PRICE LEVEL VOLATILITY AND INCOMPLETE FINANCIAL MARKETS

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Money is one of the most important elements in the modern economy, but it has a critical disadvantage; it is vulnerable to inflation and inflation uncertainty, mainly due to unstable monetary policies or market psychology (sunspots). The main focus of this dissertation is to investigate the impact of financial innovation or fiscal policies on welfare and asset prices when the economy is contaminated by inflation volatility. This dissertation consists of four essays.

The first essay investigates an economy with incomplete financial markets as described by Cass (1989), where there is typically a continuum of equilibria driven by sunspots. In this essay, I define price volatility in a natural way and take it as a parameter of a derived economy. I show that for each level of price volatility, there is a unique regular sunspot-economy. Typically, there is no Pareto ranking among the different sunspot economies. However, I consider a compensation test based on balanced lump-sum tax-transfer plans that are implemented in period 0 and denominated in money or commodities. This test reveals that lower volatility economies are generically Kaldor-Hicks superior to higher volatility economies. The findings imply that Kaldor-Hicks efficiency is achieved through sunspot-stabilizing policies.

The second essay introduces a sunspots-economy where both money and inflation-indexed bond markets are active. The model of the economy is exactly the same as Cass' (1989) GEI model with the addition of the indexed bonds. Mas-Colell (1992) and Goenka and Prechac (2006) have shown that financial markets

can be immune to sunspots by introducing real securities such as inflation-indexed bonds. However, the introduction of these real securities results in a complete shutdown of nominal financial markets. To resolve this unrealistic outcome, I assume that a transaction cost in intermediating indexed bonds exists. This paper shows that the market for money is always active with the transaction costs of indexed bonds. I also show that these bonds have a greater opportunity to be actively traded as the market has higher inflation volatility.

The third essay introduces a two-period monetary general equilibrium model with proportional transaction costs on nominal and inflation-indexed bonds. The main focus of this essay is to investigate the impact of introducing inflation-indexed bonds on nominal interest rates and the welfare of savers and borrowers. I demonstrate that this financial innovation on indexed bonds causes equilibrium interest rates of nominal securities to increase when agents have precautionary saving motives. This result implies that ignoring precautionary motives would underestimate savers' welfare gain and overestimates borrowers' welfare gain from innovation on indexed bonds. I show the main results of this paper by incorporating financial transaction costs and unstable monetary policies in the GE model and provide the rigorous proofs for existence and uniqueness of the equilibrium. I also provide the comparative statics of bond trading volume with respect to asymmetric transaction costs.

A static economy in which nominal taxes and transfers are balanced, as proposed by Balasko and Shell (1993), typically has a continuum of equilibrium money prices. The fourth essay presents a constructive example in which the set of equilibrium money prices is not connected. By allowing negative consumption as a mathematical construct, closed form solutions for equilibrium tax-adjusted income are derived.

BIOGRAPHICAL SKETCH

Minwook Kang was born and raised in Korea. He received his Bachelor of Science in electrical engineering from Seoul National University in Korea. In August 2009, he enrolled as a graduate student in the economics department of Cornell University, majoring in macroeconomics, microeconomics and international economics. He completed his Doctor of Philosophy in economics in May 2013.

To my Father and Mother

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CHAPTER 1

PRICE LEVEL VOLATILITY AND INCOMPLETE MARKETS WITH SUNSPOTS

1.1 Introduction

Sunspots provide explanations of excess volatility of both price levels and allocations.[1] Cass (1989) shows that when markets are incomplete, there is typically a continuum of sunspot equilibria. This paper shows that in the Cass GEI model, a unique regular economy exists for each given measure of price-level volatility. Although each equilibrium exhibits different levels of ex-post price volatility, the equilibria in a given economy are not, in general, Pareto-ordered by price volatility levels.[2] Therefore, this paper also addresses the following questions: 1) Is there a sensible welfare ranking that allows for comparisons among equilibria from different price-level volatilities? 2) If so, what would be the policy implications of this finding?

For question 1, I offer a compensation test based on balanced lump-sum taxes and transfers (balanced tax-transfer plans) that are implemented in period 0 and denominated by commodities or money.[3] This paper compares any two regular economies with the same economic fundamentals but with different price volatilities. Specifically, it demonstrates that there are balanced lump-sum tax plans that

This essay was presented at the Cornell-Penn State Macroeconomics Conference in Spring 2012.

[1] See Cass and Shell (1983) and Shell (1987, 2008).

[2] Goenka and Préchac (2006) and Kajii (2007) show that sunspot equilibria are not, in general, Pareto dominated by non-sunspot equilibria in incomplete financial markets

[3] Whether the taxes and transfers are denominated by commodity or money, the main results of this paper are invariant: if there exist welfare-improving commodity tax plans from one economy to another, there also exist money tax plans, and vice versa.

would allow an economy with lower volatility to be Pareto superior to one with higher volatility. In other words, a lower volatility economy is Kaldor-Hicks superior to a higher volatility economy with a compensation test based on the proposed tax plans.[4,5]

These findings have important policy implications. Several studies have suggested stabilizing policies to eliminate the effects of sunspots on incomplete markets. Four dominant policies have been proven: a) the introduction of new types of nominal securities[6], b) the introduction of as many real securities as the number of goods in each state[7], c) the indexation of nominal bonds in terms of price levels[8] and d) the introduction of options.[9] These policies immunize the economy from sunspot effects; consequently, the outcomes of the equilibria are Pareto efficient. However, efficiency does not imply that all consumers are better off. Recent sunspot literature has shown that many consumers can actually benefit from sunspots.[10] This means that the government can fail to gain consensus in adopting stabilizing policies. However, if balanced lump-sum tax plans are allowed, consumer consensus on sunspot-stabilizing policies can be achieved. Simply stated, Kaldor-Hicks efficiency is reached through sunspot-stabilizing policies.

The remainder of this paper is organized as follows: In Section 2, I introduce the general setting of the model. Section 3 presents the main results in Cass

[4] On the other hand, there are no existing plans that improve welfare in an economy with high volatility relative to one with low volatility. Therefore, the Kaldor-Hicks criterion in this paper satisfies both completeness and transitivity

[5] The concept that "economy A" is Kaldor-Hicks superior to "economy B" in this paper means that there are balanced lump-sum tax plans which make the allocations in "economy A" Pareto superior to those in "economy B". If "A" is Pareto superior to "B", "A" is also Kaldor-Hicks superior to "B". But the reverse is not necessarily true. See Kaldor (1939) and Hicks (1939).

[6] See Cass and Shell (1983) [Proposition 3] and Balasko (1983) [Theorem 1]

[7] See Mas-Colell (1992)

[8] See Goenka and Préchac (2006)

[9] See Antinolfi and Keister(1998) and Kajii(1997).

[10] See Bhattacharya, Guzman and Shell (1998), Goenka and Préchac (2006), Kajii (2007) and Cozzi, Goenka and Shell (2012).

(1989) and then show that a regular economy can be defined within the model. In Section 4, I define a measure of price-level volatility that is invariant to the choice of a numeraire price and a nominal interest rate. Given this measure, the main result of this paper–the existence of welfare improving tax plans from a low volatility economy to a high volatility one–is introduced in Section 5. The proof of the result in Section 5 is shown in Section 6. Finally, concluding remarks are presented in Section 7.

1.2 Model

There are two periods, today and tomorrow, labelled by the superscripts $t = 0, 1$. At date 1, there are two states, $s = \alpha, \beta$ having positive probabilities $0 < \pi^\alpha < 1$ and $\pi^\beta = 1 - \pi^\alpha$, respectively. There are H consumers, labelled by the subscripts $h \in H = \{1, 2, ..., H\}$.[11]

Consumer h's consumption allocation is $x_h = \left(x_h^0, x_h^{1\alpha}, x_h^{1\beta}\right) \in X = \mathbb{R}^3_{++}$ corresponding to price $p = \left(p^0, p^{1\alpha}, p^{1\beta}\right) \gg 0$. His endowment is $e_h = \left(e_h^0, e_h^{1\alpha}, e_h^{1\beta}\right) \in X$ where $e_h^{1\alpha} = e_h^{1\beta} = e_h^1$. Denote by $\mathcal{U}$ the space of $\mathcal{C}^2$ utility functions on $\mathbb{R}^2_{++}$ which are twice differentiable, strictly increasing, strictly concave. This also has the closure of indifference curves contained in $\mathbb{R}^2_{++}$ and satisfies the von Neumann-Morgenstern expected utility hypothesis. Denote by $\mathcal{A}$ the set of characteristics $(v_h, e_h) \in \mathcal{U} \times X$. Consumer h's preferences are

$$u_h(x_h) = u_h\left(x_h^0, x_h^{1\alpha}, x_h^{1\beta}\right) = \pi^\alpha v_h\left(x_h^0, x_h^{1\alpha}\right) + \pi^\beta v_h\left(x_h^0, x_h^{1\beta}\right)$$

Throughout the paper, I assume that there is an incentive for at least two of the

[11]There are two differences between Cass's model and the model here: (1) Cass allows only two households but the model here allows many households, and (2) Cass distinguishes the asset return in the α-state with that in β state and denoted them as r^α and r^β.

consumers to trade, i.e., for $h, h' \in H$,

$$\frac{\partial v_h\left(e_h^0, e_h^1\right)}{\partial x^1} / \frac{\partial v_h\left(e_h^0, e_h^1\right)}{\partial x^0} \neq \frac{\partial v_{h'}\left(e_{h'}^0, e_{h'}^1\right)}{\partial x^1} / \frac{\partial v_{h'}\left(e_{h'}^0, e_{h'}^1\right)}{\partial x^0}$$

This condition implies that the initial endowment is not Pareto efficient.

In a monetary market, there is only one financial instrument (money). m_h denotes consumer h's money holdings. An economic fundamental $\mathcal{E}$ is simply a list $(u_h, e_h) \in \mathcal{A}$, $h \in H$. Denote the space of economic fundamentals by $\mathcal{M}$ where $\mathcal{M} = \prod_{h \in H} \mathcal{A}$. The monetary equilibrium is defined as follows: there are some positive spot prices $p \gg 0$ and associated money holdings m such that each household chooses (x_h, m_h) in the optimization problem, denoted as (PA).

$$\max u_h\left(x_h^0, x_h^{1\alpha}, x_h^{1\beta}\right)$$
$$\text{subject to} \quad \begin{cases} p^0 x_h^0 + m_h \leq p^0 e_h^0 \\ p^{1\alpha} x_h^{1\alpha} \leq p^{1\alpha} e_h^1 + m_h \\ p^{1\beta} x_h^{1\beta} \leq p^{1\beta} e_h^1 + m_h \end{cases} \tag{PA}$$
$$\text{and} \quad x_h \in X$$

each market clears,

$$\sum_h x_h^0 = \sum_h e_h^0,$$
$$\sum_h x_h^{1\alpha} = \sum_h x_h^{1\beta} = \sum_h e_h$$
$$\sum_h m_h = 0.$$

1.3 Equilibrium

Several authors have demonstrated that in a general model of sunspots with incomplete markets, the set of equilibrium allocations takes on a continuum.[12] The

[12] See Cass (1989, 1991, 1992) and Siconolfi (1991). Manuelli and Peck (1992) show that in overlapping generation model with incomplete markets, a continuum of sunspot equilibria can

economy has only one asset (money) but two sunspots states, α and β, at date 1. Therefore, the equilibrium set has one degree of real indeterminacy. This suggests that by adding one "relevant" additional equation into the system of equations for the optimal solution, we can obtain a unique or finitely many equilibria.[13] Cass (1989) chooses the most "relevant" equation as the ratio of two real returns and shows that a finite number of equilibria exist for any given ratio of the two real returns. The two real returns, denoted as $\left(R^{\alpha}, R^{\beta}\right)$ in the monetary economy, are given as

$$\left(R^{\alpha}, R^{\beta}\right) = \left(\frac{1}{p^{1\alpha}}, \frac{1}{p^{1\beta}}\right).$$

The ratio of two real returns, equivalent to the ratio of two prices $p^{1\alpha}$ and $p^{1\beta}$, determines the ratio between excess demand in each state: $x^{1\alpha} - e^1$ and $x^{1\beta} - e^1$. The relationship is derived from budget constraint at date 1:

$$\begin{cases} p^{1\alpha} x_h^{1\alpha} = p^{1\alpha} e^1 + m_h \\ p^{1\beta} x_h^{1\beta} = p^{1\beta} e^1 + m_h \end{cases} \Rightarrow \frac{x_h^{1\beta} - e^1}{x_h^{1\alpha} - e^1} = \frac{p^{1\alpha}}{p^{1\beta}}.$$

Figure 1.1(a) represents the excess-demand domains of both lenders and borrowers. The figure should be three-dimensional including the excess demand of x^0, but we can imagine that the three-dimensional figure is projected onto a two-dimensional space. Once the return-line is determined, there will be a unique or finitely many equilibria point(s) for both the lender ($m_h > 0$) and the borrower ($m_h < 0$) on the line.[14] The lender's allocation point is located in the northeast area while the borrower's point is in the southwest area. By market clearing, the two points are symmetric to $(0, 0)$ assuming that there are only two consumers ($H = 2$) in the economy.

be interpreted as the limiting case of economies overreacting to small shocks to fundamentals.

[13] One requirement is that the gradient of the "relevant" equation must be linearly independent from those of the system of equations at the equilibrium prices or allocations.

[14] In the monetary market, the asset buyer and seller can be translated into the money lender and the money borrower, respectively.

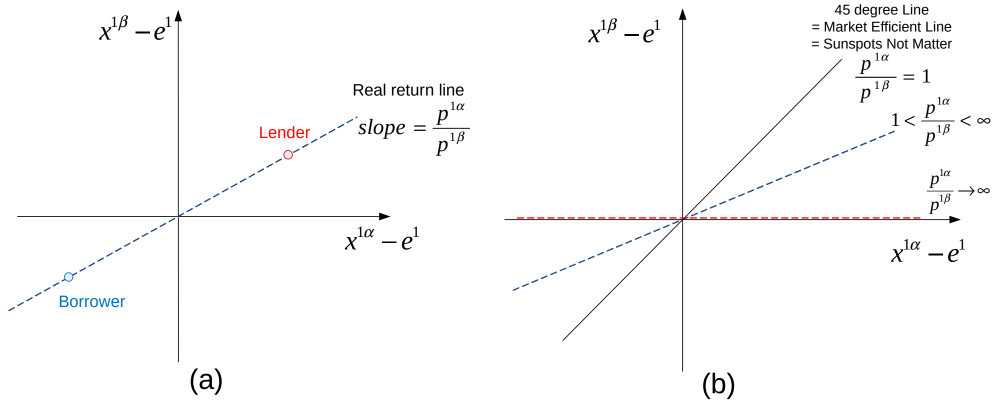

Figure 1.1: Real return lines

A relative price between $p^{1\alpha}$ and $p^{1\beta}$ is defined as

$$\mathcal{P} = \frac{p^{1\beta}}{p^{1\alpha}} \tag{1.1}$$

where $\mathcal{P} > 1$, so we can say that relative to state α (no sunspots), state β is inflationary.[15] There is no price-level volatility if $\mathcal{P} = 1$. For each fixed level of a relative price, there is the corresponding unique return-line shown in Figure 1.1. This paper later proves that a unique or finitely many equilibria exist given each ratio of a relative price. Simply stated, the economy given $\mathcal{P} \in (0, \infty)$ is regular and determinate. For the convenience of proofs and better intuition, I define the equivalent maximization problem in Definition 1. I label the original maximization problem as (PA) and the equivalent problem as (PB). The equivalent maximization problem (PB) is given as

Definition 1.1 *Given (q^0, q), consumer h's equivalent maximization problem (PB) is defined as*

$$\max U_h(z_h^0, B_h)$$

$$\textit{subject to} \;\; q^0 z_h^0 + qB_h \leq 0 \tag{PB}$$

[15] See Bhattacharya, Guzman and Shell (1998)

$$where \quad \frac{R^\beta}{R^\alpha} = \frac{1}{\mathcal{P}} \ .$$

where

$$U_h(z_h^0, B_h) = u_h(e_h^0 + z_h^0, e_h^1 + R^\alpha B_h, e_h^1 + R^\beta B_h)$$

The following lemma shows that for the given relative price $\mathcal{P} \in \mathbb{R}_{++}$, (PB) is equivalent to (PA).

Lemma 1.1 *Given* $\mathcal{P} = p^{1\beta}/p^{1\alpha}$, (PA) *and* (PB) *are equivalent where*

$$x_h^0 = e_h^0 + z_h^0, \quad x_h^{1\alpha} = e_h^1 + R^\alpha B_h, \quad x_h^{1\beta} = e_h^1 + R^\beta B_h$$
$$m_h = qB_h, \quad p^{1\alpha} = \frac{q}{R^\alpha} \quad and \quad p^{1\beta} = \frac{q}{R^\beta}$$

Proof. The argument simply involves comparing first-order conditions for the two problems. Applying the Kuhn-Tucker theorem for concave programming, the following necessary and sufficient conditions can be derived. The Lagrangian of (PA) is given by

$$\begin{aligned} L^{PA} &= u_h(x_h) + \lambda_h^0 \left(p^0 e_h^0 - p^0 x_h^0 - m_h\right) \\ &\quad + \lambda_h^{1\alpha}\left(p^{1\alpha}e_h^1 - p^{1\alpha}x_h^{1\alpha} + m_h\right) + \lambda_h^{1\beta}\left(p^{1\beta}e_h^1 - p^{1\beta}x_h^{1\beta} + m_h\right) \end{aligned}$$

The system of equations from the first-order conditions, budget constraints, and relative price are

$$\left.\begin{array}{c} \frac{\partial u_h(x_h)}{\partial x_h^0} = \lambda_h^0 p^0 \\ \frac{\partial v_h\left(x_h^0, x_h^{1s}\right)}{\partial x_h^{1s}} = \lambda_h^{1s}\frac{p^{1s}}{\pi^s} \quad \text{for } s = \alpha, \beta \\ \left(\lambda_h^{1\alpha} + \lambda_h^{1\beta}\right) = \lambda_h^0 \\ \left(\Rightarrow \frac{\pi^\alpha}{p^{1\alpha}}\frac{\partial v_h\left(x_h^0, x_h^{1\alpha}\right)}{\partial x_h^{1\alpha}} + \frac{\pi^\beta}{p^{1\beta}}\frac{\partial v_h\left(x_h, x_h^{1\beta}\right)}{\partial x_h^{1\beta}} = \lambda_h^0\right) \\ -p^0\left(x_h^0 - e_h^0\right) = m_h \\ p^{1s}\left(x_h^{1s} - e_h^1\right) = m_h \quad \text{for } s = \alpha, \beta. \\ p^{1\beta} = \mathcal{P}p^{1\alpha}. \end{array}\right\} \qquad (1.2)$$

The Lagrangian of (PB) is given by

$$L^{PB} = u_h(e_h^0 + z_h^0, e_h^1 + R^\alpha B_h, e_h^1 + R^\beta B_h) - \mu_h \left(q^0 z_h^0 + qB_h \right).$$

The system of equations from the first-order conditions and the budget constraints are

$$\left.\begin{array}{c} \frac{\partial u_h(x_h)}{\partial z_h^0} \left(= \frac{\partial u_h(x_h)}{\partial x_h^0} \right) = \mu_h q^0 \\ \pi^\alpha \frac{\partial v_h \left(e_h^0 + z_h^0, e_h^1 + R^\alpha B_h \right)}{\partial B_h} + \pi^\beta \frac{\partial v_h \left(e_h^0 + z_h^0, e_h^1 + R^\beta B_h \right)}{\partial B_h} = \mu_h q \\ \left(\Rightarrow \pi^\alpha R^\alpha \frac{\partial v_h \left(e_h^0 + z_h^0, e_h^1 + R^\alpha B_h \right)}{\partial x_h^{1\alpha}} + \pi^\beta R^\beta \frac{\partial v_h \left(e_h^0 + z_h^0, e_h^1 + R^\beta B_h \right)}{\partial x_h^{1\beta}} = \mu_h q \right) \\ q^0 z_h^0 + qB_h = 0 \\ \frac{R^\beta}{R^\alpha} = \frac{1}{\mathcal{P}} \end{array}\right\} \quad (1.3)$$

Finally, we need to show that the two systems of equations (1.2) and (1.3) are equivalent when choosing

$$p^0 = q^0, \lambda_h^0 = \mu_h,$$

$$p^{1s} = \frac{q}{R^s} \quad \text{for } s = \alpha, \beta$$

$$\lambda_h^{1s} = \pi^s \mu_h \quad \text{for } s = \alpha, \beta$$

$$x_h^0 = e_h^0 + z_h^0, \quad x_h^{1\alpha} = e_h^1 + R^\alpha B_h, \quad x_h^{1\beta} = e_h^1 + R^\beta B_h, \quad m_h = qB_h,$$

and

$$\frac{p^{1\beta}}{p^{1\alpha}} = \frac{R^\alpha}{R^\beta} = \frac{1}{\mathcal{P}}.$$

■

In the equivalent problem, $R = \left(R^\alpha, R^\beta \right)$ is a function of $\mathcal{P}$ and the mean $E[R]$ is fixed as a constant. The mean of R does not affect equilibrium allocations. Only the relative price $\mathcal{P}$ matters in equilibrium allocations. From Lemma 1, we can prove the following proposition.

Proposition 1.1 *There is a generic set of fundamentals $\mathcal{M}^* \subset \mathcal{M}$ on which any economy, specified by a pair $(\mathcal{P}, \mathcal{E}) \in \mathbb{R}_{++} \times \mathcal{M}^*$, is regular.*

Proof. In the equivalent maximization problem, the utility function is given by

$$U_h\left(z_h^0, B_h\right) = \pi^\alpha v_h(e_h^0 + z_h^0, e_h^1 + R^\alpha B_h) + \pi^\beta v_h(e_h^0 + z_h^0, e_h^1 + R^\beta B_h).$$

We can prove that $U_h\left(z_h^0, B_h\right)$ is continuous, strictly quasi-concave and strictly increasing in (z_h^0, B_h). The three properties of U_h are trivially inherited from v_h. Note that this has now become a conventional equilibrium problem with two commodities. Therefore, the economy is regular and determinate generically in $\mathcal{M}$.

Remark 1.1 *Geanakoplos and Polemarchakis (1986) proved the existence and regularity of the conventional 2-period model with real assets based on Debreu (1970). We can also use their proof directly in our case.*

■

The next step is to compare the two economies with different levels of relative prices but still with the same economic fundamentals. Figure 1.1(b) shows the relationship between relative prices and their corresponding return-lines. It is clear that as the economy has higher relative prices where $\mathcal{P} \geq 1$, the return-line will deviate more from the 45 degree line. This implies that all households, including both lenders and borrowers, will feel more risk averse in a higher relative price $\mathcal{P}$ where $\mathcal{P} \geq 1$.

1.4 Price Level Volatility

Let $\overline{\sigma} = \sigma(\widetilde{p}^1)/E(\widetilde{p}^1)$, where $\sigma(\widetilde{p}^1)$ and $E(\widetilde{p}^1)$ are respectively the standard deviation and the mean of prices. The measure is invariant to scaling changes in spot prices, which can be caused by the choice of the numeraire. Simply stated, $\overline{\sigma}$ is not affected by scaling changes in $(p^{1\alpha}, p^{1\beta})$; it depends only on the ratio $(p^{1\beta}/p^{1\alpha})$.[16] Thus, $\overline{\sigma}$ is well-defined:

$$\overline{\sigma} = \frac{\sigma(\widetilde{p}^1)}{E(\widetilde{p}^1)} \geq 0 \tag{1.4}$$

If $\overline{\sigma} = 0$, the economy is a non-sunspot economy (equivalent to certainty economy), but it is a proper sunspot economy if $\overline{\sigma} > 0$.

Now, we can construct a direct relationship between price-level volatility and its corresponding derived regular economy.

Corollary 1.1 *There is a generic set of fundamentals $\mathcal{M}^* \subset \mathcal{M}$ on which any economy, specified by a pair $(\overline{\sigma}, \mathcal{E}) \in \mathbb{R}_+ \times \mathcal{M}^*$, is regular.*

Proof. Price-level volatility $\overline{\sigma}$ is given by

$$\overline{\sigma} = \frac{\sigma(\widetilde{p}^1)}{E(\widetilde{p}^1)} = \frac{\sqrt{\pi^\alpha \pi^\beta}\left|p^{1\alpha} - p^{1\beta}\right|}{\pi^\alpha p^{1\alpha} + \pi^\beta p^{1\beta}} = \frac{\sqrt{\pi^\alpha \pi^\beta}\left|1 - \mathcal{P}\right|}{\pi^\alpha + \pi^\beta \mathcal{P}}.$$

Volatility $\overline{\sigma}$ is strictly increasing in $\mathcal{P} \geq 1$. As $\mathcal{P} \to \infty$, $\sigma(\widetilde{p}^1)/E(\widetilde{p}^1) \to \sqrt{\frac{\pi^\alpha}{\pi^\beta}}$. By Proposition 1, it is clear that there is a corresponding regular economy to each level of volatility $\overline{\sigma}$. ■

1.5 Welfare-improving lump-sum tax plans: the Compensation test

The purpose of this section is to introduce a "reasonable" way to compare economies with different levels of price volatility. A simple and direct way might

[16] It is assumed that $p^{1\beta} \geq p^{1\alpha}$ throughout this paper.

be to check for the Pareto superiority of the equilibria from comparable markets. One might conjecture that an economy with lower price volatility always admits Pareto superior equilibria compared to an economy with higher price volatility, given the same economic fundamentals. The answer can be true if the value of money is invariant in the changes of market beliefs (price-level volatility). However, the value of money is not generally the same for two different market beliefs even if both economies have the same economic fundamentals. Several papers have shown that allocations from a non-sunspot economy are not Pareto superior to those from a sunspot economy even though the former is Pareto efficient and the latter is not.[17] However, this paper shows the existence of complete welfare ranking among equilibria with different levels of price volatility by introducing a compensation test based on balanced lump-sum tax plans. The space of balanced lump-sum tax plans is defined as

$$T = \left\{ (\tau_1, ..., \tau_H) \in \mathbb{R}^H | \sum_{h \in H} \tau_h = 0 \right\}.$$

A set of welfare-improving tax plans is defined below.

Definition 1.2 $(\overline{\sigma}^A, \mathcal{E})$ *is Kaldor-Hicks superior to* $(\overline{\sigma}^B, \mathcal{E})$*, i.e.,*

$$(\overline{\sigma}^A, \mathcal{E}) \succ_{KH} (\overline{\sigma}^B, \mathcal{E}),$$

[17] One simple example in Goenka and Préchac (2006) is the case where there are two consumers who have the same expected utility functions $v_1 = v_2 = \log(x^0) + \log(x^1)$ and whose endowments are $e_1 = (1, 0)$ and $e_2 = (0, 1)$. In this case, as the price-level volatility increases, the second consumer's utility value increases. The intuition is as follows: For a high level of price volatility, the real return of assets becomes very small in an inflation state. If the inflation state is realized, the first consumer will end up getting a very small amount of the second-period good and consequently her utility will be considerably damaged. Therefore, the first consumer's demand for the asset will be higher to protect the inflation state against the low return in a higher level of price volatility. The high demand of the first consumer results in a high price (value) of the asset. Finally, the second consumer, who sells the asset to the first one, obtains more income as the asset price goes higher. This positive income effect for the second consumer outweighs the negative risk effects from the high level of price volatility. (See the Appendix for more details about the example.)

if there exist(s) $\tau \in T$ *such that the economy* $(\overline{\sigma}^A, \mathcal{E})$ *with the tax-transfer plan* τ *is Pareto superior to the economy* $(\overline{\sigma}^B, \mathcal{E})$.

The tax-transfer plan τ is applied to the economy $(\overline{\sigma}^A, \mathcal{E})$. Then, the budget constraints at date 0 is modified to be

$$p^0 x_h^0 + m_h \le p^0 e_h^0 - \tau_h \qquad\qquad h = 1, ..., H \ \cdot \tag{1.5}$$

The budget constraints in the second period are invariant. Assuming that $p^0 = 1$, we can interpret that the taxes and transfers are denominated by the first-period commodities.

The following is the main result of this paper.

Proposition 1.2 *Generically in* $\mathcal{M}$, *if* $\mathcal{E} \in \mathcal{M}$ *and* $\overline{\sigma}^A < \overline{\sigma}^B$,

$$(\overline{\sigma}^A, \mathcal{E}) \succ_{KH} (\overline{\sigma}^B, \mathcal{E})$$

and

$$(\overline{\sigma}^B, \mathcal{E}) \not\succ_{KH} (\overline{\sigma}^A, \mathcal{E}).$$

Proof. See Section 6. ■

The proposition is proven in the next section. This proposition implies that an economy with lower volatility is superior in a welfare sense to one with higher volatility, since (1) there exists a balanced tax plan which allows the former to be "superior" to the latter, and (2) at the same time, there does not exist such a plan in the other way around. In other words, a lower volatility economy is Kaldor-Hicks superior to a higher volatility economy with a compensation test based on balanced tax plans. The Kaldor-Hicks criterion in this paper satisfies both completeness and transitivity.

Even though the proof is not mathematically trivial, the intuition is simple. Although each consumer experiences different wealth changes driven by price volatility changes (price ratio changes), the aggregate wealth changes equal zero by market clear conditions. Therefore, a social planner can always find lump-sum tax plans which eliminate the wealth effects from one price volatility level to another level. With these tax plans, only the direct risk effect (the risk effect from the deviation of the asset-return line depicted in Figure **??**) will matter in consumers' utility values. (See Section 6 for more details.)

The results of Proposition 2 also make it possible to compare sunspot equilibria ($\overline{\sigma} > 0$) and non-sunspot (or certainty) equilibria ($\overline{\sigma} = 0$). Where sunspot equilibria are not Pareto dominated by certainty equilibria, the government can easily fail to arrive at a consensus in sunspot-stabilizing policies. The following corollary guarantees that there are welfare-improving tax plans from a sunspot to a non-sunspot economy, thus allowing for consensus of sunspot-stabilizing policies:

Corollary 1.2 *Generically in $\mathcal{M}$, if $\mathcal{E} \in \mathcal{M}$,*

$$(\mathcal{E}, 0) \succ_{KH} (\mathcal{E}, \overline{\sigma}) \quad if \quad \overline{\sigma} > 0$$

Proof. Trivial from Proposition 2. ■

1.6 Wealth Analysis

In this section, I investigate how utility levels respond to the changes of market beliefs. Market beliefs in price levels are associated with utility levels in two different ways. First, an increase in the relative price (assuming that $\mathcal{P} \geq 1$) can cause all of the agents to feel more risk averse, which imposes negative effects on the utility level. Second, the relative money price affects the equilibrium prices, which, of course, are factors in determining agents' utility levels. For the second effect, both possibilities of gaining or losing wealth are available. The first way is the direct (risk) effects and the second way is the indirect (income) effects. The distinction between these two effects is helpful for understanding a sunspot economy and to prove the main results in this paper. All of the analyses are based on the equivalent maximization problem (PB). I also assume that $\mathcal{P} \geq 1$ in this section.

