ESSAYS ON ECONOMICS AND COMPUTER SCIENCE

A Dissertation

Presented to the Faculty of the Graduate School

of Cornell University

in Partial Fulfillment of the Requirements for the Degree of

Doctor of Philosophy

by

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December 2021

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Cornell University 2021

This dissertation considers a number of problems in pure and applied game theory. The first chapter considers the problem of how the introduction of fines and monitoring affects welfare in a routing game. I characterize equilibria of the game and discuss network topologies in which the introduction of fines can harm those agents which are not subject to them. The second, and primary, chapter considers the computational aspects of tenable strategy sets. I characterize these set-valued solution concepts using the more familiar framework of perturbed strategies, introduce strong alternatives to the problems of verifying whether a strategy block satisfies the conditions of tenability, and provide some hardness results regarding the verification of fine tenability. Additionally, I show an inclusion relation between the concept of coarse tenability and the notion of stability introduced by Kohlberg and Mertens (1986). Finally, I show how the methods developed for tenability provide an alternative characterization for proper equilibria in bimatrix games. This characterization gives a bound on the perturbations required in the definition of proper equilibria, though such bounds cannot be computed efficiently in general. The third, and final, chapter develops a model of contracting for content creation in an oligopolistic environment of attention intermediaries. I characterize symmetric equilibria in single-homing (exclusive) and multi-homing regimes. The focus is on the trade-off between reductions in incentives offered by intermediaries and the benefits of access to additional content for consumers. I show that when the extent of

multi-homing is exogenous in the absence of exclusivity clauses, consumer surplus is always higher with multi-homing than under exclusivity, despite weaker incentives offered by platforms to content creators.

BIOGRAPHICAL SKETCH

Tyler Matthew Porter holds a Bachelor of Arts from Kent State University, double majoring in mathematics and economics. He also holds a Masters degree in economics from Cornell University. He is currently pursuing a Ph.D. in economics from Cornell University. His primary research interests are in pure and computational game theory.

For David Zamos.

ACKNOWLEDGEMENTS

I thank my committee members David Easley, Eva Tardos, Larry Blume and Tommaso Denti for their continued assistance during my time as a graduate student at Cornell University. David Easley for his expertise and excellent advising, as well as his understanding and flexibility when I would barge into his office with questions. Eva Tardos for her guidance on topics in computer science and assistance with submissions. Larry Blume for his advice on motivating ideas and brief discussions on game theory and applied mathematics. Tommaso Denti for his help with developing ideas and assumptions.

I would like to thank my closest friends within the economics department Alesandro Arcuri, Fikri Pitsuwan and Derek Lougee for always being there when I needed it. I thank Sasha Tettenhorst for excellent discussions ranging from jazz composition to pure mathematics.

I thank Michael Lovenheim, Seth Sanders and Penny Sanders for innumerable zoom meetings ahead of the job market. I wouldn't have this opportunity without them.

Finally, I thank my family for setting me on the path.

CONTENTS

CHAPTER 1

INTRODUCTION

The field of economics and computation has experienced a surge in activity in the last two decades. Computer science offers novel perspectives on economic modeling and efficiency constraints. Indeed, the field of approximate mechanism design exists in part due to the fact that, while efficient mechanisms can be shown to exist in quite general instances, exact implementation of these can be infeasible in practice. While complexity concerns provide an obvious connection between the theory of computing and economic efficiency, the field of algorithmic game theory offers insights beyond those that can be obtained by reductions.[1] In cases where a well-defined socially optimal outcome exists the price of anarchy offers a novel welfare criterion in which one can evaluate how much worse equilibria are compared to the optimum. This allows for a different perspective on the efficacy of policy interventions. Discrete structures, and particularly graphs, are becoming an increasingly integral modeling tool in microeconomics and econometrics. In this dissertation, I consider several problems which broadly reside at the interface of economics and computer science.

The first chapter considers a twist on selfish routing that allows for a network operator to impose fines on an exogenous subset of the flow. Tolling and fines have been considered a great deal in the literature on routing games and transportation systems. This paper considers a slightly different welfare problem. The focus of the paper, aside from characterizing equilibria, is whether or not the inclusion of fines into the system can yield negative externalities for

[1] I believe, and I am not alone in this opinion, that complexity concerns will play an increasingly important role in the evaluation and construction of solution concepts in the field of game theory, as well as our understanding of human behavior more broadly. von Neumann was among the first to seriously consider the connections between computers and the human brain.

those that are not subject to them. I demonstrate, using several results already established in the literature, different topologies in which such negative externalities cannot arise. I then show a class of examples in which those not subject to fines can in fact be worse off, and provide a loose sufficient condition for this to be the case. An open problem is to determine a single necessary and sufficient condition on the network topology which guarantees that those not subject to fines cannot be worse off after the introduction of fines.

The second chapter is a pure computational game theory paper. I consider the robust set-valued equilibrium concepts of coarse and fine tenability introduced in Myerson and Weibull (2015)[48]. I show how the framework of consideration-set games can be mapped into the more familiar language of strategically-perturbed games, while preserving the appropriate conditions on type distributions that the solution concepts demand. This connection allows for a number of insights. First, it allows for a direct comparison between tenability and solution concepts already established within the refinement literature. In particular, I show that in any finite game every coarsely tenable block, viewed as a subset of the space of mixed strategy profiles, contains a set that is stable under the definition of Kohlberg and Mertens (1986)[37]. Moreover, I provide strong alternatives to coarse and fine tenability using existence of refutations of a particular form. This allows for a straightforward characterization of these solution concepts in bimatrix games using the properties of a collection of polytopes. I also show how these refutations can be used to determine the complexity of verifying whether a set is finely tenable. The analysis shows that finding a refutation for a strategy block that consists of singletons for each player is NP complete. The problem of finding a refutation is the complementary problem of verifying whether a set is finely tenable, and so the verification problem for fine tenability, even in the case of singleton strategies, is CO-NP complete. Finally, I show how the methods developed in this paper can be applied to the problem of verifying proper equilibria. I demonstrate a geometric characterization of proper equilibria which has, to the best of my knowledge, not been formalized in the past, though there do exist some informal methods which implicitly use this characterization.

The final chapter is a short article on the economics of the internet, specifically media platforms. I develop a tractable parametric model of competition among attention intermediaries that design simple limited liability contracts to incentivize the production of high quality content on their platform in the face of moral hazard. I use an attention based model of consumer viewership in order to demonstrate the distortionary nature of advertising in the market, as well as allow for tractable generalizations of the model to continuous output. Each platform serves as an intermediary between content and consumers. Offering better incentives leads to more high quality content on a platform, which in turn increases demand for that platform. The focus is on the effects of exclusivity on consumer welfare. I show that under sufficient concavity restrictions on content creator membership and assuming exogenous multi-homing of content creators, consumer surplus is increasing in the extent of multi-homing despite reductions in incentives offered by platforms due to the presence of multi-homing content. This suggests that exclusivity in these markets may be harmful for consumer welfare, though a more complete picture requires allowing for endogenous multi-homing. This is a subject of further research.

