MATHEMATICAL MODELS FOR SWING OPTIONS AND SUBPRIME MORTGAGE DERIVATIVES.

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MATHEMATICAL MODELS FOR SWING OPTIONS AND SUBPRIME MORTGAGE DERIVATIVES.

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The deregulation of the energy market and the recent soaring (and possible bubble) of commodity prices motivates the first part of the thesis. We analyze a certain kind of contract in the commodity market known as swing or take-or-pay options. These contracts are American type options where the holder has multiple exercise rights. The goal is to find the optimal consumption process for the underlying commodity. We present a pricing methodology using the theory of reflected backward stochastic differential equations and the theory of Snell envelopes. Once the model is constructed, one can use numerical techniques to solve the pricing problem and compute a replicating strategy using forward contracts.

The recent burst of the real estate bubble has drawn a lot of attention to the subprime derivatives market. Existing models have proven inadequate due to their inability to account for the complexity of mortgage derivatives. Chapter 3 provides an analytical framework for understanding the mortgage market. In Chapter 4, we give a condition on the underlying securities that allows us to directly compute the loss distribution term structure of the portfolio. Then, we build a tractable model for pricing options on large credit portfolios such as Collateralized Debt Obligations of subprime Asset Backed Securities / Home Equity Loans.

BIOGRAPHICAL SKETCH

Nicolas Diener was born in Strasbourg, France but spend the first six years of his life in Oran, Algeria. Then, he moved to Paris, France for a few years and finally landed in Nice, France where he spent 10 years and graduated from the High school Lycée du Parc Imperial. He moved back to Paris and, in 2001, he received his Bachelor of Science in Mathematics from Université Pierre and Marie Curie. In 2003, he received two Masters of Science, one from the engineering school Ecole Nationale Supèrieur des Télécomunications and one in Probability and Finance from Université Pierre and Marie Curie.

In August 2004, Nicolas started his Ph.D in the School of Operations Research and Industrial Engineering, Cornell University. In August 2006, he transferred to the Center for Applied Mathematics, Cornell University, and pursue his Ph.D. under the thesis guidance of Professor Philip Protter.

To my grandmother, Lili.

To my grandparents, Edwige, George and Oscar.

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CHAPTER 1

INTRODUCTION TO BACKWARD STOCHASTIC DIFFERENTIAL EQUATIONS

The study of forward backward stochastic differential equations (FBSDEs for short) was originally motivated by stochastic optimal control theory. Decoupled FBSDEs were first introduced by Bismut [10], and in linear form Benssoussan [7] proved the well-posedness of general linear BSDEs by using a martingale representation theorem. While studying the general Pontryagin-type maximum principle for stochastic optimal controls, Pardoux and Peng [70] proved the first well-posedness result for nonlinear BSDEs. Antonelli [2] obtained in his PhD Thesis the first result on the solvability of strongly coupled FBSDEs over a ”small” time duration. Ma and Yong [65] started a systematic investigation on the well-posedness of FBSDEs over arbitrary time durations. El Karoui, Kapoudjian, Pardoux, Peng and Quenez [28] introduced the notion of reflected backward stochastic differential equations (RBSDEs) where the solution is forced to remain above a continuous process, which is considered as the lower barrier. Several methods has been established for solving FBSDEs. The *Four Step Scheme* by Ma, Protter and Yong [64] provides the explicit solution by using a quasi-linear partial differential equation. The potential of BSDEs primarily comes from areas such as stochastic control and mathematical finance.

The remainder of this chapter is organized as follows. In the first section, we will present the differences between stochastic differential equations (SDEs) and backward stochastic differential equations (BSDEs). We will then give an existence and uniqueness theorem for the solution of BSDEs. In the second section, we introduce the concept of obstacle and define a reflected BSDE. The third section starts with an introduction to asset pricing theory. We see how the theory of BSDEs can be used to solve the pricing of European contingent claims and we use reflected BSDEs to price American contingent claims. In the fourth section, we present a discretization procedure to obtain numerical solutions of BSDEs.

1.1 What is a Backward Stochastic Differential Equation?

Throughout, we will assume that we are given an underlying probability space $(\Omega; \mathcal{F}; \mathbb{F}; P)$ where $\mathbb{F} = (\mathcal{F}_t)_{t\geq 0}$. We further assume $\mathcal{F}_s \subset \mathcal{F}_t$ if $s < t$, $\mathcal{F}_0$ contains all the P-null sets of $\mathcal{F}$, and that $\bigcap_{s>t} \mathcal{F}_s \equiv \mathcal{F}_{t+} = \mathcal{F}_t$ by hypothesis. This last property is called the right continuity of the filtration. These hypotheses, taken together, are known as the usual hypotheses. When the usual hypotheses hold, it is known that every martingale has a version which is càdlàg.

A stochastic differential equation with given terminal condition does not have, in general, a non-anticipating solution. Let us consider the following example. Let the terminal condition ξ and the driver $f : \mathbb{R}_+ \times \mathbb{R}$ have the appropriate properties (we will define them later). Solving a BSDE reduces to finding $(Y_t)_{t\in[0,T]}$ such that

$$\begin{cases} -dY_t = f(t, Y_t)dt \\ Y_T = \xi \end{cases}$$

We can re-write the solution as

$$Y_t = \xi + \int_t^T f(s, Y_s)ds.$$

However if $\xi \in \mathcal{F}_T$ and we want the solution Y to be adapted, then the equation is more complex resulting in

$$Y_t = \mathbf{E}\left[\xi + \int_t^T f(s, Y_s)ds|\mathcal{F}_t\right].$$

We can re-write this equation as

$$\begin{aligned} Y_t &= \mathbf{E}\left[\xi + \int_0^T f(s, Y_s)ds | \mathcal{F}_t\right] - \int_0^t f(s, Y_s)ds \\ &= M_t - \int_0^t f(s, Y_s)ds \end{aligned}$$

where $(M_t)_{t\geq 0}$ is a martingale. Let us suppose that we are solving the BSDE on the canonical space for Brownian motion. Then, we have that the martingale representation property holds and hence there exists a predictable process Z such that

$$M_t = \alpha + \int_0^t Z_s dB_s$$

where

$$\alpha = E\{\xi\}$$

and B is Brownian motion. Let us now use a trick: we write the BSDE in the form

$$Y_T - Y_t = M_T - M_t - \int_t^T f(s, Y_s)ds,$$

or equivalently

$$Y_t = \xi + \int_t^T f(s, Y_s)ds - (M_T - M_t).$$

We can now replace M by its martingale representation given by

$$Y_t = \xi + \int_t^T f(s, Y_s)ds - \int_t^T Z_s dB_s.$$

We obtain that the solution of a BSDE is a pair of adapted processes (Y, Z) satisfying

$$\begin{cases} dY_t = f(t, Y_t)dt - Z_t dB_t \quad 0 \leq t \leq T \\ Y_T = \xi \end{cases}$$

This example provides some intuition about how to solve a simple version of a BSDE. However, we still need to prove that a solution exists for a BSDE with a more general driver f.

The following notations hold throughout the paper:

- $(\mathcal{F}_t)_{t\geq 0}$ is the augmented filtration of the standard Brownian motion $(W_t)_{t\geq 0}$
- $\mathcal{P}$ is the predictable σ field
- $\mathcal{B}$ is the Borel σ field
- $\mathbb{L}^2_T = \{X \in \mathbb{R}; X \in \mathcal{F}_T\text{-measurable such that } \mathbf{E}|X|^2 < +\infty\}$
- $\mathbb{H}^2_T = \left\{(H_t)_{0\leq t\leq T} \text{ is predictable and } \mathbf{E}\int_0^T |H_t|^2 dt < +\infty\right\}$
- $\mathcal{S}^2_T = \left\{(S_t)_{0\leq t\leq T} \text{ is progressive and } \mathbf{E}\left(\sup_{0\leq t\leq T} |S_t|^2\right) < +\infty\right\}$
- For $\beta > 0$ and $\phi \in \mathbb{H}^2_T$, $\|\phi\|_\beta = \mathbf{E}\int_0^T e^{\beta t}|\phi|^2 dt$
- $\mathbb{H}^2_{T,\beta}$ is the space $\mathbb{H}^2_T$ endowed with the norm $\| \, \|_\beta$.

Definition 1. *A driver f, mapping $\Omega \times \mathbb{R}^+ \times \mathbb{R}^d \times \mathbb{R}^{n\times d}$ onto $\mathbb{R}^d$ is said to be standard if*

- *it is $\mathcal{P} \otimes \mathcal{B}^d \otimes \mathcal{B}^{n\times d}$ measurable,*
- *$f(.,0,0) \in \mathbb{H}^2_T$,*
- *f is uniformly Lipschitz: there exists $C > 0$ such that $dP \otimes dt$ almost everywhere*

$$|f(\omega,t,y_1,z_1) - f(\omega,t,y_2,z_2)| \leq C(|y_1 - y_2| + |z_1 - z_2|) \;\; \forall (y_1,z_1), \forall (y_2,z_2).$$

Definition 2. *If the driver is standard and the terminal condition ξ is in $\mathbb{L}^2_T$, the data (f,ξ) is said to be standard.*

Definition 3. *Let data* (f, ξ) *be standard. The solution of the following backward stochastic differential equation*

$$\begin{cases} -dY_t = f(t, Y_t, Z_t)dt - Z_t dB_t \\ Y_T = \xi \end{cases} \tag{1.1}$$

is a pair $(Y, Z) \in \mathbb{H}^2_T(\mathbb{R}^d) \times \mathbb{H}^2_T(\mathbb{R}^{n\times d})$.

Theorem 1. *(Pardoux-Peng [70]) If the data* (f, g) *is standard, then Equation* (1.1) *has a unique solution.*

The proof follows the idea developed in the previous example and uses a fixed point theorem. We also need the following proposition for *a priori* estimates.

Proposition 2. *(El Karoui-Quenez [27]) Let* $((f^i, \xi^i); i = 1, 2)$ *be two standard data for BSDE and let* $((Y^i, Z^i); i = 1, 2)$ *be the associated square integrable solutions. Let* C *be a Lipschitz constant for* f^1*, and put* $\delta Y_t = Y^1_t - Y^2_t$*,* $\delta Z_t = Z^1_t - Z^2_t$ *and* $\delta_2 f_t = f^1(t, Y^2_t, Z^2_t) - f^2(t, Y^2_t, Z^2_t)$*. Then for any* $\beta > C(2 + C)$*, the following a priori estimates hold:*

$$\|\delta Y\|^2_\beta \leq T\left[e^{\beta T} E\left\{|\delta Y_T|^2\right\} + \frac{1}{\beta - 2C - C^2}\|\delta_2 f\|^2_\beta\right]$$
$$\|\delta Z\|^2_\beta \leq (2 + 2C^2 T)e^{\beta T} E\left\{|\delta Y_T|^2\right\} + \frac{2 + 2C^2 T}{\beta - 2C - C^2}\|\delta_2 f\|^2_\beta.$$

Proof. (Theorem (1)) Starting with a couple (y, z) in $\mathbb{H}^2_{T,\beta}(\mathbb{R}^d) \times \mathbb{H}^2_{T,\beta}(\mathbb{R}^{n\times d})$, we will prove that the mapping of (y, z) onto the solution (Y, Z) of the BSDE with driver $f(t, y_t, z_t)$ is a contraction and therefore a fixed point theorem can be applied to obtain the solution.

The first step is to prove the existence of the solution (Y, Z) of the BSDE with driver $f(t, y_t, z_t)$, given by

$$\begin{cases} -dY_t = f(t, y_t, z_t)dt - Z_t dB_t \\ Y_T = \xi \end{cases}$$

Let M be the square integrable martingale

$$M_t = E\left\{\int_0^T f(s, y_s, z_s)ds + \xi|\mathcal{F}_t\right\}.$$

By the martingale representation theorem for the Brownian motion, there exists a unique integrable process $Z \in \mathbb{H}_T^{2,n\times d}$ such that

$$M_t = M_0 + \int_0^t Z_s dW_s.$$

It follows that the adapted and continuous process Y given by

$$Y_t = M_t - \int_0^t f(s, y_s, z_s)ds$$

is a solution of (1.2) and is square integrable.

Let (y^1, z^1) and (y^2, z^2) be two elements in $\mathbb{H}^2_{T,\beta}(\mathbb{R}^d) \times \mathbb{H}^2_{T,\beta}(\mathbb{R}^{n\times d})$ and let (Y^1, Z^1) and (Y^2, Z^2) be the associated solution. By applying Proposition (2), we obtain

$$\|\delta Y\|^2_\beta + \|\delta Z\|^2_\beta \leq \frac{2(2+T)C^2}{\beta}\left(\|\delta y\|^2_\beta + \|\delta z\|^2_\beta\right)$$

By choosing $\beta > 2(2+T)C^2$, we see that the mapping is a contraction from $\mathbb{H}^2_{T,\beta}(\mathbb{R}^d) \times \mathbb{H}^2_{T,\beta}(\mathbb{R}^{n\times d})$ into itself, and therefore there exists a unique fixed point.

□

1.2 Reflected Backward Stochastic Differential Equations

A reflected backward stochastic differential equation is a backward stochastic differential equation where the solution is forced to stay above a given stochastic process, called the obstacle. An increasing process is introduced which pushes the solution upwards so that it remains above the obstacle. It can be shown that solutions of reflected BSDEs satisfy properties similar to the classical case. The uniqueness of solutions of reflected BSDEs comes from a comparison theorem for non-reflected BSDEs. It is then possible to obtain *a priori* estimates for the spread of the solution of two reflected BSDEs. Finally, we can show the existence of a solution.

Definition 4. *Let the data (f, ξ) be standard. Moreover, let the obstacle $(S_t)_{t\geq 0}$ be a continuous and adapted real valued process bounded in $\mathbb{L}^2$ such that $S_T \leq \xi$. Such a triple (f, ξ, S) is called standard data.*

Definition 5. *Let the data (f, ξ, S) be standard. The solution to a reflected backward stochastic differential equation is a triple $(Y_t, Z_t, K_t)_{0\leq t\leq T}$ of $\mathcal{F}_t$ progressively measurable processes taking values in $\mathbb{R} \times \mathbb{R}^d \times \mathbb{R}^+$ satisfying*

$$\begin{cases} -dY_t = f(t, Y_t, Z_t)dt - Z_t dB_t + dK_t \\ Y_T = \xi \\ Z \in \mathbb{H}^2, Y \in \mathcal{S}^2, K_T \in \mathbb{L}^2 \\ Y_t \geq S_t, \quad 0 \leq t \leq T \\ (K_t)_{t\geq 0} \textit{ is continuous increasing}, K_0 = 0 \\ \int_0^T (Y_s - S_s)dKs = 0 \end{cases} \tag{1.2}$$

Intuitively, dK_t is the amount that is added to $-dY$ so that the constraint $Y \geq S$ is satisfied. The last condition says that the addition is done in a minimal fashion; only when the constraint is saturated.

As in the classical case, we have a comparison theorem. However, the theorem differs from the classical case because the difference of two reflected BSDEs is not a reflected BSDE.

Theorem 3. *Let* (f_1, ξ^1, S^1) *and* (f^2, ξ^2, S^2) *be two sets of standard data, and let* (Y^1, Z^1, K^1) *and* (Y^2, Z^2, K^2) *be the solutions of the respective reflected BSDE. Suppose that*

- $\xi^1 \leq \xi^2$*, a.s.*
- $f^1(t, y, z) \leq f^2(t, y, z)\ dP \times dt$ *a.e.,* $\forall (y, z) \in \mathbb{R} \times \mathbb{R}^d$
- $S^1 \leq S^2$*, a.s.*

Then

$$Y_1 \leq Y_2, \ a.s.$$

Proof. This follows from a result of El-Karoui, Pardoux and Quenez found in [29]. By applying Ito's Formula to $|(Y_t^1 - Y_t^2)^+|^2$, we obtain after calculation

$$|(Y_t^1 - Y_t^2)^+|^2 \leq \overline{C} E\left\{\int_t^T |(Y_s^1 - Y_s^2)^+|^2 ds\right\}$$

where x^+ is the positive part of x (i.e. $x^+ = max(x, 0)$) and $\overline{C}$ is some constant. From Gronwall's Lemma, we conclude that $|(Y_t^1 - Y_t^2)^+|^2 = 0$ for all $0 \leq t \leq T$. □

We can deduce immediately the uniqueness result when $\xi^1 = \xi^2$, $f^1 = f^2$ and $S^1 = S^2$. It remains to show the existence of a solution to the reflected BSDE.

Theorem 4.

- *(a priori estimate) Let* $((\xi^i, f^i, S); i = 1, 2)$ *be two standard data of a reflected BSDE with the same obstacle, and let* $((Y^i, Z^i, K^i); i = 1, 2)$ *be the solution of the associated reflected BSDE. Then the a priori estimates of Proposition (2) still hold.*

- *(Existence) Let (ξ, f, S) be a set of standard data. Then the reflected BSDE of Definition (1.2) has a unique solution.*

Proof. The proof for *a priori* estimates is very similar to the one for BSDEs. The only difference comes from the process K. However, special properties of K (in particular, it is an increasing process), adapt well to computing inequalities. Uniqueness can be derived from Theorem (3). For the existence, we show in the Proposition (7) that we can build a solution to the reflected BSDE with driver $f(t, y_t, z_t)$ for (y, z) in $\mathcal{S}^2 \times \mathbb{H}^2$ by applying Skohorod's Lemma. The reminder of the proof is similar to the non reflected case (fixed point theorem). □

We will now see that the square-integrable solution Y of a reflected BSDE corresponds to the value of an optimal stopping problem.
In a deterministic framework, the formulation of the reflected BSDE corresponds to the Skohorod problem (see Revuz-Yor [72]).

Lemma 5. *(Skohorod) Let x be a real valued continuous function on $[0, \infty[$ such that $x_0 \geq 0$. There exists a unique pair (y, k) of functions on $[0, \infty[$ such that*

- $y = x + k$
- *y is positive*
- *$(k_t)_{t \geq 0}$ is continuous and increasing, $k_0 = 0$ and $\int_0^T y_t dk_t = 0$.*

The pair (y, k) is said to be the solution of the Skohorod problem. The function k is given by

$$k_t = \sup_{s \leq t} x_s^-$$

where x^- is the negative part of $x$defined by $x^- = max(-x, 0)$.

Using Lemma (5), we can re-write the increasing process K as a supremum, given by the following Proposition.

Proposition 6. *Let* $(Y_t, Z_t, K_t)_{0\leq t\leq T}$ *be a solution of Equation* (1.2)*. Then for each* $t \in [0, T]$

$$K_T - K_t = \sup_{t\leq u\leq T}\left(\xi + \int_u^T f(s, Y_s, Z_s)ds - \int_u^T Z_s dB_s - S_u\right)^{-}.$$

Proof. We obtain the desired result by first fixing $\omega \in \Omega$ and by applying Skohorod's Lemma to

$$\begin{aligned} x_t &= \left(\xi + \textstyle\int_{T-t}^T f(s, Y_s, Z_s)ds - \int_{T-t}^T Z_s dB_s - S_{T-t}\right)(\omega) \\ y_t &= (K_T - K_{T-t})(\omega) \\ k_t &= (Y_{T-t} - S_{T-t})(\omega). \end{aligned}$$

□

Finally, we obtain the following result linking reflected BSDEs to the optimal stopping problem.

Proposition 7. *Let* $(Y_t, Z_t, K_t)_{0\leq t\leq T}$ *be a solution of the reflected BSDE* (1.2)*. Then for each* $t \in [0, T]$

$$Y_t = ess \sup_{v\in\mathcal{T}_t} E\left\{\int_t^v f(s, Y_s, Z_s)ds + S_v\mathbf{1}_{\{v<T\}} + \xi\mathbf{1}_{\{v=T\}}|\mathcal{F}_t\right\}$$

where $\mathcal{T}$ *is the set of all stopping times , and*

$$\mathcal{T}_t = \{v \in \mathcal{T}; t \leq v \leq T\}.$$

Proof. Let v be in $\mathcal{T}_t$. Taking the conditional expectation in Equation (1.2) yields

$$Y_t = E\left\{\int_t^v f(s, Y_s, Z_s)ds + Y_v + K_v - K_t|\mathcal{F}_t\right\}.$$

However, K is increasing, resulting in

$$Y_t \geq E\left\{\int_t^v f(s, Y_s, Z_s)ds + S_v \mathbf{1}_{\{v<T\}} + \xi \mathbf{1}_{\{v=T\}} | \mathcal{F}_t\right\}.$$

Let us now define the following stopping times by

$$D_t = \inf\{t \leq u \leq T; Y_u = S_u\}.$$

K is continuous, starting from 0 and increasing only on the set $\{Y_t = S_t\}$. Therefore

$$K_{D_t} - K_t = 0$$

and

$$Y_t = E\left\{\int_t^{D_t} f(s, Y_s, Z_s)ds + S_{D_t} \mathbf{1}_{\{D_t<T\}} + \xi \mathbf{1}_{\{D_t=T\}} | \mathcal{F}_t\right\}.$$

□

In the linear case, that is when $f(t, y, z)$ is linear in (y, z), the solution can be written using the adjoint process. This result will be useful when applied to financial markets.

Proposition 8. *Let (β_t, γ_t) be a bounded $(\mathbb{R} \times \mathbb{R}^d)$-valued predictable process, and ϕ_t be an element of $\mathbb{H}^2_T(\mathbb{R})$. Let f be the standard generator defined by*

$$f(t, y, z) = \phi_t + \beta_t y + \gamma_t^* z$$

Let $(\Gamma_{t,s}; t \leq s \leq T)$ be the adjoint process satisfying the linear SDE given by

$$\begin{cases} d\Gamma_{t,s} = \Gamma_{t,s}\left(\beta_s ds + \gamma_s^* dB_s\right) \\ \Gamma_{t,t} = 1. \end{cases}$$

Then the unique solution $(Y_t, Z_t, K_t)_{0\leq t\leq T}$ of the reflected BSDE with driver f satisfies, for each $t \in [0,T]$,

$$Y_t = ess \sup_{v\in\mathcal{T}_t} E\left\{\int_t^v \Gamma_{t,s}\phi_s ds + \Gamma_{t,v}S_v\mathbf{1}_{\{v<T\}} + \Gamma_{t,v}\xi\mathbf{1}_{\{v=T\}}|\mathcal{F}_t\right\}.$$

This can also be written as

$$Y_t = ess \sup_{v\in\mathcal{T}_t} X_t\left(v, \widetilde{S}_v\right)$$

where $\left(X_t\left(v, \widetilde{S}_v\right); 0 \leq t \leq v\right)$ is the solution of the BSDE with generator f, terminal time v and terminal condition $\widetilde{S}_v$ where

$$\widetilde{S}_t = S_t\mathbf{1}_{\{t<T\}} + \xi\mathbf{1}_{\{t=T\}}.$$

Furthermore, the stopping time $D_t = \inf\{t \leq s \leq T; Y_s = S_s\}$ is optimal.

