodic theory of infinite particle systems (Minneapolis, Minn., 1984–1985)*, volume 8 of *IMA Vol. Math. Appl.*, pages 85–119. Springer, New York.
- Durrett, R. and Swindle, G. (1991). Are there bushes in a forest? *Stochastic Process. Appl.*, **37**(1), 19–31.

- Harris, T. E. (1972). Nearest-neighbor Markov interaction processes on multidimensional lattices. *Advances in Math.*, **9**, 66–89.
- Harris, T. E. (1974). Contact interactions on a lattice. *Ann. Probab.*, **2**, 969–988.
- Klein, A. (1994). Extinction of contact and percolation processes in a random environment. *Ann. Probab.*, **22**(3), 1227–1251.
- Liggett, T. M. (1985). *Interacting particle systems*, volume 276 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, New York.
- Liggett, T. M. (1992). The survival of one-dimensional contact processes in random environments. *Ann. Probab.*, **20**(2), 696–723.
- Liggett, T. M. (1999). *Stochastic interacting systems: contact, voter and exclusion processes*, volume 324 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin.
- Luo, X. (1992). The Richardson model in a random environment. *Stochastic Process. Appl.*, **42**(2), 283–289.
- Meester, R. and Roy, R. (1996). *Continuum percolation*, volume 119 of *Cambridge Tracts in Mathematics*. Cambridge University Press, Cambridge.

CHAPTER 3  
LIMIT THEOREMS FOR INDIVIDUAL-BASED MODELS IN  
ECONOMICS AND FINANCE\*

There is a widespread recent interest in using ideas from statistical physics to model certain types of problems in economics and finance. The main idea is to derive the macroscopic behavior of the market from the random local interactions between agents. Our purpose is to present a general framework that encompasses a broad range of models, by proving a law of large numbers and a central limit theorem for certain interacting particle systems with very general state spaces. To do this we draw inspiration from some work done in mathematical ecology and mathematical physics. The first result is proved for the system seen as a measure-valued process, while to prove the second one we will need to introduce a chain of embeddings of some abstract Banach and Hilbert spaces of test functions and prove that the fluctuations converge to the solution of a certain generalized Gaussian stochastic differential equation taking values in the dual of one of these spaces.

### 3.1 Introduction

We consider interacting particle systems of the following form. There is a fixed number  $N$  of particles, each one having a type  $w \in W$ . The particles change their types via two mechanisms. The first one corresponds simply to transitions from one type to another at some given rate. The second one involves a

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direct interaction between particles: pairs of particles interact at a certain rate and acquire new types according to some given (random) rule. We will allow these rates to depend directly on the types of the particles involved and on the distribution of the whole population on the type space.

Our purpose is to prove limit theorems, as the number of particles  $N$  goes to infinity, for the empirical random measures  $\nu_t^N$  associated to these systems.  $\nu_t^N$  is defined as follows: if  $\eta_t^N(i) \in W$  denotes the type of the  $i$ -th particle at time  $t$ , then

$$\nu_t^N = \frac{1}{N} \sum_{i=1}^N \delta_{\eta_t^N(i)},$$

where  $\delta_w$  is the probability measure on  $W$  assigning mass 1 to  $w$ .

Our first result, Theorem 3.3.1, provides a law of large numbers for  $\nu_t^N$  on a finite time interval  $[0, T]$ : the empirical measures converge in distribution to a deterministic continuous path  $\nu_t$  in the space of probability measures on  $W$ , whose evolution is described by a certain system of integro-differential equations. Theorem 3.4.1 analyzes the fluctuations of the finite system  $\nu_t^N$  around  $\nu_t$ , and provides an appropriate central limit result: the fluctuations are of order  $1/\sqrt{N}$ , and the asymptotic behavior of the process  $\sqrt{N}(\nu_t^N - \nu_t)$  has a Gaussian nature. This second result is, as could be expected, much more delicate than the first one.

In recent years there has been an increasing interest in the use of interacting particle systems to model phenomena outside their original application to statistical physics, with special attention given to models in ecology, economics, and finance. Our model is specially suited for the last two types of problems, in particular because we have assumed a constant number of particles, which may represent agents in the economy or financial market (ecological problems, on the

other hand, usually require including birth and death of particles). Particle systems were first used in this context in Föllmer (1974), and they have been used recently by many authors to analyze a variety of problems in economics and finance. The techniques that have been used are diverse, including, for instance, ideas taken from the Ising model in Föllmer (1974), the voter model in Giesecke and Weber (2004), the contact process in Huck and Kosfeld (2007), the theory of large deviations in Dai Pra, Runggaldier, Sartori, and Tolotti (2007), and the theory of queuing networks in Davis and Esparragoza-Rodriguez (2007) and Bayraktar, Horst, and Sircar (2007).

Our original motivation for this work comes precisely from financial modeling. It is related to some problems studied by Darrell Duffie and coauthors (see Examples 3.2.1 and 3.3.3) in which they derive some models from the random local interactions between the financial agents involved, based on the ideas of Duffie and Sun (2007). Our initial goal was to provide a general framework in which this type of problems could be rigorously analyzed, and in particular prove a law of large numbers for them. In our general setting,  $W$  will be allowed to be any locally compact complete separable metric space. Considering type spaces of this generality is one of the main features of our model, and it allows us to provide a unified framework to deal with models of different nature (for instance, the model in Example 3.2.1 has a finite type space and the limit solves a finite system of ordinary differential equations, while in Example 3.3.3 the type space is  $\mathbb{R}$  and the limit solves a system of uncountably many integro-differential equations).

To achieve this first goal, we based our model and techniques on ideas taken from the mathematical biology literature, and in particular on Fournier and

Méléard (2004), where the authors study a model that describes a spatial ecological system where plants disperse seeds and die at rates that depend on the local population density, and obtain a deterministic limit similar to ours. We remark that, following their ideas, our results could be extended to systems with a non-constant population by adding assumptions which allow to control the growth of the population, but we have preferred to keep this part of the problem simple.

The central limit result arose as a natural extension of this original question, but, as we already mentioned, it is much more delicate. The extra technical difficulties are related with the fact that the fluctuations of the process are signed measures (as opposed to the process  $\nu_t^N$  which takes values in a space of probability measures), and the space of signed measures is not well suited for the study of convergence in distribution. The natural topology to consider for this space in our setting, that of weak convergence, is in general not metrizable. One could try to regard this space as the Banach space dual of the space of continuous bounded functions on  $W$  and endow it with its operator norm, but this topology is too strong in general to obtain tightness for the fluctuations (observe that, in particular, the total mass of the fluctuations  $\sqrt{N}(\nu_t^N - \nu_t)$  is not a priori bounded uniformly in  $N$ ). To overcome this difficulty we will show convergence of the fluctuations as a process taking values in the dual of a suitable abstract Hilbert space of test functions. We will actually have to consider a sequence of embeddings of Banach and Hilbert spaces, which will help us in controlling the norm of the fluctuations. This approach is inspired by ideas introduced in Métivier (1987) to study weak convergence of some measure-valued processes using sequences of Sobolev embeddings. Our proof is based on Méléard (1998), where the author proves a similar central limit result for a system of interacting diffusions associated with Boltzmann equations.

The rest of the paper is organized as follows. Section 3.2 contains the description of the general model, Section 3.3 presents the law of large numbers for our system, and Section 3.4 presents the central limit theorem, together with the description of the extra assumptions and the functional analytical setting we will use to obtain it. All the proofs are contained in Section 3.5.

## 3.2 Description of the Model

### 3.2.1 Introductory example

To introduce the basic features of our model and fix some ideas, we begin by presenting one of the basic examples we have in mind.

**Example 3.2.1.** We consider the model for over-the-counter markets introduced in Duffie, Gârleanu, and Pedersen (2005). There is a “consol”, which is an asset paying dividends at a constant rate of 1, and there are  $N$  investors that can hold up to one unit of the asset. The total number of units of the asset remains constant in time, and the asset can be traded when the investors contact each other and when they are contacted by marketmakers. Each investor is characterized by whether he or she owns the asset or not, and by an intrinsic type that is “high” or “low”. Low-type investors have a holding cost when owning the asset, while high-type investors do not. These characteristics will be represented by the set of types  $W = \{ho, hn, lo, ln\}$ , where  $h$  and  $l$  designate the high- and low-type of an investor while  $o$  and  $n$  designate whether an investor owns or not the asset.

At some fixed rate  $\lambda_d$ , high-type investors change their type to low. This

means that each investor runs a Poisson process with rate  $\lambda_d$  (independent from the others), and at each event of this process the investor changes his or her intrinsic type to low (nothing happens if the investor is already of low-type). Analogously, low-type investors change to high-type at some rate  $\lambda_u$ . The meetings between agents are defined as follows: each investor decides to look for another investor at rate  $\beta$  (understood as before, i.e., at the times of the events of a Poisson process with rate  $\beta$ ), chooses the investor uniformly among the set of  $N$  investors, and tries to trade. Additionally, each investor contacts a marketmaker at rate  $\rho$ . The marketmakers pair potential buyers and sellers, and the model assumes that this pairing happens instantly. At equilibrium, the rate at which investors trade through marketmakers is  $\rho$  times the minimum between the fraction of investors willing to buy and the fraction of investors willing to sell (see Duffie *et al.* (2005) for more details). In this model, the only encounters leading to a trade are those between *hn*- and *lo*-agents, since high-type investors not owning the asset are the only ones willing to buy, while low-type investors owning the asset are the only ones willing to sell.

Theorem 3.3.1 will imply the following for this model: as  $N$  goes to infinity, the (random) evolution of the fraction of agents of each type converges to a deterministic limit which is the unique solution of the following system of ordinary differential equations:

$$\begin{aligned}
\dot{u}_{ho}(t) &= 2\beta u_{hn}(t)u_{lo}(t) + \rho \min\{u_{hn}(t), u_{lo}(t)\} + \lambda_u u_{lo}(t) - \lambda_d u_{ho}(t), \\
\dot{u}_{hn}(t) &= -2\beta u_{hn}(t)u_{lo}(t) - \rho \min\{u_{hn}(t), u_{lo}(t)\} + \lambda_u u_{ln}(t) - \lambda_d u_{hn}(t), \\
\dot{u}_{lo}(t) &= -2\beta u_{hn}(t)u_{lo}(t) - \rho \min\{u_{hn}(t), u_{lo}(t)\} - \lambda_u u_{lo}(t) + \lambda_d u_{ho}(t), \\
\dot{u}_{ln}(t) &= 2\beta u_{hn}(t)u_{lo}(t) + \rho \min\{u_{hn}(t), u_{lo}(t)\} - \lambda_u u_{ln}(t) + \lambda_d u_{hn}(t).
\end{aligned} \tag{3.1}$$

Here  $u_w(t)$  denotes the fraction of type- $w$  investors at time  $t$ . This deterministic limit corresponds to the one proposed in Duffie *et al.* (2005) for this model

(see the referred paper for the interpretation of this equations and more on this model).

### 3.2.2 Description of the General Model

We will denote by  $I_N = \{1, \dots, N\}$  the set of particles in the system. In line with our original financial motivation, we will refer to these particles as the “agents” in the system (like the investors of the aforementioned example). The possible types for the agents will be represented by a locally compact Polish (i.e., separable, complete, metrizable) space  $W$ . Given a metric space  $E$ ,  $\mathcal{P}(E)$  will denote the collection of probability measures on  $E$ , which will be endowed with the topology of weak convergence. When  $E = W$ , we will simply write  $\mathcal{P} = \mathcal{P}(W)$ . We will denote by  $\mathcal{P}_a$  the subset of  $\mathcal{P}$  consisting of purely atomic measures.

The Markov process  $\nu_t^N$  we are interested in takes values in  $\mathcal{P}_a$  and describes the evolution of the distribution of the agents over the set of types. We recall that it is defined as

$$\nu_t^N = \frac{1}{N} \sum_{i=1}^N \delta_{\eta_t^N(i)},$$

where  $\delta_w$  is the probability measure on  $W$  assigning mass 1 to  $w \in W$  and  $\eta_t^N(i)$  corresponds to the type of the agent  $i$  at time  $t$ . In other words, the vector  $\eta_t^N \in W^{I_N}$  gives the configuration of the set of agents at time  $t$ , while for any Borel subset  $A$  of  $W$ ,  $\nu_t^N(A)$  is the fraction of agents whose type is in  $A$  at time  $t$ .

The dynamics of the process is defined by the following rates:

- Each agent decides to change its type at a certain rate  $\gamma(w, \nu_t^N)$  that de-

depends on its current type  $w$  and the current distribution  $\nu_t^N$ . The new type is chosen according to a probability measure  $a(w, \nu_t^N, dw')$  on  $W$ .

- Each agent contacts each other agent at a certain rate that depends on their current types  $w_1$  and  $w_2$  and the current distribution  $\nu_t^N$ : the total rate at which a given type- $w_1$  agent contacts type- $w_2$  agents is given by  $N\lambda(w_1, w_2, \nu_t^N)\nu_t^N(\{w_2\})$ . After a pair of agents meet, they choose together a new pair of types according to a probability measure  $b(w_1, w_2, \nu_t^N, dw'_1 \otimes dw'_2)$  (not necessarily symmetric in  $w_1, w_2$ ) on  $W \times W$ .

For a fixed  $\mu \in \mathcal{P}_a$ ,  $a(w, \mu, dw')$  and  $b(w_1, w_2, \mu, dw'_1 \otimes dw'_2)$  can be interpreted, respectively, as the transition kernels of Markov chains in  $W$  and  $W \times W$ .

Let  $\mathcal{B}(W)$  be the collection of bounded measurable functions on  $W$  and  $\mathcal{C}_b(W)$  be the collection of bounded continuous functions on  $W$ . For  $\nu \in \mathcal{P}$  and  $\varphi \in \mathcal{B}(W)$  (or, more generally, any measurable function  $\varphi$ ) we write

$$\langle \nu, \varphi \rangle = \int_W \varphi d\nu.$$

Observe that

$$\langle \nu_t^N, \varphi \rangle = \frac{1}{N} \sum_{i=1}^N \varphi(\eta_t^N(i)).$$

We make the following assumption:

**Assumption A.**

(A1) The rate functions  $\gamma(w, \nu)$  and  $\lambda(w, w', \nu)$  are defined for all  $\nu \in \mathcal{P}$ . They are non-negative, measurable in  $w$  and  $w'$ , bounded respectively by constants  $\bar{\gamma}$  and  $\bar{\lambda}$ , and continuous in  $\nu$ .

(A2)  $a(w, \nu, \cdot)$  and  $b(w, w', \nu, \cdot)$  are measurable in  $w$  and  $w'$ .

(A3) The mappings

$$\begin{aligned} \nu &\longmapsto \int_W \gamma(w, \nu) a(w, \nu, \cdot) \nu(dw) \quad \text{and} \\ \nu &\longmapsto \int_W \int_W \lambda(w_1, w_2, \nu) b(w_1, w_2, \nu, \cdot) \nu(dw_2) \nu(dw_1), \end{aligned}$$

which assign to each  $\nu \in \mathcal{P}_a$  a finite measure on  $W$  and  $W \times W$ , respectively, are continuous with respect to the topology of weak convergence and Lipschitz with respect to the total variation norm: there are constants  $C_a, C_b > 0$  such that

$$\left\| \int_W \gamma(w, \nu_1) a(w, \nu_1, \cdot) \nu_1(dw) - \int_W \gamma(w, \nu_2) a(w, \nu_2, \cdot) \nu_2(dw) \right\|_{\text{TV}} \leq C_a \|\nu_1 - \nu_2\|_{\text{TV}}$$

and

$$\left\| \int_W \int_W \lambda(w_1, w_2, \nu_1) b(w_1, w_2, \nu_1, \cdot) \nu_1(dw_2) \nu_1(dw_1) - \int_W \int_W \lambda(w_1, w_2, \nu_2) b(w_1, w_2, \nu_2, \cdot) \nu_2(dw_2) \nu_2(dw_1) \right\|_{\text{TV}} \leq C_b \|\nu_1 - \nu_2\|_{\text{TV}}.$$

We recall that the total variation norm of a signed measure  $\mu$  is defined by

$$\|\mu\|_{\text{TV}} = \sup_{\varphi: \|\varphi\|_\infty \leq 1} |\langle \mu, \varphi \rangle|.$$

(A3) is satisfied, in particular, whenever the rates do not depend on  $\nu$ .

### 3.3 Law of large numbers for $\nu_t^N$

Our first result shows that the process  $\nu_t^N$  converges in distribution, as the number of agents  $N$  goes to infinity, to a deterministic limit that is characterized by a measure-valued system of differential equations (written in its weak form).

Given a metric space  $S$ , we will denote by  $D([0, T], S)$  the space of càdlàg functions  $\nu: [0, T] \rightarrow S$ , and we endow these spaces with the Skorohod topology (see Ethier and Kurtz (1986) or Billingsley (1999) for a reference on this topology and weak convergence in general). Observe that our processes  $\nu_t^N$  have paths on  $D([0, T], \mathcal{P})$  (recall that we are endowing  $\mathcal{P}$  with the topology of weak convergence, which is metrizable). We will also denote by  $C([0, T], S)$  the space of continuous functions  $\nu: [0, T] \rightarrow S$ .

**Theorem 3.3.1.** *Suppose that Assumption A holds. For any given  $T > 0$ , consider the sequence of  $\mathcal{P}$ -valued processes  $\nu_t^N$  on  $[0, T]$ , and assume that the sequence of initial distributions  $\nu_0^N$  converges in distribution to some fixed  $\nu_0 \in \mathcal{P}$ . Then the sequence  $\nu_t^N$  converges in distribution in  $D([0, T], \mathcal{P})$  to a deterministic  $\nu_t$  in  $C([0, T], \mathcal{P})$ , which is the unique solution of the following system of integro-differential equations: for every  $\varphi \in \mathcal{B}(W)$  and  $t \in [0, T]$ ,*

$$\begin{aligned} \langle \nu_t, \varphi \rangle = & \langle \nu_0, \varphi \rangle + \int_0^t \int_W \gamma(w, \nu_s) \int_W (\varphi(w') - \varphi(w)) a(w, \nu_s, dw') \nu_s(dw) ds \\ & + \int_0^t \int_W \int_W \lambda(w_1, w_2, \nu_s) \int_{W \times W} (\varphi(w'_1) + \varphi(w'_2) - \varphi(w_1) - \varphi(w_2)) \\ & \cdot b(w_1, w_2, \nu_s, dw'_1 \otimes dw'_2) \nu_s(dw_2) \nu_s(dw_1) ds. \end{aligned} \tag{S1}$$

Observe that, in particular, (S1) implies that for every Borel set  $A \subseteq W$  and almost every  $t \in [0, T]$ ,

$$\begin{aligned} \frac{d\nu_t(A)}{dt} = & - \int_A \left( \gamma(w, \nu_t) + \int_W (\lambda(w, w', \nu_t) + \lambda(w', w, \nu_t)) \nu_t(dw') \right) \nu_t(dw) \\ & + \int_W \gamma(w, \nu_t) a(w, \nu_t, A) \nu_t(dw) \\ & + \int_W \int_W \lambda(w, w', \nu_t) \left[ b(w, w', \nu_t, A \times W) + b(w, w', \nu_t, W \times A) \right] \nu_t(dw') \nu_t(dw). \end{aligned} \tag{S1'}$$

Furthermore, standard measure theory arguments allow to show that the sys-

tem (S1') actually characterizes the solution of (S1) (by approximating the test functions  $\varphi$  in (S1) by simple functions).

(S1') has an intuitive interpretation: the first term on the right side is the total rate at which agents leave the set of types  $A$ , the second term is the rate at which agents decide to change their types to a type in  $A$ , and the third term is the rate at which agents acquire types in  $A$  due to interactions between them.

The following corollary of the previous result is useful when writing and analyzing the limiting equations (S1) or (S1') (see, for instance, Example 3.3.3).

**Corollary 3.3.1.** *In the context of Theorem 3.3.1, assume that  $\nu_0$  is absolutely continuous with respect to some measure  $\mu$  on  $W$  and that the measures*

$$\int_W \gamma(w, \nu_0) a(w, \nu_0, \cdot) \nu_0(dw) \text{ and } \int_W \int_W \lambda(w_1, w_2, \nu_0) b(w_1, w_2, \nu_0, \cdot) \nu_0(dw_1) \nu_0(dw_2)$$

*are absolutely continuous with respect to  $\mu$  and  $\mu \otimes \mu$ , respectively. Then the limit  $\nu_t$  is absolutely continuous with respect to  $\mu$  for all  $t \in [0, T]$ .*

The following two examples show two different kinds of models: one with a finite type space and the other with  $W = \mathbb{R}$ . The first model is the one given in Example 3.2.1.

**Example 3.3.2** (Continuation of Example 3.2.1). To translate into our framework the model for over-the-counter markets of Duffie *et al.* (2005), we take  $W = \{ho, hn, lo, ln\}$  and consider a set of parameters  $\gamma, a, \lambda$ , and  $b$  with all but  $\lambda$  being independent of  $\nu_t^N$ . Let

$$\begin{aligned} \gamma(ho) = \gamma(hn) = \lambda_d, & \quad a(ho, \cdot) = \delta_{lo}, & \quad a(hn, \cdot) = \delta_{ln}, \\ \gamma(lo) = \gamma(ln) = \lambda_u, & \quad a(lo, \cdot) = \delta_{ho}, & \quad a(ln, \cdot) = \delta_{hn}. \end{aligned}$$

Observe that with this definition, high-type investors become low-type at rate  $\lambda_d$  and low-type investors become high-type at rate  $\lambda_u$ , just as required. For the encounters between agents we take

$$\lambda(hn, lo, \nu) = \lambda(lo, hn, \nu) = \begin{cases} \beta + \frac{\rho}{2} \frac{\nu(\{hn\}) \wedge \nu(\{lo\})}{\nu(\{hn\}) \nu(\{lo\})} & \text{if } \nu(\{hn\}) \nu(\{lo\}) > 0, \\ \beta & \text{if } \nu(\{hn\}) \nu(\{lo\}) = 0, \end{cases}$$

$$b(hn, lo, \nu, \cdot) = \delta_{(ho, ln)}, \quad \text{and} \quad b(lo, hn, \nu, \cdot) = \delta_{(ln, ho)}$$

(where  $a \wedge b = \min\{a, b\}$ ), and for all other pairs  $w_1, w_2 \in W$ ,  $\lambda(w_1, w_2, \nu) = 0$  (recall that the only encounters leading to a trade are those between  $hn$ - and  $lo$ -agents and vice versa, in which case trade always occurs). The rates  $\lambda(hn, lo, \nu)$  and  $\lambda(lo, hn, \nu)$  have two terms: the rate  $\beta$  corresponding to the rate at which  $hn$ -agents contact  $lo$ -agents, plus a second rate reflecting trades carried out via a marketmaker. The form of this second rate assures that  $hn$ - and  $lo$ -agents meet through marketmakers at the right rate of  $\rho \nu(\{hn\}) \wedge \nu(\{lo\})$ . It is not difficult to check that these parameters satisfy Assumption A, using the fact that  $x \wedge y = (x + y - |x - y|)/2$  for  $x, y \in \mathbb{R}$ .

Now let  $u_w(t) = \nu_t(\{w\})$ , where  $\nu_t$  is the limit of  $\nu_t^N$  given by Theorem 3.3.1. We need to compute the right side of (S1') with  $A = \{w\}$  for each  $w \in W$ . Take, for example,  $w = ho$ . We get

$$\begin{aligned} \dot{u}_{ho}(t) = & \lambda_u u_{lo}(t) - \lambda_d u_{ho}(t) + \beta u_{hn}(t) u_{lo}(t) + \frac{\rho}{2} u_{hn}(t) \wedge u_{lo}(t) \\ & + \beta u_{lo}(t) u_{hn}(t) + \frac{\rho}{2} u_{hn}(t) \wedge u_{lo}(t), \end{aligned}$$

which corresponds exactly to the first equation in (3.1). The other three equations follow similarly.

**Example 3.3.3.** Our second example is based on the model for information percolation in large markets introduced in Duffie and Manso (2007). We will only

describe the basic features of the model, for more details see the cited paper. There is a random variable  $X$  of concern to all agents which has two possible values, “high” or “low”. Each agent holds some information about the outcome of  $X$ , and this information is summarized in a real number  $x$  which is a sufficient statistic for the posterior probability assigned by the agent (given his or her information) to the outcome of  $X$  being high. We take these statistics as the types of the agents (so  $W = \mathbb{R}$ ). The model is set up so that these statistics satisfy the following: after a type- $x_1$  agent and a type- $x_2$  agent meet and share their information,  $x_1 + x_2$  becomes a sufficient statistic for the posterior distribution of  $X$  assigned by both agents given now their shared information.

In this model the agents change types only after contacting other agents, so we take  $\gamma \equiv 0$ , and encounters between agents occur at a constant rate  $\lambda > 0$ . The transition kernel for the types of the agents after encounters is independent of  $\nu_t^N$  and is given by

$$b(x_1, x_2, \cdot) = b(x_2, x_1, \cdot) = \delta_{(x_1+x_2, x_1+x_2)}$$

for every  $x_1, x_2 \in \mathbb{R}$ . This choice for the parameters trivially satisfies Assumption A.

To compute the limit of the process, let  $A$  be a Borel subset of  $\mathbb{R}$ . Then, since  $\gamma \equiv 0$  and  $\lambda$  is constant, (S1') gives

$$\begin{aligned} \dot{\nu}_t(A) &= -2\lambda\nu_t(A) + \lambda \int_{\mathbb{R}^2} (\delta_{(x+y, x+y)}(\mathbb{R} \times A) + \delta_{(x+y, x+y)}(A \times \mathbb{R})) \nu_t(dy) \nu_t(dx) \\ &= -2\lambda\nu_t(A) + 2\lambda \int_{\mathbb{R}^2} \delta_{x+y}(A) \nu_t(dy) \nu_t(dx) \\ &= -2\lambda\nu_t(A) + 2\lambda \int_{-\infty}^{\infty} \nu_t(A - x) \nu_t(dx), \end{aligned}$$

where  $A - x = \{y \in \mathbb{R} : y + x \in A\}$ . Therefore,

$$\dot{\nu}_t(A) = -2\lambda\nu_t(A) + 2\lambda(\nu_t * \nu_t)(A). \quad (3.3)$$

Using Corollary 3.3.1 we can write the last equation in a nicer form: if we assume that the initial condition  $\nu_0$  is absolutely continuous with respect to the Lebesgue measure, then the measures  $\nu_t$  have a density  $g_t$  with respect to the Lebesgue measure, and we obtain

$$\dot{g}_t(x) = -2\lambda g_t(x) + 2\lambda \int_{-\infty}^{\infty} g_t(z-x)g_t(z) dz = -2\lambda g_t(x) + 2\lambda(g_t * g_t)(x).$$

This is the system of integro-differential equations proposed in Duffie and Manso (2007) for this model (except for the factor of 2, which is omitted in that paper).

### 3.4 Central limit theorem for $\nu_t^N$

Theorem 3.3.1 gives the law of large numbers for  $\nu_t^N$ , in the sense that it obtains a deterministic limit for the process as the size of the market goes to infinity. We will see now that, under some additional hypotheses, we can also obtain a central limit result for our process: the fluctuations of  $\nu_t^N$  around the limit  $\nu_t$  are of order  $1/\sqrt{N}$ , and they have, asymptotically, a Gaussian nature. As we mentioned in the Introduction, this result is much more delicate than Theorem 3.3.1, and we will need to work hard to find the right setting for it.

The *fluctuations* process is defined as follows:

$$\sigma_t^N = \sqrt{N}(\nu_t^N - \nu_t).$$

$\sigma_t^N$  is a sequence of finite signed measures, and our goal is to prove that it converges to the solution of a system of stochastic differential equations driven by a Gaussian process. As we explained in the Introduction, regarding the fluctuations process as taking values in the space of signed measures, and endowing

this space with the topology of weak convergence (which corresponds to seeing the process as taking values in the Banach space dual of  $\mathcal{C}_b(W)$  topologized with the weak\* convergence) is not the right approach for this problem. The idea will be to replace the test function space  $\mathcal{C}_b(W)$  by an appropriate Hilbert space  $\mathcal{H}_1$  and regard  $\sigma_t^N$  as a linear functional acting on this space via the mapping  $\varphi \mapsto \langle \sigma_t^N, \varphi \rangle$ . In other words, we will regard  $\sigma_t^N$  as a process taking values in the dual  $\mathcal{H}_1'$  of a Hilbert space  $\mathcal{H}_1$ .

The space  $\mathcal{H}_1$  that we choose will depend on the type space  $W$ . Actually, whenever  $W$  is not finite we will not need a single space, but a chain of seven spaces embedded in a certain structure. Our goal is to handle (at least) the following four possibilities for  $W$ : a finite set,  $\mathbb{Z}^d$ , a “sufficiently smooth” compact subset of  $\mathbb{R}^d$ , and all of  $\mathbb{R}^d$ . We wish to handle these cases under a unified framework, and this will require us to abstract the necessary assumptions on our seven spaces and the parameters of the model. We will do this in Sections 3.4.1 and 3.4.2, and then in Section 3.4.3 we will explain how to apply this abstract setting to the four type spaces  $W$  that we just mentioned.

### 3.4.1 General setting

During this and the next subsection we will assume as given the collection of spaces in which our problem will be embedded, and then we will make some assumptions on the parameters of our process that will assure that they are compatible with the structure of these spaces. The idea of this part is that we will try to impose as little as possible on these spaces, leaving their definition to be specified for the different cases of type space  $W$ .

The elements we will use are the following:

- Four separable Hilbert spaces of measurable functions on  $W$ ,  $\mathcal{H}_1, \mathcal{H}_2, \mathcal{H}_3$ , and  $\mathcal{H}_4$ .
- Three Banach spaces of continuous functions on  $W$ ,  $\mathcal{C}_0, \mathcal{C}_2$ , and  $\mathcal{C}_3$ .
- Five continuous functions  $\rho_0, \rho_1, \rho_2, \rho_3, \rho_4: W \rightarrow [1, \infty)$  such that  $\rho_i \leq \rho_{i+1}$  for  $i = 0, 1, 2, 3$ ,  $\rho_i \in \mathcal{C}_i$  for  $i = 0, 2, 3$ , and for all  $w \in W$ ,  $\rho_1^p(w) \leq C\rho_4(w)$  for some  $C > 0$  and  $p > 1$  (this last requirement is very mild, as we will see in the examples below, but will be necessary in the proof of Theorem 3.4.1).

The seven spaces and the five functions introduced above must be related in a specific way. First, we assume that the following sequence of continuous embeddings holds:

$$\mathcal{C}_0 \hookrightarrow \mathcal{H}_1 \xrightarrow{\text{c}} \mathcal{H}_2 \hookrightarrow \mathcal{C}_2 \hookrightarrow \mathcal{H}_3 \hookrightarrow \mathcal{C}_3 \hookrightarrow \mathcal{H}_4, \quad (\text{B1})$$

where the c under the second arrow means that the embedding is compact. We recall that a continuous embedding  $E_1 \hookrightarrow E_2$  between two normed spaces  $E_1, E_2$  implies, in particular, that  $\|\cdot\|_{E_2} \leq C\|\cdot\|_{E_1}$  for some  $C > 0$ , while saying that the embedding is compact means that every bounded set in  $E_1$  is compact in  $E_2$ .

