

SET THEORY IN INFINITE-DIMENSIONAL VECTOR SPACES

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We study examples of set-theoretic phenomena occurring in infinite-dimensional spaces, motivated by functional analysis. This includes equivalence relations induced by ideals of operators on a Hilbert space, a new “local” Ramsey theory for block sequences in Banach spaces and countable discrete vector spaces, analogues of selective ultrafilters and coideals in these settings, and families of infinite-dimensional subspaces which have pairwise finite-dimensional intersection. We draw analogies to the structure of the infinite subsets of the natural numbers.

BIOGRAPHICAL SKETCH

Iian Bryce Smythe was born on May 30, 1989, in Winnipeg, Manitoba, Canada, to parents Russell Smythe and Sherri Smythe (née Takatsu).

Living for the first 22 years of his life in the same Winnipeg home, Iian attended the nearby Donwood Elementary School, Chief Peguis Junior High School, and River East Collegiate. In 2007, he became the first member of his immediate family to attend college or university by matriculating at the University of Manitoba. He would go on to graduate with a B.Sc.(Hons.) in Mathematics, with a minor in Philosophy, in 2011.

In 2011, Iian moved to Ithaca, New York, to pursue a Ph.D. in Mathematics at Cornell University, for which this dissertation is the culmination. In the fall of 2017, he will begin a postdoctoral appointment as a Hill Assistant Professor of Mathematics at Rutgers, the State University of New Jersey.

For Tamiko, whom I never knew.

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CHAPTER 1

INTRODUCTION

This dissertation is largely motivated by a single question: to what extent does the structure of the infinite-dimensional subspaces of a vector space resemble that of the infinite subsets of the natural numbers? We will focus on separable Banach and Hilbert spaces due to their central role in functional analysis. Our aim is to catalogue analogies and differences between these settings, viewed through the lens of set theory.

The collection $\mathcal{P}(\omega)$ of all subsets of the natural numbers ω can be viewed simultaneously as a Boolean algebra with the operations of \cap and \cup , a partial order under \subseteq , and as a Polish space, that is, a separable completely metrizable topological space, when identified via characteristic functions with the Cantor space 2^ω . The collection of finite subsets of ω , denoted by Fin or $[\omega]^{<\omega}$, forms an ideal in this Boolean algebra, so we may form the quotient $\mathcal{P}(\omega)/\text{Fin}$. The nonzero elements of $\mathcal{P}(\omega)/\text{Fin}$ correspond to infinite subsets of ω identified *modulo finite*. The collection of infinite subsets of ω is denoted by $[\omega]^\omega$.

In the case of a Hilbert space H , which will typically be separable, infinite-dimensional, and taken over the complex field, closed linear subspaces can be identified¹ with orthogonal projection operators. The collection of all projection operators is denoted by $\mathcal{P}(H)$. This suggests two natural analogues of equivalence modulo finite: Two subspaces are equivalent *modulo finite dimensions* if each is contained in a finite-dimensional expansion of the other, or equivalently², the difference of the corresponding projection operators is finite-rank.

¹3.2.13 in [72].

²Proposition 2.5.1 below.

Two subspaces are equivalent *modulo compact* if the corresponding projection operators differ by a compact operator. The *compact operators* on a Hilbert space are exactly³ the operator-norm limits of finite-rank operators and form the only⁴ nontrivial closed ideal in the C*-algebra $\mathcal{B}(H)$ of all bounded operators on H . The quotient of $\mathcal{B}(H)$ by the ideal of compact operators $\mathcal{K}(H)$ is called the *Calkin algebra*, denoted by $\mathcal{C}(H)$, an important object of study in operator algebras (e.g., [19]). The *projections in the Calkin algebra*, denoted by $\mathcal{P}(\mathcal{C}(H))$, are the self-adjoint idempotent elements of $\mathcal{C}(H)$, and coincide⁵ with the image of $\mathcal{P}(H)$ under the quotient map $\mathcal{B}(H) \rightarrow \mathcal{C}(H)$.

The projections in the Calkin algebra can be viewed as a “noncommutative” [27] [88] [90] version of $\mathcal{P}(\omega)/\text{Fin}$. This analogy is made plain by observing that the projections in ℓ^∞ , which sits as a maximal abelian self-adjoint subalgebra of $\mathcal{B}(H)$, are exactly the $\{0, 1\}$ -valued sequences and can be identified with the elements of $\mathcal{P}(\omega)$. In the quotient of ℓ^∞ by its sole nontrivial closed ideal c_0 , projections correspond to elements of $\mathcal{P}(\omega)/\text{Fin}$.

In recent years, $\mathcal{P}(\mathcal{C}(H))$ has been investigated from a set-theoretic viewpoint. This includes the structure of its automorphisms [26] [73], maximal chains [36] [89], almost orthogonal families [ibid.], almost disjoint families [11], gaps [91], and filters [13] [27]. It is this body of work that lead us to consider many of the questions below.

In descriptive set theory, the study of definable subsets of Polish spaces, we examine $\mathcal{P}(\omega)/\text{Fin}$ via the equivalence relation E_0 induced by Fin on 2^ω :

$$xE_0y \iff \exists n \forall m \geq n (x(m) = y(m)).$$

³Theorem 3.3.3 in [72].

⁴For *separable* Hilbert spaces. See E 3.3.1 and E 4.5.10 in [72].

⁵Proposition 3.1 in [88].

That the resulting quotient does not carry a reasonable Polish or Borel structure is a consequence of the fact that E_0 is not *smooth*, that is, there is no Borel way to assign real number invariants to E_0 -classes so that each real is assigned to a unique class. This fact is closely related to Vitali's well-known construction of a nonmeasurable subset of \mathbb{R} , and can be seen using either the Baire category theorem or basic measure theory⁶. Consequently, E_0 is "more complicated", as an equivalence relation, than equality on the real numbers.

A major area of research in descriptive set theory is the study of definable equivalence relations on Polish spaces via *Borel reduction* (defined in §2.2.2). This yields a notion of relative complexity; we may ask how certain equivalence relations compare in complexity to others, e.g., those induced by group actions, or notions of isomorphism for suitable classes of mathematical structures.

In Ch. 2, we establish lower bounds for the complexity of the equivalence relations given by modulo finite dimensions and modulo compact for closed subspaces, via the corresponding projection operators, on a Hilbert space. As a consequence, the former is not reducible to the orbit equivalence relation of any Polish group action and the latter not reducible to isomorphism on any class of countable mathematical structures. We also address related equivalence relations induced by these and other ideals of operators, such as the Schatten p -ideals, on larger classes of operators.

An important example of the rich combinatorial structure of the infinite subsets of ω is Ramsey's theorem [74]: whenever $[\omega]^n$, the set of n -element subsets of ω , is partitioned into finitely many pieces, there is an infinite set $x \in [\omega]^\omega$ all of whose n -element subsets are contained in one piece of the partition. The

⁶See §6.1 in [32].

corresponding statement for partitions of $[\omega]^\omega$ is false (via the Axiom of Choice). However, if the pieces of the partition are Borel, or analytic and coanalytic (i.e., continuous images of Borel sets and their complements) then the corresponding result does hold; there must be an $x \in [\omega]^\omega$ all of whose infinite subsets are contained in one piece of the partition, theorems of Galvin–Prikry [31] and Silver [83], respectively. Assuming large cardinal hypotheses⁷, these results can be extended to any “reasonably definable” partition [80], in the sense that the partition is contained in the constructible closure of the reals $L(\mathbb{R})$.

The Ramsey theory of ω is deeply intertwined with the structure of certain ultrafilters in the Boolean algebra $\mathcal{P}(\omega)$. Recall that a *filter* in $\mathcal{P}(\omega)$ is a proper nonempty subset which is closed under taking supersets and finite intersections. It is an *ultrafilter* if it is maximal with respect to these properties, or equivalently, for every $x \in \mathcal{P}(\omega)$, one of x or $\omega \setminus x$ is in the ultrafilter. These notions are dual to those of ideals and maximal ideals, respectively. Ultrafilters can be identified with finitely-additive probability measures on ω and thus provide a notion of “largeness” for subsets of ω according to whether or not they are in the ultrafilter. Of particular interest are *nonprincipal* ultrafilters, those which do not contain any singletons, or equivalently, concentrate on the infinite sets. Such ultrafilters can be viewed as ultrafilters in the quotient $\mathcal{P}(\omega)/\text{Fin}$.

An ultrafilter is *selective*⁸ if it witnesses Ramsey’s theorem: for any finite partition of $[\omega]^n$, there is a set x in the ultrafilter all of whose n -element subsets are contained in one piece of the partition. Note that in the case $n = 1$, we recover the defining property of an ultrafilter. While their existence cannot be established from the usual axioms of ZFC alone [56], selective ultrafilters exist

⁷Certain strong “axioms of infinity” not provable from the other ZFC axioms. See [48].

⁸There are many equivalent definitions, see [17].

assuming the Continuum Hypothesis CH or Martin’s Axiom MA. Mathias [64] showed that selective ultrafilters, and more general selective coideals, also witness the infinite-dimensional Ramsey theorems of Galvin–Prikry and Silver: if $[\omega]^\omega$ is partitioned into finitely many Borel, or analytic and coanalytic, pieces, then there is a set x in the ultrafilter, all of whose infinite subsets are contained in one piece of the partition. By extending these results under large cardinal hypotheses to all definable partitions, Todorčević [25] showed that selective ultrafilters are generic, in the sense of forcing, over the inner model $L(\mathbb{R})$.

What sort of infinite-dimensional Ramsey theory is there for vector spaces and Banach spaces? And what are the corresponding notions of selective ultrafilter and coideal for the subspaces of a vector space, or in the quotient of the subspaces of a Hilbert space modulo compact? These are the main topics of Ch. 3, which constitutes the most substantial component of this dissertation.

The Ramsey theory for subspaces of a Banach space was developed by Gowers [33] [34], who used it to resolve the homogeneous space problem and develop a “rough” classification program for Banach spaces. Gowers’ dichotomy is a weak analogue of the results of Galvin–Prikry and Silver mentioned above. It states that given an analytic partition of the space of all infinite block sequences⁹ of a Banach space, and an error tolerance, there is a block sequence X such that either one piece of the partition contains all of X ’s further block subsequences, or the other piece contains “many” block subsequences of X “up to error”. Implicit in this work, and later clarified by Rosendal [76], is a Ramsey theorem for block sequences in discrete infinite-dimensional vector spaces over countable fields. We isolate the corresponding analogues of selective ultrafil-

⁹See §3.2 for a definition in the discrete setting. The definition is similar for Banach spaces having a Schauder basis.

ters and coideals, in both the discrete and Banach space settings, proving that the aforementioned Ramsey-theoretic dichotomies can always be witnessed by these families. Under large cardinal hypotheses, these results are extended to partitions in $L(\mathbb{R})$, and allow us to conclude that our ultrafilters are generic over the inner model $L(\mathbb{R})$.

In the case of a Hilbert space, we use these results to prove a criterion for genericity over $L(\mathbb{R})$, under large cardinal hypotheses, for filters of subspaces (or projections) modulo compact. Implicit in work of Farah–Weaver [27] is that such filters induce certain linear functionals on the bounded operators of the Hilbert space, called pure states, which are counterexamples to a conjecture of Anderson [3]. Thus, we have given a combinatorial characterization of these counterexamples, though their existence is merely consistent with ZFC and the consistency of the conjecture remains unresolved.

Mad families, collections of infinite subsets of ω which have pairwise finite intersection and are maximal with respect to this property, are a classical object of study in the set theory of $\mathcal{P}(\omega)$. Two central questions about mad families are what sizes they may have and to what extent they can be definable, as they are usually produced via Zorn’s Lemma. For instance, the minimum cardinality of a mad family, though always uncountable, is not decidable in ZFC, being comparatively small or large in different models of set theory¹⁰. Regarding their definability, Mathias [64] drew a connection between mad families and selective coideals, and showed via the Ramsey-theoretic results mentioned above that mad families can never be analytic. Under large cardinal hypotheses, this can be extended to show that there are no reasonably definable mad families.

¹⁰E.g., Theorem 2.15 in Ch. II and Theorem 2.3 in Ch. VIII of [57].

In Ch. 4, we study *mad families of subspaces* of a discrete countable vector space, that is, collections of infinite-dimensional subspaces with pairwise finite-dimensional intersection, focusing on questions of their size and definability. We prove that their minimum cardinality cannot be decided in ZFC and that the “spectrum” of cardinalities of mad families of subspaces can be made arbitrarily large, in analogy to results for mad families on ω . We then apply the Ramsey-theoretic material in Ch. 3 to mad families of subspaces and give a partial result concerning their necessary nondefinability. The general question of whether mad families of subspaces can be analytic remains open.

A word about the organization of this dissertation: Each of the chapters can be read independently, though Ch. 4 makes use of some of the results from Ch. 3. Each chapter has its own introduction containing more specific information and background about the content therein. Much of Ch. 2 has previously appeared in the author’s publication [83], though we have added details to some of the proofs and a supplementary section. Ch. 3 is derived from the author’s preprint [82], though again we have added additional details and two supplementary sections. The material in Ch. 4 is making its first appearance in print here.

CHAPTER 2
BOREL EQUIVALENCE RELATIONS IN THE SPACE OF BOUNDED
OPERATORS

2.1 Introduction

A fundamental problem in the theory of operators on an infinite-dimensional separable complex Hilbert space is to classify a collection of operators up to some notion of equivalence, a classical example being the following:

Theorem (Weyl–von Neumann [87]). *For T and S bounded self-adjoint operators on an infinite-dimensional separable complex Hilbert space, the following are equivalent:*

- (i) *T and S are unitarily equivalent modulo compact, i.e., there is a compact operator K and a unitary operator U such that $UTU^* - S = K$.*
- (ii) *T and S have the same essential spectrum¹.*

That is, bounded self-adjoint operators are completely classified up to unitary equivalence modulo compact by their essential spectra.

The modern theory of Borel equivalence relations affords us a general framework for such results. Given a space X of objects and an equivalence relation E on X , completely classifying the elements of X up to E -equivalence amounts to finding another space Y with equivalence relation F , and specifying a map $f : X \rightarrow Y$ such that

$$xEy \iff f(x)Ff(y),$$

¹For a definition of essential spectra see p. 30 in [70].

for all $x, y \in X$. The spaces and equivalence relations should be “reasonably definable”, in the sense that the former are Polish (or standard Borel) and the latter Borel. Enforcing that the classifying map f is Borel captures that idea that f is “computing” (in a very coarse sense) an invariant for the objects in X . Such a map is called a Borel reduction of E to F and its existence or nonexistence allows us to compare the complexity of such equivalence relations. The “simplest” Borel equivalence relations are those given by equality on Polish spaces and are said to be *smooth*.

Recasting the motivating problem in this setting requires specifying a Polish or Borel structure on the collection of operators of interest, verifying that the notion of equivalence is Borel, and reducing the equivalence relation to another, preferably well-understood, equivalence relation. In the setting of the Weyl–von Neumann theorem above, we have:

Theorem (Ando–Matsuzawa [4]). *The map $T \mapsto \sigma_{\text{ess}}(T)$ is a Borel function from the space of bounded self-adjoint operators to the Effros Borel space of closed subsets of \mathbb{R} . In particular, unitary equivalence modulo compact of bounded self-adjoint operators is smooth.*

In contrast, many natural equivalence relations on classes of operators are not smooth. In fact, they exhibit a very strong form of nonclassifiability; they cannot be reduced to the isomorphism relation on any class of countable algebraic or relational structures, e.g., groups, rings, graphs, etc. Such equivalence relations are said to be *not classifiable by countable structures*. The method used to exhibit this property is Hjorth’s theory of turbulence [40]. Relevant examples are given by:

Theorem (Kechris–Sofronidis [53]). *Unitary equivalence of self-adjoint (or unitary)*

operators is not classifiable by countable structures.

Theorem (Ando–Matsuzawa [4]). *Unitary equivalence modulo compact of unbounded self-adjoint operators is not classifiable by countable structures.*

In this chapter, we present nonclassification results for collections of operators, focusing on equivalence relations induced by ideals of compact operators. The paper is arranged as follows:

In §2, we review the relevant theory of bounded operators and Borel equivalence relations. In §3, we describe Borel and Polish structures on collections of operators and in §4, establish the first of our results:

- Theorem 2.1.1.** (a) *Equivalence modulo finite rank operators (on $\mathcal{B}(H)$ and $\mathcal{K}(H)$) is a Borel equivalence relation that is not Borel reducible to the orbit equivalence relation of any Polish group action.*
- (b) *Equivalence modulo compact operators (on $\mathcal{B}(H)$ and $\mathcal{B}(H)_{\leq 1}$) is a Borel equivalence relation that is not classifiable by countable structures.*
- (c) *Equivalence modulo Schatten p -class (on $\mathcal{B}(H)$, $\mathcal{B}(H)_{\leq 1}$, and $\mathcal{K}(H)$) is a Borel equivalence relation that is not classifiable by countable structures.*

As a consequence of Theorem 2.1.1(a), we show:

Corollary 2.1.2. *The space of all finite-rank operators on H is not Polishable in either the norm or strong operator topologies.*

In §5, we restrict our attention to the projection operators $\mathcal{P}(H)$, considering the restrictions of modulo finite-rank and modulo compact. The latter provides an alternate view of the projections in the Calkin algebra. We improve upon Theorem 2.1.1 parts (a) and (c), showing:

Theorem 2.1.3. (a) *Equivalence modulo finite-rank on $\mathcal{P}(H)$ is not Borel reducible to the orbit equivalence relation of any Polish group action.*

(b) *Equivalence modulo compact on $\mathcal{P}(H)$ is not classifiable by countable structures.*

2.2 Preliminaries

2.2.1 Bounded operators on Hilbert spaces

Throughout, we fix an infinite-dimensional separable complex Hilbert space H with inner product $\langle \cdot, \cdot \rangle$. Let $\mathcal{B}(H)$ denote the set of all bounded operators on H with operator norm $\| \cdot \|$. A standard reference for the theory of $\mathcal{B}(H)$ is [72].

The *strong operator topology* is the topology induced by the family of seminorms $T \mapsto \|Tv\|$ for $v \in H$, while the *weak operator topology* is induced by the family of seminorms $T \mapsto |\langle Tv, w \rangle|$ for $v, w \in H$.

We denote by T^* the (Hermitian) adjoint of an operator $T \in \mathcal{B}(H)$. An operator $T \in \mathcal{B}(H)$ is *self-adjoint* if $T = T^*$ and *positive* if $\langle Tv, v \rangle \geq 0$ for all $v \in H$. To each operator $T \in \mathcal{B}(H)$, there is a unique positive operator $|T|$ satisfying $|T|^2 = T^*T$.

An operator $P \in \mathcal{B}(H)$ is a *projection* if $P^2 = P^* = P$. Equivalently, P is the orthogonal projection onto a closed subspace (namely, $\text{ran}(P)$) of H . Every projection is positive with $\|P\| = 1$ whenever $P \neq 0$. We denote the set of projections by $\mathcal{P}(H)$. Note that if P is a projection, and $\{f_k : k \in \omega\}$ an orthonormal basis for $\text{ran}(P)$, then for $v \in H$, $Pv = \sum_{k=0}^{\infty} \langle v, f_k \rangle f_k$.

An operator $T \in \mathcal{B}(H)$ is *diagonal* with respect to an orthonormal basis $\{e_n : n \in \omega\}$ of H if there is a sequence $\{\lambda_n : n \in \omega\}$ of complex numbers such that, for $v \in H$, $Tv = \sum_{n=0}^{\infty} \lambda_n \langle v, e_n \rangle e_n$.

An operator $T \in \mathcal{B}(H)$ is *compact* if the image of the closed unit ball of H under T has compact closure. The set of compact operators is denoted by $\mathcal{K}(H)$. T is *finite-rank* if $\text{rank}(T) = \dim(\text{ran}(T)) < \infty$ and the set of finite rank operators is denoted by $\mathcal{B}_f(H)$. It is well-known that an operator on H is compact if and only if it is a norm limit of finite-rank operators. The following characterizes which diagonal operators are compact:

Proposition 2.2.1 (3.3.5 in [72]). *If $T \in \mathcal{B}(H)$ is diagonal with respect to an orthonormal basis $\{e_n : n \in \omega\}$, say $Tv = \sum_{n=0}^{\infty} \lambda_n \langle v, e_n \rangle e_n$ for all $v \in H$, then T is compact if and only if $\lim_{n \rightarrow \infty} \lambda_n = 0$.*

It is easy to check that $\mathcal{K}(H)$ is a norm-closed, self-adjoint ideal in $\mathcal{B}(H)$, and the corresponding quotient $\mathcal{B}(H)/\mathcal{K}(H)$ is called the *Calkin algebra*.

For $1 \leq p < \infty$, the *Schatten p -class* $\mathcal{B}^p(H)$ is the set of all operators $T \in \mathcal{B}(H)$ such that for some orthonormal basis (e_n) of H , one has $\sum_{n=0}^{\infty} \langle |T|^p e_n, e_n \rangle < \infty$ (this quantity is independent of the choice of basis). Each $\mathcal{B}^p(H)$ is a self-adjoint ideal in $\mathcal{B}(H)$, which fails to be norm-closed, and $\mathcal{B}_f(H) \subsetneq \mathcal{B}^p(H) \subsetneq \mathcal{K}(H)$.

The following facts will be relevant in the sequel. We caution that the adjoint operation $T \mapsto T^*$ is not strongly continuous and multiplication is not jointly strongly continuous on all of $\mathcal{B}(H)$. For these facts, and the following lemma, see §4.6 in [72].

Lemma 2.2.2. (a) *The adjoint operation is weakly continuous on $\mathcal{B}(H)$.*

(b) *Multiplication of operators is strongly continuous when restricted to $B \times \mathcal{B}(H) \rightarrow \mathcal{B}(H)$, where B is any norm bounded subset of $\mathcal{B}(H)$.*

Lemma 2.2.3. (a) *The closed unit ball $\mathcal{B}(H)_{\leq 1}$ is strongly closed and completely metrizable in $\mathcal{B}(H)$.*

(b) *The set of self-adjoint operators $\mathcal{B}(H)_{sa}$ is strongly closed in $\mathcal{B}(H)$.*

(c) *The set of positive operators is strongly closed in $\mathcal{B}(H)$.*

(d) *The set of projections $\mathcal{P}(H)$ is strongly closed in $\mathcal{B}(H)_{\leq 1}$.*

Proof. For (a), $\mathcal{B}(H)_{\leq 1}$ is closed by an application of the uniform boundedness principle and completely metrizable by 4.6.2 in [72]. By Lemma 2.2.2(a), $\mathcal{B}(H)_{sa}$ is weakly, thus strongly, closed, showing (b). Part (c) follows from the fact that the maps $T \mapsto \langle Tv, v \rangle$, for $v \in H$, are strongly continuous. For (d), let $\mathcal{I} = \{T \in \mathcal{B}(H)_{\leq 1} : T^2 = T\}$ and $\mathcal{S} = \{T \in \mathcal{B}(H)_{\leq 1} : T^* = T\}$. By Lemma 2.2.2, \mathcal{I} and \mathcal{S} are strongly closed in $\mathcal{B}(H)_{\leq 1}$, and $\mathcal{P}(H) = \mathcal{S} \cap \mathcal{I}$. \square

2.2.2 Borel equivalence relations

A *Polish space* is a separable and completely metrizable topological space, while a *standard Borel space* is a set X together with a σ -algebra of Borel sets coming from some Polish topology on X . An equivalence relation E on X is *Borel* if $\{(x, y) \in X^2 : xEy\}$ is a Borel subset of X^2 . Given equivalence relations E and F on Polish (or standard Borel) spaces X and Y , respectively, a map $f : X \rightarrow Y$ is a *Borel reduction* of E to F if f is Borel measurable, and

$$xEy \iff f(x)Ff(y)$$

for all $x, y \in X$. Equivalently, f is a Borel map which descends to a well-defined injection $X/E \rightarrow Y/F$. In this case, we say that E is *Borel reducible* to F and

write $E \leq_B F$. If f is injective, we say that f is a *Borel embedding* of E into F and write $E \sqsubseteq_B F$. If $E \leq_B F$ and $F \leq_B E$, we write $E \equiv_B F$ and say that E and F are *Borel bireducible*. Intuitively, $E \leq_B F$ means that classifying elements of Y up to F is at least as complicated as classifying elements of X up to E , as any classification of the former yields one for the latter. $E \equiv_B F$ means that they are of equal complexity.

Example 2.2.4. If X is a Polish space, we denote by $\Delta(X)$ the equality relation on X . $\Delta(X)$ is a closed, and thus Borel, subset of X^2 .

Example 2.2.5. Identifying $2 = \{0, 1\}$ with the discrete topology, the Borel equivalence relation E_0 is defined on 2^ω by

$$(x_n)_n E_0 (y_n)_n \iff \exists m \forall n \geq m (x_n = y_n).$$

Example 2.2.6. The Borel equivalence relation E_1 is defined on \mathbb{R}^ω by

$$(x_n)_n E_1 (y_n)_n \iff \exists m \forall n \geq m (x_n = y_n).$$

A Borel equivalence relation E on a Polish space X is *smooth* if $E \leq_B \Delta(Y)$ for some Polish space Y . Smooth equivalence relations are exactly those which admit complete classification by real numbers. It is well-known that E_0 is not smooth (cf. §6.1 in [32]) and so any Borel equivalence relation to which it reduces also fails to be smooth.

A *Polish group* G is a topological group which has a Polish topology. If X is a Polish space and G acts continuously on X , i.e., the map $G \times X \rightarrow X$ given by $(g, x) \mapsto g \cdot x$ is continuous, then we say that X is a *Polish G -space* and denote by E_G (or sometimes E/G) the orbit equivalence relation

$$x E_G y \iff \exists g \in G (g \cdot x = y).$$

This equivalence relation is not Borel in general (see §9.4 of [32] for examples).

A group with a given Borel structure (e.g., a Borel subgroup of a Polish group) is *Polishable* if it can be endowed with a Polish group topology having the same Borel structure. It is easy to check that the orbit equivalence relation induced by the translation action of a Polishable (or Borel) subgroup of a Polish group is Borel. The following shows that E_1 is an obstruction to classification by orbits of Polish group actions.

Theorem 2.2.7 (Kechris–Louveau [52]). *Let G be a Polish group and X a Polish G -space. Then, $E_1 \not\leq_B E_G^X$.*

The isomorphism relation on the class of countable structures of a first-order theory, e.g., groups, rings, graphs, etc, can be represented as the orbit equivalence relation of a Polish G -space (cf. Ch. 11 of [32]), where G is a closed subgroup of S_∞ , the infinite permutation group of the natural numbers. If an equivalence relation is Borel reducible to such a relation, we say that it is *classifiable by countable structures*. Hjorth [40] isolated a dynamical property of Polish G -spaces, called *turbulence*, which implies that the corresponding orbit equivalence relation resists such classification.

Theorem 2.2.8 (Hjorth [40]). *Let X be a Polish G -space. If the action of G is turbulent, then E_G is not classifiable by countable structures.*

For our purposes, it suffices to consider examples of such actions. For a more detailed discussion of turbulence see the supplementary §2.7 below, or the references [32], [40], [51].

Lemma 2.2.9 (Proposition 3.25 in [40] and p. 35 in [51]). (a) *The translation action of c_0 on \mathbb{R}^ω is turbulent.*

(b) For $1 \leq p < \infty$, the translation actions of ℓ^p on \mathbb{R}^ω and c_0 are turbulent.

In particular, the orbit equivalence relations \mathbb{R}^ω/c_0 , \mathbb{R}^ω/ℓ^p , and c_0/ℓ^p are not classifiable by countable structures.

We consider the restrictions of \mathbb{R}^ω/c_0 and \mathbb{R}^ω/ℓ^p to the subset $[0, 1]^\omega$, and denote them by $[0, 1]^\omega/c_0$ and $[0, 1]^\omega/\ell^p$, respectively. It is evident that these are Borel equivalence relations and $[0, 1]^\omega/c_0 \sqsubseteq_B \mathbb{R}^\omega/c_0$ and $[0, 1]^\omega/\ell^p \sqsubseteq_B \mathbb{R}^\omega/\ell^p$, via the inclusion maps. Moreover:

Lemma 2.2.10 (Lemma 6.2.2 in [49], see also [71]). (a) $\mathbb{R}^\omega/c_0 \equiv_B [0, 1]^\omega/c_0$.

(b) For $1 \leq p < \infty$, $\mathbb{R}^\omega/\ell^p \equiv_B [0, 1]^\omega/\ell^p$.

In particular, $[0, 1]^\omega/c_0$ and $[0, 1]^\omega/\ell^p$ are not classifiable by countable structures.²

An alternative proof of the nonclassifiability of $[0, 1]^\omega/c_0$ and $[0, 1]^\omega/\ell^p$ is given in §2.7.

2.3 Topology and Borel structure on $\mathcal{B}(H)$

In order to study Borel equivalence relations on $\mathcal{B}(H)$ or its subsets, we must endow them with a Polish or standard Borel structure. The norm topology on $\mathcal{B}(H)$ (or $\mathcal{P}(H)$) is *not* Polish as it contains discrete subsets of size 2^{\aleph_0} : given an orthonormal basis $\{e_n : n \in \omega\}$, consider the family of projections P_x onto $\overline{\text{span}}\{e_n : n \in x\}$, for $x \subseteq \omega$. Instead, we use the strong operator topology.

Lemma 2.3.1. (a) $\mathcal{B}(H)_{\leq 1}$ and $\mathcal{P}(H)$ are Polish in the strong operator topology.

²Recent work by Hartz and Lupini [37] has developed a general theory of *turbulent Polish groupoids* in which this can be seen more directly.

(b) $\mathcal{B}(H)$ is a standard Borel space with respect to the Borel structure generated by the strong operator topology.

Proof. (a) follows from Lemma 2.2.3 and the fact that the strong operator topology is separable. For (b), note that a countable union of standard Borel spaces is standard Borel and $\mathcal{B}(H) = \bigcup_{n \geq 1} n\mathcal{B}(H)_{\leq 1}$. \square

All references to Borel subsets of, or functions on, $\mathcal{B}(H)$ will be with respect to this Borel structure, which coincides with that of the weak operator topology (as closed convex sets in $\mathcal{B}(H)$ are weakly closed if and only if they are strongly closed, Corollary 4.6.5 in [72]). We caution that $\mathcal{B}(H)$ is not Polish in the strong operator topology (it is not metrizable, see E 4.6.4 in [72]), nor is it even Polishable as a group with this Borel structure (this follows from Lemma 9.3.3 in [32]).

The equivalence relations we study below arise from the ideals $\mathcal{B}_f(H)$, $\mathcal{K}(H)$ and $\mathcal{B}^p(H)$ for $1 \leq p < \infty$, thus we will need to show that the corresponding ideal is Borel in the relevant topology.

Lemma 2.3.2. *For each $n \in \omega$, the set $\mathcal{F}_{\leq n} = \{T \in \mathcal{B}(H) : \text{rank}(T) \leq n\}$ is strongly closed in $\mathcal{B}(H)$.*

Proof. ³ Suppose that $T \in \mathcal{B}(H)$ is such that $\text{rank}(T) > n$. There are vectors $v_0, \dots, v_n \in H$ such that Tv_0, \dots, Tv_n are linearly independent, or equivalently, their Gram determinant $\det(\langle Tv_i, Tv_j \rangle_{i,j})$ is nonzero. Since the Gram determinant is continuous, there is a strongly open neighborhood of T in $\mathcal{B}(H)$ such that for all S in that neighborhood, the Gram determinant $\det(\langle Sv_i, Sv_j \rangle_{i,j})$ is

³We thank the anonymous referee for a much shortened proof of this fact.

also nonzero, and so $\text{rank}(S) > n$. Thus, the complement of $\mathcal{F}_{\leq n}$ is strongly open. \square

Proposition 2.3.3. $\mathcal{B}_f(H)$ is an F_σ set and $\mathcal{K}(H)$ is an $F_{\sigma\delta}$ set in the strong operator topology on $\mathcal{B}(H)$.

Proof. The claim for $\mathcal{B}_f(H)$ is an immediate consequence of Lemma 2.3.2. The proof for the claim regarding $\mathcal{K}(H)$ is essentially that of the more general Theorem 3.1 in [23]. Let $\{T_k\}_{k=1}^\infty$ be a norm-dense sequence in $\mathcal{K}(\mathcal{H})$, and let $B = \mathcal{B}(\mathcal{H})_{\leq 1}$. Then,

$$\mathcal{K}(\mathcal{H}) = \bigcap_{n=1}^{\infty} \left(\mathcal{K}(\mathcal{H}) + \frac{1}{n}B \right) \supseteq \bigcap_{n=1}^{\infty} \bigcup_{k=1}^{\infty} \left(T_k + \frac{1}{n}B \right) \supseteq \mathcal{K}(\mathcal{H}),$$

where the first equality is the result of $\mathcal{K}(H)$ being norm-closed in $\mathcal{B}(H)$. Since B is strongly closed, this shows that $\mathcal{K}(\mathcal{H})$ is $F_{\sigma\delta}$. \square

Lemma 2.3.4 (cf. p. 48 of [24]). *If $f : \mathbb{R} \rightarrow \mathbb{R}$ is a bounded Borel function, then the map $\mathcal{B}(H)_{sa} \rightarrow \mathcal{B}(H)_{sa}$ given by $T \mapsto f(T)$ is Borel.*

Proof. Let $(p_n)_n$ be a sequence of real polynomials converging to f pointwise, which are uniformly bounded on compact sets. It follows by basic spectral theory that, for $T \in \mathcal{B}(H)_{sa}$, $p_n(T)$ converges to $f(T)$ weakly. Thus, the map in question is a pointwise (weak) limit of Borel functions, by Lemma 2.2.2, and hence Borel. \square

Lemma 2.3.5. $\mathcal{K}(H)$ and $\mathcal{B}^p(H)$, for $1 \leq p < \infty$, are Polish spaces. In fact, they are separable Banach spaces when considered with the operator norm and p -norm, respectively.

Proof. It suffices to verify separability, which follows from the fact that each of the spaces considered contains $\mathcal{B}_f(H)$ as a dense subset. \square

Proposition 2.3.6. *For each $1 \leq p < \infty$, $\mathcal{B}^p(H)$ is a Polishable subspace of $\mathcal{K}(H)$ in the norm topology and a Borel subset of $\mathcal{B}(H)$ in the strong operator topology.*

Proof. Fix $1 \leq p < \infty$. By Lemma 2.3.5, $\mathcal{B}^p(H)$ is a separable Banach space under the p -norm. To prove that it is Polishable in $\mathcal{K}(H)$, it suffices to verify that the Borel structures in both topologies coincide. This follows from the fact that $\|T\| \leq \|T\|_p$ for $T \in \mathcal{B}^p(H)$, showing that the inclusion map $\mathcal{B}^p(H) \rightarrow \mathcal{K}(H)$ is a continuous injection.

For the second claim, the map $T \mapsto (T^*T)^{p/2} = |T|^p$ is Borel by Lemmas 2.2.2 and 2.3.4, and if $\{e_n : n \in \omega\}$ is a fixed orthonormal basis for H , then $T \in \mathcal{B}^p(H)$ if and only if there is an M , such that for all N , $\sum_{n=0}^N \langle |T|^p e_n, e_n \rangle < M$. Thus, $\mathcal{B}^p(H)$ is Borel. \square

2.4 Equivalence relations in $\mathcal{B}(H)$

As per Lemmas 2.3.1 and 2.3.5, $\mathcal{B}(H)$ will be considered as a standard Borel space with the Borel structure induced by the strong operator topology, $\mathcal{B}(H)_{\leq 1}$ a Polish space with the strong operator topology, and $\mathcal{K}(H)$ a Polish space with the norm topology. We will consider the equivalence relations on $\mathcal{B}(H)$, and their restrictions to $\mathcal{B}(H)_{\leq 1}$ and $\mathcal{K}(H)$, induced by the ideals $\mathcal{B}_f(H)$, $\mathcal{K}(H)$ and $\mathcal{B}^p(H)$ for $1 \leq p < \infty$, denoted (and named) as follows:

$$T \equiv_f S \iff T - S \in \mathcal{B}_f(H) \quad (\text{modulo finite-rank})$$

$$T \equiv_{\text{ess}} S \iff T - S \in \mathcal{K}(H) \quad (\text{modulo compact or essential equivalence})$$

$$T \equiv_p S \iff T - S \in \mathcal{B}^p(H) \quad (\text{modulo } p\text{-class}), \text{ for } 1 \leq p < \infty.$$

Fix an orthonormal basis $\{e_n : n \in \omega\}$ for H for the remainder of this section. Consider the map $\ell^\infty \rightarrow \mathcal{B}(H)$ given by $\alpha \mapsto T_\alpha$, where $T_\alpha v = \sum_{n=0}^\infty \alpha_n \langle v, e_n \rangle e_n$, for $\alpha = (\alpha_n)_n \in \ell^\infty$ and $v \in H$.

Lemma 2.4.1. (a) *The map $\alpha \mapsto T_\alpha$ is an isometric embedding $\ell^\infty \rightarrow \mathcal{B}(H)$, with respect to the usual norms on these spaces, and maps c_0 into $\mathcal{K}(H)$.*

(b) *The map $\alpha \mapsto T_\alpha$ is continuous $[0, 1]^\omega \rightarrow \mathcal{B}(H)$, where $[0, 1]^\omega$ is endowed with the product topology and $\mathcal{B}(H)$ with the strong operator topology. Its range is contained within $\mathcal{B}(H)_{\leq 1}$.*

Proof. (a) This map is the well-known isometric embedding of ℓ^∞ as diagonal multiplication operators on H (see 4.7.6 in [72]). That it maps c_0 into $\mathcal{K}(H)$ is a restatement of Proposition 2.2.1.

(b) Fix $\alpha \in [0, 1]^\omega$ and let $U = \{T \in \mathcal{B}(H) : \|(T - T_\alpha)v\| < \epsilon\}$, a subbasic open neighborhood of T_α in the strong operator topology, where $v = \sum_{n=0}^\infty a_n e_n$ and $\epsilon > 0$. Pick m such that $\sum_{n=m+1}^\infty |a_n|^2 < \epsilon^2/2$, and let

$$V = \left\{ \beta \in [0, 1]^\omega : \sum_{n=0}^m |\beta_n - \alpha_n|^2 |a_n|^2 < \epsilon^2/2 \right\}.$$

V is an open neighborhood of α in $[0, 1]^\omega$. If $\beta \in V$, then

$$\|(T_\beta - T_\alpha)v\|^2 = \sum_{n=0}^m |\beta_n - \alpha_n|^2 |a_n|^2 + \sum_{n=m+1}^\infty |\beta_n - \alpha_n|^2 |a_n|^2 < \epsilon^2,$$

showing that $T_\beta \in U$. It follows that the map is continuous. □

We can now complete the proof of Theorem 2.1.1.

Proof of Theorem 2.1.1. By Propositions 2.3.3 and 2.3.6, each of the equivalence relations under consideration is Borel in the relevant spaces. We will use restric-

tions of the map $\alpha \mapsto T_\alpha$ to different domains, which are continuous injections in all relevant cases by Lemma 2.4.1.

(a) Let $X = \prod_{n=0}^{\infty} [0, \frac{1}{n+1}]$ and consider the equivalence relation E :

$$\alpha E \beta \quad \Leftrightarrow \quad \exists m \forall n \geq m (\alpha_n = \beta_n)$$

for $\alpha, \beta \in X$. This can be identified (up to Borel bi-reducibility) with E_1 . We use the restriction of the map $\alpha \mapsto T_\alpha$ to X . By Lemma 2.4.1(a), it maps into $\mathcal{K}(H)$. Moreover, $T_\alpha - T_\beta$ is finite-rank if and only if $\alpha E_1 \beta$. Thus, $E_1 \sqsubseteq_B \equiv_f$ on $\mathcal{K}(H)$ or $\mathcal{B}(H)$, and the result follows by Theorem 2.2.7.

(b) We use the map the restriction of the map $\alpha \mapsto T_\alpha$ to $[0, 1]^\omega$, which maps into $\mathcal{B}(H)_{\leq 1}$ by Lemma 2.4.1(b). Suppose that $\alpha, \beta \in [0, 1]^\omega$, then $(T_\alpha - T_\beta)v = \sum_{n=0}^{\infty} (\alpha_n - \beta_n) \langle v, e_n \rangle e_n$, for $v \in H$. By Proposition 2.2.1, $T_\alpha - T_\beta$ is compact if and only if $\alpha - \beta \in c_0$, showing $[0, 1]^\omega / c_0 \sqsubseteq_B \equiv_{\text{ess}}$ on $\mathcal{B}(H)_{\leq 1}$ or $\mathcal{B}(H)$. The result follows by Lemma 2.2.10(a).

(c) We again use the restriction of $\alpha \mapsto T_\alpha$ to $[0, 1]^\omega$. Fix $1 \leq p < \infty$. Suppose that $\alpha, \beta \in [0, 1]^\omega$. For $x \in H$, we have that $|T_\beta - T_\alpha|^p x = \sum_{n=0}^{\infty} |\beta_n - \alpha_n|^p \langle x, e_n \rangle e_n$, and so, $\sum_{n=0}^{\infty} \langle |T_\alpha - T_\beta|^p e_n, e_n \rangle = \sum_{n=0}^{\infty} |\beta_n - \alpha_n|^p$. Thus, $\alpha - \beta \in \ell^p$ if and only if $T_\alpha - T_\beta \in \mathcal{B}^p(H)$, showing $[0, 1]^\omega / \ell^p \sqsubseteq_B \equiv_p$ on $\mathcal{B}(H)_{\leq 1}$ or $\mathcal{B}(H)$. Similarly, for the restriction to $\mathcal{K}(H)$, we use the restriction of the map $\alpha \mapsto T_\alpha$ to c_0 and obtain $c_0 / \ell^p \sqsubseteq_B \equiv_p$ on $\mathcal{K}(H)$. The results follow as in (b), using Lemma 2.2.10(b) in the $[0, 1]^\omega / \ell^p$ case and Lemma 2.2.9(a) in c_0 / ℓ^p case. \square

Proof of Corollary 2.1.2. If $\mathcal{B}_f(H)$ was Polishable in either topology, then its translation action on $\mathcal{K}(H)$ would be a Polish group action, contrary to Theorem 2.1.1(a). \square

2.5 Equivalence relations in $\mathcal{P}(H)$

Recall that $\mathcal{P}(H)$ is the set of projections in $\mathcal{B}(H)$, a Polish space in the strong operator topology by Lemma 2.3.1. Fix an orthonormal basis $\{e_n : n \in \omega\}$ for H throughout this section. For each $x \subseteq \omega$, let P_x be the projection onto the subspace $\overline{\text{span}}\{e_n : n \in x\}$. Then, for $v \in H$, $P_x v = \sum_{n \in x} \langle v, e_n \rangle e_n$. The map $x \mapsto P_x$ is called the *diagonal embedding* (with respect to this basis) and is the restriction to 2^ω of the map $\alpha \mapsto T_\alpha$ from §4.

2.5.1 Equivalence modulo finite-rank

There are two natural ways to define equivalence modulo *finite-rank* or *finite dimension* on $\mathcal{P}(H)$. One could simply restrict \equiv_f to $\mathcal{P}(H)$, or one could say that $P \equiv_{fd} Q$ if there exist finite-dimensional subspaces U and V of H such that $\text{ran}(P) \subseteq \text{ran}(Q) + U$ and $\text{ran}(Q) \subseteq \text{ran}(P) + V$. In fact, these notions coincide. We will use the fact that if V is a closed subspace of H and F a finite-dimensional subspace of H , then $V + F$ is closed (E 2.1.4 in [72]).

Proposition 2.5.1. *Let $P, Q \in \mathcal{P}(H)$. The following are equivalent:*

- (i) $P \equiv_{fd} Q$.
- (ii) *There exist finite dimensional subspaces $W \subseteq \text{ran}(P)^\perp$ and $Y \subseteq \text{ran}(Q)^\perp$ such that $\text{ran}(P) + W = \text{ran}(Q) + Y$.*
- (iii) $P \equiv_f Q$.

Proof. (i) \Rightarrow (ii): Let U and V witness $P \equiv_{fd} Q$ as in the definition. Let W and Y be the images of U and V under orthogonal projections onto $\text{ran}(P)^\perp$ and

to $\text{ran}(Q)^\perp$, respectively. Then, $\text{ran}(P) + U = \text{ran}(P) + W$ and $\text{ran}(Q) + V = \text{ran}(Q) + Y$.