Taking the total derivative in $U_h\left(z_h^0, B_h\right)$ with respect to $\mathcal{P}$, we can obtain

$$\frac{dU_h}{d\mathcal{P}} = \underbrace{\frac{\partial U_h}{\partial \mathcal{P}}}_{\text{Direct (Risk) Effect}} + \underbrace{\frac{\partial U_h}{\partial z_h^0}\frac{dz_h^0}{d\mathcal{P}} + \frac{\partial U_h}{\partial B_h}\frac{dB_h}{d\mathcal{P}}}_{\text{Trading Effect}}. \tag{1.6}$$

There are two terms in eqn (1.6). The second term, "trading effect", represents the utility value changes driven by the consumption changes. Later in this section, I show that this trading effect is the same as an indirect (income) effect, which is driven by the change of asset price q.

What is a direct risk effect represented by the first term in eqn (1.6)? As shown in Figure (1.2), as $\mathcal{P}$ increases, the lender's equilibrium point moves from A to C and the borrower's equilibrium point moves from a to c. The direct effect is the change of utility values from A(a) to B(b). Assuming that the mean of real returns is constant in relative price, i.e., "$\pi^\alpha R^\alpha(\mathcal{P}) + \pi^\beta R^\beta(\mathcal{P}) = \text{constant}$," the household has the same expected value of $\widetilde{x}^1$ in both points A(a) and B(b). In addition, since there is no trading effect from A(a) to B(b), the amount of x^0 does not change. Therefore, we can prove the following lemma:

Lemma 1.2 *The direct effect is negative.* $\left(\frac{\partial U_h}{\partial \mathcal{P}} < 0 \text{ where } \mathcal{P} \geq 1\right)$

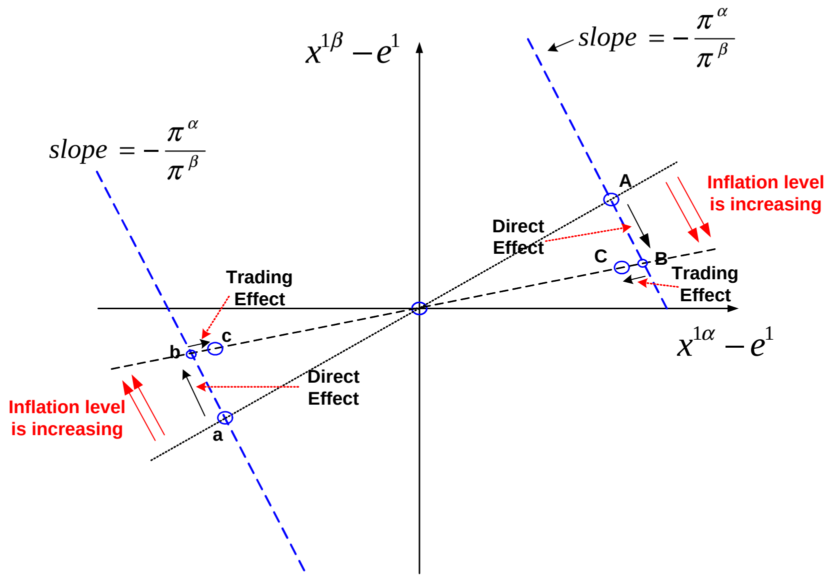

Figure 1.2: Income and direct effects

Proof. Without loss of generality, let's assume that $\pi^\alpha R^\alpha + \pi^\beta R^\beta = 1$. Since $\frac{R^\beta}{R^\alpha} = \frac{1}{\mathcal{P}}$, R^α *and* R^β *are*

$$R^\alpha = \frac{\mathcal{P}}{\mathcal{P}\pi^\alpha + \pi^\beta} \text{ and } R^\beta = \frac{1}{\mathcal{P}\pi^\alpha + \pi^\beta}.$$

$U_h\left(z_h^0, B_h\right)$ is

$$U_h = \pi^\alpha v_h\left(e_h^0 + z_h^0, e_h^1 + \frac{\mathcal{P}}{\mathcal{P}\pi^\alpha + \pi^\beta} B_h\right) + \pi^\beta v_h\left(e_h^0 + z_h^0, e_h^1 + \frac{\mathcal{P}}{\mathcal{P}\pi^\alpha + \pi^\beta} B_h\right),$$

and $\partial U / \partial \mathcal{P}$ is

$$\begin{aligned}
\frac{\partial U}{\partial \mathcal{P}} &= \pi^\alpha \frac{\partial v_h\left(x_h^0, x_h^{1\alpha}\right)}{\partial x^{1\alpha}} B_h \frac{\partial R^\alpha}{\partial \mathcal{P}} + \pi^\alpha \frac{\partial v_h\left(x_h^0, x_h^{1\beta}\right)}{\partial x^{1\beta}} B_h \frac{\partial R^\beta}{\partial \mathcal{P}} \\
&= \pi^\alpha \frac{\partial v_h\left(x_h^0, x_h^{1\alpha}\right)}{\partial x^{1\alpha}} B_h \frac{\pi^\beta}{\left(\mathcal{P}\pi^\alpha + \pi^\beta\right)^2} + \pi^\beta \frac{\partial v_h\left(x_h^0, x_h^{1\beta}\right)}{\partial x^{1\beta}} B_h \frac{-\pi^\alpha}{\left(\mathcal{P}\pi^\alpha + \pi^\beta\right)^2} \\
&= \frac{\pi^\alpha \pi^\beta}{\left(\mathcal{P}\pi^\alpha + \pi^\beta\right)^2} B_h \left(\frac{\partial v_h\left(x_h^0, x_h^{1\alpha}\right)}{\partial x^{1\alpha}} - \frac{\partial v_h\left(x_h^0, x_h^{1\beta}\right)}{\partial x^{1\beta}}\right)
\end{aligned}$$

Lender $(B_h > 0)$: Since $R^\alpha > R^\beta$ and $B_h > 0$, $x_h^{1\alpha} > x_h^{1\beta}$. Since v_h is strictly concave, $\frac{\partial v_h\left(x_h^0, x_h^{1\alpha}\right)}{\partial x^{1\alpha}} < \frac{\partial v_h\left(x_h^0, x_h^{1\beta}\right)}{\partial x^{1\beta}}$. Therefore,

$$B_h \left(\frac{\partial v_h \left(x_h^0, x_h^{1\alpha}\right)}{\partial x^{1\alpha}} - \frac{\partial v_h \left(x_h^0, x_h^{1\beta}\right)}{\partial x^{1\beta}} \right) < 0.$$

Borrower ($B_h < 0$): Since $R^\alpha > R^\beta$ and $B_h < 0$, $x_h^{1\alpha} < x_h^{1\beta}$. Since v_h is strictly concave, $\frac{\partial v_h\left(x_h^0, x_h^{1\alpha}\right)}{\partial x^{1\alpha}} > \frac{\partial v_h\left(x_h^0, x_h^{1\beta}\right)}{\partial x^{1\beta}}$. Therefore,

$$B_h \left(\frac{\partial v_h \left(x_h^0, x_h^{1\alpha}\right)}{\partial x^{1\alpha}} - \frac{\partial v_h \left(x_h^0, x_h^{1\beta}\right)}{\partial x^{1\beta}} \right) < 0.$$

■

The source of the trading effect is the income change driven by the change of prices. One interpretation is that a change in the asset price q positively or negatively affects consumer h's income, and consequently, his trading will be adjusted according to the income change. Choosing q^0 as the numeraire price, i.e., setting $q^0 = 1$, the "income effect" from the change of the asset price is defined as

$$\text{Income Effect} = -\frac{dq}{d\mathcal{P}} B_h.$$

The following lemma shows that the trading effect is actually the income effect:

Lemma 1.3 *Trading effect = Income Effect*[18]*, i.e.,*

$$\frac{1}{\lambda_h} \left(\frac{\partial U_h}{\partial z_h^0} \frac{dz_h^0}{d\mathcal{P}} + \frac{\partial U_h}{\partial B_h} \frac{dB_h}{d\mathcal{P}} \right) = -\frac{dq}{d\mathcal{P}} B_h$$

Proof. *The Lagrangian of the maximization problem is defined as*

$$L_h = U_h \left(z_h^0, B_h; \mathcal{P}\right) - \lambda_h \left(z_h^0 + qB_h \right)$$

[18] Here, "$\frac{1}{\lambda_h} \left(\frac{\partial U_h}{\partial z_h^0} \frac{dz_h^0}{dP} + \frac{\partial U_h}{\partial B_h} \frac{dB_h}{dP} \right)$" can be interpreted as a trading effect on wealth rather than on utility value.

The first order condition is

$$\frac{\partial U_h}{\partial z_h^0} = \frac{1}{q}\frac{\partial U_h}{\partial B_h} = \lambda_h > 0 \tag{1.7}$$

Taking the total derivative of $z_h^0 + qB_h = 0$ *with respect to* $\mathcal{P}$*, we get*

$$\frac{dz_h^0}{d\mathcal{P}} + \frac{dq}{d\mathcal{P}}B_h + q\frac{dB_h}{d\mathcal{P}} = 0. \tag{1.8}$$

From equations (1.7) and (1.8), we get

$$\frac{1}{\lambda_h}\left(\frac{\partial U_h}{\partial z_h^0}\frac{dz_h^0}{d\mathcal{P}} + \frac{\partial U_h}{\partial B_h}\frac{dB_h}{d\mathcal{P}}\right) = \frac{1}{\lambda_h}\left(\lambda_h\frac{dz_h^0}{d\mathcal{P}} + \lambda_h q\frac{dB_h}{d\mathcal{P}}\right) = -\frac{dq}{d\mathcal{P}}B_h.$$

■

During the transition from lower volatility to higher volatility, the direct effect is always negative. However, the income effect can be negative or positive. When the positive income effect outweighs the negative direct effect, the household's utility is increasing in relative price, creating an interesting case where the household prefers the greater price volatility.

By Lemma 3, $\frac{1}{\lambda_h}\frac{dU_h}{d\mathcal{P}}$ is decomposed into income effects (which are the same as trading effects) and direct effects. Now, the unit of both effects are "wealth/relative price($\mathcal{P}$)."[19]

$$\frac{1}{\lambda_h}\frac{dU_h}{d\mathcal{P}} = \underbrace{\frac{1}{\lambda_h}\frac{\partial U_h}{\partial \mathcal{P}}}_{\text{Direct Effect}} + \underbrace{-\frac{dq}{d\mathcal{P}}B_h}_{\text{Income Effect}}. \tag{1.9}$$

In a regular economy, the set of competitive equilibria is a continuously differentiable function of the endowment allocation. Here we assume that the endowment

[19] To the best of my knowledge, the term normalized ordinal utility, which is $\frac{1}{\lambda_h}\frac{dU_h}{d\mathcal{P}}$ here, in the welfare analysis was first suggested by Donsimoni and Polemarchakis (1994). They showed that if there is no change in the aggregate endowment in a regular economy, no endowment redistribution can affect the aggregate change in the normalized utility. They called the changes of individuals' utility through redistribution the "relative price effects" which are basically the same as the "income effects" in our model.

is a differentiable function of relative price $\mathcal{P}$. Specifically, the endowment allocation is replaced with $\left(e_h^0 - \tau_h\left(\mathcal{P}\right), e_h^1, e_h^1\right)_{h=1}^H$ where $\tau_h(\mathcal{P})$ is a differentiable function of $\mathcal{P}$. The following lemma shows how the utility levels are affected by the endowment function:

Lemma 1.4 *Assuming that the endowment is defined as* $\left(e_h^0 - \tau_h\left(\mathcal{P}\right), e_h^1, e_h^1\right)_{h=1}^H$, $\frac{1}{\lambda_h}\frac{dU_h}{d\mathcal{P}}$ *is given by*

$$\frac{1}{\lambda_h}\frac{dU_h}{d\mathcal{P}} = \underbrace{\frac{1}{\lambda_h}\frac{\partial U_h}{\partial \mathcal{P}}}_{\text{Direct (Risk) Effect}} + \underbrace{-\frac{dq}{d\mathcal{P}}B_h}_{\text{Income Effect}} - \frac{\partial \tau_h(\mathcal{P})}{\partial \mathcal{P}}$$

Proof. We can prove this lemma in the same way as in Lemma 3 by replacing eqn (1.8) with

$$\frac{dz_h^0}{d\mathcal{P}} + \frac{dq}{d\mathcal{P}}B_h + q\frac{dB_h}{d\mathcal{P}} + \frac{\partial \tau_h(\mathcal{P})}{\partial \mathcal{P}} = 0.$$

■

This Lemma is used in proving Propositions 2. $\left[\tau_h\left(\mathcal{P}\right)\right]_{h=1}^H$ can be interpreted as balanced lump-sum tax plans in Propositions 2. Finally, the proof of Proposition 2 is presented.

Proof. **(Proof of Proposition 2)** Assuming that $\mathcal{P} \geq 1$, we compare two economies: one is an economy with $\mathcal{P}^*$ and the other is an economy with $\mathcal{P}^+$ where $\mathcal{P}^+ > \mathcal{P}^* \geq 1$. We need to show that there is a possible wealth redistribution in the economy $(\mathcal{P}^*, \mathcal{E}) \in \mathbb{R}_{++} \times \mathcal{M}^*$ which leads to a Pareto superior equilibrium to that in the economy $(\mathcal{P}^+, \mathcal{E}) \in \mathbb{R}_{++} \times \mathcal{M}^*$. Without loss of generality, let's choose p^0 as a numeraire price, i.e., $p^0 = 1$. Then, consumer h's budget constraint with transfers $[\tau_h(\mathcal{P})]_{h=1}^H$ is

$$x_h^0 + m_h = e^0 - \tau_h(\mathcal{P}). \tag{1.10}$$

where $\tau_h(\mathcal{P})$ is continuously differentiable in $\mathcal{P}$.

The corresponding budget constraint of the equivalent problem (PB) is

$$z_h^0 + qB_h + \tau_h(\mathcal{P}) = 0. \tag{1.11}$$

From Lemma 4, we can get:

$$\frac{1}{\lambda_h}\frac{dU_h}{d\mathcal{P}} = \frac{1}{\lambda_h}\frac{\partial U_h}{\partial \mathcal{P}} - q\frac{dB_h}{d\mathcal{P}} - \frac{\partial \tau_h(\mathcal{P})}{\partial \mathcal{P}}$$

Designing the tax plan as

$$\frac{\partial \tau_h(\mathcal{P})}{\partial \mathcal{P}} = -\frac{dq}{d\mathcal{P}}B_h, \tag{1.12}$$

$$\tau_h(\mathcal{P}^+) = 0 \quad \text{and} \quad \tau_h(\mathcal{P}^*) = \int_{\mathcal{P}^+}^{\mathcal{P}*}\left(\frac{\partial \tau_h(\mathcal{P})}{\partial \mathcal{P}}\right)d\mathcal{P},$$

The summation of the transfers is equal to zero by the market clearing condition of B_h:

$$\begin{aligned}\textstyle\sum_h \tau_h(\mathcal{P}^*) &= \textstyle\sum_h \int_{\mathcal{P}^+}^{\mathcal{P}*}\left(\frac{\partial \tau_h(\mathcal{P})}{\partial \mathcal{P}}\right)d\mathcal{P} \\ &= \textstyle\int_{\mathcal{P}^+}^{\mathcal{P}*}\left(\sum_h \frac{\partial \tau_h(\mathcal{P})}{\partial \mathcal{P}}\right)d\mathcal{P} \\ &= \textstyle\int_{\mathcal{P}^+}^{\mathcal{P}*}\left(\frac{dq}{d\mathcal{P}}\sum_h B_h\right)d\mathcal{P} = 0.\end{aligned}$$

With the tax plan, the utility value change is the same as the risk effect:

$$\frac{1}{\lambda_h}\frac{dU_h}{d\mathcal{P}} = \frac{1}{\lambda_h}\frac{\partial U_h}{\partial \mathcal{P}}.$$

By Lemma 2, the risk effect $\frac{1}{\lambda_h}\frac{\partial U_h}{\partial \mathcal{P}}$ is negative. Therefore, with the lump-sum transfer plan $[\tau_h(\mathcal{P}^*)]_{h=1}^H$, complete welfare improvement from the relative price $\mathcal{P}^+$ to $\mathcal{P}^*$ is achieved. That is

$$\frac{1}{\lambda_h}\frac{dU_h}{d\mathcal{P}} < 0 \text{ for all } \mathcal{P} \in \left[\mathcal{P}^+, \mathcal{P}^*\right] \text{ and } h = 1, 2, 3..., H.$$

■

Remark 1.2 *The proof is based on the assumption that the set of competitive equilibria is a continuously differentiable function of the endowment allocation* $(e_h^0 - \tau_h(\mathcal{P}), e_h^1, e_h^1)_{h=1}^H$. *Therefore, the proof is not applicable to singular economies.*

Remark 1.3 *Here, I suggest a new approach for the proof of Proposition 6. The feasible allocation set is determined on the return-line of the corresponding relative price. On the feasible set, any equilibrium is constrained Pareto optimal, which is the same idea in the First Welfare Theorem. (See Diamond 1967.) Second, we know that the feasible set with the higher volatility is dominated by a set with lower volatility in the sense that for any allocation in the former, there always exists a Pareto superior allocation, which is constrained Pareto optimal. Now, the "constrained" version of the Second Welfare Theorem plays a crucial role in finishing the proof. The theorem asserts that for any "constrained" Pareto allocation, there exists a lump-sum wealth transfer to achieve the allocation in a competitive market. This completes the proof.*

In the proof, I introduce one possible tax plan: the amount of each consumer's tax or transfer is the same as that of the income effect. However, they should not necessarily be the same to satisfy welfare supremacy in an economy with a lower level of price volatility. For example, even though a greater amount of tax is collected than the individual's income effect, the consumer can still get higher utility levels in a lower level of price volatility because of the strictly positive direct risk effect. Considering the risk effect, there will be a continuum of welfare-improving tax plans satisfying the following two conditions:

$$\frac{\partial \tau_h(\mathcal{P})}{\partial \mathcal{P}} > \frac{1}{\lambda_h}\frac{\partial U_h}{\partial \mathcal{P}} - \frac{dq}{d\mathcal{P}} B_h \quad \text{for all } h.$$

and

$$\textstyle\sum_h \frac{\partial \tau_h(\mathcal{P})}{\partial \mathcal{P}} = 0.$$

1.7 Conclusion

The main difficulty in research on incomplete markets is that indeterminacy of equilibrium is usually inevitable. This problem is solved in this paper with the introduction of a measure of price-level volatility. This paper proposes a method by which indeterminate economies can be reinterpreted as derived regular economies for which price-level volatility is a parameter. I show that economies with different price volatility levels could be ranked by a compensation test based on balanced tax plans. This finding implies that sunspot-stabilizing policies are Kaldor-Hicks efficient.

CHAPTER 2

SUNSPOTS AND INFLATION-INDEXED BONDS

2.1 Introduction

Sunspots (extrinsic uncertainty) provide explanations of excess volatility for both the price level and allocations.[1] Cass and Shell's (1983) seminal paper investigates sunspot equilibria with excess volatility, by mainly focusing on market situations in which some consumers are restricted from full market participation. Cass (1989) further explains that incomplete financial markets can also allow a continuum of sunspot equilibria. Kang (2012) also shows that for each level of price volatility (inflation volatility), there is a unique regular sunspot-economy as in Cass's (1989) GEI model. This paper introduces a sunspot-economy where both money and inflation-indexed bond markets are available. The model of the economy is exactly the same as the Cass GEI model with the addition of the indexed bonds.

Several theoretical studies have attempted to understand the role of indexed bonds in an economy with inflation volatility, but the explanation about the source of inflation volatility varies within the literature. For example, Magill and Quinzii (1997) assume that inflation volatility is from monetary shocks while Geanakoplos (2005) assumes that it is from intrinsic endowment shocks. This paper assumes that inflation volatility comes from extrinsic shocks (sunspots).

With future inflation uncertainty, nominal securities are unsafe and relatively real securities will be more attractive to risk-averse consumers. Consequently, the introduction of real securities to an economy with inflation volatility can cause the

This essay was presented at the Cornell Macroeconomics Seminar in Fall 2012.

[1] See Cass and Shell (1983) and Shell (1987, 2008).

monetary market to be less active. Mas-Colell (1992) and Goenka and Préchac (2006) have separately shown that the introduction of real securities or inflation-indexed bonds to a sunspot economy can result in a complete shutdown of nominal financial markets.[2] With riskless securities, there is no incentive for consumers to take any unnecessary risks by trading in nominal assets.[3] In reality, the indexed bonds are being used as minor supplements for money, not as the main financial instrument in monetary markets. This problem can be resolved by introducing a transaction cost for intermediating indexed bonds. The transaction cost represents the inefficiency of indexed bonds in contrast to the efficiency of money as a financial instrument.[4]

This paper shows that the introduction of indexed bonds can never cause the monetary market to shut down. In contrast, the indexed bond market can be completely inactive if the transaction costs are high enough or the inflation volatility level is low enough. I also show that these bonds have a greater opportunity to be actively traded as the market has higher inflation volatility.

Recently, government inflation-indexed bonds have become available in a number of countries. This paper reveals that there is a possibility for not only governments but also financial entrepreneurs to issue these bonds. Since an economy

[2] Mas-Colell (1992) indicates that financial markets can be immune to sunspots by introducing as many real securities as the number of goods in each state. In addition, Goenka and Préchac (2006) show that the introduction of inflation-indexed bonds completely eliminates sunspot effects in incomplete markets. The introduction of these real securities results in a complete shutdown of nominal financial markets.

[3] However, with intrinsic uncertainty both riskless and nominal bonds can co-exist. (See Neumeyer, 1998.)

[4] Cozzi, Goenka and Shell (2012) have a similar idea that commodity taxes have intermidiation costs while money taxes do not. Specifically, an iceberg cost is incurred when the government uses a commodity tax system, but no cost is involved when it uses a money tax system. Nevertheless, the money taxes can be less attractive if the price-levels are unstable due to the effects of sunspots. Their paper investigates the comparative statics of whether the majority of consumers prefers commodity taxes versus money taxes when the economy is affected by sunspot signals. Their model is based on Bhattacharya, Guzman and Shell (1998).

with inflation volatility always has an arbitrage in the values of risk-free securities, financial entrepreneurs can also open the market if they have the technology to access indexed bonds at a small enough cost. When the economy has a higher level of inflation volatility, there will be a larger gap in the values of risk-free assets and therefore the indexed bond market will be friendlier for profit-seeking financial entrepreneurs.

It is counter-intuitive that the introduction of indexed bonds actually can decrease some consumers' welfare. This is due to the substitution effects: the introduction of a new asset causes the demand for money to decrease and consequently devalues money. This devaluation has a negative impact on the asset sellers'(borrowers') utility as they sell money (risky asset) at a low price. This situation could be a matter of concern for governments that need to gain consensus to adopt a new policy, but it may not apply to profit-seeking financial entrepreneurs. Because the introduction of indexed bonds does not necessarily make the economy Pareto improving, this paper considers a compensation test based on lump-sum tax-transfer plans which are denominated in period-0 money. Specifically, it demonstrates that there are balanced lump-sum tax-transfer plans that would allow the market with indexed bonds to be Pareto superior to one without indexed bonds.

This paper discusses both single-good and multi-good economies. In a single-good economy, the payoff for indexed bonds is straightforward. However, in a multi-good economy, the relative prices and the fixed commodity bundle from Consumer Price Index (CPI) generally do not agree across both the agents and the states. This problem can be resolved and the ideal payoffs of the inflation-indexed bonds can be defined if all agents' preferences are identically homothetic.

The outline of this paper is as follows. First, in Section 2, I introduce the combined market of a single-good economy in which both money and indexed bonds are available. The existence of a regular economy and the activeness of the monetary market are shown in Section 3. The impact of inflation volatility on the activeness of the indexed bond market is investigated in Section 4. Section 5 discusses the welfare implications of the introduction of the indexed bonds to the market. Section 6 introduces a multi-good model where the agents have identical homothetic preferences at both dates 0 and 1. Section 7 is a brief conclusion. Some numerical examples are introduced in the appendix.

2.2 Single-good Economy

There are two periods, today and tomorrow, labelled as subscripts $t = 0, 1$. At date 1, there are two states, $s = \alpha, \beta$ having positive probabilities $0 < \pi_\alpha < 1$ and $\pi_\beta = 1 - \pi_\alpha$, respectively. There are two consumers, labelled as superscripts $i \in I = \{1, 2\}$.[5] Consumer i's consumption allocation is $x^i = \left(x_0^i, x_\alpha^i, x_\beta^i\right) \in X = \mathbb{R}_{++}^3$ corresponding to prices $p = (p_0, p_\alpha, p_\beta) \gg 0$. His endowment is $e^i = \left(e_0^i, e_\alpha^i, e_\beta^i\right) \in X$ where $e_\alpha^i = e_\beta^i = e_1^i$. Denote by $\mathcal{U}$ the space of $\mathcal{C}^2$ utility functions on $\mathbb{R}_{++}^2$ which are twice differentiable, strictly increasing, strictly concave, having the closure of indifference curves contained in $\mathbb{R}_{++}^2$ and satisfying the von Neumann-Morgenstern expected utility hypothesis. Denote by $\mathcal{A}$ the set of characteristics $(v^i, e^i) \in \mathcal{U} \times X$. Consumer i's preferences are

$$u^i(x^i) = u^i\left(x_0^i, x_\alpha^i, x_\beta^i\right) = \pi_\alpha v^i\left(x_0^i, x_\alpha^i\right) + \pi_\beta v^i\left(x_0^i, x_\beta^i\right)$$

[5]Although the model can easily be extended to an economy with more than two consumers, I restrict the number of consumers to two because too many generalizations could cause readers to ignore the most important conclusions.

Throughout this paper, I assume that there is an incentive to trade, i.e., for consumer i and i' :

$$\frac{\partial v^i\left(e_0^i, e_1^i\right)}{\partial x_0} / \frac{\partial v^i\left(e_0^i, e_1^i\right)}{\partial x_1} \neq \frac{\partial v^{i'}\left(e_0^{i'}, e_1^{i'}\right)}{\partial x_0} / \frac{\partial v^{i'}\left(e_0^{i'}, e_1^{i'}\right)}{\partial x_1}$$

This condition implies that the initial endowment is not Pareto efficient. Since money can be considered a risky asset with sunspots, a lender and a borrower can be translated into an asset buyer and an asset seller. Let's define $I = \{B, S\}$ where B and S represent the asset buyer and the seller, respectively.

In a monetary market, there is only one financial instrument (money). m^i denotes consumer i's money holdings and its nominal return is r which is exogenously given. An economic fundamental $\mathcal{E}$ is simply a list $(u^i, e^i) \in \mathcal{A}$, $i \in I = \{1, 2\}$. Denote the space of economic fundamentals by $\mathcal{M}$ where $\mathcal{M} = \prod_{i \in I} \mathcal{A}$. Equilibrium in pure monetary markets is defined as follows: There are some positive spot prices $(p_0, p_\alpha, p_\beta) \gg 0$ and associated money holdings m such that each household is optimized, (x^i, m^i) is the solution to the maximization problem, denoted as

$$\max \quad u^i\left(x_0^i, x_\alpha^i, x_\beta^i\right)$$

$$\text{subject to} \quad \begin{cases} p_0 x_0^i + m^i \leq p_0 e_0^i \\ p_\alpha x_\alpha^i \leq p_\alpha e_1^i + r m^i \\ p_\beta x_\beta^i \leq p_\beta e_1^i + r m^i \end{cases}$$

$$\text{and} \quad x^i \in X$$

each market clears,

$$\textstyle\sum_i x_0^i = \sum_i e_0^i, \;\; \sum_i x_\alpha^i = \sum_i x_\beta^i = \sum_i e_1^i$$

$$\text{and} \quad \textstyle\sum_i m^i = 0.$$

I now introduce inflation-indexed bonds with a relative transaction cost θ. The transaction cost is relative to the trading price. For example, if $\theta = 0.1$, 10% of the purchasing price of those bonds help cover the transaction cost. Because of the existence of θ, the selling price of the bond is not the same as the purchasing price. The cost is incurred at date 1, implying that the intermediaries exchange the transaction fees with the first period good. Therefore, the market clearing condition for the first period is dependent on the amount of the indexed bonds traded in the market. I label a monetary market with indexed bonds as a combined market (CM). The equilibrium in CM is defined as

Given Prices $(p_0, p_\alpha, p_\beta, p)$, $(m^i, n^i, x_0^i, x_\alpha^i, x_\beta^i)_{h=1}^H$ solve the following maximization problem (P-CM)

$$\max u^i \left(x_0^i, x_\alpha^i, x_\beta^i\right)$$

subject to

$$p_0 x_0^i + m^i + p\,(1+\theta)\max(n^i, 0) + p\min(n^i, 0) \leq p_0 e_0^i \tag{2.1}$$

$$p_\alpha x_\alpha^i \leq p_\alpha e_1^i + p_\alpha n^i + r m^i \tag{2.2}$$

$$p_\beta x_\beta^i \leq p_\beta e_1^i + p_\beta n^i + r m^i \tag{2.3}$$

Market clear conditions are

$$\textstyle\sum_i m^i = 0, \sum_i n^i = 0$$

$$\textstyle\sum_i x_0^i + \dfrac{p}{p_0}\theta \sum_i \max(n^i, 0) = \sum_i e_0^i$$

$$\text{and } \textstyle\sum_i x_\alpha^i = \sum_i x_\beta^i = \sum_i x_1^i.$$

Here, $p\,(1+\theta)$ and p represent the purchasing and selling prices of the indexed bonds, respectively. The nominal interest rate r does not affect the equilibrium allocation in both markets. This implies that the nominal interest rate cannot be the source of the need for inflation-indexed bonds if it is deterministic. The nominal payoffs of the indexed bonds are p_α and p_β in state α and β, respectively.

