

DERIVED COMMUTING SCHEMES,
REPRESENTATION HOMOLOGY, AND
COHOMOLOGY OF LIE ALGEBRAS

A Dissertation

Presented to the Faculty of the Graduate School

of Cornell University

in Partial Fulfillment of the Requirements for the Degree of

Doctor of Philosophy

by

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August 2025

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DERIVED COMMUTING SCHEMES, REPRESENTATION HOMOLOGY, AND
COHOMOLOGY OF LIE ALGEBRAS

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Cornell University 2025

The commuting schemes of an algebraic group or a Lie algebra, along with their derived versions, play a role in many areas of mathematics. They can be viewed as derived representation schemes, whose basic algebraic invariant is representation homology. The case of reductive groups and reductive Lie algebras have been studied in the literature. In the present thesis, we focus on representation homology with coefficients in non-reductive groups and non-reductive Lie algebras.

We begin by establishing a connection between the commuting schemes $C(U_n)$ of a group scheme U_n consisting of upper triangular unipotent matrices, and the representation homology $\mathrm{HR}_*(T^2, U_n)$ of the topological torus T^2 with coefficients in the group U_n . As an outcome, we provide a criterion for determining whether commuting schemes $C(U_n)$ form a complete intersection.

We then obtain comparison theorems between different types of representation homology, and we establish an equivalence between the derived commuting scheme of a unipotent algebraic group and that of its associated nilpotent Lie algebra. Additionally, we refine the test for the formal smoothness of associative algebras via representation homology, proposed by Berest, Felder, and Ramadoss in [2].

Using the characteristic trace map linking the classical Lie algebra homology, we describe the higher-degree classes appearing in the representation homology of 2-dimensional Lie algebras with nilpotent coefficients. This result reveals connections with classical Bott–Kostant Theorem concerning the cohomology of maximal nilpotent

subalgebra of a semisimple Lie algebra.

BIOGRAPHICAL SKETCH

Guanyu Li was born in Zibo, Shandong Province, China in 1994. He spent his primary school, middle school and high school in this industrial city, and he went to Sun Yat-sen University for college in Guangzhou, China. Then he pursued his math PhD at Cornell in 2018-2025.

This thesis is dedicated to my parents.

ACKNOWLEDGEMENTS

First and foremost, I want to express my deepest gratitude to my advisor, Yuri Berest, without whom I cannot imagine making it to this point. Over the years, I have learned so much from our long discussions, as well as from his classes and lecture notes. It is no surprise that the way I think and do mathematics has been deeply influenced by him. I believe this will remain one of the greatest treasures of my academic career.

I would like to thank Mike Stillman, who helped me build the code that made computations possible. This marked the beginning and inspiration of the entire project. Taking that step into computation helped me overcome the first—and perhaps the biggest—barrier in my mathematical journey.

I owe a great deal of thanks to many professors in the Cornell Math Department, including Dan Halpern-Leistner, Allen Knutson, Inna Zakharevich, Tara Holm, Jason Manning, and Marie MacDonald. I had the privilege of taking classes from them or working as a teaching assistant in their courses.

I'm grateful to Yun Liu and Isaac Goldberg, who, as fellow students of Yuri Berest, shared many enjoyable and thought-provoking mathematical conversations with me. I'll always cherish those times. I also fondly remember the parties and Chinese dinners I shared with my fellow Chinese graduate students in the department, including Beihui Yuan, Daoji Huang, Zaoli Chen, Shiping Cao, Yichen Ma, Dexuan Hu, Yibo Ji, Ruoqi Zhang, Xuan Yao, Xingzhe Li, and Mingjun Sun. To the Algebraic Geometry and Homotopy graduate student groups: thank you all for the many discussions and your generous exchange of ideas. I greatly appreciate the time I spent with everyone in these communities.

Special thanks go to Melissa, Jim, and Elly for your help and support as GFAs. Your assistance made a real difference.

I also want to give a big hug to my girlfriend, who has become—and continues to

be—my greatest source of emotional support during the final stage of my PhD.

Lastly, though the words must cross an ocean, I want to thank my parents, who have supported me throughout my life. I owe everything to your unwavering belief in me.

TABLE OF CONTENTS

Biographical Sketch	iii
Dedication	iv
Acknowledgements	v
Table of Contents	vii
1 Introduction	1
1.1 Motivation	1
1.2 Main results	5
1.2.1 Derived commuting schemes in unipotent coefficients	5
1.2.2 Comparisons of representation homology	6
1.2.3 Characteristic maps	8
1.2.4 Open questions	9
1.3 Organization of the paper	10
1.4 Notations and conventions	10
2 Simplicial homotopy theories	12
2.1 The simplex category and simplicial objects	12
2.1.1 Simplicial groups	14
2.2 Model categories	15
2.2.1 Quillen pairs and Quillen equivalences	18
2.2.2 The Dold-Kan correspondence	20
2.2.3 Promoting model category structures	20
2.3 CDGA	22
2.4 Derived algebraic geometry models	24
2.4.1 DG (simplicial) scheme	26
3 Lie algebras and their decompositions	28
3.1 Lie algebras, semisimple Lie algebras, and triangular decompositions	28
3.1.1 Semisimple Lie algebras	28
3.1.2 Structure of complex semisimple Lie algebras	31
3.2 Universal enveloping algebra	33
4 Homological algebra	36
4.1 Lie algebra homology and cohomology	36
4.1.1 Lie (co)homology as derived functor	36
4.1.2 Chavelley-Eilenberg complex	37
4.1.3 Bott-Kostant theorem	40
4.2 Koszul Duality	41
4.2.1 Twisting cochains	41
4.2.2 Duality	42
4.2.3 Other forms of duality	43
4.3 Quillen homology	45

4.4	Functor homology	48
4.5	Other useful homological algebra results	55
5	Representation homology	57
5.1	Representation homology in terms of derived functor	57
5.1.1	Algebraic groups and Lie algebras as functors	57
5.1.2	Representation schemes	60
5.1.3	Representation homology	61
5.2	Representation homology by functor homology	64
6	Results	66
6.1	Representation homology of torus with unipotent coefficients	66
6.1.1	Representation homology of unipotent coefficients	69
6.1.2	A counterexample	71
6.1.3	A partial collection of the computational results	71
6.2	Generalization of the coefficients of representation homology	76
6.2.1	Representation homology of monoids	76
6.2.2	Representation homology of algebras	81
6.2.3	Application : formal smoothness	89
6.3	Comparisons of representation homology	94
6.3.1	Representation homology of monoids and algebras	94
6.3.2	Representation homology of algebras and Lie algebras	95
6.3.3	Representation homology of monoids and groups	98
6.3.4	Application : Derived commuting schemes of unipotent groups and their associated nilpotent Lie algebras	100
6.4	Characteristic pairing with nilpotent Lie algebra coefficient	106
6.4.1	Lie algebra characteristic pairing map	107
6.4.2	Characteristic pairing in nilpotent coefficient	110
7	Open questions	118
7.1	A unipotent analogue of Macdonald identity	118
7.2	Exceptional classes	119
7.3	Representation homology of semisimple and nilpotent coefficients	119
7.4	The character trace map for spaces	120
	Bibliography	121

CHAPTER 1

INTRODUCTION

1.1 Motivation

Given a complex reductive algebraic group G (or a reductive Lie algebra \mathfrak{g}), its commuting scheme plays a role in many areas of mathematics. Geometrically, these schemes are highly singular, and many desirable properties—such as being a complete intersection—generally do not hold. Their derived versions can be used to remedy these deficiencies by considering additional higher homological information.

The derived commuting scheme of a reductive algebraic group G (resp. reductive Lie algebra \mathfrak{g}) is a special case of the so-called derived representation scheme. As an object in derived algebraic geometry (DAG), the derived representation scheme admits different models. In the language of Toën and Vezzosi [52], one has the derived mapping stack construction $\mathbf{Map}((X, x_0), (BG, *))$. An alternative model is given by Kapranov’s moduli space of derived G -local systems, constructed via a certain injective resolution of BG [35]. As the nonabelian derived functor of the classical representation scheme, the derived representation schemes are introduced and studied in [3, 8, 9, 10]. All these models are shown to be equivalent ([8, Appendix]), and are regarded as homotopically correct derived enhancements. Their classical truncation recovers the correct moduli functor, meaning the underlying scheme $\mathrm{Rep}_G(\Gamma)$ (or $\mathrm{Rep}_{\mathfrak{g}}(\mathfrak{a})$) that parametrizes the family of representations of Γ in G (resp. representations of \mathfrak{a} in \mathfrak{g}).

Unlike the derived local system model requiring an injective resolution of BG , which is computationally inaccessible, or the derived mapping stack model relying on highly abstract technical machinery, representation homology could be defined in the language

of classical homotopy theory in the sense of [48]. This approach is more accessible to explicit computations. Let us briefly describe it. A finite dimensional algebraic group G (or a finite dimensional Lie algebra \mathfrak{g}) can be viewed as a functor $\mathbf{CommAlg} \rightarrow \mathbf{Gp}$ (resp. $\mathbf{CommAlg} \rightarrow \mathbf{Lie}_k$). This functor admits a left adjoint

$$(-)_G : \mathbf{Gp} \rightarrow \mathbf{CommAlg} \quad (\text{resp. } (-)_{\mathfrak{g}} : \mathbf{Lie} \rightarrow \mathbf{CommAlg})$$

which is called the representation functor. The adjunction extends naturally to simplicial objects as

$$(-)_G : s\mathbf{Gp} \rightleftarrows s\mathbf{CommAlg} : G \quad (\text{resp. } (-)_{\mathfrak{g}} : s\mathbf{Lie}_k \rightleftarrows s\mathbf{CommAlg} : \mathfrak{g}).$$

Then the left adjoint admits a well-behaved left total derived functor (cf [10, Lemma 3.1.], [1, Theorem 2.2.]), which we denote by $\mathbb{L}(-)_G$ (resp. $\mathbb{L}(-)_{\mathfrak{g}}$). Thus, following [8, 10], given any simplicial group Γ , we define the **representation homology** $\mathrm{HR}_i(\Gamma, G) := \pi_i(\mathbb{L}(\Gamma)_G)$ to be the homotopy groups of derived representation functor applied to Γ . A theorem of [8] asserts that

$$\mathbb{R}\mathrm{Spec}(\mathbb{L}(\Gamma)_G) \simeq \mathbf{Map}((B\Gamma, x_0), (BG, *))$$

where $\mathbb{R}\mathrm{Spec}$ stands for the (right) derived functor of classical Spec functor, in the sense of [51, 52] (see also Example 2.4.2).

In this framework, representation homology thought as an algebraic invariant of the derived representation scheme reveals genuine features of homotopy theory, rather than focusing solely on geometric aspects. For example, there are various analogues of trace maps, canonical pairings, and comparison maps naturally constructed. In some cases, representation homology of reductive group coefficients have been computed. Notably, Berest, Ramadoss, and Yeung [8] showed the local behavior of representation homology on a Riemann surface Σ_g of genus g with reductive coefficient. They conjecture that these local computations hold true globally. Precisely

Conjecture 1 ([8, Conjecture 1.]). *For a connected affine reductive group G over \mathbb{C} , the following is true:*

1. $\mathrm{HR}_i(T^2, G) = 0$ for $i > \mathrm{rank} G$, and
2. if $g \geq 2$, $\mathrm{HR}_i(\Sigma_g, G) = 0$ for $i > \dim Z(G)$.

On the other hand, the reductive group G (resp., Lie algebra \mathfrak{g}) naturally acts on the derived representation scheme $\mathrm{DRep}_G(\Gamma)$ (resp., $\mathrm{DRep}_{\mathfrak{g}}(\alpha)$), and the resulting quotient—known as the derived character scheme—is also of vital importance. Taking the quotient corresponds to forgetting the basepoint data in both the derived local system and derived mapping stack models. In the language of representation homology, this operation corresponds to taking the invariant part of the associated algebraic invariants.

Let us now dive into some details using the Lie algebra framework. In the simplest case, when $\alpha = \mathfrak{k}$ is a one-dimensional Lie algebra, we have

$$\mathrm{HR}_*(\alpha, \mathfrak{g}) \cong k[\mathfrak{g}] \quad \text{and} \quad \mathrm{HR}_*(\alpha, \mathfrak{h})^W \cong k[\mathfrak{h}]^W,$$

and the Chavelley Restriction Theorem asserts that $k[\mathfrak{g}]^{\mathfrak{g}} \rightarrow k[\mathfrak{h}]^W$ is an isomorphism. This restriction map admits a natural derived extension

$$\mathrm{HR}_*(\alpha, \mathfrak{g})^{\mathfrak{g}} \rightarrow \mathrm{HR}_*(\alpha, \mathfrak{h})^W \tag{1.1}$$

defined for any Lie algebra α , and it is natural to ask when this map is an isomorphism for finite dimensional complex reductive Lie algebras. By Berest, Felder et al. we have

Conjecture 2 ([1, Conjecture 8.1.]). *The map 1.1 is an isomorphism when $\dim \alpha = 2$.*

It is shown in [41, Theorem 1.4.7.] that the map 1.1 is an isomorphism under an technical assumption which is not published yet. By computing the weighted Euler

characteristics of both sides of 1.1, conjecturally there is a constant term identity

$$\frac{(1-qt)^l}{(1-q)^l(1-t)^l} \text{CT} \prod_{\alpha \in R} \frac{(1-qte^\alpha)(1-e^\alpha)}{(1-qe^\alpha)(1-te^\alpha)} = \sum_{w \in W} \frac{\det(1-qtw)}{\det(1-qw)\det(1-tw)} \quad (1.2)$$

where R is a system of roots of the Lie algebra \mathfrak{g} , $l := \dim_k(\mathfrak{h})$ is its rank and CT denotes the constant term of a Laurent polynomial. Specialize Equation (1.2) at $k = \mathbb{C}$ and $t = 0$ yielding a famous equation

$$\int_G \frac{\det(1-qt \text{Ad } g)}{\det(1-q \text{Ad } g)\det(1-t \text{Ad } g)} dg = \prod_{i=1}^l \frac{1}{1-q^{d_i}},$$

which exhibits the equality of the Poincaré series of both sides of the Chevalley isomorphism, where the integration is taken over a real compact form of the complex Lie group G equipped with the invariant Haar measure dg normalized.

On the other hand, the Drinfeld trace map is defined in [1] in the spirit of trace maps in K-theory. In the simplest case when \mathfrak{a} is a one-dimensional Lie algebra, the Drinfeld trace map coincides with the inverse of the Chevalley isomorphism. For models of even-dimensional spheres, it coincides with the Hopf-Koszul-Samuelson isomorphism

$$\left(\bigwedge \mathfrak{g}\right)^G \cong \bigwedge(\text{Prim } \mathfrak{g}),$$

which could be viewed as a natural ‘odd’ analogue to the Chavelley isomorphism. When the Drinfeld trace is an isomorphism, as discussed in [1], taking Poincaré series yields several well-known Macdonald’s famous (q, t) -constant term identity

$$\frac{1}{|W|} \text{CT} \prod_{n \geq 0} \prod_{\alpha \in R} \frac{1 - q^n e^\alpha}{1 - q^n t e^\alpha} = \prod_{n \geq 0} \prod_{i=1}^l \frac{(1 - q^n t)(1 - q^{n+1} t^{d_i-1})}{(1 - q^{n+1})(1 - q^n t^{d_i})} \quad (1.3)$$

proposed [43] and proven in [13]. The fact that the Drinfeld trace map is an isomorphism is essentially equivalent to the strong Macdonald conjecture, originally proposed in [28, 22] and later proven in [23].

From the results and conjectures mentioned above, it is evident that the existing discussions primarily concern reductive algebraic group schemes G (reductive Lie algebra

g). However, the construction of the derived representation scheme does not inherently require the group scheme (or the Lie algebra) to be reductive.

1. When the group (or Lie algebra) is not reductive, what can be said about the (derived) commuting scheme and the (derived) character scheme?
2. If we understand the derived representation scheme for a unipotent group (or a nilpotent Lie algebra), what can we gain in the reductive case, when the unipotent group (or nilpotent Lie algebra) comes from the reductive group (Lie algebra)?

At present, very little is known about representation homology for non-reductive groups or Lie algebras. In this thesis, we shed some light on these questions, by focusing on the opposite extreme of the reductive case—namely, unipotent groups and nilpotent Lie algebras.

1.2 Main results

1.2.1 Derived commuting schemes in unipotent coefficients

We first provide some concrete computations on representation homology of unipotent groups and their links to the underlying representation schemes.

It is generally agreed that derived algebraic geometry provides a natural and powerful generalization of classical algebraic geometry, and a derived object contains valuable geometric information. However, from a computational point of view there seems to be a gap between derived algebraic geometry and classical algebraic geometry. There are not so many examples where the numerical invariants of derived objects give geomet-

ric data of classical algebro-geometric objects. The next theorem provides one such example:

Theorem 1.2.1 (Proposition 6.1.5, Theorem 6.1.8 and Proposition 6.1.9). *Let U_n be the algebraic group of $n \times n$ upper triangular unipotent matrices over a characteristic 0 field k , and let $C(U_n)$ be the commuting scheme of U_n . Then $C(U_n)$ is a (global) complete intersection of codimension $\frac{(n-2)(n-1)}{2}$ in $U_n \times U_n$ if and only if the representation homology vanishes in degrees greater or equal than the size of matrices, i.e.*

$$HR_i(\Sigma_g, U_n) = 0 \quad \forall i \geq n. \quad (1.4)$$

Condition 1.4 holds for $n \leq 5$, so that $C(U_2), C(U_3), C(U_4), C(U_5)$ are complete intersections of correct dimensions. In contrast, $C(U_n)$ is not a complete intersection of codimension $\frac{(n-2)(n-1)}{2}$ in $U_n \times U_n$ when $n \geq 6$.

The classification is carried out in part using the Macaulay2 ([27]) package developed in [40], and in part through a theoretical proof (see Proposition 6.1.9), which is also strongly motivated by the Macaulay2 computations. The failure of being a complete intersection is related to the existence of some exceptional characteristic classes, which still remain mysterious from both geometric and representation-theoretic perspectives.

We also present a partial collection of the computational results, including a table indicating the pathology of non-complete-intersection phenomenon, and the representation homology of maximal nilpotent subalgebras of small rank semisimple algebras.

1.2.2 Comparisons of representation homology

It is not surprising that the derived commuting scheme of an algebraic group may be related to that of its Lie algebra. However, the two derived commuting schemes are not

equivalent in general when the group is reductive (see Example 6.3.17). The situation seems different for unipotent groups and their corresponding nilpotent Lie algebras. In this case, one might expect the exponential map—which is an isomorphism of schemes $\mathfrak{n} \rightarrow U$ —to compare the derived commuting schemes. Unfortunately, the exponential map does not lift to be a morphism between two derived objects naturally. Instead, we will use intermediate objects to facilitate the comparison.

There are various types of representation homology, and one should expect relationships among them. For instance, in [6], Berest and Ramadoss establish the comparison

$$\mathrm{HR}_*^{\mathrm{Alg}}(k[\Gamma], \mathrm{Mat}_n) \cong \mathrm{HR}_*^{\mathrm{Gp}}(\Gamma, \mathrm{GL}_n)$$

which serves to connect symmetric homology with representation homology. A substantial portion of this thesis is devoted to studying such comparisons between different types of representation homology. As a useful application of the comparison theorems, we prove that the derived commuting scheme of the maximal unipotent subgroup of GL_n is equivalent to the derived commuting scheme of its associated nilpotent Lie algebra (see Theorem 6.3.15).

To establish this result, we first consider the representation homology defined in [3, 1] for algebras, with more choices of coefficients. It appears unnecessary to change the coefficient from matrices to arbitrary unital algebras, since, classically, representations of an algebra are understood as maps into matrix algebras. However, this generalization becomes useful when testing for the formal smoothness of associative algebras. As a motivation, we recall a heuristic principle in noncommutative geometry proposed by Kontsevich and Rosenberg in [36]: it suggests that geometrically meaningful properties of a noncommutative algebra A should induce standard geometric properties on all its representation schemes in matrix algebras. In other words, classical representation schemes can be viewed as approximations of “noncommutative scheme” $\mathrm{Spec} A$. This

principle was refined and extended to derived representation schemes in [2]: this says that the higher representation homology measures the failure of Kontsevich–Rosenberg approximation. We provide Example 6.2.23 as an illustrative example: it exhibits a failure of smoothness that is not detected by representation homology with matrix coefficients. However, when the category of coefficients is enlarged, representation homology does reveal this failure of smoothness.

1.2.3 Characteristic maps

As the name suggests, the representations of an algebraic group (or Lie algebra) are expected to encode information about representation homology. Studied in [3, 5], there are characteristic maps

$$\mathrm{Tr}_n(A) : \mathrm{HC}_i(A) \rightarrow \mathrm{HR}_i(A, n)$$

extending the canonical trace $\mathrm{Tr}_n(A) : \mathrm{HC}_0(A) \rightarrow \mathrm{HR}_0(A, n)$ to the higher cyclic homology of an associative algebra A . In the Lie algebra setting, we construct an analogous characteristic map

$$\chi_{\mathfrak{a}}(\mathfrak{g}) : \bigoplus_{p+q=n} H_n(\mathfrak{a}; k) \otimes H^q(\mathfrak{g}; k) \rightarrow \mathrm{HR}_p(\mathfrak{a}, \mathfrak{g})$$

using Koszul duality. This map, which can be viewed as a twisted evaluation, sends (co)homology classes of Lie algebras to classes in representation homology.

In the derived commuting scheme case, the characteristic trace map turns out to be trivial for semisimple Lie algebras (see Example 6.4.3). However, in Theorem 6.4.4, we establish a connection between the derived commuting scheme of the maximal nilpotent subalgebra of a semisimple Lie algebra and the classical Bott–Kostant theorem concerning the cohomology of nilpotent Lie algebras. This implies that this map captures significant information about higher-degree classes, related to the computational results

Theorem 1.2.1. With this interpretation, we recover a connection between representation homology and the structure of the root system of the semisimple Lie algebra.

1.2.4 Open questions

Finally, we want to point out that there are still many interesting questions remaining to be studied. As we already have some information about the commuting scheme of nilpotent Lie algebras, the first one is the question that posted before, that how to deduce information of derived commuting scheme of a reductive Lie algebra from the derived commuting scheme of its maximal nilpotent Lie algebra, especially the invariant part information. More precisely, the Chavelley restriction theorem establishes an explicit description of the invariant ring $k[\mathfrak{g}^*]^{\mathfrak{g}}$, indicating that triangular decomposition plays a useful role in computations. It would therefore be valuable to understand how the representation homology of a nilpotent Lie algebra relates to that of a reductive Lie algebra.

It is shown in [4, Corollary 3.1] that the image of the characteristic map lies in the invariant part. On the other hand, Theorem 1.2.1 demonstrates that representation homology with unipotent (resp. nilpotent) coefficients often contains additional, mysterious classes. In light of the implications in [4], it would be highly desirable to better understand these “exceptional” classes from both geometric and representation-theoretic perspectives.

Moreover, constructing a multiplicative analogue of the Lie algebra characteristic pairing map would be of independent interest, particularly in the context of detecting homology classes. Finally, if a unipotent analogue of the Macdonald identities were to exist, it would be an interesting discovery.

1.3 Organization of the paper

In Chapter 2-Chapter 5, we provide necessary background, including simplicial homotopy theory, basic Lie theory, some homological algebra constructions as well as basic results, and properties of representation homology. In Section 6.1, we present the computation result of the representation homology of torus with algebraic group consisting of unipotent upper triangular matrices, and we link this computation with the geometric properties of the underlying classical commuting scheme of the algebraic group. In Section 6.2, we give a definition of representation homology of monoids equivalent to that in [7], and generalize the definition of representation homology of algebras to other coefficients. After generalizing the definitions, we give a formal smoothness test using the representation homology. In Section 6.3, we prove the comparison theorems of representation homology, and an application, we show that the derived commuting scheme of a unipotent group scheme is equivalent to that of its associated nilpotent Lie algebra. In Section 6.4, we give a representation-theoretical explanation of the homology classes shown in Proposition 6.1.5, which relates to the classical Bott-Kostant theorem. Then in Section 7.1-Section 7.4, we talk about some interesting open questions.

1.4 Notations and conventions

We shall always denote by k a commutative ring (all commutative rings contain 1). Most of the time k will be a field of characteristic 0. We will always use homological convention, i.e. the degree of differential maps are -1 , except in Section 7.1.

By \mathbf{Alg}_k we denote the category of k -algebras with 1, in which a morphism $f : A \rightarrow B$ needs to satisfy $f(1_A) = 1_B$. $\mathbf{Alg}_{k/k}$ is the category of augmented k -algebras (an

augmentation is an algebra map $\epsilon : A \rightarrow k$). This category could be interpreted as the slice category \mathbf{Alg}_k/k , which explains the notation. $\mathbf{CommAlg}_k$ is the full subcategory consisting of commutative k -algebras. \mathbf{Lie}_k is the category of Lie algebras over k , and \mathbf{DGLA}_k is the category of differential graded Lie algebras over k .

\otimes will always be \otimes_k .

For any category C , sC is the category of simplicial objects.

CHAPTER 2
SIMPLICIAL HOMOTOPY THEORIES

We give the homotopy theory background material in this chapter.

2.1 The simplex category and simplicial objects

Definition 2.1.1. The simplex category Δ is the category with

1. the objects are $\{[n]\}_{n \in \mathbb{N}}$ where $[n]$ is the set $\{0, 1, \dots, n\}$, and
2. the morphisms are the order-preserving maps, i.e.

$$\text{Hom}_{\Delta}([m], [n]) := \{f : [m] \rightarrow [n] \mid f(i) \leq f(j) \text{ if } i < j\}.$$

The category Δ is generated by a set of special morphisms called coface maps defined by

$$d_{n+1}^i : [n] \rightarrow [n+1]$$

$$k \mapsto \begin{cases} k, & k < i \\ k+1, & k \geq i \end{cases}$$

and codegeneracy maps defined by

$$s_n^i : [n+1] \rightarrow [n]$$

$$k \mapsto \begin{cases} k, & k \leq i \\ k-1, & k > i. \end{cases}$$

Definition 2.1.2. Given a category C , the category of simplicial objects in C is the category $\text{Funct}(\Delta^{op}, C)$, denoted by sC .

We call an object in $s\mathbf{Set}$ a **simplicial set**. A simplicial set X_* can be interpreted as a sequence of sets $X_0, X_1, \dots, X_n, \dots$ which could be put in a diagram

$$X_0 \begin{array}{c} \xleftarrow{d_0} \\ \xrightarrow{s_0} \\ \xleftarrow{d_1} \end{array} X_1 \begin{array}{c} \xleftarrow{s_0} \\ \xrightarrow{d_1} \\ \xleftarrow{s_1} \\ \xrightarrow{d_2} \end{array} X_2 \quad \dots ,$$

where the maps s_j, d_i are called satisfying the “simplicial identities” [26, I.1.]:

$$\begin{aligned} d_i^{[n]} d_j^{[n+1]} &= d_{j-1}^{[n]} d_i^{[n+1]}, & \forall i < j \\ s_i^{[n+1]} s_j^{[n]} &= s_{j+1}^{[n+1]} s_i^{[n]}, & \forall i \leq j \\ d_i^{[n+1]} s_j^{[n]} &= s_{j-1}^{[n-1]} d_i^{[n]}, & \forall i < j \\ d_i^{[n+1]} s_j^{[n]} &= \text{id}_{[n]}, & i = j \text{ or } i = j + 1 \\ d_i^{[n+1]} s_j^{[n]} &= s_j^{[n-1]} d_{i-1}^{[n]}, & \forall i > j + 1. \end{aligned} \tag{2.1}$$

Example 2.1.3. *There is a canonical functor $\Delta : \mathbf{\Delta} \rightarrow \mathbf{Top}$ mapping $[n]$ to the geometric n -simplex*

$$\Delta^n = \{(x_0, \dots, x_n) \in \mathbf{R}^{n+1} \mid x_0 + \dots + x_n = 1, x_i \geq 0\},$$

Given a topological space X , we can define a simplicial set SX where $(SX)_n := \text{Map}(\Delta^n, X)$ with face map

$$d_i : (SX)_{n+1} \rightarrow (SX)_n$$

is mapping $f : \Delta_{n+1} \rightarrow X$ to $f \circ \Delta(d^i) : \Delta_n \rightarrow X$, and the degeneracy map

$$s_j : (SX)_{n-1} \rightarrow (SX)_n$$

is mapping $f : \Delta_{n-1} \rightarrow X$ to $f \circ \Delta(s^j) : \Delta_n \rightarrow X$. SX is called the singular complex of X .

Definition 2.1.4. Given a simplicial set X , the geometric realization $|X|$ of X is a topological space defined by

$$|X| := \text{coeq} \left(\coprod_{f:[n] \rightarrow [m]} X_m \otimes \Delta^n \begin{array}{c} \xrightarrow{f^*} \\ \xrightarrow{f_*} \end{array} \coprod_{[n]} X_n \otimes \Delta^n \right).$$

Proposition 2.1.5. *Geometric realization is left adjoint to the singular complex functor:*

$$\text{Hom}_{\mathbf{Top}}(|X|, Y) \cong \text{Hom}_{\mathbf{sSet}}(X, \mathcal{S}Y).$$

We denote the **standard simplicial set** by $\Delta^{[n]} := \text{hom}_{\Delta}(-, [n])$. The subset of $\Delta^{[n]}$ generated by the image of $\{d_{[n]}^i : \Delta^{[n-1]} \rightarrow \Delta^{[n]} \mid 0 \leq i \leq n, i \neq k\}$ is called the simplicial horn $\Lambda_k^{[n]}$, i.e.

$$\Lambda_k^{[n]} = \bigcup_{0 \leq i \leq n, i \neq k} \partial_i \Delta^{[n]}.$$

Definition 2.1.6. A simplicial set X is called a **Kan fibration** if it has the horn-filling property, namely for any natural number n and $0 \leq k \leq n$ and a simplicial morphism $f : \Lambda_k^{[n]} \rightarrow X$, there exists (possibly non unique) map $\tilde{f} : \Delta^{[n]} \rightarrow X$ making the diagram

$$\begin{array}{ccc} \Lambda_k^{[n]} & \xrightarrow{f} & X \\ \downarrow & \nearrow \tilde{f} & \\ \Delta^{[n]} & & \end{array}$$

commutative.

Definition 2.1.7. A simplicial set X is called **reduced** if it has a single vertex, that is, $X_0 = \{*\}$. The full subcategory of \mathbf{sSet} consisting of reduced simplicial sets will be denoted \mathbf{sSet}_0 .

2.1.1 Simplicial groups

Definition 2.1.8. A simplicial group $\Gamma = \{\Gamma_n\}_{n \geq 0}$ is called **semi-free** if there is a sequence of subsets $B_n \subseteq \Gamma_n$, one in each degree, such that Γ_n is freely generated by B_n , and the set $B = \bigcup_{n \geq 0} B_n$ is closed under degeneracies of Γ , that is, $s_j(B_{n-1}) \subseteq B_n$ for all $0 \leq j \leq n-1$ and $n \geq 1$. The subset $\bar{B}_n := B_n - \bigcup_{i=0}^{n-1} s_i(B_{n-1})$ is called the set of nondegenerate generators of Γ of degree n .

