IS INEQUALITY STATISTICALLY INEVITABLE? ECONOPHYSICAL APPROACHES TO MODELING WEALTH AND INCOME DISTRIBUTIONS

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Max Greenberg

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ABSTRACT

The last twenty-five years have seen the development of a significant trend within the subfield of econophysics which attempts to model economic inequality as an emergent property of stochastic interactions among ensembles of agents. In this thesis, the literature surrounding this approach to the study of wealth and income distributions, henceforth the ”random asset exchange” literature, is thoroughly reviewed for the first time. The foundational papers of Drăgulescu & Yakovenko (2000), Chakraborti & Chakrabarti (2000), and Bouchaud & Mézard (2000) are discussed in detail, and principal canonical models within the random asset exchange literature are established. The most common variations upon these canonical models are enumerated, and significant papers within each kind of modification are introduced. The successes of such models, as well as the limitations of their underlying assumptions, are discussed, and it is argued that the literature must move in the direction of more explicit representations of economic structure and processes before it can truly be seen as explanatory.

In order to demonstrate what such a pivot could look like, a completely new model, principally inspired by the one introduced by Wright (2005), is proposed. By constructing the Markov process governing the expected motion of a representative unit of wealth, the equilibrium division of wealth between the three subpopulations posited by the model is derived. Monte Carlo simulation methods are used to further explore the behavior of the model and to establish the phase space bounds within which the model possesses a stable statistical equilibrium.

It is found that the distributions of both wealth and income across all three subpopulations are well fit by Gamma distributions and that the equilibrium class division of wealth closely matches the Markov process prediction. When taxation is introduced into the model, it is furthermore found that wealth, income, and sales taxes all have strong equalizing effects on the equilibrium wealth distribution, while payroll and turnover taxes do not. The degree of income inequality, as measured by the Gini coefficient, is not alleviated by the introduction of any form of flat taxation, consistent with the limited empirical findings on the subject. The thesis concludes with a discussion of the overall significance of the last 25 years of random asset exchange modeling and an identification of the most important areas for future work.

BIOGRAPHICAL SKETCH

Max Greenberg is a master's student in the Systems Engineering Department of Cornell University and an affiliate researcher of the Center for the Study of Inequality. Born on March 28th, 2000, Max grew up in New York City and attended Hunter College High School on the Upper East Side of Manhattan before graduating in 2018. He moved to Ithaca, NY and began his studies at Cornell University that same year, and graduated with a bachelor's degree in mathematics and economics in 2021. His research interests currently include the study of economic inequality and political economy.

This document is dedicated to my parents, William and Jennifer, and to my siblings, Samuel and Theodora.

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CHAPTER 1

INTRODUCTION

Over the last fifteen years, the question of economic inequality has become the epicenter of one of the most intense political debates in the United States. Awareness of the growing gap between rich and poor has been growing since the 2008 American bank bailouts and the 2010 *Citizens United v. FEC* Supreme Court decision, but the inequality question was decisively pushed to the forefront of American politics in September 2011 with the beginning of the Occupy Wall Street protest movement, which introduced the dichotomy of ″the 1%″ vs. ″the 99%″ to public consciousness. Though the Occupy movement did not immediately produce anything by way of practical politics, it nonetheless laid the foundation for U.S. Senator Bernie Sanders' two campaigns for president, in which he used the language of Occupy to reframe economic inequality as the result of policy choices which could be rectified through a social-democratic ″political revolution″. [81]

But the idea that economic inequality is a problem which needs to be addressed by way of policy is by no means an uncontroversial one. The majority of American adults who electorally favor the Republican Party do not believe that the current level of economic inequality in the United States is excessive. [82] The legacy of ″Reaganomics″—a colloquial term referring to the economic policy pursued by the Federal government of the United States under the tenure of former President Ronald Reagan, which was characterized by cuts to tax rates and other concessions to ″supply-side″ economic theory—remains contentious. [113] And within the Republican delegation to the United States House of Representatives, a proposal to eliminate the current bracketed income tax system and to replace it with a much higher nationwide flat sales tax, in order to dramatically lessen the tax burden on the wealthy, is gaining some traction. [56]

The controversy surrounding economic inequality is just as longstanding and just as intense within the realm of academic economics. On one hand, economists tend to be more skeptical than other social scientists of government intervention in the economy, as a great deal of emphasis is placed on the fact that, within the discipline's canonical models, free markets have no trouble arriving at socially optimal allocations of resources all on their own. On the other hand, French economist Thomas Piketty's 2013 magnum opus *Capital in the 21st Century*, which posits the imposition of a global, progressive tax on wealth as an ideal horizon for policy to attempt to approach precisely in order to rein in what he views as the excessive amount of economic inequality in the world today, has proven to be greatly influential in the popular-academic debates concerning the issue in its own right. [130] Needless to say, this debate shows no sign of abating anytime soon.

1.1 The Universality of Economic Inequality

What is indisputable, however, is that in nearly every single developed market economy, the degree of stratification between the rich and everyone else is not only staggering, but also gradually increasing. In the United States, the share of household wealth owned by the top 1% of the population by net worth grew from

29.9% in 1989 to 35.5% in 2013; meanwhile, the share of wealth owned by the bottom 50% of the population shrunk from 3.0% to 1.1% during the same period. [98] In Germany, individuals at the 90th percentile of net assets own 13 times as much wealth as the median individual and over a quarter of individuals have liabilities equal to or greater than their assets, resulting in a negative net worth. [74] While there are many countries where the degree of wealth inequality is not this extreme—the United States has one of the most unequal distributions of wealth in the world—the overall structure is strikingly similar in almost every country. [50] In every market economy for which data exists, many possess very little wealth and a few possess much.

Great inequality also governs the distribution of incomes within market economies. As reported by Horowitz *et al.* for the Pew Research Center, the share of aggregate income possessed by high-income households, defined as households with incomes greater than twice the national median, has grown from 29% in 1970 to 48% today. [82] In the same period, the share of aggregate income possessed by low-income households, defined as households with incomes less than two-thirds the national median, fell from 10% to 9% over the same period. In a similar vein, the incomes of those who are already in the top 5% of the population in terms of earnings have consistently grown the faster than all other earners over the past 40 years.

One of the most common ways to quantify the degree of inequality present in a given wealth or income distribution—or, indeed, any density distribution over a non-negative domain—is the Gini coefficient, named for Italian statistician Corrado Gini. The Gini coefficient is canonically defined by reference to the Lorenz curve, itself defined as the function $L(x)$ representing the share of some asset—say, income—held by the bottom $100x\%$ of the population. [57] The Gini coefficient is then given by twice the difference between the area under Lorenz curve of a perfectly egalitarian distribution—a straight line with a slope of 1—and the Lorenz curve of the distribution in question. Thus, the canonical formula used to calculate the Gini coefficient is:

$$G = 1 - 2\int_0^1 L(x)dx \tag{1.1}$$

When the distribution in question is defined over a finite population X consisting of N individuals sorted in increasing order, the Gini coefficient may be approximated using the computationally much faster formula:

$$G = \frac{N+1}{N} - \frac{2}{N^2\bar{X}}\sum_{i=1}^{N}(N-i+1)X_i \tag{1.2}$$

as demonstrated by Allison (1978). [3] Note that the Gini index over a discrete population does not perfectly correspond to its continuous counterpart, however, as the former has an upper bound of $1 - 1/N$. [4]

Table 1.1 displays the Gini coefficients for the wealth and income distributions of ten countries. One observes that wealth distributions are practically always ″more unequal″ than income distributions: Gini coefficients for wealth distributions tend to range between 0.5 and 0.8, while Gini coefficients for income distributions tend to range from 0.25 to 0.45. Furthermore, there is no obvious correlation between the Gini coefficients for wealth distributions and for income distri-

Gini Coefficients of Wealth and Income for Ten Countries		
Country	*Gini Coefficient of Wealth*	*Gini Coefficient of Income*
United States	0.801	0.401
France	0.730	0.311
United Kingdom	0.697	0.396
India	0.669	0.344*
Germany	0.667	0.289
Netherlands	0.650	0.298*
Australia	0.622	0.331*
Italy	0.609	0.353
Spain	0.570	0.343
China	0.550	0.420*

Table 1.1: Gini coefficients of wealth and income inequality for ten countries, all major world economies, based on data from the year 2000. Data for Gini coefficients of wealth are taken from Davies *et al.* (2009), while data for Gini coefficients of income are taken from the FRED database hosted by the Federal Reserve Bank of St. Louis. Gini coefficients of income for India, the Netherlands, and Australia are from 2004. Gini coefficient of income for China is from 2002.

butions: some countries, such as China, have coefficients relatively close in value, while other countries, such as France, have Gini coefficients for wealth over twice as high as the corresponding value for income.

Regardless of what one personally believes regarding the question of the role governments should play in redistributing wealth from the rich to the poor, the universality of the phenomenon of extreme inequality should raise eyebrows. Different countries have dramatically different approaches to welfare programs, taxation, and all other sorts of policy. Yet the distributions of wealth and income which emerge in these countries are remarkably similar in form. It follows that there must be some shared set of characteristics that account for this common structure of wealth distribution. This line of questioning points one to an often overlooked and still poorly understood aspect of economic inequality: its origin.

1.2 Pareto, Gibrat, and the Econophysicists

The nature and origin of the distribution of wealth and income in market economies has been an open problem in economics for more than a century. In 1897, the Italian civil engineer-turned-economist Vilfredo Pareto attempted an answer after noticing a striking pattern in data for land-ownership rates in Italy. Specifically, Pareto posited that income in every society was distributed according to a decreasing power law; namely:

$$p(w) \propto w^{-1-\alpha} \tag{1.3}$$

where $p(w)$ represents the probability density function of income and α is the "Pareto index." This observation has come to be known as the "weak Pareto law," with its strong counterpart including the additional claim that the Pareto index possesses a value in the range 1.5 ± 0.5. [120] But not long thereafter it became apparent that this law did not actually well characterize the entire income distribution. Instead, when low- and middle-income strata were taken into account, the data seemed to be much better fit by a right-skewed lognormal distribution:

$$p(w) = \frac{1}{w\sigma\sqrt{2\pi}} \exp\left(-\frac{(\ln(w) - \mu)^2}{2\sigma^2}\right) \tag{1.4}$$

This fact was first noticed by Robert Gibrat. [70] It is now well established that, in fact, both Pareto and Gibrat were correct: a lognormal-like distribution tends to characterize the bulk of incomes, while the Pareto distribution tends to characterize the highest 2-3% of incomes. [115]

Since these discoveries, mainstream economic theory has, broadly speaking, shied away from further attempts to impose a universal form to these distributions or to explain the processes responsible for their emergence. There are both normative and methodological reasons for this gap in the economics literature. The normative aversion, as voiced by Piketty, questions Pareto's claim that a universal law governing the right tails of income distributions even exists. [130] The methodological aversion, on the other hand, stems from the fact that most macroeconomic models make use of single, representative agents, which are ill-suited for describing heterogeneity within a population. Meanwhile, more sophisticated tools capable of addressing such questions, such as the Heterogeneous Agent New Keynesian (HANK) class of models, are still in their infancy. [2] Nonetheless, the result of this aversion was that that there has remained comparatively little in the way of literature concerning one of the most crucial questions in economics today. This gap drew the attention of, of all people, physicists interested in applying methods developed for the study of the natural sciences to questions in the social sciences in the late 1990s.

The aim of the econophysicists' models was to capture the characteristic features of empirical wealth and income distributions, as made known by extensive statistical analyses. There is now substantial evidence that the bulk of the income distribution in all capitalist countries follows an exponential distribution. [155]

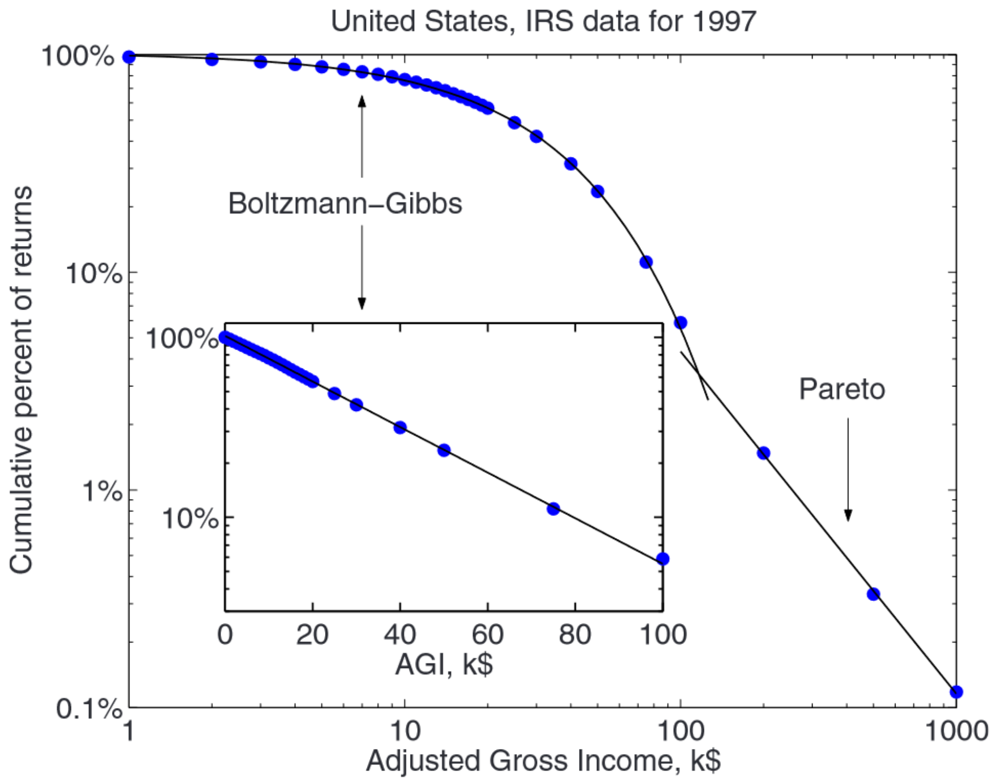

Figure 1.1: Log-log plot of U.S. income distribution in 1997, as printed in Yakovenko (2009). [165]

The right tail of the income distribution follows the aforementioned Pareto law and the left tail follows Gibrat's law. The exponential bulk and the log-normal left tail are sometimes unified in the form of the closely related Gamma distribution:

$$p(w) = \frac{\beta^{\alpha}}{\Gamma(\alpha)} w^{\alpha-1} e^{-\beta w} \tag{1.5}$$

where α is called the "shape parameter" and β the "rate parameter". However, there remains insufficient data to conclude which distribution provides the better empirical fit. [34]

Wealth distributions are unfortunately much less well understood due to a dearth of publicly available data. Rough estimates of wealth distributions in pre-capitalist societies, such as in the New Kingdom of Egypt and medieval Hungary, provide some evidence that such societies exhibited power-law distributions of wealth, but these results are far from conclusive. [1, 79] Drăgulescu & Yakovenko (2001b) used inheritance tax data to study the wealth distribution in the modern United Kingdom, which was found to have a similar structure to the UK's income distribution. [60] Further supporting this conclusion, Sinha (2006), among others, found evidence that the very wealthiest stratum of society, as measured by published "rich lists", follows a power law distribution as well. [143] These features appear to emerge even in artificial economies, with Fuchs *et al.* (2014) having observed an exponential bulk and power-law tail even in the wealth distribution across players of a massively multiplayer online game with an inbuilt system of production and trade. [66] Thus, early exchange models in the econophysics literature sought to generate distributions exhibiting both the exponential bulk and power-law tail observed in data by means of symmetric binary interactions.

The earliest paper in this lineage was Ispolatov *et al.* (1998), and shortly thereafter two papers which would ultimately become the cornerstones of the stochastic wealth distribution modeling literature—Drăgulescu & Yakovenko (2000) and Bouchaud & Mézard (2000)—emerged. [94, 59, 25] As it turned out, however, the econophysicists were not the first to approach this question in this way. The sociologist John Angle had actually published a series of papers containing a model extremely similar to Ispolatov *et al.*'s a decade earlier, though the literature had no knowledge of this fact until it was pointed out by Lux (2005). [5, 6, 7, 109] Likewise, it was realized by Patriarca *et al.* (2005) that Drăgulescu & Yakovenko's model was anticipated by a series of papers by Eleonora Bennati, which had been published in ill-known Italian economics journals in the 1980s and which had not been (and still today have not yet been) translated into English. [125, 16, 15]

Nonetheless, in the twenty-five years since Ispolatov *et al.*'s initial paper, a sizeable literature on this subject has emerged, with countless variations of the aforementioned models proposed and investigated. The literature has also become much more diverse in that time: though this subject was initially solely the domain of a subset of physicists interested in exploring economic questions, they have since been joined by researchers with backgrounds in mathematics, economics, systems science, and more.

1.3 A Survey and a Contribution

The aim of this thesis is twofold. First, it provides, for the first time, a comprehensive and thoroughgoing review of what will be referred to as the "random asset exchange" literature. While many excellent partial reviews do already exist (see Chatterjee & Chakrabarti (2007), Yakovenko & Rosser (2009), Patriarca *et al.* (2010), and Patriarca & Chakraborti (2013), just to name a few), all either have since become outdated or focus on only a delimited part of the literature. [41, 165, 121, 122] This review is the first, to the author's knowledge, that not only discusses all significant econophysical models of income inequality, but fully enumerates the most common variations upon the literature's canonical models as well.

Second, it introduces a new model which attempts to innovate upon pre-existing models by combining a similar stochastic exchange mechanism with explicit representations of firm structure and production. This modeling approach is not completely original: a small handful of papers, namely Wright (2005) and Lavička *et al.* (2010), have investigated models with similar orientations in the past. However, this new model is distinguished by its highly parametrized and adaptible nature, as well as by its ability to investigate non-equilibrium dynamics. It is argued that more attention ought to be given to ”processual” models of this type which more explicitly represent economic phenomena.

Chapter 2 corresponds to the first aim set out above, and Chapter 3 corresponds to the second. Chapter 4 concludes.

CHAPTER 2

REVIEW OF LITERATURE

In order to understand both what the econophysics literature on the subject of wealth and income inequality has achieved and where it stands to improve, one must first have a comprehensive understanding of the the development of said literature over the past 25 years. The aim of this chapter is to provide exactly that.

Section 1 introduces the main classes of random asset exchange model, with which any researcher interested in this subject should be familiar. These include the models introduced in the two canonical papers at the foundation of the random asset exchange literature—Drăgulescu & Yakovenko (2000) and Bouchaud & Mézard (2000)—and notable, mostly named, variations thereon. The terminology popularized by Hayes (2002), distinguishing between ”theft-and-fraud” (TF) style models and ”yard sale” (YS) style models is introduced, and a number of minor but nonetheless notable early models are mentioned. [78]

Section 2 provides a comprehensive overview of the most common featural adjustments made to the canonical models detailed in Section 1, including the introduction of non-conservation of wealth, adjacency networks, goal-oriented behavior, and taxation. The major papers discussing each significant modification are reviewed, and results therefrom are reported.

Finally, Section 3 assesses the achievements of random asset exchange modeling, and concludes with a discussion of the explanatory potential of such models.

2.1 The Taxonomy of Random Asset Exchange Models

Most random asset exchange models tend to fall into one of two classes. The first of these is conventionally called the ”kinetic wealth exchange” (KWE) class of model, which was popularized by Drăgulescu & Yakovenko (2000). Named such because of their similarity to thermodynamic models from the kinetic theory of gases, KWE models are typically—though not always—characterized by the following properties:

1. Pairwise exchange between agents is the primary system state transition function;
2. Total money present in the system is conserved; and
3. Total money present between all pairs of agents engaged in exchange is conserved.

These features are analogous to the role of particle collisions, conservation of energy, and conservation of momentum in the kinetic theory of gases, respectively.

The second prominent class of model, inspired by models of directed polymers rather than ideal gases, is the Bouchaud-Mézard (BM) type model, first introduced by Bouchaud & Mézard (2000). [25] In contrast to KWE-style models, BM-style models tend to be characterized instead by fixed wealth flow rates between all ”adjacent” pairs of agents, as defined by an implicit adjacency network, being the main mechanism of system evolution. Furthermore, each agent's wealth is subject to endogenous stochastic variation, leading to systemic non-conservation of wealth.

While other formulations exist, models belonging to one of these two classes represent the great majority of the literature. In this section, the most significant variations of both classes—as well as a handful of minor but nonetheless significant alternative model classes—are reviewed.

2.1.1 Kinetic wealth exchange

While their work was anticipated by Angle (1986), Bennati (1988, 1993), and Ispolatov *et al.* (1998), it was Drăgulescu & Yakovenko (2000) who are credited with first formalizing and thoroughly studying the KWE model. [5, 16, 15, 94, 59] In their initial formulation, a system of $N \gg 0$ agents with $M \gg N$ units of wealth between them is posited. Agents then engage in random pairwise exchanges, with a winner and loser being randomly selected in each pair and a transfer of wealth occurring, following the exchange rule:

$$\begin{bmatrix} w_i \\ w_j \end{bmatrix} \rightarrow \begin{bmatrix} w_i + \Delta w \\ w_j - \Delta w \end{bmatrix} \tag{2.1}$$

with $\Delta w > 0$ if agent i is the winner of the exchange, and $\Delta w < 0$ if instead j is victorious.