Second, we assume that for  $i = 1, 2, 3, 4$ , if  $\varphi \in \mathcal{H}_i$  then

$$|\varphi(w)| \leq C \|\varphi\|_{\mathcal{H}_i} \rho_i(w) \quad (\text{B2})$$

for all  $w \in W$ , for some  $C > 0$  which does not depend on  $\varphi$ . The same holds for the spaces  $\mathcal{C}_i$ : for  $i = 0, 2, 3$  and  $\varphi \in \mathcal{C}_i$ ,

$$|\varphi(w)| \leq C \|\varphi\|_{\mathcal{C}_i} \rho_i(w). \quad (\text{B3})$$

The functions  $\rho_i$  will typically appear as weighting functions in the definition of the norms of the spaces  $\mathcal{H}_i$  and  $\mathcal{C}_i$ . They will dictate the maximum growth rate allowed for functions in these spaces.

We will denote by  $\mathcal{H}_i'$  and  $\mathcal{C}_i'$  the topological duals of the spaces  $\mathcal{H}_i$  and  $\mathcal{C}_i$ , respectively, endowed with their operator norms (in particular, the spaces  $\mathcal{H}_i'$  and  $\mathcal{C}_i'$  are Hilbert and Banach spaces themselves). Observe that (B1) implies the following dual continuous embeddings:

$$\mathcal{H}_4' \hookrightarrow \mathcal{C}_3' \hookrightarrow \mathcal{H}_3' \hookrightarrow \mathcal{C}_2' \hookrightarrow \mathcal{H}_2' \xrightarrow{\mathcal{C}} \mathcal{H}_1' \hookrightarrow \mathcal{C}_0'. \quad (\text{B1}')$$

Before continuing, let us describe briefly the main ideas behind the proof of our central limit theorem, which will help explain why this is a good setting for proving convergence of the fluctuations process. What we want to prove is that  $\sigma_t^N$  converges in distribution, as a process taking values in  $\mathcal{H}_1'$ , to the solution  $\sigma_t$  of a certain stochastic differential equation (see (S2) below). The approach we will take to prove this (the proof of Theorem 3.3.1 follows an analogous line) is standard: we first prove that the sequence  $\sigma_t^N$  is tight, then we show that any limit point of this sequence satisfies the desired stochastic differential equation, and finally we prove that this equation has a unique solution (in distribution). To achieve this we will follow the line of proof of Méléard (1998). Our sequence of embeddings (B1') corresponds there to a sequence of embeddings of weighted Sobolev spaces (see (3.11) in the cited paper); in particular, we will use a very similar sequence of spaces to deal with the case  $W = \mathbb{R}^d$  in Section 3.4.3.4. One important difficulty with this approach is the following: the operator  $J_s$  associated with the drift term of our limiting equation (see (3.4)), as well as the corresponding operators  $J_s^N$  for  $\sigma_t^N$  (see (3.18)), cannot in general be taken to be bounded as operators acting on any of the spaces  $\mathcal{H}_i$ . This forces us

to introduce the spaces  $\mathcal{C}_i$ , on which (B3) plus some assumptions on the rates of the process will assure that  $J_s$  and  $J_s^N$  are bounded.

The scheme of proof will be roughly as follows. We will consider the semimartingale decomposition of the real-valued process  $\langle \sigma_t^N, \varphi \rangle$ , for  $\varphi \in \mathcal{H}_4$ , and then show that the martingale part defines a martingale in  $\mathcal{H}_4'$ . This, together with a moment estimate on the norm of the martingale part in  $\mathcal{H}_4'$  and the boundedness of the operators  $J_s^N$  in  $\mathcal{C}_3'$ , will allow us to deduce that  $\sigma_t^N$  can be seen as a semimartingale in  $\mathcal{H}_3'$ , and moreover give its semimartingale decomposition. Next, we will give a uniform estimate (in  $N$ ) of the norm of  $\sigma_t^N$  in  $\mathcal{C}_2'$ . This implies the same type of estimate in  $\mathcal{H}_2'$ , and this will allow us to obtain the tightness of  $\sigma_t^N$  in  $\mathcal{H}_1'$ . The fact that the embedding  $\mathcal{H}_2' \hookrightarrow \mathcal{H}_1'$  is compact is crucial in this step. Then we will show that all limit points of  $\sigma_t^N$  have continuous paths in  $\mathcal{H}_1'$  and they all satisfy the desired stochastic differential equation (S2). Unfortunately, it will not be possible to achieve this last part in  $\mathcal{H}_1'$ , due to the unboundedness of  $J_s$  in this space. Consequently, we are forced to embed the equation in the (bigger) space  $\mathcal{C}_0'$ . The boundedness of  $J_s$  in  $\mathcal{C}_0'$  will also allow us to obtain uniqueness for the solutions of this equation in this space, thus finishing the proof.

Our last assumption (D below) will assure that our abstract setting is compatible with the rates defining our process. Before that, we need to replace Assumptions (A1) and (A2) by stronger versions:

**Assumption C.**

(C1) There is a family of finite measures  $\{\Gamma(w, z, \cdot)\}_{w, z \in W}$  on  $W$ , whose total masses are bounded by  $\bar{\gamma}$ , such that for every  $w \in W$  and every  $\nu \in \mathcal{P}$  we

have

$$\gamma(w, \nu)a(w, \nu, dw') = \int_W \Gamma(w, z, dw') \nu(dz).$$

$\Gamma(w, z, \cdot)$  is measurable in  $w$  and continuous in  $z$ .

(C2) There is a family of measures  $\{\Lambda(w_1, w_2, z, \cdot)\}_{w_1, w_2, z \in W}$  on  $W \times W$ , whose total masses are bounded by  $\bar{\lambda}$ , such that for every  $w_1, w_2 \in W$  and every  $\nu \in \mathcal{P}$  we have

$$\lambda(w_1, w_2, \nu)b(w_1, w_2, \nu, dw'_1 \otimes dw'_2) = \int_{W \times W} \Lambda(w_1, w_2, z, dw'_1 \otimes dw'_2) \nu(dz).$$

$\Lambda(w_1, w_2, z, \cdot)$  is measurable in  $w_1$  and  $w_2$  and continuous in  $z$ .

The intuition behind this assumption is the following: the total rate at which a type- $w$  agent becomes a type- $w'$  agent is computed by averaging the effect that each agent in the market has on this rate for the given agent. Observe that, under this assumption, (A3) holds.

**Remark 3.4.1.** Assumption C has the effect of linearizing the jump rates in  $\nu$ . This turns out to be very convenient, because it will allow us to express the drift term of the stochastic differential equation describing the limiting fluctuations  $\sigma_t$  ((S2) below) as  $J_t \sigma_t$  for some  $J_t \in \mathcal{C}_0'$  (see (3.4) and (3.18)). A more general approach would be to assume that the jump kernels, seen as operators acting on  $\mathcal{C}_0'$ , are Fréchet differentiable. In that case we would need to change the form of the drift operator  $J_t$  in the limiting equation and of Assumption D below, but the proof of Theorem 3.4.1 would still work, without any major modifications. To avoid extra complications, and since all the examples we have in mind satisfy Assumption C, we will restrict ourselves to this simpler case.

We introduce the following notation: given a measurable function  $\varphi$  on  $W$ ,

let

$$\Gamma\varphi(w; z) = \int_W (\varphi(w') - \varphi(w)) \Gamma(w, z, dw') \quad \text{and}$$

$$\Lambda\varphi(w_1, w_2; z) = \int_{W \times W} (\varphi(w'_1) + \varphi(w'_2) - \varphi(w_1) - \varphi(w_2)) \Lambda(w_1, w_2, z, dw'_1 \otimes dw'_2).$$

These quantities can be thought of as the jump kernels for the process associated with the effect of a type- $z$  agent on the transition rates. Averaging these rates with respect to  $\nu_t^N(dz)$  gives the total jump kernel for the process.

**Assumption D.**

(D1) There is a  $C > 0$  such that for all  $w, z \in W$  and  $i = 0, 1, 2, 3, 4$ ,

$$\int_W \rho_i^2(w') \Gamma(w, z, dw') < C (\rho_i^2(w) + \rho_i^2(z)).$$

(D2) There is a  $C > 0$  such that for all  $w_1, w_2, z \in W$  and  $i = 0, 1, 2, 3, 4$ ,

$$\int_{W \times W} (\rho_i^2(w'_1) + \rho_i^2(w'_2)) \Lambda(w_1, w_2, z, dw'_1 \otimes dw'_2) < C (\rho_i^2(w_1) + \rho_i^2(w_2) + \rho_i^2(z)).$$

(D3) Let  $\mu_1, \mu_2 \in \mathcal{P}$  be such that  $\langle \mu_i, \rho_4^2 \rangle < \infty$  and define the following operator acting on measurable functions  $\varphi$  on  $W$ :

$$\begin{aligned} J_{\mu_1, \mu_2} \varphi(z) &= \int_W \Gamma\varphi(w; z) \mu_1(dw) + \int_W \Gamma\varphi(z; x) \mu_2(dx) \\ &\quad + \int_W \int_W \Lambda\varphi(w_1, w_2; z) \mu_1(dw_2) \mu_1(dw_1) \\ &\quad + \int_W \int_W \Lambda\varphi(w, z; x) \mu_1(dw) \mu_2(dx) \\ &\quad + \int_W \int_W \Lambda\varphi(z, w; x) \mu_2(dw) \mu_2(dx). \end{aligned}$$

Then:

- (i)  $J_{\mu_1, \mu_2}$  is a bounded operator on  $\mathcal{C}_i$ , for  $i = 0, 2, 3$ . Moreover, its norm can be bounded uniformly in  $\mu_1, \mu_2$ .

(ii) There is a  $C > 0$  such that given any  $\varphi \in \mathcal{C}_0$  and any  $\mu_1, \mu_2, \mu_3, \mu_4 \in \mathcal{P}$  satisfying  $\langle \mu_i, \rho_4^2 \rangle < \infty$ ,

$$\|(J_{\mu_1, \mu_2} - J_{\mu_3, \mu_4})\varphi\|_{\mathcal{C}_0} \leq C \|\varphi\|_{\mathcal{C}_0} (\|\mu_1 - \mu_3\|_{\mathcal{C}_2'} + \|\mu_2 - \mu_4\|_{\mathcal{C}_2'}).$$

(D1) and (D2) correspond to moment assumptions on the transition rates of the agents, and assure that the agents do not jump “too far”. (D3.i) says two things: first, that the jump kernel defined by the rates preserves the structure of the spaces  $\mathcal{C}_i$  and, second, that the resulting operator is bounded, which will imply the boundedness of the drift operators  $J_s$  and  $J_s^N$  mentioned above. (D3.ii) involves a sort of strengthening of the Lipschitz condition (A3) on the rates, and will be used to prove uniqueness for the limiting stochastic differential equation. Observe that by taking larger weighting functions  $\rho_i$ , which corresponds to taking smaller spaces of test functions  $\mathcal{H}_i$ , we add more moment assumptions on the rates of the process; on the other hand, asking for more structure on the spaces  $\mathcal{H}_i$  and  $\mathcal{C}_i$ , such as differentiability in the Euclidean case, adds more requirements on the regularity of the rates.

### 3.4.2 Statement of the theorem

For  $\xi \in \mathcal{H}_i'$  (respectively  $\mathcal{C}_i'$ ) and  $\varphi \in \mathcal{H}_i$  (respectively  $\mathcal{C}_i$ ) we will write

$$\langle \xi, \varphi \rangle = \xi(\varphi).$$

Given  $\varphi \in \mathcal{H}_1$  and  $z \in W$  define

$$\begin{aligned}
J_s \varphi(z) &= \int_W \Gamma \varphi(w; z) \nu_s(dw) + \int_W \Gamma \varphi(z; x) \nu_s(dx) \\
&\quad + \int_W \int_W \Lambda \varphi(w_1, w_2; z) \nu_s(dw_2) \nu_s(dw_1) \\
&\quad + \int_W \int_W [\Lambda \varphi(z, w; x) + \Lambda \varphi(w, z; x)] \nu_s(dw) \nu_s(dx)
\end{aligned} \tag{3.4}$$

Observe that  $J_s = J_{\nu_s, \nu_s}$ . Therefore, under moment assumptions on  $\nu_s$ , (D3.i) implies that  $J_s$  is a bounded operator on each of the spaces  $\mathcal{C}_i$ . Observe that if we integrate the first and third terms on the right side of (3.4) with respect to  $\nu_s(dz)$ , we obtain the integral term in (S1). In our central limit result, the variable  $z$  will be integrated against the limiting fluctuation process  $\sigma_t$ . The other two terms in (3.4) correspond to fluctuations arising from the dependence of the rates on its other arguments (the types of the agents involved).

The operator  $J_s$  (or, more properly, its adjoint  $J_s^*$ ) will appear in the drift term of the stochastic differential equation describing the limiting fluctuations process, which will be expressed as a Bochner integral. We recall that these integrals are an extension of the Lebesgue integral to functions taking values on a Banach space, see Section V.5 in Yosida (1995) for details.

**Theorem 3.4.1.** *Assume that Assumptions C and D hold, that (B1), (B2), and (B3) hold, and that*

$$\begin{aligned}
\sqrt{N}(\nu_0^N - \nu_0) &\implies \sigma_0, \quad \sup_{N \geq 0} \mathbb{E} \left( \left\| \sqrt{N}(\nu_0^N - \nu_0) \right\|_{\mathcal{C}_2'}^2 \right) < \infty, \\
\sup_{N \geq 0} \mathbb{E}(\langle \nu_0^N, \rho_4^2 \rangle) &< \infty, \quad \text{and} \quad \mathbb{E}(\langle \nu_0, \rho_4^2 \rangle) < \infty
\end{aligned} \tag{3.5}$$

*hold, where the convergence in distribution above is in  $\mathcal{H}_1'$ . Then the sequence of processes  $\sigma_t^N$  converges in distribution in  $D([0, T], \mathcal{H}_1')$  to a process  $\sigma_t \in C([0, T], \mathcal{H}_1')$ . This process is the unique (in distribution) solution in  $\mathcal{C}_0'$  of the following stochastic*

differential equation:

$$\sigma_t = \sigma_0 + \int_0^t J_s^* \sigma_s ds + Z_t, \quad (\text{S2})$$

where the above is a Bochner integral,  $J_s^*$  is the adjoint of the operator  $J_s$  in  $\mathcal{C}_0$ , and  $Z_t$  is a centered  $\mathcal{C}_0'$ -valued Gaussian process with quadratic covariations specified by

$$\begin{aligned} [Z.(\varphi_1), Z.(\varphi_2)]_t &= \int_0^t \int_W \int_W \int_W (\varphi_1(w') - \varphi_1(w))(\varphi_2(w') - \varphi_2(w)) \Gamma(w, z, dw') \\ &\quad \cdot \nu_s(dz) \nu_s(dw) ds \\ &\quad + \int_0^t \int_W \int_W \int_W \int_W \int_{W \times W} (\varphi_1(w'_1) + \varphi_1(w'_2) - \varphi_1(w_1) - \varphi_1(w_2)) \\ &\quad \cdot (\varphi_2(w'_1) + \varphi_2(w'_2) - \varphi_2(w_1) - \varphi_2(w_2)) \\ &\quad \cdot \Lambda(w_1, w_2, z, dw'_1 \otimes dw'_2) \nu_s(dz) \nu_s(dw_2) \nu_s(dw_1) ds \end{aligned}$$

for every  $\varphi_1, \varphi_2 \in \mathcal{C}_0$ .

We will denote by  $C_s^{\varphi_1, \varphi_2}$  the sum of the two terms inside the time integrals above, so

$$[Z.(\varphi_1), Z.(\varphi_2)]_t = \int_0^t C_s^{\varphi_1, \varphi_2} ds.$$

**Remark 3.4.2.**

1. (S2) implies, in particular, that the solution  $\sigma_t$  satisfies

$$\langle \sigma_t, \varphi \rangle = \langle \sigma_0, \varphi \rangle + \int_0^t \langle \sigma_s, J_s \varphi \rangle ds + Z_t(\varphi) \quad (\text{S2-w})$$

simultaneously for every  $\varphi \in \mathcal{C}_0$ .

2. Observe that for any  $\varphi_1, \dots, \varphi_k \in \mathcal{C}_0$ , the process

$$Z_t^{\varphi_1, \dots, \varphi_k} = (Z_t(\varphi_1), \dots, Z_t(\varphi_k))$$

is a continuous  $\mathbb{R}^k$ -valued centered martingale with deterministic quadratic covariations, so it can be represented as

$$Z_t^{\varphi_1, \dots, \varphi_k} \stackrel{d}{=} \int_0^t ([C_s]^{\varphi_1, \dots, \varphi_k})^{1/2} dB_s,$$

where  $[C_t]^{\varphi_1, \dots, \varphi_k}$  is the  $k \times k$  matrix-valued process with entries given by  $[C_t^{\varphi_1, \dots, \varphi_k}]_{ij} = C_t^{\varphi_i, \varphi_j}$ ,  $([C_t]^{\varphi_1, \dots, \varphi_k})^{1/2}$  is the square root of this matrix, and  $B_t$  is a standard  $k$ -dimensional Brownian motion. Thus, writing  $\langle \sigma_t; \varphi_1, \dots, \varphi_k \rangle = (\langle \sigma_t, \varphi_1 \rangle, \dots, \langle \sigma_t, \varphi_k \rangle)$  we have

$$\langle \sigma_t; \varphi_1, \dots, \varphi_k \rangle \stackrel{d}{=} \int_0^t \langle \sigma_s; J_s \varphi_1, \dots, J_s \varphi_k \rangle ds + \int_0^t ([C_s]^{\varphi_1, \dots, \varphi_k})^{1/2} dB_s. \quad (3.6)$$

3. The limiting fluctuations  $\sigma_t$  have zero mass in the following sense: whenever  $\mathbf{1} \in \mathcal{C}_0$  and  $\langle \sigma_0, \mathbf{1} \rangle = 0$ ,  $\langle \sigma_t, \mathbf{1} \rangle = 0$  for all  $t \in [0, T]$  almost surely. This follows from (3.6) simply by observing that, in this case,  $J_s \mathbf{1}$  and  $C_s^{\mathbf{1}, \mathbf{1}}$  are both always zero.

Before presenting concrete examples where the setting and assumptions of this section hold, we present a general condition which allows to deduce that the assumptions (3.5) on the initial distributions  $\nu_0^N$ ,  $\nu_0$ , and  $\sigma_0^N$  hold (namely, that  $\nu_0^N$  is a product measure).

**Theorem 3.4.2.** *In the setting of Theorem 3.4.1, assume that  $\nu_0^N$  is the product of  $N$  copies of a fixed probability measure  $\nu_0 \in \mathcal{P}$  (i.e.,  $\nu_0^N$  is chosen by picking the initial type of each agent independently according to  $\nu_0$ ), and that  $\mathbb{E}(\langle \nu_0, \rho_4^2 \rangle) < \infty$ . Then  $\nu_0^N$  converges in distribution in  $\mathcal{P}$  to  $\nu_0$ ,  $\sigma_0^N$  converges in distribution in  $\mathcal{H}_1'$  to a centered Gaussian  $\mathcal{H}_1'$ -valued random variable  $\sigma_0$ , and all the assumptions in (3.5) are satisfied.*

### 3.4.3 Application to concrete type spaces

In this part we will present conditions under which the assumptions of Theorem 3.4.1 are satisfied in the four cases discussed at the beginning of this section.

#### 3.4.3.1 Finite $W$

This is the easy case. The reason is that  $\mathcal{C}_b(W)$  can be identified with  $\mathbb{R}^{|W|}$ , and thus  $\sigma_t^N$  can be regarded as an  $\mathbb{R}^{|W|}$ -valued process, so most of the technical issues disappear. In particular, Theorem 3.4.1 can be proved in this case by arguments very similar to those leading to Theorem 3.3.1.

In the abstract setting of Theorem 3.4.1, it is enough to choose  $\rho_i \equiv 1$  and  $\mathcal{H}_i = \mathcal{C}_i = \mathbb{R}^{|W|} \cong \ell^2(W)$  for the right indices  $i \in \{0, 1, 2, 3, 4\}$  in each case. (B1) follows simply from the finite-dimensionality of  $\mathbb{R}^{|W|}$  and the equivalence of all norms in finite dimensions and (B2), (B3), and Assumption D are satisfied trivially.

Theorem 3.4.1 takes a simpler form in this case. Write  $W = \{w_1, \dots, w_k\}$ ,

$$\sigma_i^N(t) = \sigma_t^N(\{w_i\}), \quad f_i(\sigma) = \sum_{j=1}^k J_s \mathbf{1}_{\{w_i\}}(w_j) \sigma_j, \quad \text{and} \quad g_{ij}(t) = C_t^{\mathbf{1}_{\{w_i\}}, \mathbf{1}_{\{w_j\}}},$$

where  $\sigma$  above is in  $\mathbb{R}^k$ . Also write  $F(\sigma) = (f_1(\sigma), \dots, f_k(\sigma))$  and  $G(t) = (g_{ij}(t))_{i,j=1,\dots,k}$ . Observe that  $G(t)$  is a positive semidefinite matrix for all  $t \geq 0$ .

**Theorem 3a.** *In the above context, assume that Assumption C holds and that*

$$\sqrt{N} (\nu_0^N - \nu_0) \implies \sigma_0 \quad \text{and} \quad \sup_{N>0} \mathbb{E} \left( \left| \sqrt{N} (\nu_0^N - \nu_0) \right|^2 \right) < \infty,$$

where the probability measures  $\nu_t^N$  and  $\nu_t$  are taken here as elements of  $[0, 1]^k$  and  $\sigma_0 \in \mathbb{R}^k$ . Then the sequence of processes  $\sigma^N(t)$  converges in distribution in  $D([0, T], \mathbb{R}^k)$  to

the unique solution  $\sigma(t)$  of the following system of stochastic differential equations:

$$d\sigma(t) = F(\sigma(t)) dt + G^{1/2}(t) dB_t, \quad (\text{S2-f})$$

where  $B_t$  is a standard  $k$ -dimensional Brownian motion.

**Example 3.4.3.** This example provides a very simple model of agents changing their opinions on some issue of common interest, with rates of change depending on the “popularity” of each alternative. These opinions will be represented by  $W = \{-m, \dots, m\}$  ( $m$  can be thought of as being the strongest agreement with some idea, 0 as being neutral, and  $-m$  as being the strongest disagreement with it). Alternatively, one could think of the model as describing the locations of the agents, who move according to the density of agents at each site.

The agents move in two ways. First, each agent feels attracted to other positions proportionally to the fraction of agents occupying them. Concretely, we assume that an agent at position  $i$  goes to position  $j$  at rate  $\beta q_{i,j} \nu_t^N(\{j\})$ , where  $Q = (q_{i,j})_{i,j \in W}$  is the transition matrix of a Markov chain on  $W$ . One interpretation of these rates is that each agent decides to try to change its position at rate  $\beta$ , chooses a possible new position  $j$  according to  $Q$ , and then changes its position with probability  $\nu_t(\{j\})$  and stays put with probability  $1 - \nu_t(\{j\})$ . Second, each agent leaves its position at a rate proportional to the fraction of agents at its own position. We assume then that, in addition to the previous rates, each agent at position  $i$  goes to position  $j$  at rate  $\alpha p_{i,j} \nu_t^N(\{i\})$ , where  $P = (p_{i,j})_{i,j \in W}$  is defined analogously to  $Q$ . This can be thought of as the agent leaving its position  $i$  due to “overcrowding” at rate  $\alpha \nu_t(\{i\})$  and choosing a new position according to  $P$ . We assume for simplicity that  $p_{i,i} = q_{i,i} = 0$  for all  $i \in W$ .

We will set up the rates using the notation of Assumption C:

$$\Gamma(i, k, \{j\}) = \begin{cases} \alpha p_{i,j} & \text{if } k = i \\ \beta q_{i,j} & \text{if } k = j \\ 0 & \text{otherwise} \end{cases} \quad \text{and} \quad \Lambda \equiv 0.$$

Assume that  $\nu_0^N$  converges in distribution to some  $\nu_0 \in \mathcal{P}$ , let  $\nu_t$  be the limit given by Theorem 3.3.1 and write  $u_t(i) = \nu_t(\{i\})$ . It is easy to check that  $u_t$  satisfies

$$\frac{du_t(i)}{dt} = \alpha \sum_{j=-m}^m p_{j,i} u_t(j)^2 - \alpha u_t(i)^2 + \beta \sum_{j=-m}^m [q_{j,i} - q_{i,j}] u_t(i) u_t(j).$$

Now let  $\sigma_t$  be the limit in distribution of the fluctuations process  $\sqrt{N}(u_t^N - u_t)$ , and assume that the initial distributions  $\nu_0^N$  and  $\nu_0$  satisfy the assumptions of Theorem 3a. It easy to check as before that

$$F_i(\sigma_t) = 2\alpha \sum_{j=-m}^m p_{j,i} u_t(j) \sigma_t(j) - 2\alpha u_t(i) \sigma_t(i) + \beta \sum_{j=-m}^m [q_{j,i} - q_{i,j}] (u_t(i) \sigma_t(j) + u_t(j) \sigma_t(i)).$$

Thus, after computing the quadratic covariations we obtain the following: if  $\star$  denotes the coordinate-wise product in  $\mathbb{R}^{|W|}$  (i.e.,  $u \star v(i) = u(i)v(i)$ ) then the limiting fluctuations process  $\sigma_t$  solves

$$d\sigma_t = 2\alpha P^t(u_t \star \sigma_t) dt - 2\alpha u_t \star \sigma_t dt + \beta \left( [Q^t - Q] \sigma_t \right) \star u_t dt + \beta \left( [Q^t - Q] u_t \right) \star \sigma_t dt + \sqrt{G(t)} dB_t,$$

where  $B_t$  is a  $(2m+1)$ -dimensional standard Brownian motion and  $G(t)$  is given by

$$G_{i,j}(t) = \begin{cases} \alpha \sum_{k \neq i} p_{k,i} u_t(k)^2 + \alpha u_t(i)^2 + \beta \sum_{k \neq i} (q_{k,i} + q_{i,k}) u_t(i) u_t(k) & \text{if } i = j \\ -\alpha (p_{j,i} u_t(j)^2 + p_{i,j} u_t(i)^2) - \beta (q_{j,i} + q_{i,j}) u_t(i) u_t(j) & \text{if } i \neq j. \end{cases}$$

### 3.4.3.2 $W = \mathbb{Z}^d$

In this case  $\mathcal{C}_b(W)$  is no longer finite-dimensional and, moreover, the type space is not compact, so we will need to make use of the weighting functions  $\rho_i$ . We let  $D = \lfloor d/2 \rfloor + 1$  and take

$$\rho_i(x) = \sqrt{1 + |x|^{2iD}}.$$

Clearly, we have in this case that  $\rho_1^p \leq C\rho_4$  for  $C = p = 2$ .

Consider the following spaces:

$$\begin{aligned} \mathcal{C}_0 &= \ell^\infty(\mathbb{Z}^d) = \left\{ \varphi: \mathbb{Z}^d \rightarrow \mathbb{R} \text{ such that } \|\varphi\|_\infty < \infty \right\}, \\ \mathcal{C}_i &= \ell^{\infty, iD}(\mathbb{Z}^d) = \left\{ \varphi: \mathbb{Z}^d \rightarrow \mathbb{R} \text{ such that } \|\varphi\|_{\infty, iD} = \sup_{x \in \mathbb{Z}^d} \frac{|\varphi(x)|}{1 + |x|^{iD}} < \infty \right\} \\ &\hspace{20em} \text{(for } i = 2, 3), \\ \mathcal{H}_i &= \ell^{2, iD}(\mathbb{Z}^d) = \left\{ \varphi: \mathbb{Z}^d \rightarrow \mathbb{R} \text{ such that } \|\varphi\|_{2, iD}^2 = \sum_{x \in \mathbb{Z}^d} \frac{|\varphi(x)|^2}{1 + |x|^{2iD}} < \infty \right\} \\ &\hspace{20em} \text{(for } i = 1, 2, 3, 4), \end{aligned}$$

endowed with the norms defined within these definitions (we observe that  $\rho_i$  does not appear explicitly in the definition of the spaces  $\mathcal{C}_i$ , but the definition does not change if we replace the weighting function  $1 + |x|^{iD}$  appearing there by  $\rho_i$ ). These spaces are easily checked to be Banach (the  $\mathcal{C}_i$ ) and Hilbert (the  $\mathcal{H}_i$ ) as required. With these definitions we have the following continuous embeddings:

$$\begin{aligned} \ell^\infty(\mathbb{Z}^d) \hookrightarrow \ell^{2, D}(\mathbb{Z}^d) \xrightarrow{c} \ell^{2, 2D}(\mathbb{Z}^d) \hookrightarrow \ell^{\infty, 2D}(\mathbb{Z}^d) \hookrightarrow \ell^{2, 3D}(\mathbb{Z}^d) \\ \hookrightarrow \ell^{\infty, 3D}(\mathbb{Z}^d) \hookrightarrow \ell^{2, 4D}(\mathbb{Z}^d) \end{aligned} \tag{3.7}$$

(these embeddings will be proved in the proof of Theorem 3b).

To obtain (D1) and (D2) we will need to assume now that

$$\sum_{y \in \mathbb{Z}^d} |y|^{8D} \Gamma(x, z, \{y\}) \leq C (1 + |x|^{8D} + |z|^{8D}) \quad \text{and} \quad (3.8a)$$

$$\begin{aligned} \sum_{y_1, y_2 \in \mathbb{Z}^d} (|y_1|^{8D} + |y_2|^{8D}) \Lambda(x_1, x_2, z, \{(y_1, y_2)\}) \\ \leq C (1 + |x_1|^{8D} + |x_2|^{8D} + |z|^{8D}) \end{aligned} \quad (3.8b)$$

for all  $x_1, x_2, z \in \mathbb{Z}^d$  (the other six inequalities in (D1) and (D2) follow from these two and Jensen's inequality). We remark that in Méléard (1998) the author also needs to assume moments of order  $8D$  for the jump rates ( $8D + 2$  in her case, see  $(H'_1)$  in her paper).

**Theorem 3b.** *In the above context, suppose that Assumption C holds and that (3.5), (3.8a), and (3.8b) hold. Then the conclusion of Theorem 3.4.1 is valid, i.e.,  $\sigma_t^N$  converges in distribution in  $D([0, T], \ell^{-2,D}(\mathbb{Z}^d))$  (where  $\ell^{-2,D}(\mathbb{Z}^d)$  is the dual of  $\ell^{2,D}(\mathbb{Z}^d)$ ) to the unique solution  $\sigma_t$  of the  $(\ell^\infty(\mathbb{Z}^d))'$ -valued system given in (S2).*

We recall that the dual of  $\ell^\infty(\mathbb{Z}^d)$  can be identified with the space of finitely additive measures on  $\mathbb{Z}^d$ , and thus every  $\xi \in \ell^\infty(\mathbb{Z}^d)'$  can be represented as  $(\xi(x))_{x \in \mathbb{Z}^d}$  and we can write

$$\langle \xi, \varphi \rangle = \sum_{x \in \mathbb{Z}^d} \varphi(x) \xi(x)$$

for  $\varphi \in \ell^\infty(\mathbb{Z}^d)$ . Therefore, (S2) can be expressed in this case in a manner analogous to (S2-f).