Proof. Applying Itô's Formula to $Y_t\Gamma_{0,t}$ yields

$$Y_t\Gamma_0, t = \Gamma_{0,T}\xi \int_t^T \Gamma_{0,s}\phi_s ds + \int_t^T \Gamma_{0,s}dK_s - \int_t^T \Gamma_{0,s}(Z_s + Y_s)^* dB_s.$$

We can now use the same arguments as in Proposition (7). □

1.3 Application of linear (Reflected) Backward Stochastic Differential Equations to Finance

We will begin with an introduction to asset pricing theory mainly inspired by the article of Jarrow and Protter [48].

The setup for our continuous time asset pricing models contains two securities: one risk-less asset, the money market account (or bond) with price process R, and one risky asset, the stock, with price process S. For simplicity we limit our discussion to one dimension. Let $(a_t)_{t\geq 0}$ and $(b_t)_{t\geq 0}$ be the position at time t in the stock and the bond, respectively. We call the holdings of S and R a portfolio.

Definition 6. *The value at time t of a portfolio (a, b) is*

$$V_t(a, b) = a_t S_t + b_t R_t$$

A trading strategy in the risky asset is a predictable process $a = (a_t)_{t\geq 0}$; its economic interpretation is that at time t one holds an amount a_t of the asset. We also remark that it is reasonable that the process a be predictable; a trader's position at time t, has to be based on information obtained at times strictly before t, but not t itself.

Definition 7. *A trading strategy (a, b) is called self-financing if*

$$a_t S_t + b_t R_t = a_0 S_0 + b_0 R_0 + \int_0^t a_s dS_s + \int_0^t b_s dR_s \text{ for all } t \geq 0.$$

Indeed, we want the proceeds of the sale of the risky asset to be placed in the money market account and when we purchase risky assets, we use the cash from the money market account only to pay for the expenditure. There is no exchange of money outside of these two assets. Mathematically, this implies

$$d(a_t S_t + b_t R_t) = a_t dS_t + b_t dR_t + S_t da_t + R_t db_t.$$

However, we restrict ourselves to self-financing strategies:

$$S_t da_t + R_t db_t = 0.$$

In order to develop a methodology to compute "fair" prices of contingent claims, we need to rule out any strategy that makes a profit without risk. The standard way of modelling this mathematically is as follows.

Definition 8. *A model is arbitrage free if there does not exist a self-financing trading strategy* (a, b) *such that*

- $V_0(a, b) = 0$,
- $V_T(a, b) \geq 0$,
- $P(V_T(a, b) > 0) > 0$.

A natural question to ask now is what conditions are required on the model for the market to be free of arbitrage? Before addressing this question,the concept of pricing measures need to be introduced.

Definition 9. *A probability measure* P^* *is called an equivalent martingale measure, alternatively a risk neutral probability, if* P^* *is equivalent to* P*, and if under* P^* *the price process* S *is a* σ *martingale.*

For the rest of the chapter, we will assume that there exist a unique risk neutral probability.

1.3.1 Linear BSDEs and European Options

Now consider the case of a **European** contingent claim. It is known that we can build a replicating portfolio by investing dynamically in the underlying stock S and in a money market account. Assume that the dynamic of the stock under the risk neutral measure is

$$\frac{dS_t}{S_t} = r_t dt + \sigma_S(S_t) dB_t$$

where r_t is the instantaneous interest rate.
In order for this model to be complete, we will assume that the processes r

and σ_S are predictable and bounded. Moreover we will assume that σ_S^{-1} is also bounded.

Over a period dt, the value of the portfolio evolves according to

$$\begin{aligned} dV_t &= b_t dR_t + a_t dS_t \\ &= r_t b_t R_t dt + a_t dS_t \\ &= r_t(V_t - a_t S_t)dt + a_t dS_t \end{aligned}$$

.

Hence, the change in the value of the portfolio reduces to

$$\begin{aligned} dV_t &= r_t(V_t - a_t S_t)dt + a_t(r_t S_t dt + \sigma_S(S_t) S_t dB_t) \\ &= (r_t V_t + a_t(r_t - r_t))dt + a_t \sigma_S(S_t) S_t dB_t \end{aligned}$$

Let $Z_t = a_t \sigma_S(S_t) S_t$.

We obtain

$$\begin{aligned} dV_t &= r_t V_t dt + Z_t dB_t \\ &= -f(t, V_t, Z_t)dt + Z_t dB_t \end{aligned}$$

with $-f(t, v, z) = r_t v$.

We know the value ξ_T of the contingent claim at maturity T (for example $\xi_T = (K - S_T)^+$ for a European put). Therefore, pricing the option reduces to solving the backward SDE

$$\begin{cases} dV_t = -f(t, V_t, Z_t)dt + Z_t dB_t & 0 \leq t \leq T \\ V_T = \xi_T \end{cases}$$

1.3.2 Linear Reflected BSDEs and American Options

Definition 10. *We are given an adapted process U and an expiration time T. An American type derivative is a claim to the payoff U_τ at a stopping time $\tau \leq T$. The stopping* *time τ is chosen by the holder of the derivative and is called the exercise policy.*

Let V_t be the price of the security at time t. The objective is to find $(V_t)_{0\leq t\leq T}$. Let $V_t(\tau)$ denote the value of the security at time t if the holder follows exercise policy τ. Let us further assume, without loss of generality, that $R_t \equiv 1$. Then

$$V_t(\tau) = E^* \left\{ U_0(\tau) | \mathcal{F}_t \right\}$$

where E^* denotes expectation with respect to the equivalent martingale measure P^*. Let us recall that

$$\mathcal{T}_t = \left\{ \text{all stopping times with values in} [t, T] \right\}.$$

Definition 11. *A rational exercise policy is a solution to the optimal stopping problem*

$$V_0^* = \sup_{\tau \in \mathcal{T}_0} V_0(\tau). \tag{1.3}$$

Ideally, we want to establish a price for an American type derivative. Specifically, how much should one pay for the right to purchase U in between $[0, T]$ at a stopping rule of one's choice?

Definition 12. *A super-replicating trading strategy θ is a self-financing trading strategy such that $\theta_t S_t \geq U_t$ for all t with $0 \leq t \leq T$, where S is the price of the underlying risky security on which the American type derivative is based. (We are again assuming $R \equiv 1$.)*

Theorem 9. *Suppose the supremum in (1.3) is achieved at τ^*. Then V_0^* is a lower bound for the no arbitrage price of the American type derivative. Suppose a super replicating strategy θ exists with $\theta_0 S_0 = V_0^*$. Then, V_0^* is an upper bound for the no arbitrage price of the American type derivative.*

In order to prove the existence of super-replicating trading strategies we will use the theory of Snell envelopes.

Definition 13. *A stochastic process Y is said to be of class (D) if the collection $\mathcal{H} = \{Y_\tau : \tau \text{ is a stopping time}\}$ is uniformly integrable.*

Finally, recall some properties of the Snell envelope [24].

Theorem 10. *Let $\{(U_t),\ 0 \leq t \leq T\}$ be an optional process of class (D), then there exists a unique optional process $\{(V_t),\ 0 \leq t \leq T\}$, which is a super-martingale of class (D) such that:*

- $V \geq U$
- *if* $\left\{(V_t^{'}),\ 0 \leq t \leq T\right\}$ *is a super-martingale and* $V^{'} \geq U$*, then* $V^{'} \geq V$.

The process V is called the Snell envelope of ξ. Moreover, for every $t \in [0, T]$,

$$V_t = ess \sup_{\tau \in \mathcal{T}_t} E\left\{U_\tau | \mathcal{F}_t\right\}.$$

For a proof, please consult Meyer [24]. Now we can present the final result for American contingent claims.

Theorem 11. *Under regularity assumptions, there exists a super-replicating trading strategy θ such that $\theta_0 S_0 = V_0^*$. A rational exercise policy is*

$$\tau^* = \inf\{t > 0 : Z_t = U_t\}$$

where Z is the Snell Envelope of U under P^*.

The idea is that in the case of an **American** contingent claim, the contract gives its owner the right to exercise at any time before maturity. We cannot find a self-financing strategy that replicates the payoff so we have to use a super-replicating strategy where a consumption process is added to the usual self-financing replicating portfolio.

Indeed, we want the value of the hedging portfolio V to be always above the exercise value of the option

$$\begin{cases} dV_t = -f(t, V_t, Z_t)dt + Z_t dB_t \\ V_T = \xi_T \\ V_t \geq U_t \;\; 0 \leq t \leq T \text{ a.s.} \end{cases} \tag{1.4}$$

However, there is no reason for V_t to be above U_t and therefore Equation (1.4) does not necessarily have any solution. To solve this problem, we have to inject into the portfolio a positive quantity dK_t, the consumption, between t and $t+dt$. We want to inject continuously and in a "minimal" fashion; dK_t has to be zero when $V_t > U_t$,or equivalently, we only inject money when we have to do so (i.e. $\int_0^T (V_t - U_t)dK_t = 0$).

Therefore, the problem reduces to solving the reflected backward SDE given by

$$\begin{cases} V_t = U_T + \int_t^T f(s, V_s, Z_s)ds - \int_t^T Z_s dB_s + \int_t^T dKs \quad 0 \leq t \leq T \\ V_t \geq \xi_t \quad 0 \leq t \leq T \\ K \text{ is a continuous and increasing process starting at } 0 \\ \int_0^T (V_t - U_t)dK_t = 0 \end{cases}$$

The process V represents the value of the smallest super-hedging strategy for an American option with maturity T and payoff U. Z represents the hedge and K is the cumulative consumption necessary for the super-replication. In this case, the price of the option is given by V_0.

Remark 1. *The obstacle of a reflected backward SDE can be interpreted as the instantaneous payoff of an American option.*

1.4 Numerical Methods for Reflected Backward Stochastic Differential Equations

In the previous section, we have presented the mathematical analysis of BSDEs and we continue by studying its numerical resolution. Several numerical methods have been proposed: First, Ma et al. [64] propose the *four step scheme* to solve general FBSDEs, requiring the numerical resolution of a quasilinear parabolic PDE. Bally presents in [4] a time discretization scheme based on a Poisson net. However, extra computations of high dimensional integrals are needed. In a recent work [81], Zhang proves some regularity conditions, which allow the use of a regular deterministic time mesh. However, these methods are costly in terms of computational time or computer memory.

Another approach is regression based algorithms. The idea is to find a discretization procedure and compute the conditional expectations which appear at each discretization time.
Bouchard and Touzi [14] propose a Monte Carlo approach where the conditional expectations are computed by discretizing the space of each state variable. The authors use a general regression operator which can be derived, for instance, from kernel estimators or from the Malliavin calculus integration by parts formulas.

Gobet, Lemor and Warin [39] develop a simple algorithm, based on Monte Carlo regression on function bases, that has the particularity of requiring only one set of paths to approximate all the regression operators at each discretization (see also Lemor's thesis [60]). This approach is based on the work of Longstaff and Schwartz [62] for the pricing of Bermuda options. See also Clement, Lamberton, and Protter [19] for a proof of the convergence of such an algorithm.

We are interested in finding a numerical approximation for the solution of the decoupled forward backward stochastic differential equation given by

$$\begin{cases} dX_t = b(X_t)dt + \sigma(X_t)dB_t \quad 0 \leq t \leq T \\ X_0 = x \\ dY_t = f(t, X_t, Y_t, Z_t)dt - Z_t dB_t \quad 0 \leq t \leq T \\ Y_T = g(X_T) \end{cases}$$

with appropriate regularity assumptions for b, σ and f.

Without loss of generality, assume that the maturity T is one. We will solve the BSDE on the time interval $[0, 1]$.

Let $\pi : 0 = t_0 < t_1 < ... < t_n = 1$ be a partition of the time interval $[0, 1]$ with mesh

$$|\pi| \equiv \max_{1 \leq i \leq n} |t_i - t_{i-1}|.$$

We use the notation

$$\Delta_i^\pi = t_i - t_{i-1}$$
$$\Delta^\pi B_i = B_{t_i} - B_{t_{i-1}}.$$

The problem of discretization and simulation of the forward component X is well known (see Talay [77] or Kloden and Platen [55]). We have the classical

Euler scheme

$$X_0^\pi = x$$
$$X_{t_i}^\pi = X_{t_{i-1}}^\pi + b(X_{t_{i-1}}^\pi)\Delta_i^\pi + \sigma(X_{t_{i-1}}^\pi)\Delta^\pi B_i \quad i = 1..n.$$

Let us now consider the naive Euler discretization of the backward component (Y, Z) given by

$$\begin{aligned} Y_{t_n}^\pi &= g(X_{t_n}^\pi) \\ Z_{t_n}^\pi &= 0 \\ Y_{t_i}^\pi &= Y_{t_{i-1}}^\pi + f(t_{i-1}, X_{t_{i-1}}^\pi, Y_{t_{i-1}}^\pi, Z_{t_{i-1}}^\pi)\Delta_i^\pi - Z_{t_{i-1}}^\pi(B_{t_i} - B_{t_{i-1}}). \end{aligned} \tag{1.5}$$

Unfortunately, there is no $\mathcal{F}_{t_{i-1}}$-measurable random variables $(Y_{t_{i-1}}^\pi, Z_{t_{i-1}}^\pi)$ that solve the previous equation. The measurability issue is addressed by using conditional expectations. Indeed, $Y_{t_{i-1}}^\pi$ is $\mathcal{F}_{t_{i-1}}$-measurable and therefore

$$\begin{aligned} Y_{t_{i-1}}^\pi &= E\left\{Y_{t_i}^\pi|\mathcal{F}_{t_{i-1}}\right\} - E\Big\{f(t_{i-1}, X_{t_{i-1}}^\pi, Y_{t_{i-1}}^\pi, Z_{t_{i-1}}^\pi)\Delta_i^\pi \\ &\qquad + Z_{t_{i-1}}^\pi(B_{t_i} - B_{t_{i-1}})|\mathcal{F}_{t_{i-1}}\Big\} \\ &= E\left\{Y_{t_i}^\pi|\mathcal{F}_{t_{i-1}}\right\} - f(t_{i-1}, X_{t_{i-1}}^\pi, Y_{t_{i-1}}^\pi, Z_{t_{i-1}}^\pi)\Delta_i^\pi \\ &\qquad + E\left\{Z_{t_{i-1}}^\pi(B_{t_i} - B_{t_{i-1}})|\mathcal{F}_{t_{i-1}}\right\}. \end{aligned}$$

We used that

$$\begin{aligned} E\left\{Z_{t_{i-1}}^\pi(B_{t_i} - B_{t_{i-1}})|\mathcal{F}_{t_{i-1}}\right\} &= Z_{t_{i-1}}^\pi B_{t_i} - Z_{t_{i-1}}^\pi E\left\{B_{t_{i-1}}|\mathcal{F}_{t_{i-1}}\right\} \\ &= Z_{t_{i-1}}^\pi B_{t_i} - Z_{t_{i-1}}^\pi B_{t_i} \\ &= 0. \end{aligned}$$

For an estimate of $Z_{t_{i-1}}^\pi$, we multiply Equation (1.5) by $\Delta^\pi B_i$

$$Y_{t_i}^\pi \Delta^\pi B_i = Y_{t_{i-1}}^\pi \Delta^\pi B_i - f(t_{i-1}, X_{t_{i-1}}^\pi, Y_{t_{i-1}}^\pi, Z_{t_{i-1}}^\pi)\Delta_i^\pi \Delta^\pi B_i + Z_{t_{i-1}}^\pi(\Delta^\pi B_i)^2$$

and take the conditional expectation

$$\begin{aligned} E\left\{Y_{t_i}^\pi \Delta^\pi B_i|\mathcal{F}_{t_{i-1}}\right\} &= E\left\{Z_{t_{i-1}}^\pi(\Delta^\pi B_i)^2|\mathcal{F}_{t_{i-1}}\right\} \\ &= Z_{t_{i-1}}^\pi \Delta_i^\pi \end{aligned}$$

We used that

$$E\left\{Y^\pi_{t_{i-1}}\Delta^\pi B_i|\mathcal{F}_{t_{i-1}}\right\}=0$$
$$E\left\{f(t_{i-1},X^\pi_{t_{i-1}},Y^\pi_{t_{i-1}},Z^\pi_{t_{i-1}})\Delta^\pi_i\Delta^\pi B_i|\mathcal{F}_{t_{i-1}}\right\}=0$$

Then the following discrete time approximation

$$Y^\pi_{t_n}=g(X^\pi_{t_n})$$
$$Z^\pi_{t_{i-1}}=\frac{1}{\Delta^\pi_i}E\left\{Y^\pi_{t_i}\Delta^\pi B_i|\mathcal{F}_{t_{i-1}}\right\}$$
$$Y^\pi_{t_{i-1}}=E\left\{Y^\pi_{t_i}|\mathcal{F}_{t_{i-1}}\right\}+f(t_{i-1},X^\pi_{t_{i-1}},Y^\pi_{t_{i-1}},Z^\pi_{t_{i-1}})\Delta^\pi_i$$

is obtained.

The problem reduces now to finding a numerical approximation of the conditional expectations given by

$$E\left\{Y^\pi_{t_i}\Delta^\pi B_i|\mathcal{F}_{t_{i-1}}\right\}$$
$$E\left\{Y^\pi_{t_i}|\mathcal{F}_{t_{i-1}}\right\}$$

Using an induction argument, it can be proved that the random variables $Y^\pi_{t_i}$ and $Z^\pi_{t_i}$ are deterministic functions of $X^\pi_{t_i}$ for each $i=0,...,n$. Since the FBSDE is decoupled, the forward process X is Markov, and it follows that the conditional expectations involved can be replaced by

$$E\left\{Y^\pi_{t_i}\Delta^\pi B_i|X_{t_{i-1}}\right\}$$
$$E\left\{Y^\pi_{t_i}|X_{t_{i-1}}\right\}.$$

Let us now concentrate on computing

$$E\left\{\Phi(X_T)|X_t=x\right\} \tag{1.6}$$

or

$$E\left\{\Phi(X_T)|X_t=x\right\}=\frac{E\left\{\Phi(X_T)\delta_x(X_t)\right\}}{E\left\{\delta_x(X_t)\right\}} \tag{1.7}$$

where δ_x is the Dirac point mass in x.

1.4.1 Malliavin Calculus Based Simulation Method

This technique has been proposed by Fournie, Lasry, Lebuchoux and Lions [31], and further developed by Bouchard, Ekeland and Touzi [12]. The main idea is to use the Malliavin integration by parts formula in order to get rid of the Dirac point masses in Equation (1.7). In doing so one gets

$$E\left\{\Phi(X_T)|X_t = x\right\} = \frac{E\left\{\Phi(X_T)H_x(X_t)\Pi\right\}}{E\left\{H_x(X_t)\Pi\right\}} \tag{1.8}$$

where $H_x(.) = 1_{.\geq x}$ is the Heaviside function, and Π is some non-negative random variable. An important consequence of this formula is the fact that the associated Monte Carlo estimator:

$$E\left\{\Phi(X_T)|X_t = x\right\} = \frac{\sum_{i=1}^{n}\Phi(X_T^i)H_x(X_t^i)\Pi^i}{\sum_{i=1}^{n}H_x(X_t^i)\Pi^i}, \tag{1.9}$$

constructed from an independent sample $(X^i, \Pi^i)_{i=1,,N}$ of size N, converges at the $\sqrt{N}$-rate by the classical central limit theorem.

1.4.2 Regression Based Algorithm

For any square integrable random variable X, the conditional expectation of X_T given X_t is an $\mathbb{L}^2$ projection of X_T on the Hilbert space generated by X_t, namely $\sigma(X_t)$. We can approximate this projection by the partial sums of its decomposition on any orthonormal basis of the Hilbert space. A finite set of regression functions will be chosen to represent this basis. Since the coefficients in such an expansion are expectations of products of X_T by functions of X_t, they can be estimated from a random sample $(X^i)_{i=1,,N}$. Its use in the context of American option pricing was suggested by Longstaff and Schwartz [62], and the corresponding price estimate has been shown to be consistent by Clément, Lamberton and Protter [19].

CHAPTER 2

VALUATION OF SWING OPTIONS USING REFLECTED BACKWARD STOCHASTIC DIFFERENTIAL EQUATIONS

2.1 Introduction

Pricing methods for Swing options have been extensively studied over the past decade. The deregulation of the energy market and the recent soaring (and possible bubble) of commodity prices motivates this research. There are two main approaches to the pricing problem. The first one uses stochastic control theory. The goal is to find the optimal consumption process for the underlying commodity and to use dynamic programming techniques to compute a numerical solution. Jaillet et al [45] extend the usual binomial/trinomial method to the so-called forest of trees method and compute a numerical solution in the case where the underlying process follows a one-factor model. Barrera et al [5] use the Longstaff-Schwartz (see [62]) algorithm and neural network technics to derive the optimal consumption law. Dahlgren [21] studies the pricing of swing options as an impulse control problem and proves that it is the solution of a system of quasi-variational inequalities. Keppo [54] finds a replicating strategy using a basket of forwards and calls.

The second approach is due to Carmona and Touzi [16] and Carmona and Dayanik [15]. They use the theory of Snell envelopes and prove the existence of a sequence of optimal exercise strategies. Our approach is based on their work in the sense that we apply their idea and define a sequence of reflected backward stochastic differential equations (Reflected BSDE's in short). In this way we can understand the behavior of swing options, by providing equations for the price process as well as the actual hedging value process. These equations can be solved numerically for a broad range of underlying processes. Indeed, regression based algorithms to solve BSDE's have been recently developed and they can be applied to a variety of diffusions, unlike the numerical solution proposed in [16] which is valid only within a Black-Scholes framework. Indeed, they are using the closed form solution of the Black-Scholes model for pricing European contigent claims. Also, the problem solved in [16] does not include some of the customary features of swing options, such as the minimum and maximum number of exercises associated to a penalty function. We also give a theorem describing the exercise region of the option.

An outline is as follows: In the second section, we explain swing options. In the third section, we recall some results from arbitrage pricing theory and explain how they are related to backward SDE's and the optimal stopping problem. In the fourth section we prove that the price process of a swing option can be obtained from a special reflected backward SDE. For the sake of clarity, we start the section with a very simple type of swing option and then gradually add complexity to the structure. Finally, the fifth section is dedicated to the study of the exercise regions of swing options.