For a single-good economy, the government can always design the nominal payoffs of those bonds, guaranteeing the same purchasing power across states. However, defining the payoffs to buy the same commodity bundle across states is not clear in a multi-good economy, as discussed in Section 6.

Finally, the budget constraint set in a combined market is still convex, which is necessary for the existence of single-valued demand functions. In the next section, I investigate the existence and regularity of the equilibria in a combined market.

2.3 Equilibrium

Kang (2012) defined price-level volatility (inflation volatility) based on the relative standard deviation (RSD) of price-level. In the same way as Kang (2012), inflation volatility in this paper is defined as

$$\sigma^R = \frac{\sigma(\widetilde{p}_1/p_0)}{E(\widetilde{p}_1/p_0)} \tag{2.4}$$

where $\sigma(\widetilde{p}_1/p_0)$ and $E(\widetilde{p}_1/p_0)$ are the standard deviation and the expected value of the level of inflation, respectively.[6] Kang (2012) shows that there exists a generic set $\mathcal{M}^* \subset \mathcal{M}$ such that any economy in a pure monetary market (MM), specified as a pair $\left(\mathcal{E}, \sigma^R\right)_M \in \mathcal{M}^* \times \mathbb{R}_+$ is regular and determinate. In this section, I show that a regular economy can be defined even in a combined market as fixing the inflation volatility level σ^R. In this model, the volatility σ^R is exogenously given, which implies that the occurrence of non-fundamental volatility does not vanish nor diminish with the introduction of real securities.

[6] Price-level volatility and inflation volatility based on the relative standard deviation (SRD) have exactly the same value in a single-good economy. Section 6 shows that in a multi-good economy, the two volatility measures also have the same value if the inflation is defined by the consumer price index (CPI)

In a similar method to Kang (2012), I reinterpret nominal security (money) in terms of a real security as fixing the ratio of the price levels in two states as constant. The following relationship shows that the fixing price ratio $\mathcal{P}$ is equivalent to fixing the inflation volatility σ^R:

$$\sigma^R = \frac{\sigma(\widetilde{p}_1/p_0)}{E(\widetilde{p}_1/p_0)} = \frac{\sqrt{\pi_\alpha \pi_\beta}\,|1-\mathcal{P}|}{\pi_\alpha + \pi_\beta \mathcal{P}}.$$

For a fixed price ratio, the ratio of the two states' returns of money is fixed by the following relationship:

$$\mathcal{P} = \frac{p_\beta}{p_\alpha} = \frac{r/p_\alpha}{r/p_\beta}$$

where $\left(\frac{r}{p_\alpha}, \frac{r}{p_\beta}\right)$ is the return of money in state α and β.

Given price ratio $\mathcal{P}$, it is possible to interpret nominal security (money) as a security with a real return (R_α, R_β) satisfying that $\mathcal{P} = \frac{R_\alpha}{R_\beta}$. Then, the equivalent economy with two types of real securities[7] can be defined (Lemma 1). Finally, I show that the economy is regular and determinate (Proposition 1).

First, the equivalent maximization problem based on two types of real securities is defined as (E-CM)

$$\max u^i(e_0^i + z_0^i, e_1^i + n^i + R_\alpha B^i, e_1^i + n^i + R_\beta B^i) \tag{2.5}$$

$$\text{subject to} \quad q_0 z_0^i + qB^i + p\,(1+\theta)\max(n^i,0) + p\min(n^i,0) \leq 0 \tag{2.6}$$

$$where \quad \frac{R_\beta}{R_\alpha} = \frac{1}{\mathcal{P}} \ .$$

Lemma 2.1 *Given $\mathcal{P} = p_\beta/p_\alpha$, P-CM is equivalent to E-CM where*

$$\begin{aligned} x_0^i &= e_0^i + z_0^i, \quad x_\alpha^i = e_1^i + n^i + R_\alpha B^i, \quad x_\beta^i = e_1^i + n^i + R_\beta B^i \\ m^i &= qB^i, \quad p_0 = q_0\,(=1)\,, \quad p_\alpha = \frac{rq}{R_\alpha}, \quad p_\beta = \frac{rq}{R_\beta} \end{aligned}$$

[7] One is the equivalent real security to money and the other is the inflation-indexed bonds. In this paper, I call them a risky asset and a risk-free asset, respectively.

Proof. (i) (P-CM⇒E-CM) P-CM is defined with the given parameters $(p_0, p_\alpha, p_\beta, p, r, \mathcal{P})$ while the given parameters are $(q_0, q, R_\alpha, R_\beta, p, \mathcal{P})$ in E-CM. Defining p_0 and m^i as

$$p_0 \equiv q_0 \quad \text{and} \quad m^i \equiv qB^i,$$

it is clear that the budget constraint (eqn 2.6) in E-CM can be derived from that in P-CM. Next, Defining p_α and p_β as

$$p_\alpha \equiv \frac{rq}{R_\alpha}, \quad p_\beta \equiv \frac{rq}{R_\beta},$$

x_α^i and x_β^i can be expressed as

$$x_\alpha^i \leq e_1^i + n^i + R_\alpha B^i \text{ and } x_\beta^i \leq e_1^i + n^i + R_\beta B^i.$$

Since utility functions are strictly increasing, $\leq$ can be replaced with $=$. Finally, by the equation $x_0^i = e_0^i + z_0^i$, the utility function (eqn 2.5) in E-CM is the same as the one in P-CM.

(ii) (E-CM⇒P-CM) Defining q_0 and B^i as

$$q_0 \equiv p_0, \quad B^i \equiv \frac{m^i}{q}$$

the period-0 budget constraint (eqn 2.1) in P-CM is equivalent to that (eqn 2.6) in E-CM. Let's define x_α^i and x_β^i as

$$x_\alpha^i \equiv e_1^i + n^i + R_\alpha B^i \text{ and } x_\beta^i \equiv e_1^i + n^i + R_\beta B^i.$$

then, where

$$R_\alpha = \frac{rq}{p_\alpha}, \quad R_\beta = \frac{rq}{p_\beta},$$

x_α^i and x_β^i can be expressed as

$$p_\alpha x_\alpha^i \leq p_\alpha e_1^i + p_\alpha n^i + rm^i$$

$$p_\beta x_\beta^i \leq p_\beta e_1^i + p_\beta n^i + rm^i$$

which are the same as eqns (2.2) and (2.3), respectively. Finally, because $x_0^i \equiv e_0^i + z_0^i$, the utility function in P-CM is equivalent to that in E-CM. ■

In a competitive equilibrium, the mean of the real return (R_α, R_β) does not affect the equilibrium allocations since the value of q is adjusted according to the value of the mean. For the convenience of proofs and computation, I assume that the mean value of the real return is fixed as one, i.e., $\pi_\alpha R_\alpha + \pi_\beta R_\beta = 1$. Below, all of the analyses are based on the equivalent problem (E-CM).

Proposition 2.1 *There exists a generic set $\mathcal{M}^* \subset \mathcal{M}$ such that any economy in a combined market, specified as a triple $\left(\mathcal{E}, \sigma^R, \theta\right)_C \in \mathcal{M}^* \times \mathbb{R}^2_+$, is regular and determinate.*

Proof. In the equivalent maximization problem, the budget sets are convex and the utility function is strictly concave for all $(q_0, R_\alpha, R_\beta, q, \mathcal{P}, p) \gg 0$. Therefore, the excess demand is single valued. (The proof can be done by defining a compact subset of the budget set.)

For the proof of a regular and determinate economy, we need to define the aggregate excess demand and check if it satisfies sufficient conditions for the existence of regular economies. The market aggregate demand function is defined as

$$Z(Q) = \sum_i \begin{pmatrix} z_0^i(Q) + \theta \frac{p}{q_0} \max\left(n^i(Q), 0\right) \\ B^i(Q) \\ n^i(Q) \end{pmatrix}$$

$$\text{where } Q = (q_0, q, p)$$

This market aggregate demand function is not the same as a conventional one because of the additional term $\theta \frac{p}{p_0} \max\left(n^i(Q), 0\right)$ which is the amount of first-period goods the intermediaries charge as transaction fees.

It can be easily shown that $Z(Q)$ satisfies the five properties:

(i) continuous

(ii) homogeneous of degree zero

(iii) (Walras' law) $Q \cdot Z(Q) = 0$

(iv) There is an $s > 0$ such that $Z(Q) > (-s, -s, -s)$ for all P.

(v) If $Q^n \to Q$, where $Q \neq 0$ and $p = 0$, then $\sum_i n^i(Q) \to \infty$.

Therefore, the economy is regular and determinate. (See Debreu 1970.) ■

The existence of equilibria does not necessarily imply that either (both) an indexed bond market or (and) a monetary market are active. The main difference from previous articles[8] dealing with the combined market of money and indexed bonds in a sunspot economy is that the monetary market in my model is always active even after the introduction of indexed bonds. The following proposition shows this.

Proposition 2.2 *The monetary market is always active, i.e,* $m^i \neq 0$ *for some* $i \in I$ *if* $\theta > 0$.[9]

Proof. **(by contradiction)** Let's assume that $m^i = 0$ for $i \in \{S, B\}$, labeling the asset buyer ($B^i > 0$ or $m^i > 0$) as $i = B$ and the asset seller ($m^i < 0$) as $i = S$. Assuming that $n^B = n^S = 0$, money should be active since the initial endowment is not Pareto efficient. Next, we need to check the case where $n^B \neq 0$ and $n^S \neq 0$.

[8]See Mas-Colell (1992) and Goenka and Préchac (2006).

[9]It is proven that where $\theta = 0$, money is not traded in the market and the sunspot effects will disappear. (See Goenka and Préchac 2006.)

For this case, we can compute each individual's value of the risky-asset (money). The value is the ratio between the marginal utility of risk assets and that of the period-0 good. Let's assume that the risky asset's return (R_α, R_β) satisfies that $\pi_\alpha R_\alpha + \pi_\beta R_\beta = 1$. Then, the value (price) of the risky asset for the buyer is computed as

$$
\begin{aligned}
q_R^B &= \frac{\pi_\alpha \frac{\partial v^B\left(x_0^B, e_1^B + R_\alpha B^B\right)}{\partial B^B} + \pi_\beta R_\beta \frac{\partial v^B\left(x_0^B, e_1^B + R_\beta B^B\right)}{\partial B^B}}{\pi_\alpha \frac{\partial v^B\left(x_0^B, x_\alpha^B\right)}{\partial x_0} + \pi_\beta \frac{\partial v^B\left(x_0^B, x_\beta^B\right)}{\partial x_0}} \\
&= \frac{\pi_\alpha R_\alpha \frac{\partial v^B\left(x_0^B, x_\alpha^B\right)}{\partial x_1} + \pi_\beta R_\beta \frac{\partial v^B\left(x_0^B, x_\beta^B\right)}{\partial x_1}}{\pi_\alpha \frac{\partial v^B\left(x_0^B, x_\alpha^B\right)}{\partial x_0} + \pi_\beta \frac{\partial v^B\left(x_0^B, x_\beta^B\right)}{\partial x_0}} \quad \left(\begin{array}{c} \pi_\alpha R_\alpha + \pi_\beta R_\beta = 1 \\ \mathcal{P} = R_\alpha / R_\beta \end{array}\right)
\end{aligned}
$$

Since $m^B = 0 (B^B = 0)$, $x_\alpha^B = x_\beta^B$. Also, it is true that $\pi_\alpha R_\alpha + \pi_\beta R_\beta = \pi_\alpha + \pi_\beta$. Therefore, we get:

$$
q_R^B = \frac{\pi_\alpha \frac{\partial v^B\left(x_0^B, x_\alpha^B\right)}{\partial x_1} + \pi_\beta \frac{\partial v^B\left(x_0^B, x_\beta^B\right)}{\partial x_1}}{\pi_\alpha \frac{\partial v^B\left(x_0^B, x_\alpha^B\right)}{\partial x_0} + \pi_\beta \frac{\partial v^B\left(x_0^i, x_\beta^i\right)}{\partial x_0}}.
$$

The right part of the equation above is the same as the asset buyer's value of the indexed bonds. This is also the same as the purchasing price of the indexed bond. Therefore, we can get

$$
q_R^B = p(1 + \theta).
$$

In the same way, we can show that the asset seller's value of a risky-asset is the same as the selling price of the indexed bond:

$$
q_R^S = p.
$$

Since $q_R^B > q_R^S$, there is a price $q \in \left(q_R^S, q_R^B\right)$ in which the two consumers have an incentive to trade with money. This contradicts that $m^i = 0$ for $i = S, B$. ■

Proposition 2 shows that the market for money is still operating even with an extremely high level of inflation volatility if $\theta > 0$. However, without a transaction cost, i.e., $\theta = 0$, the existence of inflation volatility makes the market for money completely shut down, which has been proven in Goenka and Préchac (2006).

2.4 Indexed Bond Market and Inflation Volatility

Although inflation-indexed bonds are not issued in a monetary market, the values of those securities can be computed at the equilibrium. Let's define p_F^i as the value of a risk-free asset for consumer $i \in \{B, S\}$:

$$p_F^i = \frac{\pi_\alpha \frac{\partial v^i\left(x_0^i, x_\alpha^i\right)}{\partial x_1} + \pi_\beta \frac{\partial v^i\left(x_0^i, x_\beta^i\right)}{\partial x_1}}{\pi_\alpha \frac{\partial v^i\left(x_0^i, x_\alpha^i\right)}{\partial x_0} + \pi_\beta \frac{\partial v^i\left(x_0^i, x_\beta^i\right)}{\partial x_0}}$$

which is the ratio between the marginal utility of risk-free assets and that of the period-0 good. The following lemma shows that there is an arbitrage between two values p_F^B and p_F^S where "B" and "S" represent the asset buyer and the asset seller, respectively.

Lemma 2.2 *If $\sigma^R \neq 0$ in a pure monetary market, then*[10]

$$p_F^S < p_F^B.$$

Proof. q^+ represents the equilibrium price of the risky asset in a pure monetary market.

$$q^+ = \frac{\pi_\alpha R_\alpha \frac{\partial v^i\left(x_0^i, x_\alpha^i\right)}{\partial x_1} + \pi_\beta R_\beta \frac{\partial v^i\left(x_0^i, x_\beta^i\right)}{\partial x_1}}{\pi_\alpha \frac{\partial v^i\left(x_0^i, x_\alpha^i\right)}{\partial x_0} + \pi_\beta \frac{\partial v^i\left(x_0^i, x_\beta^i\right)}{\partial x_0}} \qquad \left(\begin{array}{c} \pi_\alpha R_\alpha + \pi_\beta R_\beta = 1 \\ \mathcal{P} = R_\alpha / R_\beta \end{array}\right)$$

Let's label the asset buyer ($B^i > 0$) as $i = B$ and the asset seller ($B^i < 0$) as $i = S$. Since the initial endowment allocation is not Pareto efficient, they will trade and therefore $B^B > 0$ and $B^S < 0$. First, we want to show that p_F^B is higher than q^+. Since the state β is inflationary $\mathcal{P} > 1$, we get: $R_\alpha > R_\beta$ and $x_\alpha^B > x_\beta^B$. $\frac{p_F^B}{q^+}$ is

$$\frac{p_F^B}{q^+} = \frac{\pi_\alpha \frac{\partial v^B\left(x_0^B, x_\alpha^B\right)}{\partial x_1} + \pi_\beta \frac{\partial v^B\left(x_0^B, x_\beta^B\right)}{\partial x_1}}{\pi_\alpha R_\alpha \frac{\partial v^B\left(x_0^B, x_\alpha^B\right)}{\partial x_1} + \pi_\beta R_\beta \frac{\partial v^B\left(x_0^B, x_\beta^B\right)}{\partial x_1}}.$$

[10] An economy without inflation volatility ($\sigma^R = 0$) is the same as a certainty economy. Therefore, where $\sigma^R = 0$ there is no arbitrage, i.e., $p_F^S = p_F^B$.

Since v^i is strictly concave, it is true that

$$\frac{\partial v^B\left(x_0^B, x_\alpha^B\right)}{\partial x_1} < \frac{\partial v^B\left(x_0^B, x_\beta^B\right)}{\partial x_1}.$$

Since $\pi_\alpha R_\alpha + \pi_\beta R_\beta = 1$, $\pi_\alpha + \pi_\beta = 1$ and $\pi_\alpha R_\alpha > \pi_\alpha$, $\pi_\beta R_\beta < \pi_\beta$, we can get:

$$\frac{p_F^B}{q^+} > 1.$$

In the same way, we can prove the following:

$$\frac{p_F^S}{q^+} < 1.$$

From the two inequalities, we know that $p_F^S < p_F^B$. ■

The arbitrage in the values of the risk-free securities between two consumers provides the incentive for them to accept real securities as the financial instrument in a combined market. However, the existence of the arbitrage does not necessarily imply that the indexed bonds are active. The magnitude of the arbitrage should be large enough, compared to the transaction cost, for the indexed bonds to be traded in a combined market. The following propositions and corollary show that the indexed bond markets can be active or inactive depending on the level of inflation volatility and the level of a transaction cost.

Proposition 2.3 *If $\sigma^R \neq 0$, there exists $\overline{\theta} \in \mathbb{R}_{++}$ such that the indexed bond market is active (inactive) if $\theta < \overline{\theta}$ $\left(\theta \geq \overline{\theta}\right)$, i.e.,*

$$n^i \begin{cases} \neq 0 & \text{if } \theta < \overline{\theta} \\ = 0 & \text{if } \theta \geq \overline{\theta} \end{cases} \qquad \text{for } i = S, B.$$

where

$$\overline{\theta} = \frac{p_F^B - p_F^S}{p_F^S}. \tag{2.7}$$

Proof. (i) *For* $n^i \neq 0$ *where* $\theta < \overline{\theta}$.

(Proof by contradiction) Let's assume that $n^i = 0$ for $i = S, B$. Then, the equilibrium allocations in a combined market are the same as those in a pure monetary market. If $\theta < \overline{\theta}$, there exists $p > 0$ such that $p_F^S < p < p(1+\theta) < p_F^B$. (This can be proven as follows: Let $p = p_F^S + \varepsilon$ and $\theta = \frac{p_F^B - p_F^S - \delta}{p_F^S} > 0$. Then, for all $\delta \in \left(0, p_F^B - p_F^S\right)$, there exists ε which satisfies the inequality $p_F^S \leq p < p(1+\theta) \leq p_F^B$.) This implies that the two consumers have an incentive to trade with the indexed bonds. This contradicts that $n^i = 0$.

(ii) *For* $n^i = 0$ *where* $\theta \geq \overline{\theta}$.

(Proof by contradiction) Let's assume that $n^i \neq 0$ for $i = S, B$. Assuming that $\theta \geq \overline{\theta}$, there does not exist $p > 0$ such that $p_F^S < p < p(1+\theta) < p_F^B$. This contradicts that $n^i \neq 0$ for $i = S, B$. ■

Next, it will be shown that as the market has higher inflation volatility (more uncertainty about the future price level), there are more chances for the indexed bond market to be active. In the model, the decision about whether financial entrepreneurs should enter the market (or whether the government should issue bonds) depends on the value of $\overline{\theta}$. For a smaller value of $\overline{\theta}$, there would be a smaller chance for financial entrepreneurs to survive in the market. The following proposition shows the direct connection between the value of $\overline{\theta}$ and the level of inflation volatility.

Proposition 2.4 *If* v^i *is additively separable, i.e.,* $v^i\left(x_0^i, x_1^i\right) = f^i(x_0^i) + g^i(x_1^i)$, $\overline{\theta}$ *is strictly increasing in* σ^R. *(See Appendix for the proof.)*

Proof. (See Appendix for the proof.) ■

The proposition is the extended version of lemma 1 saying that $\overline{\theta}$ is strictly positive where $\sigma^R > 0$ and equals zero where $\sigma^R = 0$. The proof for the non-separable utility functions remains open. Proposition 4 implies that as the economy has higher inflation volatility, financial entrepreneurs have a better chance of making a profit and consequently stronger incentive to enter the indexed bond market.

From Propositions 3 and 4, we derive the following result.

Corollary 2.1 *If v^i is additively separable, for any $\theta > 0$, there exists a constant $\overline{\sigma}^R \in \mathbb{R}_{++}$ such that only the monetary market is active if $\sigma^R < \overline{\sigma}^R$ while both indexed bond and monetary markets are active if $\sigma^R \geq \overline{\sigma}^R$.*

Proof. Trivial from Propositions 3 and 4. ■

2.5 Welfare

A combined market provides consumers more trading choices than a pure monetary market. Therefore, it can be expected that the equilibrium in the combined market is Pareto superior to that in the pure monetary market. However, it has been shown that financial innovations to incomplete markets do not necessarily induce the economy to be Pareto improving.[11] Financial innovations are known to affect market prices under a general equilibrium setting. These price changes negatively affect agents' real wealth and consequently, their utility values. Specifically, the introduction of the indexed bonds causes the value of money to decrease due to substitution effects, which have a negative impact on asset sellers'(borrowers') wealth.

[11] See Cass and Citanna (1998), Elul (1995) and Hart (1997) .

The following example is the case where the introduction of indexed bonds make some consumers worse off. There are two consumers who have the same expected utility functions $v^1 = v^2 = \log(x_0) + \log(x_1)$ and whose endowments are $e^1 = (10, 0)$ and $e^2 = (0, 10)$. In this case, as the level of inflation volatility increases, the second consumer's utility value increases. The intuition is as follows. For a high level of inflation volatility, the real return of assets becomes small in an inflationary state. If the inflationary state is realized, the first consumer will end up getting a small amount of the good at date 1 and consequently his utility will be particularly low. Therefore, the first consumer's demand for the asset will be higher to insure against the inflation state in a higher level of inflation volatility. The high demand of the first consumer results in a high price (value) of the risky asset (money). Finally, the second consumer, who sells the risky asset (money) to the first one, obtains more income as the asset price goes higher in a pure monetary market.

However, the introduction of new riskless securities, which are substitutes for money, has caused a considerable decrease in money demand. Therefore, the value of money in a combined market is smaller than that in a pure monetary market. A decrease in the money value negatively affects the second consumer's utility level. In this example, even though the asset seller has more trading choices in the combined market, his utility level is actually lower than that in the pure monetary market. (See the Appendix for more details about the example.)

Although some of the consumers may be negatively affected, it is not possible for all of them to be worse off by the introduction of the indexed bonds according to the revealed preferences hypothesis. The following proposition shows this.

Proposition 2.5 *(i) For any economic fundamental $\mathcal{E} \subset \mathcal{M}$, there exists at least* *one consumer who is better off. (ii) However, all consumers can be better off for some economic fundamentals in $\mathcal{M}$.*

Proof. (i) When the value of money, which is considered q in the equivalent problem, is not the same in both markets, there must be at least one consumer who will benefit from the change of the value. (The consumer should be the asset buyer (seller) if the value decreases (increases) in a combined market.) Since the consumer has more choices in the combined market, according to the revealed preference argument, the consumer must be better off.

(ii) If variation in the value of money is small enough or zero, according to revealed preference argument, all consumers can be better off. ■

The introduction of the indexed bonds can be considered a sunspot-stabilizing policy in the sense that it makes the equilibrium allocations less volatile.[12,13] However, Proposition 5 shows that the introduction of indexed bonds does not necessarily make all agents better off. This means that the government can fail to gain consensus in adopting the financial innovation–the introduction of the indexed bonds. However, Proposition 6 shows that if balanced lump-sum tax plans are allowed along with financial innovations, consumer consensus on the policy can be achieved. Although there is no clear Pareto ranking between a combined economy and a pure monetary economy, it can be shown that a combined economy is Kaldor-Hicks superior to a pure monetary economy by considering a compensation

[12]More accurately, the relative standard deviation (RSD) of the excess demand $\{x_\alpha - e_1, x_\beta - e_1; \pi_\alpha, \pi_\beta\}$ is lowered with the introduction of the indexed bonds.

[13]However, it is different from conventional sunspot-stablizing policies because it cannot completely eliminate the consumption volatility unless the transaction cost is zero. Several studies have suggested stabilizing policies to completely eliminate the effects of sunspots on incomplete markets. Three dominant policies have been shown: 1) the introduction of new types of nominal securities. (see Cass and Shell 1983 [Proposition 3] and Balasko 1983 [Theorem 1]); 2) the introduction of as many real securities as the number of goods in each state (see Mas-Colell 1992); and 3) the introduction of options. (see Antinolfi and Keister 1998 and Kajii 1997).

test based on balanced lump-sum tax-transfer plans. The space of the balanced lump-sum tax plans is defined as

$$T = \left\{ (\tau^1, ..., \tau^I) \in \mathbb{R}^I | \sum_{i \in I} \tau^i = 0 \right\}.$$

The tax-transfer plan τ is applied to either the combined economy or the pure monetary market. Then, the budget constraints at date 0 are modified to be

$$s.t \begin{cases} p_0 x_0^i + m^i + p\,(1+\theta)\max(n^i, 0) + p\min(n^i, 0) \leq p_0 e_0^i - \tau^i \\ \\ p_\alpha x_\alpha^i \leq p_\alpha e_1^i + p_\alpha n^i + r m^i \\ p_\beta x_\beta^i \leq p_\beta e_1^i + p_\beta n^i + r m^i \end{cases} \tag{P-CM}$$

The budget constraints in the second period are invariant. Assuming that $p^0 = 1$, we can interpret that the taxes and transfers are denominated by the period-0 commodity.

Definition 2.1 *Market "A" is Kaldor-Hicks superior to market "B", i.e.,*

$$market\ A \succeq_{KH} market\ B,$$

if there exist(s) $\tau \in T$ such that market A with the tax-transfer plan τ is Pareto superior to market B.

In the above definition, the tax-transfer plans are applied to only the first market (market A) but not to the second market (market B).

The following proposition shows that a combined market is “superior” to a pure monetary market based on the compensation principle.

Proposition 2.6 *For any $\mathcal{E} \in \mathcal{M}$, $\sigma^R > 0$ and $\theta > 0$,*

$$\left(\mathcal{E}, \sigma^R, \theta\right)_C \succeq_{KH} \left(\mathcal{E}, \sigma^R\right)_M$$

and

$$\left(\mathcal{E}, \sigma^R\right)_M \not\succ_{KH} \left(\mathcal{E}, \sigma^R, \theta\right)_C .$$

Proof. See the Appendix. ■

Propositions 5 and 6 shows that there is no clear Pareto ranking between monetary and combined economies but there does exist Kaldor-Hicks welfare ranking between the two economies. In the same way, it is possible to compare two types of combined economies with different levels of transaction costs. Surprisingly, there are some consumers (among asset sellers) who prefer an economy with higher transaction costs since higher transaction costs make the value of money higher, which is preferred by asset sellers (borrowers). (See the Appendix for a detailed numerical example.) In the same sense as Proposition 6, it can be shown that a combined economy with lower transaction costs is Kaldor-Hicks superior to an economy with higher costs, i.e.,

$$\left(\mathcal{E}, \sigma^R, \theta_1\right)_C \succeq_{KH} \left(\mathcal{E}, \sigma^R, \theta_2\right)_C \quad \text{if} \quad \theta_1 < \theta_2.$$

I do not put the results as a proposition since this can be considered somewhat trivial.

2.6 Multi-good Economy

The object of this section is to construct a more realistic model by extending the number of commodities in each spot from one to $L > 1$. To clearly distinguish the notations from the single-good case, I use the symbol "→" in all vectors in a multi-good economy. Consumer i's consumption allocation is $\overrightarrow{x}^i = \left(\overrightarrow{x}^i_0, \overrightarrow{x}^i_\alpha, \overrightarrow{x}^i_\beta\right) \in X = \mathbb{R}^{3L}_{++}$ where

$$\overrightarrow{x}^i_s = \left(x^i_{s1}, ..., x^i_{sl}, ..., x^i_{sL}\right) \qquad s = 0, \alpha, \beta.$$

Corresponding prices are $\overrightarrow{p} = (\overrightarrow{p}_0, \overrightarrow{p}_\alpha, \overrightarrow{p}_\beta) \in \mathbb{R}^{3L}_{++} \gg 0$ where

$$\overrightarrow{p}_s = (p_{s1}, ..., p_{sl}, ..., p_{sL}) \qquad s = 0, \alpha, \beta.$$

Consumer i's endowment is $\overrightarrow{e}^i = \left(\overrightarrow{e}^i_0, \overrightarrow{e}^i_\alpha, \overrightarrow{e}^i_\beta\right) \in X$ where $\overrightarrow{e}^i_\alpha = \overrightarrow{e}^i_\beta = \overrightarrow{e}^i_1$ and

$$\overrightarrow{e}^i_t = \left(e^i_{t1}, ..., e^i_{tl}, ..., e^i_{tL}\right) \quad t = 0, 1$$

Then, the budget constants in the multi-good economy are:

$$\overrightarrow{p}_0 \cdot \overrightarrow{x}^i_0 + m^i + p(1+\theta)\max\left(n^i, 0\right) + p\min\left(n^i, 0\right) \leq \overrightarrow{p}_0 \cdot \overrightarrow{e}^i_0$$

$$\overrightarrow{p}_\alpha \cdot \overrightarrow{x}^i_\alpha = \overrightarrow{p}_\alpha \cdot \overrightarrow{e}^i_1 + \overrightarrow{p}_\alpha \cdot \overrightarrow{w}_1 n^i + rm^i \tag{2.8}$$

$$\overrightarrow{p}_\beta \cdot \overrightarrow{x}^i_\beta = \overrightarrow{p}_\beta \cdot \overrightarrow{e}^i_1 + \overrightarrow{p}_\beta \cdot \overrightarrow{w}_1 n^i + rm^i \tag{2.9}$$

where $\overrightarrow{w}_1$ is a fixed commodity bundle from the consumer price index (CPI).

The payoffs of indexed bonds are based on nominal returns $\overrightarrow{p}_\alpha \cdot \overrightarrow{w}_1$ and $\overrightarrow{p}_\beta \cdot \overrightarrow{w}_1$ in the states α and β, respectively. The question is whether all consumers will buy the same fixed commodity bundle $\overrightarrow{w}_1$ across the states with nominal returns $\overrightarrow{p}_s \cdot \overrightarrow{w}_1$, $s = \alpha, \beta$.[14] In a single good-economy, the answer is always "yes" since consumers have no other choice except purchasing the single good. However, in a multi-good economy, it is not trivial to define the payoffs of indexed bonds,

[14] This is the same question as whether the indexed bonds have the exact same effects on the economy as real securities.

which guarantee the same purchasing power across the states, without any further restrictions on preferences.