**Example 3.4.4.** Here we consider a well-known model in mathematical biology, the Fleming-Viot process, which was originally introduced in Fleming and Viot (1979) as a stochastic model in population genetics with a constant number of individuals which keeps track of the positions of the individuals. We will actually consider the version of this model studied in Ferrari and Marić (2007).

We take as a type space  $W = \mathbb{Z}^+$  and consider an infinite matrix  $Q = (q(i, j))_{i, j \in W \cup \{0\}}$  corresponding to the transition rates of a conservative continuous-time Markov process on  $W \cup \{0\}$ , for which 0 is an absorbing state (observe that, in particular,  $q(i, i) = -\sum_{j \neq i} q(i, j)$ ). Each individual moves independently according to  $Q$ , until it gets absorbed at 0. On absorption, it chooses an individual uniformly from the population and jumps (instantaneously) to its position. We assume that the exit rates from each site are uniformly bounded, i.e.,  $\sup_{i \geq 1} \sum_{j \in (W \cup \{0\}) \setminus \{i\}} q(i, j) < \infty$  (this is so that (A1) is satisfied). The rates take the following form:

$$\Gamma(i, k, \{j\}) = \begin{cases} q(i, j) & \text{if } k \neq j \text{ and } i \neq j \\ q(i, j) + q(i, 0) & \text{if } k = j \text{ and } i \neq j \\ 0 & \text{if } i = j \end{cases} \quad \text{and} \quad \Lambda \equiv 0.$$

Observe that with this definition, the total rate at which a particle at  $i$  jumps to  $j$  when the whole population is at state  $\nu$  is given by  $q(i, j) + q(i, 0)\nu(\{j\})$ .

We will write  $u_t^N(i) = \nu_t^N(\{i\})$ . It is clear that this model satisfies the assumptions of Theorem 3.3.1. Therefore, if the initial distributions  $u_0^N$  converge, and we denote by  $u_t$  the limit given by Theorem 3.3.1, we obtain that for each  $i \geq 1$ ,

$$\frac{du_t(\{i\})}{dt} = \sum_{j \geq 1} [q(i, j) + q(i, 0)u_t(j)]u_t(i).$$

This limit was obtained in Theorem 1.2 of Ferrari and Marić (2007) (though there the convergence is proved for each fixed  $t$ ).

To study the fluctuations process we need to add the following moment assumption on  $Q$ :

$$\sum_{j \geq 1} j^8 q(i, j) \leq C(1 + i^8)$$

for some  $C > 0$  independent of  $i$ . With this, if (3.5) holds, we can apply Theorem 3.4.1. By the remark following Theorem 3b, to express the limiting system for the fluctuations process it is enough to apply (S2-w) to functions of the form  $\varphi = \mathbf{1}_i$  for each  $i \geq 1$ . Doing this, and after some algebraic manipulations, we deduce that the limiting fluctuations process  $\sigma_t$  is the unique process with paths in  $C([0, T], \ell^\infty(\mathbb{Z}^+)')$  satisfying the following stochastic differential equation:

$$d\sigma_t = Q^t \sigma_t dt + \left( \sum_{k \geq 1} Q(k, 0) \sigma_t(k) \right) u_t dt + \left( \sum_{k \geq 1} Q(k, 0) u_t(k) \right) \sigma_t dt + \sqrt{V_t} dB_t,$$

where  $B_t$  is an infinite vector of independent standard Brownian motions and  $V_t$  is given by

$$V_t(i, j) = \begin{cases} \sum_{k \neq i} [q(k, i) + q(k, 0)u_t(i)]u_t(k) \\ \quad - [q(i, i) - q(i, 0)]u_t(i) + q(i, 0)u_t(i)^2 & \text{if } i = j, \\ -q(i, j)u_t(i) - q(j, i)u_t(j) - [q(i, 0) + q(j, 0)]u_t(i)u_t(j) & \text{if } i \neq j. \end{cases}$$

### 3.4.3.3 $W = \Omega$ , a compact, sufficiently smooth subset of $\mathbb{R}^d$

Unlike the last case, the type space  $W$  is now compact, so we can simply take  $\rho_i \equiv 1$ . Nevertheless,  $W$  is not a discrete set now, and this leads us to use Sobolev spaces for our sequence of continuous embeddings:

$$\mathcal{C}^{3D}(\Omega) \hookrightarrow H^{3D}(\Omega) \xrightarrow{c} H^{2D}(\Omega) \hookrightarrow \mathcal{C}^D(\Omega) \hookrightarrow H^D(\Omega) \hookrightarrow \mathcal{C}(\Omega) \hookrightarrow L^2(\Omega)$$

(with  $D = \lfloor d/2 \rfloor + 1$  as before), where  $\mathcal{C}^k(\Omega)$  is the space of continuous functions on  $\Omega$  with  $k$  continuous derivatives, endowed with the norm

$$\|\varphi\|_{\mathcal{C}^k(\Omega)} = \sum_{|\alpha| \leq k} \sup_{x \in \Omega} |\partial^\alpha \varphi(x)|,$$

and  $H^k(\Omega)$  is the Sobolev space (with respect to the  $L^2(\Omega)$  norm) of order  $k$ , i.e., the space of functions on  $\Omega$  with  $k$  weak derivatives in  $L^2(\Omega)$ , endowed with the norm

$$\|\varphi\|_{H^k(\Omega)}^2 = \sum_{|\alpha| \leq k} \int_{\Omega} |\partial^\alpha \varphi(x)|^2 dx.$$

The above embeddings are either direct or are consequences of the usual Sobolev embedding theorems, see Theorem 4.12 of Adams (2003). For these to hold we need  $\Omega$  to be sufficiently smooth (a locally Lipschitz boundary is enough). The compact embedding  $H^{2D}(\Omega) \hookrightarrow H^D(\Omega)$  is a consequence of the Rellich–Kondrakov Theorem (see Theorem 6.3 of Adams (2003)).

In this case (D1) and (D2) hold trivially. (D3) is much more delicate, and we will just leave it stated as it is. (The assumptions (H3), (H3)', and (H3)'' of Méléard (1998) give some particular conditions which, if translated to our setting, would assure that (D3) holds. These conditions are suitable in her setting but they unfortunately rule out some interesting examples for us).

**Theorem 3c.** *In the above context, assume that Assumption and C holds, and that (D3) and (3.5) hold. Then the conclusion of Theorem 3.4.1 is valid, i.e.,  $\sigma_t^N$  converges in distribution in  $D([0, T], H^{-3D}(\Omega))$  (where  $H^{-3D}(\Omega)$  is the dual of  $H^{3D}(\Omega)$ ) to the unique solution  $\sigma_t$  of the  $(C^{3D}(\Omega))'$ -valued system given in (S2).*

#### 3.4.3.4 $W = \mathbb{R}^d$

This case combines both of the difficulties encountered before:  $W$  is neither discrete nor compact. To get around these problems we need to use now weighted Sobolev spaces. The weighting functions  $\rho_i$  are given by

$$\rho_i(x) = \sqrt{1 + |x|^{2iD+2q}},$$

where  $D = \lfloor d/2 \rfloor + 1$  and  $q \in \mathbb{N}$  (to be chosen). We consider now the spaces  $\mathcal{C}^{j,k}$  of continuous functions  $\varphi$  with continuous partial derivatives up to order  $j$  and such that  $\lim_{|x| \rightarrow \infty} |\partial^\alpha \varphi(x)| / (1 + |x|^k) = 0$  for all  $|\alpha| \leq j$ , with the norms

$$\|\varphi\|_{\mathcal{C}^{j,k}} = \sum_{|\alpha| \leq j} \sup_{x \in \mathbb{R}^d} \frac{|\partial^\alpha \varphi(x)|}{1 + |x|^k},$$

(as in Section 3.4.3.2, the weighting functions  $\rho_i$  do not appear explicitly here, but the definition does not change if we replace the term  $1 + |x|^k$  by  $\sqrt{1 + |x|^{2k}}$ ) and the weighted Sobolev spaces  $W_0^{j,k}$  (with respect to the  $L^2$  norm) defined as follows: we define the norms

$$\|\varphi\|_{W_0^{j,k}}^2 = \sum_{|\alpha| \leq j} \int_{\mathbb{R}^d} \frac{|\partial^\alpha \varphi(x)|^2}{1 + |x|^{2k}} dx$$

and let  $W_0^{j,k}$  be the closure in  $L^2$  under this norm of the space of functions of class  $\mathcal{C}^\infty$  with compact support.

The right sequence of embeddings is now the following:

$$\begin{aligned} \mathcal{C}^{3D,q} &\hookrightarrow W_0^{3D,D+q} \xrightarrow{c} W_0^{2D,2D+q} \hookrightarrow \mathcal{C}^{D,2D+q} \hookrightarrow W_0^{D,3D+q} \\ &\hookrightarrow \mathcal{C}^{0,3D+q} \hookrightarrow W_0^{0,4D+q}. \end{aligned}$$

$q \in \mathbb{N}$  can be chosen depending on the specific example being analyzed:  $q = 0$  works for many examples, but as we will see in the next example, choosing a positive  $q$  ( $q = 1$  in that case) can help, for instance, by making all constant functions be in  $\mathcal{C}^{3D,q}$ . These embeddings are, as before, either straightforward or consequences of the usual Sobolev embedding theorems (adapted now to the weighted case; see Méléard (1998), where the author uses the same type of embeddings, and see Kufner (1980) for a general discussion of weighted Sobolev spaces).

To obtain (D1) and (D2) we need to add the following moment assumptions on the rates, analogous to those we used in Theorem 3b: for all  $x, x_1, x_2, z \in \mathbb{R}^d$ ,

$$\int_{\mathbb{R}^d} |y|^{8D+2q} \Gamma(x, z, dy) \leq C (1 + |x|^{8D+2q} + |z|^{8D+2q}) \quad \text{and} \quad (3.9a)$$

$$\begin{aligned} \int_{\mathbb{R}^d \times \mathbb{R}^d} (|y_1|^{8D+2q} + |y_2|^{8D+2q}) \Lambda(x_1, x_2, z, dy_1 \otimes dy_2) \\ \leq C (1 + |x_1|^{8D+2q} + |x_2|^{8D+2q} + |z|^{8D+2q}). \end{aligned} \quad (3.9b)$$

We observe that the power  $8D + 2q$  appearing in this assumption corresponds exactly, when  $q = 1$ , to the moments of order  $8D + 2$  assumed in  $(H'_1)$  in Méléard (1998). (D3), as in the previous case, is much more involved, so we will again leave it stated as it is.

**Theorem 3d.** *In the above context, assume moreover that Assumption C holds, and that (3.5), (D3), (3.9a), and (3.9b) hold. Then the conclusion of Theorem 3.4.1 is valid, i.e.,  $\sigma_t^N$  converges in distribution in  $D([0, T], W_0^{-3D, D+q})$  (where  $W_0^{-3D, D+q}$  is the dual of  $W_0^{3D, D+q}$ ) to the unique solution  $\sigma_t$  of the  $((C^{3D, q})'$ -valued) system given in (S2).*

**Example 3.4.5** (Continuation of Example 3.3.3). In the previous section we obtained the system (3.3) that characterizes the information percolation model of Duffie and Manso (2007) by using (S1'). If we use (S1) instead we obtain

$$\frac{d}{dt} \langle \nu_t, \varphi \rangle = 2\lambda \langle \nu_t, \nu_t * \varphi \rangle - 2\lambda \langle \nu_t, \varphi \rangle$$

for all  $\varphi \in \mathcal{B}(\mathbb{R})$ , where  $(\nu_s * \varphi)(z) = \int_W \varphi(x + z) \nu_s(dx)$ .

To obtain the fluctuations limit, we need to check the assumptions of Theorem 3d. As we mentioned, we will take  $q = 1$ . Assumption C holds trivially because  $\lambda(w_1, w_2, \nu)$  and  $b(w_1, w_2, \nu, \cdot)$  do not depend on  $\nu$ . We will assume that the initial distribution of the system satisfies (3.5). (3.9a) and (3.9b) are straightforward to check in this case.

We are left checking (D3). Let  $\varphi \in \mathcal{C}^{3,1}$  and take  $\mu_1, \mu_2, \mu_3, \mu_4 \in \mathcal{P}$  having moments of order 10. We have that

$$J_{\mu_1, \mu_2} \varphi(z) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (2\varphi(w_1 + w_2) - \varphi(w_1) - \varphi(w_2)) \mu_1(dw_2) \mu_1(dw_1) \\ + \int_{-\infty}^{\infty} (2\varphi(w + z) - \varphi(w) - \varphi(z)) [\mu_1(dw) + \mu_2(dw)].$$

The first term on the right side is constant in  $z$ , so it is in  $\mathcal{C}^{3,1}$  (this is why we needed  $q = 1$  in this example). For the second term, since  $|\varphi(x)| \leq C \|\varphi\|_{\mathcal{C}^{3,1}} (1 + |x|)$  and  $\langle \mu_i, 1 + |\cdot|^{10} \rangle < \infty$ , the integral is bounded, and hence the derivatives with respect to  $z$  can be taken inside the integral, whence we get that this term is also in  $\mathcal{C}^{3,1}$ . The same argument can be repeated for  $\mathcal{C}^{1,3}$  and  $\mathcal{C}^{0,4}$ . The fact that the norm of this operator in these spaces is bounded uniformly in  $\mu_1, \mu_2$  follows from the same argument. This gives (D3.i). Using the same formula it is easy to show that

$$\|(J_{\mu_1, \mu_2} - J_{\mu_3, \mu_4})\varphi\|_{\mathcal{C}^{3,1}} \leq C \|\varphi\|_{\mathcal{C}^{3,1}} \left[ \|\mu_1 - \mu_3\|_{(\mathcal{C}^{3,1})'} + \|\mu_2 - \mu_4\|_{(\mathcal{C}^{3,1})'} \right],$$

which is stronger than (D3.ii).

We have checked all the assumptions of Theorem 3d, so we deduce that the fluctuations process  $\sigma_t^N$  converges in distribution in  $W_0^{-3,2}$  to the unique solution of (S2) (which is an equation in  $(\mathcal{C}^{3,1})'$ ). Writing down the formula for  $J_s$  in this case yields

$$\langle \sigma_s, J_s \varphi \rangle = 4\lambda \langle \sigma_s, \nu_s * \varphi \rangle - 2\lambda \langle \sigma_s, \varphi \rangle$$

for every  $\varphi \in \mathcal{C}^{3,1}$ . For the quadratic covariations we get

$$C_s^{\varphi_1, \varphi_2} = 4\lambda \langle \nu_s, \nu_s * (\varphi_1 \varphi_2) \rangle - 6\lambda \langle \nu_s, \varphi_1 \rangle \langle \nu_s, \varphi_2 \rangle + 2\lambda \langle \nu_s, \varphi_1 \varphi_2 \rangle$$

for every  $\varphi_1, \varphi_2 \in \mathcal{C}^{3,1}$ . Therefore the limiting fluctuations satisfy

$$\langle \sigma_t, \varphi \rangle = \langle \sigma_0, \varphi \rangle + \lambda \int_0^t [4 \langle \sigma_s, \nu_s * \varphi \rangle - 2 \langle \sigma_s, \varphi \rangle] ds + Z_t(\varphi),$$

with  $Z_t$  being a centered Gaussian process taking values in the dual of  $\mathcal{C}^{3,1}$  with quadratic covariations given by  $[Z(\varphi_1), Z(\varphi_2)]_t = \int_0^t C_s^{\varphi_1, \varphi_2} ds$  for each  $\varphi_1, \varphi_2 \in \mathcal{C}^{3,1}$ .

## 3.5 Proofs of the Results

Throughout this section,  $C$ ,  $C_1$ , and  $C_2$  will denote constants whose values might change from line to line.

### 3.5.1 Preliminary computations and proof of Theorem 3.3.1

Since  $\nu_t^N$  is a jump process in  $\mathcal{P}$  with bounded jump rates, its generator is given by

$$\begin{aligned} \Omega_N f(\nu) &= N \int_W \gamma(w, \nu) \int_W \Delta_N f(\nu; w; w') a(w, \nu, dw') \nu(dw) \\ &\quad + N \int_W \int_W \lambda(w_1, w_2, \nu) \int_{W \times W} \Delta_N f(\nu; w_1, w_2; w'_1 w'_2) \\ &\quad \cdot b(w_1, w_2, \nu, dw_1 \otimes dw'_2) \nu(dw_1) \nu(dw_2) \end{aligned} \quad (3.10)$$

for any bounded measurable function  $f$  on  $\mathcal{P}$ , where  $\Delta_N f(\nu; w; w') = f(\nu + N^{-1}(\delta_{w'} - \delta_w)) - f(\nu)$  and  $\Delta_N f(\nu; w_1, w_2; w'_1, w'_2) = f(\nu + N^{-1}(\delta_{w'_1} + \delta_{w'_2} - \delta_{w_1} - \delta_{w_2})) - f(\nu)$ .

Given  $\varphi \in \mathcal{B}(W)$  we get by using (3.10) and Proposition IV.1.7 of Ethier and

Kurtz (1986) for  $f(\nu) = \langle \nu, \varphi \rangle$  that

$$\begin{aligned}
\langle \nu_t^N, \varphi \rangle &= \langle \nu_0^N, \varphi \rangle + M_t^{N,\varphi} \\
&+ \int_0^t \int_W \gamma(w, \nu_s^N) \int_W (\varphi(w') - \varphi(w)) a(w, \nu_s^N, dw') \nu_s^N(dw) ds \\
&+ \int_0^t \int_W \int_W \lambda(w_1, w_2, \nu_s^N) \int_{W \times W} (\varphi(w'_1) + \varphi(w'_2) - \varphi(w_1) - \varphi(w_2)) \\
&\quad \cdot b(w_1, w_2, \nu_s^N, dw'_1 \otimes dw'_2) \nu_s^N(dw_2) \nu_s^N(dw_1) ds,
\end{aligned} \tag{3.11}$$

where  $M_t^{N,\varphi}$  is a martingale starting at 0. This formula is the key to the proof of Theorem 3.3.1 because, ignoring the martingale term, this equation has the exact form we need for obtaining (S1), and thus the idea will be to show that  $M_t^{N,\varphi}$  vanishes in the limit as  $N \rightarrow \infty$ . This follows from the fact that the quadratic variation of  $M_t^{N,\varphi}$  is of order  $O(1/N)$ . More precisely, we have the following formula: for any  $\varphi_1, \varphi_2 \in \mathcal{B}(W)$ , the predictable quadratic covariation between the martingales  $M_t^{N,\varphi_1}$  and  $M_t^{N,\varphi_2}$  is given by

$$\begin{aligned}
\left\langle M_t^{N,\varphi_1}, M_t^{N,\varphi_2} \right\rangle_t &= \frac{1}{N} \int_0^t \int_W \gamma(w, \nu_s^N) \int_W (\varphi_1(w') - \varphi_1(w)) (\varphi_2(w') - \varphi_2(w)) \\
&\quad \cdot a(w, \nu_s^N, dw') \nu_s^N(dw) ds \\
&+ \frac{1}{N} \int_0^t \int_W \int_W \lambda(w_1, w_2, \nu_s^N) \int_{W \times W} (\varphi_1(w'_1) + \varphi_1(w'_2) - \varphi_1(w_1) - \varphi_1(w_2)) \\
&\quad \cdot (\varphi_2(w'_1) + \varphi_2(w'_2) - \varphi_2(w_1) - \varphi_2(w_2)) \\
&\quad \cdot b(w_1, w_2, \nu_s^N, dw'_1 \otimes dw'_2) \nu_s^N(dw_2) \nu_s^N(dw_1) ds.
\end{aligned} \tag{3.12}$$

The proof of this formula is almost the same as that of Proposition 3.4 of Fournier and Méléard (2004) so we will omit it (there the computation is done for  $\varphi_1 = \varphi_2$ , but the generalization is straightforward, and can also be obtained by polarization).

*Proof of Theorem 3.3.1.* The proof is relatively standard, and its basic idea is the following. First one proves that the sequence of processes  $\langle \nu_t^N, \varphi \rangle$  is tight in

$D([0, T], \mathbb{R})$  for each  $\varphi \in \mathcal{C}_b(W)$ , which in turn implies the tightness of  $\nu_t^N$  in  $D([0, T], \mathcal{P})$ . The tightness of these processes follows from standard techniques and uses (3.11) and (3.12). Next, one uses a martingale argument and (3.12) to show that any limit point of  $\nu_t^N$  satisfies (S1). Finally, using Gronwall's Lemma one deduces that (S1) has a unique solution. We refer the reader to the proof of Theorem 5.3 of Fournier and Méléard (2004) for the details.  $\square$

*Proof of Corollary 3.3.1.* Denote by  $(\tau_i^N)_{i>0}$  the sequence of stopping times corresponding to the jumps of the process  $\nu_t^N$ . Let  $A$  be any Borel subset of  $W$  with  $\mu(A) = 0$  and let  $\varphi$  be any positive function in  $\mathcal{B}(W)$  whose support is contained in  $A$ . By (3.11), for every  $t \in [0, T]$  we have that

$$\begin{aligned} \mathbb{E} \left( \left\langle \nu_{t \wedge \tau_1^N}^N, \varphi \right\rangle \right) &= \mathbb{E} (\langle \nu_0, \varphi \rangle) + \mathbb{E} \left( M_{t \wedge \tau_1^N}^{N, \varphi} \right) \\ &+ \mathbb{E} \left( \int_0^{t \wedge \tau_1^N} \int_W \gamma(w, \nu_s^N) \int_W (\varphi(w') - \varphi(w)) a(w, \nu_s^N, dw') \nu_s^N(dw) ds \right) \\ &+ \mathbb{E} \left( \int_0^{t \wedge \tau_1^N} \int_W \int_W \lambda(w_1, w_2, \nu_s^N) \int_{W \times W} (\varphi(w'_1) + \varphi(w'_2) - \varphi(w_1) - \varphi(w_2)) \right. \\ &\quad \left. \cdot b(w_1, w_2, \nu_s^N, dw'_1 \otimes dw'_2) \nu_s^N(dw_2) \nu_s^N(dw_1) ds \right). \end{aligned} \tag{3.13}$$

The first term on the right side of (3.13) is 0 because the support of  $\varphi$  is contained in  $A$  and  $\nu_0(A) = 0$ . The second term is 0 by Doob's Optional Sampling Theorem. For the third term observe that for  $s < \tau_1^N$ ,  $\nu_s^N = \nu_0$ , so

$$\begin{aligned} \mathbb{E} \left( \left| \int_0^{t \wedge \tau_1^N} \int_W \gamma(w, \nu_s^N) \int_W (\varphi(w') - \varphi(w)) a(w, \nu_s^N, dw') \nu_s^N(dw) ds \right| \right) \\ \leq \bar{\gamma} \mathbb{E} \left( \int_0^{t \wedge \tau_1^N} \int_W \int_W |\varphi(w') - \varphi(w)| a(w, \nu_0, dw') \nu_0(dw) \right) \end{aligned}$$

which is 0 since  $\varphi$  is supported inside  $A$  and the measure  $\int_W a(w', \nu_0, \cdot) \nu_0(dw')$  is absolutely continuous with respect to  $\mu$ . The fourth term is 0 by analogous

reasons. We deduce that the expectation on the left side of (3.13) is 0, and therefore, since  $\varphi$  is positive,  $\langle \nu_{t \wedge \tau_1^N}^N, \varphi \rangle = 0$  with probability 1. In particular,  $\nu_{t \wedge \tau_1^N}^N$  is absolutely continuous with respect to  $\mu$  for all  $t \in [0, T]$  with probability 1.

Using the strong Markov property we obtain inductively that  $\langle \nu_{t \wedge \tau_i^N}^N, \varphi \rangle = 0$  almost surely for every  $i > 0$  and  $t \in [0, T]$ . Since the jump rates of the process are bounded, there are finitely many jumps before  $T$  with probability 1, and we deduce that  $\langle \nu_t^N, \varphi \rangle = 0$  almost surely for all  $t \in [0, T]$ . Now if  $\nu_t$  is the limit in distribution of the sequence  $\nu_t^N$  given by Theorem 3.3.1 and  $\varphi \in \mathcal{C}_b(W)$ ,  $\mathbb{E}(\langle \nu_t^N, \varphi \rangle) \rightarrow \langle \nu_t, \varphi \rangle$  as  $N \rightarrow \infty$ , so  $\langle \nu_t, \varphi \rangle = 0$  for all  $t \in [0, T]$  whenever  $\varphi$  is supported inside  $A$ , and the result follows.  $\square$

### 3.5.2 Proof of Theorem 3.4.1

We will assume throughout this part that all the assumptions of Theorem 3.4.1 hold. For simplicity, we will also assume that  $\Gamma \equiv 0$  (these terms are easier to handle and are in fact a particular case of the ones corresponding to  $\Lambda$ ).

Before getting started we recall that, by Parseval's identity, given any  $A \in \mathcal{H}_i'$  and a complete orthonormal basis  $(\phi_k)_{k \geq 0}$  of  $\mathcal{H}_i$ ,

$$\|A\|_{\mathcal{H}_i'}^2 = \sum_{k \geq 0} |A(\phi_k)|^2.$$

We will use this fact several times below.

### 3.5.2.1 Moment estimates for $\nu_t^N$ and $\nu_t$

Recall that we have assumed that  $\sup_{N>0} \mathbb{E}(\langle \nu_0^N, \rho_4^2 \rangle) < \infty$  and  $\mathbb{E}(\langle \nu_0, \rho_4^2 \rangle) < \infty$ .

We need to show that these moment assumptions propagate to  $\nu_t^N$  and  $\nu_t$ :

**Proposition 3.5.1.** *The following properties hold:*

$$\sup_{N>0} \mathbb{E} \left( \sup_{t \in [0, T]} \langle \nu_t^N, \rho_4^2 \rangle \right) < \infty, \quad \text{and} \quad (3.14a)$$

$$\sup_{t \in [0, T]} \langle \nu_t, \rho_4^2 \rangle < \infty. \quad (3.14b)$$

The proof of this result will rely on an explicit construction of the process in terms of Poisson point measures. This is similar to what is done in Section 2.2 of Fournier and Méléard (2004) (though we will need to use a more abstract approach because our type spaces are not necessarily Euclidean), so we will only sketch the main ideas.

We fix  $N > 0$  and consider the following random objects, defined on a sufficiently large probability space: a  $\mathcal{P}$ -valued random variable  $\nu_0^N$  (corresponding to the initial distribution) and a Poisson point measure  $Q(ds, di, dj, du, d\theta)$  on  $[0, T] \times I_N \times I_N \times [0, 1] \times [0, 1]$  with intensity measure  $(\bar{\lambda}/N) ds di dj du d\theta$ . We also consider a Blackwell-Dubins representation  $\varrho$  of  $\mathcal{P}(W \times W)$  with respect to a uniform random variable on  $[0, 1]$ , i.e., a continuous function  $\varrho: \mathcal{P}(W \times W) \times [0, 1] \rightarrow W \times W$  such that  $\varrho(\xi, \cdot)$  has distribution  $\xi$  (with respect to the Lebesgue measure on  $[0, 1]$ ) for all  $\xi \in \mathcal{P}(W \times W)$  and  $\varrho(\cdot, u)$  is continuous for almost every  $u \in [0, 1]$  (see Blackwell and Dubins (1983) for the existence of such a function). This gives us an abstract way to use a uniform random variable to pick the pairs of types to which agents go after interacting. Finally, we introduce the following notation:  $\eta^i(\nu_t^N)$  will denote the  $i$ -th type, with respect to some fixed total order of

$W$ , appearing in  $\nu_t^N$  (we recall that, under the axiom of choice, any set can be well-ordered, and hence totally ordered; moreover, this ordering can be taken to be measurable because  $W$ , being a Polish space, is measurably isomorphic to  $[0, 1]$ ). With this definition, choosing a type uniformly from  $\nu_t^N$  is the same as choosing  $i$  uniformly from  $I_N$  and considering the type given by  $\eta^i(\nu_t^N)$ . Our process can be represented then as follows:

$$\begin{aligned} \nu_t^N = \nu_0^N + \int_0^t \int_{I_N} \int_{I_N} \int_0^1 \int_0^1 \frac{1}{N} & \left[ \delta_{\varrho^1(b(\eta^i(\nu_{s-}^N), \eta^j(\nu_{s-}^N), \nu_{s-}^N, \cdot), u)} \right. \\ & \left. + \delta_{\varrho^2(b(\eta^i(\nu_{s-}^N), \eta^j(\nu_{s-}^N), \nu_{s-}^N, \cdot), u)} - \delta_{\eta^i(\nu_{s-}^N)} - \delta_{\eta^j(\nu_{s-}^N)} \right] \\ & \cdot \mathbf{1}_{\theta \leq \lambda(\eta^i(\nu_{s-}^N), \eta^j(\nu_{s-}^N), \nu_{s-}^N) / \bar{\lambda}} Q(ds, di, dj, du, d\theta), \end{aligned}$$

where  $\varrho^1$  and  $\varrho^2$  are the first and second components of  $\varrho$  (see Definition 2.5 in Fournier and Méléard (2004) for more details on this construction).

*Proof of Proposition 3.5.1.* Since  $\langle \nu_t^N, \rho_4^2 \rangle = \langle \nu_0^N, \rho_4^2 \rangle + \sum_{0 \leq s \leq t} [\langle \nu_s^N - \nu_{s-}^N, \rho_4^2 \rangle]$ , it is easy to deduce from the last equation that

$$\begin{aligned} \langle \nu_t^N, \rho_4^2 \rangle = \langle \nu_0^N, \rho_4^2 \rangle + \int_0^t \int_{I_N} \int_{I_N} \int_0^1 \int_0^1 \frac{1}{N} & \left[ \rho_4^2(\varrho^1(b(\eta^i(\nu_{s-}^N), \eta^j(\nu_{s-}^N), \nu_{s-}^N, \cdot), u)) \right. \\ & \left. + \rho_4^2(\varrho^2(b(\eta^i(\nu_{s-}^N), \eta^j(\nu_{s-}^N), \nu_{s-}^N, \cdot), u)) - \rho_4^2(\eta^i(\nu_{s-}^N)) - \rho_4^2(\eta^j(\nu_{s-}^N)) \right] \\ & \cdot \mathbf{1}_{\theta \leq \lambda(\eta^i(\nu_{s-}^N), \eta^j(\nu_{s-}^N), \nu_{s-}^N) / \bar{\lambda}} Q(ds, di, dj, du, d\theta). \end{aligned}$$

Taking expectations and ignoring the (positive) terms being subtracted we ob-

tain

$$\begin{aligned}
\mathbb{E} \left( \sup_{t \in [0, T]} \langle \nu_t^N, \rho_4^2 \rangle \right) &\leq \mathbb{E}(\langle \nu_0^N, \rho_4^2 \rangle) + \frac{1}{N^2} \int_0^T \mathbb{E} \left( \sum_{i=1}^N \sum_{j=1}^N \lambda(\eta^i(\nu_{s-}^N), \eta^j(\nu_{s-}^N), \nu_{s-}^N) \right. \\
&\quad \cdot \int_0^1 \left[ \rho_4^2(\varrho^1(b(\eta^i(\nu_{s-}^N), \eta^j(\nu_{s-}^N), \nu_{s-}^N, \cdot), u)) \right. \\
&\quad \left. \left. + \rho_4^2(\varrho^2(b(\eta^i(\nu_{s-}^N), \eta^j(\nu_{s-}^N), \nu_{s-}^N, \cdot), u)) \right] du \right) ds \\
&= \mathbb{E}(\langle \nu_0^N, \rho_4^2 \rangle) + \int_0^T \mathbb{E} \left( \int_W \int_W \int_W \int_{W \times W} [\rho_4^2(w'_1) + \rho_4^2(w'_2)] \Lambda(w_1, w_2, z, dw'_1 \otimes dw'_2) \right. \\
&\quad \left. \cdot \nu_s^N(dz) \nu_s^N(dw_2) \nu_s^N(dw_1) \right) ds \\
&\leq \mathbb{E}(\langle \nu_0^N, \rho_4^2 \rangle) + C \int_0^T \mathbb{E} \left( \int_W \int_W \int_W [\rho_4^2(w_1) + \rho_4^2(w_2) + \rho_4^2(z)] \right. \\
&\quad \left. \cdot \nu_s^N(dz) \nu_s^N(dw_2) \nu_s^N(dw_1) \right) ds \\
&\leq \mathbb{E}(\langle \nu_0^N, \rho_4^2 \rangle) + C \int_0^T \mathbb{E} \left( \sup_{s \in [0, t]} \langle \nu_s^N, \rho_4^2 \rangle \right) ds,
\end{aligned}$$

where we used (D2) in the second inequality. By hypothesis  $\mathbb{E}(\langle \nu_0^N, \rho_4^2 \rangle)$  is bounded uniformly in  $N$ , so by Gronwall's Lemma we deduce that

$$\mathbb{E} \left( \sup_{t \in [0, T]} \langle \nu_t^N, \rho_4^2 \rangle \right) \leq C_1 e^{C_2 T}, \tag{3.15}$$

with  $C_1$  and  $C_2$  being independent of  $N$ , whence (3.14a) follows.

