

On the Distribution of Land Equivalent Ratios[†]

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SUMMARY

The land equivalent ratio (LER) has been defined as a measure of efficiency of an intercrop or mixture. The LER compares land areas required under single or sole cropping to give the yields obtained from the component crops of the mixture. Values greater than one indicate intercropping to be more efficient than sole cropping in terms of land use, while values less than one indicate a loss in efficiency due to intercropping. There are many practical and statistical difficulties with the forms of LER's in the current literature. Two forms that overcome some of these difficulties are described in the present paper. Their conditional and unconditional distributions are given. These ideas are extended to intercropping involving a mixture of k crops.

Keywords: INTERCROPPING; MIXTURES; RATIOS OF RANDOM VARIABLES; LINEAR COMBINATIONS OF VARIABLES; MULTIVARIATE; CONDITIONAL DISTRIBUTIONS; UNCONDITIONAL DISTRIBUTIONS

1. INTRODUCTION

The land equivalent ratio (LER) described by de Wit and van den Bergh (1965) and Willey and Osiru (1972) is a measure of the efficiency of an intercrop or mixture.

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The LER is the relative land area required under sole cropping to give the yields obtained from the component crops of the mixture. If the LER is greater than one, the intercrop or mixture is advantageous; if the LER is less than one, intercropping is disadvantageous.

Mead and Riley (1981, pages 475, 496-8, and 507) point out that little is known about the distribution of the land equivalent ratio. Standardization by some "optimal" values is suggested as one way to alleviate some of the difficulties. To illustrate, consider the following forms of land equivalent ratios:

$$X_{hij} = \frac{Y_{hijm}}{Y_{hiis}} + \frac{Z_{hijm}}{Z_{hjjs}}, \quad (1)$$

$$X'_{.ij} = \frac{\bar{Y}_{.ijm}}{\bar{Y}_{.iis}} + \frac{\bar{Z}_{.ijm}}{\bar{Z}_{.jjs}}, \quad (2)$$

$$X_{hij}^* = \frac{Y_{hijm}}{\bar{Y}_{...s}} + \frac{Z_{hijm}}{\bar{Z}_{...s}}, \quad (3)$$

$$X_{hij}^- = \frac{Y_{hijm}}{\bar{Y}_{.ios}} + \frac{Z_{hijm}}{\bar{Z}_{.jos}}, \quad (4)$$

and

$$X_{hij}^+ = \frac{Y_{hijm}}{\bar{Y}_{es}} + \frac{Z_{hijm}}{\bar{Z}_{es}}, \quad (5)$$

where there are $i=1,2,\dots,I$ cultivars of crop one, $j=1,2,\dots,J$ cultivars of crop two, Y_{hijm} is the yield response per unit area for the i th cultivar of crop one in mixture ij in the h th replicate, $h=1,2,\dots,H$, Z_{hijm} is the yield response per unit area of the j th cultivar of crop two in mixture ij in replicate h , Y_{hiis} and Z_{hjjs} are the corresponding yield responses per unit area for sole crops in replicate h , $\bar{Y}_{.iis}$, $\bar{Y}_{.ijm}$, $\bar{Z}_{.jjs}$, and $\bar{Z}_{.ijm}$ are arithmetic means over replicates, $\bar{Y}_{...s}$ and $\bar{Z}_{...s}$ are means for

sole crops one and two, respectively, over all replicates and cultivars of each crop, $\bar{Y}_{.ios}$ and $\bar{Z}_{.jos}$ denote the "optimal" responses referred to by Mead and Riley (1981), and \bar{Y}_{es} and \bar{Z}_{es} are mean responses for sole crops one and two, respectively, external to the experiment. The latter might be farmers' yields of the two crops or means of a series of experiments on sole crops. There are other forms of LER's, but the above suffice for this discussion.