(ii) \Rightarrow (iii): For W and Y as in (ii), let R be the projection onto W and R' the projection onto Y . Since W is orthogonal to $\text{ran}(P)$, $P + R$ is the projection onto $\text{ran}(P) + W$. Likewise $Q + R'$ is the projection onto $\text{ran}(Q) + Y$. Thus, $P + R = Q + R'$, and so $P - Q = R' - R$, a finite rank operator.

(iii) \Rightarrow (i): Suppose that $P - Q = A$ where $A \in \mathcal{B}_f(A)$. Then,

$$\text{ran}(P) = \text{ran}(Q + A) \subseteq \text{ran}(Q) + \text{ran}(A) = \text{ran}(Q) + \text{ran}(A),$$

and likewise,

$$\text{ran}(Q) = \text{ran}(P - A) \subseteq \text{ran}(P) + \text{ran}(A) = \text{ran}(P) + \text{ran}(A).$$

Since $\text{ran}(A)$ is finite-dimensional, it follows that $P \equiv_{fd} Q$. □

Consequently, we will use \equiv_f for this (Borel, by Proposition 2.3.3) equivalence relation on $\mathcal{P}(H)$. It is easy to see that the diagonal embedding witnesses the nonsmoothness of \equiv_f on $\mathcal{P}(H)$: Given $x, y \in 2^\omega$,

$$(P_x - P_y)v = \sum_{n=0}^{\infty} (x_n - y_n) \langle v, e_n \rangle e_n$$

for all $v \in H$, and this operator is finite rank if and only if all but finitely many of the terms $x_n - y_n$ are 0.

To show that \equiv_f restricted to $\mathcal{P}(H)$ is of higher complexity, we define a new map $[0, 1]^\omega \rightarrow \mathcal{P}(H)$ given by $\alpha \mapsto P_\alpha$ as follows: For each $\alpha = (\alpha_n)_n \in [0, 1]^\omega$, let P_α be the projection onto $\overline{\text{span}}\{e_{2n} + \alpha_n e_{2n+1} : n \in \omega\}$. Note that the range of this map is not commutative and, in particular, not simultaneously diagonalizable by some basis.

Lemma 2.5.2. *The map $[0, 1]^\omega \rightarrow \mathcal{P}(H)$ given by $\alpha \mapsto P_\alpha$ is a continuous injection.*

Proof. First we show that $\alpha \mapsto P_\alpha$ is injective. Let $\alpha, \beta \in [0, 1]^\omega$ with $\alpha \neq \beta$, so $\alpha_k \neq \beta_k$ for some k . In order to show that $P_\alpha \neq P_\beta$, it suffices to show that $P_\alpha(e_{2k} + \beta_k e_{2k+1}) \neq e_{2k} + \beta_k e_{2k+1} = P_\beta(e_{2k} + \beta_k e_{2k+1})$. Note that

$$P_\alpha(e_{2k} + \beta_k e_{2k+1}) = \frac{1 + \alpha_k \beta_k}{1 + \alpha_k^2} (e_{2k} + \alpha_k e_{2k+1}).$$

By linear independence of e_{2k} and e_{2k+1} , the right hand side of the displayed equation is equal to the input on the left hand side if and only if $\alpha_k = \beta_k$. Thus, $P_\alpha \neq P_\beta$.

To see that the map is continuous,⁴ for each $n \in \omega$ and $\alpha \in [0, 1]^\omega$, let $P_{n,\alpha}$ be the projection of H onto $\text{span}\{e_{2n} + \alpha_n e_{2n+1}\}$. It is clear that for each n , the map $[0, 1]^\omega \rightarrow \mathcal{P}(H)$ given by $\alpha \mapsto P_{n,\alpha}$ is strongly continuous, and $P_\alpha = \bigoplus_{n \in \omega} P_{n,\alpha}$. To see that $\alpha \mapsto P_\alpha$ is strongly continuous, let $\alpha_k \rightarrow \alpha$ in $[0, 1]^\omega$, and v be a unit vector. By density and the fact that $\|P_{n,\alpha}\| \leq 1$ for all n and α , it suffices to consider v in the (algebraic) direct sum $\bigoplus_n \text{span}\{e_{2n}, e_{2n+1}\}$, in which case $\|(P_{\alpha_k} - P_\alpha)v\| \rightarrow 0$ follows from the strong continuity of each of the factors $P_{n,\alpha}$. \square

For $\alpha \in [0, 1]^\omega$, the vectors $\frac{1}{\sqrt{1+\alpha_n^2}}(e_{2n} + \alpha_n e_{2n+1})$, $n \in \omega$, form an orthonormal basis for $\text{ran}(P_\alpha)$. Thus, we can write,

$$\begin{aligned} P_\alpha v &= \sum_{n=0}^{\infty} \left\langle \sum_{k=0}^{\infty} a_k e_k, \frac{1}{\sqrt{1+\alpha_n^2}}(e_{2n} + \alpha_n e_{2n+1}) \right\rangle \frac{1}{\sqrt{1+\alpha_n^2}}(e_{2n} + \alpha_n e_{2n+1}) \\ &= \sum_{n=0}^{\infty} \frac{a_{2n} + a_{2n+1} \alpha_n}{1 + \alpha_n^2} (e_{2n} + \alpha_n e_{2n+1}), \end{aligned}$$

⁴We thank the anonymous referee for a much shortened proof of this fact.

and

$$(P_\alpha - P_\beta)v = \sum_{n=0}^{\infty} \left[\frac{a_{2n} + a_{2n+1}\alpha_n}{1 + \alpha_n^2} - \frac{a_{2n} + a_{2n+1}\beta_n}{1 + \beta_n^2} \right] e_{2n} \\ + \sum_{n=0}^{\infty} \left[\frac{a_{2n}\alpha_n + a_{2n+1}\alpha_n^2}{1 + \alpha_n^2} - \frac{a_{2n}\beta_n + a_{2n+1}\beta_n^2}{1 + \beta_n^2} \right] e_{2n+1},$$

for $\alpha, \beta \in [0, 1]^\omega$ and $v = \sum_{n=0}^{\infty} a_n e_n \in H$.

Since we must consider the difference $P_\alpha - P_\beta$ several times in what follows, it will be useful to have it in a canonical form. Denote by T_0, T_1, T_2 and T_3 the diagonal operators

$$T_0v = \sum_{n=0}^{\infty} \left[\frac{1}{1 + \alpha_n^2} - \frac{1}{1 + \beta_n^2} \right] a_{2n} e_{2n}, \\ T_1v = \sum_{n=0}^{\infty} \left[\frac{\alpha_n}{1 + \alpha_n^2} - \frac{\beta_n}{1 + \beta_n^2} \right] a_{2n+1} e_{2n+1}, \\ T_2v = \sum_{n=0}^{\infty} \left[\frac{\alpha_n}{1 + \alpha_n^2} - \frac{\beta_n}{1 + \beta_n^2} \right] a_{2n} e_{2n}, \\ T_3v = \sum_{n=0}^{\infty} \left[\frac{\alpha_n^2}{1 + \alpha_n^2} - \frac{\beta_n^2}{1 + \beta_n^2} \right] a_{2n+1} e_{2n+1},$$

and by S_0 and S_1 the operators

$$S_0v = \sum_{n=0}^{\infty} a_{2n+1} e_{2n} \quad \text{and} \quad S_1v = \sum_{n=0}^{\infty} a_{2n} e_{2n+1},$$

for $v = \sum_{n=0}^{\infty} a_n e_n$. Each of the aforementioned operators is bounded, and by collecting terms, one can show that

$$P_\alpha - P_\beta = T_0 + S_0T_1 + S_1T_2 + T_3. \quad (2.1)$$

We can now prove Theorem 2.1.3(a).

Proof of Theorem 2.1.3(a). ⁵ By Lemma 2.5.2, the map $\alpha \mapsto P_\alpha$ is a continuous injection. Represent E_1 on $[0, 1]^\omega$ by $\alpha E_1 \beta \Leftrightarrow \exists m \forall n \geq m (\alpha_n = \beta_n)$. As above, for

⁵The author is indebted to Ilijas Farah for suggesting this result and ideas of its proof.

$\alpha, \beta \in [0, 1]^\omega$, we have the representation $P_\alpha - P_\beta = T_0 + S_0T_1 + S_1T_2 + T_3$. Clearly, if $\alpha E_1 \beta$, then all but finitely many of the coefficients (which are independent of v) $\left[\frac{1}{1+\alpha_n^2} - \frac{1}{1+\beta_n^2} \right]$, $\left[\frac{\alpha_n}{1+\alpha_n^2} - \frac{\beta_n}{1+\beta_n^2} \right]$ and $\left[\frac{\alpha_n^2}{1+\alpha_n^2} - \frac{\beta_n^2}{1+\beta_n^2} \right]$ will be 0, showing that $P_\alpha - P_\beta$ has finite rank.

Conversely, suppose that $P_\alpha - P_\beta$ has finite rank. It follows that the operator $T = T_0 + S_0T_1$, given by

$$Tv = \sum_{n=0}^{\infty} \left[\frac{1}{1+\alpha_n^2} - \frac{1}{1+\beta_n^2} \right] a_{2n} e_{2n} + \sum_{n=0}^{\infty} \left[\frac{\alpha_n}{1+\alpha_n^2} - \frac{\beta_n}{1+\beta_n^2} \right] a_{2n+1} e_{2n}$$

for $v = \sum_{n=0}^{\infty} a_n e_n$, is of finite rank. Using vectors of the form $\sum_{n=0}^{\infty} a_{2n} e_{2n}$ and $\sum_{n=0}^{\infty} a_{2n+1} e_{2n+1}$ it is easy to see that in order for T to be finite rank, all but finitely many of the terms $\left[\frac{1}{1+\alpha_n^2} - \frac{1}{1+\beta_n^2} \right]$, and $\left[\frac{\alpha_n}{1+\alpha_n^2} - \frac{\beta_n}{1+\beta_n^2} \right]$ are 0. Since $\alpha_n \geq 0$ and $\beta_n \geq 0$, $\frac{1}{1+\alpha_n^2} - \frac{1}{1+\beta_n^2} = 0$ if and only if $\alpha_n = \beta_n$. Thus, $\alpha E_1 \beta$, and so $E_1 \sqsubseteq_B \equiv_f$ on $\mathcal{P}(H)$. The result follows by Theorem 2.2.7. \square

2.5.2 Essential equivalence

The last equivalence relation we wish to study is the restriction of \equiv_{ess} to $\mathcal{P}(H)$. The quotient of $\mathcal{P}(H)$ by this relation can be identified with the set of projections in Calkin algebra $\mathcal{B}(H)/\mathcal{K}(H)$, by Proposition 3.1 in [88].

We note that, although a projection is compact if and only if it is of finite rank, this is not true of the difference of two projections. In particular, \equiv_{ess} does not coincide with \equiv_f on $\mathcal{P}(H)$. However, as before, the diagonal embedding witnesses the nonsmoothness of \equiv_{ess} .

To prove Theorem 2.1.3(b), we will again use the map $\alpha \mapsto P_\alpha$ used to prove Theorem 2.1.3(a).

Proof of Theorem 2.1.3(b). We claim that the map $\alpha \mapsto P_\alpha$ is a reduction of $[0, 1]^\omega / c_0$ to \equiv_{ess} , from which the result will follow by Lemma 2.2.10. Let $\alpha, \beta \in [0, 1]^\omega$ and suppose that $\alpha - \beta \in c_0$. We will use the representation of $P_\alpha - P_\beta$ in equation (2.1). By Proposition 2.2.1, and the inequalities

$$\begin{aligned} \left| \frac{1}{1 + \alpha_n^2} - \frac{1}{1 + \beta_n^2} \right| &= \left| \frac{\beta_n^2 - \alpha_n^2}{(1 + \alpha_n^2)(1 + \beta_n^2)} \right| \leq |\beta_n - \alpha_n| |\beta_n + \alpha_n|, \\ \left| \frac{\alpha_n}{1 + \alpha_n^2} - \frac{\beta_n}{1 + \beta_n^2} \right| &= \left| \frac{\alpha_n + \alpha_n \beta_n^2 - \beta_n - \alpha_n^2 \beta_n}{(1 + \alpha_n^2)(1 + \beta_n^2)} \right| \\ &\leq |\alpha_n - \beta_n| + |\alpha_n| |\beta_n - \alpha_n| |\beta_n|, \\ \left| \frac{\alpha_n^2}{1 + \alpha_n^2} - \frac{\beta_n^2}{1 + \beta_n^2} \right| &= \left| \frac{\alpha_n^2 - \beta_n^2}{(1 + \alpha_n^2)(1 + \beta_n^2)} \right| \leq |\beta_n - \alpha_n| |\beta_n + \alpha_n|, \end{aligned}$$

we have that T_0, T_1, T_2 and T_3 are compact. Since the compact operators form an ideal, $S_0 T_1$ and $S_1 T_2$ are also compact, and thus so is $P_\alpha - P_\beta$.

Conversely, suppose that $P_\alpha - P_\beta$ is compact. We will use that if an operator is compact, then it is weak-norm continuous on the closed unit ball of H (3.3.3 in [72]). Since the sequence e_m converges *weakly* to 0 as $m \rightarrow \infty$, it follows that $(P_\alpha - P_\beta)e_{2m}$ and $(P_\alpha - P_\beta)e_{2m+1}$ converge *in norm* to 0. Observe that

$$\begin{aligned} (P_\alpha - P_\beta)e_{2m} &= \left[\frac{1}{1 + \alpha_m^2} - \frac{1}{1 + \beta_m^2} \right] e_{2m} + \left[\frac{\alpha_m}{1 + \alpha_m^2} - \frac{\beta_m}{1 + \beta_m^2} \right] e_{2m+1}, \\ (P_\alpha - P_\beta)e_{2m+1} &= \left[\frac{\alpha_m}{1 + \alpha_m^2} - \frac{\beta_m}{1 + \beta_m^2} \right] e_{2m} + \left[\frac{\alpha_m^2}{1 + \alpha_m^2} - \frac{\beta_m^2}{1 + \beta_m^2} \right] e_{2m+1}. \end{aligned}$$

Thus,

$$\begin{aligned} \|(P_\alpha - P_\beta)e_{2m}\|^2 &= \left| \frac{1}{1 + \alpha_m^2} - \frac{1}{1 + \beta_m^2} \right|^2 + \left| \frac{\alpha_m}{1 + \alpha_m^2} - \frac{\beta_m}{1 + \beta_m^2} \right|^2, \\ \|(P_\alpha - P_\beta)e_{2m+1}\|^2 &= \left| \frac{\alpha_m}{1 + \alpha_m^2} - \frac{\beta_m}{1 + \beta_m^2} \right|^2 + \left| \frac{\alpha_m^2}{1 + \alpha_m^2} - \frac{\beta_m^2}{1 + \beta_m^2} \right|^2 \end{aligned}$$

and both converge to 0 as $m \rightarrow \infty$. Using the inequalities

$$\begin{aligned} \left| \frac{1}{1 + \alpha_m^2} - \frac{1}{1 + \beta_m^2} \right| &= \left| \frac{\beta_m^2 - \alpha_m^2}{(1 + \alpha_m^2)(1 + \beta_m^2)} \right| \geq \frac{1}{4} |\alpha_m - \beta_m| |\alpha_m + \beta_m|, \\ \left| \frac{\alpha_m}{1 + \alpha_m^2} - \frac{\beta_m}{1 + \beta_m^2} \right| &= \left| \frac{\alpha_m + \alpha_m \beta_m^2 - \beta_m - \alpha_m^2 \beta_m}{(1 + \alpha_m^2)(1 + \beta_m^2)} \right| \\ &\geq \frac{1}{4} |\alpha_m - \beta_m| |1 - \alpha_m \beta_m|, \end{aligned}$$

the quantities on the right must also converge to 0. For any m , since $\alpha_m, \beta_m \in [0, 1]$, we have that $\alpha_m + \beta_m \geq \sqrt{2\alpha_m \beta_m} \geq \alpha_m \beta_m$ and so

$$|\alpha_m + \beta_m| + |1 - \alpha_m \beta_m| = \alpha_m + \beta_m + 1 - \alpha_m \beta_m \geq 1.$$

Thus,

$$|\alpha_m - \beta_m| |\alpha_m + \beta_m| + |\alpha_m - \beta_m| |1 - \alpha_m \beta_m| \geq |\alpha_m - \beta_m|,$$

and so $\alpha_m - \beta_m$ converges to 0, as claimed. \square

2.6 Further questions

We have seen in the proof of Theorem 2.1.1 that the equivalence relations $[0, 1]^\omega / c_0$ and $[0, 1]^\omega / \ell^p$ are Borel reducible to \equiv_{ess} and \equiv_p for $1 \leq p < \infty$, respectively. We may think of \equiv_{ess} and \equiv_p as noncommutative analogues of \mathbb{R}^ω / c_0 and $\mathbb{R}^\omega / \ell^p$, and ask whether they are of the same complexity:

Question. *Are the equivalence relations \equiv_{ess} and \equiv_p on $\mathcal{B}(H)$ (or $\mathcal{P}(H)$) Borel reducible to \mathbb{R}^ω / c_0 and $\mathbb{R}^\omega / \ell^p$ for $1 \leq p < \infty$, respectively?*

The Weyl–von Neumann theorem and the work of Ando–Matsuzawa [4] show that unitary equivalence modulo compact on bounded self-adjoint operators is smooth. The refinement of this given by unitary equivalence modulo Schatten p -class has also been studied; see [20]. We ask:

Question. *What is the Borel complexity of unitary equivalence modulo Schatten p -class? Is it smooth? Is it classifiable by countable structures?*

2.7 Supplementary material: Turbulence

In what follows, we give a more detailed discussion of turbulence and an alternate proof of the nonclassifiability by countable structures of $[0, 1]^\omega/c_0$ and $[0, 1]^\omega/\ell^p$, the crucial part of Lemma 2.2.10 used in the proofs of Theorems 2.1.1 (parts (b) and (c)) and 2.1.3. This work predates the appearance of [37], which yields even more direct proofs of these facts.

Let X be a Polish G -space. For $U \subseteq X$ open and $V \subseteq G$ a symmetric open neighborhood of the identity, the (U, V) -local orbit $\mathcal{O}(x, U, V)$ of a point $x \in U$ is the collection of all points $y \in U$ such that there are $g_0, \dots, g_k \in V$ and $x = x_0, \dots, x_{k+1} = y \in U$ with $x_{i+1} = g_i \cdot x_i$ for $i \leq k$. For such an X and G , we say that the action of G is *turbulent* if every orbit is dense, every orbit is meager, and every (U, V) -local orbit is *somewhere dense*, i.e., for every such U, V and x , $\overline{\mathcal{O}(x, U, V)}$ has nonempty interior.

The following examples encompass Lemma 2.2.9.

Example 2.7.1. We say that a subgroup G of the additive group \mathbb{R}^ω is *strongly dense* if for every finite sequence (x_0, \dots, x_n) of real numbers, there is a $y = (y_0, y_1, \dots) \in G$ such that $y_i = x_i$ for $i \leq n$. If G is a proper, Polishable, and strongly dense subgroup of \mathbb{R}^ω , then the translation action of G on \mathbb{R}^ω is turbulent (Proposition 3.25 in [40]). The subgroups c_0 and ℓ^p for $1 \leq p < \infty$ are easily seen to be strongly dense.

Example 2.7.2. Let X be a separable Frechet space. If Y is a proper, Polishable, dense subspace of X , then the translation action of Y on X is turbulent (see p. 35 in [51]). Examples of such pairs (X, Y) include (\mathbb{R}^ω, c_0) and $(\mathbb{R}^\omega, \ell^p)$, as well as $(L^p([0, 1]), C([0, 1]))$ and (c_0, ℓ^p) for $1 \leq p < \infty$.

By considering equivalence relations on \mathbb{T}^ω induced by actions of c_0 and ℓ^p , where \mathbb{T} is the unit circle $\{z \in \mathbb{C} : |z| = 1\}$, we will establish more directly the nonclassifiability of $[0, 1]^\omega/c_0$ and $[0, 1]^\omega/\ell^p$ and give new equivalents up to Borel bireducibility.

As in Example 2.7.1, a subgroup G of \mathbb{T}^ω is *strongly dense* if for all finite sequences (z_0, \dots, z_n) of unit complex numbers, there is a $g = (g_0, g_1, \dots) \in G$ such that $g_i = z_i$ for $i \leq n$. The proof of the following is modeled on the corresponding result for strongly dense subgroups of \mathbb{R}^ω .

Lemma 2.7.3. *If G is a proper, Polishable, and strongly dense subgroup of \mathbb{T}^ω , then the translation action of G on \mathbb{T}^ω is turbulent.*

Proof. Let G be as described. Clearly every orbit is dense. That G , and hence every orbit, is meager follows from G being proper and Borel by Pettis' Theorem (Theorem 2.3.2 in [32]).

It remains to verify that every local orbit is somewhere dense. Let $U \subseteq \mathbb{T}^\omega$ be open and $x \in U$. We may assume that the first m factors of U are arcs about x_j , for $j < m$, and the remaining factors are all of \mathbb{T} . Let $V \subseteq G$ be an open neighborhood of $1 = (1, 1, 1, \dots)$. Take $y \in U$ arbitrary and let $U_0 \subseteq U$ be an open neighborhood of y whose first $M \geq m$ factors are neighborhoods of the corresponding coordinates of y , the rest being all of \mathbb{T} . We claim that $U_0 \cap \mathcal{O}(x, U, V) \neq \emptyset$, showing that $\overline{\mathcal{O}(x, U, V)} = U$.

Consider the projection $\pi_M : G \rightarrow \mathbb{T}^M : g \mapsto g \upharpoonright M$. Since G is Polishable, strongly dense, and π_M is a Baire measurable (in fact, Borel) homomorphism, Pettis' Theorem implies that π_M is both continuous and open, where G has its Polish topology and \mathbb{T}^M the product topology. Let $W = \pi_M(V)$.

For $j < M$, pick $\xi_j \in (-\pi, \pi]$ such that $e^{i\xi_j}x_j = y_j$, and moreover, the arc $[0, 1] \rightarrow \mathbb{T} : t \mapsto e^{it\xi_j}x_j$ is entirely contained in the j th factor of U . This can be done by our assumptions on U . Pick an integer $N \geq 1$ large enough so that $w = (e^{i\xi_0/N}, \dots, e^{i\xi_M/N}) \in W$ and let $g \in V$ be such that $\pi_M(g) = w$. Then, each of x, gx, g^2x, \dots, g^Nx is in U and $g^Nx \in U_0 \cap \mathcal{O}(x, U, V)$. \square

For G a subgroup of \mathbb{R}^ω , consider the map

$$\varphi_G : G \rightarrow \mathbb{T}^\omega : (\alpha_n)_n \mapsto (e^{i\alpha_n})_n.$$

Lemma 2.7.4. *For G a Polishable subgroup of \mathbb{R}^ω and φ_G as above:*

- (a) φ_G is a continuous group homomorphism.
- (b) $\text{ran}(\varphi_G)$ is a Polishable subgroup of \mathbb{T}^ω .
- (c) If G is strongly dense in \mathbb{R}^ω , then $\text{ran}(\varphi_G)$ is strongly dense in \mathbb{T}^ω .

Proof. (a) φ_G is clearly a homomorphism and Borel on G . By compatibility of the topology, and Pettis' Theorem (Theorem 2.3.2 in [32]), it is continuous.

(b) Let $K_G = \ker(\varphi_G)$, a closed subgroup of G . Let τ be the Polish group topology on $\text{ran}(\varphi_G)$ making the induced map $G/K_G \rightarrow \text{ran}(\varphi_G)$ a topological group isomorphism. Let $\varphi = \varphi_{\mathbb{R}^\omega} : \mathbb{R}^\omega \rightarrow \mathbb{T}^\omega$ and $K = \ker(\varphi)$. Then, $\iota : g + K_G \mapsto g + K$ is a well-defined, injective, continuous group homomorphism making the fol-

lowing diagram commute:

$$\begin{array}{ccc}
 G & \xrightarrow{\subseteq} & \mathbb{R}^\omega \\
 \downarrow & & \downarrow \\
 G/K_G & \xrightarrow{\iota} & \mathbb{R}^\omega/K \\
 \cong \downarrow & & \downarrow \cong \\
 \text{ran}(\varphi_G) & \xrightarrow{\subseteq} & \mathbb{T}^\omega
 \end{array}$$

If $B \subseteq G/K_G$ is Borel, then $\iota(B)$ is Borel in \mathbb{R}^ω/K being a continuous injective image of a Borel set. Passing through the topological isomorphisms $G/K_G \cong \text{ran}(\varphi_G)$ and $\mathbb{R}^\omega/K \cong \mathbb{T}^\omega$, we have that every τ -Borel subset of $\text{ran}(\varphi_G)$ is Borel in \mathbb{T}^ω , verifying compatibility of the Borel structure.

(c) This claim is obvious. □

The subgroups of \mathbb{T}^ω in which we are interested are those arising as $\text{ran}(\varphi_G)$, where G is one of c_0 or ℓ^p , for $1 \leq p < \infty$. Denote by

$$G_0 = \text{ran}(\varphi_{c_0}) \quad \text{and} \quad G_p = \text{ran}(\varphi_{\ell^p}).$$

Observe that $G_p \subseteq G_0$ and G_0 is proper in \mathbb{T}^ω . The actions of these subgroups by translation are orbit equivalent to actions of c_0 and ℓ^p given by

$$(\alpha_n)_n \cdot (e^{i\theta_n})_n = e^{i(\theta_n + \alpha_n)},$$

for $(\alpha_n)_n$ in c_0 and ℓ^p , respectively.

Proposition 2.7.5. *G_0 and G_p for $1 \leq p < \infty$, are strongly dense, Polishable subgroups of \mathbb{T}^ω which act turbulently on \mathbb{T}^ω by translation.*

Proof. This is immediate from Lemma 2.7.4 and Lemma 2.7.3. □

Our goal for the remainder of this section is to show:

Proposition 2.7.6. (a) $[0, 1]^\omega/c_0 \equiv_B \mathbb{T}^\omega/G_0$.

(b) For $1 \leq p < \infty$, $[0, 1]^\omega/\ell^p \equiv_B \mathbb{T}^\omega/G_p$.

In particular, $[0, 1]^\omega/c_0$ and $[0, 1]^\omega/\ell^p$ are not classifiable by countable structures.

Proof. Fix $1 \leq p < \infty$ throughout. To see that $[0, 1]^\omega/c_0 \sqsubseteq_B \mathbb{T}^\omega/G_0$ and $[0, 1]^\omega/\ell^p \sqsubseteq_B \mathbb{T}^\omega/G_p$, observe that both embeddings are witnessed by the map $f : [0, 1]^\omega \rightarrow \mathbb{T}^\omega$ given by

$$f((\alpha_n)_n) = (e^{i\pi/2(\alpha_n)})_n.$$

For G a subgroup of \mathbb{R}^ω , let $([0, 1]^\omega)^2/G \times G$ denote the equivalence relation E on $([0, 1]^\omega)^2$ given by

$$xEy \iff x - y \in G \times G,$$

for $x, y \in ([0, 1]^\omega)^2$.

We claim that $([0, 1]^\omega)^2/c_0 \times c_0 \sqsubseteq_B [0, 1]^\omega/c_0$ and $([0, 1]^\omega)^2/\ell^p \times \ell^p \sqsubseteq_B [0, 1]^\omega/\ell^p$. Again, both embeddings are witnessed by the same map $g : ([0, 1]^\omega)^2 \rightarrow [0, 1]^\omega$ given by

$$g((\alpha_n^0)_n, (\alpha_n^1)_n) = (\alpha_0^0, \alpha_0^1, \alpha_1^0, \alpha_1^1, \dots).$$

Lastly, we claim that \mathbb{T}^ω/G_0 and \mathbb{T}^ω/G_p are continuously embeddable into $([-1, 1]^\omega)^2/c_0 \times c_0$ and $([-1, 1]^\omega)^2/\ell^p \times \ell^p$, respectively, which suffices since the latter are clearly continuously biembeddable with $([0, 1]^\omega)^2/c_0 \times c_0$ and $([0, 1]^\omega)^2/\ell^p \times \ell^p$. These are witnessed by the map $h : \mathbb{T}^\omega \rightarrow ([-1, 1]^\omega)^2$ given by

$$h((z_n)_n) = ((\operatorname{Re} z_n)_n, (\operatorname{Im} z_n)_n).$$

Composing these reductions together yields the result. \square

CHAPTER 3

A LOCAL RAMSEY THEORY FOR BLOCK SEQUENCES

3.1 Introduction

Ramsey-theoretic techniques have a long history of use in Banach space theory¹. Most relevant for the present work is Gowers' dichotomy for infinite block sequences in Banach spaces:

Theorem (Gowers [33] [34]). *Let B be an infinite-dimensional Banach space with a Schauder basis. If \mathbb{A} is an analytic set of normalized block sequences, then for any $\Delta > 0$, there is a block sequence Y such that either:*

- (i) *every normalized block subsequence of Y is in \mathbb{A}^c , or*
- (ii) *It has a strategy in the Gowers game $G^*[Y]$ for playing into \mathbb{A}_Δ .*

Loosely speaking (the rigorous definitions will be given later), Gowers' result says that for \mathbb{A} as above, there is a block sequence Y such that either all of Y 's normalized block subsequences are disjoint from \mathbb{A} , or there is a wealth of normalized block subsequences of Y which are within a small perturbation of \mathbb{A} . This result was used, together with work of Komorowski and Tomczak-Jaegerman [55], to solve (affirmatively) the homogeneous space problem².

In the setting of a discrete countably infinite-dimensional vector space E over a countable field, Rosendal isolated an "exact" version of Gowers' dichotomy which yields a simplified proof of the above result:

¹For examples see, e.g., [5].

²Banach's homogeneous space problem [8] asks whether ℓ^2 is the only infinite-dimensional Banach space isomorphic to all of its closed infinite-dimensional subspaces, up to isomorphism.

Theorem (Rosendal [76]). *If \mathbb{A} is an analytic set of block sequences in E , then there is a block sequence Y such that either:*

- (i) *I has a strategy in the infinite asymptotic game $F[Y]$ for playing into \mathbb{A}^c , or*
- (ii) *II has a strategy in Gowers game $G[Y]$ for playing into \mathbb{A} .*

These dichotomies are analogues, in the Banach space and vector space settings, respectively, of the following result for partitions of $[\omega]^\omega$, the set of infinite subsets of the natural numbers.

Theorem (Silver [81]). *If $\mathbb{A} \subseteq [\omega]^\omega$ is analytic, then there is an infinite set y with either all of its further infinite subsets disjoint from, or contained in, \mathbb{A} .*

While the theory of topological Ramsey spaces, in the sense of [84], encompasses many variations on this result, the dichotomies of Gowers and Rosendal highlighted above do not fall into this framework.

An important generalization of Silver's theorem is the following "local" Ramsey theorem, showing that the witness y in the conclusion can always be found in a given selective coideal.³

Theorem (Mathias [64]). *Let $\mathcal{H} \subseteq [\omega]^\omega$ be a selective coideal. If $\mathbb{A} \subseteq [\omega]^\omega$ is analytic, then there is an infinite set $y \in \mathcal{H}$ with either all of its further infinite subsets disjoint from, or contained in, \mathbb{A} .*

By passing to the generic extension resulting from the Lévy collapse of a Mahlo cardinal, Mathias strengthened⁴ this result to all partitions \mathbb{A} which are

³A *selective coideal* is a collection of subsets of ω which is the complement of a nontrivial (i.e., contains all finite sets) ideal of sets and is closed under taking certain diagonalizations. See [64] or [84] for definitions.

⁴In ZFC alone, Mathias' and Silver's theorem are sharp in the sense that assuming $\mathbf{V} = \mathbf{L}$, one can easily construct Σ_2^1 counterexamples.

“reasonably definable”, that is, in the constructible closure of the reals $L(\mathbb{R})$. Later work of Farah and Todorćevic [25] generalized this to semiselective coideals and showed that under stronger large cardinal hypotheses the passage to a forcing extension is not necessary. The corresponding extension of Silver’s theorem to all partitions in $L(\mathbb{R})$ is due to Shelah and Woodin [80]. Similar results have been recently developed for topological Ramsey spaces [21] [65].

The upshot of obtaining these local results is two-fold: we clearly isolate the combinatorial properties which enable the original dichotomies, and we obtain greater control over the witnesses to said dichotomies.

This latter point was used by Todorćevic [25] to characterize, under large cardinal hypotheses, selective ultrafilters as being exactly those which are generic for $([\omega]^\omega, \subseteq^*)$ over $L(\mathbb{R})$. Such ultrafilters are said to possess “complete combinatorics”, following Blass and Laflamme [58] who used this phrase to describe ultrafilters which are generic over $L(\mathbb{R})$ after collapsing a Mahlo cardinal. We instead ask for genericity over $L(\mathbb{R})$ of the ground model, at the expense of stronger large cardinal hypotheses.

Using [76] as a starting point, we develop local versions of Gowers’ and Rosendal’s dichotomies. When E is a countably infinite-dimensional vector space over some countable field, we isolate (p^+) -families of block sequences in §3.2, and in §3.3 establish our local form of Rosendal’s dichotomy:

Theorem 3.1.1. *Let \mathcal{H} be a (p^+) -family of block sequences in E . If \mathbb{A} is an analytic set of block sequences and $X \in \mathcal{H}$, then there is a $Y \in \mathcal{H} \upharpoonright X$ such that either:*

- (i) *I has a strategy in $F[Y]$ for playing into \mathbb{A}^c , or*
- (ii) *II has a strategy in $G[Y]$ for playing into \mathbb{A} .*

Stronger properties of families are discussed in §3.4, notably *strategic* families. The existence of filters with these properties is considered in §3.5 and §3.6, where it is proved to be independent of ZFC.

In §3.7 we show that, under large cardinal hypothesis, strategic (p^+) -filters have complete combinatorics for infinite block sequences with the block subsequence ordering and generalize Theorem 3.1.1 to partitions in $\mathbf{L}(\mathbb{R})$ (the corresponding extension of Gowers' original result is due to López-Abad [61], see also [7]). This requires an analysis of a Mathias-like notion of forcing used to build generic block sequences.

Theorem 3.1.2. *Assume that there is a supercompact cardinal. A filter \mathcal{F} of block sequences in E is generic over $\mathbf{L}(\mathbb{R})$ for the partial ordering of block sequences if and only if it is a strategic (p^+) -filter.*

Theorem 3.1.3. *Assume that there is a supercompact cardinal. Let \mathcal{H} be a strategic (p^+) -family of block sequences in E . If \mathbb{A} is a set of block sequences in $\mathbf{L}(\mathbb{R})$ and $X \in \mathcal{H}$, then there is a $Y \in \mathcal{H} \upharpoonright X$ such that either:*

- (i) *I has a strategy in $F[Y]$ for playing into \mathbb{A}^c , or*
- (ii) *II has a strategy in $G[Y]$ for playing into \mathbb{A} .*

In §3.8 we consider normed vector spaces and Banach spaces. For an infinite-dimensional separable Banach space B with a Schauder basis, we isolate the notion of *spread (p^*) -families* and establish the following local form of Gowers' dichotomy and its extension to $\mathbf{L}(\mathbb{R})$:

Theorem 3.1.4. *Let \mathcal{H} be a spread (p^*) -family of normalized block sequences in B which is invariant under small perturbations. If \mathbb{A} is an analytic set of normalized block sequences and $X \in \mathcal{H}$, then for any $\Delta > 0$, there is a $Y \in \mathcal{H} \upharpoonright X$ such that either:*

- (i) every normalized block subsequence of Y is in \mathbb{A}^c , or
- (ii) I has a strategy in $G^*[Y]$ for playing into \mathbb{A}_Δ .

Theorem 3.1.5. *Assume that there is a supercompact cardinal. Let \mathcal{H} be a strategic (p^*) -family of normalized block sequences in B which is invariant under small perturbations. If \mathbb{A} is a set of normalized block sequences in $\mathbf{L}(\mathbb{R})$ and $X \in \mathcal{H}$, then for any $\Delta > 0$, there is a $Y \in \mathcal{H} \upharpoonright X$ such that either:*

- (i) every normalized block subsequence of Y is in \mathbb{A}^c , or
- (ii) I has a strategy in $G^*[Y]$ for playing into \mathbb{A}_Δ .

It is our hope that Theorem 3.1.4 will afford new applications of the techniques introduced by Gowers in [34] to obtain block sequences in Banach spaces with simultaneous properties, some captured by the target set \mathbb{A} , while others are captured by the family \mathcal{H} .

In §3.9 we apply these results to the study of the projections in the Calkin algebra, the quotient of the bounded operators $\mathcal{B}(H)$ on a Hilbert space H by the compact operators. The natural ordering on projections in the Calkin algebra induces an ordering \leq_{ess} on $\mathcal{P}_\infty(H)$, the infinite-rank projections in $\mathcal{B}(H)$. We give a version of Theorem 3.1.2 for filters in this ordering:

Theorem 3.1.6. *Assume that there is a supercompact cardinal. A filter \mathcal{G} in $(\mathcal{P}_\infty(H), \leq_{\text{ess}})$ is $\mathbf{L}(\mathbb{R})$ -generic if and only if projections onto block subspaces are \leq_{ess} -dense in \mathcal{G} and the associated family of normalized block sequences in H is a strategic (p^*) -family.*

Generic filters for $(\mathcal{P}_\infty(H), \leq_{\text{ess}})$ induce *pure states* on $\mathcal{B}(H)$, via the theory *quantum filters* introduced by Farah and Weaver [27]. By work implicit in [27]

(we present a proof in the supplementary §3.12), these generic pure states are not pure on any atomic maximal abelian self-adjoint subalgebra, and are thus counterexamples to a conjecture of Anderson [3]. We show that any family satisfying the hypotheses of Theorem 3.1.4 and generating a pure state on $\mathcal{B}(H)$ produces such a counterexample. We caution that our counterexamples remain beyond ZFC.

Theorem 3.1.7. *A spread (p^*) -family \mathcal{H} of normalized block sequences in H which is \leq_{ess} -centered induces a singular pure state ρ on $\mathcal{B}(H)$ which is not pure on any atomic maximal abelian self-adjoint subalgebra.*

§3.10 contains questions for future investigation. §3.11 and §3.12 contain supplementary material on restricted Gowers games and generic pure states, respectively.

An effort has been made to keep the set-theoretic prerequisites for understanding this chapter to a minimum with the hope that the material, particularly in §3.3 and §3.8, may be used for further applications in Banach space and operator theory. We assume a familiarity with the basic properties of Polish spaces, Borel sets, and analytic sets (as covered in [50]) throughout. We only make explicit use of the method of forcing and large cardinal hypotheses in §3.5 and §3.7, with occasional reference back to that material in §3.8 and §3.9. The Banach space prerequisites amount to little more than a familiarity with basic sequences (as covered in the first sections of [2]).

3.2 Families of block sequences

Fix a countable field F , a countably infinite-dimensional F -vector space E , and an F -basis (e_n) for E . We will typically think of F as a subfield of \mathbb{R} or \mathbb{C} , however we do not need any such restrictions and may even allow F to be finite. Given $v \in E$, say with $v = \sum_{n=0}^N a_n e_n$, let $\text{supp}(v) = \{n \in \omega : a_n \neq 0\}$, the *support* of v . We write $n < v$ if $n < \min(\text{supp}(v))$ and $v < w$ if $\max(\text{supp}(v)) < \min(\text{supp}(w))$.

We say that a (finite or infinite) sequence (x_n) of nonzero vectors in E is a *block sequence* (with respect to (e_n)) if for all n , $x_n < x_{n+1}$. If $\vec{x} = (x_0, \dots, x_n)$ is a finite block sequence, let $\text{supp}(\vec{x}) = \bigcup_{i=0}^n \text{supp}(x_i)$, and for X any block sequence, let $\langle X \rangle = \text{span}(X) \setminus \{0\}$, a *block subspace*. We will abuse notation and write E for $E \setminus \{0\}$ and use “vector” to mean nonzero vector.

Let $\text{bb}^\infty(E)$ be the collection of all infinite block sequences in E , which we consider as a subspace of the product E^ω , where E has the discrete topology. It is easy to check that $\text{bb}^\infty(E)$ is a G_δ subset of E^ω and thus a Polish space. Let $\text{bb}^{<\infty}(E)$ be the collection of all finite block sequences in E .

For $X = (x_n)$ and $Y = (y_n)$ in $\text{bb}^\infty(E)$, we write $X \preceq Y$ if (x_n) is a block sequence with respect to (y_n) , called a *block subsequence* of Y , or equivalently (for block sequences), $\langle X \rangle \subseteq \langle Y \rangle$. We write $X \preceq^* Y$ if for some m , $X/m \preceq Y$, where X/m is the tail of X with supports above m . For $\vec{x} \in \text{bb}^{<\infty}(E)$, write X/\vec{x} for $X/\max(\text{supp}(\vec{x}))$. Note that the orderings \preceq and \preceq^* fail to be antisymmetric, but are reflexive and transitive.

We will make repeated use of the following order-theoretic notions: A subset

D of a preorder (P, \leq) (i.e., \leq is reflexive and transitive) is *dense* if for all $p \in P$, there is a $q \in D$ with $q \leq p$. It is, moreover, *dense open*, if whenever $q \leq p \in D$, then $q \in D$. Elements p and q in P are *compatible* if they have a common lower bound in P and *incompatible* otherwise.

Compatibility in $(\text{bb}^\infty(E), \preceq)$ is equivalent to that in $(\text{bb}^\infty(E), \preceq^*)$ and we write $X \perp Y$ when X and Y are incompatible. The following observation shows that $(\text{bb}^\infty(E), \preceq)$ can be identified with a dense suborder of the lattice of all infinite-dimensional subspaces of E . In particular, X and Y are compatible if and only if $\langle X \rangle \cap \langle Y \rangle$ is infinite-dimensional.

Lemma 3.2.1. *If X is an infinite-dimensional subspace of E , then X contains an infinite block sequence.*

Proof. By taking appropriate linear combinations, one can show that for any N , X contains an infinite-dimensional subspace whose supports are above N . From this, it is easy to inductively construct a block sequence in X . \square

Throughout, when we speak of a *family* $\mathcal{H} \subseteq \text{bb}^\infty(E)$, we mean a nonempty subset which is closed upwards with respect to \preceq^* . For $X \in \mathcal{H}$, we denote by $\mathcal{H} \upharpoonright X = \{Y \in \mathcal{H} : Y \preceq X\}$. A *filter* $\mathcal{F} \subseteq \text{bb}^\infty(E)$ is a family such that for every $X, Y \in \mathcal{F}$, there is a $Z \in \mathcal{F}$ with $Z \preceq X$ and $Z \preceq Y$.

Definition 3.2.2. (a) Given a descending sequence $X_0 \succeq X_1 \succeq \dots$ in $\text{bb}^\infty(E)$, we call $Y \in \text{bb}^\infty(E)$ a *diagonalization* of (X_n) if for all n , $Y \preceq^* X_n$.

(b) Given a sequence (\mathcal{D}_n) of subsets of $\text{bb}^\infty(E)$, we call Y a *diagonalization* of (\mathcal{D}_n) if for each n , there is an $X_n \in \mathcal{D}_n$ such that $Y \preceq^* X_n$.

For $\mathcal{H} \subseteq \text{bb}^\infty(E)$, a set \mathcal{D} is \preceq -*dense (open)* in \mathcal{H} if $\mathcal{D} \cap \mathcal{H}$ is.

Definition 3.2.3. A family $\mathcal{H} \subseteq \text{bb}^\infty(E)$ is a (p) -family, or has the (p) -property, if whenever $X_0 \succeq X_1 \succeq \dots$ is a decreasing sequence with each $X_n \in \mathcal{H}$, there is a diagonalization $Y \in \mathcal{H}$ of (X_n) .

It is easy to see that $\text{bb}^\infty(E)$ itself is a (p) -family. We note that every (p) -family \mathcal{H} contains a diagonalization of any given sequence (\mathcal{D}_n) of \preceq -dense open subsets in \mathcal{H} : build a decreasing sequence (X_n) in \mathcal{H} with each $X_n \in \mathcal{D}_n$, then any diagonalization $Y \in \mathcal{H}$ of (X_n) will be a diagonalization of (\mathcal{D}_n) . This can be done below any given $X \in \mathcal{H}$, so the set of such diagonalizations is \preceq -dense in \mathcal{H} . This latter property, which could be called the *weak (p) -property*, is sufficient for all of the results in §3.3, and in particular, for Theorem 3.1.1.