To deal with this issue, it is necessary to define the Consumer price index (CPI) in this paper. The payoffs of TIPS[15] is based on the CPI. CPI consists of two factors: the fixed commodity bundle and the price index. These factors should be estimated in advance to design the optimal indexed bonds. The weighted factor is the vector of the reference consumption-bundle. It can be defined as

$$\overrightarrow{w}_t = (w_{t1}, w_{t2}, ..., w_{tl}, ..., w_{tL}) \quad \text{where } t = 0, 1$$

where w_{tl} represents the quantity of good l at date t.

The second factor in CPI is the (relative) price index. It can be defined as

$$\overrightarrow{pi}_t = (pi_{t1}, pi_{t2}, ..., pi_{tl}, ..., pi_{tL}) \quad \text{where } t = 0, 1.$$

where pi_{tl} represents the price of good l at date t.

To design TIPS payoffs guaranteeing the same purchasing power across agents and states, the following conditions should be satisfied

(1) There exists a common fixed commodity bundle at date 1 for all agents

(2) There exists a common price index at date 1 across states.

To get a consistent price index and commodity bundle, additional restrictions on utility functions and endowments are required. One advantage of our model is that the endowments across states are identical and consequently there are no relative-price fluctuations from endowment shocks. Therefore, we do not need any

[15] Treasury Inflation Protected Securities (TIPS) are U.S. inflation-indexed bonds whose coupons and principal are adjusted according to the evolution of the consumer price index. In the paper, inflation-indexed bonds and Treasury inflation-protected securities represent the same thing.

more restrictions on endowments. However, we still need an additional assumption on utility functions:

Assumption 2.1 *Utility functions u^i, $i \in I$ are weakly separable across states and identically homothetic within states. That is, there exist functions v^i for all $i \in I$ and f such that*[16]

$$v^i : \mathbb{R}^2 \to \mathbb{R}, \qquad i \in I$$

which are smooth, strictly increasing and strictly concave, and functions

$$f : \mathbb{R}^L_+ \to \mathbb{R}$$

which are smooth, strictly increasing, strictly concave, having the closure of indifference curves contained in $\mathbb{R}^L_{++}$ and homogeneous of degree one. Then, u^i is defined as

$$u^i\left(\overrightarrow{x}^i_0, \overrightarrow{x}^i_\alpha, \overrightarrow{x}^i_\beta\right) = \pi_\alpha v^i\left(f\left(\overrightarrow{x}^i_0\right), f\left(\overrightarrow{x}^i_\alpha\right)\right) + \pi_\beta v^i\left(f\left(\overrightarrow{x}^i_0\right), f\left(\overrightarrow{x}^i_\beta\right)\right).$$

The following lemma shows that a common commodity bundle and price index exist at both date 0 and 1.

Lemma 2.3 *If Assumption 1 holds, the price index and the fixed commodity bundle can be expressed as*

$$\overrightarrow{pi}_t = c^{pi}_t \nabla f\left(e_t\right) \quad \textit{and} \quad \overrightarrow{w}_t = c^w_t \overrightarrow{e}_t \qquad t = 0, 1$$

where

$$\overrightarrow{e}_t = \left(e_{t1}, ..., e_{tL}\right) = \textstyle\sum_i e^i_t.$$

[16] The two theoretical articles for the inflation-indexed bonds from Geankoplos (2005) and Magill and Quinzii (1997) used almost the same utiliy format. Both articles assumed a homothetic embedded function $f\left(x^i\right)$ at date 1 and utility v^i based on utility f.

and

$$c_t^{pi}, c_t^{w} > 0$$

Proof. Since (1) all the agents' spot-utility functions $f(\cdot)$ are identical, and (2) the aggregate endowment in each spot is deterministic, there exists a representative agent at each spot with a utility function $f(\cdot)$ at both date 0 and 1 and whose endowment is $\overrightarrow{e}_t\,(=\sum_i e_t^i)$ at date t. Then, $\overrightarrow{pi}_t$ and $\nabla f(\overrightarrow{e}_t)$ are collinear by first-order conditions of the representative agents. Since $f(\cdot)$ is homothetic, $\overrightarrow{x}_t^i$ and $\overrightarrow{e}_t$ are collinear for all $i \in I$. Therefore, the fixed commodity bundle can be defined as $\overrightarrow{w}_t = c_t^w \overrightarrow{e}_t$. ■

$\overrightarrow{pi}_t$ and $\overrightarrow{w}_t$ are estimated by the government to design the payoffs of the indexed bonds. c_t^{pi} and c_t^w are positive constant, and their values do not affect the equilibrium allocation since the price of the indexed bond, "p" is automatically adjusted according to c_t^{pi} and c_t^w. Therefore, the government can assign any values to c_t^{pi} and c_t^w. For the convenience of notation, I normalize $\overrightarrow{w}_t$ as $(1, w_{t2}, ..., w_{tl}, ..., w_{tL})$ and $\overrightarrow{pi}_t$ as satisfying that $\overrightarrow{pi}_t \cdot \overrightarrow{w}_t = 1$.

Remark 2.1 *Geanakoplos 2005 also assumed a homogeneous degree of one $f(x_s^i)$ which is defined in the commodity space at date 1. In his model, there exists intrinsic uncertainty on endowments and therefore, the aggregate endowment across states is not identical. In that case, it is not possible to derive any common CPI price and commodity bundle, even with the homothetic utility functions. However, he pointed out that the payoffs which guarantee the same utility in terms of the embedded utility $f(x_s^i)$ still can be achievable. This is due to the homogeneity of degree one of $f(x_s^i)$. In my model, the payoffs of those bonds guarantee both the same CPI commodity bundle and the same utility since there is no intrinsic uncertainty.*

Lemma 3 shows that common relative prices $\overrightarrow{pi}_1$ exist across the states. Therefore, the price in each state can be expressed as

$$\overrightarrow{p}_\alpha = p_\alpha \overrightarrow{pi}_1 \quad \text{and} \quad \overrightarrow{p}_\beta = p_\beta \overrightarrow{pi}_1,$$

where p_α are p_β scalars, which are determined endogenously. In the same way, the prices in period 0 can be expressed as $\overrightarrow{p}_0 = p_0 \overrightarrow{pi}_0$. Then, consumer i's budget constraints can be modified as:

$$\begin{aligned} p_0 \overrightarrow{pi}_0 \cdot \overrightarrow{x}^i_0 + m^i + p(1+\theta)\max\left(n^i, 0\right) + p\min\left(n^i, 0\right) &\leq p_0 \overrightarrow{pi}_0 \cdot \overrightarrow{e}^i_0 \\ p_\alpha \overrightarrow{pi}_1 \cdot \overrightarrow{x}^i_\alpha &= p_\alpha \overrightarrow{pi}_1 \cdot \left(\overrightarrow{e}^i_1 + n^i \overrightarrow{w}_1\right) + rm^i \\ p_\beta \overrightarrow{pi}_1 \cdot \overrightarrow{x}^i_\beta &= p_\beta \overrightarrow{pi}_1 \cdot \left(\overrightarrow{e}^i_1 + n^i \overrightarrow{w}_1\right) + rm^i \end{aligned} \tag{2.10}$$

where

$$\overrightarrow{p} = (\overrightarrow{p}_0, \overrightarrow{p}_\alpha, \overrightarrow{p}_\beta) = \left(p_0 \overrightarrow{pi}_0, p_\alpha \overrightarrow{pi}_1, p_\beta \overrightarrow{pi}_1\right).$$

In this section, p_0, p_α are p_β scalars, which are determined endogenously, but they do not represent the prices of any specific good. The return of the indexed bonds can be interpreted as real or nominal returns. $\overrightarrow{w}_1$ is the consumption bundle vector of the real return of the bonds while $p_\alpha \overrightarrow{pi}_1 \cdot \overrightarrow{w}_1$ and $p_\beta \overrightarrow{pi}_1 \cdot \overrightarrow{w}_1$ are the nominal returns, which have the same purchasing power to have the consumption bundle $\overrightarrow{w}_1$ across the states. Whether the returns are delivered in terms of commodities or money, it does not affect the equilibrium allocations nor prices. The values of the two scalars p_α and p_β depend on price-level volatility; however, the price index ratio $\overrightarrow{pi}_1$ and the commodity bundle $\overrightarrow{w}_1$ do not. Therefore, the government can perfectly estimate the weighted factor $\overrightarrow{w}_1$ and the price index $\overrightarrow{pi}_1$ even without any information about price volatility σ^R.

Next, we need to define price-level volatility and inflation volatility in a multi-good economy. Since the relative prices are the same across states, the price-level volatility can be computed with the same formula as that in the single-good economy:

$$\begin{aligned}\sigma^R &= \frac{\sigma(\widetilde{p}_{1l})}{E(\widetilde{p}_{1l})} = \frac{\sigma(\widetilde{p}_1)}{E(\widetilde{p}_1)} \text{ for all } l \in L \\ &= \frac{\sqrt{\pi_\alpha \pi_\beta}\,|p_\alpha - p_\beta|}{\pi_\alpha p_\alpha + \pi_\beta p_\beta} = \frac{\sqrt{\pi_\alpha \pi_\beta}\,|1-\mathcal{P}|}{\pi_\alpha + \pi_\beta \mathcal{P}}\end{aligned}$$

where $\widetilde{p}_1 = \{p_\alpha, p_\beta; \pi_\alpha, \pi_\beta\}$ is a random variable and $\mathcal{P} = \frac{p_\beta}{p_\alpha} \geq 1$.

At a fixed value of price-level volatility σ^R, the indeterminacy is eliminated even in a multi-good economy.[17] Next, I show that a multi-good economy can be mathematically equivalent to that of a single-good economy if Assumption 1 holds. With the fixed commodity bundle and the price index, the utility maximization problem can be re-defined in terms of only the first good ($l = 1$) in each spot. Then, the maximization problem of the multi-good economy is equivalent to that of the single-good economy. In this case, the utility function can be expressed with only the period-0 good ($l = 1$) :

$$v^i\left(f(x_0^i), f(x_s^i)\right) = v^i\left(x_{01}^i f(\overrightarrow{w}_0), x_{s1}^i f(\overrightarrow{w}_1)\right),$$

since f is the homogeneous degree of one and $x_0^i = x_{01}^i \overrightarrow{w}_0$ and $x_s^i = x_{s1}^i \overrightarrow{w}_1$ for $s = \alpha, \beta$.

Also, with the relation $\overrightarrow{pi}_t \cdot \overrightarrow{w}_t = 1$, the budget constraints in (2.10) can be replaced with

$$p_0 x_{01}^i + m^i + p(1+\theta)\max\left(n^i, 0\right) + p\min\left(n^i, 0\right) \leq p_0 e_{01}^i$$

$$p_\alpha x_{\alpha 1}^i = p_\alpha\left(\overrightarrow{pi}_1 \cdot \overrightarrow{e}_1 + n^i\right) + r m^i \tag{2.11}$$

[17] Cass (1992) indicated that in an incomplete financial market of a conventional two-period model, sunspots can generate $D = S - J$ nominal and real indeterminacy with S sunspot states and $0 < J < S$ distinct types of bonds. Therefore, the set of the equilibria, without specifying the level of inflation volatility, has one degree of indeterminancy. In my model, as fixing the level of inflation volatility, one degree of indeterminacy can be eliminated.

$$p_\beta x^i_{\beta 1} = p_\beta \left(\overrightarrow{pi}_1 \cdot \overrightarrow{e}_1 + n^i\right) + rm^i.$$

Then, the maximization problem is exactly the same as that of the single-good economy with a modification in endowment $\left(\overrightarrow{pi}_0 \cdot \overrightarrow{e}^i_0, \overrightarrow{pi}_1 \cdot \overrightarrow{e}^i_1\right)$ and utility $v^i\left(x^i_{01} f(\overrightarrow{w}_0), x^i_{s1} f(\overrightarrow{w}_1)\right)$.

Finally, we need one more clarification about market clearing conditions in period0. In a single-good economy, it is clear that the intermediaries use the revenue from transaction fees to purchase a single good since there is no alternative substitute except the good. However, in a multi-good economy, we need to specify the intermediaries' utility function to determine the corresponding quantity of commodity choices at date 0 from the nominal quantity of the intermediaries's revenue. It would be reasonable to assume that the intermediaries' utility function at date 0 is the same as $f(x_0)$. Then, the intermediaries' aggregate budget constraint in the spot market at date 0 is

$$\overrightarrow{p}_0 \cdot \overrightarrow{x}_0 = p\theta \textstyle\sum_i \max(n^i, 0). \tag{2.12}$$

According to the assumption that the intermediaries' utility function at date 0 is the same $f(\overrightarrow{x}_0)$, $\overrightarrow{x}_0$ is collinear with $\overrightarrow{w}_0$. Assuming that $\overrightarrow{x}_0 = s\overrightarrow{w}_0$, we get the value of s from eqn. (2.12):

$$\overrightarrow{p}_0 \cdot s\overrightarrow{w}_0 = p\theta \textstyle\sum_i \max(n^i, 0) \Rightarrow s = \dfrac{p\theta \sum_i \max(n^i, 0)}{\overrightarrow{p}_0 \cdot \overrightarrow{w}_0}.$$

Therefore, the intermediates's demand function $\overrightarrow{x}_0$ can be derived as

$$\begin{aligned} \overrightarrow{x}_0 &= \frac{p\theta \sum_i \max(n^i, 0)}{\overrightarrow{p}_0 \cdot \overrightarrow{w}_0} \overrightarrow{w}_0 = \frac{p\theta \sum_i \max(n^i, 0)}{p_0 \overrightarrow{pi}_0 \cdot \overrightarrow{w}_0} \overrightarrow{w}_0 \\ &= \frac{p}{p_0} \theta \textstyle\sum_i \max(n^i, 0) \overrightarrow{w}_0. \end{aligned} \tag{2.13}$$

From eqn (2.13), we know that $\overrightarrow{x}_0$ is homogeneous degree one in prices, which is a necessary property in proving the existence and regularity of an economy.

Finally, the market clear conditions can be expressed as

$$\sum_i m^i = 0, \sum_i n^i = 0$$

$$\sum_i x_{01}^i + \frac{p}{p_0}\theta \sum_i \max(n^i, 0) = \sum_i e_{0i}^i$$

which are equivalent to those in a single-good economy.[18]

Therefore, all of the results including Propositions 1-6 can be directly applied for a multi-good case.

Proposition 2.7 *If Assumption 1 holds in a multi-good economy, the following are true:*

(i) A regular economy given price-level volatility σ^R is defined. (Proposition 1)

(ii) Money is always traded if $\theta > 0$. (Proposition 2)

(iii) There exists a $\overline{\theta}$ such that the indexed bonds are (not) traded if $\theta < \overline{\theta}$ $\left(\theta > \overline{\theta}\right)$. (Proposition 3)

(iv) $\overline{\theta}$ strictly increases in price-level volatility σ^R if v^i is additively separable. (Proposition 4)

(v) (+) There exists at least one consumer who is better off. (++)All consumers can be better off for some economic fundamentals. (Proposition 5)

(vi) A combined market is "superior" to a (pure) monetary market based on the compensation principal (Proposition 6).

[18] The market clearing condition for period-0 good can be derived from the following equations:

$$\sum_i \overrightarrow{x}_0^i + \frac{p}{p_0}\theta \sum_i \max(n^i, 0)\overrightarrow{w}_0 = \sum_i \overrightarrow{e}_0^i,$$

$$\overrightarrow{x}_0^i = x_{01}^i \overrightarrow{w}_0 \quad \text{and} \quad \sum_i \overrightarrow{e}_0^i = \left(\sum_i e_{01}^i\right)\overrightarrow{w}_0$$

The definition of price-level volatility is still clear in a multi-good economy since there is no variation in relative prices across states. In contrast, the inflation level between two periods can vary depending on which commodity is considered when defining inflation because $\overrightarrow{pi}_0$ and $\overrightarrow{pi}_1$ are not necessarily linearly independent. Nevertheless, there is no ambiguity in determining the inflation volatility based on the CPI. The inflation from the CPI is defined as

$$\widetilde{\Pi} = c\frac{\overrightarrow{p}_1 \cdot \overrightarrow{w}_1}{\overrightarrow{p}_0 \cdot \overrightarrow{w}_0}$$

where $\overrightarrow{p}_1$ $(= \{\overrightarrow{p}_\alpha, \overrightarrow{p}_\beta; \pi_\alpha, \pi_\beta\})$ is a random variable vector and c is some positive constant. With the two equations: $\overrightarrow{pi}_t \cdot \overrightarrow{w}_t = 1$ and $\overrightarrow{p}_s = p_s \overrightarrow{pi}_t \cdot \overrightarrow{w}_t$ for $s = \alpha, \beta$, Π can be expressed as

$$\widetilde{\Pi} = c\frac{\overrightarrow{p}_1 \cdot \overrightarrow{w}_1}{\overrightarrow{p}_0 \cdot \overrightarrow{w}_0} = c'\frac{\widetilde{p}_1}{p_0}.$$

where $\widetilde{p}_1 = \{p_\alpha, p_\beta; \pi_\alpha, \pi_\beta\}$ is a random variable and c' is some positive constant. Finally, we can show that the inflation volatility based on the CPI is the same as the price-level volatility σ^R.

$$\frac{\sigma\left(\widetilde{\Pi}\right)}{E\left(\widetilde{\Pi}\right)} = \frac{\sigma\left(\widetilde{p}_1\right)}{E\left(\widetilde{p}_1\right)} = \sigma^R$$

2.7 Concluding remarks

This paper shows that inflation-indexed bonds can be more actively traded in an economy with higher inflation volatility, which is exogenously given. In the same way, we can show that the uncertainty of a nominal interest rate can provide room for inflation-indexed bonds.[19] This idea is basically the same as in Magill and Quinzii (1997). Using a two-period general equilibrium model, they show that the

[19] To show this, the nominal returns in two states should be distinguished, i.e., $r_\alpha \neq r_\beta$.

Central Bank's imperfect process of controlling the money supply can cause the economy to prefer indexed bonds. While inflation volatility is based on market psychology, volatility in the nominal interest rate is caused by an inconsistent monetary policy.

Recently, a significant number of such inflation-indexed bonds have been issued in many countries. These bonds are taking up a more important position as widely accepted financial instruments. The findings of the study can provide a better understanding of these bonds from a theoretical point of view.

CHAPTER 3

INFLATION-INDEXED BONDS AND NOMINAL BONDS: FINANCIAL INNOVATION AND ASSET TRADING

3.1 Introduction

Economists have long advocated for the creation of inflation-indexed bonds, mainly based on the premise that indexed bonds are beneficial substitutes for nominal assets, whose payoffs are contaminated by inflation volatility. However, this paper demonstrates that borrowers would be even worse off with the introduction of indexed bonds, as this situation causes equilibrium interest rates of nominal securities to increase when agents have precautionary saving motives. In an economy without indexed bonds, agents face income uncertainty from fluctuations in real payoffs of nominal securities. The income risks from nominal securities prompt agents to have a demand for precautionary savings, which drives up the equilibrium price of nominal securities. However, the introduction of indexed bonds relaxes the precautionary savings demand for nominal securities in two ways: it decreases income risks as agents trade in indexed bonds (risk-free asset) and also can be a substitute for nominal securities for the purpose of precautionary savings. The decrease in demand for nominal securities, by innovation on indexed bonds, results in a decrease in the equilibrium price of nominal securities (equivalently, an increase in a return of the securities). To clearly understand the impact of precautionary motives on a change in the equilibrium price through financial innovation, I use quasi-linear utility functions to suppress the impact of marginal utility of a date-0 good to be constant because *relative* asset prices also depends

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on the marginal utility of the date-0 good.

This paper also demonstrates the separate roles of risk aversion effects and precautionary effects in financial innovation. The effects of risk aversion in financial innovation on indexed bonds induces the trading volume of nominal securities to be driven down but surprisingly has no impact on the equilibrium nominal interest rates. This result implies that using mean-variance preferences based on the Capital Asset Price Model (CAPM), where agents lack precautionary savings motives, overestimates the borrowers' welfare gains but underestimates the savers' gains by financial innovation on indexed bonds. More importantly, as CAPM-based utility functions minimize the impact of financial innovation on existing asset prices, Pareto improvement with the introduction of indexed bonds is guaranteed, as shown in Geonakoplos (2005) and Magill and Quinzii (1997).

To understand the impact of financial innovation, the model needs to ensure the coexistence of the two types of bonds. However in a standard frictionless GE model, the introduction of inflation-indexed bonds results in a complete shut-down of nominal security markets if the economy is contaminated with inflation uncertainty. The assumption of frictionless asset markets leads to this empirically-unsupported result because agents have no incentive to take any unnecessary risks by trading in nominal securities when riskless assets such as indexed bonds are feasible.[1] Therefore, this paper assumes proportional transaction costs in asset trading. Under the assumption that transaction costs of indexed bonds are higher than those of nominal bonds, this paper shows that both types of securities would

[1] For example, illiquidity of Treasury Inflation-Protected Securities (TIPS) relative to nominal securities is believed to result in high TIPS yields with a liquidity premium. D'Amico, Kim and Wei (2009), and Pflueger and Viceira (2011) empirically estimated a liquidity premium in TIPS yields ranging from 0.4% to 1% in each year since TIPS were first issued in 1996. See Shen (2006), D'Amico, Kim and Wei (2009), Fleming and Krishnan (2009) for a detailed empirical analysis of TIPS' liquidity risks, yields, and trading volumes. See also Campbell, Shiller and Viceira (2009); Dudley, Roush and Ezer (2009); Gürkaynak, Sack, and Wright (2010).

be active in the market.

The theoretical background of the invention of inflation-indexed bonds is based on the fact that a monetary economy cannot be free from inflation fluctuations and thus nominal securities should have an inflation risk premium. To explain how inflation and inflation uncertainty are generated from monetary shocks, the model in this paper incorporates the quantity theory of money, in which price levels are directly affected by the money supply. Specifically, agents must sell their entire endowment for money in each period, which also is assumed in Diamond (1984), Lucas (1980; 1990), and Magill and Quinzii (1992).[2] Then, the price level is no longer a free variable in the economy but endogenously determined by the quantity of money and the total aggregate endowment.

In this paper, preferences and endowments are assumed to be state independent while the distinction among states is characterized by the different amounts of money supply. Without any real economic fundamental fluctuations across states, there is no reason for the central bank to provide different quantities of money. Therefore, any monetary shocks in this model can be interpreted as a source of "instability" in a monetary policy.

This paper also derives intuitive results about the impact of monetary instability (equivalently, the impact of inflation volatility) on financial markets and welfare.[3] As the level of monetary instability is increasing, the maximum possible transaction cost for indexed bonds being active is also increasing. In addition, when the economy with the mean-variance preferences has a more unstable mon-

[2]Lucas (1980) and Magill and Quinzii (1992) interpreted an endowment as endowed labor, which cannot be consumed directly by the endowment owner but can be transformed into consumption goods with a constant return. In addition, this paper assumes that the endowment is also used as transaction costs for operating financial and commodity markets.

[3]In this paper, the measure of monetrary instatibility is defined as a *relative* standard deviation of monetary shocks. See Section 5.1.

etary policy, the trading volume of nominal bonds decreases while that of indexed bonds increases.

The rest of the paper is organized as follows. Section 2 introduces the main model of this paper based on the quantity theory of money. Section 3 deals with the existence and uniqueness issues of equilibrium. Section 4 shows the comparative statistics of the active/inactive bonds market in the space of transaction costs. Section 5 shows the impact of monetary policy instability on bond market trading volumes. Section 6 investigates how financial innovation on indexed bonds affects the return of nominal securities and welfare based on the theory of precautionary savings. Section 7 concludes.

3.2 Economy

3.2.1 Preference and Endowment

There are two periods, date-0 and date-1, labelled as subscripts $t = 0, 1$. At date 1, there are two states, $s = \alpha, \beta$ having positive probabilities $0 < \pi_\alpha < 1$ and $\pi_\beta = 1 - \pi_\alpha$, respectively. There are two agents, labelled as superscripts $i \in I = \{1, 2\}$. Agent i's consumption allocation is $x^i = \left(x_0^i, x_\alpha^i, x_\beta^i\right) \in X = \mathbb{R}_{++}^3$ corresponding to prices $p = (p_0, p_\alpha, p_\beta) \gg 0$. His endowment is $e^i = \left(e_0^i, e_\alpha^i, e_\beta^i\right) \in X$ where $e_\alpha^i = e_\beta^i = e_1^i$. Denote by U the set of two agents' utility functions which are twice differentiable, strictly increasing, strictly concave, having the closure of indifference curves contained in $\mathbb{R}_{++}^2$ and satisfying the von Neumann-Morgenstern expected utility hypothesis. Denote by e the set of two agents' endowments. Agent i's preferences are

$$u^i(x^i) = u^i\left(x_0^i, x_\alpha^i, x_\beta^i\right) = \pi_\alpha v^i\left(x_0^i, x_\alpha^i\right) + \pi_\beta v^i\left(x_0^i, x_\beta^i\right) \tag{3.1}$$

3.2.2 Monetary policy and Price level

In this paper, monetary shocks are the only source of inflation uncertainty.[4] The following monetary economy model was initially introduced in Magill and Quinzii (1992). The price levels are determined by the quantity of money supply (quantity theory of money) and the aggregate endowment. Throughout this paper, we assume that the money supply at date 1 is large enough to maintain the equilibrium nominal interest rate not to be negative.

The monetary policy is characterized as (M_0, M_α, M_β) which represents the quantity of the money supply from the Central Exchange in each state. In subperiod 1 at state s, the Central Exchange injects a quantity of money M_s. Agents can hold the money in the exchange with endowments. At this step, the price level of the economy is determined by the following equation:

$$M_s = p_s \sum_{i\in I} e_s^i \quad \Leftrightarrow \quad p_s = \frac{M_s}{\sum_{i\in I} e_s^i} \tag{3.2}$$

$$where \quad s = 0, \alpha, \beta$$

For the given the price level p_s by eqn (3.2), the amount of money holdings for each agent is determined by the following equation:

$$m_s^i = p_s e_s^i \tag{3.3}$$

[4]Kang (2012a) shows that the GEI model with pure inside money, whose model was originally developed by Cass (1989; 1992) can have inflation uncertainty triggered by sunspots.

As Magill and Quinzii (2002) indicated, the initial endowment e_s^i also can be interpreted as the amount of labor that agent i holds. Agent i exchanges the labor for money through the Central Exchange. The labor can be used to produce a consumption good with linear technology and also to operate financial/commodity markets. The amount of labor used in operating the financial markets represents the financial transaction costs, which are introduced in the next section.

3.2.3 Security and commodity markets

There are two types of bonds: a nominal bond with a nominal return, and an indexed bond whose payoffs are adjusted by the price level. The transaction costs in the security markets are incurred at date 0; hence the purchasing prices of securities are higher than the selling prices. The prices and payoffs of the two bond markets are summarized in Table 1 below.

Table 3.1: Security transaction costs

	transaction cost	buying price	selling price	nominal return
nominal bond	θ_n	$q_n\,(1+\theta_n)$	q_n	$(1,1)$
indexed bond	θ_d	$q_d\,(1+\theta_d)$	q_d	(p_α, p_β)

The amount of transaction costs for obtaining one unit of security is proportional to the purchasing or selling prices. For example, if the purchasing price is \$100 and the relative transaction cost θ is 5%, then \$5 (5% of \$100) is incurred as the transaction fee at date 0. To define the market clearing condition, we need to assume that the date-0 endowment is used up for operating the financial transaction.[5] There is also a proportional transaction cost θ_c in the commodity market. The transaction cost in the commodity market is not the main interest in this paper, but it should be considered because it is usually believed that operating commodity markets is more costly than operating financial markets. Denote by θ the set of transaction costs $(\theta_n, \theta_d, \theta_c)$

At date 0, agents make portfolio and consumption decisions with the money they hold, m_0^i for $i \in I$. The quantity of money an agent i holds, m_0^i is determined by agent i's endowment and price level. (See eqn (3.3).) In the notation, m_s^i represents the amount of agent i's money holding before assessing the security markets while $m_s'^i$ represents the money holding after the security markets but before accessing the commodity markets. The amount of money holding at each subperiod of date-0 and 1 is summarized in Table 2 below.

Table 3.2: Time line

Time line		*security market*		*commodity market*	
money holding	m_s^i		$m_s'^i$		$m_s'^i$ date 0 0 date 1

The amount of money holding immediately before agents access the commodity market is determine by their portfolio decision:

$$m_s'^i = m_0^i - \left\{ \begin{array}{l} q_n\left(1+\theta_n\right)\max\left(z_n^i, 0\right) + q_n \min\left(z_n^i, 0\right) \\ +q_p\left(1+\theta_p\right)\max\left(z_d^i, 0\right) + q_p \min\left(z_d^i, 0\right) \\ +m_{01}^i \end{array} \right\} \quad (3.4)$$

m_{01}^i is the amount of money stored at date 0 to use at date 1, which is not

[5] Foley (1970) also assumed real resource (endowment) costs in the operation of commodity markets and analyzed the consequences of the costs.

allowed to be negative. Agents would have no incentive to carry money balances from date-0 to date 1, if the borrowing nominal interest rate was not negative, i.e., $\frac{1}{q_n(1+\theta_n)} > 1$. A positive nominal interest rate can be achieved if the money supply at date 1 is large enough compared to that at date 0. In this paper, we assume that the Central Exchange has a monetary plan to maintain a positive nominal interest rate. With this assumption, there is no incentive for agents to hold money at date 0, i.e., $m^i_{01} = 0$.

Agent i accesses to the commodity markets with the amount of money m'^i_0 :

$$m'^i_0 = p_0 (1 + \theta_c) x^i_0 \tag{3.5}$$

where θ_c is a proportional transaction cost in the commodity market. Combining eqns (3.3), (3.4), and (3.5), we can get the following simple budget constraint at date 0:

$$\begin{aligned} & p_0 (1 + \theta_c) x^i_0 \\ & +q_n (1 + \theta_n) \max\left(z^i_n, 0\right) + q_n \min\left(z^i_n, 0\right) \\ & +q_d (1 + \theta_d) \max\left(z^i_d, 0\right) + q_d \min\left(z^i_d, 0\right) \\ = \; & p_0 e^i_0 \end{aligned} \tag{3.6}$$

In the same way, at date 1 the amount of money holding immediately before agents' access to commodity market is determine by the portfolio payoffs:

$$m'^i_s = m^i_s + z^i_n + p_s z^i_d, \quad s = \alpha, \beta \tag{3.7}$$

Agent i accesses to the commodity markets with the amount of money m'^i_s :

$$p_s (1 + \theta_c) x^i_s = m'^i_s, \quad s = \alpha, \beta \tag{3.8}$$

Combining eqns (3.3), (3.7), and (3.8), we get the following budget constraint:

$$p_s (1 + \theta_c) x_s^i = p_s e_s^i + z_n^i + p_s z_d^i, \quad s = \alpha, \beta \tag{3.9}$$

3.2.4 Definition of Equilibrium

With the assumption that the money supply at date 1 is large enough to maintain a positive nominal interest rate, we can define the equilibrium of the economy from eqns (3.2), (3.6), and (3.9).