The Kan loop group construction provides an important class of semi-free simplicial groups that arise naturally from reduced simplicial sets:

Theorem 2.1.9. *There is an adjunction*

$$\mathbb{G} : s\mathbf{Set}_0 \rightleftarrows s\mathbf{Gp} : \bar{W}, \quad (2.2)$$

where \mathbb{G} is called the Kan loop group functor and \bar{W} is the classifying simplicial complex.

Furthermore the Kan loop group $\mathbb{G}X$ of any reduced simplicial set X is semi-free.

We refer to [26, Chapter V.] for detailed construction and proof.

2.2 Model categories

The purpose of the model category theory is to provide a framework categorical homotopy theory. The idea is to introduce the notion of maps that are “weak equivalences” instead of isomorphisms. There are special morphisms called fibrations and cofibrations, which could be viewed as the existence of injective or projective resolutions.

Definition 2.2.1. A (closed) **model category** is a category \mathcal{M} equipped with three distinguished classes of morphisms WE, Fib and Cof, satisfying

1. \mathcal{M} is (finitely) complete and cocomplete.
2. 2-out-of-3 property: if any two of f, g and $g \circ f$ are in WE, then so is the third.
3. If f is a retract of g and if g is in WE, Fib or Cof, then f belongs to the same family. By definition, f is a retract of g if there is a commutative diagram

$$\begin{array}{ccccc}
X_1 & \xrightarrow{s_1} & Y_1 & \xrightarrow{r_1} & X_1 \\
\downarrow f & & \downarrow g & & \downarrow f \\
X_2 & \xrightarrow{s_2} & Y_2 & \xrightarrow{r_2} & X_1,
\end{array}$$

such that $s_j r_j = \text{id}_{X_j}$, $j = 1, 2$.

4. Given a commutative diagram

$$\begin{array}{ccc}
A & \xrightarrow{g} & E \\
i \downarrow & \nearrow h & \downarrow p \\
X & \xrightarrow{f} & B,
\end{array}$$

in \mathcal{M} , where $i \in \text{Cof}$ and $p \in \text{Fib}$, if either i or p belongs to WE , then there exists a lift $h : X \rightarrow E$ which makes the whole diagram commutative.

5. Any morphism $f : A \rightarrow B$ in \mathcal{M} has factorizations $f = q \circ i = p \circ j$ where $p, q \in \text{Fib}$, $i, j \in \text{Cof}$, and $i, p \in \text{WE}$.

The morphisms in WE , Fib and Cof are called **weak equivalence**, **fibration**, and **cofibration** respectively. They are denoted by $\xrightarrow{\sim}$, \twoheadrightarrow and \hookrightarrow respectively. An object $X \in \mathcal{M}$ (resp. $A \in \mathcal{M}$) is called a **fibrant object** (resp. **cofibrant object**) if $X \rightarrow *$ is a fibration (resp. $\emptyset \rightarrow A$ is a cofibration). A fibration (cofibration) is called **acyclic** if it also belongs to WE .

Example 2.2.2. *Top* is a model category, where

1. *weak equivalences are the weak homotopy equivalences, namely the maps $f : X \rightarrow Y$ whose induced maps $\pi_i(f) : \pi_i(X, x) \rightarrow \pi_i(Y, f(x))$ are isomorphisms for all i and for all $x \in X$,*
2. *fibrations are the Serre fibrations,*
3. *cofibrations are the retracts of cell-gluing maps. Namely, for the diagram*

$$\begin{array}{ccccc}
A_2 & \xrightarrow{s} & A_1 & \xrightarrow{r} & A_2 \\
\downarrow j_2 & & \downarrow j_1 & & \downarrow j_2 \\
X_2 & \xrightarrow{i} & X_1 & \xrightarrow{p} & X_2
\end{array}$$

where $r \circ s = \text{id}_{A_2}$ and $p \circ i = \text{id}_{X_2}$, if $j_1 : A_1 \hookrightarrow X_1$ is a relative cellular map, then $j_2 : A_2 \rightarrow X_2$ is a cofibration.

One notices that CW complexes are fibrant objects in this model category.

Example 2.2.3. $s\text{Set}$ is a model category, where

1. weak equivalences are Quillen weak equivalences, i.e. maps $f : X \rightarrow Y$ s.t. $|f| : |X| \rightarrow |Y|$ is a weak equivalence in **Top**,
2. cofibrations are levelwise injections,
3. fibrations are Kan fibrations.

Example 2.2.4 (Projective model structure). $\mathbf{Com}_{\geq 0}(k\text{-Mod})$ is a model category where

1. weak equivalences are quasi-isomorphisms,
2. cofibrations are the maps $\{i_n : X_n \rightarrow Y_n\}_{n \in \mathbb{Z}}$ such that i_n is injective with projective cokernel for all n ,
3. fibrations are the maps $\{p_n : X_n \rightarrow Y_n\}_{n \in \mathbb{Z}}$ such that p_n is surjective for all n .

There is also a model structure on $\mathbf{Com}_{\bullet}(k\text{-Mod})$, the category of unbounded complexes.

Example 2.2.5. Given a model category \mathcal{M} with $C \in \text{ob } \mathcal{M}$, then the slice category \mathcal{M}/C admits a natural model structure:

1. $f_{|C} : X_{|C} \rightarrow Y_{|C}$ is a weak equivalence if and only if $f : X \rightarrow Y$ is a weak equivalence in \mathcal{M} .
2. $i_{|C} : A_{|C} \rightarrow X_{|C}$ is a cofibration if and only if $i : A \rightarrow X$ is a cofibration in \mathcal{M} .
3. $p_{|C} : E_{|C} \rightarrow B_{|C}$ is a fibration if and only if $p : E \rightarrow B$ is a fibration in \mathcal{M} .

2.2.1 Quillen pairs and Quillen equivalences

Lemma 2.2.6. *Given two model categories \mathcal{M}, \mathcal{N} and a pair of adjoint functors*

$$F : \mathcal{M} \rightleftarrows \mathcal{N} : G,$$

the following are equivalent:

1. F preserves cofibrations and acyclic cofibrations.
2. G preserves fibrations and acyclic fibrations.
3. F preserves cofibrations and G preserves fibrations.

A pair of functors (F, G) satisfying the conditions of Lemma 2.2.6 is called a **Quillen pair**. The main result is the following theorem about the existence of derived adjunctions:

Theorem 2.2.7 ([19]). *Let $F : \mathcal{M} \rightleftarrows \mathcal{N} : G$ be a Quillen pair, then the total derived functors $\mathbb{L}F$ and $\mathbb{R}G$ exists and form an adjoint pair*

$$\mathbb{L}F : \mathrm{Ho}(\mathcal{M}) \rightleftarrows \mathrm{Ho}(\mathcal{N}) : \mathbb{R}G.$$

The left functor can be computed by

$$\mathbb{L}F(A) = \gamma F(QA), \quad \mathbb{L}F(f) = \gamma F(\tilde{f}),$$

where $QA \rightarrow A$ is a cofibrant replacement of A and \tilde{f} is a lift of f .

Theorem 2.2.8. *Let $F : \mathcal{M} \rightleftarrows \mathcal{N} : G$ be a Quillen pair, then $(\mathbb{L}F, \mathbb{R}G)$ is an equivalence of categories if and only if the following condition holds: for each cofibrant object $A \in \mathcal{M}$ and fibrant object $X \in \mathcal{N}$,*

Example 2.2.9. *The adjunction*

$$|-| : \mathbf{sSet} \rightleftarrows \mathbf{Top} : S$$

mentioned in Proposition 2.1.5 is a Quillen equivalence.

Example 2.2.10. *For a pointed topological space $(X, *)$, its Eilenberg subcomplex of $\bar{S}_*(X)$ is defined by*

$$\bar{S}_n(X) := \{f : \Delta^n \rightarrow X \mid f(v_i) = * \text{ for all vertices } v_i \in \Delta^n\}.$$

If X is connected, the natural inclusion $\bar{S}_(X) \hookrightarrow S_*(X)$ is a weak equivalence of simplicial sets. Further, if we restrict functor \bar{S} to the category $\mathbf{Top}_{0,*}$ of connected pointed spaces, we get the pair of adjoint functors (cf Definition 2.1.7)*

$$|-| : \mathbf{sSet}_0 \rightleftarrows \mathbf{Top}_{0,*} : \bar{S}$$

which is also a Quillen equivalence.

Example 2.2.11. *There is a model category structure on \mathbf{sGp} such that semi-free simplicial groups are cofibrant objects in the model category \mathbf{sGp} . The Kan loop group adjunction $\mathbb{G} : \mathbf{sSet}_0 \rightleftarrows \mathbf{sGp} : \bar{W}$ (Equation (2.2)) is also a Quillen equivalence, where \mathbb{G} is the Kan loop group functor and \bar{W} is the classifying simplicial complex.*

Combined with Example 2.2.10 we have a sequence of equivalences of homotopy categories

$$\mathrm{Ho}(\mathbf{Top}_{0,*}) \simeq \mathrm{Ho}(\mathbf{sSet}_0) \simeq \mathrm{Ho}(\mathbf{sGp}). \quad (2.3)$$

This means giving a simplicial group is the same as giving a pointed, connected topological group.

2.2.2 The Dold-Kan correspondence

Theorem 2.2.12 ([26, Corollary III.2.3, Corollary III.2.7.]). *The category of simplicial k -modules is equivalent to the category of non-negatively graded chain complexes of k -modules. Namely, there is a Quillen equivalence*

$$N : s(k - \mathbf{Mod}) \rightleftarrows \mathbf{Com}_{\geq 0}(k - \mathbf{Mod}) : \Gamma$$

when N is the normalisation functor and Γ is its right adjoint.

Thus for any simplicial k -module $M \in s(k - \mathbf{Mod})$, we define the homotopy group $\pi_*(M)$ to be the homology group of its normalisation $N(M)$, i.e.

$$\pi_*(M) := H_*(N(M)).$$

Remark 2.2.13. *The Dold-Kan correspondence has monoidal versions, one of which says that the normalization functor is symmetric monoidal with respect to the shuffle map. Hence it takes commutative simplicial rings to commutative (in the graded sense) differential graded ring. See [50].*

2.2.3 Promoting model category structures

Given an adjunction

$$F : \mathcal{C} \rightleftarrows \mathcal{D} : G$$

where one side is a model category. If there are more structure on the model category and the functors satisfy some certain properties, then the model structure can be promoted to the other side ([25, Theorem 3.6.] for cofibrantly generated model categories and [26, Theorem II.5.1.] for simplicial model category). We will give an example here.

Definition 2.2.14 ([25]). A model category \mathcal{M} is said to be cofibrantly generated if there are sets of morphisms I and J so that

1. the source of every morphism in I is small for the class of all cofibrations and $f : X \rightarrow Y$ is an acyclic fibration if and only if f has the right lifting property with respect to all morphisms of I ; and
2. the source of every morphism in J is small with respect to the class of all acyclic cofibrations and $f : X \rightarrow Y$ is a fibration if and only if f has the right lifting property with respect to all morphisms of J .

The set I and the set J generate the cofibrations and the acyclic cofibrations respectively. Here, “generate” means the cofibrations are the smallest class of maps that contains I and is closed under coproducts, cobase change, sequential colimits, and retracts.

Example 2.2.15. *The model category $s\mathbf{Set}$ with Quillen model structure is cofibrantly generated:*

1. *generating cofibrations the boundary inclusions $\partial\Delta^{[n]} \hookrightarrow \Delta^{[n]}$;*
2. *generating acyclic cofibrations the horn inclusions $\Lambda_i^{[n]} \hookrightarrow \Delta^{[n]}$.*

Example 2.2.16 ([48, Part II, Section 4, Theorem 4 and Remarks]). *Consider the adjunction*

$$\mathbb{F} : s\mathbf{Set} \rightleftarrows s\mathbf{Mod} : U$$

where U is the forgetful functor and \mathbb{F} stands for the free monoid functor. Then $s\mathbf{Mon}$ admits a model structure such that

1. *weak equivalences are the weak equivalences between the underlying simplicial sets,*

2. *fibrations are the fibrations between the underlying simplicial sets, and*
3. *cofibrations are the retracts of relative semifree monoids.*

2.3 CDGA

In this section, k will be a field.

By a DG algebra we mean a \mathbb{Z} -graded unital associative k -algebra equipped with a differential of degree -1 . We write \mathbf{DGA}_k for the category of all such algebras and denote by \mathbf{CDGA}_k the full subcategory consisting of commutative DG algebras.

Theorem 2.3.1 ([25, Theorem 3.6., Example 3.7.], [29, Theorem 4.1.1., Remark 4.2.]).
The category \mathbf{DGA}_k and \mathbf{CDGA}_k (when $\text{char } k = 0$) inherit model structures from $\mathbf{Com}_\bullet(k - \mathbf{Mod})$ in which

1. *weak equivalences are quasi-isomorphisms,*
2. *fibrations are levelwise surjective maps, and*
3. *cofibrations are the morphisms having the left-lifting property with respect to acyclic fibrations.*

A detailed proof could be found in [33]. Similar to Example 2.2.4, there is a model structure on the category of non-negatively graded DG algebras. We need to introduce some terminology before stating the result.

The full subcategories consisting of non-negatively graded DG algebras of \mathbf{DGA}_k and \mathbf{CDGA}_k are denoted by $\mathbf{DGA}_k^{\geq 0}$ and $\mathbf{CDGA}_k^{\geq 0}$.

A DG algebra $A \in \mathbf{DGA}_k^{\geq 0}$ is called **semi-free** if its underlying algebra is isomorphic to the tensor algebra $T_k V$ of a graded k -vector space. We say that a DG homomorphism $f : A \rightarrow B$ is a semi-free extension if there is an isomorphism $B \cong A \otimes T_k V$ and the composition of f with this isomorphism is the canonical map $A \hookrightarrow A \otimes T_k V$.

Similarly, a commutative DG algebra $A \in \mathbf{CDGA}_k^{\geq 0}$ is called **semi-free** if its underlying algebra is isomorphic to the symmetric algebra $\Lambda_k V$ of a graded k -vector space. A DG homomorphism $f : A \rightarrow B$ between two commutative DG algebras is a semi-free extension if there is an isomorphism $B \cong A \otimes \Lambda_k V$ and the composition of f with this isomorphism is the canonical map $A \hookrightarrow A \otimes \Lambda_k V$.

Theorem 2.3.2. *Let k be a field of characteristic 0. The category $\mathbf{DGA}_k^{\geq 0}$ and $\mathbf{CDGA}_k^{\geq 0}$ have model structures with*

1. *weak equivalences are quasi-isomorphisms,*
2. *fibrations are the maps which are surjective in all positive degrees, and*
3. *cofibrations are retracts of semi-free homomorphisms.*

The model structure on $\mathbf{CDGA}_k^{\geq 0}$ is well-known. A detailed proof could be found in [24, Chapter V.]. Similarly, A detailed proof for the model structure on $\mathbf{DGA}_k^{\geq 0}$ could be found in [45].

Remark 2.3.3. *It is essential in the $\mathbf{CDGA}_k^{\geq 0}$ case that the characteristic of k is 0. It is even unknown if there exists a model structure when the characteristic is positive.*

However, for $\mathbf{DGA}_k^{\geq 0}$, it is known that model structure on $\mathbf{DGA}_k^{\geq 0}$ is Quillen equivalent to simplicial associative algebras over an arbitrary commutative ring k ([50, Theorem 1.1.]).

2.4 Derived algebraic geometry models

Historically, the notion of derived moduli spaces goes back to the work of Beilinson, Deligne, Drinfeld, and Kontsevich. They introduced the principle known as the *hidden smoothness*, which predicts that natural moduli spaces should admit derived enhancements that are quasi-smooth.

One of the key motivations behind derived algebraic geometry is to generalize the notion of a classical moduli space—traditionally defined as a functor on the category of commutative rings to sets or groupoids. In contrast, a derived moduli space is defined as a functor on the category of simplicial commutative rings, with values extended from sets (or groupoids) to simplicial sets (or ∞ -groupoids).

Example 2.4.1. *The pullback of schemes $X \times_Z Y$ is characterized by a universal property among schemes with maps to X and Y , such that the composite maps to Z agree. It is built locally by the tensor product operation on commutative rings.*

The derived fibered product $X \times_Z^h Y$ is characterized by an analogous universal property: the commutativity is replaced by homotopy commutativity and the uniqueness is also up to homotopy (see [44, Section 2.] for a detailed definition). It can be built locally by the derived tensor product operation on simplicial commutative algebras.

Example 2.4.2. *A derived stack is a simplicial presheaf on $d\mathbf{Aff}_k := (\mathbf{sCommAlg}_k)^{\text{op}}$, i.e. a functor $F : \mathbf{sCommAlg}_k \rightarrow \mathbf{sSet}$, that is objectwise fibrant, preserves equivalences and satisfies a certain hyperdescent property ([51, Definition 4.6.6.],[52, Definition 2.2.2.14.]).*

1. *To each derived affine scheme $X \in d\mathbf{Aff}_k$, one can associate a simplicial presheaf*

defined by

$$\mathbb{R}h_X := \text{Map}_{d\mathbf{Aff}_k}(-, X) : s\mathbf{CommAlg}_k \rightarrow s\mathbf{Set}$$

where $\text{Map}_{d\mathbf{Aff}_k}$ is the homotopy mapping space in $d\mathbf{Aff}_k$.

2. The derived stack associated to a simplicial derived affine scheme X_\bullet is the stackification of the simplicial presheaf

$$|X_\bullet|^{\text{pre}} : d\mathbf{Aff}_k^{\text{op}} \xrightarrow{\mathbb{R}h_{X_\bullet}} s(s\mathbf{Set}) \xrightarrow{\text{Diag}} s\mathbf{Set}$$

where $\mathbb{R}h_{X_\bullet}$ is to apply the derived Yoneda functor levelwise. We denote it by $|X_\bullet|^{\text{st}}$.

3. BG is the derived stack associated to the simplicial derived affine scheme $B_\bullet G$ defined in the usual way by $B_n G = G \times \cdots \times G$.
4. For any simplicial set $X \in s\mathbf{Set}$, define the mapping stack $\text{Map}(X, BG)$ to be the simplicial presheaf

$$\text{Map}(X, BG) : d\mathbf{Aff}_k^{\text{op}} \rightarrow s\mathbf{Set}$$

$$A \mapsto \underline{\text{Hom}}(X, BG(A)),$$

where $\underline{\text{Hom}}(X, Y)$ is the simplicial set $\underline{\text{Hom}}(X, Y)_n := \text{Hom}_{s\mathbf{Set}}(X \times \Delta^{[n]}, Y)$.

5. BG is canonically pointed and therefore one can define the pointed mapping stack

$$\text{Map}((X, *), (BG, *)) : d\mathbf{Aff}_k^{\text{op}} \rightarrow s\mathbf{Set}$$

$$A \mapsto \underline{\text{Hom}}((X, *), (BG(A), *)).$$

It is important to notice that

$$\mathbb{R}h_{\text{Spec}(\mathbb{L}(\Gamma)_G)} \simeq \mathbf{Map}((\Gamma, x_0), (BG, *))$$

and

$$\mathbb{R}h_{\text{Spec}(\mathbb{L}(\Gamma)_G^c)} \simeq \mathbf{Map}(\Gamma, BG)$$

([8, Appendix]).

2.4.1 DG (simplicial) scheme

The “upgraded” language of derived moduli spaces has many advantages. However, the abstract nature of this framework makes both its foundations and computations difficult, sometimes even inaccessible. As a result, it is often more practical to introduce alternative models formulated in more elementary and concrete terms.

Definition 2.4.3. A **simplicial scheme** $X = (X_0, \mathcal{O}_{X_\bullet})$ is a pair where X_0 is an ordinary scheme and \mathcal{O}_X is quasi-coherent sheaf of simplicial commutative algebras.

The monoidal Dold-Kan correspondence (Remark 2.2.13) indicates that when $\text{char } k = 0$ we can use the category \mathbf{CDGA}_k instead of using the category $s\mathbf{CommAlg}_k$ whose objects are much larger and difficult to control. Therefore, we will use the notion of DG schemes when possible, although almost all the concepts and results have simplicial versions.

For more details, we refer to [15, Section 2]. We want to warn the readers that, unlike [15], we use the homological convention, due to the connection with simplicial objects.

1. A DG scheme $X = (X_0, \mathcal{O}_{X_\bullet})$ is **affine** if X_0 is affine.
2. In the affine case, we write $\pi_0(X) := \text{Spec } H_0(\mathcal{O}_{X_\bullet})$ and identify $\pi_0(X)$ a closed subscheme of X_0 .
3. A DG scheme is called **smooth** if X_0 is a smooth scheme, and \mathcal{O}_{X_\bullet} is locally isomorphic to the graded symmetric algebra

$$\mathcal{O}_{X_\bullet} \cong \Lambda_{\mathcal{O}_{X_0}}(E_\bullet)$$

as a sheaf of graded \mathcal{O}_{X_0} -algebras, where E_\bullet is a graded \mathcal{O}_{X_0} -module whose components E_i are finite rank locally free sheaves on X_0 .

Definition 2.4.4. Given a DG scheme X and a closed k -point $x \in X_0$, we define the **DG tangent space** $(T_x X)_\bullet$ at x to be the derivation complex

$$(T_x X)_\bullet := \underline{\text{Der}}(O_{X,\bullet}, k_x),$$

where $k_x := O_{X,\bullet}/\mathfrak{m}_x$ as an $O_{X,\bullet}$ -module. The homology groups of this complex are denoted

$$\pi_i(X, x) := H_i(T_x X)$$

and called the **derived tangent spaces** of X at x .

Dually the **DG cotangent space** $(T_x^* X)_\bullet$ at $x \in X_0$ is defined by taking the complex of Kähler differentials

$$(T_x^* X)_\bullet := \Omega_{\text{comm}}^1(O_{X,\bullet})_x \cong \mathfrak{m}_x/\mathfrak{m}_x^2.$$

The following result is an analogue to the Whitehead theorem in classical topology.

Proposition 2.4.5 ([34, Proposition 1.3.]). *Let $f : X \rightarrow Y$ be a morphism of smooth DG schemes. Then f is a quasi-isomorphism if and only if*

1. $\pi_0(f) : \pi_0(X) \rightarrow \pi_0(Y)$ is an isomorphism of schemes, and
2. for every closed point $x \in X_0$, the differential $d_x f$ induces linear isomorphisms

$$\pi_i(X, x) \xrightarrow{\sim} \pi_i(Y, f(x)), \quad \forall i \geq 0.$$

CHAPTER 3

LIE ALGEBRAS AND THEIR DECOMPOSITIONS

Throughout this chapter, we will assume k be a field.

3.1 Lie algebras, semisimple Lie algebras, and triangular decompositions

3.1.1 Semisimple Lie algebras

Definition 3.1.1. A subspace J of \mathfrak{g} is called a Lie **subalgebra** if $[x, y] \in J$ for all $x, y \in J$. A subspace I of \mathfrak{g} is called an **ideal** if $[x, y] \in I$ for all $x \in I$ and $y \in \mathfrak{g}$.

Example 3.1.2. Any vector space V can be made into a Lie algebra trivially by defining $[x, y] = 0$. This Lie algebra is called **abelian**.

Definition 3.1.3.

1. Any linear map δ on a Lie algebra \mathfrak{g} is called a **derivation** if it satisfies $\delta([x, y]) = [x, \delta(y)] + [\delta(x), y]$ for $x, y \in \mathfrak{g}$.
2. Let x be any element of the Lie algebra \mathfrak{g} . We define the map $\text{ad}_x : \mathfrak{g} \rightarrow \mathfrak{g}$ by $\text{ad}_x(y) = [x, y]$. This map is linear but not a homomorphism. However, it is a derivation by Jacobi identity.
3. Now, we consider the map $\text{ad} : \mathfrak{g} \rightarrow \mathfrak{gl}(\mathfrak{g})$ defined by $\text{ad}(x) = \text{ad}_x$. This map is a homomorphism, i.e, $\text{ad}_{[x, y]} = [\text{ad}_x, \text{ad}_y]$, called the **adjoint representation** of \mathfrak{g} .
4. The set of derivations $\text{Der } \mathfrak{g}$ of \mathfrak{g} is a subalgebra of $\mathfrak{gl}(\mathfrak{g})$. Moreover, $\text{ad}(\mathfrak{g})$ is an ideal of $\text{Der } \mathfrak{g}$.

Definition 3.1.4. A non-abelian Lie algebra \mathfrak{g} is called simple if it has no ideals other than \mathfrak{g} and 0 .

Definition 3.1.5. Let \mathfrak{g} be a Lie algebra.

1. The **derived series** of \mathfrak{g} is defined to be the sequence of ideals $\mathfrak{g}^{(0)} = \mathfrak{g}$, and $\mathfrak{g}^{(n+1)} = [\mathfrak{g}^{(n)}, \mathfrak{g}^{(n)}]$.
2. The **lower central series** of \mathfrak{g} is defined to be the sequence of ideals $\mathfrak{g}^0 = \mathfrak{g}$, and $\mathfrak{g}^{n+1} = [\mathfrak{g}, \mathfrak{g}^n]$.
3. \mathfrak{g} is called solvable if $\mathfrak{g}^{(n)} = 0$ for some n .
4. \mathfrak{g} is called nilpotent if $\mathfrak{g}^n = 0$ for some n .

By induction, $\mathfrak{g}^{(n)} \subseteq \mathfrak{g}^n$ for any n , therefore any nilpotent algebra is solvable.

Lemma 3.1.6. *1. Subalgebras and homomorphic images of a solvable (resp. nilpotent) Lie algebra are solvable (resp. nilpotent).*

2. *If I is a solvable ideal of a Lie algebra \mathfrak{g} such that \mathfrak{g}/I is solvable, then \mathfrak{g} is solvable.*
3. *If a Lie algebra \mathfrak{g} is nilpotent and nonzero, then the center $Z(\mathfrak{g}) \neq 0$.*
4. *If a Lie algebra \mathfrak{g} such that $\mathfrak{g}/Z(\mathfrak{g})$ is nilpotent, then \mathfrak{g} is nilpotent.*

Theorem 3.1.7 (Engel). *A finite dimensional Lie algebra \mathfrak{g} is nilpotent if every element is ad-nilpotent*

Example 3.1.8. *1. $[\mathfrak{sl}(n; \mathbb{C}), \mathfrak{sl}(n; \mathbb{C})] = \mathfrak{sl}(n; \mathbb{C})$, and $\mathfrak{sl}(n; \mathbb{C})$ is a simple Lie algebra for $n \geq 2$.*

2. $[\mathfrak{gl}(n; \mathbb{C}), \mathfrak{gl}(n; \mathbb{C})] = \mathfrak{sl}(n; \mathbb{C})$.

Definition 3.1.9. Let \mathfrak{g} be a Lie algebra.

1. The unique maximal solvable ideal of \mathfrak{g} is called the **radical** of \mathfrak{g} , denoted by $\text{Rad } \mathfrak{g}$.

2. \mathfrak{g} is called **semisimple**, if $\text{Rad } \mathfrak{g} = 0$. In other words, \mathfrak{g} is semisimple if it does not contain any nonzero solvable ideal.
3. An element $x \in \text{End}(V)$ is called **semisimple** if the roots of its minimal polynomial are all distinct.

Lemma 3.1.10. *Let \mathfrak{g} be a solvable Lie algebra with a representation $\rho : \mathfrak{g} \rightarrow \mathfrak{gl}(V)$. Then there is a common eigenvector of all $\rho(x)$, $x \in \mathfrak{g}$.*

Theorem 3.1.11 (Lie). *Let \mathfrak{g} be a solvable Lie algebra and $\rho : \mathfrak{g} \rightarrow \mathfrak{gl}(V)$ be a representation. Then there exists a basis \mathcal{B} of V such that $[\rho(x)]_{\mathcal{B}}$ is upper triangular for any $x \in \mathfrak{g}$.*

Theorem 3.1.12 (Cartan's Criterion). *A Lie algebra \mathfrak{g} is solvable if and only if $\text{Tr}(\text{ad}_x \cdot \text{ad}_y) = 0$ for all $x \in [\mathfrak{g}, \mathfrak{g}]$ and $y \in \mathfrak{g}$.*

Theorem 3.1.13 (Jordan decomposition). *Let k be an algebraically closed field. Let $T : V \rightarrow V$ be a linear map where V is a finite dimensional vector space over k . Then, the followings hold*

1. *There are unique linear maps T_s and T_n such that $T = T_s + T_n$, where T_s is semisimple, T_n is nilpotent and $T_s \cdot T_n = T_n \cdot T_s$.*
2. *There are polynomials $p, q \in k[x]$ such that $p(T) = T_s$ and $q(T) = T_n$. In particular, T_s and T_n commute with any linear map commuting with T .*

Definition 3.1.14. The symmetric bilinear form $\kappa : \mathfrak{g} \times \mathfrak{g} \rightarrow k$ defined by

$$\kappa(x, y) = \text{Tr}(\text{ad}_x \cdot \text{ad}_y).$$

is called the **Killing form**.

The Killing form satisfies $\kappa(x, [y, z]) = \kappa([x, y], z)$.

Proposition 3.1.15. *If κ is the Killing form of \mathfrak{g} and I is an ideal of \mathfrak{g} , then the Killing form κ_I of I coincides with $\kappa : I \times I$.*

Corollary 3.1.15.1 (Cartan's Criterion). *An ideal I of a lie algebra \mathfrak{g} is solvable if and only if $\kappa(x, y) = 0$ for all $x \in [\mathfrak{g}, \mathfrak{g}]$ and $y \in \mathfrak{g}$.*

Definition 3.1.16. The set $\{x \in \mathfrak{g} \mid \kappa(x, y) = 0 \forall y \in \mathfrak{g}\}$ is called the **radical** of the Killing form κ .

The radical of the Killing form is an ideal.

Proposition 3.1.17. *Finite dimensional Lie algebra \mathfrak{g} is semisimple if and only if it has no nonzero abelian ideal. In particular, $Z(\mathfrak{g}) = 0$ whenever \mathfrak{g} is semisimple.*

Theorem 3.1.18 (Cartan's Criterion). *A finite dimensional Lie algebra \mathfrak{g} is semisimple if and only if its Killing form is non-degenerate.*

Theorem 3.1.19. *If \mathfrak{g} is a semisimple Lie algebra, then there are ideals I_1, \dots, I_k so that*

$$\mathfrak{g} = I_1 \oplus \dots \oplus I_k$$

and each I_i is a simple Lie algebra. In fact, I_i 's are the only simple ideals of \mathfrak{g} .

Theorem 3.1.20. *For a semisimple Lie algebra \mathfrak{g} , $\text{ad}_{\mathfrak{g}} = \text{Der } \mathfrak{g}$.*

3.1.2 Structure of complex semisimple Lie algebras

We will assume $k = \mathbb{C}$ for this subsection, and \mathfrak{g} a finite dimensional Lie algebra over k .