Recall that $\mathcal{H} \subseteq [\omega]^\omega$ is a *coideal* if it contains all cofinite sets, is closed upwards with respect to \subseteq , and whenever $Y_0 \cup Y_1 \in \mathcal{H}$, then one of Y_0 or Y_1 is also in \mathcal{H} . This last property asserts that \mathcal{H} witnesses the pigeonhole principle. In our setting, provided $|F| > 2$,⁵ the “obvious” formulation of the pigeonhole principle is simply false, as the following example shows:

Example 3.2.4.⁶ Consider the case when $F \subseteq \mathbb{R}$. Similar examples can be constructed whenever $|F| > 2$, cf. Theorem 7 in [59]. For a vector $x \in E$ define the *oscillation* $\text{osc}(x)$ as the number of times the sign of the nonzero coefficients of x alternate in its expansion with respect to (e_n) . So, $\text{osc}(e_0 - e_1 + e_2) = 2$, $\text{osc}(e_2 + e_4 - e_5 + e_7 - e_{10}) = 3$, etc.

Define $A_0 \subseteq E$ (respectively, $A_1 \subseteq E$) to be the set of all $x \in E$ such that $\text{osc}(x)$ is even (respectively, odd), and let $\mathbb{A}_i = \{(x_n) : x_0 \in A_i\}$ for $i = 0, 1$.

⁵When F is a finite field of order 2, such a pigeonhole principle for block subspaces does hold; this is essentially Hindman’s Theorem [39].

⁶The author would like to thank Jordi López-Abad for pointing out this example which has the advantage of being well-defined at the level of the spanned subspaces.

The \mathbb{A}_i are clopen sets which partition $\text{bb}^\infty(E)$. Moreover, the pair $\mathbb{A}_0, \mathbb{A}_1$ is *asymptotic*, that is, for any $X \in \text{bb}^\infty(E)$ and $i = 0, 1$, there is $Y_i \preceq X$ such that $Y_i \in \mathbb{A}_i$. To see this, suppose that $X = (x_n)$ is such that $X \in \mathbb{A}_0$, so $\text{osc}(x_0)$ is even. If $\text{osc}(x_1)$ is odd, then $(x_n)_{n \geq 1} \preceq X$ and in \mathbb{A}_1 . If $\text{osc}(x_1)$ is even, then let $x = x_0 - x_1$ if the signs of the last nonzero coefficient in x_0 and the first in x_1 agree, and $x = x_0 + x_1$ otherwise. In either case, $\text{osc}(x) = \text{osc}(x_0) + \text{osc}(x_1) + 1$, so (x, x_2, x_3, \dots) is in \mathbb{A}_1 .

The following is a weak analogue of the pigeonhole property of coideals.

Definition 3.2.5. Let $\mathcal{H} \subseteq \text{bb}^\infty(E)$ be a family.

- (a) A subset $\mathcal{D} \subseteq \text{bb}^\infty(E)$ is \mathcal{H} -dense below some $X \in \mathcal{H}$ if for every $Y \in \mathcal{H} \upharpoonright X$, there is a $Z \preceq Y$ with $Z \in \mathcal{D}$. A set $D \subseteq E$ is \mathcal{H} -dense below X if $\{Z : \langle Z \rangle \subseteq D\}$ is.
- (b) \mathcal{H} is *full* if whenever $D \subseteq E$ (not necessarily a subspace) and $X \in \mathcal{H}$ are such that D is \mathcal{H} -dense below X , there is a $Z \in \mathcal{H} \upharpoonright X$ with $\langle Z \rangle \subseteq D$.

Fullness allows one to upgrade $\{Z : \langle Z \rangle \subseteq D\}$ being \mathcal{H} -dense below X to being \preceq -dense (open) below X in \mathcal{H} . Obviously $\text{bb}^\infty(E)$ itself is a full family. If the family in question is a filter \mathcal{F} , we may simplify the definition of fullness by replacing X with (e_n) (or any condition in \mathcal{F}). We note that any full filter is maximal; this can be seen by applying the definition of fullness when D is a block subspace.

A family in $\text{bb}^\infty(E)$ which is full and has the (p) -property will be called a (p^+) -family. Likewise for (p^+) -filter.

Lemma 3.2.6. (a) For $X_0 \succeq X_1 \succeq \dots$ in $\text{bb}^\infty(E)$, the set

$$\mathcal{D}_{(X_n)} = \{Y : Y \text{ is a diagonalization of } (X_n) \text{ or } \exists n(Y \perp X_n)\}$$

is \preceq -dense open.

(b) For $D \subseteq E$ and $X \in \text{bb}^\infty(E)$, the set

$$\mathcal{D}_{D,X} = \{Z : \langle Z \rangle \subseteq D \text{ or } \forall V \preceq X (\langle V \rangle \subseteq D \rightarrow V \perp Z)\}$$

is \preceq -dense open below X .

Proof. (a) Take $Y \in \text{bb}^\infty(E)$ which is compatible with all of the X_n . We can build a diagonalization $X = (x_n) \preceq Y$ by picking vectors $x_n \in \langle X_n \rangle \cap \langle Y \rangle$ with $x_n < x_{n+1}$.

(b) Take $Y \preceq X$. If there is no $Z \preceq Y$ such that $\langle Z \rangle \subseteq D$, then for any $V \preceq X$ with $\langle V \rangle \subseteq D$, it must be that $V \perp Y$, as otherwise any Z witnessing the compatibility of V and Y would satisfy $\langle Z \rangle \subseteq D$. \square

Lemma 3.2.6 will be used to construct (p^+) -filters §5. We will see in Corollary 3.6.6 that the existence of full filters is independent of ZFC.

3.3 Games with vectors and a local Rosendal dichotomy

The *Gowers game* [33] [34] played below $X \in \text{bb}^\infty(E)$, denoted $G[X]$, is defined as follows: Two players, I and II, alternate with I going first and playing block sequences $X_k \preceq X$, and II responding with vectors $y_k \in \langle X_k \rangle$ subject to the constraint $y_k < y_{k+1}$.

I	X_0	X_1	X_2	\dots
II	y_0	y_1	y_2	\dots

The block sequence (y_k) is the *outcome* of a play of the game. Given $\vec{x} \in \text{bb}^{<\infty}(E)$ and $X \in \text{bb}^\infty(E)$, the game $G[\vec{x}, X]$ is defined exactly as $G[X]$ except that II is restricted to playing vectors above \vec{x} and the outcome is $\vec{x}^\wedge(y_k)$.

A *strategy* for II in $G[\vec{x}, X]$ is a function α taking sequences (X_0, \dots, X_k) of possible prior moves by I to vectors $y \in \langle X_k \rangle$, with $\vec{x} < \alpha(X_0, \dots, X_{k-1}) < y$, for all k . Given a set $\mathbb{A} \subseteq \text{bb}^\infty(E)$, we say that α is a strategy in $G[\vec{x}, X]$ for *playing into* \mathbb{A} if whenever II follows α (that is, at each turn, given as input I's prior moves, they play the output of α), the resulting outcome lies in \mathbb{A} . These notions are defined likewise for I.

The *infinite asymptotic game* [75] [76] played below X , denoted $F[X]$, is defined in a similar fashion: Two players, I and II, alternate with I going first and playing natural numbers n_k , and II responding with vectors $y_k \in \langle X/n_k \rangle$ subject to the constraint $y_k < y_{k+1}$.

I	n_0	n_1	n_2	\dots
II	y_0	y_1	y_2	\dots

Again, (y_k) is the *outcome* of a play of the game. The game $F[\vec{x}, X]$ is defined as above, as are *strategies* for I and II, and the notion of having a strategy for *playing into* a set.

It is important to note that plays of $F[\vec{x}, X]$ can be considered as plays of $G[\vec{x}, X]$ where I is restricted to playing tail block subsequences of X . Consequently, if II has a strategy in $G[\vec{x}, X]$ for playing into a set \mathbb{A} , then II has such a strategy in $F[\vec{x}, X]$ as well. Similarly, if I has a strategy in $F[\vec{x}, X]$ for playing into \mathbb{A} , then they have such a strategy in $G[\vec{x}, X]$.

The following generalizes the notion of *strategically Ramsey* given in [76], where \mathcal{H} was taken to be all of $\text{bb}^\infty(E)$.

Definition 3.3.1. For $\mathcal{H} \subseteq \text{bb}^\infty(E)$ a family, we say that a subset $\mathbb{A} \subseteq \text{bb}^\infty(E)$ is *\mathcal{H} -strategically Ramsey* if for all $\vec{y} \in \text{bb}^{<\infty}(E)$ and $X \in \mathcal{H}$, there is a $Y \in \mathcal{H} \upharpoonright X$ such that either:

- (i) I has a strategy in $F[\vec{y}, Y]$ for playing into \mathbb{A}^c , or
- (ii) II has a strategy in $G[\vec{y}, Y]$ for playing into \mathbb{A} .

Note that consequences (i) and (ii) are mutually exclusive by our comments above. The key fact about \mathcal{H} -strategically Ramsey sets is that the witness, Y in the above definition, can be found in \mathcal{H} .

Our goal for the remainder of this section is to outline the proof that, for any (p^+) -family \mathcal{H} , analytic sets are \mathcal{H} -strategically Ramsey, thereby establishing Theorem 3.1.1. Much of what follows closely hews to [76].

Definition 3.3.2. Let \mathcal{H} be a family and $\mathbb{A} \subseteq \text{bb}^\infty(E)$ be given. For $\vec{y} \in \text{bb}^{<\infty}(E)$ and $Y \in \mathcal{H}$, we say that

- (1) (\vec{y}, Y) is *good* (for \mathbb{A}) if II has a strategy in $G[\vec{y}, Y]$ for playing into \mathbb{A} ,
- (2) (\vec{y}, Y) is *bad* (for \mathbb{A}) if for all $Z \in \mathcal{H} \upharpoonright Y$, (\vec{y}, Z) is not good.
- (3) (\vec{y}, Y) is *worse* (for \mathbb{A}) if it is bad and there is an n such that for every $v \in \langle Y/n \rangle$, $(\vec{y} \frown v, Y)$ is bad.

Reference to \mathbb{A} and \mathcal{H} will be suppressed where understood.

Lemma 3.3.3. *If \mathcal{H} is a (p^+) -family and $\mathbb{A} \subseteq \text{bb}^\infty(E)$, then for every $\vec{x} \in \text{bb}^{<\infty}(E)$ and $X \in \mathcal{H}$, there is a $Y \in \mathcal{H} \upharpoonright X$ such that either:*

- (i) (\vec{x}, Y) is good, or
- (ii) I has a strategy in $F[\vec{x}, Y]$ for playing into

$$\{(z_n) : \forall n(\vec{x} \hat{\ } (z_0, \dots, z_n), Y) \text{ is worse}\}.$$

Proof. Observe that if (\vec{y}, Y) is good/bad/worse and $Z \preceq^* Y$ is in \mathcal{H} , then (\vec{y}, Z) is also good/bad/worse. It is immediate that for each \vec{y} , the set

$$\mathcal{D}_{\vec{y}} = \{Y \in \mathcal{H} : (\vec{y}, Y) \text{ is either good or bad}\}$$

is \preceq -dense open in \mathcal{H} .

Claim. If (\vec{y}, Y) is bad, then for all $Z \in \mathcal{H} \upharpoonright Y$, there is a $V \preceq Z$ such that for all $x \in \langle V/\vec{y} \rangle$, $(\vec{y} \hat{\ } x, Y)$ is not good.

Proof of claim. Let (\vec{y}, Y) be bad. Towards a contradiction, suppose that there is some $Z \in \mathcal{H} \upharpoonright Y$ such that for all $V \preceq Z$, there is an $x \in \langle V/\vec{y} \rangle$ such that $(\vec{y} \hat{\ } x, Y)$ is good. We claim that (\vec{y}, Z) is good. If I plays $V \preceq Z$, then by supposition there is some $x \in \langle V/\vec{y} \rangle$ such that $(\vec{y} \hat{\ } x, Z)$ is good. Let II play that x and from then on follow the strategy given from $(\vec{y} \hat{\ } x, Z)$ being good. This is contrary to (\vec{y}, Y) being bad. □_(claim.)

Claim. For each \vec{y} , the set

$$\mathcal{E}_{\vec{y}} = \{Z \in \mathcal{H} : (\vec{y}, Z) \text{ is either good or worse}\}$$

is \preceq -dense open in \mathcal{H} .

Proof of claim. Fix \vec{y} and let $Y \in \mathcal{H}$. Since the sets $\mathcal{D}_{\vec{x}}$ are dense in \mathcal{H} and there are only countably many \vec{x} , the (p) -property allows us to diagonalize all of them within \mathcal{H} and assume that for all \vec{x} , (\vec{x}, Y) is either good or bad. Suppose that

(\vec{y}, Y) is bad. Let $D = \{x : (\vec{y} \hat{\ } x, Y) \text{ is not good}\}$. By the previous claim, D is \mathcal{H} -dense below Y . Since \mathcal{H} is full, there is a $Z \in \mathcal{H} \upharpoonright Y$ such that $\langle Z \rangle \subseteq D$. If $z \in \langle Z \rangle$, then $(\vec{y} \hat{\ } z, Z)$ is not good, hence bad, by our choice of Y . Thus, (\vec{y}, Z) is worse. □_(claim.)

We can now prove the lemma. By the previous claim, we have a $Y \in \mathcal{H} \upharpoonright X$ so that for all \vec{y} , $(\vec{x} \hat{\ } \vec{y}, Y)$ is either good or worse. If (\vec{x}, Y) is good, we're done, so suppose that (\vec{x}, Y) is worse. We will describe a strategy for I in $F[\vec{x}, Y]$: Suppose that at some point in the game (z_0, \dots, z_k) has been played by II so that $(\vec{x} \hat{\ } (z_0, \dots, z_k), Y)$ is worse. Then, there is some n such that for all $z \in \langle Y \rangle$, if $n < z$, then $(\vec{x} \hat{\ } (z_0, \dots, z_k) \hat{\ } z, Y)$ is bad, hence worse. Let I play n . □

Lemma 3.3.4 (cf. Lemma 2 in [76]). *Let $\mathcal{H} \subseteq \text{bb}^\infty(E)$ a (p^+) -family. Then, open sets are \mathcal{H} -strategically Ramsey.*

Proof. Let $\mathbb{A} \subseteq \text{bb}^\infty(E)$ be open. Given $\vec{x} \in \text{bb}^{<\infty}(E)$ and $X \in \mathcal{H}$, by Lemma 3.3.3, there is a $Y \in \mathcal{H} \upharpoonright X$ such that either (\vec{x}, Y) is good, in which case we're done, or I has a strategy in $F[\vec{x}, Y]$ to play (z_n) such that for all n , $(\vec{x} \hat{\ } (z_0, \dots, z_n), Y)$ is worse. In the latter case, if I follows this strategy, as II builds (z_n) , for no m can II have a strategy in $G[\vec{x} \hat{\ } (z_0, \dots, z_m), Y]$ to play in \mathbb{A} . Since \mathbb{A} is open, this means that $\vec{x} \hat{\ } (z_0, z_1, \dots) \notin \mathbb{A}$ and I has a strategy for playing into \mathbb{A}^c . □

Lemma 3.3.5 (cf. Lemma 4 in [76]). *Let $\mathcal{H} \subseteq \text{bb}^\infty(E)$ be a (p^+) -family. Suppose that $\mathbb{A}_n \subseteq \text{bb}^\infty(E)$ for $n \in \omega$ and $\mathbb{A} = \bigcup_{n \in \omega} \mathbb{A}_n$. Let \vec{x} and $X \in \mathcal{H}$ be given. Then, there is a $Y \in \mathcal{H} \upharpoonright X$ such that either:*

(i) *I has a strategy in $F[\vec{x}, Y]$ for playing into \mathbb{A}^c , or*

(ii) *II has a strategy in $G[Y]$ for playing (z_k) for which there is some n such that for every $V \in \mathcal{H} \upharpoonright Y$, I has no strategy in $F[\vec{x}^\wedge(z_0, \dots, z_n), V]$ for playing into \mathbb{A}_n^c .*

Proof. For $Y \in \mathcal{H}$, $\vec{y} \in \text{bb}^{<\infty}(E)$ and $n \in \omega$, we say (\vec{y}, n) *accepts* Y if I has a strategy in $F[\vec{y}, Y]$ for playing into \mathbb{A}_n^c and (\vec{y}, n) *rejects* Y if for all $Z \in \mathcal{H} \upharpoonright Y$, (\vec{y}, n) does not accept Z . Both acceptance and rejection are \preceq^* -hereditary and the sets

$$\mathcal{D}_{\vec{y}, n} = \{Y : (\vec{y}, n) \text{ accepts or rejects } Y\}$$

are clearly \preceq -dense open in \mathcal{H} . By the (p) -property, we can find $Y \in \mathcal{H} \upharpoonright X$ such that for all \vec{y} and n , (\vec{y}, n) either accepts or rejects Y . Put

$$R = \{(z_k) : \exists n(\vec{x}^\wedge(z_0, \dots, z_n), n) \text{ rejects } Y\},$$

and notice that R is open in $\text{bb}^\infty(E)$. By Theorem 3.3.4, we may assume that Y is such that either I has a strategy for $F[Y]$ to play into R^c , or II has a strategy in $G[Y]$ for playing into R . The latter implies (ii) directly, so assume the former, that is, I has a strategy in $F[Y]$ to play (z_k) such that for all n , I has a strategy $\sigma_{(z_0, \dots, z_n)}$ in $F[\vec{x}^\wedge(z_0, \dots, z_n), Y]$ to play into \mathbb{A}_n^c . We describe a strategy for I in $F[\vec{x}, Y]$ for playing into \mathbb{A}^c : If at stage n , (z_0, \dots, z_{n-1}) has been played thus far, then I responds with

$$\max\{\sigma(z_0, \dots, z_{n-1}), \sigma_{(z_0)}(z_1, \dots, z_{n-1}), \dots, \sigma_{(z_0, \dots, z_{n-1})}(\emptyset)\}$$

If (z_n) is an outcome of this strategy, then for all n , we have ensured that $\vec{x}^\wedge(z_0, \dots, z_{n-1})^\wedge(z_n, z_{n+1}, \dots) \in \mathbb{A}_n^c$. Thus, $\vec{x}^\wedge(z_n) \in \bigcap_n \mathbb{A}_n^c = \mathbb{A}^c$. \square

Proof of Theorem 3.1.1. This proof will show that analytic sets are \mathcal{H} -strategically Ramsey, though for simplicity, we consider when $\vec{x} = \emptyset$. Let $X \in \mathcal{H}$ be given.

Let $F : \omega^\omega \rightarrow \mathbb{A}$ be a continuous surjection and for each $s \in \omega^{<\omega}$, let $\mathbb{A}_s = F''(\mathcal{N}_s)$ where $\mathcal{N}_s = \{\alpha \in \omega^\omega : s \subseteq \alpha\}$. Note that $\mathbb{A}_s = \bigcup_n \mathbb{A}_{s \frown n}$.

Let $R(s, \vec{x}, Y)$ (for $Y \in \mathcal{H}$) be the set of all (z_k) for which there is some n such that for every $V \in \mathcal{H} \upharpoonright Y$, I has no strategy in $F[\vec{x} \frown (z_0, \dots, z_n), V]$ for playing into $\mathbb{A}_{s \frown n}^c$. By Lemma 3.3.5 and the (p) -property, there is an $Y \in \mathcal{H} \upharpoonright X$ such that for all \vec{x} and $s \in \omega^{<\omega}$, either:

- (i) I has a strategy in $F[\vec{x}, Y]$ for playing into \mathbb{A}_s^c , or
- (ii) II has a strategy in $G[Y]$ for playing into $R(s, \vec{x}, X)$.

Suppose that I has no strategy in $F[Y]$ to play into $\mathbb{A}^c = \mathbb{A}_\emptyset^c$. We will describe a strategy for II in $G[Y]$ for playing into \mathbb{A} : As II has a strategy in $G[Y]$ for playing into $R(\emptyset, \emptyset, Y)$, they follows this strategy until (z_0, \dots, z_{n_0}) has been played such that I does not have strategy in $F[(z_0, \dots, z_{n_0}), Y]$ for playing into $\mathbb{A}_{s \frown n_0}$. By the assumption on Y , II must have a strategy in $G[Y]$ for playing into $R((n_0), (z_0, \dots, z_{n_0}), Y)$. II follows this strategy until a further $(z_{n_0+1}, \dots, z_{n_0+n_1+1})$ has been played so that I does not have a strategy in $F[(z_0, \dots, z_{n_0}, \dots, z_{n_0+n_1+1}), Y]$ for playing into $\mathbb{A}_{s \frown n_0 \frown n_1}^c$.

Continuing in this fashion, and letting $m_k = (\sum_{j \leq k} n_j) + k$, the outcome of the game will be a sequence $Z = (z_0, z_1, \dots, z_{m_0}, \dots, z_{m_1}, \dots)$ such that for all k , I does not have a strategy in $F[(z_0, \dots, z_{m_k}), Y]$ for playing into $\mathbb{A}_{(n_0, \dots, n_k)}^c$. In particular, for all k , there is some $Z^k \sqsupseteq (z_0, \dots, z_{m_k})$ in $\mathbb{A}_{(n_0, \dots, n_k)} = F''(\mathcal{N}_{(n_0, \dots, n_k)})$. Take $\beta_k \in \mathcal{N}_{(n_0, \dots, n_k)}$ such that $F(\beta_k) = Z^k$. Then, $\beta_k \rightarrow (n_0, n_1, \dots) \in \omega^\omega$ and $Z^k \rightarrow Z$, so by the continuity of F , we have that $Z = F(n_0, n_1, \dots) \in \mathbb{A}$. \square

Theorem 3.1.1 is consistently sharp and necessarily asymmetric, as there is a coanalytic counterexample (for $\mathcal{H} = \text{bb}^\infty(E)$) in L [61].⁷ In particular, the collection of \mathcal{H} -strategically Ramsey sets may fail to be a σ -algebra. It is, however, closed under countable unions. Again, the proof is nearly identical to that of the corresponding result in [76].

Theorem 3.3.6 (cf. Theorem 9 in [76]). *Let $\mathcal{H} \subseteq \text{bb}^\infty(E)$ be a (p^+) -family. Then, the collection of \mathcal{H} -strategically Ramsey sets is closed under countable unions.*

Proof. Suppose that \mathbb{A}_n is \mathcal{H} -strategically Ramsey for each n and let $\mathbb{A} = \bigcup_n \mathbb{A}_n$. Let \vec{x} and $X \in \mathcal{H}$ be given. For each \vec{y} and n , the set of all $Y \in \mathcal{H} \upharpoonright X$ such that either I has a strategy in $F[\vec{y}, Y]$ for playing into \mathbb{A}_n^c , or II has a strategy in $G[\vec{x}, Y]$ for playing into \mathbb{A}_n , is \preceq -dense open below X in \mathcal{H} . By the (p) -property, find $Y \in \mathcal{H} \upharpoonright X$ have this property for all \vec{y} and n . By Lemma 3.3.5, there is a $Z \in \mathcal{H} \upharpoonright Y$ such that either

- (i) I has a strategy in $F[\vec{x}, Y]$ for playing into \mathbb{A}^c , or
- (ii) II has a strategy in $G[Y]$ for playing (z_k) for which there is some n such that for every $V \in \mathcal{H} \upharpoonright Y$, I has no strategy in $F[\vec{x}^\frown(z_0, \dots, z_n), V]$ for playing into \mathbb{A}_n^c .

In the first case, we are done. By our choice of Y , (ii) implies that II has a strategy in $G[Z]$ for playing (z_k) such that for some n , II has a strategy in $G[\vec{x}^\frown(z_0, \dots, z_n), Z]$ for playing into \mathbb{A}_n . Pasting these two strategies together yields a strategy for II in $G[\vec{x}, Z]$ for playing into \mathbb{A} . \square

We note that fullness is a necessary assumption for our results:

⁷This counterexample is to Gowers' theorem, but the discussion in §5 of [76] shows that this also yields a counterexample to Rosendal's dichotomy.

Proposition 3.3.7. *If $\mathcal{H} \subseteq \text{bb}^\infty(E)$ is a family for which clopen sets are \mathcal{H} -strategically Ramsey, then \mathcal{H} is full.*

Proof. Given $D \subseteq E$, \mathcal{H} -dense below some $X \in \mathcal{H}$, let $\mathbb{D} = \{(z_n) : z_0 \in D\}$, a clopen subset of $\text{bb}^\infty(E)$. For no $Y \in \mathcal{H} \upharpoonright X$ can II have a strategy into \mathbb{D}^c : Consider the round of $G[Y]$ where I starts by playing some $Z \preceq Y$ with $\langle Z \rangle \subseteq D$. Since \mathbb{D}^c is \mathcal{H} -strategically Ramsey, there is a $Y \in \mathcal{H} \upharpoonright X$ such that I has a strategy σ in $F[Y]$ for playing into \mathbb{D} . Let $Z = Y/\sigma(\emptyset) \in \mathcal{H}$. Since σ is a strategy for playing into \mathbb{D} , $\langle Z \rangle \subseteq D$. \square

3.4 Stronger properties of families

If an element Y in a family \mathcal{H} witnesses Theorem 3.1.1, then either \mathbb{A}^c or \mathbb{A} is \mathcal{H} -dense below Y , depending on which half of the dichotomy holds. However, it would be desirable to ensure that \mathcal{H} itself meets whichever one of \mathbb{A}^c or \mathbb{A} the conclusion of the dichotomy provides. To this end, we consider stronger properties of families, the first of which is based on the original definition of selectivity (or being “happy”) in [64].

Definition 3.4.1. (a) For $(X_{\vec{x}})_{\vec{x} \in \text{bb}^{<\infty}(E)}$ generating a filter in $\text{bb}^\infty(E)$, we say that

$X \in \text{bb}^\infty(E)$ *strongly diagonalizes* $(X_{\vec{x}})$ if $X/\vec{x} \preceq X_{\vec{x}}$ whenever $\vec{x} \sqsubseteq X$.

(b) A family $\mathcal{H} \subseteq \text{bb}^\infty(E)$ is a *strong (p) -family*, or has the *strong (p) -property*, if whenever $(X_{\vec{x}})_{\vec{x} \in \text{bb}^{<\infty}(E)}$ generates a filter in \mathcal{H} , there is an $Y \in \mathcal{H}$ which strongly diagonalizes $(X_{\vec{x}})$.

The strong (p) -property implies the (p) -property: Take $X_0 \succeq X_1 \succeq \dots$ in \mathcal{H} and define $X_{\vec{x}} = X_{|\vec{x}|}$ for $\vec{x} \in \text{bb}^{<\infty}(E)$. Any X strongly diagonalizing $(X_{\vec{x}})$ will

diagonalize (X_n) .

As in Lemma 3.2.6, it will be useful for constructing families with the strong (p) -property to know that it corresponds to certain \preceq -dense sets.

Lemma 3.4.2. *For $(X_{\vec{x}})_{\vec{x} \in \text{bb}^{<\infty}(E)}$ generating a filter in $\text{bb}^\infty(E)$, the set*

$$\{Y : Y \text{ is a strong diagonalization of } (X_{\vec{x}})_{\vec{x} \in \text{bb}^{<\infty}(E)}, \text{ or} \\ \{Y\} \cup (X_{\vec{x}})_{\vec{x} \in \text{bb}^{<\infty}(E)} \text{ does not generate a filter}\}$$

is \preceq -dense.

Proof. Fix $X \in \text{bb}^\infty(E)$ and suppose that $\{X\} \cup (X_{\vec{x}})$ generates a filter. We build a $Y \preceq X$ which strongly diagonalizes $(X_{\vec{x}})$: Pick any $y_0 \in \langle X \rangle \cap \langle X_\emptyset \rangle$. Since X , X_\emptyset and $X_{(y_0)}$ generate a filter, there is a $y_1 \in \langle X \rangle \cap \langle X_\emptyset \rangle \cap \langle X_{(y_0)} \rangle$ with $y_0 < y_1$. Continue in this fashion. \square

The following result connects the strong (p) -property to the infinite asymptotic game and is based on a characterization of selective ultrafilters (Theorem 4.5.3 in [9]). In the case that \mathcal{F} is a strong (p) -filter, one can strengthen this result by replacing $F[X]$ with the *restricted Gowers game* $G_{\mathcal{F}}[X]$, wherein I must play elements of \mathcal{F} ; for this and other material on the restricted Gowers games, see the supplementary §3.11.

Theorem 3.4.3. *If $\mathcal{H} \subseteq \text{bb}^\infty(E)$ is a strong (p) -family, then for no $X \in \mathcal{H}$ does I have a strategy in $F[X]$ for playing into \mathcal{H}^c .*

Proof. Let σ be a strategy for I in $F[X]$ for playing into \mathcal{H}^c , where $X \in \mathcal{H}$. Towards a contradiction, suppose that \mathcal{H} has the strong (p) -property. Define sets $\mathcal{A}_{\vec{x}} \subseteq \mathcal{H}$ as follows: $\mathcal{A}_\emptyset = \{X/\sigma(\emptyset)\}$ and inductively, for $\vec{x} = (x_0, \dots, x_{n-1})$, $\mathcal{A}_{\vec{x}}$

is the set of all X/m where m is played by I following σ in the first n rounds of $F[X]$ as II plays x_0, x_1, \dots, x_{n-1} . In the case that elements of a given \vec{x} fail to be valid moves for II against σ , let $\mathcal{A}_{\vec{x}} = \mathcal{A}_{\vec{x}'}$ where \vec{x}' is the maximal initial segment of \vec{x} consisting of valid moves. Then, for all \vec{x} , $\mathcal{A}_{\vec{x}}$ is finite and $\mathcal{A}_{\vec{x}} \subseteq \mathcal{A}_{\vec{y}}$ whenever $\vec{x} \sqsubseteq \vec{y}$.

For each \vec{x} , let $M_{\vec{x}} = \max\{m : X/m \in \mathcal{A}_{\vec{x}}\}$ and $Y_{\vec{x}} = X/M_{\vec{x}}$. Clearly $(Y_{\vec{x}})$ generates a filter in \mathcal{H} . By the strong (p) -property, there is a $Y = (y_n) \in \mathcal{H} \upharpoonright X$ such that $Y/\vec{y} \preceq Y_{\vec{y}}$ for all $\vec{y} \sqsubseteq Y$.

Consider the play of $F[X]$ wherein I follows σ and II plays y_0, y_1 , and so on. We claim that this is a valid sequence of moves for II. Note that $y_0 \in \langle Y/\emptyset \rangle \subseteq \langle Y_\emptyset \rangle \subseteq \langle X/\sigma(\emptyset) \rangle$, so y_0 is a valid move. Inductively, suppose that (y_0, \dots, y_k) is a valid sequence of moves. We have $y_{k+1} \in \langle Y/(y_0, \dots, y_k) \rangle \subseteq \langle Y_{(y_0, \dots, y_k)} \rangle \subseteq \langle X/\sigma(y_0, \dots, y_k) \rangle$, where the last containment uses our induction hypothesis. Thus, y_{k+1} is a valid move. Since the resulting outcome of this play is in \mathcal{H} , we have a contradiction. \square

Equivalently, Theorem 3.4.3 says that if \mathcal{H} is a strong (p) -family and σ is a strategy for I in $F[X]$, where $X \in \mathcal{H}$, then there is an outcome of σ in \mathcal{H} .

Lemma 3.4.4. *If $\mathbb{D} \subseteq \text{bb}^\infty(E)$ is \preceq -dense open below $X \in \text{bb}^\infty(E)$, then*

- (a) *II has a strategy in $F[X]$ for playing into \mathbb{D} , and*
- (b) *I has a strategy in $G[X]$ for playing into \mathbb{D} .*

Proof. For $F[X]$, take $Y \preceq X$ in \mathbb{D} and let II always play vectors in Y . For $G[X]$, take $Y \preceq X$ in \mathbb{D} and let I simply play Y repeatedly. \square

It follows from Lemma 3.4.4, and Theorems 3.1.1 and 3.4.3, that whenever $\mathcal{H} \subseteq \text{bb}^\infty(E)$ is a strong (p^+) -family and \mathbb{D} is a coanalytic \preceq -dense open set, then $\mathcal{H} \cap \mathbb{D} \neq \emptyset$. In particular, strong (p^+) -families meet all \preceq -dense open Borel sets. This is a special case of Theorem 3.1.2. The following definition is a counterpart to Theorem 3.4.3 for Π in $G[X]$.

Definition 3.4.5. A family $\mathcal{H} \subseteq \text{bb}^\infty(E)$ is *strategic* if whenever $X \in \mathcal{H}$ and α is a strategy for Π in $G[X]$, there is an outcome of α which is in \mathcal{H} .

As above, if $\mathcal{H} \subseteq \text{bb}^\infty(E)$ is a strategic (p^+) -family and $\mathbb{D} \subseteq \text{bb}^\infty(E)$ is an analytic \preceq -dense open set, then $\mathbb{D} \cap \mathcal{H} \neq \emptyset$. As a consequence for (p^+) -filters, being strategic subsumes the strong (p) -property.

Lemma 3.4.6. *If $\mathcal{F} \subseteq \text{bb}^\infty(E)$ is a strategic (p^+) -filter, then \mathcal{F} is also a strong (p) -filter.*

Proof. Suppose that \mathcal{F} is as described and $(X_{\vec{x}})_{\vec{x} \in \text{bb}^{<\infty}(E)}$ is contained \mathcal{F} . Let \mathcal{D} be the set given in Lemma 3.4.2, so that the \preceq -downwards closure of \mathcal{D} is a \preceq -dense open set. Moreover, \mathcal{D} is easily seen to be Borel and its \preceq -downwards closure analytic. By the comments above, it follows that $\mathcal{F} \cap \mathcal{D} \neq \emptyset$ and any $Y \in \mathcal{F} \cap \mathcal{D}$ must be a strong diagonalization of $(X_{\vec{x}})$. \square

In §3.5 we will construct (under set-theoretic hypotheses) strategic (p^+) -filters. To this end, we again need to know that certain sets are \preceq -dense, but also that there are not “too many” of them. If α is a strategy for Π in $G[X]$, then the set of outcomes which result from α , denoted by $[\alpha, X]$, is \preceq -dense below X . However, as strategies are functions from finite sequences in $\text{bb}^\infty(E)$ to vectors, there are $2^{2^{\aleph_0}}$ many of them.

One way to resolve this is to “finitize” the Gowers game as in [6]: given $X \in \text{bb}^\infty(E)$, the *finite-dimensional Gowers game below X* , denoted by $G_f[X]$, is defined as follows: Two players, I and II, alternate with I going first and playing a nonzero vector $x_0^{(0)} \in \langle X \rangle$. II responds with either a nonzero $y_0 \in \langle x_0^{(0)} \rangle$ or 0. If II plays y_0 , then the game “restarts” with I playing a nonzero vector $x_0^{(1)} \in \langle X \rangle$. If II plays 0, then I must play a nonzero vector $x_1^{(0)} \in \langle X/x_0^{(0)} \rangle$, to which II again responds with either a nonzero vector $y_0 \in \langle x_0^{(0)}, x_1^{(0)} \rangle$ or 0, and so on. The nonzero plays of II are required to satisfy $y_n < y_{n+1}$ and the outcome is the sequence (y_n) . The notion of strategy for II in $G_f[X]$ is defined in the obvious way (with the requirement that the outcome must be infinite) and we denote by $[\alpha, X]_f$ the corresponding set of outcomes.

Lemma 3.4.7. *If α is a strategy for II in $G[X]$, then there is a strategy α' for II in $G_f[X]$ such that $[\alpha', X]_f \subseteq [\alpha, X]$. Moreover, $[\alpha', X]_f$ is still \preceq -dense below X .*

Proof. The proof is identical to the (\Leftarrow) direction of Theorem 1.2 in [6]. □

It is easy to see that strategies α for II in $G_f[X]$ are coded by reals and $[\alpha, X]_f$ is an analytic set. This will suffice for our constructions in §3.5. Lemma 3.11.8 in the supplementary material is a strengthening of this fact.

3.5 Constructions of filters in $\text{bb}^\infty(E)$

In this section we show how to construct filters $\mathcal{F} \subseteq \text{bb}^\infty(E)$ having all of the properties discussed in §3.2 and §3.4. These constructions use either assumptions about certain “cardinal invariants” which hold consistently with ZFC, or

via the method of forcing (see [15] for examples). We will see in Corollary 3.6.6 that we cannot hope for a construction in ZFC alone.

Definition 3.5.1. (a) A *tower* (of length κ) in $\text{bb}^\infty(E)$ is a sequence $(X_\alpha)_{\alpha < \kappa}$ such that $\alpha < \beta < \kappa$ implies $X_\beta \preceq^* X_\alpha$ and $X_\alpha \not\preceq^* X_\beta$, and there is no $X \in \text{bb}^\infty(E)$ with $X \preceq^* X_\alpha$ for all $\alpha < \kappa$.

(b) \mathfrak{t}^* is the minimum length of a tower in $\text{bb}^\infty(E)$.

\mathfrak{t}^* is a regular cardinal and, moreover, uncountable as $\text{bb}^\infty(E)$ has the (p) -property. Thus, the Continuum Hypothesis (CH) implies that $\mathfrak{t}^* = 2^{\aleph_0}$.

We use the following notational conventions for versions of Martin's Axiom (see Ch. II of [57]): for an uncountable cardinal $\kappa < 2^{\aleph_0}$, $\text{MA}(\kappa)$ is the forcing axiom for meeting κ -many dense subsets of a ccc poset, MA is $\forall \kappa < 2^{\aleph_0} (\text{MA}(\kappa))$, and $\text{MA}(\sigma\text{-centered})$ is MA restricted to σ -centered posets.

Lemma 3.5.2 (Lemma 5 in [29]). ($\text{MA}(\sigma\text{-centered})$) *If $\mathcal{L} \subseteq \text{bb}^\infty(E)$ is linearly ordered with respect to \preceq^* and $|\mathcal{L}| < 2^{\aleph_0}$, then there is a Y such that $Y \preceq^* X$ for all $X \in \mathcal{L}$. In particular, $\mathfrak{t}^* = 2^{\aleph_0}$.*

Consequently, the following theorem holds under CH or $\text{MA}(\sigma\text{-centered})$.

Theorem 3.5.3. ($\mathfrak{t}^* = 2^{\aleph_0}$) *There exists a strategic (p^+) -filter in $\text{bb}^\infty(E)$.*

Proof. Fix enumerations:

- (i) $\{X_\xi : \xi < 2^{\aleph_0}\} = \text{bb}^\infty(E)$,
- (ii) $\{(X_n^\xi) : \xi < 2^{\aleph_0}\}$ of all \preceq^* -decreasing sequences (X_n^ξ) in $\text{bb}^\infty(E)$,
- (iii) $\{D_\xi : \xi < 2^{\aleph_0}\}$ of all subsets D_ξ of E ,
- (iv) $\{[\alpha_\xi, X_\xi]_f : \xi < 2^{\aleph_0}\}$ of all sets $[\alpha, X]_f$ of outcomes of α , where α is a strategy for II in $G_f[X]$.

This can be done in (i) and (ii) since $|\text{bb}^\infty(E)| = 2^{\aleph_0}$, in (iii) since E is countable, and in (iv) since the strategies α are coded by reals.

Define sets, for $\xi, \gamma < 2^{\aleph_0}$, with $\langle \cdot, \cdot \rangle$ a bijection $2^{\aleph_0} \times 2^{\aleph_0} \rightarrow 2^{\aleph_0}$,

$$\mathcal{D}_\xi = \{Y : Y \text{ is a diagonalization of } (X_n^\xi) \text{ or } \exists n(Y \perp X_n^\xi)\}$$

$$\mathcal{F}_{\langle \xi, \gamma \rangle} = \{Y : \langle Y \rangle \subseteq D_\xi \text{ or } \forall V \preceq X_\gamma (\langle V \rangle \subseteq D_\xi \Rightarrow V \perp Y)\}$$

$$\mathcal{S}_\xi = \{Y : Y \in [\alpha_\xi, X_\xi]_f \text{ or } Y \perp X_\xi\}.$$

Note that the first two sets above are \preceq -dense in $\text{bb}^\infty(E)$ by Lemma 3.2.6 and the third is \preceq -dense by Lemma 3.4.7.

We construct a \preceq^* -descending chain (Y_η) of length 2^{\aleph_0} in $\text{bb}^\infty(E)$ by transfinite induction on η . For $\eta = 0$, pick Y_0 below conditions in each of \mathcal{D}_0 , \mathcal{F}_0 , and \mathcal{S}_0 . If we have already defined Y_β for all $\beta < \eta$, pick Y_η below each Y_β for $\beta < \eta$ and below conditions in each of \mathcal{D}_η , \mathcal{F}_η , and \mathcal{S}_η . This is possible since $t^* > \eta$.

Let \mathcal{F} be the filter generated by $\{Y_\eta : \eta < 2^{\aleph_0}\}$ in $\text{bb}^\infty(E)$. To see that \mathcal{F} is a (p) -filter, suppose that (X_n^ξ) is a \preceq^* -decreasing sequence in \mathcal{F} . Let $Y \in \mathcal{F} \cap \mathcal{D}_\xi$. It cannot be the case that $Y \perp X_n^\xi$ for any n , as \mathcal{F} is a filter, so Y must be a diagonalization of (X_n^ξ) . Similarly, using the sets \mathcal{S}_ξ , \mathcal{F} is strategic.

To see that \mathcal{F} is full, suppose $D_\xi \subseteq E$ and $X_\gamma \in \mathcal{F}$ are such that D_ξ is \mathcal{F} -dense below X_γ . Take $Y \in \mathcal{F} \cap \mathcal{F}_{\langle \xi, \gamma \rangle} \neq \emptyset$ below X_γ . By assumption, there is a Y' below both Y and X_γ such that $\langle Y' \rangle \subseteq D_\xi$, but obviously it cannot be that $Y' \perp Y$. Thus, it must be that $\langle Y \rangle \subseteq D_\xi$. \square

The next result allows us to obtain (p^+) -filters generically by forcing with $(\text{bb}^\infty(E), \preceq^*)$. Since the dense sets involved are all definable in a simple way

from real parameters, they are contained in $\mathbf{L}(\mathbb{R})$. In particular, this establishes (without any large cardinals) the (\Rightarrow) direction of Theorem 3.1.2.

Theorem 3.5.4. *For $\mathcal{H} \subseteq \text{bb}^\infty(E)$ a (p^+) -family, forcing with (\mathcal{H}, \preceq^*) adds no new reals and if $\mathcal{G} \subseteq \mathcal{H}$ is $\mathbf{L}(\mathbb{R})$ -generic for (\mathcal{H}, \preceq^*) , \mathcal{G} will be a (p^+) -filter. If \mathcal{H} is strategic (respectively, has the strong (p) -property), then \mathcal{G} will also be strategic (respectively, have the strong (p) -property).*

Proof. \mathcal{H} being a (p) -family implies that (\mathcal{H}, \preceq^*) is σ -closed and thus adds no new reals. We use this fact implicitly in what follows. Let \mathcal{G} be as described. To see that \mathcal{G} is full, let $D \subseteq E$ be \mathcal{G} -dense below some $X \in \mathcal{G}$. Translating this into the forcing language, there must be an $X' \in \mathcal{G}$, which we may assume is below X , with

$$X' \Vdash_{\mathcal{H}} \forall Y \in \dot{\mathcal{G}} \upharpoonright X \exists Z \preceq Y (\langle Z \rangle \subseteq \check{D}).$$

We claim that the set $\mathcal{D} = \{Z : \langle Z \rangle \subseteq D\}$ is \preceq -dense below X' in \mathcal{H} . If not, then by fullness of \mathcal{H} , D must fail to be \mathcal{H} -dense below X' . That is, there is some $Y \in \mathcal{H} \upharpoonright X'$ with no $Z \preceq Y$ such that $\langle Z \rangle \subseteq D$. Then, Y fails to force the statement in the displayed line above, contrary to $Y \preceq X'$. Since $X' \in \mathcal{G}$ and \mathcal{D} is \preceq -dense below X' in \mathcal{H} , $\mathcal{G} \cap \mathcal{D} \neq \emptyset$, showing that \mathcal{G} is full. The remainder of the proof consists of observing that the relevant \preceq -dense sets in Lemmas 3.2.6, 3.4.2, and 3.4.7 are \preceq -dense in \mathcal{H} under these hypotheses. \square

3.6 Connections to filters on a countable set

In this section, we relate the filters discussed thus far to filters of subsets of a countable set. In our case, the countable set will be $E \setminus \{0\}$, but we will call

these filters on E . Whenever we write $\langle X \rangle$ in what follows, it will be understood that $X \in \text{bb}^\infty(E)$, unless otherwise specified.