An equilibrium of the economy $\mathcal{E}(U, e, M, \theta)$ is a pair of actions and prices $\left((x^i, z^i)_{i=1}^I, (p, q)\right)$, where $p = (p_0, p_\alpha, p_\beta)$ and $q = (q_n, q_d)$ such that

(1) $x^i \in \arg\max \{U^i(x^i) | x^i \in B^i(p, q, e^i, \theta)\}$, for $i \in I$.

(2) (Monetary market clearing condition)

$$p_s \sum_{i \in I} e_s^i = M_s, \;\; s = 0, \alpha, \beta$$

(3) (Security market clearing condition)

$$\sum_{i \in I} z_n^i = 0 \quad \text{and} \quad \sum_{i \in I} z_d^i = 0.$$

(4) (date-0 commodity market clearing condition)

$$p_0 \sum_{i \in I} (1 + \theta_c) x_0^i + q_n \sum_{i \in I} \max(z_n^i, 0) + q_d \sum_{i \in I} \max(z_d^i, 0) = p_0 \sum_{i \in I} e_0^i$$

(5) (date-1 commodity market clearing condition)

$$\sum_{i \in I} e_1^i = (1 + \theta_c) \sum_{i \in I} x_\alpha^i = (1 + \theta_c) \sum_{i \in I} x_\beta^i$$

where the budget constraint is defined as

$$B^i(p,q,e^i,\theta) = \left\{ \begin{array}{l} x^i \in \mathbb{R}^3_+/\{0\}| \\ \;\; p_0\,(1+\theta_c)\,x^i_0 \\ \;\; +q_n\,(1+\theta_n)\max\left(z^i_n,0\right) + q_n \min\left(z^i_n,0\right) \\ \;\; +q_d\,(1+\theta_d)\max\left(z^i_d,0\right) + q_d \min\left(z^i_d,0\right) \\ \;\; = p_0 e^i_0 \\ p_s\,(1+\theta_c)\,x^i_s = p_s e^i_s + z^i_n + p_s z^i_d, \quad s=\alpha,\beta \end{array} \right\} \tag{3.10}$$

3.3 Existence and Uniqueness

3.3.1 Overview

Because of the presence of security transaction costs, the budget constraint is not differentiable when there is no trade in a security. If the equilibrium is on the kink point (non-differentiable point of the budget constraint), the price of the asset is not locally unique; thus Debreu's (1970) approach for regular economies is not applicable. However, this difficult can be resolved by the general equilibrium (GE) property that if there is no trade in a security whose net supply is zero, the equilibrium allocations without the security is identical to those with the original economy (Property 1). Based on this idea, we define four different economies: an economy with both types of bonds, an economy only with a nominal security, an economy only with an indexed security and an economy with no securities. I show that each economy is regular when the feasible bonds in each economy are traded. To prove this, I borrow the methodology which was used to prove regular economics in Debreu (1970). The next step, which is introduced in Section 4, is to check whether the security markets are really active or inactive, i.e., to check which "economy" is "true" among the four economies. For this step, I use another GE property that if a security market is not active, there is no available security prices (including buying and selling prices) for the security being tradable in equilibrium (Property 2). Specifically, under the assumption that the indexed bond market is inactive, we can derive equilibrium allocations by Property 1. Then, we can compute the agents' values (potentially prices) of the indexed bond.[6] Finally, we can check whether the gap between the two agents' values of the indexed bond is large enough so that there are no available buying and selling prices for the indexed bond being tradable. If the gap is not large enough, the assumption that the indexed bond markets are inactive is wrong and thus the equilibrium should be derived from the "different" economy *with* an indexed bond market.

The complete proof of the existence and local uniqueness is done in both this section and Section 4. As mentioned above, we assume that there are four "different" economies in this section. Section 4 shows that given the financial transaction costs (θ_n, θ_d), only one economy from the four "different" economies is "true" and eventually there are four different active/inactive regions of security markets in the space of transaction costs (θ_n, θ_d). See Figure 3.1.

Before going on to the main proof of this section, I introduce a simple trick to deal with the commodity transaction cost for the convenience of math.

3.3.2 Commodity transaction costs

For the convenience of the existence and uniqueness proofs, it would be necessary to simplify the mathematical steps associated with the commodity transaction

[6] These values are the ratio of the marginal utility of the indexed bonds, relative to the marginal utility of date-0 good.

costs. Defining the utility function $\overline{v}^i$ as

$$\overline{v}^i\left(\overline{x}_0^i, \overline{x}_1^i\right) = v^i\left(\frac{\overline{x}_0^i}{1+\theta_c}, \frac{\overline{x}_1^i}{1+\theta_c}\right) \tag{3.11}$$

$$where\ \left(\overline{x}_0^i, \overline{x}_1^i\right) = (1+\theta_c)\left(x_0^i, x_1^i\right) \text{ or}$$

$$\left(\overline{x}_0^i, \overline{x}_\alpha^i, \overline{x}_\beta^i\right) = (1+\theta_c)\left(x_0^i, x_\alpha^i, x_\beta^i\right), \tag{3.12}$$

the original maximization problem becomes identical to that of $\overline{v}^i\left(\overline{x}_0^i, \overline{x}_1^i\right)$ and the zero commodity transaction cost $\theta_c = 0$. $\overline{v}^i$ still satisfies the properties such as monotonicity and concavity, which are described in Section 2-1. Therefore, we can solve the equilibrium with $\overline{v}^i$ and $\left(\overline{x}_0^i, \overline{x}_\alpha^i, \overline{x}_\beta^i\right)$ and recover the original equilibrium allocations $\left(x_0^i, x_\alpha^i, x_\beta^i\right)$ from eqn (3.12).[7] For simplification of the notations, from this point in this paper, v^i and $\left(x_0^i, x_\alpha^i, x_\beta^i\right)$ are assumed to be $\overline{v}^i$ and $\left(\overline{x}_0^i, \overline{x}_\alpha^i, \overline{x}_\beta^i\right)$, respectively. Then, the economy can be defined with a zero commodity transaction cost.

3.3.3 Existence and uniqueness

By suppressing the commodity transaction cost as zero, agent i' maximization is as follows:

$$\max_{x_0^i, z_n^i, z_d^i} \pi_\alpha v^i\left(x_0^i, e_1^i + \frac{1}{p_\alpha} z_n^i + z_d^i\right) + \pi_\beta v^i\left(x_0^i, e_1^i + \frac{1}{p_\beta} z_n^i + z_d^i\right) \tag{3.13}$$

[7] In addition, If v^i is homothetic, θ_c does not have real effects on equilibrium asset prices and trading volumes.

$$\text{s.t} \quad \begin{aligned} & p_0 x_0^i \\ & + q_n (1+\theta_n) \max(z_n^i, 0) + q_n \min(z_n^i, 0) \\ & + q_d (1+\theta_d) \max(z_d^i, 0) + q_d \min(z_d^i, 0) \\ & = p_0 e_0^i \end{aligned} \tag{3.14}$$

where

$$p_s = \frac{M_s}{\sum_{i \in I} e_s^i}, \quad s = 0, \alpha, \beta.$$

In the maximization problem described in eqns (3.13) and (3.14), agent i chooses (x_0^i, z_n^i, z_d^i) at date 0. Based on the three variables for the optimal choices (x_0^i, z_n^i, z_d^i), we can define the following three market clearing conditions:

$$\sum_{i \in I} e_0^i = \sum_{i \in I} x_0^i - \frac{q_n}{p_0} \theta_n \sum_{i \in I} \max\left(z_n^i, 0\right) - \frac{q_d}{p_0} \theta_d \sum_{i \in I} \max\left(z_d^i, 0\right), \tag{3.15}$$

$$\sum_{i \in I} z_n^i = 0 \text{ and } \sum_{i \in I} z_d^i = 0 \tag{3.16}$$

Finally, the original economy is simplified as the economy with optimal choices (x_0^i, z_n^i, z_d^i) given prices (p_0, q_n, q_d) and market clearing conditions for (x_0, z_n, z_d).[8] Then, we can define the market aggregate demand functions of (x_0^i, z_n^i, z_d^i) given prices (p_0, q_n, q_d), which are shown in the proof of Proposition 1 in this section.

Although the budget constraints are not differentiable at $z_n^i = 0$ and $z_d^i = 0$, the budget constraint sets are connected and convex, hence each agent's maximization problem has unique equilibrium allocations given any prices. Therefore, it can be shown that where $z_n^i \neq 0$ and $z_d^i \neq 0$, the economy has a finite number of local unique equilibrium prices and allocations. When one of the security markets is

[8] The commodity market clearing conditions for x_α^i and x_β^i are automatically satisfied if the security market clearing conditions in eqn (3.16) hold.

inactive (the equilibrium is at the kink), for example $z_n^i = 0$, the equilibrium price q_n is not locally unique. Even in this case, all the equilibrium allocations and all the prices except the price of the inactive security are locally unique. The proof of Poposition1 is based on the GE property that if a financial market is inactive, the financial market has no real impact on the economy. In other words, the economy with an inactive financial market has the same equilibrium allocations with the economy without the financial market.

Proposition 3.1 *The economy* $\mathcal{E}(U, e, M, \theta)$

(1) is regular if $z_n^i \neq 0$ *and* $z_d^i \neq 0$.

(2) has locally unique equilibrium allocations and prices except q_n *if* $z_n^i = 0$ *and* $z_d^i \neq 0$.

(3) has locally unique equilibrium allocations and prices except q_n *if* $z_n^i \neq 0$ *and* $z_d^i = 0$.

(4) has locally unique equilibrium allocations and prices except (q_n, q_d) *if* $z_n^i = 0$ *and* $z_d^i = 0$.

Proof. Case (1): For the proof of a regular economy, we need to define the aggregate excess demand and check that it satisfies sufficient conditions for the economy to be regular. The market aggregate demand function is defined as

$$Z(P) = \begin{pmatrix} \sum_{i \in I} z_0^i(P) - \frac{q_n}{p_0}\theta_n \sum_{i \in I} \max(z_n^i, 0) - \frac{q_n}{p_0}\theta_d \sum_{i \in I} \max(z_n^i, 0) \\ z_n^i(P) \\ z_d^i(P) \end{pmatrix}$$

$$\text{where} \quad P = (p_0, q_n, q_d) \quad \text{and} \quad z_0^i(P) = x_0^i(P) - e_0^i.$$

In the market aggregate demand functions, $\frac{q_n}{p_0}\theta_n \sum_{i \in I} \max(z_n^i, 0)$ and $\frac{q_d}{p_0}\theta_d \sum_{i \in I} \max(z_d^i, 0)$ represents the amount of the transaction costs in the two financial markets, respectively. It can be easily shown that $Z(P)$ satisfies the five properties:

(i) continuous

(ii) homogeneous of degree zero

(iii) (Walras' law) $P \cdot Z(P) = 0$

(iv) There is an $s > 0$ such that $Z(P) > (-s, -s, -s)$ for all P.

(v) If $P^n \to P$, where $P \neq 0$ and $q_n = 0$, then $\sum_{i \in I} z_n^i(P) \to \infty$.

Therefore, the economy is regular and determinate. (See Debreu 1970.)

Case (2): Since the indexed bond market is inactive, the equilibrium allocations from the original economy are the same as those in the economy without the indexed bond market. Under the economy without the indexed bond market, the market aggregate demand function is defined as

$$Z(P) = \begin{pmatrix} \sum_{i \in I} z_0^i(P) - \frac{q_n}{p_0}\theta_n \sum_{i \in I} \max(z_n^i, 0) \\ z_n^i(P) \end{pmatrix},$$

$$\text{where} \quad P = (p_0, q_n) \quad \text{and} \quad z_0^i(P) = x_0^i(P) - e_0^i.$$

Where $z_n^i \neq 0$, $Z(P)$ satisfies that the five properties and a regular economy can be defined.

Case (3): We can prove it in the same way as (2).

Case (4): The proof is trivial. In this case, the two types of security markets are totally inactive and thus agents consume their own endowments. ■

A regular economy does not always guarantee globally unique equilibrium but in most cases including an economy with log-linear preferences or with quasi-linear preferences, which are the two main numerical examples in this paper, a globally unique equilibrium does exist. Proposition 1 shows the existence and local uniqueness of the equilibrium allocations in the four distinct cases but does not indicate under what conditions the equilibrium belongs out of the four cases. The next section answers that question.

3.4 Active and Inactive financial markets

3.4.1 Overview

The main results in this section are as follows. Figure 3.1 represents the four different regions in the space of two transaction costs. There is a 45 degree line which bisects region 1 and region 3. If the transaction costs are in region 1, both nominal and indexed bond markets are active. In region 3, the indexed bond market is active while the nominal bond market is inactive. At the top end of the 45 degree line, there is an "L"-shape hall, in which both security markets are inactive. In the upper-left region above the 45-degree line, there is an increasing curve which bisects region 1 and region 2. The 45-degree line and the "L"-shape hall is not affected by the monetary policy. In contrast, the 1-2 bisecting curve is shifting up as the measure of monetary instability σ_M^R [9] is increasing, which is proven in Section 5. Under the perfectly stable monetary policy, i.e., $\sigma_M^R = 0$, the curve converges to the 45-degree line and thus the region 1 is eliminated. Propositions 2-4 in this section show that the four regions are determined in the

[9] See Section 5.1 for the definition of monetary instatibility σ_M^R.

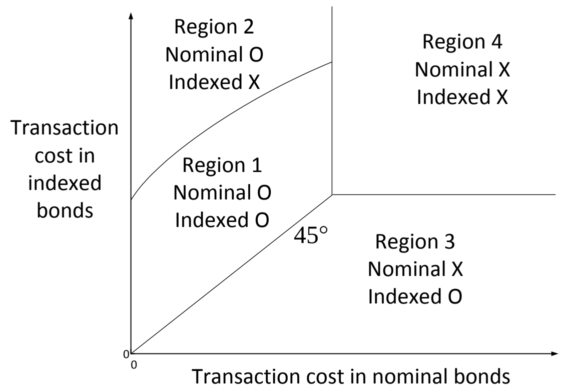

Figure 3.1: Transaction costs and asset trading

way described in Figure B.1.

Before going on the proof, we define an equivalent asset to the nominal security for the convenience of proofs. The new risky asset b^i has endogenously determined selling and purchasing prices q_b and $(1+\theta_d)\,q_b$ and returns (r_α, r_β) which are uniquely determined by price levels at date 1 through the following relations:

$$(r_\alpha, r_\beta) = \frac{1}{\frac{\pi_\alpha}{p_\alpha} + \frac{\pi_\beta}{p_\beta}} \left(\frac{1}{p_\alpha}, \frac{1}{p_\beta} \right). \tag{3.17}$$

The expected real return of the equivalent bonds is "1" which is also the real return of the indexed bond. Replacing the nominal bond with the new asset does not change the equilibrium allocations because the ratio of returns across the states are identical. The prices of the new asset and the nominal security have the following relations:

$$q_n = q_b \left(\frac{\pi_\alpha}{p_\alpha} + \frac{\pi_\beta}{p_\beta} \right)$$

Through this paper, I use the equivalent asset b instead of z_n for computational convenience.

3.4.2 Case (4) - "L-shape hall"

In this model, agents *have to* access to the commodity markets to get consumption goods but they may not need to access financial markets if their initial endowment is "near" Pareto optimal allocation. This section investigate the condition when at least one type of financial market is active. Assuming that there are no transaction costs, i.e., $\theta_n = \theta_d = 0$, the condition for active financial markets is that the endowment allocation is not Pareto optimal. That is, there are two agents i and i' such that

$$\frac{\partial v^i\left(e_0^i, e_1^i\right)}{\partial x_1} / \frac{\partial v^i\left(e_0^i, e_1^i\right)}{\partial x_0} \neq \frac{\partial v^{i'}\left(e_0^{i'}, e_1^{i'}\right)}{\partial x_1} / \frac{\partial v^{i'}\left(e_0^{i'}, e_1^{i'}\right)}{\partial x_0}. \tag{3.18}$$

For an economy with financial transaction costs, we need a more strict condition than condition (3.18). The following proposition indicates the condition that at least one type of security markets is active.

Proposition 3.2 *At least one type of security market is active if and only if*

$$\max_{i,i' \in I} \left[\frac{\frac{\partial v^i\left(e_0^i, e_1^i\right)}{\partial x_1} / \frac{\partial v^i\left(e_0^i, e_1^i\right)}{\partial x_0}}{\frac{\partial v^{i'}\left(e_0^{i'}, e_1^{i'}\right)}{\partial x_1} / \frac{\partial v^{i'}\left(e_0^{i'}, e_1^{i'}\right)}{\partial x_0}} \right] > 1 + \min\left(\theta_n, \theta_d\right) \tag{3.19}$$

Proof. At the endowment allocation, agent i's marginal utility of a nominal security b^i is given by

$$\pi_\alpha \frac{\partial v^i\left(e_0^i, e_1^i + r_\alpha b^i\right)}{\partial b^i}|_{b^i=0} + \pi_\beta \frac{\partial v^i\left(e_0^i, e_1^i + r_\beta b^i\right)}{\partial b^i}|_{b^i=0},$$

which has the same value as $\partial v^i\left(e_0^i, e_1^i\right)/\partial x_1$ since $\pi_\alpha r_\alpha + \pi_\beta r_\beta = 1$. In the same way, we can show that the marginal utility of an indexed bond at the endowment allocation is also the same as $\partial v^i\left(e_0^i, e_1^i\right)/\partial x_1$. Then, the marginal utility of the nominal bond (and the indexed bond), relative to that of the date-0 good, for the agent i and agent $i^{'}$ are given as

$$w^i = \frac{\partial v^i\left(e_0^i, e_1^i\right)}{\partial x_1} / \frac{\partial v^i\left(e_0^i, e_1^i\right)}{\partial x_0} \text{ and} \tag{3.20}$$

$$w^{i^{'}} = \frac{\partial v^{i'}\left(e_0^{i'}, e_1^{i'}\right)}{\partial x_1} / \frac{\partial v^{i'}\left(e_0^{i'}, e_1^{i'}\right)}{\partial x_0} \tag{3.21}$$

Without loss of generality, we assume that w^i is higher than $w^{i'}$. Now, we need to check whether they have an incentive to get access to security markets. If there exists a selling price which is higher than $w^{i'} \times p_0$ and a buying price which is lower than $w^i \times p_0$ for at least one of the two types of security markets, the security market will be active. Assuming that the price of the security is q, the necessary condition for the active security market is that

$$w^{i'} < \frac{q}{p_0} < \frac{q}{p_0}(1+\theta) < w^i. \tag{3.22}$$

The necessary and sufficient condition for the existence of the value of q, which satisfies inequality (3.22) is that

$$\frac{w^i}{w^{i'}} > (1+\theta) \tag{3.23}$$

If inequality (3.23) is violated for both θ_n and θ_d, the two types of securities are completely inactive. Therefore, the necessary and sufficient condition for at least one type of security market is operating in that

$$\frac{w^i}{w^{i'}} > \min\left(1+\theta_n, 1+\theta_d\right)$$

which is equivalent to condition (3.19). ■

If $\theta_n = \theta_d = 0$, condition (3.19) is equivalent to the condition that the initial endowment is not Pareto optimal, which is condition (3.18). As the value of $\min\left(\theta_n, \theta_d\right)$ is higher, the area for non-active financial markets is larger in the

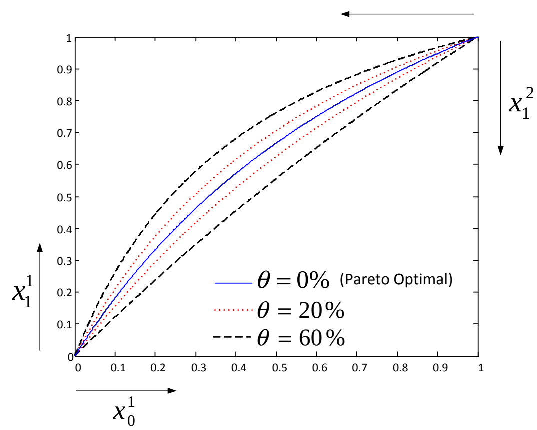

Figure 3.2: Active asset markets

Edgeworth box in Figure B.2.[10] The endowment area, in which financial markets are inactive, is the area in between the two same-shaped curves. If $\min(\theta_n, \theta_d)$ is zero, i.e., if the asset markets are frictionless, the inactive-area is the same as Pareto optimal allocations, which is the single solid curve in Figure 3.2. As $\min(\theta_n, \theta_d)$ increases, the inactive-area also increases. Through this paper, I assume that the condition in Proposition 2 holds; thus, agents have access to at least one type of financial markets.

3.4.3 Cases (1) and (3) - "the 45 degree line"

This section shows that if the transaction cost of an indexed bond is the same as that of a nominal bond (the 45-degree line in Figure 3.1), only the indexed bond market is active but the nominal bond market is not. In contrast, if a nominal bond has lower transaction costs than an indexed bond (above the 45-degree line in Figure B.1), the nominal bond market does not shut down. The following

[10]The example is based on the following economy: $u^1\left(x_0^1, x_1^1\right) = \frac{1}{2}\log x_0^1 + \frac{1}{2}\log x_1^1$; $u^2\left(x_0^2, x_1^2\right) = \frac{2}{3}\log x_0^2 + \frac{1}{3}\log x_1^2$; $e_0^1 + e_0^2 = e_1^1 + e_1^2 = 1$.

proposition addresses these issues.

Proposition 3.3 *(a) If $\theta_n < \theta_d$ (above the 45-degree line in Figure 1), a nominal bond market is active (b) If $\theta_n \geq \theta_d$ (below the 45-degree line in Figure 1), a nominal bond market is inactive while an indexed bond market is active.*

Proof. **(a)($\theta_n < \theta_d$). (by contradiction)** We assume that the nominal security market is inactive, i.e., $z_n^i = 0$ (i.e., $b^i = 0$) for all $i \in I$. Under the assumption that condition (3.19) holds, the indexed bond market should be active. Then, it is true that $z_d^i \neq 0$ for all $i \in I$. Among the two agents in the economy, one is a saver with $z_d^s > 0$ and the other is a borrower with $z_d^b < 0$, where the superscript s and b represent a saver and a borrower, respectively.

For the equilibrium allocation[11], we can compute the value of the nominal security b^i. The saver's relative value (price) of the nominal security, denoted as "q_r^s/p_0" where p_0 is the price level at date 0, is computed as

$$\begin{aligned} \frac{q_r^s}{p_0} &= \frac{\pi_\alpha \frac{\partial v^s\left(x_0^s, e_1^s + r_\alpha b^s\right)}{\partial b^s} + \pi_\beta \frac{\partial v^s\left(x_0^s, e_1^s + r_\beta b^s\right)}{\partial b^s}}{\pi_\alpha \frac{\partial v^s\left(x_0^s, x_\alpha^s\right)}{\partial x_0} + \pi_\beta \frac{\partial v^s\left(x_0^s, x_\beta^s\right)}{\partial x_0}} \\ &= \frac{\pi_\alpha r_\alpha \frac{\partial v^s\left(x_0^s, x_\alpha^s\right)}{\partial x_1} + \pi_\beta r_\beta \frac{\partial v^s\left(x_0^s, x_\beta^s\right)}{\partial x_1}}{\pi_\alpha \frac{\partial v^s\left(x_0^s, x_\alpha^s\right)}{\partial x_0} + \pi_\beta \frac{\partial v^s\left(x_0^s, x_\beta^s\right)}{\partial x_0}} = \frac{\frac{\partial v^s\left(x_0^s, x_\beta^s\right)}{\partial x_1}}{\frac{\partial v^S\left(x_0^s, x_\beta^s\right)}{\partial x_0}} \end{aligned} \tag{3.24}$$

since $b^i = 0$ and $x_\alpha^s = x_\beta^s$. q_r^s/p_0 is the same as the purchasing price of an indexed bond relative to the date-0 good price because

$$\frac{q_d(1+\theta_d)}{p_0} = \frac{\frac{\partial v^s\left(x_0^s, x_\beta^s\right)}{\partial x_1}}{\frac{\partial v^S\left(x_0^s, x_\beta^s\right)}{\partial x_0}}. \tag{3.25}$$

From eqns (3.24) and (3.25), we can get

$$q_r^s = q_d(1+\theta_d). \tag{3.26}$$

[11] This equilibrium allocation is from an economy without a nominal security market. Check case 3 in Section 3.

In the same way, we can show that the borrower's value of the nominal security is the same as the selling price of the indexed bond:

$$q_r^b = q_d. \tag{3.27}$$

Since $q_r^s = q_r^b (1 + \theta_d)$ and $\theta_n < \theta_d$, there is a price $q_b \in \left(q_r^b, q_r^s\right)$ satisfying that

$$q_r^b < q_b < q_b (1 + \theta_n) < q_r^s. \tag{3.28}$$

The inequality $q_r^b < q_b$ implies that the borrower is willing to sell the nominal security at the price of q_b and the other inequality $q_b (1 + \theta_n) < q_r^s$ implies that the saver is willing to buy the nominal security at the price of $q_b (1 + \theta_n)$. This contradicts that there is no tradable price for the nominal security. Therefore, the assumption that a nominal security market is inactive is wrong. ■

Proof. **(b)**$(\theta_n \geq \theta_d)$. In the same way in proof (a), we can derive q_r^s and q_r^b in an economy where a nominal security is not feasible. Then, we can shows that if $\theta_n \geq \theta_d$, there is no available price $q_b \in \left(q_r^b, q_r^s\right)$ which satisfies that (1) $q_r^s = q_r^b (1 + \theta_d)$ and (2) $q_r^b < q_b < q_b (1 + \theta_n) < q_r^s$. This implies that even if a nominal security is introduced, the security market is not active if $\theta_n < \theta_d$. Under the assumption that condition (3.19) holds, the indexed bond market should be active. ■

Proposition 3 also indicates that if no transaction cost is assumed, i.e., $\theta_n = \theta_d = 0$, there will be zero demand or supply for the nominal security. That implies that the introduction of an indexed bond, in a standard frictionless GE model, causes a complete shutdown of nominal security markets. However, empirical evidences show that a trading volume of a nominal bond is not significantly affected by the introduction of an inflation-indexed bond.

3.4.4 Cases (1) and (2) - "the curve"

In Figure (3.1), there is a curve bisecting region 1 and region 2. Where the transaction cost (θ_n, θ_d) is in region 1, both nominal and indexed security markets are active. However, as (θ_n, θ_d) is approaching the curve, the trading volume of the indexed bond converges at zero. In this section, I prove that there exists such a curve if the monetary policy is unstable, i.e., $M_\alpha \neq M_\beta$. For the proof of the main result in this section, we assume that an indexed bond is infeasible, which is Case 2 in Proposition 2. Even under the assumption that indexed bond market does not exist, the values of those bonds can be computed at the equilibrium allocations of the economy. Let's define q_f^i as the value of a risk-free asset for agent $i \in \{1, 2\}$:

$$q_f^i = \frac{\pi_\alpha \frac{\partial v^i\left(x_0^i, x_\alpha^i\right)}{\partial x_1} + \pi_\beta \frac{\partial v^i\left(x_0^i, x_\beta^i\right)}{\partial x_1}}{\pi_\alpha \frac{\partial v^i\left(x_0^i, x_\alpha^i\right)}{\partial x_0} + \pi_\beta \frac{\partial v^i\left(x_0^i, x_\beta^i\right)}{\partial x_0}} \times p_0,$$

which is the ratio between the marginal utility of risk-free assets and that of the period-0 good. The following lemma shows that there is an arbitrage between two values q_f^s and q_f^b where "s" and "b" represent a saver and a borrower, respectively.

Lemma 3.1 *If $M_\alpha \neq M_\beta$, then*[12]

$$q_f^s > q_f^b \left(1 + \theta_n\right).$$

Proof. q_b^+ and $q_b^+ \left(1 + \theta_n\right)$ represents the equilibrium selling and buying prices of the nominal security b^i, respectively, in a pure monetary market. At the equilibrium allocation, q_b^+ and $q_b^+ \left(1 + \theta_n\right)$ are

$$q_b^+ = \frac{\pi_\alpha r_\alpha \frac{\partial v^b\left(x_0^b, x_\alpha^b\right)}{\partial x_1} + \pi_\beta r_\beta \frac{\partial v^b\left(x_0^b, x_\beta^b\right)}{\partial x_1}}{\pi_\alpha \frac{\partial v^b\left(x_0^b, x_\alpha^b\right)}{\partial x_0} + \pi_\beta \frac{\partial v^b\left(x_0^b, x_\beta^b\right)}{\partial x_0}} \times p_0.$$

[12] Where $M_\alpha = M_\beta$, there is no arbitrage, i.e., $q_f^s = q_f^b \left(1 + \theta_n\right)$ .An economy without monetary volatility $(M_\alpha = M_\beta)$ is equivalent to a certaintyeconomy.

and

$$q_b^+ (1+\theta_n) = \frac{\pi_\alpha r_\alpha \frac{\partial v^s\left(x_0^s, x_\alpha^s\right)}{\partial x_1} + \pi_\beta r_\beta \frac{\partial v^s\left(x_0^s, x_\beta^s\right)}{\partial x_1}}{\pi_\alpha \frac{\partial v^s\left(x_0^s, x_\alpha^s\right)}{\partial x_0} + \pi_\beta \frac{\partial v^s\left(x_0^s, x_\beta^s\right)}{\partial x_0}} \times p_0.$$

where the superscripts s and b represent the saver and the borrower, respectively. Under the assumption that condition (3.19) holds, agents will trade the nominal security and therefore $b^s > 0$ and $b^b < 0$. First, we want to show that q_f^s is higher than $q_b^+ (1+\theta_n)$. $\frac{q_f^s}{q_b^+ (1+\theta_n)}$ is

$$\frac{q_f^s}{q_b^+ (1+\theta_n)} = \frac{\pi_\alpha \frac{\partial v^s\left(x_0^s, x_\alpha^s\right)}{\partial x_1} + \pi_\beta \frac{\partial v^s\left(x_0^s, x_\beta^s\right)}{\partial x_1}}{\pi_\alpha r_\alpha \frac{\partial v^s\left(x_0^s, x_\alpha^s\right)}{\partial x_1} + \pi_\beta r_\beta \frac{\partial v^s\left(x_0^s, x_\beta^s\right)}{\partial x_1}}. \tag{3.29}$$

Without loss of generality, we assume that state β is inflationary, i.e., $M_\beta > M_\alpha$. Then, we get the following two inequalities: $r_\alpha > r_\beta$ and thus, $x_\alpha^s > x_\beta^s$. Since v^i is strictly concave and $x_\alpha^s > x_\beta^s$, we can get the following relation:

$$\frac{\partial v^s\left(x_0^s, x_\alpha^s\right)}{\partial x_1} < \frac{\partial v^s\left(x_0^s, x_\beta^s\right)}{\partial x_1}. \tag{3.30}$$

Since $\pi_\alpha r_\alpha + \pi_\beta r_\beta = 1$, $\pi_\alpha + \pi_\beta = 1$ and $\pi_\alpha r_\alpha > \pi_\alpha$, $\pi_\beta r_\beta < \pi_\beta$, from eqns (3.29) and (3.30), we can get:

$$\frac{q_f^s}{q_b^+ (1+\theta_n)} > 1. \tag{3.31}$$

In the same way, we can prove the following:

$$\frac{q_f^b}{q_b^+} < 1. \tag{3.32}$$

From inequalities (3.31) and (3.32), we know that $q_f^s > q_f^b (1+\theta_n)$. ■

The arbitrage in the values of a riskless security between two agents provides the incentive for them to accept indexed bonds as a financial instrument. However, the existence of the arbitrage does not necessarily imply that the indexed bonds markets are active. The magnitude of the arbitrage should be large enough for an indexed bond to be traded in a financial market. The following proposition shows that the indexed bond market can be active or inactive depending on the level of inflation volatility and the level of transaction costs.