Using equation (1), as for example Wijesinha et al. (1982) have done, is rife with statistical difficulties. The X_{hij} are correlated with different correlations because of the dependence of the denominators on i and j . The distribution of X_{hij} involves a sum of ratios of random variables, and Y_{hiis} or Z_{hjjs} could take on the value of zero. Furthermore, a simulation study of LER's performed at EMBRAPA and the University of Brasilia (Luiz Hernan Roderiguez Castro, personal communication, 1982) has demonstrated that the X_{hij} deviate considerably from normality. From a practical viewpoint, comparisons among means of the X_{hij} over all replicates cannot be made because of the differences in the denominators as i and j vary. For example, the value of the LER tells us little about the components of the LER, as the following two pairs of LER's illustrate:

$$\frac{10}{20} + \frac{20}{30} = \frac{7}{6} = \frac{100}{200} + \frac{200}{300} \quad \text{and} \quad \frac{10}{10} + \frac{150}{300} = \frac{3}{2} = \frac{30}{40} + \frac{120}{160} .$$

A comparison of any set of the above four LER's would be meaningless in practical terms.

$X'_{.ij}$ in (2) and X_{hij}^- in (4) have difficulties similar to those discussed above; however, if $\bar{Y}_{.ios}$ and $\bar{Z}_{.jos}$ in (4) are replaced by values that do not depend upon the particular cultivars in mixture ij , then LER's can be compared. This appears to be what Mead and Riley (1981) imply in their section on standardization.

Comparisons can be made using means of X_{hij}^* in (3) and X_{hij}^+ in (5).

These comparisons do have practical interpretations. Furthermore, these versions of the LER are closely connected to other quantities of interest, as the LER can be viewed as a linear combination $k_1 Y_{hijm} + k_2 Z_{hijm}$ of crop responses. The coefficients k_1 and k_2 also could be crop values; protein, calorie, or carbohydrate conversion factors; farmer's values; coefficients obtained from a multivariate analysis; etc. Note that the ratio of coefficients k_2/k_1 and the linear combination $Y_{hijm} + (k_2/k_1) Z_{hijm}$ may be easier than the LER to think about on practical grounds. For example, the cash value of beans and maize may fluctuate considerably while the ratio of bean prices to maize prices remains quite stable at about 4:1. The same is true for ratios of mean yields of two crops. In equation (5) one might use $\bar{Y}_{es}/\bar{Z}_{es}$ and $Y_{hijm} + (\bar{Y}_{es}/\bar{Z}_{es})Z_{hijm}$ in this fashion.

2. CONDITIONAL DISTRIBUTIONS OF X_{hij}^* AND X_{hij}^+

Assume that the pair (Y_{hijm}, Z_{hijm}) has a bivariate normal distribution $BVN(\mu_{Ym}, \mu_{Zm}; \sigma_{Ym}^2, \sigma_{Zm}^2; \rho)$ under the null hypothesis for effects. It is known from multivariate theory that for any constants k_1 and k_2 , the linear combination $k_1 Y_{hijm} + k_2 Z_{hijm}$ has a univariate normal distribution $N(k_1 \mu_{Ym} + k_2 \mu_{Zm}, k_1^2 \sigma_{Ym}^2 + k_2^2 \sigma_{Zm}^2 + 2k_1 k_2 \rho \sigma_{Ym} \sigma_{Zm})$.

If the sole crop means \bar{Y}_{es} and \bar{Z}_{es} of (5), external to the experiment, are available, the distribution of $X_{hij}^+ = Y_{hijm}/\bar{Y}_{es} + Z_{hijm}/\bar{Z}_{es}$ conditional on \bar{Y}_{es} and \bar{Z}_{es} is univariate normal, since X_{hij}^+ is a linear combination of the type just described. Under the null hypothesis, the parameters of the distribution of X_{hij}^+ conditional on \bar{Y}_{es} and \bar{Z}_{es} are easily determined as $N(\mu_{Ym}/\bar{Y}_{es} + \mu_{Zm}/\bar{Z}_{es}, \sigma_{Ym}^2/\bar{Y}_{es}^2 + \sigma_{Zm}^2/\bar{Z}_{es}^2 + 2\rho\sigma_{Ym}\sigma_{Zm}/\bar{Y}_{es}\bar{Z}_{es})$. This conditional approach is especially appropriate when precise information on sole crop means can be obtained and when the values \bar{Y}_{es} and \bar{Z}_{es} are not different for each cultivar i and each cultivar j . Previous series of surveys or experiments may have

determined the sole crop yields to a high degree of accuracy so that they can be regarded as constants.