CHAPTER 2
MONITORING IN SELFISH ROUTING GAMES

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I consider a model of nonatomic selfish routing with agents of different types and a network operator. In particular, I suppose that some fixed fraction of the commuters are either in violation of a traffic contract or are unwanted by the network operator, who in turn has the ability and authority to impose additional costs via fines on these types of agents. This model is motivated by several situations that arise in transportation systems. The primary applications are in fare evasion in public transportation networks, the location of mobile weigh stations for the monitoring of freight transportation, and the location of law-enforcement for traffic violations. In this paper, I characterize properties of equilibria in terms of the potential externalities that can arise as a result of the inclusion of fines in the presence of heterogeneous agents. A discussion of the network topologies in which the inclusion of a revenue maximizing network operator cannot harm those not subject to fines at an equilibrium follows.

2.1 Introduction

Recently there has been a growing literature on network interdiction and pricing games, wherein a player or group of players seek to interdict along or increase the cost of paths used by agents (attackers) whose objective is to traverse a network. Several papers in this literature [34, 53] consider models without congestion where a single network operator places fines on edges in a network in order to maximize revenue. On the other hand, models of network pricing [1, 33] that include congestion often consider agents with homogeneous cost functions. While the contributions of these papers and other models of network interdiction are interesting in their own right, the motivations behind these models often involve situations in which honest agents travel alongside those being interdicted. This paper seeks to address how the inclusion of security to interdict a fraction of the flow affects the congestion experienced by the honest agents traversing the network.

The primary applications are in fare evasion in public transportation networks, the location of mobile weigh stations for the monitoring of freight transportation, and the location of law-enforcement and speed traps for traffic violations. While other applications, such as the location of DUI checkpoints and the implementation of random checkpoints to monitor borders, may seem relevant as well, we will see that the model described below excludes certain assumptions that are significant in these situations. The model of this paper is stylized, and attempting to make statements about more general situations of network pricing and interdiction would call for additional modeling assumptions in order to justify them.

2.1.1 Related Literature

This paper is primarily related to the literature on pricing and fines in network routing games. The most related papers to this one are Correa et al. (2017)[34] and Borndorfer et al. (2015)[53]. Correa et al. (2017)[34] consider several variants on a leader-follower framework for fare inspection and fines in public transportation. In their model, the split of agent types is determined endogenously by the fare, fines and inspection strategy of the leader. The model presented in this paper assumes that the masses of evaders and honest agents are exogenously given. This model also assumes that the fine received upon being caught is exogenously given, and that the network authority issuing fines to evaders does not have a first-mover advantage. At the same time, I allow for congestion effects to be present within the network since the primary interest of this paper is to determine the latency externalities experienced by honest agents as a consequence of the inclusion of fines for evaders. In a similar vein, Borndorfer et al. (2015)[53] consider a model of network spot-checking games without congestion in which inspectors are to be distributed across a network to hand out fines to agents evading fares. They show that equilibria can be computed using linear programming and place bounds on the difference in revenue obtained between a setting in which the network authority can move first and when the game is simultaneous moves, referred to as the price of spite. The monopoly setting of Acemoglu and Ozaglar (2007)[1] is also somewhat similar to the model in this paper, but they restrict attention to parallel links and assume that edge costs are homogeneous among players.

This paper is also related to the literature on tolling in routing games with heterogeneous agents. In particular, one can view the agents in this model as trading off latency and money in different ways as in Cole, Dodis and Roughgarden (2003)[55]. In this model, I consider an extreme case in which one group, the honest agents, do not consider tolls at all when making routing decisions (this corresponds to $\alpha = 0$ for these agents in their model). Furthermore, the goal of the fines in this model are to maximize revenue rather than induce optimal flows. The heterogeneous agents models of Cole, Lianeas, and Nikolova (2018)[54] and Meir and Parkes (2018)[42] are also quite similar to this paper. The two types of agents in this model can indeed be viewed as a special case of their models, but the idiosyncratic component of θ_b type preferences are endogenously determined by the network operator in this paper.

A number of other papers consider a graph-theoretic approach to the inefficiencies that can arise in selfish routing games. In particular, we will make use of the definitions and results from Chen, Diao, and Hu (2016)[69].

2.2 Model

I consider a finite, connected, directed multigraph $G = (V, E)$ and a source-sink pair (s,t) with $s, t \in V$. A source node, also called an origin, is the starting point for some set of agents. A sink, also called a destination, is a terminal node for a set of agents. An agent with source-sink pair (s, t) seeks to route from node s to node t on a shortest path. I will restrict attention to a single-source single-sink setup for this paper.

Definition 1 *A path from node s to node t is a sequence of edges* $\{e_1, e_2, e_3, ..., e_n\}$ *and connecting nodes* $\{s, v_1, v_2, ..., v_{n-1}, t\}$ *such that* $e_1 = (s, v_1)$, $e_j = (v_{j-1}, v_j)$ *for* $j \in$

$2, 3, ..., n-1$ *and* $e_n = (v_{n-1}, t)$*. That is, a path is a sequence of edges and intermediate nodes that connects node s to node t in the graph G. A path is simple if none of the nodes in the sequence are repeated. I will restrict attention to simple paths in this paper.*

Denote the set of simple paths from source s to sink t by P. I will suppose that total mass 1 of agents seek to route from source to sink. When agents route from source to sink along the paths available to them, one obtains a flow f on the graph G.

Definition 2 *A flow on graph G is a function* $f : P \to \mathbb{R}_+$ *that associates to each simple path connecting a source-sink pair a mass of agents routed along that path. The function f must satisfy*

$$\sum_{p \in P} f(p) = 1$$

The amount of flow on an edge is given, with some abuse of notation, by

$$f(e) := \sum_{\{p:\ e \in p\}} f(p)$$

where p is a path from a source to a sink. Associated with each edge is a latency function $C_e : \mathbb{R}_+ \to \mathbb{R}_+$ providing the latency to an agent for using edge e as a function of the flow on edge e. I will assume that C_e is nondecreasing and continuous for every edge e. Under flow f, an agent taking path p incurs a latency cost

$$C_p(f) = \sum_{e \in p} C_e(f(e))$$

This cost is the same for any agent taking this path. There are three kinds of agents in this model. There are agents of type θ_g, "good" types, which are expected cost minimizers and seek only to route from source to sink on a path of lowest cost. There are also agents of type θ_b, "bad" types, which are also expected cost minimizers. Agents of type θ_b have a similar objective as agents of type θ_g, but with one difference. When an agent of type θ_b passes through an edge which I will call "secure", the agent incurs a cost $\kappa \in \mathbb{R}_{++}$. Thus the cost to these agents of choosing any path is given by the cost that would be incurred by a θ_g type, with an additional cost κ of passing through a secure edge if one exists along the path. Agents of type θ_b are "nonadaptive" using the terminology of Correa et al. (2017) in the sense that paths are chosen ex ante and are not revised by θ_b types if they are caught during travel.The set of secure edges is determined by a single agent, referred to as security. I think of the fine level κ to be fixed and exogeneous, with security randomizing over edges in the digraph with the objective of receiving as much revenue from θ_b types as possible. I suppose that security chooses one edge in expectation.