Definition 3.6.1. A filter \mathcal{F} on E is a *block filter* if it has a base consisting of sets of the form $\langle X \rangle$.

It is tempting to define a *block ultrafilter* on E to be a block filter on E which is also an ultrafilter. However, unless F is a finite field of order 2, such objects do not exist: Let \mathcal{F} be a block filter on E . For $A_0, A_1 \subseteq E$ given in Example 3.2.4, note that $E = A_0 \cup A_1$. But, for every $X \in \text{bb}^\infty(E)$, $\langle X \rangle \cap A_0 \neq \emptyset$ and $\langle X \rangle \cap A_1 \neq \emptyset$, so neither set can be in \mathcal{F} .

Since nonprincipal ultrafilters on a countable C set always fail to have the Baire property as subsets of 2^C , the following observation also shows that, when F is infinite, block filters can never be ultrafilters, nor even nonmeager filters.

Proposition 3.6.2. *If F is an infinite field, then the set $\mathbb{S} = \{A \subseteq E : A \text{ contains a nonzero subspace}\}$ is meager as a subset of 2^E .*

Proof. We construct a partition (I_n) of E into nonempty finite sets such that whenever $v \in I_n$, for some n , then for any $m > n$, there is a scalar λ such that $\lambda v \in I_m$. Enumerate both E and the scalar field F , and build I_{n+1} to contain the next new element of E and some new scalar multiple of each element of I_n . Then, if $A \subseteq E$ is any nonzero subspace, $A \cap I_n \neq \emptyset$ implies $A \cap I_m \neq \emptyset$ for all $m \geq n$.

Observe that the sets $F_n = \{A \subseteq E : \forall m \geq n (A \cap I_m \neq \emptyset)\}$ are closed and nowhere dense in 2^E , since $\bigcup_{m \geq n} \{A \subseteq E : A \cap I_m = \emptyset\}$ is dense open. Then, the set \mathbb{S} above is contained in $\bigcup_{n \in \omega} F_n$ and is thus meager. \square

Let FIN be the set of nonempty finite subsets of ω . An ultrafilter \mathcal{U} on FIN is said to be an *ordered union ultrafilter* [16] if it has a base consisting of sets of the form $\langle X \rangle = \{x_{n_0} \cup \cdots \cup x_{n_k} : n_0 < \cdots < n_k\}$, where $X = (x_n)$ is a block sequence in FIN (that is, for all n , $\max(x_n) < \min(x_{n+1})$). The set of infinite block sequences in FIN is denoted by $\text{FIN}^{[\infty]}$. We have, perhaps, overloaded the notation $\langle X \rangle$, but its intended interpretation should be clear from context. If $X = (x_n) \in \text{bb}^\infty(E)$, denote by $\tilde{X} = (\text{supp}(x_n)) \in \text{FIN}^{[\infty]}$.

If F is a finite field of order 2, then $E \setminus \{0\}$ can be identified with FIN via each vector's support. Sums of vectors in block position corresponds to unions of their supports. As a consequence of Hindman's Theorem (Corollary 3.3 in [39]), one can construct (under additional hypotheses such as CH or MA) ordered union ultrafilters on FIN , or equivalently in this case, block ultrafilters on E .

For the remainder of this section we will consider a general countable field F . The map which takes a vector to its support will provide the connection between this general setting and FIN .

Definition 3.6.3. Let \mathcal{F} be a block filter on E .

- (a) A subset $D \subseteq E$ is \mathcal{F} -dense if for every $\langle X \rangle \in \mathcal{F}$, there is a $Z \preceq X$ with $\langle Z \rangle \subseteq D$.
- (b) \mathcal{F} is *full* if whenever $D \subseteq E$ is \mathcal{F} -dense, we have that $D \in \mathcal{F}$.

As in the case for filters in $\text{bb}^\infty(E)$, every full block filter on E is maximal with respect to containment amongst block filters.

The map $s : X \mapsto \langle X \rangle$ takes block sequences to subsets of E . It is straightforward to show that the image of a (full) filter in $\text{bb}^\infty(E)$ under s generates a (full) block filter on E and that the inverse image of a (full) block filter on E is a

(full) filter in $\text{bb}^\infty(E)$. By Theorem 3.5.3 (or Theorem 3.5.4), it is consistent that such filters exist.

Theorem 3.6.4. *Suppose that \mathcal{F} is a full block filter on E and let*

$$\text{supp}(\mathcal{F}) = \{A \subseteq \text{FIN} : \exists F \in \mathcal{F} (A \supseteq \{\text{supp}(v) : v \in F\})\}.$$

Then, $\text{supp}(\mathcal{F})$ is an ordered union ultrafilter on FIN .

Proof. Let $A, B \in \text{supp}(\mathcal{F})$, say with $A \supseteq \{\text{supp}(v) : v \in F\}$ and $B \supseteq \{\text{supp}(v) : v \in G\}$, for $F, G \in \mathcal{F}$. Then,

$$\begin{aligned} A \cap B &\supseteq \{s : \exists v \in F \exists w \in G (s = \text{supp}(v) = \text{supp}(w))\} \\ &\supseteq \{\text{supp}(v) : v \in F \cap G\}, \end{aligned}$$

which is in $\text{supp}(\mathcal{F})$, as $F \cap G \in \mathcal{F}$. Since $\text{supp}(\mathcal{F})$ is upwards closed by definition, we have that $\text{supp}(\mathcal{F})$ is a filter on FIN . As \mathcal{F} is a block filter, it follows that $\text{supp}(\mathcal{F})$ has a base consisting of sets $\langle \tilde{X} \rangle$ for $X \in \text{bb}^\infty(E)$.

It remains to show that $\text{supp}(\mathcal{F})$ is an ultrafilter. Take $A \subseteq \text{FIN}$ such that for all $B \in \text{supp}(\mathcal{F})$, $A \cap B \neq \emptyset$. Let

$$D_0 = \{v \in E : \text{supp}(v) \in A\}$$

$$D_1 = \{v \in E : \text{supp}(v) \notin A\}.$$

Towards a contradiction, suppose that for all $\langle X \rangle \in \mathcal{F}$, there is a $\langle Z \rangle \subseteq \langle X \rangle$ with $\langle Z \rangle \subseteq D_1$. Since \mathcal{F} is full, there is a $\langle Z \rangle \in \mathcal{F}$ with $\langle Z \rangle \subseteq D_1$. Then $\langle \tilde{Z} \rangle \in \text{supp}(\mathcal{F})$, but $A \cap \langle \tilde{Z} \rangle = \emptyset$, a contradiction.

Thus, there is some $\langle X \rangle \in \mathcal{F}$ such that for no $\langle Z \rangle \subseteq \langle X \rangle$ is $\langle Z \rangle \subseteq D_1$. Take $\langle Y \rangle \in \mathcal{F} \upharpoonright \langle X \rangle$. By Hindman's Theorem applied to $\langle \tilde{Y} \rangle$, there is a $\tilde{Z} \in \text{FIN}^{[\infty]}$ such that $\langle \tilde{Z} \rangle \subseteq \langle \tilde{Y} \rangle$ and either (i) $\langle \tilde{Z} \rangle \subseteq A$, or (ii) $\langle \tilde{Z} \rangle \subseteq \langle \tilde{Y} \rangle \setminus A$.

Take any $Z \preceq Y$ in $\text{bb}^\infty(E)$ whose supports agree with \tilde{Z} , then if (ii) holds, $\langle Z \rangle \subseteq D_1$, contrary to what we know about $\langle X \rangle$. Thus, $\langle \tilde{Z} \rangle \subseteq A$ and $\langle Z \rangle \subseteq D_0$. Since $\langle Y \rangle \in \mathcal{F} \upharpoonright \langle X \rangle$ was arbitrary, we have that D_0 is \mathcal{F} -dense. As \mathcal{F} is full, we can find a $\langle Z \rangle \in \mathcal{F}$ with $\langle Z \rangle \subseteq D_0$. Then, $\langle \tilde{Z} \rangle \in \text{supp}(\mathcal{F})$ and $\langle \tilde{Z} \rangle \subseteq A$, so $A \in \text{supp}(\mathcal{F})$. \square

As a consequence Theorem 3.6.4 and the Corollary on p. 87 of [16] we have:

Corollary 3.6.5. *If \mathcal{F} is a full filter on E , then*

$$\min(\mathcal{F}) = \{\{n = \min(\text{supp}(v)) : v \in F\} : F \in \mathcal{F}\},$$

$$\max(\mathcal{F}) = \{\{n = \max(\text{supp}(v)) : v \in F\} : F \in \mathcal{F}\}$$

are selective ultrafilters on ω .

As it is consistent that there are no selective ultrafilters [56], we have:

Corollary 3.6.6. *The existence of full block filters on E , and thus full filters in $\text{bb}^\infty(E)$, is independent of ZFC.*

An ordered union ultrafilter \mathcal{U} on FIN is *stable* [14] if whenever $(\langle X_n \rangle)_{n \in \omega}$ is contained in \mathcal{U} , for $X_n \in \text{FIN}^{[\infty]}$, there is an $\langle X \rangle \in \mathcal{U}$ with $\langle X \rangle \subseteq^* \langle X_n \rangle$ for all n . Much as selective ultrafilters on ω provide local witnesses to Silver's theorem, stable ordered union ultrafilters on FIN witness a theorem of Milliken [67] on analytic partitions of $\text{FIN}^{[\infty]}$. It is easy to see, given Theorem 3.6.4, that (p^+) -filters in $\text{bb}^\infty(E)$ induce stable ordered union ultrafilters on FIN. See [21], [65], and [93] for (equivalent) alternate definitions of "selective ultrafilter" on FIN.

3.7 Extending to universally Baire sets and $\mathbf{L}(\mathbb{R})$

In this section, we show that under additional set-theoretic hypotheses, Theorem 3.1.1 can be extended beyond the analytic sets to obtain Theorems 3.1.2 and 3.1.3, provided the families involved are strategic. We begin by noting the following result:

Theorem 3.7.1 (Rosendal [76]). *(MA(\aleph_1)) A union of \aleph_1 -many strategically Ramsey sets is strategically Ramsey.*

The above theorem, plus existing results in the literature, yields:

Theorem 3.7.2. *Assume that there is a supercompact cardinal.⁸ Every subset of $\text{bb}^\infty(E)$ in $\mathbf{L}(\mathbb{R})$ is strategically Ramsey.⁹*

Proof. We follow the proof of Theorem 4 in [61]. The existence of a supercompact cardinal implies that $\mathbf{L}(\mathbb{R})$ is a *Solovay model* in the sense of [22] and Lemma 4.4 of the same reference shows that every set of reals in such a model is a union of \aleph_1 -many analytic sets. By Theorem 3.7.1, under MA(\aleph_1) a union of \aleph_1 -many strategically Ramsey sets is again strategically Ramsey. Since supercompactness implies [80] that $\mathbf{L}(\mathbb{R})^{\mathbf{V}[G]}$ is elementarily equivalent to $\mathbf{L}(\mathbb{R})$ for any set-forcing extension $\mathbf{V}[G]$, and one can force MA(\aleph_1) in a way which preserves \aleph_1 , the same is true in $\mathbf{L}(\mathbb{R})$. As analytic sets are strategically Ramsey by Theorem 3.1.1, every set in $\mathbf{L}(\mathbb{R})$ is as well. □

⁸Throughout this chapter, the assumption of supercompactness can be weakened to the existence of a proper class of Woodin cardinals, see [60]. We use supercompactness due to its central role in the literature and verbal brevity.

⁹Noé de Rancourt has obtained a different proof of this result using methods inspired by determinacy considerations.

Assuming $\text{MA}(\aleph_1) + 2^{\aleph_0} = \aleph_2$, a minor modification of the proof of Theorem 3.5.3 allows one to obtain a strategic (p^+) -filter \mathcal{F} which is \aleph_2 -closed: every \preceq^* -decreasing sequence of length $\leq \aleph_1$ in \mathcal{F} has a \preceq^* -lower bound in \mathcal{F} . By aping the proof of Theorem 3.7.1, one can prove that the union of \aleph_1 -many \mathcal{F} -strategically Ramsey sets is \mathcal{F} -strategically Ramsey for such an \mathcal{F} . In particular, under these hypotheses, Σ_2^1 subsets of $\text{bb}^\infty(E)$ are \mathcal{F} -strategically Ramsey.

The goal of the remainder of this section is to get all subsets of $\text{bb}^\infty(E)$ in $\mathbf{L}(\mathbb{R})$ to be \mathcal{H} -strategically Ramsey, for any strategic (p^+) -family \mathcal{H} .

Following [69], given a notion of forcing Q and a complete metric space (X, d) , we say that a Q -name \dot{x} is a *nice Q -name for an element of \dot{X}* if there is a countable collection \mathcal{D} of dense subsets of Q such that $\dot{x}(G)$ (the interpretation of \dot{x} by G) is an element of X whenever G is a \mathcal{D} -generic filter for Q . One can show that if \dot{y} is a Q -name and $p \Vdash_Q \dot{y} \in \dot{X}$, then there is a nice Q -name \dot{x} for an element of \dot{X} such that $p \Vdash_Q \dot{y} = \dot{x}$.

A subset $A \subseteq X$ is *universally Baire* if whenever Q is a notion of forcing, there is a Q -name \dot{A} such that for every nice Q -name \dot{x} for an element of \dot{X} , there is a countable collection \mathcal{D} of dense subsets of Q such that

- (1) $\{q \in Q : q \text{ decides } \dot{x} \in \dot{A}\}$ is in \mathcal{D} ,
- (2) whenever G is \mathcal{D} -generic for Q , $\dot{x}(G)$ is in X and $\dot{x}(G)$ is in A if and only if there is a $q \in G$ such that $q \Vdash \dot{x} \in \dot{A}$.

The following result will be the main tool for going beyond the analytic sets.¹⁰

Theorem 3.7.3 (Feng–Magidor–Woodin [28]). *Assume that there is a supercompact cardinal. Every set of reals in $\mathbf{L}(\mathbb{R})$ is universally Baire.*

¹⁰The same reference proves that analytic sets are universally Baire in ZFC.

Consider the following variant of the infinite asymptotic game: If $A \subseteq E$ is an infinite-dimensional subspace of E , we define $F[A]$ to be the game in which I plays natural numbers n_k , which we assume are increasing, and II plays vectors $y_k \in A$ subject to the constraint $n_k < y_k < y_{k+1}$. By Lemma 3.2.1, this is well-defined. One can define *outcome*, *strategies*, and the game $F[\vec{x}, A]$ exactly as in §3.3. Note that the game $F[\vec{x}, \langle X \rangle]$ in this sense, where $X \in \text{bb}^\infty(E)$, coincides with $F[\vec{x}, X]$ from §3.3, and we will denote it as such.

Suppose that σ is a strategy for I in $F[A]$ and τ a strategy for I in $F[B]$, where $B \subseteq A$ are infinite-dimensional subspaces. We write $\tau \geq \sigma$ if for all \vec{y} in the domain of τ , $\tau(\vec{y}) \geq \sigma(\vec{y})$ ($\sigma(\vec{y})$ is well-defined by induction). Observe that if $\tau \geq \sigma$, then whenever (y_n) is an outcome of $F[B]$ where I follows τ , then it is also an outcome of $F[A]$ where I follows σ . In particular, if σ is a strategy for playing into a set \mathbb{A} , then so is τ .

If σ is a strategy for I in $F[A]$ and $B \subseteq A$ as above, then denote by $\sigma \upharpoonright B$ the restriction of σ to the part of its domain contained in B , a strategy for I in $F[B]$. Clearly, $\sigma \upharpoonright B \geq \sigma$. Let ε be the strategy in $F[E]$ where I plays n on the n th move. Then, for all A and strategies σ for I in $F[A]$, we have that $\sigma \geq \varepsilon \upharpoonright A$.

Definition 3.7.4. Let \mathbb{P} be the set of all triples (\vec{x}, A, σ) , where $\vec{x} \in \text{bb}^{<\infty}(E)$, A is an infinite-dimensional subspace of E and σ is a strategy for I in $F[\vec{x}, A]$. We say that $(\vec{y}, B, \tau) \leq (\vec{x}, A, \sigma)$ if

- (i) $\vec{y} = \vec{x} \hat{\ } (y_0, \dots, y_{k-1})$ where y_0, \dots, y_{k-1} are the first k moves by II in a round of $F[\vec{x}, A]$ where I follows σ ,
- (ii) $B \subseteq A$,
- (iii) $\tau(\cdot) \geq \sigma((y_0, \dots, y_k) \hat{\ } \cdot)$.

The ordering \leq on \mathbb{P} is reflexive and transitive, though fails to be antisymmetric. We treat \mathbb{P} as a notion of forcing. Note that \mathbb{P} has a maximal element, namely $(\emptyset, E, \varepsilon)$. If $X \in \text{bb}^\infty(E)$, we write (\vec{x}, X, σ) for $(\vec{x}, \langle X \rangle, \sigma)$. If $\mathcal{H} \subseteq \text{bb}^\infty(E)$ is a family, let

$$\mathbb{P}(\mathcal{H}) = \{(\vec{x}, A, \sigma) \in \mathbb{P} : \exists X \in \mathcal{H} (\langle X \rangle \subseteq A)\},$$

a suborder of \mathbb{P} . Note that if $\mathcal{H} \subseteq \text{bb}^\infty(E)$ is a family, then the set of conditions (\vec{x}, X, σ) where $X \in \mathcal{H}$ is dense in $\mathbb{P}(\mathcal{H})$.

For $(\vec{x}, A, \sigma) \in \mathbb{P}$, let

$$[\vec{x}, A, \sigma] = \{Y \in \text{bb}^\infty(E) : Y \text{ is an outcome of } F[\vec{x}, A] \text{ where I follows } \sigma\}.$$

We collect some basic properties of \mathbb{P} in the following lemma:

- Lemma 3.7.5.** (a) *If $(\vec{y}, B, \tau) \leq (\vec{x}, A, \sigma)$ in \mathbb{P} , then $[\vec{y}, B, \tau] \subseteq [\vec{x}, A, \sigma]$. Conversely, if $[\vec{y}, B, \tau] \subseteq [\vec{x}, A, \sigma]$, then (\vec{y}, B, τ) is below (\vec{x}, A, σ) in the separative quotient of \mathbb{P} .*
- (b) *If $(\vec{x}, A, \sigma) \in \mathbb{P}$, then the set $[\vec{x}, A, \sigma]$ is (topologically) closed.*
- (c) *If $\mathcal{F} \subseteq \text{bb}^\infty(E)$ is a filter, then $\mathbb{P}(\mathcal{F})$ is σ -centered.*

Proof. (a) The first part follows from our observations about the ordering on strategies for I. For the converse, suppose that $[\vec{y}, B, \tau] \subseteq [\vec{x}, A, \sigma]$. Then, every outcome of $F[\vec{y}, B]$ where I follows τ is an outcome of $F[\vec{x}, A]$ where I follows σ . In particular, $\vec{y} = \vec{x} \frown (y_0, \dots, y_{k-1})$ where y_0, \dots, y_{k-1} are the first k moves by II in a round of $F[\vec{x}, A]$ where I follows σ .

We claim that $B/m \subseteq A$, where $m = \max\{\text{supp}(\vec{y}), \tau(\emptyset)\}$ and $B/m = \{y \in B : y > m\}$. To see this, note that for any $y \in B/m$, there is an outcome $\vec{y} \frown y \frown Z \in [\vec{y}, B, \tau]$ and thus in $[\vec{x}, A, \sigma]$. In particular, $y \in A$.

By our choice of m , $\tau \upharpoonright B/m = \tau$. So, $(\vec{y}, B/m, \tau) \leq (\vec{x}, A, \sigma)$ and the sets of extensions of $(\vec{y}, B/m, \tau)$ and (\vec{y}, B, τ) coincide. Thus, their images in the separative quotient of \mathbb{P} coincide.

(b) If $Y = (y_n) \notin [\vec{x}, A, \sigma]$, then either $\vec{x} \not\sqsubseteq Y$, or there is some least n such that y_n is not a valid response to $\sigma(y_0, \dots, y_{n-1})$, i.e., $y_n \notin A$ or $y_n \not\prec \sigma(y_0, \dots, y_{n-1})$. As E is discrete, these are open conditions.

(c) Suppose that (\vec{x}, A, σ) and (\vec{x}, B, τ) are both in $\mathbb{P}(\mathcal{F})$. There are $X, Y \in \mathcal{F}$ with $\langle X \rangle \subseteq A$ and $\langle Y \rangle \subseteq B$. Since \mathcal{F} is a filter, there is a $Z \in \mathcal{F}$ below both. Let ρ be the strategy for I in $F[Z]$ given by $\rho(\vec{z}) = \max\{\sigma(\vec{z}), \tau(\vec{z})\}$. Then, $(\vec{x}, Z, \rho) \in \mathbb{P}(\mathcal{F})$ and extends both (\vec{x}, A, σ) and (\vec{x}, B, τ) . Since there are only countably many such \vec{x} , this shows that $\mathbb{P}(\mathcal{F})$ is σ -centered. \square

Given a family $\mathcal{H} \subseteq \text{bb}^\infty(E)$ and a sufficiently generic filter G for $\mathbb{P}(\mathcal{H})$, we denote by $X_{\text{gen}}(G)$ the *generic block sequence* determined by G ,

$$X_{\text{gen}}(G) = \bigcup \{ \vec{x} : \exists (\vec{x}, A, \sigma) \in G \}.$$

In what follows, G will be \mathcal{D} -generic for some countable collection of dense sets \mathcal{D} coming from the definition of universally Baire, and so G can be taken to be in \mathbf{V} . Any such \mathcal{D} will ensure that $X_{\text{gen}}(G)$ is infinite. We write \dot{X}_{gen} to be a nice (as defined above) $\mathbb{P}(\mathcal{H})$ -name for this block sequence.

Lemma 3.7.6. *Let $\mathcal{F} \subseteq \text{bb}^\infty(E)$ be a filter, \mathcal{D} a collection of dense subsets of $\mathbb{P}(\mathcal{F})$, and G a \mathcal{D} -generic filter for $\mathbb{P}(\mathcal{F})$. For $X = X_{\text{gen}}(G)$, the set*

$$G(X) = \{ (\vec{x}, A, \sigma) \in \mathbb{P}(\mathcal{F}) : X \in [\vec{x}, A, \sigma] \}$$

is a \mathcal{D} -generic filter for $\mathbb{P}(\mathcal{F})$ which contains G and $X_{\text{gen}}(G(X)) = X$.

Proof. By Lemma 3.7.5(a), $G(X)$ is closed upwards. If $(\vec{x}, A, \sigma) \in G$, then one can build a decreasing sequence $(\vec{x}_n, A_n, \sigma_n)$ in G with $(\vec{x}_0, A_0, \sigma_0) = (\vec{x}, A, \sigma)$, $|\vec{x}_n| \rightarrow \infty$ as $n \rightarrow \infty$, and X the union of the \vec{x}_n . By construction, X must be in $[\vec{x}, A, \sigma]$. This shows that $G \subseteq G(X)$, and consequently the latter is \mathcal{D} -generic. It remains to show that $G(X)$ is a filter. Take $(\vec{x}, A, \sigma), (\vec{y}, B, \tau) \in G(X)$. As X has both \vec{x} and \vec{y} as an initial segment, one must be an initial segment of the other, say $\vec{x} \sqsubseteq \vec{y}$, and the part of \vec{y} above \vec{x} is a sequence of moves by Π against σ . As \mathcal{F} is a filter, there is a $Y \in \mathcal{F}$ with $\langle Y \rangle \subseteq A \cap B$. Let ρ be the strategy for I in $F[A \cap B]$ given by $\rho(\vec{v}) = \max\{\sigma(\vec{v}), \tau(\vec{v})\}$, for \vec{v} in its domain. Then, $(\vec{y}, A \cap B, \rho)$ is below both (\vec{x}, A, σ) and (\vec{y}, B, τ) . Moreover, $X \in [\vec{y}, A \cap B, \rho]$, and so $(\vec{y}, A \cap B, \rho) \in G(X)$. That $X_{\text{gen}}(G(X)) = X$ is clear. \square

A consequence of Lemma 3.7.6 is that if G is generic for $\mathbb{P}(\mathcal{F})$ over a model of a sufficient fragment of ZFC, then $G(X) = G$, though we will not make use of this here.

Lemma 3.7.7. *Let $\mathcal{F} \subseteq \text{bb}^\infty(E)$ be a filter and \mathcal{D} a countable collection of dense open subsets of $\mathbb{P}(\mathcal{F})$.*

(a) *For any $(\vec{x}, A, \sigma) \in \mathbb{P}(\mathcal{F})$, the set*

$$\mathbb{G}_{\mathcal{D}, (\vec{x}, A, \sigma)} = \{X_{\text{gen}}(G) : G \text{ a } \mathcal{D}\text{-generic filter for } \mathbb{P}(\mathcal{F}) \text{ with } (\vec{x}, A, \sigma) \in G\}$$

is an $F_{\sigma\delta}$ subset of $\text{bb}^\infty(E)$.

(b) *If $X \in \mathcal{F}$, then for no $Y \in \mathcal{F} \upharpoonright X$ does I have a strategy in $F[\vec{x}, Y]$ for playing into*

$$(\mathbb{G}_{\mathcal{D}, (\vec{x}, X, \sigma)})^c.$$

(c) *If \mathcal{F} is a (p^+) -filter and $X \in \mathcal{F}$, then there is a $Y \in \mathcal{F} \upharpoonright X$ for which Π has a strategy in $G[\vec{x}, Y]$ for playing into $\mathbb{G}_{\mathcal{D}, (\vec{x}, X, \sigma)}$.*

Proof. (a) Enumerate $\mathcal{D} = \{D_n : n \in \omega\}$. Since $\mathbb{P}(\mathcal{F})$ is ccc by Lemma 3.7.5(c), each D_n contains a countable maximal antichain A_n below (\vec{x}, A, σ) . We claim that

$$\mathbb{G}_{\mathcal{D},(\vec{x},A,\sigma)} = \bigcap_{n \in \omega} \bigcup \{[\vec{y}, B, \tau] : (\vec{y}, B, \tau) \in A_n\},$$

which is $F_{\sigma\delta}$, as each set $[\vec{y}, Y, \tau]$ is closed by Lemma 3.7.5(b).

If $X = X_{\text{gen}}(G)$ where G is a \mathcal{D} -generic filter with $(\vec{x}, A, \sigma) \in G$, then for each n , $G \cap A_n \neq \emptyset$, say with $(\vec{y}_n, B_n, \tau_n) \in G \cap A_n$. By Lemma 3.7.6, for each n , $X \in [\vec{y}_n, B_n, \tau_n]$, and so X is in the set on right hand side of the above displayed line. For the reverse inclusion, suppose that X is in set on the right hand side. Then, by Lemma 3.7.6, $G(X)$ is a \mathcal{D} -generic filter containing (\vec{x}, A, σ) for which $X_{\text{gen}}(G(X)) = X$, and so $X \in \mathbb{G}_{\mathcal{D},(\vec{x},A,\sigma)}$.

(b) Let $X \in \mathcal{F}$ and $Y \in \mathcal{F} \upharpoonright X$ be given. Towards a contradiction, suppose that ρ is a strategy for I in $F[\vec{x}, Y]$ for playing into $(\mathbb{G}_{\mathcal{D},(\vec{x},X,\sigma)})^c$. We may assume $\rho \geq \sigma \upharpoonright Y$. Consider the following play of $F[\vec{x}, Y]$: I plays $\rho(\emptyset) = n_0$. Pick

$$p_0 = (\vec{x} \wedge (y_0^0, \dots, y_{k_0}^0), B^0, \rho^0) \leq (\vec{x}, Y, \rho) \leq (\vec{x}, X, \sigma)$$

in D_0 and let II play $y_0 = y_0^0$. Note that this is a valid move by definition of \leq in $\mathbb{P}(\mathcal{F})$. Next, I plays $\rho(y_0) = n_1$. Pick

$$p_1 = (\vec{x} \wedge (y_0^0, \dots, y_{k_0}^0) \wedge (y_0^1, \dots, y_{k_1}^1), B^1, \rho^1) \leq (\vec{x} \wedge (y_0^0, \dots, y_{k_0}^0), B^0, \rho^0)$$

in D_1 and let II play $y_1 = y_1^0$ if $k_0 \geq 1$, and $y_1 = y_0^1$ otherwise. Continuing in this fashion, we build an outcome (y_n) . Observe that (y_n) must be in $\mathbb{G}_{\mathcal{D},(\vec{x},X,\sigma)}$: the conditions p_n picked in D_n above form a \mathcal{D} -generic chain in $\mathbb{P}(\mathcal{F})$ below (\vec{x}, X, σ) , thus generate a \mathcal{D} -generic filter G with $X_{\text{gen}}(G) = (y_n)$ and $(\vec{x}, X, \sigma) \in G$. This contradicts our choice of ρ .

(c) follows from (a) and (b) by an application of Theorem 3.1.1. \square

Lemma 3.7.8. *Let $\mathcal{F} \subseteq \text{bb}^\infty(E)$ be a (p^+) -filter. If $\mathbb{A} \subseteq \text{bb}^\infty(E)$ is universally Baire, then for any $\vec{x} \in \text{bb}^{<\infty}(E)$ and $X \in \mathcal{F}$, there is a $Y \in \mathcal{F} \upharpoonright X$ such that II has a strategy in $G[\vec{x}, Y]$ for playing into one of \mathbb{A} or \mathbb{A}^c .*

Proof. Let $X \in \mathcal{F}$ be given. We may assume that \vec{x} is \emptyset . Recall, for $\vec{y} \in \text{bb}^{<\infty}(E)$ and $Y \in \mathcal{F}$, Definition 3.3.2 of (\vec{y}, Y) being good/bad/worse (for the set \mathbb{A}). By Lemma 3.3.3, there is a $Y \in \mathcal{F} \upharpoonright X$ such that either (\emptyset, Y) is good or I has a strategy σ in $F[Y]$ to play into the set

$$\{(z_n) : \forall n(z_0, \dots, z_n, Y) \text{ is worse}\}.$$

In the former case we're done, so we assume the latter.

Since \mathbb{A} is universally Baire, we may let $\dot{\mathbb{A}}$ be a $\mathbb{P}(\mathcal{F})$ -name for \mathbb{A} and \mathcal{D} countable collection of dense open subsets of $\mathbb{P}(\mathcal{F})$ such that

- (i) $\{q \in \mathbb{P}(\mathcal{F}) : q \text{ decides } \dot{X}_{\text{gen}} \in \dot{\mathbb{A}}\}$ is in \mathcal{D} , and
- (ii) whenever G is \mathcal{D} -generic in $\mathbb{P}(\mathcal{F})$, $X_{\text{gen}}(G)$ is in $\text{bb}^\infty(E)$ and $X_{\text{gen}}(G)$ is in \mathbb{A} if and only if there is a $q \in G$ such that $q \Vdash_{\mathbb{P}(\mathcal{F})} \dot{X}_{\text{gen}} \in \dot{\mathbb{A}}$.

Thus, if G is \mathcal{D} -generic for $\mathbb{P}(\mathcal{F})$, contains (\emptyset, Y, σ) , and $(\emptyset, Y, \sigma) \Vdash_{\mathbb{P}(\mathcal{F})} \dot{X}_{\text{gen}} \notin \dot{\mathbb{A}}$, then $X_{\text{gen}}(G) \notin \mathbb{A}$. We claim that $(\emptyset, Y, \sigma) \Vdash_{\mathbb{P}(\mathcal{F})} \dot{X}_{\text{gen}} \notin \dot{\mathbb{A}}$.

Suppose not, then there is a $(\vec{y}, Z, \tau) \leq (\emptyset, Y, \sigma)$, with $Z \in \mathcal{F}$, such that $(\vec{y}, Z, \tau) \Vdash_{\mathbb{P}(\mathcal{F})} \dot{X}_{\text{gen}} \in \dot{\mathbb{A}}$. Applying Lemma 3.7.7(c), take $W \in \mathcal{F} \upharpoonright Z$ such that II has a strategy α in $G[\vec{y}, W]$ for playing into $\mathbb{G}_{\mathcal{D}, (\vec{y}, Z, \tau)}$. We claim that $\mathbb{G}_{\mathcal{D}, (\vec{y}, Z, \tau)} \subseteq \mathbb{A}$. Let (z_n) be in $\mathbb{G}_{\mathcal{D}, (\vec{y}, Z, \tau)}$. Take G a \mathcal{D} -generic filter for which $(z_n) = X_{\text{gen}}(G)$ and $(\vec{y}, Z, \tau) \in G$. Since $(\vec{y}, Z, \tau) \Vdash_{\mathbb{P}(\mathcal{F})} \dot{X}_{\text{gen}} \in \dot{\mathbb{A}}$, we have that $(z_n) \in \mathbb{A}$. Thus, α is a strategy for II in $G[\vec{y}, W]$ for playing into \mathbb{A} . This, however, contradicts the fact that σ ensures that (\vec{y}, Z) is bad.

Thus, $(\emptyset, Y, \sigma) \Vdash_{\mathbb{P}(\mathcal{F})} \dot{X}_{\text{gen}} \notin \dot{\mathbb{A}}$. Then, exactly as in the preceding paragraph, we may find $W \in \mathcal{F} \upharpoonright Y$ such that II has a strategy in $G[W]$ for playing into $\mathbb{G}_{\mathcal{D},(\emptyset, Y, \sigma)}$ and thus into \mathbb{A}^c . \square

While the symmetric result in Lemma 3.7.8 is appealing on its own, and applies to all analytic sets (being universally Baire [28]) in ZFC, it is not a true “dichotomy” as II can easily have strategies into both \mathbb{A} and \mathbb{A}^c .

One consequence of Lemma 3.7.7 and the proof of Lemma 3.7.8 is that, given (p^+) -filter \mathcal{F} and a universally Baire set $\mathbb{A} \subseteq \text{bb}^\infty(E)$, there is always an $X \in \mathcal{F}$ such that one of \mathbb{A} or \mathbb{A}^c contains an $F_{\sigma\delta}$ set \preceq -dense below X .

We can now complete the proofs of Theorems 3.1.2 and 3.1.3.

Proof of Theorem 3.1.2. We have already proven the (\Rightarrow) direction in Theorem 3.5.4. For the remaining direction, let $\mathbb{D} \subseteq \text{bb}^\infty(E)$ be a \preceq -dense open set which is in $\mathbf{L}(\mathbb{R})$ and thus universally Baire by Theorem 3.7.3. By Lemma 3.7.8, there is an $X \in \mathcal{F}$ such that II has a strategy in $G[X]$ for playing into either \mathbb{D} or \mathbb{D}^c . By Lemma 3.4.4, the latter can never occur. Thus, II has a strategy in $G[X]$ for playing into \mathbb{D} . Since \mathcal{F} is strategic, there is a play by this strategy, say Z , with $Z \in \mathbb{D} \cap \mathcal{F} \neq \emptyset$. \square

Lemma 3.7.9. *Assume that there is a supercompact cardinal. Let $\mathcal{F} \subseteq \text{bb}^\infty(E)$ be a strategic (p^+) -filter. Every subset of $\text{bb}^\infty(E)$ in $\mathbf{L}(\mathbb{R})$ is \mathcal{F} -strategically Ramsey.*

Proof. Let $\mathbb{A} \subseteq \text{bb}^\infty(E)$ be in $\mathbf{L}(\mathbb{R})$, and fix $\vec{x} \in \text{bb}^{<\infty}(E)$ and $X \in \mathcal{F}$. By Theorem 3.7.2, the set of all $Y \preceq X$ witnessing that \mathbb{A} is strategically Ramsey is \preceq -dense below X and in $\mathbf{L}(\mathbb{R})$. Since \mathcal{F} is $\mathbf{L}(\mathbb{R})$ -generic, \mathcal{F} must contain such a Y . \square

Proof of Theorem 3.1.3. Let $\mathbb{A} \subseteq \text{bb}^\infty(E)$ be in $\mathbf{L}(\mathbb{R})$, and fix $\vec{x} \in \text{bb}^{<\infty}(E)$ and $X \in \mathcal{H}$. Let \mathcal{G} be \mathbf{V} -generic for (\mathcal{H}, \preceq^*) and contain X . By Theorem 3.5.4, \mathcal{G} is a strategic (p^+) -filter in $\mathbf{V}[\mathcal{G}]$. By Lemma 3.7.9, there is a $Y \in \mathcal{G} \upharpoonright X$ witnessing that \mathbb{A} is \mathcal{G} -strategically Ramsey in $\mathbf{V}[\mathcal{G}]$. Since forcing with (\mathcal{H}, \preceq^*) adds no new reals, Y witnesses that \mathbb{A} is \mathcal{H} -strategically Ramsey in \mathbf{V} . \square

We end this section with a discussion of Theorem 3.1.2. The phenomena expressed in this result says that large cardinals allows us to “upgrade” genericity for a collection of relatively simple sets to all definable sets.

In the case of $([\omega]^\omega, \subseteq^*)$, meeting the (topologically) closed dense sets

$$\{X \in [\omega]^\omega : [X]^2 \subseteq A \text{ or } [X]^2 \cap A = \emptyset\},$$

for $A \subseteq [\omega]^2$, ensures that the filter is a selective ultrafilter (cf. Theorem 4.9 in [17]), thus $\mathbf{L}(\mathbb{R})$ -generic for $([\omega]^\omega, \subseteq^*)$ under large cardinal hypotheses [25].

In the case of $(\text{bb}^\infty(E), \preceq^*)$, the situation appears more complicated. It is useful here to think about \mathcal{G} -dense sets, rather than arbitrary dense sets. This presents no harm due to the general fact that a filter G in a poset Q is M -generic if and only if it meets every G -dense subset of Q which is in M . Here, G -dense has the same meaning we gave for families in $\text{bb}^\infty(E)$ in §3.2, and M any model of a sufficient fragment of ZFC.

By definition, a filter $\mathcal{G} \subseteq \text{bb}^\infty(E)$ is a strategic (p^+) -filter if and only if it meets the sets

$$\{Y : Y \text{ is a diagonalization of } (X_n)\},$$

$$\{Y : \langle Y \rangle \subseteq D\}, \text{ and}$$

$$\{Y : Y \text{ is an outcome of } \alpha \text{ in } G[X]\},$$

where (X_n) is a \preceq -decreasing sequence in \mathcal{G} , D a \mathcal{G} -dense subset of E , and α a strategy for Π in $G[X]$, with $X \in \mathcal{G}$. Each of these sets is \mathcal{G} -dense. Moreover, the first set is easily seen to be $F_{\sigma\delta}$, the second is closed, and the third can be refined to a closed dense set by Lemma 3.4.7 and Lemma 6.4 in [30] (see also Lemma 3.11.8 below). Thus, Theorem 3.1.2 tells us that, under large cardinal hypotheses, we can upgrade genericity for \mathcal{G} -dense $F_{\sigma\delta}$ sets to genericity for all definable dense sets.

3.8 Normed spaces and a local Gowers dichotomy

We now consider the case when E is a countably infinite-dimensional normed vector space, with *normalized* basis (e_n) (i.e., $\|e_n\| = 1$ for all n), over a countable subfield F of \mathbb{R} (or \mathbb{C}) so that the norm takes values in F . If V is a subspace of E , let $S(V) = \{x \in V : \|x\| = 1\}$.

Let $\text{bb}_1^\infty(E) = \{(x_n) \in \text{bb}^\infty(E) : \forall n(\|x_n\| = 1)\}$ and $\text{bb}_1^{<\infty}(E) = \{\vec{x} \in \text{bb}^{<\infty}(E) : \forall n < |\vec{x}|(\|x_n\| = 1)\}$. For $X \in \text{bb}^\infty(E)$, let $[X] = \{Y \in \text{bb}_1^\infty(E) : Y \preceq X\}$. Taking E discrete, $\text{bb}_1^\infty(E)$ is a closed subset of the Polish space $\text{bb}^\infty(E)$, thus itself Polish.

For $X = (x_n), Y = (y_n) \in \text{bb}_1^\infty(E)$ and $\Delta = (\delta_n)$ a sequence of positive real numbers, written $\Delta > 0$, we write $d(X, Y) \leq \Delta$ if for all n , $\|x_n - y_n\| \leq \delta_n$. Given $\mathbb{A} \subseteq \text{bb}_1^\infty(E)$ and $\Delta > 0$, let

$$\mathbb{A}_\Delta = \{Y \in \text{bb}_1^\infty(E) : \exists X \in \mathbb{A}(d(X, Y) \leq \Delta)\},$$

the Δ -*expansion* of \mathbb{A} . We collect a few useful properties of Δ -expansions in a lemma which will be used tacitly in what follows. The proof is left to the reader.

Lemma 3.8.1. *Let $\mathbb{A} \subseteq \text{bb}_1^\infty(E)$ and $\Delta > 0$.*

- (a) *If $\mathbb{A} = \bigcup_{i \in I} \mathbb{A}_i$, then $\mathbb{A}_\Delta = \bigcup_{i \in I} (\mathbb{A}_i)_\Delta$.*
- (b) *If \mathbb{A} is analytic, then so is \mathbb{A}_Δ .*
- (c) *$(\mathbb{A}_\Delta)^c \subseteq ((\mathbb{A}_\Delta)^c)_\Delta \subseteq \mathbb{A}^c$.*
- (d) *If $0 < \Gamma \leq \Delta/2$, then $((\mathbb{A}_\Delta)^c)_\Gamma \subseteq (\mathbb{A}_\Gamma)^c$.*

□

The notions of *family*, *filter*, *fullness*, *(p)-property*, etc, in $\text{bb}_1^\infty(E)$, are defined exactly as for $\text{bb}^\infty(E)$ in §3.2. Moreover, all of the results established in the previous sections could have been carried out in $\text{bb}_1^\infty(E)$ in the event that E is normed. The only necessary modification is that in the games $G[\vec{x}, X]$ and $F[\vec{x}, X]$, the two players must play normalized block sequences and vectors, respectively. This will be assumed in what follows.

For $D \subseteq S(E)$ and $\epsilon > 0$, let

$$D_\epsilon = \{x \in S(E) : \exists y \in D(\|x - y\| \leq \epsilon)\}.$$

We weaken the notion of fullness to the following approximate version.¹¹

Definition 3.8.2. A family $\mathcal{H} \subseteq \text{bb}_1^\infty(E)$ is *almost full* if whenever $D \subseteq S(E)$ and $X \in \mathcal{H}$ are such that D is \mathcal{H} -dense below X (that is, for all $Y \in \mathcal{H} \upharpoonright X$, there is a $Z \preceq Y$ with $S(\langle Z \rangle) \subseteq D$), then for any $\epsilon > 0$, there is a $Z \in \mathcal{H} \upharpoonright X$ with $S(\langle Z \rangle) \subseteq D_\epsilon$.

If a family has the *(p)-property* and is almost full we call it a *(p*)-family*.

Likewise for *(p*)-filter*, *strategic (p*)-family*, etc.

¹¹While this hampers our ability to reuse results from §3.3 and §3.7, we hope that it will enable further applications. An elementary proof of Proposition 3.8.20, without the hypothesis of being “strategic”, would greatly simplify the situation in the cases of interest.

Definition 3.8.3. Given a family $\mathcal{H} \subseteq \text{bb}_1^\infty(E)$, a set $\mathbb{A} \subseteq \text{bb}_1^\infty(E)$ is \mathcal{H} -weakly Ramsey if for every $\Delta > 0$ and $X \in \mathcal{H}$, there is a $Y \in \mathcal{H} \upharpoonright X$ such that either:

- (i) $[Y] \subseteq \mathbb{A}^c$, or
- (ii) II has a strategy in $G[Y]$ for playing into \mathbb{A}_Δ .

The first goal of this section is to show that for certain (p^*) -families \mathcal{H} , analytic sets in $\text{bb}_1^\infty(E)$ are \mathcal{H} -weakly Ramsey. We begin with variants of Lemmas 3.3.3 and 3.3.4, and Theorem 3.1.1, for (p^*) -families. Since dealing with both families and Δ -expansions requires some care, we include proofs of these results. As in §3.3, these arguments are very similar to those in [76].