The conditioning method can be applied to any of the forms in equations (1) through (5), as long as one is willing to condition on the values in the denominators. This would not be useful for (1), (2), and (4) but would be for (3) and (5). Experimenters would often be willing to condition on the ratio of sole crop mean yields, $\bar{Y}_{\dots s} / \bar{Z}_{\dots s}$, obtained in the experiment. In some circumstances, they might be willing to condition on the individual sole crop cultivar means or their ratio $\bar{Y}_{\dots s} / \bar{Z}_{\dots s}$ in equations (2) and (4). Standard univariate statistical procedures may be utilized for these conditional situations.

3. UNCONDITIONAL DISTRIBUTIONS OF X_{hij}^{**} AND X_{hij}^+

Assume that each pair (Y_{hijm}, Z_{hijm}) has a bivariate normal distribution $BVN(\mu_{Ym}, \mu_{Zm}; \sigma_{Ym}^2, \sigma_{Zm}^2; \rho)$, each Y_{hiis} is normal $N(\mu_{Ys}, \sigma_{Ys}^2)$, each Z_{hjjs} is normal $N(\mu_{Zs}, \sigma_{Zs}^2)$, and these 3H random variables are independent. (Note that Y_{hiis} , Z_{hjjs} , and the pair (Y_{hijm}, Z_{hijm}) will be independent in completely randomized and randomized complete block designs when there is no competition among experimental units or if sufficient border material is used to eliminate competition.) The distribution of X_{hij}^{**} in (3) will now be derived.

It is immediate that $\bar{Y}_{\dots s} \sim N(\mu_{Ys}, \sigma_{Ys}^2 / IH)$, $\bar{Z}_{\dots s} \sim N(\mu_{Zs}, \sigma_{Zs}^2 / JH)$, and these are independent of each other and of (Y_{hijm}, Z_{hijm}) . For notational convenience, define $Y \equiv \bar{Y}_{\dots s}$, $Z \equiv \bar{Z}_{\dots s}$, $U \equiv Y_{hijm}$, $V \equiv Z_{hijm}$, and let $\mu_Y(\mu_Z, \mu_U, \mu_V)$ and $\sigma_Y^2(\sigma_Z^2, \sigma_U^2, \sigma_V^2)$ denote the mean and variance of $Y(Z, U, V)$. Define $X \equiv U/Y + V/Z$ and $W \equiv Z/Y$, so X is the LER. It can be described as a linear combination of two correlated components, each of which has a double noncentral t distribution (see Johnson and Kotz, 1970, p. 213).

The transformation from variables U, V, Y, Z to X, U, W, Y can be shown routinely

to be one-to-one from R^4 onto itself, with Jacobian $J = wy^2$. The joint density of U, V, Y, Z is the product of normal densities

$$f_{U, V, Y, Z}(u, v, y, z) = (2\pi\sqrt{1-\rho^2}\sigma_U\sigma_V)^{-1} \\ \times \exp[-\frac{1}{2}(1-\rho^2)^{-1}\{(u-\mu_U)^2/\sigma_U^2+(v-\mu_V)^2/\sigma_V^2-2\rho(u-\mu_U)(v-\mu_V)/\sigma_U\sigma_V\}] \\ \times (\sqrt{2\pi}\sigma_Y)^{-1} \exp[-\frac{1}{2}(y-\mu_Y)^2/\sigma_Y^2] (\sqrt{2\pi}\sigma_Z)^{-1} \exp[-\frac{1}{2}(z-\mu_Z)^2/\sigma_Z^2] . \quad (6)$$