This is a simultaneous moves game. I suppose that the relative weights of θ_g and θ_b types are described by $w(\theta_g) \in [0, 1]$ and $w(\theta_b) \in [0, 1]$ respectively, with the requirement that $w(\theta_g) + w(\theta_b) = 1$. I will restrict attention to deterministic flows in this paper, requiring that agents of type θ_g or θ_b are choosing paths with probability 1 (playing pure strategies)[1]. A pure strategy for security is to secure a single edge $e \in E$. Security's set of mixed strategies is then the set of all probability distributions over edges in G, here denoted by ΔE. Outcomes of this game are then described by a triple (f_g, f_b, a_s), where f_g is a flow of value $w(\theta_g)$, f_b is a flow of value $w(\theta_b)$, and $a_s \in \Delta E$ is a randomization over edges for security. The probability that edge e is secured is denoted by $a_s(e)$. I will denote the total flow by f, so that $f = f_g + f_b$.

[1]One can do the usual "purification" of commuter mixed strategies when latency functions are linear.

Given an outcome (f_g, f_b, a_s), the cost to a θ_g types to taking path p is just the latency:

$$C_p^g(f) = \sum_{e \in p} C_e(f(e))$$

The cost to a θ_b types to taking path p is latency plus expected fine, given by

$$C_p^b(f, a_s) = \sum_{e \in p} (C_e(f(e)) + a_s(e)\kappa)$$

The payoff to security is given by

$$\pi(f, a_s) = \sum_{e \in E} \kappa a_s(e) f_b(e)$$

2.2.1 Equilibria

I consider Nash equilibria of this game in which θ_g and θ_b types play pure strategies.[2] Throughout, I refer to an equilibrium of this form as a *Security Equilibrium*.

Definition 3 *A Security Equilibrium is a triple (f_g, f_b, a_s) such that:*

- *for any path p such that $f_g(p) > 0$*

$$\sum_{e \in p} C_e(f(e)) \leq \sum_{e \in \hat{p}} C_e(f(e)) \quad \forall \hat{p} \in P$$

- *for any path p such that $f_b(p) > 0$*

$$\sum_{e \in p} C_e(f(e)) + a_s(e)\kappa \leq \sum_{e \in \hat{p}} C_e(f(e)) + a_s(e)\kappa \quad \forall \hat{p} \in P$$

- $\sum_{p \in P} f_g(p) = w(\theta_g)$

- $\sum_{p \in P} f_b(p) = w(\theta_b)$

[2] Existence of equilibria when allowing for mixed strategies is an application of the results of Mas-Colell (1984).[41]

- $\forall p \in P$, $f_g(p), f_b(p) \geq 0$
- a_s *solves the following linear program given the flow* f_b

$$\max_{\{a_s \in \triangle E\}} \sum_{e \in E} a_s(e) f_b(e)$$

$$s.t. \quad \sum_{e \in E} a_s(e) = 1 \qquad (LP)$$

$$a_s(e) \geq 0 \ \forall e \in E$$

One can characterize security equilibria of this game as solutions to a collection of convex programs. In particular, one can use the potential function representation of the games induced by the decisions of the other players to obtain the best responses of the θ_g and θ_b type agents. Beckmann (1956)[40] was the first to utilize the potential function to characterize equilibria in traffic problems. The method of using potential functions in games induced by the decisions of other types of players has also been used before (see Babaioff, Kleinberg, and Papadimitrou (2007)[45] for one example). In particular, given f_b, the best response of the θ_g type agents are the solutions to the following convex program:

$$\min_{\{f_g\}} \sum_{e \in E} \int_0^{f_g(e)} C_e(f_b(e) + t)dt$$

$$s.t. \quad \sum_{p} f_g(p) = w(\theta_g) \qquad (CP1)$$

$$f_g(p) \geq 0 \quad \forall p \in P$$

One can also use a similar approach to obtain best responses for agents of type θ_b given f_g and a_s. The best response for θ_b types is obtained by solving the following convex program:

$$\min_{\{f_b\}} \sum_{e \in E} \int_0^{f_b(e)} C_e(f_g(e) + t) + a_s(e)\kappa \, dt$$

$$s.t. \quad \sum_{p} f_b(p) = w(\theta_b) \qquad (CP2)$$

$$f_b(p) \geq 0 \ \ \forall p \in P$$

Lemma 2.2.1 *If a tuple (f_g, f_b, a_s) is such that:*

- *given f_b, f_g solves (CP1) defined above*
- *given f_g and a_s, f_b solves (CP2) defined above*
- *given f_b, a_s solves (LP)*

then (f_g, f_b, a_s) constitutes a secrutiy equilibrium of this game.

Proof To prove the result, I'll show that that the optimality conditions of the above convex programs imply the conditions laid out in the definition of security equilibria. Consider the optimality conditions of (CP1) given f_b, in particular

$$\sum_{e \in p} C_e(f(e)) = -\lambda + \gamma_p \ \ \forall p$$

$$\sum_{p} f_g(p) = w(\theta_g)$$

$$f_g(p) \geq 0 \ \ \forall p \in P$$

$$\gamma_p f_g(p) = 0 \ \ \forall p \in P$$

$$\gamma_p \geq 0 \ \ \forall p \in P$$

where γ_p are the Lagrange multipliers associated with the non-negativity constraints of $f_g(p)$ and λ is the Lagrange multiplier associated with the equality constraint. Note that constraint qualification is satisfied. By the complementary slackness conditions, if $f_g(p) > 0$ then it must be that $\gamma_p = 0$. This immediately implies (from the first order condition) that if $f_g(p) > 0$ then

$$\sum_{e \in p} C_e(f(e)) = -\lambda \leq \sum_{e \in \hat{p}} C_e(f(e)) \;\; \forall \hat{p} \in P$$

Which is the same as the first condition for security equilibrium.

Let's now consider solutions to (CP2) given f_g. The optimality conditions for (CP2) are:

$$\sum_{e \in p} C_e(f(e)) + a_s(e)\kappa = -\lambda + \gamma_p \;\; \forall p \in P$$

$$\sum_{p} f_b(p) = w(\theta_b)$$

$$f_b(p) \geq 0 \;\; \forall p \in P$$

$$\gamma_p f_b(p) = 0 \;\; \forall p \in P$$

$$\gamma_p \geq 0 \;\; \forall p \in P$$

where γ_p are the Lagrange multipliers associated with the non-negativity constraints and λ is the Lagrange multiplier associated with the equality constraint. Note that constraint qualification is satisfied. By the complementary slackness conditions, if $f_b(p) > 0$ then $\gamma_p = 0$. We then obtain that if $f_b(p) > 0$,

$$\sum_{e \in p} C_e(f(e)) + a_s(e)\kappa = -\lambda \leq \sum_{e \in \hat{p}} C_e(f(e)) + a_s(e)\kappa \;\; \forall \hat{p} \in P$$

which is the second condition for security equilibrium. Finally, note that if a_s solves (LP) given f_b, then the final condition of security equilibrium is satisfied.