Proposition 3.4 *If the monetary policy is unstable, i.e., $M_\alpha \neq M_\beta$, there is a constant $\overline{\theta}_d$ such that if $\theta_d < \overline{\theta}_d$ ($\theta_d \geq \overline{\theta}_d$), an indexed bond market is active (inactive) where*

$$\overline{\theta}_d = \frac{q_f^s - q_f^b}{q_f^b} > \theta_n.$$

Proof. First, we can prove that $\frac{q_f^s - q_f^b}{q_f^b} > \theta_n$ from Lemma 1.

(i) Proof for $z_d^i \neq 0$ (active indexed bond market) where $\theta_d < \overline{\theta}_d$.

(Proof by contradiction) Let's assume that $z_d^i = 0$ for $i \in s, b$. Because $\theta_d < \overline{\theta}_d \left(= \frac{q_f^s - q_f^b}{q_f^b}\right)$, there exists $q_d > 0$ such that $q_f^b < q_d < q_d (1 + \theta_d) < q_f^s$. (This can be proven as follows: Let $q_d = q_f^b + \varepsilon$ and $\theta_d = \frac{q_f^s - q_f^b - \delta}{q_f^s} > 0$. Then, for all $\delta \in \left(0, q_f^s - q_f^b\right)$, there exists $\varepsilon > 0$ which satisfies the inequality $q_f^b < q_d < q_d (1 + \theta_d) < q_f^s$.) This implies that the two agents have an incentive to trade the indexed bond. This contradicts that $z_d^i = 0$.

(ii) Proof for $z_d^i = 0$ (inactive indexed bond) *where* $\theta_d \geq \overline{\theta}_d$.

(Proof by contradiction) Let's assume that $z_d^i \neq 0$ for $i \in s, b$. Because $\theta_d \geq \overline{\theta}_d$, there does not exist $q_d > 0$ such that $q_f^b < q_d < q_d (1 + \theta_d) < q_f^s$. This contradicts that $z_n^i \neq 0$ for $i \in s, b$. ■

Proposition 4 indicates that the curve bisecting regions 1 and 3 exists above the 45-degree line in Figure 3.1. This implies that if the two types of bonds have the same level of transaction costs, the nominal bonds markets would completely shut down but the indexed bonds markets do not. Several empirical papers have suggested that the indexed bonds markets have higher liquidity costs than the nominal bond market, and consequently the trading volume of the indexed bonds are considerably smaller than that of the nominal security. Proposition 4 supports this empirical result as showing that the trading volume of indexed bonds is low for a higher transaction.

3.5 Unstable monetary policy and Financial markets

3.5.1 Measure of monetary instability

This section proposes a measure of monetary instability and investigates how the trading volumes of bonds are affected by the measure. The measure is defined as

$$\sigma_M^R\left(\widetilde{M}\right)=\frac{\sigma\left(\widetilde{M}\right)}{E\left(\widetilde{M}\right)} \tag{3.33}$$

where $\widetilde{M}=\{M_\alpha, M_\beta; \pi_\alpha, \pi_\beta\}$. $\sigma\left(\widetilde{M}\right)$ and $E\left(\widetilde{M}\right)$ represent the standard deviation and the expected value of $\widetilde{M}$, respectively. There is a reason I define the measure of monetary uncertainty in terms of a *relative* standard deviation. In the model of this paper, any proportional changes in money supply does not change the equilibrium allocation. Specifically, an economy $\mathcal{E}(U,e,M,\theta)$ has the same equilibrium allocations with an economy $\mathcal{E}(U,e,\alpha M,\theta)$ for any $\alpha>0$. The relative standard deviation is the measure which is also invariant with any proportional changes in the random variable, i.e., $\sigma_M^R\left(\widetilde{M}\right)=\sigma_M^R\left(\alpha\widetilde{M}\right)$ for any $\alpha>0$. Therefore, we can define the economy with the measure of monetary instability σ_M^R instead of M.

Because there is no real economic fundamental fluctuations in the model, the relative standard deviation of inflation uncertainty has the same value as the measure of monetary instability. Where inflation is $i_s = p_s/p_0 - 1$, $s = \alpha,\beta$, we can define inflation volatility σ_i^R as

$$\sigma_i^R\left(\widetilde{i}\right) = \frac{\sigma\left(\widetilde{i}\right)}{E\left(\widetilde{i}\right)}. \tag{3.34}$$

where $\widetilde{i} = \{p_\alpha/p_0 - 1, p_\beta/p_0 - 1; \pi_\alpha, \pi_\beta\}$. The value of inflation volatility σ_i^R is exactly the same as that of monetary instability σ_M^R in the model of this paper.

3.5.2 The impact of monetary instability on four regions

In this section, we assume that the vNM utility function $v^i(x_0^i, x_1^i)$ is time additively separable. It is intuitive that as the monetary policy is more unstable, i.e., σ_M^R is increasing, an indexed bond has more chance to be active while a nominal bond less chance. The following proposition shows that the cut-off curve bisecting region 1 and region 2 in Figure B.1 is shifting up as the monetary policy becomes more unstable. That means that the maximum possible transaction cost for the indexed bond market being operated is increasing in the measure of monetary policy instability σ_M^R.

Proposition 3.5 *If v^i is time additively separable, i.e., $v^i\left(x_0^i, x_1^i\right) = f^i(x_0^i) + g^i(x_1^i)$ for all $i \in I$, as σ_M^R is increasing, $\overline{\theta}_d$ is increasing.*

Proof. For the proof of this proposition, we need the following Lemma.

Lemma 3.2 *In an economy without an indexed bond market, x_α^s is increasing but x_β^s is decreasing in σ_M^R where the superscript "s" represents a saver. (See the Appendix for the proof.)*

Remembering that q_f^i is defined as

$$q_f^i = \frac{\pi_\alpha g^{i'}\left(x_\alpha^i\right) + \pi_\beta g^{i'}\left(x_\beta^i\right)}{f^{i'}\left(x_0^i\right)} \times p_0$$

then, $\frac{q_f^b}{q^+}$ and $\frac{q_f^s}{q^+(1+\theta_n)}$ are

$$\frac{q_f^b}{q^+} = \frac{\pi_\alpha g^{b'}\left(x_\alpha^b\right) + \pi_\beta g^{b'}\left(x_\beta^b\right)}{\pi_\alpha r_\alpha g^{b'}\left(x_\alpha^b\right) + \pi_\beta r_\beta g^{b'}\left(x_\beta^b\right)}$$

and

$$\frac{q_f^s}{q^+\left(1+\theta_n\right)} = \frac{\pi_\alpha g^{s'}\left(x_\alpha^s\right) + \pi_\beta g^{s'}\left(x_\beta^s\right)}{\pi_\alpha r_\alpha g^{s'}\left(x_\alpha^s\right) + \pi_\beta r_\beta g^{s'}\left(x_\beta^s\right)}.$$

We need to show that $\frac{d}{d\mathcal{P}}\left(\frac{q_f^b}{q^+}\right) < 0$ and $\frac{d}{d\mathcal{P}}\left(\frac{q_f^s}{q^+}\right) > 0$. There are four variables $r_\alpha\left(\mathcal{P}\right), r_\beta\left(\mathcal{P}\right), x_\alpha^i\left(\mathcal{P}\right)$ and $x_\beta^i\left(\mathcal{P}\right)$ which should be considered in the total derivatives of $\frac{q_f^b}{q^+}$ and $\frac{q_f^s}{q^+}$. The total derivatives can be divided into two parts:

$$\frac{d}{d\mathcal{P}}\left(\frac{q_f^s}{q^+\left(1+\theta_n\right)}\right) = \frac{d}{d\mathcal{P}}\left(\frac{\pi_\alpha g^{s'}\left(x_\alpha^s\right) + \pi_\beta g^{s'}\left(x_\beta^s\right)}{\pi_\alpha r_\alpha g^{s'}\left(x_\alpha^s\right) + \pi_\beta r_\beta g^{s'}\left(x_\beta^s\right)}\right)_{x_\alpha^s, xs=\text{constant}}$$
$$+\frac{d}{d\mathcal{P}}\left(\frac{\pi_\alpha g^{s'}\left(x_\alpha^s\right) + \pi_\beta g^{s'}\left(x_\beta^s\right)}{\pi_\alpha r_\alpha g^{s'}\left(x_\alpha^s\right) + \pi_\beta r_\beta g^{s'}\left(x_\beta^s\right)}\right)_{r_\alpha, r_\beta=\text{constant}}. \quad (3.35)$$

Since g^i is strictly concave and $x_\alpha^s > x_\beta^s$ for the saver, $g^{s'}\left(x_\alpha^s\right) < g^{s'}\left(x_\beta^s\right)$. Because $\pi_\alpha r_\alpha\left(\mathcal{P}\right)$ and $\pi_\beta r_\beta\left(\mathcal{P}\right)$ increases and decreases in $\mathcal{P}$, respectively, and $\pi_\alpha r_\alpha\left(\mathcal{P}\right) + \pi_\beta r_\beta\left(\mathcal{P}\right)$ is constant in $\mathcal{P}$,

$$\frac{d}{d\mathcal{P}}\left(\pi_\alpha r_\alpha g^{s'}\left(x_\alpha^s\right) + \pi_\beta r_\beta g^{s'}\left(x_\beta^s\right)\right)_{x_\alpha^i, x_\beta^i=\text{constant}} < 0.$$

Therefore, the first term in eqn (3.35) is strictly positive.

The second term can be expressed as

$$T(G) = \frac{\pi_\alpha + \pi_\beta G}{\pi_\alpha r_\alpha + \pi_\beta r_\beta G} \quad \text{where} \quad G = \frac{g^{s'}\left(x_\beta^s\right)}{g^{s'}\left(x_\alpha^s\right)}$$

$T(G)$ is increasing in G since $\pi_\beta > \pi_\beta r_\beta$ and $\pi_\alpha + \pi_\beta = \pi_\alpha r_\alpha + \pi_\beta r_\beta$. $\frac{dT}{d\mathcal{P}}$ is positive since $\frac{dx_\alpha^s}{d\mathcal{P}} > 0$, $\frac{dx_\beta^s}{d\mathcal{P}} < 0$ (by lemma) and strictly concavity of $g^{'}$. Thus, the second term also increases in $\mathcal{P}$ and $\frac{d}{d\mathcal{P}}\left(\frac{q_f^s}{q^+(1+\theta_n)}\right) > 0$.

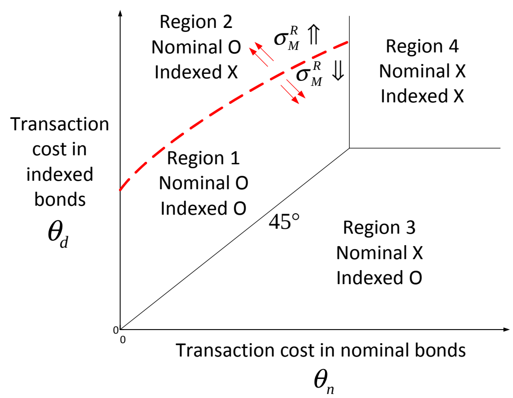

Figure 3.3: Unstable monetary policy and asset trading

We can prove that $\frac{d}{d\mathcal{P}}\left(\frac{q_f^b}{q^+}\right) < 0$ in the same way. Finally,

$$\frac{d}{d\mathcal{P}}\left(\frac{q_f^s - q_f^b}{q_f^b}\right) = \frac{d}{d\mathcal{P}}\left(\frac{q_f^s}{q_f^b}\right) = \frac{d}{d\mathcal{P}}\left(\frac{q_f^s}{q^+}\right) / \frac{d}{d\mathcal{P}}\left(\frac{q_f^b}{q^+}\right) > 0.$$

■

Proposition 5 indicates that when monetary policy is more unstable, region 1 is enlarging while region 2 is shrinking as the curve bisecting regions 1 and 2 is shifting up. In contrast, regions 3 and 4 are invariant with the change in the monetary policy. See Figure 3.3.

3.5.3 Bond trading volume and Quasi-linear quadratic preferences

This section discusses how the trading volume of bonds are affected by the measure of monetary instability σ_M^R. Trading volume of nominal and indexed bonds are defined as $q_n z_n^s$ and $q_d z_d^s$ respectively where the superscript "s" represents the saver. Since there are only two agents in this model, the saver's asset holdings, z_n^s

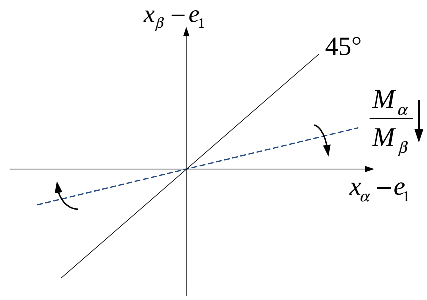

Figure 3.4: Inflation lines

and z_d^s directly represents the trading volumes of nominal and indexed bonds.

As the economy has higher excess inflation volatility, agents feel more risk averse in trading nominal securities. Therefore, it can be expected that the trading volume of a nominal bond would decrease while that of an indexed bond would increases as the measure of monetary instability increases. However, this is not typically true in a general equilibrium model because asset prices are not invariant with the changes in a monetary policy. Since asset trading volumes are also affected by asset prices, the trading volumes are not necessarily increasing or decreasing in the measure of monetary instability σ_M^R. To make equilibrium asset prices not to be affected by the monetary policy, in this section we assume that preferences are represented by quasi-linear quadratic preferences. These preferences suppress the motives of precautionary savings and the effects of the marginal utility of date-0 goods when the economy experiences changes in a monetary policy. In Figure 3.4, the dashed line represents the feasible allocations line by the nominal security, in the space of excess demand for date-1 good. The slope of the line is r_β/r_α which is the same as M_α/M_β. As σ_M^R increases, i.e., M_α/M_β decreases where $M_\beta > M_\alpha$, the feasible allocation line deviates more from the 45-degree line. Thus, both savers and borrowers would feel more risk averse in trading in a nominal

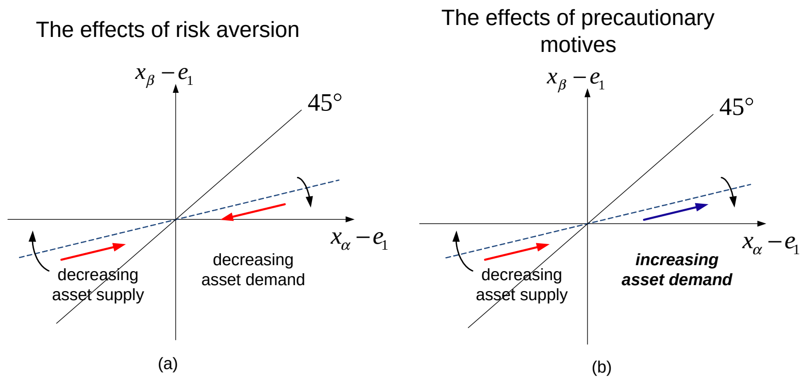

Figure 3.5: Inflation volatility and risk effects

security. By the increase in the effects of risk aversion, both the saver and the borrower have incentives to decrease the demand and supply for a nominal bond, respectively. (See Figure 3.5(a)) When both asset supply and demand curves, in the supply/demand analysis, are shifting to the left at the same magnitude, there will be a decrease in trading volume of the asset but no change in the asset price.

In contrast, the motives for precautionary savings make both agents to have additional demand (i.e., negative supply) when they face income risks that are originated from the uncertain real payoffs of nominal securities. As the monetary policy is more unstable, the line becomes more deviated from the 45-degree line and consequently agents, who have assets or liabilities in nominal securities, will face higher income risks. These income risks increase the demand (or negative supply) for precautionary savings, thus the nominal asset prices are increasing, as well. (See Figure 3.5(b).) The changes in asset prices through the effects of precautionary savings affect agents' optimal decisions on securities. Therefore, if agents are motivated to have precautionary savings, i.e., the third derivative of marginal utility is positive, it is not obvious how the trading volume of the assets will be affected by an unstable monetary policy.

I provide a more detailed explanation about why the prices are fixed under quasi-linear quadratic preferences in Section 7. The following proposition shows that in an economy with quasi-linear quadratic preferences, the trading volume of bonds monotonically increases or decreases as the monetary policy becomes more unstable.

Proposition 3.6 *If* $v^i(x_0^i, x_1^i) = \lambda_0^i x_0^i + g^i(x_1^i)$ *where* $(g^i)' > 0, (g^i)'' < 0$ *and* $(g^i)''' = 0$*, for any given transaction costs* (θ_n, θ_d)*, as* σ_M^R *increases, the trading volume of a nominal bond decreases while the trading volume of an indexed bond increases.*

Proof. It can be proven directly from eqns (C.20) and (C.21) in the Appendix. ■

3.6 Financial innovation–the introduction of an indexed bond

3.6.1 Precautionary savings and nominal interest rates

Precautionary savings are extra savings caused by future income being random rather than determinate. In a partial equilibrium model, Leland (1968) and Sandmo (1970) have demonstrated that when marginal utility is convex, agents are motivated to have precautionary savings. This concept is important to understand the impact of financial innovation on the prices of existing assets.[13] Several

[13] In a standard GE model, financial innovation refers to the introduction of a new security. In the economy with financial transaction costs, financial innovation can be interpreted as that the transaction cost of the "new" assets is decreasing from an inactive region to an active region.

research papers have suggested that the precautionary motive can be a main cause for an increase in the equilibrium interest rate when new securities are introduced in the market. Weil (1993) shows that if marginal utility is convex and the endowment is symmetric across agents and states, then the financial innovation, which completes the market, will result in an decrease in the existing security prices. In addition, Elul (1997) and Willen (2005) shows that financial innovation can result in an increase in the equilibrium interest rate when the marginal utility of exponential preferences is convex. Even though it has been shown that precautionary savings effects, in general, lead to a decrease in the equilibrium interest rate, it is not obvious to derive an unambiguous conclusion that financial innovation always decreases the nominal interest rate even if the marginal utility is convex. This is because the precautionary saving motive is not the only source of changes in nominal interest rates. The marginal utility from date-0 goods and intrinsic (endowment) uncertainty can matter in the changes of existing asset prices. The economy in this paper does not have intrinsic shocks; therefore, we only need to care about the marginal utility of date-0 good. To eliminate the effect of the marginal utility of date-0 good, we define the following quasi-linear functions:

$$v^i\left(x_0^i, x_1^i\right) = \lambda_0^i x_0^i + g^i(x_1^i)$$

where $\lambda^i > 0, (g^i)' > 0, (g^i)'' < 0$. With these preferences, the relative asset prices to date-0 good are invariant with the level of consumption at date 0.

The following proposition shows that if the marginal utility is convex, i.e., agents have precautionary saving motives, then the introduction of an indexed bond results in an increase in the nominal interest rate.

Proposition 3.7 *If $(g^i)''' > 0$ for all $i \in I$, the introduction of an indexed bond induces borrowing and lending interest rates to increase.*

Proof. The equilibrium purchasing and selling prices of the nominal bond, in an economy without an indexed bond, are given by

$$q_b (1 + \theta_n) = \frac{\pi_\alpha r_\alpha g^{s'} (x^s_\alpha) + \pi_\beta r_\beta g^{s'} \left(x^s_\beta\right)}{\lambda^s_0} \tag{3.36}$$

and

$$q_b = \frac{\pi_\alpha r_\alpha g^{b'} \left(x^b_\alpha\right) + \pi_\beta r_\beta g^{b'} \left(x^b_\beta\right)}{\lambda^b_0} \tag{3.37}$$

where superscripts s and b represent a saver and a borrower, respectively. In Figure 3.6, the dashed line with a slope p_α/p_β represents a set of feasible allocations through nominal securities. The points a and a^* represent the equilibrium excess demands of the saver and the borrower, respectively. Market clearing conditions imply that the two points are symmetrical to $(0, 0)$. By the introduction of indexed bonds, the equilibrium allocation moves into some point in the area between the 45-degree line and the feasible allocation line. Let's define the random variable $\widetilde{y} = \left\{x^s_\alpha - e^s_1, x^s_\beta - e^s_1; \pi_\alpha r_\alpha, \pi_\beta r_\beta\right\}$, where $\left(x^s_\alpha, x^s_\beta\right)$ is the saver's equilibrium allocation in the absence of an indexed bond market. Then, $\pi_\alpha r_\alpha g^{s'} (x^s_\alpha)+\pi_\beta r_\beta g^{s'} \left(x^s_\beta\right)$ in eqn (3.36) and $\pi_\alpha r_\alpha g^{b'} \left(x^b_\alpha\right)+\pi_\beta r_\beta g^{b'} \left(x^b_\beta\right)$ in eqn (3.37) can be expressed as $Eg' \left(e^s_1 + \widetilde{y}\right)$ and $Eg' \left(e^b_1 - \widetilde{y}\right)$, respectively.

Proof. In the same way, we can define $\widetilde{y}^+ = \left\{x^{s+}_\alpha - e^s_1, x^{s+}_\beta - e^s_1; \pi_\alpha r_\alpha, \pi_\beta r_\beta\right\}$ where $\left(x^{s+}_\alpha, x^{s+}_\beta\right)$ is the saver's equilibrium allocation with the indexed bond markets. The point $\left(x^{s+}_\alpha - e^s_1, x^{s+}_\beta - e^s_1\right)$ is located in the area between the 45-degree line and the feasible allocation line.

For the convenience of proof, we assume a iso-mean dotted line $(\overline{ab})$ in Figure 3.6. If a random variable $\widetilde{t}$ is defined with variables $(x_\alpha - e_1, x_\beta - e_1)$ on the dotted line and with probabilities $\pi_\alpha r_\alpha$ and $\pi_\beta r_\beta$, then $E\left[\widetilde{t}\right] = E\left[\widetilde{y}\right]$. The variance of $\widetilde{t}$ is strictly decreasing as the point $(x_\alpha - e_1, x_\beta - e_1)$ is moving from point a to point b. Therefore, $\widetilde{y}$ is a mean-preserving spread of $\widetilde{t}$.

Figure 3.6: Mean preserving lines

In the same way, we can define the dotted line $(\overline{a^*b^*})$ in quadrant III. Along the line $(\overline{a^*b^*})$, we can define the random variable $\widetilde{y}^*$ in the same way. The line $\overline{ab}$ and $\overline{a^*b^*}$ are symmetrical to (0,0).

We need to show that for any $\widetilde{y}^+$ in the region A or B, either that $Eg^{s'}(e_1^s + \widetilde{y}^+) < Eg^{s'}(e_1^s + \widetilde{y})$ or that $Eg^{b'}\left(e_1^b - \widetilde{y}^+\right) < Eg^{b'}\left(e_1^b - \widetilde{y}\right)$ is true. Then, by eqns (3.36) and (3.37), we can finalize the proof that the financial innovation induces both q_b and to $q_b(1+\theta_n)$ increase.

Case 1 $(\widetilde{y} \in A)$. Since $(g^i)'' < 0$, there is a $\widetilde{t}\,(< \widetilde{y})$ on $\overline{ab}$ such that $Eg^{s\prime}(e_1^s + \widetilde{y}^+) < Eg^{s\prime}\left(e_1^s + \widetilde{t}\right)$. We know that $Eg^{s\prime}\left(e_1^s + \widetilde{t}\right) < Eg^{s\prime}(e_1^s + \widetilde{y})$ because $(g^i)''' > 0$ and $\widetilde{y}$ is a mean-preserving spread of $\widetilde{t}$. Therefore, it is true that $Eg^{s\prime}(e_1^s + \widetilde{y}^+) < Eg^{s\prime}(e_1^s + \widetilde{y})$.

Case 2: $(\widetilde{y} \in B)$. By market clearing conditions, it is true that $(-\widetilde{y}^+) \in B^*$. Since $(g^i)'' < 0$, there is a $\widetilde{t}^*\,(< -\widetilde{y}^+)$ on $\overline{a^*b^*}$ such that $Eg^{b\prime}\left(e_1^b - \widetilde{y}^+\right) < Eg^{b\prime}\left(e_1^b - \widetilde{t}^*\right)$. We know that $Eg^{b\prime}\left(e_1^b - \widetilde{t}^*\right) < Eg^{b\prime}\left(e_1^b - \widetilde{y}\right)$ because $(g^i)''' > 0$ and

$-\widetilde{y}$ is a mean-preserving spread of $\widetilde{t}^{*}$. Therefore, it is true that $Eg^{b\prime}\left(e_1^b - \widetilde{y}^{+}\right) < Eg^{b\prime}\left(e_1^b - \widetilde{y}\right)$.

By Cases 1 and 2, we know that either $Eg^{s\prime}\left(e_1^s + \widetilde{y}\right)$ or $Eg^{b\prime}\left(e_1^b - \widetilde{y}\right)$ decrease by the introduction of indexed bonds; thus, q_b decreases by eqns (3.36) and (3.37), as well.■ ■

If a monetary policy is unstable, the real payoffs of nominal securities is uncertain. Therefore, the agents who trade nominal securities will face future income risks and thus, have precautionary savings motives. These precautionary motives cause the asset demand to increase while the asset supply decreases and eventually results in an increase in the equilibrium asset price. However, the introduction of an indexed bond decreases the precautionary demand for nominal securities in two ways. First, the indexed bond is a risk-safe asset which does not transfer any income uncertainty. Second, it can be a substitute for a nominal bond for precautionary savings. Therefore, the introduction of an indexed bond drives down the equilibrium price of the nominal security, and thus drives up the equilibrium nominal interest rate.

In contrast, the effect of risk aversion is clearly distinct from that of precautionary saving. With the introduction of the indexed bonds, the effects of risk aversion provides both savers and borrowers with an incentive to decrease the trading volume of the risky asset which is a nominal security. (See Figure 3.7(a).) Therefore, in terms of risk aversion effects, both demand and supply simultaneously decrease by the indexed bonds; thus, the equilibrium prices are barely affected or completely unaffected. However, in terms of precautionary saving effect, the introduction of an indexed bond prompts agent to decrease their precautionary savings. (See Figure 3.7(b).) Consequently, both demand and supply of the nominal securities decrease,

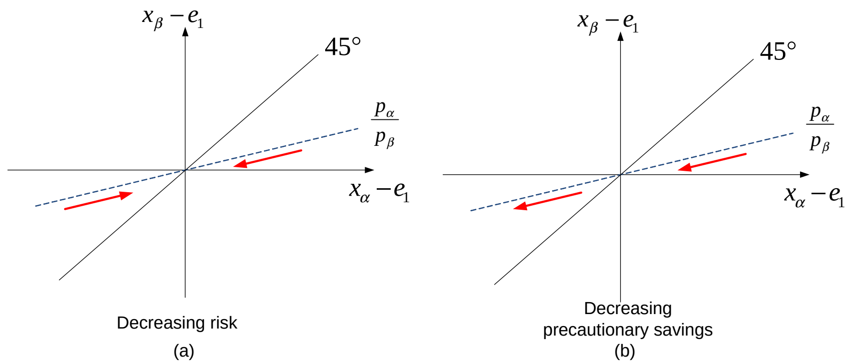

Figure 3.7: Risk aversion and precautionary saving effects

thus the equilibrium price (nominal interest rate) of the securities decreases (increases). Proposition 7 and the following corollary show that the main cause of a rise in the equilibrium interest rates after the introduction of an indexed bond is a precautionary savings motive but not the effect of risk aversion. The following corollary clearly suggests that the risk aversion does not affect the changes in asset prices through financial innovation.

Corollary 3.1 *By the introduction of an indexed bond, the nominal interest rates*

$$\begin{array}{rl} \textit{increase} & if \ \ (g^i)''' > 0, \ i \in I. \\ \textit{are unchanged} & if \ \ (g^i)''' = 0, \ i \in I. \\ \textit{decrease} & if \ \ (g^i)''' < 0, \ i \in I. \end{array}$$

Proof. For the case of $(g^i)''' < 0$, we can prove it in the same way as Proposition 7. For the case of $(g^i)''' = 0$, see Lemma 3 in Section 7. ■

Corollary 1 shows that if agents lack precautionary savings motives, there is no change in equilibrium asset prices through financial innovation. As mentioned above, the risk aversion effects are associated with the changes in asset trading volume but not with those in equilibrium prices.

3.6.2 Financial innovation and Welfare

It has been a controversial issue, starting from Hart's (1972) seminal work, whether financial innovation improves welfare. Financial innovation typically has real effects on equilibrium allocations, and thus on welfare when markets are incomplete. The welfare effects of a new security can be decomposed into two distinct components. One is a portfolio effect, providing agents more portfolio choices and the other is a price effect, the change of existing asset prices. The first effect always has a positive impact on welfare. Without any changes in equilibrium prices of existing assets, the introduction of a new asset provides agents with a better opportunity to distribute income across states. However, because of the price effect, financial innovation does not guarantee Pareto improvement in the economy. When the introduction of an indexed bond drives down the prices of nominal securities, savers reap the benefits from the lower interest rates while borrowers get lost. If the price effect is great enough compared to the portfolio effect, the borrowers would even be worse off through financial innovation. (See such an example in the Appendix.) The following corollary addresses this issue.