Applying the Jacobian and the inverse transformation $v = xwy - uw$, $z = wy$ gives

$$f_{X, U, W, Y}(x, u, w, y) = (2\pi)^{-2} (\sigma_U\sigma_V\sigma_Y\sigma_Z\sqrt{1-\rho^2})^{-1} |w|y^2 \exp[-S-T] , \quad (7)$$

where

$$S = \frac{1}{2}(1-\rho^2)^{-1}\{(u-\mu_U)^2/\sigma_U^2+(xwy-uw-\mu_V)^2/\sigma_V^2-2\rho(u-\mu_U)(xwy-uw-\mu_V)/\sigma_U\sigma_V\}$$

and

$$T = \frac{1}{2}(y-\mu_Y)^2/\sigma_Y^2 + \frac{1}{2}(wy-\mu_Z)^2/\sigma_Z^2 .$$

Two easily derived integral formulas will be helpful:

$$\int_{-\infty}^{\infty} \exp[-(cu^2+bu+a)] du = \sqrt{\pi/c} \exp[(b^2-4ac)/4c] \quad (8)$$

and

$$\int_{-\infty}^{\infty} u^2 \exp[-(cu^2+bu+a)] du = \sqrt{2\pi}(2c)^{-5/2}(2c+b^2) \exp[(b^2-4ac)/4c] , \quad (9)$$

where a , b , and c are constants and $c > 0$. The first of these can be used to integrate out u , which appears only in the exponent S . This exponent can be rewritten as $S = cu^2 + bu + a$ with

$$c = \frac{1}{2}(1-\rho^2)^{-1}\{1/\sigma_U^2+w^2/\sigma_V^2+2\rho w/\sigma_U\sigma_V\} = \frac{1}{2}(1-\rho^2)^{-1}\sigma_U^{-2}\sigma_V^{-2}\{\sigma_V^2+\sigma_U^2w^2+2\rho\sigma_U\sigma_Vw\} ,$$

$$b = -(1-\rho^2)^{-1}\{\mu_U/\sigma_U^2+w(xwy-\mu_V)/\sigma_V^2+\rho(xwy-\mu_V)/\sigma_U\sigma_V+\rho\mu_Uw/\sigma_U\sigma_V\} ,$$

and

$$a = \frac{1}{2}(1-\rho^2)^{-1}\{\mu_U^2/\sigma_U^2+(xwy-\mu_V)^2/\sigma_V^2+2\rho\mu_U(xwy-\mu_V)/\sigma_U\sigma_V\} .$$

Substituting these values into (8) shows, after much algebra, that

$$\int_{-\infty}^{\infty} \exp[-S] du = [2\pi\sigma_U^2\sigma_V^2(1-\rho^2)]^{\frac{1}{2}} r^{-1} \exp\left[-\frac{1}{2}(xwy-\mu_V-\mu_Uw)^2/r^2\right]$$

where $r^2 = \sigma_V^2 + \sigma_U^2w^2 + 2\rho\sigma_U\sigma_Vw$, and therefore

$$\begin{aligned} f_{X,W,Y}(x,w,y) &= (2\pi)^{-3/2} \sigma_Y^{-1} \sigma_Z^{-1} r^{-1} |w| y^2 \\ &\quad \times \exp\left[-\frac{1}{2}\left\{(xwy-\mu_V-\mu_Uw)^2/r^2 + (y-\mu_Y)^2/\sigma_Y^2 + (wy-\mu_Z)^2/\sigma_Z^2\right\}\right] \\ &= (2\pi)^{-3/2} \sigma_Y^{-1} \sigma_Z^{-1} r^{-1} |w| y^2 \exp[-(c'y^2 + b'y + a')] \end{aligned} \quad (10)$$

where

$$c' = \frac{1}{2} \left\{ x^2w^2/r^2 + 1/\sigma_Y^2 + w^2/\sigma_Z^2 \right\} ,$$

$$b' = -xw(\mu_V + \mu_Uw)/r^2 - \mu_Y/\sigma_Y^2 - \mu_Zw/\sigma_Z^2 ,$$

and

$$a' = \frac{1}{2} \left\{ (\mu_V + \mu_Uw)^2/r^2 + \mu_Y^2/\sigma_Y^2 + \mu_Z^2/\sigma_Z^2 \right\} .$$