Drăgulescu & Yakovenko showed that, so long as Δw is chosen so that the exchange process was time-reversal symmetric, then the distribution of money among agents converges to the entropy-maximizing exponential distribution:

$$p(w) = \frac{1}{T} \exp\left(-\frac{w}{T}\right) \tag{2.2}$$

where $T = \langle w \rangle = \frac{M}{N}$ represents the average wealth held by agents—analogous to temperature in the equivalent thermodynamic system. This result proves to be extremely robust, not varying with one's choice of time-reversal symmetric exchange rule or underlying adjacency network. [102]

The differences between Drăgulescu & Yakovenko's model and those of Angle, Ispolatov *et al.*, and Bennati are subtle. In both Angle's initial model (the "one-parameter inequality process," or OPIP) and Ispolatov *et al.*'s multiplicative-random exchange model, $\Delta w = \varepsilon w_{loser}$, such that the exchange rule becomes:

$$\begin{bmatrix} w_i \\ w_j \end{bmatrix} \to \begin{bmatrix} w_i + \varepsilon w_j \\ (1-\varepsilon) w_j \end{bmatrix} \tag{2.3}$$

if agent i wins the exchange. The sole difference between these two formulations is that Angle (1986) draws ε from a uniform distribution before each exchange, whereas Ispolatov *et al* (1998) define ε as a fixed parameter. Unlike the exchange rules investigated by Drăgulescu & Yakovenko (2000), both of these models break time symmetry and produce identical distributions which are very well-approximated by, but not exactly given by, Gamma distributions. [7]

In both Ispolatov *et al.*'s additive-random exchange model and Bennati's model, on the other hand, agents exchange constant, quantized amounts of wealth, equivalent under rescaling to $\Delta w = 1$. In Ispolatov *et al.* (1998), agents

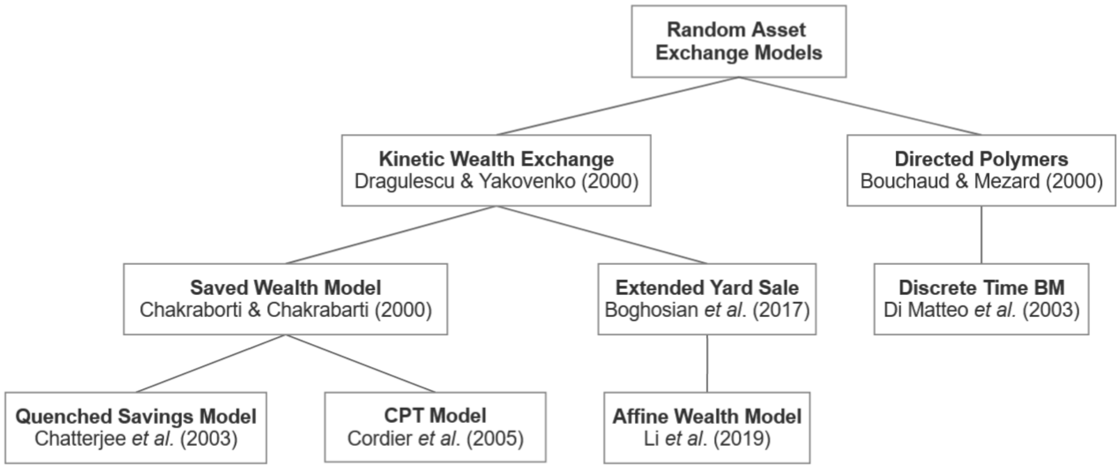

Figure 2.1: "Family tree" of some major papers in the RAE literature, where a "child" node represents a model formulated as a variation of the "parent" node's model.

with 0 wealth are removed from the system entirely, causing all the wealth in the system to eventually be accumulated by a single agent (a phenomenon termed "condensation"). In Bennati (1988), however, agents with 0 wealth are permitted to win, but not to lose, exchanges, identical to the provision in the constant exchange rule discussed by Drăgulescu & Yakovenko. For that reason, the KWE model with time reversal-symmetric exchange rule is sometimes referred to as the Bennati-Drăgulescu-Yakovenko (BDY) model. [165]

Saving propensity

An extension of Drăgulescu & Yakovenko's initial model proposed contemporaneously with its initial publication was investigated by Chakraborti & Chakrabarti

(2000), which introduced a "saving propensity" parameter λ. [35] Called the CC model (or, more rarely, the "saved wealth" model), its system dynamics are characterized by the fact that, for $\lambda \in [0, 1)$, every agent engages in multiplicative exchange with only a fraction $1 - \lambda$ of their total wealth. The exchange rule in such models thus becomes:

$$\begin{bmatrix} w_i(t+1) \\ w_j(t+1) \end{bmatrix} = \begin{bmatrix} \lambda + \varepsilon(1-\lambda) & \varepsilon(1-\lambda) \\ (1-\varepsilon)(1-\lambda) & \lambda + (1-\varepsilon)(1-\lambda) \end{bmatrix} \begin{bmatrix} w_i(t) \\ w_j(t) \end{bmatrix} \tag{2.4}$$

where ε is drawn from a uniform distribution on [0, 1] at every exchange.

Curiously, this slight modification dramatically changes the equilibrium distribution of money among agents within the system as the mode of the distribution (the "most likely agent wealth") becomes non-zero, approaching $T = \frac{M}{N}$ (an egalitarian distribution) as λ approaches 1. Gupta (2006) observes that this departure from the entropy-maximizing distribution is a consequence of the fact that the introduction of the saving propensity parameter λ results in the system transition matrix becoming non-singular. [76] Patriarca *et al.* (2004a, b) demonstrates that the resultant distribution is extremely well fit by the standard Gamma distribution:

$$p(w) = \frac{1}{\Gamma(n)} \left(n \cdot \frac{w}{T}\right)^{n-1} \exp\left(-n \cdot \frac{w}{T}\right) \tag{2.5}$$

where $n = 1 + 3\lambda/(1-\lambda)$. [124, 123] The fit is not exact, however, as the distributions differ in their fourth moments. [101]

The CC model is extremely influential in the random asset exchange literature, and it itself has two major variations which must be mentioned. The first, introduced by Chatterjee *et al.* (2003), defines the saving propensity parameter to be heterogenously distributed throughout the population; rather than being identical for all agents, each agent i has their own individual saving propensity $\lambda_i \in [0, 1)$ drawn from the uniform distribution during model initialization. [42] The exchange rule of this model, called the CCM model, is thus:

$$\begin{bmatrix} w_i(t+1) \\ w_j(t+1) \end{bmatrix} = \begin{bmatrix} \lambda_i + \varepsilon(1-\lambda_i) & \varepsilon(1-\lambda_j) \\ (1-\varepsilon)(1-\lambda_i) & \lambda_j + (1-\varepsilon)(1-\lambda_j) \end{bmatrix} \begin{bmatrix} w_i(t) \\ w_j(t) \end{bmatrix} \tag{2.6}$$

The steady-state distribution then exhibits a Gamma-like bulk, as in the CC model, as well as a right tail well fit by a power law with Pareto parameter $\alpha = 1$; this power law is robust for any distribution of saving propensity of the form $\rho(\lambda) \approx |\lambda_0 - \lambda|^{\alpha}$, or for uniform distributions within a restricted range $\lambda_i \in [a, b] \subset [0, 1)$. [43]

One well-known (and, arguably, unrealistic) aspect of the CCM model is that average agent wealth is highly correlated with their saving parameter, such that the agents who save nearly all of their money in every transaction invariably become the wealthiest. This remains the case even if a significant bias in favor of poorer agents is introduced, because thrifty agents in the CCM model always stand to gain much more than they lose from every transaction. [117] This aspect of the model also explains the surprising appearance of the Pareto tail, which is actually somewhat illusory: the right tail of the equilibrium distribution of the CCM model is constituted by the overlapping exponential tails of the exponential distributions corresponding to the subpopulations with the highest saving parameters. [125]

Another significant drawback of the CCM model, as demonstrated by Repetowicz *et al.* (2005), is that, while the right tails of the steady state distributions change from approximately Pareto with index 1 to exponential as the distribution of λ_i narrows, the empirical value of $\alpha \approx 1.5$ is never reached. [134] However, there are a number of ways to modify the CCM model and recover such a regime: Repetowicz *et al.* (2006), for instance, note that introducing modified wealth parameters with memory—$\hat{w}_i(t) = w_i(t) + \gamma w_i(q)$, with $\gamma \in (0, 1)$ and $q < t$—before each transaction and applying the CC exchange rule thereto does permit Pareto tails with indices $\alpha > 1$ to be obtained. [135] Likewise, Bisi (2017) demonstrates that replacing the saving propensity parameter with a bounded, global function of an agent's wealth $\gamma(w_i)$ also permits superunitary Pareto indices. [18]

The second variation, introduced by Cordier *et al.* (2005) and usually called the CPT model, investigates the CC model with an additional stochastic growth term:

$$\begin{bmatrix} w_i(t+1) \\ w_j(t+1) \end{bmatrix} = \begin{bmatrix} (1-\lambda)+\eta_i & \lambda \\ \lambda & (1-\lambda)+\eta_j \end{bmatrix} \begin{bmatrix} w_i(t) \\ w_j(t) \end{bmatrix} \tag{2.7}$$

Where η_i and η_j are independent and identically distributed variables with mean 0 and variance σ^2. [47] As in the BDY, CC, and CCM models, debts are not permitted, so a transaction only takes place so long as neither agent is reduced to a negative level of wealth. Because η_i and η_j are uncorrelated, total wealth is now only preserved in the mean. The CPT model produces an inverse Gamma steady state:

$$p(w) = \frac{(\alpha-1)^\alpha}{\Gamma(\alpha)} \cdot w^{-1-\alpha} \cdot \exp\left(-\frac{\alpha-1}{w}\right) \tag{2.8}$$

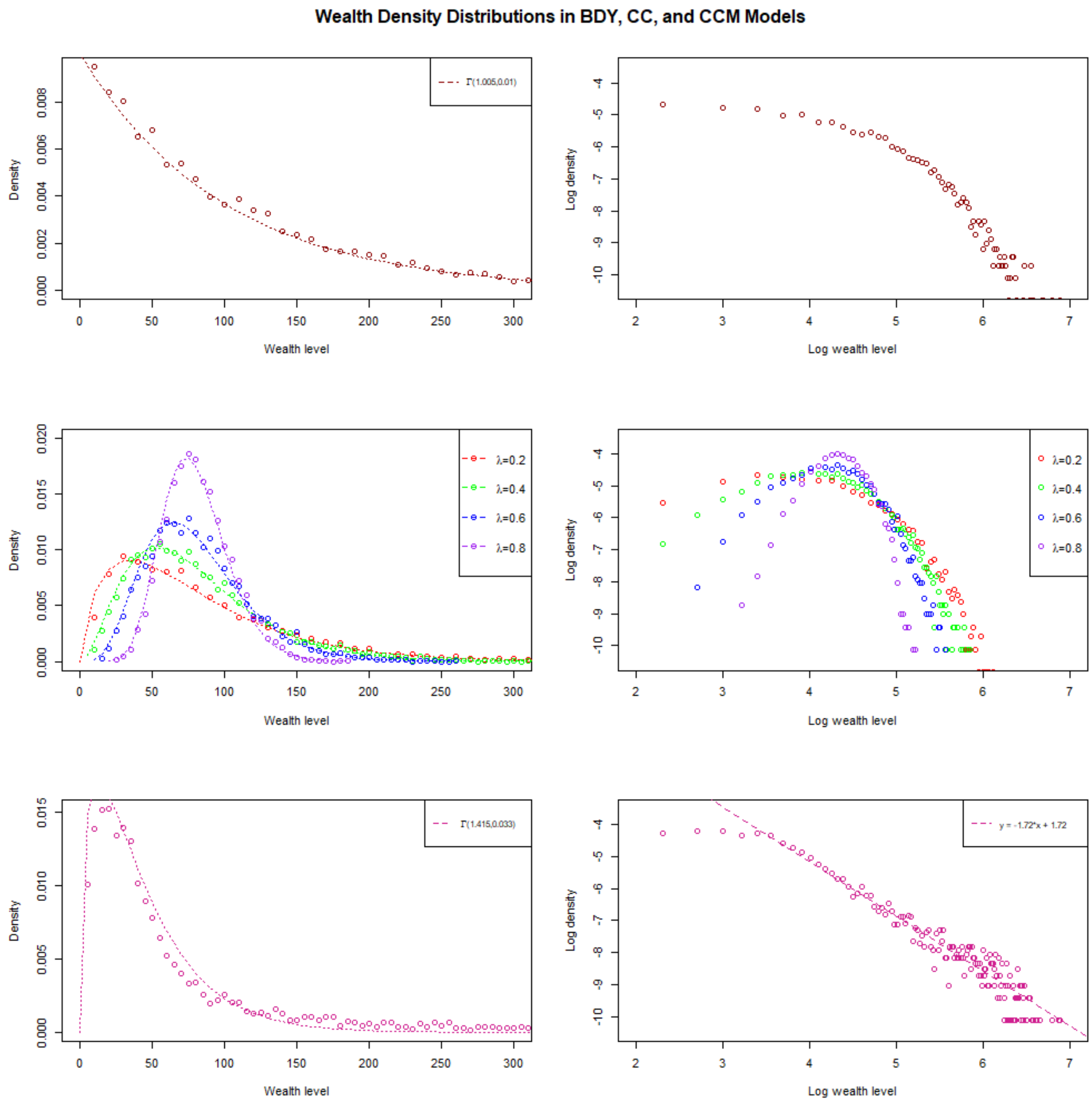

Figure 2.2: Stationary distributions produced by the BDY (left), CC (middle), and CCM (right) models, with best-fit gamma curves shown. All simulations were performed with the parameterization $N = 5000$ and $\langle w \rangle = 100$ over 10^5 iterations.

where the Pareto index $\alpha = 1 + \frac{2\lambda}{\sigma^2}$.

The CPT model, like the CCM model, is quite flexible and has been studied in a variety of other contexts. Düring & Toscani (2008) employ a CPT model with quenched saving propensities to study international transactions, representing countries as subpopulations with different saving propensities. [63] Bisi & Spiga (2010) consider a variation on the CPT wherein the amount of wealth an agent receives from his trading partner is also subject to stochastic fluctuations. [19] More recently, Zhou *et al.* (2021) investigate the effect of introducing a non-Maxwellian (i.e. wealth-varying) collision kernel in the CPT model. [168]

Yard sale model

Following the terminology of Hayes (2002), binary exchange models in which the transfer amount is proportional to the wealth of the loser are commonly referred to as ”theft and fraud” (TF) models, while those in which the transfer amount is proportional to the wealth of the poorer agent are referred to as ”yard sale” (YS) models. That is, the YS model posits an exchanged quantity Δw of the form:

$$\Delta w \propto \min\{w_i, w_j\} \tag{2.9}$$

The advantage of the YS model is that, from a strategic perspective, agents are not disincentivized from engaging in trade, as the expected value of an exchange is always 0. This is in contrast to the TF model, which is so-named precisely because the expected value of an exchange is always negative for the richer agent. If agents were allowed to choose whether or not to engage in a given exchange, a TF economy would immediately freeze as soon as a wealth differential appeared. The principal drawback of the YS model, however, is that it is now well-known that the unmodified YS model always exhibits condensation, though a non-degenerate equilibrium can be recovered if the probability of winning a given exchange is biased in favor of the less wealthy agent, if a mechanism for redistributing wealth from richer agents to poorer ones is introduced, or if extremal dynamics are coupled to the system. [23, 142, 32, 10]

Moukarzel *et al.* (2007) demonstrated that, in the case of the YS model where the proportion of the poorer agent's wealth at stake in each transaction is a fixed constant f, a sufficient bias of the probability p towards the poorer agent alone was sufficient to avoid condensation. [116] In particular, the critical probability p^* above which the system does not condense was found to be:

$$p^* = \frac{\log\left(\frac{1}{1-f}\right)}{\log\left(\frac{1+f}{1-f}\right)} \tag{2.10}$$

Based on this result, Bustos-Guajardo & Moukarzel (2012) studied an extension of the YS model on an adjacency network, such that exchanges may only take place between adjacent agents. They found that the value of the critical probability remains the same regardless of the choice of network. [30] In fact, most system dynamics in the stable phase of the system are independent of the choice of network. However, certain dynamical aspects of the system (such as time required for the system to fully condense) do differ from the fully-connected case in the unstable (i.e. condensing) region. This is not entirely surprising, seeing as the number of agents to whom the wealth will condense is directly determined by the underlying network; instead of one agent accumulating all the money, the distribution condenses to a set of ″locally rich agents,″ sometimes termed the ″oligarchy.″

Redistribution in the YS model was examined by Boghosian (2014a), in which a mechanism by which, at each time step, χ percent of each agent's wealth was confiscated and subsequently redistributed uniformly among the population. [21] Introducing this mechanism not only prevented condensation, but also produced a gamma-like steady state distribution with a Pareto-like tail. The dynamics of this mechanism were studied in more detail by Boghosian *et al.* (2017) and Devitt-Lee *et al.* (2018), in which it was combined with an bias in exchanges in favor of the wealthy, called the ″wealth-attained advantage.″ [22, 52] In this variation, termed the ″extended yard sale″ (EYS) model, the wealthier agent wins a given exchange with probability $p = rT(w_i - w_j)$, where T is the average wealth of the system and w_i is the wealth of the richer agent. The wealth-attained advantage formally acts as a net tax on the non-oligarchy, while the redistribution acts as a net tax on the oligarchy; once the poor-to-rich flux of the redistributive mechanism was eclipsed by the rich-to-poor flux of the wealthy agents' advantage, the system moves from a subcritical to a supercritical state and the inequality of the resulting wealth distribution, as measured by the Gini coefficient, begins increasing rapidly. This acts as a second-order phase transition within the system. One additional variation of the EYS model, the ″affine wealth″ (AW) model, was introduced by Li *et al.* (2019). [104] The AW model permits negative wealth by defining a debt limit Δ, adding Δ to the wealths of both agents before each exchange, and subtracting

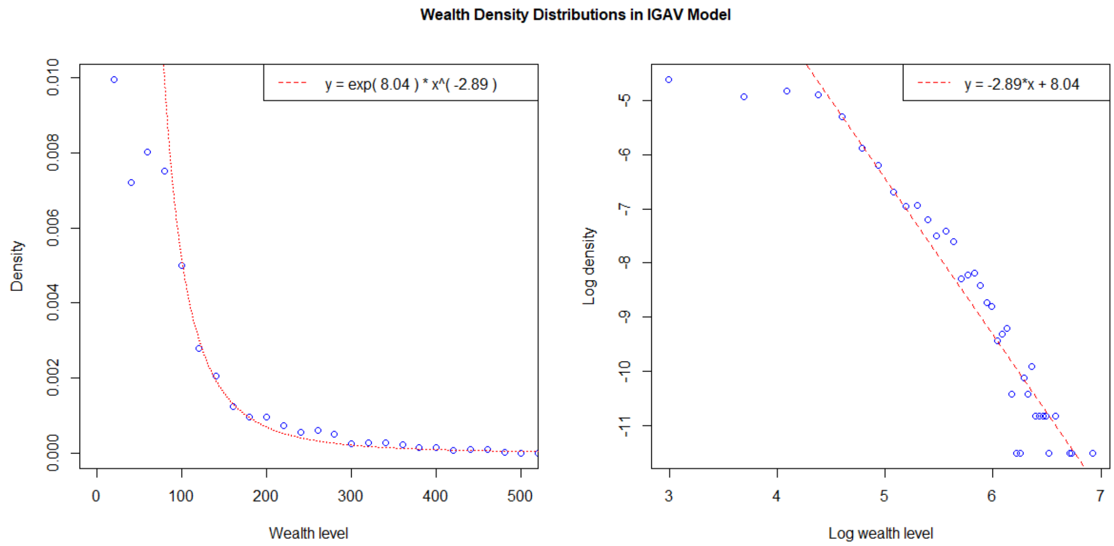

Figure 2.3: Stationary distribution produced by the IGAV model with quenched savings propensities and a bias function per Eq. (12). Simulation was performed with the parameterization $N = 5000$ and $\langle w \rangle = 100$ over 10^6 iterations.

Δ once the exchange is complete. The AW model provides a remarkably good fit to the U.S. wealth distribution, as reported by the U.S. Survey of Consumer Finances.

It is also worth mentioning the variation on the YS model first formulated by Iglesias *et al.* (2004). [92] This model, sometimes referred to as the IGAV model, sees agents with wealths w_i and w_j and saving parameters λ_i and λ_j exchange quantities $\Delta w_{ij} = \min\left\{(1-\lambda_i)w_i, (1-\lambda_j)w_j\right\}$, where the bias in favor of the poorer agent is defined as per Scafetta *et al.* (2002) [137]:

$$p = \frac{1}{2} + f \cdot \frac{w_1 - w_2}{w_1 + w_2} \tag{2.11}$$

The asymmetry flux index $f \in [0, 1/2]$ essentially defines the degree of social protection offered to the poor. A number of similar models which modify Iglesias *et al.*'s exchange rule were studied by Caon *et al.* (2007). [31] Recently, Neñer & Laguna (2021a) showed that, in the IGAV model, the richest agents are not necessarily the thriftiest. [117] Instead, the saving propensity λ_i^* that maximizes equilibrium average wealth lies in the interval $(0, 1)$, increasing with f.