Example 3.1.21. *If \mathfrak{g} is semisimple and $\rho : \mathfrak{g} \rightarrow \text{gl}(V)$ is a representation, then $\rho(\mathfrak{g}) \subseteq \text{sl}(V)$.*

Theorem 3.1.22 (Weyl). *If \mathfrak{g} is a semisimple Lie algebra, then any finite dimensional \mathfrak{g} -module is completely reducible. In other words, any finite dimensional \mathfrak{g} -module is a direct sum of irreducible \mathfrak{g} -modules.*

Definition 3.1.23. A subalgebra \mathfrak{h} of a Lie algebra \mathfrak{g} is called a **Cartan subalgebra** if \mathfrak{h} is nilpotent and $\mathfrak{h} = N_{\mathfrak{g}}(\mathfrak{h})$ where $N_{\mathfrak{g}}(\mathfrak{h}) = \{x \in \mathfrak{g} \mid [\mathfrak{h}, x] \subseteq \mathfrak{h}\}$. Here $N_{\mathfrak{g}}(\mathfrak{h})$ is called the normalizer of \mathfrak{h} in \mathfrak{g} and it is the largest subalgebra of \mathfrak{g} in which \mathfrak{h} is an ideal.

Theorem 3.1.24. *Every finite dimensional Lie algebra \mathfrak{g} has a Cartan subalgebra. Moreover, given any two Cartan subalgebras $\mathfrak{h}_1, \mathfrak{h}_2$ there exists $\sigma \in \text{Aut}(\mathfrak{g})$ such that $\sigma(\mathfrak{h}_1) = \mathfrak{h}_2$.*

Although we will not provide the proof, the idea is based on the following definition.

Definition 3.1.25. Let \mathfrak{t} be a subalgebra of a semisimple Lie algebra \mathfrak{g} . \mathfrak{t} is called **toral** if it contains only ad-semisimple elements.

Theorem 3.1.26. *Every semisimple Lie algebra has a toral subalgebra.*

Theorem 3.1.27. *Toral subalgebra of a semisimple Lie algebra is abelian.*

Theorem 3.1.28. *Maximal toral subalgebra \mathfrak{t} is self-centralizing. Therefore maximal toral subalgebra of a semisimple Lie algebra is a Cartan subalgebra.*

Definition 3.1.29. Let \mathfrak{h} be a Cartan subalgebra of a semisimple Lie algebra \mathfrak{g} . A nonzero functional $\alpha : \mathfrak{h} \rightarrow k$ is called a **root** of \mathfrak{g} with respect to \mathfrak{h} if

$$\mathfrak{g}_{\alpha} := \{x \in \mathfrak{g} \mid [h, x] = \alpha(x) \forall h \in \mathfrak{h}\}.$$

Now we can state the root space decomposition theorem.

Theorem 3.1.30. *Let \mathfrak{g} be a complex semisimple Lie algebra, and let \mathfrak{h} be a fixed Cartan subalgebra of \mathfrak{g} . Then there is a subset R of \mathfrak{h}^* consisting of roots with respect to \mathfrak{h} , such that*

1. $\mathfrak{g} = \mathfrak{h} \oplus \sum_{\alpha \in R} \mathfrak{g}_\alpha$.
2. \mathfrak{g}_α is 1-dimensional, so is the subspace $\mathfrak{h}_\alpha := [\mathfrak{g}_\alpha, -\mathfrak{g}_\alpha]$.
3. If $\alpha, \beta \in R$ and $\alpha + \beta \neq 0$, then

$$\mathfrak{g}_{\alpha+\beta} = [\mathfrak{g}_\alpha, \mathfrak{g}_\beta].$$

The subset $R \subset \mathfrak{h}^*$ is called a (the) **root system** of \mathfrak{g} . It carries the information of the symmetries of \mathfrak{h}^* .

Definition 3.1.31. Let R be a root system. A subset $\Delta = \{\alpha_i\}$ of R is called a base if Δ is a basis of V and each root α can be written as $\alpha = \sum_{i \in \Delta} k_i \alpha_i$ with integral coefficients k_i all nonnegative or nonpositive. Any root in Δ is called positive simple. $\alpha \in R$ is positive (respectively negative) if all k_i are positive (respectively negative). Any root in $-\Delta$ is called negative simple.

Definition 3.1.32. Let R be a root system of V . The subgroup of automorphisms of V generated by the reflections $\{s_\alpha\}_{\alpha \in R}$ is called the **Weyl group** of R . The **length** of an element $w \in W$ is the smallest number k such that $w = s_{\alpha_1} \cdots s_{\alpha_k}$, $\alpha_i \in \Delta$, where Δ is a chosen base.

The classification of complex semisimple Lie algebra can be reduced to the classification of *irreducible*¹ root systems, which is a much easier combinatorial object.

3.2 Universal enveloping algebra

Definition 3.2.1. Given a Lie algebra \mathfrak{g} , a universal enveloping algebra of \mathfrak{g} is an associative k -algebra $U\mathfrak{g}$ together with a map $i : \mathfrak{g} \rightarrow U\mathfrak{g}$ satisfying

$$i([x, y]) = i(x)i(y) - i(y)i(x),$$

¹A root system is irreducible if it is not a disjoint union of two.

s.t. for any other k -linear map $f : \mathfrak{g} \rightarrow A$ satisfying $f([x, y]) = f(x)f(y) - f(y)f(x)$, where A is an associative k -algebra, there is a unique algebra homomorphism $\tilde{f} : U\mathfrak{g} \rightarrow A$ with

$$\tilde{f} \circ i = f,$$

i.e. the diagram

$$\begin{array}{ccc} \mathfrak{g} & \xrightarrow{i} & U\mathfrak{g} \\ & \searrow f & \downarrow \tilde{f} \\ & & A \end{array}$$

commutes.

The universal enveloping algebra $U\mathfrak{g}$ of \mathfrak{g} gives the left functor of the adjunction

$$U : \mathbf{Lie}_k \rightleftarrows k\text{-}\mathbf{Alg} : (-)_{\text{Lie}}$$

such that

$$\text{Hom}_{k\text{-}\mathbf{Alg}}(U\mathfrak{g}, A) \cong \text{Hom}_{\mathbf{Lie}_k}(\mathfrak{g}, A_{\text{Lie}}).$$

Recall that for any k -VS, the tensor algebra

$$T_k V := \bigoplus_{n \in \mathbb{N}} V^{\otimes n}$$

and the symmetric algebra $\text{Sym}_k V := T_k V / \langle a \otimes b - b \otimes a \rangle$, with the construction

$$U\mathfrak{g} := T\mathfrak{g} / \langle a \otimes b - b \otimes a - [a, b] \rangle,$$

where $\langle a \otimes b - b \otimes a - [a, b] \rangle$ is the 2-sided ideal in the tensor algebra. Furthermore, there is an augmentation

$$\epsilon : U\mathfrak{g} \rightarrow k$$

induced by the augmentation of the tensor algebra.

Theorem 3.2.2 (Poincare-Birkhoff-Witt). *For any basis $\{x_i \mid i \in I\}$ of a Lie algebra \mathfrak{g} with ordered index set I , the monomial*

$$x_{i_1}^{e_1} \cdots x_{i_n}^{e_n} \quad (\text{with } i_1 < \cdots < i_n, \text{ and integers } e_i > 0)$$

form a basis for the enveloping algebra $U\mathfrak{g}$.

Corollary 3.2.2.1. *The natural map of a Lie algebra to its universal enveloping algebra is an injection.*

Remark 3.2.3. *A consequence of PBW is the existence of **free Lie algebra**. Given any k -linear space V , the free Lie algebra $\mathbb{L}_k V$ is the Lie subalgebra of $T_k V$ (with commutator bracket) generated by V . The inclusion $\mathbb{L}_k V \hookrightarrow T_k V$ extends to an isomorphism of associative algebras*

$$U\mathbb{L}_k V \cong T_k V.$$

CHAPTER 4
HOMOLOGICAL ALGEBRA

4.1 Lie algebra homology and cohomology

4.1.1 Lie (co)homology as derived functor

The reference for this subsection is [54].

Definition 4.1.1. Given a \mathfrak{g} -module M , we have two functors

1. The invariant submodule $M^{\mathfrak{g}}$ of M

$$M^{\mathfrak{g}} := \{m \in M \mid [x, m] = 0 \quad \forall x \in \mathfrak{g}\},$$

denoted by $-^{\mathfrak{g}}$.

2. The coinvariants $M_{\mathfrak{g}}$ of M , $M_{\mathfrak{g}} = M/\mathfrak{g}M$, denoted by $-_{\mathfrak{g}}$.

$M^{\mathfrak{g}}$ is the maximal trivial \mathfrak{g} -submodule of M , and so that $-^{\mathfrak{g}}$ is right adjoint to the trivial \mathfrak{g} -module functor, which is a left exact functor. Similarly $M_{\mathfrak{g}}$ is the largest quotient module of M that is trivial, and so that $-_{\mathfrak{g}}$ is left adjoint to the trivial \mathfrak{g} -module functor.

Definition 4.1.2. We write $H_*(\mathfrak{g}, M)$ or $H_*^{\text{Lie}}(\mathfrak{g}, M)$ for the left derived functors $\mathbb{L}_*(-_{\mathfrak{g}})(M)$ of $-_{\mathfrak{g}}$ and call them the *homology groups* of \mathfrak{g} with coefficients in M . By definition, $H_*^{\text{Lie}}(\mathfrak{g}, M) = M_{\mathfrak{g}}$.

Dually, we write $H^*(\mathfrak{g}, M)$ or $H_{\text{Lie}}^*(\mathfrak{g}, M)$ for the right derived functors $\mathbb{R}_*(-^{\mathfrak{g}})(M)$ of $-^{\mathfrak{g}}$ and call them the *cohomology groups* of \mathfrak{g} with coefficients in M . By definition, $H_{\text{Lie}}^*(\mathfrak{g}, M) = M^{\mathfrak{g}}$.

Theorem 4.1.3. *Let M be a \mathfrak{g} -module, then*

$$H_*(\mathfrak{g}, M) \cong \text{Tor}_*^{U\mathfrak{g}}(k, M)$$

and

$$H^*(\mathfrak{g}, M) \cong \text{Ext}_{U\mathfrak{g}}^*(k, M).$$

Proof.

$$k \otimes_{U\mathfrak{g}} M = (U\mathfrak{g}/\text{Ker } \epsilon) \otimes_{U\mathfrak{g}} M \cong M/\text{Ker } \epsilon \cdot M = M/\mathfrak{g}M = M_{\mathfrak{g}}$$

and

$$\text{Hom}_{U\mathfrak{g}}(k, M) = \text{Hom}_{U\mathfrak{g}}(k, M) = M^{\mathfrak{g}}.$$

□

Proposition 4.1.4. *For any Lie algebra \mathfrak{g} , $H_1(\mathfrak{g}, k) = \mathfrak{g}_{\text{ab}}$.*

4.1.2 Chavelley-Eilenberg complex

Chavelley and Eilenberg [14] give a resolution explicitly, and we will briefly summarise it here.

Given a Lie algebra \mathfrak{g} over k and a \mathfrak{g} -module M . Then the complex $C_*(\mathfrak{g}; M) := (M \otimes_k \bigwedge_{i=1}^n \mathfrak{g}, \delta_n)$ with the boundary map defined by

$$\begin{aligned} \delta_n : M \otimes_k \bigwedge_{i=1}^n \mathfrak{g} &\rightarrow M \otimes_k \bigwedge_{i=1}^{n-1} \mathfrak{g} \\ m \otimes a_1 \wedge \cdots \wedge a_n &\mapsto \sum_{i=1}^n (-1)^i m a_i \otimes a_1 \wedge \cdots \wedge \hat{a}_i \wedge \cdots \wedge a_n \\ &\quad + \sum_{1 \leq i < j \leq n} (-1)^{i+j-1} m \otimes [a_i, a_j] \wedge a_1 \wedge \cdots \wedge \hat{a}_i \wedge \cdots \wedge \hat{a}_j \wedge \cdots \wedge a_n, \end{aligned} \tag{4.1}$$

is called the Chavelley-Eilenberg chain complex of \mathfrak{g} with coefficient M , where

$$\delta_1 : M \otimes_k \mathfrak{g} \rightarrow M$$

$$m \otimes a \mapsto ma.$$

In particular, we write $C_*(\mathfrak{g})$ for $C_*(\mathfrak{g}; U\mathfrak{g})$.

By convention, $\wedge^0 \mathfrak{g} = k$ and $\wedge^1 \mathfrak{g} = \mathfrak{g}$, therefore there is a natural augmented map

$$\epsilon : C_0(\mathfrak{g}) = U\mathfrak{g} \rightarrow k$$

$$a \in \mathfrak{g} \mapsto 0$$

that is zero-extended to a map of complexes.

Theorem 4.1.5 ([54, Theorem 7.]). $\epsilon : C_*(\mathfrak{g}) \rightarrow k$ is a projective resolution of \mathfrak{g} -module k .

Corollary 4.1.5.1 (Chavelley-Eilenberg). *If M is a right \mathfrak{g} -module, then the homology modules $H^*(\mathfrak{g}; M)$ are the homology of the chain complex*

$$M \otimes_{U\mathfrak{g}} C^*(\mathfrak{g}) = M \otimes_{U\mathfrak{g}} U\mathfrak{g} \otimes_k \bigwedge \mathfrak{g} = M \otimes_k \bigwedge \mathfrak{g}.$$

If M is a left \mathfrak{g} -module, then the cohomology modules $H^(\mathfrak{g}; M)$ are the cohomology of the cochain complex*

$$\mathrm{Hom}_{\mathfrak{g}}(C(\mathfrak{g}), M) = \mathrm{Hom}_{\mathfrak{g}}(U\mathfrak{g} \otimes_k \bigwedge \mathfrak{g}, M) = \mathrm{Hom}_k(\bigwedge \mathfrak{g}, M).$$

In this complex, an n -cochain $f : \wedge^n \mathfrak{g} \rightarrow M$ is just an alternating k -multilinear function $f(x_1, \dots, x_n)$ of n variables in \mathfrak{g} , taking values in M . The coboundary δf of such an n -cochain is the $(n+1)$ -cochain

$$\begin{aligned} \delta f(x_1, \dots, x_{n+1}) &= \sum (-1)^{i+1} x_i f(x_1, \dots, \hat{x}_i, \dots, x_{n+1}) \\ &+ \sum (-1)^{i+j} f([x_i, x_j], x_1, \dots, \hat{x}_i, \dots, \hat{x}_j, \dots, x_{n+1}). \end{aligned}$$

Example 4.1.6. In particular, for trivial coefficient $M = k$, the boundary map Equation (4.1) becomes

$$\delta_n : \bigwedge_{i=1}^n \mathfrak{g} \rightarrow \bigwedge_{i=1}^{n-1} \mathfrak{g}$$

$$a_1 \wedge \cdots \wedge a_n \mapsto + \sum_{1 \leq i < j \leq n} (-1)^{i+j-1} [a_i, a_j] \wedge a_1 \wedge \cdots \wedge \hat{a}_i \wedge \cdots \wedge \hat{a}_j \wedge \cdots \wedge a_n,$$

and similarly, the coboundary map

Remark 4.1.7. For a graded k -vector space V , we use the following notation

$$\bigwedge V := \text{Sym}_k(V[1]) = k \oplus V_0 \oplus (V_1 \oplus V_0 \wedge V_0) \oplus (V_2 \oplus V_1 \wedge V_1 \oplus V_0 \wedge V_0 \wedge V_0) \oplus \cdots .$$

Thus we have an isomorphism of graded vector spaces $C^*(\mathfrak{g}) = \text{Sym}(\mathfrak{g}^*[-1])$.

Example 4.1.8. Let k^n be the abelian Lie algebra of dimension n . By the discussion in Example 4.1.6, we know that the boundary map is 0 by the commutativity. Thus the homology is simply

$$H_*(k^n) = \bigwedge k^n, \quad (4.2)$$

which is the same as $H_*(T^n; k)$.

Let \mathfrak{n}_3 be the nilpotent algebra of 3×3 strict upper triangular matrices, with generators

$$x := \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad y := \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \quad z := \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

and relations $[x, y] = z$ (all others are zero!). Thus, the boundary map δ is

$$x \wedge y \wedge z \mapsto [x, y] \wedge z = 0, \quad x \wedge y \mapsto z, \quad \text{all other terms} \mapsto 0.$$

Thus

$$H_*(\mathfrak{n}_3) = k \oplus k^2 \oplus k^2 \oplus k. \quad (4.3)$$

Example 4.1.9. Consider the cohomology of k^n and \mathfrak{n}_3 respectively. They could be computed by taking cohomology of the dual complex. Assume that $\text{char } k = 0$.

The dual complex of $C(k^n)$ is itself since all the differentials are 0, hence

$$H^*(k^n) = \bigwedge k^n. \quad (4.4)$$

On the other hand, the dual complex of \mathfrak{n}_3 is

$$k \rightarrow kx^* + ky^* + kz^* \xrightarrow{d} k(x \wedge y)^* + k(y \wedge z)^* + k(z \wedge x)^* \rightarrow k(x \wedge y \wedge z)^*$$

where d is the map

$$z^* \mapsto (x \wedge y)^*, \text{ all other terms } \mapsto 0.$$

Hence the cohomology is

$$H^*(\mathfrak{n}_3) = k \oplus k^2 \oplus k^2 \oplus k, \quad (4.5)$$

in particular $H^1 = k \cdot x^* + k \cdot y^*$.

4.1.3 Bott-Kostant theorem

Example 4.1.9 is a special case of the celebrated Bott-Kostant theorem, which states the general structure of the cohomology group of the maximal nilpotent subalgebra of a complex semisimple Lie algebra:

Theorem 4.1.10. Let \mathfrak{g} be a complex semisimple Lie algebra and \mathfrak{n} be a maximal nilpotent subalgebra. Let V be a finite dimensional irreducible \mathfrak{g} -module. Then

$$\dim H^i(\mathfrak{n}, V) = |W^{(i)}|,$$

where $W^{(i)}$ is the subset of the Weyl group consisting of all elements of length i .

This theorem was first studied in the sequence of papers [12, 37, 38]. Later, proofs using different technologies were kept found, for instance, in [11], the authors introduce the category \mathcal{O} and the so-called BGG resolution in this category, which provides a proof with pure homological algebra language; in [20], the authors explain the harmonic forms on the flag manifold G/B introduced in [37, 38] using (Bruhat) Poisson structure.

4.2 Koszul Duality

We will assume that k be a field of characteristic 0 in this subsection.

Let \mathbf{DGC}_k denote the category of coassociative counital DG coalgebras over k . We shall often work with augmented coassociative counital DG coalgebras C which are conilpotent in the sense that

$$\bar{C} = \bigcup_{n \geq 2} \text{Ker}(C \xrightarrow{\Delta(n)} C^{\otimes n} \twoheadrightarrow \bar{C}^{\otimes n})$$

where $\Delta(n)$ denotes the n -th iteration of the comultiplication map $\Delta_C : C \rightarrow C \otimes C$ and \bar{C} is the cokernel of the augmentation map $\epsilon_C : k \rightarrow C$. We denote the category of such coalgebras by $\mathbf{DGC}_{k/k}$.

4.2.1 Twisting cochains

Given an algebra $R \in \mathbf{DGA}_{k/k}$ and a coalgebra $C \in \mathbf{DGC}_{k/k}$, we define a twisting cochain $\tau : C \rightarrow R$ to be a linear map of degree -1 satisfying

$$d_R \tau + \tau d_C + m_R(\tau \otimes \tau) \Delta_C = 0, \quad \tau \epsilon_C = 0, \quad \epsilon_R \tau = 0.$$

We write $\text{Tw}(C, R)$ for the set of all twisting cochains from C to R . For a fixed algebra R , the functor

$$\text{Tw}(-, R) : \mathbf{DGC}_{k/k} \rightarrow \mathbf{Set}, \quad C \mapsto \text{Tw}(C, R),$$

is representable; the corresponding coalgebra $B(R) \in \mathbf{DGC}_{k/k}$ is called the bar construction of R : it is defined as the tensor coalgebra $T_k(\bar{R}[1])$ with differential lifting d_R and m_R . Dually, for a fixed coalgebra C , the functor

$$\text{Tw}(C, -) : \mathbf{DGA}_{k/k} \rightarrow \mathbf{Set}, \quad C \mapsto \text{Tw}(C, R),$$

is corepresentable; the corresponding algebra $\Omega(C) \in \mathbf{DGA}_{k/k}$ is called the cobar construction of C : it is defined as the tensor algebra $T_k(C[-1])$ with differential lifting d_C and Δ_C . Thus, we have canonical isomorphisms

$$\text{Hom}_{\mathbf{DGA}_{k/k}}(\Omega C, A) \cong \text{Tw}(C, A) \cong \text{Hom}_{\mathbf{DGC}_{k/k}}(C, B(A)),$$

which means

$$\Omega : \mathbf{DGC}_{k/k} \rightleftarrows \mathbf{DGA}_{k/k} : B \tag{4.6}$$

forms an adjunction.

Theorem 4.2.1. *Both $\mathbf{DGA}_{k/k}$ and $\mathbf{DGC}_{k/k}$ carry model structures ([30]). The adjunction 4.6 is a Quillen equivalence ([42]).*

4.2.2 Duality

Let $R \in \mathbf{DGA}_{k/k}$ and $C \in \mathbf{DGC}_{k/k}$. Let $\mathbf{Mod} - R$ denote the category of right DG modules over R , and dually let $\mathbf{Comod} - C$ denote the category of right DG comodules over C which are conilpotent. Given a twisting cochain $\tau \in \text{Tw}(C, R)$ one can define the functors

$$- \otimes_{\tau} C : \mathbf{Mod} - R \rightleftarrows \mathbf{Comod} - C : - \otimes_{\tau} R \tag{4.7}$$

called the **twisted tensor products**.

Theorem 4.2.2 ([42, Theorem 2.3.1.]). *For $\tau \in \text{Tw}(C, R)$, the following are equivalent:*

1. *the adjunction 4.7 induces a derived equivalence.*
2. *the complex $C \otimes_{\tau} R$ is acyclic;*
3. *the complex $R \otimes_{\tau} C$ is acyclic;*
4. *the morphism $\Omega(C) \rightarrow R$ corresponding to τ is a quasi-isomorphism in $\mathbf{DGA}_{k/k}$;*
5. *the morphism $C \rightarrow \bar{R}$ corresponding to τ is a quasi-isomorphism in $\mathbf{DGC}_{k/k}$.*

If furthermore any of the conditions holds, the DG algebra R is determined by C up to isomorphism in $\text{Ho}(\mathbf{DGA}_{k/k})$ and the DG coalgebra C is determined by R up to isomorphism in $\text{Ho}(\mathbf{DGC}_{k/k})$.

A twisting cochain $\tau \in \text{Tw}(C, R)$ satisfying the conditions of Theorem 4.2.2 is called acyclic. In this case, the DG coalgebra C is called Koszul dual to the DG algebra R and R is called Koszul dual to C .

4.2.3 Other forms of duality

Theorem 4.2.3. *Let $\mathbf{DGCC}_{k/k}$ be the category of cocommutative conilpotent coaugmented coalgebras over k . Then there is a Quillen equivalence*

$$\Omega_{\text{com}} : \mathbf{DGCC}_{k/k} \rightleftarrows \mathbf{DGLA}_k : \mathbf{B}_{\text{Lie}},$$

where $\mathbf{B}_{\text{Lie}}\alpha := \text{Sym}^c(\alpha[1])$ and $\Omega_{\text{com}}C := \mathbb{L}(\bar{C}[-1])$.

Example 4.2.4. Let \mathfrak{a} be the 2-dimensional abelian Lie algebra over k , concentrated at degree 0. And $\mathfrak{a}[1]$ is concentrated at degree 1 (recall that $\mathfrak{a}[1]_i = \mathfrak{a}_{i-1}$). Then

$$C = k \oplus \mathfrak{a}[1] \oplus \Lambda^2(\mathfrak{a}[1])$$

and

$$\bar{C} = \mathfrak{a}[1] \oplus \Lambda^2(\mathfrak{a}[1]),$$

so

$$\bar{C}[-1] = \mathfrak{a} \oplus \Lambda^2(\mathfrak{a})[1].$$

Write x, y the basis of \mathfrak{a} and t the basis of $\Lambda^2(\mathfrak{a})$, then

$$\mathbb{L}(\bar{C}[-1]) = \mathbb{L}_k \langle x, y, t \mid dt = [x, y] \rangle \quad (4.8)$$

and

$$U\mathbb{L}(\bar{C}[-1]) = k \langle x, y, t \mid dt = [x, y] \rangle.$$

To really see the differential map in Example 4.2.4, we need to carefully check the construction of the Quillen functor Ω_{com} .

Consider from the associative cobar construction, where given a coalgebra C , the cobar-bar construction

$$\Omega(C) := T_k(\bar{C}[-1])$$

with differential $d = d_0 + d_1$, where

$$d_0 = d_{C[-1]} : C[-1] \rightarrow C[-2]$$

is the inner differential on C and $d_1 : \bar{C}[-1] \rightarrow \bar{C}[-1] \otimes \bar{C}[-1]$ is given by

$$\begin{array}{ccc} \bar{C} & \xrightarrow{\bar{\Delta}_C} & \bar{C} \otimes \bar{C} \\ \uparrow [+1] & & \downarrow [-1] \otimes [-1] \\ \bar{C}[-1] & \xrightarrow{d_1} & \bar{C}[-1] \otimes \bar{C}[-1]. \end{array}$$

Suppose $c \in \bar{C}$ such that $\bar{\Delta}_C(c) = \sum a_i \otimes b_i$. The cocommutativity implies that $\bar{\Delta}_C(c) = \tau \bar{\Delta}_C(c) = \sum (-1)^{|a_i||b_i|} b_i \otimes a_i$ and so

$$\bar{\Delta}_C(c) = \frac{1}{2} \sum (a_i \otimes b_i + (-1)^{|a_i||b_i|} b_i \otimes a_i).$$

Plug it in we have that

$$\begin{aligned} d_1(c[-1]) &= ([-1] \otimes [-1]) \circ \bar{\Delta}_C(c) \\ &= \frac{1}{2} \sum (-1)^{|a_i|} \left(a_i[-1] \otimes b_i[-1] - (-1)^{(|a_i|-1)(|b_i|-1)} b_i[-1] \otimes a_i[-1] \right) \\ &= \frac{1}{2} \sum (-1)^{|a_i|} [a_i[-1], b_i[-1]]. \end{aligned}$$

We now know the differential map in Example 4.2.4, and also

Theorem 4.2.5. $\mathbb{L}_k(\bar{C}[-1], d = d_0 + d_1)$ is a DG Lie algebra with universal enveloping algebra ΩC .

Theorem 4.2.6 (Universal Property, [21, Lemma 22.1.]). *Suppose $C = k \oplus \bar{C}$ is a primitively cogenerated cocommutative graded coalgebra. Then any linear map of degree zero, $f : \bar{C} \rightarrow V$ lifts to a unique morphism of graded coalgebras $\tilde{f} : C \rightarrow \Lambda V$ such that*

$$\begin{array}{ccc} \bar{C} & \xrightarrow{\tilde{f}} & \bar{\Lambda}V \\ & \searrow f & \downarrow \text{pr} \\ & & V, \end{array}$$

where ΛV has the coalgebra structure $\Delta(x) = 1 \otimes x + x \otimes 1$.

4.3 Quillen homology

Definition 4.3.1. Given a category C , an **abelian object** A in C is an object where $\text{Hom}_C(-, A)$ is naturally an abelian group. i.e. the functor

$$\text{Hom}_C(-, A) : C \rightarrow \mathbf{Set}$$

factors through \mathbf{Ab} via the forgetful functor $U : \mathbf{Ab} \rightarrow \mathbf{Set}$.

We denote C^{ab} the category of abelian objects, with morphisms preserving the abelian group structures. Following [48, Section II.5.], we assume that the forgetful functor $i : C^{\text{ab}} \rightarrow C$ has a left adjoint $\text{Ab} : C \rightarrow C^{\text{ab}}$, and there is a model structure on C^{ab} such that

$$\text{Ab} : C \rightleftarrows C^{\text{ab}} : i$$

is a Quillen pair.

Definition 4.3.2. By Theorem 2.2.7, in this case the left derived functor $\mathbb{L}\text{Ab}$ exists, which is called the **Quillen homology** of C .

Example 4.3.3. Let $C = \mathbf{DGLA}_k$ be the category of DG Lie algebras over k . This category has a natural model structure, with weak equivalences being quasi-isomorphisms ([48, Part II, Section 5.]). The abelian objects in C are just the abelian Lie algebras (i.e., the DG Lie algebras with zero bracket). The category C^{ab} can thus be identified with $\text{Com}(k)$. The abelianization functor $\text{Ab} : \mathbf{DGLA}_k \rightarrow \text{Com}(k)$ is given by $\mathfrak{g} \mapsto \mathfrak{g}/[\mathfrak{g}, \mathfrak{g}]$. If \mathfrak{g} is an ordinary Lie algebra, and $L \xrightarrow{\sim} \mathfrak{g}$ is a cofibrant resolution of \mathfrak{g} in \mathbf{DGLA}_k ,

$$H_i(L/[L, L]) \cong H_{i+1}(\mathfrak{g}; k),$$

Thus, the Quillen homology of \mathfrak{g} agrees with the usual Lie algebra homology with trivial coefficients.

Example 4.3.4. If $C = \mathbf{DGA}_k$ be the category of associative DG algebras. Unlike in \mathbf{DGLA}_k , the only abelian object in \mathbf{DGA}_k is the zero algebra. However, if we fix an algebra $A \in \mathbf{DGA}_k$ and consider the category $C = \mathbf{DGA}_k/A$ of algebras over A , then it is easy to show that C^{ab} is equivalent to the (abelian) category $A - \mathbf{DGBimod}$ of DG bimodules over A . The equivalence is given by the semi-direct product construction

$$A \ltimes - : A - \mathbf{DGBimod} \rightarrow \mathbf{DGA}_k/A,$$

assigning to a bimodule M the DG algebra $A \ltimes M$ together with the canonical projection $A \ltimes M \rightarrow A$.

$$\mathrm{Hom}_{\mathbf{DGA}_k}(B, A \ltimes M) \cong \mathrm{Der}_k(B, M)$$

On the other hand, for any A -bimodule M , there is a natural isomorphism

$$\mathrm{Der}_k(B, M) \cong \mathrm{Hom}_{\mathbf{DGA}_k}(\Omega_k^1(B/A), M)$$

where $\Omega_k^1(B/A) := A \otimes_B \Omega_k^1(B) \otimes_B A$ and $\Omega_k^1(B)$ denotes the kernel of the multiplication map $B \otimes B \rightarrow B$. Thus, for $C = \mathbf{DGA}_k/A$, we have a Quillen pair

$$\Omega_k^1(-/A) : \mathbf{DGA}_k/A \rightleftarrows A - \mathbf{DGBimod} : A \ltimes -.$$

If A is an ordinary k -algebra, the Quillen homology of C essentially coincides with Hochschild homology:

$$H_n(\mathbb{L}\Omega_k^1(B/A)) = \begin{cases} \Omega_k^1(B/A) & n = 0 \\ \mathrm{Tor}_{n+1}^B(A, A) & n \geq 1. \end{cases} \quad (4.9)$$

The derived abelianization functor $\mathbb{L}\Omega_k^1(A)$ evaluated at id_A is called the **noncommutative cotangent complex** of A .