Definition 3.8.4. Given a family $\mathcal{H} \subseteq \text{bb}_1^\infty(E)$, $\mathbb{A} \subseteq \text{bb}_1^\infty(H)$ and $\Delta > 0$, for $\vec{y} \in \text{bb}_1^{<\infty}(E)$ and $Y \in \mathcal{H}$, we say the pair (\vec{y}, Y) is Δ -good/ Δ -bad/ Δ -worse if it is good/bad/worse for the set \mathbb{A}_Δ (in the sense of Definition 3.3.2). Further:

- (1) (\vec{y}, Y) is Δ^* -good if it is $\Delta(|\vec{y}|)$ -good,
- (2) (\vec{y}, Y) is Δ^* -bad if it is $\Delta(|\vec{y}|)$ -bad,
- (3) (\vec{y}, Y) is Δ^* -worse if it is Δ^* -bad and there is a n such that for all $v \in S(\langle Y/n \rangle)$, $(\vec{y} \hat{\ } v, Y)$ is Δ^* -bad.

Here, $\Delta(m) = (\delta_0/2, \delta_1/2, \dots, \delta_{m-1}/2, \delta_m, \delta_{m+1}, \dots)$.

Note that Δ^* -good implies Δ -good and Δ^* -bad implies $\Delta/2$ -bad.

Lemma 3.8.5. If \mathcal{H} is a (p^*) -family and $\mathbb{A} \subseteq \text{bb}^\infty(E)$, then for every $\vec{x} \in \text{bb}^{<\infty}(E)$, $X \in \mathcal{H}$ and $\Delta > 0$, there is a $Y \in \mathcal{H} \upharpoonright X$ such that either

- (i) (\vec{x}, Y) is Δ -good, or
- (ii) I has a strategy in $F[\vec{x}, Y]$ for playing into

$$\{(z_n) : \forall n(\vec{x} \hat{\ } (z_0, \dots, z_n), Y) \text{ is } \Delta^*\text{-worse}\}.$$

Proof. Let $\mathcal{H}, \mathbb{A}, X \in \mathcal{H}$ and $\Delta > 0$ be given. As in the proof of Lemma 3.3.3, for any \vec{y} and $\Gamma > 0$, the set

$$\mathcal{D}_{\vec{y}}^{\Gamma} = \{Y : (\vec{y}, Y) \text{ is } \Gamma\text{-good or } \Gamma\text{-bad}\}$$

is \preceq -dense open in \mathcal{H} , and if (\vec{y}, Y) is Γ -bad, then for every $V \in \mathcal{H} \upharpoonright Y$, there is a $Z \preceq V$ such that for all $x \in S(\langle Z \rangle)$, $(\vec{y} \frown x, Y)$ is not Γ -good.

Claim. For any $\vec{y} \in \text{bb}_1^{\leq \infty}(E)$, the set

$$\mathcal{E}_{\vec{y}} = \{Y : (\vec{y}, Y) \text{ is } \Delta^*\text{-good or } \Delta^*\text{-worse}\}$$

is \preceq -dense open in \mathcal{H} .

Proof of claim. Let $Y \in \mathcal{H}$. By diagonalizing over the sets $\mathcal{D}_{\vec{z}}^{\Delta(|\vec{z}|)}$, we may assume that for all \vec{z} , (\vec{z}, Y) is Δ^* -good or Δ^* -bad. Assume that (\vec{y}, Y) is Δ^* -bad. Let

$$D = \{x \in S(E) : (\vec{y} \frown x, Y) \text{ is not } \Delta(|\vec{y}|)\text{-good}\}.$$

Take $\epsilon = \delta_{|\vec{y}|}/2$. By almost fullness, there is a $Z \in \mathcal{H} \upharpoonright Y$ such that $S(\langle Z \rangle) \subseteq D_{\epsilon}$. Given $z \in S(\langle Z \rangle)$, pick $z' \in D$ with $\|z - z'\| \leq \epsilon$. If $(\vec{y} \frown z, Z)$ is Δ^* -good, then there is a strategy α for Π in $G[\vec{y} \frown z, Z]$ for playing into $\mathbb{A}_{\Delta(|\vec{y}|+1)}$. We may assume that all plays according to α are above z and z' , so we can treat α as a strategy α' for Π in $G[\vec{y} \frown z', Z]$. If $\vec{y} \frown z' \frown W$ is an outcome of α' , then $\vec{y} \frown z \frown W$ is an outcome of α , and thus in $\mathbb{A}_{\Delta(|\vec{y}|+1)}$. By our choice of ϵ , it follows that $\vec{y} \frown z' \frown W$ is in $\mathbb{A}_{\Delta(|\vec{y}|)}$. Then, $(\vec{y} \frown z', Z)$ is $\Delta(|\vec{y}|)$ -good, contradicting that $z' \in D$. Thus, $(\vec{y} \frown z, Z)$ is Δ^* -bad, and (\vec{y}, Z) is Δ^* -worse. □_(claim.)

Returning to the proof of the lemma, we consider the case when $\vec{x} = \emptyset$. By the claim, we can find $Y \in \mathcal{H} \upharpoonright X$ such that for all \vec{y} , (\vec{y}, Y) is either Δ^* -good or Δ^* -worse. If (\emptyset, Y) is Δ^* -good, we're done, so assume that it is Δ^* -worse. In this case, we define a strategy for I in $F[Y]$ for playing into $\{(z_n) : \forall n(z_0, \dots, z_n, Y) \text{ is } \Delta^*\text{-worse}\}$ exactly as in the proof of Lemma 3.3.3. □

Lemma 3.8.6 (cf. Lemma 2 in [76]). *Let $\mathcal{H} \subseteq \text{bb}_1^\infty(E)$ be a (p^*) -family. Given $\mathbb{A} \subseteq \text{bb}_1^\infty(E)$ open, $x \in \text{bb}_1^{<\infty}(E)$, $X \in \mathcal{H}$, and $\Delta > 0$, there is a $Y \in \mathcal{H} \upharpoonright Y$ such that either:*

- (i) *I has a strategy in $F[\vec{x}, Y]$ for playing into $(\mathbb{A}_{\Delta/2})^c$, or*
- (ii) *II has a strategy in $G[\vec{x}, Y]$ for playing into \mathbb{A}_Δ .*

Proof. The proof is similar to Lemma 3.3.4, using Lemma 3.8.5. □

Lemma 3.8.7 (cf. Lemma 4 in [76]). *Let $\mathcal{H} \subseteq \text{bb}_1^\infty(E)$ be a (p^*) -family. Suppose that $\mathbb{A} = \bigcup_{n \in \omega} \mathbb{A}_n$, each $\mathbb{A}_n \subseteq \text{bb}_1^\infty(E)$. Let $\vec{x}, X \in \mathcal{H}$, and $\Delta > 0$ be given. Then, there is a $Y \in \mathcal{H} \upharpoonright X$ such that either:*

- (i) *I has a strategy in $F[\vec{x}, Y]$ for playing into $(\mathbb{A}_{\Delta/2})^c$, or*
- (ii) *II has a strategy in $G[Y]$ for playing (z_k) for which there is some n such that for every $V \in \mathcal{H} \upharpoonright Y$, I has no strategy in $F[\vec{x}^\wedge(z_0, \dots, z_n), V]$ for playing into $((\mathbb{A}_n)_\Delta)^c$.*

Proof. For $Y \in \mathcal{H}$, $\vec{y} \in \text{bb}^{<\infty}(E)$, and $n \in \omega$, we say (\vec{y}, n) Γ -accepts Y if I has a strategy in $F[\vec{y}, Y]$ for playing into $((\mathbb{A}_n)_\Gamma)^c$ and (\vec{y}, n) Γ -rejects Y if for all $Z \in \mathcal{H} \upharpoonright Y$, (\vec{y}, n) does not Γ -accept Z . Both acceptance and rejection are \preceq^* -hereditary in \mathcal{H} , and the sets

$$\mathcal{D}_{\vec{y}, n}^\Gamma = \{Y : (\vec{y}, n) \Gamma\text{-accepts or } \Gamma\text{-rejects } Y\}$$

are clearly \preceq -dense open in \mathcal{H} . By the (p) -property, we can find $Y \in \mathcal{H} \upharpoonright X$ such that for all \vec{y} and n , (\vec{y}, n) either $\Delta/2$ -accepts or $\Delta/2$ -rejects Y . Put

$$R = \{(z_k) : \exists n(\vec{x}^\wedge(z_0, \dots, z_n), n) \Delta/2\text{-rejects } Y\},$$

and notice that R is open in $\text{bb}_1^\infty(E)$. By Lemma 3.8.6, there is $Y' \in \mathcal{H} \upharpoonright Y$ such that either II has a strategy in $G[Y']$ for playing into $R_{\Delta/2}$, or I has a strategy

in $F[Y']$ for playing into $(R_{\Delta/4})^c \subseteq R^c$. In the first case, suppose that (z_k) is an outcome of II' 's strategy. Then, there is (z'_k) with $\|z_k - z'_k\| \leq \delta_k/2$ for all k , and an n such that $(\vec{x}^\wedge(z'_0, \dots, z'_n), n)$ $\Delta/2$ -rejects Y . We claim $(\vec{x}^\wedge(z_0, \dots, z_n), n)$ Δ -rejects Y . If not, then for some $Z \in \mathcal{H} \upharpoonright Y$, I has a strategy in $F[\vec{x}^\wedge(z_0, \dots, z_n), Z]$ for playing into $((\mathbb{A}_n)_\Delta)^c$. This yields a strategy for I in $F[\vec{x}^\wedge(z'_0, \dots, z'_n), Z]$ for playing into $((\mathbb{A}_n)_\Delta)^c$. By Lemma 3.8.1(d), $((\mathbb{A}_n)_\Delta)^c_{\Delta/2} \subseteq ((\mathbb{A}_n)_{\Delta/2})^c$, and so $(\vec{x}^\wedge(z'_0, \dots, z'_n), n)$ fails to $\Delta/2$ -reject Y , a contradiction. Thus, (z_k) is as desired for (ii).

Suppose that I has a strategy σ in $F[Y]$ for playing into $(R_{\Delta/4})^c \subseteq R^c$. In particular, I plays (z_k) such that for all n , I has a strategy $\sigma_{(z_0, \dots, z_n)}$ in $F[\vec{x}^\wedge(z_0, \dots, z_n), Y]$ to play into $((\mathbb{A}_n)_{\Delta/2})^c$. As in the proof of Lemma 4 in [75], we successively put more strategies for I into play and obtain a strategy for playing into $\bigcap_n ((\mathbb{A}_n)_{\Delta/2})^c = (\mathbb{A}_{\Delta/2})^c$. \square

Theorem 3.8.8 (cf. Theorem 5 in [76]). *Let $\mathcal{H} \subseteq \text{bb}_1^\infty(E)$ be a (p^*) -family. If $\mathbb{A} \subseteq \text{bb}_1^\infty(E)$ is analytic, $\Delta > 0$, $\vec{x} \in \text{bb}_1^{<\infty}(E)$, and $X \in \mathcal{H}$, then there is a $Y \in \mathcal{H} \upharpoonright Y$ such that either:*

- (i) *I has a strategy in $F[\vec{x}, Y]$ for playing into $(\mathbb{A}_{\Delta/2})^c$, or*
- (ii) *II has a strategy in $G[\vec{x}, Y]$ for playing into \mathbb{A}_Δ .*

Proof. We consider the case when $\vec{x} = \emptyset$. Let $F : \omega^\omega \rightarrow \mathbb{A}$ be a continuous surjection and for each $s \in \omega^{<\omega}$, let $\mathbb{A}_s = F''(\mathcal{N}_s)$ where, again, $\mathcal{N}_s = \{\alpha \in \omega^\omega : s \subseteq \alpha\}$. Note that $\mathbb{A}_s = \bigcup_n \mathbb{A}_{s \frown n}$.

Let $R(s, \vec{x}, Y)$ (for $Y \in \mathcal{H}$) be the set of all (z_k) for which there is an n such that for all $Z \in \mathcal{H} \upharpoonright Y$, I has no strategy in $F[\vec{x}^\wedge(z_0, \dots, z_n), Z]$ for playing into $((\mathbb{A}_{s \frown n})_\Delta)^c$. By Lemma 3.8.7 and the (p) -property, there is an $Y \in \mathcal{H} \upharpoonright X$ such

that for all \vec{x} and $s \in \omega^{<\omega}$, either

- (i) I has a strategy in $F[\vec{x}, Y]$ for playing into $((\mathbb{A}_s)_{\Delta/2})^c$, or
- (ii) II has a strategy in $G[Y]$ for playing into $R(s, \vec{x}, X)$.

Suppose I has no strategy in $F[Y]$ for playing into $(\mathbb{A}_{\Delta/2})^c = ((\mathbb{A}_{\emptyset})_{\Delta/2})^c$. We will describe a strategy for II in $G[Y]$ for playing into \mathbb{A}_{Δ} : As II has a strategy in $G[Y]$ for playing into $R(\emptyset, \emptyset, Y)$, they follow this strategy until (z_0, \dots, z_{n_0}) has been played such that I has no strategy in $F[(z_0, \dots, z_{n_0}), Y]$ for playing into $((\mathbb{A}_{s \frown n_0})_{\Delta})^c$. By the assumption on Y , II must have a strategy in $G[Y]$ to play in $R((n_0), (z_0, \dots, z_{n_0}), Y)$. II follows this until a further $(z_{n_0+1}, \dots, z_{n_0+n_1+1})$ has been played so that I has no strategy in $F[(z_0, \dots, z_{n_0}, \dots, z_{n_0+n_1+1}), Y]$ for playing into $((\mathbb{A}_{s \frown n_0 \frown n_1})_{\Delta})^c$.

We continue in this fashion, exactly as in the proof of Theorem 5 in [76], so that the outcome $Z = (z_n)$ satisfies that for all k , with $m_k = (\sum_{j \leq k} n_j) + k$, there is some $Z^k \sqsupseteq (z_0, \dots, z_{m_k})$ in $(\mathbb{A}_{(n_0, \dots, n_k)})_{\Delta} = (F''(\mathcal{N}_{(n_0, \dots, n_k)}))_{\Delta}$. Continuity of F ensures that, for $\alpha = (n_0, n_1, \dots)$, $d(F(\alpha), Z) \leq \Delta$. \square

The following result provides the link between strategically Ramsey sets and weakly Ramsey sets.

Theorem 3.8.9 (Rosendal [76]). *Suppose that, for some $X \in \text{bb}_1^{\infty}(E)$, I has a strategy in $F[X]$ to play into some set $\mathbb{A} \subseteq \text{bb}_1^{\infty}(E)$. Then, for any $\Delta > 0$, there is a sequence of finite intervals $I_0 < I_1 < \dots$ in ω such whenever $Y = (y_n) \preceq X$ and $\forall n \exists m (I_0 < y_n < I_m < y_{n+1})$, we have that $Y \in \mathbb{A}_{\Delta}$.*

Inspired by this theorem, we define:

Definition 3.8.10. A family $\mathcal{H} \subseteq \text{bb}_1^{\infty}(E)$ is *spread* if whenever $X = (x_n) \in \mathcal{H}$

and $I_0 < I_1 < \dots$ is a sequence of intervals in ω , there is a $Y = (y_n) \in \mathcal{H} \upharpoonright X$ such that $\forall n \exists m (I_0 < y_n < I_m < y_{n+1})$.

We will see in Proposition 3.8.26 below that this property is analogous to the (q^+) -property for coideals on ω .

Lemma 3.8.11. *Given a sequence of intervals $I_0 < I_1 < \dots$ in ω , the set*

$$\{(y_n) : \forall n \exists m (I_0 < y_n < I_m < y_{n+1})\}.$$

is \preceq -dense open in $\text{bb}_1^\infty(E)$.

Proof. Let $X = (x_n)$ be given. We thin down X to obtain a desired Y : Let y_0 be the first x_n with $I_0 < x_n$. If I_m is the first interval above y_0 , let y_1 be the first x_k with $I_m < x_k$. Continue in this fashion. This verifies that the set in question is \preceq -dense.

To see that it is, moreover, open, let $Z = (z_n) \preceq Y = (y_n)$, with Y satisfying $\forall n \exists m (I_0 < y_n < I_m < y_{n+1})$. For each n , clearly $I_0 < z_n$, and moreover, if y_k is the last vector from Y used in the support of z_n , and I_m is such that $y_k < I_m < y_{k+1}$, we have that $z_n < I_m < z_{n+1}$. \square

Clearly, $\text{bb}_1^\infty(E)$ itself is spread. As in §3.5, one can build spread filters (which are full, almost full, strategic, etc) under the additional set-theoretic hypotheses of Theorem 3.5.3 or by forcing. We note that being strategic suffices:

Lemma 3.8.12. *If $\mathcal{H} \subseteq \text{bb}_1^\infty(E)$ is a strategic family, then it is spread.*

Proof. Fix $X \in \mathcal{H}$ and let $I_0 < I_1 < \dots$ be an increasing sequence of intervals in ω . Consider the following strategy α for Π in $G[X]$: At move 0, we ensure that

II plays above $y_0 > I_0$. Inductively, regardless of I's prior moves, at move n we ensure that II plays $y_n > I_m$, where I_m is the first interval entirely above y_{n-1} . This describes a valid strategy for II and any outcome of this strategy in \mathcal{H} will witness that \mathcal{H} is spread. \square

Theorem 3.8.13 (cf. Theorem 7 in [76]). *Let $\mathcal{H} \subseteq \text{bb}_1^\infty(E)$ be a spread (p^*) -family. Then, every analytic set is \mathcal{H} -weakly Ramsey.*

Proof. Let $\mathbb{A} \subseteq \text{bb}_1^\infty(E)$ be analytic. Fix $X \in \mathcal{H}$ and $\Delta > 0$. By Theorem 3.8.8, there is $Y \in \mathcal{H} \upharpoonright X$ such that either I has a strategy in $F[Y]$ for playing into $(\mathbb{A}_{\Delta/2})^c$, or II has a strategy in $G[Y]$ for playing into \mathbb{A}_Δ . In the latter case, we're done, so assume the former. Theorem 3.8.9 and \mathcal{H} being spread implies that there is some $Z \in \mathcal{H} \upharpoonright Y$ with $[Z] \subseteq ((\mathbb{A}_{\Delta/2})^c)_{\Delta/2} \subseteq \mathbb{A}^c$, where the last containment is given by Lemma 3.8.1(c). \square

In order to extend to sets in $\mathbf{L}(\mathbb{R})$, we will use the following analogue of Lemma 3.7.8.

Lemma 3.8.14. *Let $\mathcal{F} \subseteq \text{bb}_1^\infty(E)$ be a (p^*) -filter. If $\mathbb{A} \subseteq \text{bb}_1^\infty(E)$ is such that continuous images of \mathbb{A} are universally Baire, then for any $X \in \mathcal{F}$ and $\Delta > 0$, there is a $Y \in \mathcal{F} \upharpoonright X$ for which II has a strategy in $G[Y]$ for playing into one of $(\mathbb{A}_{\Delta/8})^c$ or \mathbb{A}_Δ .*

Proof. Let $X \in \mathcal{F}$ and $\Delta > 0$. By Lemma 3.8.5, there is a $Y \in \mathcal{F} \upharpoonright X$ such that either (\emptyset, Y) is Δ -good or I has a strategy σ in $F[Y]$ for playing into

$$\{(z_n) : \forall n (z_0, \dots, z_n, Y) \text{ is } \Delta/2\text{-bad}\}.$$

In the former case, we're done, so assume the latter.

By hypothesis, \mathbb{A}_Γ is universally Baire for all Γ . In particular, we may let $\dot{\mathbb{A}}_{\Delta/4}$ be a $\mathbb{P}(\mathcal{F})$ -name for $\mathbb{A}_{\Delta/4}$ and \mathcal{D} a countable collection of dense open subsets of $\mathbb{P}(\mathcal{F})$ such that

- (i) $\{q \in \mathbb{P}(\mathcal{F}) : q \text{ decides } \dot{X}_{\text{gen}} \in \dot{A}\}$ is in \mathcal{D} , and
- (ii) whenever G is \mathcal{D} -generic in $\mathbb{P}(\mathcal{F})$, \dot{X}_{gen} is in $\text{bb}_1^\infty(E)$ and $\dot{X}_{\text{gen}}(G)$ is in $\mathbb{A}_{\Delta/4}$ if and only if there is a $q \in G$ such that $q \Vdash_{\mathbb{P}(\mathcal{F})} \dot{X}_{\text{gen}} \in \dot{\mathbb{A}}_{\Delta/4}$.

We claim that $(\emptyset, Y, \sigma) \Vdash_{\mathbb{P}(\mathcal{F})} \dot{X}_{\text{gen}} \notin \dot{\mathbb{A}}_{\Delta/4}$.

Suppose not, then there is a $(\vec{y}, Z, \tau) \leq (\emptyset, Y, \sigma)$, with $Z \in \mathcal{F}$, such that $(\vec{y}, Z, \tau) \Vdash_{\mathbb{P}(\mathcal{F})} \dot{X}_{\text{gen}} \in \dot{\mathbb{A}}_{\Delta/4}$. Applying Lemma 3.7.7(b) and Theorem 3.8.8, there is a $W \in \mathcal{F} \upharpoonright Z$ such that II has a strategy α in $G[\vec{y}, W]$ for playing into $(\mathbb{G}_{\mathcal{D}, (\vec{y}, Z, \tau)})_{\Delta/4}$. As in the proof of Lemma 3.7.8, $\mathbb{G}_{\mathcal{D}, (\vec{y}, Z, \tau)} \subseteq \mathbb{A}_{\Delta/4}$, so α is a strategy for II in $G[\vec{y}, W]$ for playing into $\mathbb{A}_{\Delta/2}$. This, however, contradicts the fact that σ ensures (\vec{y}, Z) is $\Delta/2$ -bad.

Thus, $(\emptyset, Y, \sigma) \Vdash_{\mathbb{P}(\mathcal{F})} \dot{X}_{\text{gen}} \notin \dot{\mathbb{A}}_{\Delta/4}$. But then, exactly as in the preceding paragraph, we may find $W \in \mathcal{F} \upharpoonright Y$ such that II has a strategy in $G[W]$ for playing into $(\mathbb{G}_{\mathcal{D}, (\emptyset, Y, \sigma)})_{\Delta/8}$, and thus into $((\mathbb{A}_{\Delta/4})^c)_{\Delta/8} \subseteq (\mathbb{A}_{\Delta/8})^c$, where the last containment follows from Lemma 3.8.1(d). \square

In what follows, we strengthen the hypotheses on the basis (e_n) , asserting that there is some $K > 0$ such that for all $m \leq n$ and scalars (a_k) ,

$$\left\| \sum_{k \leq m} a_k e_k \right\| \leq K \left\| \sum_{k \leq n} a_k e_k \right\|.$$

This is equivalent to (e_n) being a Schauder basis of the completion \overline{E} of E , cf. Proposition 1.1.9 [2]. The infimum of all such K as above is called the *basis constant* of (e_n) . We will use that if $v = \sum a_n e_n$ in E , then $\sup_n |a_n| \leq 2K \|v\|$.

Lemma 3.8.15. *For any $\Delta > 0$, there is a $\Gamma > 0$ such that whenever $X = (x_n)$, $X' = (x'_n) \in \text{bb}_1^\infty(E)$ satisfy $d(X', X) \leq \Gamma$, then $[X'] \subseteq [X]_\Delta$. In fact, if $Y' \in [X']$, then $\tilde{Y} \in [X]$ and $d(Y', \tilde{Y}) \leq \Delta$, where \tilde{Y} is the normalization of the image of Y' under the linear map extending $x'_n \mapsto x_n$.*

Proof. Let $\Delta > 0$. If K is the basis constant of (e_n) , then by Lemma 1.3.5 in [2], the basis constant of X is $\leq K$. Pick $\Gamma = (\gamma_n) > 0$ with $\sum_{n \geq m} \gamma_n \leq \min\{1/6K, \delta_m/8K\}$. For $X' = (x'_n)$ with $d(X', X) \leq \Gamma$, consider the map on the completions $T : \overline{\langle X \rangle} \rightarrow \overline{\langle X' \rangle}$ extending $x_n \mapsto x'_n$. T is a bounded linear isomorphism, as whenever $v = \sum a_n x_n \in \overline{\langle X \rangle}$,

$$\begin{aligned} \|Tv\| - \|v\| &\leq \|Tv - v\| \leq \left\| \sum a_n x'_n - \sum a_n x_n \right\| \leq \sup_n |a_n| \sum \|x'_n - x_n\| \\ &\leq 2K\|v\| \sum \|x'_n - x_n\| \leq 1/3\|v\|, \end{aligned}$$

and so $\|T\| \leq 4/3$. Using that $1/\|T^{-1}\| = \inf_{\|v\|=1} \|Tv\|$, we also have

$$\|T^{-1}\| \leq \frac{1}{1 - \sup_{\|v\|=1} \|v - Tv\|} \leq \frac{1}{1 - 1/3} = 3/2.$$

As the basis constant for X' is also $\leq K$, for $v' = \sum_{n \geq m} a_n x'_n \in \overline{\langle X' \rangle}$, we have

$$\begin{aligned} \|T^{-1}v' - v'\| &\leq \left\| \sum_{n \geq m} a_n x_n - \sum_{n \geq m} a_n x'_n \right\| \leq \sup_n |a_n| \sum_{n \geq m} \|x_n - x'_n\| \\ &\leq 2K\|v'\| \sum_{n \geq m} \|x_n - x'_n\| \leq \delta_m/4\|v'\|. \end{aligned}$$

If v' is a unit vector, then we moreover have that

$$\left| 1 - \frac{1}{\|T^{-1}v'\|} \right| \leq \|T\| \|T^{-1}v' - v'\| \leq (4/3)(\delta_m/4) \leq \delta_m/3.$$

For $Y' = (y'_m) \in [X']$, we claim $d(Y', \tilde{Y}) \leq \Delta$, where \tilde{Y} is the normalization of $Y = (y_m) = (T^{-1}(y'_m))$. Observe that

$$\|y_m - \frac{1}{\|y_m\|} y_m\| \leq \left| 1 - \frac{1}{\|T^{-1}(y'_m)\|} \right| \|T^{-1}(y'_m)\| \leq (\delta_m/3)(3/2) = \delta_m/2.$$

Thus, for all m ,

$$\|y'_m - \frac{1}{\|y_m\|}y_m\| = \|y'_m - y_m\| + \|y_m - \frac{1}{\|y_m\|}y_m\| \leq \delta_m.$$

□

The following lemma expresses the uniform continuity of the games $F[X]$ and $G[X]$.

Lemma 3.8.16. *Let $\mathbb{A} \subseteq \text{bb}_1^\infty(E)$ and $\Delta > 0$. There is a $\Gamma > 0$ such that whenever $X \in \text{bb}_1^\infty(E)$ is such that I (respectively, II) has a strategy in $F[X]$ (respectively, $G[X]$) for playing into \mathbb{A} and $d(X, X') \leq \Gamma$, then I (respectively, II) has a strategy in $F[X']$ (respectively, $G[X']$) for playing into \mathbb{A}_Δ .*

Proof. Take $\Gamma > 0$ as in Lemma 3.8.15. Suppose I has a strategy σ in $F[X]$ for playing into \mathbb{A} and $d(X, X') \leq \Gamma$. We define a strategy σ' for I in $F[X']$. Let $\sigma'(\emptyset) = \sigma(\emptyset)$. Inductively, suppose that $\sigma'(y'_0, \dots, y'_k)$ has been defined and is equal to $\sigma(y_0, \dots, y_k)$, where y_0, \dots, y_k is a valid play by II in $F[X]$ against σ , and $\|y'_i - y_i\| \leq \gamma_i$ for $0 \leq i \leq k$. Suppose that $y'_{k+1} > \sigma'(y'_0, \dots, y'_k)$ in $S(\langle X' \rangle)$. By our choice of Γ , there is a $y_{k+1} > \sigma'(y'_0, \dots, y'_k) = \sigma(y_0, \dots, y_k)$ in $S(\langle X \rangle)$ with $\|y'_{k+1} - y_{k+1}\| \leq \gamma_{k+1}$. Let $\sigma'(y'_0, \dots, y'_k, y'_{k+1}) = \sigma(y_0, \dots, y_k, y_{k+1})$. It follows that σ' is a strategy for playing into \mathbb{A}_Δ .

Suppose that II has a strategy α in $G[X]$ for playing into \mathbb{A} and $d(X, X') \leq \Gamma$. Let $T : \langle X \rangle \rightarrow \langle X' \rangle$ be as in the proof of Lemma 3.8.15. We define a strategy α' for II in $G[X']$. Suppose that I begins by playing $Y'_0 \in [X']$. Let $\alpha'(Y'_0) = \widetilde{T}(\alpha(\widetilde{T}^{-1}(Y'_0)))$, where \widetilde{T} and \widetilde{T}^{-1} indicate taking normalizations. Continue in this fashion. Then, α is a strategy for playing into \mathbb{A}_Δ . □

Theorem 3.8.17. *Assume that there is a supercompact cardinal. Let $\mathcal{F} \subseteq \text{bb}_1^\infty(E)$ be a strategic (p^*) -filter. Then, every set $\mathbb{A} \subseteq \text{bb}_1^\infty(E)$ in $\mathbf{L}(\mathbb{R})$ is \mathcal{F} -weakly Ramsey.*

Proof. Let $\mathbb{A} \subseteq \text{bb}_1^\infty(E)$ be in $\mathbf{L}(\mathbb{R})$, $X \in \mathcal{F}$, and $\Delta > 0$. By Theorem 3.7.2, the set \mathbb{D} of all $Y \preceq X$ such that either I has a strategy in $F[Y]$ for playing into $(\mathbb{A}_{\Delta/2})^c$, or II has a strategy in $G[Y]$ for playing into $\mathbb{A}_{\Delta/2}$, is \preceq -dense open and in $\mathbf{L}(\mathbb{R})$. By Lemmas 3.4.4 and 3.8.14, there is a $Y \in \mathcal{F} \upharpoonright X$ such that II has a strategy for playing into \mathbb{D}_Γ , where Γ is as in Lemma 3.8.16, applied to $\Delta/4$. Since \mathcal{F} is strategic, there is a $Z \in \mathcal{F} \upharpoonright Y$ which is in \mathbb{D}_Γ . By our choice of Γ , then either I has a strategy in $F[Z]$ for playing into $((\mathbb{A}_{\Delta/2})^c)_{\Delta/4} \subseteq (\mathbb{A}_{\Delta/2})^c$, or II has a strategy in $G[Z]$ for playing into \mathbb{A}_Δ . In the latter case, we're done, and in the former case, we need only apply Theorem 3.8.9 and Lemma 3.8.12. \square

We will use the following analogue of Theorem 3.5.4, whose proof is similar and left to the reader.

Lemma 3.8.18. *For $\mathcal{H} \subseteq \text{bb}_1^\infty(E)$ a (p^*) -family, forcing with (\mathcal{H}, \preceq^*) adds no new reals and if $\mathcal{G} \subseteq \mathcal{H}$ is $\mathbf{L}(\mathbb{R})$ -generic for (\mathcal{H}, \preceq^*) , \mathcal{G} will be a (p^*) -filter. If \mathcal{H} is strategic (respectively, spread), then \mathcal{G} will also be strategic (respectively, spread).*

Theorem 3.8.19. *Assume that there is a supercompact cardinal. Let $\mathcal{H} \subseteq \text{bb}_1^\infty(E)$ be a strategic (p^*) -family. Then, every set $\mathbb{A} \subseteq \text{bb}_1^\infty(E)$ in $\mathbf{L}(\mathbb{R})$ is \mathcal{H} -weakly Ramsey.*

Proof. The proof is similar to that of Theorem 3.1.3, using Lemma 3.8.18 and Theorem 3.8.17. \square

Some of the above can be simplified in the case when the family \mathcal{H} in question is *invariant under small perturbations*, that is, there is some $\Delta > 0$ so that $\mathcal{H}_\Delta = \mathcal{H}$. The reason lies in the following fact:

Proposition 3.8.20. *If \mathcal{H} is a strategic (p^*) -family which is invariant under small perturbations, then \mathcal{H} is a (p^+) -family.*

*Proof.*¹² Let $D \subseteq S(E)$ be \mathcal{H} -dense below some $X \in \mathcal{H}$ and put $\mathbb{D} = \{Y \preceq X : S(\langle Y \rangle) \subseteq D\}$. Take $\Delta > 0$ so that $\mathcal{H}_\Delta = \mathcal{H}$. Note that \mathbb{D} is closed and thus it and its continuous images are universally Baire. Let \mathcal{G} be a \mathbf{V} -generic filter for (\mathcal{H}, \preceq^*) which contains X , so that by Lemma 3.8.18, \mathcal{G} is a strategic (p^*) -filter in $\mathbf{V}[\mathcal{G}]$. By Lemma 3.8.14 in $\mathbf{V}[\mathcal{G}]$, there is a $Y \in \mathcal{G} \upharpoonright X$ so that II has a strategy in $G[Y]$ for playing into one of $(\mathbb{D}_{\Delta/8})^c$ or \mathbb{D}_Δ . However, as I has a strategy in $G[Y]$ for playing into \mathbb{D} and $(\mathbb{D}_{\Delta/8})^c \subseteq \mathbb{D}^c$ by Lemma 3.8.1(c), II 's strategy must be for playing into \mathbb{D}_Δ . Since forcing with (\mathcal{H}, \preceq^*) added no new reals, such a strategy must exist in \mathbf{V} (we are using Lemma 3.4.7 implicitly here). As \mathcal{H} is strategic and $\mathcal{H}_\Delta = \mathcal{H}$, we have that $\mathcal{H} \cap \mathbb{D} \neq \emptyset$, showing that \mathcal{H} is full. \square

We now extend these principles to Banach spaces. In what follows, B is a Banach space with normalized Schauder basis (e_n) . We say that a countable subfield F of \mathbb{R} (or \mathbb{C}) is *suitable* if the norm on E_F , the F -span of (e_n) , takes values in F . Let $\langle X \rangle_F$ the F -span of $X \in \text{bb}^\infty(E_F)$. If V is a subspace of B , again let $S(V) = \{v \in V : \|v\| = 1\}$.

Let $\text{bb}_1^\infty(B)$ be the set of all infinite block sequences (with respect to (e_n)) in B , which we endow with the Polish topology inherited from B^ω . The relations \preceq and \preceq^* extend to $\text{bb}_1^\infty(B)$. For $Y \in \text{bb}_1^\infty(B)$, let $[Y]^* = \{Z \in \text{bb}_1^\infty(B) : Z \preceq Y\}$. We denote by $G^*[Y]$ the *Gowers game* defined as before, except that the players may now play real (complex) block sequences and block vectors. The notions of *family*, (p) -*family*, *spread*, and *strategic* are defined as before, with appropriate modifications for real (complex) scalars.

Strategic families in $\text{bb}_1^\infty(B)$ arise naturally from strategic families in

¹²We suspect that an elementary proof of this result can be found and that “strategic” can be relaxed to “spread”.

$\text{bb}_1^\infty(E_F)$: Given a strategic $\mathcal{H} \subseteq \text{bb}_1^\infty(E_F)$, if $\widehat{\mathcal{H}}$ is invariant under small perturbations and equal to the \preceq -upwards closure of \mathcal{H}_Δ (taken in $\text{bb}_1^\infty(B)$) for some small $\Delta > 0$, then $\widehat{\mathcal{H}}$ is strategic. This follows from the fact that Lemma 3.8.15 and the proof of Lemma 3.8.16 can be carried out in B .

Definition 3.8.21. We say that \mathcal{H} is *almost full* if whenever $D \subseteq S(B)$ is closed and \mathcal{H} -dense below some $X \in \mathcal{H}$ (that is, for all $Y \in \mathcal{H} \upharpoonright X$, there is a $Z \preceq Y$ with $S(\langle Z \rangle) \subseteq D$), then for any $\epsilon > 0$, there is a $Y \in \mathcal{H} \upharpoonright X$ with $S(\overline{\langle Y \rangle}) \subseteq D_\epsilon$.

Again, we call an almost full (p) -family in $\text{bb}_1^\infty(B)$ a (p^*) -family. The meaning should be clear from context.

Definition 3.8.22. Given a family $\mathcal{H} \subseteq \text{bb}_1^\infty(B)$, a set $\mathbb{A} \subseteq \text{bb}_1^\infty(B)$ is \mathcal{H} -weakly Ramsey if for every $\Delta > 0$ and $X \in \mathcal{H}$, there is a $Y \in \mathcal{H} \upharpoonright X$ such that either

- (i) $[Y]^* \subseteq \mathbb{A}^c$, or
- (ii) II has a strategy in $G^*[Y]$ for playing into \mathbb{A}_Δ .

Proving Theorems 3.1.4 and 3.1.5 amounts to showing that for spread (respectively, strategic) (p^*) -families $\mathcal{H} \subseteq \text{bb}_1^\infty(B)$ which are invariant under small perturbations, analytic (respectively, $\mathbf{L}(\mathbb{R})$) sets are \mathcal{H} -weakly Ramsey.

Lemma 3.8.23. Let F be a suitable subfield of \mathbb{R} (or \mathbb{C}). If $X_0 \succeq X_1 \succeq X_2 \succeq \dots$ is a \preceq -decreasing sequence in $\text{bb}_1^\infty(E_F)$, $X \in \text{bb}_1^\infty(B)$ is such that $X \preceq X_n$ for all n , and $\Delta > 0$, then there is an $X' \in \text{bb}_1^\infty(E_F)$ with $X' \in [X]_\Delta$, and $X' \preceq^* X_n$ for all n .

Proof. Let (X_n) , X and $\Delta > 0$ be as described, say with $X = (x_n)$. We construct $X' = (x'_n)$ as follows: There is an $M_0 \in \omega$ so that $\overline{\langle X/M_0 \rangle_F} \subseteq \overline{\langle X_0 \rangle_F}$. Let x_{n_0} be the first entry of X/M_0 . Pick a unit vector $x'_0 \in \langle X_0 \rangle_F$ such that $d(x_{n_0}, x'_0) \leq \delta_0$. Continue inductively. At stage k , we have chosen $M_0 < \dots < M_k$ and $x'_0 < \dots <$

x'_k so that if x_{n_i} is the first entry of X/M_i , then $x'_i \in \langle X_i \rangle_F$ and $d(x_{n_i}, x'_i) \leq \delta_i$, for $i \leq k$. By construction, $X'/n \preceq X_n$ for all n , and $X' \in [X]_\Delta$. \square

Lemma 3.8.24. *If $\mathcal{H} \subseteq \text{bb}_1^\infty(B)$ is a (p^*) -family which is invariant under small perturbations, then $\mathcal{H} \cap \text{bb}_1^\infty(E_F)$ is a (p^*) -family for any suitable subfield F of \mathbb{R} (or \mathbb{C}). If \mathcal{H} is spread (strategic), then so is $\mathcal{H} \cap \text{bb}_1^\infty(E_F)$.*

Proof. Let \mathcal{H} and F be as described and put $\tilde{\mathcal{H}} = \mathcal{H} \cap \text{bb}_1^\infty(E_F)$. Lemma 3.8.23 implies that $\tilde{\mathcal{H}}$ is a (p) -family. To see that $\tilde{\mathcal{H}}$ is almost full, let $D \subseteq S(E_F)$ be $\tilde{\mathcal{H}}$ -dense below $X \in \tilde{\mathcal{H}}$ and take $\epsilon > 0$. Consider $\overline{D_{\epsilon/3}} \subseteq S(B)$. For $\Delta = (\epsilon/3, \epsilon/3, \dots)$, let Γ be as in Lemma 3.8.15. For any $Y \in \mathcal{H} \upharpoonright X$, there is a $Y' \in \tilde{\mathcal{H}} \upharpoonright X$ with $d(Y, Y') \leq \Gamma$ and a $Z' \preceq Y'$ with $S(\langle Z' \rangle) \subseteq D$. By our choice of Γ , there is a $Z \in [Y]^*$ with $S(\langle Z \rangle) \subseteq D_{\epsilon/3}$ and so $S(\langle \overline{Z} \rangle) \subseteq \overline{D_{\epsilon/3}}$. Thus, $\overline{D_{\epsilon/3}}$ is \mathcal{H} -dense below X . By almost fullness of \mathcal{H} , there is a $W \in \mathcal{H} \upharpoonright X$ with $S(\langle \overline{W} \rangle) \subseteq (\overline{D_{\epsilon/3}})_{\epsilon/3}$. Then, one can find a $W' \in \tilde{\mathcal{H}} \upharpoonright X$ with $S(\langle W' \rangle) \subseteq D_\epsilon$, showing that $\tilde{\mathcal{H}}$ is almost full.

To see that \mathcal{H} being strategic implies that $\tilde{\mathcal{H}}$ is strategic, let α be a strategy for Π in $G[X]$, with $X \in \tilde{\mathcal{H}}$. Define a strategy α' in $G^*[X]$ which is equal to α on their shared domain and otherwise plays so that the outcomes are sufficiently small (using Lemma 3.8.15 and our assumption about \mathcal{H}) perturbations of outcomes of α . Then, if any outcome of α' is in \mathcal{H} , an outcome of α must be in $\tilde{\mathcal{H}}$. The proof for being spread is left to the reader. \square

Proof of Theorem 3.1.4. Suppose that $\mathbb{A} \subseteq \text{bb}_1^\infty(B)$ is analytic, $\Delta > 0$, and $X \in \mathcal{H}$ is such that for no $Y \in \mathcal{H} \upharpoonright X$ is $[Y]^* \subseteq \mathbb{A}^c$. Let F be a suitable field for (e_n) . Let $\tilde{\mathcal{H}} = \mathcal{H} \cap \text{bb}_1^\infty(E_F)$. If there was some $Y \in \tilde{\mathcal{H}} \upharpoonright X$ with $[Y] \subseteq (\mathbb{A}_{\Delta/3})^c \cap \text{bb}_1^\infty(E_F)$, then $[Y]^* \subseteq ((\mathbb{A}_{\Delta/3})^c)_{\Delta/3} \subseteq \mathbb{A}^c$, contrary to our assumption. Thus, by Lemma 3.8.24 and Theorem 3.8.13, there is a $Y \in \tilde{\mathcal{H}} \upharpoonright X$ such that Π has a strategy in

$G[Y]$ for playing into $\mathbb{A}_{\Delta/2} \cap \text{bb}_1^\infty(E_F)$. Easy perturbation arguments show that Π has a strategy in $G^*[Y]$ for playing into \mathbb{A}_Δ . \square

Proof of Theorem 3.1.5. The proof is similar to that of Theorem 3.1.4, using Theorem 3.8.19, or alternatively, Proposition 3.8.20 and Theorem 3.1.3. \square

The following is an analytical example of a strategic (p^*) -family, which, though trivial in the sense that it is hereditary, we hope suggests further applications:

Example 3.8.25. Suppose that B contains a normalized block sequence X equivalent to the standard basis of c_0 or ℓ^p for $1 \leq p < \infty$. Let \mathcal{H} be the set of all block sequences in B which have a further block subsequence equivalent to X . Then, \mathcal{H} is a strategic (p^*) -family which is invariant under small perturbations. This follows from the block homogeneity of the spaces c_0 and ℓ^p , Lemma 2.1.1 in [2].

We end this section by noting that being spread is analogous to the following property of coideals: A coideal \mathcal{H} in $[\omega]^\omega$ has the (q^+) -property if for every $x \in \mathcal{H}$ and partition $x = \bigcup_m I_m$ into finite sets, there exists a $y \in \mathcal{H} \upharpoonright x$ such that $\forall m (|y \cap I_m| \leq 1)$. Let's say (temporarily) that a coideal \mathcal{H} in $[\omega]^\omega$ is *spread* if for every $x \in \mathcal{H}$ and sequence of finite sets $I_0 < I_1 < I_2 < \dots$ in ω , there exists a $y \in \mathcal{H} \upharpoonright x$ such that $\forall n \exists m (I_0 < y_n < I_m < y_{n+1})$, where (y_n) is the increasing enumeration of y . In fact, this is equivalent to the (q^+) -property.

Proposition 3.8.26. *For a coideal \mathcal{H} in $[\omega]^\omega$, the following are equivalent.*

- (i) \mathcal{H} has the (q^+) -property.
- (ii) For every $x \in \mathcal{H}$ and sequence of finite sets $I_0 < I_1 < I_2 < \dots$ in ω , there exists a $y \in \mathcal{H} \upharpoonright x$ such that $\forall m (|y \cap I_m| \leq 1)$

(iii) \mathcal{H} is spread.

Proof. (i \Rightarrow ii): This is trivial.