To get (3.14b), write  $(\rho_4^2 \wedge L)(w) = \rho_4^2(w) \wedge L$ , and observe that, since  $\rho_4^2 \wedge L \in \mathcal{C}_b(W)$ , Theorem 3.3.1 implies that  $\lim_{N \rightarrow \infty} \mathbb{E}(\sup_{t \in [0, T]} \langle \nu_t^N, \rho_4^2 \wedge L \rangle) = \sup_{t \in [0, T]} \langle \nu_t, \rho_4^2 \wedge L \rangle$ , so by (3.15),

$$\sup_{t \in [0, T]} \langle \nu_t, \rho_4^2 \wedge L \rangle \leq C_1 e^{C_2 T}.$$

Using the monotone convergence theorem it is easy to check that  $\sup_{s \in [0, T]} \langle \nu_s, \rho_4^2 \wedge L \rangle \rightarrow \sup_{s \in [0, T]} \langle \nu_s, \rho_4^2 \rangle$  as  $L \rightarrow \infty$ , and thus (3.14b) follows.  $\square$

For most of this section we will continue ignoring the type-process  $\eta_t^N$ , working instead with the empirical distribution process  $\nu_t^N$  we are interested in. However, we will need to consider  $\eta_t^N$  directly in Step 2 of the proof of Theorem 3.4.1, and we will need to use a moment estimate similar to (3.14a) for this process. Observe that statement of the theorem (and that of Theorem 3.3.1) makes no assumption on the distribution of  $\eta_0^N$ , but instead only deals with the initial empirical distribution  $\nu_0^N$ . Therefore we are free to choose  $\eta_0^N$  in any way compatible with  $\nu_0^N$ . For convenience we can construct  $\eta_0^N$  in the following way: assuming  $\nu_0^N$  takes a specific value  $\bar{\nu}_0^N \in \mathcal{P}_a$ , choose  $\eta_0^N(1)$  uniformly from  $\bar{\nu}_0^N$  and then inductively choose  $\eta_0^N(i)$  uniformly from the remaining  $N - i + 1$  individuals, i.e., from  $[N\bar{\nu}_0^N - \delta_{\eta_0^N(1)} - \dots - \delta_{\eta_0^N(i-1)}]/(N - i + 1)$ . It is clear then that, with this choice,  $\eta_0^N$  is exchangeable and  $\frac{1}{N} \sum_{i=1}^N \delta_{\eta_0^N(i)} = \bar{\nu}_0^N$  as required. Moreover, given any  $i \in I_N$ ,  $\mathbb{E}(\rho_4^2(\eta_0^N(i))) = \mathbb{E}(\langle \nu_0^N, \rho_4^2 \rangle)$ , and thus the moment assumption that we made on  $\nu_0^N$  can be rewritten as  $\sup_{N>0} \sup_{i \in I_N} \mathbb{E}(\rho_4^2(\eta_0^N(i))) < \infty$  for all  $i \in I_N$ . The proof of (3.14a) can then be adapted (by modifying slightly the explicit construction we made of  $\nu_t^N$  to deal with  $\eta_t^N$ ) to obtain

$$\sup_{N>0} \sup_{i \in I_N} \mathbb{E} \left( \sup_{t \in [0, T]} \rho_4^2(\eta_t^N(i)) \right) < \infty. \quad (3.16)$$

(We remark that the proof of this estimate uses (3.14a) itself).

### 3.5.2.2 Extension of $\langle \nu_t^N, \cdot \rangle$ and $\langle \nu_t, \cdot \rangle$ to $\mathcal{H}_4'$

The  $\mathcal{P}$ -valued process  $\nu_t^N$  can be seen as a linear functional on  $\mathcal{B}(W)$  via the mapping  $\varphi \mapsto \langle \nu_t^N, \varphi \rangle$ , and the same can be done for  $\nu_t$ . However, since  $\mathcal{H}_4$  consists of measurable but not necessarily bounded functions, the integrals  $\langle \nu_t^N, \varphi \rangle$  and  $\langle \nu_t, \varphi \rangle$  may diverge. Our first task will be to show that these integrals are finite and, moreover, that  $\nu_t^N$  (and  $\nu_t$ ) can be seen as taking values in  $\mathcal{H}_4'$  (and thus

also in all the other dual spaces we are considering). A consequence of this will be that  $\sigma_t^N$  is well defined as an  $\mathcal{H}_4'$ -valued process.

**Proposition 3.5.2.** *The mapping  $\varphi \in \mathcal{H}_4 \mapsto \langle \nu_t^N, \varphi \rangle$  is in  $\mathcal{H}_4'$  almost surely for every  $t \in [0, T]$  and  $N > 0$ . Analogously, the mapping  $\varphi \in \mathcal{H}_4 \mapsto \langle \nu_t, \varphi \rangle$  is in  $\mathcal{H}_4'$  for every  $t \in [0, T]$ .*

Furthermore,  $\nu_t$  satisfies (S1) for every  $\varphi \in \mathcal{H}_4$ , while  $\nu_t^N$  satisfies (3.11) for every  $\varphi \in \mathcal{H}_4$  almost surely. In particular, given any  $\varphi \in \mathcal{H}_4$ ,  $M_t^{N, \varphi}$  is a martingale starting at 0 such that the predictable quadratic covariations  $\langle M^{N, \varphi_1}, M^{N, \varphi_2} \rangle_t$  are the ones given by the formula in (3.12) for all  $\varphi_1, \varphi_2 \in \mathcal{H}_4$ .

*Proof.* We are only going to prove the assertions for  $\nu_t^N$ , the ones for  $\nu_t$  can be checked similarly (and more easily).

The first claim follows directly from (B2) and Proposition 3.5.1: for  $\varphi \in \mathcal{H}_4$ ,

$$|\langle \nu_t^N, \varphi \rangle| \leq \int_W |\varphi(w)| \nu_t^N(dw) \leq C \|\varphi\|_{\mathcal{H}_4} \int_W \rho_4(w) \nu_t^N(dw) \leq C \|\varphi\|_{\mathcal{H}_4} \sqrt{\langle \nu_t^N, \rho_4^2 \rangle},$$

and the term inside the square root is almost surely bounded by (3.14a), so the mapping  $\varphi \in \mathcal{H}_4 \mapsto \langle \nu_t^N, \varphi \rangle$  is continuous almost surely.

Next we need to show that  $\langle \nu_t^N, \varphi \rangle$  satisfies (3.11) for all  $\varphi \in \mathcal{H}_4$ . That is, we need to show that the formula

$$M_t^{N, \varphi} = \langle \nu_t^N, \varphi \rangle - \langle \nu_0^N, \varphi \rangle - \int_0^t \int_W \int_W \int_W \Lambda \varphi(w_1, w_2; z) \nu_s^N(dz) \nu_s^N(dw_2) \nu_s^N(dw_1) ds$$

defines a martingale for each  $\varphi \in \mathcal{H}_4$ . Let  $\varphi \in \mathcal{H}_4$  and  $m > 0$  and write  $(\varphi \wedge m)(w) = \varphi(w) \wedge m$ .  $\varphi \wedge m$  is in  $\mathcal{B}(W)$ , so  $M_t^{N, \varphi \wedge m}$  is a martingale. We deduce that given any  $0 \leq s_1 \leq \dots \leq s_k < s < t$  and any continuous bounded functions

$\psi_1, \dots, \psi_k$  on  $\mathcal{H}_4$ , if we let

$$X^m = \psi_1(\nu_{s_1}^N) \cdots \psi_k(\nu_{s_k}^N) [M_t^{N, \varphi \wedge m} - M_s^{N, \varphi \wedge m}],$$

then  $\mathbb{E}(X^m) = 0$ . Using the monotone convergence theorem one can show that  $X^m \rightarrow \psi_1(\nu_{s_1}^N) \cdots \psi_k(\nu_{s_k}^N) [M_t^{N, \varphi} - M_s^{N, \varphi}]$  as  $m \rightarrow \infty$ . On the other hand, the sequence  $(X^m)_{m>0}$  is uniformly integrable. Indeed, using (B2) and (3.14a) one can show that

$$\mathbb{E} \left( \left| \psi_1(\nu_{s_1}^N) \cdots \psi_k(\nu_{s_k}^N) [M_t^{N, \varphi \wedge m} - M_s^{N, \varphi \wedge m}] \right|^2 \right) \leq Ct^2 \mathbb{E} \left( \sup_{r \in [0, t]} \langle \nu_r^N, \rho_4^2 \rangle \right) < \infty.$$

We deduce that

$$\mathbb{E}(\psi_1(\nu_{s_1}^N) \cdots \psi_k(\nu_{s_k}^N) [M_t^{N, \varphi} - M_s^{N, \varphi}]) = \lim_{m \rightarrow \infty} \mathbb{E}(X^m) = 0,$$

which implies that  $M_t^{N, \varphi}$  is a martingale. The fact that  $\langle M^{N, \varphi_1}, M^{N, \varphi_2} \rangle_t$  has the right form follows from the same arguments as those for (3.12) (here we need to replace  $\varphi_1$  and  $\varphi_2$  by  $\varphi_1^m$  and  $\varphi_2^m$  and then take  $m \rightarrow \infty$  as above).  $\square$

### 3.5.2.3 The drift term

By Proposition 3.5.2, we have now that the fluctuations process  $\sigma_t^N$  is well defined as a process taking values in  $\mathcal{H}_4'$  and it satisfies

$$\begin{aligned} \langle \sigma_t^N, \varphi \rangle &= \sqrt{N} \langle \nu_0^N - \nu_0, \varphi \rangle + \sqrt{N} M_t^{N, \varphi} \\ &\quad + \sqrt{N} \int_0^t \int_W \int_W \int_W \Lambda \varphi(w_1, w_2; z) \left[ \nu_s^N(dz) \nu_s^N(dw_2) \nu_s^N(dw_1) \right. \\ &\quad \left. - \nu_s(dz) \nu_s(dw_2) \nu_s(dw_1) \right] ds \end{aligned}$$

for every  $\varphi \in \mathcal{H}_4$ . The integral term can be rewritten as

$$\begin{aligned} \int_0^t \int_W \int_W \int_W \Lambda \varphi(w_1, w_2; z) \left[ \sigma_s^N(dz) \nu_s^N(dw_2) \nu_s^N(dw_1) \right. \\ \left. + \nu_s(dz) (\sigma_s^N(dw_2) \nu_s^N(dw_1) + \nu_s(dw_2) \sigma_s^N(dw_1)) \right] ds. \end{aligned}$$

Therefore,

$$\langle \sigma_t^N, \varphi \rangle = \sqrt{N} \langle \nu_0^N - \nu_0, \varphi \rangle + \sqrt{N} M_t^{N, \varphi} + \int_0^t \langle \sigma_s^N, J_s^N \varphi \rangle ds \quad (3.17)$$

for each  $\varphi \in \mathcal{H}_4$ , where

$$\begin{aligned} J_s^N \varphi(z) &= \int_W \int_W \Lambda \varphi(w_1, w_2; z) \nu_s^N(dw_2) \nu_s^N(dw_1) \\ &+ \int_W \int_W \Lambda \varphi(w, z; x) \nu_s^N(dw) \nu_s(dx) + \int_W \int_W \Lambda \varphi(z, w; x) \nu_s(dw) \nu_s(dx). \end{aligned} \quad (3.18)$$

Observe that  $J_s^N = J_{\nu_s^N, \nu_s}$  and  $J_s = J_{\nu_s, \nu_s}$ , where the operators  $J_{\mu_1, \mu_2}$  are the ones defined in Assumption D. Hence (D3) and Proposition 3.5.1 imply that  $J_s^N$  and  $J_s$  are bounded linear operators on each space  $\mathcal{C}_i$  ( $i = 0, 2, 3$ ) and, moreover, for all  $\varphi \in \mathcal{C}_i$ ,

$$\|J_s^N \varphi\|_{\mathcal{C}_i} \leq C \|\varphi\|_{\mathcal{C}_i} \quad \text{and} \quad \|J_s \varphi\|_{\mathcal{C}_i} \leq C \|\varphi\|_{\mathcal{C}_i}, \quad (3.19)$$

almost surely for some constant  $C > 0$  independent of  $N$  and  $s$ . Similarly, given any  $\varphi \in \mathcal{C}_0$ ,

$$\|(J_s^N - J_s) \varphi\|_{\mathcal{C}_0} \leq C \|\varphi\|_{\mathcal{C}_0} \|\nu_s^N - \nu_s\|_{\mathcal{C}_2'}, \quad (3.20)$$

almost surely for some constant  $C > 0$  independent of  $N$  and  $s$ .

#### 3.5.2.4 Uniform estimate for the martingale term in $\mathcal{H}_4'$

Proposition 3.5.2 implies that the martingale term  $M_t^{N, \varphi}$  is well defined for all  $\varphi \in \mathcal{H}_4$ . We will denote by  $M_t^N$  the bounded linear functional on  $\mathcal{H}_4$  given by  $M_t^N(\varphi) = M_t^{N, \varphi}$ .

**Theorem 3.5.3.**  $\sqrt{N} M_t^N$  is a càdlàg square integrable martingale in  $\mathcal{H}_4'$ , whose Doob–Meyer process  $(\langle \sqrt{N} M^N \rangle_t(\varphi_1))(\varphi_2) = N \langle \sqrt{N} M^N(\varphi_1), \sqrt{N} M^N(\varphi_2) \rangle_t$  (which is a

linear operator from  $\mathcal{H}_4$  to  $\mathcal{H}_4'$ ) can be obtained from the formula in (3.12). Moreover,

$$\sup_{N>0} \mathbb{E} \left( \sup_{t \in [0, T]} \left\| \sqrt{N} M_t^N \right\|_{\mathcal{H}_4'}^2 \right) < \infty.$$

*Proof.* We already know, by Proposition 3.5.2, that  $\sqrt{N} M_t^N$  is a martingale in  $\mathcal{H}_4'$  with the right Doob–Meyer process. The fact that the paths of  $\sqrt{N} M_t^N$  are in  $D([0, T], \mathcal{H}_4')$  can be checked by the same arguments as those in the proof of Corollary 3.8 in Méléard (1998). So we only need to show the last assertion. Let  $(\phi_k)_{k \geq 0}$  be an orthonormal complete basis of  $\mathcal{H}_4$ . We observe that, by (B2), if  $\chi_w \in \mathcal{H}_4'$  is defined by  $\chi_w(\varphi) = \varphi(w)$  then

$$\sum_{k \geq 0} \phi_k^2(w) = \|\chi_w\|_{\mathcal{H}_4'}^2 \leq C \rho_4^2(w).$$

Thus by Proposition 3.5.2 and Doob's inequality,

$$\begin{aligned} \mathbb{E} \left( \sup_{t \in [0, T]} \left\| \sqrt{N} M_t^N \right\|_{\mathcal{H}_4'}^2 \right) &\leq \mathbb{E} \left( \sum_{k \geq 0} \sup_{t \in [0, T]} N \left| M_t^{N, \phi_k} \right|^2 \right) \\ &\leq 4 \sum_{k \geq 0} \mathbb{E} (N \langle M^{N, \phi_k}, M^{N, \phi_k} \rangle_T) \\ &= 4 \mathbb{E} \left( \int_0^T \int_W \int_W \int_W \int_{W \times W} \sum_{k \geq 0} (\phi_k(w'_1) - \phi_k(w_1) + \phi_k(w'_2) - \phi_k(w_2))^2 \right. \\ &\quad \left. \cdot \Lambda(w_1, w_2, z, dw'_1 \otimes dw'_2) \nu_s^N(dz) \nu_s^N(dw_2) \nu_s^N(dw_1) ds \right) \\ &\leq C \int_0^T \mathbb{E} \left( \int_W \int_W \int_W \int_{W \times W} (\rho_4^2(w_1) + \rho_4^2(w_2) + \rho_4^2(w'_1) + \rho_4^2(w'_2)) \right. \\ &\quad \left. \cdot \Lambda(w_1, w_2, z, dw'_1 \otimes dw'_2) \nu_s^N(dz) \nu_s^N(dw_2) \nu_s^N(dw_1) \right) ds \\ &\leq C \int_0^T \mathbb{E} \left( \int_W \int_W \int_W (2\rho_4^2(w_1) + 2\rho_4^2(w_2) + \rho_4^2(z)) \right. \\ &\quad \left. \cdot \nu_s^N(dz) \nu_s^N(dw_2) \nu_s^N(dw_1) \right) ds \\ &\leq C \int_0^T \mathbb{E} (\langle \nu_s^N, \rho_4^2 \rangle) ds. \end{aligned}$$

The last integral is bounded, uniformly in  $N$ , by Proposition 3.5.1. □

### 3.5.2.5 Evolution equation for $\sigma_t^N$ in $\mathcal{H}_3'$

Recall that our goal is to prove convergence of  $\sigma_t^N$  in  $D([0, T], \mathcal{H}_1')$ . Therefore, a necessary previous step is to make sense of (3.17) as an equation in  $\mathcal{H}_1'$ . We will actually need to show something stronger:  $\sigma_t^N$  can be seen as a semimartingale in  $\mathcal{H}_3'$ , whose semimartingale decomposition takes the form suggested by (3.17). We need the following simple result first (for its proof see Proposition 3.4 of Méléard (1998)):

**Lemma 3.5.4.** *For every  $N > 0$  there is a constant  $C(N) > 0$  such that*

$$\sup_{t \in [0, T]} \mathbb{E}(\|\sigma_t^N\|_{\mathcal{H}_3'}) \leq C(N).$$

Recall that under our assumptions,  $J_s^N$  need not be (and in general is not) a bounded operator on  $\mathcal{H}_3$ , nor on any other  $\mathcal{H}_i$ , and in fact  $J_s^N(\mathcal{H}_i)$  need not even be contained in  $\mathcal{H}_i$ , so it does not make complete sense to speak of  $(J_s^N)^*$  as the adjoint operator of  $J_s^N$ . Nevertheless, for convenience we will abuse notation by writing  $(J_s^N)^* \sigma_s^N$  to denote the linear functional defined by the following mapping:

$$\varphi \in \mathcal{H}_3 \longmapsto (J_s^N)^* \sigma_s^N(\varphi) = \langle \sigma_s^N, J_s^N \varphi \rangle \in \mathbb{R}.$$

Part of the proof of the following result will consist in showing that  $(J_s^N)^* \sigma_s^N$  is actually in  $\mathcal{H}_3'$ .

**Proposition 3.5.5.** *For each  $N > 0$ ,  $\sigma_t^N$  is an  $\mathcal{H}_3'$ -valued semimartingale, and its Doob–Meyer decomposition is given by*

$$\sigma_t^N = \sigma_0^N + \sqrt{N} M_t^N + \int_0^t (J_s^N)^* \sigma_s^N ds, \tag{3.21}$$

where the above is a Bochner integral in  $\mathcal{H}_3'$ .

*Proof.* By Theorem 3.5.3 and the embedding  $\mathcal{H}_4' \hookrightarrow \mathcal{H}_3'$ ,  $\sqrt{N}M_t^N$  is an  $\mathcal{H}_3'$ -valued martingale. Thus, by (3.17), the only thing we need to show is that the integral term makes sense as a Bochner integral in  $\mathcal{H}_3'$ . The first step in doing this is to show that  $(J_s^N)^* \sigma_s^N \in \mathcal{H}_3'$  for all  $s \in [0, T]$ . That is, we need to show that there is a  $C > 0$  such that

$$|\langle \sigma_s^N, J_s^N \varphi \rangle| \leq C \|\varphi\|_{\mathcal{H}_3} \quad (3.22)$$

for all  $\varphi \in \mathcal{H}_3$ . Observe that by (3.19) and the embedding  $\mathcal{H}_3 \hookrightarrow \mathcal{C}_3$ ,  $J_s^N \varphi \in \mathcal{C}_3$  for  $\varphi \in \mathcal{H}_3$ , and thus

$$|\langle \sigma_s^N, J_s^N \varphi \rangle| \leq \|\sigma_s^N\|_{\mathcal{C}_3'} \|J_s^N \varphi\|_{\mathcal{C}_3} \leq C \|\sigma_s^N\|_{\mathcal{C}_3'} \|\varphi\|_{\mathcal{C}_3} \leq C \|\sigma_s^N\|_{\mathcal{C}_3'} \|\varphi\|_{\mathcal{H}_3}$$

for such a function  $\varphi$  by (B1), so (3.22) holds almost surely by Lemma 3.5.4 and (B1').

To see that the Bochner integral is (almost surely) well defined, we recall (see Section V.5 in Yosida (1995)) that it is enough to prove that: (i) given any function  $F$  in the dual of  $\mathcal{H}_3'$ , the mapping  $s \mapsto F((J_s^N)^* \sigma_s^N)$  is measurable; and (ii)  $\int_0^T \|(J_s^N)^* \sigma_s^N\|_{\mathcal{H}_3'} ds < \infty$ . (i) is satisfied by the continuity assumptions on the parameters and (ii) follows from (3.22), using the fact that the constant  $C$  there can be chosen uniformly in  $s$ .  $\square$

We omit the proof of the following corollary (see Corollary 3.8 of Méléard (1998)):

**Corollary 3.5.6.** *For any  $N > 0$ , the process  $\sigma_t^N$  has paths in  $D([0, T], \mathcal{H}_3')$ .*

### 3.5.2.6 Uniform estimate for $\sigma_t^N$ on $\mathcal{C}_2'$

Having given sense to equation (3.21) in  $\mathcal{H}_3'$ , we can now give a uniform estimate for  $\sigma_t^N$  in  $\mathcal{C}_2'$ . This will be crucial for obtaining the tightness of  $\sigma_t^N$  in the proof of Theorem 3.4.1.

**Theorem 3.5.7.**

$$\sup_{N>0} \sup_{t \in [0, T]} \mathbb{E} \left( \|\sigma_t^N\|_{\mathcal{C}_2'}^2 \right) < \infty.$$

*Proof.* By (3.21) and the embedding  $\mathcal{H}_3' \hookrightarrow \mathcal{C}_2'$ ,

$$\mathbb{E} \left( \|\sigma_t^N\|_{\mathcal{C}_2'}^2 \right) \leq 2\mathbb{E} \left( \|\sigma_0^N\|_{\mathcal{C}_2'}^2 \right) + 2\mathbb{E} \left( \left\| \sqrt{N} M_t^N \right\|_{\mathcal{C}_2'}^2 \right) + 2\mathbb{E} \left( \left\| \int_0^t (J_s^N)^* \sigma_s^N ds \right\|_{\mathcal{C}_2'}^2 \right).$$

The first expectation on the right side is bounded uniformly in  $N$  by (3.5), and the same holds for the second one by (B1') and Theorem 3.5.3. For the last expectation we have

$$\begin{aligned} \mathbb{E} \left( \left\| \int_0^t (J_s^N)^* \sigma_s^N ds \right\|_{\mathcal{C}_2'}^2 \right) &\leq \mathbb{E} \left( \left[ \int_0^t \left\| (J_s^N)^* \sigma_s^N \right\|_{\mathcal{C}_2'} ds \right]^2 \right) \\ &\leq T \int_0^t \mathbb{E} \left( \left\| (J_s^N)^* \sigma_s^N \right\|_{\mathcal{C}_2'}^2 \right) ds \leq CT \int_0^T \mathbb{E} \left( \sup_{s \in [0, t]} \left\| \sigma_s^N \right\|_{\mathcal{C}_2'}^2 \right) dt, \end{aligned}$$

where we used Corollary V.5.1 of Yosida (1995) in the first inequality and (3.19) in the last one. Thus by Gronwall's Lemma we get  $\mathbb{E} \left( \sup_{t \in [0, T]} \left\| \sigma_t^N \right\|_{\mathcal{C}_2'}^2 \right) \leq C_1 e^{C_2 T}$ , uniformly in  $N$ , and the result follows.  $\square$

### 3.5.2.7 Proof of the theorem

We are finally ready to prove Theorem 3.4.1.

*Proof of Theorem 3.4.1.* As before, we will proceed in several steps.

**Step 1.** Our first goal is to show that the sequence of processes  $\sigma_t^N$  is tight in  $D([0, T], \mathcal{H}_1')$ . By Aldous' criterion (which we take from Theorem 2.2.2 in Joffe and Métivier (1986) and the corollary that precedes it in page 34), we need to prove that the following two conditions hold:

(t1) For every rational  $t \in [0, T]$  and every  $\varepsilon > 0$ , there is a compact  $K \subseteq \mathcal{H}_1'$  such that

$$\sup_{N>0} \mathbb{P}(\sigma_t^N \notin K) \leq \varepsilon.$$

(t2) If  $\mathfrak{T}_T^N$  is the collection of stopping times with respect to the natural filtration associated to  $\sigma_t^N$  that are almost surely bounded by  $T$ , then for every  $\varepsilon > 0$ ,

$$\lim_{r \rightarrow 0} \limsup_{N \rightarrow \infty} \sup_{\substack{s < r \\ \tau \in \mathfrak{T}_T^N}} \mathbb{P}(\|\sigma_{\tau+s}^N - \sigma_\tau^N\|_{\mathcal{H}_1'} > \varepsilon) = 0.$$

Observe that since the embedding of  $\mathcal{H}_2'$  into  $\mathcal{H}_1'$  is compact, (t1) will follow once we show that for any  $\varepsilon > 0$  and  $t \in [0, T]$  there is an  $L > 0$  such that  $\sup_{N>0} \mathbb{P}(\|\sigma_t^N\|_{\mathcal{H}_2'} > L) < \varepsilon$ . This follows directly from Markov's inequality, (B1'), and Theorem 3.5.7, since given any  $\varepsilon > 0$ ,

$$\sup_{N>0} \mathbb{P}(\|\sigma_t^N\|_{\mathcal{H}_2'} > L) \leq \frac{1}{L^2} \sup_{N>0} \mathbb{E}(\|\sigma_t^N\|_{\mathcal{H}_2'}^2) \leq \frac{1}{L^2} \sup_{N>0} \mathbb{E}(\|\sigma_t^N\|_{\mathcal{C}_2'}^2) < \varepsilon$$

for large enough  $L$ .

To obtain (t2) we will use the semimartingale decomposition of  $\sigma_t^N$  in  $\mathcal{H}_3'$  given in Proposition 3.5.5, i.e.,  $\sigma_t^N = \sigma_0^N + \sqrt{N}M_t^N + \int_0^t (J_s^N)^* \sigma_s^N ds$ . By Rebolledo's criterion (see Corollary 2.3.3 in Joffe and Métivier (1986)), (t2) is obtained for the martingale term  $\sqrt{N}M_t^N$  if it is proved for the trace of its Doob-Meyer process  $\langle\langle \sqrt{N}M^N \rangle\rangle_t$  in  $\mathcal{H}_1$ , and thus for  $\sigma_t^N$  if it is proved moreover for the finite variation term  $\int_0^t (J_s^N)^* \sigma_s^N ds$  ( $\sigma_0^N$  is tight by hypothesis).

We start with the martingale part. Let  $\tau$  be a stopping time bounded by  $T$  and let  $s > 0$ . Let  $(\phi_k)_{k \geq 0}$  be an orthonormal complete basis of  $\mathcal{H}_1$ . Using the same calculations as in the proof of Theorem 3.5.3 we get

$$\begin{aligned} & \mathbb{E} \left( \left| \text{tr}_{\mathcal{H}_1} \langle \sqrt{N} M^N \rangle_{\tau+s} - \text{tr}_{\mathcal{H}_1} \langle \sqrt{N} M^N \rangle_{\tau} \right| \right) \\ &= \mathbb{E} \left( \int_{\tau}^{\tau+s} \int_W \int_W \int_W \int_{W \times W} \sum_{k \geq 0} (\phi_k(w'_1) - \phi_k(w_1) + \phi_k(w'_2) - \phi_k(w_2))^2 \right. \\ & \quad \left. \cdot \Lambda(w_1, w_2, z, dw'_1 \otimes dw'_2) \nu_s^N(dz) \nu_s^N(dw_2) \nu_s^N(dw_1) \right) \\ & \leq Cs, \end{aligned}$$

uniformly in  $N$ . Thus by Markov's inequality,

$$\mathbb{P} \left( \left| \text{tr}_{\mathcal{H}_1} \langle \sqrt{N} M^N \rangle_t - \text{tr}_{\mathcal{H}_1} \langle \sqrt{N} M^N \rangle_{\tau} \right| > \varepsilon \right) \leq \frac{1}{\varepsilon} Cs,$$

whence (t2) follows for the martingale term.

For the integral term we have that

$$\begin{aligned} & \mathbb{E} \left( \left\| \int_0^{\tau+s} (J_r^N)^* \sigma_r^N dr - \int_0^{\tau} (J_r^N)^* \sigma_r^N dr \right\|_{\mathcal{H}_1'} \right) \leq \mathbb{E} \left( \int_{\tau}^{\tau+s} \|(J_r^N)^* \sigma_r^N\|_{\mathcal{C}_2'} dr \right) \\ & \leq C \int_{\tau}^{\tau+s} \mathbb{E} \left( \|\sigma_r^N\|_{\mathcal{C}_2'} \right) dr \leq Cs \sup_{r \in [0, T]} \sqrt{\mathbb{E}(\|\sigma_r^N\|_{\mathcal{C}_2'}^2)} \end{aligned}$$

for some  $C > 0$ , uniformly in  $N$ , where we used Corollary V.5.1 of Yosida (1995) as before and (B1') in the first inequality and (3.19) in the second one. Using Markov's inequality as before and Theorem 3.5.7 we obtain (t2) for the integral term.