If furthermore the model structure on C^{ab} is stable, i.e. C^{ab} is pointed and there is an invertible suspension functor $\Sigma : \mathrm{Ho}(C^{\mathrm{ab}}) \rightarrow \mathrm{Ho}(C^{\mathrm{ab}})$ making $\mathrm{Ho}(C^{\mathrm{ab}})$ a triangulated category ([31, Section 7.1.]), we can define the dual notion:

Definition 4.3.5. Following the notations and assumptions, for any $X \in C$ and $A \in C^{\mathrm{ab}}$, we call

$$H_C^n(X, A) := \mathrm{Hom}_{\mathrm{Ho}(C^{\mathrm{ab}})}(\mathbb{L}\mathrm{Ab}(X), \Sigma^{-n}A)$$

the **Quillen cohomology** of X with coefficient in A .

Example 4.3.6. Fix an algebra $A \in \mathbf{CDGA}_k$ and consider the category $C = \mathbf{CDGA}_k/A$ of algebras over A . As in Example 4.3.4, there is a Quillen pair

$$\Omega_c^1(-/A) : \mathbf{CDGA}_k/A \rightleftarrows A - \mathbf{DGMod} : A \times -.$$

Here the abelianization functor is given by $\Omega_c^1(B/A) := A \otimes_B \Omega_c^1(B)$, where $\Omega_c^1(B)$ is the module of Kähler differentials of the commutative k -algebra B . The corresponding derived functor $\mathbb{L}\Omega_c^1(A)$ evaluated at the identity morphism of A is usually denoted $\mathbb{L}_{k/A}$ and called the cotangent complex of A . This is an object in the homotopy category $\mathbf{Ho}(\mathbf{CDGA}_k)$, which can be computed by the formula $\mathbb{L}_{k/A} = A \otimes_R \Omega_c^1(R)$, where $Q \rightarrow A$ is a cofibrant resolution of A . The homology of the cotangent complex is called the André-Quillen homology of A .

4.4 Functor homology

Given a small category C , let $C - \mathbf{Mod}$ (resp. $\mathbf{Mod} - C$) be the category of covariant (resp. contravariant) functors from C to $R - \mathbf{Mod}$. This is apparently an abelian category.

Example 4.4.1. Given k -algebra A , consider the category \underline{A} with one object $*$ and $\text{hom}_{\underline{A}}(*, *) = A$ where the composition is the multiplication.

Let \underline{M} be a right \underline{A} -module, i.e. a functor $\underline{M} : \underline{A}^{\text{op}} \rightarrow R - \mathbf{Mod}$. Now the functor gives an object $M := \underline{M}(*)$ in $R - \mathbf{Mod}$ and for any morphisms $a, b \in A = \text{hom}_{\underline{A}}(*, *)$, denote by

$$\underline{M}(a) : M \rightarrow M$$

$$m \mapsto m \cdot a,$$

then the functoriality implies

$$m \cdot (ab) = \underline{M}(ab)(m) = \underline{M}(b)\underline{M}(a)(m) = \underline{M}(b)(m \cdot a) = (m \cdot a) \cdot b,$$

which means M is a right A -module. On the other hand, a right A -module gives in this way a right \underline{A} -module.

Example 4.4.2. Let \mathcal{C} be a small category and \mathcal{D} be a cocomplete category. Consider the 2-functor

$$S : \mathcal{C}^\circ \times \mathcal{C} \rightarrow \mathcal{D},$$

then the coequalizer of f_* and f^*

$$\coprod_{f:A_0 \rightarrow A_1} S(A_0, A_1) \rightrightarrows \coprod_{A \in \mathbf{ob} \mathcal{C}} S(A, A)$$

is called the coend of S , where f_* is the composition $S(A_0, A_1) \xrightarrow{S(f, \text{id})} S(A_1, A_1) \hookrightarrow \prod_{A \in \mathbf{ob} \mathcal{C}} S(A, A)$ and f^* is the composition $S(A_0, A_1) \xrightarrow{S(\text{id}, f)} S(A_0, A_0) \hookrightarrow \prod_{A \in \mathbf{ob} \mathcal{C}} S(A, A)$. We denote it by $\int_{A \in \mathbf{ob} \mathcal{C}} S(A, A)$.

Given right A -module M and left A -module N , we have a natural functor

$$\begin{aligned} S : \underline{A}^\circ \times \underline{A} &\rightarrow \mathbf{Ab} \\ (*, *) &\mapsto M \otimes_{\mathbb{Z}} N \\ (f, g) &\mapsto f \otimes g. \end{aligned}$$

We can verify that $\int_{\underline{A}} S = M \otimes_{\mathbb{Z}} N$.

We need a notion of PROP to do the homological algebra over.

Definition 4.4.3. Given a monoidal small category \mathcal{C} , we call it a PROP if the objects in \mathcal{C} corresponds to \mathbb{N} (so that the objects are denoted by $\{[n]\}_{n \in \mathbb{N}}$), with the monoidal structure coincides with the addition of natural numbers, namely

$$[n] \otimes [m] = [n + m].$$

Example 4.4.4. Let \mathbf{FinSet}_* be the category of finite pointed sets, i.e.

1. objects in \mathbf{FinSet}_* are $\{\langle n \rangle := \{0, 1, \dots, n\}\}_{n \in \mathbb{N}}$ where $0 \in \langle n \rangle$ is the base point,
2. morphisms in \mathbf{FinSet}_* are the maps that preserve the base point, and
3. the monoidal structure on \mathbf{FinSet}_* is given by the wedge product, i.e.

$$\langle n \rangle \wedge \langle m \rangle = \langle n + m \rangle,$$

clearly this is a PROP.

Example 4.4.5. Let \mathfrak{G} be the category of finitely generated free groups, whose objects are denoted by $\langle n \rangle$, $n \in \mathbb{N}$. The monoidal structure is given by free product.

Let k be a field. A commutative Hopf algebra H over k defines the (strong monoidal) covariant functor $\mathfrak{G} \rightarrow \mathbf{Vec}_k$ by $n \mapsto H^{\otimes n}$, which we denote by \underline{H} . The assignment $H \mapsto \underline{H}$ gives an equivalence between the category of commutative Hopf algebras over k and the category of k -algebras over the PROP \mathfrak{G} .

Notice that a cocommutative Hopf algebra is a right module over \mathfrak{G} and a commutative Hopf algebra is a left module over \mathfrak{G} .

Example 4.4.6. Let k be a field and let \mathfrak{A}_k be the category of finitely generated free associative algebras over k , whose objects are denoted by $\langle n \rangle$, $n \in \mathbb{N}$. The monoidal structure is given by free product.

Example 4.4.7. We have a canonical left \mathfrak{G} -module \mathbf{lin}_k , defined by

$$\begin{aligned} \mathbf{lin}_k : \mathfrak{G} &\xrightarrow{(-)_{\text{ab}}} \mathbf{Ab} \xrightarrow{- \otimes k} k - \mathbf{Mod} \\ \langle n \rangle &\longmapsto \mathbb{Z}^n \longmapsto k^n. \end{aligned}$$

Lemma 4.4.8. Functors $R[\text{hom}_C(\langle n \rangle, -)]$ and $R[\text{hom}_C(-, \langle n \rangle)]$ are projective.

Proof. Given left C -modules F, G with surjective maps $\tau : F \Rightarrow G$, then we need to verify that for any map $\alpha : R[\text{hom}_C(\langle n \rangle, -)]$, there exists

$$\begin{array}{ccc}
& & R[\text{hom}_C(\langle n \rangle, -)] \\
& \swarrow \tilde{\alpha} & \downarrow \alpha \\
F & \xrightarrow{\tau} & G \longrightarrow 0.
\end{array}$$

Notice that for any left modules $H, K : C \Rightarrow R\text{-Mod}$, $\text{Nat}(H, K)$ has a natural k -module structure given by

$$(\alpha + \beta)_A := \alpha_A + \beta_A,$$

$$(r\alpha)_A := r\alpha_A.$$

Let $h^n := R[\text{hom}_C(\langle n \rangle, -)]$ be the free R -module generated by the set $\text{hom}_C(\langle n \rangle, -)$, hence by Yoneda lemma

$$\Phi : \text{Nat}(h^n, G) \xrightarrow{\cong} G(\langle n \rangle) : \Psi$$

$$\alpha \mapsto \alpha_{\langle n \rangle}(\text{id}_{\langle n \rangle})$$

$$\eta^m \mapsto m$$

is an isomorphism of R -modules, where η^m is the natural transformation

$$\eta^m : \text{hom}_C(A, -) \Rightarrow F$$

$$\eta_B^m : \text{hom}_C(A, B) \rightarrow F(B)$$

$$h \mapsto F(h)(m).$$

This means under the Yoneda isomorphism, α corresponds to an element m in $G(\langle n \rangle)$, and since

$$\tau_{\langle n \rangle} : F(\langle n \rangle) \rightarrow G(\langle n \rangle)$$

is surjective, there exists $\tilde{m} \in F(\langle n \rangle)$ such that $\tau_{\langle n \rangle}(\tilde{m}) = m$. Let $\tilde{\alpha}$ be the natural transformation corresponding to \tilde{m} under Yoneda, and the naturality of Yoneda implies the commutativity.

Similar for the functor $R[\text{hom}_C(-, \langle n \rangle)]$. □

Therefore, we denote by $h_A := R[\text{hom}_C(-, A)]$ and $h^A := R[\text{hom}_C(A, -)]$ for an object A in C .

Proposition 4.4.9. *For a PROP C , $C - \mathbf{Mod}$ has sufficiently many projectives and injectives.*

Proof idea. Consider

$$\bigoplus_{\substack{A \in \text{ob } C \\ a \in F(A)}} h_A \Rightarrow F,$$

where the natural transformation $h_A \Rightarrow F$ is given by $a \in F(A)$ under Yoneda. The direct sum is projective by Lemma 4.4.8, and the surjectiveness is due to Yoneda lemma. \square

We can construct a 2-functor

$$- \otimes_C - : \mathbf{Mod} - C \times C - \mathbf{Mod} \rightarrow R - \mathbf{Mod},$$

so that

$$G \otimes_C F = \left(\bigoplus_{A \in \text{ob } C} G(A) \otimes_k F(A) \right) \Bigg/ \langle \langle (G(f)(x)) \otimes y - x \otimes (F(f)(y)) \rangle \rangle_{\substack{x \in G(B), y \in F(A), \\ A, B \in \text{ob } C, \\ f \in \text{hom}_C(A, B)}} \rangle$$

or equivalently

$$F \otimes_C G \cong \text{coeq} \left[\bigoplus_{f \in \text{hom}_C(A, B)} G(B) \otimes_R F(A) \xrightleftharpoons[f_*]{f^*} \bigoplus_{A \in \text{ob } C} G(A) \otimes_R F(A) \right],$$

where for any $x \in G(B), y \in F(A)$, $f^*(x \otimes y) := x \otimes F(f)(y)$, $f_*(x \otimes y) := G(f)(x) \otimes y$.

Example 4.4.10. *Let k be the constant module (sending every morphism in C to id_k), then*

$$F \otimes_C R \cong \text{colim}_C F.$$

For any diagram $F : J \rightarrow \mathcal{D}$,

$$\text{colim}_J F \cong \text{coeq} \left[\prod_{f \in \text{mor } J} F(\text{dom } f) \xrightleftharpoons[f_*]{f^*} \prod_{j \in J} F(j) \right],$$

where f^* is induced by $\text{id} : F(\text{dom } f) \rightarrow F(\text{dom } f)$ and f_* is induced by $F(f) : F(\text{dom } f) \xrightarrow{f} F(\text{codom } f)$. Thus

$$F \otimes_C R \cong \text{coeq} \left[\bigoplus_{f \in \text{hom}_C(A,B)} F(B) \otimes_R R \begin{array}{c} \xrightarrow{f^*} \\ \xrightarrow{f_*} \end{array} \bigoplus_{A \in \text{ob } C} F(A) \otimes_R R \right],$$

and by definition f^* is induced by $\text{id} : F(\text{dom } f) \rightarrow F(\text{dom } f)$ and f_* is induced by $F(f) : F(\text{dom } f) \xrightarrow{f} F(\text{codom } f)$. Notice that the coproduct in $k - \mathbf{Mod}$ is tensor product, we finish the proof.

Example 4.4.11. We can see that

$$F \otimes_C h^n \cong F(\langle n \rangle)$$

by Yoneda lemma.

Specifically, for any $M \in C - \mathbf{Mod}$ and $N \in \mathbf{Mod} - C$, $\text{Tor}_i^C(M, N)$ makes perfect sense.

There are several interesting applications of functor homology, and we will list some of these here.

Example 4.4.12. Let \mathbf{BG} be the classifying category of a (discrete) group G , then

$$\text{Tor}_i^{\mathbf{BG}}(k, M) \cong H_i(G; M)$$

where M is a left G -module. This is because the bar-resolution is also a resolution in \mathbf{BG} -modules.

Lemma 4.4.13 (Berest). Let lin_k be the composition of functors

$$\text{lin}_k : \mathfrak{G} \hookrightarrow \mathbf{Gp} \xrightarrow{\text{ab}} \mathbf{Ab} \xrightarrow{-\otimes_{\mathbb{Z}} k} k - \mathbf{Vec},$$

then for an affine group scheme G

$$H^{i+1}(G; k) \cong \text{Ext}_{\mathfrak{G}}^i(\text{lin}_k, \mathcal{O}(G)).$$

Proof. For a (discrete) group Γ , denote by $C_\bullet(\mathrm{B}\Gamma, k)$ the standard chain complex

$$C_n(\mathrm{B}\Gamma, k) = k[\Gamma \times \cdots \times \Gamma]$$

and define

$$C_\bullet^+(\mathrm{B}\Gamma, k) := C_\bullet(\mathrm{B}\Gamma, k)[1].$$

Then

$$H_0(C_\bullet^+(\mathrm{B}\Gamma, k)) \cong H_1(\Gamma; k) \cong \Gamma_{\mathrm{ab}} =: \mathrm{lin}_k(\Gamma).$$

Furthermore, when Γ is free, $H_i(C_\bullet^+(\mathrm{B}\Gamma, k)) = 0$ for $i \geq 1$, in which case $\epsilon : C_\bullet^+(\mathrm{B}\Gamma, k) \rightarrow \mathrm{lin}_k(\Gamma)$ is a quasi-isomorphism. We also notice that

$$k[\Gamma \times \cdots \times \Gamma] \cong h_n(\Gamma),$$

hence we have a projective resolution

$$\epsilon : h_{\bullet+1} \rightarrow \mathrm{lin}_k$$

in the category $\mathfrak{G} - \mathbf{Mod}$ (Lemma 4.4.8).

Applying $\mathrm{Hom}_{\mathfrak{G}}(-, \mathcal{O}(G))$ to this resolution, we have an identification

$$\mathrm{Hom}_{\mathfrak{G}}(h_{n+1}, \mathcal{O}(G)) \cong \mathcal{O}(G \times \cdots \times G)$$

by Yoneda lemma. The corresponding complex is the classical Hochschild complex $C_{\mathrm{Hoch}}^{\bullet+1}(G, k)$ which computes the (rational) cohomology $H^{*+1}(G; k)$ of an affine group scheme G ([32]). \square

Similarly, we could also have

$$\mathrm{Tor}_i^{\mathfrak{G}}(k[\Gamma], \mathrm{lin}_k) \cong H_{i+1}(\Gamma; k) \tag{4.10}$$

Example 4.4.14. A left \mathfrak{A}_k -module is a k -algebra.

Proposition 4.4.15 ([17, Proposition 1.1.]). *Let $p : \mathcal{C} \rightarrow \mathcal{D}$ be a functor between small categories. Then there is an adjunction*

$$\mathbb{L}p_! : D(\mathcal{C} - \mathbf{Mod}) \rightleftarrows D(\mathcal{D} - \mathbf{Mod}) : p^*,$$

which induces isomorphism of complexes

$$p^*N \otimes_{\mathcal{C}}^{\mathbb{L}} M \cong N \otimes_{\mathcal{D}}^{\mathbb{L}} \mathbb{L}p_!M$$

in $D(k - \mathbf{Mod})$.

4.5 Other useful homological algebra results

Definition 4.5.1. Let $F \in \mathbf{Alg}_k$ be a smooth algebra (see Definition 6.2.18), and let J be a finitely generated 2-sided ideal of F . The algebra $A = F/J$ (or the pair $J \subseteq F$) is called a **noncommutative complete intersection** (NCCI) if J/J^2 is a projective bimodule over A .

The following definition is the analogue of Koszul complex in associative algebras.

Definition 4.5.2. Let A be an associative k -algebra. Let $S = \{a_i\} \subseteq A$ be a homogeneous subset of an algebra A . Let us equip the algebra $A\langle\tilde{S}\rangle$ with a supplementary grading (“homological grading”) by putting $|\tilde{a}_i| = 1$ and $|A| = 0$. The differentiation d satisfying $d(\tilde{a}_i) = a_i$, $d(A) = 0$, with

$$d(ab) = d(a)b + (-1)^{|a|}ad(b)$$

(where the elements a and b are homogeneous in both gradings) makes $A\langle\tilde{S}\rangle$ a differential graded algebra, which is called the **Shafarevich complex** and denoted by $\mathrm{Sh}(S, A)$.

Theorem 4.5.3 ([46, Theorem 2.4., Theorem 2.19.]). *Let $S \subseteq A$ be a homogeneous subset consisting of elements of positive degree that minimally generates an ideal $I \subseteq A$. The following statements are equivalent:*

1. $H_i(\text{Sh}(S, A)) = 0$ for $i \geq 0$;
2. $H_1(\text{Sh}(S, A)) = 0$;
3. *the map $A/I\langle \tilde{S} \rangle \rightarrow \text{gr } A$ is an isomorphism of algebras.*

Lemma 4.5.4. *Given a k -algebra A . Suppose homology functor $H_i(-)$ commutes with $\text{Hom}_k(-, k)$, then*

$$\text{Tor}_i^A(M, N)^* \cong \text{Ext}_A^i(M, N^*)$$

for all $i \geq 0$.

Proof. For two rings R, S and a right R -module M , an $R - S$ -bimodule N and a right S -module P , one has

$$\text{Hom}_S(M \otimes_R N, P) \cong \text{Hom}_R(M, \text{Hom}_S(N, P)).$$

Now by definition, take a projective resolution $P_\bullet \rightarrow M$,

$$\begin{aligned} \text{Tor}_i^A(M, N)^* &= (H_i(P_\bullet \otimes_A N))^* \\ &\cong H^i((P_\bullet \otimes_A N)^*) \\ &= H^i(\text{Hom}_A(P_\bullet \otimes_A N, k)) \\ &\cong H^i(\text{Hom}_A(P_\bullet, \text{Hom}_k(N, k))) \\ &= \text{Ext}_A^i(M, N^*). \end{aligned}$$

This argument is formal, which is also valid for functor homology. □

CHAPTER 5
REPRESENTATION HOMOLOGY

5.1 Representation homology in terms of derived functor

5.1.1 Algebraic groups and Lie algebras as functors

Definition 5.1.1. An **affine group scheme** G is a group object in the category \mathbf{Sch}_k which is affine as a scheme. We shall denote by $O(G)$ the coordinate ring of G .

In this paper, we will use the functor of points interpretation of a group scheme, which says a scheme $G \in \mathbf{Sch}_k$ is a group scheme if and only if by viewing it a functor $\mathbf{CommAlg}_k \rightarrow \mathbf{Set}$, it admits factorization

$$\mathbf{CommAlg}_k \rightarrow \mathbf{Gp} \xrightarrow{U} \mathbf{Set}$$

where U is the forgetful functor $\mathbf{Gp} \rightarrow \mathbf{Set}$. This is equivalent to saying that for each $A \in \mathbf{CommAlg}_k$, there is a functorial way of assigning the set $G(A)$ a group structure.

Example 5.1.2. *The basic example is the general linear group GL_n , the affine (group) scheme such that for any commutative k -algebra R ,*

$$GL_n(R) := \left\{ M = \begin{bmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,n} \\ a_{2,1} & a_{2,2} & \cdots & a_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n,1} & a_{n,2} & \cdots & a_{n,n} \end{bmatrix} \middle| a_{i,j} \in R, \det M \in R^\times \right\}.$$

Its coordinate ring is $k[x_{i,j}, \det^{-1}]_{1 \leq i,j \leq n}$, with comultiplication

$$\Delta(x_{i,j}) = \sum_{t=1}^n x_{i,t} \otimes x_{t,j}. \tag{5.1}$$

We will concentrate on the following two types of group schemes.

Definition 5.1.3. Let U_n be the affine (group) scheme such that for any k -algebra R ,

$$U_n(R) := \left\{ \left[\begin{array}{ccccc} 1 & a_{1,2} & a_{1,3} & \cdots & a_{1,n} \\ 0 & 1 & a_{2,3} & \cdots & a_{2,n} \\ 0 & 0 & 1 & \cdots & a_{3,n} \\ 0 & 0 & 0 & \ddots & \vdots \\ 0 & 0 & 0 & 0 & 1 \end{array} \right] \middle| a_{i,j} \in R \right\}.$$

The coordinate ring of U_n is the polynomial algebra $k[x_{i,j}]_{1 \leq i < j \leq n}$. We call it the **unipotent $n \times n$ -matrix group scheme**. We call the matrix $X \in U_n(k[x_{i,j}]_{1 \leq i < j \leq n})$ of the form

$$X := \begin{bmatrix} 1 & x_{1,2} & x_{1,3} & \cdots & x_{1,n} \\ 0 & 1 & x_{2,3} & \cdots & x_{2,n} \\ 0 & 0 & 1 & \cdots & x_{3,n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix}, \quad (5.2)$$

the generic **coordinate matrix of U_n** and also write $\mathcal{O}(U_n) = k[X] := k[x_{i,j}]_{1 \leq i < j \leq n}$.

Definition 5.1.4. Similarly, let B_n be the affine (group) scheme such that for any k -algebra R ,

$$B_n(A) := \left\{ \left[\begin{array}{ccccc} a_{1,1} & a_{1,2} & a_{1,3} & \cdots & a_{1,n} \\ 0 & a_{2,2} & a_{2,3} & \cdots & a_{2,n} \\ 0 & 0 & a_{3,3} & \cdots & a_{3,n} \\ 0 & 0 & 0 & \ddots & \vdots \\ 0 & 0 & 0 & 0 & a_{n,n} \end{array} \right] \middle| a_{i,j} \in R, a_{1,1} \cdots a_{n,n} \in R^\times \right\}.$$

The coordinate ring is $\frac{k[x_{i,j}, t]_{1 \leq i \leq j \leq n}}{t \cdot x_{1,1} \cdots x_{n,n} = 1}$. We call it the **Borel $n \times n$ -matrix group**

scheme. We call the matrix $X \in B_n \left(\frac{k[x_{i,j}, t]_{1 \leq i \leq j \leq n}}{t \cdot x_{1,1} \cdots x_{n,n} = 1} \right)$ of the form

$$X := \begin{bmatrix} x_{1,1} & x_{1,2} & x_{1,3} & \cdots & x_{1,n} \\ 0 & x_{2,2} & x_{2,3} & \cdots & x_{2,n} \\ 0 & 0 & x_{3,3} & \cdots & x_{3,n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & x_{n,n} \end{bmatrix}. \quad (5.3)$$

the **coordinate matrix** of B_n and write $O(B_n) = k[X] := \frac{k[x_{i,j}, t]_{1 \leq i \leq j \leq n}}{t \cdot a_{1,1} \cdots a_{n,n} = 1}$.

In general, one could define the internal grading by

$$\deg x_{i,j} = j - i, \quad (5.4)$$

which is a constant in the diagonal direction, and preserved by the multiplication. Since there is only positive degrees, one sets

$$\deg 0 = -\infty.$$

Lemma 5.1.5. 1. *The grading (5.4) makes $O(U_n)$ and $O(B_n)$ graded Hopf algebras.*

2. (a) *The kernel of the counit map $\epsilon_{U_n} : O(U_n) \rightarrow k$ is generated by a regular sequence $\{x_{i,j}\}_{1 \leq i < j \leq n}$.*

(b) *The kernel of the counit map $\epsilon_{B_n} : O(B_n) \rightarrow k$ is generated by a regular sequence $\{x_{i,j} - \delta_j^i\}_{1 \leq i \leq j \leq n}$ where $\delta_j^i = \begin{cases} 1 & i = j \\ 0 & \text{otherwise} \end{cases}$ is the Kronecker delta.*

We will refer to [39] for the proof.

Remark 5.1.6. *There are similar results for Lie algebras. A finite dimensional Lie algebra \mathfrak{g} by viewing it a functor $\mathbf{CommAlg}_k \rightarrow \mathbf{Set}$, has a natural factorization*

$$\mathbf{CommAlg}_k \rightarrow \mathbf{Gp} \xrightarrow{U} \mathbf{Set},$$

where $\mathfrak{g}(A)$ is defined by $\mathfrak{g}(A) := \mathfrak{g} \otimes A$ as vector spaces and $[\xi \otimes a, \eta \otimes b] := [\xi, \eta] \otimes ab$. The resulting Lie algebra $\mathfrak{g}(A)$ can be understood by coefficient changing, from k to A .

The similar results for finite dimensional associative algebras will be in Section 6.2.2.

5.1.2 Representation schemes

Given an affine group scheme G over k , by viewing it a functor, one has an adjunction

$$(-)_G : \mathbf{Gp} \rightleftarrows \mathbf{CommAlg}_k : G \quad (5.5)$$

where the left functor $(-)_G : \mathbf{Gp} \rightarrow \mathbf{CommAlg}_k$ exists since a group homomorphism only satisfies algebraic equations. To any group Γ , the scheme $\text{Rep}_G(\Gamma) := \text{Spec}(\Gamma)_G$ associated to the ring $(\Gamma)_G$ is called the **representation scheme**.

Example 5.1.7. By definition, for any group Γ and any group scheme G ,

$$\text{Hom}_{\mathbf{CommAlg}_k}((\Gamma)_G, R) \cong \text{Hom}_{\mathbf{Gp}}(\Gamma, G(R)).$$

1. Consider when $\Gamma = 1$ the trivial group, then $\text{Hom}_{\mathbf{Gp}}(\Gamma, G(R))$ consists of only one element. Thus $(\Gamma)_G = k$ since $\text{Hom}_{\mathbf{CommAlg}_k}((\Gamma)_G, R)$ consists of only one element for any k -algebra R .
2. Consider when $\Gamma = \mathbb{Z}$ the trivial group, then $\text{Hom}_{\mathbf{Gp}}(\Gamma, G(R)) \cong G(R)$ and in order to have $\text{Hom}_{\mathbf{CommAlg}_k}((\Gamma)_G, R) \cong G(R)$ for all R , by definition $(\Gamma)_G \cong O(G)$.
3. Consider when $\Gamma = \mathbb{Z} \times \mathbb{Z}$, then

$$\text{Hom}_{\mathbf{Gp}}(\mathbb{Z} \times \mathbb{Z}, G(R)) = \{(a, b) \in G(R) \times G(R) \mid ab = ba\}.$$

This means $(\mathbb{Z} \times \mathbb{Z})_G$ is the coordinate ring of $C(G)$, and the representation scheme is a (wide) generalization of the commuting scheme.

The way one generally gets such a group Γ is by taking the fundamental group of some topological space. For instance, the group $\mathbb{Z} \times \mathbb{Z}$ is $\pi_1(T^2)$ where T^2 is the topological 2-torus.

Definition 5.1.8. Given an affine group scheme G over k with coordinate ring $\mathcal{O}(G)$ and a positive integer $g \geq 1$, **the commuting scheme of genus g** is the scheme

$C_g(G) : \mathbf{CommAlg}_k \rightarrow \mathbf{Set}$

$$R \mapsto \left\{ (a_1, b_1, \dots, a_g, b_g) \in \prod_{i=1}^{2g} G(R) \mid [a_1, b_1][a_2, b_2] \cdots [a_g, b_g] = 1 \right\}.$$

Notice that when $g = 1$, the higher genus commuting scheme goes back to the classical commuting scheme. The reason why we give it such a name will be in Example 6.1.3.

5.1.3 Representation homology

Next we extend the adjunction (Equation (5.5)) to simplicially, getting another adjunction

$$(-)_G : s\mathbf{Gp} \rightleftarrows s\mathbf{CommAlg}_k : \mathbf{G}. \quad (5.6)$$

Unfortunately, this is not a Quillen pair. However, one still has

Lemma 5.1.9 ([10, Lemma 3.1]). *The functor $(-)_G$ in (Equation (5.6)) maps the weak equivalences between cofibrant objects in $s\mathbf{Gp}$ to weak equivalences in $s\mathbf{CommAlg}_k$, and hence has a total left derived functor*

$$\mathbb{L}(-)_G : \mathbf{Ho}(s\mathbf{Gp}) \rightarrow \mathbf{Ho}(s\mathbf{CommAlg}_k).$$

As a consequence, the left derived functor $\mathbb{L}(-)_G$ can be computed by $\Gamma \mapsto (Q\Gamma)_G$, where $Q\Gamma$ is a cofibrant replacement of Γ in $s\mathbf{Gp}$.

We shall also denote the derived functor by $O(\text{DRep}_G(\Gamma)) : \text{Ho}(\mathbf{sGp}) \rightarrow \text{Ho}(\mathbf{sCommAlg}_k)$ and call the corresponding (derived) scheme $\text{DRep}_G(\Gamma)$ the **derived representation scheme**.

Notice that we have a chain of Quillen equivalences (see also Equation (2.3))

$$\text{Ho}(\mathbf{Top}_{0,*}) \simeq \text{Ho}(\mathbf{sSet}_0) \simeq \text{Ho}(\mathbf{sGp}),$$

which leads to the definition

Definition 5.1.10. Given a (pointed, connected) topological space X and a group scheme G over k , the representation homology $HR_*(X, G)$ of X with coefficient G is the homotopy group

$$\pi_*((\mathbb{G}(X))_G).$$

Example 5.1.11. Let $G = \mathbb{G}_a$ be the additive group. Then for any group $\Gamma \in \mathbf{Gr}$, one has

$$\text{Hom}_{\mathbf{Gr}}(\Gamma, \mathbb{G}_a(A)) \cong \text{Hom}_{\mathbf{Ab}}(\Gamma_{\text{ab}}, \mathbb{G}_a(A)) \cong \text{Hom}_{k\text{-CommAlg}}(\text{Sym}_k(\Gamma_{\text{ab}} \otimes_{\mathbb{Z}} k), A),$$

where the first isomorphism is because $\mathbb{G}_a(A)$ is an abelian group, and the second isomorphism is the free-forgetful adjunction.

Also, for any reduced simplicial set X , $\mathbb{G}X$ is a canonical simplicial model for $|X|$, so

$$HR_*(X, \mathbb{G}_a) \cong \pi_*((\mathbb{G}X)_{\mathbb{G}_a}).$$

Applying this we have

$$\begin{aligned}
\mathrm{HR}_*(X, \mathbb{G}_a) &\cong \pi_* \mathrm{Sym}_k((\mathbb{G}X)_{\mathrm{ab}} \otimes_{\mathbb{Z}} k) \\
&\cong \mathrm{Sym}_k\left(\bigoplus_{n \geq 0} \pi_n(\mathbb{G}X)_{\mathrm{ab}} \otimes_{\mathbb{Z}} k\right) \\
&\cong \mathrm{Sym}_k\left(\bigoplus_{n \geq 0} H_{n+1}(X, \mathbb{Z}) \otimes_{\mathbb{Z}} k\right) \\
&\cong \mathrm{Sym}_k\left(\bigoplus_{n \geq 0} H_{n+1}(X, k)\right)
\end{aligned}$$

where Sym_k is the graded symmetric product over k and $\pi_*(\mathbb{G}X)_{\mathrm{ab}} \cong H_{*+1}(X, \mathbb{Z})$.