Evidently then existence of security equilibrium boils down to showing existence of such a triple. The next result uses standard methods from game theory to show existence of security equilibria.

Proposition 2.2.2 *Suppose that the latency on each edge C_e is a nonnegative, increasing, continuous function of the load on that edge. Then the game defined above has a Wardrop equilibrium.*

Proof Define by $F(w(\theta_g))$ to be the collection of all flows on paths from source to sink with value $w(\theta_g)$. Define the analogous collection of flows $F(w(\theta_b))$. These are finite dimensional simplices. Recall that $\triangle E$ is the collection of all probability distributions over edges. It is also a finite dimensional simplex. Thus $F(w(\theta_g)) \times F(w(\theta_b)) \times \triangle E$ is closed and convex. Consider the mapping $\Phi : (F(w(\theta_g)) \times F(w(\theta_b)) \times \triangle E) \to 2^{\{F(w(\theta_g)) \times F(w(\theta_b)) \times \triangle E\}}$ defined by

$$\Phi(f_g, f_b, a_s) = B_g(f_b) \times B_b(f_g, a_s) \times B_s(f_b)$$

where $B_g(f_b))$ is the set of solutions to (CP1) given f_b, $B_b(f_g, a_s)$ is the set of solutions to (CP2) given f_g and a_s, and $B_s(f_b)$ is the set of solutions to (LP) given f_b. To complete the proof, follow the usual steps of verifying that Φ satisfies the conditions of Kakutani's fixed point theorem by using the properties of solutions to the above optimization problems. It is easy to see that the solutions to each problem are nonempty (Weierstrass), convex (by convexity of the objective function), compact (since the objective function is continuous and the constraint set is compact). We get upper-hemicontinuity of Φ by applying Berge's (Minimum) Theorem.

It turns out that one can collapse the two potential functions for θ_g and θ_b types into a single potential function by modifying the graph G appropriately. A consequence of this is the following lemma.

Lemma 2.2.3 *Suppose that the latency function on each edge is a strictly increasing, nonnegative, and continuous function of the load on that edge. Then for a fixed decision* *of security a_s, the payoffs of any of the mutual best responses of θ_g and θ_b types are unique. That is, for a fixed a_s and κ, the (f_g, f_b) that are such that*

- *given f_b, f_g solves (CP1) as above*
- *given f_g and a_s, f_b solves (CP2) as above*

induce payoffs that are unique for the θ_g and θ_b types.

Proof The proof relies on a useful reformulation of the problem.[3] For each edge $e = (v_i, v_j)$ in the graph G, insert an additional node v_{ij} and create a directed edge from v_i to v_{ij} and another from v_{ij} to v_j. Remove the edge $e = (v_i, v_j)$. Finally, construct an additional edge from v_{ij} to v_j. We will allow all types to pass through the edge (v_i, v_{ij}), and the cost function for this edge is equal to the cost function of the deleted edge $e = (v_i, v_j)$. Now, for the two edges from v_{ij} to v_j, one will have constant cost of 0 and permits only θ_g types to pass. The other edge has constant cost equal to the expected fine that was placed on edge $e = (v_i, v_j)$, and permits only bad types to pass. Performing this change to every edge yields a new graph, call this graph $\hat{G} = (\hat{V}, \hat{E})$. Now, using the new graph and the additional constraints imposed on flows of each type, we collapse the two potential functions in Lemma 2.1 into a single potential function. In particular, we have that security equilibria are exactly the flows that satisfy security's linear program in addition to the potential function below. In what follows, we define P_g as the set of paths that can be used by θ_g types in the new graph, and P_b the set of paths that can be used by θ_b types in the new graph.

$$min_{\{f_g, f_b\}} \sum_{e \in \hat{E}} \int_0^{f(e)} C_e(t) + a_s(e)\kappa dt$$

[3]This reformulation was suggested by Eva Tardos.

$$s.t. \quad \sum_{p \in P_g} f_g(p) = w(\theta_g)$$

$$\sum_{p \in P_b} f_b(p) = w(\theta_b)$$

$$f_b(p), f_g(p) \geq 0, \quad \forall p \in P$$

Note that on any given edge the objective function will take on one of three forms, in particular one of $C_e(t)$, $a_s(e)\kappa$, or 0. Using the fact that this is a convex program when latency functions are nondecreasing, we obtain that θ_g and θ_b types have unique payoffs given the decision of security. [4]

An important remark about the previous lemma is that this implies that a Stackelberg setting in which the network operator moves first and chooses a randomization is well-defined as long as latency functions are strictly increasing on every edge. That is, a given randomization for the network operator will result in a unique level of revenue. To see why, note that the (strict) convexity of the potential function and the fact that latency functions are strictly increasing implies that not only the latency (and fine) on any given edge in $\hat{G}$ is unique for any solution to the above convex program, but also the exact flow on any given edge in G. This implies uniqueness of the total flow at the resulting equilibrium. Note that the exact composition of this total flow in terms of θ_g and θ_b can vary, but I claim that it does not vary in a way that is payoff-relevant. With uniqueness of total flow established, this pins down a unique level of revenue using the uniqueness of payoffs to θ_b types.

[4]The existence of a potential function for this game is a consequence of Theorem 4.4 of Farokhi et al. (2014)[21].

2.3 Results

This section is devoted to examining the properties of security equilibria and how they relate to the equilibrium of the game without security. In particular, we are interested in comparing the latency incurred by θ_g types at equilibria of the security game as above to the latency incurred by all agents at equilibria of the regular selfish routing setting on the same graph with the same cost functions. As a preliminary, Wardrop equilibria[67] of the regular selfish routing game in this setting are given by solutions to the following convex program:

$$min_{\{f\}} \sum_{e \in E} \int_0^{f(e)} C_e(t)dt$$

$$s.t. \quad \sum_p f(p) = 1$$

$$f(p) \geq 0 \quad \forall p \in P$$

Nash equilibria of the selfish routing game have the property that all paths with positive flow have weakly less latency than any other path. That is if $f(p) > 0$, then $\sum_{e \in p} C_e(f(e)) \leq \sum_{e \in \hat{p}} C_e(f(e))$ for any $\hat{p} \in P$. For an explanation of the properties of Nash equilibria in nonatomic selfish routing games, see Roughgarden and Tardos (2001) [57].

I begin by discussing some basic properties of security equilibria. Throughout this section, I will use the notation of:

- $P_s^g(f)$: The set of paths such that if $p \in P_s^g(f)$, then $\sum_{e \in p} C_e(f(e)) \leq \sum_{e \in p'} C_e(f(e))$ for all p'.
- $P_s^b(f)$: The set of paths such that if $p \in P_s^b(f)$, then $\sum_{e \in p} C_e(f(e)) + a_s(e)\kappa \leq \sum_{e \in p'} C_e(f(e)) + a_s(e)$ for all p'.