**Step 2.** We have now that every subsequence of  $\sigma_t^N$  has a further subsequence which converges in distribution in  $D([0, T], \mathcal{H}_1')$ . Consider a convergent subsequence of  $\sigma_t^N$ , which we will still denote by  $\sigma_t^N$ , and let  $\sigma_t$  be its limit in  $D([0, T], \mathcal{H}_1')$ . Observe that the only jumps of  $\sigma_t^N$  are those coming from  $\nu_t^N$

and, with probability 1, at most two agents jump at the same time. Suppose that there is a jump at time  $t$ , involving agents  $i$  and  $j$ . Then given  $\varphi \in \mathcal{H}_1$ ,

$$\begin{aligned} |\langle \sigma_t^N, \varphi \rangle - \langle \sigma_{t-}^N, \varphi \rangle| &= \frac{1}{\sqrt{N}} |\varphi(\eta_t^N(i)) + \varphi(\eta_t^N(j)) - \varphi(\eta_{t-}^N(i)) - \varphi(\eta_{t-}^N(j))| \\ &\leq \frac{C}{\sqrt{N}} \|\varphi\|_{\mathcal{H}_1} \left[ \sup_{r \in [0, t]} \rho_1(\eta_r^N(i)) + \sup_{r \in [0, t]} \rho_1(\eta_r^N(j)) \right] \end{aligned}$$

by (B2). We deduce by (3.16) that

$$\mathbb{E} \left( \sup_{s \in [0, t]} \|\sigma_s^N - \sigma_{s-}^N\|_{\mathcal{H}_1'}^2 \right) \leq \frac{C}{N} \quad (3.24)$$

and hence  $\sup_{s \in [0, t]} \|\sigma_s^N - \sigma_{s-}^N\|_{\mathcal{H}_1'}$  converges in probability to 0 as  $N \rightarrow \infty$ . Therefore,  $\sigma_t$  is almost surely strongly continuous by Proposition 3.26 of Jacod and Shiryaev (1987). That is, we have shown that every limit point of  $\sigma_t^N$  is (almost surely) in  $C([0, T], \mathcal{H}_1')$ .

**Step 3.** Our next goal is to prove that the sequence of martingales  $\sqrt{N}M_t^N$  converges in distribution in  $D([0, T], \mathcal{H}_1')$  to the centered Gaussian process  $Z_t$  defined in the statement of the theorem. That is, we need to show that given any  $\varphi_1, \varphi_2 \in \mathcal{H}_1$ , the sequence of  $\mathbb{R}^2$ -valued martingales  $\sqrt{N}M_t^{N, (\varphi_1, \varphi_2)} = (\sqrt{N}M_t^{N, \varphi_1}, \sqrt{N}M_t^{N, \varphi_2})$  converges in distribution to  $(Z_t(\varphi_1), Z_t(\varphi_2))$ .

By (3.21),  $\sqrt{N}M_t^N$  and  $\sigma_t^N$  have the same jumps, and thus (3.24) implies that

$$\mathbb{E} \left( \sup_{s \in [0, t]} \left| \sqrt{N}M_s^{N, (\varphi_1, \varphi_2)} - \sqrt{N}M_{s-}^{N, (\varphi_1, \varphi_2)} \right|^2 \right) \xrightarrow{N \rightarrow \infty} 0. \quad (3.25)$$

On the other hand, we claim that for every  $\varphi_1, \varphi_2 \in \mathcal{H}_1$ ,

$$\lim_{N \rightarrow \infty} \mathbb{E} \left( \left\langle \sqrt{N}M^{N, \varphi_1}, \sqrt{N}M^{N, \varphi_2} \right\rangle_t \right) = \int_0^t C_s^{\varphi_1, \varphi_2} ds. \quad (3.26)$$

(3.25) and (3.26) imply that  $\sqrt{N}M_t^{N, (\varphi_1, \varphi_2)}$  satisfies the hypotheses of the Martingale Central Limit Theorem (see Theorem VII.1.4 in Ethier and Kurtz (1986)) so,

assuming that (3.26) holds, we get that  $\sqrt{N}M_t^{N,(\varphi_1,\varphi_2)}$  converges in distribution in  $D([0, T], \mathbb{R}^2)$  to  $(Z_t(\varphi_1), Z_t(\varphi_2))$ .

To prove (3.26) it is enough to consider the case  $\varphi_1 = \varphi_2 = \varphi$ , the general case follows by polarization. Given  $\mu \in D([0, T], \mathcal{H}_1')$  let

$$\begin{aligned} \Psi_t(\mu) = \int_0^t \int_W \int_W \int_W \int_{W \times W} (\varphi(w'_1) + \varphi(w'_2) - \varphi(w_1) - \varphi(w_2))^2 \Lambda(w_1, w_2, z, dw'_1 \otimes dw'_2) \\ \cdot \mu_s(dz) \mu_s(dw_2) \mu_s(dw_1) ds. \end{aligned}$$

Then we need to prove that  $\lim_{N \rightarrow \infty} \mathbb{E}(\Psi_t(\nu^N)) = \Psi_t(\nu)$ . Let  $p > 1$  be the exponent we assumed to be such that  $\rho_1^p \leq C\rho_4$  for some  $C > 0$ . Repeating the calculations in the proof of Theorem 3.5.3 and using Jensen's inequality we get that

$$|\Psi_t(\nu^N)|^p \leq \left[ C_1 t \|\varphi\|_{\mathcal{H}_1}^2 \sup_{s \in [0, t]} \langle \nu_s^N, \rho_1^2 \rangle \right]^p \leq C_2 t^p \|\varphi\|_{\mathcal{H}_1}^{2p} \sup_{s \in [0, t]} \langle \nu_s^N, \rho_4^2 \rangle.$$

Thus Proposition 3.5.1 implies that the sequence  $(\Psi_t(\nu^N))_{N > 0}$  is uniformly integrable, whence we deduce the desired convergence.

**Step 4.** As in Step 2, let  $\sigma_t$  be a limit point of  $\sigma_t^N$ . Observe that by the embedding  $\mathcal{H}_1' \hookrightarrow \mathcal{C}_0'$ ,  $\sigma_t^N$  converges in distribution to  $\sigma_t$  in  $D([0, T], \mathcal{C}_0')$ . We want to prove now that  $\sigma_t$  satisfies (S2-w).

Fix  $\varphi \in \mathcal{C}_0$ . By (3.21),

$$\begin{aligned} \langle \sigma_t, \varphi \rangle - \langle \sigma_0, \varphi \rangle - \int_0^t \langle \sigma_s, J_s \varphi \rangle ds - Z_t(\varphi) \\ = \left[ \sqrt{N}M_t^{N,\varphi} - Z_t(\varphi) \right] + [\langle \sigma_t, \varphi \rangle - \langle \sigma_t^N, \varphi \rangle] + [\langle \sigma_0^N, \varphi \rangle - \langle \sigma_0, \varphi \rangle] \\ + \int_0^t [\langle \sigma_s^N, J_s^N \varphi \rangle - \langle \sigma_s^N, J_s \varphi \rangle] ds + \int_0^t [\langle \sigma_s^N, J_s \varphi \rangle - \langle \sigma_s, J_s \varphi \rangle] ds, \end{aligned} \tag{3.27}$$

so we need to show that the right side converges in distribution to 0 as  $N \rightarrow \infty$ . The first term goes to 0 by the previous step. The next two go to 0 because  $\sigma_t$  is a limit point of  $\sigma_t^N$  and, since  $J_s \varphi \in \mathcal{C}_0$ , the last term goes to 0 for the same reason.

To show that the remaining term in (3.27) also goes to 0 in distribution, it is enough to show that

$$\mathbb{E} \left( \left| \int_0^t \langle \sigma_s^N, (J_s^N - J_s) \varphi \rangle ds \right| \right) \xrightarrow{N \rightarrow \infty} 0. \quad (3.28)$$

Since, by (3.19),  $J_s^N - J_s$  maps  $\mathcal{C}_0$  into itself, we get by using (B1') and (3.20) that

$$\begin{aligned} |\langle \sigma_s^N, (J_s^N - J_s) \varphi \rangle| &\leq \|\sigma_s^N\|_{\mathcal{C}_0'} \|(J_s^N - J_s) \varphi\|_{\mathcal{C}_0} \leq C \|\sigma_s^N\|_{\mathcal{C}_2'} \|\varphi\|_{\mathcal{C}_0} \|\nu_s^N - \nu_s\|_{\mathcal{C}_2'} \\ &= \frac{C}{\sqrt{N}} \|\varphi\|_{\mathcal{C}_0} \|\sigma_s^N\|_{\mathcal{C}_2'}^2. \end{aligned}$$

(3.28) now follows from this bound and Theorem 3.5.7.

**Step 5.** We have shown in Step 4 that if  $\sigma_t$  is any accumulation point of  $\sigma_t^N$ , then  $\sigma_t$  satisfies (S2-w) for every  $\varphi \in \mathcal{C}_0$ . To see that the limit points of  $\sigma_t^N$  actually solve (S2), the only thing left to show is that the integral term in (S2) makes sense as a Bochner integral in  $\mathcal{C}_0'$ . This can be verified by repeating the arguments of the proof of Proposition 3.5.5.

**Step 6.** We want to prove now pathwise uniqueness for the solutions of (S2). Fix a centered Gaussian process  $Z_t$  in  $\mathcal{C}_0'$  with the right covariance structure and suppose that  $\sigma_t^1, \sigma_t^2 \in \mathcal{C}_0$  are two solutions of (S2) for this choice of  $Z_t$ . Then  $\sigma_t^1 - \sigma_t^2 = \int_0^t (J_s^* \sigma_s^1 - J_s^* \sigma_s^2) ds$ , so

$$\sup_{t \in [0, T]} \|\sigma_t^1 - \sigma_t^2\|_{\mathcal{C}_0'} \leq \int_0^T \sup_{s \in [0, t]} \|J_s^* (\sigma_s^1 - \sigma_s^2)\|_{\mathcal{C}_0'} dt.$$

By (3.19),  $J_s$  is a bounded operator on  $\mathcal{C}_0$ , and thus so is  $J_s^*$  as an operator on  $\mathcal{C}_0'$ . Moreover,  $\|J_s^*\|_{\mathcal{C}_0'}$  can be bounded uniformly in  $s$ . Thus

$$\mathbb{E}\left(\sup_{t \in [0, T]} \|\sigma_t^1 - \sigma_t^2\|_{\mathcal{C}_0'}\right) \leq C \int_0^t \mathbb{E}\left(\sup_{s \in [0, t]} \|\sigma_s^1 - \sigma_s^2\|_{\mathcal{C}_0'}\right) dt,$$

and Gronwall's Lemma implies that  $\sigma_t^1 = \sigma_t^2$  for all  $t \in [0, T]$  almost surely, so the pathwise uniqueness for (S2) follows.

**Step 7.** We have now that any accumulation point  $\sigma_t$  of the sequence  $\sigma_t^N$  satisfies equation (S2), which has a unique pathwise solution. The last thing left to show is the uniqueness in law for the solutions of this equation. Since we have pathwise uniqueness, this can be obtained by adapting the Yamada–Watanabe Theorem to our setting (see Theorem IX.1.7 of Revuz and Yor (1999)). The proof works in the same way assuming we can construct regular conditional probabilities in  $D([0, T], \mathcal{C}_0')$ , which is possible in any complete metric space (see Theorem I.4.12 of Durrett (1996)). This (together with the embedding  $\mathcal{H}_1' \hookrightarrow \mathcal{C}_0'$ ) implies that (S2) determines a unique process in  $C([0, T], \mathcal{H}_1')$ .  $\square$

### 3.5.3 Proof of Theorems 3.4.2 and 3a-3d

*Proof of Theorem 3.4.2.* There are three conditions to check. The first one,  $\sigma_0^N \implies \sigma_0$  in  $\mathcal{H}_1'$ , follows directly from applying the Central Limit Theorem in  $\mathbb{R}$  to each of the processes  $\langle \sigma_0^N, \varphi \rangle$  for  $\varphi \in \mathcal{H}_1$ , while the condition  $\sup_{N>0} \mathbb{E}(\langle \nu_0^N, \rho_4^2 \rangle) < \infty$  is straightforward. For the remaining one we can prove something stronger, namely that  $\sup_{N>0} \mathbb{E}(\|\sigma_0^N\|_{\mathcal{H}_4'}^2) < \infty$ . In fact, if  $(\phi_k)_{k \geq 0}$  is a complete orthonormal basis of  $\mathcal{H}_4$  and  $\eta_0^N$  is chosen by picking the type  $\eta_0^N(i)$  of each agent  $i \in I_N$

independently according to  $\nu_0$  then

$$\mathbb{E}\left(\|\sigma_0^N\|_{\mathcal{H}_4'}^2\right) = \mathbb{E}\left(\sum_{k \geq 0} \langle \sigma_0^N, \phi_k \rangle^2\right) = \frac{1}{N} \sum_{k \geq 0} \mathbb{E}\left(\left[\sum_{i=1}^N [\phi_k(\eta_0^N(i)) - \langle \nu_0, \phi_k \rangle]\right]^2\right).$$

A simple computation and (B2) (see the proof of Proposition 3.5 in Méléard (1998)) show that this is bounded by  $\mathbb{E}(\langle \nu_0^N, \rho_4^2 \rangle) + \langle \nu_0, \rho_4^2 \rangle$ , which is in turn bounded by some  $C < \infty$  uniformly in  $N$ , so the result follows.  $\square$

For Theorems 3a (finite  $W$ ), 3c ( $W = \Omega \subseteq \mathbb{R}^d$  smooth and compact), and 3d ( $W = \mathbb{R}^d$ ), we already explained why the assumptions of Theorem 3.4.1 hold, so the results follow directly from that theorem (together with (3.6) when  $W$  is finite). We are left with the case  $W = \mathbb{Z}^d$ .

*Proof of Theorem 3b.* Let  $\varphi \in \ell^\infty(\mathbb{Z}^d)$ . Then

$$\|\varphi\|_{2,D}^2 = \sum_{x \in \mathbb{Z}^d} \frac{\varphi(x)^2}{1 + |x|^{2D}} \leq C \|\varphi\|_\infty^2,$$

where we used the fact that  $2D > d$  implies that  $\sum_{x \in \mathbb{Z}^d} (1 + |x|^{2D})^{-1} < \infty$ . This gives the embedding  $\ell^\infty(\mathbb{Z}^d) \hookrightarrow \ell^{2,D}(\mathbb{Z}^d)$ . The other continuous embeddings in (3.7) are similar. To see that the embedding  $\ell^{2,D}(\mathbb{Z}^d) \hookrightarrow \ell^{2,2D}$  is compact, observe that the family  $(e_y)_{y \in \mathbb{Z}^d} \subseteq \ell^{2,D}(\mathbb{Z}^d)$  defined by  $e_y(x) = \sqrt{1 + |x|^{2D}} \mathbf{1}_{x=y}$  defines an orthonormal complete basis of  $\ell^{2,D}(\mathbb{Z}^d)$  and, using the same fact as above,

$$\sum_{y \in \mathbb{Z}^d} \|e_y\|_{2,2D}^2 = \sum_{y \in \mathbb{Z}^d} \frac{1 + |y|^{2D}}{1 + |y|^{4D}} < \infty,$$

so the embedding is Hilbert–Schmidt, and hence compact. (B2) and (B3) follow directly from the definition of the spaces in this case.

(D1) and (D2) for  $\rho_4^2$  are precisely what is assumed in Theorem 3b, and using this and Jensen’s inequality we get the same estimates for  $\rho_1^2$ ,  $\rho_2^2$ , and  $\rho_3^2$ . We are

left checking (D3). For simplicity we will assume here that  $\Lambda \equiv 0$ . For (D3.i), the case  $\mathcal{C}_0 = \ell^\infty(\mathbb{Z}^d)$  is straightforward. Now if  $\langle \mu_i, 1 + |\cdot|^{8D} \rangle < \infty$ ,  $i = 1, 2$ , and  $\varphi \in \ell^{\infty, 2D}(\mathbb{Z}^d)$ ,

$$\begin{aligned}
\left| \frac{J_{\mu_1, \mu_2} \varphi(z)}{1 + |z|^{2D}} \right| &= \frac{1}{1 + |z|^{2D}} \sum_{x \in \mathbb{Z}^d} \sum_{y \in \mathbb{Z}^d} (\varphi(y) - \varphi(x)) \Gamma(x, z, \{y\}) \mu_1(\{x\}) \\
&\quad + \frac{1}{1 + |z|^{2D}} \sum_{x \in \mathbb{Z}^d} \sum_{y \in \mathbb{Z}^d} (\varphi(y) - \varphi(z)) \Gamma(z, x, \{y\}) \mu_2(\{x\}) \\
&\leq \frac{C \|\varphi\|_{\infty, 2D}}{1 + |z|^{2D}} \left[ \sum_{x \in \mathbb{Z}^d} \sum_{y \in \mathbb{Z}^d} (1 + |y|^{2D}) \Gamma(x, z, \{y\}) (\mu_1(\{x\}) + \mu_2(\{x\})) \right. \\
&\quad \left. + \sum_{x \in \mathbb{Z}^d} (1 + |x|^{2D}) \mu_1(\{x\}) + 1 + |z|^{2D} \right] \\
&\leq \frac{C \|\varphi\|_{\infty, 2D}}{1 + |z|^{2D}} \left[ 1 + |z|^{2D} + \sum_{x \in \mathbb{Z}^d} (1 + |x|^{2D}) \mu_2(\{x\}) \right] \leq C \|\varphi\|_{\infty, 2D}
\end{aligned}$$

uniformly in  $z$ , where we used (3.8a) with a power of  $2D$  instead of  $8D$ . We deduce that  $\|J_{\mu_1, \mu_2} \varphi\|_{\infty, 2D} \leq C \|\varphi\|_{\infty, 2D}$  as required. The proof for  $\ell^{\infty, 3D}(\mathbb{Z}^d)$  is similar. For (D3.ii), consider  $\varphi \in \ell^\infty(\mathbb{Z}^d)$  and  $\mu_1, \mu_2, \mu_3, \mu_4 \in \mathcal{P}$ . Then

$$\begin{aligned}
|(J_{\mu_1, \mu_2} - J_{\mu_3, \mu_4})\varphi(z)| &= \left| \int_W \Gamma\varphi(w; z) [\mu_1(dw) - \mu_3(dw)] \right. \\
&\quad \left. + \int_W \Gamma\varphi(z; w) [\mu_2(dw) - \mu_4(dw)] \right| \\
&\leq \|\Gamma\varphi(\cdot; z)\|_\infty \|\mu_1 - \mu_3\|_{\ell^\infty(\mathbb{Z}^d)'} + \|\Gamma\varphi(z; \cdot)\|_\infty \|\mu_2 - \mu_4\|_{\ell^\infty(\mathbb{Z}^d)'} .
\end{aligned}$$

Now  $\|\Gamma\varphi(\cdot; z)\|_\infty$  and  $\|\Gamma\varphi(z; \cdot)\|_\infty$  are both bounded by  $4\bar{\lambda} \|\varphi\|_\infty$ , so we get

$$\|(J_{\mu_1, \mu_2} - J_{\mu_3, \mu_4})\varphi\|_\infty \leq 4\bar{\lambda} \|\varphi\|_\infty \left[ \|\mu_1 - \mu_3\|_{\ell^\infty, 2d(\mathbb{Z}^d)'} + \|\mu_2 - \mu_4\|_{\ell^\infty, 2d(\mathbb{Z}^d)'} \right]$$

as required.  $\square$

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## REFERENCES

- Adams, R. A. (2003). *Sobolev Spaces*. Academic Press, second edition.
- Bayraktar, E., Horst, U., and Sircar, S. (2007). Queueing theoretic approaches to financial price fluctuations. *arXiv:math/0703832*.
- Billingsley, P. (1999). *Convergence of probability measures*. Wiley Series in Probability and Statistics: Probability and Statistics. John Wiley & Sons Inc., New York, second edition.
- Blackwell, D. and Dubins, L. E. (1983). An extension of Skorohod's almost sure representation theorem. *Proc. Amer. Math. Soc.*, **89**(4), 691–692.
- Dai Pra, P., Runggaldier, W. J., Sartori, E., and Tolotti, M. (2007). Large portfolio losses; a dynamic contagion model. *arXiv:0704.1348v1 [math.PR]*.
- Davis, M. H. A. and Esparragoza-Rodriguez, J. C. (2007). Large portfolio credit risk modeling. *Int. J. Theor. Appl. Finance*, **10**(4), 653–678.
- Duffie, D. and Manso, G. (2007). Information percolation in large markets. *American Economic Review, Papers and Proceedings*, **97**, 203–209.
- Duffie, D. and Sun, Y. (2007). Existence of independent random matching. *Ann. Appl. Probab.*, **17**(1), 386–419.
- Duffie, D., Gârleanu, N., and Pedersen, L. H. (2005). Over-the-counter markets. *Econometrica*, **73**(6), 1815–1847.
- Durrett, R. (1996). *Probability: theory and examples*. Duxbury Press, Belmont, CA, second edition.

- Ethier, S. N. and Kurtz, T. G. (1986). *Markov processes: characterization and convergence*. Wiley Series in Probability and Mathematical Statistics. John Wiley & Sons Inc., New York.
- Ferrari, P. A. and Marić, N. (2007). Quasi stationary distributions and Fleming-Viot processes in countable spaces. *Electron. J. Probab.*, **12**, no. 24, 684–702 (electronic).
- Fleming, W. H. and Viot, M. (1979). Some measure-valued Markov processes in population genetics theory. *Indiana Univ. Math. J.*, **28**(5), 817–843.
- Föllmer, H. (1974). Random economies with many interacting agents. *J. Math. Econom.*, **1**(1), 51–62.
- Fournier, N. and Méléard, S. (2004). A microscopic probabilistic description of a locally regulated population and macroscopic approximations. *Ann. Appl. Probab.*, **14**(4), 1880–1919.
- Giesecke, K. and Weber, S. (2004). Cyclical correlations, credit contagion, and portfolio losses. *Journal of Banking and Finance*, **28**(12), 3009–3036.
- Huck, S. and Kosfeld, M. (2007). The dynamics of neighbourhood watch and norm enforcement. *The Economic Journal*, **117**(516), 270–286.
- Jacod, J. and Shiryaev, A. N. (1987). *Limit theorems for stochastic processes*, volume 288 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin.
- Joffe, A. and Métivier, M. (1986). Weak convergence of sequences of semi-martingales with applications to multitype branching processes. *Adv. in Appl. Probab.*, **18**(1), 20–65.

- Kufner, A. (1980). *Weighted Sobolev Spaces*. Leipzig.
- Méléard, S. (1998). Convergence of the fluctuations for interacting diffusions with jumps associated with Boltzmann equations. *Stochastics Stochastics Rep.*, **63**(3-4), 195–225.
- Métivier, M. (1987). Weak convergence of measure valued processes using Sobolev-embedding techniques. In *Stochastic partial differential equations and applications (Trento, 1985)*, volume 1236 of *Lecture Notes in Math.*, pages 172–183. Springer, Berlin.
- Revuz, D. and Yor, M. (1999). *Continuous martingales and Brownian motion*, volume 293 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin, third edition.
- Yosida, K. (1995). *Functional analysis*. Classics in Mathematics. Springer-Verlag, Berlin. Reprint of the sixth (1980) edition.

## CHAPTER 4

### CHAOS IN A SPATIAL EPIDEMIC MODEL \*

We investigate an interacting particle system inspired by the gypsy moth, whose populations grow until they become sufficiently dense so that an epidemic reduces them to a low level. We consider this process on a random 3-regular graph and on the  $d$ -dimensional lattice and torus, with  $d \geq 2$ . On the finite graphs with global dispersal or with a dispersal radius that grows with the number of sites, we prove convergence to a dynamical system that is chaotic for some parameter values. We conjecture that on the infinite lattice with a fixed finite dispersal distance, distant parts of the lattice oscillate out of phase so there is a unique non-trivial stationary distribution.

#### 4.1 Introduction

The inspiration for this paper arose almost twenty years ago. The first author had recently moved to Ithaca, New York and the Northeastern United States was in the midst of a gypsy moth infestation. For all of one summer, he and his wife destroyed egg masses, picked larvae off of trees, and put bands of sticky tape to keep the larvae from climbing the trees. When the next summer came, the outlook for their trees seemed bleak, but suddenly all of the larvae were dead or deformed, a victim of the nuclear polyhedrosis virus, which spreads through the gypsy moth population once it becomes sufficiently dense.

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To model this process we use dynamics that occur in discrete time with each site in some graph  $G_N$  either occupied or vacant. The number of nodes in  $G_N$  will be an increasing function of  $N$  which tends to infinity. Two processes occur alternately: growth and epidemic.

*Growth.* Gypsy moths lay dormant in the winter as eggs, so no occupied site survives to the next time period but gives birth to a mean  $\beta > 1$  number of individuals. Each individual born at  $x$  is sent to a site randomly chosen from  $\mathcal{N}_N(x) \subseteq G_N$ , the *growth neighborhood* of  $x$ , which contains all of the nearest neighbors of  $x$  in the graph but in general will be larger.

*Epidemic.* With a small probability  $\alpha_N$  an infection lands at each site. If the site  $x$  is occupied an infection starts which spreads from  $x$  to all of its occupied neighbors in the graph and continues until all sites in the connected component of occupied sites containing  $x$  are wiped out (observe that the larger the cluster of occupied sites, the more likely it is to be wiped out by the epidemic). It is assumed that the epidemic occurs rapidly so it is completed before the next growing season.

Our goal is to study this process on a random 3-regular graph and on a discrete torus of dimension  $d \geq 2$ . The second graph is more realistic from a biological point of view, but the first one is easier to deal with because explicit formulas are available. In both cases, infections will be transmitted along edges connecting neighbors. Observe that if we assume that  $\alpha_N \rightarrow 0$  then only components with  $O(1/\alpha_N)$  sites will be affected by epidemics. In site percolation on an regular tree of degree 3 and on  $\mathbb{Z}^d$  there is phase transition from all components small to the existence of an infinite component at some density  $p_c$ . On the random 3-regular graph and the torus this phase transition produces one giant

component of size  $O(n)$ . Thus we expect that the density of occupied sites will increase until  $p > p_c$ , at which point a large epidemic occurs and reduces the density to a low level and the cycle begins again. We will show that in some cases this leads to chaotic behavior of the densities.

### 4.1.1 Mean-field growth on a random 3-regular graph

To work our way up to proving results about this system and the corresponding process on the torus we begin with the case in which  $G_N$  is a random 3-regular graph with  $N$  nodes, that is, a graph chosen at random from the set of graphs with  $N$  vertices all of which have degree 3 ( $N$  must be even). We will denote this random graph by  $R_N$  and we will condition on the event that  $R_N$  is connected. It is known, see Janson, Łuczak, and Rucinski (2000), that the probability that  $R_N$  is connected tends to 1. We choose this graph, not because it reflects reality, but because  $R_N$  is locally a tree, so we have explicit formulas for the percolation probabilities. To have a simple process in which the number of occupied sites at the beginning of the growing season is a Markov process, we let  $\mathcal{N}_N(x) = R_N$  for all  $x$ . As we will see, in the limit as  $N \rightarrow \infty$  the result is a very interesting dynamical system.

To guess what this limiting system must be, observe that if we assume that the density of occupied sites before the growth step is  $p$ , so the expected number of occupied sites is  $pN$ , then the expected density after the birth step is

$$f_N(p) = 1 - \left(1 - \frac{\beta}{N}\right)^{pN} \approx f(p) = 1 - e^{-\beta p}.$$

Now the random 3-regular graph looks locally like a tree in which each vertex has degree 3 (we will refer to this tree as the 3-tree). Proceeding heuristically,

in the limit  $N \rightarrow \infty$ , each occupied site survives the epidemic if and only if it is not in the giant component of the percolation process on the 3-tree defined by declaring open the sites that are occupied after the growth step. Thus if the density before the epidemic is  $p$ , the density  $g_T(p)$  after the epidemic (the  $T$  in the subscript is for tree) is exactly the probability that the origin is open in this percolation process but it does not percolate. The threshold for the existence of a giant component is  $p_c = 1/2$ , so if  $p \leq 1/2$  then  $g_T(p) = p$ .

To compute the density for  $p > 1/2$  we need to compute the percolation probability on the 3-tree. Throughout the rest of the paper, whenever we say percolation we mean the event that the origin is an infinite cluster of occupied sites. We start by noting that for site percolation on the binary tree (which is an infinite rooted tree where each vertex has 2 descendants, so all vertices have degree 3 except for the root which has degree 2) the percolation probability  $\theta_{\text{bin}}(p)$  satisfies

$$\theta_{\text{bin}}(p) = p(1 - (1 - \theta_{\text{bin}}(p))^2)$$

since for this event to occur the origin must be occupied and percolation must occur from one of the two neighbors. Solving gives

$$\theta_{\text{bin}}(p) = \frac{2p - 1}{p} = 2 - \frac{1}{p}.$$

On the 3-tree the probability of percolation is then

$$\theta_T(p) = p(1 - (1 - \theta_{\text{bin}}(p))^3)$$

since the site must be occupied and percolation must occur from one of the three neighbors. Thus for  $p \in (1/2, 1]$

$$g_T(p) = \mathbb{P}(0 \text{ is occupied, } |C_0| < \infty) = p - \theta_T(p) = p \left( \frac{1}{p} - 1 \right)^3 = \frac{(1-p)^3}{p^2}.$$

Let  $a_0$  be the solution of  $1 - e^{-\beta a_0} = 1/2$  (i.e.,  $a_0 = (\log 2)/\beta$ ). Combining the formulas for  $f$  and  $g_T$  we see that the limiting dynamical system should be the one defined by the function

$$h_T(p) = g_T(f(p)) = \begin{cases} 1 - e^{-\beta p} & 0 \leq p \leq a_0 \\ \frac{e^{-3\beta p}}{(1 - e^{-\beta p})^2} & a_0 < p \leq 1. \end{cases}$$

Observe that  $h_T$  is continuous in  $[0, 1]$ .

We are interested in properties of the iterates of  $h_T(p)$ :

- If  $\beta \leq 1$  then  $f(p) < p$  for all  $p > 0$  and thus  $h_T^k(p)$  decreases to 0 as  $k \rightarrow \infty$ .
- If  $\beta > 1$  then starting from a small positive  $p$ ,  $f^k(p)$  increases to a unique fixed point  $p^*$ . If  $p^* \leq 1/2$  then we never get an epidemic and  $h_T^k(p)$  increases to the same fixed point.
- $1/2$  is a fixed point when  $e^{-\beta/2} = 1/2$ , i.e.,  $\beta = 2 \log 2$ . When  $\beta > 2 \log 2$ , we let  $a_1 = h_T(1/2) = e^{-3\beta/2}/(1 - e^{-\beta/2})^2$ . Eventually the iterates of  $h_T$  lie in the interval  $[a_1, 1/2]$ , and once they reach this interval, they stay there (see Figure 4.1).

Hence if  $\beta \leq \beta_c = 2 \log 2$ ,  $h_T(p) = f(p)$  for all  $p$  and the epidemic part of the dynamics is not seen in the limiting system. If  $\beta > \beta_c$  then  $h_T(p) < 1/2 < f(p)$  for  $p \geq a_0$ .