Corollary 3.2 *If* $(g^i)''' > 0$ *for all* $i \in I$, *the introduction of an indexed bond makes the saver (the agent who holds a positive amount of nominal securities) better off while the borrower (the agent who holds a negative amount of nominal securities) can be better or worse off.*

Proof. For the fixed nominal interest rate, the introduction of an indexed bond make agents better off by the revealed preferences hypothesis. When the nominal interest rate increases, i.e., the price of nominal securities decreases, the saver can be better off. Therefore, it is trivial that the saver will be better off through the introduction of indexed bonds. In contrast, the borrower is not necessarily made better off because of the negative impact of the price effect. ■

Corollary 2 implies that the innovation on indexed bonds, in general, is more beneficial for savers than borrowers. Borrowers are even worse off by the innovation. Nevertheless, financial innovations cannot make everyone worse off unless the equilibrium set is unique or finite.[14] In other words, there is at least one agent who is better off by the introduction of a new asset.

3.6.3 Introduction of a nominal bond

It is also interesting to consider the situation where a nominal bond is introduced in the economy where an indexed bond and money were previously available as financial instruments. In this case, the introduction of a nominal security triggers the precautionary saving motives, which is the opposite situation to the case in the previous section. Therefore, the price of an indexed bond increases by the introduction of the nominal security, and equivalently the returns of the indexed bonds decreases.

Proposition 3.8 *If* $(g^i)''' > 0$ *for all* $i \in I$, *the introduction of a nominal bond in an economy with money and an indexed bond make the returns of the indexed bond decrease.*

Proof. It can be proven in the same way as the proof of Proposition 7. In this case, with the introduction of a new asset, agents' consumption is more unbalanced

[14] If the economy has real indeterminacy, especially in GEI with nominal securities, financial innovations make all agents be worse off. See Cass and Citanna (1998) and Elul (1995).

across states. By the innovation on a nominal bond, the excess demands, previously located on the 45-degree line, now move to the area between the 45-degree line and the feasible allocation line. ■

Several research papers have suggested that the introduction of a new asset, in general, decreases the precautionary saving motives, and thus drives down the asset prices. However, this paper shows that a new asset can even boost the motives of precautionary saving. In this case, the financial innovation drives down the existing asset prices. The following two tables summarize the main results in Section 6 when agents have quasi-linear utility function. For the numerical examples for the three cases ($g''' > 0, g''' = 0, g''' < 0$), see the Appendix.

Table 3.3: The impact of financial innovation on interest rates

	nominal interest rate after introduction of an indexed bond	real interest rate after introduction of a nominal bond
$g''' > \mathbf{0}$	**up**	**down**
$g''' = 0$ $(CAPM)$	fixed	fixed
$g''' < 0$	down	up

3.7 Concluding remarks

The presence of proportional transaction costs in asset markets can affect asset market trading significantly. This paper demonstrates that nominal security markets could completely shut down under the assumption of frictionless asset markets

Table 3.4: Precautionary savings and welfare

		welfare after introduction of indexed bonds	welfare after introduction of nominal bonds
$g''' > 0$	saver	better off	*better* or *worse*
	borrower	*better* or *worse*	better off
$g''' = 0$ $(CAMP)$	saver	better off	better off
	borrower	better off	better off
$g''' < 0$	saver	*better* or *worse*	better off
	borrower	better off	*better* or *worse*

but are active if the transaction cost of the nominal bonds is lower than that of the indexed bonds. These results provide theoretical explanations for several empirical findings that the higher liquidity costs of TIPS compared to those of nominal securities is associated with low trading volume of TIPS markets. This paper also indicates that incorporating the capital asset pricing model (CAPM) in a general equilibrium framework can be misleading in understanding the welfare effect of financial innovation. The precautionary saving motives, which are ignored in CAPM, can affect equilibrium nominal interest rates significantly when financial innovation occurs; the precautionary motives force the equilibrium nominal interest rates to increase when an indexed bond is introduced to the economy. Therefore, borrowers can be made even worse by the innovation on indexed bonds.

In this paper, real economic fundamental shocks are suppressed and the only source of inflation volatility is an unstable monetary policy. In general, the source of inflation uncertainty can be classified into non-economic fundamental shocks (monetary shocks) and real fundamental shocks, which can be simply understood by Fisher's famous equation: $MV = PQ$. The price level P is affected by Q (real GDP) and M (money supply). The issue is that inflation volatility from real shocks can be even "beneficial" to the economy since the shocks induce real returns of nominal securities to be positively correlated with economic-wide shocks. In short, price-level volatility from real economic fundamentals can be interpreted as a result of efficient market allocations. This is a similar idea to relative price fluctuations from sector-specific shocks in complete markets, which help the economy achieve Pareto-optimal allocations. The main concern of inflation volatility for economists is not the volatility which arises as a result of real shocks, but that from non-real economic fundamentals such as monetary shocks or sunspots.

This paper assumes that the inflation volatility is from an unstable monetary policy. However, the volatility can be also from other types of non-economic fundamentals such as sunspots. The possibility of inflation volatility triggered by sunspots is shown in a general equilibrium model with incomplete markets (GEI) in Cass (1989;1992).[15] In the Cass GEI model, the monetary economy is defined by pure inside money, the net supply of which is zero. In contrast, "outside" money, the quantity of which is determined by the central bank, is assumed in this model. To incorporate the two types of non-real fundamental uncertainty into one model is still a remaining question.

The model in this paper can be easily extended to a model with many consumers and many states. The one difficulty in the extension is the question of how to define the measure of monetary instability if there are more than two states at date 1. Each equilibrium allocation corresponds to a unique level of relative standard deviation of money supply fluctuations in the two-state model. However, with

[15] Kang (2012b) investigated how the asset trading volumes and welfare can be affected by the introduction of indexed bonds when the economy has inflation uncertainty triggered by sunspots.

more than two states, the equilibrium allocation is not uniquely determined by the relative standard deviation unless all consumers have mean-variance preferences.

CHAPTER 4

NON-CONNECTEDNESS OF THE SET OF EQUILIBRIUM MONEY PRICES IN THE STATIC ECONOMY: A CONSTRUCTIVE EXAMPLE

4.1 Introduction

For the static general equilibrium model with lump-sum taxes and transfers denominated in money, Balasko and Shell (1993) show that there is a continuum of money prices for which a monetary equilibrium exists at each price of money. They also suggest the possibility that the set of equilibrium money prices is not connected, because some taxpayers become bankrupt as the price of money increases in a certain range; however, as the price of money increases beyond that range, consumers become free from bankruptcy. Peck (1987) constructs an example of an economy with three consumers and two commodities in which the set of equilibrium money prices is not connected. Based on Peck's example, Garratt (1992) shows that this connectedness property can depend on the choice of the numeraire. However, the examples of Peck (1987) and Garratt (1992) do not specify the functional form of utility. This paper suggests three specific utility functions for which such a paradoxical outcome can be derived. Using the concept of unrestricted equilibrium which allows negative consumption, closed form solutions for equilibrium prices are derived.

The existence of such a non-connected set implies that some consumers can be better off by paying higher taxes, especially when their income is recovering from bankruptcy. To understand this paradoxical result, it is necessary to distinguish

The modified version of this essay is forthcoming at *Macroeconomic Dynamics*.

the direct effect from the indirect (relative price) effect of tax plans on consumers' income. The magnitude of the direct effect depends on the price of money; as the price of money goes up, the amount of taxes also go up. An increase in the money price can also affect the equilibrium relative price. The income change from the relative price is the indirect effect. For taxpayers, the direct effect always decreases their utility as the price of money increases, while the indirect effect can increase or decrease their utility. If the *negative* direct effect is dominated by the *positive* indirect effect, bankrupt taxpayers can be free from bankruptcy and, consequently, monetary equilibrium can be restored.

These paradoxical results are similar to the "three-agents transfer paradox," where tax-transfer plans make taxpayers better off or recipients worse off.[1] A specific example in a general equilibrium model was first suggested by Gale (1974), and has been developed by Guesnerie and Laffont (1978), Chichilnisky (1980), Leonard and Manning (1983), Yano (1983), Jones (1984), and Kang and Ye (2012). Even though taxes and transfers in the transfer paradox are denominated in commodities but not money, its main outcomes are almost the same as that of the economy with nominal taxes and transfers. However, constructing an example for a non-connected set of equilibrium money prices is more challenging than the transfer paradox, because in the former case we must have bankruptcy occur in the interior of the set of money prices. On the other hand, in the case of the transfer paradox, the tax-adjusted income change is not necessarily a convex or concave function of taxes and the consumer does not need to be bankrupt. Therefore, if a non-connected equilibrium money price set exists for an economy, we can always construct the transfer paradox example for the same preferences and endowments.

[1] Yano (1983) distinguished the *weak* transfer paradox, where a donor becomes better off or a recipient becomes worse off, from the *strong* transfer paradox, where the welfare of both the donor and the recipient changes in the paradoxical direction. Based on his definition, our example is classified as the weak transfer paradox, where donors (taxpayers) can be better off.

The reverse is not true, however.

The rest of the paper is organized as follows. Section 2 describes the basic setting of the economy of this paper. Section 3 introduces a constructive example. The paradoxical results of the example are shown in Section 4. Section 5 concludes.

4.2 Economy

Balasko and Shell (1993) and Peck (1987) indicate that we must have at least two commodities and at least three consumers to get the paradoxical outcomes. Let the number of commodities be two and the number of consumers be three. The vector of commodity prices given by $p = (p^1, p^2)$ has commodity one as numeraire, so $p^1 = 1$. We denote $x_h = (x_h^1, x_h^2) \in \mathbb{R}_+^2$ for the consumption plan for consumer $h \in H$ where $H = \{1, 2, 3\}$. Let $\omega_h = (\omega_h^1, \omega_h^2) \in \mathbb{R}_+^2$ for the commodity endowments for consumer h.[2] Next, the fiscal policy is defined by the vector $\tau = (\tau_1, \tau_2, \tau_3)$. The fiscal policy is balanced, i.e., $\sum\limits_{h \in H} \tau_h = 0$. The price of money in terms of commodity 1 is $p^m \geq 0$. Let u_h consumer h's preferences defined on $\left[0, \sum\limits_{h \in H} \omega_h^1\right] \times \left[0, \sum\limits_{h \in H} \omega_h^2\right]$ where $\sum\limits_{h \in H} \omega_h^1$ and $\sum\limits_{h \in H} \omega_h^2$ are aggregate endowments of commodities 1 and 2, respectively.

A competitive equilibrium of the economy $\mathcal{E}(u, \omega, \tau)$ is a pair of actions and prices (x, q) where $x = (x_1, x_2, x_3)$ and $q = (p, p^m)$, such that

(i) for $h \in H$,

$$x_h = \arg\max_{x_h^1, x_h^2} \quad u_h\left(x_h^1, x_h^2\right) \tag{4.1}$$

[2] In Balasko and Shell (1993), the space of endowment is defined as $\mathbb{R}_{++}^2$ instead of $\mathbb{R}_+^2$.

$$s.t \;\; x_h^1 + p^2 x_h^2 \le \omega_h^1 + p^2 \omega_h^2 - p^m \tau_h \tag{4.2}$$

$$\text{and } x_h \in \mathbb{R}_+^2,$$

and

(ii) (commodity market clearing conditions)

$$\sum_{h\in H} x_h^1 = \sum_{h\in H} \omega_h^1 \;\text{ and }\; \sum_{h\in H} x_h^2 = \sum_{h\in H} \omega_h^2. \tag{4.3}$$

4.3 Example

Let endowments be

$$\omega_1 = \left(\omega_1^1, \omega_1^2\right) = \left(0, \frac{1}{2}\right) \tag{4.4}$$

$$\omega_2 = \left(\omega_2^1, \omega_2^2\right) = \left(\frac{4}{5}, \frac{1}{2}\right) \tag{4.5}$$

$$\omega_3 = \left(\omega_3^1, \omega_3^2\right) = \left(\frac{1}{5}, 0\right) \tag{4.6}$$

and let the tax-transfer policy be given by

$$\tau = (\tau_1, \tau_2, \tau_3) = \left(\frac{2}{5}, \frac{3}{5}, -1\right).$$

The tax policy is balanced, i.e., $\tau_1 + \tau_2 + \tau_3 = 0$. Consumers 1 and 2 are taxpayers, while consumer 3 is a recipient.

The preferences of the three consumers are given by

$$u_1\left(x_1^1, x_1^2\right) = \min\left(x_1^1, x_1^2\right) \tag{4.7}$$

$$u_2\left(x_2^1, x_2^2\right) = \min\left(x_2^1, x_2^2\right) \tag{4.8}$$

and

$$u_3\left(x_3^1, x_3^2\right) = -\frac{\left(1 - x_3^1\right)^2}{2} + \frac{1}{10}\log x_3^2 \tag{4.9}$$

This paper chooses the two taxpayers' utilities as Leontief preferences for three reasons. First, the choice is simply for computational convenience. With these Leontief utilities, we get the following two equations, $x_1^1 = x_1^2$ and $x_2^1 = x_2^2$, which are necessary to get closed form solutions. Second, the Leontief utilities are limiting cases of the CES utilities. Therefore, if a "non-connectedness" example exists for Leontief utilities, such an example also exists for smooth CES utilities by the continuity property. Smoothness of utilities is one of the regularity assumptions that is also assumed in Balasko and Shell (1983). Finally, Leontief utilities are also well-defined for negative values of consumption. Therefore, as allowing negative consumption, we can derive a set of *unrestricted* equilibria. In contrast to a competitive equilibrium which is only defined on the positive orthant, an unrestricted equilibrium is defined on both the negative and positive orthant. The set of unrestricted equilibria includes that of competitive equilibria. The former has a proper interval, but the latter is a non-connected set in this economy.

Because the two taxpayers have Leontief preferences, the commodity price, p^2, is determined by the marginal rate of substitution (MRS) of the recipient's utility. If the equilibrium commodity price, p^2, is a convex "function" of equilibrium money price, p^m, and the "function" has a minimum value for some value of p^m, it may be possible that consumer 1's income is also convex in p^m because consumer 1 is endowed with more of the second good, i.e., $\omega_h^1 << \omega_h^2$ from equation (4.4). In this case, as p^m is increasing from zero, p^2 is decreasing. Thus, the taxpayers' income from endowment is decreasing and he eventually becomes bankrupt. However, as p^m increases further, p^2 actually increases and the taxpayer becomes free from bankruptcy.[3] To design such a case, where p^2 is a convex "function" of p^m, the MRS of the recipient (consumer 3) should be convex in his income. As consumer 3's income is increasing, his consumption will be increasing also. Because the two other consumers have Leontief preferences, consumer 3 always consumes an equal amount of the two goods in equilibrium. Therefore, we need to check the MRS of the preference at the equal consumption plan, i.e., $x_3^1 = x_3^2 \in (0, 1]$. Based on these ideas, we propose the following equation for the MRS of the recipient's utility at the equal consumption:

$$\begin{aligned} MRS_{(x_3^1, x_3^2)} &= \frac{\partial u_3(y,y)/\partial x_3^1}{\partial u_3(y,y)/\partial x_3^2} = ay\,(1-y) \quad \text{where} \quad a > 0. \\ &= 1/p^2 \end{aligned} \tag{4.10}$$

where $y = x_3^1 = x_3^2$. The MRS is concave in consumption and has a maximum value at $y = 1/2$. Choosing $a = 10$, the utility function in equation (4.9) satisfies equation (4.10). In addition, the utility function is strictly increasing and strictly concave in $(0, 1]^2$. Figure 4.1 shows the indifference curves of consumer 3's utility function and their tangent lines at the 45 degree line in which the slopes of tangent lines are equal to $1/p^2$ (MRS).

4.4 Result

Because consumer 1's and consumer 2's optimal consumption satisfies the equations $x_2^1 = x_2^2$ and $x_3^1 = x_3^2$, by the commodity market clearing conditions, we get the following equation:

$$x_3^1 = x_3^2 \tag{4.11}$$

[3] We can find the opposite situation where the taxpayer is endowed with more of the first good, and the relative price p^2 is a concave function of the price of money.

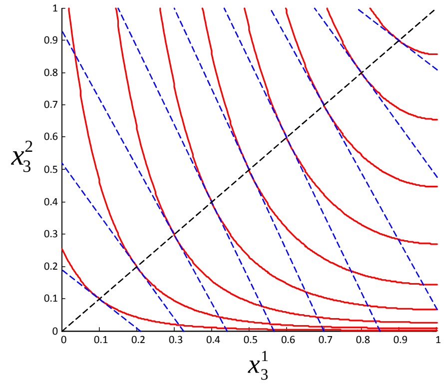

Figure 4.1: Consumer 3's indifference curves

From the first order conditions of consumer 3's maximization problem, we get:

$$\frac{1}{10x_3^2\left(1-x_3^1\right)}=p^2. \tag{4.12}$$

The budget constraint of consumer 3 is that

$$x_3^1+p^2x_3^2=\frac{1}{5}+p^m. \tag{4.13}$$

From equations (4.11-4.13), we derive equilibrium relative price p^2 as a corresponding function of p^m. That is,

$$p^2\left(p^m\right)=\frac{1-15p^m+25\left(p^m\right)^2+\left(5p^m+1\right)\sqrt{25\left(p^m\right)^2-40p^m+26}}{50p^m+5}, \tag{4.14}$$

where both p^2 and p^m are endogenous. For any nonnegative p^m, there is a unique unrestricted equilibrium, so p^m is indexing the unrestricted equilibrium and $p^2\left(p^m\right)$ is the corresponding function indicating the value of p^2.

$p^2\left(p^m\right)$ is strictly convex and has a minimum value at $p^m=1/2$, which is shown in Figure B.1. From equation (4.14), each consumer h's income $\omega_h^1+p^2\omega_h^2-p^m\tau_h$ can be derived as a function of p^m. Figure B.2 shows consumer 1's and consumer 2's income, given each equilibrium money price. For values of the money price

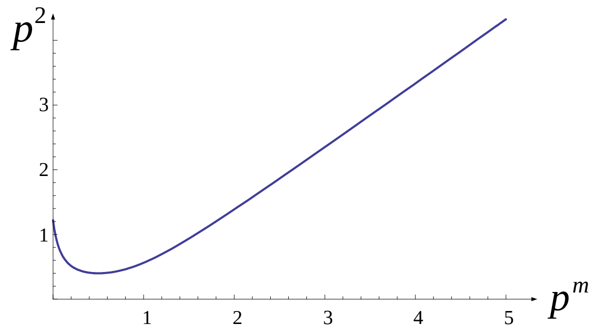

Figure 4.2: A relative price and money price

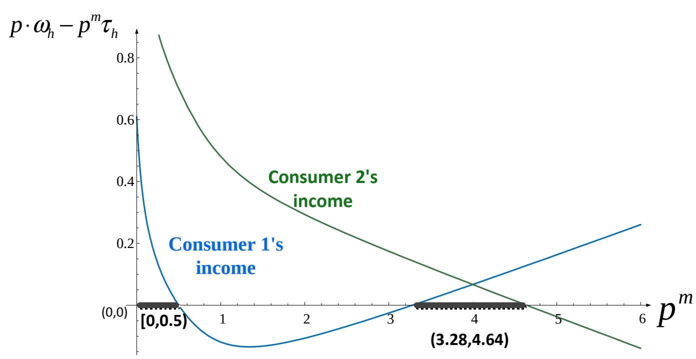

Figure 4.3: Consumers' income and money price

in the interval $[0, 0.5)$, two consumers' incomes are strictly positive and, thus, the monetary equilibrium is defined. However, for $p^m \in [0.5, 3.28]$, consumer 1 is bankrupt and no monetary equilibrium is defined. For $p^m \in (3.28, 4.64)$, consumer 1 becomes solvent again. If the money price is higher than 4.64, consumer 2 becomes bankrupt. Therefore, the set of equilibrium money prices is defined as the union of two disconnected sets: $[0, 0.5) \cup (3.28, 4.64)$.

The paradoxical outcome that consumer 1 can be better off despite paying more taxes occurs for money prices in the interval $(3.28, 4.64)$. Consumer 1's utility can increase with the price of money only if the direct (tax) effect is dominated by the indirect (relative price) effect. This counterintuitive result is similar to that of the

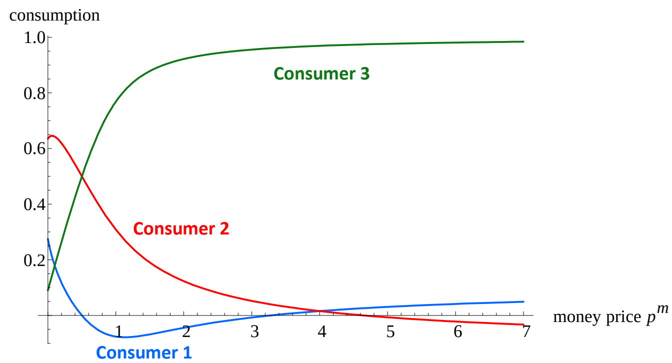

Figure 4.4: Consumption and money price

"transfer paradox" in the economy with taxes and transfers denominated with a commodity. The transfer paradox occurs when a donor (taxpayer) is better off or a recipient is worse off, or both. With the same preferences and endowments but with commodity 1 taxes instead of money taxes, we can derive the same graphs in Figures B.1-4.4 by replacing the X-axis with the commodity-1 tax.

In Figure 4.4, the consumption of the three consumers is plotted allowing for negative consumption. The set of money prices that guarantees positive consumption is the same as that with positive incomes.

4.5 Conclusion

This paper presents a constructive example where a set of equilibrium money prices is not connected using two Leontief utilities and one smooth utility. To facilitate the analysis of how the bankrupted taxpayer can be recovered by the change of money prices, this paper introduces a *unrestricted* equilibrium that allows negative tax-adjusted income. The choice of Leontief functions for the two taxpayers' utility makes it possible to define the unrestricted equilibria.[4] This paper demonstrates that the change in relative price, which is driven by the MRS of the smooth utility, leads to the paradoxical result that taxpayers can be better off by paying higher taxes. Another contribution of this paper is to show that the three-agents transfer paradox, first proposed by Gale (1974), is similar to the non-connectedness example: higher taxes can be beneficial for some agents if the positive indirect effect from relative commodity prices dominates the negative the tax effect.

[4]In general, the negative values in utility function are not even mathematically-defined, especially with a smooth CES utility. The one exception is when the CES utility's elasticity of substitution is 1/2.

APPENDIX A

CHAPTER 1 OF APPENDIX

There are two consumers with endowments of $(1, 0)$ and $(0, 1)$, respectively. Their expected utility functions are the same as $\log(x_0) + \log(x_1)$. State probabilities are $\pi_\alpha = \pi_\beta = 0.5$. Figure A.1(a) shows a continuum of equilibrium allocations in the space of excess demand of $x^{1\alpha}$ and $x^{1\beta}$ where $0 \leq \overline{\sigma} \leq 0.6$.[1] The lender's ($m_h > 0$) allocations are located in the northeast area while the borrower's ($m_h < 0$) allocations are in the southwest area. In Figure A.1(b), the two consumers' utility levels are plotted with price-level volatility $\overline{\sigma}$. In this example, the borrower's utility level is higher with higher price volatility while the lender's is the opposite. In Figure A.2, the solid lines represent a continuum of equilibrium allocations and utility levels without any tax plan while the dashed line represents those with the proposed tax plans, as described in eqn (1.12). Figure A.2b clearly shows that the welfare is improving from higher price volatility to lower volatility with the proposed tax plans.

[1] The figure should be three-dimensional including the excess demand of x^0, but the three-dimensional figure is projected onto a two-dimensional space.

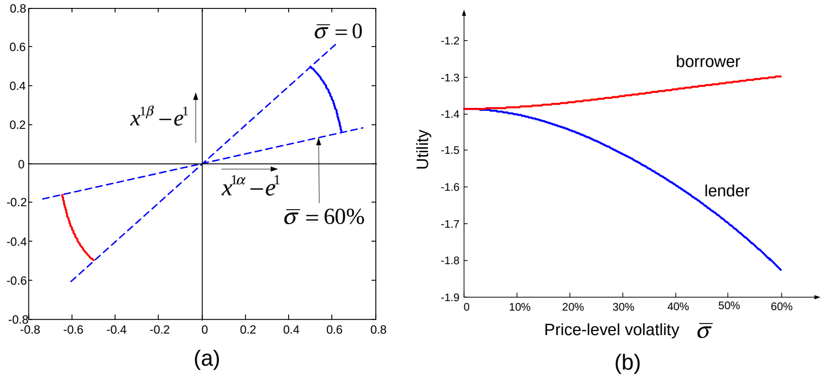

Figure A.1: A continuum of equilibrium allocations

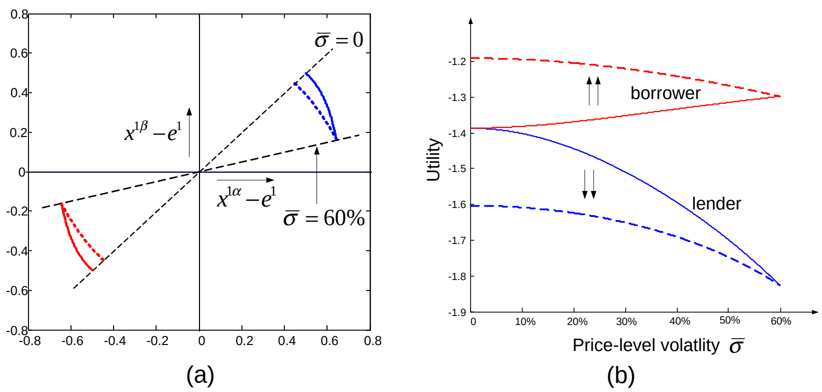

Figure A.2: The proposed tax-transfer plans

APPENDIX B

CHAPTER 2 OF APPENDIX

B.1 Proof of Proposition 5

Lemma B.1 *x_{α}^{B} is increasing but x_{β}^{B} is decreasing in $\mathcal{P} > 1$. (Subscript "B" represents the asset buyer.)*

Proof. There are four cases:

Case 1 : both x_{α}^{i} and x_{β}^{i} increase in $\mathcal{P} > 1$.

Case 2 : x_{α}^{i} increases but x_{β}^{i} decreases in $\mathcal{P} > 1.(*)$

Case 3 : x_{α}^{i} decreases but x_{β}^{i} increases in $\mathcal{P} > 1$.

Case 4 : both x_{α}^{i} and x_{β}^{i} decrease in $\mathcal{P} > 1$.

First, Case 3 is ruled out since $x_{\beta}^{i}/x_{\alpha}^{i}$ must increase in $\mathcal{P} > 1$. Then, we need to show that Case 1 and 4 are not true.

Assuming that Case 1 is true: for $1 < \mathcal{P} < \mathcal{P}'$, $x_{\alpha}^{B}(\mathcal{P}) < x_{\alpha}^{B}(\mathcal{P}')$ and $x_{\beta}^{B}(\mathcal{P}) < x_{\beta}^{B}(\mathcal{P}')$. By market clearing conditions, we get: $x_{S}^{1\alpha}(\mathcal{P}) > x_{S}^{1\alpha}(\mathcal{P}')$ and $x_{S}^{1\beta}(\mathcal{P}) > x_{S}^{1\beta}(\mathcal{P}')$.

Choosing $q_0 = 1$, the price of a risky asset is given by

$$\begin{aligned} q^{+} &= \frac{\pi_{\alpha} R_{\alpha} g^{S\prime}\left(x_{S}^{1\alpha}\right) + \pi_{\beta} R_{\beta} g^{S\prime}\left(x_{S}^{1\beta}\right)}{f^{S\prime}\left(x_{0}^{S}\right)} \\ &= \frac{\pi_{\alpha} R_{\alpha} g^{B\prime}\left(x_{\alpha}^{B}\right) + \pi_{\beta} R_{\beta} g^{B\prime}\left(x_{\beta}^{B}\right)}{f^{B\prime}\left(x_{0}^{B}\right)}. \end{aligned} \tag{B.1}$$

By $x_{\alpha}^{B}(\mathcal{P}) < x_{\alpha}^{B}(\mathcal{P}')$,$x_{\beta}^{B}(\mathcal{P}) < x_{\beta}^{B}(\mathcal{P}')$ and strictly concavity of g_h, $\pi_\alpha R_\alpha g^{S\prime}\left(x_S^{1\alpha}\right) + \pi_\beta R_\beta g^{S\prime}\left(x_S^{1\beta}\right)$ decreases in $\mathcal{P}$. In the same way, $\pi_\alpha R_\alpha g^{B\prime}\left(x_\alpha^B\right) + \pi_\beta R_\beta g^{B\prime}\left(x_\beta^B\right)$ increases in $\mathcal{P}$. Therefore, by eqn (B.1), the market clearing condition, $x_0^S + x_0^B = e_0^S + e_0^B$ and strict concavity of f^i, the following inequalities must be satisfied

$$x_0^B(\mathcal{P}) < x_0^B(\mathcal{P}') \text{ and } x_0^S(\mathcal{P}) > x_0^S(\mathcal{P}').$$

The asset buyer's budget constraints are

$$x_0^B(\mathcal{P}) + q^+(\mathcal{P})B^B(\mathcal{P}) = 0 \quad \text{and}$$

$$x_0^B(\mathcal{P}') + q(\mathcal{P}')B^B(\mathcal{P}') = 0$$

Since $x_0^B(\mathcal{P}) < x_0^B(\mathcal{P}')$ and $B^B(\mathcal{P}) < B^B(\mathcal{P}')$,

$$q^+(\mathcal{P}) > q^+(\mathcal{P}') \tag{B.2}$$

The asset seller's budget constraints are

$$x_0^S(\mathcal{P}) + q^+(\mathcal{P})B^S(\mathcal{P}) = 0 \quad \text{and}$$

$$x_0^S(\mathcal{P}') + q^+(\mathcal{P}')B^S(\mathcal{P}') = 0$$

Since $x_0^S(\mathcal{P}) > x_0^S(\mathcal{P}')$ and $B^S(\mathcal{P}) > B^S(\mathcal{P}')$,

$$q^+(\mathcal{P}) < q^+(\mathcal{P}') \tag{B.3}$$

Inequalities (B.2) and (B.3) contradict each other.