Immediate Exchange

Heinsalu & Patriarca (2014) introduce a variation of the BDY model meant to more explicitly model the dynamics of barter economies. [80] While Drăgulescu & Yakovenko examined, among others, the TF exchange rule:

$$\begin{bmatrix} w_i(t+1) \\ w_j(t+1) \end{bmatrix} = \begin{bmatrix} \varepsilon & \varepsilon \\ 1-\varepsilon & 1-\varepsilon \end{bmatrix} \begin{bmatrix} w_i(t) \\ w_j(t) \end{bmatrix} \tag{2.12}$$

where ε is a uniform random variable with mean 0.5, Heinsalu & Patriarca consider the rule:

$$\begin{bmatrix} w_i(t+1) \\ w_j(t+1) \end{bmatrix} = \begin{bmatrix} 1-\varepsilon_i & \varepsilon_j \\ \varepsilon_i & 1-\varepsilon_j \end{bmatrix} \begin{bmatrix} w_i(t) \\ w_j(t) \end{bmatrix} \tag{2.13}$$

where ε_i and ε_j are i.i.d. uniform random variables with mean 0.5. This modification, called the "immediate exchange" (IE) model, changes the system from a pure TF one, where wealth flows unidirectionally and, on average, from richer to poorer agents, to one in which wealth flows bidirectionally. In the general IE model, transactions have some probability μ of occurring unidirectionally in the manner of Angle (1986). Katriel (2014) proved that the pure IE model with $\mu = 0$

has a steady state distribution $p(w)$ which is an exact Gamma distribution with a shape parameter of 2, meaning $\lim_{x\to 0} p(x) = 0$, $p(x)$ has a non-zero mode, and the right tail is well-approximated by a Pareto distribution with $\alpha = 1$. [96]

2.1.2 Bouchaud-Mézard models

Unlike the KWE model, the BM model does not make use of agents pairing up and engaging in binary transactions with a winner and a loser; rather, the rate of exchange between agents are defined by a fixed adjacency matrix $\mathbf{J}$, each entry of which J_{ij} represents the "cash flow rate" from agent j to agent$i$. In Bouchaud & Mézard's original paper, each agent in the population of size N has two sources of income—stochastic returns from investments and sales of a product to other agents—and one source of expenses—purchases of products from other agents. Thus, the income of agent i is given by:

$$\frac{\mathrm{d}w_i}{\mathrm{d}t} = \eta_i(t)w_i(t) + \sum_{j\neq i} J_{ij}w_j(t) - \sum_{j\neq i} J_{ji}w_i(t) \tag{2.14}$$

where η_i is a Gaussian random variable with variance $2\sigma^2$. [25] Notably, the BM model has no restriction on total wealth being conserved. The simplest case, in which all rates of exchange are equalized such that $J_{ij} = \frac{J}{N}$, lends itself well to a mean-field approximation, which produces an equilibrium distribution:

$$p(w) = \frac{(\mu - 1)^{\mu}}{\Gamma[\mu]} \cdot w^{-1-\mu} \exp\left(-\frac{\mu - 1}{w}\right) \tag{2.15}$$

where $\mu = 1 + \frac{J}{\sigma^2}$. However, as shown by Medo (2009), the mean-field approximation is time-limited; for any finite number of agents, the BM model on a complete graph will eventually exhibit wealth condensation and the probability that a given agent will have wealth less than any finite fraction of total wealth grows to 1. [112]

Further investigation into this class of model demonstrated that the resulting distribution is also sensitive to the nature of the underlying network defining the non-zero entries of the transaction matrix **J**. Souma *et al.* (2001) demonstrated through simulation that defining **J** on a small-world network—where each agent neighbors only 0.1% of the population—leads to distributions which are best fit by a combination of log-normal and power-law distributions. [149] Garlaschelli & Loffredo (2004, 2008) likewise showed that it is possible to retrieve a realistic mixed log-normal-power law distribution by simulating the model on a simple heterogeneous network with a small number of ”hub” agents, and that the BM model on a homogeneous network is able to reproduce either a log-normal or a power law distribution—but not both—depending on the average number of adjacencies per agent. [67, 68] Ma *et al.* (2013) simulated the BM model on a partially connected network and found the generalized inverse Gamma (GIGa) distribution provided the best fit to the steady state. [110]

Though the original BM model is a continuous-time model, a number of authors have studied similar models in discrete time as well. Di Matteo *et al.* (2003), for example, considers the variation [53]:

$$\Delta w_i(t) = w_i(t+1) - w_i(t) = A_i(t) + B_i(t)w_t + \sum_{j\neq i} Q_{j\to i}(t)w_j(t) - \sum_{j\neq i} Q_{i\to j}(t)w_i(t) \quad (2.16)$$

For the purposes of their analysis, additive noise $A_i(t)$ is assumed to be Gaussian with mean zero, and multiplicative noise $B_i(t) = 0$. Additionally, each agent i is

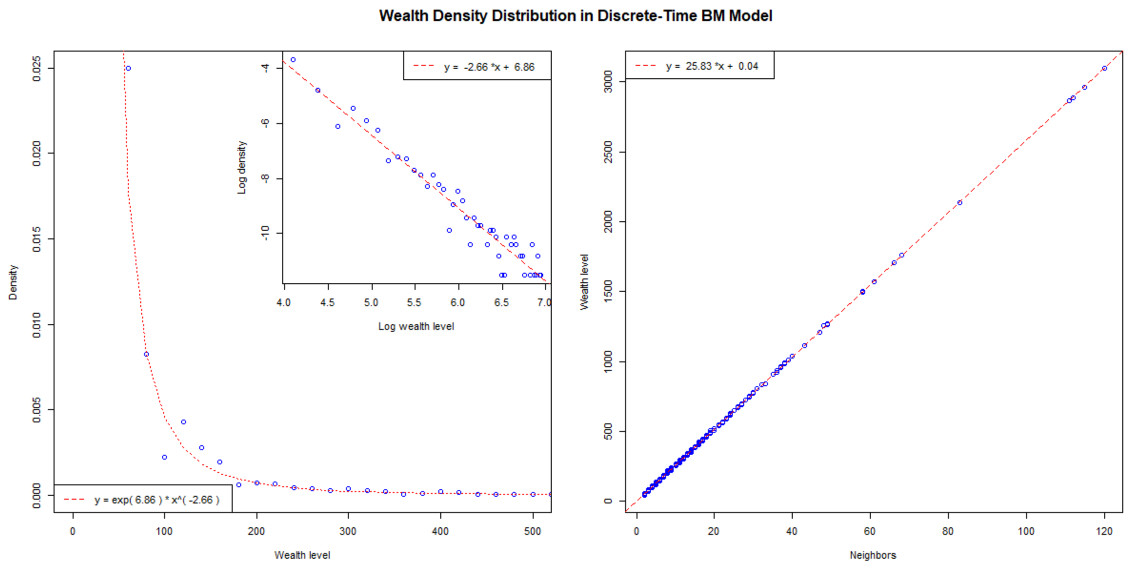

Figure 2.4: Stationary distribution produced by the discrete-time BM model on a Barabási–Albert scale-free network, as described by Di Matteo *et al.* (2003). Simulation was performed with the parameterization $N = 5000$, $\langle w \rangle = 100$, $q_0 = 0.1$, and $E\left[A_i(t)^2\right] = 1$ over 10^5 iterations.

assumed to split a fixed share q_0 of their wealth evenly with all of their neighbors $j \in \mathcal{I}_i$, where $|\rangle| = z_i$. Thus, $Q_{i\to j}(t) = \frac{q_0}{z_i}$ if $j \in \mathcal{I}_i$ and 0 otherwise. Their restricted system dynamics thus become:

$$w_i(t+1) - w_i(t) = A_i(t) - q_0 w_i(t) + \sum_{j\in\mathcal{I}_i} \frac{q_0}{z_j} w_j(t) \tag{2.17}$$

Simulating this time series produces results dependent on the choice of adjacency network, the most notable being that scale-free networks produce power-law distributions. In this case, the equilibrium wealth level of a given node is nearly perfectly correlated with the number of neighbors it has in the specified network, as shown in Figure 4c.

Scafetta *et al.* (2004) propose another discrete-time variation of Bouchaud & Mézard's model, motivated by a dissatisfaction with the formulation of wealth transfer via exchange as it appears in their original paper, which sees a constant wealth flux from rich to poor. [138] This is not necessarily realistic as wealth should only be transferred in exchange if an agent buys an asset for a price different than its value; such a model cannot explain wealth inequality under the assumption of perfect pricing. Thus, Scafetta *et al.* propose a model in which the wealth of agent i is given by:

$$w_i(t+1) = w_i(t) + r_i\xi(t)w_i(t) + \sum_{j\neq i} w_{i\to j}(t) \tag{2.18}$$

where $r_i = V\Pi_i > 0$ is the "individual investment index," given as the product of the global investment index and the proportion of wealth actually invested by agent i, $\xi(t)$ is a Gaussian random variable representing return on investment, and $w_{i\to j}(t)$ represents the flow of wealth from agent j to agent i in period t, which is assumed to be Gaussian with mean $\mu = fh\frac{w_i - w_j}{w_i + w_j}\min\{w_i, w_j\}$ and standard deviation $\sigma = h\min\{w_i, w_j\}$.

Varying f, h, and r then allowed the authors to tune the strengths of different system dynamics. If $h > 0$ and $f = r = 0$ (the symmetric trade-only model), wealth condensation occurs. If $f, h > 0$ and $r = 0$ (the asymmetric trade-only model), a Gamma-like distribution is observed. Finally, if $f, h, r > 0$ (the asymmetric trade-investment model), a Gamma-like distribution with a power-law tail is observed.

Various other modifications to the BM model have been studied as well. Huang (2004) extends the BM model to negative wealth levels, and Torregrossa

& Toscani (2017) prove analytically that a unique steady state with support on the entire real number line exists. [86, 157] Johnston *et al.* (2005) imposes the additional restriction of conservation of wealth, finding that wealth condensation still occurs for high values of μ. [95] Finally Ichinomiya (2012a, b) relaxes Bouchaud & Mézard's mean field assumption to adiabatic and independent assumptions, drawn from quantum mechanics. [88, 89] The power law-like tail is reproduced and condensation is seen to take place at a higher J than the mean-field case would indicate, though the Pareto index obtained is smaller than those empirically observed. [90]

2.1.3 Minor formulations

While the two principal model classes of the random asset exchange literature are KWE model and the BM model, a variety of less influential formulations, which will be briefly mentioned, also exist.

A simple formulation which has nonetheless been significant in the economics literature is the multiplicative stochastic process (MSP), which was studied by the economists Robert Gibrat and D.G. Champernowne. [68] It would however be a stretch to say that the MSP model is truly a type of random asset *exchange* model as they are primarily characterized by the lack of exchange or any other sort of interaction between agents. Such models essentially represent agents' wealths in terms of independent random walks, but nonetheless are able to capture some essential characteristics of observed distributions. The simplest model in this vein is the pure MSP $w(t+1) = \lambda(t)w(t)$, where $\lambda(t)$ is a Gaussian random variable. It is straightforward to show that the distribution of wealth among an ensemble of agents whose wealth evolution is governed by a pure MSP will follow a lognormal distribution, though variations which include additive noise (identical to the Kesten process from the biological sciences) and ”minimum wage”-style boundary constraints can also reproduce power law tails. [148] For examples of such models, see Biham *et al.* (1998), Huang & Solomon (2001), Souma & Nirei (2005), and Basu & Mohanty (2008). [17, 87, 150, 14]

Another class of model which studied in the early days of the RAE literature in particular is the Generalized Lotka-Volterra (GLV) model, which also has its origins in the biological sciences. The original Lotka-Volterra process, studied by Biham *et al.* (1998), is given by:

$$w_i(t+1) = \lambda(t)w_i(t) + a\bar{w}(t) - bw_i(t)\bar{w}(t) \tag{2.19}$$

where λ is a time-dependent random variable and $\bar{w}(t)$ is the average wealth in the system. [17] The inclusion of the $\bar{w}(t)$ terms represents a form of indirect interaction between agents: much like in the mean-field approximation of the BM model, instead of including specific ”inter-species interaction” terms $b_{ij}w_i(t)w_j(t$, all interactions are assumed to be symmetrical: $b_{ij} = b/N$.

The generalized form of this model was introduced by Solomon & Richmond

(2001, 2002), and follows [147, 146]:

$$\Delta w_i(t) = w_i(t+1) - w_i(t) = (\varepsilon_i(t)\sigma_i + c_i(w_1, w_2, \ldots, w_N, t))\, w_i(t) + a_i \sum_i b_j w_j(t) \quad (2.20)$$

where ϵ_i is a stochastic variable such that $E[\epsilon_i] = 0$ and $E[\epsilon_i^2] = 1$, c_i represents endogenous and exogenous dynamics in returns, and a_i and b_i represent arbitrary redistributions of wealth among agents. The restrictions on ϵ can be made without loss of generality thanks to the c_i term. Under certain assumptions, this model also produced mixed exponential-Pareto distributions. However, this model ultimately faded in popularity due to the difficulty it has accurately representing the left tail of income distributions, as well as the lack of economic justification for some of its terms. [134]

2.2 Notable Modifications

While the papers discussed above serve as the foundation for the random asset exchange literature, a vast number of variations upon these canonical models have been formulated and studied. In this section, an overview of the most common of these variations is provided and a number of key papers in each category are summarized.

2.2.1 Non-conservation of wealth

One of the primary criticisms leveled against the original KWE models is that the assumption of total conservation of wealth, made by analogy with the conservation of energy in ideal gas models, is highly unrealistic. In real economies, wealth is constantly being created and destroyed—not just by means of production and consumption, but even by the constant issuing and repaying of loans. Thus, a number of modifications to the conservative KWE model have attempted to represent this fact. Most such models can be classified into one of two types: models which, like the CPT model, conserve wealth in the mean, and models which tie wealth to a fixed global influx rate.

Bisi *et al.* (2009) and Bassetti & Toscani (2010) both consider models of the first type. [20, 12] The latter considers the non-conservative exchange rule:

$$\begin{bmatrix} w_i(t+1) \\ w_j(t+1) \end{bmatrix} = \begin{bmatrix} \varepsilon_i & \varepsilon_i \\ \varepsilon_j & \varepsilon_j \end{bmatrix} \begin{bmatrix} w_i(t) \\ w_j(t) \end{bmatrix} \tag{2.21}$$

where ε_i and ε_i are i.i.d. and $E[\varepsilon_i + \varepsilon_j] = 1$. Bassetti *et al.* (2014) considers a class of similar lotteries and demonstrates they tend to produce inverse-Gamma steady states. [13]

Slanina (2004) was the first to consider a non-conservative model of the second type, in which a constant inflow of wealth from outside the system of interacting agents is permitted. [144] As in other formulations, the model sees pairs of agents i

and j chosen at random to engage in a transfer of wealth, defined by the dynamics:

$$\begin{bmatrix} w_i(t+1) \\ w_j(t+1) \end{bmatrix} = \begin{bmatrix} 1-\lambda+\epsilon & \lambda \\ \lambda & 1-\lambda+\epsilon \end{bmatrix} \begin{bmatrix} w_i(t) \\ w_j(t) \end{bmatrix} \tag{2.22}$$

where $\lambda \in [0,1]$ represents the wealth exchanged between two interacting agents and $\epsilon > 0$ represents the rate at which exogenous wealth flows into the system. Slanina's model produces a Gamma-like equilibrium distribution with a Pareto tail with an index $\alpha \sim 1 + \frac{2\lambda}{\epsilon^2}$. Coelho *et al.* (2008) extended this model, redefining $\lambda(w_i)$ as a piecewise function taking on two different values depending on which side of a pre-specified wealth threshold $n\bar{w}(t)$ an agent's wealth $w_i(t)$ fell:

$$\begin{bmatrix} w_i(t+1) \\ w_j(t+1) \end{bmatrix} = \begin{bmatrix} 1-\lambda(w_i(t))+\epsilon & \lambda(w_j(t)) \\ \lambda(w_i(t)) & 1-\lambda(w_j(t))+\epsilon \end{bmatrix} \begin{bmatrix} w_i(t) \\ w_j(t) \end{bmatrix} \tag{2.23}$$

This modification reproduced a double power-law regime, a phenomenon observed when comparing the right tail of income from tax data to estimates for the capital gains of a country's very wealthiest individuals. [46]

A number of non-conservative models have dynamics which attempt to more directly model the process of money creation through borrowing. For example, Chen *et al.* (2013) consider a random exchange model in which agents who would otherwise reach zero wealth are permitted to borrow money from a central bank, which in turn can issue loans with no interest up to a certain global debt limit. [44] This process of money creation (issue of loans) and annihilation (paying back of loans) leads to a system in which the money supply grows logarithmically. Schmitt *et al.* (2014) introduces a similar system of money creation and analyzes the non-local effect that issuing credit has on the rest of the system; though the recipient of the loan clearly benefits, the effects of the increase in the money supply quickly propagate and all agents suffer the resultant inflationary effects. [139]

Recently, Liu *et al.* (2021) and Klein *et al.* (2021) introduced a generalization of the unaltered YS model which permits growth in the money supply over time, which they call the ”Growth, Exchange, and Distribution” (GED) model. [107, 99] Each time-step, total wealth $W(t)$ is increased by a factor of $1 + \mu$, and the wealth influx $\mu W(t)$ is distributed among agents such that agent i receives $w_i^\lambda/(\sum_j w_j^\lambda)$. For subunitary values of λ, poorer agents disproportionately benefit from the growth in the money supply and a quasi-stationary distribution exists; otherwise, the system exhibits wealth condensation as in the unaltered YS model. The system dynamics at play here are quite similar to the model of Vallejos *et al.* (2018), in which growth surplus is apportioned according to a more indirect ”wealth power” parameter. [160]

2.2.2 Adjacency networks and preferential attachment

It is a notable and well-established result that the significance of the introduction of adjacency networks into random asset exchange models depends heavily on the specific model formulation. While, for instance, the specific nature of the network has a decisive effect on the steady state wealth distribution in BM-style models, the opposite tends to be true for KWE-style models. Networks of exchange are an important aspect of real economic systems, and as such there has been a significant effort to study the effect they have on various types of RAE models.

Interestingly, models characterized by unidirectional exchange exhibit greater sensitivity to network structure than bidirectional exchange models. Chatterjee (2009), for example, introduce a toy model in which agents exchange fixed fractions of their wealth on a directed network characterized by a disorder parameter p. [39] Higher values of p produced networks where more agents had similar incoming and outgoing connections, and the distributions obtained were more Gamma-like, as opposed to the Boltzmann-like distributions obtained from lower values of p. Martínez-Martínez & López-Ruiz (2013) study a unidirectional model with random exchange fractions, meant to represent payments on a non-complete graph. [111] This ”directed random exchange” (DRE) model thus has the exchange rule:

$$\begin{bmatrix} w_i(t+1) \\ w_j(t+1) \end{bmatrix} = \begin{bmatrix} \varepsilon & 0 \\ 1-\varepsilon & 1 \end{bmatrix} \begin{bmatrix} w_i(t) \\ w_j(t) \end{bmatrix} \tag{2.24}$$

As with Chatterjee (2009), the choice of adjacency network affects the equilibrium distribution of the DRE model. For the fully-connected case, Katriel (2015) demonstrates that the equilibrium distribution $p(x)$ is again exactly Gamma, with shape parameter $\frac{1}{2}$. [97] Notably, this implies that p possesses a singularity at 0, explaining why Martínez-Martínez & López-Ruiz observed a condensation-like phenomenon even on fully-connected networks.

Sánchez *et al.* (2007) investigate a model in which agents populate a one-dimensional lattice. Each agent's wealth grows in a deterministic fashion as a product of a linear ”natural growth” term and an exponential ”control” term, which retards growth as the difference between an agent's wealth and the average wealth of its neighbors increases. [153] While this system produced a pure power law distribution, González-Estévez *et al.* (2008, 2009) later demonstrated that, for different values of the system's endogenous parameters or a rearrangement of agents' neighborhoods, either a Boltzmann-Gibbs or a Pareto distribution could be obtained. [72, 71]

A handful of models have included the additional possibility of agents exchanging connections or positions on a lattice as well as units of wealth. Gusman *et al.* (2005) define an IGAV model on a random network in which the winner of an exchange is rewarded with additional connections on the network, producing a power law regime. [77] Aydiner *et al.* (2019) examine a CCM-style bidirectional exchange model on a one-dimensional lattice, with the twist that some fraction of agents exchange lattice position each iteration of the simulation. [9] Fernandes & Tempere (2020) likewise consider a variation of the CC model in which agents on a two-dimensional lattice randomly switch positions on the lattice such that the average wealth difference between neighboring nodes is reduced. [65] This ultimately results in perfect wealth segregation and uniformly higher inequality.

The dynamics of wealth exchange coupled with extremal dynamics was thoroughly studied in the ”conservative exchange market” (CEM) model of Pianegonda *et al.* (2003), Iglesias *et al.* (2003), and Pianegonda & Iglesias (2004).

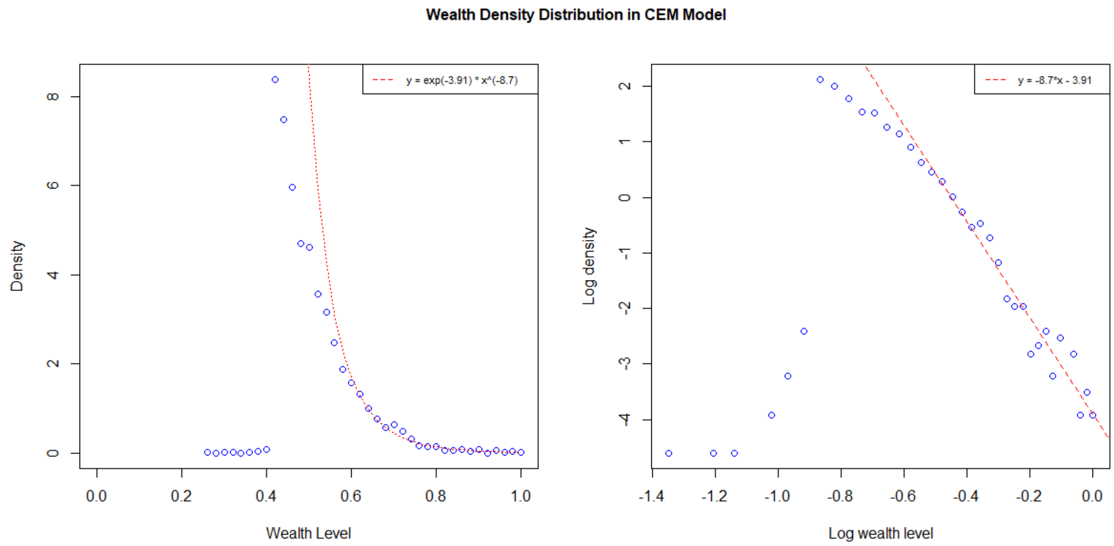

Figure 2.5: Stationary distribution produced by the CEM model, with the best log-linear fit of the right tail of the distribution shown. Simulation was performed with $N = 5000$ over 10^7 iterations.

[129, 93, 128] The CEM model populates a lattice with agents who possess wealth levels in the range $[0, 1]$, and each time step sees the poorest agent's wealth randomly re-randomized at the expense or benefit of its two closest neighbors. The selection rule in this model induces self-organizing behavior such that almost all agents end up with wealth levels above a "poverty line," which proved to be higher in the restricted lattice case than in the fully-connected case. This model has been extended by a number of follow-up papers over the years. Iglesias *et al.* (2010) used this model to compare two different redistribution schemes, and Ghosh *et al.* (2011) considered its mean-field approximation. [91, 69] Chakraborty *et al.* (2012) and Braunstein *et al.* (2013) studied the same dynamics on various other networks. [38, 27]

A concept closely related to adjacency is that of preferential attachment, which defines the likelihood of two agents interacting as a function of endogenous variables. The variable chosen is usually wealth, representing the fact that, in real economies, both the rich and the poor tend to interact more often with people of similar socioeconomic status to themselves. Because of its non-discrete nature, preferential attachment can allow for somewhat more dynamic interactions than adjacency networks can, permitting agents who become wealthy to access the networks of the rich and not totally disallowing chance rich-poor interactions. In this sense, preferential attachment is a more general formulation than adjacency.

Laguna *et al.* (2005) studied the effect of this phenomenon on the IGAV model by imposing the restriction that a given agent is only permitted to interact with another agent if the difference between their two wealth levels is less than a given threshold value u. [100] Large values of u more or less replicate the IGAV model and small values freeze the system entirely, as one would expect. Intermediate values, however, produced a self-organizing separation within the distribution of wealth, with a gap separating rich agents from poor ones spontaneously arising. This bimodal distribution persisted even for high values of the poor-bias parameter f.

Chakraborty & Manna (2010) introduce a model with simple preferential attachment behavior, such that richer agents engage in exchange more frequently. [37] That is, the probability that agent i is selected as the first trader is proportional w_i^α, and the probability that j is selected as the second trader is proportional

w_j^{β}. The limit as either exponent goes to infinity yields purely extremal mechanics, while $\alpha = \beta = 0$ is the CCM model. Goswami & Sen (2014) defines a more complicated attachment function, wherein the probability of a given pair of agents (i, j) interacting depends on i's total wealth, the difference in wealth between i and j, and the number of past interactions between agents i and j. [73] The strength of each factor is modulated by a corresponding exponent, and, when applied to the classical BDY model, the choice of modulation has a significant effect on the Pareto index of the steady-state distribution.

2.2.3 Goods and rationality

Despite their reductive nature, many of the simplifying assumptions discussed above are not uncommon to find in the economics literature as well. Many neoclassical models of simple "exchange economies" study the distribution of endowed and conserved assets absent wealth creation, and comparatively few consider the effect of exchange networks or other kinds of barriers to freely-associating exchange between agents (an important source of imperfect competition and thus market inefficiency). Rather, the main distinction between RAE-style models and those found in the mainstream economics literature lies in the fact that the models preferred in the latter typically study ensembles of rational (i.e. utility-maximizing) agents exchanging goods, with money exchange being an implicit consequence of goods exchange. A number of attempts have been made to partially bridge this difference between these two literatures by introducing goods and rationality into RAE models.