(ii \Rightarrow iii): Let $x \in \mathcal{H}$ and $I_0 < I_1 < I_2 < \dots$ be a sequence of finite sets in ω . Let $y \in \mathcal{H} \upharpoonright x$ be as in (ii). We may assume that $I_0 < y_0$. We partition $y = u \cup v$ as follows: $u_n = y_{2n}$ and $v_n = y_{2n+1}$ for all n , where (y_n) is the increasing enumeration of y . For every n , since $u_n = y_{2n}, y_{2n+1}$, and $u_{n+1} = y_{2n+1}$ must be contained in three distinct I_k 's (in order), the middle one must separate u_n and u_{n+1} , that is, there is an m such that $I_0 < u_n < I_m < u_{n+1}$. Similarly for the v_n . Since \mathcal{H} is coideal, one of u or v must be in \mathcal{H} .

(iii \Rightarrow i): Let $x \in \mathcal{H}$ and $x = \bigcup_m I_m$ be a partition of x into finite sets. We define an interval partition $\omega = \bigcup_k J_k$ as follows: $J_0 = [0, \max I_0]$. Let J_1 be the smallest interval immediately above J_0 such that $J_0 \cup J_1$ covers all I_m for which $I_m \cap J_0 \neq \emptyset$. Continue in this fashion. J_{k+1} is the smallest interval immediately above J_k such that $J_0 \cup \dots \cup J_k \cup J_{k+1}$ covers all I_m for which $I_m \cap (J_0 \cup \dots \cup J_k) \neq \emptyset$. Let $y \in \mathcal{H} \upharpoonright x$ be as in the definition of spread applied to $J_0 < J_1 < \dots$. Towards a contradiction, suppose that $y_i < y_j$ are both in some I_m . Let n be the least such that $I_m \subseteq J_0 \cup \dots \cup J_n$. We may assume $n > 0$ (otherwise, we are done). Since n is the least such, $I_m \cap J_{n-1} \neq \emptyset$, but (if $n > 1$) $I_m \cap (J_0 \cup \dots \cup J_{n-2}) = \emptyset$. Thus, $I_m \subseteq J_{n-1} \cup J_n$. But then, y_i and y_j are in adjacent intervals, contrary to y witnessing that \mathcal{H} is spread. \square

3.9 Pure states and projections in the Calkin algebra

Given a Banach space, one might wish to develop a notion of forcing with block sequences “modulo small perturbation” and then prove an analogue of Theorem 3.1.2, characterizing $L(\mathbb{R})$ -generic filters.¹³ We focus on a particular variant of this which is of significant interest.

Let H be a complex infinite-dimensional separable Hilbert space with orthonormal basis (e_n) . Note that any normalized block sequence (with respect to (e_n)) is necessarily orthonormal. Throughout, E will denote the $\overline{\mathbb{Q}}$ -linear span of (e_n) in H , where $\overline{\mathbb{Q}}$ is the algebraic closure of \mathbb{Q} , and $\text{bb}_1^\infty(E)$ the space of infinite normalized block sequences in E . For $X \in \text{bb}_1^\infty(E)$, $\langle X \rangle$ is the $\overline{\mathbb{Q}}$ -span of X .

For $X \in \text{bb}_1^\infty(E)$, let P_X be the orthogonal projection onto $\overline{\langle X \rangle}$. Note that, for $X, Y \in \text{bb}_1^\infty(E)$, $X \preceq Y$ if and only if $P_X \leq P_Y$ in the usual ordering of projections (that is, $P \leq Q$ if $\text{ran}(P) \subseteq \text{ran}(Q)$, or equivalently, $PQ = P$). We call such projections *block projections*.

Let $\mathcal{B}(H)$ be the C^* -algebra of bounded operators on H and $\mathcal{K}(H)$ the ideal of compact operators on H . The quotient $\mathcal{C}(H) = \mathcal{B}(H)/\mathcal{K}(H)$ is also a C^* -algebra, called the *Calkin algebra*. We write $\pi : \mathcal{B}(H) \rightarrow \mathcal{C}(H)$ for the quotient map.

Denote by $\mathcal{P}(H)$ (respectively, $\mathcal{P}_\infty(H)$) the set of (respectively, infinite-rank) projections in $\mathcal{B}(H)$, and by $\mathcal{P}(\mathcal{C}(H))$ (respectively, $\mathcal{P}(\mathcal{C}(H))^+$) the set of (respectively, nonzero) projections, i.e., self-adjoint idempotents, in $\mathcal{C}(H)$. By Proposition 3.1 in [88], $\mathcal{P}(\mathcal{C}(H)) = \pi(\mathcal{P}(H))$. The ordering \leq on $\mathcal{P}(\mathcal{C}(H))$ is inherited from the ordering on $\mathcal{P}(H)$.

¹³There are obstacles to this being a meaningful endeavor in general, e.g., in a hereditarily indecomposable Banach space, the collection of *all* infinite-dimensional subspaces modulo small perturbations forms a filter, cf. (iii) on p. 820 of [34], and is thus trivial as a forcing notion.

- Definition 3.9.1.** (a) For projections $P, Q \in \mathcal{P}(H)$, we write $P \leq_{\text{ess}} Q$ if $\pi(P) \leq \pi(Q)$ in $\mathcal{P}(\mathcal{C}(H))$ and $P \equiv_{\text{ess}} Q$ if $\pi(P) = \pi(Q)$.
- (b) For $X, Y \in \text{bb}_1^\infty(E)$, we write $X \leq_{\text{ess}} Y$ if $P_X \leq_{\text{ess}} P_Y$ and $X \equiv_{\text{ess}} Y$ if $P_X \equiv_{\text{ess}} P_Y$.

The last sentence of the following lemma requires a slight modification of the original proof and is left to the reader.

Lemma 3.9.2 (Proposition 3.3 in [88]). *For P and Q projections on H , the following are equivalent:*

- (i) $P \leq_{\text{ess}} Q$.
- (ii) *For every $\epsilon > 0$, there is a finite-codimensional subspace V of $\text{ran}(P)$ such that every unit vector $v \in V$ satisfies $d(v, \text{ran}(Q)) \leq \epsilon$.*

In the event that P and Q are block projections, one can replace “finite-codimensional subspace” in (ii) with “tail subspace”.

Lemma 3.9.3. *Suppose that $\Delta = (\delta_n) > 0$ is summable and P and Q are projections on H whose ranges have orthonormal bases (x_n) and (y_n) respectively. If for all n , $\|x_n - y_n\| \leq \delta_n$, then $P \equiv_{\text{ess}} Q$.*

Proof. Assuming that for all n , $\|x_n - y_n\| \leq \delta_n$, we will show that $P \leq_{\text{ess}} Q$. The result follows by symmetry. Let $\epsilon > 0$ and choose an N such that $\sum_{n \geq N} \delta_n \leq \epsilon$. Let $V = \overline{\langle (x_n)_{n \geq N} \rangle}$, a finite-codimensional subspace of $\text{ran}(P)$. If $v \in V$ is a unit vector, say with $v = \sum_{n \geq N} a_n x_n$, then for $y = \sum_{n \geq N} a_n y_n \in \text{ran}(Q)$, we have

$$\|v - y\| = \left\| \sum_{n \geq N} a_n (x_n - y_n) \right\| \leq \sum_{n \geq N} \|a_n (x_n - y_n)\| \leq \sum_{n \geq N} \|a_n\| \delta_n \leq \epsilon.$$

The claim follows by Lemma 3.9.2. □

It follows that \equiv_{ess} -invariant families in $\text{bb}_1^\infty(E)$ are invariant under small perturbations. The following observation can be proved using Lemma 3.9.3 and standard manipulations with basic sequences (cf. Proposition 1.3.10 in [2]).

Lemma 3.9.4. *The set of block projections is dense in $(\mathcal{P}_\infty(H), \leq_{\text{ess}})$.* □

Thus, $(\mathcal{P}(\mathcal{C}(H))^+, \leq)$, $(\mathcal{P}_\infty(H), \leq_{\text{ess}})$, and $(\text{bb}_1^\infty(E), \leq_{\text{ess}})$ are equivalent as notions of forcing. It is for this reason that we focus on $(\text{bb}_1^\infty(E), \leq_{\text{ess}})$.

Lemma 3.9.5. *If $X_0 \succeq X_1 \succeq X_2 \succeq \dots$ is a decreasing sequence in $\text{bb}_1^\infty(E)$ and $X \in \text{bb}_1^\infty(E)$ is such that $X \leq_{\text{ess}} X_n$ for all n , then there is an $X' \leq_{\text{ess}} X$ such that $X' \preceq^* X_n$ for all n .*

Proof. This can be proved using Lemmas 3.9.2 and 3.9.3 in a way similar to the proof of Lemma 3.8.23. □

Clearly, any \preceq -dense subset of $\text{bb}_1^\infty(E)$ is also \leq_{ess} -dense; the following lemma is a partial converse to this.¹⁴

Lemma 3.9.6. *If $\mathcal{D} \subseteq \text{bb}_1^\infty(E)$ is \leq_{ess} -dense open, then it is \preceq -dense open.*

Proof. Suppose $\mathcal{D} \subseteq \text{bb}_1^\infty(E)$ is \leq_{ess} -dense open. Given any $X \in \text{bb}_1^\infty(E)$, there is a $Y \in \mathcal{D}$ with $Y \leq_{\text{ess}} X$. Applying Lemma 3.9.5 with $X_n = X$ for all n , there is a $Y' \leq_{\text{ess}} Y$ with $Y' \preceq X$. Then, $Y' \in \mathcal{D}$. □

We can now establish Theorem 3.1.6, an analogue of Theorem 3.1.2 for $(\mathcal{P}_\infty(E), \leq_{\text{ess}})$. We first prove a more general result.

¹⁴Lemma 3.9.6 implies that if \mathcal{G} is a generic filter for $(\text{bb}_1^\infty(E), \preceq^*)$, then its \leq_{ess} -upwards closure is a generic filter for $(\text{bb}_1^\infty(E), \leq_{\text{ess}})$.

Theorem 3.9.7. (a) If \mathcal{G} is an $\mathbf{L}(\mathbb{R})$ -generic filter for $(\text{bb}_1^\infty(E), \leq_{\text{ess}})$, then \mathcal{G} is a strategic (p^+) -family.

(b) Assume that there is a supercompact cardinal. If $\mathcal{G} \subseteq \text{bb}_1^\infty(E)$ is a strategic (p^*) -family which is also a \leq_{ess} -filter, then \mathcal{G} is $\mathbf{L}(\mathbb{R})$ -generic for $(\text{bb}_1^\infty(E), \leq_{\text{ess}})$.

Proof. (a) Let \mathcal{G} be as described. Clearly, it is a family. To see that it is full, suppose that $D \subseteq S(E)$ is \mathcal{G} -dense below some $X \in \mathcal{G}$. Let

$$\mathcal{D}_0 = \{Z : \langle Z \rangle \subseteq D \text{ or } \forall V \preceq X (\langle V \rangle \subseteq D \rightarrow V \perp Z)\},$$

where \perp denotes incompatibility with respect to \preceq . \mathcal{D}_0 is \preceq -dense open by Lemma 3.2.6, thus \leq_{ess} -dense as well, and clearly in $\mathbf{L}(\mathbb{R})$, so there is a $Z \in \mathcal{D}_0 \cap (\mathcal{G} \upharpoonright X)$. Then, there is a $Z' \preceq Z \preceq X$ with $S(\langle Z' \rangle) \subseteq D$, so we have that $\langle Z \rangle \subseteq D$, showing that \mathcal{G} is full.

To see that \mathcal{G} is a (p) -family, suppose $X_0 \succeq X_1 \succeq X_2 \succeq \dots$ in \mathcal{G} . Let

$$\mathcal{D}_1 = \{Y : \forall n (Y \preceq^* X_n) \text{ or } \exists n (Y \perp_{\text{ess}} X_n)\},$$

where \perp_{ess} denotes incompatibility with respect to \leq_{ess} . We want to show that \mathcal{D}_1 is \leq_{ess} -dense. The set

$$\mathcal{D}'_1 = \{Y : \forall n (Y \leq_{\text{ess}} X_n) \text{ or } \exists n (Y \perp_{\text{ess}} X_n)\}$$

is \leq_{ess} -dense open. Then, given any X , we can find a $Y \in \mathcal{D}'_1$ which is $\leq_{\text{ess}} X$. If $Y \perp_{\text{ess}} X_n$ for some n , we're done. Otherwise, $Y \leq_{\text{ess}} X_n$ for all n , and we can apply Lemma 3.9.5 to find a $Y' \leq_{\text{ess}} Y$ with $Y' \preceq^* X_n$ for all n . Such a Y' is in \mathcal{D}_1 , verifying that this set is \leq_{ess} -dense. As \mathcal{D}_1 is in $\mathbf{L}(\mathbb{R})$, $\mathcal{G} \cap \mathcal{D}_1 \neq \emptyset$, and anything in this intersection must be a diagonalization of (X_n) . It is likewise easy to see that \mathcal{G} must be strategic.

(b) Let $\mathcal{D} \subseteq \text{bb}_1^\infty(E)$ be \leq_{ess} -dense open and in $\mathbf{L}(\mathbb{R})$. By Lemma 3.9.6, \mathcal{D} is also \preceq -dense open. For $\Delta > 0$ summable, $\mathcal{D}_\Delta = \mathcal{D}$ by Lemma 3.9.3. Thus, by Theorem 3.8.13, there is an $X \in \mathcal{H}$ such Π has a strategy for playing into \mathcal{D} . Since \mathcal{G} is strategic, it follows that $\mathcal{G} \cap \mathcal{D} \neq \emptyset$. \square

Proof of Theorem 3.1.6. The (\Rightarrow) direction is proved by a straightforward verification of the relevant sets being \preceq -dense open, thus \leq_{ess} -dense by Lemma 3.9.6. The (\Leftarrow) direction follows from Theorem 3.9.7(b) or Theorem 3.1.5. \square

We conclude this section by describing a hoped-for application of our machinery and its limitations. A *state* τ on $\mathcal{B}(H)$ is a linear functional on $\mathcal{B}(H)$ which is *positive*, that is, $\tau(T^*T) \geq 0$ for all T , and satisfies $\tau(I) = 1$, where I is the identity operator. The set of states forms a weak*-compact convex subset of the dual of $\mathcal{B}(H)$ and thus has extreme points, called *pure states*. These definitions generalize to arbitrary unital C*-algebras, including $\mathcal{C}(H)$.

A state on $\mathcal{B}(H)$ is *singular* if it vanishes on $\mathcal{K}(H)$. Composing with the quotient map $\pi : \mathcal{B}(H) \rightarrow \mathcal{C}(H)$ yields a bijective correspondence between singular pure states on $\mathcal{B}(H)$ and pure states on $\mathcal{C}(H)$.

For any choice of orthonormal basis (f_k) for H , and any ultrafilter \mathcal{U} on ω , the functional defined by $\tau_{\mathcal{U}}(T) = \lim_{k \rightarrow \mathcal{U}} \langle T f_k, f_k \rangle$ is a pure state which is singular if and only if \mathcal{U} is nonprincipal (cf. Theorem 4.21 and Example 6.1 in [27]). Such pure states are said to be *diagonalizable*. On an abelian C*-algebra, pure states coincide with characters, so the aforementioned $\tau_{\mathcal{U}}$ restricts to a pure state on the atomic maximal abelian self-adjoint subalgebra (or *masa*) generated by the rank 1 projections corresponding to the f_k . The following problem asks to what extent this is true of all pure states:

Problem (Kadison–Singer [47]). *Does every pure state on $\mathcal{B}(H)$ restrict to a pure state on some (atomic or continuous)¹⁵ masa?*

Anderson conjectured that not only is the answer to this question “yes”, but that every pure state is of the form $\tau_{\mathcal{U}}$, for some choice of orthonormal basis (f_k) and ultrafilter \mathcal{U} :

Conjecture (Anderson [3]). *Every pure state on $\mathcal{B}(H)$ is diagonalizable.*

Akemann and Weaver [1] showed that the above problem of Kadison and Singer has a negative answer, and thus Anderson’s conjecture is false, assuming CH. It remains an open question whether Anderson’s conjecture is consistent with ZFC.

By the recent positive solution [62] to the Kadison–Singer problem regarding *extensions* of pure states (which differs from the above), Anderson’s conjecture is equivalent to saying that every pure state on $\mathcal{B}(H)$ restricts to a pure state on some atomic masa.

Following [13], we say that a subset $\mathcal{F} \subseteq \mathcal{P}(\mathcal{C}(H))^+$ is *centered*¹⁶ if every finite subset of \mathcal{F} has a lower bound in $\mathcal{P}(\mathcal{C}(H))^+$. \mathcal{F} is *linked* if every pair of elements in \mathcal{F} has a lower bound in $\mathcal{P}(\mathcal{C}(H))^+$. *Maximal centered* has the obvious meaning. Similarly for \leq_{ess} -centered, \leq_{ess} -linked, and *maximal \leq_{ess} -centered* for subsets of $\text{bb}_1^\infty(E)$.

Theorem 3.9.8 (Farah–Weaver, Theorem 6.42 in [27]). *There is a bijective correspondence between nonsingular pure states τ on $\mathcal{B}(H)$ and maximal centered subsets of $\mathcal{P}(\mathcal{C}(H))^+$ via $\tau \mapsto \mathcal{F}_\tau = \{p \in \mathcal{P}(\mathcal{C}(H))^+ : \tau(p) = 1\}$.*

¹⁵A *continuous masa* in $\mathcal{B}(H)$ is one isomorphic to $L^\infty([0, 1])$ acting by diagonal operators on $H = L^2[0, 1]$.

¹⁶These were called *quantum filters* by Farah and Weaver [27].

If $\mathcal{F} = \mathcal{F}_\tau$ as above and τ fails to be diagonalizable, we say that \mathcal{F} yields a counterexample to Anderson's conjecture.

Theorem 3.9.9 (essentially Farah–Weaver, cf. Theorem 6.46 in [27]). *If \mathcal{G} is \mathbf{V} -generic for $\mathcal{P}(\mathcal{C}(H))^+$, then \mathcal{G} is a maximal centered set which yields a counterexample to Anderson's conjecture.*

We will present a proof of this result in the supplementary §3.12. We note that the proof result uses much less than full genericity, or even genericity over $\mathbf{L}(\mathbb{R})$. By considering the complexity of the dense sets involved in the proof, we obtain Theorem 3.1.7:

Proof of Theorem 3.1.7. Let $\mathcal{H} \subseteq \text{bb}_1^\infty(E)$ be a spread (p^*) -family which is \leq_{ess} -centered and $\widehat{\mathcal{H}}$ the upwards closure of $\pi(\mathcal{H})$ in $\mathcal{P}(\mathcal{C}(H))^+$. First, we claim that $\widehat{\mathcal{H}}$ is a maximal centered set. Clearly, $\widehat{\mathcal{H}}$ is centered. For maximality, let $p \in \mathcal{P}(\mathcal{C}(H))^+$ be such that p is compatible with every finite subset of $\widehat{\mathcal{H}}$. Let $P \in \mathcal{P}(H)$ be such that $\pi(P) = p$ and define

$$\mathcal{D}_P = \{X : P_X \leq_{\text{ess}} P \text{ or } P_X \perp_{\text{ess}} P\},$$

which is a coanalytic and \leq_{ess} -dense open subset of $\text{bb}_1^\infty(H)$. By Lemma 3.9.6, \mathcal{D}_P is \preceq -dense open, so by Theorem 3.8.13, we can find a $Y \in \mathcal{H} \upharpoonright X$ with $Y \in \mathcal{D}_P$. It must then be the case that $P_Y \leq_{\text{ess}} P$ and so $p \in \widehat{\mathcal{H}}$.

To see that $\widehat{\mathcal{H}}$ yields a counterexample to Anderson's conjecture, we refer to §3.12 (or Theorem 6.46 in [27]) and omit the details here except to note that it suffices to show that \mathcal{H} meets the \leq_{ess} -dense open sets

$$\mathcal{D}_{\vec{J}} = \{X \in \text{bb}_1^\infty(E) : \forall n (\|P_{J_n \cup J_{n+1}}^{(f_k)} P_X\| < 1/2)\},$$

where $\vec{J} = (J_n)$ is a partition of ω into finite intervals J_n and $P_J^{(f_k)}$ denotes the orthogonal projection onto $\overline{\text{span}}\{f_k : k \in J\}$, for (f_k) an orthonormal basis of H . These sets are easily seen to be Borel and meeting them with \mathcal{H} uses the combination of Lemma 3.9.6 and Theorem 3.8.13 as before. \square

For spread (p^*) -families, being \leq_{ess} -linked implies being a \leq_{ess} -filter:

Lemma 3.9.10. *Let $\mathcal{H} \subseteq \text{bb}_1^\infty(E)$ be a spread (p^*) -family which is, moreover, \leq_{ess} -linked. Then, \mathcal{H} is a \leq_{ess} -filter.*

Proof. Let $X, Y \in \mathcal{H}$, and consider the set

$$\mathcal{D} = \{Z : (Z \leq_{\text{ess}} X \text{ and } Z \leq_{\text{ess}} Y) \text{ or } (Z \perp_{\text{ess}} X \text{ or } Z \perp_{\text{ess}} Y)\}.$$

It is easy to check that \mathcal{D} is coanalytic. Clearly \mathcal{D} is \leq_{ess} -dense open, thus \preceq -dense open by Lemma 3.9.6. By Theorem 3.8.13, there is a $Z \in \mathcal{H}$ with $Z \in \mathcal{D}$. Since \mathcal{D} is \leq_{ess} -linked, we must have that $Z \leq_{\text{ess}} X$ and $Z \leq_{\text{ess}} Y$. \square

By Lemma 3.9.10, the maximal centered sets in Theorem 3.1.7 are also filters in $\mathcal{P}(\mathcal{C}(H))^+$. The following result of Bice, using Shelah's model without p -points (VI. §4 in [77]), presents an obstacle to ZFC constructions.

Theorem 3.9.11 (Bice [13]). *It is consistent with ZFC that no maximal centered set in $\mathcal{P}(\mathcal{C}(H))^+$ is a filter.*

Consequently, we have:

Corollary 3.9.12. *It is consistent with ZFC that no spread (p^*) -family in $\text{bb}_1^\infty(E)$ can be \leq_{ess} -linked, and in particular, that there are no spread (p^*) -filters.*

3.10 Further questions

Despite our constructions, under additional hypotheses, of (p^+) -filters, there remains a lack of examples of interesting, purely analytical (p^+) and (p^*) -families, Example 3.8.25 notwithstanding.

Question. *Are there naturally occurring nontrivial (ZFC) examples of (p^+) or (p^*) families of block sequences?*

While Theorem 3.1.6 does give a criterion for $\mathbf{L}(\mathbb{R})$ -genericity for filters of projections in the Calkin algebra, it would be desirable to have a such criterion expressed in the language of C^* -algebras.

Question. *Can the local Ramsey theory of block sequences in a (separable, infinite-dimensional, complex) Hilbert space be described in C^* -algebraic terms? Under large cardinals, is there a C^* -algebraic characterization of $\mathbf{L}(\mathbb{R})$ -generic filters in the projections in the Calkin algebra?*

Lastly, as the sufficient conditions described in Theorem 3.1.7 for producing a counterexample to Anderson's conjecture cannot be satisfied in Shelah's model without p -points, the status of Anderson's conjecture in that model appears to be a natural test question.

Question. *Does Anderson's conjecture hold in Shelah's model without p -points?*

3.11 Supplementary material: Restricted Gowers games

Given a family $\mathcal{H} \subseteq \text{bb}^\infty(E)$, $\vec{x} \in \text{bb}^{<\infty}(E)$, and $X \in \mathcal{H}$, the *restricted Gowers game* $G_{\mathcal{H}}[\vec{x}, X]$ is defined like the Gowers game, except that player I is required

always play elements of $\mathcal{H} \upharpoonright X$. Note that if $\mathcal{H} = \text{bb}^\infty(E)$, then the restricted game coincides with the original Gowers game, while if \mathcal{H} consists only of block sequences whose spans are of finite codimension, then the restricted game coincides with the infinite asymptotic game. In general, the restricted Gowers games are an intermediate between these two other games.

The advantage of using the restricted games is that we obtain a local Rosendal-type dichotomy, as in Theorem 3.1.1, without the need for fullness of the family. This comes at the price of a weaker conclusion for player II.

What follows is a parallel series of arguments to those in §3.3, culminating in Theorem 3.11.5, the restricted form of Theorem 3.1.1. Similar results, in an “abstract” framework, have been obtained independently by Noé de Rancourt.

Definition 3.11.1. Let \mathcal{H} be a family and $\mathbb{A} \subseteq \text{bb}^\infty(E)$ be given. For $\vec{y} \in \text{bb}^{<\infty}(E)$ and $Y \in \mathcal{H}$, we say that

- (1) (\vec{y}, Y) is \mathcal{H} -good (for \mathbb{A}) if II has a strategy in $G_{\mathcal{H}}[\vec{y}, Y]$ for playing into \mathbb{A} ,
- (2) (\vec{y}, Y) is \mathcal{H} -bad (for \mathbb{A}) if for all $Z \in \mathcal{H} \upharpoonright Y$, (\vec{y}, Z) is not \mathcal{H} -good.
- (3) (\vec{y}, Y) is \mathcal{H} -worse (for \mathbb{A}) if it is \mathcal{H} -bad and there is an n such that for every $v \in \langle Y/n \rangle$, $(\vec{y} \frown v, Y)$ is \mathcal{H} -bad.

Reference to \mathbb{A} will be suppressed where understood.

Lemma 3.11.2. If \mathcal{H} is a (p) -family and $\mathbb{A} \subseteq \text{bb}^\infty(E)$, then for every $\vec{x} \in \text{bb}^{<\infty}(E)$ and $X \in \mathcal{H}$, there is a $Y \in \mathcal{H} \upharpoonright X$ such that either:

- (i) (\vec{x}, Y) is \mathcal{H} -good, or
- (ii) I has a strategy in $F[\vec{x}, Y]$ for playing into

$$\{(z_n) : \forall n (\vec{x} \frown (z_0, \dots, z_n), Y) \text{ is } \mathcal{H}\text{-worse}\}.$$

Proof. Observe that if (\vec{y}, Y) is \mathcal{H} -good/bad/worse and $Z \preceq^* Y$ is in \mathcal{H} , then (\vec{y}, Z) is also \mathcal{H} -good/bad/worse. It is immediate that for each \vec{y} , the set

$$\mathcal{D}_{\vec{y}} = \{Y \in \mathcal{H} : (\vec{y}, Y) \text{ is either } \mathcal{H}\text{-good or } \mathcal{H}\text{-bad}\}$$

is \preceq -dense open in \mathcal{H} .

Claim. If (\vec{y}, Y) is \mathcal{H} -bad, then there is a $Z \in \mathcal{H} \upharpoonright Y$ such that for all $z \in \langle Z/\vec{y} \rangle$, $(\vec{y} \frown z, Y)$ is not \mathcal{H} -good.

Proof of claim. Let (\vec{y}, Y) be \mathcal{H} -bad. Towards a contradiction, suppose that for all $Z \in \mathcal{H} \upharpoonright Y$, there is an $z \in \langle Z/\vec{y} \rangle$ such that $(\vec{y} \frown z, Y)$ is \mathcal{H} -good. We claim that (\vec{y}, Y) is \mathcal{H} -good. If I plays $Z \in \mathcal{H} \upharpoonright Y$, then by supposition there is some $z \in \langle Z/\vec{y} \rangle$ such that $(\vec{y} \frown z, Y)$ is \mathcal{H} -good. Let II play that z and from then on follow the strategy given from $(\vec{y} \frown z, Y)$ being \mathcal{H} -good. This is contrary to (\vec{y}, Y) being \mathcal{H} -bad. □_(claim.)

Claim. For each \vec{y} , the set

$$\mathcal{E}_{\vec{y}} = \{Z \in \mathcal{H} : (\vec{y}, Z) \text{ is either } \mathcal{H}\text{-good or } \mathcal{H}\text{-worse}\}$$

is \preceq -dense open in \mathcal{H} .

Proof of claim. Fix \vec{y} and let $Y \in \mathcal{H}$. Since the sets $\mathcal{D}_{\vec{x}}$ are dense in \mathcal{H} and there are only countably many \vec{x} , the (p) -property allows us to diagonalize all of them within \mathcal{H} and assume that for all \vec{x} , (\vec{x}, Y) is either \mathcal{H} -good or \mathcal{H} -bad. Suppose that (\vec{y}, Y) is \mathcal{H} -bad. By the previous claim, there is a $Z \in \mathcal{H} \upharpoonright Y$ such that if $z \in \langle Z \rangle$, then $(\vec{y} \frown z, Z)$ is not \mathcal{H} -good, hence \mathcal{H} -bad, by our choice of Y . Thus, (\vec{y}, Z) is \mathcal{H} -worse. □_(claim.)

We can now prove the lemma. By the previous claim, we have a $Y \in \mathcal{H} \upharpoonright X$ so that for all \vec{y} , $(\vec{x} \frown \vec{y}, Y)$ is either \mathcal{H} -good or \mathcal{H} -worse. If (\vec{x}, Y) is \mathcal{H} -good, we're

done, so suppose that (\vec{x}, Y) is \mathcal{H} -worse. We will describe a strategy for I in $F[\vec{x}, Y]$: Suppose that at some point in the game (z_0, \dots, z_k) has been played by II so that $(\vec{x}^\wedge(z_0, \dots, z_k), Y)$ is \mathcal{H} -worse. Then, there is some n such that for all $z \in \langle Y \rangle$, if $n < z$, then $(\vec{x}^\wedge(z_0, \dots, z_k)^\wedge z, Y)$ is \mathcal{H} -bad, hence \mathcal{H} -worse. Let I play that n . □

The proofs of the following lemmas, and of Theorem 3.11.5, are completely identical to those in §3.3 and so we omit them.

Lemma 3.11.3. *Let $\mathcal{H} \subseteq \text{bb}^\infty(E)$ be a (p) -family and $\mathbb{A} \subseteq \text{bb}^\infty(E)$ open. Then, for any $X \in \mathcal{H}$ and $\vec{x} \in \text{bb}^\infty(E)$, there is a $Y \in \mathcal{H} \upharpoonright X$ such that either:*

- (i) *I has a strategy in $F[\vec{x}, Y]$ for playing into \mathbb{A}^c , or*
- (ii) *II has a strategy in $G_{\mathcal{H}}[\vec{x}, Y]$ for playing into \mathbb{A} .*

□

Lemma 3.11.4. *Let $\mathcal{H} \subseteq \text{bb}^\infty(E)$ be a (p) -family. Suppose that $\mathbb{A}_n \subseteq \text{bb}^\infty(E)$ for $n \in \omega$, and $\mathbb{A} = \bigcup_{n \in \omega} \mathbb{A}_n$. Let \vec{x} and $X \in \mathcal{H}$ be given. Then, there is a $Y \in \mathcal{H} \upharpoonright X$ such that either:*

- (i) *I has a strategy in $F[\vec{x}, Y]$ for playing into \mathbb{A}^c , or*
- (ii) *II has a strategy in $G_{\mathcal{H}}[Y]$ for playing (z_k) for which there is some n such that for every $V \in \mathcal{H} \upharpoonright Y$, I has no strategy in $F[\vec{x}^\wedge(z_0, \dots, z_n), V]$ for playing into \mathbb{A}_n^c .*

□

Theorem 3.11.5. *Let $\mathcal{H} \subseteq \text{bb}^\infty(E)$ be a (p) -family and $\mathbb{A} \subseteq \text{bb}^\infty(E)$ analytic. Then, for any $X \in \mathcal{H}$ and $\vec{x} \in \text{bb}^\infty(E)$, there is a $Y \in \mathcal{H} \upharpoonright X$ such that either:*

- (i) *I has a strategy in $F[\vec{x}, Y]$ for playing into \mathbb{A}^c , or*

(ii) *I has a strategy in $G_{\mathcal{H}}[\vec{x}, Y]$ for playing into \mathbb{A} .*

□

We believe that similar arguments as to those in §3.7 can be used to extend these results to sets in $L(\mathbb{R})$, under suitable large cardinal hypotheses, though we will not attempt to do so here. Noé de Rancourt has obtained results in this direction from determinacy considerations.

Likewise, by adapting the arguments in §3.8, one should be able to obtain restricted Gowers-type dichotomies for suitably invariant spread (p) -families in normed and Banach spaces.

For the remainder of this section, we will instead consider some of the combinatorial properties of (p) -families, using the restricted Gowers games. The following result, connecting the (strong) (p) -property to the restricted Gowers games, is a strengthening of Theorem 3.4.3.

Theorem 3.11.6. *Let $\mathcal{F} \subseteq \text{bb}^\infty(E)$ be a filter.*

- (a) *If \mathcal{F} does not have the (p) -property, then for every $X \in \mathcal{F}$ I has a strategy in $G_{\mathcal{F}}[X]$ for playing into \mathcal{F}^c .*
- (b) *If \mathcal{F} has the strong (p) -property, then for no $X \in \mathcal{F}$ does I have a strategy in $G_{\mathcal{F}}[X]$ for playing into \mathcal{F}^c .*

Proof. (a) Suppose that \mathcal{F} does not have the (p) -property, so there is a decreasing sequence $X_0 \succeq X_1 \succeq X_2 \succeq \dots$ in \mathcal{F} such that there is no $X \in \mathcal{F}$ with $X \preceq^* X_n$ for all n . Take $X \in \mathcal{F}$ arbitrary. We can build a decreasing sequence (Y_n) in \mathcal{F} such that for all n , $Y_n \preceq X$ and $Y_n \preceq X_n$: take $Y_0 \in \mathcal{F}$ witnessing compatibility of X and X_0 , $Y_1 \in \mathcal{F}$ witnessing compatibility of Y_0 and X_1 , and so on. Note

that any diagonalization of (Y_n) is a diagonalization of (X_n) . Thus, in the game $G_{\mathcal{F}}[X]$, if I plays Y_n on the n th move, no outcome can be in \mathcal{F} . This describes a strategy for I as desired.

(b) Let σ be a strategy for I in $G_{\mathcal{F}}[X]$ for playing into \mathcal{F}^c , where $X \in \mathcal{F}$, and suppose towards a contradiction that \mathcal{F} has the strong (p) -property. Define sets $\mathcal{A}_{\vec{x}} \subseteq \mathcal{F}$ as follows: $\mathcal{A}_{\emptyset} = \{\sigma(\emptyset)\}$ and in general, $\mathcal{A}_{\vec{x}}$ is the set of all $Y \in \mathcal{F}$ played by I, when I follows σ and $\vec{x} = (x_0, \dots, x_{n-1})$ are the first n moves by II. Note that some elements of a given \vec{x} may not be valid moves for II, in which case $\mathcal{A}_{\vec{x}} = \mathcal{A}_{\vec{x}'}$ where \vec{x}' is the maximal initial segment of \vec{x} consisting of valid moves. Then, for all \vec{x} , $\mathcal{A}_{\vec{x}}$ is finite, $\mathcal{A}_{\vec{x}} \subseteq \mathcal{A}_{\vec{y}}$ whenever $\vec{x} \sqsubseteq \vec{y}$.

For each \vec{x} , pick $Y_{\vec{x}} \in \mathcal{F}$ such that for all $Y \in \mathcal{A}_{\vec{x}}$, $Y_{\vec{x}} \preceq Y$. As \mathcal{F} is a filter, $(Y_{\vec{x}})$ generates a filter. By the strong (p) -property, there is a $Y = (y_n) \in \mathcal{F}$ (which we may assume is $\preceq X$) and such that $Y/\vec{y} \preceq Y_{\vec{y}}$ for all $\vec{y} \sqsubseteq Y$.

Consider the play of $G_{\mathcal{F}}[X]$ wherein I follows σ , and II plays y_0, y_1 , etc. This is a valid play by II by our choice of Y : $y_0 \in \langle Y_{\emptyset} \rangle \subseteq \langle \sigma(\emptyset) \rangle$, $y_1 \in \langle Y/(y_0) \rangle \subseteq \langle Y_{(y_0)} \rangle \subseteq \langle \sigma(y_0) \rangle$, etc. The resulting outcome is Y , which is in \mathcal{F} , yielding a contradiction to our assumption about σ . \square

What is the relationship between fullness and the restricted Gowers games? One answer to this question, relevant for the material in Ch. 4, is given by considering a generalization of being strategic.

Definition 3.11.7. A family $\mathcal{H} \subseteq \text{bb}^{\infty}(E)$ is *+-strategic* if whenever $X \in \mathcal{H}$ and α is a strategy for II in $G_{\mathcal{H}}[X]$, there is an outcome of α in \mathcal{H} .

A tree $T \subseteq \text{bb}^{<\infty}(E)$ is a subset which is downwards closed with respect to

initial segments. Standard arguments show that $[T]$, the set of infinite branches through T (i.e., sequences $(y_n) \in \text{bb}^\infty(E)$ such that $(y_0, \dots, y_n) \in T$ for all n), is a closed subset of $\text{bb}^\infty(E)$.

Lemma 3.11.8 (cf. Lemma 6.4 in [30]). *Let \mathcal{H} be a family, $X \in \mathcal{H}$, and α a strategy for II in $G_{\mathcal{H}}[X]$. Then, there is a tree $T \subseteq \text{bb}^{<\infty}(E)$ such that:*

- (i) $[T] \subseteq [\alpha]$, and
- (ii) *whenever $(y_0, \dots, y_n) \in T$ and $Y \in \mathcal{H} \upharpoonright X$, there is a $y \in \langle Y \rangle$ so that $(y_0, \dots, y_n, y) \in T$. In particular, $[T]$ is \mathcal{H} -dense below X .*

Proof. We will define a pair of trees $T \subseteq \text{bb}^{<\infty}(E)$ and $S \subseteq \mathcal{H}^{<\infty}$ as follows: Put $\emptyset \in T$ and S . The first level of T consists of all $(y) \in \text{bb}^{<\infty}(E)$ of length 1 such that y is a “first move” by II according to α . That is, there some $Y \in \mathcal{H} \upharpoonright X$ so that $\alpha(Y) = y$. For each such y , pick a unique $Y \in \mathcal{H} \upharpoonright X$ in its preimage under α ; these comprise the first level of S .

Inductively, having put $(y_0, \dots, y_n) \in T$ and $(Y_0, \dots, Y_n) \in S$ with $\alpha(Y_0, \dots, Y_i) = y_i$ for $i \leq n$, we put (y_0, \dots, y_n, y) if there is some $Y \in \mathcal{H} \upharpoonright X$ with $\alpha(Y_0, \dots, Y_n, Y) = y$. Choose some $Y \in \mathcal{H} \upharpoonright X$ with this property and put (Y_0, \dots, Y_n, Y) into S . This completes the construction.

Clearly, $[T] \subseteq [\alpha]$. To see that $[T]$ satisfies (ii), let $(y_0, \dots, y_n) \in T$ and $Y \in \mathcal{H} \upharpoonright X$. Let $(Y_0, \dots, Y_n) \in S$ be such that $\alpha(Y_0, \dots, Y_i) = y_i$ for $i \leq n$, and put $y_{n+1} = \alpha(Y_0, \dots, Y_n, Y)$. By construction, there is some Y_{n+1} with $(Y_0, \dots, Y_n, Y_{n+1}) \in S$ and $y_{n+1} = \alpha(Y_0, \dots, Y_n, Y_{n+1})$. \square

Theorem 3.11.9. *For a family \mathcal{H} , the following are equivalent:*

- (i) \mathcal{H} is a +-strategic (p)-family.

(ii) \mathcal{H} is a strategic (p^+) -family.

Proof. (i \Rightarrow ii) Let \mathcal{H} be a +-strategic (p) -family. First observe that +-strategic implies strategic: Given any $X \in \mathcal{H}$ and a strategy α for II in $G[X]$, let α' be the restriction of α to $G_{\mathcal{H}}[X]$ (in the obvious sense). Since \mathcal{H} is +-strategic, there is an outcome of α' , and thus of α , in \mathcal{H} .

To see that such an \mathcal{H} is full, let $D \subseteq E$ and $X \in \mathcal{H}$ be such that D is \mathcal{H} -dense below X . Let $\mathcal{D} = \{Y : \langle Y \rangle \subseteq D\}$, a closed set. By Theorem 3.11.5, there is a $Y \in \mathcal{H} \upharpoonright X$ such that either I has a strategy in $F[Y]$ for playing into \mathcal{D}^c , or II has a strategy in $G_{\mathcal{H}}[Y]$ for playing in \mathcal{D} . However, the former is impossible: pick some $Z \preceq Y$ in \mathcal{D} and let II always play elements of $\langle Z \rangle$. As \mathcal{H} is +-strategic, there is then some outcome of II's strategy in $G_{\mathcal{H}}[Y]$ in \mathcal{H} , showing that $(\mathcal{H} \upharpoonright X) \cap \mathcal{D} \neq \emptyset$ and verifying fullness.

(ii \Rightarrow i) Let \mathcal{H} be a strategic (p^+) -family. We must prove that \mathcal{H} is +-strategic. Let $X \in \mathcal{H}$ and α be a strategy for II in $G_{\mathcal{H}}[X]$. Let $T \subseteq \text{bb}^{<\infty}(E)$ be as in Lemma 3.11.8. By Theorem 3.1.1, there is a $Y \in \mathcal{H} \upharpoonright X$ such that either I has a strategy in $F[Y]$ for playing into $[T]^c$, or II has a strategy in $G[Y]$ for playing into $[T]$. The former is impossible as II has a strategy in $F[Y]$ for playing into $[T]$: Inductively apply the property in Lemma 3.11.8(ii) to the tail block sequences played by I in $F[Y]$. Thus, II has a strategy in $G[Y]$ for playing into $[T]$. As \mathcal{H} is strategic, there is some outcome of this strategy, thus some element of $[\alpha]$, in \mathcal{H} . \square

3.12 Supplementary material: Generic pure states and the Calkin algebra

This section is devoted to giving a full proof of Theorem 3.9.9. While this result is not stated as such in [27], it is implicit, and what follows is derived from that reference. Note that $\mathcal{P}(\mathcal{C}(H))^+$ is σ -closed by Lemma 3.9.5.

Recall that $\mathcal{F} \subseteq \mathcal{P}(\mathcal{C}(H))^+$ is *centered* if every finite subset of \mathcal{F} has a lower bound in $\mathcal{P}(\mathcal{C}(H))^+$.

Lemma 3.12.1 (Bice [13]). *For $p, q \in \mathcal{P}(\mathcal{C}(H))^+$, p and q are compatible if and only if $\|pq\| = 1$.*

Lemma 3.12.2. *Let \mathcal{G} be a \mathbf{V} -generic filter for $\mathcal{P}(\mathcal{C}(H))^+$. Then, \mathcal{G} is a maximal centered set in $\mathbf{V}[\mathcal{G}]$.*

Proof. Since \mathcal{G} is a filter, it is centered, so it suffices to show that \mathcal{G} is maximal. We claim that for every $p \in \mathcal{P}(\mathcal{C}(H))^+$, either $p \in \mathcal{G}$, or there is a $q \in \mathcal{G}$ such that $\|pq\| < 1$. Since $\mathcal{P}(\mathcal{C}(H))^+$ is separative [12] (i.e., whenever $q \not\leq p$, there is an $r \leq q$ which is incompatible with p), the set

$$\mathcal{D}_p = \{q \in \mathcal{P}(\mathcal{C}(H))^+ : q \leq p\} \cup \{q \in \mathcal{P}(\mathcal{C}(H))^+ : q \text{ is incompatible with } p\}$$

is dense, so the claim follows by Lemma 3.12.1. If $\mathcal{G} \subsetneq \mathcal{F}$, for some centered set \mathcal{F} and $p \in \mathcal{F} \setminus \mathcal{G}$, then there is a $q \in \mathcal{G}$ with $\|pq\| < 1$, but this is impossible since $q, p \in \mathcal{F}$. \square

Lemma 3.12.3. *Let $\mathcal{F} = \mathcal{F}_\rho$ be a maximal centered set in $\mathcal{P}(\mathcal{C}(H))$ for ρ a pure state on $\mathcal{B}(H)$, $\vec{f} = (f_k)_{k \in \omega}$ an orthonormal basis for H , and $\omega = \bigcup_{j=1}^n A_j$ a finite partition of ω . If there exists $q \in \mathcal{F}$ such that $\|\pi(P_{A_j}^{\vec{f}})q\| < 1$ for all j , then ρ is not diagonalized by $(f_k)_{k \in \omega}$.*

Proof. Suppose that $\rho = \rho_{\mathcal{U}}$ for an ultrafilter \mathcal{U} on ω . Then, there is some j for which $A_j \in \mathcal{U}$, and thus

$$\rho(P_{A_j}^{\vec{f}}) = \lim_{k \rightarrow \mathcal{U}} \langle P_{A_j}^{\vec{f}} f_k, f_k \rangle = 1.$$

Hence, $\pi(P_{A_j}^{\vec{f}}) \in \mathcal{F}$. But then, the existence of a $q \in \mathcal{F}$ for which $\|\pi(P_{A_j}^{\vec{f}})q\| < 1$ contradicts the \mathcal{F} being centered. \square

Lemma 3.12.4. *Let $\vec{e} = (e_n)_{n \in \omega}$ and $\vec{f} = (f_k)_{k \in \omega}$ be orthonormal bases of H , and $\epsilon > 0$. Then, there is a partition $\omega = \bigcup_{n \in \omega} J_n$ into finite intervals J_n such that for all $k \in \omega$, there is an $n = n(k)$ for which f_k is within $\epsilon/2^n$ of $\text{span}\{e_i : i \in J_n \cup J_{n+1}\}$.*

Proof. Choose J_0 to be a finite initial segment of ω . Choose K_0 such that for all $k \geq K_0$, $\|P_{J_0}^{\vec{e}} f_k\| < \epsilon/4$; this can be done since the f_k converge to 0 weakly in H . Choose M_0 such that for all $k \in J_0 \cup K_0$, $\|P_{(M_0, \infty)}^{\vec{e}} f_k\| < \epsilon/4$. Let J_1 be a finite interval immediately above J_0 and such that $J_0 \cup J_1$ covers $\{0, \dots, M_0\}$. Then, $J_0 \cup J_1$ “works” for all f_k with $k \in J_0$ or having “large support” in J_0 .