Figure 4.2 shows the orbits of the system as a function of  $\beta$ . We plot  $h_T^k(p)$  for  $501 \leq k \leq 550$  to remove the initial transient. Note that the system proceeds directly from a stable fixed point to a “chaotic phase” rather than via period doubling bifurcations of the type occurring in the quadratic maps  $rx(1 - x)$ . To say in what sense the behavior is chaotic, we will use two results of the theory

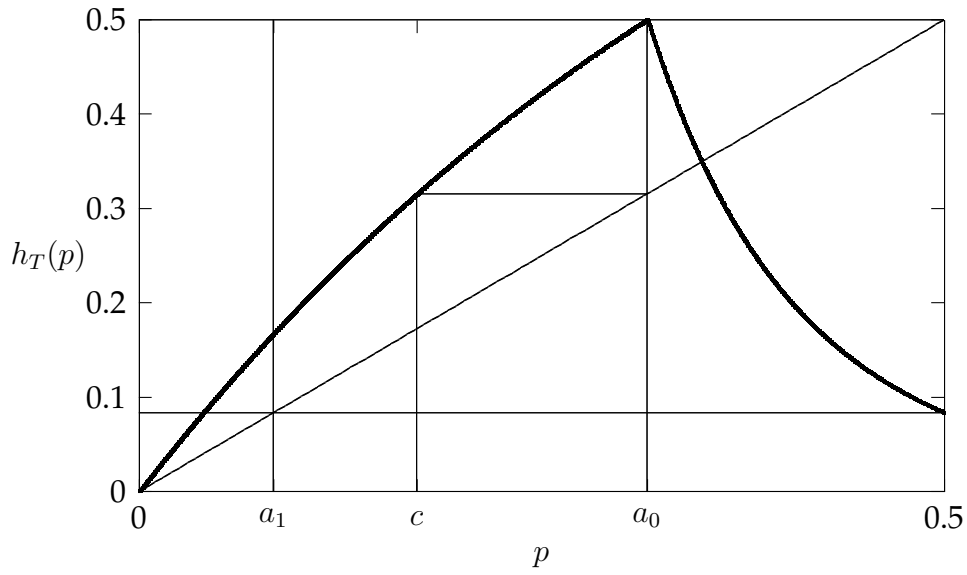


Figure 4.1: Graph of  $h_T$  with  $\beta = 2 \log 3$ . The point  $c = h_T^{-1}(a_0)$  will play a role in the proof of Theorem 4.1.1.

of discrete time dynamical systems. The first result, which we include here for convenience, is commonly referred to as “period three implies chaos”:

**Proposition 4.1.1** (Theorem 1 in Li and Yorke (1975)). *Let  $F: J \rightarrow J$  be a continuous function on a real interval  $J$  and assume that there is point  $a \in J$  such that*

$$F^3(a) \leq a < F(a) < F^2(a).$$

*Then*

- (a) *For every  $k = 1, 2, \dots$  there is a point in  $J$  of period  $k$ , i.e., a point  $r \in J$  such that  $F^k(r) = r$  but  $F^j(r) \neq r$  for  $0 < j < k$ .*
- (b) *There is an uncountable set  $S \subseteq J$  containing no periodic points such that*
  - (b.i) *For every  $p, q \in S, p \neq q,$*

$$\limsup_{N \rightarrow \infty} |F^N(p) - F^N(q)| > 0$$

and

$$\liminf_{N \rightarrow \infty} |F^N(p) - F^N(q)| = 0.$$

(b.ii) For every  $p \in S$  and any periodic point  $q \in J$ ,

$$\limsup_{N \rightarrow \infty} |F^N(p) - F^N(q)| > 0.$$

We will say that  $F$  is *chaotic* if  $F$  satisfies the conditions (a) and (b) above. (b.ii) rules out convergence to periodic orbits, while (b.i) shows that all the points in  $S$  have different limiting behaviors.

**Theorem 4.1.1.**

- (a) The dynamical system defined by the function  $h_T : [a_1, 1/2] \rightarrow [a_1, 1/2]$  is chaotic for every  $\beta > 2 \log 2$ .
- (b) If  $\beta \in (2 \log 2, 2.48]$  then the system has an invariant measure,  $\mu = \mu \circ h_T^{-1}$ , which is absolutely continuous with respect to the Lebesgue measure.

Simulations suggest that (b) actually holds for all  $\beta > 2 \log(2)$ .

Now we come back to the process running on  $R_N$ . We will denote our process by  $\eta_k^N$ , with  $\eta_k^N(i) = 1$  if  $i$  is occupied at time  $k$  and  $\eta_k^N(i) = 0$  if not. The density of occupied sites at time  $k$  will be denoted by  $\rho_k^N$ :

$$\rho_k^N = \frac{1}{N} |\eta_k^N| = \frac{1}{N} \sum_{i=1}^N \eta_k^N(i). \quad (4.1)$$

The initial distribution  $\eta_0^N$  of the process will always be assumed to be a product measure with some density  $p \in [0, 1]$  (so, in particular,  $\rho_0^N$  converges in probability to  $p$ ). In the preceding discussion we argued heuristically that  $\rho_k^N$  converges to the deterministic system defined by  $h_T$ . The next result shows that this is indeed the case:

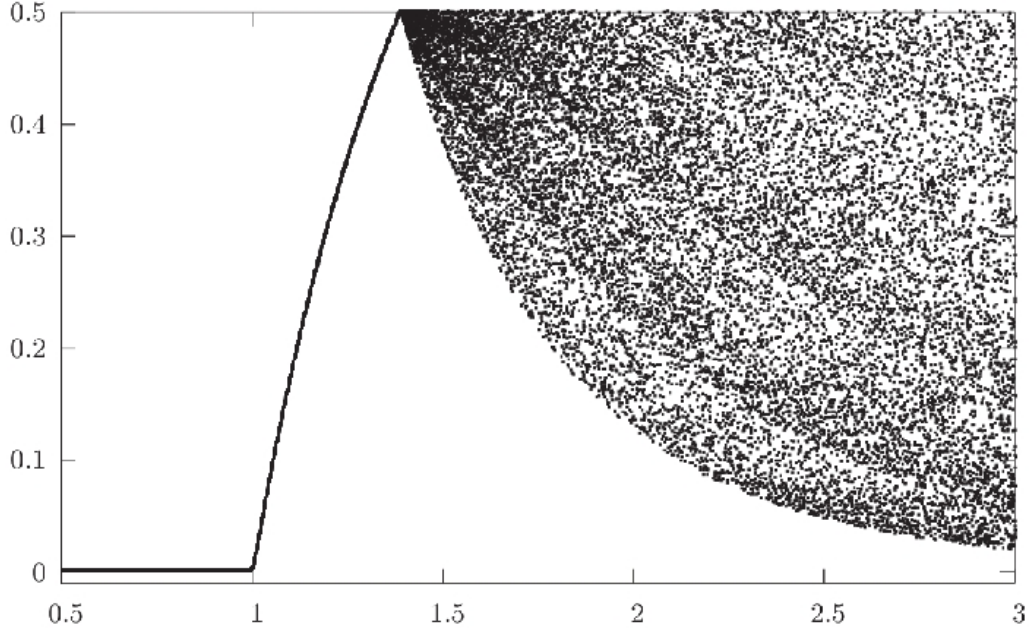


Figure 4.2: Orbits of the system  $(h_T^k(p))_{k \geq 0}$  started at  $p = 0.1$ . The  $x$ -axis has the values of  $\beta$  used in the simulations, while the  $y$ -axis has  $h_T^k(p)$  for  $k = 501, \dots, 550$ .

**Theorem 4.1.2.** *Assume that  $G_N = R_N$  and that the infection probability of the epidemic satisfies*

$$\alpha_N \log_2 N \xrightarrow{N \rightarrow \infty} \infty.$$

*Then the process  $(\rho_k^N)_{k \geq 0}$  converges in distribution to the (deterministic) orbit, starting at  $p$ , of the dynamical system associated to  $h_T$ .*

The above convergence means that  $(\rho_k^N)_{k \geq 0}$  converges in distribution to a deterministic process whose paths are given by the orbits  $(h_T^k(p))_{k \geq 0}$ .

### 4.1.2 Local growth on the $d$ -dimensional torus

Turning now to a more realistic setting, we consider the process running on the  $d$ -dimensional torus  $(\mathbb{Z} \bmod N)^d$ , for  $d \geq 2$ , which we will denote by  $\mathbb{T}_N$ . The

case  $d = 2$  is the one relevant to gypsy moths, but it is no harder to prove our results in general.

To add some more realism and make our process more interesting, we will take now the growth neighborhoods  $\mathcal{N}_N(x)$  to be smaller than  $\mathbb{T}_N$ . We let

$$\mathcal{N}_N(x) = \{y \in \mathbb{T}_N : 0 < \|y - x\|_\infty \leq r_N\}$$

(here the difference  $y - x$  is computed modulo  $N$ ) and take the range  $r_N$  to be such that  $r_N \rightarrow \infty$ . (We remark that on  $\mathbb{T}_N$  we are considering the  $L^1$  distance; in particular, two points  $x, y \in \mathbb{T}_N$  are neighbors if  $\|x - y\|_1 = 1$ ).

We start as before by guessing what the limiting system should be. To do this we will assume for a moment that  $r_N = \infty$  for all  $N$ , so we are back in the case of mean-field growth of the previous subsection. The growth step behaves exactly as before: if  $p$  is the density of occupied sites before the growth step, then the density after is

$$f_{N^2}(p) = 1 - \left(1 - \frac{\beta}{N^2}\right)^{pN^2} \approx f(p) = 1 - e^{-\beta p}.$$

The behavior of the epidemic step in the limit  $N \rightarrow \infty$  is analogous to the one in the random 3-regular graph: if  $p$  is the density of occupied sites before the epidemic, then the density  $g_L(p)$  after (here the subscript  $L$  is for lattice) is the probability that the origin is open but does not percolate in a site percolation process in  $\mathbb{Z}^d$ .

Unlike the case of percolation on the 3-tree, we do not have an explicit formula available for the percolation probability in  $\mathbb{Z}^d$ , but we still know some qualitative properties. Letting  $C_0$  be the percolation cluster containing the origin and

$$\theta_L(p) = \mathbb{P}(|C_0| = \infty)$$

we have that there is a  $p_c \in (0, 1)$  ( $p_c \approx 0.593$  in  $d = 2$ ) such that  $\theta_L(p) = 0$  for  $p \leq p_c$ ,  $\theta_L(p)$  is strictly increasing for  $p > p_c$ , and  $\theta_L(p)$  is infinitely differentiable at every  $p \neq p_c$  (see Theorem 8.92 of Grimmett (1999)). We also have that

$$g_L(p) = \mathbb{P}(0 < |C_0| < \infty) = \mathbb{P}(|C_0| < \infty) - \mathbb{P}(|C_0| = 0) = p - \theta_L(p),$$

so  $g_L(p)$  is infinitely differentiable at  $p \neq p_c$  and  $g_L(p) = p$  for  $p \leq p_c$ .

As before we let  $h_L(p) = g_L(f(p))$  and  $\beta_c$  be the value of  $\beta$  solving  $p_c = 1 - e^{-\beta p_c}$ , that is,

$$\beta_c = \frac{1}{p_c} \log\left(\frac{1}{1 - p_c}\right)$$

( $\beta_c \approx 1.516$  in  $d = 2$ ). Observe that  $g_L(p) \in (0, 1)$  for  $p \in (0, 1)$  so, in particular,  $h_L(p) > 0$  for  $p > 0$ . Our next result holds under an hypothesis on the percolation function which might seem strange at a first look, but which holds in  $d = 2$  and is expected to also hold in  $3 \leq d < 6$ .

**Theorem 4.1.3.** *Suppose that*

$$\lim_{p \downarrow p_c} \theta'_L(p) = \infty. \tag{4.2}$$

*Then there is an  $\varepsilon > 0$  such that for every  $\beta \in (\beta_c, \beta_c + \varepsilon)$  the dynamical system  $(h_L^k(p))_{k \geq 0}$  has an invariant measure which is absolutely continuous with respect to the Lebesgue measure.*

We believe (and simulations suggest) that the result holds for all  $\beta > \beta_c$ . As Yuval Peres pointed out to us, it is easy to show that (4.2) holds in  $d = 2$  using Russo's formula and the fact that the expected number of pivotal sites goes to infinity as  $p \downarrow p_c$  in two dimensions. This argument would obviously work in other dimensions too if we knew that the expected number of pivotal sites blows up at  $p_c$ . This should be the case in  $3 \leq d < 6$  because it is expected that

$\theta_L(p) \approx C(p - p_c)^\gamma$  as  $p \downarrow p_c$  with  $\gamma < 1$  in  $d < 6$ ,  $\gamma = 1$  in  $d > 6$ , and with logarithmic corrections in  $d = 6$  (see, e.g., Chapter 9 of Grimmett (1999)).

Our next goal is to show that the process  $\rho_k^N$  on the torus  $\mathbb{T}_N$  converges to the deterministic orbit of the dynamical system defined by  $h_L$ . The processes  $\eta_k^N$  and  $\rho_k^N$  are defined in this case exactly as for the random 3-regular graph, see (4.1) and the preceding lines. If we consider the case of mean-field growth (i.e.,  $\mathcal{N}_N(x) = \mathbb{T}_N$ ) then the result follows from the same arguments as those we will use to prove Theorem 4.1.2 (the proof is actually simpler because we do not have to prove that the torus looks locally like  $\mathbb{Z}^d$ ). Figure 4.3 shows part of the trajectories of  $\rho_k^N$  in the case of mean-field growth. But, as we mentioned, we want to deal with the more general case  $\mathcal{N}_N(x) = \{y \in \mathbb{T}_N : 0 < \|y - x\|_\infty \leq r_N\}$  with  $r_N \rightarrow \infty$ . The result does not seem to be true if we do not take  $r_N \rightarrow \infty$ . As Figure 4.4 shows, the graph of  $\{(\rho_k^N, \rho_{k+1}^N), k \geq 0\}$  does not correspond to any function. This difficulty disappears as  $N \rightarrow \infty$  if we take  $r_N \rightarrow \infty$  at an appropriate rate.

We will assume the following on  $\alpha_N$  and  $r_N$ :

$$\begin{aligned} \frac{r_N}{N} &\longrightarrow 0 \quad \text{and} \\ \alpha_N r_N &\longrightarrow \infty. \end{aligned}$$

For instance, we could take  $r_N = N^\gamma$  and  $\alpha_N = N^{-\delta}$  for some  $0 < \delta < \gamma < 1$ .

**Theorem 4.1.4.** *Assume that  $G_N = \mathbb{T}_N$ , with  $d \geq 2$ , and that the number of individuals to which each occupied site gives birth to during the growing season is a Poisson random variable with mean  $\beta$ . Then the process  $(\rho_k^N)_{k \geq 0}$  converges in distribution to the (deterministic) orbit, starting at  $p$ , of the dynamical system associated to  $h_L$ .*

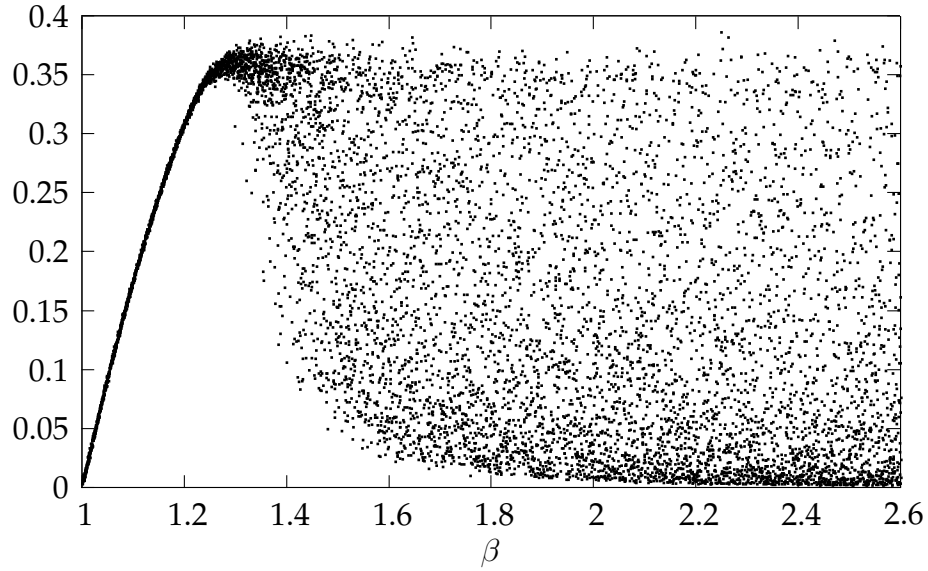


Figure 4.3: Density process  $(\rho_k^N(p))_{k \geq 0}$  running on the 2-dimensional torus with mean-field growth, depicted for  $k = 501, \dots, 550$  for different values of the parameter  $\beta$  (similar to Figure 4.2). Here  $N = 500$ .

### 4.1.3 Local growth on $\mathbb{Z}^d$

We now consider the case in which  $r_N$  is constant. Figure 4.5 shows that when  $r_N = 5$  the fluctuations in the density of occupied sites decrease as the system size increases. Figure 4.6 shows a picture of the process running on the torus of size  $450 \times 450$  with  $r_N = 5$ . As this picture suggests the density stays constant because different parts of the lattice oscillate out of phase.

**Theorem 4.1.5.** *Consider the process running in  $\mathbb{Z}^d$  with  $d \geq 2$ . If  $r_N = L$  and  $L$  is sufficiently large then there is a nontrivial stationary distribution.*

*Sketch of the proof.* The key to the proof is that the density of occupied sites after growth is at most  $f(1) = 1 - e^{-\beta}$  so after the epidemic there will be a positive density of occupied sites. Let  $\delta = (1 - e^{-\beta})e^{-4\beta}$  be the probability that a site is occupied and has four vacant neighbors. Divide space into squares of side  $L/2$

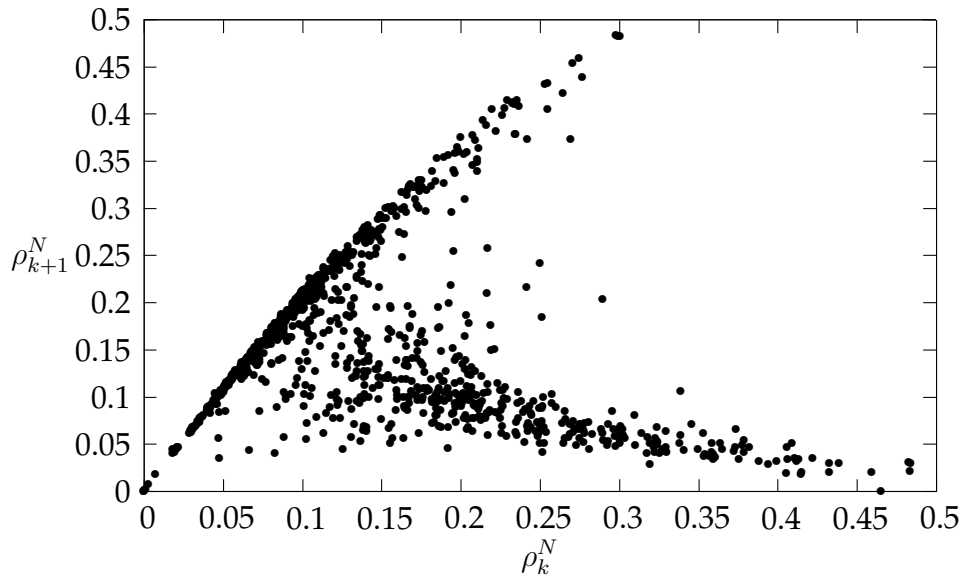


Figure 4.4: Graph of  $\rho_k^N$  against  $\rho_{k+1}^N$  on the 2-dimensional torus with  $N = 750$  and  $r_N = 50$ . The graph clearly does not correspond to a function.

and declare that the square is occupied if at least a fraction  $\delta/2$  of the sites are. If  $L$  is large enough and  $T$  is chosen suitably then the set of occupied sites at time  $nT$  dominates oriented percolation with  $p$  close to 1 and the result follows from standard “block construction” arguments (for an account of this method see, for instance, Durrett (1995)). By order of the Associate Editor further details are left to the reader. □

The remainder of the paper is devoted to proofs. The proof of Theorem 4.1.1 is given in Section 4.2. If you get bored with all of the algebra and calculus involved you can skip to Section 4.3 where the proof of Theorem 4.1.2 is given. The proof of Theorem 4.1.3 given in Section 4.4 and the more complicated proof of Theorem 4.1.4 in Section 4.5 rely on ideas from Sections 4.2 and 4.3, but are independent of each other.

The authors would like to thank referee Nicolas Lanchier for his careful read-

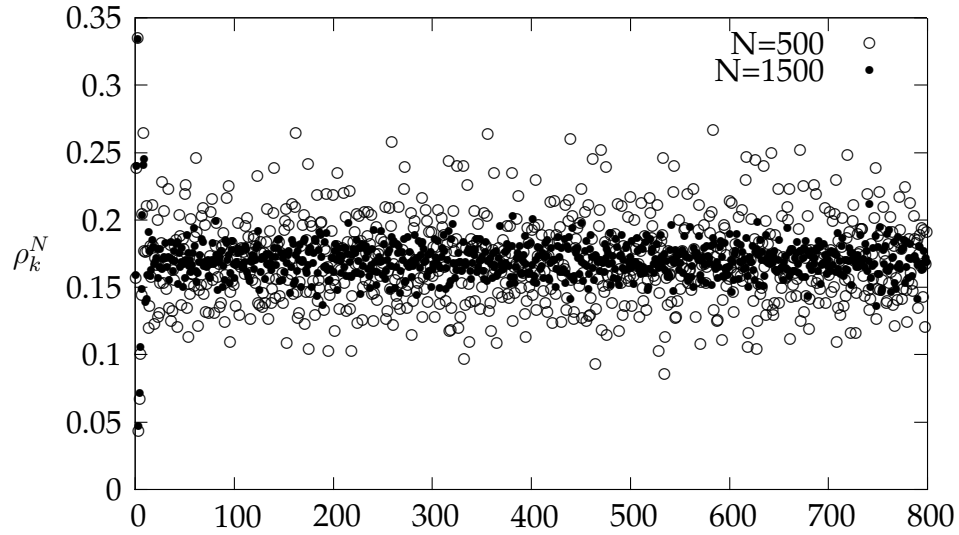


Figure 4.5: Sequence of densities  $\rho_k^N$  of the process running on the 2-dimensional torus with local interactions in the epidemic step for  $N = 500$  and  $N = 1500$ , both with  $r_N = 5$ . As this graph suggests, the fluctuations of the density process get small as  $N$  grows if the range  $r_N$  is held fixed.

ing of the paper which resulted in a number of corrections and clarifications.

## 4.2 Proof of Theorem 4.1.1

*Proof.* By Proposition 4.1.1, to obtain (a) it is enough to prove that there is a point  $c \in [a_1, 1/2]$  such that

$$h_T^3(c) \leq c < h_T(c) < h_T^2(c).$$

In our case we can take

$$c = f^{-1}(a_0) = \frac{1}{\beta} \log \left( \frac{\beta}{\beta - \log 2} \right)$$



Figure 4.6: State of the process at time 200 on a torus of size  $450 \times 450$  (black dots are occupied). In this simulation,  $\beta = 2.25$ ,  $r_N = 5$ , and the infection probability at each site is  $5 \cdot 10^{-6}$ . This picture corresponds to an intermediate state of the process, after an epidemic event wiped out a big cluster but the process has had time to grow back.

(see Figure 4.1). Observe that since  $a_0 < 1/2$ ,  $c = \beta^{-1} \log((1 - a_0)^{-1}) < \beta^{-1} \log 2 = a_0$ . Hence

$$\begin{aligned} h_T(c) &= f(c) = a_0, \\ h_T^2(c) &= f(a_0) = \frac{1}{2}, \quad \text{and} \\ h_T^3(c) &= h_T(1/2) = a_1. \end{aligned}$$

It is clear then that  $c < h_T(c) < h_T^2(c)$ . To see that  $h_T^3(c) \leq c$  we need to show that  $a_1 \leq f^{-1}(a_0)$ , i.e., that

$$\frac{e^{-3\beta/2}}{(1 - e^{-\beta/2})^2} \leq \frac{1}{\beta} \log\left(\frac{\beta}{\beta - \log 2}\right),$$

or, equivalently, that

$$\phi_1(\beta) = \exp\left(\frac{\beta e^{-3\beta/2}}{(1 - e^{-\beta/2})^2}\right) \leq \phi_2(\beta) = \frac{\beta}{\beta - \log 2} \quad (4.3)$$

for all  $\beta > 2 \log 2$ . If you look at the picture of these two functions it seems clear that the inequality holds, but the proof is not as simple as the picture suggests. We will divide it into two parts.

First, assume that  $\beta \in (2 \log 2, 1.75]$ . We will show that

$$\phi_1(\beta) \leq 4 - \frac{\beta}{\log 2} \leq \phi_2(\beta). \quad (4.4)$$

To get the first inequality let

$$\sigma(\beta) = \frac{\beta e^{-3\beta/2}}{(1 - e^{-\beta/2})^2}.$$

A simple calculation gives

$$\sigma''(\beta) = \frac{9e^\beta - 4e^{\beta/2} + 1}{4e^{5\beta/2} - 16e^{2\beta} + 24e^{3\beta/2} - 16e^\beta + 4e^{\beta/2}},$$

and we claim that this quotient is positive. Indeed, it is easy to see that the numerator is positive, while putting  $a = e^{\beta/2}$  the denominator becomes  $4a^5 - 16a^4 + 24a^3 - 16a^2 + 4a$ , so dividing by  $4a$  we need to show that

$$w(a) = a^4 - 4a^3 + 6a^2 - 4a + 1 > 0$$

for all  $a > 2$ . Observe that  $w'(a) = 4a^3 - 12a^2 + 12a - 4$ , so  $w'(2) = 4$ , while  $w''(a) = 12(a - 1)^2 > 0$ , so  $w'(a) > 0$  for all  $a > 2$ . Since  $w(2) = 1$  we deduce that  $w(a) > 0$  for all  $a > 2$  as required. Hence  $\sigma$  is convex, and thus so is  $\phi_1 = \exp(\sigma(\cdot))$ . Since

$$\phi_1(2 \log(2)) = 2 = 4 - \frac{2 \log 2}{\log 2} \quad \text{and} \quad \phi_1(1.75) \approx 1.4518 < 4 - \frac{1.75}{\log 2} \approx 1.4753,$$

the convexity of  $\phi_1$  gives the desired inequality.

To get the second inequality in (4.4), observe that

$$\phi_2(2 \log 2) = 2 = 4 - \frac{2 \log 2}{\log 2} \quad \text{and} \quad \phi_2'(2 \log 2) = -\frac{1}{\log 2}.$$

Therefore, since this last quantity is exactly the slope of the line appearing in the middle term of (4.4) and since  $\phi_2$  is strictly convex, we deduce that  $\phi_2'(\beta)$  is larger than this slope for every  $\beta > 2 \log 2$  and thus the inequality holds.

Now we assume that  $\beta > 1.75$ . Using the Taylor expansion of the functions  $1/(1-x)$  and  $e^x$  about  $x = 0$  we get that (4.3) is equivalent to

$$\sum_{n \geq 0} \left( \frac{\log 2}{\beta} \right)^n \geq \sum_{n \geq 0} \frac{1}{n!} \left( \frac{\beta e^{-3\beta/2}}{(1 - e^{-\beta/2})^2} \right)^n,$$

so it is enough to show that

$$\left( \frac{\log 2}{\beta} \right)^n \geq \frac{1}{n!} \left( \frac{\beta e^{-3\beta/2}}{(1 - e^{-\beta/2})^2} \right)^n$$

for all  $n \geq 0$  and  $\beta > 1.75$ . The inequality holds trivially for  $n = 0$ , so by induction it is enough to prove that

$$\frac{\log 2}{\beta} \geq \frac{1}{n} \frac{\beta e^{-3\beta/2}}{(1 - e^{-\beta/2})^2}$$

for all  $n \geq 1$  or, equivalently, for  $n = 1$ . That is, we need to show that

$$\beta^2 \frac{e^{-3\beta/2}}{(1 - e^{-\beta/2})^2} \leq \log 2 \tag{4.5}$$

for all  $\beta > 1.75$ . To see that this holds we observe that the derivative of the left side with respect to  $\beta$  is

$$-\frac{\beta e^{-\beta/2} (3\beta e^{\beta/2} - 4e^{\beta/2} - \beta + 4)}{2(e^{\beta/2} - 1)^3}.$$

We claim that this quotient is negative for  $\beta > 1.75$ . Indeed, the denominator is clearly positive, so we only need to show that

$$w(\beta) = 3\beta e^{\beta/2} - 4e^{\beta/2} - \beta + 2 > 0$$

for  $\beta > 1.75$ . This is easy, because  $w'(\beta) = 3e^{\beta/2}(1 + \beta/2) - 2e^{\beta/2} - 1 > e^{\beta/2} - 1 > 0$  and  $w(1.75) \approx 5.28$ . Thus the left side of the (4.5) is decreasing in  $\beta$ , and then the

inequality holds because its value at  $\beta = 1.75$  is approximately  $0.6523 < \log 2$ . This finishes the proof of (a).

To get (b) it is enough to show by Lasota and Yorke (1973) that

$$\inf_{p \in [a_1, 1/2] \setminus \{a_0\}} |(h_T^3)'(p)| > 1 \quad (4.6)$$

for  $\beta \in (2 \log 2, 2.48]$ . The idea of the proof is the following. We find an explicit formula for  $(h_T^3)'$  and use it to compute numerically its infimum on  $[a_1, 1/2] \setminus \{a_0\}$  for every  $\beta$  in a certain grid of  $(2 \log 2, 2.48]$ . Due to monotonicity properties of the derivative of  $h_T$  the numerical computation of the infimum is exact (up to floating-point numerical errors which are small enough for our purposes) for any fixed  $\beta$ . We then show that  $(h_T^3)'$ , as a function of  $\beta$ , has a Lipschitz constant that ensures that the infimum is larger than 1 for every  $\beta$  between subsequent points in the grid. We will do this step by step.

We begin by computing  $(h_T^3)'$ . For  $p \in [a_1, a_0)$ ,  $h_T'(p) = f'(p) = \beta e^{-\beta p}$ , while for  $p \in (a_0, 1/2]$ ,

$$h_T'(p) = \frac{-3\beta e^{-3\beta p}}{(1 - e^{-\beta p})^2} - 2 \frac{e^{-3\beta p}}{(1 - e^{-\beta p})^3} \beta e^{-\beta p} = \frac{e^{-3\beta p}}{(1 - e^{-\beta p})^3} [-3\beta + \beta e^{-\beta p}].$$

This gives an explicit formula for  $h_T'$ . On the other hand,

$$(h_T^3)'(p) = h_T'(h_T^2(p)) h_T'(h_T(p)) h_T'(p). \quad (4.7)$$

Putting these two formulas together we get an explicit expression for  $(h_T^3)'$ .