Chakraborti *et al.* (2001) studies a model with both a fixed commodity supply Q and money supply M distributed among a population of agents. [36] These agents first seek to ensure their level of goods q_i exceeds some subsistence level q_0, and afterwards seek to maximize their money holdings m_i; thus, agents with $q_i > q_0$ find agents with $q_j < q_0$ to sell their excess goods to at a fixed price of 1. Not surprisingly, the steady-state distribution of this system is found to be sensitive to the global quantities Q and N; if the commodity supply is limited ($Q/N < q_0$), some fraction of agents will necessarily fall below subsistence level, while if the money supply is limited, agents lack the ability to redistribute the commodity supply in an efficient manner. A similar model with stochastic price fluctuations is considered by Chatterjee & Chakrabarti (2006), in which wealth is taken to be the sum of money and commodity holdings. [40] In both models, the money distribution exhibits a Pareto tail with index 1 while the commodity distribution is exponential so long as neither Q nor M is restricted.

Silver *et al.* (2002) considers a model with a more sophisticated utility function, in which agents possess stochastically time-varying Cobb-Douglas utility functions of the form [141]:

$$u_{i,t}(a_{i,t}, w_{i,t}) = (a_{i,t})^{f_{i,t}}(w_{i,t} - a_{i,t})^{1-f_{i,t}} \tag{2.25}$$

where $a_{i,t}$ represents agent i's holdings of the money commodity at time t, $w_{i,t} - a_{i,t}$ represents agent i's holdings of non-money commodities at time t, and $f_{i,t}$ is a random variable independently and identically distributed across both indices. In an approach highly reminiscent of the derivation of the equilibrium of the canonical Arrow-Debreu model in economics, each agent chooses to re-allocate their wealth between money and non-money commodities in such a way that maximizes $u_{i,t}$ subject to supply constraints. Simulations of this system produce a wealth distribution well-fit by a Gamma distribution with a shape parameter of 1 and a scale parameter of $1/\alpha$, where α represents the global supply of the money commodity.

However, not every exchange model with goods is paired with rational agents. Ausloos & Pękalski (2007) consider a model with money, goods, and completely stochastic agent behavior. [8] Each time step, one agent decides via coin toss whether to purchase a nonzero number of goods. If so, he randomly selects a fraction of his money to spend and taps another agent to sell to him. If this second agent has enough goods to sell and has a desire to sell (again decided via coin toss), the exchange takes place. This model produces a distribution of wealth which interpolates between two power laws as time progresses, while the distribution of goods follows a static power-law. In general, those agents that are rich in terms of money are poor in terms of goods, and vice versa.

Another interesting line of research has concerned itself with defining traditional macroeconomic ensembles which produce equivalent results to RAE models. For example, Chakrabarti & Chakrabarti (2009) demonstrates that the dynamics of the CCM model can be replicated in a neoclassical framework with rational agents producing differentiated goods and trading in order to maximize time-varying Cobb-Douglas utility functions for goods and money. [33] In this case, the stochastic nature of exchange in the CCM model is represented by random variations in agents' utility functions in the analog model. Tao (2015) derives the entropy-maximizing exponential distribution as the statistical equilibrium of an Arrow-Debreu market system populated by agents with such time-varying utility functions. [154] More recently, Quevedo & Quimbay (2020) have extended this formulation to permit agents to save a portion s of goods possessed, naturally leading to an equivalent non-conservative RAE model. [132]

2.2.4 Strategies

Another approach to modeling ″smarter″ agent behavior attempts to integrate game-theoretic or machine learning dynamics into RAE models. The exact nature of this integration can take various forms, including bilateral agreement, strategic heterogeneity, and behavioral evolution, just to name a few.

In Heinsalu & Patriarca's original paper introducing the immediate exchange model, the authors consider the effect of introducing an acceptance criterion—a probabilistic factor defining the odds a given agent will agree to engage in a given transaction as a function of the difference in wealths between the agent and his partner, with both agents needing to agree to a transaction for it to take place. [80] In both the BDY and IE models, the choice of any symmetrical acceptance criterion (whether linear, exponential, etc.) only impacts the time of relaxation to equilibrium, but not the shape equilibrium itself. Asymmetrical decision criteria cause the equilibrium distribution to lose its universal form and to depend instead on the structure of the rule chosen. For the CC model, however, introducing even a symmetric criterion causes the equilibrium to lose its Gamma-like shape.

Sun *et al.* (2008) investigate a KWE model in which each agent can follow one of four strategies, chosen at random before the simulation begins. [152] The exchange rule between two agents depends on their strategy and the strategy of their partner: two of the strategies are passive and tend towards equalizing the wealth of the two agents, while the other two are aggressive and tend towards classical theft-and-fraud exchange. As with Heinsalu & Patriarca (2014), the introduction of heterogeneous trading strategies leads to a steady-state distribution which depends heavily on the model parameters, specifically those defining the rate of success of the aggressive strategies against the passive strategies.

Heterogeneity in strategies is often studied alongside dynamics for updating agents' strategies, representing a rudimentary form of learning. Hu *et al.* (2006, 2007, 2008), for example, consider a model in which agents begin as either cooperators or defectors and play a series of prisoner's dilemma or snowdrift-style games with their neighbors. [83, 84, 85] After each game, an agent identifies the strategy of its richest neighbor and adopts it with some probability defined by his most recent payout, leading on average to more successful strategies propagating throughout the network. In a similar vein, da Silva & de Figueirêdo (2014) investigate an adaptive variation of the CCM model in which each agent i has a fixed probability γ_i of being able to update their savings parameter according to a pre-defined rule each time step. [48] Neñer & Laguna (2021b) study a variation on a poor-biased YS model with non-zero saving propensity, in which a fraction of agents are subjected to a genetic evolutionary algorithm after each Monte Carlo simulation step to update their exchange parameters, which approach the optimal values determined by Neñer & Laguna (2021a). [118, 117] A BM-style model coupled with game-theoretic dynamics is extensively analytically studied by Degond *et al.* (2014). [51]

2.2.5 Class division

As mentioned above, one of the key features of empirical income distributions which RAE models attempt to capture is the bifurcation of the overall distribution into distinct exponential and Pareto ("thermal" and "superthermal," following the terminology of Silva & Yakovenko (2004) [140]) components. While some models attempt to replicate this two-regime behavior while preserving homogeneity of system dynamics (e.g. by distributing a behavioral parameter throughout the population or imposing a specific network structure), a number of authors have instead sought an explanation by means of a bifurcation in system dynamics for agents with large wealth. It is very natural to identify the exponential bulk of the income distribution with labor income and the power law tail with capital gains, as done by Silva & Yakovenko (2004), seeing as Pareto's original observations came from data for property incomes. [140] In this way, asymmetric system dynamics represent the fact that, in real economies, the rich do indeed have access to economic mechanisms not available to the majority of the population. [114]

Simple models which have this class division "baked in" are easily able to replicate two-regime structures of income. Yarlagadda & Das (2005) and Das & Yarlagadda (2005), for instance, introduce a model in which trading dynamics differ for agents with wealths on either side of a fixed wealth threshold. [166, 49] Poorer agents engage in bilateral exchange exactly as in the model of Chakraborti & Chakrabarti (2000), while richer agents engage in exchange—with a different saving parameter—against the system-totality, representing forms of leverage only available to the wealthy. Quevedo & Quimbay (2020) also study a trading model in which a fixed fraction of the population acts as "producers," who employ the remainder of the population as "workers." [132] Producers trade wealth and pay their associated workers a portion of the exchanged quantity, creating two differently-shaped Gamma distributions for producer and worker income which, when combined, create a clear two-regime distribution.

Lim & Min (2020) consider the case in which the CCM model is partitioned into two classes by a wealth percentile threshold and a "solidarity effect" among agents below said threshold is introduced. [105] If two agents belong to the same class, then exchange proceeds according to the familiar CCM system dynamics. But if the agents belong to different classes then the lower-class agent gathers "partners" equal to some fraction of the size of the class, and wins a fraction of the upper-class agent's wealth with a probability equal to the percent wealth his coalition possesses in the exchange. This solidarity factor turns out to be crucial for the generation of a realistic wealth distribution, as without it the middle income stratum collapses and one obtains a bimodal distribution, as with Laguna *et al.* (2005).

Imposing a fixed boundary differentiating the upper class from the lower is not necessarily the best approach here, however, as analysis has shown that the ”superthermal” component of the income distribution is highly volatile, fluctuating in size with the stochastic movements of financial markets. [140] A number of models consequently attempt to capture this out-of-equilibrium aspect of the distribution’s right tail by setting class boundaries dynamically. Russo (2014) investigates a model without exchange in which a new wealth percentile threshold defining the size of the upper class is chosen from the uniform distribution at each time step. [136] Agents above that threshold then see their wealth augmented by a multiplicative stochastic process, while agents below it have their wealth augmented by an additive stochastic process. A different approach is forwarded by Smerlak (2016), who constructs a Markov process defining transition probabilities between a finite number of stratified classes. [145] Agents in higher classes derive proportionally greater amounts of income from a multiplicative process subject to shocks, and consequently exhibit much greater fluctuations in wealth compared to the majority of agents, who persist at low levels of wealth indefinitely.

Finally, the author of this thesis wishes to highlight here the unique and striking ”social architecture” (SA) model of Wright (2005), which sees agents spontaneously self-organize into three distinct classes. [163] Wright defines an ensemble with three types of agents—employers, employees, and the unemployed—and in each iteration, an agent *i* is randomly chosen to be ”active”. The activities agent *i* engages in depends on its status: if *i* is an employer, it pays as many of its employees as it can afford; if *i* is an employee, it receives a wage and spends it on consumption goods produced by an employer; and if agent *i* is unemployed, a random (wealthy) agent is chosen to hire *i*, assuming their level of wealth is sufficient to pay *i*'s wages. Although the initial conditions of the simulation posited complete equality of agents (all agents began with equal wealth and no employer or employees), the population quickly restructured itself into a three-class regime with a distribution of wealth characterized by an exponential bulk and a Pareto tail. The exact nature of this distribution becomes clear when disaggregated for class: the wealth of ”employee” agents was completely governed by an exponential distribution, while that of ”employer” agents was well-fit by a power law. This result is consonant the argument forwarded by Montroll & Shlesinger (1982) and in contradistinction to explanations of the two-regime distribution which rely on endogenous differences between agents. Unfortunately, Wright's model has seen few direct extensions, though a similar self-organizing model was studied in Lavička *et al.* (2010). [103]

2.2.6 Redistributive policy

A good deal of attention has also been dedicated to the potential usefulness of RAE models in the analysis of the efficiency of redistributive mechanisms. Early

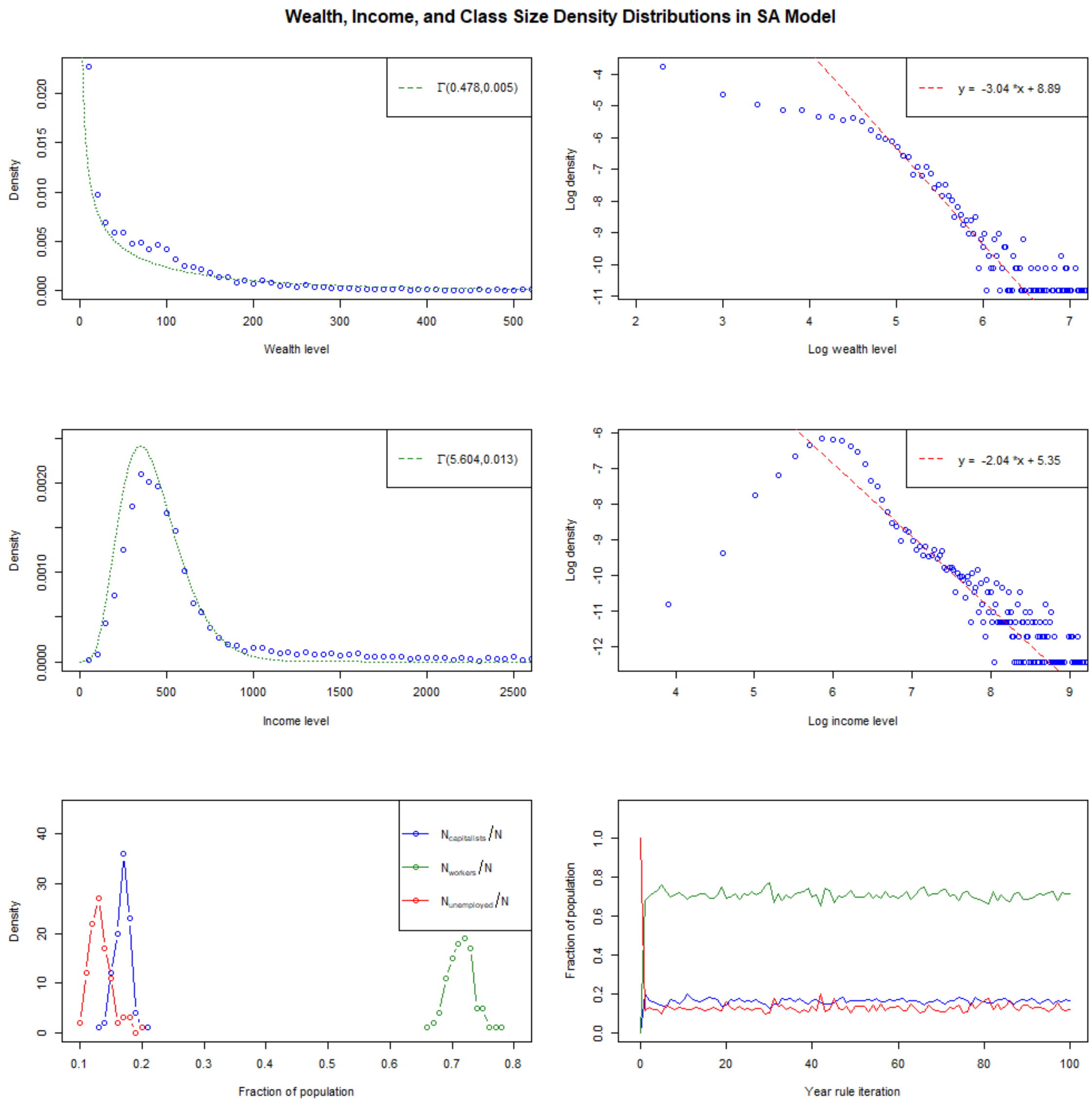

Figure 2.6: Stationary distributions of wealth, income, and class size produced by the SA model. Simulation was performed with the parameterization $N = 5000$, $\langle w \rangle = 100$, $[w_a, w_b] = [10, 90]$, over 10^2 "year rule" iterations ($6 \cdot 10^6$ time-steps).

studies such as Guala (2008) and Toscani (2009) considered the effect of a simple ”income tax,” in which a fixed fraction is withdrawn from each exchange by an external body and uniformly redistributed, on mean-conservative KWE models, which was found to not alter the exponential nature of the steady state distribution. [75, 158] Diniz & Mendes (2012) extend this result to multiple different taxation rules on a CC model, representing both income taxes (taxes on transaction amounts) and wealth taxes (taxes on wealth level). [55] Bouleau & Chorro (2017) contrast the effect of income and wealth taxes on YS-like models, demonstrating analytically that the income taxes alone are not sufficient to prevent condensation. [26] Similarly, Burda *et al.* (2019) investigate the dynamics of a BM-style model with the parameterization $J < 0$, which would normally cause the system to condense, paired with a redistributive mechanism. [29] A sufficiently strong mechanism succeeded in preventing condensation and recovering a heavy-tailed wealth distribution, with a multimodal critical phase also being observed. A number of non-standard redistribution rules in a YS model were examined by Lima *et al.* (2022). [106]

Recently, however, interest within the literature has grown around the problem of identifying optimal tax rates in models, often borrowing techniques from control theory to do so. Bouchaud (2015) extends the BM model to permit a wealth tax capable of reallocating wealth between the private and public sectors, with different growth rate parameters. [24] By maximizing expected economic growth, an optimal tax rate in the interval $(0, 1)$ is obtained for growth rate differences within an intermediate range. Düring *et al.* (2018) develop a finite-horizon model predictive control mechanism for the CPT model to derive the feasible tax regime which minimizes a cost function representing some metric of inequality, and consider various objective functions and redistribution schemes. [62] Zhou & Lai (2022) investigate a novel model of individual wealth growth and formulate both an additive and a multiplicative control mechanism to modulate the excessive growth of the right tail of the wealth distribution of the ensemble. [167] Lastly, Wang *et al.* (2022) pairs a CPT model with an evolutionary description of agents' decision-making competence—which feeds back into their saving propensities—and a model predictive control mechanism to reduce reduce inequality. [162]

2.2.7 Miscellanea

Though the above list enumerates the most widely-studied modifications to RAE models, it should by no means be considered exhaustive. The flexibility of the random asset exchange framework makes it easy to introduce new system dynamics and isolate the effects of a given modification. Just to name a few examples, Pareschi & Toscani (2014) investigate the effect of variable agent knowledge on the CPT model, obtaining the intriguing result that the most knowledgeable agents tend not to be the richest ones; Trigaux (2005) examines the effect of introducing altruistic behavior to a subpopulation and finds a very strong equalizing effect when combined with redistribution; Coelho *et al.* (2005) and Patrício & Araújo (2021) model the propagation of wealth on a generational network to study the stratifying effect of inheritance; and Dimarco *et al.* (2020) use a class-based framework to characterize the effect of pandemics on wealth inequality. [119, 159, 45, 126, 54].

The RAE literature has also given rise to a number of proposals for new analytical techniques. Ballante *et al.* (2020) demonstrate that fitting the distribution of saving propensities to real-time economic data in a generalized CCM model via statistical sampling may be useful as a leading indicator of economic stressors which have the potential to increase inequality. [11] Luquini *et al.* (2020) establish a formal equivalence between KWE models and population-based random search algorithms in computer science, and speculate that said formulation could ultimately be used as a benchmark model in cybernetics. [108] Finally, dos Santos *et al.* (2022) propose a computational technique by which the crossover point between the exponential and Pareto regimes can be identified within data sets of real income distributions, aiding in the empirical study of economic inequality. [58]

2.3 Discussion

From the ambition and breadth of recent publications such as those mentioned above, it is clear that random asset exchange modeling is being increasingly recognized as a highly versatile tool which has the potential to find wide application even beyond its original use as a descriptive econophysical model. It is also clear that, in seeking to explain the characteristic features of wealth and income dis-

Notable Papers				
Model Feature	*Theft and Fraud*	*Yard Sale*	*Bouchaud-Mézard*	*Other*
Canonical	Angle (1986) Bennati (1988) Ispolatov *et al.* (1998) Drăgulescu & Yakovenko (2000) Chakraborti & Chakrabarti (2000) Chatterjee *et al.* (2003)	Hayes (2002) Iglesias *et al.* (2004) Caon *et al.* (2007) Moukarzel *et al.* (2007)	Bouchaud & Mézard (2000)	Biham *et al.* (1998) Solomon & Richmond (2001, 2002)
Non-cons.	Slanina (2004) Cordier *et al.* (2005) Coelho *et al.* (2008) Bisi *et al.* (2009) Bassetti & Toscani (2010) Chen *et al.* (2013) Schmitt *et al.* (2014)	Liu *et al.* (2021) Klein *et al.* (2021)		Heinsalu & Patriarca (2014)
Networks	Chatterjee (2009) Martínez-Martínez & López-Ruíz (2013) Aydiner *et al.* (2019) Fernandes & Tempere (2020)	Gusman *et al.* (2005) Laguna *et al.* (2005) Guajardo & Moukarzel (2012)	Souma *et al.* (2001) Di Matteo *et al.* (2003) Scafetta *et al.* (2004) Garlaschelli & Loffredo (2004) Ma *et al.* (2013)	Pianegonda *et al.* (2003) Sánchez *et al.* (2007)
Goods	Chakraborti *et al.* (2001) Chatterjee & Chakrabarti (2006)			Ausloos & Pękalski (2007)
Rationality	Chakrabarti & Chakrabarti (2009) Tao (2015) Quevedo & Quimbay (2020)			Silver *et al.* (2002)
Strategies	Sun *et al.* (2008) da Silva & de Figueirêdo (2014)	Neñer & Laguna (2021b)	Degond *et al.* (2014)	Hu *et al.* (2006)
Class div.	Yarlagadda & Das (2005) Lim & Min (2020)			Wright (2005) Lavička *et al.* (2010) Russo (2014) Smerlak (2016)
Redist.	Guala (2008) Toscani (2009) Diniz & Mendes (2012)	Boghosian (2014a) Boghosian *et al.* (2017) Bouleau & Chorro (2017) Düring *et al.* (2018) Lima *et al.* (2022) Wang *et al.* (2022) Li *et al.* (2019)	Bouchaud (2015) Burda *et al.* (2019)	Zhou & Lai (2022)

Table 2.1: Notable papers in the random asset exchange literature, disaggregated by formulation type and prominent features.

tributions, such models have highlighted the existence of a number of more fundamental economic phenomena underlying those features, such as the inherently diffusive nature of exchange economies and the emergence of apparent power laws from overlapping exponentials.

But does the random asset exchange literature, as it currently stands, provide an adequate explanation for the emergence of the distributional features which it posits to be universal? That is to say, is it able to identify and describe the concrete mechanisms, common to all economies, which generate said features? The degree of simplification present in these models forces one to answer in the negative. The last twenty-five years of research have produced a number of fascinating results which provide a solid foundation for future work, but, as argued by Reddy (2020), the literature would need to pivot towards a processual account in order to approach a *bona fide* explanation. [133]

It is helpful here to briefly discuss the relationship between the distribution of wealth and the distribution of income, the nature of which the literature has not consistently grasped. Wealth and income are two linked but quite distinct quantities. Wealth can take a wide variety of forms—money, consumption goods, real estate, debts, and even information or skills can all be considered forms of wealth. Income, on the other hand, typically refers to the amount of currency received by an individual in a given time period, prior to expenses. This money is of course added to one's existing wealth, but the straightforward relation of income as the time-derivative of wealth only holds under the simplifying assumption that wealth is not subject to any endogenous changes: that is, no articles of wealth are consumed, fluctuate in value, are traded for articles of differing value, etc.

For the most part, random asset exchange models are concerned with the distribution of an undifferentiated, non-consumable, exchangeable asset—usually a stand-in for money—throughout an ensemble of agents. Thus, the distributions of said asset throughout the population are best interpreted as wealth distributions, and it is inapt to compare them to empirical distributions of income. In fact, Xu *et al.* (2010) note that reconstructing time-series of agents' income within canonical KWE models actually produces income distributions which are Gaussian, as opposed to exponential, directly contrary to the available data. [164]

Ultimately, this confusion has its roots in a deeper problem: the literature still lacks a modeling framework which, by virtue of concretely representing economic structure, would allow "true" income asymmetries to be easily recovered. To illustrate this point, consider the differences between the processes which give rise to the distribution of wealth among a community of financial traders and among an entire macroeconomy, which the literature tends to model interchangeably.