Choose $K_1 > K_0$ such that for all $k \geq K_1$, $\|P_{J_0 \cup J_1}^{\vec{e}} f_k\| < \epsilon/8$. Choose $M_1 > M_0$ such that for all $k \in J_0 \cup J_1 \cup K_1$, $\|P_{(M_1, \infty)}^{\vec{e}} f_k\| < \epsilon/8$. Let J_2 be such that $J_0 \cup J_1 \cup J_2$ covers $\{0, \dots, M_1\}$. If f_k is such that $k > K_0$ is in J_1 or below K_1 , then $J_1 \cup J_2$ works for f_k . Continue in this fashion. \square

Fix an orthonormal basis $\vec{e} = (e_n)_{n \in \omega}$. For $\vec{J} = (J_n)_{n \in \omega}$ a partition of ω into finite intervals, let

$$\mathcal{D}_{\vec{J}} = \{q \in \mathcal{P}(\mathcal{C}(H))^+ : \exists Q \in \mathcal{P}(H) (q = \pi(Q) \text{ and } \forall n \in \omega (\|P_{J_n \cup J_{n+1}}^{\vec{e}} Q\| < 1/2))\}$$

Lemma 3.12.5. *The sets $\mathcal{D}_{\vec{J}}$ are dense in $\mathcal{P}(\mathcal{C}(H))^+$.*

Proof. Omitted. (Not difficult.) \square

Proof of Theorem 3.9.9. Let \mathcal{G} be \mathbf{V} -generic for $\mathcal{P}(\mathcal{C}(H))^+$ and let $\rho_{\mathcal{G}}$ be the corresponding pure state in $\mathbf{V}[\mathcal{G}]$. Let $\vec{f} = (f_k)_{k \in \omega}$ be an orthonormal basis for H . We will show that $\rho_{\mathcal{G}}$ is not diagonalized by $(f_k)_{k \in \omega}$.

Using Lemma 3.12.4, for an $\epsilon > 0$ to be specified later, pick a partition $\vec{J} = (J_n)_{n \in \omega}$ such that for all k , there is an $n(k)$ such that f_k is within $\epsilon/2^{n(k)}$ of $\text{span}\{e_i : i \in J_{n(k)} \cup J_{n(k)+1}\}$. We may assume (by the proof of Lemma 3.12.4) that the $n(k)$ are nondecreasing in k and that the assignment $k \mapsto n(k)$ maps onto ω . By Lemma 3.12.5 and the fact that $\mathcal{P}(\mathcal{C}(H))^+$ is σ -closed, $\mathcal{G} \cap \mathcal{D}_{\vec{J}} \neq \emptyset$.

Define $A_j = \{k : n(k) \equiv j \pmod{2}\}$ for $j \in \{0, 1\}$. We claim that if $q \in \mathcal{G} \cap \mathcal{D}_{\vec{J}}$ and $Q \in \mathcal{P}(H)$ witnesses $q \in \mathcal{D}_{\vec{J}}$, then $\|\pi(P_{A_j}^{\vec{f}})q\| \leq \|P_{A_j}^{\vec{f}}Q\| < 1$ for each j . This will complete the proof, by Lemma 3.12.3.

Fix $j \in \{0, 1\}$ and let $v = \sum_{k \in A_j} a_k f_k$ be a unit vector in $\text{ran}(P_{A_j}^{\vec{f}})$. Let $v' = \sum_{k \in A_j} a_k P_{J_{n(k)} \cup J_{n(k)+1}}^{\vec{e}} f_k$. By our choice of Q , $\|QP_{J_{n(k)} \cup J_{n(k)+1}}^{\vec{e}} f_k\| \leq 1/2$. By choosing ϵ above sufficiently small, we can ensure that $\|Qv\| \leq 1 - \delta$ for some $\delta > 0$ which doesn't depend on v . Since v was an arbitrary unit vector in $\text{ran}(P_{A_j}^{\vec{f}})$, this completes the proof. \square

CHAPTER 4
MADNESS IN VECTOR SPACES

4.1 Introduction

In this final chapter, we will consider questions regarding “almost disjoint” families of subspaces of a vector space, to be defined below.

Recall that two infinite subsets x and y of ω are said to be *almost disjoint* if $x \cap y$ is finite. A collection $\mathcal{A} \subseteq [\omega]^\omega$ is an *almost disjoint family* if its elements are pairwise almost disjoint and is a *maximal almost disjoint family*, or *mad family*, if it is not properly contained in another such family. While any finite (almost) partition of ω forms a mad family, we will be interested in infinite mad families.

The following facts about mad families on ω are well-known: every almost disjoint family is contained in a mad family and every infinite almost disjoint family is uncountable. The former is a standard application of Zorn’s Lemma, while the later a straightforward diagonalization.

A large almost disjoint family can be obtained as follows: Identifying ω with $2^{<\omega}$, consider

$$\mathcal{A} = \{ \{x \upharpoonright n : n \in \omega\} : x \in 2^\omega \}. \quad (4.1)$$

It is easy to see that \mathcal{A} is almost disjoint and of size \mathfrak{c} , thus can be extended to a mad family of size \mathfrak{c} . Note that \mathcal{A} is (topologically) closed as it is a homeomorphic image of 2^ω .

Two fundamental questions about infinite mad families are:

1. How big (or small) can they be?
2. How definable can they be?

One way of addressing question 1 is to determine the value¹ of the cardinal invariant

$$\mathfrak{a} = \min\{|\mathcal{A}| : \mathcal{A} \text{ is an infinite mad family}\}.$$

By our comments above, $\aleph_1 \leq \mathfrak{a} \leq \mathfrak{c}$, however, the value of \mathfrak{a} cannot be decided in ZFC. For instance, both CH and MA (Martin and Solovay [63]) imply that $\mathfrak{a} = \mathfrak{c}$, and thus, consistently $\aleph_1 < \mathfrak{a} = \mathfrak{c}$, while Kunen [57] showed that in the model obtained by adding \aleph_2 -many Cohen reals to a model of CH, $\aleph_1 = \mathfrak{a} < \mathfrak{c} = \aleph_2$. Hrušák [45] showed² that the latter also holds in the model obtained by adding \aleph_2 -many Sacks reals iteratively to a model of CH. A more sophisticated version of question 1 might ask for the “spectrum” of cardinalities between \aleph_1 and \mathfrak{c} that mad families can possess. This was partially answered by Hechler [38], who produced a method for obtaining arbitrarily large continuum and, simultaneously, mad families of all cardinalities κ for $\aleph_0 \leq \kappa \leq \mathfrak{c}$. While beyond the scope of our investigations here, these questions have been the focus of much deep work in recent decades, notably Brendle’s [18], which establishes the consistency of $\mathfrak{a} = \aleph_\omega$, Shelah’s [78], which establishes the consistency of $\mathfrak{d} < \mathfrak{a}$, and Shelah and Spinas’ [79], which gives a nearly-sharp characterization of possible mad spectra.

Question 2 above seeks to understand in what sense mad families are non-constructive objects. A well-known result of Mathias [64] says that an infinite mad family can never be analytic (i.e., the continuous image of a Borel set). Un-

¹This could mean which \aleph_α is such that $\mathfrak{a} = \aleph_\alpha$, or how \mathfrak{a} relates to other well-studied cardinal invariants (cf. [15]) between \aleph_1 and \mathfrak{c} .

²Given the comments in [45], this result was likely known long before.

der large cardinal hypotheses, this can be pushed further to show that there are no definable mad families at all, in the sense that there are none in $L(\mathbb{R})$ (see [25], [64], [86], and for a consistency result without large cardinals, [43]). Thus, the nonconstructive methods used to obtain mad families are, in some sense, necessary. We note that Mathias' result is sharp; Miller [66] proved that there is a coanalytic (i.e., the complement of an analytic set) mad family assuming $V = L$, work later refined by Törnquist [85].

This chapter is concerned with an analogue of mad families arising in infinite-dimensional vector spaces. As in Ch. 3, let E be a countably infinite-dimensional vector space over a countable field F and (e_n) a fixed F -basis for E . There is no harm in thinking of E as $\bigoplus_n F$ and e_n as the n th unit coordinate vector. When we speak of *subspaces* of E , we will mean infinite-dimensional linear subspaces, unless otherwise specified.

Definition 4.1.1. We say that subspaces X and Y of E are *almost disjoint* if $X \cap Y$ is finite-dimensional.

Definition 4.1.2. A collection \mathcal{A} of subspaces of E is an *almost disjoint family of subspaces* if its elements are pairwise almost disjoint and is a *maximal almost disjoint family of subspaces*, or *mad family of subspaces*, if it is not properly contained in another such family.

While the topic of almost disjoint families of subspaces seems very natural, it appears to have been little studied except for a paper by Kolman [54], wherein they are called "almost disjoint packings".³ We seek to reintroduce this topic.

³Several proofs in [54] appear to use a stronger property than almost disjointness, namely that whenever $X_0, \dots, X_n \in \mathcal{A}$ are distinct, then $X_i \cap (\sum_{j \neq i} X_j)$ is finite-dimensional. It is easy to construct almost disjoint families of subspaces for which this fails, e.g., $X_0 = \text{span}\{e_{2n} : n \in \omega\}$, $X_1 = \text{span}\{e_{2n+1} : n \in \omega\}$, and $X_2 = \text{span}\{e_{2n} + e_{2n+1} : n \in \omega\}$. This can be extended to an infinite almost disjoint family by our Proposition 4.2.5. As such, we reprove several of the results from §3 of [54] below.

We begin with the following easy facts:

Proposition 4.1.3. *Every almost disjoint family of subspaces is contained in a mad family of subspaces.*

Proof. This is a standard Zorn’s Lemma argument. □

Proposition 4.1.4. *There is an almost disjoint family of subspaces, and thus a mad family of subspaces, of size \mathfrak{c} .*

Proof. Let \mathcal{A} be an almost disjoint family on ω of size \mathfrak{c} , as in (4.1) above. Consider the injective map $x \mapsto \text{span}\{e_n : n \in x\}$. The image of \mathcal{A} under this map is easily seen to be an almost disjoint family of subspaces. □

Note that any nontrivial almost disjoint family of subspaces contained in the image of the “diagonal” map $x \mapsto \text{span}\{e_n : n \in x\}$ used above fails to be maximal: $\text{span}\{e_{2n} + e_{2n+1} : n \in \omega\}$ will be disjoint from every subspace having infinite codimension in this image.

In light of the above questions regarding mad families on ω , we ask the analogous questions for infinite mad families of subspaces:

1. How big (or small) can they be? In particular, what is

$$\mathfrak{a}_{\text{vec}} = \min\{|\mathcal{A}| : \mathcal{A} \text{ is an infinite mad family of subspaces}\}^4$$

2. How definable can they be?

Two related notions have been studied in the setting of an infinite-dimensional separable Hilbert space, namely that of “almost orthogonal” and

⁴Technically, $\mathfrak{a}_{\text{vec}}$ might depend on the field F , though we will suppress this throughout.

“almost disjoint” families of closed subspaces, where “almost” is measured by considering the corresponding projection operators modulo the compact operators. Results concerning question 1 in these settings were obtained in papers of Wofsey [89] and Bice [11], respectively. While not directly related⁵ to our setting, these papers provide both motivation for, and ideas used in, the results in §4.2 below.

When F is a finite field of order 2, vectors may be identified with their supports and sums of vectors in block position correspond to unions of the corresponding supports. This is then the setting of Hindman’s Theorem [39] on disjoint unions of finite subsets of ω and will be relevant in §4.3, where Ramsey theory enters the picture. However, the particular choice of F will not affect our main results.

This chapter is organized as follows: In §4.2, we consider issues of cardinality and address question 1, establishing analogues of the results of Martin and Solovay, Kunen, Hechler, and Hrušák mentioned above. In §4.3, we consider issues of definability. We discuss a failed attempt at a solution to question 2 in the case when F is a finite field (in particular, when $|F| = 2$) and then use the results from Ch. 3 to give a partial solution for “fully mad” families of subspaces. The existence of such families is established under suitable set-theoretic hypotheses. We conclude in §4.4 with further remarks and open questions.

⁵Almost orthogonal families of closed subspaces of Hilbert space appear more closely related to almost disjoint families on ω than does our setting. For instance, countable almost orthogonal families arise as images of countable almost disjoint families on ω via the “diagonal map” (cf. Lemma 5.34 in [27]), and, consistently [89], some mad families on ω remain maximal when passed through this map. Less is understood about the notion of almost disjointness for closed subspaces, e.g., it remains open whether the corresponding cardinal invariant is \aleph_1 in ZFC.

4.2 Cardinality

We recall various notions for (sequences of) vectors defined in Ch. 3. For $x \in E$, the *support* of x is given by

$$\text{supp}(x) = \{n \in \omega : x = \sum a_i e_i \Rightarrow a_n \neq 0\}.$$

For nonzero vectors, we write $x < y$ if $\max(\text{supp}(x)) < \min(\text{supp}(y))$ and say that a (finite or infinite) sequence of nonzero vectors (x_n) is a *block sequence* if $x_n < x_{n+1}$ for all n . A space spanned by an infinite block sequence is a *block subspace*. Throughout, all subspaces are assumed to be infinite-dimensional unless otherwise specified. To deal with general subspaces, the following definition will be useful:

Definition 4.2.1. A sequence (x_n) of nonzero vectors in E is in *reduced echelon form* if the matrix whose n th row is given by x_n , expressed with respect to the basis (e_n) , is in reduced echelon form.

As all vectors have finite support, this definition is unambiguous even for infinite sequences. Note that row reduction of an infinite matrix with finitely-supported rows will always converge coordinatewise to an infinite reduced echelon form matrix. It follows that every subspace has a (unique) basis in reduced echelon form, and by passing to a sufficiently “spread out” subsequence, that every subspace contains a block subspace (cf. Lemma 3.2.1).

Given a subspace Y and an $M \in \omega$, we write Y/M for all those vectors in Y with supports above M . This is always a subspace of Y . Given a vector x , we write Y/x for $Y/\max(\text{supp}(x))$. The following lemma will be key to much of what follows.

Lemma 4.2.2. *Let Y be a subspace of E and $x_0 < \dots < x_m$ a finite block sequence in E . Then, there is an M such that whenever $x > M$,*

$$\langle x_0, \dots, x_m, x \rangle \cap Y = \begin{cases} \langle x_0, \dots, x_m \rangle \cap Y & \text{if } x \notin Y, \\ (\langle x_0, \dots, x_m \rangle \cap Y) + \langle x \rangle & \text{if } x \in Y. \end{cases}$$

Proof. Let (y_n) be a basis for Y in reduced echelon form, $K = \max(\text{supp}(x_m))$, and N minimal such that

$$\left(\bigcup_{n>N} \text{supp}(y_n) \right) \cap [0, K] = \emptyset.$$

Such an N exists as (y_n) is in reduced echelon form. Let

$$M = \max \left\{ \max \left(\bigcup_{n \leq N} \text{supp}(y_n) \right), K \right\}.$$

We claim that M is as desired. Take $x > M$ and suppose that

$$v = \lambda_0 x_0 + \dots + \lambda_m x_m + \lambda x \in Y.$$

Write

$$\alpha_0 y_0 + \dots + \alpha_n y_k = \lambda_0 x_0 + \dots + \lambda_m x_m + \lambda x.$$

Case 1: $x \notin Y$. We suppose that $\lambda \neq 0$ and proceed towards a contradiction. Note that $k > N$ as $x > M$ and the λ_i 's are not all 0. It follows that $\bigcup_{n \leq N} \text{supp}(y_n)$ overlaps with $[0, K]$ and $\bigcup_{N < n \leq k} \text{supp}(y_n)$ is strictly above K . We claim that

$$\alpha_0 y_0 + \dots + \alpha_j y_j = \lambda_0 x_0 + \dots + \lambda_m x_m$$

for some $j \leq N$, which implies $x \in Y$, a contradiction. To see this, note that in order for this to fail, there must be some $\ell > N$, with $\alpha_\ell \neq 0$ and y_ℓ having support overlapping with that of some y_j , for $j \leq N$. But then, as the y_n are in reduced echelon form, the leading coefficient (when expressed with respect

to (e_n) of $\alpha_\ell y_\ell$ occurs in v , while being both below x and above x_m , which is absurd.

Case 2: $x \in Y$. The same argument shows that if $\lambda \neq 0$, then either the λ_i 's are all 0, in which case $v = \lambda x$, or $v = \alpha_0 y_0 + \cdots + \alpha_N y_N + \lambda x$. In either case, $v \in (\langle x_0, \dots, x_m \rangle \cap Y) + \langle x \rangle$. \square

Lemma 4.2.3. *Suppose that Y_0, \dots, Y_n, Y_{n+1} are pairwise disjoint subspaces of E and $x_0 < \cdots < x_n$ vectors such that each $x_k \in Y_k$, $\langle x_0, \dots, x_n \rangle \cap Y_k = \langle x_k \rangle$ for $k \leq n$ and $\langle x_0, \dots, x_n \rangle \cap Y_{n+1} = \{0\}$. Then, there is an M such that for any $x_{n+1} \in Y_{n+1}/M$, $\langle x_0, \dots, x_n, x_{n+1} \rangle \cap Y_k = \langle x_k \rangle$ for $k \leq n + 1$.*

Proof. By repeatedly applying Lemma 4.2.2, we can obtain an increasing sequence M_k , for $k \leq n$, such that for any $x \in Y_{n+1}/M_k$,

$$\langle x_0, \dots, x_n, x \rangle \cap Y_k = \langle x_0, \dots, x_n \rangle \cap Y_k = \langle x_k \rangle.$$

A further application of Lemma 4.2.2 yields an $M_{n+1} \geq M_n$ so that whenever $x \in Y_{n+1}/M_{n+1}$,

$$\langle x_0, \dots, x_n, x \rangle \cap Y_{n+1} = \langle x_0, \dots, x_n \rangle \cap Y_{n+1} + \langle x \rangle = \langle x \rangle.$$

Then, $M = M_{n+1}$ is as desired. \square

Lemma 4.2.4. *Suppose that Y_0, \dots, Y_n, Y_{n+1} are pairwise disjoint subspaces and $x_0 < \cdots < x_m$ vectors such that $\langle x_0, \dots, x_m \rangle \cap Y_k = \{0\}$ for $k \leq n + 1$. Then, there is an $x > x_m$ such that $\langle x_0, \dots, x_m, x \rangle \cap Y_k = \{0\}$ for $k \leq n + 1$.*

Proof. By applying Lemma 4.2.2 $n + 1$ times, we obtain an M so that whenever $x > M$ and not in any of the Y_k 's, $\langle x_0, \dots, x_m, x \rangle \cap Y_k = \{0\}$ for $k \leq n + 1$. To find such an x , one can use Lemma 4.2.3 repeatedly to build $x'_0 < \cdots < x'_{n+1}$ above

M and satisfying $\langle x'_0, \dots, x'_n \rangle \cap Y_k = \langle x'_k \rangle$ for $k \leq n+1$. Then, $x = x'_0 + \dots + x'_{n+1}$ is not in $\langle Y_k \rangle$ for $k \leq n+1$ and is as desired. \square

If X is a finite-codimensional subspace, then $\{X\}$ is always a mad family of size 1. These are, in fact, the only countable mad families of subspaces.

Proposition 4.2.5. *Let \mathcal{A} be a maximal almost disjoint family of subspaces of size ≥ 2 . Then, \mathcal{A} is uncountable.⁶*

Proof. Suppose first that $\mathcal{A} = \{Y_0, \dots, Y_n, Y_{n+1}\}$ is a finite almost disjoint family. By replacing each Y_k with a relatively finite-codimensional subspace, we may assume that they are pairwise disjoint. Pick an x_0 not in any of the Y_k 's, which can be done as in the proof of Lemma 4.2.4. By repeatedly applying Lemma 4.2.4, we can build an infinite block sequence (x_m) such that for each m and $k \leq n+1$, $\langle x_0, \dots, x_m \rangle \cap Y_k = \{0\}$. Then, $\langle (x_m) \rangle$ witnesses that \mathcal{A} fails to be maximal.

Suppose that $\mathcal{A} = \{Y_n : n \in \omega\}$ is a countably infinite almost disjoint family. Again, by passing to finite-codimensional subspaces, we may assume that the Y_k are pairwise disjoint. Pick a nonzero $x_0 \in Y_0$. By repeatedly applying Lemma 4.2.3, we can build an infinite block sequence (x_m) such that for each n , $\langle (x_m) \rangle \cap Y_n = \langle x_n \rangle$, so again, \mathcal{A} fails to be maximal. \square

Thus, under CH every mad family of subspaces is of size \mathfrak{c} . The following shows that this also holds under $\text{MA}(\sigma\text{-centered})$ (cf. Theorem 3.5 in [54]).

Theorem 4.2.6. ($\text{MA}_\kappa(\sigma\text{-centered})$) *Every infinite mad family of subspaces has cardinality greater than κ .*

⁶When \mathcal{A} is finite, this is a special case of the fact that an infinite-dimensional vector space cannot be written as a finite union of proper subspaces. This can be proved using Lemma 4.2.2.

Proof. Let \mathcal{A} be an infinite almost disjoint family of subspaces. Define a poset \mathbb{P} to be all pairs (s, F) where s is a finite reduced echelon form block sequence in E and F a finite subset of \mathcal{A} . We order elements of \mathbb{P} by $(s', F') \leq (s, F)$ if $s' \supseteq s$, $F' \supseteq F$, and $\forall X \in F' (\langle s' \rangle \cap X \subseteq \langle s \rangle)$. Note that if $(s, F'), (s, F) \in \mathbb{P}$, for a fixed s , then $(s, F' \cup F) \in \mathbb{P}$ and extends both conditions. As there are only countably many such s , this shows that \mathbb{P} is σ -centered. If G is a filter in \mathbb{P} , then we let $X_G = \bigcup \{s : \exists F ((s, F) \in G)\}$.

Observe that if $X \in \mathcal{A}$, then the set $D_X = \{(s, F) \in \mathbb{P} : X \in F\}$ is dense, and if $D_X \in G$, then $X_G \cap X$ is finite dimensional. For $n \in \omega$, let $E_n = \{(s, F) \in \mathbb{P} : |s| \geq n\}$. In order to see that the sets E_n are dense, it suffices to show that a given (s, F) in \mathbb{P} can be extended to an $(s \hat{\ } x, F)$ in \mathbb{P} . This can be accomplished by using Lemma 4.2.2 to obtain an M for which whenever $x > M$ and not in $\bigcup F$, $\langle s \hat{\ } x \rangle \cap X = \langle s \rangle \cap X$ for each (of the finitely many) $X \in F$. Then, for any such x , $(s \hat{\ } x, F) \leq (s, F)$.

If $|\mathcal{A}| \leq \kappa$, by $\text{MA}_\kappa(\sigma\text{-centered})$, there is a filter $G \subseteq \mathbb{P}$ which meets the sets D_X and E_n , for $X \in \mathcal{A}$ and $n \in \omega$, and X_G witnesses that \mathcal{A} fails to be maximal. \square

Let \mathbb{C} be *Cohen forcing*, the set of all finite partial functions with $\text{dom}(p) \subseteq \omega$ and $\text{ran}(p) \subseteq 2$, ordered by extension. By the *Cohen model*, we mean the generic extension of a model of CH obtained by a finite support iteration of Cohen forcing of length ω_2 . Theorem 4.2.7 is stated as Theorem 3.7 in [54], however the proof given is just a reference to [57]. We give a complete proof here.

Theorem 4.2.7. *In the Cohen model, there is a maximal almost disjoint family of (block) subspaces of size \aleph_1 .*

Proof. We follow the proof of the corresponding result for mad families of subsets of ω , Theorem 2.3 in Ch. VIII of [57]. We define a maximal almost disjoint family $\mathcal{A} = \{X_\xi : \xi < \omega_1\}$ of block subspaces having the property that it remain maximal after adding a single Cohen real. By standard properties of Cohen forcing (Lemma 2.2 in Ch. VIII of [57]), this suffices.

Using CH in the ground model, let (p_ξ, τ_ξ) for $\omega \leq \xi < \omega_1$ enumerate all pairs (p, τ) such that $p \in \mathbb{C}$ and τ is a nice \mathbb{C} -name for a subset of E (in the sense of Definition 5.11 in Ch. VII of [57]). We recursively pick block subspaces X_ξ as follows: Let X_n , $n < \omega$, be any sequence of almost disjoint block subspaces. If $\omega \leq \xi < \omega_1$, and we have chosen X_η for all $\eta < \xi$, choose X_ξ almost disjoint from each of the (countably many) X_η for $\eta < \xi$ and so that if

$$p_\xi \Vdash_{\mathbb{C}} \tau_\xi \text{ is a subspace and } \forall \eta < \xi \dim(\tau_\xi \cap \check{X}_\eta) < \infty \quad (4.2)$$

then

$$\forall n \forall q \leq p_\xi \exists r \leq q \exists v > n (v \in X_\xi \text{ and } r \Vdash_{\mathbb{C}} \check{v} \in \tau_\xi).$$

To see that X_ξ can be chosen, assume that (4.2) holds. Let Y_i enumerate $\{X_\eta : \eta < \xi\}$ and let q_i enumerate $\{q : q \leq p_\xi\}$. By (4.2), for each i , $q_i \Vdash_{\mathbb{C}} \dim(\tau_\xi \cap \check{Y}_i) < \infty$. We construct $r_i \in \mathbb{C}$ and $x_i \in E$ inductively in i . Pick $r_0 \leq q_0$ and x_0 a nonzero vector so that $r_0 \Vdash_{\mathbb{C}} \check{x}_0 \in \tau_\xi \setminus \check{Y}_0$. Having chosen r_0, \dots, r_n and $x_0 < \dots < x_n$ so that $r_i \leq q_i$ and

$$r_i \Vdash_{\mathbb{C}} \check{x}_i \in \tau_\xi \wedge \forall k \leq i (\langle \check{x}_0, \dots, \check{x}_i \rangle \cap \check{Y}_k = \{0\}),$$

apply Lemma 4.2.2 to find $r_{n+1} \leq q_{n+1}$ and $x_{n+1} > x_n$ so that

$$r_{n+1} \Vdash_{\mathbb{C}} \check{x}_{n+1} \in \tau_\xi \wedge \forall k \leq n+1 (\langle \check{x}_0, \dots, \check{x}_n, \check{x}_{n+1} \rangle \cap \check{Y}_k = \{0\}).$$

Let $X_\xi = \langle (x_n) \rangle$.

Clearly \mathcal{A} is an almost disjoint family. It suffices to show that it is maximal in $V[G]$, where G is V -generic for \mathbb{C} . Towards a contradiction, suppose that for some (p_ξ, τ_ξ) with $p_\xi \in G$,

$$p_\xi \Vdash_{\mathbb{C}} \tau_\xi \text{ is a subspace and } \forall X \in \mathcal{A} (\dim(\tau_\xi \cap X) < \infty).$$

In particular, (4.2) holds at ξ . But $p_\xi \Vdash_{\mathbb{C}} \dim(\tau_\xi \cap \check{X}_\xi) < \infty$, so there is a $q \leq p_\xi$ and an N so that $q \Vdash_{\mathbb{C}} \tau_\xi \cap \check{X}_\xi \subseteq \langle \check{e}_0, \dots, \check{e}_N \rangle$, contradicting that

$$\exists r \leq q \exists x > N (x \in X_\xi \wedge r \Vdash_{\mathbb{C}} \check{x} \in \tau_\xi).$$

□

Given an uncountable regular cardinal κ , let

$$\mathcal{D}_\kappa = \{(\alpha, \beta) \in \kappa \times \kappa : \alpha \text{ is an uncountable limit ordinal and } \beta < \alpha\}.$$

Let \mathbb{Q}_κ be the set of all functions $p : F_p \times n_p \rightarrow E$ where $F_p \in [\mathcal{D}_\kappa]^{<\omega}$, $n_p \in \omega$, and for each $(\alpha, \beta) \in F_p$, $(p(\alpha, \beta, 0), \dots, p(\alpha, \beta, n_p - 1))$ is a block sequence in E . We say $q \leq p$ if $q \supseteq p$ and whenever $(\alpha, \beta), (\alpha, \gamma) \in F_p$ with $\beta \neq \gamma$, we have that

$$\langle \langle (q(\alpha, \beta, i))_{i < n_q} \rangle \cap \langle \langle (q(\alpha, \gamma, i))_{i < n_q} \rangle = \langle \langle (p(\alpha, \beta, j))_{j < n_p} \rangle \cap \langle \langle (p(\alpha, \gamma, j))_{j < n_p} \rangle.$$

Theorem 4.2.8. *Let κ be an uncountable regular cardinal. If G is \mathbf{V} -generic for \mathbb{Q}_κ , then in $\mathbf{V}[G]$, for every uncountable cardinal $\lambda < \kappa$ there is a mad family of subspaces of E of cardinality λ . In this model, $\kappa \leq \mathfrak{c} \leq (\kappa^{\aleph_0})^{\mathbf{V}}$.*

Typically, $\kappa = \kappa^{\aleph_0}$ and so $\mathfrak{c} = \kappa$ in the extension. In particular, it is consistent that $\mathfrak{c} > \aleph_2$ (or even $\mathfrak{c} > \aleph_{\omega_1}$, etc) and for every uncountable cardinal $\lambda \leq \mathfrak{c}$, there is a mad family of size λ . We will proceed with a series of lemmas.

Lemma 4.2.9. \mathbb{Q}_κ is ccc and $|\mathbb{Q}_\kappa| = \kappa^{\aleph_0}$.

Proof. Suppose that $\{p_\xi : \xi < \aleph_1\} \subseteq \mathbb{Q}_\kappa$. By thinning down, we may assume that there is some fixed n for which $n_{p_\xi} = n$ for all $\xi < \aleph_1$. By the Δ -system lemma, we may further thin down so that the F_{p_ξ} form a Δ -system, that is, there is some finite set $R \subseteq \mathcal{D}_\kappa$ for which $F_{p_\xi} \cap F_{p_\eta} = R$ for all $\xi \neq \eta < \aleph_1$. But as there are only countably many functions $R \times n \rightarrow E$, uncountably many of the p_ξ agree on $R \times n$. Given such p_ξ and p_η , it is then immediate that $q = p_\xi \cup p_\eta$ is a common extension. That $|\mathbb{Q}_\kappa| = \kappa^{\aleph_0}$ is clear. \square

Lemma 4.2.10. *Let $p \in \mathbb{Q}_\kappa$. For any $(\alpha, \beta) \in \mathcal{D}_\kappa$, there is a $q \leq p$ with $(\alpha, \beta) \in F_q$.*

Proof. If $(\alpha, \beta) \notin F_p$, we can define $q \leq p$ so that $F_q = F_p \cup \{(\alpha, \beta)\}$, $n_q = n_p$, and $(q(\alpha, \beta, 0), \dots, q(\alpha, \beta, n_q - 1))$ any block sequence in E whatsoever. \square

Lemma 4.2.11. *Let $p \in \mathbb{Q}_\kappa$. For any $M > 0$, there is a $q \leq p$ so that $n_q = n_p + 1$ and $q(\alpha, \beta, n_p) > M$ for all $(\alpha, \beta) \in F_q$.*

Proof. Let $q(\alpha, \beta, i) = p(\alpha, \beta, i)$ for $i < n_p$ and $(\alpha, \beta) \in F_p$, as required. Fix α occurring as a first coordinate in F_p . Enumerate by β_0, \dots, β_k those β with $(\alpha, \beta) \in F_p$. Let $Y_j = \langle p(\alpha, \beta_j, 0), \dots, p(\alpha, \beta_j, n_p - 1) \rangle$ for $j \leq k$. By repeated applications of Lemma 4.2.2 (technically we are applying it to a finite-dimensional space Y , however the lemma remains true by an even simpler proof), there is an $N_0 \geq M$ so that whenever $x > N_0$ and not in Y_j ,

$$\langle q(\alpha, \beta_0, 0), \dots, q(\alpha, \beta_0, n_p - 1), x \rangle \cap Y_j = Y_0 \cap Y_j,$$

for $0 < j \leq k$. Let $q(\alpha, \beta_0, n_p)$ be any vector $x > N_0$ and not in $\bigcup_{j \leq k} Y_j$. Let $Y'_0 = \langle q(\alpha, \beta_0, 0), \dots, q(\alpha, \beta_0, n_p - 1), q(\alpha, \beta_0, n_p) \rangle$.

Continue in this fashion, choosing $N_\ell \geq M$ so that whenever $x > N_\ell$ and not

in Y'_i or Y_j ,

$$\langle q(\alpha, \beta_\ell, 0), \dots, q(\alpha, \beta_\ell, n_p - 1), x \rangle \cap Y'_i = Y_\ell \cap Y'_i = Y_\ell \cap Y_i,$$

and

$$\langle q(\alpha, \beta_\ell, 0), \dots, q(\alpha, \beta_\ell, n_p - 1), x \rangle \cap Y_j = Y_\ell \cap Y_j,$$

for $i < \ell$ and $\ell < j \leq k$. Let $q(\alpha, \beta_\ell, n_p)$ be any vector $x > N_\ell$ and not in $\bigcup_{i < \ell} Y'_i \cup \bigcup_{\ell < j \leq k} Y_j$. At the end of the construction, q will be a condition with domain $F_p \times (n_p + 1)$ extending p and having the desired property. \square

Proof of Theorem 4.2.8. Let G be \mathbf{V} -generic for \mathbb{Q}_κ . By Lemmas 4.2.10 and 4.2.11, $\bigcup G : \mathcal{D}_\kappa \times \omega \rightarrow E$ so that for each $(\alpha, \beta) \in \mathcal{D}_\kappa$, $G_{\alpha, \beta}(\cdot) = \bigcup G(\alpha, \beta, \cdot)$ is an infinite block sequence in E .

Given an uncountable limit $\alpha < \kappa$, we claim that $\langle G_{\alpha, \beta} \rangle \cap \langle G_{\alpha, \gamma} \rangle$ is finite-dimensional, for $\beta \neq \gamma < \alpha$. Let $p \in \mathbb{Q}_\kappa$ be given with $(\alpha, \beta), (\alpha, \gamma) \in F_p$. By the definition of \leq in \mathbb{Q}_κ , we have that

$$p \Vdash \langle \dot{G}_{\alpha, \beta} \rangle \cap \langle \dot{G}_{\alpha, \gamma} \rangle = \langle (\check{p}(\alpha, \beta, i))_{j < n_p} \rangle \cap \langle (\check{p}(\alpha, \gamma, i))_{j < n_p} \rangle.$$

Thus, $\langle G_{\alpha, \beta} \rangle \cap \langle G_{\alpha, \gamma} \rangle$ is forced to be finite-dimensional and $\mathcal{A}_\alpha = \{\langle G_{\alpha, \beta} \rangle : \beta < \alpha\}$ is an almost disjoint family of subspaces. As \mathbb{Q}_κ preserves cardinals, $|\mathcal{A}_\alpha| = |\alpha|$. It remains to show that each \mathcal{A}_α is maximal.

Fix α as above and let τ be a nice \mathbb{Q}_κ -name for a subset of E . As \mathbb{Q}_κ is ccc, there is a countable set of conditions $A \subseteq \mathbb{Q}_\kappa$ which decides which vectors are in τ and whether τ is a (infinite-dimensional) subspace. That is, if $p \Vdash \check{v} \in \tau$, for some $v \in E$ and $p \in \mathbb{Q}_\kappa$, then there is a $q \in A$ with $q \Vdash \check{v} \in \tau$, and likewise if $p \Vdash \tau$ is a subspace. A is contained in

$$\mathbb{Q}_{\kappa, S} = \{p \in \mathbb{Q}_\kappa : (\alpha, \gamma) \in F_p \Rightarrow \gamma \in S\}$$

for some countable $S \subseteq \alpha$. Suppose that

$$p \Vdash \tau \text{ is a subspace of } E \text{ and } \forall \gamma \in \check{S}(\dim(\tau \cap \langle \dot{G}_{\alpha, \gamma} \rangle) < \infty)$$

for $p \in \mathbb{Q}_{\kappa, S}$. Fix $\xi \in \alpha \setminus S$. We claim that for all $M > 0$, the set of conditions $q \in \mathbb{Q}_{\kappa}$ such that

$$q \Vdash \exists v > M(v \in \tau \cap \langle \dot{G}_{\alpha, \xi} \rangle)$$

is dense below p . Let $p' \leq p$. We may assume that $(\alpha, \xi) \in F_{p'}$. Let $p'' = p' \upharpoonright (\{(\alpha, \gamma) : \gamma \in S\} \times n_{p'}) \in \mathbb{Q}_{\kappa, S}$. Then, $p'' \leq p$, and so

$$p'' \Vdash \tau \text{ is a subspace of } E \text{ and } \forall \gamma \in \check{S}(\dim(\tau \cap \langle \dot{G}_{\alpha, \gamma} \rangle) < \infty)$$

By Lemmas 4.2.2 and 4.2.11, there is a $p''' \leq p''$ in $\mathbb{Q}_{\kappa, S}$ and a $v > M$ so that

$$p''' \Vdash \check{v} \in \tau \wedge \forall (\alpha, \gamma) \in \check{F}_{p''}(\check{v} \in \langle \dot{G}_{\alpha, \gamma} \rangle),$$

and moreover, there is a condition $q \in \mathbb{Q}_{\kappa}$ so that $F_q = F_{p'} \cup F_{p'''}$, $n_q = n_{p'''} + 1$, $q(\alpha, \xi, n_{p'''}) = v$, and $q \leq p'$. But then,

$$q \Vdash \exists v > M(v \in \tau \cap \langle \dot{G}_{\alpha, \xi} \rangle),$$

as claimed. Thus, \mathcal{A}_{α} is forced to be a mad family of subspaces.

That $\mathfrak{c} \leq \kappa^{\aleph_0}$ in $\mathbf{V}[G]$ follows from standard facts about ccc forcing (cf. Lemma 5.13 of Ch. VII in [57]). \square

Recall that *Sacks forcing* \mathbb{S} is the collection of all perfect subtrees of $2^{<\omega}$, ordered by inclusion. \mathbb{S} enjoys the *Sacks property*: whenever $p \in \mathbb{S}$ and \dot{g} is an \mathbb{S} -name for an element of ω^{ω} , there is a $q \leq p$ and a function $F : \omega \rightarrow \mathcal{P}(\omega)$ such that for all n , $|F(n)| \leq 2^n$ and $q \Vdash \forall n(\dot{g}(n) \in F(n))$. In particular, \mathbb{S} is ω^{ω} -*bounding*: every element of ω^{ω} in the generic extension is bounded by some element of the ground model.

By the *Sacks model*, we mean the generic extension of a model of CH obtained by forcing with a countable support iteration of Sacks forcing of length ω_2 . Our Theorem 4.2.13 below can be seen as a corollary of Theorem 4.2.7 (or Theorem 4.2.8) and a general theorem of Zapletal (Theorem 0.2 in [92]). However, we wish to give the construction explicitly and avoid the use of large cardinals, which are used in Zapletal's result.

Theorem 4.2.12. (CH) *If \mathbb{P} is a proper⁷ poset of size \aleph_1 having the Sacks property, then there is a \mathbb{P} -indestructible mad family of (block) subspaces.*

Proof. Using CH and properness, we can construct a sequence of pairs (p_ξ, τ_ξ) , $\xi < \omega_1$, so that:

- (i) τ_ξ is a nice \mathbb{P} -name for an element of $\text{bb}^\infty(E)$ with all antichains occurring in τ_ξ countable, and
- (ii) $p_\xi \in \mathbb{P}$ is such that for any such τ and $p \in \mathbb{P}$ with $p \Vdash \tau \in \text{bb}^\infty(E)$, there is a ξ such that $p_\xi \leq p$ and $p_\xi \Vdash \tau = \tau_\xi$.

We construct a family of block sequences $\mathcal{A} = \{X_\alpha : \alpha < \omega_1\}$ recursively as follows: Begin by letting $\{X_i : i \in \omega\}$ be any almost disjoint family of block sequences (i.e., the corresponding subspaces are almost disjoint).

At stage $\alpha \geq \omega$: If

$$p_\alpha \not\Vdash \forall \xi < \alpha (\dim(\langle \tau_\alpha \rangle \cap \langle \check{X}_\xi \rangle) < \infty),$$

then choose X_α to be any block sequence almost disjoint from all of the X_ξ for $\xi < \alpha$. Otherwise, enumerate by (\dot{v}_n) and (\dot{I}_n) \mathbb{P} -names for vectors (in block

⁷We will not say much explicitly about properness here, except to note that \mathbb{S} is proper. See [68] for more on this subject.

position) and intervals containing their supports, respectively, which are forced by p_α to make up τ_α . Enumerate α as $(\xi_n)_{n < \omega}$.

As the X_{ξ_n} are almost disjoint, there is an $f \in \omega^\omega$ so that for all n , $X_{\xi_0}/f(0), \dots, X_{\xi_n}/f(n)$ are disjoint. By our assumption on p_α , there is a \mathbb{P} -name \dot{g} for an element of ω^ω so that

$$p_\alpha \Vdash \forall n (\langle \tau_\alpha / \dot{g}(n) \rangle \cap \langle \check{X}_{\xi_n} \rangle = \{0\}).$$

Claim. If X_0, \dots, X_n, X_{n+1} are disjoint block sequences and $x_0 < \dots < x_n$ so that for all $k \leq n$, $\langle x_0, \dots, x_n \rangle \cap \langle X_k \rangle = \{0\}$, then there is an M so that whenever $x > M$ and not in any of $\langle X_0 \rangle, \dots, \langle X_n \rangle, \langle X_{n+1} \rangle$, then for all $k \leq n+1$, $\langle x_0, \dots, x_n, x \rangle \cap \langle X_k \rangle = \{0\}$.

Proof of claim. See the proof of Lemma 4.2.4. □

By the claim, there is a \mathbb{P} -name \dot{h} for an element of ω^ω so that p_α forces that “whenever $i_0 < \dots < i_n$ and $\dot{h}(0) < \dot{v}_{i_0}, \dots, \dot{h}(n) < \dot{v}_{i_n}$, then $\forall k \leq n \langle \dot{v}_{i_0}, \dots, \dot{v}_{i_n} \rangle \cap \langle \check{X}_{\xi_k} / \dot{f}(k) \rangle = \{0\}$ ”.

As \mathbb{P} is ω^ω -bounding, there is a $p \leq p_\alpha$ and a function $m \in \omega^\omega$ so that

$$p \Vdash \forall n (m(n) \geq \max\{\check{f}(n), \dot{g}(n), \dot{h}(n)\}),$$

and so p forces that m shares the properties of f , g , and h as above. Further, by ω^ω -bounding, there is an increasing sequence of intervals $(J_n)_{n < \omega}$, and a $p' \leq p$, so that

$$p' \Vdash \forall n \exists m (\dot{I}_m \subseteq J_n).$$

Choose a further increasing sequence of intervals $(K_n)_{n < \omega}$ so that K_n contains at least 2^n many intervals of the form J_m , all of which are above $m(n)$.