2.2 Swing Options

Swing contracts are designed to provide the purchaser flexibility on the timing of delivery and the quantity of a specified commodity. Usually the buyer of the option is interested in buying a fixed quantity of the commodity periodically between the starting time of the contract and the maturity time T. The swing contract gives him the right to change ("swing") the periodic fixed amount delivered, to a new amount with the restriction that this volume of the commodity remains between some pre-specified boundaries v_{min} and v_{Max}. These swings have to be exercised at some specified dates and their number is limited.

In order to avoid the natural optimal strategy to exercise all the rights at the same time, a minimum time period, or refraction time, must elapse between two consecutive exercises. Indeed, Jaillet et al [45] proved that if it is optimal to exercise one right, then all the rights should be exercised at the same optimal time. The refraction time can depend on the number or the size of the swing but it is pre-specified at the beginning of the contract.

The last characteristic of a typical contract is an insurance for the seller of a minimal and a maximal volume of commodity to be delivered over the total time period. Those constraints may be violated subject to the payment of a penalty.

Although swing options are similar to American options, the existence of a refraction period makes their pricing non-trivial. Indeed, we can see it by looking at the two extreme cases of swing options: the one-swing (one exercise right) and the full-swing (as many exercises as the overall contract time divided by the refracting period). A one-swing reduces to a "simple" American option while a full-swing is a combination of European options (there is no extra flexibility in choosing the optimal exercise strategy). We start to see that depending on the available number of exercises, the structure of the problem becomes more and more complex. This is due to the strongly coupled feature embedded in a swing option.

A swing has two components: one representing the fixed volume delivered periodically and the other one representing the extra volume bought at each exercise. The first component has no options embedded in it; it is determined at the beginning of the swing. It is a group of forward contracts of fixed volume v_{min} (one contract for each delivery period). Since we can compute the value of the fixed components independently of the floating one, we can assume without loss of generality that the value of this component is zero, or equivalently that the minimal volume to be delivered periodically is zero: $v_{min} = 0$.

When it is optimal to exercise a right, it has been proved by Jaillet et al [45] and independently by Rodrigez [73] that the option owner should should maximize his take of the underlying commodity. This behavior is known as exercising in a "bang bang" fashion. When the discounted profit of exercising a right is greater than the expected value of the penalty function, a maximizing agent should exercise as much as possible. Therefore, we will restrain the exercise strategy as a single possible pay-off at each exercise.

We will define a swing contract as an option where the purchaser is given the right to exercise between n and N times before maturity T with the constraint that the buyer of the contract has to wait for a refracting time δ between two successive exercises. He cannot exercise more than N times or less than n times unless he pays a penalty $\mathcal{P}_T$ at maturity. We assume that $\mathcal{P}_T$ is $\mathcal{F}_T$ measurable. At each exercise time τ_i, he receives a payoff ξ_{τ_i}. The forward prices with maturity T are given by $S = (S(t,T))_{0 \leq t \leq T}$ and $(r_t)_{0 \leq t \leq T}$ denotes the short rate.

In the general case, the payoff of the option is:

$$\text{Pay-off(Swing)} = \sum_{i\geq 1} \xi_{\tau_i} \mathbf{1}_{\{\tau_i \leq T\}} + \mathcal{P}_T$$

where the set of exercise strategies τ_i is defined in the following section.

A typical penalty function is of the form

$$\mathcal{P}_T = \mathcal{P}^{min}(S_T) \left(n - \sum_{i\geq 1} \mathbf{1}_{\{\tau_i < T\}} \right)_+ + \mathcal{P}^{Max}(S_T) \left(\sum_{i\geq 1} \mathbf{1}_{\{\tau_i < T\}} - N \right)_+$$

where $\{. \mapsto \mathcal{P}^{min}(.)\}$ and $\{. \mapsto \mathcal{P}^{Max}(.)\}$ are two deterministic functions representing, respectively, the penalty for not using enough exercise rights and for using too many exercise rights.

According to asset pricing theory, the price of this option is given by:

$$\sup_{\tau_i} \mathbf{E} \left[\sum_{i\geq 1} \xi_{\tau_i} e^{-\int_0^{\tau_i} r_s ds} \mathbf{1}_{\{\tau_i \leq T\}} + \mathcal{P}_T e^{-\int_0^T r_s ds} \right] \tag{2.1}$$

The penalty $\mathcal{P}_T$ can be decomposed into three parts. If we look at the first n rights, for every single right not exercised before maturity, a penalty $\mathcal{P}^{min}(S_T)$ has to be paid. Then, the next $N - n$ rights can be exercised or not without involving any penalty. Every right exercised after that will cost the option owner a penalty $\mathcal{P}^{Max}(S_T)$ (this is the third part). We will write $\tau_i = \infty$ if the i^{th} right is not used. We can rewrite the swing option pay-off as follows:

$$\begin{aligned} \text{Pay-off(Swing)} &= \sum_{i=1}^{n} \left(\xi_{\tau_i} \mathbf{1}_{\{\tau_i < T\}} + \mathcal{P}^{min}(S_T) \mathbf{1}_{\{\tau_i = \infty\}} \right) \\ &\quad + \sum_{i=n+1}^{N} \xi_{\tau_i} + \sum_{i\geq N+1} \xi_{\tau_i} + \mathcal{P}^{Max}(S_T) \end{aligned}$$

or

$$\text{Pay-off(Swing)} = \text{Pay-off(Swing)}_1 + \text{Pay-off(Swing)}_2 + \text{Pay-off(Swing)}_3$$

with

$$\text{Pay-off(Swing)}_1 = \sum_{i=1}^{n} \left(\xi_{\tau_i} \mathbf{1}_{\{\tau_i < T\}} + \mathcal{P}^{min}(S_T) \mathbf{1}_{\{\tau_i = \infty\}} \right)$$

$$\text{Pay-off(Swing)}_2 = \sum_{i=n+1}^{N} \xi_{\tau_i}$$

$$\text{Pay-off(Swing)}_3 = \sum_{i \geq N+1} \left(\xi_{\tau_i} + \mathcal{P}^{Max}(S_T) \right)$$

In the next two sections, we concentrate on the valuation of swing options with pay-off of type 1. Later we show how to extend the methodology to options with pay-offs 2 and 3.

We will refer to $\mathcal{P}_T$ as the penalty to be paid at time T for every exercise right not used (i.e., $\mathcal{P}^{min}(S_T) = \mathcal{P}_T$). The price of the type 1 swing option is given by:

$$\sup_{\tau_i} \mathbf{E} \left[\sum_{i=1}^{n} \xi_{\tau_i} e^{-\int_0^{\tau_i} rsds} \mathbf{1}_{\{\tau_i < T\}} + \mathcal{P}_T e^{-\int_0^T rsds} \mathbf{1}_{\{\tau_i = \infty\}} \right] \tag{2.2}$$

We see that the pricing of swing contracts is an optimal stopping problem. In the following sections of this chapter, we explain how to use Reflected Backward SDE's to solve optimal stopping problems.

2.3 Notation and Useful Theorems

Let $\mathbf{T} > 0$ be a fixed and finite time horizon. Let $B = (B_t)_{t\geq 0}$ be a standard Brownian motion defined on $(\Omega, \mathbb{F}, \mathbb{P})$, a filtered probability space satisfying the "usual hypotheses." (See [71] for a definition of the *usual hypotheses*.) $\mathbb{F} = (\mathcal{F}_t)_{0\leq t\leq \mathbf{T}}$ is the natural filtration of B and $\mathbb{P}$ is the risk neutral probability. We are not dealing here with incomplete markets; we assume that the pricing measure exists and is unique.

Swing options are typical contracts on the energy and power market: electricity, natural gas, etc. However, not all such underlyings can be treated in the same fashion. Because of its non-storability, we cannot evaluate contracts on electricity by using a traditional cash and carry strategy. However, we can price derivatives by building a (super)replicating portfolio, including forward contracts and a risk free money market account (see for example Jarrow and Turnbull [49]). Therefore, in order for the following analysis to be as general as possible, the underlying of the swing option will be a forward contract on some commodity with expiration date greater than the maturity of the swing option.

Let $S = (S(t,T))_{0\leq t\leq T}$ and $(r_t)_{0\leq t\leq T}$ be respectively the $\mathbb{F}$ adapted forward contract with maturity T and the short rate. For simplicity of notation, we will refer to $S(t,T)$ as S_t. A swing contract will pay $\phi(S_\tau)$ at exercise time τ for some deterministic function ϕ.

Let $\mathcal{T}$ be the set of all $\mathbb{F}$ stopping times bounded by T and denote $\mathcal{T}_t = \{\tau \in \mathcal{T} : \;\; t \leq \tau \leq T\}$.

Let $\mathcal{S}_t = \{\tau_1 < \tau_2 < ..., \;\; \tau_i > \tau_{i-1} + \delta, \;\; \tau_i \in \mathcal{T}_t \bigcup \{\infty\}\}$ be the set of all admissible exercise strategies for a swing option. We say that $\tau_i = \infty$ when the i^{th} right is not exercised.

Finally, let us define the following sets:

$$\mathbb{L}^2 = \left\{X \in \mathbb{R}; X \in \mathcal{F}_T\text{-measurable such that } \mathbf{E}|X_t|^2 < +\infty\right\}$$

$$\mathbb{H}^2 = \left\{(H_t)_{0\leq t\leq T} \text{ is predictable and } \mathbf{E}\int_0^T |H_t|^2 dt < +\infty\right\}$$

$$\mathbb{S}^2 = \left\{(S_t)_{0\leq t\leq T} \text{ is progressive and } \mathbf{E}\left(\sup_{0\leq t\leq T} |S_t|^2\right) < +\infty\right\}$$

The next two theorems are essential in the proof of our result. Proofs of Theorems 12 and 13 can be found in [27]. The first theorem establishes the existence and uniqueness of a solution. The second is a comparison theorem.

Theorem 12. *Let the obstacle $\xi = (\xi_t)_{0\leq t\leq T}$ be a continuous process in $\mathbb{S}^2$. There exists a unique $\mathcal{F}$ progressively measurable process $\{(Y_t, Z_t, K_t)_{0\leq t\leq T}\}$ in $\mathbb{S}^2 \times \mathbb{H}^2 \times \mathbb{L}^2$ satisfying the following Reflected BSDE:*

$$\begin{cases} Y_t = \xi_T \vee \mathcal{P}_T + \int_t^T r_s Y_s ds - \int_t^T Z_s dB_s + K_T - K_t \quad 0 \leq t \leq T \\ Y_t \geq \xi_t \quad 0 \leq t \leq T \\ K \textit{ is continuous, increasing, and starts at 0, } \mathcal{P}_T \textit{ is the penalty, and:} \\ \int_0^T (Y_t - \xi_t) dK_t = 0 \end{cases} \tag{2.3}$$

Theorem 13. *Let ξ, ξ' be two obstacles in $\mathbb{S}^2$ satisfying*

$$\xi_t \leq \xi_t^{'} \;\; 0 \leq t \leq T \textit{ a.s.}$$

Let (Y,Z,K) be the solution of the Reflected BSDE with obstacle ξ, let (Y',Z',K') be the solution of the Reflected BSDE with obstacle ξ'. Then

$$Y_t \leq Y_t^{'} \;\; 0 \leq t \leq T \text{ a.s.}$$

For clarity in the proof, we will assume that there is no interest rate, i.e. $r_t = 0,\ \forall t \in [0,T]$.

Let us look at the relation between Snell envelopes and Reflected Backward SDE's. This is well known (see for example [28]), but we will use the techniques of proof repeatedly in what follows, so we include a brief treatment of the fundamental issues. We begin with two lemmas:

Lemma 14. *Let $(Y_t)_{0\leq t\leq T}$ be the solution of the following Reflected BSDE:*

$$\begin{cases} Y_t = \xi_T - \int_t^T Z_s^Y dB_s + K_T^Y - K_t^Y \quad 0 \leq t \leq T \\ Y_t \geq \xi_t \quad 0 \leq t \leq T \\ K^Y \text{ is a continuous and increasing process starting at 0} \\ \int_0^T (Y_t - \xi_t) dK_t^Y = 0 \end{cases} \tag{2.4}$$

then

$$Y_t \geq \mathbf{E}\left[\xi_v | \mathcal{F}_t\right] \text{ for every stopping time } v \in \mathcal{T}_t.$$

Proof. From (2.3) we have

$$Y_t = Y_v - \int_t^v Z_s dB_s + K_v - K_t$$

We know that Z is an adapted $\mathbb{H}^2$ process (see Theorem 12) and therefore $\int_t^v Z_s dB_s$ is a martingale starting at 0. We deduce

$$Y_t = \mathbf{E}\left[Y_v + K_v - K_t | \mathcal{F}_t\right]$$

However, K is an increasing process: $K_v - K_t \geq 0$ and Y dominates ξ: $Y_t \geq \xi_t, 0 \leq t \leq T$. Therefore,

$$Y_t \geq \mathbf{E}\left[\xi_v | \mathcal{F}_t\right]$$

□

Lemma 15. *Let $(Y_t)_{0 \leq t \leq T}$ be the solution of the following Reflected BSDE:*

$$\begin{cases} Y_t = \xi_T - \int_t^T Z_s^Y dB_s + K_T^Y - K_t^Y \quad 0 \leq t \leq T \\ Y_t \geq \xi_t \quad 0 \leq t \leq T \\ K^Y \text{ is a continuous and increasing process starting at 0} \\ \int_0^T (Y_t - \xi_t) dK_t^Y = 0 \end{cases} \tag{2.5}$$

then for all $t \in [0, T]$

$$Y_t = \mathbf{E}\left[\xi_\tau | \mathcal{F}_t\right] \text{ with } \tau = \inf\{t \leq v \leq T : Y_v = \xi_v\}$$

Proof. We have seen that for every stopping time v such that $t \leq v \leq T$

$$Y_t = Y_v - \int_t^v Z_s dB_s + K_v - K_t$$

Let the stopping time $\tau = \inf\{t \leq v \leq T : Y_v = \xi_v\}$. We know that

$$\int_0^\tau (Y_s - \xi_s) dKs = 0$$

and

$$(Y_t - \xi_t) > 0 \quad \text{for all} t < \tau$$

We deduce that $K_\tau - K_t = 0$.

Substituting in Equation (2.3), we obtain

$$Y_t = Y_\tau - \int_t^\tau Z_s dB_s$$

and

$$Y_t = \mathbf{E}\left[Y_\tau|\mathcal{F}_t\right]$$

□

Theorem 16. *Let $\{(Y_t, Z_t, K_t)_{0\leq t\leq T}\}$ be the solution of the following Reflected BSDE:*

$$\begin{cases} Y_t = \xi_T - \int_t^T Z_s dB_s + K_T - K_t \quad 0 \leq t \leq T \\ Y_t \geq \xi_t \quad 0 \leq t \leq T \\ K \textit{ is a continuous and increasing process starting at 0} \\ \int_0^T (X_t - \xi_t) dK_t = 0 \end{cases} \tag{2.6}$$

then

$$Y_t = ess\sup_{\tau\in\mathcal{T}_t} \mathbf{E}\left[\xi_\tau|\mathcal{F}_t\right] \tag{2.7}$$

Proof. The proof is a simple combination of the two previous lemmas. □

Remark 2. *We can see in Theorem* 16 *that the obstacle of a reflected backward SDE can be interpreted as the instantaneous payoff of an American option. The solution $\{(Y_t)_{0\leq t\leq T}\}$ of the equation gives the price of the option. The optimal exercise time is the first time the price of the option equals the payoff (see Lemma* 15*). This is well known, of course; see for example [28].*

2.4 Valuation of Multiple Stopping Options

2.4.1 Simple Case

Let us first study a simplified version of the problem: The buyer of the option can exercise up to N rights, he has to wait at least δ units of time between any two exercises. Any remaining rights can be exercised at maturity. This case corresponds to $n = N$ and $\mathcal{P}_T = 0$ in the general framework with no additional exercise rights. The pay-off profile of this option is similar to the general case:

$$\text{Pay-off(N-Swing)} = \sum_{i=1}^{N} \xi_{\tau_i} \quad \tau_i \in \mathcal{S}_t, i = 1..N$$

Where $\mathcal{S}_t = \{\tau_i \in \mathcal{T}_t, \tau_{i+1} - \tau_i \geq \delta, i = 2..N\} \bigcup \{\tau_i = T\}$ is the modified set of all possible exercise strategies.

Let us first look at the case of a single exercise right. At time t, the price of the option is the price of an American contingent claim:

$$Y_t = ess \sup_{\tau \in \mathcal{T}_t} \mathbf{E}\left[\xi_\tau | \mathcal{F}_t\right]$$

Theorem 16 tells us that Y is the solution of the following Reflected BSDE:

$$
\begin{cases}
Y_t = \xi_T - \int_t^T Z_s^Y dB_s + K_T^Y - K_t^Y \quad 0 \le t \le T \\
Y_t \ge \xi_t \quad 0 \le t \le T \\
K^Y \text{ is a continuous and increasing process starting at } 0 \\
\int_0^T (Y_t - \xi_t) dK_t^Y = 0
\end{cases}
\tag{2.8}
$$

We want to extend the previous idea to the case of 2-swing options (i.e., swing option with 2 exercise rights). We will prove that the price process of this option can be obtained as the solution of a reflected BSDE by choosing the obstacle correctly.

We have seen that the obstacle of a reflected backward SDE can be interpreted as the instantaneous payoff of an American option. If we want to use reflected BSDE's to price a swing option, we have to evaluate it as an optimal stopping problem with only one exercise. We need to embed the remaining exercise right as an option delivered at the first exercise time. However, this embedded option is itself a swing option with one exercise right less, and in the case of $N = 2$, it is an American option.

Therefore, the value of the swing option at the first exercise time τ_1 is the instantaneous payoff at that time (i.e. the usual payoff ξ_{τ_1}) plus the value of the right to exercise one more time before maturity. The financial value of this embedded right is the price at time τ_1 of an American option starting at time $\tau_1 + \delta$ (the owner of the option has to wait at least δ units of time before exercising his second right) with pay-off $(. \mapsto \xi_.)$ and maturity T. We know from the previous section that this price is given by $\mathbf{E}\left[Y_{t+\delta} | \mathcal{F}_t\right]$, where Y_t is the solution of (2.8).

Hence, by defining a new obstacle

$$\zeta_\tau = \xi_t + \mathbf{E}\left[Y_{t+\delta}|\mathcal{F}_t\right]\mathbf{1}_{\{0\leq t\leq T-\delta\}} + \mathbf{E}\left[\xi_T|\mathcal{F}_t\right]\mathbf{1}_{\{T-\delta<t\leq T\}}$$

we can introduce the process $\left((X_t, Z^X, K_t^X)_{0\leq t\leq T}\right)$ which is the solution of:

$$\begin{cases} X_t = \zeta_T - \int_t^T Z_s^X dB_s + K_T^X - K_t^X \quad 0\leq t\leq T \\ X_t \geq \zeta_t \quad 0\leq t\leq T \\ K^X \text{ is a continuous and increasing process starting at 0} \\ \int_0^T (X_t - \zeta_t)dK_t^X = 0 \end{cases} \tag{2.9}$$

We want to prove that the solution of (2.9) gives us the price of the 2-swing option.

Remark 3. *The process ζ is continuous on $[0,T]$:*

$$\zeta_{T-\delta} = \mathbf{E}\left[Y_T|\mathcal{F}_{T-\delta}\right] = \mathbf{E}\left[\xi_T|\mathcal{F}_{T-\delta}\right] = \lim_{t\searrow T}\mathbf{E}\left[\xi_t|\mathcal{F}_{T-\delta}\right] = \lim_{t\searrow T}\zeta_{t+\delta} \tag{2.10}$$

Lemma 17. *Let $\left((X_t, Z^X, K_t^X)_{0\leq t\leq T}\right)$ be the solution of:*

$$\begin{cases} X_t = \zeta_T - \int_t^T Z_s^X dB_s + K_T^X - K_t^X \quad 0\leq t\leq T \\ X_t \geq \zeta_t \quad 0\leq t\leq T \\ K^X \textit{ is a continuous and increasing process starting at 0} \\ \int_0^T (X_t - \zeta_t)dK_t^X = 0 \end{cases} \tag{2.11}$$

then

$X_t \geq \mathbf{E}\left[\xi_u + \xi_v|\mathcal{F}_t\right]$ *for all stopping times u and v such that $t\leq u < u+\delta\leq v$.*

Proof. We know from Lemma 14 that for every stoping time $v \in \mathcal{T}_t$, we have

$$X_t \geq \mathbf{E}\left[\zeta_u | \mathcal{F}_t\right]$$

However

$$\mathbf{E}\left[\zeta_u | \mathcal{F}_t\right] = \mathbf{E}\left[\xi_u + \mathbf{E}\left[Y_{u+\delta} | \mathcal{F}_u\right] | \mathcal{F}_t\right] = \mathbf{E}\left[\xi_u + Y_{u+\delta} | \mathcal{F}_t\right]$$

and

$$Y_{u+\delta} \geq \mathbf{E}\left[\xi_v | \mathcal{F}_{u+\delta}\right]$$

for every stopping time v such that $u + \delta \leq v \leq T$. □

We are ready to state the main result.