Now observe that  $h_T'$  is decreasing in  $[a_1, a_0)$  and increasing in  $(a_0, 1/2]$ . Indeed,  $h_T''(p) = f''(p) = -\beta^2 e^{-\beta p} < 0$  on the first interval, while on the second one  $h_T''(p) = g_T''(f(p)) f'(p)^2 + g_T'(f(p)) f''(p)$ , so since  $f' > 0$ ,  $f'' < 0$ ,

$$g_T'(p) = \left(\frac{1}{p} - 1\right)^3 + 3p \left(\frac{1}{p} - 1\right)^2 \left(\frac{-1}{p^2}\right) = -\left(1 + \frac{2}{p}\right) \left(\frac{1}{p} - 1\right)^2 < 0,$$

and

$$g_T''(p) = \frac{2}{p^2} \left( \frac{1}{p} - 1 \right)^2 - \left( 1 + \frac{2}{p} \right) \left( \frac{1}{p} - 1 \right) \left( \frac{-1}{p^2} \right) > 0,$$

we get that  $h_T''(p) > 0$  for  $p \in (a_0, 1/2]$ . This means by (4.7) that  $(h_T^3)'$  is monotone in each interval of constancy of its sign. These intervals are given by the partition of  $[a_1, 1/2]$  defined by the preimage of  $a_0$  under  $h_T^3$ . We deduce that

$$\inf_{p \in [a_1, 1/2] \setminus \{a_0\}} |(h_T^3)'(p)| = \inf_{p \in h_T^{-3}(a_0) \cup \{a_1, 1/2\}} \min \{ |(h_T^3)'(p^-)|, |(h_T^3)'(p^+)| \},$$

where the superscripts  $-$  and  $+$  denote left and right derivatives respectively.

Using this observation we can compute numerically the infimum in (4.6) for any given  $\beta$ . We did this for every  $\beta$  in a grid of width  $2 \cdot 10^{-6}$  of  $(2 \log 2, 2.48]$ , and we obtained that the infimum is larger than 1.002 at each of these values of  $\beta$ .

Figure 4.7 shows a graph of the values obtained.

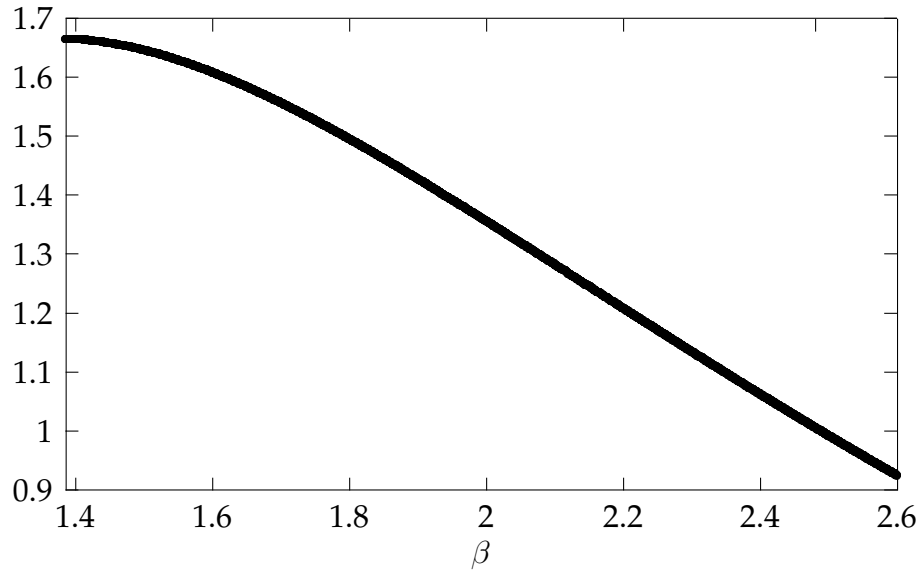


Figure 4.7: Infimum of  $|(h_T^3)'(p)|$  on the relevant interval for  $\beta \in (2 \log 2, 2.6)$ . The computation was done for each  $\beta$  on a grid of width  $2 \cdot 10^{-6}$  on this interval, as explained within the proof of Theorem 4.1.1. The infimum lies above 1.02 for  $\beta \in (2 \log 2, 2.48]$ .

The last step is to make sure that the infimum in (4.6) stays above 1 for

every  $\beta \in (2 \log 2, 2.48]$ . We will write  $h_T(p, \beta)$  to indicate the dependence of  $h_T(p)$  on the value of the parameter  $\beta$ . Our goal is to find a bound for  $\left| \frac{\partial^2}{\partial \beta \partial p} h_T^3(p, \beta) \right|$ . Observe that by the product rule and (4.7), if  $\left| \frac{\partial}{\partial p} h_T(p, \beta) \right| \leq M_1$  and  $\left| \frac{\partial^2}{\partial \beta \partial p} h_T(p, \beta) \right| \leq M_2$  for all  $\beta \in (2 \log 2, 2.48]$  and  $p \in [a_1, 1/2] \setminus \{a_0\}$  then

$$\left| \frac{\partial^2}{\partial \beta \partial p} h_T^3(p, \beta) \right| \leq 3M_1^2 M_2 \quad (4.8)$$

for all such  $\beta$  and  $p$ . We already computed  $\left| \frac{\partial}{\partial p} h_T(p, \beta) \right|$ . For  $p \in [a_1, a_0)$ , it equals  $\beta e^{-\beta p}$  which is smaller than 2.48 for each  $\beta \leq 2.48$ . For  $p \in (a_0, 1/2]$  we know that  $h'_T$  is negative and increasing, so

$$\begin{aligned} \left| \frac{\partial}{\partial p} h_T(p, \beta) \right| &\leq \left| \frac{\partial}{\partial p} h_T\left(\frac{1}{2}, \beta\right) \right| = \left| \frac{e^{-3\beta/2}}{(1 - e^{-\beta/2})^3} [-3\beta + \beta e^{-\beta/2}] \right| \\ &\leq \frac{e^{-3 \cdot 2.48/2}}{2^{-3}} \cdot 4 \cdot 2.48 \approx 1.923. \end{aligned}$$

Thus if we take  $M_1 = 2.48$  the desired inequality holds. Now for  $p \in [a_1, a_0)$ ,

$$\left| \frac{\partial^2}{\partial \beta \partial p} h_T(p, \beta) \right| = \left| \frac{d}{d\beta} (\beta e^{-\beta p}) \right| = |(1 - \beta^2) e^{-\beta p}| \leq 1.$$

For  $p \in (a_0, 1/2]$ ,

$$\begin{aligned} \left| \frac{\partial^2}{\partial \beta \partial p} h_T(p, \beta) \right| &= \left| \frac{d}{d\beta} \left( \frac{e^{-3\beta p}}{(1 - e^{-\beta p})^3} [-3\beta + \beta e^{-\beta p}] \right) \right| \\ &= \frac{e^{-\beta p}}{(1 - e^{-\beta p})^4} |(9\beta p - 3)e^{-2\beta p} + (4 - 4\beta p)e^{-3\beta p} + (\beta p - 1)e^{-4\beta p}| \\ &\leq \frac{e^{-\beta a_0}}{(1 - e^{-\beta/2})^4} (14\beta p + 8) \leq \frac{2^{-1}}{(1 - e^{-2.48/2})^4} (14 \cdot 2.48/2 + 8) \approx 49.73 \end{aligned}$$

if  $\beta \in (2 \log 2, 2.48]$ . Thus if we take  $M_2 = 49.73$  we get by (4.8) that

$$\left| \frac{\partial}{\partial \beta} h_T^3(p, \beta) \right| \leq 917.6.$$

The bound we just obtained implies that for any fixed  $p \in [a_1, 1/2] \setminus \{a_0\}$  the function  $\beta \mapsto \frac{d}{dp} h_T(p, \beta)$  is Lipschitz and its Lipschitz constant is at most 917.6. Now fix  $\beta \in (2 \log 2, 2.48]$  and let  $\beta'$  be the point in the grid of  $(2 \log 2, 2.48]$  on

which we computed the infimum in (4.6) which is immediatly before  $\beta$ . Then for any  $p \in [a_1, 1/2] \setminus \{a_0\}$ ,

$$\begin{aligned} \left| \frac{\partial}{\partial p} h_T^3(p, \beta) \right| &\geq \left| \frac{\partial}{\partial p} h_T^3(p, \beta') \right| - \left| \frac{\partial}{\partial p} h_T^3(p, \beta) - \frac{\partial}{\partial p} h_T^3(p, \beta') \right| \\ &\geq 1.002 - 917.6|\beta - \beta'| \geq 1.002 - 917.6 \cdot 2 \cdot 10^{-6} \approx 1.0001. \end{aligned}$$

This finishes the proof of (4.6). □

### 4.3 Proof of Theorem 4.1.2

To prove this result it will be enough to study the one-step transition probabilities for  $\rho_k^N$ . Recall that in the growth step, since here  $\mathcal{N}_N(x) = G_N$ , every site becomes occupied with probability  $1 - (1 - \beta/N)^{pN} \approx 1 - e^{-\beta p}$ , where  $p$  is the starting density of occupied sites. For simplicity we will assume that the occupation probability of each site after the growth step is exactly  $1 - e^{-\beta p}$ , and then in the proof of the theorem we will say how to remove this assumption.

Abusing notation, we will also let  $\eta_k^N$  stand for the set of occupied sites in the process.  $\eta_{k+1/2}^N$  will denote the intermediate state of the process between  $\eta_k^N$  and  $\eta_{k+1}^N$  after the growth part of the dynamics has been run but before running the epidemic. We will denote by  $\{0, \dots, N-1\}$  the set of nodes of  $R_N$ .  $B(i, r)$  will denote the set of sites in  $R_N$  at distance at most  $r$  from  $i$  (here the distance between two points  $i$  and  $j$  is defined as the number of edges in the shortest path going from  $i$  to  $j$ ).

Let  $\tilde{\eta}_1^N$  be the set of occupied sites after the epidemic is run on  $\eta_{1/2}^N$  ignoring infections coming from a distance greater than  $(\log_2 N)/5$ . Define  $\tilde{\rho}_1^N = |\tilde{\eta}_1^N|/N$ .

Recall that we are assuming that

$$\alpha_N \log_2 N \longrightarrow \infty.$$

**Lemma 4.3.1.**

$$\mathbb{E}(|\tilde{\rho}_1^N - \rho_1^N|) \xrightarrow{N \rightarrow \infty} 0$$

uniformly in the initial density  $p$ .

*Proof.* By translation invariance, and observing that  $\tilde{\eta}_1^N(i) \geq \eta_1^N(i)$  for all  $i \in R_N$ ,

$$\begin{aligned} \mathbb{E}(|\tilde{\rho}_1^N - \rho_1^N|) &\leq \frac{1}{N} \sum_{i \in R_N} \mathbb{E}(|\tilde{\eta}_1^N(i) - \eta_1^N(i)|) = \mathbb{P}(0 \in \tilde{\eta}_1^N) - \mathbb{P}(0 \in \eta_1^N) \\ &= \mathbb{P}(0 \in \tilde{\eta}_1^N \setminus \eta_1^N) \leq (1 - \alpha_N)^{\frac{1}{5} \log_2 N} \approx e^{-\frac{1}{5} \alpha_N \log_2 N} \longrightarrow 0. \end{aligned}$$

The second inequality above follows from the fact that if 0 is in  $\tilde{\eta}_1^N$  but not in  $\eta_1^N$ , then there must be an open path in  $\eta_{1/2}^N$  going from 0 to  $\partial B(0, (\log_2 N)/5)$ , and all sites in this path must have not been infected.  $\square$

Now let

$$H_N = \{i \in R_N : B(i, (\log_2 N)/5) \text{ is a finite 3-tree}\}.$$

By a finite 3-tree we mean a finite tree where all nodes have degree 3 except for the leaves which have degree 1. The next lemma says that  $R_N$  looks locally like a 3-tree:

**Lemma 4.3.2.**

$$\mathbb{E}\left(\frac{1}{N} |R_N \setminus H_N|\right) = \mathbb{P}(0 \notin H_N) \xrightarrow{N \rightarrow \infty} 0.$$

*Proof.* A random 3-regular graph is a special case of a graph with a fixed degree distribution and can be studied using techniques in Section 3.2 of Durrett

(2007). To explore the subgraph  $B(0, (\log_2 N)/5)$  of  $R_N$ , let  $R_0 = \emptyset$ ,  $A_0 = \{0\}$ , and  $U_0 = \{1, \dots, N-1\}$ . These are called the removed, active, and unexplored sites respectively. If  $A_n \neq \emptyset$  then to go from time  $n$  to  $n+1$  we pick a site  $i_n$  from  $A_n$  according to some given rule and let

$$\begin{aligned} R_{n+1} &= R_n \cup \{i_n\} \\ A_{n+1} &= (A_n \setminus \{i_n\}) \cup \{j \in U_n : j \sim i\} \\ U_{n+1} &= U_n \setminus \{j \in U_n : j \sim i\}, \end{aligned}$$

where  $j \sim i$  here denotes that  $j$  and  $i$  are neighbors. For  $n \leq 3N^{1/5}/2$ ,  $|A_n| \leq 3N^{1/5}/2 + 2$ , so the probability of a collision (i.e., that when we examine the neighbors of  $i_n$  we see a site already in  $A_n$ ) at some time is at most

$$2 \cdot \frac{3}{2} N^{1/5} \frac{3N^{1/5}/2 + 2}{N} \longrightarrow 0.$$

Now suppose that when choosing the sites  $i_n$  we choose those at distance 1 from 0 first, then those at distance 2, etcetera. Then by time  $3N^{1/5}/2$  we will have investigated all points within distance  $(\log_2 N)/5$  of 0, and if we see no collision, then we will know that the subgraph  $B(0, (\log_2 N)/5)$  is a tree.  $\square$

**Lemma 4.3.3.** *Let  $C_0$  be the cluster containing the origin in a site percolation process on the 3-tree, and let  $\mathbb{P}_p$  denote the law of this process when each site is retained independently with probability  $p \in [0, 1]$ . Then for any  $k_N \uparrow \infty$ ,*

$$\sup_{p \in [0, 1]} |\mathbb{P}_p(\text{diam}(C_0) < \infty) - \mathbb{P}_p(\text{diam}(C_0) \leq k_N)| \xrightarrow{N \rightarrow \infty} 0.$$

*Proof.* The result follows from the fact that any increasing sequence of continuous functions on  $[0, 1]$  which converges pointwise to a continuous function on  $[0, 1]$  actually converges uniformly to that function (see, for instance, Theorem

7.13 in Rudin (1976)). We only need to observe that  $\mathbb{P}_p(\text{diam}(C_0) < \infty)$  and  $\mathbb{P}_p(\text{diam}(C_0) \leq k_N)$  are continuous on  $[0, 1]$  as functions of  $p$ , and the latter is increasing in  $N$  and converges pointwise to the former as  $N \rightarrow \infty$ .  $\square$

**Lemma 4.3.4.**

$$\mathbb{E}\left(\frac{1}{N}|\tilde{\eta}_1^N \cap H_N|\right) \xrightarrow{N \rightarrow \infty} h_T(p)$$

*uniformly in the initial density  $p$ .*

*Proof.* Observe that since  $0 \in \tilde{\eta}_1^N$  implies that  $0 \in \tilde{\eta}_{1/2}^N = \eta_{1/2}^N$ ,

$$\begin{aligned} \mathbb{E}\left(\frac{1}{N}|\tilde{\eta}_1^N \cap H_N|\right) &= \mathbb{P}\left(0 \in \tilde{\eta}_1^N \mid 0 \in H_N \cap \eta_{1/2}^N\right) \mathbb{P}(0 \in H_N) \mathbb{P}(0 \in \eta_{1/2}^N). \end{aligned} \quad (4.9)$$

By Lemma 4.3.2,  $\mathbb{P}(0 \in H_N) \rightarrow 1$  uniformly in  $p$ , while by our assumption,  $\mathbb{P}(0 \in \eta_{1/2}^N) = 1 - e^{-\beta p}$ .

For the other term on the right side of (4.9), we only need to look at the configuration of  $\eta_{1/2}^N$  inside  $B(0, (\log_2 N)/5)$ , on which, conditional on the event  $\{0 \in H_N\}$ , the graph looks like a finite 3-tree. Thus, we can construct the random variables  $(\tilde{\eta}_1^N(0))_{N>0}$  conditioned on  $\{0 \in H_N \cap \eta_{1/2}^N\}$  on a common probability space in the following way. Let  $\mathfrak{T}$  be the set of sites in an infinite (rooted) 3-tree and consider a site percolation process on  $\mathfrak{T}$  with each site being open, independently, with probability  $1 - e^{-\beta p}$ . We will call  $C_0$  the corresponding percolation cluster containing 0. We also consider a collection  $(B_i^N)_{i \in \mathfrak{T}, N>0}$  of independent Bernoulli random variables with  $\mathbb{P}(B_i^N = 1) = \alpha_N$ . With this, the random variable  $\tilde{\eta}_1^N(0)$ , conditional on the event  $\{0 \in H_N \cap \eta_{1/2}^N\}$ , can be constructed as

$$\tilde{\eta}_1^N(0) = \begin{cases} 1 & \text{if } B_i^N = 0 \text{ for all } i \in C_0 \cap B(0, (\log_2 N)/5), \\ 0 & \text{otherwise.} \end{cases}$$

It is clear that this construction gives the right conditional distribution for  $\tilde{\eta}_1^N(0)$ .

Now let  $l_N = \log_2(\alpha_N^{-1/2})$ . Observe that  $l_N < (\log_2 N)/5$  for large  $N$ , so we have that

$$\begin{aligned}
& \mathbb{P}\left(0 \in \tilde{\eta}_1^N \mid 0 \in H_N \cap \eta_{1/2}^N\right) \\
&= \mathbb{P}\left(0 \in \tilde{\eta}_1^N, \text{diam}(C_0) \leq l_N \mid 0 \in H_N \cap \eta_{1/2}^N\right) \\
&+ \mathbb{P}\left(0 \in \tilde{\eta}_1^N, l_N < \text{diam}(C_0) \leq \frac{1}{5} \log_2 N \mid 0 \in H_N \cap \eta_{1/2}^N\right) \\
&+ \mathbb{P}\left(0 \in \tilde{\eta}_1^N, \text{diam}(C_0) > \frac{1}{5} \log_2 N \mid 0 \in H_N \cap \eta_{1/2}^N\right).
\end{aligned} \tag{4.10}$$

For the first probability on the right side we have that

$$\begin{aligned}
& \mathbb{P}\left(0 \in \tilde{\eta}_1^N, \text{diam}(C_0) \leq l_N \mid 0 \in H_N \cap \eta_{1/2}^N\right) \\
&\leq \mathbb{P}(0 < \text{diam}(C_0) \leq l_N \mid 0 \in H_N \cap \eta_{1/2}^N) \\
&\longrightarrow \mathbb{P}(0 < \text{diam}(C_0) < \infty \mid 0 \text{ is open}) = \frac{g_T(1 - e^{-\beta p})}{1 - e^{-\beta p}}.
\end{aligned}$$

This convergence is uniform in  $p$  thanks to Lemma 4.3.3. On the other hand, since any subset of  $\mathfrak{T}$  with diameter  $n$  has at most  $1 + 3 \cdot 2^{n-1} < 3 \cdot 2^n$  nodes, we get that

$$\begin{aligned}
& \mathbb{P}\left(0 \in \tilde{\eta}_1^N, \text{diam}(C_0) \leq l_N \mid 0 \in H_N \cap \eta_{1/2}^N\right) \\
&= \mathbb{P}(B_i^N = 0 \forall i \in C_0, \text{diam}(C_0) \leq l_N \mid 0 \in H_N \cap \eta_{1/2}^N) \\
&= \mathbb{E}\left((1 - \alpha_N)^{|C_0|}, 0 < \text{diam}(C_0) \leq l_N \mid 0 \in H_N \cap \eta_{1/2}^N\right) \\
&\geq (1 - \alpha_N)^{3\alpha_N^{-1/2}} \mathbb{P}(0 < \text{diam}(C_0) \leq l_N \mid 0 \in H_N \cap \eta_{1/2}^N) \\
&\longrightarrow \frac{g_T(1 - e^{-\beta p})}{1 - e^{-\beta p}}
\end{aligned}$$

by the same reason as above and because  $(1 - \alpha_N)^{3\alpha_N^{-1/2}} \approx e^{-3\sqrt{\alpha_N}} \rightarrow 1$ . We

deduce that

$$\mathbb{P}\left(0 \in \tilde{\eta}_1^N, \text{diam}(C_0) \leq l_N \mid 0 \in H_N \cap \eta_{1/2}^N\right) \longrightarrow \frac{g_T(1 - e^{-\beta p})}{1 - e^{-\beta p}}$$

uniformly in  $p$ . For the second probability on the right side of (4.10) we have that, since  $\mathbb{P}\left(0 \in H_N \cap \eta_{1/2}^N\right) \geq C = (1 - e^{-\beta p})/2$  for large enough  $N$ ,

$$\begin{aligned} & \mathbb{P}\left(0 \in \tilde{\eta}_1^N, l_N < \text{diam}(C_0) \leq \frac{1}{5} \log_2 N \mid 0 \in H_N \cap \eta_{1/2}^N\right) \\ & \leq C^{-1} \mathbb{P}\left(l_N < \text{diam}(C_0) \leq \frac{1}{5} \log_2 N\right) \\ & = C^{-1} [\mathbb{P}(\text{diam}(C_0) > l_N) - \mathbb{P}(\text{diam}(C_0) = \infty)] \\ & \quad - C^{-1} \left[ \mathbb{P}\left(\text{diam}(C_0) > \frac{1}{5} \log_2 N\right) - \mathbb{P}(\text{diam}(C_0) = \infty) \right] \\ & \longrightarrow 0 \end{aligned}$$

uniformly in  $p$ , again by Lemma 4.3.3. For the last probability in (4.10) we simply observe that

$$\begin{aligned} & \mathbb{P}\left(0 \in \tilde{\eta}_1^N, \text{diam}(C_0) > \frac{1}{5} \log_2 N \mid 0 \in H_N \cap \eta_{1/2}^N\right) \\ & \leq (1 - \alpha_N)^{\frac{1}{5} \log_2 N} \approx e^{-\frac{1}{5} \alpha_N \log_2 N} \longrightarrow 0. \end{aligned}$$

The previous calculations and (4.10) imply that

$$\mathbb{P}(0 \in \tilde{\eta}_1^N \mid 0 \in H_N \cap \eta_{1/2}^N) \longrightarrow \frac{g_T(1 - e^{-\beta p})}{1 - e^{-\beta p}}$$

uniformly in  $p$ . Putting this together with (4.9) we get the result.  $\square$

*Proof of Theorem 4.1.2.* By Karr (1975), it is enough to prove that  $\rho_0^N \Rightarrow p$  and that given any sequence  $p_N$  in  $[0, 1]$  converging to some  $p' \in [0, 1]$ , the sequence  $\rho_1^N$ , with  $\eta_0^N$  started at a product measure of density  $p_N$ , converges weakly (or, equivalently, in probability) to  $h_T(p')$ .

The first part is straightforward. For the second part we will assume, for simplicity, that  $p_N = p'$  for all  $N$  and, moreover, that each site is occupied with probability  $1 - e^{-\beta p'}$  after the growing season. The general case follows from the facts that  $1 - (1 - \beta/N)^{p'N}$  converges uniformly as  $N \rightarrow \infty$  to  $1 - e^{-\beta p'}$  for  $p' \in [0, 1]$  and that, by the preceding lemmas, all the convergences we will prove below are uniform on the initial density  $p$ .

Observe that by Markov's inequality, given any  $\varepsilon > 0$

$$\mathbb{P}(|\rho_1^N - h_T(p')| > \varepsilon) \leq \frac{1}{\varepsilon} \mathbb{E}(|\rho_1^N - h_T(p')|),$$

so

$$\begin{aligned} \mathbb{P}(|\rho_1^N - h_T(p')| > \varepsilon) &\leq \frac{1}{\varepsilon} \mathbb{E}(|\rho_1^N - \tilde{\rho}_1^N|) + \frac{1}{\varepsilon} \mathbb{E} \left( \left| \tilde{\rho}_1^N - \frac{1}{N} |\tilde{\eta}_1^N \cap H_N| \right| \right) \\ &\quad + \frac{1}{\varepsilon} \mathbb{E} \left( \left| \frac{1}{N} |\tilde{\eta}_1^N \cap H_N| - \mathbb{E} \left( \frac{1}{N} |\tilde{\eta}_1^N \cap H_N| \right) \right| \right) \quad (4.11) \\ &\quad + \frac{1}{\varepsilon} \left| \mathbb{E} \left( \frac{1}{N} |\tilde{\eta}_1^N \cap H_N| \right) - h_T(p') \right|. \end{aligned}$$

Lemmas 4.3.1 and 4.3.4 imply that the first and last terms on the right side of the inequality go to 0 as  $N \rightarrow \infty$ . The second one also goes to 0 since, using Lemma 4.3.2,

$$\mathbb{E} \left( \left| \tilde{\rho}_1^N - \frac{1}{N} |\tilde{\eta}_1^N \cap H_N| \right| \right) \leq \mathbb{E} \left( \frac{1}{N} |R_N \setminus H_N| \right) \longrightarrow 0.$$

To deal with the third term, observe that

$$\begin{aligned} \text{Var}(|\tilde{\eta}_1^N \cap H_N|) &= \sum_{i=0}^{N-1} \sum_{j=0}^{N-1} \text{Cov}(\mathbf{1}_{i \in \tilde{\eta}_1^N \cap H_N}, \mathbf{1}_{j \in \tilde{\eta}_1^N \cap H_N}) \\ &\leq |\{(i, j) \in H_N \times H_N : B(i, (\log_2 N)/5) \cap B(j, (\log_2 N)/5) \neq \emptyset\}| \\ &= |\{(i, j) \in H_N \times H_N : |i - j| \leq 2(\log_2 N)/5\}| \leq 2N \cdot N^{2/5}. \end{aligned}$$

Hence, by Jensen's inequality,

$$\begin{aligned} \mathbb{E} \left( \left| \frac{1}{N} |\tilde{\eta}_1^N \cap H_N| - \mathbb{E} \left( \frac{1}{N} |\tilde{\eta}_1^N \cap H_N| \right) \right| \right)^2 &\leq \text{Var} \left( \frac{1}{N} |\tilde{\eta}_1^N \cap H_N| \right) \\ &\leq \frac{2N \cdot N^{2/5}}{N^2} \longrightarrow 0. \end{aligned}$$

We deduce from (4.11) that  $\rho_1^N$  converges in probability to  $h_T(p')$ .  $\square$

#### 4.4 Proof of Theorem 4.1.3

*Proof.* As in the case of the 3-tree, we let  $a_0$  be the solution of  $f(a_0) = p_c$  (i.e.,  $a_0 = \log(1/(1 - p_c))/\beta$ ) and  $a_1 = h_L(p_c)$  (see Figure 4.1 for a sketch of these values in the case of the 3-tree). It is enough to prove, by Lasota and Yorke (1973), that there is a  $K \in \mathbb{N}$  such that

$$\inf_{p \in [a_1, p_c] \setminus \{a_0\}} |(h_T^K)'(p)| > 1. \quad (4.12)$$

Fix any  $\beta_1 > \beta_c$ . Since  $a_1$  is bounded away from 0 for  $\beta \in (\beta_c, \beta_1)$ , there is a  $K \in \mathbb{N}$  such that  $\min\{k \in \mathbb{N} : f^k(a_1) > p_c\} \leq K - 1$  for any such  $\beta$ . In particular, since  $a_0$  is always less than  $p_c$  we deduce that given any  $\beta \in (\beta_c, \beta_1)$  and any  $p \in [a_1, p_c]$ , the  $K$ -tuple  $(p, h_L(p), \dots, h_L^{K-1}(p))$  contains at least one point in  $(a_0, p_c]$ .

Now recall that  $f'' < 0$ , so  $f'$  attains its minimum on the interval  $[a_1, a_0]$  at  $a_0$ , and at this point its value is  $\beta(1 - p_c)$ . Thus for every  $\beta \in (\beta_c, \beta_1)$ , this minimum is larger than  $\beta_c(1 - p_c)$ . Since  $g_L(p) = p$  for  $p \in [a_1, a_0]$  we deduce that

$$|h_L(p)| \geq \beta_c(1 - p_c) \quad \text{for all } p \in [a_1, a_0].$$

Now using the fact that  $a_0 \uparrow p_c$  as  $\beta \downarrow \beta_c$ , we can choose given any  $\varepsilon > 0$  a  $\beta_2 \in (\beta_c, \beta_1)$  so that  $f(p_c) - p_c = f(p_c) - f(a_0) < \varepsilon$  for any  $\beta \in (\beta_c, \beta_2)$ . Since (4.2)

implies that

$$g'_L(p) = 1 - \theta'_L(p) \xrightarrow{p \downarrow p_c} -\infty,$$

we can choose a small enough  $\varepsilon$ , so that

$$|h'_L(p)| = |g'_L(f(p))||f(p)| > \max\{[\beta_c(1 - p_c)]^{-(K-1)}, 1\} \quad \text{for all } p \in (a_0, p_c],$$

and thus this inequality holds for all  $\beta \in (\beta_c, \beta_2)$ .

Putting the previous arguments together with the fact that

$$(h_L^K)'(p) = h'_L(h_L^{K-1}(p))h'_L(h_L^{K-2}(p)) \cdots h'_L(p)$$

we deduce that (4.12) holds for all  $\beta \in (\beta_c, \beta_2)$ . □

## 4.5 Proof of Theorem 4.1.4

Given  $i \in \mathbb{T}_N$  and  $m \in \mathbb{N}$  we will write

$$B(i, m) = \{j \in \mathbb{T}_N : \|i - j\|_\infty \leq m\} \quad \text{and} \quad V(m) = (2m + 1)^d = |B(i, m)|$$

(here and in what follows all differences  $i - j$  for  $i, j \in \mathbb{T}_N$  are computed modulo  $N$ ). Define, for  $k \in \mathbb{N}$ ,

$$d_k^N(i) = \frac{1}{V(r_N)} \sum_{\|j-i\|_\infty \leq r_N} \eta_k^N(j) \quad \text{and}$$

$$G_k^N(\varepsilon) = \{i \in \mathbb{T}_N : |d_k^N(i) - h_L^k(p)| < \varepsilon\}.$$

$d_k^N(i)$  is the density of occupied sites in the growth neighborhood of  $i$ , while  $G_k^N(\varepsilon)$  can be thought of as the set of “good sites at time  $k$ ”, where a site is said to be good at time  $k$  if the density of occupied sites in its growth neighborhood at that time is close to the desired value  $h_L^k(p)$ . The proof of Theorem 4.1.4 will depend on the following proposition:

**Proposition 4.5.1.** Fix  $\varepsilon_1, \varepsilon_2 > 0$  and  $k \in \mathbb{N}$  and assume that

$$\frac{1}{N^d} \mathbb{E}(|\mathbb{T}_N \setminus G_k^N(\delta_1)|) < \delta_2. \quad (4.13)$$

Then if  $\delta_1$  and  $\delta_2$  are small enough and  $N$  is large enough,

$$\frac{1}{N^d} \mathbb{E}(|\mathbb{T}_N \setminus G_{k+1}^N(\varepsilon_1)|) < \varepsilon_2.$$

This result will allow us to give an inductive proof of Theorem 4.1.4. We will need thus the following lemma:

**Lemma 4.5.2.** Given any  $\delta > 0$ ,

$$\frac{1}{N^d} \mathbb{E}(|\mathbb{T}_N \setminus G_0^N(\delta)|) \xrightarrow{N \rightarrow \infty} 0.$$

*Proof.* By translation invariance,

$$\mathbb{E}(|\mathbb{T}_N \setminus G_0^N(\delta)|) = \sum_{i \in \mathbb{T}_N} \mathbb{P}(i \notin G_0^N(\delta)) = N^d \mathbb{P}(|d_0^N(0) - p| \geq \delta).$$

Since  $\mathbb{E}(d_0^N(0)) = p$ , Chebyshev's inequality and the fact that (by definition)  $V(r_N)d_0^N(0)$  is the sum of  $V(r_N)$  independent Bernoulli random variables with success probability  $p$  imply that

$$\mathbb{P}(|d_0^N(0) - p| \geq \delta) \leq \frac{1}{\delta^2 V(r_N)^2} V(r_N) p(1-p),$$

so

$$\frac{1}{N^d} \mathbb{E}(|\mathbb{T}_N \setminus G_0^N(\delta)|) \leq \frac{1}{\delta^2 V(r_N)} p(1-p) \longrightarrow 0. \quad \square$$

Now we turn to the proof of Proposition 4.5.1. Many parts in the argument will be similar to those in the proof of Theorem 4.1.2 and the lemmas that preceded it, so we will skip some details. We begin with some preliminary results.