Remark 5.1.12. Continue the discussion in Remark 5.1.6, there are similar results and constructions for Lie algebras. We shall assume that k is always a field of characteristic 0 when talking about Lie algebras. There is an extended adjunction

$$(-)_{\mathfrak{g}} : \mathbf{DGLA}_k \rightleftarrows \mathbf{CDGA}_k : \mathfrak{g} \quad (5.7)$$

which is analogous to Equation (5.5). This adjunction is a Quillen pair ([1, Theorem 6.5.]), and the total left derived functor $\mathbb{L}(-)_{\mathfrak{g}}$ exists. For a given DG Lie algebra \mathfrak{a} , we now define the representation homology of \mathfrak{a} in \mathfrak{g} ([9]) by

$$\mathrm{HR}_*(\mathfrak{a}, \mathfrak{g}) := H_*(\mathbb{L}(\mathfrak{a})_{\mathfrak{g}}).$$

The similar results and constructions for finite dimensional associative algebras will be in Section 6.2.2.

Example 5.1.13. Consider k as a trivial Lie algebra, then it is the functor

$$\begin{aligned}
k - \mathbf{CommAlg} &\rightarrow \mathbf{Lie}_k \\
B &\mapsto B \otimes k = B
\end{aligned}$$

with trivial Lie algebra structure since B is commutative. Then one can see that

$$(\mathfrak{a})_k \cong \mathrm{Sym}_k(\mathfrak{a}_{\mathrm{ab}})$$

since

$$\mathrm{Hom}_{\mathbf{CommAlg}_k}(\mathrm{Sym}_k(\mathfrak{a}_{\mathrm{ab}}), B) = \mathrm{Hom}_{k\text{-Vec}}(\mathfrak{a}_{\mathrm{ab}}, B) = \mathrm{Hom}_{\mathbf{Lie}_k}(\mathfrak{a}, B),$$

where the last equality comes from the trivial Lie structure.

Thus, $\mathrm{HR}_*(\mathfrak{a}, k) \cong \mathrm{Sym}(H_{*+1}(\mathfrak{a}; k))$ for the similar reasons in Example 5.1.11.

5.2 Representation homology by functor homology

There is also a functor homology interpretation inspired by the higher Hochschild homology defined in [47].

Let k be a field. Recall from Example 4.4.5, a commutative Hopf algebra H defines a functor $\underline{H} : \mathfrak{G} \rightarrow \mathbf{Vec}_k$. Since H is a commutative Hopf algebra, it can be viewed as a functor $\underline{H} : \mathfrak{G} \rightarrow \mathbf{CommAlg}_k$. Via left Kan extension, there exists a canonical functor $\mathbf{FreeGp} \rightarrow \mathbf{CommAlg}_k$

$$\begin{array}{ccc} \mathfrak{G} & \xrightarrow{\underline{H}} & \mathbf{CommAlg}_k \\ \downarrow & \nearrow & \\ \mathbf{FreeGp} & & \end{array}$$

which will also be denoted by \underline{H} .

Let X be a reduced simplicial set (or equivalently, a pointed connected topological space). Recall that the Kan loop group construction gives a semifree simplicial group $\mathbb{G}X$ (Theorem 2.1.9), therefore given a commutative Hopf algebra H , we can consider the composition of functors

$$\Delta^{\mathrm{op}} \xrightarrow{\mathbb{G}X} \mathbf{FreeGp} \xrightarrow{\underline{H}} \mathbf{CommAlg}_k,$$

which defines a simplicial commutative algebra $\underline{H}(\mathbb{G}X)$.

Definition 5.2.1. The representation homology of X in H is defined by

$$\mathrm{HR}_*(X, H) := \pi_*(\underline{H}(\mathbb{G}X)) = H_*(N(\underline{H}(\mathbb{G}X))).$$

The following proposition compares two definitions.

Proposition 5.2.2 ([10, Proposition 4.1.]). *Let G be an affine group scheme defined over k with coordinate ring $\mathcal{O}(G)$. Then, for any $X \in s\mathbf{Set}_0$, there is a natural isomorphism of graded commutative algebras*

$$\mathrm{HR}_*(X, \mathcal{O}(G)) \cong \mathrm{HR}_*(X, G).$$

In particular, $\mathrm{HR}_0(X, \mathcal{O}(G)) \cong \pi_1(X)_G$, where $\pi_1(X)$ is the fundamental group of X .

One can also use Tor groups to define the representation homology:

Proposition 5.2.3 ([10, Theorem 4.2.]). *For any $X \in s\mathbf{Set}_0$ and any commutative Hopf algebra H , there is a natural isomorphism of graded commutative algebras*

$$\mathrm{Tor}_i^{\mathbb{G}}(k[\Gamma], H) \cong \mathrm{HR}_i(\Gamma, H).$$

The representation homology of monoids can be defined in the same pattern using functor homology. We refer to [7] for the details.

CHAPTER 6
RESULTS

6.1 Representation homology of torus with unipotent coefficients

We begin our results part by some general computations.

Example 6.1.1 (See also [10, Section 6.1.2.]). *Given a positive integer g , let Σ_g be the closed orientable surface of genus g . Then one has a homotopy colimit diagram in \mathbf{Top}_**

$$\Sigma_g \simeq \text{hocolim} \left[\{*\} \leftarrow S^1 \xrightarrow{\alpha} \bigvee_{i=1}^{2g} S^1 \right]$$

where the map α is the attaching map of the 2-cell onto $\bigvee_{i=1}^{2g} S^1$. In terms of the generators of fundamental groups, we could denote the map α^g by

$$\begin{aligned} \alpha : S_c^1 &\rightarrow \bigvee_{i=1}^g (S_{a_i}^1 \vee S_{b_i}^1) \\ c &\mapsto [a_1, b_1][a_2, b_2] \cdots [a_g, b_g] \end{aligned}$$

where c is a generator of $\pi_1(S^1)$ and a_i, b_i are the generators of $\pi_1(S_{a_i}^1)$ and $\pi_1(S_{b_i}^1)$. Then by applying the Kan loop group construction (see Equation (2.2)), one get a simplicial group model for T^2 :

$$\mathbb{G}(\Sigma_g) \simeq \text{hocolim} \left[1 \leftarrow \mathbb{F}_1 \xrightarrow{\alpha^g} \mathbb{F}_{2g} \right].$$

Since the representation homology commutes with homotopy colimit ([10, Theorem 3.2.]), by Example 5.1.7 one applies the functor $\mathbb{L}(-)_G$ and has

$$\text{HR}_*(\Sigma_g, G) \simeq \text{hocolim} [k \xleftarrow{\epsilon} \mathcal{O}(G) \xrightarrow{\alpha_*^g} \mathcal{O}(G^{2g})]$$

where the map $\epsilon : \mathcal{O}(G) \rightarrow k$ is induced by the unit map of the group G and $\alpha_*^g : \mathcal{O}(G) \rightarrow \mathcal{O}(G^{2g})$ is

$$\alpha_*^g(f)(x_1, y_1, \dots, x_g, y_g) := f([x_1, y_1][x_2, y_2] \cdots [x_g, y_g]), \quad (6.1)$$

or equivalently

$$\begin{aligned}\alpha_g^* : G^{2g}(R) &\rightarrow G(R) \\ (x_1, y_1, \dots, x_g, y_g) &\mapsto [x_1, y_1][x_2, y_2] \cdots [x_g, y_g].\end{aligned}$$

In the category $s\mathbf{CommAlg}_k$, the colimit

$$\operatorname{colim} [k \xleftarrow{\epsilon} \mathcal{O}(G) \xrightarrow{\alpha_g^*} \mathcal{O}(G^{2g})]$$

is

$$\mathcal{O}(G)^{\otimes 2g} \otimes_{\mathcal{O}(G)} k \tag{6.2}$$

where $\mathcal{O}(G)^{\otimes 2g}$ and k are viewed as $\mathcal{O}(G)$ -modules via α_g^* and ϵ . Now by viewing the homotopy colimit as the derived functor of colimit,

$$\operatorname{hocolim} [k \xleftarrow{\epsilon} \mathcal{O}(G) \xrightarrow{\alpha_g^*} \mathcal{O}(G^{2g})] \simeq \mathcal{O}(G)^{\otimes 2g} \otimes_{\mathcal{O}(G)}^{\mathbb{L}} k,$$

where $-\otimes_{\mathcal{O}(G)}^{\mathbb{L}} -$ is the left derived functor of $-\otimes_{\mathcal{O}(G)} -$.

Summarising Example 6.1.1, we have

Lemma 6.1.2. *Let Σ_g be the Riemann surface of genus g .*

$$\operatorname{HR}_*(\Sigma_g, G) \cong \operatorname{H}_*(\mathcal{O}(G)^{\otimes 2g} \otimes_{\mathcal{O}(G)}^{\mathbb{L}} k)$$

where $-\otimes_{\mathcal{O}(G)}^{\mathbb{L}} -$ is the left derived functor of $-\otimes_{\mathcal{O}(G)} -$ in $s\mathbf{CommAlg}_k$.

Example 6.1.3. *One could show that $\operatorname{HR}_0(X, G) \cong (\pi_1(X))_G$ by comparing the universal mapping properties (see also [10, (3.6)]). In fact, for any k -algebra R ,*

$$\begin{aligned}\operatorname{Hom}_{\mathbf{CommAlg}_k}((\pi_1(X))_G, R) &\cong \operatorname{Hom}_{\mathbf{Gp}}(\pi_1(X), G(R)) \\ &\cong \operatorname{Hom}_{s\mathbf{Gp}}(\mathbb{G}(X), G(R)) \\ &\cong \operatorname{Hom}_{s\mathbf{CommAlg}_k}((\mathbb{G}(X))_G, R) \\ &\cong \operatorname{Hom}_{\mathbf{CommAlg}_k}(\operatorname{H}_0((\mathbb{G}(X))_G), R)\end{aligned}$$

where the second and the forth equality are because $G(R)$ and R are viewed as simplicial objects concentrated on 0-th degree, i.e. all higher elements are degenerate. In particular one has

$$\mathrm{HR}_0(\Sigma_g, G) \cong \mathcal{O}(C_g(G)) \cong \mathcal{O}(G)^{\otimes 2g} \otimes_{\mathcal{O}(G)} k.$$

This means the representation homology of Riemann surface of genus g is a derived version of the commuting scheme of higher genus g .

To see from the definition that $\mathcal{O}(G)^{\otimes 2g} \otimes_{\mathcal{O}(G)} k$ is the coordinate ring, by the universal property of colimit,

$$\begin{aligned} & \mathrm{Hom}_{\mathbf{CommAlg}_k}(\mathcal{O}(G)^{\otimes 2g} \otimes_{\mathcal{O}(G)} k, R) \\ & \cong \{(f, h) \in \mathrm{Hom}_{\mathbf{CommAlg}_k}(\mathcal{O}(G)^{\otimes 2g}, R) \times \mathrm{Hom}_{\mathbf{CommAlg}_k}(k, R) \mid f\alpha_*^g = h\epsilon\} \\ & = \{(f_1, \dots, f_{2g}, h) \in G^{2g}(R) \times e(R) \mid (f_1, \dots, f_{2g})\alpha_*^g = h\epsilon\}. \end{aligned}$$

By Equation (6.1), $(f_1, \dots, f_{2g})\alpha_*^g = [f_1, f_2] \cdots [f_{2g-1}, f_{2g}]$, and $e(R)$ consists of only the identity element, so

$$\begin{aligned} & \{(f_1, \dots, f_{2g}, h) \in G^{2g}(R) \times e(R) \mid (f_1, \dots, f_{2g})\alpha_*^g = h\epsilon\} \\ & = \{(f_1, \dots, f_{2g}) \in G^{2g}(R) \mid [f_1, f_2] \cdots [f_{2g-1}, f_{2g}] = 1\}. \end{aligned}$$

Remark 6.1.4. When $\mathrm{char} k = 0$, there is a model structure on \mathbf{CDGA}_k . Now there is a Quillen equivalence

$$N^* : \mathbf{CDGA}_k \rightleftarrows \mathbf{sCommAlg}_k : N,$$

by Dold-Kan correspondence Theorem 2.2.12, hence

$$\begin{aligned} \mathrm{HR}_*(\Sigma_g, G) & := \mathrm{H}_*\left(N(\mathrm{hocolim} [k \xleftarrow{\epsilon} \mathcal{O}(G) \xrightarrow{\alpha_*^g} \mathcal{O}(G^{2g})])\right) \\ & \cong \mathrm{H}_*\left(\mathrm{hocolim} [N(k) \xleftarrow{\epsilon} N(\mathcal{O}(G)) \xrightarrow{\alpha_*^g} N(\mathcal{O}(G^{2g}))])\right) \\ & \cong \mathrm{H}_*\left(\mathrm{hocolim} [k \xleftarrow{\epsilon} \mathcal{O}(G) \xrightarrow{\alpha_*^g} \mathcal{O}(G^{2g})]) \right) \end{aligned}$$

where the last homotopy colimit is taken in the category \mathbf{CDGA}_k . This means (when $\text{char } k = 0$ as the assumption at the beginning) the homotopy colimit $\mathcal{O}(G)^{\otimes 2g} \otimes_{\mathcal{O}(G)}^{\mathbb{L}} k$ in $s\mathbf{CommAlg}_k$ could be computed in \mathbf{CDGA}_k .

6.1.1 Representation homology of unipotent coefficients

The following propositions state the connection between the vanishing result of representation homology and the geometric properties of the underlying scheme.

Proposition 6.1.5. *Let U_n be the affine group scheme consisting of unipotent upper triangular matrices, and let Σ_g be the closed orientable surface of genus g . Then the following are equivalent:*

1. *The representation homology $\text{HR}_i(\Sigma_g, U_n)$ vanishes in dimension greater or equal than n , namely*

$$\text{HR}_i(\Sigma_g, U_n) = 0 \quad \forall i \geq n.$$

2. *There is an isomorphism of graded algebras*

$$\text{HR}_*(\Sigma_g, U_n) \cong \text{HR}_0(\Sigma_g, U_n) \otimes \text{Sym}_k(T_1, \dots, T_{n-1}) \quad (6.3)$$

where Sym_k is the graded symmetric algebra over k and T_i is of homological degree 1.

3. *The commuting scheme of higher genus $C_g(U_n)$ is a complete intersection.*

And similarly

Proposition 6.1.6. *Let B_n be the affine group scheme consisting of invertible upper triangular matrices, and let Σ_g be the closed orientable surface of genus g . Then the following are equivalent:*

1. The representation homology $\mathrm{HR}_i(\Sigma_g, U_n)$ vanishes in dimension greater than n , namely

$$\mathrm{HR}_i(\Sigma_g, B_n) = 0 \quad \forall i > n.$$

2. There is an isomorphism of graded algebras

$$\mathrm{HR}_*(\Sigma_g, B_n) \cong \mathrm{HR}_0(\Sigma_g, B_n) \otimes_k \mathrm{Sym}_k(T_1, \dots, T_n) \quad (6.4)$$

where Sym_k is the graded symmetric product over k and T_i is of homological degree 1.

3. The commuting scheme of higher genus $C_g(B_n)$ is a complete intersection of codimension $\frac{(n-1)n}{2}$.

In the propositions, we have not only the vector space structure of $\mathrm{HR}_*(\Sigma_g, U_n)$ (resp. $\mathrm{HR}_*(\Sigma_g, B_n)$), but also the algebra structure on it when the vanishing condition holds.

Remark 6.1.7. *It seems from Equation (6.3) and Equation (6.4), one can expect that the higher representation homologies are determined by the 0-th homology. However, this is generally not true. One reason is that the representation homology depends on the homotopy type of the space, while the classical representation scheme $\mathrm{Rep}_G(\pi_1(X))$ depends only on the fundamental group. Also, it is a coincidence that we have Equation (6.3), mainly because we take coefficients in unipotent groups. If we take classical reductive group, things will be very different, for instance see Example 6.3.17. We will also provide a counterexample in Proposition 6.1.9.*

Now we can state our main result:

Theorem 6.1.8. *$C(U_2), C(U_3), C(U_4), C(U_5), C(B_2), C(B_3)$ are complete intersections. $C(U_n)$ is not a complete intersection when $n \geq 6$.*

6.1.2 A counterexample

Proposition 6.1.9. $C(U_6)$ is not a complete intersection of codimension 10 in $U_6 \times U_6$.

We refer to [39] for the detailed proof of Proposition 6.1.5, Proposition 6.1.6 and Proposition 6.1.9.

Remark 6.1.10. The proof of Proposition 6.1.9 also shows that U_n for $n \geq 6$ are not complete intersection of the correct dimension, since for bigger n , the general matrix of U_n contains a 6×6 matrix as its submatrix.

6.1.3 A partial collection of the computational results

In this section, we will put together some computational tools and results of the representation homology with unipotent or nilpotent coefficients.

The computations presented in this section form the hints and inspirations for this project. We carefully organize the code and release it as a Macaulay2 package. The code could be found in [40], and we hope that the package will be part of Macaulay2 as well.

Example 6.1.11. We first present the code to check when $n = 2, 3, 4, 5$ the condition

$$HR_i(\Sigma_1, U_n) = 0 \quad \forall i \geq n, \quad (6.5)$$

then by Proposition 6.1.5, $C(U_2), C(U_3), C(U_4), C(U_5)$ are complete intersections.

```
i1 : needsPackage "RepHomology";
```

```
i2 : surfaceRepHomologyGroup(2, 1, GroupType=>"Unipotent")
```

```

H_0 = (QQ[x_{1,1,2}, y_{1,1,2}]
      1
H_1 = (QQ[x_{1,1,2}, y_{1,1,2}]
      1
o2 = ...
o2 : List
i3 : surfaceRepHomologyGroup(3, 1, GroupType=>"Unipotent")
H_0 = cokernel | x_{(1,2,3)}y_{(1,1,2)}-x_{(1,1,2)}y_{(1,2,3)} |
H_1 = cokernel | x_{(1,2,3)}y_{(1,1,2)}-x_{(1,1,2)}y_{(1,2,3)} 0
|
| 0 x_{(1,2,3)}y_{(1,1,2)}-x_{(1,1,2)}y_{(1,2,3)}
H_2 = cokernel | x_{(1,2,3)}y_{(1,1,2)}-x_{(1,1,2)}y_{(1,2,3)} |
o3 = ...
o3 : List
i4 : surfaceRepHomologyGroup(4, 1, GroupType=>"Unipotent")
H_0 = cokernel | x_{(1,3,4)}y_{(1,2,3)}-x_{(1,2,3)}y_{(1,3,4)} ... |
H_1 = cokernel | x_{(1,3,4)}y_{(1,2,3)}-x_{(1,2,3)}y_{(1,3,4)} ... |
| 0 ... |
| 0 ... |
H_2 = cokernel | x_{(1,3,4)}y_{(1,2,3)}-x_{(1,2,3)}y_{(1,3,4)} ... |
| 0 ... |
| 0 ... |
H_3 = cokernel | x_{(1,3,4)}y_{(1,2,3)}-x_{(1,2,3)}y_{(1,3,4)} ... |
o4 = ...
o4 : List

```

```

i5 : surfaceRepHomologyGroup(5, 1, GroupType=>"Unipotent")
H_0 = cokernel | x_(1,4,5)y_(1,3,4)-x_(1,3,4)y_(1,4,5) ... |
H_1 = cokernel | x_(1,4,5)y_(1,3,4)-x_(1,3,4)y_(1,4,5) ... |
          | 0 ... |
          | 0 ... |
          | 0 ... |
H_2 = cokernel | x_(1,4,5)y_(1,3,4)-x_(1,3,4)y_(1,4,5) ... |
          | 0 ... |
          | 0 ... |
          | 0 ... |
          | 0 ... |
          | 0 ... |
H_3 = cokernel | x_(1,4,5)y_(1,3,4)-x_(1,3,4)y_(1,4,5) ... |
          | 0 ... |
          | 0 ... |
          | 0 ... |
H_4 = cokernel | x_(1,4,5)y_(1,3,4)-x_(1,3,4)y_(1,4,5) ... |
o5 = ...
o5 : List

```

We thus verify the condition (6.5).

Example 6.1.12. *This example gives a list all the computations of representation homology of (closed) Riemann surfaces, with maximal unipotent subgroup coefficient:*

Table 6.1: Representation homology of surfaces with maximal unipotent coefficient

Vanishing bound for $HR_i(\Sigma_g, U_n)$	$n = 2$	$n = 3$	$n = 4$	$n = 5$	$n = 6$
$g = 1$	$i = 2$	$i = 3$	$i = 4$	$i = 5$	$i > 6$
$g = 2$	$i = 2$	$i = 3$	$i = 4$	$i = 5$	$i \geq 6$
$g = 3$	$i = 2$	$i = 3$	$i = 4$		
$g = 4$	$i = 2$	$i = 3$	$i = 4$		
$g = 5$	$i = 2$	$i = 3$	$i = 4$		
$g = 6$	$i = 2$	$i = 3$	$i = 4$		
$g = 7$	$i = 2$	$i = 3$	$i = 4$		
$g = 8$	$i = 2$	$i = 3$	$i = 4$		
$g = 9$	$i = 2$	$i = 3$	$i = 4$		

Unfortunately, for higher ranks and genus, the calculations go beyond the capability of computers. Therefore, we leave those terms blank in the table.

At beginning, it really seems that there is a uniform pattern suggested by the table. However, later more work were done and some pathologies were found (Proposition 6.1.9), which corresponds to the unexpected growth of the vanishing bound in the table.

The calculation in Example 6.1.12 relies on certain Koszul resolutions, which inspires a similar computation for Lie algebras, as the infinitesimal approximation of the representation homology of groups. In fact, calculations suggest that the infinitesimal approximation is isomorphic to the original representation homology of groups, which will be shown in Theorem 6.3.15.

In the infinitesimal approximation case, the coefficient is \mathfrak{n}_n , the Lie algebra of all

strictly upper triangular (nilpotent) matrices. This coefficient is the maximal nilpotent subalgebra of the type \mathfrak{a} semisimple Lie algebra, and it is natural to ask the calculation of that with maximal nilpotent subalgebra coefficient of the type semisimple Lie algebras.

Example 6.1.13. *We are also able to compute the representation of 2-dimensional abelian Lie algebra with maximal nilpotent subalgebras of complex semisimple Lie algebras. Let l be the rank of the semisimple Lie algebra \mathfrak{g} .*

Table 6.2: Multiple types nilpotent Lie algebras ($HR_* = 0$ for $* > N$)

Vanishing bound for $HR_i(k \oplus k, \mathfrak{n}(\mathfrak{g}))$	$l = 2$	$l = 3$	$l = 4$	$l = 5$	$l = 6$
Type \mathfrak{a}_n	$N = 2$	$N = 3$	$N = 4$	$N > 5$	
Type \mathfrak{d}_n	$N = 2$	$N = 3$	$N = 5$		
Type \mathfrak{b}_n	$N = 3$	$N = 4$			
Type \mathfrak{c}_n	$N = 2$	$N = 4$			
Type \mathfrak{g}_2	$N = 3$				

The red marks, as in Table 6.1, indicate the existence of “exceptional classes”, which equivalent to the non-complete intersection property of the underlying commuting schemes.

We have further explanations of the table in Theorem 6.4.4 and Example 6.4.5.

6.2 Generalization of the coefficients of representation homology

6.2.1 Representation homology of monoids

In [6], the authors discuss representation homology of monoids, defined via functor homology (see also Chapter 5). We first give another definition using derived functors and compare it with the functor homology definition analogous to [10]. We will use both of the definitions to do comparisons.

Lemma 6.2.1. *Given an affine algebraic monoid M (meaning an affine scheme $\mathbf{CommAlg}_k \rightarrow \mathbf{Set}$ factors through \mathbf{Mon} via the forgetful functor $U : \mathbf{Mon} \rightarrow \mathbf{Set}$), the functor*

$$\begin{aligned} M : \mathbf{CommAlg}_k &\rightarrow \mathbf{Mon} \\ A &\mapsto M(A) \end{aligned}$$

admits a left adjoint

$$(-)_M^{\mathbf{Mon}} : \mathbf{Mon} \rightarrow \mathbf{CommAlg}_k.$$

Proof. Then for an arbitrary monoid N , one has

$$\mathrm{hom}_{\mathbf{Set}}(N, M(A)) \cong \mathrm{hom}_{k\text{-Vec}}(k[N], M(A))$$

where $k[N]$ is the k -vector space spanned by N . Notice that

$$\mathrm{hom}_{\mathbf{Mon}}(N, M(A)) = \{\varphi \in \mathrm{hom}_{\mathbf{Set}}(N, M(A)) \mid \varphi(xy) = \varphi(x)\varphi(y), \varphi(1) = 1\},$$

and

$$M(A) = \mathrm{hom}_{k\text{-CommAlg}}(O(M), A)$$

where the multiplication is given by

$$\begin{aligned} \text{Hom}_{k\text{-CommAlg}}(\mathcal{O}(M), A) \times \text{Hom}_{k\text{-CommAlg}}(\mathcal{O}(M), A) &\rightarrow \text{Hom}_{k\text{-CommAlg}}(\mathcal{O}(M), A) \\ (g, h) &\mapsto \nabla_A(g \otimes h)\Delta \end{aligned}$$

where $g \otimes h$ is the k -algebra homomorphism $\mathcal{O}(M) \otimes \mathcal{O}(M) \rightarrow A \otimes A$, and the identity element is $\mathcal{O}(M) \rightarrow k \rightarrow A$. Thus

$$\text{Hom}_{\text{Mon}}(N, M(A)) = \{\varphi \in \text{hom}_{\text{Set}}(N, M(A)) \mid \varphi(xy) = \nabla_A(\varphi(x) \otimes \varphi(y))\Delta, \varphi(e_N) = e_M\} \quad (6.6)$$

where $\nabla_A : A \otimes A \rightarrow A$ is the multiplication in A .

On the other hand, one has

$$\text{hom}_{k\text{-CommAlg}}(\mathcal{O}(M), A) = \{\varphi \in \text{hom}_k(\mathcal{O}(M), A) \mid \varphi(fg) = \nabla_A(\varphi(f), \varphi(g)), \varphi(1) = 1\} \quad (6.7)$$

and the tensor-hom adjunction

$$\text{Hom}_k(k[N] \otimes \mathcal{O}(M), A) \cong \text{Hom}_k(k[N], \text{Hom}_k(\mathcal{O}(M), A)),$$

where $\varphi \in \text{Hom}_k(k[N], \text{Hom}_k(\mathcal{O}(M), A))$ corresponds to

$$\begin{aligned} \varphi^\# : k[N] \otimes \mathcal{O}(M) &\rightarrow A \\ \gamma \otimes f &\mapsto \varphi(\gamma)(f) \end{aligned}$$

and the free-forgetful adjunction

$$\text{Hom}_{k\text{-CommAlg}}(\text{Sym}_k(k[N] \otimes \mathcal{O}(M)), A) \cong \text{Hom}_k(k[N] \otimes \mathcal{O}(M), A)$$

where $\text{Sym}_k V$ is the symmetric free algebra of V over k . We then use equations 6.6 and 6.7 to find the exact subset of $\text{Hom}_{k\text{-CommAlg}}(\text{Sym}_k(k[N] \otimes \mathcal{O}(M)), A)$ that corresponds to $\text{hom}_{\text{Mon}}(N, M(A))$.

Given any $\gamma \otimes f_1 f_2 \in k[N] \otimes \mathcal{O}(M)$, one has by Equation (6.7)

$$\varphi^\sharp(\gamma \otimes f_1 f_2) = (\varphi(\gamma)(f_1))(\varphi(\gamma)(f_2)) = \varphi^\sharp(\gamma \otimes f_1)\varphi^\sharp(\gamma \otimes f_2)$$

and

$$\varphi^\sharp(\gamma \otimes 1) = 1,$$

in terms of elements in $\text{Hom}_{k\text{-CommAlg}}(\text{Sym}_k(k[N] \otimes \mathcal{O}(M)), A)$, they corresponds to maps satisfying

$$\tilde{\varphi}^\sharp(\gamma \otimes f_1 f_2 - (\gamma \otimes f_1)(\gamma \otimes f_2)) = 0$$

and

$$\tilde{\varphi}^\sharp(\gamma \otimes 1 - 1) = 0.$$

By Equation (6.6), one has

$$\varphi^\sharp(\gamma_1 \gamma_2 \otimes f) = (\varphi(\gamma_1)\varphi(\gamma_2))(f) = \nabla_A(\varphi(\gamma_1) \otimes \varphi(\gamma_2))\Delta(f),$$

if one denotes by $f^{(1)} \otimes f^{(2)} := \Delta(f)$, then these φ^\sharp corresponds to

$$\tilde{\varphi}^\sharp(\gamma_1 \gamma_2 \otimes f - (\gamma_1 \otimes f^{(1)})(\gamma_2 \otimes f^{(2)})) = 0$$

in $\text{Hom}_{k\text{-CommAlg}}(\text{Sym}_k(k[N] \otimes \mathcal{O}(M)), A)$; and there is another equation

$$\varphi^\sharp(1 \otimes f) = e_M(f),$$

which corresponds to elements in $\text{Hom}_{k\text{-CommAlg}}(\text{Sym}_k(k[N] \otimes \mathcal{O}(M)), A)$.

All these equations gives a (closed) subfunctor $\text{Rep}_M(N)$, hence $(N)_M$ has a representation

$$\frac{\text{Sym}_k(k[N] \otimes \mathcal{O}(M))}{\begin{array}{c} \gamma \otimes f_1 f_2 - (\gamma \otimes f_1)(\gamma \otimes f_2), \\ \gamma \otimes 1 - 1, \\ \gamma_1 \gamma_2 \otimes f - (\gamma_1 \otimes f^{(1)})(\gamma_2 \otimes f^{(2)}), \\ 1 \otimes f - e_M(f) \end{array}}.$$

□

Extending the adjunction

$$(-)_M^{\text{Mon}} : \mathbf{Mon} \rightleftarrows \mathbf{CommAlg}_k : M$$

simplicially yields another adjunction

$$(-)_M^{\text{Mon}} : s\mathbf{Mon} \rightleftarrows s\mathbf{CommAlg}_k : M. \quad (6.8)$$

Lemma 6.2.2. *The adjunction 6.8 is a Quillen adjunction, hence the total left derived functor $\mathbb{L}(-)_M^{\text{Mon}}$ exists.*

Proof. It suffices to show that $(-)_M^{\text{Mon}}$ preserves cofibrations and M preserves fibrations (Lemma 2.2.6). □

Definition 6.2.3. Let M be an affine algebraic monoid and let N be a monoid. We call $\text{HR}_*^{\text{Mon}}(N, M) := \pi_*(\mathbb{L}(N)_M^{\text{Mon}})$ the **representation homology** of N in M .