Theorem 18. *Let* $\left((X_t, Z^X, K_t^X)_{0 \leq t \leq T}\right)$ *be the solution of:*

$$\begin{cases} X_t = \zeta_T - \int_t^T Z_s^X dB_s + K_T^X - K_t^X \quad 0 \leq t \leq T \\ X_t \geq \zeta_t \quad 0 \leq t \leq T \\ K^X \textit{ is a continuous and increasing process starting at 0} \\ \int_0^T (X_t - \zeta_t) dK_t^X = 0 \end{cases} \tag{2.12}$$

then

$$X_t = ess \sup_{\tau_1, \tau_2 \in \mathcal{S}_t} \mathbf{E}\left[\xi_{\tau_1} + \xi_{\tau_2} | \mathcal{F}_t\right]$$

Proof. Let $P_t = ess \sup_{\mathcal{S}_t} \mathbf{E}\left[\xi_{\tau_1} + \xi_{\tau_2} | \mathcal{F}_t\right]$

We know (Lemma 17) that $X_t \geq P_t$, and by applying Lemma 15 to the reflected BSDE (2.13), we have

$$X_t = \mathbf{E}\left[\zeta_\tau | \mathcal{F}_t\right]$$

with

$$\tau = \inf\{t \leq u \leq T : X_u = \zeta_u\}.$$

Recall that

$$\zeta_\tau = \xi_t + \mathbf{E}\left[Y_{t+\delta}|\mathcal{F}_t\right]\mathbf{1}_{\{0\leq t\leq T-\delta\}} + \mathbf{E}\left[\xi_T|\mathcal{F}_t\right]\mathbf{1}_{\{T-\delta<t\leq T\}}.$$

We obtain

$$X_t = \mathbf{E}\left[\xi_\tau + \mathbf{E}\left[Y_{\tau+\delta}|\mathcal{F}_\tau\right]\mathbf{1}_{\{0\leq\tau\leq T-\delta\}} + \mathbf{E}\left[\xi_T|\mathcal{F}_\tau\right]\mathbf{1}_{\{T-\delta<\tau\leq T\}}|\mathcal{F}_t\right],$$

or equivalently

$$X_t = \mathbf{E}\left[\xi_\tau + Y_{\tau+\delta}\mathbf{1}_{\{0\leq\tau\leq T-\delta\}} + \xi_T\mathbf{1}_{\{T-\delta<\tau\leq T\}}|\mathcal{F}_t\right].$$

By applying Lemma 15 to the reflected BSDE (2.5), we have

$$Y_{\tau+\delta} = \mathbf{E}\left[\xi_{\tilde{\tau}}|\mathcal{F}_{\tau+\delta}\right]$$

with

$$\tilde{\tau} = \inf\{\tau+\delta \leq v \leq T : Y_v = \xi_v\}.$$

Note that $(\tau,\tilde{\tau}) \in \mathcal{S}_t$ and $X_t = \mathbf{E}\left[\xi_\tau + \xi_{\tilde{\tau}}|\mathcal{F}_t\right]$:

$$\begin{aligned} X_t &= \mathbf{E}\left[\xi_\tau + Y_{\tau+\delta}\mathbf{1}_{\{0\leq\tau\leq T-\delta\}} + \xi_T\mathbf{1}_{\{T-\delta<\tau\leq T\}}|\mathcal{F}_t\right] \\ &= \mathbf{E}\left[\xi_\tau + \mathbf{E}\left[\xi_{\tilde{\tau}}|\mathcal{F}_{\tau+\delta}\right]\mathbf{1}_{\{0\leq\tau\leq T-\delta\}} + \xi_T\mathbf{1}_{\{T-\delta<\tau\leq T\}}|\mathcal{F}_t\right] \\ &= \mathbf{E}\left[\xi_\tau + \xi_{\tilde{\tau}}\mathbf{1}_{\{0\leq\tau\leq T-\delta\}} + \xi_T\mathbf{1}_{\{T-\delta<\tau\leq T\}}|\mathcal{F}_t\right] \end{aligned}$$

We conclude that

$$X_t = ess\sup_{\mathcal{S}_t}\mathbf{E}\left[\xi_{\tau_1} + \xi_{\tau_2}|\mathcal{F}_t\right]$$

□

Remark 4. *It is straightforward to generalize this theorem to the case of N exercise times.*

2.4.2 General Case: $\mathcal{P}_T \neq 0$

We have previously seen that to use reflected Backward SDEs to price a 2-swing option, it is convenient to evaluate it as an optimal stopping problem with only one exercise. We need to embed the remaining exercise right as an option delivered at the first exercise time. However, this embedded option is an American option (with only one exercise right).

Therefore, the value of the 2-swing option at the first exercise τ_1 is the instantaneous payoff at that time (i.e. the usual payoff ξ_{τ_1}) plus the value of the right to exercise one more time before maturity. The financial value of this embedded right is the price at time τ_1 of a 1-swing option starting at time $\tau_1 + \delta$ (the purchaser has to wait at least δ units of time before exercising the second right) with pay-off $(. \mapsto \xi_.)$ and maturity T. We know that this value is given by $\mathbf{E}\left[Y_{t+\delta}|\mathcal{F}_t\right]$, where

$$Y_t = ess \sup_{\tau \in \mathcal{T}_t} \mathbf{E}\left[\xi_\tau|\mathcal{F}_t\right]$$

and

$$\xi_t = \phi(S_t)\mathbf{1}_{\{0 \leq t < T\}} + (\phi(S_t) \vee \mathcal{P}_T)\mathbf{1}_{\{t=T\}}$$

If the first right is exercised at time τ_1 before $T - \delta$, the second right can be exercised at any time τ_2 between $\tau_1 + \delta$ and T. The value of this right at time τ_1 is given by $\mathbf{E}\left[Y_{\tau_1+\delta}|\mathcal{F}_{\tau_1}\right]$. If the first right is exercised after $T - \delta$, the second right cannot be exercised anymore and a penalty $\mathcal{P}_T$ has to be paid at maturity T. The value of this penalty at time t is given by $\mathbf{E}\left[\mathcal{P}_T|\mathcal{F}_t\right]$.

Next we define a new obstacle ζ representing the effective instantaneous payoff of a 2-swing option:

$$\zeta_t = \xi_t + \mathbf{E}\left[Y_{t+\delta}|\mathcal{F}_t\right]\mathbf{1}_{\{0\le t\le T-\delta\}} + \mathbf{E}\left[\mathcal{P}_T|\mathcal{F}_t\right]\mathbf{1}_{\{T-\delta<t\le T\}}.$$

Remark 5. *ζ is continuous in $[0,T-\delta]\cup(T-\delta,T]$ but*

$$\zeta_{T-\delta} = \xi_{T-\delta} + \mathbf{E}\left[\xi_T \vee \mathcal{P}_T|\mathcal{F}_{T-\delta}\right] \ge \xi_{T-\delta} + \mathbf{E}\left[\mathcal{P}_T|\mathcal{F}_{T-\delta}\right] = \zeta_{(T-\delta)^+}.$$

Where $\zeta_{(T-\delta)^+} = \lim_{t\searrow T-\delta}\zeta_t$, the right continuous version of ζ at time $T-\delta$

Because of the discontinuity, we cannot use the same techniques we used to prove the result in the simple case. Instead we have:

Theorem 19. *Let $\zeta = (\zeta_t)_{0\le t<T}$ be the obstacle defined above, then there exists a unique $\mathcal{F}$ progressively measurable process $\{(X_t, Z_t, K_t)_{0\le t\le T}\}$ in $\mathbb{S}^2\times\mathbb{H}^2\times\mathbb{L}^2$, and satisfying the following Reflected BSDE:*

$$\begin{cases} X_t = \zeta_T - \int_t^T Z_s dB_s + K_T - K_t \quad 0\le t\le T \\ X_t \ge \zeta_t \quad 0\le t\le T \\ K = K^c + K^d \\ K^c \text{ is a continuous and increasing process starting at 0} \\ (X_t-\zeta_t)dK_t^c = 0 \quad 0\le t\le T \\ K_t^d = -\mathbf{1}_{\{t\ge T-\delta\}}\max\left(\zeta_{T-\delta} - X_{(T-\delta)^+}, 0\right) \end{cases} \tag{2.13}$$

Moreover, for all $t\in[0,T]$

$$X_t = ess\sup_{\tau\in\mathcal{T}_t}\mathbf{E}\left[\zeta_\tau|\mathcal{F}_t\right] \tag{2.14}$$

Remark 6. *In the previous Theorem, the process K^d is a pure jump process that can be written as:*

$$K_t^d = -\mathbf{1}_{\{t \geq T-\delta\}} \left(\zeta_{T-\delta} - X_{(T-\delta)^+}\right)^+ = -(X_t - X_{t^+})^+$$

The sketch of the proof is as follows:

- We will first construct $\left\{(X_t^{T-\delta,T}, Z_t^{T-\delta,T}, K_t^{T-\delta,T}),\ T-\delta < t \leq T\right\}$, the solution of equation (2.13) on $(T-\delta, T]$.
- We will then construct $\left\{(X_t^{0,T-\delta}, Z_t^{0,T-\delta}, K_t^{0,T-\delta}),\ 0 \leq t \leq T-\delta\right\}$, the solution of equation (2.13) on $[0, T-\delta]$ with final condition $X_{T-\delta}^{0,T-\delta} = \zeta_{T-\delta} \vee X_{(T-\delta)^+}^{T-\delta,T}$
- With the two previous solutions, we will define a process on $[0,T]$ and we will use the properties of the Snell envelope to prove that this process solves (2.14) on $[0,T]$, and verifies (2.13).

This proof is inspired by the work of Hamadène [41].

Proof. The process ζ is continuous on $(T-\delta, T]$, therefore by Theorem 12 we have the existence of the solution $\left\{(X_t^{T-\delta,T}, Z_t^{T-\delta,T}, K_t^{T-\delta,T}),\ T-\delta < t \leq T\right\}$ in $\mathbb{S}^2 \times \mathbb{H}^2 \times \mathbb{L}^2$ of the following Reflected BSDE:

$$\begin{cases} dX_t = -Z_s dB_s - dK_t \quad T-\delta < t \leq T \\ X_T = \zeta_T \\ X_t \geq \zeta_t \quad T-\delta < t \leq T \\ K \text{ is a continuous and increasing process starting at } 0 \\ (Y_t - \zeta_t) dK_t = 0 \quad T-\delta < t \leq T \end{cases}$$

We can now define a new final condition $\zeta_{T-\delta} \vee X^{T-\delta,T}_{(T-\delta)^+}$.

The process ζ is continuous on $[0, T-\delta]$, therefore by Theorem 12 we have the existence of the solution $\left\{(X^{0,T-\delta}_t, Z^{0,T-\delta}_t, K^{0,T-\delta}_t),\ 0 \leq t \leq T-\delta\right\}$ in $\mathbb{S}^2 \times \mathbb{H}^2 \times \mathbb{L}^2$ of the following Reflected BSDE:

$$\begin{cases} dX_t = -Z_s dB_s - dK_t \quad 0 \leq t \leq T-\delta \\ X_{T-\delta} = \zeta_{T-\delta} \vee X^{T-\delta,T}_{T-\delta} \\ X_t \geq \zeta_t \quad 0 \leq t \leq T-\delta \\ K \text{ is a continuous and increasing process starting at } 0 \\ (X_t - \zeta_t) dK_t = 0 \quad 0 \leq t \leq T-\delta \end{cases}$$

Let us define $\{(X_t, Z_t, K_t),\ 0 \leq t \leq T\}$ such that:

$$X_t = X^{0,T-\delta}_t \mathbf{1}_{\{0 \leq t \leq T-\delta\}} + X^{T-\delta,T}_t \mathbf{1}_{\{T-\delta < t \leq T\}}$$

$$Z_t = Z^{0,T-\delta}_t \mathbf{1}_{\{0 \leq t \leq T-\delta\}} + Z^{T-\delta,T}_t \mathbf{1}_{\{T-\delta < t \leq T\}}$$

$$K^c_t = K^{0,T-\delta}_{t \wedge (T-\delta)} + K^{T-\delta,T}_t \mathbf{1}_{\{T-\delta < t \leq T\}}$$

$$K^d_t = -\mathbf{1}_{\{t \geq T-\delta\}} \left(\zeta_{T-\delta} - X^{T-\delta,T}_{(T-\delta)^+}\right)^+$$

It is easy to see that $\{(X_t, Z_t, K_t)_{0 \leq t \leq T}\}$ is in $\mathbb{S}^2 \times \mathbb{H}^2 \times \mathbb{L}^2$ and that it is a solution of (2.13).

Let $J(\zeta)_t = ess \sup_{\tau \in \mathcal{T}_t} \mathbf{E}\left[\zeta_\tau | \mathcal{F}_t\right]$. It remains to prove that $X_t = J(\zeta)_t$ for all $t \in [0,T]$.

By applying Theorem 16 we have that for all $t \in (T-\delta, T]$:

$$X_t^{T-\delta,T} = ess \sup_{\tau \in \mathcal{T}_t} \mathbf{E}\left[\zeta_\tau | \mathcal{F}_t\right]$$

and for every $t \in [0, T-\delta]$:

$$X_t^{0,T-\delta} = ess \sup_{\tau \in \mathcal{T}_t^{T-\delta}} \mathbf{E}\left[\zeta_\tau \mathbf{1}_{\{\tau < T-\delta\}} + (\zeta_{T-\delta} \vee X_{T-\delta}^{T-\delta,T})\mathbf{1}_{\{\tau = T-\delta\}} | \mathcal{F}_t\right]$$

where $\mathcal{T}_t^{T-\delta} = \{\tau \in \mathcal{T}_t; \tau \leq T-\delta\}$.

The processes $\left\{(X_t^{0,T-\delta})_{0 \leq t \leq T-\delta}\right\}$ and $\left\{(X_t^{T-\delta,T})_{T-\delta < t \leq T}\right\}$ are supermartingales (properties of the Snell Envelope). Therefore $\{(X_t)_{0 \leq t \leq T}\}$ is a supermartingale.

The process $J(\zeta)$ is the Snell envelope of ζ, and therefore it's the smallest supermartingale that dominates ζ (Theorem 10). However, the process X has been defined such that $X_t \geq \zeta_t$, for all $t \in [0,T]$. We can conclude using again Theorem 10 that $X \geq J(\zeta)$.

We already know that $J(\zeta) = X$ on $(T-\delta, T]$, we want to prove now that $J(\zeta) \geq X$ on $[0, T-\delta]$. By Theorem 13, we have that X is the Snell envelope of

$$\zeta_t \mathbf{1}_{\{0 \leq t < T-\delta\}} + (\zeta_{T-\delta} \vee X_{(T-\delta)^+})\mathbf{1}_{\{t = T-\delta\}}$$

and is the smallest supermartingale that dominates it.

By construction, we have that $J(\zeta)_t \geq \zeta_t$ for $t \in [0, T-\delta)$ and $J(\zeta)_{T-\delta} \geq J(\zeta)_{(T-\delta)^+}$.

Indeed, $\mathcal{T}_{T-\delta} \supset \mathcal{T}_t$ for $t > T-\delta$ and

$$J(\zeta)_{T-\delta} = ess \sup_{\tau \in \mathcal{T}_{T-\delta}} \mathbf{E}\left[\zeta_\tau | \mathcal{F}_t\right] \geq \lim_{t > (T-\delta)^+} ess \sup_{\tau \in \mathcal{T}_t} \mathbf{E}\left[\zeta_\tau | \mathcal{F}_t\right] = J(\zeta)_{(T-\delta)^+}$$

Therefore, for $t \in [0, T-\delta)$, $J(\zeta)_t \geq \zeta_t \mathbf{1}_{\{0 \leq t < T-\delta\}} + (\zeta_{T-\delta} \vee X_{(T-\delta)^+})\mathbf{1}_{\{t = T-\delta\}}$.

We conclude by recalling that $J(\zeta)$ is a supermartingale.

□

We show that the solution of (2.13) is indeed the price process of a 2-swing option.

Theorem 20. *Let $\zeta = (\zeta_t)_{0 \leq t < T}$ be the obstacle defined above, and let $\{(X_t, Z_t, K_t)_{0 \leq t \leq T}\}$ in $\mathbb{S}^2 \times \mathbb{H}^2 \times \mathbb{L}^2$ be the solution of the following Reflected BSDE:*

$$\begin{cases} dX_t = -Z_s dB_s - dK_t \quad 0 \leq t \leq T-\delta \\ X_{T-\delta} = \zeta_{T-\delta} \vee X_{T-\delta}^{T-\delta, T} \\ X_t \geq \zeta_t \quad 0 \leq t \leq T-\delta \\ K \text{ is a continuous and increasing process starting at 0} \\ (X_t - \zeta_t) dK_t = 0 \quad 0 \leq t \leq T-\delta \end{cases}$$

Then for all $t \in [0, T]$,

$$X_t = ess \sup_{\tau_1, \tau_2 \in \mathcal{S}_t} \mathbf{E}\left[\xi_{\tau_1} + \xi_{\tau_2} + \xi_{\tau_2} | \mathcal{F}_t\right] \tag{2.15}$$

Proof. We already know that for all $t \in [0, T]$ and all $\tau \in \mathcal{T}_t$:

$$X_t \geq \mathbf{E}\left[\zeta_\tau|\mathcal{F}_t\right] = \mathbf{E}\left[\xi_\tau + Y_{\tau+\delta}\mathbf{1}_{\{\tau \leq T-\delta\}} + \mathcal{P}_T\mathbf{1}_{\{\tau > T-\delta\}}|\mathcal{F}_t\right]$$

The process Y is the Snell envelope of ξ and therefore $Y_{\tau+\delta} \geq \mathbf{E}\left[\xi_\nu|\mathcal{F}_{\tau+\delta}\right]$ for all $\nu \in \mathcal{T}_{\tau+\delta}$.

We obtain that, for every $(\tau, \nu) \in \mathcal{S}_t$:

$$X_t \geq \mathbf{E}\left[\xi_\tau + \xi_\nu\mathbf{1}_{\{\tau \leq T-\delta\}} + \mathcal{P}_T\mathbf{1}_{\{\tau > T-\delta\}}|\mathcal{F}_t\right]$$

and therefore

$$X_t \geq ess \sup_{\tau_1, \tau_2 \in \mathcal{S}_t} \mathbf{E}\left[\xi_\tau + \xi_\nu\mathbf{1}_{\{\tau \leq T-\delta\}} + \mathcal{P}_T\mathbf{1}_{\{\tau > T-\delta\}}|\mathcal{F}_t\right]$$

Let $D_t^1 = \inf\{u \leq t \; : \; X_t = \zeta_t\} \wedge T$. From equation (1), we have that:

$$X_t = X_{D_t^1} - \int_t^{D_t^1} Z_s dB_s + K_{D_t^1} - K_t$$

We know from equation (2.13) that $K^c_{D_t^1} - K^c_t = 0$, and we want to prove that $K_{D_t^1} - K_t = 0$.

On $\{D_t^1 = T\}$, we have that $X_T > \zeta_T$, $\mathcal{P}_T > \xi_T$ and therefore K is continuous (cf the proof of Lemma 15).

On $\{T - \delta < D_t^1 < T\}$, we have that $X_{T-\delta} > \zeta_{T-\delta}$ and therefore K is continuous(cf the proof of Lemma 15).

On $\{D_t^1 \leq T-\delta\}$, we have that K is continuous on $[0, D_t^1]$.

Knowing that the process Z is in $\mathbb{H}^2$, we can conclude that:

$X_t = \mathbf{E}\left[X_{D_t^1}|\mathcal{F}_t\right] = \mathbf{E}\left[\zeta_{D_t^1}|\mathcal{F}_t\right] = \mathbf{E}\left[\xi_{D_t^1} + Y_{D_t^1+\delta}\mathbf{1}_{\{D_t^1\leq T-\delta\}} + \mathcal{P}_T\mathbf{1}_{\{D_t^1>T-\delta\}}|\mathcal{F}_t\right]$

From Lemma (14), we have that $Y_{D_t^1+\delta} = \mathbf{E}\left[\xi_{D_t^2}|\mathcal{F}_t\right]$ with

$$D_t^2 = \inf\{u \geq D_t^1+\delta \;:\; Y_t = \xi_t\} \wedge T$$

We obtain

$$X_t = \mathbf{E}\left[\xi_{D_t^1} + \xi_{D_t^2}\mathbf{1}_{\{D_t^1\leq T-\delta\}} + \mathcal{P}_T\mathbf{1}_{\{D_t^1>T-\delta\}}|\mathcal{F}_t\right]$$

and $(D_t^1, D_t^2) \in \mathcal{S}_t$.

□

Remark 7. *We can easily generalize this theorem to the case of N exercise times.*

2.5 Exercise Region of Swing Options

We proved that the value of the 2-swing option is given by:

$$X_t = \underset{\tau_1\in\mathcal{S}}{ess\sup}\, \mathbf{E}\left[\zeta_{\tau_1}|\mathcal{F}_t\right]$$

and the value of the 1-swing option is given by:

$$Y_t = \underset{\tau_2 \in \mathcal{S}}{ess\sup} \, \mathbf{E}\left[\xi_{\tau_2} | \mathcal{F}_t\right]$$

From now on, let us assume that ξ is of the form:

$$\xi_t = \phi(S_t) + (\mathcal{P}_T - \phi(S_T))_+ \mathbf{1}_{\{t=T\}}$$

with ϕ being the pay-off function and S being the underlying forward contract. Let us define the two exercise strategies:

$$\tau_2^*(t) = \inf\{u \geq t : X_u = \zeta_u\}$$

and

$$\tau_1^*(t) = \inf\{u \geq t : Y_u = \xi_u\}$$

We proved in the previous section that $\tau_2^*(t)$ is the optimal exercise time of the first right of a 2-swing option after time t and $\tau_1^*(t)$ is the optimal exercise time of a 1-swing option after time t.

Therefore, if we sell a 1-swing option to a client A and a 2-swing option to a client B (with the same underlying and pay-off profile), then client A will exercise his first right at time $\tau_2^*(t)$ and client B will exercise at time $\tau_1^*(t)$. A natural question is: Who will exercise first? We will see that because client A has one more exercise right, it will always be optimal for him to exercise before client B.

Proposition 21. *It is always optimal to exercise the 2-swing option before the 1-swing option:* $\tau_2^*(t) \leq \tau_1^*(t) \;\; \forall t \in [0, T]$

Proof. From the properties of Snell envelopes, we know that the process $\{Y_t, 0 \leq t \leq T\}$ is a supermartingale that dominates $\{(\xi_t), 0 \leq t \leq T\}$.

We need to prove that

$$M_t^{\text{sup}} = \mathbf{E}\left[Y_{t+\delta}|\mathcal{F}_t\right]\mathbf{1}_{\{0\leq t\leq T-\delta\}} + \mathbf{E}\left[\xi_T|\mathcal{F}_t\right]\mathbf{1}_{\{T-\delta<t\leq T\}}$$

is also a supermartingale.