- $E_s(a_s)$: The set of edges such that $a_s(e) > 0$

I would like to point out the difference between P_g, the paths available to θ_g types in the extended graph $\hat{G}$ used in the proof of Lemma 2.3, and $P_s^g(f)$ defined here. The next lemma simply states that θ_b types all experience the same total expected costs at equilibrium, and that this expected cost is strictly higher than θ_b types. The proof is straightforward.

Lemma 2.3.1 *At any security equilibrium* (f_g, f_b, a_s)*,* $\exists \alpha \in (0, 1]$ *such that for any* $p \in P_s^b(f)$ *and any* $\hat{p} \in P_s^g(f)$

$$\sum_{e \in p} C_e(f(e)) + a_s(e)\kappa = \sum_{e \in \hat{p}} C_e(f(e)) + \alpha\kappa$$

In other words, if $C_b^*(f_g, f_b, a_s)$ *denotes the cost experienced by any* θ_b *types at a security equilibrium, and* $C_g^*(f_g, f_b, a_s))$ *denotes the cost experienced by any* θ_g *type at a security equilibrium, then*

$$C_b^*(f_g, f_b, a_s) = C_g^*(f_g, f_b, a_s) + \alpha\kappa$$

for some $\alpha \in (0, 1]$*.*

Proof Consider any edge $e_s \in E_s$. Using the fact that a_s is a best response, there must be at least one path $p^* \in P_s^b(f)$ and an edge $e^* \in P_s^b(f) \cap E_s(a_s)$. Then the cost of taking path p^* for a θ_b type is

$$\sum_{e \in p^*} C_e(f(e)) + a_s(e)\kappa > \sum_{e \in p^*} C_e(f(e)) \geq \sum_{e \in p} C_e(f(e)) \ \ \forall p : f_g(p) > 0$$

. The result then follows from the fact that all θ_b must be indifferent between any path in $P_s^b(f)$ and all θ_g must be indifferent between any path in $P_s^g(f)$ at the equilibrium (f_g, f_b, a_s).

This easily implies the following result which characterizes in part the equilibrium assignment of security probabilities.

Corollary 2.3.2 *Fix a security equilibrium* (f_g, f_b, a_s)*. Consider the subgraph* $G(f_g)$ *of* G *formed by the set* $P_s^g(f)$ *of shortest latency paths. Claim: The set* $E_s(a_s)$ *of secure edges forms a source-sink cut of* $G(f_g)$*.*

Proof It suffices to show that for every path $p \in P_s^g(f)$, $\exists e \in p$ such that $e \in E_s$. Suppose, seeking contradiction, that there is a path $p^* \in P_s^g(f)$ such that there is no secure edge along p^*. Since $p^* \in P_s^g(f)$

$$\sum_{e \in p^*} C_e(f(e)) \leq \sum_{e \in p} C_e f(e)) \;\; \forall p \in P$$

By assumption,

$$\sum_{e \in p^*} C_e(f(e)) + a_s(e)\kappa = \sum_{e \in p^*} C_e(f(e))$$

Now pick some edge $e_s \in E_s$ and a corresponding path $p_b \in P_s^b(f)$ such that $e_s \in p_b$. This implies that

$$\sum_{e \in p_b} C_e(f(e)) + a_s(e)\kappa > \sum_{e \in p_b} C_e(f(e)) \geq \sum_{e \in p^*} C_e(f(e)) + a_s(e)\kappa$$

Since $f_b(p_b) > 0$, we have that not all θ_b are choosing shortest paths according to their cost function. A contradiction of the equilibrium assumption.

The next result takes the previous corollary a step further by using the objectives of the network operator.

Lemma 2.3.3 *Consider a security equilibrium* (f_g, f_b, a_s) *and the corresponding sets of paths* $P_s^g(f)$ *and* $P_s^b(f)$*. Claim:* $P_s^g(f) \subseteq P_s^b(f)$*.*

Proof Suppose, seeking contradiction, that there is some path $p \in p^g_s(f)$ with $p \notin P^b_s(f)$. Subpath optimality must then imply that there is some subpath $\{e_1, e_2, ...e_n\}$ such that $f_b(e_i) = 0$ for $i = 1, ..., n$ and $e_i \in p$ for $i = 1, ..., n$. Choose a maximal such subpath along p and note that this subpath cannot equal p itself due to corollary 3.2 above. Suppose that this maximal subpath connects the vertices u and v. Then since $p \notin P^b_s(f)$ but $p \in P^g_s(f)$, there must be some edge $e^* \in E_s \cap p$ that lies between u and v along p. But the fact that $e^* \in E_s$ implies that $f_b(e^*) > 0$. Thus we obtain a contradiction.

In the following results, I will refer frequently refer to f^* as an equilibrium of the selfish routing game without security. I will also use f_g, f_b and f to be θ_g flow, θ_b flow and total flow respectively at a security equilibrium. Note that for the flow f^*, one could in principle split θ_g and θ_b types in any way that results in the same total flow. I will refer to f^*_b and f^*_g to be the particular split of θ_g and θ_b types chosen for an equilibrium flow of the game without security. We have that $f^*_g + f^*_b = f^*$ by construction. One interesting component of the following results is that they hold regardless of how one chooses to split the types. Given a security equilibrium (f_g, f_b, a_s), let $C^*_g(f_g)$ denote the common latency experienced by θ_g types. Similarly, let $C^*_b(f_b)$ be the common cost experienced by θ_b types. Finally, let $C^*(f^*)$ denote the common latency experienced by all agents at an equilibrium of the game without security.

Proposition 2.3.4 *Suppose that (f_g, f_b, a_s) is a security equilibrium and f^* any equilibrium flow of the game without security (with any split f^*_g and f^*_b of θ_g and θ_b types among paths). Then*

$$\sum_{e \in E} a_s(e) f_b(e) \leq \sum_{e \in E} a_s(e) f^*_b$$

That is to say, security obtains more revenue from security equilibrium randomization a_s under any equilibrium flow of the game without security than in the security equilibrium (f_g, f_b, a_s).

Proof The proof relies on the collapsed potential function from Lemma 2.3. Using the fact that, given a_s, (f_g, f_b) is a global minimum of

$$\sum_{e\in\hat{E}} \int_0^{f(e)} C_e(t) + a_s(e)\kappa dt$$

in the extended graph $\hat{G} = (\hat{V}, \hat{E})$, one has that

$$\sum_{e\in\hat{E}} \int_0^{f(e)} C_e(t) + a_s(e)\kappa dt \leq \sum_{e\in\hat{E}} \int_0^{f^*(e)} C_e(t) + a_s(e)\kappa dt$$

Which implies that

$$\sum_{e\in P_g\cap P_b} \int_0^{f(e)} C_e(t)dt + \sum_{e\in P_b\setminus P_g} f(e)a_s(e)\kappa \leq \sum_{e\in P_g\cap P_b} \int_0^{f^*(e)} C_e(t)dt + \sum_{e\in P_b\setminus P_g} f^*(e)a_s(e)\kappa \tag{2.1}$$

On the other hand, the equilibrium flow f^* of the game without security is a global minimum of the potential function

$$\sum_{e\in\hat{E}} \int_0^{f^*(e)} C_e(t)dt$$

Therefore

$$\sum_{e\in P_g\cap P_b} \int_0^{f(e)} C_e(t)dt - \sum_{e\in P^g\cap P^b} \int_0^{f^*(e)} C_e(t)dt \geq 0$$

Combining this fact with inequality (1), we have

$$\sum_{e\in P_b\setminus P_g} a_s(e)\kappa[f(e) - f^*(e)] \leq 0$$

Which completes the proof using the equivalence of payoffs between G and $\hat{G}$.