In the first case, assuming insider trading does not occur, all individuals have access to the same system dynamics and information, and the only possible mechanisms for the time-evolution of wealth are asset exchange between equal agents and stochastic growth in asset value. In the second case, however, the picture is considerably different. There exist structural asymmetries between individuals, such as that between an employer and an employee, which completely alter the income paths available to an individual; there exist power asymmetries in exchange, as economic interactions predominantly take place not between two equal individuals, but rather between individuals and price-setting firms; and there exist asymmetries in time, as production continually increases the total amount of wealth present in society. It is not at all obvious that systems of both types should have the same processes underlying their distributions of wealth and income. Instead, the fundamental differences between exchange-only systems and macroeconomic systems must be explicitly represented in our approach if these models are to move from being merely insightful to being truly explanatory.

CHAPTER 3
A NOVEL MODEL FORMULATION

In the previous chapter, it was argued that the random asset exchange literature must obtain a greater understanding of the concrete economic processes underlying the distribution of wealth in real macroeconomies—and must begin designing models premised on such an understanding—if it seeks greater explanatory power. In this chapter, a novel agent-based model formulation, in the vein of the models published by Wright (2005) and Lavička *et al.* (2010), will be proposed in order to demonstrate what such a pivot in the literature could look like.

The emphasis of this model, however, is somewhat different from these earlier inspirations. Wright named his model the ”social architecture” model, and his aim was to examine exactly how this architecture—the forms and functions of the relationships which exist between agents within a capitalist political-economic framework—leads to complex, self-organizing behavior *over a period of time*. The fact that each iteration in Wright's model is identified as a ”month” underscores the fact that Wright was not interested in investigating a fixed moment of economic activity, but the long-term dynamics of an economic system over time. This model, however, will take the opposite approach. Our ensemble of agents will begin already divided among three subpopulations, with no possibility of one agent moving from one subpopulation to another. Prices, technology, and consumption habits will likewise be fixed for the duration of each simulation. This is because the aim of this model is not to study the temporal dynamics of a production-exchange system, but rather to determine, given certain economic parameterizations, whether an equilibrium exists in the model's phase space and, if so, how severe economic inequality is at said equilibrium. In this way, if Wright's model can be said to be a *dynamic* processual model, ours can be said to be *static*.

The remainder of this chapter proceeds as follows: Section 1 introduces the model which will be our object of study, Section 2 presents the derivation of a useful analytical result with respect to said model, and Section 3 presents the results of a series of simulations for different choices of parameterization. Section 4 concludes with a discussion of the advantages and drawbacks of this model, and suggests potential directions for future work.

3.1 Model Definition

Let $\mathcal{P}$ represent a population of of N agents. Each agent belongs to one of three subpopulations: producers of consumption goods $\mathcal{S}_1$, producers of intermediate goods $\mathcal{S}_2$, and workers $\mathcal{W}$. Every agent in $\mathcal{P}$ is in exactly one of the above groups. That is, $\mathcal{S}_1 \cap \mathcal{S}_2 = \mathcal{S}_1 \cap \mathcal{W} = \mathcal{S}_2 \cap \mathcal{W} = \{\}$ and $\mathcal{S}_1 \cup \mathcal{S}_2 \cup \mathcal{W} = \mathcal{P}$. At each time step t, each agent $i \in \mathcal{P}$ has a well-defined wealth level $W_i(t) \geq 0$ and income level $I_i(t) \geq 0$.

Each agent in subpopulations $\mathcal{S}_1$ and $\mathcal{S}_2$, collectively referred to as "capitalists" $C = \mathcal{S}_1 \cup \mathcal{S}_2$, additionally owns their own firm. At each time step, each firm $j \in C$ has a well-defined stock level $S_j(t) \geq 0$. For $j \in \mathcal{S}_1$, stock is enumerated in terms of consumption goods, which are homogeneous and sold at a constant price $p > 0$. For $j \in \mathcal{S}_2$, stock is enumerated in terms of intermediate goods, which are also homogeneous and sold at a constant price $r > 0$. For the "workers" $k \in \mathcal{W}$, $S_k(t)$ is enumerated in terms of labor supplied by each agent, which may be purchased by capitalists at a constant wage rate $w > 0$ up to some limit J_{max}. Unlike goods, labor stock is automatically replenished every time step. For the purposes of this model, goods are counted as wealth neither before nor after sale, such that the total level of wealth in the system remains constant over time.

The principal economic circuit represented in this model is the alternating process between production and consumption. Production is carried out by firms, which have homogeneous Cobb-Douglas production functions $Y(K, L) = AK^{\gamma}L^{1-\gamma}$, where K represents capital input and L represents labor input. Consumption is carried out by all agents regardless of subpopulation.

Lastly, flat taxes of five different types are represented: wealth, income, turnover, payroll, and sales. Two of these taxes—wealth and income—are extracted explicitly from each agent after each production-consumption cycle. The remaining three taxes—turnover, levied against purchases of capital; payroll, levied against purchases of labor; and sales, levied against purchases of consumption goods—are extracted implicitly at the point of purchase by means of an asymmetry between the unit price the buyer faces in the market and the unit revenue the seller receives in the event of a purchase. The revenue accrued by each tax is consolidated by an external authority—interpreted as the state—and redistributed equally throughout the population—meant to represent the effects of social spending. The levels of each of the five taxes are denoted wtr, itr, ttr, ptr, and str respectively.

Explicitly, the model defines the following rules:

Initialization Rule

1. Define population variables N, q_{cap}, and q_{s1}.
2. Define endowment variables W_{init}, S_{init}, and J_{max}.
3. Define production variables A and γ.
4. Define price variables r, w, and p.
5. Define tax variables wtr, itr, ttr, ptr, and str.
6. Set $\mathcal{P} = \{1, 2, \ldots N\}$
7. Set $N_c = \lceil N q_{cap} \rceil$ and $C = \{1, 2, \ldots N_c\}$.
8. Set $N_{s1} = \lceil N_c q_{s1} \rceil$ and $\mathcal{S}_1 = \{1, 2, \ldots N_{s1}\}$
9. Set $N_{s2} = N_c - N_{s1}$ and $\mathcal{S}_2 = C - \mathcal{S}_1$.
10. Set $N_w = N - N_c$ and $\mathcal{W} = \mathcal{P} - C$.
11. Set $W(0) = W_{init} e^N$, where e^N represents the N-dimensional vector of ones.
12. Set $S(0) = \begin{bmatrix} S_{init} e^{N_c} & J_{max} e^{N_w} \end{bmatrix}$.
13. Set $r_s = \frac{r}{1+ttr}$, $w_s = \frac{w}{1+ptr}$, and $p_s = \frac{p}{1+str}$.

Budget Selection Rule:

1. For each $i \in \mathcal{P}$, draw x_i from the random uniform distribution on $[0, 1]$. This represents each agent's allocated total expenditure for the current time step.

2. For each $i \in \mathcal{C}$, draw y_i from the same uniform distribution. This represents each producer's division of expenditures between investment and consumption.

3. For each $i \in \mathcal{C}$, solve the production maximization problem:

$$\begin{aligned} \max_{K_i, L_i} \quad & A K_i^{\gamma} L_i^{1-\gamma} \\ s.t. \quad & rK_i + wL_i \leq x_i y_i W_i \\ & K_i, L_i \geq 0 \end{aligned} \tag{3.1}$$

   The solution (K_i^*, L_i^*) of this problem represents the most efficient allocation of investment between capital and labor.

The solution to said maximization problem, conveniently, has a simple closed form solution. Given $r, w > 0$, the optimal quantities of capital and labor to purchase are:

$$\begin{aligned} K_i^* &= \frac{\gamma}{r} \cdot x_i y_i W_i \\ L_i^* &= \frac{1-\gamma}{w} \cdot x_i y_i W_i \end{aligned} \tag{3.2}$$

This result may be obtained straightforwardly by the method of Lagrange multipliers; for a full derivation, see Appendix A.1.

Purchase Rule:

1. Define the set of buyers $\mathscr{B} \subseteq \mathcal{P}$ and the set of sellers $\mathscr{S} \subset \mathcal{P}$.

2. Define any additional caps on purchases made from a single seller c^+.

3. Define the $|\mathscr{B}|$-dimensional vector R of resource to buy for each $i \in \mathscr{B}$

4. Define the unit price π and the unit revenue π_s of said resource.

5. Initialize the N-dimensional consumption, income, and expenditure vectors $C, I, E = 0$.

6. Generate a random permutation s of indices $i \in \mathscr{B}$.

7. Iterating through $i \in \mathscr{B}$:

   (a) Generate an random permutation r, independent from s, for indices $j \in \mathscr{S}$. Iterating through $j \in \mathscr{S}$:

      i. Set $Q_{i,j} = \min\{R_{s(i)}, S_{r(j)} - C_{r(j)}, c^+\}$.

      ii. Decrease $R_{s(i)}$ by $Q_{i,j}$.

      iii. Increase $C_{r(j)}$ by $Q_{i,j}$.

      iv. Increase $I_{r(j)}$ by $\pi_s Q_{i,j}$.

      v. Increase $E_{s(i)}$ by $\pi Q_{i,j}$

   (b) Repeat steps i.-v. until $R_i = 0$ or $\sum_{j \in \mathscr{S}} (S_j - C_j) = 0$.

8. Return the consumption, income, and expenditure vectors.

In the context of this model, wealth is always strictly conserved, as no agent can become wealthier without taking wealth from another. Income, however, is only conserved in expectation.

The following three rules define how **Purchase Rule** is performed in order to accomplish the requisite transfers of capital, labor, and consumption goods from sellers to buyers:

Capital Acquisition Rule:

1. Run **Purchase Rule** with:

$$\mathscr{B} = \mathcal{C} \quad \mathscr{S} = \mathcal{S}_2 \quad R = K^*$$
$$\pi = r \quad \pi_s = r_s \quad c^+ = \{\}$$

2. Define the output vectors of **Purchase Rule** as C^k, I^k, and E^k respectively.

Labor Acquisition Rule:

1. Run **Purchase Rule** with:

$$\mathscr{B} = \mathcal{C} \quad \mathscr{S} = \mathcal{W} \quad R = L^*$$
$$\pi = w \quad \pi_s = w_s \quad c^+ = \{1\}$$

2. Define the output vectors of **Purchase Rule** as C^ℓ, I^ℓ, and E^ℓ respectively.

Consumption Rule:

1. For $i \in \mathcal{C}$, set $C_i = x_i(1 - y_i)W_i/p_b$. For $i \in \mathcal{W}$, set $C_i = x_i W_i/p_b$.
2. Run **Purchase Rule** with:

$$\mathscr{B} = \mathcal{S}_1 \quad \mathscr{S} = \mathcal{P} \quad R = C$$
$$\pi = p \quad \pi_s = p_s \quad c^+ = \{\}$$

3. Define the output vectors of **Purchase Rule** as C^c, I^c, and E^c respectively.

Upon completion of all three rounds of purchases, the income vector $I(t) = I^k(t) + I^\ell(t) + I^c(t)$ becomes well-defined.

Production Rule:

1. Set $\Pi_i = A(E_i^k/r_b)^\gamma(E_i^\ell/w_b)^{1-\gamma}$ for $i \in C$ and $\Pi_i = 0$ for $i \in \mathcal{W}$.

The total amount of capital and labor available to capitalist i to use in production is reverse-engineered from expenditures, rather than consumption, because the latter is associated with the indices of the sellers, so as to appropriately calculate their remaining stock. Conversely, expenditures are associated with the indices of the buyers of a given resource.

System Update Rule:

1. Increase W by $\sum_{i\in\{k,\ell,c\}}(I^i - E^i)$.
2. Increase S by $\Pi - \sum_{i\in\{k,\ell,c\}} C^i$

Note that the "implicit" taxes—sales, payroll, and turnover—are imposed during this step, as, when at least one of str, ptr, or ttr is nonzero, then $I^i - E^i \leq 0$.

Taxation Rule:

1. Set turnover tax collected $TTC = E^k - I^k$.

2. Set payroll tax collected $PTC = E^{\ell} - I^{\ell}$
3. Set sales tax collected $STC = E^{c} - I^{c}$
4. Set income tax collected $ITC = itr \cdot (I^{k} + I^{\ell} + I^{c})$.
5. Reduce W by $ITC \cdot e$.
6. Set wealth tax collected $WTC = wtr \cdot W$.
7. Reduce W by $WTC \cdot e$.
8. Define the redistribution quantity $Q_{\Delta} = (WTC + ITC + TTC + PTC + STC) \cdot e$
9. Define the redistribution vector $\Delta = \frac{Q_{\Delta}}{N} e$.
10. Increase W by Δ.

The choice to levy the wealth tax after the income tax is ultimately arbitrary. However, the order in which the two explicit taxes are imposed is in general not interchangeable.

The complete model then operates as follows:

Step Rule:

1. Run **Budget Selection Rule**.
2. Run **Capital Acquisition Rule**.
3. Run **Labor Acquisition Rule**.
4. Run **Consumption Rule**.

5. Run **Production Rule**.

6. Run **System Update Rule**.

7. Run **Taxation Rule**.

Simulation Rule:

1. Define a parameterization $P = \{N, q_{cap}, q_{s1}, W_{init}, S_{init}, J_{max}, A, r, w, p, wtr, itr, ttr, ptr, str\}$ and a time horizon $T \gg 0$.

2. Run **Initialization Rule** with parameterization P.

3. Run **Step Rule** T times.

Based on this simulation rule, a number of emergent systemic variables can be measured. Most importantly for the purposes of this model, both the final wealth distribution $W(T)$ and the income distribution $I(T) = I^k(T) + I^\ell(T) + I^c(T)$ are well-defined. The degree of inequality present within these distributions is represented by the Gini coefficient, which is calculated as per equation (1.2). The mean levels of wealth and income for each class, as well as the wealth and income decile thresholds are also recorded. Finally, the unemployment rate is defined as:

$$U(t) = \frac{\sum_{i \in \mathcal{W}} \mathbb{1}_{I_i^\ell(t)=0}}{N} \tag{3.3}$$

That is, the unemployment rate in period t is given as the fraction of the total population represented by agents in the working class who received 0 income in period t. By investigating the responses of these variables to multiple different parameterizations, the full complexity of this model may be illustrated.

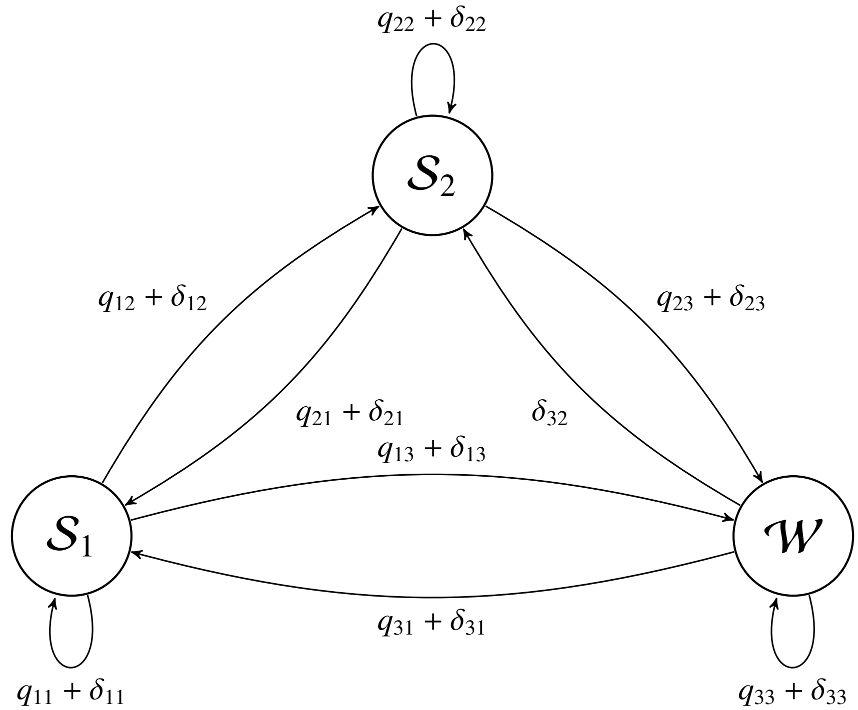

Figure 3.1: The transition probability graph implied by the model, where q_{ij} represents the (i, j)th entry in the pre-redistribution transition matrix $Q = \left(\frac{1}{2}I + X - T_I\right)(I - T_W)$ and δ_{ij} represents the (i, j)th entry in the redistribution matrix $\Delta = (I - Q)R$.

3.2 Analytical Results

The highly structural nature of this model make it difficult to derive analytical expressions for its emergent features, such as the equilibrium distributions of wealth and income. It is nonetheless possible to derive two particularly important expressions, namely the division of gross wealth and income between $\mathcal{S}_1$, $\mathcal{S}_2$, and $\mathcal{W}$. This may be done by constructing the Markov process governing the path of a representative dollar through the system.

We begin by defining the matrix X, representing the expected income obtained by each sector as a result of expenditures from a given other sector. For the full derivation of this matrix, see Appendix A.2. The income matrix is given as:

$$X = \begin{bmatrix} \frac{1}{4(1+str)} & \frac{\gamma}{4(1+ttr)} & \frac{1-\gamma}{4(1+ptr)} \\ \frac{1}{4(1+str)} & \frac{\gamma}{4(1+ttr)} & \frac{1-\gamma}{4(1+ptr)} \\ \frac{1}{2(1+str)} & 0 & 0 \end{bmatrix}$$

Naturally, income tax may be represented as a constant fraction of this matrix; that is:

$$T_I = itr \cdot X$$

Wealth is taxed at a constant rate across all three subpopulations:

$$T_W = wtr \cdot I$$

And finally, wealth is redistributed across all three subpopulations in accordance with the fraction of the population N_i/N they represent:

$$R = \frac{1}{N} \begin{bmatrix} N_{S1} & N_{S2} & N_W \\ N_{S1} & N_{S2} & N_W \\ N_{S1} & N_{S2} & N_W \end{bmatrix}$$

Note that, for any division of wealth $w_{initial} \in \mathbb{R}^3_{\geq 0}$ such that $|w_{initial}|_1 = 1$, the amount of wealth held by each subpopulation after all exchanges are performed and taxes are levied is given by:

$$\begin{aligned} w_{remaining} &= w_{initial}\left(\text{savings} + \text{income} - \text{taxed income}\right) \cdot (1 - \text{wealth tax rate}) \\ &= w_{initial}\left(\frac{1}{2}I + X - T_I\right)(I - T_W) \end{aligned}$$

It follows that the vector $\Delta = w_{initial} - w_{remaining}$ represents the fraction of total system wealth collected by taxation, disaggregated by subpopulation, and the vector ΔR represents that same fraction redistributed proportionally across the three subpopulations. Thus, the entire probability transition matrix may be written as:

$$\begin{aligned} P &= \left(\frac{1}{2}I + X - T_I\right)(I - T_W) + \left(I - \left(\frac{1}{2}I + X - T_I\right)(I - T_W)\right)R \\ &= (1 - wtr)\left(\frac{1}{2}I + (1 - itr)X\right)(I - R) + R \end{aligned} \tag{3.4}$$

Since P has non-zero diagonal elements, the Markov process represented by P is clearly aperiodic. Furthermore, the directed graph $G(P)$ representing non-zero transfer probabilities, as represented in Figure 3.1, is strongly connected even in the case that all tax rates are 0; hence the process is irreducible as well. It is clear from these facts that the Markov process constructed above will have a unique steady state division of wealth $\Pi_W \in \mathbb{R}^3_{\geq 0}$, which is explicitly defined to be the vector solving:

$$\Pi_W = \Pi_W P \tag{3.5}$$

And the pre-tax steady-state income $\Pi_I \in \mathbb{R}^3_{\geq 0}$, in terms of fractions of total system wealth, is given by:

$$\Pi_I = \Pi_W X \tag{3.6}$$

In general, the closed-form expressions of Π_W and Π_I are quite large and unwieldy. However, in the case without taxation, the steady-state vectors depend only on the capital intensiveness of production γ:

$$\begin{aligned} \Pi_W &= \begin{bmatrix} \frac{2-\gamma}{3-\gamma} & \frac{\gamma}{3-\gamma} & \frac{1-\gamma}{3-\gamma} \end{bmatrix} \\ \Pi_I &= \begin{bmatrix} \frac{2-\gamma}{6-2\gamma} & \frac{\gamma}{6-2\gamma} & \frac{1-\gamma}{6-2\gamma} \end{bmatrix} \end{aligned} \tag{3.7}$$

Note that, under conditions of 0 taxation, the working class cannot acquire more than half of total system wealth regardless of choice of parameterization or subpopulation size.

3.3 Simulation Results

Due to the analytical intractability of the model presented above, simulation is the principal tool available for studying its emergent properties. Unless otherwise stated, all simulations made use of a baseline parameterization with a total population of size $N = 5000$, 10% of whom are sector 1 capitalists and 10% of whom are sector 2 capitalists, and an initial endowment of 100 units of wealth per agent; the production function is set to $\Pi(K, L) = 8\sqrt{KL}$ (that is, $A = 8$ and $\gamma = 0.5$), and prices to $[r, w, p] = [6, 6, 8]$. No limit is imposed on the amount of labor a single agent can supply, and all tax rates are set to 0. All simulations were performed using Python 3.11.

No condensation phenomena are observed in this baseline paramterization for at least the first 100,000 simulation steps. It also exhibits extremely quick relaxation times, with most implicit variables reaching their equilibrium range of values within 3-4 simulation steps.

3.3.1 Baseline parameterization

The baseline parameterization produces distributions of wealth which are highly unequal between subpopulations. Excluding the first five simulation steps to account for time to relaxation, the working class holds on average 99204 units of wealth, translating into 24.8 units of wealth per agent or 19.8% of total system wealth. Sector 1 holds on average 301269 units of wealth, translating into 603.5 units of wealth per agent or 60.3% of total system wealth. Sector 2 holds on average 99526 units of wealth, translating into 199.1 units of wealth per agent or 19.9% of total system wealth. This is in close agreement with the analytically derived division of system wealth between subpopulations, which predicted that the working class would accrue 20% of system wealth, sector 1 would accrue 60% of system wealth, and sector 2 would accrue 20% of system wealth.