Notice that $O(M)$ is a commutative bialgebra. The reason is similar to that of the fact that $O(G)$ is a commutative Hopf algebra for an algebraic group scheme G . Given a commutative bialgebra B , one can construct a functor

$$\begin{aligned} \underline{B} : \mathfrak{M} &\rightarrow k\text{-}\mathbf{CommAlg} \\ \langle n \rangle &\mapsto B^{\otimes n}. \end{aligned}$$

The following lemma establishes the representation functor and the tensor product over the PROP \mathfrak{M} .

Lemma 6.2.4. *Following the notations, given an algebraic monoid M , there is a natural isomorphism of commutative algebras*

$$(N)_M \cong k[N] \otimes_{\mathfrak{M}} \underline{O(M)}$$

for any monoid N .

Proof. By definition,

$$k[N] \otimes_{\mathfrak{M}} \underline{\mathcal{O}(M)} = \text{coeq} \left[\bigoplus_{f:\langle m \rangle \rightarrow \langle n \rangle} k[N]^{\otimes n} \otimes \mathcal{O}(M)^{\otimes m} \begin{array}{c} \xrightarrow{f^*} \\ \xrightarrow{f_*} \end{array} \bigoplus_{n \in \mathbb{N}} k[N]^{\otimes n} \otimes \mathcal{O}(M)^{\otimes n} \right], \quad (6.9)$$

where $f^* : x \otimes y \mapsto x \otimes \mathcal{O}(M)(f)(y)$ and $f_* : x \otimes y \mapsto k[N](f)(x) \otimes y$.

Let $B := \mathcal{O}(M)$. Consider the diagram

$$\begin{array}{ccc} \mathfrak{M} & \xrightarrow{B} & k - \mathbf{CommAlg} \xrightarrow{U} k - \mathbf{Mod} \\ & \searrow i & \\ & & \mathbf{Mon} \simeq \mathbf{Funct}^{\otimes}(\mathfrak{M}^{\text{op}}, \mathbf{Set}), \end{array}$$

the inclusion functor i admits a left Kan extension $\mathcal{L}_i U \underline{B}$, which can be computed on objects $N \in \mathbf{Mon}$ by colimit diagram

$$\mathcal{L}_i U \underline{B}(N) \cong \text{colim}[i/N \hookrightarrow \mathbf{Mon} \xrightarrow{B} k - \mathbf{CommAlg} \rightarrow k - \mathbf{Mod}],$$

and this colimit is computed by coequalizer

$$\text{coeq} \left[\bigsqcup_{f:\langle m \rangle \rightarrow \langle n \rangle \in i/N} \underline{B}(\langle m \rangle) \begin{array}{c} \xrightarrow{f^*} \\ \xrightarrow{f_*} \end{array} \bigsqcup_{\langle n \rangle \in i/N} \underline{B}(\langle n \rangle) \right].$$

Notice that an object in i/N is the same as a map $\langle n \rangle \rightarrow N$, which is the same as n elements in N , thus

$$\bigsqcup_{\langle n \rangle \in i/N} \underline{B}(\langle n \rangle) = \bigsqcup_{\langle n \rangle \in \mathfrak{M}} k[N]^{\otimes n} \otimes \underline{B}(\langle n \rangle)$$

in $k - \mathbf{Mod}$. Similarly, a morphism $f : \langle m \rangle \rightarrow \langle n \rangle \in i/N$ is the same as a morphism $f : \langle m \rangle \rightarrow \langle n \rangle \in \mathfrak{M}$ with a morphism $\langle n \rangle \rightarrow N \in \mathbf{Mon}$, thus

$$\bigsqcup_{f:\langle m \rangle \rightarrow \langle n \rangle \in i/N} \underline{B}(\langle m \rangle) = \bigsqcup_{f:\langle m \rangle \rightarrow \langle n \rangle} k[N]^{\otimes n} \otimes \mathcal{O}(M)^{\otimes m}$$

in $k - \mathbf{Mod}$. Under these identifications, we see that $k[N] \otimes_{\mathfrak{M}} \underline{\mathcal{O}(M)} = \mathcal{L}_i U \underline{B}(N)$ as vector spaces. There is a natural commutative algebra structure on the coequalizer 6.9

($\mathcal{O}(M)$ is a left $\mathfrak{C}\text{omm}$ -module), hence this identification is as commutative algebras. Thus $k[N] \otimes_{\mathfrak{M}} \underline{\mathcal{O}(M)}$ represents the left Kan extension.

This left Kan extension has a right adjoint, which maps a commutative algebra A to the functor

$$R(A) : \langle n \rangle \mapsto \text{Hom}_{k\text{-CommAlg}}(\underline{B}(\langle n \rangle), A) := \prod_n \text{Hom}_{k\text{-CommAlg}}(B, A).$$

This functor corresponds to the monoid $\text{Hom}_{k\text{-CommAlg}}(B, A)$. Both functors $(-)_M$ and $k[-] \otimes_{\mathfrak{M}} \underline{\mathcal{O}(M)}$ are the left adjoints, so they are isomorphic. \square

We now have a proposition analogous to Proposition 5.2.2 for representation homology of monoids.

Proposition 6.2.5. *Let M be an affine algebraic monoid and let N be a monoid. Then*

$$\text{HR}_*^{\text{Mon}}(N, M) \cong \text{Tor}_*^{\mathfrak{M}}(k[N], \underline{\mathcal{O}(M)}).$$

This means that the definition of the representation homology of monoids in [6] coincide with Definition 6.2.3.

Proof. Let $q : Q \rightarrow N$ be a semifree replacement in $s\mathbf{Mon}$ and one easily sees that $k[q] : k[Q] \rightarrow k[N]$ be a cofibrant replacement in $s(k - \mathbf{Vec})$. Similar to the proof of [10, Lemma 4.1.], $k[q]$ gives a resolution in $\mathbf{Mod} - \mathfrak{M}$, which proves the statement. \square

6.2.2 Representation homology of algebras

In [3, 1], there are representation homology defined for algebras and augmented algebras. In these cases, the coefficients are chosen to be matrix algebras. We need to generalize the definitions to have more choices of coefficients, so that we can compare representation homology of different types.

Unital case

Given a (counital) coalgebra (C, Δ, ϵ) and an (unital) algebra (A, μ, u) , there is an algebra structure on the set

$$\text{Hom}_k(C, A)$$

whose multiplication is given by $(f, g) \mapsto f * g$ where

$$f * g := C \xrightarrow{\Delta} C \otimes C \xrightarrow{f \otimes g} A \otimes A \xrightarrow{\mu} A$$

and the unit is given by the map $u \circ \epsilon : C \rightarrow k \rightarrow A$. This algebra is called the **convolution algebra**.

Given an (unital) algebra (A, μ, u) , then for any unital algebra B , $A \otimes B$ carries an algebra structure by the multiplication

$$(a \otimes b) \cdot (c \otimes d) := ac \otimes bd$$

and the unit is $1 \otimes 1$.

In general, for a finite dimensional algebra A (with basis $\{\delta_i\}$), there is a natural isomorphism of algebras

$$\text{Hom}_k(A^*, B) \cong A \otimes B$$

where the map is given by

$$f \mapsto \sum_i \delta_i \otimes f(\delta^i). \quad (6.10)$$

where δ^i denotes the dual element of δ_i .

To verify that this is an algebra morphism, one checks that on the left hand side,

$$f * g(\delta^i) = \mu(f \otimes g)\Delta(\delta^i) = S_{jk}^i \delta_i \otimes f(\delta^j)g(\delta^k),$$

where S_{jk}^i are the structure constants, and on the right hand side

$$\left(\sum_i \delta_i \otimes f(\delta^i) \right) \left(\sum_j \delta_j \otimes g(\delta^j) \right) = \delta_i \delta_j \otimes f(\delta^i) g(\delta^j) = S_{ij}^k \delta_k \otimes f(\delta^i) g(\delta^j).$$

Equation (6.10) suggests that this isomorphism is independent of the choice of basis.

Example 6.2.6. Let M_n be the (finite dimensional) algebra of $n \times n$ matrices, and let M_n^* be the linear dual of M_n . Then it is a counital coalgebra with comultiplication given by

$$\Delta(E_{i,j}^*) = \sum_{k=1}^n E_{i,k}^* \otimes E_{k,j}^*.$$

Then, there is an isomorphism

$$\text{Hom}_k(M_n^*, A) \cong M_n \otimes A$$

$$f \mapsto \sum_{ij} E_{ij} \otimes f(E_{ij}^*).$$

Proposition 6.2.7. Given a finite dimensional (counital) coalgebra C , the functor

$$\text{Hom}_k(C, -) : \mathbf{Alg}_k \rightarrow \mathbf{Alg}_k$$

admits a left adjoint

$$\overset{c}{\sqrt{-}} : \mathbf{Alg}_k \rightarrow \mathbf{Alg}_k.$$

Proof. One has

$$\begin{aligned} \text{Hom}_{\mathbf{Alg}_k}(A, \text{Hom}_k(C, B)) &\subseteq \text{Hom}_k(A, \text{Hom}_k(C, B)) \\ &\cong \text{Hom}_k(A \otimes C, B) \\ &\cong \text{Hom}_{\mathbf{Alg}_k}(T_k(A \otimes C), B) \end{aligned}$$

where the subset consists of all maps satisfying

$$f(ab)(c) = \mu_B(f(a) \otimes f(b))\Delta(c)$$

and $f(1_A) = u \circ \epsilon$. Under the tensor-hom correspondence, it is the subset

$$\{g(ab \otimes c) = g(a \otimes c^{(1)})g(b \otimes c^{(2)})\} \quad \forall a, b \in A, c \in C \quad (6.11)$$

and $g(1_A \otimes c) = u(\epsilon(c))$, hence

$$\sqrt[c]{A} \cong T_k(A \otimes C)/J \quad (6.12)$$

where J is the two-sided ideal generated by elements

$$ab \otimes c - (a \otimes c^{(1)}) \otimes (b \otimes c^{(2)}), \quad 1_A \otimes c - \epsilon(c).$$

□

Corollary 6.2.7.1. *Restricted onto the subcategory $\mathbf{CommAlg}_k$, the functor*

$$\mathrm{Hom}_k(C, -) : \mathbf{CommAlg}_k \rightarrow \mathbf{Alg}_k$$

admits a left adjoint

$$(-)_C : \mathbf{Alg}_k \rightarrow \mathbf{CommAlg}_k$$

by $(-)_C := (\sqrt[c]{-})_{\mathrm{ab}}$.

Theorem 6.2.8. *The adjunction in Corollary 6.2.7.1 extends to an adjunction*

$$(-)_C : \mathbf{DGA}_k \rightarrow \mathbf{CDGA}_k : \mathrm{Hom}_k(C, -) \quad (6.13)$$

which is a Quillen pair.

Then one could define

$$\mathrm{HR}_*(A, C) := \mathrm{H}_*(\mathbb{L}(A)_C)$$

for a finite dimensional coalgebra C , or

$$\mathrm{HR}_*(A, M) := \mathrm{H}_*(\mathbb{L}(A)_{M^*})$$

for a finite dimensional algebra M , called the **representation homology of A with coefficient C (or M)**.

Proof. By Equation (6.10), $\text{Hom}_k(C, -)$ is isomorphic to a tensor product, which is an exact functor and preserves surjective maps. Hence it preserves fibrations and acyclic fibrations. \square

Example 6.2.9. When $C = M_n^*$, where M_n is the finite-dimensional matrix algebra of size n , $\text{HR}_*(A, C)$ goes back to the original representation homology of algebras, defined in [3], i.e.

$$\text{HR}_*(A, C) = \text{HR}_*(A, n).$$

Similar to [3, Theorem 2.5.], one could show that $\text{HR}_0(A, C) \cong A_C$:

Proposition 6.2.10. Given any finite dimensional (counital) coalgebra C , then for any $A \in \mathbf{Alg}_k$,

$$\text{HR}_0(A, C) \cong A_C.$$

Proof. Let $q : Q \rightarrow A$ be a fixed cofibrant replacement in $\mathbf{DGA}_k^{\geq 0}$. Then there are natural isomorphisms

$$\begin{aligned} \text{Hom}_{\mathbf{CommAlg}_k}(A_C, B) &= \text{Hom}_{\mathbf{Alg}_k}(A, \text{Hom}_k(C, B)) \\ &= \text{Hom}_{\mathbf{DGA}_k^{\geq 0}}(Q, \text{Hom}_k(C, B)) \\ &= \text{Hom}_{\mathbf{CDGA}_k^{\geq 0}}(Q_C, B) \\ &= \text{Hom}_{\mathbf{CommAlg}_k}(H_0(Q_C), B), \end{aligned}$$

where the second and the third follows from the fact that $\text{Hom}_k(C, B)$ and C are concentrated in degree 0. By Yoneda lemma, we finish the proof. \square

Nonunital case

With the same setting of convolution algebras, if C is coaugmented by $\eta : k \rightarrow C$ and A is augmented by $\epsilon : A \rightarrow k$, the convolution algebra $\text{Hom}_k(C, A)$ is also augmented via

the map

$$f \mapsto \epsilon(f(\eta(1))). \quad (6.14)$$

On the other hand, if the coalgebra C is non-counital, then $\text{Hom}_k(C, A)$ is also non-unital.

Similar to Corollary 6.2.7.1, we have the following

Proposition 6.2.11. *Given any (possibly nonunital) k -algebra I , the functor*

$$\begin{aligned} I : \mathbf{CommAlg}_k &\rightarrow \mathbf{Alg}_k^{\text{non}} \\ A &\mapsto \text{Hom}_k(I^*, A) \cong I \otimes A \end{aligned}$$

gives rise to an adjunction

$$(-)_{I^*} : \mathbf{Alg}_k^{\text{non}} \rightleftarrows \mathbf{CommAlg}_k : I \quad (6.15)$$

where the multiplication of $I \otimes A$ is given by

$$(x \otimes a) \cdot (y \otimes b) := (xy) \otimes (ab).$$

Notice that the functor $I \otimes -$ is isomorphic to $\text{Hom}_k(I^*, -)$ with the convolution algebra structure.

Proof. Let $C := I^*$. One has

$$\begin{aligned} \text{Hom}_{\mathbf{Alg}_k^{\text{non}}}(A, \text{Hom}_k(C, B)) &\subseteq \text{Hom}_k(A, \text{Hom}_k(C, B)) \\ &\cong \text{Hom}_k(A \otimes C, B) \\ &\cong \text{Hom}_{\mathbf{Alg}_k^{\text{non}}}(\overline{T}_k(A \otimes C), B) \end{aligned}$$

where the subset consists of all maps satisfying

$$f(ab)(c) = \mu_B(f(a) \otimes f(b))\Delta(c).$$

Under the tensor-hom correspondence, it is the subset

$$\{g(ab \otimes c) = g(a \otimes c^{(1)})g(b \otimes c^{(2)})\} \quad \forall a, b \in A, c \in C, \quad (6.16)$$

hence

$$(A)_{I^*} \cong (T_k(A \otimes C)/J)_{\text{ab}} \quad (6.17)$$

where J is the two-sided ideal generated by elements

$$ab \otimes c - (a \otimes c^{(1)}) \otimes (b \otimes c^{(2)}).$$

□

Remark 6.2.12. *Via the adjunction¹*

$$(-)_+ : \mathbf{Alg}_k^{\text{non}} \rightleftarrows \mathbf{Alg}_{k/k} : \overline{-}, \quad (6.18)$$

one can switch between augmented algebras and nonunital algebras. More precisely, one has the chain of adjunctions:

$$\mathbf{Alg}_{k/k} \begin{array}{c} \xrightarrow{\overline{-}} \\ \xleftarrow{(-)_+} \end{array} \mathbf{Alg}_k^{\text{non}} \begin{array}{c} \xrightarrow{(-)_{I^*}} \\ \xleftarrow{I} \end{array} \mathbf{CommAlg}_k.$$

The functor I could be extended to a functor $\mathbf{Alg}_k^{\text{non}} \rightarrow \mathbf{Alg}_k^{\text{non}}$, which under the adjunction Equation (6.18) is the construction \mathcal{M}' in [1, Lemma 2.5.] by setting $I = \mathbf{M}_d$, the algebra of matrices. Then it could be seen that construction $(-)_{I^}$ in Proposition 6.2.11 coincides with the construction in [1, Lemma 2.5.]. However, we want a scheme in our setting, hence we make the choice of Proposition 6.2.11 as stated.*

Theorem 6.2.13. *The adjunction in Proposition 6.2.11 extends to an adjunction*

$$(-)_{I^*} : \mathbf{DGA}_k^{\text{non}} \rightleftarrows \mathbf{CDGA}_k : I \quad (6.19)$$

which is a Quillen pair.

¹Notice that this pair of adjunctions is not an equivalence, but $(-)_{+}$ is both the left and right adjoint of $\overline{-}$, when k is a field.

Then one could define

$$\mathrm{HR}_*(A, I) := \mathrm{H}_*(\mathbb{L}(A)_{I^*})$$

for a finite dimensional algebra I , called the **representation homology of A with coefficient I** .

Proof. By the construction in Proposition 6.2.11, the functor I is a tensor product on the underlying complex, which is an exact functor and preserves surjective maps. Hence it preserves fibrations and acyclic fibrations. \square

Abelianization of representation homology

Given any $A \in \mathbf{DGA}_k$, a finite dimensional algebra M , and DG-bimodule $N \in A - \mathbf{DGBiMod}$, define

$$N_M^{\mathrm{ab}} := N \otimes_{A^e} (M \otimes (A)_M). \quad (6.20)$$

This is a functor

$$(-)_M^{\mathrm{ab}} : A - \mathbf{DGBiMod} \rightarrow A_M - \mathbf{DGMod},$$

which is a generalization to the Van den Bergh's functor defined in [53, Section 3.3.].

As a generalization to [3, Lemma 5.1.], we have

Lemma 6.2.14. *For any $N \in A - \mathbf{DGBiMod}$, $L \in A_M - \mathbf{DGMod}$, there is a canonical isomorphism of complexes*

$$\underline{\mathrm{Hom}}_{A_M}((N)_M^{\mathrm{ab}}, L) \cong \underline{\mathrm{Hom}}_{A^e}(N, M \otimes L).$$

Theorem 6.2.15. *The functor 6.20 is the abelianization of the left functor of the adjunction 6.13.*

- Theorem 6.2.16.** 1. The pair of functors $(-)_M^{\text{ab}} : A - \mathbf{DGBiMod} \rightleftarrows A_M - \mathbf{DGMod} : M \otimes -$ forms a Quillen pair.
2. The right functor $M \otimes -$ is exact and the left functor $(-)_M^{\text{ab}}$ admits a total left derived functor.

And eventually we have an analogue of [2, Proposition 7.]:

Proposition 6.2.17. *There is a canonical isomorphism*

$$\pi_i(\mathbf{DRep}_M(A)) = \begin{cases} \text{Der}(A, M) & i = 0 \\ \mathbf{HH}^{i+1}(A, M) & i \geq 1 \end{cases}$$

where $\mathbf{HH}^{i+1}(A, M)$ denotes the Hochschild cohomology of the representation $\rho : A \rightarrow M$.

All the results in this part have the same proof as their original results mentioned in [3, 2].

6.2.3 Application : formal smoothness

Theoretically matrix coefficient already tells us many information. As in the heuristic principle in noncommutative geometry proposed by Kontsevich and Rosenberg in [36] suggests, that geometrically meaningful properties of a noncommutative algebra A should induce standard geometric properties on all its representation schemes in matrix algebras. One might then have the question: Why do we need arbitrary finite dimensional algebra coefficient? In order to elaborate the necessity, we need the notion of formal smoothness, and we will establish a formal smoothness test finer than that in [2].

Definition 6.2.18 ([16, 36]). An associative algebra A is called **formally smooth** (or quasi-free) if either of the following equivalent conditions holds:

1. A has cohomological dimension ≤ 1 with respect to Hochschild cohomology.
2. The universal bimodule $\Omega_{A/k}^1$ of derivations is a projective bimodule.
3. A satisfies the lifting property with respect to nilpotent extensions in \mathbf{Alg}_k , namely for every algebra homomorphism $A \rightarrow B/I$ where $I \subseteq B$ is a nilpotent ideal, there is an algebra homomorphism $\tilde{f} : A \rightarrow B$ inducing f .

Lemma 6.2.19. *Let M be a finite dimensional algebra. For any commutative k -algebra B and nilpotent ideal I , the two-sided ideal $\langle M(I) \rangle$ is also nilpotent, where*

$$\langle M(I) \rangle = \text{Ker}(M(B) \rightarrow M(B/I)).$$

Proof. First, every finite dimensional algebra M can be embedded into a matrix algebra $\text{End}_k(M)$ (because every element $a \in M$ is a k -linear map $x \mapsto ax$), and hence we have a diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \langle M(I) \rangle & \longrightarrow & M(B) & \longrightarrow & M(B/I) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \langle \text{End}_k(M)(I) \rangle & \longrightarrow & \text{End}_k(M)(B) & \longrightarrow & \text{End}_k(M)(B/I) & \longrightarrow & 0 \end{array}$$

The map $\langle M(I) \rangle \rightarrow \langle \text{End}_k(M)(I) \rangle$ is injective by 5-lemma, so it suffices to show the result for $M = M_d$, the matrix algebra.

This is automatically true by the structure of matrix multiplication, namely

$$M_d(I) = \{d \times d \text{ matrices consisting of entries in } I\}.$$

□

Proposition 6.2.20. *If A is a smooth (associate) algebra, then $(A)_M$ is a smooth commutative algebra for any finite dimensional algebra M .*

Proof. To check $(A)_M$ is a smooth algebra, it suffices to check for any nilpotent ideal I of B , the map

$$\mathrm{Hom}_{k\text{-CommAlg}}((A)_M, B) \rightarrow \mathrm{Hom}_{k\text{-CommAlg}}((A)_M, B/I) \quad (6.21)$$

is surjective. Via the adjunction in Corollary 6.2.7.1, Equation (6.21) is the same as

$$\mathrm{Hom}_{k\text{-Alg}}(A, M(B)) \rightarrow \mathrm{Hom}_{k\text{-Alg}}(A, M(B)/M(I)), \quad (6.22)$$

and its surjectiveness is due to the smoothness of A . \square

Theorem 6.2.21. *For any algebraic algebra, the representation homology*

$$\mathrm{HR}_i^{\mathrm{Alg}}(A, M)$$

is 0 for $i > 0$ provided A is a smooth (associate) algebra.

Proof. Let $q : Q \rightarrow A$ be a semifree resolution of A in $\mathbf{DGA}_k^{\geq 0}$. Applying the functor $(-)_M^*$, we have a map

$$(q)_{M^*} : (Q)_{M^*} \rightarrow (A)_{M^*}$$

of DG commutative algebras (by viewing $(A)_{M^*}$ a DG algebra concentrated in degree 0). By Proposition 6.2.10, $(q)_{M^*}$ is an isomorphism on the underlying schemes of two DG schemes. By Proposition 6.2.17, for any $\rho \in \mathrm{Rep}_M(A)$,

$$\pi_i(\mathrm{DRep}_M(A), \rho) \cong \mathrm{HH}^{i+1}(A, M), \quad i \geq 1$$

then it follows that $\pi_i(\mathrm{DRep}_M(A), \rho) = 0$ for $i \geq 1$ since A is formally smooth. Therefore the differential dp_ρ is a quasi-isomorphism of tangent spaces for each point. By Proposition 6.2.20 and Proposition 2.4.5, the map $(q)_{M^*}$ is a quasi-isomorphism, which finishes the proof. \square

One of the applications of Theorem 6.2.21 is the following computation, which will be used in the comparison Theorem 6.3.8.

Proposition 6.2.22. *Let k be a field of characteristic 0 and let M be a finite dimensional algebra. Let $p : \mathfrak{G} \rightarrow \mathfrak{M}$ be the embedding of categories. There is an isomorphism of left \mathfrak{G} -modules in the derived category $D(\mathfrak{G} - \mathbf{Mod})$:*

$$\mathbb{L}p_! \underline{O}(M) \cong \underline{O}(M^\times), \quad (6.23)$$

where M^\times is the maximal subgroup of M .

Proof. For each $n \geq 0$, there are natural isomorphisms in $D(k - \mathbf{Mod})$

$$\mathbb{L}p_! \underline{O}(M)(\langle n \rangle) \cong \underline{k}\langle n \rangle \otimes_{\mathfrak{G}}^{\mathbb{L}} \mathbb{L}p_! \underline{O}(M) \cong p^* \underline{k}\langle n \rangle \otimes_{\mathfrak{M}}^{\mathbb{L}} \underline{O}(M)$$

where the first isomorphism follows that fact that $k\langle n \rangle$ is a projective object by Lemma 4.4.8, and the second isomorphism is by Proposition 4.4.15. Now by Proposition 6.3.1 and Proposition 6.2.5,² we have

$$H_*(p^* \underline{k}\langle n \rangle \otimes_{\mathfrak{M}}^{\mathbb{L}} \underline{O}(M)) =: HR_*^{\text{Mon}}(p^* \langle n \rangle, M) \cong HR_*^{\text{Alg}}(k\langle x_1^{\pm 1}, \dots, x_n^{\pm 1} \rangle, M).$$

By [16, Proposition 5.3. (2)], the algebra $k\langle x_1^{\pm 1}, \dots, x_n^{\pm 1} \rangle$ is formally smooth, hence by Theorem 6.2.21

$$HR_i^{\text{Alg}}(k\langle x_1^{\pm 1}, \dots, x_n^{\pm 1} \rangle, M) = \begin{cases} 0 & i > 0, \\ \underline{O}(M^\times)^{\otimes n} & i = 0. \end{cases}$$

It follows that $\mathbb{L}p_! \underline{O}(M)(\langle n \rangle) \cong \underline{O}(M^\times)^{\otimes n} \cong \underline{O}(M)(\langle n \rangle)$ for all $n \geq 0$. This implies Equation (6.23), which completes the proof. \square

In [2, 6.2.2.], we see that the quantum polynomial ring in 2 variables is not formal smooth, but all the higher representation homology in matrices coefficients vanish. Next example indicates that with more choices of coefficients, one could use representation homology to describe the failure to be smooth.

²Theoretically Proposition 6.2.22 should be put after Proposition 6.3.1, but we want to emphasize the application here. The proof of Proposition 6.3.1 does not rely on this Proposition, so logically it is still good.

Example 6.2.23 (Generalization to [2, 6.2.2.]). Let $k[x, y]_q$ be the quantum polynomial ring in two variables, i.e.

$$k[x, y]_q = \frac{k\langle x, y \rangle}{\langle xy - qyx \rangle}$$

where $q \in k^\times$ is not a root of unity. Its Shafarevich resolution is given by $k\langle x, y, t \mid dt = xy - qyx \rangle$.

Let's consider

$$\mathrm{HR}_*(k[x, y]_q, (n_3)_+).$$

We notice that

$$\begin{bmatrix} s & x & z \\ 0 & s & y \\ 0 & 0 & s \end{bmatrix} \begin{bmatrix} t & u & w \\ 0 & t & v \\ 0 & 0 & t \end{bmatrix} - q \begin{bmatrix} t & u & w \\ 0 & t & v \\ 0 & 0 & t \end{bmatrix} \begin{bmatrix} s & x & z \\ 0 & s & y \\ 0 & 0 & s \end{bmatrix} = (1 - q) \begin{bmatrix} st & sx + tu & sz + uy + tw \\ 0 & st & sy + tv \\ 0 & 0 & st \end{bmatrix}.$$

and so we can compute the representation homology by the Koszul complex

$$k \left[\begin{array}{c} s, t, x, y, z, u, v, w, t_1, t_2, t_3, t_4 \\ \left| \begin{array}{l} dt_1 = (1 - q)st, dt_2 = (1 - q)(sx + tu), \\ dt_3 = (1 - q)(sy + tv), dt_4 = (1 - q)(sz + uy + tw) \end{array} \right. \end{array} \right]. \quad (6.24)$$

Using Macaulay2, one can easily check that this Koszul complex has nontrivial homology group.

This immediately means that $k[x, y]_q$ is not smooth by Theorem 6.2.21.

This leads to the following:

Conjecture 3. An associative algebra over k is formally smooth if and only if for all finite dimensional algebra M ,

$$\mathrm{HR}_i^{\mathrm{Alg}}(A, M)$$

is 0 for $i > 0$.

6.3 Comparisons of representation homology

6.3.1 Representation homology of monoids and algebras

Proposition 6.3.1. *Let N be a monoid and let A be a finite dimensional associative algebra. Suppose $U : \mathbf{Alg}_k \rightarrow \mathbf{Mon}$ be the forgetful functor ignoring the additive structure on an associative algebra, with the left adjoint*

$$k[-] : \mathbf{Mon} \rightarrow \mathbf{Alg}_k,$$

then

$$\mathrm{HR}_*^{\mathrm{Mon}}(N, U(A)) \cong \mathrm{HR}_*^{\mathrm{Alg}}(k[N], A).$$

Proof. Notice that we have an adjunction

$$k[-] : \mathbf{Mon} \rightleftarrows k\text{-}\mathbf{Alg} : U$$

where U is the forgetful functor which keeps the multiplicative structure of an associative algebra. Given any commutative k -algebra B , we have

$$\begin{aligned} \mathrm{Hom}_{k\text{-}\mathbf{CommAlg}}((N)_{U(A)}^{\mathrm{Mon}}, B) &\cong \mathrm{Hom}_{\mathbf{Mon}}(N, U(A)(B)) \\ &\cong \mathrm{Hom}_{\mathbf{Mon}}(N, U(A \otimes B)) \\ &\cong \mathrm{Hom}_{k\text{-}\mathbf{Alg}}(k[N], A \otimes B) \\ &\cong \mathrm{Hom}_{k\text{-}\mathbf{CommAlg}}((k[N])_A^{\mathrm{Alg}}, B). \end{aligned}$$

This means there is a natural isomorphism

$$(N)_{U(A)}^{\mathrm{Mon}} \cong (k[N])_A^{\mathrm{Alg}} \tag{6.25}$$

by Yoneda lemma.

If $Q \rightarrow N$ is a cofibrant replacement of N in $s\mathbf{Mon}$, then $k[Q] \rightarrow k[N]$ is a cofibrant replacement in $s\mathbf{Alg}_k$. This is sufficient to conclude. \square

6.3.2 Representation homology of algebras and Lie algebras

There is an adjunction

$$U : \mathbf{Lie}_k \rightleftarrows \mathbf{Alg}_k : (-)_{\text{Lie}} \quad (6.26)$$

where U is the universal enveloping algebra and $(-)_{\text{Lie}}$ is the Lie algebra with the same vector space and $[a, b] := ab - ba$. Namely, there are natural isomorphisms

$$\text{Hom}_{\mathbf{Alg}_k}(U\mathfrak{g}, A) \cong \text{Hom}_{\mathbf{Lie}_k}(\mathfrak{g}, A_{\text{Lie}}).$$

This extends to a Quillen pair

$$U : \mathbf{DGLA}_k \rightleftarrows \mathbf{DGA}_k : (-)_{\text{Lie}}$$

because the functor $(-)_{\text{Lie}}$ does nothing to the underlying space and the model structure depends only on the underlying complex structure (see e.g. [21]).