The previous result seems to suggest that θ_b types are contorting themselves, at least to some extent, off of equilibrium paths in order to avoid security. This leads to the question of whether or not the costs of θ_g types always improve when one adds security to the game. This question is answered in the positive for series-parallel graphs using known results, but is not true in general.

Proposition 2.3.5 *Suppose that latency functions are nondecreasing, nonnegative, and continuous functions. Then if G is series-parallel, θ_g types haves weakly less latency at any security equilibrium than at a Nash flow. That is, if (f_g, f_b, a_s) is a security equilibrium and f^* is a Nash flow, then*

$$C_g^*(f_g, f_b, a_s) \leq C^*(f^*)$$

Proof The proof is an application of Lemma 1 from Cole, Lianeas, and Nikolova (2018)[54].[5] Since the equilibrium flow f^* and the security equilibrium flow f have the same value, there is a path P such that for all $e \in P$, $f^*(e) \geq f(e)$ and $f^*(e) > 0$. Thus we have

$$C_g^*(f) \leq \sum_{e \in P} C_e(f(e)) \leq \sum_{e \in P} C_e(f^*(e)) = C^*(f^*)$$

The previous result uses almost no structure of equilibria in the proof. Indeed, it is only using the properties of flows on series parallel graphs and the fact that θ_g types are latency minimizers. This begs the question of which topologies cannot induce any negative externalities for those not subject to fines. Before exploring this, I give a simple result that characterizes weakly dominated edges for the network operator.

[5]This is a consequence of a well-known results by Milchtaich (2006)[43].

Lemma 2.3.6 *Denote by $P(e_i)$ as the collection of paths that pass through edge e_i. Suppose that for two edges e_1 and e_2 that $P(e_1) \subset P(e_2)$. If $a_s(e_1) > 0$ at a security equilibrium (f_g, f_b, a_s), then $f_b(p) = 0$ for any path $p \in P(e_2) \setminus P(e_1)$. Furthermore, if there is a security equilibrium (f_g, f_b, a_s) in which $a_s(e_1) > 0$, then there is also a security equilibrium $(f_g^{'}, f_b^{'}, a_s^{'})$ such that $a_s^{'}(e_2) > 0$ and in which the payoffs to θ_g and θ_b types are unchanged.*

Proof The proof is simple. Suppose at an equilibrium (f_g, f_b, a_s) that $a_s(e_1) > 0$. Since $P(e_2)$ contains $P(e_1)$, it must be that $f_b(p) = 0$ for any path $p \in P(e_2) \setminus P(e_1)$ since a_s must be a weak best response for security. For the second statement, simply note that shifting all of the probability from edge e_1 to edge e_2 for security must not change their payoffs by the first part of the lemma and the fact that (f_g, f_b, a_s) is assumed to be an equilibrium with $a_s(e_1) > 0$. Note that this shifting of security cannot change the payoffs for any paths used by θ_b types, and can only increase the cost of paths not used by θ_b types. Thus θ_b types choose not to switch paths after the change in security, keeping f_g fixed. Finally, the shift in security causes no change in latency to θ_g types. Hence, $f_g^{'} = f_g$ and $f_b^{'} = f_b$.

One sees here that one can restrict attention to security equilibria in which the network operator uses only undominated edges. That is, edges $e_i \in E$ such that there is no $e_j \in E$ with $P(e_i) \subset P(e_j)$. Using this fact, I begin to look into which network topologies are such that θ_g types cannot be worse off at a security equilibrium for any assignment of cost functions in the class that I consider.

The next example shows that on the familiar Braess graph, θ_g types can be no worse off for any choice of cost functions.

Example Suppose G is the Braess graph depicted below. Then for any security equilibrium (f_g, f_b, a_s) and any equilibrium flow f^* of the game without security,

$$C_g^*(f_g, f_b, a_s) \leq C^*(f^*)$$

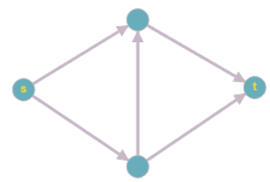

Figure 2.1: Braess Graph

Note that this graph has only two undominated edges, both of which lie on the crossover path from s to t. The inefficiencies in the Braess graph can occur when there are agents using both parallel $s - t$ paths, as well as a positive mass using the crossover path. This is a standard example. Note in this model that if θ_g types are using both of these parallel paths, corollary 3.2 implies that both of the undominated edges carry a positive expected fine. This increased cost disincentivizes those subject to the fines, the θ_b types, from inducing this inefficiency. While some mass of agents may still use the crossover path, this configuration of undominated edges guarantees that the latency on a shortest path cannot deteriorate relative to the equilibrium without fines. That is to say, θ_g types cannot be worse off.

A natural question is whether or not the inclusion of security can ever hurt those that are not subject to the fines. The answer is yes, the risk of fines can cause those subject to them to take paths that increase the latency of all shortest paths. The counterexample is simple. All we need to do is modify the Braess graph above so that there exist undominated edges that do not lie on the crossover path.

Example Consider the figure below. In this graph, all edges are undominated.

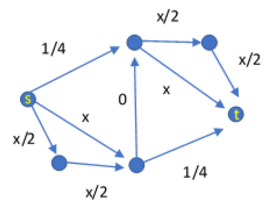

Figure 2.2: Modified Braess Graph

Fix $w(\theta_g) = 0.2$, $w(\theta_b) = 0.8$, and $\kappa = \frac{1}{10}$. At the equilibrium without fines, the latency experienced by all agents is $\frac{1}{2}$. Now, there is a security equilibrium in which each edge with constant cost $\frac{1}{4}$ has expected fine $\frac{\kappa}{2}$. At this security equilibrium, a mass of 0.1 θ_g types will traverse each of the upper and lower sections of the graph individually, and not cross over the middle edge. A mass of 0.2 θ_b types will choose to cross over, while a mass of 0.3 θ_b types will follow the upper and lower paths individually and not cross over. This leads to a the latency on fastest paths to be $\frac{3}{10} + \frac{1}{4} = \frac{11}{20} > \frac{1}{2}$.

This example is useful in that if the graph in example 3.8 is embedded in a particular way in some graph G, then we know that we can assign a vector of cost functions to G that result in θ_g types being worse off after the addition of fines. This is made more formal in the next lemma. Before moving on to the lemma, I provide a definition of an $s - t$ paradox from Chen, Diao, and Hu (2016)[69].