High degrees of inequality exist within each subpopulation as well. While the median worker holds 26.4 units of wealth on average, workers in the bottom decile hold no more than 8.8 units of wealth, while workers in the top decile hold at least 97.8 units of wealth. At the final simulation step, the very poorest worker held only 0.04 units of wealth, while the very richest held 124.3 units. Likewise, for capitalists in sectors 1 and 2, the bottom decile thresholds are 205.9 units and 32.2 units, the middle decile thresholds are 608.5 units 195.1 units, and the top decile thresholds are 2484.9 units 1088.0 units respectively; the very poorest agents hold 80.8 and 1.3 units of wealth, and the very richest hold 3234.8 and 1146.9 units of wealth. This high degree of intra-class stratification, along with the disproportionate division of wealth between classes, produces a realistic global distribution of wealth; the system measures an average Gini coefficient for wealth inequality of about 0.73, in line with the range of country-level Gini coefficients of wealth inequality in developed countries estimated by Davies *et al.* (2009). [50]

Figure 3.2 shows the size distribution of wealth within each of the three subpopulations. All three are globally well-fit by Gamma distributions with shape parameters greater than 1 and rate parameters less than 1, implying that the probability density distribution for wealth within each subpopulation is 0 at the origin. Notably, the wealth distributions for both capitalist subpopulations have extremely small best fit rate parameters—0.008 for sector 1 and 0.007 for sector 2. Because of this, the right tails of these distributions *appear* to be well-fit by a power law. This, however, is illusory, as the component of the log-density distribution to the right of the peak is characterized by the dominance of the Gamma distribution's exponential term, and not its polynomial term.

A similar situation is observed when it comes to the size distributions of income for each subpopulation. Workers receive on average 12.4 units of wealth each simulation step as income, while sector 1 capitalists receive 300.0 units and sector 2 capitalists receive 99.2 units. In the final simulation step, 1421 out of 4000 workers, or about 35.5% of the working class, received an income no greater than the "minimum wage" rate of 6 units. Conversely, the worker with the highest income received 59.8 units of wealth as income.

The disparity of intra-class income distributions is even greater within the two

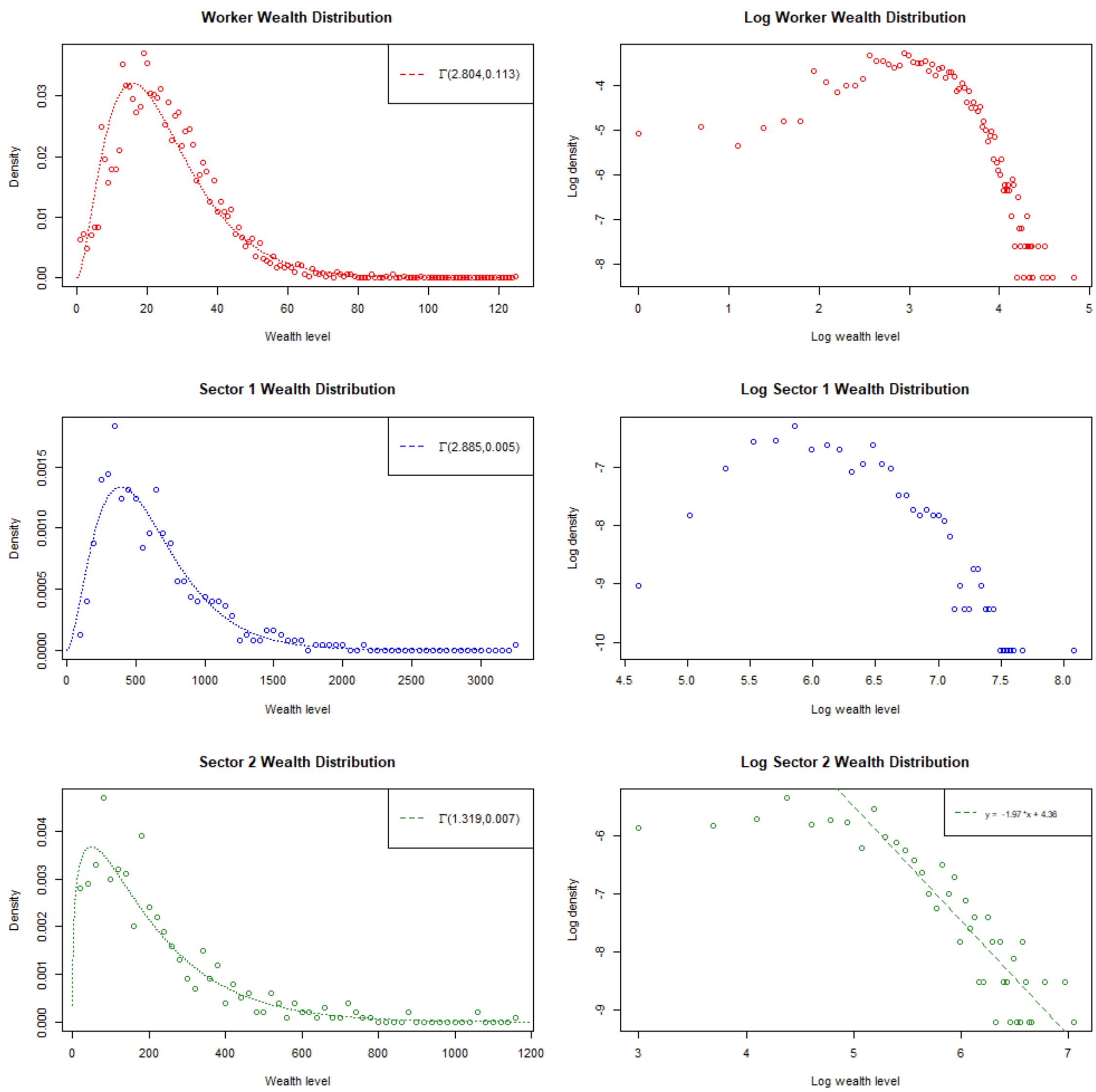

Figure 3.2: Wealth distributions for each of the three subpopulations after $T = 100$ simulation steps.

capitalist subpopuluations. The income thresholds defining the upper limit of the bottom decile are 80.9 for sector 1 and X for sector 2, while the thresholds for the bottom limit of the top decile are 1890.2 for sector 1 and Y for sector 2. In the final simulation step, the lowest incomes observed were 0 for both sectors, while the highest were 2507.2 in sector 1 and 1005.4 in sector 2. This produces a Gini coefficient for global income distribution of 0.76, substantially higher than the recorded value for any country on Earth, as reported by the World Bank. [156]

It is worth noting that, in this model, incomes are mostly quantized. Workers only receive income in the form of wages, which are almost always multiples of the wage rate w. Likewise, sector 1 capitalists receive income from sales of consumer goods, priced at p, and sector 2 capitalists from sales of production goods, priced at r. Incomes which are not strictly multiples of prices are observed because, during the execution of the **Purchase Rule**, purchases of non-integer units of a given resource are allowed, including non-integer units of labor. This fact is nonetheless particularly apparent in the income distribution for workers, which exhibits spikes at w, $2w$, and so on, and has very low density in between. For that reason, bins of width w are used to generate the size distribution of labor incomes.

Figure 3.3 shows the size distribution of income within each of the three subpopulations. All are, once again, well fit by Gamma curves with subunitary rate parameters, with the rate parameters for the two capitalist subpopulations being identical to their corresponding best Gamma fits for wealth. More interestingly, however, the best-fit Gamma curve for the distribution of income within sector 2

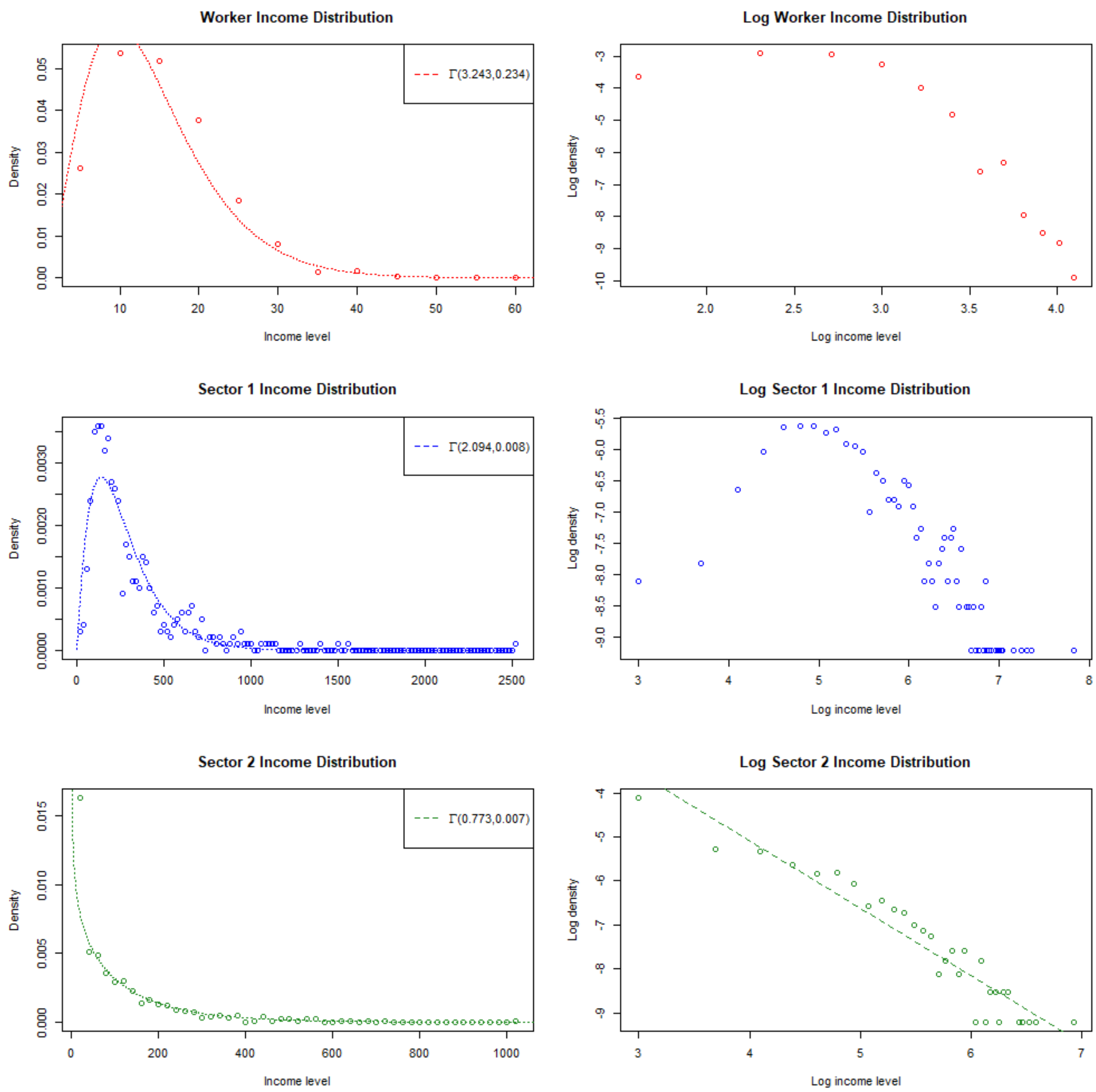

Figure 3.3: Income distributions for each of the three subpopulations after $T = 100$ simulation steps.

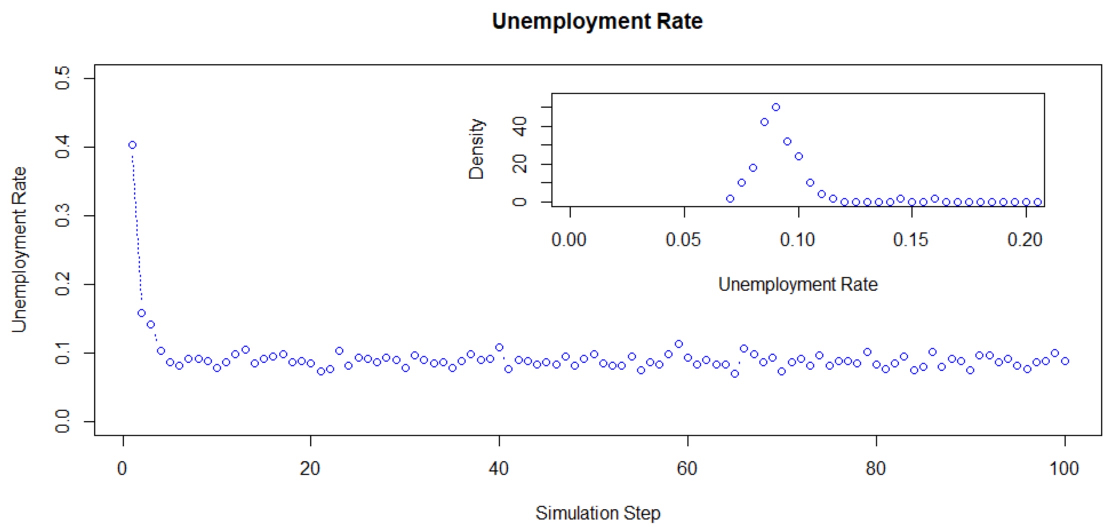

Figure 3.4: Time series of unemployment over the course of a single simulation. Inset: frequency distribution of unemployment rates observed over the course of the same simulation.

has a subunitary shape parameter, and consequently said distribution has a singularity at 0. The significance of this fact lies in the implication that the distribution of income within sector 2 specifically may be *globally* approximated by a power law relationship. This fact becomes even more evident upon looking at the log density distribution, which is well fit by a straight line with slope $m \approx -1.53$. The reason why the shape parameter for the income distribution within sector 2 is subunitary, while the corresponding parameter for the income distribution among sector 1 is over twice as large, is not at all clear.

Finally, the percentage of the population consisting of workers with an income of 0 in a given simulation step defines the unemployment rate at that step. Figure 3.4 shows the time evolution of the unemployment rate over the course of a single simulation. Given the fact that firms select workers to hire completely stochastically, it is a little surprising that the unemployment rate maintains both a non-zero mean and a relatively low variance—specifically, the unemployment rate hovers around a mean of 8.8% and exhibits a standard deviation of 0.82 percentage points. The rate of relaxation to this overall region is also remarkably quick: after an initial period of high unemployment as capitalist wealth—and consequently expenditures on labor—remain below equilibrium values, the unemployment rate exhibits no significant fluctuations from step 4 onwards.

3.3.2 Varying economic parameters

Studying the emergent behaviors of this model on a single, arbitrarily chosen parameterization, however, can only tell us so much. Acquiring a more general intuition for its capabilities and limitations requires testing many different combinations of parameters. Unfortunately, due to the relatively long time required to complete a single simulation, it is not feasible to simulate every combination of parameter values within set ranges; the best that can be done is to vary two or three parameters in tandem.

What follows is a brief analysis of some notable results obtained from a series of three searches of a restricted parameter space. The first was performed by varying q_{cap} (the percentage of ensemble population in C) from 0.05 to 0.95 by increments of 0.05 and varying q_{s1} (the percentage of capitalists in subpopulation

$\mathcal{S}_1$) from 0.25 to 0.75 by increments of 0.05. The second was performed by varying productivity A from 1 to 10 by increments of 0.25 and varying capital intensiveness γ from 0.05 to 0.95 by increments of 0.05. Finally, the third was performed by varying r, w, and p from 1 to 10 by increments of 1.

One thing which becomes immediately obvious upon testing different parameterizations is that there clearly exist emergent, implicit conditions for the existence of a statistical equilibrium within the system. For certain extremal parameter values, wealth condensation phenomena, in which the total wealth possessed by a subpopulation falls to 0, are observed.

This occurs, for instance, with the parameterizations ($q_{cap} = 0.05, q_{S1} \geq 0.7$). The reason why these parameterizations results in condensation isn't entirely clear, but it likely has to do with the viability of production in sector 2. Said paramterizations result in systems with no more than 75 agents in sector 2, and with such a small population size, random fluctuations could much more easily result in a premature bankruptcy—in this case, between simulation steps 9 and 13 specifically.

Condensation also occurs for certain values of capital intensiveness γ when productivity $A < 5$. However, the structure of the disequilibrium parameterizations is more complex, with extremal values of γ being more stable than values of γ closer to 0.5. For example, when $A = 4.75$, condensation occurs only for $\gamma = 0.45$; when $A = 4$, condensation occurs for $\gamma \in [0.25, 0.75]$; and when $A = 2$, condensation occurs for $\gamma > 0.05$. This somewhat unexpected structure is a result of the implicit condition for profit to be possible within the system. In order for the profit function $\pi = p\left(AK^{\gamma}L^{1-\gamma}\right) - rK - wL$ to be able to take on positive values, it is required that:

$$A > \frac{r^{\gamma}w^{1-\gamma}}{p\gamma^{\gamma}(1-\gamma)^{1-\gamma}} \tag{3.8}$$

The right hand side of this inequality, which defines the threshold productivity below which profit becomes impossible, has a maximum at $\gamma = \frac{r}{r+w}$, which, in this case, equals 0.5. This is why, when varying A and γ, extremal values of γ are less likely to condense: if production relies disproportionately heavily on either labor or capital, then the productivity threshold which must be surpassed is indeed lower. For the full derivation of this profitability condition, see Appendix A.3.

Finally, condensation is again observed for certain ranges of price vectors, namely those in which the price of labor is significantly higher than either the price of capital or the price of consumption goods, as well (to a lesser extent) those in which the price of capital is significantly greater than the price of consumption goods. Where the exact contours of condensation lie is, again, related to the profitability condition derived in Appendix A.3. Figure 3.5 portrays these regions of condensation in (p, r)-space for four discrete values of w, using the fraction of total system wealth accumulated by the working class as a proxy.

Worker wealth may serve such purpose because, in this model, complete dispossession of both sector 1 and sector 2 is the only possible wealth condensation pattern. This is because the income of the capitalist subpopulations is predicated on production, a process requiring wealth input from both sectors. If sector 1 cap-

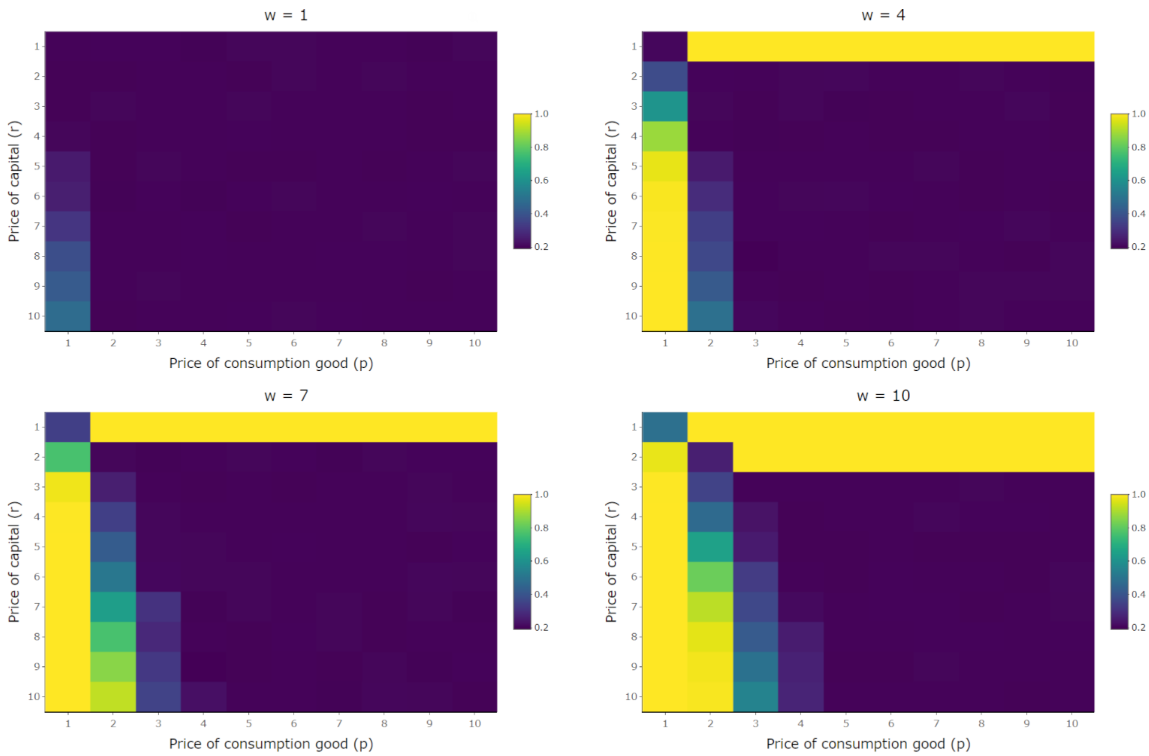

Figure 3.5: Average percentage of total system wealth held by subpopulation $\mathcal{W}$ over the last 10 iterations of various simulations, characterized by different choices of price parameterizations. Values significantly higher than 0.2 indicate parameterizations which likely exhibit long-term wealth condensation, and values close to 1 indicate parameterizations which condense almost completely within the 100 iterations simulated.

italists run out of money, then no wealth flows into sector 2 from the outside and purchases of labor will eventually bleed it dry; if sector 2 capitalists run out of money, then capital reserves will dry up and sector 1 will lose the ability to produce goods profitably. Workers, on the other hand, get all of their income from purchases of labor, and although this is dependent on wealth input from at least one capitalist sector, the lack of such funds implies that the working class already possesses all system wealth!

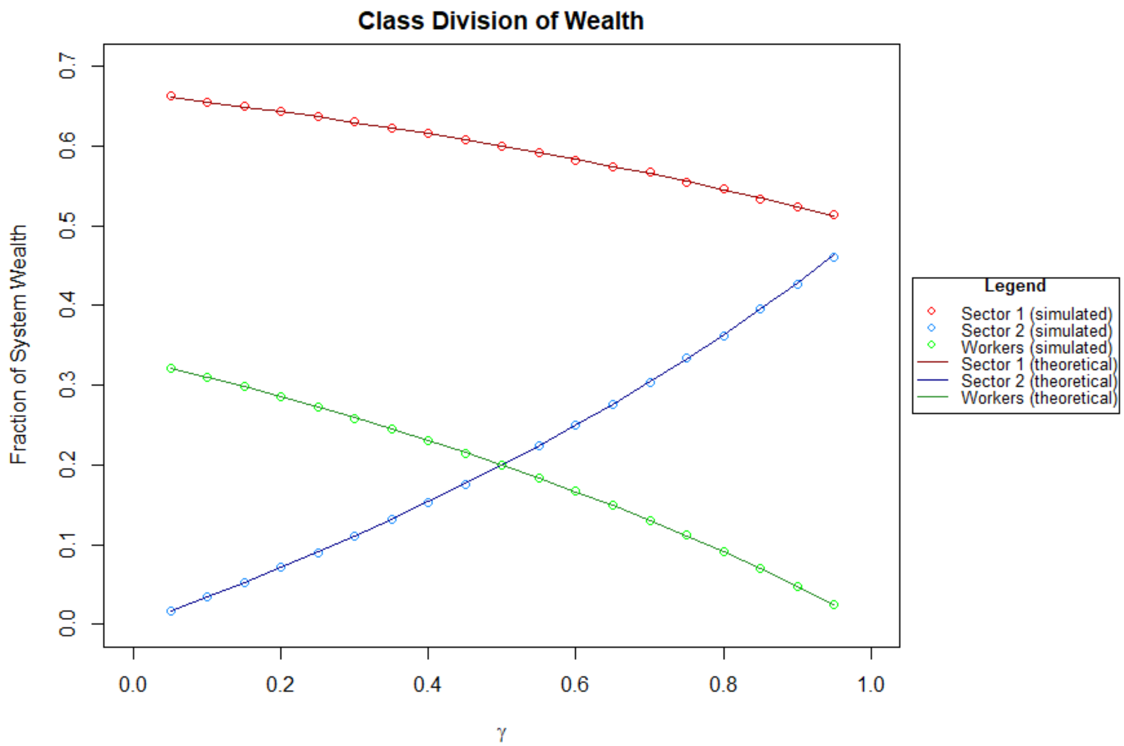

Figure 3.6: Average fraction of total system wealth held by the three subpopulations over the last 95 periods of simulation as a function of capital intensivity γ. Dots represent simulation results for $A = 8$, and solid lines represent theoretical predictions.