Given a finite dimensional algebra A and a commutative algebra B , we claim that

$$A_{\text{Lie}} \otimes B \cong (A \otimes B)_{\text{Lie}} \quad (6.27)$$

as Lie algebras. It suffices to show that the identity map $a \otimes b \mapsto a \otimes b$ preserves the Lie bracket. On the left hand side,

$$[a \otimes b, c \otimes d] := [a, c] \otimes (bd),$$

and on the right hand side,

$$[a \otimes b, c \otimes d] := (a \otimes b)(c \otimes d) - (c \otimes d)(a \otimes b) = (ac) \otimes (bd) - (ca) \otimes (db) = [a, c] \otimes (bd).$$

Theorem 6.3.2. *Given a Lie algebra \mathfrak{a} and a finite dimensional k -algebra A , there is a natural isomorphism*

$$\text{HR}_*^{\text{Lie}}(\mathfrak{a}, A_{\text{Lie}}) \cong \text{HR}_*^{\text{Alg}}(U\mathfrak{a}, A).$$

Proof. Given a Lie algebra \mathfrak{a} , a commutative algebra B and a finite dimensional algebra A , we have

$$\begin{aligned}
\mathrm{Hom}_{\mathbf{CommAlg}_k}((\mathfrak{a})_{A_{\mathrm{Lie}}}, B) &\cong \mathrm{Hom}_{\mathbf{Lie}_k}(\mathfrak{a}, A_{\mathrm{Lie}} \otimes B) \\
&\cong \mathrm{Hom}_{\mathbf{Lie}_k}(\mathfrak{a}, (A \otimes B)_{\mathrm{Lie}}) \\
&\cong \mathrm{Hom}_{\mathbf{Alg}_k}(U\mathfrak{a}, A \otimes B) \\
&\cong \mathrm{Hom}_{\mathbf{Alg}_k}(U\mathfrak{a}, \mathrm{Hom}_k(A^*, B)) \\
&\cong \mathrm{Hom}_{\mathbf{CommAlg}_k}((U\mathfrak{a})_A, B).
\end{aligned}$$

This means there is a natural isomorphism of representation varieties by Yoneda lemma

$$(\mathfrak{a})_{A_{\mathrm{Lie}}} \cong (U\mathfrak{a})_A,$$

which implies an isomorphism of functors

$$(-)_{A_{\mathrm{Lie}}} \cong (U-)_A : \mathbf{Lie}_k \rightarrow \mathbf{CommAlg}_k. \quad (6.28)$$

By [21] Theorem 21.7 and Equation (6.27), U preserves quasi-isomorphisms and cofibrant objects, hence if $Q\mathfrak{a} \rightarrow \mathfrak{a}$ is a cofibrant replacement, then so is $UQ\mathfrak{a} \rightarrow U\mathfrak{a}$.

Then by Equation (6.28)

$$\mathrm{HR}_*^{\mathrm{Alg}}(U\mathfrak{a}, A) = \mathrm{H}_*((UQ\mathfrak{a})_A) = \mathrm{H}_*((Q\mathfrak{a})_{A_{\mathrm{Lie}}}) = \mathrm{HR}_*^{\mathrm{Lie}}(\mathfrak{a}, A_{\mathrm{Lie}}).$$

□

Corollary 6.3.2.1. *For any Lie algebra \mathfrak{a} ,*

$$\mathrm{HR}_*^{\mathrm{Lie}}(\mathfrak{a}, \mathfrak{gl}_n) \cong \mathrm{HR}_*^{\mathrm{Alg}}(U\mathfrak{a}, k^n).$$

Corollary 6.3.2.2. *Let \mathfrak{b}_n be the nonstrict upper triangular Lie algebra, then for any Lie algebra \mathfrak{a} ,*

$$\mathrm{HR}_*^{\mathrm{Lie}}(\mathfrak{a}, \mathfrak{b}_n) \cong \mathrm{HR}_*^{\mathrm{Alg}}(U\mathfrak{a}, \mathfrak{b}_n)$$

where \mathfrak{b}_n is the algebra of all nonstrict upper triangular matrices.

Now we turn to the non-unital case.

Lemma 6.3.3. *Similar to Equation (6.26), there is an adjunction*

$$\overline{U} : \mathbf{Lie}_k \rightleftarrows \mathbf{Alg}_k^{\text{non}} : (-)_{\text{Lie}}$$

namely there are natural isomorphisms

$$\text{Hom}_{\mathbf{Alg}_k^{\text{non}}}(\overline{U}\mathfrak{g}, I) \cong \text{Hom}_{\mathbf{Lie}_k}(\mathfrak{g}, I_{\text{Lie}}).$$

Proof. $\text{Hom}_{\mathbf{Lie}_k}(\mathfrak{g}, I_{\text{Lie}})$ is the subset of

$$\text{Hom}_k(\mathfrak{g}, I) \cong \text{Hom}_{\mathbf{Alg}_k^{\text{non}}}(\overline{T}\mathfrak{g}, I)$$

consisting of elements satisfying

$$\varphi([\alpha, \beta]) = \varphi(\alpha)\varphi(\beta) - \varphi(\beta)\varphi(\alpha). \quad (6.29)$$

Let J be the non-unital ideal generated by elements $\alpha \otimes \beta - \beta \otimes \alpha - [\alpha, \beta]$, then $\text{Hom}_{\mathbf{Alg}_k^{\text{non}}}(\overline{T}\mathfrak{g}/J, I)$ is the corresponding subset. \square

Now given a coLie algebra \mathfrak{G} , as the discussion after Theorem 6.5 in [1], one has a Quillen pair

$$(-)_{\mathfrak{G}} : \mathbf{DGLA}_k \rightleftarrows \mathbf{CDGA}_k : \text{Hom}_k(\mathfrak{G}, \overline{-}).$$

Given a finite dimensional nonunital algebra I , for any Lie algebra \mathfrak{a} and any commutative algebra B , we have

$$\begin{aligned} \text{Hom}_{\mathbf{CommAlg}_k}((\mathfrak{a})_{I_{\text{Lie}}}, B) &\cong \text{Hom}_{\mathbf{Lie}_k}(\mathfrak{a}, I_{\text{Lie}} \otimes B) \\ &\cong \text{Hom}_{\mathbf{Lie}_k}(\mathfrak{a}, (I \otimes B)_{\text{Lie}}) \\ &\cong \text{Hom}_{\mathbf{Alg}_k^{\text{non}}}(\overline{U}\mathfrak{a}, I \otimes B) \\ &\cong \text{Hom}_{\mathbf{CommAlg}_k}((\overline{U}\mathfrak{a})_I, B). \end{aligned}$$

This means there is a natural isomorphism of representation varieties by Yoneda lemma

$$(\alpha)_{I_{\text{Lie}}} \cong (U\alpha)_I,$$

which implies an isomorphism of functors

$$(-)_{I_{\text{Lie}}} \cong (\overline{U}-)_I : \mathbf{Lie}_k \rightarrow \mathbf{CommAlg}_k. \quad (6.30)$$

Theorem 6.3.4. *Given a Lie algebra α and a finite dimensional nonunital k -algebra I , there is a natural isomorphism*

$$HR_*^{\text{Lie}}(\alpha, I_{\text{Lie}}) \cong HR_*^{\text{Alg}_{\text{non}}}(\overline{U}\alpha, I).$$

Corollary 6.3.4.1. *Let \mathfrak{n}_n be the strict upper triangular Lie algebra, then for any Lie algebra α ,*

$$HR_*^{\text{Lie}}(\alpha, \mathfrak{n}_n) \cong HR_*^{\text{Alg}_{\text{non}}}(\overline{U}\alpha, \text{nil}_n)$$

where nil_n is the algebra of all strict upper triangular matrices.

6.3.3 Representation homology of monoids and groups

The forgetful functor $U : \mathbf{Gp} \rightarrow \mathbf{Mon}$ has both left and right adjoints, and

Proposition 6.3.5. *Given a monoid M and an affine group scheme G , there is a natural isomorphism*

$$HR_*^{\text{Mon}}(M, G) \cong HR_*^{\text{Gp}}(L(M), G),$$

where $L(M)$ is the localization of the monoid M .

Proof. Notice that we have an adjunction

$$L : \mathbf{Mon} \rightleftarrows \mathbf{Gp} : U$$

where U is the forgetful functor which keeps the multiplicative structure of a group. Given any commutative k -algebra B , we have

$$\begin{aligned} \mathrm{Hom}_{k\text{-CommAlg}}((N)_{U(G)}^{\mathrm{Mon}}, B) &\cong \mathrm{Hom}_{\mathbf{Mon}}(N, U(G)(B)) \\ &\cong \mathrm{Hom}_{\mathbf{Mon}}(N, U(G(B))) \\ &\cong \mathrm{Hom}_{\mathbf{Gp}}(L(N), G(B)) \\ &\cong \mathrm{Hom}_{k\text{-CommAlg}}((L(N))_G^{\mathrm{Gp}}, B). \end{aligned}$$

This means there is a natural isomorphism

$$(N)_{U(G)}^{\mathrm{Mon}} \cong (L(N))_G^{\mathrm{Gp}} \quad (6.31)$$

by Yoneda lemma.

If $Q \rightarrow N$ is a cofibrant replacement of N in $s\mathbf{Mon}$, then $L(Q) \rightarrow L(N)$ is a cofibrant replacement in $s\mathbf{Gp}$ since the localization of a free monoid is a free group. \square

Remark 6.3.6. *Many of the comparison theorem (Proposition 6.3.1, Theorem 6.3.2, Proposition 6.3.5) could be generalized to the following: Given an algebraic theories S, \mathcal{T} and a pair of adjunction $F : S \rightleftarrows \mathcal{T} : G$, such that*

- F sends a cofibrant object in sS to a cofibrant object in $s\mathcal{T}$, and
- G is “essentially forgetful” on objects,

then there is a canonical isomorphism

$$\mathrm{HR}_*^{\mathcal{T}}(F(A), B) \cong \mathrm{HR}_*^S(A, G(B))$$

when both sides are defined. We still prefer the concrete forms mentioned earlier.

Proposition 6.3.7. *Given a simplicial group Γ and an algebraic monoid,*

$$\mathrm{HR}_*^{\mathrm{Mon}}(\Gamma, M) \cong \mathrm{HR}_*^{\mathrm{Gp}}(\Gamma, M^\times),$$

where M^\times is the maximal subgroup of units.

This is a generalization of [6, Proposition 2.2]. The proof follows as well.

Proof. We have

$$\mathrm{HR}_*^{\mathrm{Mon}}(\Gamma, M) := \mathrm{Tor}_*^{\mathfrak{M}}(k[\Gamma], \underline{Q}(M)) = \mathrm{Tor}_*^{\mathfrak{M}}(p^*k[\Gamma], \underline{Q}(M)) = \mathrm{Tor}_*^{\mathfrak{G}}(k[\Gamma], p_! \underline{Q}(M))$$

where the last equality is due to Proposition 4.4.15. Then by Proposition 6.2.22, we deduce the result. \square

Combining Proposition 6.3.1 and Proposition 6.3.7, we have:

Theorem 6.3.8. *Given a finite dimensional algebra A over characteristic 0 field k and a simplicial group Γ , there is a natural isomorphism*

$$\mathrm{HR}_*^{\mathrm{Alg}}(k[\Gamma], A) \cong \mathrm{HR}_*^{\mathrm{Gp}}(\Gamma, A^\times),$$

where A^\times is the maximal subgroup of units.

Proof. We have

$$\mathrm{HR}_*^{\mathrm{Alg}}(k[\Gamma], A) \cong \mathrm{HR}_*^{\mathrm{Mon}}(\Gamma, A) \cong \mathrm{HR}_*^{\mathrm{Gp}}(\Gamma, A^\times)$$

where the first isomorphism is by Proposition 6.3.1 and the second isomorphism is by Proposition 6.3.7. \square

6.3.4 Application : Derived commuting schemes of unipotent groups and their associated nilpotent Lie algebras

One of the most important applications of the comparison theorems is about the derived commuting scheme of a unipotent algebraic group scheme U with the derived commuting scheme of its Lie algebra \mathfrak{n} . For this section, U will always be a unipotent algebraic group scheme and \mathfrak{n} denotes its (nilpotent) Lie algebra.

Since \mathfrak{n} can be embedded into the algebra of strict upper triangular matrices and its Lie algebra structure comes from a nonunital algebra structure, we will denote also by \mathfrak{n} the nonunital algebra. \mathfrak{n}_+ is then the augmentation of \mathfrak{n} .

It is worthwhile to point out that most the comparison theorems proven above cannot be directly applied here. We need some modifications (Lemma 6.3.10, Lemma 6.3.12) before applying the theorems.

The idea is to find a middle scheme in which both are isomorphic to a closed subscheme of the middle scheme. On the scheme level, we have a diagram

$$\mathrm{Spec}(\mathfrak{a})_{\mathfrak{n}} \hookrightarrow \mathrm{Spec}(k[x, y])_{\mathfrak{n}_+} \longleftarrow \mathrm{Spec}(\mathbb{Z} \times \mathbb{Z})_U,$$

and one can *arrange* the two subsets to be the same. Then if we can show that these representation schemes are functorial on objects, then applying on a resolution will finish the comparison.

After explaining the idea, we will start the proof by defining the middle subscheme. To do this, we need to introduce a notion:

Definition 6.3.9. Given any commutative algebra B , an element $x \in \mathfrak{n}_+(B)$ is called **universally unipotent**³ if

$$x - 1 \in \mathfrak{n}(B) = \mathfrak{n} \otimes B \subseteq \mathfrak{n}_+ \otimes B = \mathfrak{n}_+(B),$$

namely $x - 1$ in the nilpotent part of the algebra $\mathfrak{n}_+(B)$.

Lemma 6.3.10. *Given any monoid M , there is a subscheme $\mathrm{Spec}(k[M])_{\mathfrak{n}_+^{\mathrm{Uni}}}$ of $\mathrm{Spec}(k[M])_{\mathfrak{n}_+}$ satisfying the following property:*

³This name comes from the fact that the element is functorially unipotent in B .

(P) For any commutative algebra B , $\text{Hom}_{k\text{-CommAlg}}((k[M])_{\mathfrak{n}_+^{\text{Uni}}}, B)$ consists of maps in

$$\text{Hom}_{k\text{-CommAlg}}((k[M])_{\mathfrak{n}_+}, B) = \text{Hom}_{\text{Alg}_k}(k[M], \mathfrak{n}_+(B)) = \text{Hom}_{\text{Mon}}(M, \mathfrak{n}_+(B)) \quad (6.32)$$

such that $f : M \rightarrow \mathfrak{n}_+(B)$ maps elements in M to universally unipotent elements.

Furthermore, this subscheme $(k[M])_{\mathfrak{n}_+^{\text{Uni}}}$ is isomorphic to $(L(M))_U$, where $L(M)$ is the localization of M .

Proof. We shall notice that for any commutative algebra B , the subset

$$\{f \in \text{Hom}_{\text{Mon}}(M, \mathfrak{n}_+(B)) \mid f(m) \text{ is universally unipotent for all } m \in M\} \quad (6.33)$$

of 6.32 is equal to

$$\text{Hom}_{\mathbf{Gp}}(L(M), U(B)).$$

This is because $U(B)$ is a subset of $\mathfrak{n}_+(B)$ consisting of all unipotent elements, and it is a group. A map $f : M \rightarrow U(B)$ between two monoids could be lifted to a map $\tilde{f} : L(M) \rightarrow U(B)$ and this correspondence is 1-1, because L is the left adjoint of the forgetful functor $\mathbf{Gp} \rightarrow \mathbf{Mon}$.

Since

$$\text{Hom}_{\mathbf{Gp}}(L(M), U(B)) = \text{Hom}_{k\text{-CommAlg}}((L(M))_U, B) = \text{Spec}((L(M))_U(B)),$$

the subset 6.33 defines an affine subscheme of $\text{Spec}(k[M])_{\mathfrak{n}_+}$, and we denote it by $\text{Spec}(k[M])_{\mathfrak{n}_+^{\text{Uni}}}$. □

Lemma 6.3.11. *The subscheme $\text{Spec}(k[M])_{\mathfrak{n}_+^{\text{Uni}}}$ is functorial on M .*

Proof. We need to prove for any map $\varphi : M \rightarrow N$ between monoids, there is a canonical map of commutative algebras

$$\varphi_* : (k[M])_{\mathfrak{n}_+^{\text{Uni}}} \rightarrow (k[N])_{\mathfrak{n}_+^{\text{Uni}}}.$$

This is equivalent to give a map of set

$$\varphi_B^* : \text{Hom}_{k\text{-CommAlg}}((k[N])_{\mathfrak{n}_+^{\text{Uni}}}, B) \rightarrow \text{Hom}_{k\text{-CommAlg}}((k[M])_{\mathfrak{n}_+^{\text{Uni}}}, B) \quad (6.34)$$

for any given commutative algebra B .

By the description of property (P) in Lemma 6.3.10, the map 6.34 is defined by post-composing φ . The functoriality is then clear. \square

Lemma 6.3.12. *Given any Lie algebra \mathfrak{a} with a fixed basis \mathcal{B} , there is a subscheme $\text{Spec}(U\mathfrak{a})_{\mathfrak{n}_+^{\text{Uni}}}$ of $\text{Spec}(U\mathfrak{a})_{\mathfrak{n}_+}$ satisfying the following property:*

- For any commutative algebra B , $\text{Hom}_{k\text{-CommAlg}}((k[M])_{\mathfrak{n}_+^{\text{Uni}}}, B)$ consists of maps in

$$\text{Hom}_{k\text{-CommAlg}}((U\mathfrak{a})_{\mathfrak{n}_+}, B) = \text{Hom}_{\text{Alg}_k}(U\mathfrak{a}, \mathfrak{n}_+(B)) = \text{Hom}_{\text{Lie}_k}(\mathfrak{a}, (\mathfrak{n}_+(B))_{\text{Lie}}) \quad (6.35)$$

such that $f : \mathfrak{a} \rightarrow \mathfrak{n}_+(B)$ maps elements in \mathcal{B} to universally unipotent elements in $\mathfrak{n}_+(B)$.

Furthermore, this subscheme $(U\mathfrak{a})_{\mathfrak{n}_+^{\text{Uni}}}$ is isomorphic to $(\mathfrak{a})_{\mathfrak{n}_+}$.

Proof. It can be observed that

$$(\mathfrak{n}_+)_{\text{Lie}} \cong \mathfrak{n} \oplus k$$

and so

$$(\mathfrak{n}_+(B))_{\text{Lie}} \cong \mathfrak{n}(B) \oplus B_{\text{Lie}}$$

for any commutative algebra B . Thus, the subset

$$\{f \in \text{Hom}_{\text{Lie}_k}(\mathfrak{a}, \mathfrak{n}_+(B)) \mid f(m) \text{ is universally unipotent for all } m \in \mathcal{B}\}$$

corresponds to

$$\{f \in \text{Hom}_{\text{Lie}_k}(\mathfrak{a}, \mathfrak{n}_+(B)) = \text{Hom}_{\text{Lie}_k}(\mathfrak{a}, \mathfrak{n}(B) \oplus B_{\text{Lie}}) \mid \text{Im } f \subseteq \mathfrak{n}(B)\},$$

where this set is clearly $\text{Hom}_{\text{Lie}_k}(\mathfrak{a}, \mathfrak{n}(B))$. \square

Remark 6.3.13. *This lemma works because the relations of $U\mathfrak{a}$ are generated by commutators. One of the category of examples come from Lie algebras, and the fundamental group of Riemann surfaces give another set of examples. For fundamental groups of non-orientable topological surfaces, their representation homology with unipotent group coefficient has strict higher vanishing property, i.e. $HR_i = 0$ for all $i > 0$. It should be emphasized that the computational result holds true not only for the fundamental group of non-orientable surfaces, but also for one-generator group with the generator is generated NOT ONLY by commutators.*

Proposition 6.3.14. *Using the terminologies in Lemma 6.3.10 and Lemma 6.3.12, when $U\mathfrak{a} \cong k[M]$ and the basis \mathcal{B} of \mathfrak{a} is a subset of M (in this case \mathcal{B} generates M), then $\text{Spec}(k[M])_{\mathfrak{n}_+^{\text{uni}}}$ and $\text{Spec}(U\mathfrak{a})_{\mathfrak{n}_+^{\text{uni}}}$ can be arranged as the same subscheme of $\text{Spec}(k[M])_{\mathfrak{n}_+} = \text{Spec}(U\mathfrak{a})_{\mathfrak{n}_+}$.*

Proof. It suffices to show that the constructions

$$\{f \in \text{Hom}_{\text{Mon}}(M, \mathfrak{n}_+(B)) \mid f(m) \text{ is universally unipotent for all } m \in M\}$$

in Lemma 6.3.10 and the constructions

$$\{f \in \text{Hom}_{\text{Lie}_k}(\mathfrak{a}, \mathfrak{n}_+(B)) \mid f(m) \text{ is universally unipotent for all } m \in \mathcal{B}\}$$

in Lemma 6.3.12 coincide for any commutative k -algebra B .

By the assumption, $U\mathfrak{a} \cong k[M]$ and so

$$\text{Hom}_{\text{Lie}_k}(\mathfrak{a}, \mathfrak{n}_+(B)) = \text{Hom}_{\text{Alg}_k}(U\mathfrak{a}, \mathfrak{n}_+(B)) = \text{Hom}_{\text{Alg}_k}(k[M], \mathfrak{n}_+(B)).$$

Now since \mathcal{B} generates M as a monoid, $f(m)$ is universally unipotent for all $m \in \mathcal{B}$ if and only if $f(m)$ is universally unipotent for all $m \in M$. Therefore two subsets coincide. \square

Theorem 6.3.15. *Given a unipotent algebraic group U , denote its associated nilpotent Lie algebra by \mathfrak{n} . Then*

$$\mathrm{HR}_*(T^2, U) \cong \mathrm{HR}_*(\mathfrak{a}, \mathfrak{n}),$$

where $\mathfrak{a} = k \oplus k$ is the 2-dimensional abelian Lie algebra.

Proof. Let Q be the cofibrant resolution of $\mathbb{N} \times \mathbb{N}$ in $s\mathbf{Mon}$, given by

- $Q_0 = \mathbb{N}x * \mathbb{N}y$,
- $Q_1 = \mathbb{N}x * \mathbb{N}y * \mathbb{N}t$, such that $d_0(x) = d_1(x) = x$, $d_0(y) = d_1(y) = y$, and $d_0(t) = xy$, $d_1(t) = yx$,
- all other elements are degeneracy, and
- $q : Q \rightarrow \mathbb{N} \times \mathbb{N}$ is given by $q_0 : \mathbb{N} * \mathbb{N} \rightarrow \mathbb{N} \times \mathbb{N}$.

This is apparently a cofibrant replacement of $\mathbb{N} \times \mathbb{N}$ in $s\mathbf{Mon}$, and $k[q] : k[Q] \rightarrow k[x, y]$ is a cofibrant replacement in $s\mathbf{Alg}_k$. Applying the functor $(k[-])_{\mathfrak{n}_+}$ and $(U-)_{\mathfrak{n}_+}$ on this resolution. In this case, two functors yield the same complex. Then taking the homology groups gives the isomorphism. \square

Corollary 6.3.15.1. *We have an isomorphism of commutative algebra*

$$(k \oplus k)_{\mathfrak{n}} \cong (\mathbb{Z} \times \mathbb{Z})_U,$$

which means the commuting scheme of a unipotent algebra group is isomorphic to that of its associated nilpotent Lie algebra.

Proof.

$$(k \oplus k)_{\mathfrak{n}} = \mathrm{HR}_0(k \oplus k, \mathfrak{n}) \cong \mathrm{HR}_0(\mathbb{Z} \times \mathbb{Z}, U) = (\mathbb{Z} \times \mathbb{Z})_U.$$

\square

Remark 6.3.16. *Lemma 6.3.10 implies that it is possible to find the analogue of $\mathrm{HR}_*(\mathfrak{a}, \mathfrak{n})$ in terms of representation homology of associative algebras. In this case, we need to find a relation $r \in k\langle a_1, b_1, a_2, b_2, \dots, a_g, b_g \rangle$, which is $[a_1, b_1] \cdots [a_g, b_g] = 1$ after taking localization.*

Theorem 6.3.15 does not hold true for reductive groups and their Lie algebras:

Example 6.3.17. *Consider \mathbb{G}_m , and we know that $\mathrm{Lie}(\mathbb{G}_m) = k$.*

For the Lie algebra representation homology, it is clear to see that $\mathrm{HR}_^{\mathrm{Lie}}(k \oplus k, k) \cong k[x, y, t; \deg t = 1]$ by Example 5.1.13.*

We have a general computation of $\mathrm{HR}_(\Gamma, \mathbb{G}_m)$ by [7, Lemma 4.1., Theorem 4.1.], however, in the case when $\Gamma = \mathbb{Z} \times \mathbb{Z}$, this object can be understood explicitly. Since \mathbb{G}_m is an abelian group, the commuting equation is redundant, so the underlying scheme is $\mathbb{G}_m \times \mathbb{G}_m$, and the derived scheme*

$$\mathrm{HR}_*(\Gamma, \mathbb{G}_m) \cong \mathrm{Sym}_k(\mathcal{O}(\mathbb{G}_m) \oplus \mathcal{O}(\mathbb{G}_m) \oplus \mathcal{O}(\mathbb{G}_m)[1]) = k[x^{\pm 1}, y^{\pm 1}, t^{\pm 1}; \deg t = 1].$$

These two representation homology groups are clearly not equal.

Remark 6.3.18. *[9, Theorem 3.1] connects the representation homology of groups and that of Lie algebras for simply connected space, and [9, Conjecture 1] for a general space.*

6.4 Characteristic pairing with nilpotent Lie algebra coefficient

For this section, we will always assume $k = \mathbb{C}$.

6.4.1 Lie algebra characteristic pairing map

Let \mathfrak{a} be a Lie algebra, and let \mathfrak{g} be a finite dimensional Lie algebra. Given any representation $\rho : \mathfrak{a} \rightarrow \mathfrak{g}$, then one has a pairing map

$$\langle -, - \rangle_\rho : \bigoplus_p H_p(\mathfrak{a}; k) \otimes H^p(\mathfrak{g}; k) \rightarrow k \quad (6.36)$$

defined by

$$\langle \bar{a}, \bar{x} \rangle_\rho = \bar{x}(\overline{\rho(a)}).$$

The adjunction

$$(-)_\mathfrak{g} : \mathbf{Lie} \rightleftarrows \mathbf{CommAlg}_{\mathfrak{g}_k/k} : \mathfrak{g}(-)$$

means that we have a natural isomorphism

$$\mathrm{Hom}_{\mathbf{Lie}}(\mathfrak{a}, \mathfrak{g}(\bar{B})) \cong \mathrm{Hom}_{\mathbf{CommAlg}_{\mathfrak{g}_k/k}}(\mathfrak{a}_\mathfrak{g}, B)$$

Putting $B = \mathfrak{a}_\mathfrak{g}$ and applying this isomorphism on $\mathrm{id}_{\mathfrak{a}_\mathfrak{g}}$, we have the universal representation

$$\pi : \mathfrak{a} \rightarrow \mathfrak{g}(\bar{\mathfrak{a}}_\mathfrak{g}) \rightarrow \mathfrak{g}(\mathfrak{a}_\mathfrak{g}), \quad (6.37)$$

and this induces a composite map on CE complex

$$C_\bullet^{\mathrm{CE}}(\mathfrak{a}; k) \xrightarrow{\pi_*} C_\bullet^{\mathrm{CE}}(\mathfrak{g}(\mathfrak{a}_\mathfrak{g}); k) \xrightarrow{\sim} \mathfrak{a}_\mathfrak{g} \otimes C_\bullet^{\mathrm{CE}}(\mathfrak{g}; k).$$

Apply the functor $H_p(-)$ we have

$$H_p(\mathfrak{a}; k) \rightarrow \mathfrak{a}_\mathfrak{g} \otimes H_p(\mathfrak{g}; k),$$

then composing with the evaluation $H_p(\mathfrak{g}; k) \otimes H^p(\mathfrak{g}; k) \rightarrow k$, we have the pairing

$$\chi_\mathfrak{a}(\mathfrak{g}) : \bigoplus_p H_p(\mathfrak{a}; k) \otimes H^p(\mathfrak{g}; k) \rightarrow \mathfrak{a}_\mathfrak{g}, \quad (6.38)$$

so that for each $a \in H_p(\mathfrak{a}; k)$, $x \in H^p(\mathfrak{g}; k)$,

$$\chi_{\mathfrak{a}}(\mathfrak{g})(a, x) := \langle \pi(a), x \rangle. \quad (6.39)$$

We call Equation (6.38) the characteristic map.

Example 6.4.1. *In this example, \mathfrak{a} will be the 2-dimensional abelian Lie algebra, s.t. $\mathfrak{a} = k \cdot a \oplus k \cdot b$, and $\mathfrak{g} = \mathfrak{n}_3$, i.e. Lie algebra consisting of 3×3 strict upper triangular matrices. We give \mathfrak{n}_3 a presentation $\mathfrak{n}_3 = k \cdot x \oplus k \cdot y \oplus k \cdot z$ where*

$$x := \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad y := \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \quad z := \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Now

$$\mathfrak{a}_{\mathfrak{g}} = \frac{k[X, Y, Z, U, V, W]}{[\dots, \dots] = 0},$$

where $\begin{pmatrix} 0 & X & Z \\ 0 & 0 & Y \\ 0 & 0 & 0 \end{pmatrix}$ and $\begin{pmatrix} 0 & U & W \\ 0 & 0 & V \\ 0 & 0 & 0 \end{pmatrix}$ are the generic matrices and the only relation is $XV - YU$. Then the universal representation

$$\mathfrak{a} \rightarrow \mathfrak{g}(\mathfrak{a}_{\mathfrak{g}}) = \mathfrak{g} \otimes \mathfrak{a}_{\mathfrak{g}}$$

is

$$a \mapsto x \otimes X + y \otimes Y + z \otimes Z$$

$$b \mapsto x \otimes U + y \otimes V + z \otimes W.$$

Now take $a \in H_1(\mathfrak{a})$ and $x^* \in H^1(\mathfrak{g})$, then as in Equation (6.39),

$$\begin{aligned} \chi_{\mathfrak{a}}(\mathfrak{g})(a, x^*) &:= \langle \pi(a), x^* \rangle \\ &= \langle x \otimes X + y \otimes Y + z \otimes Z, x^* \rangle \\ &= X, \end{aligned}$$

and similarly $\chi_{\mathfrak{a}}(\mathfrak{g})(b, x^*) = U$. This means that Z, W are not in the image of $\chi_{1,1}$.

Similarly, take $a \wedge b \in H_1(\mathfrak{a})$ and $(x \wedge z)^* \in H^1(\mathfrak{g})$, then

$$\begin{aligned} \chi_{\mathfrak{a}}(\mathfrak{g})(a \wedge b, (x \wedge z)^*) &:= \langle \pi(a \wedge b), (x \wedge z)^* \rangle \\ &= \langle (x \otimes X + y \otimes Y + z \otimes Z) \wedge (x \otimes U + y \otimes V + z \otimes W), (x \wedge z)^* \rangle \\ &= XW - ZU. \end{aligned}$$

Thus, the characteristic map is not surjective.

In order to get the derived pairing map, we need to derive the universal representation π and thus getting the derived version of Equation (6.37).