We can show that Case 4 is not true in the same way. ■

$\frac{p_h^F}{q^+}$ is

$$\frac{p_h^F}{q^+} = \frac{\pi_\alpha g^{i\prime}\left(x_\alpha^i\right) + \pi_\beta g^{i\prime}\left(x_\beta^i\right)}{\pi_\alpha R_\alpha g^{i\prime}\left(x_\alpha^i\right) + \pi_\beta R_\beta g^{i\prime}\left(x_\beta^i\right)}.$$

We need to show that $\frac{d}{d\mathcal{P}}\left(\frac{p_F^B}{q^+}\right) > 0$ and $\frac{d}{d\mathcal{P}}\left(\frac{p_F^S}{q^+}\right) < 0$. There are four variables $R_\alpha(\mathcal{P}), R_\beta(\mathcal{P}), x_\alpha^i(\mathcal{P})$ and $x_\beta^i(\mathcal{P})$ which should be considered in the total derivative. The total derivative can be divided into two parts:

$$\begin{aligned}\frac{d}{d\mathcal{P}}\left(\frac{p_F^B}{q^+}\right) &= \frac{d}{d\mathcal{P}}\left(\frac{\pi_\alpha g^{B\prime}\left(x_\alpha^B\right) + \pi_\beta g^{B\prime}\left(x_\beta^B\right)}{\pi_\alpha R_\alpha g^{B\prime}\left(x_\alpha^B\right) + \pi_\beta R_\beta g^{B\prime}\left(x_\beta^B\right)}\right)_{x_\alpha^i, x_\beta^i = \text{constant}} \\ &\quad + \frac{d}{d\mathcal{P}}\left(\frac{\pi_\alpha g^{B\prime}\left(x_\alpha^B\right) + \pi_\beta g^{B\prime}\left(x_\beta^B\right)}{\pi_\alpha R_\alpha g^{B\prime}\left(x_\alpha^B\right) + \pi_\beta R_\beta g^{B\prime}\left(x_\beta^B\right)}\right)_{R_\alpha, R_\beta = \text{constant}}.\end{aligned}$$

Since g^i is strictly concave and $x_\alpha^B > x_\beta^B$ for the asset buyer, $g^{B\prime}\left(x_\alpha^B\right) < g^{B\prime}\left(x_\beta^B\right)$. Since $\pi_\alpha R_\alpha(\mathcal{P})$ and $\pi_\beta R_\beta(\mathcal{P})$ increases and decreases in $\mathcal{P}$, respectively, and $\pi_\alpha R_\alpha(\mathcal{P}) + \pi_\beta R_\beta(\mathcal{P})$ is constant in $\mathcal{P}$,

$$\frac{d}{d\mathcal{P}}\left(\pi_\alpha R_\alpha g^{B\prime}\left(x_\alpha^B\right) + \pi_\beta R_\beta g^{B\prime}\left(x_\beta^B\right)\right)_{R_\alpha, R_\beta = \text{constant}} < 0.$$

Therefore, the first term is strictly positive.

The second term can be expressed as

$$T(G) = \frac{\pi_\alpha + \pi_\beta G}{\pi_\alpha R_\alpha + \pi_\beta R_\beta G} \text{ where } G = \frac{g^{B\prime}\left(x_\beta^B\right)}{g^{B\prime}\left(x_\alpha^B\right)}$$

$T(G)$ is increasing in G since $\pi_\beta > \pi_\beta R_\beta$. $\frac{dT}{d\mathcal{P}}$ is positive since $\frac{dx_\alpha^B}{d\mathcal{P}} > 0, \frac{dx_\beta^B}{d\mathcal{P}} < 0$ (by lemma) and strictly concavity of g'. Thus, the second term also increases in $\mathcal{P}$ and $\frac{d}{d\mathcal{P}}\left(\frac{p_F^B}{q^+}\right) > 0$.

We can prove that $\frac{d}{d\mathcal{P}}\left(\frac{p_F^S}{q^+}\right) < 0$ in the same way. Finally,

$$\frac{d}{d\mathcal{P}}\left(\frac{p_F^B - p_F^S}{p_F^S}\right) = \frac{d}{d\mathcal{P}}\left(\frac{p_F^B}{p_F^S}\right) = \frac{d}{d\mathcal{P}}\left(\frac{p_F^B}{q^+}\right) / \frac{d}{d\mathcal{P}}\left(\frac{p_F^S}{q^+}\right) > 0.$$

End of Proof.

B.2 Proof of Proposition 6

The budget sets in a pure monetary economy is defined as

$$B_+^i(q) = \left\{ \left(z^i, B^i\right) | z^i + qB^i \leq 0 \right\}.$$

The budget sets in a combined economy is defines as

$$B_*^i\left(q, p, \tau^i\right) = \left\{ \left(z^i, B^i, n^i\right) | z^i + qB^i + p\max(n^i, 0) + p(1+\theta)\min(n^i, 0) \leq \tau^i \right\}.$$

We need to show that for any equilibrium of the money market $\left(z_+^i, B_+^i\right)$, there exists a lump-sum transfer plan $\sum_{i \in I} \tau^i = 0$ such that $\left(z_+^i, B_+^i, 0\right) \in B_*^i\left(q, p, \tau^i\right)$ for all $i \in I$ and any q (since we do not know about the equilibrium price q^* after a tax plan, we need to prove it for any price $q \in \mathbb{R}_{++}$). Then, the allocation $\left(z_+^i, B_+^i, 0\right)_{i=1}^I$ is also affordable in the combined market. That means that the combined market is at least weakly Pareto superior to the money market by the revealed preferences hypothesis.

Assuming that the equilibrium price and allocations in the monetary market are q_+ and $(z_+^i, B_+^i)_{i=1}^I$, the following equation is satisfied

$$z_+^i + q_+ B_+^i = 0 \text{ for all } i = 1, ..., I \tag{B.4}$$

In the combined market, we need to show the existence of τ^i for $i = 1, .., I$ such that $\sum_i \tau^i = 0$ and $\left(z_+^i, B_+^i, 0\right) \in B_*^i\left(q, p, \tau^i\right)$ for any q. Then,

$$z_+^i + qB_+^i = \tau^i \text{ for all } i = 1, ..., I \tag{B.5}$$

subtracting eqn (B.5) with (B.4), we get

$$\left(q - q_+\right) B_+^i = \tau^i$$

That means that if $\tau^i = (q - q_+) B^i_+$, $(z^i_+, B^i_+, 0) \in B^i_+ (q, p, \tau^i)$. Also, we can prove that $\sum_i \tau^i = 0$ by market clearing:

$$\sum_i \tau^i = \sum_i (q - q_+) B^i_+ = (q - q_+) \sum_i B^i_+ = 0.$$

End of Proof.

B.3 Example

There are two consumers B and S with endowments of $(10, 0)$ and $(0, 10)$, respectively. Their expected utility functions are the same as $\log(x_0) + \log(x_1)$. State probabilities are $\pi_\alpha = \pi_\beta = 0.5$. Assuming that the inflation level $\mathcal{P} = 2$, the corresponding price-level (inflation) volatility is $\sigma^R = 33.3\%$ by eqn (2.4). In this case, it is computed that $\overline{\theta} = 23\%$ by eqn (2.7). Assuming that the transaction cost is 10% ($\theta = 0.1$) in the combined market, the equilibrium outcomes are summarized in the table.

Table B.1: Monetary and combined markets

	Monetary Market		**Combined Market**	
	Buyer	Seller	Buyer	Seller
Utility Level	3.07	3.26	3.19	3.18
$money(m^i)$	0.5	-0.5	0.212	-0.212
$bond(n^i)$	-		0.276	-0.276
(p_α, p_β)	$(0.820, 1.640)$		$(0.744, 1.488)$	
$(p, p + \theta)$	-		$(0.948, 1.048)$	
q	1.093		0.992	

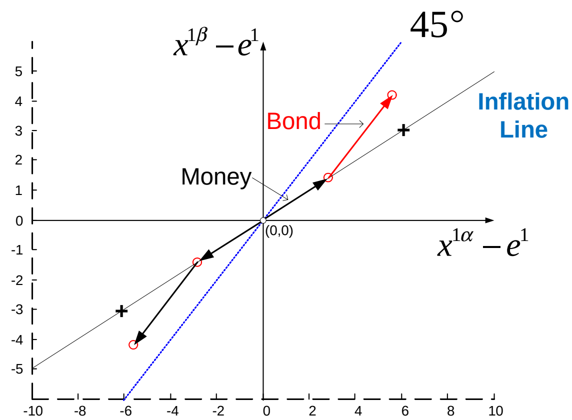

Figure B.1: Inflation line and asset trading

The equilibrium prices in the monetary market are higher than those in the combined market. Higher price levels imply a higher value of money in period 0. As explained in section 5, the higher value of money is from the asset buyer's higher demand for money. In a combined market, the demand for money is lower because of the substitution effects from the introduction of indexed bonds. Consequently, the value of money is not as high as that in a pure monetary market. The change of money value after the introduction of indexed bonds causes the asset seller's utility to decrease from 3.26 to 3.18. (See the table.)

Figure B.1 represents the equilibrium allocations in the space of excess demand. The figure should be three dimensional including the excess demand of x_0 but we can imagine the original three-dimensional figure is projected onto a two-dimensional one. The buyer's allocation is located in the northeast quadrant while the seller's is in the southwest quadrant. The allocations are symmetric to (0,0) by market clearing conditions. "+" and "o" represent the equilibrium allocations in pure monetary and combined markets, respectively. From the figure, we know that the equilibrium allocations move closer to the 45 degree line in the combined

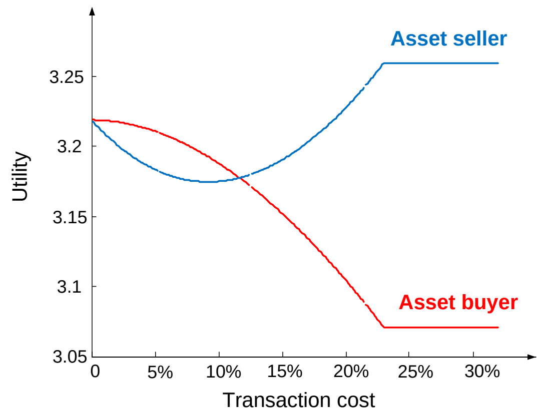

Figure B.2: Transaction cost and welfare

market although both markets have the same volatility level.

Figure B.2 shows how two consumers utility change in transaction costs. Surprisingly, the asset seller's utility is increasing in transaction cost θ if $\theta > 10\%$. Higher transaction costs make the demand for money increase by the substitution effects. Therefore, the price of money goes up and consequently, the asset buyer has more income from selling the money (risky asset) at a higher price. If the transaction cost is higher than 23%, the indexed bond market is inactive. In that case, the equilibrium allocations in a combined economy $\left(\mathcal{E}, \sigma^R, \theta\right)_C$ are identical to those in a pure monetary economy $\left(\mathcal{E}, \sigma^R\right)_M$.

APPENDIX C

CHAPTER 3 OF APPENDIX

C.1 Proof of Lemma 1

Proof. Without loss of generality, we assume that $M_\alpha < M_\beta$. Then, the state β is inflationary while the state α is deflationary, i.e., $p_\alpha < p_\beta$. Let's define $\mathcal{P}$ as the ratio between p_α and p_β: $\mathcal{P} = p_\beta/p_\alpha$. $\mathcal{P}$ is strictly increasing in σ_M^R. There are four possible changes in consumption at date 1 as $\mathcal{P}$ is increasing.

There are four cases:

Case 1 : both x_α^s and x_β^s increase in $\mathcal{P} > 1$.

Case 2 : x_α^s increases but x_β^s decreases in $\mathcal{P} > 1.(*)$

Case 3 : x_α^s decreases but x_β^s increases in $\mathcal{P} > 1$.

Case 4 : both x_α^s and x_β^s decrease in $\mathcal{P} > 1$.

First, Case 3 is ruled out since x_β^s/x_α^s must increase in $\mathcal{P} > 1$. Then, we need to show that Cases 1 and 4 are not true.

Assuming that Case 1 is true: for $1 < \mathcal{P} < \mathcal{P}'$, $x_\alpha^s(\mathcal{P}) < x_\alpha^s(\mathcal{P}')$ and $x_\beta^s(\mathcal{P}) < x_\beta^s(\mathcal{P}')$. By market clearing conditions, we get: $x_\alpha^b(\mathcal{P}) > x_\alpha^b(\mathcal{P}')$ and $x_\beta^b(\mathcal{P}) > x_\beta^b(\mathcal{P}')$.

The selling and buying prices of the (equivalent) nominal security is given by

$$q_b^+ = \frac{\pi_\alpha r_\alpha g^{b'}\left(x_\alpha^b\right) + \pi_\beta r_\beta g^{b'}\left(x_\beta^b\right)}{f^{s'}\left(x_0^s\right)} \tag{C.1}$$

$$q_b^+ (1+\theta_n) = \frac{\pi_\alpha r_\alpha g^{s'}(x_\alpha^s) + \pi_\beta r_\beta g^{s'}(x_\beta^s)}{f^{s'}(x_0^b)}. \tag{C.2}$$

By $x_\alpha^s(\mathcal{P}) < x_\alpha^s(\mathcal{P}')$, $x_\beta^s(\mathcal{P}) < x_\beta^s(\mathcal{P}')$ and strict concavity of g^i, $\pi_\alpha r_\alpha g^{b'}(x_\alpha^b) + \pi_\beta r_\beta g^{b'}(x_\beta^b)$ decreases in $\mathcal{P}$. In the same way, $\pi_\alpha r_\alpha g^{s'}(x_\alpha^s) + \pi_\beta r_\beta g^{s'}(x_\beta^s)$ increases in $\mathcal{P}$. Therefore, by eqns (C.1) and (C.2), the market clearing condition, $x_0^s + x_0^b = e_0^s + e_0^b$ and strict concavity of f^i, the following inequalities must be satisfied

$$x_0^s(\mathcal{P}) < x_0^s(\mathcal{P}')\text{and}x_0^b(\mathcal{P}) > x_0^b(\mathcal{P}').$$

The saver's budget constraints, where an indexed bond is inactive, are

$$x_0^s(\mathcal{P}) + q^+(\mathcal{P})(1+\theta_n)\, b^s(\mathcal{P}) = 0 \ \text{ and}$$

$$x_0^s(\mathcal{P}') + q^+(\mathcal{P}')(1+\theta_n)\, b^s(\mathcal{P}') = 0$$

Since $x_0^s(\mathcal{P}) < x_0^s(\mathcal{P}')$ and $b^b(\mathcal{P}) < b^b(\mathcal{P}')$,

$$q^+(\mathcal{P}) > q^+(\mathcal{P}') \tag{C.3}$$

The borrower's budget constraints are

$$x_0^b(\mathcal{P}) + q^+(\mathcal{P}) b^b(\mathcal{P}) = 0 \ \text{ and}$$

$$x_0^b(\mathcal{P}') + q^+(\mathcal{P}') b^b(\mathcal{P}') = 0$$

Since $x_0^b(\mathcal{P}) > x_0^b(\mathcal{P}')$ and $b^b(\mathcal{P}) > b^b(\mathcal{P}')$,

$$q^+(\mathcal{P}) < q^+(\mathcal{P}') \tag{C.4}$$

Inequalities (C.3) and (C.4) contradict each other.

We can show that Case 4 is not true in the same way.

C.2 Quasi-linear quadratic preferences

C.2.1 Asset prices

Agent i's utility is given by

$$u^i\left(x^i\right) = \lambda_0^i x_0^i - \frac{1}{2} E\left(\alpha^i - \widetilde{x}_s^i\right)^2, i = 1, ..., I \tag{C.5}$$

where $\widetilde{x}_s^i = \left\{x_\alpha^i, x_\beta^i; \pi_\alpha, \pi_\beta\right\}$.

The mean and variance of the random variable $\widetilde{x}_s^i$ is m^i and σ^{i2}. Then, eqn (C.5) can be expressed as

$$u^i\left(x^i\right) = \lambda_0^i x_0^i + \alpha^i m^i + \frac{1}{2} m^{i2} - \frac{1}{2}\sigma^{i2} - \alpha^{i2}, i = 1, ..., I$$

m^i and σ^{i2} is given by

$$m^i = e_1^i + b^i + z_d^i \tag{C.6}$$

$$\sigma^{i2} = var\left(e_1^i + b^i \widetilde{r} + z_d^i\right) = z_n^2 \sigma_{\widetilde{r}}. \tag{C.7}$$

where $\widetilde{r} = \{r_\alpha, r_\beta; \pi_\alpha, \pi_\beta\}$ which is the same as $\left\{\frac{\frac{1}{M_\alpha}}{\frac{\pi_\alpha}{M_\alpha} + \frac{\pi_\beta}{M_\beta}}, \frac{\frac{1}{M_\beta}}{\frac{\pi_\alpha}{M_\alpha} + \frac{\pi_\beta}{M_\beta}}; \pi_\alpha, \pi_\beta\right\}$. Given a monetary policy M, $\sigma_{\widetilde{r}}$ is computed as

$$\sigma_{\widetilde{r}} = \frac{\sqrt{\pi_\alpha \pi_\beta}\left(\frac{M_\beta}{M_\alpha} - 1\right)}{\pi_\alpha \frac{M_\beta}{M_\alpha} + \pi_\beta} \geq 1 \text{ where } M_\beta > M_\alpha. \tag{C.8}$$

From eqn (C.8), we know that as a monetary policy is more unstable, i.e., $\frac{M_\beta}{M_\alpha}$ is increasing, $\sigma_{\widetilde{r}}$ is increasing. Therefore, both $\sigma_{\widetilde{r}}$ and $\overline{\sigma}$ are increasing in $\frac{M_\beta}{M_\alpha}$.

Then, utility can be expressed as

$$\begin{aligned} u^i\left(x^i\right) &= \lambda_0^i x_0^i + \left(\alpha^i - e_1^i\right)\left(b^i + z_d^i\right) + \frac{1}{2}\left(b^i + z_d^i\right)^2 - \frac{1}{2} b^{i2} \sigma_{\widetilde{r}} \\ &\quad + \text{constant}. \end{aligned}$$

By the first order conditions, we can get the following equations:

$$\lambda^i q_b = \left(\alpha^i - e_1^i\right) + \left(b^i + z_d^i\right) - b^i \sigma_{\widetilde{r}} \text{ if } \ i = b \tag{C.9}$$

$$\lambda^i q_b \left(1 + \theta_n\right) = \left(\alpha^i - e_1^i\right) + \left(b^i + z_d^i\right) - b^i \sigma_{\widetilde{r}} \text{ if } \ i = s \tag{C.10}$$

$$\lambda^i q_d = \left(\alpha^i - e_1^i\right) + \left(b^i + z_d^i\right) \quad \text{if} \quad i = b \tag{C.11}$$

$$\lambda^i q_d \left(1 + \theta_d\right) = \left(\alpha^i - e_1^i\right) + \left(b^i + z_d^i\right) \quad \text{if} \quad i = s \tag{C.12}$$

where where λ^i is the marginal utility of income for agent i, i.e., $\lambda^i = \lambda_0^i / p_0$.

By the asset market clearing conditions $\sum\limits_{i \in I} b^i = 0$ and $\sum\limits_{i \in I} z_d^i = 0$, and eqns (C.9)-(C.12), the prices of q_b and q_d can be derived as

$$q_b = \frac{p_0 \sum\limits_{i \in I} \left(\alpha^i - e_1^i\right)}{\lambda_0^b + \left(1 + \theta_n\right) \lambda_0^s} \tag{C.13}$$

and

$$q_d = \frac{p_0 \sum\limits_{i \in I} \left(\alpha^i - e_1^i\right)}{\lambda_0^b + \left(1 + \theta_d\right) \lambda_0^s}. \tag{C.14}$$

From eqns (C.13) and (C.14), we know that $q_b > q_d$ and $\left(1 + \theta_b\right) q_b < \left(1 + \theta_d\right) q_d$ if and only if $\theta_n < \theta_d$.

C.2.2 Trading volume

The equivalent maximization for savers is given as

$$\begin{aligned} u^i\left(x^i\right) &= \lambda_0^i x_0^i + \left(\alpha^i - e_1^i\right)\left(z_n + z_d\right) + \frac{1}{2}\left(z_n + z_d\right)^2 - \frac{1}{2} z_n^2 \sigma_{\widetilde{r}} \\ &\quad + \text{constant} \end{aligned}$$

subject to

$$p_0 x_0^i + q_b b^i + q_d z_d^i = p_0 e_0.$$

First order conditions of the borrower are given as:

$$\lambda_0^b = \lambda^b p_0 \quad \left(\lambda^b = \lambda_0^b / p_0 \right), \tag{C.15}$$

$$\left(\alpha^b - e_1^b\right) + \left(b^b + z_d^b\right) = \lambda^b q_d, \tag{C.16}$$

and

$$\left(\alpha^b - e_1^b\right) + \left(b^b + z_d^b\right) - b^b \sigma_{\widetilde{r}} = \lambda^b q_b \tag{C.17}$$

From eqns (C.15), (C.16) and (C.17), z_n^b and z_d^b can be derived as a function of $(q_n, q_d, \sigma_{\widetilde{r}})$, where the superscript b represents a borrower:

$$b^b = \frac{\lambda_0^b \left(q_b - q_n\right)}{p_0 \sigma_{\widetilde{r}}^2} \tag{C.18}$$

$$\begin{aligned} z_d^b &= \lambda^b q_d - \left(\alpha^b - e_1^b\right) - z_n \\ &= \frac{\lambda_0^b}{p_0} q_d - \left(\alpha^b - e_1^b\right) - \frac{\lambda_0^b \left(q_b - q_n\right)}{p_0 \sigma_{\widetilde{r}}^2} \end{aligned} \tag{C.19}$$

By the asset market clearing conditions, b^s and z_d^s can be derived as:

$$b^s = \frac{\lambda_0^b \left(q_b - q_d\right)}{p_0 \sigma_{\widetilde{r}}^2} \tag{C.20}$$

$$z_d^s = -\frac{\lambda_0^b}{p_0} q_d + \left(\alpha^b - e_1^b\right) + \frac{\lambda_0^b \left(q_b - q_d\right)}{p_0 \sigma_{\widetilde{r}}^2} \tag{C.21}$$

The necessary condition for the indexed bond being active is that $\theta_n < \theta_d$, which implies that $q_b > q_d$ by eqns (C.13) and (C.14). From eqns (C.20) and (C.21), we know that $q_b b^s$ ($= q_b b^s$, the trading volume of the nominal bond) is decreasing in $\sigma_{\widetilde{r}}^2$ while $q_d z_d^s$ (the trading volume of the indexed bond) is increasing.

C.3 Numerical Example

C.3.1 Log-linear preferences

Preferences, endowment and transaction costs in the example are given as

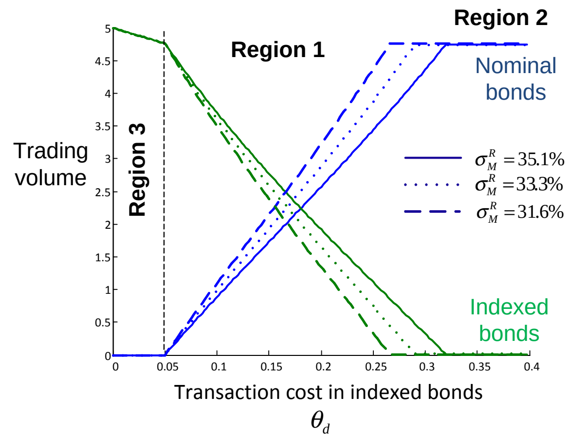

Figure C.1: Unstable monetary policy and asset trading

$$v^1(x_0, x_1) = v^2(x_0, x_1) = \ln x_0 + \ln x_1$$

$$e^1 = (10, 0), e^2 = (0, 10)$$

$$\theta_n = 5\%.$$

$$\pi^\alpha = \pi^\beta = 0.5$$

where there are two agents in the economy. The numerical examples in this paper have no connection to empirical data. The Y-axis in Figure B.4 represents the trading volumes of nominal and indexed bonds for the given transaction costs of an indexed bond. Where the value of θ_d is smaller than 5%, the indexed bond market is inactive. In region 1, the trading volumes of the nominal bond are monotonically increasing in θ_d while the trading volume of indexed bonds are monotonically decreasing in θ_d. Figure C.1 also shows that as the economy has more unstable monetary policy, the trading volume of the nominal security is decreasing while that of the indexed security is increasing.

Figure C.2 shows that the saver's and borrower's utility changes as the transaction costs of the indexed bond is increasing (or decreasing) where inflation volatility is given as 33.3%. There are three regions in Figure C.2. (See also Figure 1.) In re-

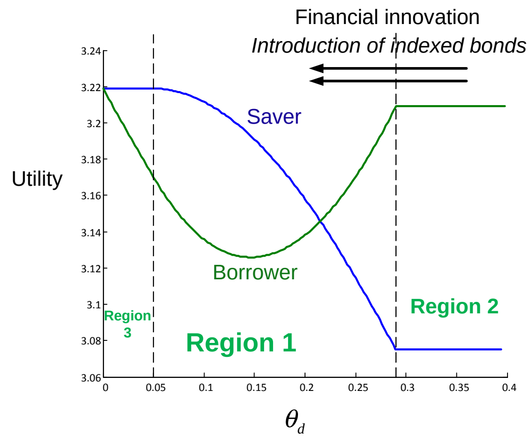

Figure C.2: Financial innovation and welfare

gion 1, both nominal and indexed bond markets are active. In region 2, the indexed bond market is inactive. In Figure C.2 the borrower's utility is not increasing as the transaction costs of indexed bonds are decreasing. Section 6 indicates that this paradoxical result is due to the motives for precautionary savings. The introduction of indexed bonds can be interpreted that the economy is moving from region 2 to region 1. From Figure C.2, we know that the introduction of the indexed bonds can make the borrower even worse off.

C.3.2 Quasi-linear preferences

Section 6 discusses the impact of financial innovation on the nominal interest rates and welfare based on quasi-linear preferences. I suggest two numerical examples to show how precautionary savings motives affect the nominal interest rate through financial innovation on an indexed bond. Once again, the numerical examples in this paper have no connection to empirical data.

In the first economy, agents have positive precautionary saving effects, i.e.,

$v_{222} > 0$. The preferences are given by

$$v^s = x_0 + \ln x_1$$
$$v^b = x_0 + \ln x_1 + 8$$

where $I = \{s, b\}$.

In the second economy, agents have negative precautionary saving effects, i.e., $v_{222} > 0$. The preferences are given by

$$v^s = 15x_0 + 36x_1 - \tfrac{1}{3}x_1^3$$
$$v^b = 15x_0 + 36x_1 - \tfrac{1}{3}x_1^3 + 40$$

The endowments of the two economies are the same as $e^s = (10, 0)$ and $e^b = (0, 10)$. In both economics, the monetary policy is given as $(M_0, M_\alpha, M_\beta) = (1, 2/3, 4/3)$ where $\pi_\alpha = \pi_\beta = 0.5$.

Figure C.3(a) and C.3(b) shows the numerical simulation of the relationship between the equilibrium nominal interest rates and the indexed bond's transaction costs. Figure C.3 shows that the financial innovation on the indexed bond affects the nominal interest rate in totally opposite directions in the two economies having positive and negative precautionary motives, respectively. The innovation makes the saver better off while the borrower worse off in the economy having positive precautionary motives but vise versa in the other economy, which is shown in Figure C.4.

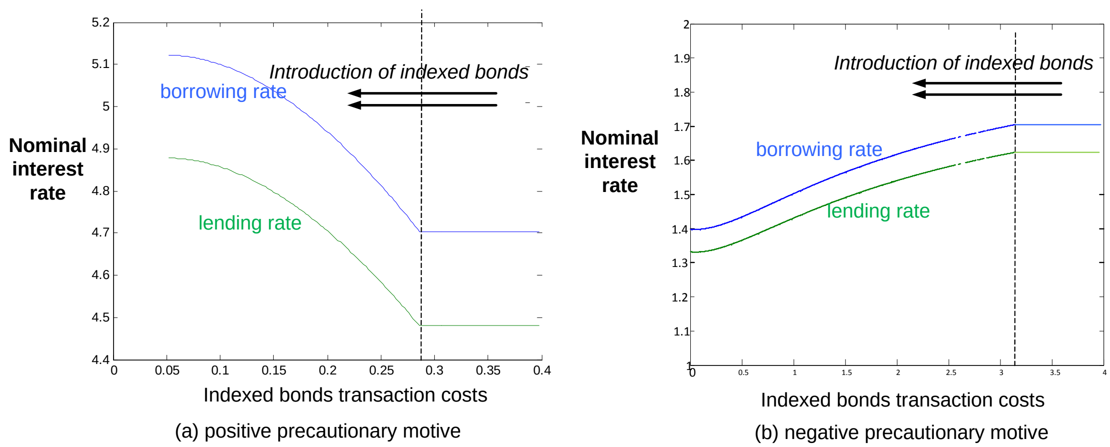

Figure C.3: Financial innovation and nominal interest rates

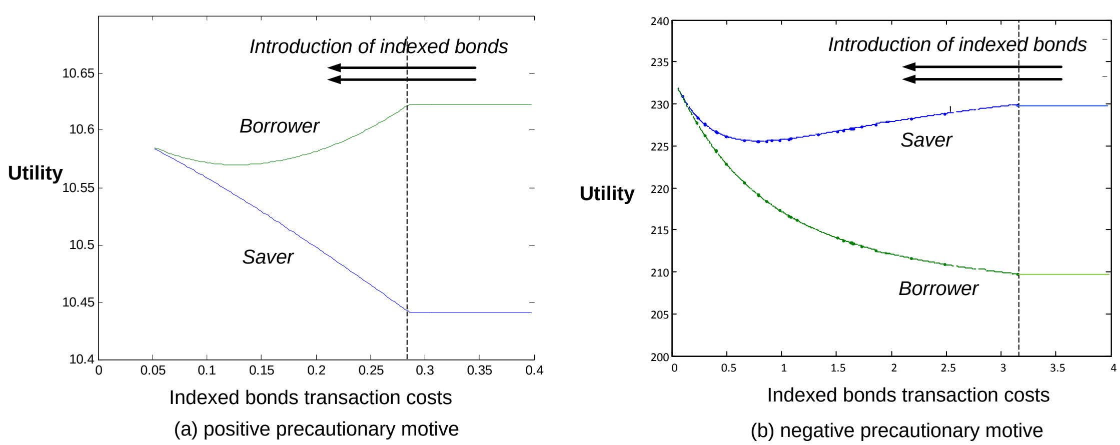

Figure C.4: Financial innovation and welfare

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