In addition to the condensation phenomenon, the theoretical relationship between equilibrium class division of wealth and capital intensiveness of production derived in §3.2 is borne out by simulation. Figure 3.6 exhibits the fraction of total system wealth held by each of the three subpopulations for $A = 8$ and various values of γ, as well as the rational functions defining the state probabilities of a representative dollar. Since the chosen productivity value guarantees condensation does not occur, an extremely high level of agreement is found between the simulated and predicted values.

Aside from the surprising results above, many emergent variables in the model prove to be invariant with respect to choice of certain parameters. For instance, the unemployment rate is completely unaffected by the value of q_{S1} except when condensation occurs, as one would expect. Likewise, within equilibrium-supporting regions, the choice of price vector (r, w, p) does not change the division of class wealth or income since, as in most economic models, expenditures are calculated independently of prices.

In some cases, however, such invariance is evidence of a lack of sophistication on the model's part. An example of such is the fact that the division of class wealth is invariant with respect to productivity A. This result follows from the nature of the profit-maximization heuristic, which assumes that the goods market will be in equilibrium, i.e. there will exist sufficient demand to account for any increase in the quantity of goods produced. However, since prices and wages are fixed, this will not generally be true. Instead, higher values of A should exhibit out-of-equilibrium dynamics in which the amount of labor and capital required by capitalists to produce according to their expected market share decreases, in turn reducing the share of system wealth held by the working class. Indeed, this would occur if agent behavior were "smarter" (maximizing expected profit with full knowledge of the probability distribution of sales), but this simpler model is not able to capture that kind of behavior.

3.3.3 Varying tax rates

It is now time to investigate how this model responds to the introduction of non-zero tax rates. For each tax—wealth, income, sales, payroll, and turnover—simulations were performed at 101 different values between 0 and 1 inclusive. For the purposes of this analysis, if one tax rate is non-zero, then all other taxes are set to 0. Nonetheless, in principle, this model is fully capable of handling parameterizations where all five tax rates are non-zero.

Investigating the effect of wealth and income taxes on the class division of wealth, visualized in Figure 3.7, one sees that both trend towards equalizing the mean wealth held by members of each of the three subpopulations. Notably, but perhaps not surprisingly, the wealth tax performs this function significantly more efficiently than the income tax, in accordance with earlier results in the literature. However, interesting and unexpected behavior is observed for high values of both taxes.

In the case of the wealth tax, starting at around $wtr \approx 0.52$, the share of system wealth held by sector 1 decreases below the theoretically predicted value, and it remains so until $wtr = 1$, when the two values again converge. The opposite occurs for the share of system wealth held by the working class, which exceeds its equilibrium value over that same range. Sector 2 exhibits no out-of-equilibrium behavior at any value of wtr.

In the case of the income tax, a similar dip occurs for sector 1 beginning around

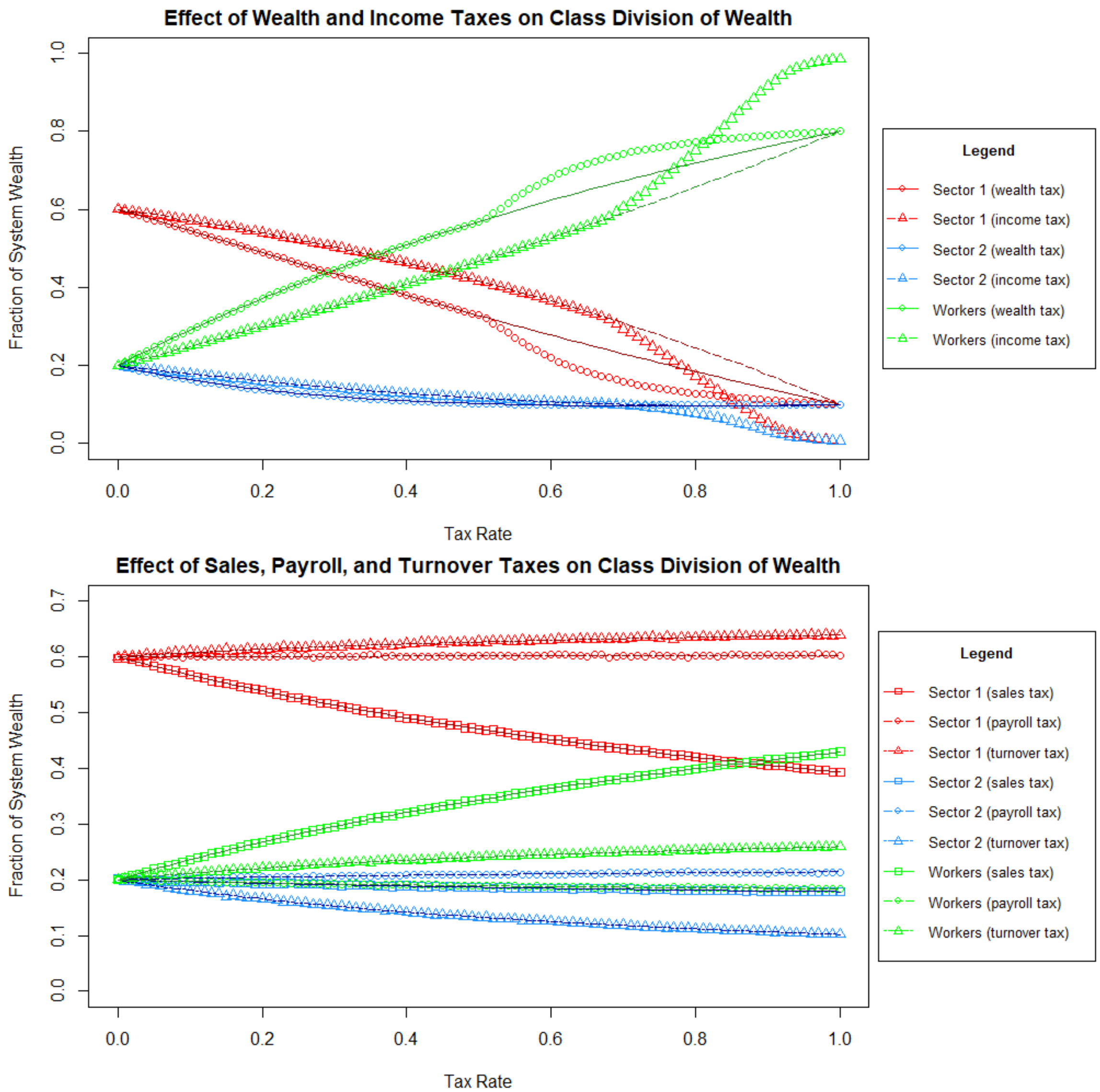

Figure 3.7: Average fraction of total system wealth held by the three subpopulations over the last 95 periods of simulation, as a function of various tax rates. Dots represent simulation results, and solid lines represent theoretical predictions.

itr ≈ 0.69. However, instead of reconverging towards the predicted value, sector 1 wealth goes to 0 as *itr* approaches 1. The same phenomenon is observed for sector 2, while wealth held by the working class approaches total system wealth as *itr* approaches 1, again representing complete wealth condensation.

The unexpected kinks in the wealth share curves for sector 1 and the working class appear to be caused by a phase shift within the intra-class distribution of income for sector 1. As sector 1 becomes poorer on average, a point is eventually reached where the number of agents who make the fewest sales are no longer able to accumulate wealth faster than the wealth tax takes it away from them becomes non-negligible. Even though, after a decrease in wealth, they may engage in profitable production at a smaller scale, their products represent a smaller portion of goods on the market; their income decreases in turn, and, in a vicious cycle, the tax decreases their wealth even further. Thus, even though the gross income of sector 1 increases with *wtr* (capitalists, who are made poorer by the wealth tax, spend a quarter of their wealth on consumption goods in expectation, while workers, who are made richer, spend half), the fraction of poor agents in sector 1 increases faster. For $wtr < 0.43$, the number of agents in sector 1 with incomes less than 12 (sufficient to purchase 1 unit of labor and 1 unit of capital in the next period) is always 0, and the mean income of sector 1 increases with taxation. At $wtr \approx 0.43$, however, a clear phase shift occurs as the mean income of sector 1 dramatically drops and the number of impoverished producers in sector 1 quickly grows: at $wtr = 0.52$, the number of such agents exceeds 30 for the first time, and around $wtr = 0.91$, the number of such agents plateaus around 60, representing 12% of the population of sector 1.

A similar process is responsible for the condensation observed for high income tax rates. It is likely not coincidental that out-of-equilibrium effects begin to take place at similar wealth thresholds: the wealth tax rate of 0.52 corresponds to sector 1 possessing 30.7% of system wealth, while the income tax rate of 0.69 corresponds to sector 1 possessing 30.6% of system wealth. That the class division of wealth subject to income taxes corresponds to its equilibrium values over a larger domain than when subject to wealth taxes, then, would simply follow from the fact that income taxes redistribute wealth less efficiently.

Finally, the reason why income taxes induce full condensation, while wealth taxes induce controlled out-of-equilibrium dynamics, has to do with the fact that the ability of an income tax to redistribute wealth is bounded by the ”velocity of money,” in this case the percent of system wealth which is actually traded in a given simulation step. Thus, if a large fraction of agents in sector 1 become impoverished and stops producing, the amount of wealth redistributed each simulation step decreases, and, instead of becoming controlled as in the case of wealth taxation, the process of condensation accelerates. Using the language of dynamical systems, one can say that, under income taxation, the state represented by full condensation is an attractor, while under wealth taxation, it is not.

For the remaining three taxes, it is clear that only sales tax has a large effect on the division of wealth among subpopulations. As the sales tax rate increases, the share of wealth held by sector 1 decreases while the share of wealth held by the working class increases; the share of wealth held by sector 2 is essentially unchanged. At $str = 1$, both sectors 1 and 2 remain disproportionately wealthy, but the working class collectively becomes wealthier than either of them. The effect of turnover tax slightly is to slightly increase the wealth shares of sector 1 and the working class at the expense of sector 2, and the effect of payroll tax is negligible.

The theoretical predictions made by the Markov representation of class expenditures once again agree extremely well with our simulation results up until the phase transition points induced by wealth and income taxes. Since no such points exist for sales, payroll, and turnover taxes (or for income tax with respect to sector 2, for that matter), their effect on the class division of wealth is globally well-fit by the Markov representation's predictions.

The effects of taxation upon the Gini coefficients measuring the degree of global wealth and income inequality within the system behave somewhat more strangely. The Gini coefficient of wealth inequality, G_W, decreases monotonically with wtr until reaching 0—perfect equality—at $wtr = 1$. It also displays the same kink around $wtr \approx 0.52$ as above. Subject to income taxation, however, G_W only decreases until it reaches the value of approximately 0.31 at $itr \approx 0.8$, beyond which point it rebounds and plateaus at $G_I \approx 0.45$. This occurs because, at $itr = 0.84$, the fraction of total system wealth held by the working class exceeds $0.8 = N_W/N$ for the first time, and further condensation only makes the working class more disproportionately wealthy, increasing inequality.

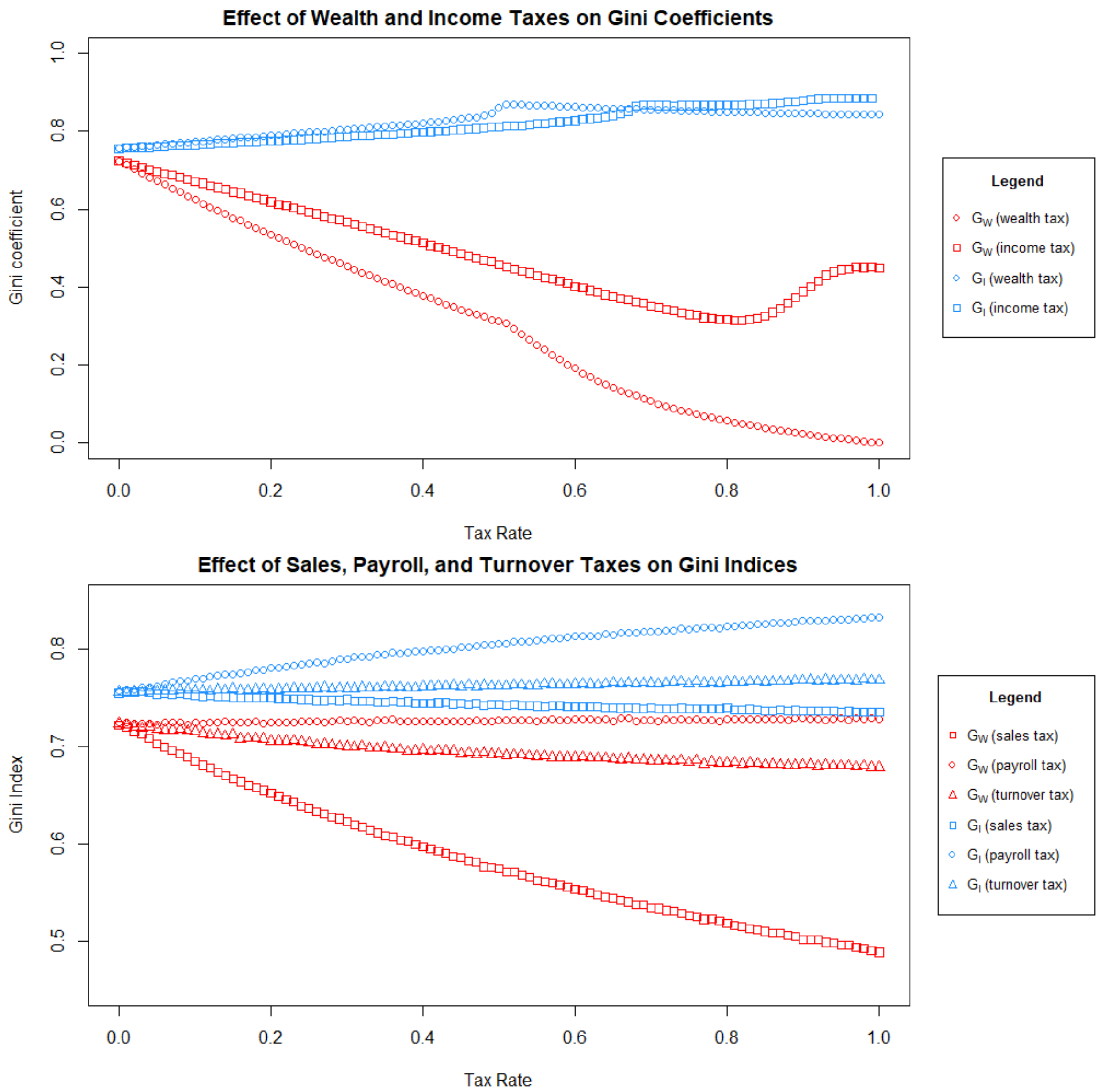

Figure 3.8: Average value of the Gini coefficients calculated for the global wealth and income distributions of the system over the last 95 periods of simulation, as a function of various tax rates.

Somewhat counterintuitively, neither wealth taxes nor income taxes reduce the Gini coefficient of income inequality G_I. Increasing *wtr* actually *increases* G_I until the wealth tax phase transition point, beyond which it plateaus at $G_I \approx 0.84$. Increasing *itr* likewise increases G_I until $itr = 1$, at which point the system fully condenses and G_I vanishes because all incomes are identically 0.

It is not at all obvious why this is the case. One may hypothesize that increase of G_I in response to an increasing wealth tax rate likely arises from the impoverishing effect said tax has on agents in the left tail of the sector 1 wealth distribution; as the number of agents in sector 1 failing to stay consistently solvent grows, the "spread" within the income distribution of sector 1 becomes wider and global income inequality increases. Conversely, once the size of this group stabilizes, further increasing *wtr* has no effect on global inequality. This explanation would also account for the fact that G_I increases monotonically with *itr*, as no such stabilization of the fraction of insolvent capitalists occurs for the reasons outlined above. However, this hypothetical causal chain is by no means a certain or complete explanation, and is unlikely to correspond to meaningful economic behavior—first of all, because few, if any, countries have ever had such high levels of taxation, and because this static model, which is not capable of representing the movement of agents between classes, fails to capture the directional behavior towards a new equilibrium that would occur in a more realistic model.

Sales, payroll, and turnover taxes, as in the determination of class division of wealth, have more predictable effects. Sales tax has a surprisingly strong negative effect on the Gini coefficient of wealth inequality, while payroll taxes have a moderate positive effect on the Gini coefficient of income inequality. All other effects are negligible.

3.4 Flat taxation and income inequality—what does data say?

In general, empirical testing of the results obtained above by varying the economic parameters of this model is difficult for a number of reasons. Identification in the social sciences is difficult even in the best cases, and the macroeconomic data available on inequality often do not provide such auspicious circumstances. However, one clear prediction made by this model which can be supported by recent analyses is the effect of flat income taxation on income inequality. According to the model above, any amount of flat income taxation actually *increases* the Gini coefficient of income inequality compared to the steady state obtained without taxation, even as it reduces the degree of wealth inequality. But does this prediction hold up empirically?

In order to answer this question, one must find a suitable ”natural experiment,” in which the effect of a given policy treatment can be isolated. Efremidze & Salayeva (2021) do exactly this by investigating the effect the imposition of flat income taxation in a number of countries transitioning from socialism to capitalism in the 1990s had on the Gini coefficients of income inequality for said countries. [64] Even after controlling for many potentially confounding variables, both political and economic in nature, it was found that flat income taxation has had a statistically significant positive effect on both the UN and World Bank Gini coefficient measurements in the given countries. This effect was observed within two years of the introduction of flat taxation, with the impact becoming clearer by the fourth or fifth year thence.

There exist a handful of smaller-scale studies which investigate the effect of introducing flat income taxation within a single country, but these studies are often inadequate for our purposes because often the new flat taxation policy is replacing a pre-existing progressive one. However, there is one country a flat tax was introduced where before there was no unified and functional tax code, closely mimicking a proper "natural experiment": the Russian Federation.

Prior to 2001, the Russian personal income tax (PIT) system was characterized by three marginal rates increasing in income. [61] However, this system suffered from low rates of enforcement, as evidenced by the dramatic increase in collections observed after the switch to a more strictly enforced flat PIT system was made in 2001. The flat PIT system, which levied a 13% income tax on all individuals regardless of income level, remained in place for 20 years, until being abolished in 2021.

A small number of studies have investigated the effect of the flat PIT system on income inequality in Russia, though unfortunately the majority are not available in English. Pugachev (2022) finds that this system of flat income taxation exacerbated income disparities between regions in Russia. [131] During the period 2005-2020, urban and northern regions remained much wealthier than rural and southern regions, and various indicators of income dispersion—including the Gini coefficient, the ratio between the 9th and 1st income decile, and the ratio between the 5th and 1st income quintile—saw no significant reduction. While greater analysis of the Russian experience of flat income taxation is clearly needed, the available data supports the conclusion that flat income taxation is not an effective means of reducing income inequality.

3.5 Discussion

Having defined, analyzed, and simulated this model, a number of clear advantages over most previous random asset exchange models have been demonstrated. Firstly and significantly, this model is capable of producing realistic wealth *and* income distributions. As previously mentioned, this is not the case for most random asset exchange models with homogeneous agents, which tend to produce Gaussian income distributions. Instead, the introduction of some representation of firm structure and production seems to be necessary to have both distributions be right-skewed.

The features of this model also have much more meaningful correlates with economic phenomena. Unlike in simpler random asset exchange models, this model incorporates production, with the result that its dynamics are distinguished from those of a pure exchange system (such as a financial market) and the close relationship between production and exchange can be incorporated into one's analysis.

Furthermore, this more realistic approach to stochastic economic modeling allows to ask questions which previous random asset exchange models could not answer. For example, when an exogenous change occurs within firms' production function (a so-called "technology shock") or consumers' spending patterns change dramatically (a "demand shock"), one would expect that both short- and long-term effects may manifest in the system's emergent parameter values, and indeed many canonical models in economics seek to explain what such effects may be. Econophysical models which do not model the production-exchange circuit, on the other hand, are not capable of investigating such questions.

But the additional complexity introduced by this model brings with it a number of drawbacks. The most immediately salient of these is that the dynamics of this model are significantly more difficult to explore analytically as most variables one would want to investigate in this model—unemployment, Gini coefficients, and so on—depend on the shape of the intra-class wealth distributions, which themselves resist easy categorization. This fact leaves simulation as the primary means of investigating the dynamics of this model.

This downside to the model is compounded by the fact that it takes significantly longer to simulate in comparison to BDY-style models. With the model's **Purchase Rule** including two nested iterations through large portions of the system's total population, this model has a time complexity of $O(N^2)$. The original

BDY model, on the other hand, only has a time complexity of $O(N)$ if the number of iterations performed is normalized such that every agent acts in expectation T times over the course of the simulation.

Although this model represents a significant step forward in terms of economic realism, it still contains a number of simplifying assumptions which work against its purpose of elucidating the dynamism of the production-exchange circuit. Among these are the assumption of fixed and uniform prices, the assumption of fixed money supply, and the assumption that agent behavior (including capitalist agent behavior) taking into account nothing beside current wealth level. The last of these three assumption in particular must be relaxed if such models are to be able to fully reflect the complex interdependencies of economic systems.

It is nonetheless clear that further iterations upon this class of processual random asset exchange models have the potential to be powerful analytical tools, and that, as the literature's interest in employing random asset exchange models in policy analysis grows, deeper investigation of this class of models is warranted. In the context of this specific model, more work ought to be done to determine under what conditions certain parameterizations in this model unexpectedly condense. That is, what are the exact, analytical relationships between system variables that cause the fully condensed state to become an attractor within the system's phase space?

More generally, the analysis above shows that modeling production and firm structure alone is insufficient to recover "true" power law regimes within income distributions. Despite this fact, the model seems to indicate that income derived from the sale of produced goods by a small subsection of the population is predisposed to being characterized by Gamma distributions with subunitary shape parameters. This result is itself significant because, as was shown, when such Gamma distributions happen to have small rate parameters as well, the resulting curves are analytically nearly indistinguishable from power laws—a phenomenon similar to what Perline (2005) termed ”log-normal power law mimicry,” in which data which is in actuality governed by right-skewed exponential-form distributions may falsely appear, over a truncated domain, to be governed by power laws. [127] Thus, it would likely be worthwhile to investigate the conditions under which Gamma distributions of this type are likely to emerge.