Throughout this part, and until the proof of Theorem 4.1.4, we fix  $k, \delta_1, \delta_2, \varepsilon_1, \varepsilon_2$  and assume that (4.13) holds.

Observe that since each occupied site  $i$  sends a  $\text{Poisson}[\beta]$  number of births during the growing season, each to a site chosen randomly from  $B(i, r_N)$ , we can equivalently think of each occupied site  $i$  as sending a  $\text{Poisson}[\beta/V(r_N)]$  number of births to each of its  $V(r_N)$  neighbors at a distance smaller than  $r_N$ . Hence during the growing season, each site  $i$  receives  $\sum_{\|j-i\|_\infty \leq r_N} \eta_k^N(j) Y_{j,i}$  births, where  $(Y_{i,j})_{i,j \in \mathbb{T}_N}$  are i.i.d.  $\text{Poisson}[\beta/V(r_N)]$  random variables. Conditional on  $d_k^N(i)$ , this last sum is distributed as a  $\text{Poisson}[d_k^N(i)\beta]$  random variable. We deduce that we can regard the growing season as taking place as follows:

Given  $\eta_k^N$ , each  $i$  will be in  $\eta_{k+1/2}^N$  with probability equal to the probability that a  $\text{Poisson}[d_k^N(i)\beta]$  random variable is positive, that is, with probability  $1 - e^{-\beta d_k^N(i)}$ .

The Poisson random variables above are taken to be independent of each other.

Let  $l_N = \sqrt{r_N/\alpha_N}$  and observe that

$$\frac{l_N}{r_N} = \frac{1}{\sqrt{\alpha_N r_N}} \longrightarrow 0 \quad \text{and} \quad \alpha_N l_N = \sqrt{\alpha_N r_N} \longrightarrow \infty.$$

We let  $\tilde{\eta}_{k+1}^N$  be the configuration obtained from  $\eta_{k+1/2}^N$  by ignoring infections coming from a distance greater than  $l_N$ .

**Lemma 4.5.3.**

$$\frac{1}{N^d} \sum_{i \in \mathbb{T}_N} \mathbb{E}(|\eta_{k+1}^N(i) - \tilde{\eta}_{k+1}^N(i)|) \xrightarrow{N \rightarrow \infty} 0.$$

*In particular,*

$$\mathbb{E}(|\rho_{k+1}^N - \tilde{\rho}_{k+1}^N|) \longrightarrow 0.$$

*Proof.* By translation invariance, and repeating the arguments of the proof of Lemma 4.3.1, we get that

$$\begin{aligned} \frac{1}{N^d} \sum_{i \in \mathbb{T}_N} \mathbb{E}(|\eta_{k+1}^N(i) - \tilde{\eta}_{k+1}^N(i)|) &= \frac{1}{N^d} \sum_{i \in \mathbb{T}_N} \mathbb{P}(\eta_{k+1}^N(i) \neq \tilde{\eta}_{k+1}^N(i)) \\ &= \mathbb{P}(0 \in \tilde{\eta}_{k+1}^N \setminus \eta_{k+1}^N) \leq (1 - \alpha_N)^{l_N} \approx e^{-\alpha_N l_N} \longrightarrow 0. \quad \square \end{aligned}$$

Before continuing, it is useful to give an explicit construction of the random variable  $\tilde{\eta}_{k+1}^N(0)$ . Consider a collection  $X = (X(i))_{i \in \mathbb{Z}^d}$  of i.i.d. random variables with uniform distribution in  $[0, 1]$  and, given  $\eta_k^N$ , construct  $\eta_{k+1/2}^N$  as follows:

$$\eta_{k+1/2}^N(i) = \mathbf{1}_{X(i) > e^{-\beta d_k^N(i)}}.$$

Observe that with this choice,  $\mathbb{P}(\eta_{k+1/2}^N(i) = 1) = 1 - e^{-\beta d_k^N(i)}$  as required. We will call  $C_0^N$  the open cluster in  $\eta_{k+1/2}^N$  containing 0. Define  $(B_i^N)_{i \in \mathbb{Z}^d, N > 0}$  as in Section 4.3 and set

$$\tilde{\eta}_{k+1}^N(0) = \begin{cases} 1 & \text{if } \tilde{\eta}_{k+1/2}^N(0) = 1 \text{ and } B_i^N = 0 \text{ for all } i \in C_0^N \cap B(0, l_N), \\ 0 & \text{otherwise.} \end{cases}$$

This construction gives the right distribution for  $\tilde{\eta}_{k+1}^N(0)$ .

We introduce another modification of  $\eta_{k+1}^N$ : let  $\hat{\eta}_{k+1}^N$  be the configuration obtained from  $\eta_k^N$  in the same way as  $\tilde{\eta}_{k+1}^N$ , except that in the growing season we replace  $\eta_{k+1/2}^N$  by the configuration  $\hat{\eta}_{k+1/2}^N$  defined by

$$\hat{\eta}_{k+1/2}^N(i) = \mathbf{1}_{X(i) > e^{-\beta h_L^k(p)}}$$

(using the same family of variables  $X$ ). That is,  $\hat{\eta}_{k+1/2}^N$  corresponds to running the growth step as if the density of occupied sites in the ball of radius  $r_N$  around each site was exactly  $h_L^k(p)$ .  $\hat{\rho}_k^N$  will denote the density of occupied sites in this modified process, i.e.,  $\hat{\rho}_k^N = |\hat{\eta}_k^N|/N^d$ . We will call  $C_0$  the open cluster containing

0 in the site percolation process in all of  $\mathbb{Z}^d$  constructed from the collection of random variables  $X$  with each site being open with probability  $1 - e^{-\beta h_L^k(p)}$ .

**Lemma 4.5.4.** *Given any  $\varepsilon > 0$ , if  $\delta_1$  and  $\delta_2$  are small enough, then*

$$\mathbb{E}(|\tilde{\rho}_{k+1}^N - \hat{\rho}_{k+1}^N|) \leq \varepsilon.$$

*Proof.* The idea behind the proof of this result is the following. By (4.13), the density of occupied sites is close to  $h_L^k(p)$  around most sites. If this holds for some site  $i$ , then in a box around  $i$  the density must still be close to this. We then prove the result by comparing  $\tilde{\eta}_{k+1}^N$  and  $\hat{\eta}_{k+1}^N$  with processes in which the outcome of the growth step is replaced by product measures of slightly smaller and slightly larger densities.

To get started we observe that

$$\begin{aligned} \mathbb{E}(|\tilde{\rho}_{k+1}^N - \hat{\rho}_{k+1}^N|) &\leq \frac{1}{N^d} \sum_{i \in \mathbb{T}_N} \mathbb{E}(|\tilde{\eta}_{k+1}^N(i) - \hat{\eta}_{k+1}^N(i)|) = \mathbb{P}(\tilde{\eta}_{k+1}^N(0) \neq \hat{\eta}_{k+1}^N(0)) \\ &\leq \mathbb{P}(\tilde{\eta}_{k+1}^N(0) \neq \hat{\eta}_{k+1}^N(0), 0 \in G_k^N(\delta_1)) + \mathbb{P}(0 \notin G_k^N(\delta_1)) \\ &\leq \mathbb{P}(\tilde{\eta}_{k+1}^N(0) \neq \hat{\eta}_{k+1}^N(0), 0 \in G_k^N(\delta_1)) + \delta_2, \end{aligned} \quad (4.14)$$

where in last bound we used (4.13). To deal with the last probability we first observe that given any  $i \in B(0, l_N)$ ,

$$\begin{aligned} d_k^N(i) &= \frac{1}{V(r_N)} \sum_{j \in B(i, r_N)} \eta_k^N(j) = d_k^N(0) + \frac{1}{V(r_N)} \sum_{j \in B(i, r_N) \setminus B(0, r_N)} \eta_k^N(j) \\ &\quad - \frac{1}{V(r_N)} \sum_{j \in B(0, r_N) \setminus B(i, r_N)} \eta_k^N(j) \\ &\leq d_k^N(0) + \frac{|B(i, r_N) \setminus B(0, r_N)|}{V(r_N)}, \end{aligned}$$

and thus, since the cardinality in the last term is largest when  $i$  is at any of the

$2^d$  corners of the hypercube  $B(0, l_N)$ , we have that for some  $C > 0$

$$|d_k^N(i) - d_k^N(0)| \leq C \frac{r_N^{d-1} l_N}{V(r_N)} \approx \frac{l_N}{r_N} \longrightarrow 0.$$

We deduce that

$$\begin{aligned} & \mathbb{P}(\tilde{\eta}_{k+1}^N(0) \neq \hat{\eta}_{k+1}^N(0), 0 \in G_k^N(\delta_1)) \\ & \leq \mathbb{P}(\tilde{\eta}_{k+1}^N(0) \neq \hat{\eta}_{k+1}^N(0), |d_k^N(i) - h_L^k(p)| \leq 2\delta_1 \forall i \in B(0, l_N), 0 \in G_k^N(\delta_1)) \\ & \quad + \mathbb{P}(|d_k^N(i) - h_L^k(p)| > 2\delta_1 \text{ for some } i \in B(0, l_N), 0 \in G_k^N(\delta_1)) \\ & \leq \mathbb{P}(\tilde{\eta}_{k+1}^N(0) \neq \hat{\eta}_{k+1}^N(0), |d_k^N(i) - h_L^k(p)| \leq 2\delta_1 \forall i \in B(0, l_N)) \\ & \quad + \mathbb{P}(|d_k^N(i) - d_k^N(0)| > \delta_1 \text{ for some } i \in B(0, l_N)) \\ & \quad + \mathbb{P}(|d_k^N(0) - h_L^k(p)| > \delta_1, 0 \in G_k^N(\delta_1)) \\ & = \mathbb{P}(\tilde{\eta}_{k+1}^N(0) \neq \hat{\eta}_{k+1}^N(0), |d_k^N(i) - h_L^k(p)| \leq 2\delta_1 \forall i \in B(0, l_N)) \end{aligned}$$

for large enough  $N$ .

Next, we introduce the following notation:  $\xi_{1/2}^q$  will be the set of open sites in a site percolation process in  $\mathbb{Z}^d$  with each site being open with probability  $1 - e^{-\beta q}$  for  $q \in [0, 1]$  constructed from the family of random variables  $X$ . In other words, we put  $\xi_{1/2}^q(i) = \mathbf{1}_{X(i) > e^{-\beta q}}$  for each  $i \in \mathbb{Z}^d$ . We also let  $\xi_1^{q,N} \subseteq \mathbb{T}_N$  be the configuration obtained after running the epidemic step on  $\xi_{1/2}^q \cap \mathbb{T}_N$  (this is done on the torus  $\mathbb{T}_N$ , so we take into account the periodic boundary conditions of the torus while running the epidemic), using the variables  $(B_i^N)_{i \in \mathbb{T}_N}$ , and ignoring infections coming from a distance greater than  $l_N$ . Observe that with these definitions,  $\hat{\eta}_{k+1/2}^N = \xi_{1/2}^{h_L^k(p)} \cap \mathbb{T}_N$  and  $\hat{\eta}_{k+1}^N = \xi_1^{h_L^k(p), N}$ . The key fact is the

following:

$$\begin{aligned}
& \mathbb{P}(\tilde{\eta}_{k+1}^N(0) \neq \hat{\eta}_{k+1}^N(0), |d_k^N(i) - h_L^k(p)| \leq 2\delta_1 \forall i \in B(0, l_N)) \\
& \leq \mathbb{P}\left(\xi_1^{h_L^k(p)+2\delta_1, N}(0) = 0, \xi_1^{h_L^k(p), N}(0) = 1\right) \\
& \quad + \mathbb{P}\left(\xi_{1/2}^{h_L^k(p)-2\delta_1}(0) = 0, \xi_{1/2}^{h_L^k(p)}(0) = 1\right) \tag{4.15} \\
& \quad + \mathbb{P}\left(\xi_1^{h_L^k(p)-2\delta_1, N}(0) = 1, \xi_1^{h_L^k(p), N}(0) = 0\right) \\
& \quad + \mathbb{P}\left(\xi_{1/2}^{h_L^k(p)+2\delta_1}(0) = 1, \xi_{1/2}^{h_L^k(p)-2\delta_1}(0) = 0\right).
\end{aligned}$$

To see that this is true observe that  $|d_k^N(i) - h_L^k(p)| \leq 2\delta_1$  for all  $i \in B(0, l_N)$  implies that

$$1 - e^{-\beta(h_L^k(p)-2\delta_1)} \leq 1 - e^{-\beta d_k^N(i)} \leq 1 - e^{-\beta(h_L^k(p)+2\delta_1)}$$

for all  $i \in B(0, l_N)$ , and thus

$$\xi_{1/2}^{h_L^k(p)-2\delta_1} \cap B(0, l_N) \subseteq C_0^N \cap B(0, l_N) \subseteq \xi_{1/2}^{h_L^k(p)+2\delta_1} \cap B(0, l_N).$$

Assuming this, we have that  $\tilde{\eta}_{k+1}^N(0) = 0$  and  $\hat{\eta}_{k+1}^N(0) = 1$  implies that  $\xi_1^{h_L^k(p), N}(0) = \xi_{1/2}^{h_L^k(p)}(0) = 1$ , and either  $\tilde{\eta}_{k+1/2}^N(0) = 0$ , which implies that  $\xi_{1/2}^{h_L^k(p)-2\delta_1}(0) = 0$ , or  $\tilde{\eta}_{k+1/2}^N(0) = 1$  but there is an infection in  $C_0^N \cap B(0, l_N)$ , which implies that  $\xi_1^{h_L^k(p)+2\delta_1, N}(0) = 0$ . Similarly,  $\tilde{\eta}_{k+1}^N(0) = 1$  and  $\hat{\eta}_{k+1}^N = 0$  implies that  $\xi_{1/2}^{h_L^k(p)+2\delta_1}(0) = 1$ ,  $\xi_1^{h_L^k(p), N} = 0$ , and there is no infection in  $C_0^N \cap B(0, l_N)$ , and thus  $\xi_1^{h_L^k(p)-2\delta_1, N}(0) = 1$  whenever  $\xi_{1/2}^{h_L^k(p)-2\delta_1}(0) = 1$ .

To finish the proof we need to bound the probabilities on the right side of (4.15). For the first one, since  $\xi_{1/2}^{h_L^k(p)-2\delta_1} \subseteq \xi_{1/2}^{h_L^k(p)} \subseteq \xi_{1/2}^{h_L^k(p)+2\delta_1}$ , we have that if  $\#\xi$  denotes the size of the cluster containing 0 in the configuration given by  $\xi$ , then

$$\begin{aligned}
& \mathbb{P}\left(\xi_1^{h_L^k(p)+2\delta_1, N}(0) = 0, \xi_1^{h_L^k(p), N}(0) = 1\right) \\
& \leq \mathbb{P}\left(\xi_{1/2}^{h_L^k(p)+2\delta_1, N}(0) = 0, \xi_{1/2}^{h_L^k(p), N}(0) = 1, \#\xi_{1/2}^{h_L^k(p)+2\delta_1} < \infty\right) \\
& + \mathbb{P}\left(\xi_1^{h_L^k(p), N}(0) = 1, \#\xi_{1/2}^{h_L^k(p)} = \infty\right) + \mathbb{P}\left(\#\xi_{1/2}^{h_L^k(p)} < \#\xi_{1/2}^{h_L^k(p)+2\delta_1} = \infty\right).
\end{aligned}$$

The first probability on the right side is bounded by

$$\begin{aligned} \mathbb{P}\left(\xi_1^{h_L^k(p)+2\delta_1, N}(0) = 0, \xi_{1/2}^{h_L^k(p)+2\delta_1}(0) = 1, \#\xi_{1/2}^{h_L^k(p)+2\delta_1} < \infty\right) \\ \leq \mathbb{E}\left(1 - (1 - \alpha_N)^{\#\xi_{1/2}^{h_L^k(p)+2\delta_1}}, \#\xi_{1/2}^{h_L^k(p)+2\delta_1} < \infty\right), \end{aligned} \quad (4.16)$$

which goes to 0 by the dominated convergence theorem. The second one goes to 0 as well because it is bounded by  $(1 - \alpha_N)^{l_N} \approx e^{-\alpha_N l_N}$ . The third one equals

$$\theta_L(h_L^k(p) + 2\delta_1) - \theta_L(h_L^k(p)),$$

which is less than  $\varepsilon/2$  for small enough  $\delta_1$  by the (uniform) continuity of the percolation probability  $\theta_L(p)$  for  $p \in [0, 1]$ . The other two probabilities on the right side of (4.15) can be bounded similarly, yielding

$$\mathbb{P}(\tilde{\eta}_{k+1}^N(0) \neq \hat{\eta}_{k+1}^N(0), 0 \in G_k^N(\delta_1)) < \varepsilon$$

for large enough  $N$  and small enough  $\delta_1$ . Putting this together with (4.14) gives the result.  $\square$

**Lemma 4.5.5.**

$$|\mathbb{E}(\hat{\rho}_{k+1}^N) - h_L^{k+1}(p)| \longrightarrow 0.$$

*Proof.* This proof is similar to that of Lemma 4.3.4. First we observe that

$$\begin{aligned} \mathbb{E}(\hat{\rho}_{k+1}^N) &= \mathbb{P}(0 \in \hat{\eta}_{k+1}^N | 0 \in \hat{\eta}_{k+1/2}^N) \mathbb{P}(0 \in \hat{\eta}_{k+1/2}^N) \\ &= \mathbb{P}(0 \in \hat{\eta}_{k+1}^N | 0 \in \hat{\eta}_{k+1/2}^N) [1 - e^{-\beta h_L^k(p)}] \end{aligned} \quad (4.17)$$

and

$$\mathbb{P}(0 \in \hat{\eta}_{k+1}^N, \text{diam}(C_0) = \infty | 0 \in \hat{\eta}_{k+1/2}^N) \leq (1 - \alpha_N)^{l_N} \approx e^{-\alpha_N l_N} \longrightarrow 0. \quad (4.18)$$

Now

$$\begin{aligned}
& \mathbb{P}(0 \in \widehat{\eta}_{k+1}^N, \text{diam}(C_0) < \infty | 0 \in \widehat{\eta}_{k+1/2}^N) \\
&= \mathbb{P}(0 \in \widehat{\eta}_{k+1}^N, \text{diam}(C_0) \leq l_N | 0 \in \widehat{\eta}_{k+1/2}^N) \\
&\quad + \mathbb{P}(0 \in \widehat{\eta}_{k+1}^N, l_N < \text{diam}(C_0) < \infty | 0 \in \widehat{\eta}_{k+1/2}^N) \quad (4.19)
\end{aligned}$$

and, trivially,

$$\begin{aligned}
& \mathbb{P}(0 \in \widehat{\eta}_{k+1}^N, l_N < \text{diam}(C_0) < \infty | 0 \in \widehat{\eta}_{k+1/2}^N) \\
&\leq \mathbb{P}(l_N < \text{diam}(C_0) < \infty | 0 \in \widehat{\eta}_{k+1/2}^N) \longrightarrow 0. \quad (4.20)
\end{aligned}$$

On the other hand,

$$\begin{aligned}
& \mathbb{P}(0 \in \widehat{\eta}_{k+1}^N, \text{diam}(C_0) \leq l_N | 0 \in \widehat{\eta}_{k+1/2}^N) \\
&= \mathbb{P}(B_i^N = 0 \forall i \in C_0 \cap B(0, l_N), \text{diam}(C_0) \leq l_N | 0 \text{ is open}) \\
&= \mathbb{E}((1 - \alpha_N)^{|C_0 \cap B(0, l_N)|}, \text{diam}(C_0) \leq l_N | 0 \text{ is open}) \\
&= \mathbb{P}(\text{diam}(C_0) \leq l_N | 0 \text{ is open}) \\
&\quad - \mathbb{E}(1 - (1 - \alpha_N)^{|C_0 \cap B(0, l_N)|}, \text{diam}(C_0) \leq l_N | 0 \text{ is open}).
\end{aligned}$$

The second expectation is positive and bounded from above by

$$\mathbb{E}(1 - (1 - \alpha_N)^{|C_0|}, |C_0| < \infty | 0 \text{ is open}),$$

so it goes to 0 as  $N \rightarrow \infty$  by the dominated convergence theorem as in (4.16).

Thus

$$\begin{aligned}
& \lim_{N \rightarrow \infty} \mathbb{P}(0 \in \widehat{\eta}_{k+1}^N, \text{diam}(C_0) \leq l_N | 0 \in \widehat{\eta}_{k+1/2}^N) = \mathbb{P}(\text{diam}(C_0) < \infty | 0 \text{ is open}) \\
&= \frac{\mathbb{P}(0 < \text{diam}(C_0) < \infty)}{1 - e^{-\beta h_L^k(p)}} = \frac{g_L(1 - e^{-\beta h_L^k(p)})}{1 - e^{-\beta h_L^k(p)}}.
\end{aligned}$$

Putting this together with (4.19) and (4.20) we get that

$$\left| \mathbb{P}(0 \in \widehat{\eta}_{k+1}^N, \text{diam}(C_0) < \infty | 0 \in \widehat{\eta}_{k+1/2}^N) - \frac{h_L^{k+1}(p)}{1 - e^{-\beta h_L^k(p)}} \right| \longrightarrow 0,$$

and thus by (4.17) and (4.18) we obtain

$$|\mathbb{E}(\widehat{\rho}_{k+1}^N) - h_L^{k+1}(p)| \longrightarrow 0$$

as required.  $\square$

*Proof of Proposition 4.5.1.*

$$\begin{aligned} \frac{1}{N^d} \mathbb{E}(|\mathbb{T}_N \setminus G_{k+1}^N(\varepsilon_1)|) &= \mathbb{P}(0 \notin G_{k+1}^N(\varepsilon_1)) = \mathbb{P}(|d_{k+1}^N(0) - h_L^{k+1}(p)| \geq \varepsilon_1) \\ &\leq \frac{1}{\varepsilon_1} \mathbb{E}(|d_{k+1}^N(0) - h_L^{k+1}(p)|). \end{aligned}$$

Hence

$$\begin{aligned} \frac{1}{N^d} \mathbb{E}(|\mathbb{T}_N \setminus G_{k+1}^N(\varepsilon_1)|) &\leq \frac{1}{\varepsilon_1} \left[ \mathbb{E}(|d_{k+1}^N(0) - \widetilde{d}_{k+1}^N(0)|) + \mathbb{E}(|\widetilde{d}_{k+1}^N(0) - \mathbb{E}(\widetilde{\rho}_{k+1}^N)|) \right. \\ &\quad \left. + \mathbb{E}(|\mathbb{E}(\widetilde{\rho}_{k+1}^N) - \widetilde{\rho}_{k+1}^N|) + \mathbb{E}(|\widetilde{\rho}_{k+1}^N - h_L^{k+1}(p)|) \right], \end{aligned} \quad (4.21)$$

where  $\widetilde{d}_{k+1}^N(0) = \frac{1}{V(r_N)} \sum_{\|j\|_\infty \leq r_N} \widetilde{\eta}_{k+1}^N(j)$

For fixed  $\varepsilon > 0$  we want to show that each of the expectations on the right side of the last inequality can be bounded by  $\varepsilon$  if  $N$  is large enough and  $\delta_1$  and  $\delta_2$  are small enough. The bound for the last one follows directly from the triangle inequality and Lemmas 4.5.4 and 4.5.5.

For the first one we have by translation invariance that

$$\begin{aligned}
\mathbb{E}\left(\left|d_{k+1}^N(0) - \tilde{d}_{k+1}^N(0)\right|\right) &\leq \frac{1}{V(r_N)} \sum_{\|j\|_\infty \leq r_N} \mathbb{E}\left(|\eta_{k+1}^N(j) - \tilde{\eta}_{k+1}^N(j)|\right) \\
&= \frac{1}{N^d V(r_N)} \sum_{i \in \mathbb{T}_N} \sum_{j \in B(i, r_N)} \mathbb{E}\left(|\eta_{k+1}^N(j) - \tilde{\eta}_{k+1}^N(j)|\right) \\
&= \frac{1}{N^d V(r_N)} \sum_{j \in \mathbb{T}_N} \mathbb{E}\left(\sum_{i \in B(j, r_N)} |\eta_{k+1}^N(j) - \tilde{\eta}_{k+1}^N(j)|\right) \\
&= \frac{1}{N^d} \sum_{j \in \mathbb{T}_N} \mathbb{E}\left(|\eta_{k+1}^N(j) - \tilde{\eta}_{k+1}^N(j)|\right) < \varepsilon
\end{aligned}$$

for large enough  $N$  by Lemma 4.5.3.

For the second one we first observe that, again by translation invariance,

$$\mathbb{E}(\tilde{d}_{k+1}^N(0)) = \mathbb{E}(\tilde{\rho}_{k+1}^N). \text{ Hence}$$

$$\begin{aligned}
\mathbb{E}\left(\left|\tilde{d}_{k+1}^N(0) - \mathbb{E}(\tilde{\rho}_{k+1}^N)\right|\right)^2 &\leq \text{Var}(\tilde{d}_{k+1}^N(0)) \\
&= \frac{1}{V(r_N)^2} \sum_{i, j \in B(0, r_N)} \text{Cov}(\tilde{\eta}_{k+1}^N(i), \tilde{\eta}_{k+1}^N(j)) \tag{4.22} \\
&\leq \frac{1}{V(r_N)^2} |\{i, j \in B(0, r_N) : \|i - j\|_\infty \leq l_N\}| \approx \frac{V(l_N)}{V(r_N)} \longrightarrow 0.
\end{aligned}$$

The bound for the third expectation on the right side of (4.21) follows from the exact same argument as previous one. We deduce that

$$\frac{1}{N^d} \mathbb{E}\left(|\mathbb{T}_N \setminus G_{k+1}^N(\varepsilon_1)|\right) \leq \frac{4\varepsilon}{\varepsilon_1}$$

for large enough  $N$ , and thus choosing  $\varepsilon < \varepsilon_1 \varepsilon_2 / 4$  gives the result.  $\square$

*Proof of Theorem 4.1.4.* Since  $[0, 1]$  is compact, it is enough to prove the convergence of the finite dimensional distributions of  $\rho_k^N$ , and since our limit is deterministic, we only need to prove that

$$\mathbb{P}\left(|\rho_k^N - h_L^k(p)| > \varepsilon\right) \xrightarrow{N \rightarrow \infty} 0 \tag{4.23}$$

for every  $k \geq 0$  and  $\varepsilon > 0$ . Proceeding as in the proof of Theorem 4.1.2 we have that

$$\begin{aligned} \mathbb{P}(|\rho_k^N - h_L^k(p)| > \varepsilon) &\leq \frac{1}{\varepsilon} \mathbb{E}(|\rho_k^N - \tilde{\rho}_k^N|) + \frac{1}{\varepsilon} \mathbb{E}(|\tilde{\rho}_k^N - \hat{\rho}_k^N|) \\ &\quad + \frac{1}{\varepsilon} \mathbb{E}(|\hat{\rho}_k^N - \mathbb{E}(\hat{\rho}_k^N)|) + \frac{1}{\varepsilon} |\mathbb{E}(\hat{\rho}_k^N) - h_L(p)|. \end{aligned} \quad (4.24)$$

By Lemmas 4.5.3, 4.5.4, and 4.5.5, given any  $v > 0$  there are constants  $\delta_1^{k-1}, \delta_2^{k-1} > 0$  such that

$$\frac{V(l_N)}{N^d} \mathbb{E}(|\mathbb{T}_N \setminus G_{k-1}^N(\delta_1^{k-1})|) < \delta_2^{k-1} \quad (4.25)$$

implies that the first, second, and last terms on the right side of (4.24) are each bounded by  $v\varepsilon$  for large enough  $N$ . The third term is also less than  $v\varepsilon$  for large  $N$ , which follows from repeating again the argument in (4.22). We deduce that

$$\mathbb{P}(|\rho_k^N - h_L^k(p)| > \varepsilon) < 4v \quad (4.26)$$

for large enough  $N$  provided that (4.25) holds.

Similarly, Proposition 4.5.1 implies that (4.25) will hold provided that

$$\frac{V(l_N)}{N^d} \mathbb{E}(|\mathbb{T}_N \setminus G_{k-2}^N(\delta_1^{k-2})|) < \delta_2^{k-2}$$

for some  $\delta_1^{k-2}, \delta_2^{k-2} > 0$ . Repeating this procedure inductively we deduce that (4.26) holds provided that

$$\frac{V(l_N)}{N^d} \mathbb{E}(|\mathbb{T}_N \setminus G_0^N(\delta_1^0)|) < \delta_2^0$$

for some small  $\delta_1^0, \delta_2^0 > 0$ , which holds for large enough  $N$  by Lemma 4.5.2, and thus (4.23) follows.  $\square$

## REFERENCES

- Durrett, R. (1995). Ten lectures on particle systems. In *Lectures on probability theory (Saint-Flour, 1993)*, volume 1608 of *Lecture Notes in Math.*, pages 97–201. Springer, Berlin.
- Durrett, R. (2007). *Random graph dynamics*. Cambridge Series in Statistical and Probabilistic Mathematics. Cambridge University Press, Cambridge.
- Grimmett, G. (1999). *Percolation*, volume 321 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin, second edition.
- Janson, S., Łuczak, T., and Rucinski, A. (2000). *Random graphs*. Wiley-Interscience Series in Discrete Mathematics and Optimization. Wiley-Interscience, New York.
- Karr, A. F. (1975). Weak convergence of a sequence of Markov chains. *Z. Wahrscheinlichkeitstheorie und Verw. Gebiete*, **33**(1), 41–48.
- Kesten, H. and Zhang, Y. (1987). Strict inequalities for some critical exponents in two-dimensional percolation. *J. Statist. Phys.*, **46**(5-6), 1031–1055.
- Lasota, A. and Yorke, J. A. (1973). On the existence of invariant measures for piecewise monotonic transformations. *Trans. Amer. Math. Soc.*, **186**, 481–488 (1974).
- Li, T. Y. and Yorke, J. A. (1975). Period three implies chaos. *Amer. Math. Monthly*, **82**(10), 985–992.
- Rudin, W. (1976). *Principles of mathematical analysis*. McGraw-Hill Book Co., New York, third edition. International Series in Pure and Applied Mathematics.