Let $h > 0$,

$$\begin{aligned}
\mathbf{E}\left[M_{t+h}^{\text{sup}}|\mathcal{F}_t\right] &= \mathbf{E}\left[Y_{t+\delta+h}|\mathcal{F}_t\right]\mathbf{1}_{\{t+h\leq T-\delta\}} + \mathbf{E}\left[\xi_T|\mathcal{F}_t\right]\mathbf{1}_{\{T-\delta<t+h\}} \\
&= \mathbf{E}\left[Y_{t+\delta+h}|\mathcal{F}_t\right]\left(\mathbf{1}_{\{t\leq T-\delta\}} - \mathbf{1}_{\{t\leq T-\delta\ <t+h\}}\right) \\
&\quad + \mathbf{E}\left[\xi_T|\mathcal{F}_t\right]\left(\mathbf{1}_{\{T-\delta<t\}} - \mathbf{1}_{\{t<T-\delta<t+h\}}\right)
\end{aligned}$$

However, we defined Y such that:

$$Y_{t+\delta+h}\mathbf{1}_{\{t\leq T-\delta\ <t+h\}} = \mathbf{E}\left[\xi_T|\mathcal{F}_t\right]\mathbf{1}_{\{t<T-\delta<t+h\}}$$

Therefore:

$$\begin{aligned}
\mathbf{E}\left[M_{t+h}^{sup}|\mathcal{F}_t\right] &= \mathbf{E}\left[Y_{t+\delta+h}|\mathcal{F}_t\right]\mathbf{1}_{\{t\leq T-\delta\}} + \mathbf{E}\left[\xi_T|\mathcal{F}_t\right]\mathbf{1}_{\{T-\delta<t\}} \\
&\geq \mathbf{E}\left[Y_{t+\delta}|\mathcal{F}_t\right]\mathbf{1}_{\{t\leq T-\delta\}} + \mathbf{E}\left[\xi_T|\mathcal{F}_t\right]\mathbf{1}_{\{T-\delta<t\}} \\
&\geq M_t^{\text{sup}}
\end{aligned}$$

We have that $Y_t + M_t^{\sup}$ is a supermartingale and

$$\begin{aligned} Y_t + M_t^{\sup} &\geq \xi_t + M_t^{\sup} \\ &\geq \xi_t + \mathbf{E}\left[Y_{t+\delta}|\mathcal{F}_t\right]\mathbf{1}_{\{0\leq t\leq T-\delta\}} + \mathbf{E}\left[\xi_T|\mathcal{F}_t\right]\mathbf{1}_{\{T-\delta<t\leq T\}} \\ &\geq \zeta_t \end{aligned}$$

However, $\{(X_t), 0 \leq t \leq T\}$ is the Snell envelope of $\{\zeta_t, 0 \leq t \leq T\}$ and therefore, it is the smallest supermartingale that dominates it.

We conclude that

$$Y_t + M_t^{\sup} \geq X_t \geq \zeta_t$$

From the definition of $\tau_1^*(t)$, we have that

$$Y_{\tau_1^*(t)} = \xi_{\tau_1^*(t)}$$

Substituting this into (1), we obtain

$$Y_{\tau_1^*(t)} + M_{\tau_1^*(t)}^{\sup} \geq X_{\tau_1^*(t)} \geq \zeta_{\tau_1^*(t)}$$

or equivalently

$$Y_{\tau_1^*(t)} + M_{\tau_1^*(t)}^{\sup} \geq X_{\tau_1^*(t)} \geq \xi_{\tau_1^*(t)} + M_{\tau_1^*(t)}^{\sup}$$

We conclude that

$$X_{\tau_1^*(t)} = \xi_{\tau_1^*(t)} + M^{\sup}_{\tau_1^*(t)} = \zeta_{\tau_1^*(t)}$$

However

$$\tau_2^*(t) = \inf\{u \geq t : X_u = \zeta_u\}$$

and therefore

$$\tau_2^*(t) \leq \tau_1^*(t)$$

□

The exercise region is the set of pairs (s,t) such that the expected reward at time t of the option is equal to its intrinsic value or value for immediate exercise (s is the value of the underlying forward at time t).

In the case of a 1-swing option, we obtain:

$$\mathcal{E}_1 = \left\{(s,t) \in \mathbf{R} \times \mathbf{R}^+ :\ Y(s,t) = \Phi(s)\right\}$$

Where

$$Y(t,s) = \sup_{t \leq \tau \leq T} \mathbf{E}\left[\Phi(S_\tau) | S_t = s\right]$$

The exercise region of a 2-swing option is defined by

$$\mathcal{E}_2 = \left\{(s,t) \in \mathbf{R} \times \mathbf{R}^+ :\ X(s,t) = \Phi(x) + \sup_{t+\delta \leq \eta \leq T} \mathbf{E}\left[\Phi(S_\eta) | S_t = s\right]\right\}$$

Where

$$X(t,x) = \sup_{\substack{t \le \tau \le T \\ \tau+\delta \le \eta \le T}} \mathbf{E}\left[\Phi(S_\tau) + \Phi(S_\eta) | S_t = s\right]$$

Proposition 22. *The exercise region of a 1-swing option is included in the exercise region of a 2-swing option: $\mathcal{E}_1 \subset \mathcal{E}_2$*

Proof. Let $S_{t_0} = s_0$, and $(s_0, t_0) \in \mathcal{E}_1$, we want to prove that $(s_0, t_0) \in \mathcal{E}_2$.

Let $\tau_1^*(t_0)$ be the optimal stopping time for the 1-swing option starting at time t_0. We know that the optimal stopping time is the first time where the process $\{(\Phi(S_t)), t_0 \le t \le T\}$ hits its Snell envelope $\{(Y_t), t_0 \le t \le T\}$:

$$\tau_1^*(t_0) = \inf\{t \ge t_0 : Y_t = \Phi(S_t)\}$$

However $(s_0, t_0) \in \mathcal{E}_1$ is equivalent to $Y_{t_0} = \Phi(s_0)$, therefore $\tau_1^*(t_0) = t_0$.

Now let $\tau_2^*(t_0)$ be the optimal stopping time for the 2-swing option starting at time t_0.

The previous Proposition tells us that $\tau_2^*(t_0) \le \tau_1^*(t_0)$.

However, $t_0 \le \tau_2^*(t_0)$ and $\tau_1^*(t_0) = t_0$, therefore $\tau_2^*(t_0) = t_0$.

We conclude by recalling that $\tau_2^*(t_0)$ is the first hitting time of the process $\left\{(\Phi(S_t) + \sup_{t+\delta \le \eta \le T} \mathbf{E}\left[\Phi(S_\eta)|\mathcal{F}_t\right]), t_0 \le t \le T\right\}$

with Snell envelope $\{(X_t), t_0 \le t \le T\}$ and therefore

$$X_{\tau_2^*(t_0)} = \Phi(S_{\tau_2^*(t_0)}) + \sup_{\tau_2^*(t_0)+\delta \le \eta \le T} \mathbf{E}\left[\Phi(S_\eta) | S_{\tau_2^*(t_0)}\right]$$

or equivalently

$$X_{t_0} = \Phi(S_{t_0}) + \sup_{t_0+\delta \le \eta \le T} \mathbf{E}\left[\Phi(S_\eta) | S_{t_0} = s_0\right]$$

and $(t_0, s_0) \in \mathcal{E}_2$ □

Remark 8. *We can easily generalize this theorem to the case of N exercise times.*

In this chapter, we have modelled swing options using RBSDEs. We developed an embedding procedure amenable to numerical solutions. This approach is more general than Carmona and Touzi [16] and allows for the computation of a hedging strategy.

CHAPTER 3

CREDIT RISK MODELLING

3.1 Introduction

There are two approaches to modelling defaults in a portfolio context: static and dynamic. In the static approach, the total number of defaults is calculated for a fixed time period. Examples of this approach include Moodys Binomial Expansion method, factor models, the CreditMetrics modelling framework, Davis and Lo [25], Schönbucher [75] and Gupton et al. [40]. From a risk perspective, knowing the probable number of defaults within a portfolio over a certain period of time is useful, however, in order to price basket credit derivatives, we are interested in the timing and identity of the defaults as well as the number. In the dynamic approach, the default processes of the individual obligors within a portfolio are modelled in the same way as in the case of a single firm. Overlaid on this is the default dependence structure which derives from both the specification of the individual default processes and their inter-relationship. As for the single-firm case, credit risk models are usually classified either as structural models or reduced-form models (also called intensity-based models). These two approaches, structural and reduced-form, represent two extreme cases: the default time is modelled as a predictable stopping time (the first moment when the firm's value hits some barrier, as in Black and Cox [11]), or by a totally inaccessible stopping time (defined by its intensity, as in Jarrow and Turnbull [50]). However, many authors (see, for instance, Dufie and Lando [25], Giesecke [33], Jarrow and Protter [47], or Jeanblanc and Valchev [52]), proved that the properties of default time are related to the available information of the modeler or his capacity to observed directly the value of the firm or the default triggering barrier.

3.1.1 Structural vs Reduced-form Model

The original structural model dates back to the early seventies and the papers of Black and Scholes [13] and Merton [67]. Their work seeks to relate credit events to economic fundamentals by modelling the dynamics of the assets of a firm with default occurring if the value of the firm drops below some threshold level. They consider a continuous time model with maturity T. They have a complete filtered probability space $(\Omega, \mathcal{H}, \mathbb{H}, P)$ satisfying the usual hypotheses. $\mathbb{H}$ is the information held by the modeler evaluating the credit risk of the firm. The firm's asset value is denoted by V_t and they assume that $\sigma(V_s, s \leq t) \subset \mathcal{H}_t$. In their model, the firm's asset value V_t follow a diffusion process and the default time of the firm is defined by τ such that

$$\tau = T\mathbf{1}_{\{V_T < L\}} + \infty \mathbf{1}_{\{V_T \geq L\}}$$

It is easy to see that in that framework, the default of the firm occurs only at maturity. Black and Cox [11] proposed the first version of what are now known as first passage models, relaxing the assumption that default can only occur at maturity. They define a time dependent deterministic barrier v_t and the following default time

$$\tau = \inf \{t \geq 0,\ V_t < v_t\}$$

However, in both these models it is possible to define a sequence of stopping times τ^n announcing τ. Indeed,

$$\tau^n = \inf \left\{ t \geq 0,\ V_t < v_t + \frac{1}{n} \right\}$$

are $\mathbb{H}$-stopping times and

$$\lim_{n\to\infty} \tau^n = \tau \ \ a.s.$$

Hull and White [43], [44] and Avellaneda and Zhu [3] propose structural models, but rather than modelling the value of the firm, they consider a credit or default index: any process that is a measure of the firms financial health and can trigger a default. A further approach based on the firm value methodology, considered by Leland [57], Leland and Toft [59] and Mella-Barral and Perraudin [66] looks at the question of optimal capital structure. Bankruptcy is assumed to be an endogenous event triggered by the equity-holders to maximize equity value. Moodys KMV uses Merton's framework for credit modelling as the motivation of a "distance-to-default" statistic that is then used in conjunction with Moodys database of historical default information to assess default probabilities. The approach is known as the KMV model and is widely used in the market. For a given firm, the statistic is then used to give an expected default frequency by calibration with historical default data. For further details, see www.moodyskmv.com.

The biggest problem with the diffusion models outlined above is the fact that credit spreads, in particular those for short-maturity bonds, are far too low. This comes from the predictable feature of the stopping time defined. Model bond prices also converge smoothly to their default levels rather than drop precipitously at or around the time of default, as tends to happen in practice. Since both characteristics are due to the predictable nature of default in these models, the obvious way to improve results is to introduce an element of unpredictability or uncertainty into the model formulation. Different ways of doing this include introducing an unpredictable jump term, making the barrier random, assuming that information available to bondholders is incomplete.

The addition of a jump term is one way to introduce default unpredictability into the basic structural model. Another is to assume that information regarding firm value and/or the level of the default barrier is incomplete from the perspective of bondholders. This idea was first introduced by Duffie and Lando [25] who built on the work by Leland and Toft [59] to assume that whereas management and equity-holders have full knowledge of the asset process and act to maximize the value of equity by setting an optimal default threshold, bondholders receive only noisy reports of firm value at discrete times. Kusuoka [53] and Coculescu et al. [20] consider extensions to this framework in which bond investors receive the noisy asset reports continuously, while Çetin et al. [17] assume that rather than the bondholders seeing management's information set plus noise, managers restrict the information that is available and the market therefore sees a reduction in management's information set. Frey and Schmidt [32] consider the situation in which firm value is filtered from discretely observed news and in each of these approaches, the framework admits a default intensity that can be used for pricing. Giesecke [34], Giesecke and Goldberg [37] and Giesecke [36] take a slightly different approach and assume that while default is a publicly observable event, either the value of the firm or the level of the default barrier (or both) is unknown. Indeed, they suppose that the default barrier is a random variable η defined on the underlying probability space (Ω, P). The default occurs at time τ where

$$\tau = \inf \{t \geq 0,\ V_t < \eta\}$$

If the random barrier η is independent of $\mathbb{H}$, they prove that the stopping time τ is not predictable anymore and has an intensity process. For each possible scenario, Giesecke [36] considers the relationship between the incomplete information framework and the existence of a default intensity in detail. Schmidt and Novikov [74] considers a generalization that allows for jumps in both the value of the firm and the default barrier. Default is modelled as the first hitting time of a stochastic and unobservable default barrier, admitting a default intensity that can be used for pricing.

Intensity models (also known as reduced-form models and first proposed by Jarrow and Turnbull [50]) have received considerably more attention in the multi-firm setting than structural models, particularly amongst practitioners. One way to introduce dependence is through correlated intensity processes, with the rationale that default intensities are driven by common macroeconomic variables (see, for example, Duffie and Singleton [26]). Jarrow and Yu [51] link default intensities at the individual obligor level, rather than looking at the whole portfolio as in Davis and Lo [22]. Related to the intensity models, by far the most popular multi-asset models used by the market in recent years have been those using copulas. Copula functions enable the distributions of marginal default times to be specified separately from the dependence structure, enabling easy implementation and calibration. Initiating from the paper by Li [61], there has been a huge amount of work done applying copulas to finance. Nelsen [69], Schönbucher [75] and Cherubini et al. [18] provide a full and rigorous overview of the mathematics of copula functions and their application to financial modelling. Giesecke and Weber [38] associate default contagion with the local dependence of firms on their business partners. By representing firms as nodes on a lattice, contagion is incorporated by making each company's financial health dependent on the state of connected companies.

In the rest of the thesis, we concentrate on intensity models for large credit portfolio and more specifically, for portfolios of subprime mortgages. In the next section, we recall properties of reduced-form models. In the last section, we describe derivative markets for mortgages.

3.2 Pricing Credit Derivatives in the Reduced-form Model

Let us consider a complete filtered probability space $(\Omega, \mathcal{H}, \mathbb{H}, P)$ satisfying the usual hypotheses and a fixed time horizon T. The filtration $\mathbb{H}$ represents the information available in the market. The starting point of the intensity approach is the knowledge of a default time τ such that τ is an $\mathbb{H}$-stopping time. A stopping time is a non-negative random variable such that the process $(\tau \wedge t)$ is $\mathcal{H}_t$ adapted. The intensity is defined as any non-negative and $\mathbb{H}$-adapted process λ such that

$$M_t = \mathbf{1}_{\{\tau \leq t\}} - \int_0^{t\wedge\tau} \lambda_s ds$$

is an $\mathbb{H}$-martingale. We will see in the next chapter that the existence of the intensity relies on the fact that $(\mathbf{1}_{\{\tau \leq t\}})$ is an increasing process, therefore a submartingale and can be written as a martingale M plus a predictable increasing process A.

Remark 9. *The intensity is not well defined after time τ. If λ is an intensity, for any non-negative predictable process (g_t), the following process*

$$\widetilde{\lambda}_t = \lambda_t \mathbf{1}_{\{\tau \geq t\}} + g_t \mathbf{1}_{\{\tau < t\}}$$

is also an intensity.

The following proposition is the building block for the pricing of credit derivatives. We include the proof for completeness (see [8]).

Proposition 23. *Let X be $\mathcal{H}_T$-adapted and in L^2, then*

$$E\left\{X\mathbf{1}_{\{\tau>T\}}|\mathcal{H}_t\right\} = \mathbf{1}_{\{\tau>t\}}E\left\{Xe^{-\int_t^T \lambda_s ds}|\mathcal{H}_t\right\}.$$

Proof. Define

$$N_t = \mathbf{1}_{\{\tau\leq t\}}$$

and

$$L_t = (1-N_t)\exp\left(\int_0^t \lambda_s ds\right)$$

We first need to prove that (L_t) is a martingale. From Itô's formula, we obtain

$$\begin{aligned} dL_t &= (1-N_{t-})d\exp\left(\int_0^t \lambda_s ds\right) - \exp\left(\int_0^t \lambda_s ds\right)dN_t \\ &= (1-N_{t-})\lambda_t\exp\left(\int_0^t \lambda_s ds\right)dt - \exp\left(\int_0^t \lambda_s ds\right)dN_t \\ &= \exp\left(\int_0^t \lambda_s ds\right)((1-N_{t-})\lambda_t dt - dN_t) \\ &= -\exp\left(\int_0^t \lambda_s ds\right)dM_t \end{aligned}$$

However, (M_t) is a martingale and therefore (L_t) is a martingale. Define

$$Y_t = E\left\{Xe^{-\int_0^T \lambda_s ds}|\mathcal{H}_t\right\}$$

and

$$\begin{aligned} U_t &= (1-N_t)\exp\left(\int_0^t \lambda_s ds\right)E\left\{Xe^{-\int_t^T \lambda_s ds}|\mathcal{H}_t\right\} \\ &= L_tY_t. \end{aligned}$$

We can first remark that (Y_t) is a martingale. We can assume that it is continuous at τ. Indeed, we can always define (λ_t) after τ so that $\Delta Y_\tau = Y_{\tau-} - Y_\tau = 0$.

We are proving that (U_t) is a martingale by applying Itô's Formula. We obtain

$$\begin{aligned} dU_t &= L_{t-}dY_t + Y_{t-}dL_t + d[L,Y]_t \\ &= L_{t-}dY_t + Y_{t-}dL_t - \Delta Y_t \\ &= L_{t-}dY_t + Y_{t-}dL_t. \end{aligned}$$

Therefore,

$$E\left\{U_T|\mathcal{H}_t\right\} = U_t$$

or equivalently

$$E\left\{X\mathbf{1}_{\{\tau>T\}}|\mathcal{H}_t\right\} = \mathbf{1}_{\{\tau>t\}}E\left\{Xe^{-\int_t^T \lambda_s ds}|\mathcal{H}_t\right\}.$$

□

Assume that interest rates are null. A defaultable bond with maturity T, default time τ, payment dates (T_i) and coupon C_i has the following cash flow

$$\sum_{i=1}^{n} C_i\mathbf{1}_{\{\tau>T_i\}}.$$

According to the previous proposition, knowing the intensity of the default process is sufficient to compute the following present value PV_t of the defaultable bond at time t

$$\begin{aligned} PV_t &= E\left\{\sum_{i=1}^{n} C_i\mathbf{1}_{\{\tau>T_i\}}|\mathcal{H}_t\right\} \\ &= \sum_{i=1}^{n} E\left\{C_ie^{-\int_t^{T_i}\lambda_s ds}\Big|\,\mathcal{H}_t\right\} \end{aligned}$$

Assume now that we have N defaultable bonds with same maturity T, same payment dates $(T_i)_{i=1..n}$ and same payment coupon $(C_i)_{i=1..n}$ at each payment dates. Each bond has a default time $\left(\tau^k\right)_{k=1..N}$. We want to price a Collateralized Debt Obligation (CDO). CDOs are a type structured credit product. CDOs are constructed from a portfolio of fixed-income assets. These assets are divided into different tranches: senior tranches (rated AAA), mezzanine tranches (AA to BB), and equity tranches (unrated). Losses are applied in reverse order of seniority and so junior tranches offer higher coupons to compensate for the added default risk. Define the following cumulative loss process at time t

$$L_t = \frac{1}{N}\sum_{k=1}^{N} \mathbf{1}_{\{\tau^k \leq t\}}$$

For a CDO with lower attachment point A_l and upper attachment point A_u (i.e., a CDO on a tranche $[A_l, A_u]$), a coupon C is paid at each payment date as long as the total number of default is less than $A_l \times N$. After $A_l \times N$ number of bonds have defaulted, only a fraction of the coupon is paid. This fraction corresponds to the upper attachment point minus the cumulative loss amount. A standard tranche $[A_l, A_u]$ is usually composed of two legs. The asset leg consists in the payments of loss amounts in between $[A_l, A_u]$, at the time they occur. The coupon leg corresponds to the payment of a fixed coupon on the outstanding notional, for each maturity date T_i, plus an accrued coupon paid when a loss occurs, on a notional that corresponds to the loss amounts in $[A_l, A_u]$. We obtain the following stream of payments for the coupon leg

$$\sum_{i=1}^{n} C\left(\mathbf{1}_{\{L_{T_i} < A_l\}} + \frac{A_u - L_{T_i}}{A_u - A_l}\mathbf{1}_{\{A_l < L_{T_i} < A_u\}}\right)$$

A CDO can be seen as a Put spread on the cumulated loss. A Put spread with strike A on the cumulated loss corresponds to a equity tranche $[0, A]$, so that a tranche $[A_l, A_u]$ can be seen as the difference between the tranches $[0, A_u]$ and $[0, A_l]$. Therefore, for the rest of the thesis, we will only consider Puts, to allow for simpler notations.

We will consider a CDO of maturity T and strike K with the following present value at time t

$$PV_t = \sum_{i=1}^{n} E\left\{ (K - L_{T_i})_+ \middle| \mathcal{H}_t \right\} \tag{3.1}$$

3.3 Mortgages

There are three types of Mortgages: Fixed-Rate Mortgages (FRMs for short) are mortgages where the rate is fixed over loan‘s term. The payments are fixed and the mortgage is fully amortizing (i.e., the loan is paid off at maturity). Adjustable Rate Mortgages (ARMs for short) have payments and rates pegged to prevailing market interest rates. The last category of mortgages is a combination of FRMs and ARMs. Non-Agency Hybrid ARMs started in the early 1990s but have only been heavily securitized in the past decade. During the initial period of the mortgage, the borrower pays a fixed rate. After this period, the loan becomes an ARM. Mortgages typically amortize over 30 years and have annual rate adjustments historically pegged to 1-year constant maturity treasury rate index and LIBOR.

While the primary mortgage market (usually thrifts) deals with the issuance of new mortgages, the secondary mortgage market is the financial market where mortgages are sold by, and transferred from, one investor or speculator to another. An important activity in the secondary market is the securitization or the transformation of mortgages into mortgage-backed securities (MBS). It is the process by which illiquid financial assets and liabilities are transformed into capital market instruments. After World War II, the creation of an active secondary market for trading mortgages was a major policy goal. It resulted in the creation of a group of government (Ginnie Mae) and quasi-government (Fannie Mae, Freddie Mac) agencies. It created a more competitive market by encouraging new firms to enter the mortgage origination business resulting in lower mortgage rates. It also allowed thrifts to securitize and sell mortgage portfolios to balance their mortgage holdings with their deposit funds.