Let $L \rightarrow \mathfrak{a}$ be the resolution $L = \Omega_{\text{Comm}}(C_{\bullet}^{\text{CE}}(\mathfrak{a}; k))$. Then the (derived) universal representation

$$\pi : L \rightarrow \mathfrak{g}(L_{\mathfrak{g}})$$

corresponds to (under the cobar-bar adjunction)

$$\tau : C_{\bullet}^{\text{CE}}(\mathfrak{a}; k) \rightarrow C_{\bullet}^{\text{CE}}(\mathfrak{g}(L_{\mathfrak{g}}); k)$$

defined by

$$\tau = \eta_{C^{\text{CE}}(\mathfrak{a})} \circ C^{\text{CE}}(\pi) \tag{6.40}$$

where $H_*(L_{\mathfrak{g}})$ computes the representation homology $HR_*(\mathfrak{a}, \mathfrak{g})$.

The composed map of complexes

$$C_{\bullet}^{\text{CE}}(\mathfrak{a}; k) \xrightarrow{\tau} C_{\bullet}^{\text{CE}}(\mathfrak{g}(L_{\mathfrak{g}}); k) \rightarrow L_{\mathfrak{g}} \otimes C_{\bullet}^{\text{CE}}(\mathfrak{g}; k)$$

yields a map on homologies

$$H_n(\mathfrak{a}; k) \rightarrow \bigoplus HR_p(\mathfrak{a}, \mathfrak{g}) \otimes H_q(\mathfrak{g}; k),$$

then composing with the evaluation $H_q(\mathfrak{g}; k) \otimes H^q(\mathfrak{g}; k) \rightarrow k$, we have the pairing

$$\chi_{\mathfrak{a}}(\mathfrak{g}) : \bigoplus H_n(\mathfrak{a}; k) \otimes H^q(\mathfrak{g}; k) \rightarrow HR_p(\mathfrak{a}, \mathfrak{g}),$$

so that for each $a \in H_n(\mathfrak{a}; k)$, $x \in H^q(\mathfrak{g}; k)$,

$$\chi_{\mathfrak{a}}(\mathfrak{g})(a, x) := \langle \tau(a), x \rangle. \quad (6.41)$$

Example 6.4.2. *Let \mathfrak{a} be the 2-dimensional abelian Lie algebra, then*

$$H_*(\mathfrak{a}) = \text{Sym}_k(\mathfrak{a}[1]).$$

Thus the only nontrivial higher map is $\chi_{2,1}$ in this case.

6.4.2 Characteristic pairing in nilpotent coefficient

Example 6.4.3. *We start by letting \mathfrak{g} be a complex semisimple Lie algebra. Then by [49]*

$$H^*(\mathfrak{g}; k) = [H^*(G/B; k) \otimes H^*(\mathfrak{h}; k)]^W$$

where, in particular

$$H^*(\mathfrak{g}; k) = \Lambda_k(t_{2m_1+1}, \dots, t_{2m_r+1})$$

where the generator t_{2m_i+1} is of degree $2m_i + 1$ and m_1, \dots, m_r are the degrees of homogeneous generators of $\text{Sym}_k(\mathfrak{h}^)^W$. As a consequence,*

$$H^1(\mathfrak{g}; k) = 0$$

and thus $\chi_{2,1} : H_2(\mathfrak{a}; k) \otimes H^1(\mathfrak{g}; k) \rightarrow HR_1(\mathfrak{a}, \mathfrak{g})$ is always trivial when \mathfrak{g} is semisimple.

However in the nilpotent case, where the cohomology groups $H^*(\mathfrak{n}; k)$ are given by the classical Bott-Kostant theorem, things are different and the characteristic map $\chi_{2,1}$ gives us plenty of information:

Theorem 6.4.4. *Let \mathfrak{a} be the 2-dimensional abelian Lie algebra. Let \mathfrak{g} be a complex semisimple Lie algebra, with triangular decomposition*

$$\mathfrak{g} = \mathfrak{h} \oplus \sum_{\alpha \in R} \mathfrak{g}^\alpha$$

and $\mathfrak{g}^\alpha = \{x \in \mathfrak{g} \mid [H, x] = \alpha(H)x, \forall H \in \mathfrak{h}\}$, where R is its root system and W is its Weyl group. Choose a base $\Delta = \{\alpha_1, \dots, \alpha_r\}$,

$$W^{(1)} := \{w \in W \mid l(w) = 1\} = \{s_{\alpha_1}, \dots, s_{\alpha_r}\},$$

where $s_\alpha = x \mapsto x - 2\frac{(x, \alpha)}{(\alpha, \alpha)}\alpha$ is the reflection associated to α and l is the length function.

Let $\mathfrak{n} = \sum_{\alpha \in R_+} \mathfrak{g}^\alpha$ be the positive maximal nilpotent subalgebra of \mathfrak{g} , then by Bott-Kostant

$$H^1(\mathfrak{n}) \cong \bigoplus_{\alpha_i \in \Delta} (\mathfrak{g}^{\alpha_i})^*[1].$$

The characteristic pairing map

$$\chi_{\mathfrak{a}(\mathfrak{n})_{2,1}} : H_2(\mathfrak{a}; k) \otimes H^1(\mathfrak{n}; k) \rightarrow \mathrm{HR}_1(\mathfrak{a}, \mathfrak{n}) \quad (6.42)$$

gives nontrivial distinct homology classes $T_1, \dots, T_r \in \mathrm{HR}_1(\mathfrak{a}, \mathfrak{n})$.

Proof. Let $0 \neq \xi_\alpha \in \mathfrak{g}^\alpha$. Then there is a basis $\{\xi_\alpha\}_{\alpha \in R_+}$ of \mathfrak{n} . Then $\{\xi_\alpha^*\}_{\alpha \in R_+}$ is a basis of \mathfrak{n}^* , and we also use $x_\alpha := \xi_\alpha^*$ in the coordinate ring $k[\mathfrak{n}] = \mathrm{Sym}_k(\mathfrak{n}^*)$.

We first show that $H^1(\mathfrak{n}; k) \cong \bigoplus_{\alpha_i \in \Delta} (\mathfrak{g}^{\alpha_i})^*[1]$, via Chavelley-Eilenberg complex. By definition, we have the cochain complex

$$\dots \leftarrow \bigwedge^2 \mathfrak{n}^* \xleftarrow{d^1} \mathfrak{n}^* \xleftarrow{d^0} k \rightarrow 0, \quad (6.43)$$

to compute the cohomology $H^*(\mathfrak{n}; k)$, where $d^0 = 0$ and $d^1(f)(\xi, \eta) = f([\xi, \eta])$ for any $f \in \mathfrak{n}^*$. If we can show that $H^1(\mathfrak{n}; k) \cong \bigoplus_{\alpha_i \in \Delta} (\mathfrak{g}^{\alpha_i})^*[1]$, then by Bott-Kostant theorem (Theorem 4.1.10), this is an equality by dimension counting.

Now, under this base, the elements in \mathfrak{g}^{α_i} has weight 1, and the elements in other components have strictly higher weights. This means, the weight of $[\xi, \eta]$ for any $\xi, \eta \in \mathfrak{n}$ is at least 2, so $\xi_{\alpha_i}^*([\xi, \eta])$ is always 0. This means $d(\xi_{\alpha_i}^*) = 0$ for $\alpha_i \in \Delta$ and we complete the calculation of $H^1(\mathfrak{n}; k)$.

Let $C = C(\mathfrak{n}; k)$ and let \mathcal{L} be the free Lie algebra

$$\mathcal{L} = \Omega_{\text{Comm}}(C) = \mathbb{L}_k \langle a, b, t \rangle$$

with $\deg a = \deg b = 0$, $\deg t = 1$ and $dt = [a, b]$, as the replacement of \mathfrak{a} . Then we have a presentation (Koszul complex) in \mathbf{CDGA}_k

$$\mathcal{L}_{\mathfrak{n}} = k[\mathfrak{n} \times \mathfrak{n}] \otimes \wedge \mathfrak{n}^* = k[a \cdot x_{\alpha}, b \cdot x_{\alpha}, t \cdot x_{\alpha}; \quad \alpha \in R_+], \quad (6.44)$$

where $\deg a \cdot x_{\alpha} = \deg b \cdot x_{\alpha} = 0$ and $\deg t \cdot x_{\alpha} = 1$. Then the (derived) universal representation

$$\pi : \mathcal{L} \rightarrow \mathfrak{n}(\mathcal{L}_{\mathfrak{n}}) = \mathcal{L}_{\mathfrak{n}} \otimes \mathfrak{n}$$

maps t to $\sum_{\alpha \in R_+} t \cdot x_{\alpha} \otimes \xi_{\alpha}$. This corresponds to the composite map $\tau : C \rightarrow C(\mathfrak{n}(\mathcal{L}_{\mathfrak{n}}); k) = \mathcal{L}_{\mathfrak{n}} \otimes C(\mathfrak{n}; k)$ (Equation (6.40)) sends $a \wedge b$ to

$$\tau(a \wedge b) = \sum_{\alpha \in R_+} t \cdot x_{\alpha} \otimes \xi_{\alpha} \quad (6.45)$$

since $\tau = C(\pi \circ \eta_C)$ under the cobar-bar adjunction where $\eta : \text{id} \Rightarrow C \circ \Omega_{\text{Comm}}$ is the unit.

By the definition of characteristic pairing map and 6.45,

$$\begin{aligned} \chi_{\mathfrak{a}}(\mathfrak{n})(a \wedge b, \xi_{\alpha_i}^*) &:= \langle \tau(a \wedge b), \xi_{\alpha_i}^* \rangle \\ &= \left\langle \sum_{\alpha \in R_+} t \cdot x_{\alpha} \otimes \xi_{\alpha}, \xi_{\alpha_i}^* \right\rangle \\ &= t \cdot x_{\alpha_i}, \end{aligned}$$

then it suffices to show that $T_i := t \cdot x_{\alpha_i}$ goes to 0 in the Koszul complex \mathcal{L}_n .

Denote by ∂ the differentials in the Koszul complex \mathcal{L}_n (Equation 6.44). The first differential $\partial_1 : t \cdot k[n \times n] \otimes n^* \rightarrow k[n \times n]$ is

$$(\partial_1(t \cdot f \otimes \varphi))(\xi, \eta) = f(\xi, \eta)\varphi([\xi, \eta]) \quad (6.46)$$

for arbitrary $\xi, \eta \in n$. Again by weight counting, when $\varphi = \xi_{\alpha_i}^*$, the right hand side is always 0.

Finally, to see that they are different homology classes, we first see that ∂_2 is determined by ∂_1 and Leibniz rule. Namely,

$$\partial_2(t \cdot (k\varphi) \wedge t \cdot (h\psi)) = kh[\partial_1(t\varphi) \wedge t\psi - t\varphi \wedge \partial_1(t\psi)].$$

Suppose there exist $f_i \in k[n \times n]$ such that

$$\sum_{\alpha_i \in \Delta} f_i \cdot T_i = 0 \in \text{HR}_1(\mathfrak{a}, n).$$

This means there exists $t \cdot (k\varphi) \wedge t \cdot (h\psi)$ satisfying

$$\sum_{\alpha_i \in \Delta} f_i \cdot T_i = \partial_2(t \cdot (k\varphi) \wedge t \cdot (h\psi)) = kh[\partial_1(t\varphi) \wedge t\psi - t\varphi \wedge \partial_1(t\psi)]. \quad (6.47)$$

Evaluate Equation (6.47) at (ξ, η, α_i) , then we have

$$f_i(\xi, \eta) = k(\xi, \eta)h(\xi, \eta) \left\{ \varphi([\xi, \eta])\psi(\alpha_i) - \varphi(\alpha_i)\psi([\xi, \eta]) \right\}.$$

Plug it back in Equation (6.47) yielding

$$\sum_{\alpha_i \in \Delta} \left\{ \varphi([\xi, \eta])\psi(\alpha_i) - \varphi(\alpha_i)\psi([\xi, \eta]) \right\} t x_{\alpha_i} = \partial_1(t\varphi) \wedge t\psi - t\varphi \wedge \partial_1(t\psi). \quad (6.48)$$

For $\alpha \in R_+$ which is not simple, evaluate Equation (6.48) at ξ_α gives 0 on the left hand side, hence φ and ψ are linear combinations of x_{α_i} 's. Then $\partial_1(t\varphi) = \partial_1(t\psi) = 0$ since $t \cdot x_{\alpha_i}$ is closed, hence right hand side of Equation (6.48) is 0. Thus we only have the trivial relation. Therefore the proof is completed. \square

As an example, in the type \mathfrak{a}_n case, we have the image linking a previous computational result before:

Corollary 6.4.4.1. *Let $\mathfrak{g} = \mathfrak{sl}_{n+1}$ be the Lie algebra of type \mathfrak{a}_n and $\mathfrak{n} + \mathfrak{h} + \mathfrak{n}_-$ be its standard decomposition. Let $\Delta = \{\alpha_i := e_{i+1} - e_i\}$ be the standard basis of its root system R so that the triangular decomposition*

$$\mathfrak{g} = \mathfrak{h} \oplus \sum_{\alpha \in R} \mathfrak{g}^\alpha$$

is $\mathfrak{g}^{\alpha_i} = \mathbb{C}E_{i,i+1}$ and $\xi_{\alpha_i} = E_{i,i+1}$. The Weyl group is

$$W = S_n = \langle s_1, \dots, s_n \rangle$$

where s_i is the $i \leftrightarrow i + 1$. Then under this presentation, the characteristic pairing map

$$\chi_{\mathfrak{a}}(\mathfrak{n})_{2,1} : H_2(\mathfrak{a}; k) \otimes H^1(\mathfrak{n}; k) \rightarrow HR_1(\mathfrak{a}, \mathfrak{n})$$

gives all the classes T_1, \dots, T_n , which is the Lie algebraic analogue of the result in [39]. (See also Theorem 6.3.15)

For small rank semisimple algebras, the classes are all the higher algebra generators of $HR_*(\mathfrak{a}, \mathfrak{n})$. However, when the rank is high enough, for instance \mathfrak{sl}_6 and \mathfrak{g}_2 , computations show that there are extra homology class in HR_1 .

Example 6.4.5. *Let \mathfrak{g} be the semisimple Lie algebra of \mathfrak{g}_2 type, and \mathfrak{n} be the maximal nilpotent subalgebra with respect to a choice of positive roots. According to [18], a*

general matrix of \mathfrak{n} could be presented as a 7×7 matrix

$$\begin{pmatrix} 0 & 0 & 0 & -2u & -2z & -2x & 0 \\ z & 0 & y & w & 0 & -u & 0 \\ x & 0 & 0 & v & u & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & x & 0 & 0 & 0 \\ 0 & 0 & 0 & -z & -y & 0 & 0 \\ u & -x & z & 0 & -w & -v & 0 \end{pmatrix}.$$

Note that there are 18 nonzero terms in a general matrix, but only 6 of which are independent. Suppose (a choice of) positive roots of \mathfrak{g}_2 are denoted by

$$\Phi = \{\alpha, \beta, \alpha + \beta, 2\alpha + \beta, 3\alpha + \beta, 3\alpha + 2\beta\},$$

where α, β form a base, then

$$\{x, y, z, u, v, w\}$$

are the corresponding linear functions with respect to Φ .

Take two general matrices

$$X = \begin{pmatrix} 0 & 0 & 0 & -2u & -2z & -2x & 0 \\ z & 0 & y & w & 0 & -u & 0 \\ x & 0 & 0 & v & u & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & x & 0 & 0 & 0 \\ 0 & 0 & 0 & -z & -y & 0 & 0 \\ u & -x & z & 0 & -w & -v & 0 \end{pmatrix} \text{ and } A = \begin{pmatrix} 0 & 0 & 0 & -2a & -2f & -2d & 0 \\ f & 0 & e & c & 0 & -a & 0 \\ d & 0 & 0 & b & a & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & d & 0 & 0 & 0 \\ 0 & 0 & 0 & -f & -e & 0 & 0 \\ a & -d & f & 0 & -c & -b & 0 \end{pmatrix},$$

(6.49)

and we notice that

$$[X, A] = \begin{pmatrix} 0 & 0 & 0 & -4zd + 4xf & -2yd + 2xe & \mathbf{0} & 0 \\ yd - xe & 0 & \mathbf{0} & -3za + yb - ve + 3uf & 0 & -2zd + 2xf & 0 \\ \mathbf{0} & 0 & 0 & -3xa + 3ud & 2zd - 2xf & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \mathbf{0} & 0 & 0 & 0 \\ 0 & 0 & 0 & -yd + xe & \mathbf{0} & 0 & 0 \\ 2zd - 2xf & \mathbf{0} & yd - xe & 0 & 3za - yb + ve - 3uf & 3xa - 3ud & 0 \end{pmatrix} \quad (6.50)$$

From Equation (6.50), the coordinate ring of the commuting scheme of \mathfrak{n} is

$$\frac{k[a, b, c, d, e, f, u, v, w, x, y, z]}{(zd - xf, yd - xe, 3za - yb + ve - 3uf, xa - ud)}.$$

It is crucial to mention that the positions corresponding to the base elements α and β (colored in red and blue respectively) are 0.

The Koszul complex $\mathcal{L}_{\mathfrak{n}}$ (Equation (6.44)) is

$$k[X, A, T; dT = [X, A]], \quad \deg X = \deg A = 0, \deg T = 1],$$

and given by Equation (6.50), the explicit formula is (recall that there are 6)

$$k \left[a, \dots, f, u, \dots, z, t_1, \dots, t_6 \left| \begin{array}{l} dt_1 = dt_2 = 0, dt_3 = yd - xe, dt_4 = zd - xf \\ dt_5 = xa - ud, dt_6 = 3za - yb + ve - 3uf \end{array} \right. \right] \quad (6.51)$$

There are two classes t_1, t_2 in Equation (6.51) which live in the homology groups (since they are sent to 0 by the differential), and these give the two homology classes in $\mathrm{HR}_1(k \oplus k, \mathfrak{n})$ that are described in Theorem 6.4.4. One could notice that the classes t_1, t_2 lie in the entries exactly corresponding to the base elements α, β , and this correspondence is the statement of Theorem 6.4.4.

On the other hand, verified with Macaulay2, the ideal

$$(zd - xf, yd - xe, xa - ud)$$

in $k[a, b, c, d, e, f, u, v, w, x, y, z]$ does not form a regular sequence, hence there exists extra homology class. In fact, one can verify that

$$\mathrm{HR}_3(k \oplus k, n) \neq 0 \text{ and } \mathrm{HR}_4(k \oplus k, n) = 0$$

with the code given in [40].

CHAPTER 7
OPEN QUESTIONS

In this chapter, we shall discuss the possible directions for the future research.

7.1 A unipotent analogue of Macdonald identity

As mentioned in Chapter 1, the Macdonald identity (Equation (1.3)) admits an interpretation via representation homology with reductive coefficients. Notably, the construction of representation homology does not require the coefficient to be reductive. This naturally raises the question of whether an analogue of Equation (1.3) can be obtained by replacing the coefficients with unipotent groups or nilpotent Lie algebras.

Rational homotopy theory (see [21]) provides effective models for describing simply connected spaces, and [9, Theorem 3.1] indicates that the Lie model computation will give the desired result.

Example 7.1.1. *The free Lie algebra*

$$L(v) := \begin{cases} k \cdot v & \text{if } |v| \text{ is odd,} \\ k \cdot v \oplus k \cdot [v, v] & \text{if } |v| \text{ is even.} \end{cases}$$

models the topological sphere and similarly

$$L(v, w) := k \cdot v \oplus k \cdot w$$

be the Lie model of $S^m \times S^l$ ($|v| = m - 1$ and $w = l - 1$). Therefore the CDGA

$$k[X, Y, T; dT = [X, Y], \deg X = m, \deg Y = l, \deg T = m + l + 1]$$

is the complex computing corresponding homology, where X, Y are generic matrices of the coefficient.

We will implement the computations in Example 7.1.1 first to collect numerical evidences.

7.2 Exceptional classes

As discussed in Table 6.1, Table 6.2, Proposition 6.1.9, and Example 6.4.5, there are exceptional homology classes in $HR_*(k \oplus k, \mathfrak{n})$ which measures the failure of the underlying commuting scheme of \mathfrak{n} being a complete intersection. We have shown in Theorem 6.4.4 that the homology classes in $HR_*(k \oplus k, \mathfrak{n})$ can be interpreted by representation theory, and it will be very nice if we can explain the exceptional classes via a similar argument.

Our candidate is the Drinfeld trace map constructed in [1]. It is shown in [4] that the Drinfeld trace may be viewed as an “extension” of the characteristic pairing (evaluated at a Chern-Simons class) to $HC_{\bullet}^{(p)}(\mathfrak{a})$.

7.3 Representation homology of semisimple and nilpotent coefficients

Recall that the Chavelley Restriction Theorem in Chapter 1, which admits an interpretation using representation, i.e. Equation (1.1) induced by the restriction map $(\mathfrak{h} \hookrightarrow \mathfrak{g})^*$

$$HR_*(\mathfrak{a}, \mathfrak{g})^{\mathfrak{g}} \rightarrow HR_*(\mathfrak{a}, \mathfrak{h})^W$$

is an isomorphism for $\mathfrak{a} = k$ and any semisimple Lie algebra \mathfrak{g} .

This restriction map may not always be an isomorphism for arbitrary \mathfrak{a} , however one could expect that the map is still capable to give us information of the invariant part, or

the whole representation homology.

On the other hand, map 1.1 only uses the Cartan subalgebra of the triangular decomposition

$$\mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+,$$

where the nilpotent parts is neglected. As we have computed the representation homology with nilpotent coefficients, it is natural to expect that the restriction map may derive interesting information of the representation homology with semisimple coefficient, from that of nilpotent coefficient.

7.4 The character trace map for spaces

Inspired by Theorem 6.3.15 and Theorem 6.4.4, one can expect that for representation homology of groups, the construction(s) of character trace map also exists. We hope to give several equivalent constructions and compare the character trace map in group case with that in Lie algebra case. In particular, we hope to have a similar result in the group case to Theorem 6.4.4.

BIBLIOGRAPHY

- [1] Yuri Berest, Giovanni Felder, Sasha Patotski, Ajay C. Ramadoss, and Thomas Willwacher. Representation homology, Lie algebra cohomology and the derived Harish-Chandra homomorphism. *J. Eur. Math. Soc. (JEMS)*, 19(9):2811–2893, 2017.
- [2] Yuri Berest, Giovanni Felder, and Ajay Ramadoss. Derived representation schemes and noncommutative geometry. In *Expository lectures on representation theory*, volume 607 of *Contemp. Math.*, pages 113–162. Amer. Math. Soc., Providence, RI, 2014.
- [3] Yuri Berest, George Khachatryan, and Ajay Ramadoss. Derived representation schemes and cyclic homology. *Adv. Math.*, 245:625–689, 2013.
- [4] Yuri Berest, Guanyu Li, and Ajay C. Ramadoss. Characteristic pairing with nilpotent lie algebra coefficient. In preparation.
- [5] Yuri Berest and Ajay Ramadoss. Stable representation homology and Koszul duality. *J. Reine Angew. Math.*, 715:143–187, 2016.
- [6] Yuri Berest and Ajay C. Ramadoss. Symmetric homology and representation homology. *Trans. Amer. Math. Soc.*, 376(9):6475–6496, 2023.
- [7] Yuri Berest and Ajay C. Ramadoss. Derived character maps of group representations. *Algebr. Geom. Topol.*, 24(9):4991–5044, 2024.
- [8] Yuri Berest, Ajay C. Ramadoss, and Wai-kit Yeung. Vanishing theorems for representation homology and the derived cotangent complex. *Algebr. Geom. Topol.*, 19(1):281–339, 2019.
- [9] Yuri Berest, Ajay C. Ramadoss, and Wai-Kit Yeung. Representation homology of simply connected spaces. *J. Topol.*, 15(2):692–744, 2022.
- [10] Yuri Berest, Ajay C. Ramadoss, and Wai-Kit Yeung. Representation homology of topological spaces. *Int. Math. Res. Not. IMRN*, (6):4093–4180, 2022.
- [11] I. N. Bernstein, I. M. Gelfand, and S. I. Gelfand. Differential operators on the base affine space and a study of \mathfrak{g} -modules. In *Lie groups and their representations (Proc. Summer School, Bolyai János Math. Soc., Budapest, 1971)*, pages 21–64. Halsted Press, New York-Toronto, Ont., 1975.

- [12] Raoul Bott. Homogeneous vector bundles. *Ann. of Math. (2)*, 66:203–248, 1957.
- [13] Ivan Cherednik. Double affine Hecke algebras and Macdonald’s conjectures. *Ann. of Math. (2)*, 141(1):191–216, 1995.
- [14] Claude Chevalley and Samuel Eilenberg. Cohomology theory of Lie groups and Lie algebras. *Trans. Amer. Math. Soc.*, 63:85–124, 1948.
- [15] Ionuț Ciocan-Fontanine and Mikhail Kapranov. Derived Quot schemes. *Ann. Sci. École Norm. Sup. (4)*, 34(3):403–440, 2001.
- [16] Joachim Cuntz and Daniel Quillen. Algebra extensions and nonsingularity. *J. Amer. Math. Soc.*, 8(2):251–289, 1995.
- [17] Aurélien Djament. Décomposition de Hodge pour l’homologie stable des groupes d’automorphismes des groupes libres. *Compos. Math.*, 155(9):1794–1844, 2019.
- [18] Cristina Draper Fontanals. Notes on G_2 : the Lie algebra and the Lie group. *Differential Geom. Appl.*, 57:23–74, 2018.
- [19] W. G. Dwyer and J. Spaliński. Homotopy theories and model categories. In *Handbook of algebraic topology*, pages 73–126. North-Holland, Amsterdam, 1995.
- [20] Sam Evens and Jiang-Hua Lu. Poisson harmonic forms, Kostant harmonic forms, and the S^1 -equivariant cohomology of K/T . *Adv. Math.*, 142(2):171–220, 1999.
- [21] Yves Félix, Stephen Halperin, and Jean-Claude Thomas. *Rational homotopy theory*, volume 205 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 2001.
- [22] B. L. Feĭgin. On the cohomology of the Lie algebra of vector fields and of the current algebra. volume 7, pages 49–62. 1988. Selected translations.
- [23] Susanna Fishel, Ian Grojnowski, and Constantin Teleman. The strong Macdonald conjecture and Hodge theory on the loop Grassmannian. *Ann. of Math. (2)*, 168(1):175–220, 2008.
- [24] Sergei I. Gelfand and Yuri I. Manin. *Methods of homological algebra*. Springer-Verlag, Berlin, 1996. Translated from the 1988 Russian original.
- [25] Paul Goerss and Kristen Schemmerhorn. Model categories and simplicial methods.

In *Interactions between homotopy theory and algebra*, volume 436 of *Contemp. Math.*, pages 3–49. Amer. Math. Soc., Providence, RI, 2007.

- [26] Paul G. Goerss and John F. Jardine. *Simplicial homotopy theory*. Modern Birkhäuser Classics. Birkhäuser Verlag, Basel, 2009. Reprint of the 1999 edition [MR1711612].
- [27] Daniel R. Grayson and Michael E. Stillman. Macaulay2, a software system for research in algebraic geometry. Available at <http://www2.macaulay2.com>.
- [28] Phil Hanlon. Cyclic homology and the Macdonald conjectures. *Invent. Math.*, 86(1):131–159, 1986.
- [29] Vladimir Hinich. Homological algebra of homotopy algebras. *Comm. Algebra*, 25(10):3291–3323, 1997.
- [30] Vladimir Hinich. DG coalgebras as formal stacks. *J. Pure Appl. Algebra*, 162(2-3):209–250, 2001.
- [31] Mark Hovey. *Model categories*, volume 63 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 1999.
- [32] Jens Carsten Jantzen. *Representations of algebraic groups*, volume 107 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, second edition, 2003.
- [33] J. F. Jardine. A closed model structure for differential graded algebras. In *Cyclic cohomology and noncommutative geometry (Waterloo, ON, 1995)*, volume 17 of *Fields Inst. Commun.*, pages 55–58. Amer. Math. Soc., Providence, RI, 1997.
- [34] M. Kapranov. Noncommutative geometry based on commutator expansions. *J. Reine Angew. Math.*, 505:73–118, 1998.
- [35] M. Kapranov. Injective resolutions of BG and derived moduli spaces of local systems. *J. Pure Appl. Algebra*, 155(2-3):167–179, 2001.
- [36] Maxim Kontsevich and Alexander L. Rosenberg. Noncommutative smooth spaces. In *The Gelfand Mathematical Seminars, 1996–1999*, Gelfand Math. Sem., pages 85–108. Birkhäuser Boston, Boston, MA, 2000.
- [37] Bertram Kostant. Lie algebra cohomology and the generalized Borel-Weil theorem. *Ann. of Math. (2)*, 74:329–387, 1961.

- [38] Bertram Kostant. Lie algebra cohomology and generalized Schubert cells. *Ann. of Math. (2)*, 77:72–144, 1963.
- [39] Guanyu Li. Commuting varieties of upper triangular matrices and representation homology, 2024. arXiv:2403.13953.
- [40] Guanyu Li. A step towards computational derived algebraic geometry: The RepHomology package for macaulay2, 2024. arXiv:2410.18383.
- [41] Penghui Li, David Nadler, and Zhiwei Yun. Functions on the commuting stack via Langlands duality. *Ann. of Math. (2)*, 200(2):609–748, 2024.
- [42] Jean-Louis Loday and Bruno Vallette. *Algebraic operads*, volume 346 of *Grundlehren der mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer, Heidelberg, 2012.
- [43] I. G. Macdonald. Some conjectures for root systems. *SIAM J. Math. Anal.*, 13(6):988–1007, 1982.
- [44] Michael Mather. Pull-backs in homotopy theory. *Canadian J. Math.*, 28(2):225–263, 1976.
- [45] Hans J. Munkholm. DGA algebras as a Quillen model category. Relations to shm maps. *J. Pure Appl. Algebra*, 13(3):221–232, 1978.
- [46] D. I. Piontkovskii. Graded algebras and their differentially graded extensions. *Sovrem. Mat. Prilozh.*, (30):65–100, 2005.
- [47] Teimuraz Pirashvili. Hodge decomposition for higher order Hochschild homology. *Ann. Sci. École Norm. Sup. (4)*, 33(2):151–179, 2000.
- [48] Daniel Quillen. Rational homotopy theory. *Ann. of Math. (2)*, 90:205–295, 1969.
- [49] Mark Reeder. On the cohomology of compact Lie groups. *Enseign. Math. (2)*, 41(3-4):181–200, 1995.
- [50] Stefan Schwede and Brooke Shipley. Equivalences of monoidal model categories. *Algebr. Geom. Topol.*, 3:287–334, 2003.
- [51] Bertrand Toën and Gabriele Vezzosi. Homotopical algebraic geometry. I. Topos theory. *Adv. Math.*, 193(2):257–372, 2005.

- [52] Bertrand Toën and Gabriele Vezzosi. Homotopical algebraic geometry. II. Geometric stacks and applications. *Mem. Amer. Math. Soc.*, 193(902):x+224, 2008.
- [53] Michel Van den Bergh. Non-commutative quasi-Hamiltonian spaces. In *Poisson geometry in mathematics and physics*, volume 450 of *Contemp. Math.*, pages 273–299. Amer. Math. Soc., Providence, RI, 2008.
- [54] Charles A. Weibel. *An introduction to homological algebra*, volume 38 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 1994.