By the Sacks property, there is a $p'' \leq p$ and a function F with domain ω so that for each n , $|F(n)| \leq 2^n$ and each element of $F(n)$ is a collection of vectors in E , in block position, so that

$$p'' \Vdash \forall n (\{\dot{v}_k : \dot{I}_k \subseteq \check{K}_n\} \in \check{F}(n)),$$

and for all n and $A \in F(n)$, there is a $q \leq p''$ with

$$q \Vdash \{\dot{v}_k : \dot{I}_k \subseteq \check{K}_n\} = \check{A}.$$

For each n , let $A_0, \dots, A_{|F(n)|-1}$ enumerate $F(n)$. We pick vectors u_n^0 recursively as follows: Let u_n^0 be the first element of A_0 . Having defined $u_n^0 < \dots < u_n^j$, with $u_n^i \in A_i$, choose u_n^{j+1} to be the first element of A_{j+1} with support above u_n^j . Note that this can be done as each A_k must contain elements with supports in each of 2^n distinct intervals J_m . Let $X_\alpha = (u_0^0, \dots, u_0^{|F(0)|-1}, u_1^0, \dots, u_1^{|F(1)|-1}, \dots)$. Observe that our choice of m ensures that X_α is a block sequence and is almost disjoint from each X_ξ for $\xi < \alpha$. That

$$p'' \Vdash \dim(\langle \tau_\alpha \rangle \cap \langle X_\alpha \rangle) = \infty$$

is ensured by the construction. It is then easy to show that $\mathcal{A} = \{X_\alpha : \alpha < \omega_1\}$ is forced to be mad by any condition in \mathbb{P} . \square

Theorem 4.2.13. *In the Sacks model, there is a mad family of (block) subspaces of size \aleph_1 .*

Proof. This is proved using Theorem 4.2.12, exactly as Theorem III.2 in [45], which the reader may consult for details. \square

We note that it follows directly from Lemma 4.2.12 that in the model obtained by forcing over a model of CH with the “side-by-side” (i.e., countable

support product of) Sacks forcing [10] of length ω_2 , there is a mad family of subspaces of size \aleph_1 . This is because any reals added in the side-by-side model are added by a product of ω_1 many copies of Sacks forcing.

4.3 Definability and fullness

In [64], Mathias showed that there are no analytic mad families on ω . His proof proceeds by showing that, given an infinite almost disjoint family \mathcal{A} on ω , the set \mathcal{H} of subsets of ω not covered by a finite union of elements of \mathcal{A} is a selective coideal. Were \mathcal{A} analytic, an application of the main Ramsey-theoretic dichotomy of [64] shows that there must be an infinite set $x \in \mathcal{H}$ none of whose infinite subsets are in the \subseteq -downwards closure of \mathcal{A} . Such an x witnesses that \mathcal{A} fails to be maximal.

We would like to replicate this strategy to prove that there are no infinite analytic mad families of subspaces of E . This naïve approach has several issues, which we discuss below.

Let's first consider the setting where F is a finite field, in which case almost disjoint *subspaces* of E are also almost disjoint as *subsets* of E . This suggests the following proof strategy: Suppose that \mathcal{A} is an infinite almost disjoint family of subspaces of E and let \mathcal{H} be the collection of all subsets of E which are not covered by a union of finitely many elements of \mathcal{A} . As above, \mathcal{H} is a selective coideal of subsets of E . Applying Mathias' theorem [64], we obtain an infinite subset $X \in \mathcal{H}$ all of whose further subsets are disjoint from the downwards closure of \mathcal{A} . If \mathcal{A} were maximal, then we would obtain the desired contradiction provided X contains a subspace. However, there is no a priori reason why X

ought to contain a subspace.

In the event that F is a finite field of order 2, hope is provided by Hindman's theorem [39], one formulation of which says that the collection \mathcal{B} of all subsets of E which contain a block subspace is a coideal. It would suffice, then, to show that $\mathcal{H} \cap \mathcal{B}$ is a selective coideal. As the union of two ideals (in any ring) is an ideal if and only if one contains the other, we would need to have that $\mathcal{H} \subseteq \mathcal{B}$ (clearly, $\mathcal{B} \not\subseteq \mathcal{H}$). Unfortunately, this is *never* true: take $X \in \mathcal{H}$ which has infinite intersection with infinitely many elements of \mathcal{A} and build a block sequence Y in X with the same property. Taken as a set, Y contains no subspaces. This argument can be adapted to show that the family of block sequences in E whose spans are in \mathcal{H} fails to be a coideal in the associated Ramsey space of all block sequences, in the sense of [21].

We now turn to a proof strategy based on the Ramsey-theoretic results in Ch. 3. Recall that $\text{bb}^\infty(E)$ denotes the (Polish) space of all infinite block sequences in E . For $X, Y \in \text{bb}^\infty(E)$, we write $X \preceq Y$ if $\langle X \rangle \subseteq \langle Y \rangle$, and $X \preceq^* Y$ if $X/n \preceq Y$ for some n . A nonempty subset of $\text{bb}^\infty(E)$ is a *family* if it is closed upwards with respect to \preceq^* . We refer the reader to Ch. 3 for the definitions of the *(p)-property* (Definition 3.2.3), *fullness* (Definition 3.2.5), the games $F[X]$ and $G[X]$ (the beginning of §3.3), and being *strategic* (Definition 3.4.5).

In what follows, if \mathcal{A} is an infinite almost disjoint family of subspaces of E , we let

$$\mathcal{H}(\mathcal{A}) = \{(x_n) \in \text{bb}^\infty(E) : \exists^\infty Y \in \mathcal{A} (\dim(\langle (x_n) \rangle \cap Y) = \infty)\}.$$

Note that $\mathcal{H}(\mathcal{A})$ is always nonempty, as it contains (e_n) , is closed upwards with respect to \preceq^* , and is thus a family. We will let $\overline{\mathcal{A}}$ denote the \preceq^* -downwards closure of \mathcal{A} . Note that $\overline{\mathcal{A}} \cap \mathcal{H}(\mathcal{A}) = \emptyset$.

Lemma 4.3.1. *If \mathcal{A} is an infinite almost disjoint family of subspaces of E , then for any $X \in \mathcal{H}(\mathcal{A})$,*

- (a) *I and II have strategies in $G[X]$ and $F[X]$, respectively, for playing into $\mathcal{H}(\mathcal{A})$.*
- (b) *If \mathcal{A} is maximal, then I and II have strategies in $G[X]$ and $F[X]$, respectively, for playing into $\overline{\mathcal{A}}$.*

Proof. (a) Fix an enumeration (Y_n) of a countably infinite subset of \mathcal{A} , each Y_n having infinite-dimensional intersection with $\langle X \rangle$, in such a way that each element is repeated infinitely often. To see that I has a strategy in $G[X]$ for playing into $\mathcal{H}(\mathcal{A})$, let I play a block subspace of $\langle X \rangle \cap Y_n$ on their n th move. The resulting outcome will have infinitely many entries in each Y_n and is thus in $\mathcal{H}(\mathcal{A})$. To see that II has a strategy in $F[X]$ for playing into $\mathcal{H}(\mathcal{A})$, let II play the first element of Y_n they can on their n th move.

(b) Suppose that \mathcal{A} is maximal. Take $Y \in \mathcal{A}$ having infinite-dimensional intersection with $\langle X \rangle$. To see that I has a strategy in $G[X]$ for playing into $\overline{\mathcal{A}}$, let I play, repeatedly, any block subspace Z contained in $\langle X \rangle \cap Y$. The resulting outcome will be below Y . To see that II has a strategy in $F[X]$ for playing into $\overline{\mathcal{A}}$, observe that so long as II plays in Y , which they may always do, the outcome will be below Y . □

Lemma 4.3.2. *For X a subspace, Y a block subspace, and $z_0 < \dots < z_\ell$, if $X \subseteq Y + \langle z_0, \dots, z_\ell \rangle$, then there is an M such that $X/M \subseteq Y$.*

Proof. Let (y_n) be a block basis for Y . Let $N = \max_{i \leq \ell}(\text{supp}(z_i))$ and suppose that y_0, \dots, y_k are those basis vectors in Y whose supports are not above N . Let $M = \max\{N, \max(\text{supp}(y_k))\}$. We claim that $X/M \subseteq Y$. Take $x \in X/M$. By assumption, $x = y + w$ where $y \in Y$ and $w \in \langle z_0, \dots, z_\ell \rangle$. Write $y = y' + y''$,

where $y' \in \langle y_0, \dots, y_k \rangle$ and $y'' \in \langle y_{k+1}, y_{k+2}, \dots \rangle$, so that $x - y'' = y' + w$. If either side of this equation is nonzero, then $\text{supp}(x - y'') > M$, but $\text{supp}(y' + w) \leq M$, a contradiction. Thus, $x = y'' \in Y$. \square

Lemma 4.3.3. *If \mathcal{A} is an infinite mad family of subspaces, then $\mathcal{H}(\mathcal{A})$ is strategic and has the (p) -property.*

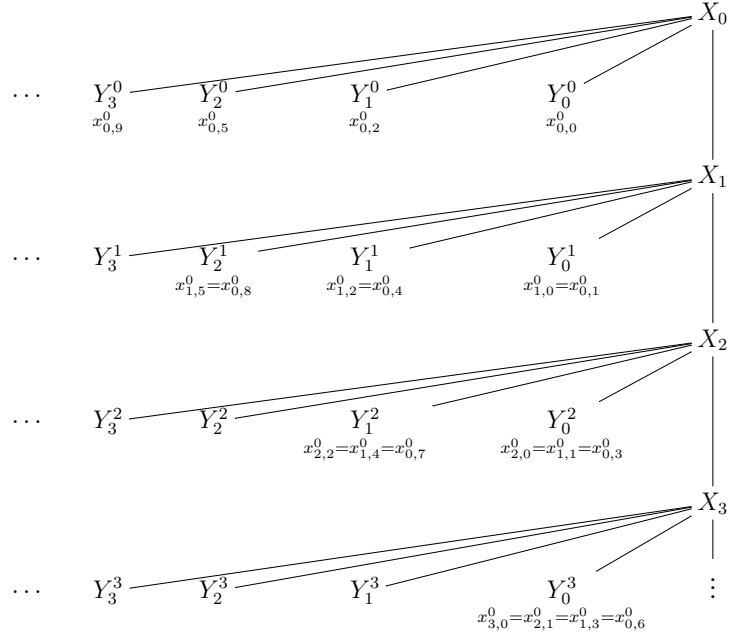
Proof. That $\mathcal{H}(\mathcal{A})$ is strategic is immediate from Lemma 4.3.1(a), as for any $X \in \mathcal{H}(\mathcal{A})$, we may let I use their strategy in $G[X]$ for playing into $\mathcal{H}(\mathcal{A})$.

To see that $\mathcal{H}(\mathcal{A})$ has the (p) -property, let $X_0 \succeq X_1 \succeq X_2 \succeq \dots$ be a decreasing sequence contained within $\mathcal{H}(\mathcal{A})$. Let $X^0 \in \text{bb}^\infty(E)$ be such that $X^0/n \preceq X_n$ for all n and take $Y^0 \in \mathcal{A}$ having infinite-dimensional intersection with $\langle X^0 \rangle$. Following the proof of the corresponding fact for mad families on ω (Proposition 0.7 in [64]), we will construct sequences (X^m) and (Y^m) in $\text{bb}^\infty(E)$ where each Y^m is a distinct element of \mathcal{A} , $\langle X^m \rangle$ has infinite-dimensional intersection with Y^m , and X^m diagonalizes (in the sense of Definition 3.2.2) (X_n) with $X^m/n \preceq X_n$ for all n .

For each n , construct a countably infinite pairwise disjoint family of block sequences \mathcal{A}_n below X_n such that

- (i) for all $Y \in \mathcal{A}_n$, there is a $Y' \in \mathcal{A}$ with $\langle Y \rangle \subseteq Y'$, and
- (ii) for all $Y \in \mathcal{A}_n$, $\langle Y \rangle$ is disjoint from Y^0 .

This can be accomplished as $X_n \in \mathcal{H}(\mathcal{A})$; simply take a countably infinite $\mathcal{A}'_n \subseteq \mathcal{A}$ not containing Y^0 , all of whose elements have infinite-dimensional intersection with $\langle X_n \rangle$, and let \mathcal{A}_n be a set of block bases of subspaces witnessing this. Pairwise disjointness and disjointness from Y^0 , for elements in \mathcal{A}_n , can



be ensured by passing to tail block sequences. Enumerate $\mathcal{A}_n = \{Y_i^n : i \in \omega\}$ in such a way that each block sequence is repeated infinitely often.

Next, we build a decreasing sequence $X_0^0 \supseteq X_1^0 \supseteq X_2^0 \supseteq \dots$ in $\mathcal{H}(\mathcal{A})$ such that for each n , $X_n^0 \supseteq X_{n+1}^0$, and $\langle X_n^0 \rangle$ is almost disjoint from Y^0 . We will denote by $X_n^0 = (x_{n,i}^0)_{i \in \omega}$.

Let $x_{0,0}^0$ be the first entry of Y_0^0 . There must be a nonzero $x \in \langle Y_0^1 \rangle$ above $x_{0,0}^0$ such that no linear combination of x and $x_{0,0}^0$ is in Y^0 , otherwise $Y_0^1 \supseteq^* Y^0$ by Lemma 4.3.2. Let $x_{1,0}^0 = x_{0,1}^0 \in Y_0^1$ be such a vector. We continue in this fashion, following the diagram above, with $X_0^0 = (x_{0,n}^0)$, $X_1^0 = (x_{1,n}^0)$, $X_2^0 = (x_{2,n}^0)$, $X_3^0 = (x_{3,n}^0)$, and so on.

That is, let $x_{0,2}^0 \in Y_1^0$ be a vector above $x_{0,1}^0$ such that no linear combination of it with $x_{0,0}^0$ and $x_{0,1}^0$ lies in Y^0 . Next, let $x_{0,3}^0 = x_{1,1}^0 = x_{2,0}^0 \in \langle Y_0^2 \rangle$ be a vector above $x_{0,2}^0$ such that no linear combination of it with $x_{0,0}^0$, $x_{0,1}^0$ and $x_{0,2}^0$ lies in Y^0 . And so on.

By construction, $X_0^0 \succeq X_1^0 \succeq X_2^0 \succeq \dots$ as each X_n^0 is a subsequence of the previous ones, and each $\langle X_n^0 \rangle$ is disjoint from Y^0 . Moreover, each $\langle X_n^0 \rangle$ has infinite-dimensional intersection with $\langle Y \rangle$, for each $Y \in \mathcal{A}_n$, and $X_n^0 \in \mathcal{H}(\mathcal{A})$. Let X^1 diagonalize (X_n^0) , again with $X^1/n \preceq X_n^0$, and thus diagonalizes the original (X_n) as well with the same error. Let $Y^1 \in \mathcal{A}$ have infinite-dimensional intersection with $\langle X^1 \rangle$. Note that we must have $Y^1 \neq Y^0$.

We continue this process to obtain (X^m) and (Y^m) as desired. Let $i : \omega \rightarrow \omega$ be an everywhere infinity-to-one surjection (i.e., for all $m \in \omega$, $i^{-1}(m)$ is infinite) and consider the sequence of pairs $(X^{i(m)}, Y^{i(m)})$. Construct $X = (x_m)$ so that each $x_m \in \langle X^{i(m)}/m \rangle \cap Y^{i(m)}$. Then, $X \in \mathcal{H}(\mathcal{A})$, and moreover, for all n , if $x \in \langle X/n \rangle$, then x is a linear combination of elements of $X^{i(m_0)}/n, \dots, X^{i(m_k)}/n$, each of which is $\preceq X_n$. So, $X/n \preceq X_n$ for all n . \square

Theorem 4.3.4. *Let \mathcal{A} be an infinite mad family of subspaces and assume that, moreover, $\mathcal{H}(\mathcal{A})$ is full.*

- (a) *If \mathcal{A} is analytic, then \mathcal{A} fails to be maximal.*
- (b) *(Assume that there is a supercompact cardinal.) If \mathcal{A} is in $\mathbf{L}(\mathbb{R})$, then \mathcal{A} fails to be maximal.*

Proof. For (a), suppose that \mathcal{A} was analytic, in which case $\overline{\mathcal{A}}$ is also analytic. By Theorem 3.1.1, there is a $X \in \mathcal{H}(\mathcal{A})$ such that either I has a strategy in $F[X]$ for playing into $\overline{\mathcal{A}}^c$, or II has a strategy in $G[Y]$ for playing into $\overline{\mathcal{A}}$. However, the latter contradicts Lemma 4.3.1(a), while the former contradicts Lemma 4.3.1(b) in the event that \mathcal{A} is maximal. Thus, \mathcal{A} cannot be maximal. Part (b) is proved similarly, using Theorem 3.1.3. \square

In light of this result, we make the following definition:

Definition 4.3.5. A mad family \mathcal{A} of subspaces is *fully mad* if $\mathcal{H}(\mathcal{A})$ is full.

Must a mad family of subspaces be full? Unfortunately, we are only able to show that, consistently, there are such mad families. It remains an open question whether such families exist in ZFC, though we suspect that not every mad family has this property.

It will be useful in what follows to note that if $\mathcal{A} \subseteq \mathcal{B}$ are infinite almost disjoint families of subspaces, then $\mathcal{H}(\mathcal{A}) \subseteq \mathcal{H}(\mathcal{B})$. Recall that $\mathfrak{a}_{\text{vec}}$ is the minimal cardinality of an infinite mad family of subspaces and so the hypothesis of the theorem below holds under CH or MA(σ -centered) by Proposition 4.2.5 and Theorem 4.2.6.

Theorem 4.3.6. ($\mathfrak{a}_{\text{vec}} = \mathfrak{c}$) *There is a fully mad family \mathcal{A} of (block) subspaces.*

Proof. We will define $\mathcal{A} = \bigcup_{\alpha < \mathfrak{c}} \mathcal{A}_\alpha$ via transfinite recursion on \mathfrak{c} . Enumerate by $\{X_\alpha : \alpha < \mathfrak{c}\}$ and $\{D_\alpha : \alpha < \mathfrak{c}\}$ all elements of $\text{bb}^\infty(E)$ and subsets of E , respectively, ensuring that the enumeration X_α repeats each $X \in \text{bb}^\infty(E)$ cofinally often. Fix a bijection $\langle \cdot, \cdot \rangle : \mathfrak{c} \times \mathfrak{c} \rightarrow \mathfrak{c}$.

Begin by letting \mathcal{A}_0 be any countably infinite almost disjoint family of block subspaces. Given $\alpha < \mathfrak{c}$, suppose that for $\beta < \alpha$, \mathcal{A}_β has been defined to be an infinite almost disjoint family of block subspaces with size $\leq |\beta| + \aleph_0$, and that $\mathcal{A}_\beta \subseteq \mathcal{A}_\gamma$ for $\beta \leq \gamma < \alpha$. We define \mathcal{A}_α as follows:

Put $\mathcal{A}'_\alpha = \bigcup_{\beta < \alpha} \mathcal{A}_\beta$. If $\langle X_\alpha \rangle$ is almost disjoint from every element of \mathcal{A}'_α , then put $\mathcal{A}''_\alpha = \mathcal{A}'_\alpha \cup \{\langle X_\alpha \rangle\}$. If not, put $\mathcal{A}''_\alpha = \mathcal{A}'_\alpha$. Say $\alpha = \langle \gamma, \delta \rangle$. If $X_\gamma \notin \mathcal{H}(\mathcal{A}''_\alpha)$, then let $\mathcal{A}_\alpha = \mathcal{A}''_\alpha$. Otherwise, let \mathcal{C} be the collection of elements of \mathcal{A}''_α with which X_γ has infinite-dimensional intersection and consider the following cases:

Case 1: There is a $Z \preceq X_\gamma$ such that $\langle Z \rangle$ is almost disjoint from each $Y \in \mathcal{C}$ and is contained in D_δ . In this case, let \mathcal{B} be a countably infinite almost disjoint family of subspaces below Z . Note that if $V \in \mathcal{B}$ is compatible with some $Y \in \mathcal{A}''_\alpha$, then X_γ must be compatible with that Y , so $Y \in \mathcal{C}$, but this yields a contradiction as $\langle Z \rangle$ must be almost disjoint from such a Y . Let $\mathcal{A}_\alpha = \mathcal{A}''_\alpha \cup \mathcal{B}$, an almost disjoint family by the preceding argument. Then, $Z \in \mathcal{H}(\mathcal{A}_\alpha)$.

Case 2: For every $Y \preceq X_\gamma$ such that $\langle Y \rangle$ is almost disjoint from every element of \mathcal{C} , there is no $Z \preceq Y$ with $\langle Z \rangle \subseteq D_\delta$. Note that if this fails, we are in Case 1. As $|\mathcal{C}| \leq |\alpha| + \aleph_0 < \mathfrak{c} = \mathfrak{a}_{\text{vec}}$, there is a $Y \preceq X_\gamma$ with $\langle Y \rangle$ almost disjoint from each element of \mathcal{C} . Here we are looking at the collection of subspaces $W \cap \langle X_\gamma \rangle$ for $W \in \mathcal{C}$ within $\langle X_\gamma \rangle$ which is, of course, isomorphic to E . Let \mathcal{B} be a countably infinite almost disjoint family below Y , and let $\mathcal{A}_\alpha = \mathcal{A}''_\alpha \cup \mathcal{B}$, an almost disjoint family by the same argument as in Case 1. Then, $Y \in \mathcal{H}(\mathcal{A}_\alpha)$.

We claim that $\mathcal{A} = \bigcup_{\alpha < \mathfrak{c}} \mathcal{A}_\alpha$ is as desired. Note that $\mathcal{H}(\mathcal{A}) = \bigcup_{\alpha < \mathfrak{c}} \mathcal{H}(\mathcal{A}_\alpha)$, as whenever $X \in \mathcal{H}(\mathcal{A})$, a countably infinite subset of \mathcal{A} all compatible with X must occur in some initial \mathcal{A}_α , as $\text{cf}(\mathfrak{c}) > \aleph_0$. Clearly, \mathcal{A} is a mad family. To verify fullness, let $D \subseteq E$ and $X \in \mathcal{H}(\mathcal{A})$, and suppose that for every $Y \in \mathcal{H}(\mathcal{A}) \upharpoonright X$, there is a $Z \preceq Y$ with $\langle Z \rangle \subseteq D$. We may take $\alpha < \mathfrak{c}$ large enough so that $\alpha = \langle \gamma, \delta \rangle$, $X = X_\gamma$, $D = D_\delta$, and $X_\gamma \in \mathcal{H}(\mathcal{A}''_\alpha)$, for \mathcal{A}''_α as in the construction above. If Case 1 occurred for this α , then there is a $Z \in \mathcal{H}(\mathcal{A}_\alpha) \upharpoonright X \subseteq \mathcal{H}(\mathcal{A}) \upharpoonright X$ with $\langle Z \rangle \subseteq D$. If Case 2 occurred for this α , then there is an $Y \in \mathcal{H}(\mathcal{A}_\alpha) \upharpoonright X \subseteq \mathcal{H}(\mathcal{A}) \upharpoonright X$ having no $Z \preceq Y$ with $\langle Z \rangle \subseteq D$, contrary to assumption. Thus, there is a $Z \in \mathcal{H}(\mathcal{A}) \upharpoonright X$ with $\langle Z \rangle \subseteq D$, as required. \square

The above proof can be adapted to show how to generically add a fully mad family of subspaces \mathcal{G} : Let \mathbb{P} be the collection of all countably infinite almost

disjoint families of subspaces, ordered by reverse inclusion. It is easy to see that \mathbb{P} is σ -closed and if G is \mathbf{V} -generic for \mathbb{P} , then $\mathcal{G} = \bigcup G$ is a mad family of subspaces. The arguments in Cases 1 and 2 above show that, for $\mathcal{A} \in \mathbb{P}$, $X \in \mathcal{H}(\mathcal{A})$, and $D \subseteq E$, the set of all $\mathcal{B} \in \mathbb{P}$ such that $\mathcal{H}(\mathcal{B})$ “witnesses fullness for X and D ” is dense below \mathcal{A} . In the language of [35], our proof shows that if $\mathfrak{a}_{\text{vec}} = \mathfrak{c}$, then fully mad families of subspaces *exist generically*.

What can we say about analytic mad families of subspaces in the absence of fullness? Recall that, for a family $\mathcal{H} \subseteq \text{bb}^\infty(E)$ and $X \in \mathcal{H}$, the game $G_{\mathcal{H}}[X]$ is the variant of $G[X]$ in which I is restricted to playing elements of $\mathcal{H} \upharpoonright X$. As a consequence of Theorem 3.11.5 and Lemma 4.3.3, we have the following:

Theorem 4.3.7. *Let \mathcal{A} be an infinite mad family of subspaces. If \mathcal{A} is analytic, then there is an $Y \in \mathcal{H}(\mathcal{A})$ such that II has a strategy in $G_{\mathcal{H}(\mathcal{A})}[Y]$ for playing into $\overline{\mathcal{A}}$.*

Note that were $\mathcal{H}(\mathcal{A})$ to be *+-strategic* (in the sense §3.11), that is, whenever α is a strategy for II in $G_{\mathcal{H}(\mathcal{A})}[X]$, for some $X \in \mathcal{H}(\mathcal{A})$, then there is an outcome of α in $\mathcal{H}(\mathcal{A})$, then the conclusion of the above theorem would yield a contradiction. However, as was shown in Theorem 3.11.9, this is equivalent to $\mathcal{H}(\mathcal{A})$ being full.

This observation suggests that fully mad families of subspaces are analogous to a notion studied by Hrušák in [44] (see also [35]) for almost disjoint families on ω : A mad family \mathcal{A} on ω is *+-Ramsey*⁸ if whenever $T \subseteq \omega^{<\omega}$ is an $\mathcal{I}(\mathcal{A})^+$ -branching tree, there is a branch $b \in [T]$ such that $\text{ran}(b) \in \mathcal{I}(\mathcal{A})^+$. Here, given an ideal \mathcal{I} , an \mathcal{I}^+ -branching tree T is one for which each successor set $\{n : s^\frown(n) \in T\} \in \mathcal{I}^+$, $\mathcal{I}(\mathcal{A})$ is the ideal generated by \mathcal{A} , and $\mathcal{I}(\mathcal{A})^+ = \mathcal{P}(\omega) \setminus \mathcal{I}(\mathcal{A})$ the corresponding coideal.

⁸A closer analogue to being *+-Ramsey* would replace player II with player I in the definition of *+-strategic*, however this does not seem relevant to the present situation.

4.4 Further remarks and open questions

Building on our work in §4.2, there are many questions one could investigate regarding the possible cardinalities of mad families of subspaces in analogy with mad families on ω . However, finding differences in these respective structures may be more rewarding, beginning with the following:

Question. *Can we separate \mathfrak{a} from $\mathfrak{a}_{\text{vec}}$? That is, are either of $\mathfrak{a} < \mathfrak{a}_{\text{vec}}$ or $\mathfrak{a}_{\text{vec}} < \mathfrak{a}$ consistent with ZFC?*

We suspect that [46] will be a relevant reference for this investigation. More generally, it would be interesting to develop any (in ZFC, or just consistently) relationships between $\mathfrak{a}_{\text{vec}}$ and other well-studied cardinal invariants, e.g., \mathfrak{a} , \mathfrak{b} , \mathfrak{d} , \mathfrak{a}_e , \mathfrak{a}_g , etc.

Our work in §4.3 raises the following questions:

Question. *Does there exist an analytic mad family of subspaces of E ?*

In particular, when F is a finite field of order 2, this is equivalent to asking whether there exist an analytic mad family of block subspaces of FIN. If such counterexamples exist, the recent remarkable work by Horowitz and Shelah on definable maximal eventually different families of functions [42] and cofinitary groups [41] may be a starting point.

Question. *Is it consistent with ZFC that there are no fully mad families of subspaces?*

This question appears related to the, still open [35], question of whether it is consistent with ZFC that there are no +-Ramsey mad families on ω .

BIBLIOGRAPHY

- [1] Charles Akemann and Nik Weaver, *$\mathcal{B}(H)$ has a pure state that is not multiplicative on any masa*, Proc. Natl. Acad. Sci. USA **105** (2008), no. 14, 5313–5314. MR 2403097
- [2] Fernando Albiac and Nigel J. Kalton, *Topics in Banach space theory*, Graduate Texts in Mathematics, vol. 233, Springer, New York, 2006. MR 2192298
- [3] Joel Anderson, *A conjecture concerning the pure states of $\mathcal{B}(H)$ and a related theorem*, Topics in modern operator theory (Timișoara/Herculane, 1980), Operator Theory: Adv. Appl., vol. 2, Birkhäuser, Basel-Boston, Mass., 1981, pp. 27–43. MR 672813
- [4] Hiroshi Ando and Yasumichi Matsuzawa, *The Weyl–von Neumann theorem and Borel complexity of unitary equivalence modulo compacts of self-adjoint operators*, Proc. Roy. Soc. Edinburgh Sect. A **145** (2015), no. 6, 1115–1144. MR 3427601
- [5] Spiros A. Argyros and Stevo Todorčević, *Ramsey methods in analysis*, Advanced Courses in Mathematics. CRM Barcelona, Birkhäuser Verlag, Basel, 2005. MR 2145246
- [6] Joan Bagaria and Jordi López-Abad, *Weakly Ramsey sets in Banach spaces*, Adv. Math. **160** (2001), no. 2, 133–174. MR 1839387
- [7] ———, *Determinacy and weakly Ramsey sets in Banach spaces*, Trans. Amer. Math. Soc. **354** (2002), no. 4, 1327–1349 (electronic). MR 1873008
- [8] Stefan Banach, *Théorie des opérations linéaires*, Monografie Matematyczne, Tom 1, Warsaw, 1932.
- [9] Tomek Bartoszyński and Haim Judah, *Set theory: On the structure of the real line*, A K Peters, Ltd., Wellesley, MA, 1995. MR 1350295
- [10] James E. Baumgartner, *Sacks forcing and the total failure of Martin’s axiom*, Topology Appl. **19** (1985), no. 3, 211–225. MR 794485
- [11] Tristan M. Bice, *MAD families of projections on l^2 and real-valued functions on ω* , Arch. Math. Logic **50** (2011), no. 7-8, 791–801. MR 2847547

- [12] ———, *The order on projections in C^* -algebras of real rank zero*, Bull. Pol. Acad. Sci. Math. **60** (2012), no. 1, 37–58. MR 2901386
- [13] ———, *Filters in C^* -algebras*, Canad. J. Math. **65** (2013), no. 3, 485–509. MR 3043037
- [14] Andreas Blass, *Ultrafilters related to Hindman's finite-unions theorem and its extensions*, Logic and combinatorics (Arcata, Calif., 1985), Contemp. Math., vol. 65, Amer. Math. Soc., Providence, RI, 1987, pp. 89–124. MR 891244
- [15] ———, *Combinatorial cardinal characteristics of the continuum*, Handbook of set theory. Vol. 1, Springer, Dordrecht, 2010, pp. 395–489. MR 2768685
- [16] Andreas Blass and Neil Hindman, *On strongly summable ultrafilters and union ultrafilters*, Trans. Amer. Math. Soc. **304** (1987), no. 1, 83–97. MR 906807
- [17] David Booth, *Ultrafilters on a countable set*, Ann. Math. Logic **2** (1970/1971), no. 1, 1–24. MR 0277371
- [18] Jörg Brendle, *The almost-disjointness number may have countable cofinality*, Trans. Amer. Math. Soc. **355** (2003), no. 7, 2633–2649. MR 1975392
- [19] Lawrence G. Brown, Ronald G. Douglas, and Peter A. Fillmore, *Unitary equivalence modulo the compact operators and extensions of C^* -algebras*, Proceedings of a Conference on Operator Theory (Dalhousie Univ., Halifax, N.S., 1973), Springer, Berlin, 1973, pp. 58–128. Lecture Notes in Math., Vol. 345. MR 0380478
- [20] Richard W. Carey and Joel David Pincus, *Unitary equivalence modulo the trace class for self-adjoint operators*, Amer. J. Math. **98** (1976), no. 2, 481–514. MR 0420323
- [21] Carlos A. Di Prisco, José G. Mijares, and Jesús Nieto, *Local ramsey theory. An abstract approach*, To appear in MLQ Math. Log. Q.
- [22] Carlos A. Di Prisco and Stevo Todorčević, *Souslin partitions of products of finite sets*, Adv. Math. **176** (2003), no. 1, 145–173. MR 1978344
- [23] Gerald A. Edgar, *Measurability in a Banach space*, Indiana Univ. Math. J. **26** (1977), no. 4, 663–677. MR 0487448

- [24] John Ernest, *Charting the operator terrain*, Mem. Amer. Math. Soc. **6** (1976), no. 171, iii+207. MR 0463941
- [25] Ilijas Farah, *Semiselective coideals*, Mathematika **45** (1998), no. 1, 79–103. MR 1644345
- [26] ———, *All automorphisms of the Calkin algebra are inner*, Ann. of Math. (2) **173** (2011), no. 2, 619–661. MR 2776359
- [27] Ilijas Farah and Eric Wofsey, *Set theory and operator algebras*, Appalachian Set Theory 2006-2012 (James Cummings and Ernest Schimmerling, eds.), Lecture Notes Series, no. 406, London Mathematical Society, 2012, pp. 63–120.
- [28] Qi Feng, Menachem Magidor, and W. Hugh Woodin, *Universally Baire sets of reals*, Set theory of the continuum (Berkeley, CA, 1989), Math. Sci. Res. Inst. Publ., vol. 26, Springer, New York, 1992, pp. 203–242. MR 1233821
- [29] Valentin Ferenczi and Christian Rosendal, *Ergodic Banach spaces*, Adv. Math. **195** (2005), no. 1, 259–282. MR 2145797
- [30] ———, *Banach spaces without minimal subspaces*, J. Funct. Anal. **257** (2009), no. 1, 149–193. MR 2523338
- [31] Fred Galvin and Karel Prikry, *Borel sets and Ramsey's theorem*, J. Symbolic Logic **38** (1973), 193–198. MR 0337630
- [32] Su Gao, *Invariant descriptive set theory*, Pure and Applied Mathematics, vol. 293, CRC Press, Boca Raton, FL, 2009. MR 2455198
- [33] W. T. Gowers, *A new dichotomy for Banach spaces*, Geom. Funct. Anal. **6** (1996), no. 6, 1083–1093. MR 1421876
- [34] ———, *An infinite Ramsey theorem and some Banach-space dichotomies*, Ann. of Math. (2) **156** (2002), no. 3, 797–833. MR 1954235
- [35] Osvaldo Guzmán-González, Michael Hrušák, Carlos Azarel Martínez-Ranero, and Ulises Ariet Ramos-García, *Generic existence of MAD families*, J. Symb. Log. **82** (2017), no. 1, 303–316. MR 3631288
- [36] Don Hadwin, *Maximal nests in the Calkin algebra*, Proc. Amer. Math. Soc. **126** (1998), no. 4, 1109–1113. MR 1443829

- [37] Michael Hartz and Martino Lupini, *The classification problem for operator algebraic varieties and their multiplier algebras*, To appear in Trans. Amer. Math. Soc.
- [38] Stephen H. Hechler, *Short complete nested sequences in $\beta N \setminus N$ and small maximal almost-disjoint families*, General Topology and Appl. **2** (1972), 139–149. MR 0307913
- [39] Neil Hindman, *Finite sums from sequences within cells of a partition of \mathbb{N}* , J. Combinatorial Theory Ser. A **17** (1974), 1–11. MR 0349574
- [40] Greg Hjorth, *Classification and orbit equivalence relations*, Mathematical Surveys and Monographs, vol. 75, Amer. Math. Soc., Providence, RI, 2000. MR 1725642
- [41] Haim Horowitz and Saharon Shelah, *A Borel maximal cofinality group*, Preprint, 2016.
- [42] ———, *A Borel maximal eventually different family*, Preprint, 2016.
- [43] ———, *Can you take Törnquist’s inaccessible away?*, Preprint, 2016.
- [44] Michael Hrušák, *Selectivity of almost disjoint families*, Acta Univ. Carolin. Math. Phys. **41** (2000), no. 2, 13–21. MR 1802331
- [45] ———, *Life in the Sacks model*, Acta Univ. Carolin. Math. Phys. **42** (2001), no. 2, 43–58, 29th Winter School on Abstract Analysis (Lhota nad Rohanovem/Zahrádky u České Lípy, 2001). MR 1900391
- [46] Michael Hrušák, Juris Steprans, and Yi Zhang, *Cofinitary groups, almost disjoint and dominating families*, J. Symbolic Logic **66** (2001), no. 3, 1259–1276. MR 1856740
- [47] Richard V. Kadison and I. M. Singer, *Extensions of pure states*, Amer. J. Math. **81** (1959), 383–400. MR 0123922
- [48] Akihiro Kanamori, *The higher infinite*, second ed., Springer Monographs in Mathematics, Springer-Verlag, Berlin, 2003. MR 1994835
- [49] Vladimir Kanovei, *Borel equivalence relations*, University Lecture Amer. Math. Soc. Series, vol. 44, Amer. Math. Soc., Providence, RI, 2008, Structure and classification. MR 2441635

- [50] Alexander S. Kechris, *Classical descriptive set theory*, Graduate Texts in Mathematics, vol. 156, Springer-Verlag, New York, 1995. MR 1321597
- [51] ———, *Actions of Polish groups and classification problems*, Analysis and logic (Mons, 1997), London Math. Soc. Lecture Note Ser., vol. 262, Cambridge Univ. Press, 2002, pp. 115–187. MR 1967835
- [52] Alexander S. Kechris and Alain Louveau, *The classification of hypersmooth Borel equivalence relations*, J. Amer. Math. Soc. **10** (1997), no. 1, 215–242. MR 1396895
- [53] Alexander S. Kechris and Nikolaos E. Sofronidis, *A strong generic ergodicity property of unitary and self-adjoint operators*, Ergodic Theory Dynam. Systems **21** (2001), no. 5, 1459–1479. MR 1855842
- [54] Oren Kolman, *Almost disjoint families: an application to linear algebra*, Electron. J. Linear Algebra **7** (2000), 41–52. MR 1756262
- [55] Ryszard A. Komorowski and Nicole Tomczak-Jaegermann, *Banach spaces without local unconditional structure*, Israel J. Math. **89** (1995), no. 1-3, 205–226. MR 1324462
- [56] Kenneth Kunen, *Some points in βN* , Math. Proc. Cambridge Philos. Soc. **80** (1976), no. 3, 385–398. MR 0427070
- [57] ———, *Set theory*, Studies in Logic and the Foundations of Mathematics, vol. 102, North-Holland, Amsterdam, 1980. MR 597342
- [58] Claude Laflamme, *Forcing with filters and complete combinatorics*, Ann. Pure Appl. Logic **42** (1989), no. 2, 125–163. MR 996504
- [59] Claude Laflamme, Lionel Nguyen Van Thé, Maurice Pouzet, and Norbert Sauer, *Partitions and indivisibility properties of countable dimensional vector spaces*, J. Combin. Theory Ser. A **118** (2011), no. 1, 67–77. MR 2737185
- [60] Paul B. Larson, *The stationary tower: Notes on a course by w. hugh woodin*, University Lecture Series, vol. 32, Amer. Math. Soc., Providence, RI, 2004. MR 2069032
- [61] Jordi López-Abad, *Coding into Ramsey sets*, Math. Ann. **332** (2005), no. 4, 775–794. MR 2179777

- [62] Adam W. Marcus, Daniel A. Spielman, and Nikhil Srivastava, *Interlacing families II: Mixed characteristic polynomials and the Kadison-Singer problem*, *Ann. of Math. (2)* **182** (2015), no. 1, 327–350. MR 3374963
- [63] Donald A. Martin and Robert M. Solovay, *Internal Cohen extensions*, *Ann. Math. Logic* **2** (1970), no. 2, 143–178. MR 0270904
- [64] A. R. D. Mathias, *Happy families*, *Ann. Math. Logic* **12** (1977), no. 1, 59–111. MR 0491197
- [65] José G. Mijares, *A notion of selective ultrafilter corresponding to topological Ramsey spaces*, *MLQ Math. Log. Q.* **53** (2007), no. 3, 255–267. MR 2330595
- [66] Arnold W. Miller, *Infinite combinatorics and definability*, *Ann. Pure Appl. Logic* **41** (1989), no. 2, 179–203. MR 983001
- [67] Keith R. Milliken, *Ramsey's theorem with sums or unions*, *J. Combinatorial Theory Ser. A* **18** (1975), 276–290. MR 0373906
- [68] Justin Tatch Moore, *The Proper Forcing Axiom: A tutorial*, Young Set Theory Workshop 2010, Raach, Austria, 2010.
- [69] ———, *The method of forcing*, In preparation, 2016.
- [70] Gerard J. Murphy, *C^* -algebras and operator theory*, Academic Press, Inc., Boston, MA, 1990. MR 1074574
- [71] Michael Ray Oliver, *Borel cardinalities below c_0* , *Proc. Amer. Math. Soc.* **134** (2006), no. 8, 2419–2425. MR 2213716
- [72] Gert K. Pedersen, *Analysis now*, Graduate Texts in Mathematics, vol. 118, Springer-Verlag, New York, 1989. MR 971256
- [73] N. Christopher Phillips and Nik Weaver, *The Calkin algebra has outer automorphisms*, *Duke Math. J.* **139** (2007), no. 1, 185–202. MR 2322680
- [74] Frank P. Ramsey, *On a Problem of Formal Logic*, *Proc. London Math. Soc.* **S2-30** (1930), no. 1, 264. MR 1576401
- [75] Christian Rosendal, *Infinite asymptotic games*, *Ann. Inst. Fourier (Grenoble)* **59** (2009), no. 4, 1359–1384. MR 2566964

- [76] ———, *An exact Ramsey principle for block sequences*, *Collect. Math.* **61** (2010), no. 1, 25–36. MR 2604856
- [77] Saharon Shelah, *Proper and improper forcing*, second ed., *Perspectives in Mathematical Logic*, Springer-Verlag, Berlin, 1998. MR 1623206
- [78] ———, *Two cardinal invariants of the continuum ($\mathfrak{d} < \mathfrak{a}$) and FS linearly ordered iterated forcing*, *Acta Math.* **192** (2004), no. 2, 187–223. MR 2096454
- [79] Saharon Shelah and Otmar Spinas, *Mad spectra*, *J. Symbolic Logic* **80** (2015), no. 3, 901–916. MR 3395354
- [80] Saharon Shelah and W. Hugh Woodin, *Large cardinals imply that every reasonably definable set of reals is Lebesgue measurable*, *Israel J. Math.* **70** (1990), no. 3, 381–394. MR 1074499
- [81] Jack Silver, *Every analytic set is Ramsey*, *J. Symbolic Logic* **35** (1970), 60–64. MR 0332480
- [82] Iian B. Smythe, *A local ramsey theory for block sequences*, Submitted, 2016.
- [83] Iian B. Smythe, *Borel equivalence relations in the space of bounded operators*, *Fund. Math.* **237** (2017), no. 1, 31–45. MR 3606397
- [84] Stevo Todorčević, *Introduction to Ramsey spaces*, *Annals of Mathematics Studies*, vol. 174, Princeton University Press, Princeton, NJ, 2010. MR 260381
- [85] Asger Törnquist, Σ_2^1 and Π_1^1 mad families, *J. Symbolic Logic* **78** (2013), no. 4, 1181–1182. MR 3156517
- [86] ———, *Definability and almost disjoint families*, Preprint, 2015.
- [87] John von Neumann, *Charakterisierung des spektrums eines integraloperators*, *Actualités Sci. Indust* **229** (1935), no. 212, 38–55.
- [88] Nik Weaver, *Set theory and C^* -algebras*, *Bull. Symbolic Logic* **13** (2007), no. 1, 1–20. MR 2300900
- [89] Eric Wofsey, $P(\omega)/\text{fin}$ and projections in the Calkin algebra, *Proc. Amer. Math. Soc.* **136** (2008), no. 2, 719–726. MR 2358514

- [90] Beatriz Zamora-Aviles, *The structure of order ideals and gaps in the Calkin algebra*, 2010, Ph.D. Thesis – York University (Canada). MR 2782268
- [91] ———, *Gaps in the poset of projections in the Calkin algebra*, *Israel J. Math.* **202** (2014), no. 1, 105–115. MR 3265315
- [92] Jindřich Zapletal, *Isolating cardinal invariants*, *J. Math. Log.* **3** (2003), no. 1, 143–162. MR 1978947
- [93] Yuan Yuan Zheng, *Selective ultrafilters on FIN*, To appear in *Proc. Amer. Math. Soc.*