MBS are instruments similar to traditional corporate bonds. Once a loan is made, it is often sold to one of the three agencies which in turn is used to form mortgage pools. Pools are held by trustees as collateral for pass-through MBS or participating certificates (PC). Each pass-through conveys ownership of an individual interest in the mortgage pool. Investors are forwarded their share of interest and principal collected from mortgages. Agency pools have agency guarantee that investors will receive timely payment of their share of principal thus reducing the riskiness of such certificates. The advantages of holding PCs as opposed to holding individual loans is that one can make investments of varying size and there exists an active market for trading PCs.

Collateralized Mortgage Obligations (CMOs) were the first major innovation in MBS security and were designed by Freddie Mac in June 1983. The long maturity nature of pass-throughs was seen as unfavorable by many investors who are often interested in shorter-termed securities. CMO is a sequential structure that works like a CDO of pass-through MBS. It is a multiple class (tranche) security collateralized by one mortgage pool. It directs mortgage pool cash flow into different bonds or tranches with different maturities and prioritizes the order of principal prepayment between the bonds. The short maturity tranche will receive all pool principal payments until it is retired (i.e., until the bond holders principal is returned entirely). While the short tranche is active, all other tranches will receive *interest only* payments on their principal. When the first tranche is retired, the second tranche becomes the next retiring tranche and the second tranche holder will receive all pool interest.

The last type of MBS are stripped MBS. It is a structure that slices up the interest and principal components of the fully amortizing mortgage payments. Interest Only (IO) bondholders have claims to the interest component only, while the Principal Only (PO) bondholders have claims to the principal component only. In general, this interest and principal structure can be applied to CMO tranches also.

In the analysis of Mortgage-Backed Securities (MBS) or other mortgage derivatives, it is important to understand the borrowers prepayment behavior. Since the bonds cash flow stream varies due to prepayment, much of the investment characteristics of such securities are also dependent on prepayment. In general, the payments can be affected by refinancing and default. Refinancing is typically the result of: changes in mortgage rates (under interest rate certainty or uncertainty), borrower relocation, or for equity extraction.

CHAPTER 4

LARGE CREDIT PORTFOLIO

The valuation of credit derivatives changed the focus of many credit risk models. Instead of developing a model for pricing defaultable bonds, and because these bonds are taken as input to derive prices for more exotic derivative securities, new models have to be developed that have this degree of flexibility.

This paper provides a model for pricing Collateralized Default Obligations (CDOs) for subprime Asset Backed Securities (ABS)/ Home Equity Loans (HELs). Subprime borrowers are usually defined as those with a FICO (Fair Isaacs & Co.) score of 650 or less, a DTI (Debt to Income Ratio) of 40% or more, and a LTV (Loan to Value) ratio of 80% or more.

To price the CDO liabilities, one must first model the underlying ABS Deals and price the ABS liabilities. An HEL loan pool faces the risks associated with default, loss severity at default, and prepayment. These risks are influenced by housing prices, interest rates, the health of the economy and idiosyncratic factors.

For pricing a derivative with a portfolio of defaultable securities as the underlying, we have to compute the loss distribution term structure of the portfolio. That can be done in two ways. We can model each security and add a correlations structure. This approach is called ”bottom up”. Alternatively, we can model the whole term structure of cash flows generated by the portfolio for any given time before the maturity of the contract. This method is referred to as ”top down”. The main advantage of the latter is the computational cost reduction in numerical applications.

In this chapter, we will find condition on the underlying security that will allow us to use a top down approach. Indeed, we start with a ”loan level analysis” that describes each loans, and we conclude by giving conditions for shifting to a ”deal level analysis” where the loss distribution of the pool is modelled directly.

Following the work of Andersen, Piterbarg and Sidenius [1], Bennani and Dahan [6] and Schönbucher [76], we develop a term structure model for the forward default rates.

In the first section, we describe a residential ABS deal. In the second section, we study some properties of default processes. We will then prove in the third section two convergence theorems and give justifications for the deal level analysis. In the fourth section, we define the forward loss process and give the no arbitrage conditions. The last section, we conclude by stating our asymptotical Theorem and we give a pricing method for CDOs of Subprime ABS.

4.1 Description of an ABS Deal

To motivate the modelling structure, let us consider a generic ABS deal. The deal consists of a large number loans (usually more than a thousand). Each loan has a default time and a prepayment time. These are competing risks in the sense that if either occurs, the loan terminates. At default, a fraction of the loan is lost and the investor receives the recovery value of the loan.

4.1.1 The Asset Side

An ABS deal consists of an HEL loan pool of approximately 7000 mortgages. The mortgages are predominately of two types:

1) Hybrid ARMs. They have a (teaser) fixed rate for 2 or 3 (or 5) years. At the reset date, it switches to a floating rate. The maturity is typically 40 years. These are denoted, for example, as 2/38 (2 years fixed, 38 years floating). The floating rate is 6 month Libor plus a spread.

2) Hybrid IO ARMs. These are ARMs, but they do not amortize principal for 2 or 3 years. Only interest payments occur during this initial period. After the reset date, both interest and principal amortization takes place. The floating rate is Libor 6 months plus a spread.

Both types of loans usually have prepayment penalties for 2 or 3 years. A typical prepayment penalty is 6 months interest on 80% of the prepaid balance.

Both types of loans will typically have caps (three types: initial, periodic, lifetime) and floors on the floating rate payments.

4.1.2 The Liability Side

Issued using the HEL loan pool as collateral are a collection of bonds and equity. The bonds are floating paying 1 month Libor plus a spread. The bonds have an AFC (available funds cap), i.e. they pay min(Libor + spread, WAC) where WAC is the Weighted Average Coupon of the HEL loan pool.

For an ABS deal, the interest and principal is allocated from the top to the bottom on the liability side. Losses occur from the bottom up. This is called the cash flow "waterfall." The waterfall cash flows may be obtained using INTEX.

4.2 Preliminaries

Let us consider a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and a fixed time horizon T. In this thesis, we will deal with two kinds of information represented by two filtrations $\mathbb{F}$ and $\mathbb{G}$.
Let $\mathbb{F} = (\mathcal{F}_t)_{0\leq t\leq T}$ with $\mathcal{F}_t = \sigma(W_s, s \leq t)$ represent the information from asset prices and other economical factors.

The second filtration $\mathbb{G} = (\mathcal{G}_t)_{0\leq t\leq T}$ is generated by default processes. Let us consider n mortgages with $(\tau^i)_{i=1..n}$ their associated default times. We assume that $(\tau^i)_{i=1..n}$ are positive $\mathcal{F}$ measurable random times. Let $\mathcal{G}_t^i = \sigma(\tau^i \wedge s,\ s \leq t)$ be the smallest filtration making τ^i a stopping time for each i, with $\mathcal{G}_t^i$ containing all the $(\mathcal{F}, \mathbb{P})$ null sets, $\forall t \geq 0$.

Finally, we can build the enlarged filtration $\mathbb{H}$ representing the information available in the market: $\mathcal{H}_t^{0,i} = \mathcal{F}_t \bigvee \mathcal{G}_t^i$ is the market filtration with only one mortgage and $\mathcal{H}_t^0 = \mathcal{F}_t \bigvee_{i=1}^{\infty} \mathcal{G}_t^i$ is the market filtration including all the mortgages. These filtrations need not be right continuous. Therefore we define:

$$\mathcal{H}_t^i = \bigcap_{u>t} \mathcal{H}_u^{0,i}$$

and

$$\mathcal{H}_t = \bigcap_{u>t} \mathcal{H}_u^0$$

We will study the properties of a default process for a single name. We fix i

for the rest of the section and define

$$N_t^i = \mathbf{1}_{(\tau^i \leq t)}$$

Definition 14. *A stopping time τ is predictable if there exists a sequence (τ^n) that announce τ (i.e., τ^n is increasing, $\tau^n < \tau$ on $\{\tau > 0\}$ and $\lim_{n\to\infty} \tau^n = \tau a.s.$).*

Definition 15. *A stopping time τ is totally inaccessible if for every predictable stopping time θ, $P\{\tau = \theta < \infty\} = 0$.*

Definition 16. *A process Z is of class (D) if the set*

$$\{Z_\tau : \tau \text{ is a finite stopping time } \}$$

is uniformly integrable.

The process $(N_t^i)_{t\geq 0}$ is a càdlàg $\mathbb{H}^i$ sub-martingale of class (D). Let $N^i = M^i + A^i$ be its Doob-Meyer decomposition (i.e., $(A_t^i)_{t\geq 0}$ is the unique continuous, increasing and $\mathbb{H}^i$ predictable process such that $N_t^i - A_t^i$ is an $\mathbb{H}^i$ martingale). We have the following Laplacian approximation theorem

Theorem 24. *(Meyer) If we define*

$$A_t^i(h) = \int_0^t \frac{N_s^i - E\left\{N_{s+h}^i | \mathcal{H}_s\right\}}{h} ds,$$

Then for any stopping time τ, the compensator of N^i is

$$A_t^i = \lim_{h\to 0} A_t^i(h),$$

in the sense of the weak topology $\sigma(L^1, L^\infty)$.

We will see later the appropriate conditions on τ^i so that its $\mathbb{H}^i$ compensator is absolutely continuous and therefore can be written as

$$A_t^i = \int_0^{t\wedge\tau^i} \lambda_s^i ds$$

with λ^i being $\mathbb{H}$ adapted a priori. λ will be interpreted as the Radon-Nikodym derivative of A^i with respect to the Lebesgue Measure.

Theorem (26) is a conditional version of a Dellacherie's classical result [23]. Since it is technically new, we provide a proof. We begin with the following lemma:

Lemma 25. *If* $Y \in L^1(\mathcal{F})$*, then*

$$E\left\{Y|\mathcal{F}_t \vee \mathcal{G}_t^i\right\} = E\left\{Y|\mathcal{F}_t, \tau^i\right\} \mathbf{1}_{\{\tau^i \le t\}} + \frac{E\left\{Y\mathbf{1}_{\{\tau^i > t\}}|\mathcal{F}_t\right\}}{P(\tau^i > t|\mathcal{F}_t)}\mathbf{1}_{\{\tau^i > t\}}$$

Proof. Let us first remark that $\mathcal{G}_t^i = \sigma(\tau^i \wedge t)$. Therefore, there exists a Borel measurable function f on $\mathbf{R} \times \mathbf{R}^\infty$ such that

$$E\left\{Y|\mathcal{F}_t \vee \mathcal{G}_t^i\right\} = f(\tau^i \wedge t, W_.)$$

with $W_.$ being the whole process between 0 and t.

We can now rewrite

$$\begin{aligned} E\left\{Y|\mathcal{F}_t \vee \mathcal{G}_t^i\right\} &= f(\tau^i \wedge t, W_.) \\ &= f(\tau^i \wedge t, W_.)\mathbf{1}_{\{\tau^i \le t\}} + f(\tau^i \wedge t, W_.)\mathbf{1}_{\{\tau^i > t\}} \\ &= f(\tau^i, W_.)\mathbf{1}_{\{\tau^i \le t\}} + f(t, W_.)\mathbf{1}_{\{\tau^i > t\}} \end{aligned}$$

We obtain that

$$\begin{aligned} f(t, W_.)\mathbf{1}_{\{\tau^i > t\}} &= E\left\{Y|\mathcal{F}_t \vee \mathcal{G}_t^i\right\} \mathbf{1}_{\{\tau^i > t\}} \\ &= E\left\{Y\mathbf{1}_{\{\tau^i > t\}}|\mathcal{F}_t \vee \mathcal{G}_t^i\right\} \end{aligned}$$

We can now take the conditional expectation

$$\begin{aligned} E\left\{f(t, W_.)\mathbf{1}_{\{\tau^i > t\}}|\mathcal{F}_t\right\} &= f(t, W_.)E\left\{\mathbf{1}_{\{\tau^i > t\}}|\mathcal{F}_t\right\} \\ &= E\left\{Y\mathbf{1}_{\{\tau^i > t\}}|\mathcal{F}_t\right\} \end{aligned}$$

Finally

$$f(t, W_\cdot) \;=\; \frac{E\left\{Y\mathbf{1}_{\{\tau^i>t\}}|\mathcal{F}_t\right\}}{P(\tau^i > t|\mathcal{F}_t)}$$

A similar argument gives us

$$f(\tau^i, W_\cdot) \;=\; E\left\{Y|\mathcal{F}_t, \tau^i\right\}$$

$\square$

We can now prove the following theorem. It is an extension of Dellacherie's theorem [23] to the case of conditional probability density functions.

Theorem 26. *Let $P(\tau^i = 0|\mathcal{F}_t) = 0$ and $P(\tau^i > t|\mathcal{F}_t) > 0$, for each $t > 0$. Then the $\mathbb{H}^i$ compensator of N^i is given by*

$$A_t^i = \int_0^{t\wedge\tau^i} \frac{1}{1 - F^i(s-)} dF^i(s),$$

where F^i is the $\mathbb{F}$ conditional cumulative distribution function of τ^i:

$$F^i(t) = P(\tau^i \leq t|\mathcal{F}_t)$$

Proof. The process N^i is purely discontinuous with jump size one, therefore its $\mathbb{H}^i$ compensator is equal to its conditional quadratic variation:

$$A_t^i = \langle N^i\rangle_t$$

Fix $t_0 > 0$ and let π_n be a sequence of partitions of $[0, t_0]$ with $\lim_{n\to\infty} mesh(\pi_n) = 0$. For all $t \in [0, t_0]$, we know that the conditional compensator is obtained by

$$\langle N^i\rangle_t = \lim_{n\to\infty} \sum_{\pi_n} E\left\{(N^i_{t_{k+1}} - N^i_{t_k})^2|\mathcal{H}^i_{t_k}\right\}$$

where the limit is taken in the weak topology $\sigma(L^1, L^\infty)$ (see [23] for details about this topology).

The previous lemma gives us

$$\begin{aligned} E\left\{(N^i_{t_{k+1}} - N^i_{t_k})^2 | \mathcal{H}^i_{t_k}\right\} &= E\left\{(N^i_{t_{k+1}} - N^i_{t_k})^2 | \mathcal{F}_{t_k}, \tau^i\right\} \mathbf{1}_{\{\tau^i \leq t_k\}} \\ &+ \frac{E\left\{(N^i_{t_{k+1}} - N^i_{t_k})^2 \mathbf{1}_{\{\tau^i > t_k\}} | \mathcal{F}_{t_k}\right\}}{P(\tau^i > t_k | \mathcal{F}_{t_k})} \mathbf{1}_{\{\tau^i > t_k\}} \end{aligned}$$

Let us remark that

$$\begin{aligned} (N^i_{t_{k+1}} - N^i_{t_k})^2 &= \left(\mathbf{1}_{\{t_{k+1} \geq \tau^i > t_k\}}\right)^2 \\ &= \mathbf{1}_{\{t_{k+1} \geq \tau^i > t_k\}} \\ &= N^i_{t_{k+1}} - N^i_{t_k} \end{aligned}$$

and

$$(N^i_{t_{k+1}} - N^i_{t_k})^2 \mathbf{1}_{\{\tau^i > t_k\}} = N^i_{t_{k+1}} - N^i_{t_k}$$

and

$$(N^i_{t_{k+1}} - N^i_{t_k})^2 \mathbf{1}_{\{\tau^i \leq t_k\}} = 0$$

Finally, we obtain

$$\begin{aligned} E\left\{(N^i_{t_{k+1}} - N^i_{t_k})^2 | \mathcal{H}^i_{t_k}\right\} &= \frac{E\left\{N^i_{t_{k+1}} - N^i_{t_k} | \mathcal{F}_{t_k}\right\}}{P(\tau^i > t_k | \mathcal{F}_{t_k})} \mathbf{1}_{\{\tau^i > t_k\}} \\ &= \frac{F^i(t_{k+1}) - F^i(t_k)}{1 - F^i(t_k)} \mathbf{1}_{\{\tau^i > t_k\}} \end{aligned}$$

□

In his thesis, Yan Zeng [79] obtains necessary and sufficient conditions for the cumulative density function F^i to be absolutely continuous. His result generalizes the classical condition of Ethier and Kurtz [30].

Theorem 27. *(Zeng) Let A be an increasing (not necessarily adapted) and integrable measurable process, with $A_0 = 0$. Let $\widetilde{A}$ be the compensator of A . Then $d\widetilde{A} \ll dt$ if and only if there exists an increasing and integrable measurable process D with with $A_0 = 0$, such that $d\widetilde{D} \ll dt$ and $\forall t, h \geq 0$,*

$$E\left\{A_{t+h} - At|\mathcal{H}_t\right\} \leq E\left\{D_{t+h} - Dt|\mathcal{H}_t\right\}$$

For the rest of the chapter, we will assume that the condition of Theorem (27) is satisfy for each compensator A^i. For each i, we call λ^i the Radon-Nikodym derivative of the A^i with respect to Lebesgue measure. Therefore, we have the following $\mathbb{H}^i$ compensator for N^i:

$$A^i_t = \int_0^{t\wedge\tau^i} \lambda^i_s ds$$

We proved in Lemma (25) that we can assume that λ^i is $\mathbb{F}$-adapted.

From Theorem (26), we have that

$$\frac{dF^i(t)}{1 - F^i(t)} = \lambda^i_t dt,$$

or equivalently

$$dZ^i(t) = -\lambda^i_t Z^i_t dt$$

with

$$Z^i(t) = P(\tau^i > t|\mathcal{F}_t).$$

4.3 Loss Process

Define the following cumulative loss process

$$\Lambda^n_t = \frac{1}{n}\sum_{i=1}^{n} N^i_t$$

We want to study the asymptotic behavior of loss processes and find conditions for modelling at the deal level. We need the two following processes:

$$
\begin{aligned}
X_t^n &= \frac{1}{n}\sum_{i=1}^{n}(N_t^i - \int_0^{t\wedge\tau^i} \lambda_s^i ds) \\
Y_t^n &= \frac{1}{\sqrt{n}}\sum_{i=1}^{n}(N_t^i - \int_0^{t\wedge\tau^i} \lambda_s^i ds) \\
&= \sqrt{n}X_t^n
\end{aligned}
$$

We will first prove that the process X^n converges to zero in $\mathbb{L}^2$. We will then study the rate of convergence via a central limit theorem for Y^n.

From now on, we will assume that the default times $(\tau^i)_{i>0}$ are $\mathbb{F}$ conditionally independent.

This is implies that for every sequence $(i_1 < ... < i_n) \in \mathbf{N}^n$, for all $(t, u) \in \mathbf{R}_+ \times \mathbf{R}$, we have

$$
E\left\{\exp(iu\sum_{k=1}^{n} N_t^{i_k})|\mathcal{F}_t\right\} = \prod_{k=1}^{n} E\left\{\exp(iuN_t^{i_k})|\mathcal{F}_t\right\}
$$

Remark 10. *This assumption implies the hypothesis (H) which reads*

(H) *Every $\mathbb{F}$-local martingale is an $\mathbb{H}$-local martingale.*

This can be written in any of the equivalent forms (see [9]):

Lemma 28. *Assume that $\mathbb{H} = \mathbb{F} \vee \mathbb{G}$, where $\mathbb{F}$ is any filtration and $\mathbb{G}$ is generated by the process $\mathbf{1}_{\{\tau\le t\}}$. Then the following conditions are equivalent to the hypothesis (H):*

- *For any $t, h \in \mathbb{R}^+$, we have*

$$
P(\tau \le t|\mathcal{F}_t) = P(\tau \le t|\mathcal{F}_{t+h})
$$

- *For any $t \in \mathbb{R}^+$, we have*

$$P(\tau \le t|\mathcal{F}_t) = P(\tau \le t|\mathcal{F}_\infty)$$

Let $Z_t^i = P(\tau^i > t|\mathcal{F}_t)$. We proved that the process Z^i follows the following stochastic differential equation:

$$\begin{cases} dZ_t^i = -\lambda_t^i Z_t^i dt \\ Z_0^i = P(\tau^i > 0|\mathcal{F}_0) = 1. \end{cases}$$

Therefore, we have an explicit solution for Z^i:

$$Z_t^i = \exp(-\int_0^t \lambda_s^i ds)$$

We can now prove our first result.