This model also indicates that wealth and income inequality, as measured by their respective Gini coefficients, can behave quite differently in response to identical taxation regimes. This in itself is an interesting and counter-intuitive result which ought to make us refine the way economic inequality is thought about. When one attempts to compare the degree of inequality across two countries, for example, which metric is more informative?

As has been shown above, a model parameterization with a non-zero wealth tax and zero income tax will have a significantly lower Gini coefficient of wealth inequality but a slightly higher Gini coefficient of income inequality as compared to the baseline parameterization. Even so, it probably is not reasonable to say that our toy ensemble has become less egalitarian as a result of this change. This is because, when people talk about ”economic inequality” colloquially, they tend to have in mind inequality in living standards. And although an increase in living standards is almost always precipitated by an increase in income, the converse is not necessarily true; a low-income population receiving an increase in income will fare no better if the increase is outpaced by any increase in expenses faced by said population. Thus, this model suggests that more thought should be given to the limitations of using the Gini coefficient of income distribution as the sole metric of inequality, and that alternative metrics, such as the Gini coefficient of wealth distribution, be more widely adopted to that end.

CHAPTER 4

CONCLUSION

The term *econophysics*—derived by comparison to the earlier terms such as *astrophysics*, *geophysics*, and *biophysics*—was coined in 1995 by American physicist H.E. Stanley to describe the utilization of techniques developed in statistical physics to explore phenomena normally considered to fall under the purview of economics. [165] Stanley's argument for the utility of such an approach was as follows. Physics studies processes consisting of very large quantities of microscopic interactions, each of which are individually governed by Newton's laws. Similarly, economics studies processes consisting of very large quantities of financial interactions, each of which are individually governed by profit maximization, supply and demand, etc. However, in physics, these processes are not modelled by naively summing all the particles and calculating their effects on the system according to classical mechanics, but rather by invoking a scaling law to convert the motions of billions of particles into one collective process, governed by the well-understood laws of statistical mechanics. [151] Could a similar approach perhaps be used with respect to economic processes?

This question was not unprecedented on Stanley's part. Physicists had been influenced by problems in economics since the early 20th century at the latest, when Louis Bachelier independently derived the equations governing Brownian motion from observations of the Parisian stock market; likewise, economists have been interested in the methods employed by physicists since the time of Adam

Smith. [28] However, Stanley's proposal retroactively acquired a great deal of significance by virtue of the flurry of activity it set off. Over the next few years, there appeared for the first time econophysical conferences, econophysical analyses of stock market fluctuations, and, as the reader is now surely well aware, econophysical models of wealth and income distributions. [165]

This thesis provided, for the first time, a truly comprehensive review of this literature, termed the random asset exchange literature, charting its course of development from its inception to its current standing. While acknowledging the innovative and anticipatory papers by Angle, Bennati, and Ispolatov *et al.*, the inauguration of this literature was associated with the seminal papers Drăgulescu & Yakovenko (2000), Chakraborti & Chakrabarti (2000), and Bouchaud & Mézard (2000). This in turn permitted the identification of a number of "canonical" models, which have since acted as the foundation for the rest of the literature to build upon: these included both discrete-time, wealth-conserving models such as the BDY, CC, and CCM models; discrete-time, non-conservative models such as the CPT model; poor-biased "yard sale" models such as the IGAV, EYS, and AW models; and continuous-time models such as the BM model. Minor model formulations such as the generalized Lotka-Volterra model were also discussed.

Based on these aforementioned models, the full scope of variation within these model classes was illustrated. A number of types of modification were characterized by alterations of the way in which the pair of agents to interact is chosen: in this class are included the imposition of adjacency networks onto the agent population or the definition of non-Maxwellian system collision kernels. Other variations altered the dynamics of the pairwise interaction itself, for example by introducing exogenous wealth influx or heuristics for self-advantaging behavior. The final class of modification include models which introduce entirely new objects and system dynamics alongside the pre-existing exchange system. As these models often attempt to emulate more canonical models within economics, such introductions often take the form of a circulating good supply, commodity production, firm structure, or some combination thereof. This last category was highlighted as producing particularly interesting emergent behaviors, with any loss of parsimony being largely negated by the fact that the additional complications do correspond to real economic processes.

More recently, the literature has turned towards studying random asset exchange models with redistributive mechanisms. In particular, various authors have used such models to compare the ability of income taxes—which most countries collect—and wealth taxes—which most countries do not collect—in counteracting economic inequality. Others have framed the question as decision problems, wherein the level of taxation is chosen to optimize a specific objective—such as economic growth—or as model predictive control problems. Unlike many extensions of the random asset exchange literature, the conclusions extrapolated from these studies have actual implications for policy.

The first chapter concluded with an assessment of the achievements and drawbacks of the literature as a whole. It was argued that, on one hand, random asset exchange models provide deep insight into the nature of the contribution to economic inequality made by the laws of probability, and are capable of reproducing realistically-shaped asset distributions from relatively elegant exchange rules. But on the other hand, the literature is still a far way off from making the qualitative leap from *providing insight* to *providing an explanation*. The literature, at this point, has begun to be held back by retaining those modeling assumptions which are more appropriate for physics than for economics, and it is argued that the literature moving in the direction of more directly modeling economic phenomena and processes is likely to significantly help in that regard. This latter point becomes particularly salient as the literature makes the aforementioned pivot towards more policy-oriented analysis.

To that end, a new model, most directly inspired by the models of Wright and Lavička *et al.*, was then introduced. This model was characterized by the existence of three static populations—producers of consumption goods, producers of capital goods, and workers—which interact stochastically in order to manufacture and purchase commodities. The model was then reinterpreted as a Markov process, representing the path of a representative unit of wealth as it gets exchanged between the three classes. The Markov process was solved and the equilibrium division of wealth and income between the three populations was derived.

The behavior of the model at a chosen baseline parameterization was then investigated by means of Monte Carlo simulation. It was deduced that, for the chosen parameterization, non-degenerate stationary distributions of wealth and income exist, both of which exhibit high degrees of inequality and are well fit by Gamma distributions. The model also exhibited a stable, non-zero unemployment rate. Evidence for the existence of power-law tails in both the wealth and income distributions was inconclusive, though the distribution of income governing the population of capital goods producers was globally well fit by a power law.

When the model's parameters were allowed to vary, it was found that the region within the system's phase space in which statistical equilibria exist has relatively tight bounds. No non-degenerate equilibria exist, for example, for parameterizations with very low numbers of capital goods producers or for certain combinations of productivity A and capital intensiveness γ. In the latter instance, the lack of equilibrium was shown to be a result of the fact that firms' expected profit dips below zero, and the threshold value of A below which all firms are insolvent is highest for $\gamma = 0.5$. Similar out-of-equilibrium dynamics were observed for price vectors with large differences between the highest and lowest entries. The theoretically-predicted division of wealth between populations was additionally confirmed by simulation.

Finally, simulations were performed to investigate the effect of introducing taxation into the system. Both wealth and income taxes were found to have strong equalizing effects upon the distribution of wealth, though condensation was observed for large values of both parameters. Furthermore, the highly counter-intuitive result that, within the context of this model, taxation *increases* income inequality as measured by the Gini coefficient of distribution of income was obtained and discussed. The second chapter concluded with an assessment of the novel results obtained from this model and a confirmation of the thesis forwarded by Perline (2005), that relying purely on visual tests to distinguish between power law relationships and other similarly-shaped distributions is an unreliable method at best. [127] Both more accurate statistical tests and more accurate data for income distributions' right tails are therefore needed.

Furthermore, it has become clear in the course of this investigation that the random asset exchange modeling literature, and the contribution thereto represented by this thesis, have a number of bigger-picture implications. First, all of the models discussed indicate that a large proportion of observed economic inequality is the result of the inherently diffusive (entropy-increasing) nature of exchange itself. While some authors, such as Venkatasubramanian *et al.* (2015), take this to mean that the "natural," entropy-maximizing level of inequality is by definition fair, such a conclusion is far too strong and veers into the territory of naturalistic fallacies. [161] Instead, the conclusion one ought to draw from this cardinal result of the random asset exchange literature clearly depends on one's own (subjective) beliefs concerning the "ideal" level of inequality—however that is determined—as compared to the level currently prevailing. For champions of relatively unrestrained capitalism, who have argued that inequality has an important stimulus effect in the economy by encouraging people to work harder in the hopes of achieving better economic outcomes, the implications of said result are quite positive: statistics naturally guarantee such inequality without market-distorting conditions such as the formation of monopolies or the institutionalization of economic thievery! But even this implication is not exactly groundbreaking. There are few who question whether capitalism is capable of generating extreme inequality as is, after all. On the other hand, for those policy makers who aim to reduce the degree of economic inequality present in modern societies, the corresponding implication may be somewhat more dismal. What these models show in their case is that altering government policies to make market economies operate more "fairly" by, for example, introducing progressive taxation, heavily regulating key industries, and so on can only do so much. At the end of the day, large scale regimes of wealth redistribution will be necessary in order to reduce inequality below the level that is endogenous to exchange-based systems.

The above likely represents the most significant insight provided by the random asset exchange literature, which is non-trivial. However, econophysical models are not without their own problems. Most are still incapable of replicating all of the characteristic features of wealth and income distributions. For wealth distributions, as has been discussed, these include non-negligible segments of the population with non-positive wealth and possibly a power law right tail; for income distributions, these include an exponential or log-normal bulk, and an at least apparent power law tail with exponent between -2 and -3. But the blame for this shortcoming cannot be laid solely at the feet of these simple models. One should not expect to replicate real distributions of wealth and income perfectly given the fact that they abstract away from almost all concrete economic processes. As mentioned above, they already, in their current form, serve as excellent demonstrations of the role random chance plays in generating the inequalities observed in market economies. But if an attempt is to be made to improve these models in order to more closely approach the much-desired *explanation* for the role of stochastic processes in generating economic inequality—and not approach the best fit for the data sets available—some attempts show more promise than others.

Models which impose specific distributions on an endogenous parameter throughout the population (thriftiness, size of social network, etc.) clearly have the capability of producing any desired distribution, but such results have far less explanatory power seeing as they merely defer the question to one level lower. If one observes a given distribution of wealth because there exists an underlying distribution of a certain behavioral parameter, why is this parameter distributed the way it is throughout the population? One ultimately returns to Pareto's unsatisfying explanation for his own law and finds oneself no closer to actually understanding the crux of the issue. Instead, the most promise is shown by what have been called throughout this thesis "processual" models, which introduce the elements of production and class relationships into the mix as fundamental processes of economic systems. This approach reflects concrete asymmetries in the economy, increases the degree of economic realism present within the models, and permits the identification of different sections of the income distribution with different social positions. However, as the model introduced in this thesis also showed, the shape of the distributions which emerge from these kinds of models appear to be just as implementation-dependent as other types of models, making the aforementioned identification somewhat weaker.

Nonetheless, the promise of such an approach is real, and the author argues that literature ought to move in the direction of developing more sophisticated models specifically in this ”processual” vein. And though this thesis has to this point very much not been in dialogue with the economics literature, it stands to reason that economists could benefit just as much from moving in an econophysical direction and giving more thought to the consequences of introducing heterogeneity by means of partially stochastic agent behavior. Given that many of the canonical models in economics rely heavily on the assumption of ”representative” agents who, if provided identical endowments, will by definition have identical economic outcomes, the immense inequality in outcomes observed across populations of real individuals, who all have relatively comparable abilities to perform labor and to innovate, is difficult to explain. Such a pivot could therefore prove useful on the other end of the as-of-yet unbridged gap between the two approaches and bring greater clarity to both.

And such clarity is indeed sorely needed. Per Horowitz *et al.* once again, 61% of American adults believe that there exists too much inequality in the United States today. [82] Of that 61%, 81% believe that this problem will require either major policy interventions or a complete restructuring of the economy to address. There exists a clear political will, at least in the U.S., to reduce the degree of inequality that has been allowed to develop over the past few decades. But despite that fact, the same survey demonstrated that there exists no consensus on what the major contributors to economic inequality even are. Needless to say, determining what policies would be required to create a more egalitarian society requires a clear understanding of the principal processes responsible for its generation, and much work remains to be done on the part of both econophysicists and economists before such a satisfactory understanding is reached.

APPENDIX A

DERIVATIONS FOR CHAPTER 3

A.1 Optimal capital allocation

We write the Lagrangian of the maximization problem:

$$\mathscr{L} = AK_i^{\gamma}L_i^{1-\gamma} - \lambda(x_i y_i W_i - rK_i - wL_i) \tag{A.1}$$

Taking the derivative with respect to both inputs produces the first-order conditions:

$$\begin{aligned} A\gamma K_i^{\gamma-1}L_i^{1-\gamma} &= \lambda r \\ A(1-\gamma)K_i^{\gamma}L_i^{-\gamma} &= \lambda w \\ rK_i + wL_I &= x_i y_i W_i \end{aligned} \tag{A.2}$$

Eliminating λ produces the reduced system:

$$\begin{aligned} \gamma w L_i &= (1-\gamma) r K_i \\ rK_i + wL_I &= x_i y_i W_i \end{aligned} \tag{A.3}$$

Which solves as:

$$\begin{aligned} K_i &= \frac{\gamma}{r} \cdot x_i y_i W_i \\ L_i &= \frac{1-\gamma}{w} \cdot x_i y_i W_i \end{aligned} \tag{A.4}$$

A.2 Derivation of transition matrix

Let $EQ_{\mathcal{I}}^{a}$ represent the expected number of purchases made of resource a by subpopulation $\mathcal{I}$. Let $W_{\mathcal{I}}$ represent the total wealth held by subpopulation $\mathcal{I}$ and let W represent total system wealth.

From subpopulations $\mathcal{I} \in \{\mathcal{S}_1, \mathcal{S}_2\}$, sector 1 receives a fraction of subpopulation wealth X_{i1} defined by:

$$\begin{aligned} X_{i1} &= \frac{1}{W_{\mathcal{I}}} p_s EQ_{\mathcal{I}}^{c} \\ &= \frac{1}{W_{\mathcal{I}}} \frac{p}{1+str} \cdot \mathbb{E}\left[\frac{1}{p} \sum_{j\in\mathcal{I}} x_j(1-y_j)W_j\right] \\ &= \frac{1}{W_{\mathcal{I}}} \frac{1}{1+str} \sum_{j\in\mathcal{I}} \mathbb{E}[x_j]\mathbb{E}[1-y_j]W_j \\ &= \frac{1}{4(1+str)} \end{aligned} \tag{A.5}$$

From subpopulation $\mathcal{W}$, sector 1 receives:

$$\begin{aligned} X_{31} &= \frac{1}{W_{\mathcal{W}}} p_s EQ_{\mathcal{W}}^{c} \\ &= \frac{1}{W_{\mathcal{W}}} \frac{p}{1+str} \cdot \mathbb{E}\left[\frac{1}{p} \sum_{j\in\mathcal{W}} x_j W_j\right] \\ &= \frac{1}{W_{\mathcal{W}}} \frac{1}{1+str} \sum_{j\in\mathcal{W}} \mathbb{E}[x_j]W_j \\ &= \frac{1}{2(1+str)} \end{aligned} \tag{A.6}$$

The same process may be used to find the entries in columns 2 and 3. From subpopulations $\mathcal{I} \in \{\mathcal{S}_1, \mathcal{S}_2\}$, sector 2 receives:

$$X_{i2} = \frac{1}{W_{\mathcal{I}}} r_s EQ_{\mathcal{I}}^{k} = \frac{1}{W_{\mathcal{I}}} \frac{r}{1+ttr} \cdot \mathbb{E}\left[\frac{\gamma}{p} \sum_{j\in\mathcal{I}} x_j y_j W_j\right] = \frac{\gamma}{4(1+ttr)} \tag{A.7}$$

From subpopulations $\mathcal{I} \in \{\mathcal{S}_1, \mathcal{S}_2\}$, the working class receives:

$$X_{i3} = \frac{1}{W_{\mathcal{I}}} w_s E Q_{\mathcal{I}}^{\ell} = \frac{1}{W_{\mathcal{I}}} \frac{\ell}{1+ptr} \cdot \mathbb{E}\left[\frac{1-\gamma}{p} \sum_{j \in \mathcal{I}} x_j y_j W_j\right] = \frac{1-\gamma}{4(1+ptr)} \tag{A.8}$$

Finally, neither sector 2 producers nor the working class receives any income from purchases made by workers.

A.3 Profitability condition

Assuming supply meets demand, total system profit is:

$$\pi = pAK_{total}^{\gamma} L_{total}^{1-\gamma} - rK_{total} - wL_{total} \tag{A.9}$$

where K_{total} represents total capital expenditures and L_{total} represents total labor expenditures. According to the output maximizing heuristic in Appendix A.1:

$$\begin{aligned} K_{total} &= \frac{\gamma}{r} W_C \\ L_{total} &= \frac{1-\gamma}{w} W_C \end{aligned} \tag{A.10}$$

Plugging these into equation (B.9) produces the expression:

$$\pi = pA\left(\frac{\gamma}{r}\right)^{\gamma} \left(\frac{1-\gamma}{w}\right)^{1-\gamma} W_C - W_C \tag{A.11}$$

In order for profit to be possible, it is necessary that:

$$\begin{aligned} \pi = pA\left(\frac{\gamma}{r}\right)^{\gamma} \left(\frac{1-\gamma}{w}\right)^{1-\gamma} W_C - W_C &> 0 \\ pA\left(\frac{\gamma}{r}\right)^{\gamma} \left(\frac{1-\gamma}{w}\right)^{1-\gamma} W_C &> 1 \\ A &> \frac{r^{\gamma} w^{1-\gamma}}{p\gamma^{\gamma}(1-\gamma)^{1-\gamma}} \end{aligned} \tag{A.12}$$

Define the threshold level of productivity required for profit to be possible $A_{\pi=0}(\gamma) = \frac{r^\gamma w^{1-\gamma}}{p\gamma^\gamma(1-\gamma)^{1-\gamma}}$. $A_{\pi=0}$ varies with γ as:

$$\begin{aligned}
\frac{dA_{\pi=0}}{d\gamma} &= A_{\pi=0}(\gamma)\frac{d}{d\gamma}\ln\left(A_{\pi=0}(\gamma)\right) \\
&= \frac{r^\gamma w^{1-\gamma}}{p\gamma^\gamma(1-\gamma)^{1-\gamma}} \cdot \frac{d}{d\gamma}\left(\gamma\ln(r) + (1-\gamma)\ln(w) - \ln(p) - \gamma\ln(\gamma) - (1-\gamma)\ln(1-\gamma)\right) \\
&= \frac{r^\gamma w^{1-\gamma}}{p\gamma^\gamma(1-\gamma)^{1-\gamma}} \cdot \left(\ln(r) - \ln(w) - \ln(\gamma) - 1 + \ln(1-\gamma) + 1\right) \\
&= A_{\pi=0}(\gamma) \cdot \ln\left(\frac{r(1-\gamma)}{w\gamma}\right)
\end{aligned} \tag{A.13}$$

Equation (B.13) has a root at:

$$\begin{aligned}
\frac{dA_{\pi=0}}{d\gamma} = A_{\pi=0}(\gamma) \cdot \ln\left(\frac{r(1-\gamma)}{w\gamma}\right) &= 0 \\
\ln\left(\frac{r(1-\gamma)}{w\gamma}\right) &= 0 \\
\frac{r(1-\gamma)}{w\gamma} &= 1 \\
\gamma &= \frac{r}{r+w}
\end{aligned} \tag{A.14}$$

Furthermore, the second derivative of $A_{\pi=0}$ is:

$$\begin{aligned}
\frac{d^2 A_{\pi=0}}{d\gamma^2} &= \left(\frac{dA_{\pi=0}}{d\gamma}\right) \cdot \frac{d}{d\gamma} \ln\left(\frac{dA_{\pi=0}}{d\gamma}\right) \\
&= \frac{dA_{\pi=0}}{d\gamma} \cdot \frac{d}{d\gamma} \ln\left(\frac{r^{\gamma} w^{1-\gamma}}{p\gamma^{\gamma}(1-\gamma)^{1-\gamma}} \cdot \ln\left(\frac{r(1-\gamma)}{w\gamma}\right)\right) \\
&= \frac{dA_{\pi=0}}{d\gamma} \cdot \left(\ln\left(\frac{r(1-\gamma)}{w\gamma}\right) + \frac{d}{d\gamma} \ln\left(\ln\left(\frac{r(1-\gamma)}{w\gamma}\right)\right)\right) \\
&= \frac{dA_{\pi=0}}{d\gamma} \cdot \left(\ln\left(\frac{r(1-\gamma)}{w\gamma}\right) - \frac{1}{\ln\left(\frac{r(1-\gamma)}{w\gamma}\right)} \frac{1}{\frac{r(1-\gamma)}{w\gamma}} \left(\frac{r}{w\gamma} + \frac{r(1-\gamma)}{w\gamma^2}\right)\right) \qquad \text{(A.15)} \\
&= A_{\pi=0}(\gamma) \cdot \ln\left(\frac{r(1-\gamma)}{w\gamma}\right) \cdot \left(\ln\left(\frac{r(1-\gamma)}{w\gamma}\right) - \frac{1}{\ln\left(\frac{r(1-\gamma)}{w\gamma}\right)} \left(\frac{1}{1-x} + \frac{1}{x}\right)\right) \\
&= A_{\pi=0}(\gamma) \cdot \left(\ln\left(\frac{r(1-\gamma)}{w\gamma}\right)^2 - \frac{1}{\gamma(1-\gamma)}\right)
\end{aligned}$$

At $\gamma = \frac{r}{r+w}$, equation (B.15) evaluates as:

$$\begin{aligned}
\frac{d^2 A_{\pi=0}}{d\gamma^2}\left(\frac{r}{r+w}\right) &= A_{\pi=0}\left(\frac{r}{r+w}\right) \cdot \left(\ln\left(\frac{r\left(1-\frac{r}{r+w}\right)}{w\frac{r}{r+w}}\right)^2 - \frac{1}{\frac{r}{r+w}\left(1-\frac{r}{r+w}\right)}\right) \\
&= A_{\pi=0}\left(\frac{r}{r+w}\right) \cdot \left(\ln\left(\frac{r(r+w)-r^2}{rw}\right)^2 - \frac{(r+w)^2}{r\,(r+w-r)}\right) \qquad \text{(A.16)} \\
&= A_{\pi=0}\left(\frac{r}{r+w}\right) \cdot \frac{-(r+w)^2}{rw} < 0
\end{aligned}$$

Thus, $A_{\pi=0}$ has a maximum at $\gamma = \frac{r}{r+w}$.

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