Using (9) to integrate y out of (10) gives, after more algebra,

$$\begin{aligned} f_{X,W}(x,w) &= (2\pi)^{-1} \sigma_Y^{-1} \sigma_Z^{-1} r^{-1} |w| (x^2w^2/r^2 + 1/\sigma_Y^2 + w^2/\sigma_Z^2)^{-5/2} \\ &\quad \times \left\{ x^2w^2/r^2 + 1/\sigma_Y^2 + w^2/\sigma_Z^2 + [xw(\mu_V + \mu_Uw)/r^2 + \mu_Y/\sigma_Y^2 + \mu_Zw/\sigma_Z^2]^2 \right\} \\ &\quad \times \exp \left[-\frac{1}{2} \frac{\left[\sigma_Z^2(\mu_Yxw - \mu_Uw - \mu_V)^2 + \sigma_Y^2w^2(\mu_Uw + \mu_V - x\mu_Z)^2 + r^2(\mu_Yw - \mu_Z)^2 \right]}{\sigma_Y^2\sigma_Z^2x^2w^2 + \sigma_Z^2r^2 + \sigma_Y^2r^2w^2} \right] . \end{aligned}$$

This expression can be integrated over w to give

$$f_X(x) = \int_{-\infty}^{\infty} f_{X,W}(x,w) dw .$$

The integral is analytically intractable, but it can be evaluated numerically when the values of μ_U , μ_V , μ_Y , μ_Z , σ_U^2 , σ_V^2 , σ_Y^2 , σ_Z^2 are known. For large $|w|$, $f_{X,W}(x,w)$ is $O(w^{-3})$.

This derivation of the distribution of X_{hij}^* also yields the distribution of X_{hij}^+ when the external crop means \bar{Y}_{es} and \bar{Z}_{es} are normal $N(\mu_{Yes}, \sigma_{Yes}^2)$ and $N(\mu_{Zes}, \sigma_{Zes}^2)$, respectively, each pair (Y_{hijm}, Z_{hijm}) is bivariate normal $BVN(\mu_{Ym}, \mu_{Zm}; \sigma_{Ym}^2, \sigma_{Zm}^2; \rho)$, and these $H+2$ random variables are independent. With \bar{Y}_{es} replacing $\bar{Y}_{...s}$ as Y and \bar{Z}_{es} replacing $\bar{Z}_{...s}$ as Z , the rest of the analysis remains valid without further change, leading to the distribution of X_{hij}^+ .

4. EXTENSIONS TO MIXTURES OF k CROPS

Consider the following forms of the LER for mixtures of k crops with individual responses available for each crop of the mixture:

$$X_{hij}^* = \frac{Y_{hijm}}{\bar{Y}_{...s}} + \frac{Z_{hijm}}{\bar{Z}_{...s}} + \frac{W_{hijm}}{\bar{W}_{...s}} + \dots \quad (11)$$

and

$$X_{hij}^+ = \frac{Y_{hijm}}{\bar{Y}_{es}} + \frac{Z_{hijm}}{\bar{Z}_{es}} + \frac{W_{hijm}}{\bar{W}_{es}} + \dots, \quad (12)$$

where W_{hijm} , $\bar{W}_{...s}$, and \bar{W}_{es} , etc., are defined in a manner similar to the other quantities in equations (3) and (5) and where there are k such responses combined in a linear manner for X_{hij}^* and for X_{hij}^+ . All that was said for conditional distributions for two variables holds here as well for k variables. Conditionally, X_{hij}^* and X_{hij}^+ are univariate normal under the null hypothesis and when $(Y_{hijm}, Z_{hijm}, W_{hijm}, \dots)$ have a k -dimensional multivariate normal distribution. The unconditional distribution of X_{hij}^* can be obtained in principle from an extension of the method of Section 3 for two variates; however, the mathematical complexity of the computation is markedly greater.

5. CONCLUDING REMARKS

There are experimental situations in which the distributional assumptions of the preceding sections are satisfied, e.g., if there is uniform behavior among the cultivars of each crop, both as sole crops and as components of an intercrop, and if the field layout insures that no competition exists among experimental units. On the other hand, there are other situations in which these assumptions are an oversimplification of the true state of affairs. The correlation ρ may depend on the mixture ij , and perhaps on the replicate if soil type, drainage, and similar factors vary. The means and variances, i.e., the μ 's and σ^2 's, may also depend on the mixture and perhaps on the replicate. Further work is needed to analyze the behavior of land equivalent ratios under more general conditions.

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