Theorem 29. *If, for all $i \in \mathbf{N}$, $\lambda^i \in L^2(\mathcal{F})$, then for all $t \in \mathbf{R}_+$,*

$$\lim_{n\to\infty} E\left\{(X_t^n)^2\right\} = 0$$

Proof. First remark that

$$\begin{aligned} E\left\{A_t^i|\mathcal{F}_t\right\} &= E\left\{\int_0^{t\wedge\tau^i} \lambda_s^i ds|\mathcal{F}_t\right\} \\ &= \int_0^t \lambda_s^i E\left\{\mathbf{1}_{\{\tau^i>s\}}|\mathcal{F}_t\right\} ds. \end{aligned}$$

We assumed (H-hypothesis) that

$$E\left\{\mathbf{1}_{\{\tau^i>s\}}|\mathcal{F}_t\right\} = E\left\{\mathbf{1}_{\{\tau^i>s\}}|\mathcal{F}_s\right\},$$

and therefore we obtain

$$\begin{aligned} E\left\{A_t^i|\mathcal{F}_t\right\} &= \int_0^t \lambda_s^i E\left\{\mathbf{1}_{\{\tau^i>s\}}|\mathcal{F}_s\right\} ds. \\ &= \int_0^t \lambda_s^i Z_s^i ds \end{aligned}$$

Let $Z_t^i = P(\tau^i > t|\mathcal{F}_t)$. We proved previously that

$$Z_t^i = \exp(-\int_0^t \lambda_s^i ds)$$

We obtain

$$\begin{aligned} E\left\{A_t^i|\mathcal{F}_t\right\} &= \int_0^t \lambda_s^i E\left\{\mathbf{1}_{\{\tau^i>s\}}|\mathcal{F}_s\right\} ds \\ &= \int_0^t \lambda_s^i Z_s^i ds \\ &= -\int_0^t dZ_s^i \\ &= 1 - Z_t^i \\ &= E\left\{1 - \mathbf{1}_{\{\tau^i>t\}}|\mathcal{F}_t\right\} \\ &= E\left\{N_t^i|\mathcal{F}_t\right\} \end{aligned}$$

Therefore

$$E\left\{N_t^i - A_t^i|\mathcal{F}_t\right\} = 0$$

Compute now

$$\begin{aligned} E\left\{(X_t^n)^2|\mathcal{F}_t\right\} &= \frac{1}{n^2}\sum_{i=1}^n E\left\{(N_t^i - \int_0^{t\wedge\tau^i} \lambda_s^i ds)^2|\mathcal{F}_t\right\} \\ &+ \frac{1}{n^2}\sum_{i\neq j} E\left\{(N_t^i - \int_0^{t\wedge\tau^i} \lambda_s^i ds)(N_t^j - \int_0^{t\wedge\tau^j} \lambda_s^j ds)|\mathcal{F}_t\right\} \end{aligned}$$

However, τ^i and τ^j are $\mathbb{F}$-conditionally independent and therefore we have

$$\begin{aligned} E\left\{(X_t^n)^2|\mathcal{F}_t\right\} &= \frac{1}{n} E\left\{(N_t^1 - \int_0^{t\wedge\tau^1} \lambda_s^1 ds)^2|\mathcal{F}_t\right\} \\ &+ \frac{1}{n^2}\sum_{i\neq j} E\left\{N_t^i - \int_0^{t\wedge\tau^i} \lambda_s^i ds|\mathcal{F}_t\right\} E\left\{N_t^j - \int_0^{t\wedge\tau^j} \lambda_s^j ds|\mathcal{F}_t\right\} \end{aligned}$$

We proved that for all $i \in \mathbf{N}$

$$E\left\{N_t^i - \int_0^{t\wedge\tau^i} \lambda_s^i ds|\mathcal{F}_t\right\} = 0$$

For every i, λ^i is a positive process in $\mathbb{L}^2$. Therefore

$$
\begin{aligned}
E\left\{(N_t^1 - \int_0^{t\wedge\tau^1} \lambda_s^1 ds)^2\right\} &\leq 2E\left\{(N_t^1)^2\right\} + 2E\left\{(\int_0^{t\wedge\tau^1} \lambda_s^1 ds)^2\right\} \\
&\leq 2 + 2E\left\{(\int_0^{T} \lambda_s^1 ds)^2\right\} \\
&\leq 2 + 2\|\lambda\|_2
\end{aligned}
$$

We obtain

$$
\begin{aligned}
\lim_{n\to\infty} E\left\{(X_t^n)^2 | \mathcal{F}_t\right\} &= \lim_{n\to\infty} \frac{1}{n} E\left\{(N_t^1 - \int_0^{t\wedge\tau^1} \lambda_s^1 ds)^2 | \mathcal{F}_t\right\} \\
&= 0
\end{aligned}
$$

We can conclude that X_t^n converges to 0 conditionally in $L^2(\mathcal{F})$. □

We will now study the rate of convergence of the process X^n. We want to prove that the sequence $(Y_t^n)_{t\geq 0}$ converges weakly to a process $(Y_t)_{t\geq 0}$. Before defining what weak convergence means for stochastic processes, we have to define the appropriate topology. Let us first remark that $(Y_t^n)_{t\geq 0}$ is a sequence of processes with càdlàg sample paths, therefore they are defining a mapping from (Ω, F) to $\mathbb{D}(\mathbb{R})$. We need a topology on $\mathbb{D}(\mathbb{R})$ that allow us to apply classical results and unfortunately the local uniform topology is not separable. The Skorokhod topology is such that $\mathbb{D}(\mathbb{R})$ is a Polish space.

Proving convergence in this topology is usually done in a two step procedure initiated by Prokhorov:

(a) Prove that the sequence $(Y_n)_{n>0}$ is tight

(b) Prove that the distribution of $(Y_t)_{t\geq 0}$ is the only possible limit for the distribution of the sequence $(Y_t^n)_{t\geq 0}$.

The first point is usually hard to deal with. However, we have the following

Proposition 30. *A sequence $(\alpha_n)_{n>0}$ converges to a continuous function α for the Skorokhod topology if and only if it converges to α for the local uniform topology. Moreover, if the limit points of the distribution of the sequence $(\alpha_n)_{n>0}$ are the laws of a continuous process, the sequence $(X_n)_{n>0}$ is tight.*

In order to prove (b) we can use the following lemma

Lemma 31. *If two càdlàg processes X and Y have the same finite dimensional distributions on a dense subset of $\mathbb{R}_+$, then they have same distribution.*

We need to show that there exists a process $(Y_t)_{t\geq 0}$ such that

$$\lim_{n\to\infty} E\left\{e^{iu_1 Y^n_{t_1}} e^{iu_2 Y^n_{t_2}} ... e^{iu_k Y^n_{t_k}} | \mathcal{F}_\infty\right\} = E\left\{e^{iu_1 Y_{t_1}} e^{iu_2 Y_{t_2}} ... e^{iu_k Y_{t_k}} | \mathcal{F}_\infty\right\}$$

for all $k \in \mathbb{N},\ (u_1, u_2, ..., u_k)\ \in \mathbb{R}^k,\ (t_1, t_2, ..., t_k)\ \in \mathbb{R}^k_+\ st\ t_1 < t_2 < ... < t_k$

Remark that conditionally on $\mathcal{F}_\infty$, $(Y^n_t)_{t\geq 0}$ is a process with independent increments. Therefore (see Jacod and Shiryaev [46]), we need to show only that

$$E\left\{e^{iuY^n_t} | \mathcal{F}_\infty\right\} \longrightarrow E\left\{e^{iuY_t} | \mathcal{F}_\infty\right\} \text{ for all } u\ \in \mathbb{R},\ t\ \in \mathbb{R}_+.$$

We can now state our result

Proposition 32. *If $\frac{1}{n}\sum_{i=1}^n e^{-\int_0^t \lambda^i_s ds}$ has a limit for all $t \in \mathcal{R}_+$, then, the sequence $(Y^n)_{n>0}$ converges weakly to a time-changed Brownian motion.*

Proof. First compute the following conditional characteristic function

$$
\begin{aligned}
& E\left\{\exp iu(1_{\{\tau^i\leq t\}} - \int_0^{t\wedge\tau^i} \lambda_s^i ds)|\mathcal{F}_\infty\right\} \\
= & E\left\{\exp iu(1_{\{\tau^i\leq t\}} - \int_0^{t\wedge\tau^i} \lambda_s^i ds)|\mathcal{F}_t\right\} \\
= & E\left\{\exp iu(1_{\{\tau^i\leq t\}} - \int_0^{t\wedge\tau^i} \lambda_s^i ds)1_{\{\tau\leq t\}}|\mathcal{F}_t\right\} \\
+ & E\left\{\exp iu(1_{\{\tau^i\leq t\}} - \int_0^{t\wedge\tau^i} \lambda_s^i ds)1_{\{\tau^i> t\}}|\mathcal{F}_t\right\} \\
= & E\left\{\exp iu(1 - \int_0^{\tau^i} \lambda_s^i ds)1_{\{\tau^i\leq t\}}|\mathcal{F}_t\right\} + E\left\{\exp(-iu\int_0^{t} \lambda_s^i ds)1_{\{\tau^i> t\}}|\mathcal{F}_t\right\} \\
= & E\left\{\exp iu(1 - \int_0^{\tau^i} \lambda_s^i ds)1_{\{\tau^i\leq t\}}|\mathcal{F}_t\right\} + E\left\{1_{\{\tau^i> t\}}|\mathcal{F}_t\right\}\exp(-iu\int_0^{t} \lambda_s^i ds)
\end{aligned}
$$

However, we assumed that τ^i has a conditional cumulative density function that is absolutely continuous with respect to the Lebesgue measure and therefore

$$
P\left(\tau^i > t|\mathcal{F}_t\right) = \exp\left(\int_0^t \lambda_s^i ds\right)
$$

We obtain

$$
\begin{aligned}
& E\left\{\exp iu(1_{\{\tau^i\leq t\}} - \int_0^{t\wedge\tau^i} \lambda_s^i ds)|\mathcal{F}_\infty\right\} \\
= & \int_0^\infty \exp iu(1 - \int_0^x \lambda_s^i ds)1_{\{x<t\}}\left(\lambda_x^i \exp - \int_0^x \lambda_s^i ds\right) dx \\
+ & \exp(-\int_0^t \lambda_s^i ds)\ \exp(-iu\int_0^t \lambda_s^i ds) \\
= & e^{iu}\int_0^t \lambda_x^i \exp\left(-(1+iu)\int_0^x \lambda_s^i ds\right) dx + \exp\left(-(1+iu)\int_0^t \lambda_s^i ds\right) \\
= & e^{iu}\frac{1}{1+iu}\left(1 - \exp -(1+iu)\int_0^t \lambda_s^i ds\right) + \exp\left(-(1+iu)\int_0^t \lambda_s^i ds\right)
\end{aligned}
$$

Define

$$
G_t(u) = E\left\{\exp iu(1_{\{\tau^i<t\}} - \int_0^{t\wedge\tau^i} \lambda_s^i ds)|\mathcal{F}_t\right\}
$$

If we do a Taylor expansion in u, we obtain the following

$$\begin{aligned}
G_t(u) &= e^{iu}\frac{1}{1+iu}\left(1-e^{-(1+iu)\int_0^t \lambda_s^i ds}\right)+e^{-(1+iu)\int_0^t \lambda_s^i ds}\\
&= (1+iu-\frac{1}{2}u^2)(1-iu-\frac{1}{2}u^2)\left(1-e^{-\int_0^t \lambda_s^i ds}\right)+e^{-\int_0^t \lambda_s^i ds}+o(u^2)\\
&= (1-u^2)\left(1-e^{-\int_0^t \lambda_s^i ds}\right)+e^{-\int_0^t \lambda_s^i ds}+o(u^2)\\
&= 1-u^2\left(1-e^{-\int_0^t \lambda_s^i ds}\right)+o(u^2).
\end{aligned}$$

The characteristic function of Y_t^n is now

$$\begin{aligned}
& E\left\{e^{iuY_t^n}|\mathcal{F}_\infty\right\}\\
&= E\left\{\exp iu\frac{1}{\sqrt{n}}\sum_{i=1}^n\left(1_{\{\tau^i\le t\}}-\int_0^{t\wedge\tau^i}\lambda_s^i ds\right)|\mathcal{F}_t\right\}\\
&= \prod_{i=1}^n E\left\{\exp i\frac{u}{\sqrt{n}}\left(1_{\{\tau^i\le t\}}-\int_0^{t\wedge\tau^i}\lambda_s^i ds\right)|\mathcal{F}_t\right\}\\
&= \prod_{i=1}^n G(\frac{u}{\sqrt{n}})\\
&= \prod_{i=1}^n\left(1-\frac{u^2}{n}\left(1-e^{-\int_0^t \lambda_s^i ds}\right)+o(\frac{1}{n})\right)\\
&= \exp\sum_{i=1}^n\log\left(1-\frac{u^2}{n}\left(1-e^{-\int_0^t \lambda_s^i ds}\right)+o(\frac{1}{n})\right)\\
&= \exp-\frac{u^2}{n}\sum_{i=1}^n\left(1-e^{-\int_0^t \lambda_s^i ds}\right)+o(1)\\
&= \exp-u^2\left(1-\frac{1}{n}\sum_{i=1}^n e^{-\int_0^t \lambda_s^i ds}\right)+o(1).
\end{aligned}$$

We can conclude that if the following limit exist for all t

$$h(t)=\lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^n e^{-\int_0^t \lambda_s^i ds},$$

then

$$\lim_{n\to\infty}E\left\{e^{iuY_t^n}|\mathcal{F}_\infty\right\}=\exp-u^2\left(1-h(t)\right).$$

The process h is decreasing. Indeed, the intensities λ^i are positive and therefore the function

$$t \mapsto e^{-\int_0^t \lambda_s^i ds}$$

is decreasing.

$t \mapsto 1 - h(t)$ is now an increasing function and therefore

$$Y_t^n \hookrightarrow B_{1-h(t)}$$

(i.e., Y^n converges weakly to a time changed Brownian motion). □

4.4 Forward Default Rates

A zero-coupon bond with maturity T is a contract that guaranties its owner the payment of one dollar at time T. We will denote its price at time $t \in [0, T]$ by $B(t, T)$. Let us assume that we have in our market zero-coupon bonds for all maturities in $[0, T]$. The forward interest rate process $(f(t, T))_{t \in [0,T]}$ is defined by

$$f(t, T) = -\frac{\partial \log B(t, T)}{\partial T},$$

when the derivative exists (for example, if $T \mapsto f(t, T)$ is continuous for all t in $\mathbb{R}_+$). The short interest rate is defined by

$$r_t = f(t, t).$$

Moreover, we will assume that we have a family of local volatilities $(\Gamma(t, T))_{t \in [0,T]}$ for the zero-coupon bonds such that

$$\begin{cases} dB(t, T) = B(t, T) \left(r_t + \Gamma(t, T) dW_t\right) \\ B(T, T) = 1 \end{cases} \tag{4.1}$$

Assumption 1. *The volatility function is continuous, uniformly bounded, differentiable with respect to the maturity T and has a uniformly bounded derivative.*

Define the volatility derivative by

$$\frac{\partial \Gamma(t,T)}{\partial T} = \gamma(t,T)$$

Remark 11. *Assumption (1) also implies the existence of the forward rate in (4.1).*

By constructing a model with no arbitrage, Heath, Jarrow and Morton [42] obtain the following equations for the forward rates

$$\begin{cases} df(t,T) = \gamma(t,T)\Gamma(t,T)dt - \gamma(t,T)dW_t \\ f(t,t) = r_t. \end{cases} \tag{4.2}$$

We will now find similar conditions for a defaultable claim. A defaultable zero-coupon bonds has a payoff of $\mathbf{1}_{\{\tau^i > T\}}$ at time T. The price at time t of this bond is given by $\overline{B}^i(t,T)\mathbf{1}_{\{\tau^i > t\}}$. $\overline{B}^i(t,T)$ is the price of a defaultable zero-coupon bond at time t assuming that the bond hasn't default yet. Therefore $\overline{B}^i(t,T)$ is positive. On the set $\{\tau^i > t\}$, $\overline{B}^i(t,T)$ doesn't need to be defined. We can now define the defaultable forward rate process $(\overline{f}^i(t,T))_{t\in[0,T]}$ by

$$\overline{f}^i(t,T) = -\frac{\partial \log \overline{B}^i(t,T)}{\partial T}.$$

when the derivative exists (for example, if $T \mapsto \overline{f}^i(t,T)$ is continuous for all t in $\mathbb{R}_+$). We obtain equivalently

$$\overline{B}^i(t,T) = e^{-\int_t^T \overline{f}^i(t,s)ds}$$

Moreover, assume that we have a family of local volatilities $(V^i(t,T))_{t\in[0,T]}$ for the defaultable zero-coupon bonds such that

$$\begin{cases} d\overline{B}^i(t,T) = \overline{B}^i(t,T)\left(\overline{\mu}_t^i + V^i(t,T)dW_t\right) \\ \overline{B}^i(T,T) = 1. \end{cases} \tag{4.3}$$

Assumption 2. *The volatility function is continuous, uniformly bounded, differentiable with respect to the maturity T and has a uniformly bounded derivative.*

Proposition 33. *The value at time t of the defaultable zero-coupon bond with payoff $\mathbf{1}_{\{\tau^i>T\}}$ at time T is given by $\mathbf{1}_{\{\tau^i>T\}}\overline{B}^i(t,T)$ with*

$$\begin{aligned} \overline{B}^i(t,T) = \ & \tfrac{\overline{B}^i(0,T)}{\overline{B}^i(0,t)} \exp\Big(-\tfrac{1}{2}\textstyle\int_0^t(|V^i(s,T)|^2 - |V^i(s,t)|^2)ds \\ & + \textstyle\int_0^t(V^i(s,T) - V^i(s,t))dW_s\Big). \end{aligned} \tag{4.4}$$

Proof. The solution of Equation (4.3) is given by

$$\begin{aligned} \overline{B}^i(t,T) = \ & \overline{B}^i(0,T)\exp\Big(-\textstyle\int_0^t(\mu_s^i)ds - \tfrac{1}{2}\int_0^t |V^i(s,T)|^2 ds \\ & + \textstyle\int_0^t V^i(s,T)dW_s\Big). \end{aligned} \tag{4.5}$$

We can now evaluate Equation (4.5) at time $T = t$. We obtain

$$\begin{aligned} \overline{B}^i(t,t) = \ & \overline{B}^i(0,t)\exp\Big(-\textstyle\int_0^t(\mu_s^i)ds - \tfrac{1}{2}\int_0^t |V^i(s,t)|^2 ds \\ & + \textstyle\int_0^t V^i(s,t)dW_s\Big). \end{aligned} \tag{4.6}$$

We know that $\overline{B}^i(t,t) = 1$. We conclude by observing that Equation (4.4) is the quotient of Equation (4.5) and Equation (4.6). □

Remark 12. *From Equation (4.4) and today's curve of defaultable prices $\left(\overline{B}^i(0,t)\right)_{t\in[0,T]}$, we can compute the value of a defaultable bond at any time t in $[0,T]$ on the event $\mathbf{1}_{\{\tau^i>t\}}$ (i.e., assuming that the bond hasn't defaulted yet).*

Define the volatility derivative by

$$\frac{\partial V^i(t,T)}{\partial T} = \sigma^i(t,T)$$

Proposition 34. *The forward default rates are the solution of the following SDE*

$$\begin{cases} d\overline{f}^i(t,T) = \sigma^i(t,T)V^i(t,T)dt - \sigma^i(t,T)dW_t \\ \overline{f}^i(t,t) = r_t + \lambda^i_t. \end{cases} \tag{4.7}$$

Proof. The solution of Equation (4.3) is given by

$$\overline{B}^i(t,T) = \overline{B}^i(0,T)\exp\Big(-\int_0^t(\mu^i_s)ds - \tfrac{1}{2}\int_0^t |V^i(s,T)|^2 ds + \int_0^t V^i(s,T)dW_s\Big)$$

We can now compute the forward default rate

$$\begin{aligned} \overline{f}^i(t,T) &= -\frac{\partial}{\partial T}\log\overline{B}^i(t,T) \\ &= -\frac{\partial}{\partial T}\log\overline{B}^i(0,T) - \frac{\partial}{\partial T}\Big(-\int_0^t(\mu^i_s)ds - \frac{1}{2}\int_0^t |V^i(s,T)|^2 ds \\ &\quad + \int_0^t V^i(s,T)dW_s\Big) \\ &= \overline{f}^i(0,T) + \int_0^t \frac{\partial V^i(s,T)}{\partial T}V^i(s,T)ds - \int_0^t \frac{\partial V^i(s,T)}{\partial T}dW_s \end{aligned}$$

We obtain

$$\overline{f}^i(t,T) = \overline{f}^i(0,T) + \int_0^t \sigma^i(s,T)V^i(s,T)ds - \int_0^t \sigma^i(s,T)dW_s$$

We want now to prove that $\overline{f}^i(t,t) = r_t + \lambda^i_t$. However, we can compute μ^i from $\overline{B}^i$.

According to arbitrage pricing theory, the price at time t of a contingent claim with payoff $\mathbf{1}_{\{\tau^i>T\}}$ at time T is given by

$$E\left\{\mathbf{1}_{\{\tau^i>T\}}e^{-\int_t^T r_s ds}|\mathcal{H}_t\right\}.$$

We proved earlier that we obtain

$$E\left\{\mathbf{1}_{\{\tau^i>T\}}e^{-\int_t^T r_s ds}|\mathcal{H}_t\right\} = \mathbf{1}_{\{\tau^i>t\}}E\left\{e^{-\int_t^T (r_s+\lambda_s^i)ds}|\mathcal{F}_t\right\}.$$

We can deduce that

$$\overline{B}^i(t,T) = E\left\{e^{-\int_t^T (r_s+\lambda_s^i)ds}|\mathcal{F}_t\right\} \tag{4.8}$$

on the set $\{\tau^i > t\}$.