Definition 4 *We call G an $s-t$ paradox if $G = P_1 \cup P_2 \cup P_3$ is the union of three paths P_1, P_2, and P_3 such that:*

- *P_1 is an $s-t$ path going through distinct vertices a, u, v, b in this order*

- *P_2 is an $a - v$ path with $V(P_2) \cap V(P_1) = \{a, v\}$*
- *P_3 is a $u - b$ path with $V(P_3) \cap V(P_1) = \{u, b\}$ and $V(P_3) \cap V(P_2) = \varnothing$*

Lemma 2.3.7 *Suppose that a graph G contains an $s - t$ paradox $\hat{G}$ with corresponding paths P_1, P_2, P_3 and vertices a, u, v, b. Suppose that G has the following properties:*

- *In addition to P_1, there is another $a - u$ path P_4 that is edge-disjoint from P_1 and such that $V(P_4) \cap V(P_2) = \{a\}$*
- *In addition to P_1, there is another $v - t$ path P_5 that is edge-disjoint from P_1 and such that $V(P_5) \cap V(P_3) = b$*
- *There exist three pairwise edge-disjoint $s-a$ paths and three pairwise edge-disjoint $b - t$ paths.*

Then there is an assignment of latency functions, type-weights, and fine level such that there is an equilibrium of the security game (f_g, f_b, a_s) with

$$C_g^*(f_g, f_b, a_s) > C^*(f^*)$$

Proof The proof uses the fact that the graph in example 3.8 is a embedded in G, as well as the fact that there are no bottlenecks in key locations. Select three edge-disjoint $s - a$ paths and three edge-disjoint $b - t$ paths. Also determine the relevant paradox paths P_1, P_2, P_3, P_4, and P_5 as above. For any edge not on the aforementioned collection of paths, give a latency function of excessively high constant cost so that these edges will never be used at an equilibrium. What we're left with is essentially a segment that resembles example 3.8 as well as three paths at each end of the segment that allow us to distribute the flow of θ_b types enough so that they do not incur security. The result then follows from example 3.8.

The conditions from the previous lemma are extremely narrow. Future work will be devoted to tightening this condition and pinning down network topologies in which θ_g types do not experience any negative externalities from the presence of security.

Up until this point, I've considered whether or not the inclusion of security can improve latency for θ_g type agents. One natural question is just how much of an improvement can be made. The next example shows that θ_g types can experience arbitrarily less latency than at the equilibrium without security.

Example Let $m \in \mathbb{N}$, $m \geq 2$. Consider the graph given by m-1 parallel links. Suppose that $m - 2$ of these links have constant cost m, and the final link has cost function $C_e(x) = x$. Let $w(\theta_g) = \frac{1}{m}$ and $w(\theta_b) = \frac{m-1}{m}$. The Nash equilibrium in the usual selfish routing game is for all agents to to route over the link with cost function x and obtain a cost of 1. Now consider the security game over the same graph with $\kappa = m - \frac{2}{m}$. Then we can find an equilibrium of the security game in which there are $\frac{1}{m}$ θ_b types on each edge, $\frac{1}{m}$ θ_g types on the edge with cost function x, and security chooses the edge with cost function x with certainty. Then all θ_b types incur cost m, and θ_g types incur cost $\frac{2}{m}$. Allowing m to tend to infinity yields the result that θ_g types can be arbitrarily better off.

Note that this required the mass of θ_g types to be small in order to achieve large improvements in latency for them. It is another application of Lemma 1 of Cole, Lianeas, and Nikolova (2018)[54] that θ_g types can be no better off in a security equilibrium than at an equilibrium of a routing game on the same graph with mass equal to $w(\theta_g)$ (when θ_g types are playing by themselves).

2.4 Conclusion and Future Directions

We have seen above that in several cases, adding security to the game cannot hurt the θ_g types. The primary question that remains is whether or not this is always the case. In principle, one could work toward placing bounds on how much the welfare of θ_g types can differ between equilibria of the two games. These bounds will depend on the relative weights on types as well as the properties of the graph and cost functions. There are many ways to go about this. Perhaps the most promising is to use the fact that the equilibrium flows considered in this game are ϵ-approximate Nash equilibria as defined in Christodoulou et al. (2011)[23]. These equilibria have their own price of anarchy values (the worst-case ratio of social welfare at a Nash equilibrium and a social optimum). While these values take into account the welfare of the entire flow, it's possible that one can still leverage these results based on the value of ϵ and $w(\theta_g)$.

One could also allow security to choose the level of fine κ associated with being caught in order to maximize revenue. As a note, a solution to the problem of choosing a randomization over edges and a κ would not exist if one were to allow for $\kappa \in [0, \infty)$. The reasoning is because there is a positive lower bound to the amount of flow security is guaranteed to catch since they can simply form an s-t cut of the graph G. Then security obtains at least $\frac{\kappa w(\theta_b)}{|C|}$ with certainty, where $|C|$ is the number of edges in the cut. Thus there would be no solution if one were to allow unbounded κ. This is likely the reason network pricing models often include elastic demand or place restrictions on the edges which can be priced. The question of maximizing revenue for $\kappa \in [0, \bar{\kappa}]$ is interesting however. Comparative statics involving the mass of θ_b passing over secure edges at equilibrium would be quite helpful in solving this problem.

Another shortcoming of this model is that I have assumed the presence of security does not slow down θ_g agents at all. This was done for simplicity, but anybody who has waited to be screened at an airport or DUI checkpoint knows that screening agents takes time. On the other hand, ticket inspectors on trains in public transit often impose no additional latency on agents.

Finally, it would be interesting to characterize the differences between the settings of security moving as a leader and the game in its current state of simultaneous moves. Determining how much more revenue can be obtained by security as a result of moving first is interesting. Furthermore, the potential for differences in revenue to security also has implications on the latency externalities experienced by θ_g types in the two settings. Future drafts of this paper will seek to address these questions.

CHAPTER 3

ON THE CHARACTERISTICS AND VERIFICATION OF TENABLE STRATEGY SETS IN BIMATRIX GAMES

ESSAYS ON ECONOMICS AND COMPUTER SCIENCE
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I consider the robust set-valued equilibrium concepts *coarse tenability* and *fine tenability* introduced in Myerson and Weibull (2015)[48]. I show how their framework of *consideration-set* games maps onto the more familiar framework of strategy perturbations. This allows me to compare these set-valued concepts to other objects from the equilibrium refinement literature, as well as provide methods to verify whether strategy sets satisfy these robustness properties in bimatrix games. I provide complexity results for the verification of *fine tenability* and show how the methods developed for verifying this concept can be applied to proper equilibria.