By definition of the forward default rate, we have that

$$\overline{f}^i(t,t) = \lim_{T\to t} -\frac{\partial \log \overline{B}^i(t,T)}{\partial T}.$$

From Equation (4.8), we obtain

$$-\frac{\partial \log \overline{B}^i(t,T)}{\partial T} = \frac{E\left\{(r_T+\lambda_T^i)e^{-\int_t^T (r_s+\lambda_s^i)ds}|\mathcal{F}_t\right\}}{E\left\{e^{-\int_t^T (r_s+\lambda_s^i)ds}|\mathcal{F}_t\right\}}.$$

We know that r and λ^i are uniformly bounded and therefore

$$\lim_{T\to t} e^{-\int_t^T (r_s+\lambda_s^i)ds} = 1.$$

Moreover, r and λ^i are $\mathbb{F}$-adapted and therefore

$$\begin{aligned}
\lim_{T\to t} -\frac{\partial \log \overline{B}^i(t,T)}{\partial T} &= \lim_{T\to t} \frac{E\left\{(r_T+\lambda_T^i)e^{-\int_t^T (r_s+\lambda_s^i)ds}|\mathcal{F}_t\right\}}{E\left\{e^{-\int_t^T (r_s+\lambda_s^i)ds}|\mathcal{F}_t\right\}} \\
&= E\left\{(r_t+\lambda_t^i)|\mathcal{F}_t\right\} \\
&= r_t + \lambda_t^i
\end{aligned}$$

□

Define the discounted loss process

$$L^n(t,T) = E\left\{\frac{1}{n}\sum_{i=1}^{n}\mathbf{1}_{\{\tau^i\leq T\}}e^{-\int_t^T r_s ds}|\mathcal{F}_t\right\}$$

and the discounted survival process

$$S^n(t,T) \quad = \quad = E\left\{\frac{1}{n}\sum_{i=1}^{n}\mathbf{1}_{\{\tau^i>T\}}e^{-\int_t^T r_s ds}|\mathcal{F}_t\right\}$$

We proved that

$$\begin{aligned} S^n(t,T) \quad &= \quad = \frac{1}{n}\sum_{i=1}^{n}\mathbf{1}_{\{\tau^i>t\}}E\left\{e^{-\int_t^T r_s ds}e^{-\int_t^T \lambda_s^i ds}|\mathcal{F}_t\right\} \\ &= \quad = \frac{1}{n}\sum_{i=1}^{n}\mathbf{1}_{\{\tau^i>t\}}e^{-\int_t^T \overline{f}^i(t,s)ds} \end{aligned}$$

We have the following convergence result

Proposition 35. *If λ^i is in L^2 for each $i \in \mathbb{N}$, then*

$$S^n(t,T) - \frac{1}{n}\sum_{i=1}^{n}e^{-\int_0^t \lambda^i ds}e^{-\int_t^T \overline{f}^i(t,s)ds}$$

converges to zero in L^2

Proof. First compute

$$
\begin{aligned}
& E\left\{\left(S^n(t,T) - \frac{1}{n}\sum_{i=1}^{n} e^{-\int_0^t \lambda^i ds} e^{-\int_t^T \overline{f}^i(t,s)ds}\right)^2 |\mathcal{F}_t\right\} \\
= \; & \frac{1}{n^2}\sum_{i=1}^{n} E\left\{\left(\mathbf{1}_{\{\tau^i>T\}} - e^{-\int_0^t \lambda^i ds} e^{-\int_t^T \overline{f}^i(t,s)ds}\right)^2 |\mathcal{F}_t\right\} \\
+ \; & \sum_{i\neq j} E\left\{\left(\mathbf{1}_{\{\tau^i>T\}} - e^{-\int_0^t \lambda^i ds} e^{-\int_t^T \overline{f}^i(t,s)ds}\right)\right. \\
& \left.\left(\mathbf{1}_{\{\tau^i>T\}} - e^{-\int_0^t \lambda^j ds} e^{-\int_t^T \overline{f}^j(t,s)ds}\right) |\mathcal{F}_t\right\} \\
= \; & \frac{1}{n^2}\sum_{i=1}^{n} E\left\{\left(\mathbf{1}_{\{\tau^i>T\}} - e^{-\int_0^t \lambda^i ds} e^{-\int_t^T \overline{f}^i(t,s)ds}\right)^2 |\mathcal{F}_t\right\} \\
+ \; & \sum_{i\neq j} E\left\{\left(\mathbf{1}_{\{\tau^i>T\}} - e^{-\int_0^t \lambda^i ds} e^{-\int_t^T \overline{f}^i(t,s)ds}\right) |\mathcal{F}_t\right\} \\
& E\left\{\left(\mathbf{1}_{\{\tau^i>T\}} - e^{-\int_0^t \lambda^j ds} e^{-\int_t^T \overline{f}^j(t,s)ds}\right) |\mathcal{F}_t\right\}
\end{aligned}
$$

We assumed that the λ^i are in L^2 and therefore

$$
E\left\{\left(\mathbf{1}_{\{\tau^i>T\}} - e^{-\int_0^t \lambda^i ds} e^{-\int_t^T \overline{f}^i(t,s)ds}\right)^2 |\mathcal{F}_t\right\}
$$

is bounded and

$$
\begin{aligned}
& E\left\{\left(\mathbf{1}_{\{\tau^i>T\}} - e^{-\int_0^t \lambda^j ds} e^{-\int_t^T \overline{f}^j(t,s)ds}\right) |\mathcal{F}_t\right\} \\
= \; & E\left\{E\left\{\left(\mathbf{1}_{\{\tau^i>T\}} - e^{-\int_0^t \lambda^j ds} e^{-\int_t^T \overline{f}^j(t,s)ds}\right) |\mathcal{H}_t\right\} |\mathcal{F}_t\right\} \\
= \; & E\left\{\mathbf{1}_{\{\tau^i>t\}} E\left\{e^{-\int_t^T \lambda^i_s ds} |\mathcal{H}_t\right\} - e^{-\int_0^t \lambda^j ds} e^{-\int_t^T \overline{f}^j(t,s)ds} |\mathcal{F}_t\right\} \\
= \; & E\left\{\mathbf{1}_{\{\tau^i>t\}} E\left\{e^{-\int_t^T \lambda^i_s ds} |\mathcal{F}_t\right\} - e^{-\int_0^t \lambda^j ds} e^{-\int_t^T \overline{f}^j(t,s)ds} |\mathcal{F}_t\right\} \\
= \; & E\left\{e^{-\int_0^t \lambda^j ds} e^{-\int_t^T \overline{f}^j(t,s)ds} - e^{-\int_0^t \lambda^j ds} e^{-\int_t^T \overline{f}^j(t,s)ds} |\mathcal{F}_t\right\} \\
= \; & 0.
\end{aligned}
$$

We used that

$$
E\left\{\mathbf{1}_{\{\tau^i>t\}} - e^{-\int_0^t \lambda^j ds} |\mathcal{F}_t\right\} = 0
$$

and

$$E\left\{e^{-\int_t^T \lambda_s^i ds}|\mathcal{F}_t\right\} = e^{-\int_t^T \overline{f}^j(t,s)ds}.$$

□

4.5 The Asymptotic Model

The pool of mortgages is divided into K buckets or homogeneous sub-pools of mortgages of size n_k. The mortgages are sorted according to their FICO scores, DTI and LTV. We can therefore assume that for each k, $0 \leq k \leq K$, each mortgage in the sub-pool k has the same probability of default (respectively the same probability of prepayment).

Define by $\tau^{i,k}$ the default time of mortgage i in pool k.

Assumption 3. *For each k, $0 \leq f \leq K$, the default times $(\tau^{i,k})_{i>0}$ are identically distributed or equivalently*

$$P(\tau^{i,k} > t|\mathcal{F}_t) = e^{-\int_0^t \lambda_s^k ds}, \quad 0 \leq k \leq K, 0 \leq i \leq n_k.$$

This assumption is consistent with the way we are building our portfolio and implies that the default rates in pool k do not depend on i anymore. We obtain

$$\begin{aligned} \overline{B}^{i,k}(t,T) &= E\left\{e^{-\int_t^T (r_s+\lambda_s^{i,k})ds}|\mathcal{F}_t\right\} \\ &= E\left\{e^{-\int_t^T (r_s+\lambda_s^{k})ds}|\mathcal{F}_t\right\} \end{aligned}$$

and therefore

$$\begin{aligned} \overline{f}^{i,k}(t,T) &= -\frac{\partial \log \overline{B}^{i,k}(t,T)}{\partial T} \\ &= -\frac{\partial \log \overline{B}^{k}(t,T)}{\partial T} \\ &= \overline{f}^{k}(t,T). \end{aligned}$$

We have the following corollary for Theorem (29)

Corollary 1. *If $\lambda^k \in L^2(\mathcal{F})$, then for all $t \in \mathbf{R}_+$,*

$$\lim_{n_k \to \infty} \frac{1}{n_k} \sum_{i=1}^{n_k} N_t^{i,k} = 1 - \exp(-\int_0^t \lambda_s^k ds)$$

where the convergence is in L^2.

Proof. Remark that

$$\begin{aligned} E\left\{\frac{1}{n_k}\sum_{i=1}^{n_k}\int_0^{t\wedge\tau^{i,k}} \lambda_s^{i,k}|\mathcal{F}_t\right\} &= E\left\{\int_0^t \lambda_s^k e^{-\int_0^t \lambda_s^k ds}|\mathcal{F}_t\right\} \\ &= 1 - \exp(-\int_0^t \lambda_s^k ds) \end{aligned}$$

□

Another important result concerns the pricing of Put options on the cumulative loss process. Indeed, our objective is to price CDOs of subprime ABS and we proved in the previous chapter that we can reduce this problem to pricing Put spreads on the cumulative loss process (see Equation (3.1)).

Corollary 2. *If λ^k is in $L^2(\mathcal{F})$, then for all $t \in \mathbf{R}_+$,*

$$\begin{aligned} &\lim_{n_k \to \infty} E\left\{\left(K - \frac{1}{n_k}\sum_{i=1}^{n_k} N_t^{i,k}\right)_+ e^{-\int_t^T r_s ds}\middle| \mathcal{H}_t\right\} \\ = \;& E\left\{\left(K - 1 + \exp\left(-\int_0^t \lambda_s^k ds\right)\right)_+ e^{-\int_t^T r_s ds}\middle| \mathcal{F}_t\right\} \end{aligned}$$

where the convergence is conditional in L^2.

Proof. For $x, y, K \in \mathbb{R}$, we have the following

$$\begin{aligned}(K-x)_+ - (K-y)_+ &= (K-y+y-x)_+ - (K-y)_+ \\ &= (K-y+y-x)\mathbf{1}_{\{K>x\}} - (K-y)\mathbf{1}_{\{K>y\}} \\ &= (K-y)(\mathbf{1}_{\{K>x\}} - \mathbf{1}_{\{K>y\}}) + (y-x)\mathbf{1}_{\{K>x\}} \\ &= (K-y)\mathbf{1}_{\{K\in[x,y]\}} + (y-x)\mathbf{1}_{\{K>x\}}\end{aligned}$$

Using this result, we obtain

$$\begin{aligned}(K-\Lambda_T^{n_k})_+ - (K-\Lambda_T)_+ &= (K-\Lambda_T)\mathbf{1}_{\{K\in[min(\Lambda_T,\Lambda_T^{n_k}),max(\Lambda_T,\Lambda_T^{n_k})]\}} \\ &\quad +(\Lambda_T-\Lambda_T^{n_k})\mathbf{1}_{\{K>\Lambda_T^{n_k}\}}\end{aligned}$$

where

$$\Lambda_T^{n_k} = \frac{1}{n_k}\sum_{i=1}^{n_k} N_t^{i,k}$$

and

$$\Lambda_T = 1 - \exp\left(-\int_0^t \lambda_s^k ds\right).$$

We can now take the expectation. We obtain

$$\begin{aligned}& E\left\{(K-\Lambda_T^{n_k})_+ - (K-\Lambda_T)_+ | \mathcal{H}_t\right\} \\ = \; & E\left((K-\Lambda_T)\mathbf{1}_{\{K\in[min(\Lambda_T,\Lambda_T^{n_k}),max(\Lambda_T,\Lambda_T^{n_k})]\}}|\mathcal{H}_t\right) \\ & +E\left\{\Lambda_T-\Lambda_T^{n_k}|\mathcal{H}_t\right\}\end{aligned}$$

We proved that $\Lambda_T^{n_k}$ converge to Λ_T in L^2. Therefore:

$$\mathbf{1}_{\{K\in[min(\Lambda_T,\Lambda_T^{n_k}),max(\Lambda_T,\Lambda_T^{n_k})]\}} \longrightarrow 0 \text{ in } L^2$$

and

$$E\left\{\Lambda_T - \Lambda_T^{n_k} | \mathcal{H}_t\right\} \longrightarrow 0 \text{ in } L^2$$

However, Λ_t is $\mathbb{F}$-adapted. Therefore we have

$$E\{(K - \Lambda_T)_+ | \mathcal{H}_t\} = E\{(K - \Lambda_T)_+ | \mathcal{F}_t\}$$

□

Remark 13. *The same results can be obtained for prepayment rates.*

In this chapter, we gave a condition on the underlying mortgages default rates that allow us to directly compute the loss distribution term structure of the portfolio. Then, we built a tractable model for pricing Put options on the cumulative loss process.

BIBLIOGRAPHY

[1] Andersen, L.B.G., Piterbarg, V. and Sidenius, J., (2008). *A new framework for dynamic credit portfolio loss modeling*. International Journal of Theoretical and Applied Finance, 11(2):163-197.

[2] Antonelli, F., (1990). *Backward Forward Stochastic Differential Equations*, Annals of Probability 3 3, 777-793.

[3] Avellaneda, M. and Zhu, J. (2001). *Modeling the distance-to-default process of a firm*. Working Paper.

[4] Bally, V., (1997). *Approximation scheme for solutions of BSDE*. In Backward stochastic differential equations (Paris, 19951996), volume 364 of Pitman Res. Notes Math. Ser., pages 177191. Longman, Harlow.

[5] Barrera-Esteve, C., Bergeret, F., Dossal, C., Gobet, E., Meziou, A., Munos, R. and Reboul-Salze, D., (2006). *Numerical Methods for the Pricing of Swing Options: A Stochastic Control Approach*, Methodology And Computing In Applied Probability, **8**, 517-540.

[6] Bennani, N. and Dahan, D., (2004). *An extended creditv market model*. Stochastic Finance International Conference, Lisbon.

[7] Benssoussan, A., (1983). *Stochastic maximum principle for distributed parameter systems*. J. Franklin Inst., 315, 387-406.

[8] Bielecki, T. R. and Rutkowski, M., (2002). *Credit Risk: Modeling, Valuation and Hedging*. Springer-Verlag, Berlin.

[9] Bielecki, T. R., Jeanblanc, M. and Rutkowski, M., (2007). *Pricing and trading credit default swaps in a hazard process model*. Working Paper.

[10] Bismut, J. M., (1973). *Theorie probabiliste du controle des diffusions*. Mem. Amer. Math. Soc. 176.

[11] Black, F. and Cox, J., (1976). *Valuing corporate securities: Some effects of bond indenture provisions*. Journal of Finance, 31:351367.

[12] Bouchard, B., Ekeland, I. and Touzi, N., (2002). *On the Malliavin approach to Monte Carlo approximation of conditional expectations*. Finance and Stochastics.

[13] Black, F. and Scholes, M., (1973). *The pricing of options and corporate liabilities*, Journal of Political Economy, 81:63754, 1973.

[14] Bouchard, B. and Touzi, N., (2004). *Discrete time approximation and Monte Carlo simulation of backward stochastic differential equations*, Stochastic Process. Appl. **111**, 175206.

[15] Carmona, R. and Dayanik S., (2007). *Optimal multiple-stopping of linear diffusions*, Mathematics of Operations Research to appear.

[16] Carmona, R. and N. Touzi, (2008). *Optimal Multiple Stopping and Valuation of Swing Options*, Mathematical Finance **18**, 239-268.

[17] Çetin, U., Jarrow, R., Protter, P. and Yildirim, Y., (2004). *Modeling credit risk with partial information*, Annals of Applied Probability, 14 (3), 1167-1178.

[18] Cherubini, U., Luciano, E. and Vecchiato, W., (2004). *Copula methods in finance*. Wiley.

[19] Clement, E., Lamberton, D. and Protter P., (2002). *An analysis of a least squares regression method for American option pricing*, Finance and Stochastics, 6, 449-472.

[20] Coculescu, D., Geman, H. and Jeanblanc, M., (2006). *Valuation of default sensitive claims under imperfect information*. Working Paper.

[21] Dahlgren, M., (2005). *A Continuous Time Model to Price Commodity-Based Swing Options*, Review of Derivatives Research, **8**, 27-47.

[22] Davis, M. and Lo, V., (2001). *Infectious defaults*. Quantitative Finance, 1(4):382387.

[23] Dellacherie, C., (1970). *Capacite et Processus Stochastiques*, Springer-Verlag, Heidelberg.

[24] Dellacherie, C. and Meyer, P.A., (1978). *Probabilities and potential*, North-Holland.

[25] Duffie, D. and D. Lando, (2001). *Term Structures of Credit Spreads with Incomplete Accounting Information*, Econometrica, Vol. 69-3, pp. 633-664.

[26] Duffie, D. and Singleton K., (1999). *Simulating correlated defaults*. Working Paper, Stanford University.

[27] El Karoui, N. and Quenez, M.C., (1996). *Non-linear Pricing Theory and Backward Stochastic Differential Equations*, Financial Mathematics (ed: W.J. Runggaldier), Lecture Notes in Mathematics **1656**, Springer Verlag, 191246.

[28] El Karoui, N., Kapoudjian, C., Pardoux, E., Peng, S. and M.C. Quenez (1997). *Reflected Solutions of Backward SDEs and Related Obstacle Problems for PDEs*, Ann. Prob., Vol. 25, **2**, 702-737.

[29] El Karoui, N., Peng, S.G. and Quenez, M.C., (1997). *Backward stochastic differential equations in finance*. Math. Finance, 7(1):171.

[30] Ethier, S. N. and Kurtz, T. G., (1986) *Markov processes: Characterization and convergence*. John Wiley & Sons.

[31] Fournie, E., Lasry, J.-M, Lebuchoux, J. Lions, P.-L, Touzi, T., (1999). *Applications of Malliavin calculus to Monte-Carlo methods in finance*. Fin. and Stoch. 3, 391412.

[32] Frey, R. and Schmidt, T., (2006). *Pricing corporate securities under noisy asset information*. Working Paper.

[33] Giesecke, K., (2002). *Default compensator, incomplete information, and the term structure of credit spreads*. Working Paper, Humboldt-Universitat zu Berlin.

[34] Giesecke, K., (2003). *Successive correlated defaults: Pricing trends and simulation*. Computing in Economics and Finance, 247, 2003. K.

[35] Giesecke, K., (2004). *Credit risk modeling and valuation: An introduction*. Working Paper, Cornell University.

[36] Giesecke, K., (2006). *Default and information*. Journal of Economic Dynamics and Control, 30(11):22812303.

[37] Giesecke, K. and Goldberg, L. (2004). *Forecasting default in the face of uncertainty*. Journal of Derivatives, 12(1):1125.

[38] Giesecke, K. and Weber, S., (2004). *Cyclical correlations, credit contagion, and portfolio losses*. Journal of Banking and Finance, 28(12):30093036, .

[39] Gobet, E., Lemor, J.P. and Warin, X., (2005). *A regression-based Monte Carlo method to solve backward stochastic differential equations*. Ann. Appl. Probab. Volume 15, Number 3 (2005), 2172-2202.

[40] Gupton, G., Finger, C. and Bhatia. M., (1997). *Creditmetrics*. Technical Document, J. P. Morgan.

[41] Hamadène, S., (2002). *Reflected BSDEs with Discontinuous Barrier and Application*, Stochastics and Stochastic Reports, vol.74 (3-4), pp.571-596

[42] Heath, D., Jarrow, R. A. and Morton, A., (1992). *Bond Pricing and the Term Structure of Interest Rates: A New Methodology for Contingent Claims Valuation*. Econometrica, Econometric Society, vol. 60(1), pages 77-105.

[43] Hull, J., and White, A., (1995). *The impact of default risk on the prices of options and other derivative securities*. Journal of Banking and Finance, 19(2):299322,

[44] Hull, J., and White, A., (2001). *Valuing credit default swaps II: Modeling default correlations*. Journal of Derivatives, 8 No. 3:1222.

[45] Jaillet, P., Ronn, E.I. and Tompaidis S., (2004). *Valuation of Commodity-Based Swing Options*, Management Science, **50**, 909–921.

[46] Jacob, J. and Shiryaev, A. N., (2003). *Limit Theorems for Stochastic Processes*. 2nd Ed., Springer.

[47] Jarrow, R. A. and Protter, P., (2004). *Structural versus reduced form models: a new information based perspective*. Journal of Investment Management, 2 (2), 1-10.

[48] Jarrow, R. A. and Protter, P, (2006). *An Introduction to Financial Asset Pricing*, to appear in the Elsevier Handbook of Financial Engineering.

[49] Jarrow, R. A. and Turnbull, S., (2000). *Derivative Securities*, South-Western College Publishing.

[50] Jarrow, R. A. and Turnbull, S., (1995). *Pricing derivatives on financial securities subject to credit risk*. Journal of Finance, 50(1):5385.

[51] Jarrow, R. A. and Yu, F., (2001). *Counterparty risk and the pricing of defaultable securities*. Journal of Finance, 56(5):17651799.

[52] Jeanblanc, M. and Valchev, S., (2005). *Partial Information and Hazard Process*. International Journal of Theoretical and Applied Finance 8, 807-838.

[53] Kusuoka, S., (1999). *A remark on default risk models. Advances in Mathematical Economics*, 1:6982.

[54] Keppo, J., (2004). *Pricing of Electricity Swing Options*, Journal of Derivatives, **11**, 26-43.

[55] Kloden, P.E. and Platen, E., (1992). *Numerical Solution of Stochastic Differential Equations*, Springer, Berlin.

[56] Lando, D., (2004). *Credit Risk Modeling*, Princeton University Press.

[57] Leland, Hayne E., (1994). *Corporate debt value, bond covenants, and optimal capital structure.* Journal of Finance 49 (September): 121352.

[58] Leland, Hayne E., (2004). *Predictions of expected default frequencies in structural models of debt* Journal of Investment Management 2, 520.

[59] Leland, Hayne E. and Toft, K., (1996). *Optimal capital structure, endogenous bankruptcy and the term structure of credit spreads*. Journal of Finance, 51:9871019.

[60] Lemor, J.P., (2005). *Ph.D. thesis*, Ecole Polytechnique.

[61] Li, D. (2000). *On default correlation: A copula function approach*. Journal of Fixed Income, 9:4354.

[62] Longstaff, F.A. and Schwartz, E.S., (2001). *Valuing American options by simulation: a simple least squares approach*. The Review of Financial Studies, 14:113147.

[63] Longstaff, F.A. and Schwartz, E.S., (1995). *A simple approach to valuing risky fixed and floating rate debt.* Journal of Finance, 50, 789-819.

[64] Ma, J., Protter, P. and Yong, J., (1994). *Solving forward-backward stochastic differential equations explicitlya four step scheme*. Probab. Theory Related Fields, 98(3):339359.

[65] Ma, J. and Yong, J., (1993). *Solvability of Forward Backward SDEs and the nodal*

set of hamilton-Jacobi-Bellman equations. Chinese Annals of Mathematics 16, 279-298.

[66] Mella-Barral, P. and Perraudin, W., (1997). *Strategic debt service.* Journal of Finance, 52(2):531556.

[67] Merton, R.C. (1974). *On the pricing of corporate debts: the risk structure of interest rates*, Journal of Finance, 29, 449-470.

[68] Merton, R.C. (1976). *Option pricing when underlying stock returns are discontinuous*. Journal of Financial Economics, 3:125144.

[69] Nelsen, R., (1999). *An Introduction to Copulas*. Springer.

[70] Pardoux, E. and Peng, S.G., (1990). *Adapted solution of a backward stochastic differential equation*. Systems Control Lett., 14(1):5561.

[71] Protter, P., (2005). *Stochastic Integration and Differential Equations: Second Edition, Version 2.1*, Springer-Verlag, Heidelberg.

[72] Revuz, D., Yor, M., (1994). *Continuous Martingales and Brownian Motion, Second edition*. Springer MR 1303781 — Zbl 0804.60001

[73] Rodriguez, J., (2005). *Ph.D. thesis*, Cornell University

[74] Schmidt, T. and Novikov, A. (2008). *A structural model with unobserved default boundary*. Applied Mathematical Finance. Volume 15, Issue 2, 183-2003.

[75] Schonbucher, P., (2003). *Credit derivatives pricing models*. Wiley.

[76] Schonbucher, P., (2006). *Portfolio losses and the term structure of loss transition rates: a new methodology for the pricing of portfolio credit derivatives*. Working Paper

[77] Talay, D., (1996). *Probabilistic Numerical Methods for PDEs: Element of Analysis*. Lecture Notes in Mathematics 1627.

[78] Vasicek, O., (1977). *An equilibrium characterization of the term structure*. Journal of Financial Economics, 5:177188.

[79] Zeng, Y., (2005). *Compensators of Stopping Times*, Ph.D. thesis, Cornell University

[80] Zhou, C., (2001). *The term structure of credit spreads with jump risk*. Journal of Banking & Finance, 25:20152040.

[81] Zhang, J., (2004). *A numerical scheme for BSDEs*. Ann. Appl. Probab., 14(1):459 488.