3.1 Introduction

Since the seminal works of von Neumann and Morganstern (1944)[66] and Nash (1951)[49], game theoretic methods have been brought to bear on problems of strategic interaction of every variety. The need for solution concepts with more cutting power than Nash equilibrium has led to the vast literature on equilibrium refinements. Many of these refinements come with different levels of agent sophistication or particular game forms in mind. In this paper, I study the characteristics and provide methods for the verification of the robust set-valued equilibrium concepts introduced in Myerson and Weibull (2015)[48]. These concepts are motivated by, on the one hand, models of evolutionary game theory in which a population of players is called upon to engage in strategic interaction with norms and historical precedent playing a role; and, on the other hand models of rationalizable choice and inattention. A central idea is that within a population a particular convention arises in which only a certain collection of strategies for players in various type roles are ever played. A natural question then is which conventions survive the test of time? In order to answer this question, Myerson and Weibull (2015)[48] discuss multiple solution concepts that describe when a convention is robust against small deviations from convention within the population. This paper is part of the growing literature on equilibrium refinement computation. I do not attempt to survey the entirety of this literature here, but I will briefly discuss those works which are most related to this one.

Few papers deal with the computation or verification of set-valued equilibrium concepts. Benisch et al. (2010)[9] provide methods to compute the minimal sets which are closed under rational behavior (CURB) of Basu and Weibull

(1991)[7] in polynomial time by building such sets upon individual strategies. For finite games of 3 or more players, it was shown in Klimm et al. (2011)[36] that the problem of computing a minimal CURB set as well as the problem of verifying whether a given set is a minimal CURB set are both NP-complete. In this same paper, it is shown that computing strong CURB sets (in which correlated strategies are allowed) can be done in polynomial time by modifying the methods of Benisch et al. (2010)[9]. Brandt and Brill (2016)[11] discuss the computation of several set-valued solution concepts which can be expressed using the dominance structures defined in Duggan and Le Breton (2014)[19]. Jansen and Vermeulen (2001)[31] provide a framework for verifying whether a collection of equilibria satisfy *stability* as defined by Kohlberg and Mertens (1986)[37]. I adapt their framework in Section 4 below for the current problem.

Much of the literature on the computation of equilibria and equilibrium refinements focuses on point-valued concepts. The problem of computing a Nash equilibrium in a bimatrix game is known to be PPAD-complete due to a series of results by Chen and Deng (2006)[15] and Daskalakis et al. (2006)[18]. Connitzer and Sandholm (2008)[17] extend the results of Gilboa and Zemel(1989)[24] to show that many problems such as approximating the maximum social welfare at any Nash equilibrium are hard even for symmetric bimatrix games. For papers discussing the complexity and computation of equilibrium refinements in finite games see Miltersen et al. (2010) [44], Hansen et al. (2010)[28], and Hansen (2017)[26] and references therein.

It is important to note that the complexity and computation of proper equilibria is particularly relevant to the discussion in this paper. The concept of *fine tenability*, formally defined later, is quite similar to proper equilibrium in that it imposes a rationality requirement on trembles. I show that the exact methods I develop for fine tenability can also be applied to the verification of proper equilibria in bimatrix games. Hansen and Lund (2018)[27] show that the complexity of verifying the conditions imposed by proper equilibrium is NP-complete in bimatrix games. I show a similar result for verifying the conditions of fine tenability by providing a nontrivial modification of their reduction. Sørensen (2012)[63] showed that the problem of finding a single proper equilibrium in a bimatrix game can be done by applying Lemke's algorithm to a linear complementarity problem of polynomial size, and hence is PPAD-complete. The discussion in sections 5 and 6 of this paper provide an answer to the question of how small ϵ needs to be to guarantee that an *ϵ-proper* equilibrium profile is indeed close to a proper equilibrium, though this bound in general cannot be determined efficiently. This was briefly mentioned as an issue in Sørensen (2012)[63], although the method proposed in that paper did not require this information. Belhaiza et al. (2012)[8] propose ad-hoc methods for computing proper equilibria by first solving a mixed 0-1 quadratic program and analytically verifying solution candidates. They note that the informal analytical approach of iteratively satisfying the criteria of *ϵ-properness* was introduced by Myerson (1991)[47]. This paper provides a formal generalization of this method using the language of hyperplane arrangements as it applies to the problem of verifying fine tenability. This characterization is, to the best of my knowledge, not fleshed out in any of the previous literature, and sheds light on the geometry of fine tenability and proper equilibria. This characterization allows for a clean discussion of the complexity of verifying fine tenability, and also provides explicit bounds for the $\bar{\epsilon}$ used in the definition of fine tenability (See definition 3 below) in bimatrix games.

3.2 Preliminaries

I consider finite games of complete information defined as $G = (N, S, u)$. The set N denotes the set of players, with $|N| = n$. The set $S = \times_{i=1}^{N} S_i$ denotes the set of pure strategy profiles, with S_i denoting the set of pure strategies for player i. The mapping $u_i : S \to \mathbb{R}$ assigns to every pure strategy profile s a payoff for player i. The space of mixed strategies for player i is given by ΔS_i, the collection of probability distributions over strategies in S_i, with generic elements given by σ_i. The set of mixed strategy profiles is denoted $\Sigma = \times_{i=1}^{N} \Delta S_i$, with generic element σ. For a mixed strategy profile, the payoff to player i is given by,

$$u_i(\sigma) = \sum_{s \in S} \sigma(s) u_i(s)$$

where $\sigma(s) = \Pi_{i=1}^{N} \sigma_i(s_i)$. With the usual abuse of notation, $u_i(s_i, s_{-i})$ denotes the payoff of player i from playing $s_i \in S_i$ when all other players play according to $s \in S$. Similarly, $u_i(s_i, \sigma_{-i})$ is defined as the payoff to player i when she plays pure strategy s_i while all other players randomize according to $\sigma \in \Sigma$.

For a given mixed strategy σ_i, let $Y_i(\sigma_i) = \{s_i \in S_i : \sigma_i(s_i) > 0\}$ denote the *carrier* of σ_i. For a mixed strategy profile σ, let $BR_i(\sigma) = \{s_i \in S_i : u_i(s_i, \sigma_{-i}) \geq u_i(s_i^{'}, \sigma_{-i}) \ \ \forall s_i^{'} \in S_i\}$ denote the set of best responses for player i to the profile σ. A profile σ is a *Nash Equilibrium* of $G = (N, S, u)$ if and only if for each player i, $Y_i(\sigma_i) \subseteq BR_i(\sigma)$.

In this paper, I discuss the robust strategy blocks introduced in Myerson and Weibull (2015)[48]. In particular, I am interested in the verification of blocks that are said to be *coarsely tenable* and *finely tenable*. I make no nondegeneracy assumptions about the finite game in question. Before proceeding to the relevant equilibrium concepts, I introduce the framework of *consideration-set games* from Myerson and Weibull (2015)[48].

For a set E, let $\mathbb{P}(E)$ denote the power set of E. Let C_i denote a generic element of $\mathbb{P}(S_i)$. Every set C_i is referred to as a "type" for player i. If a player is of type C_i, then strategies within C_i are precisely the strategies that player i considers using. All strategies outside of C_i are played with probability 0 if player i is of type C_i. There is a probability distribution μ_i over types for each player. That is, $\mu_i(C_i)$ denotes the probability that player i is of type C_i. These distributions are independent for each player. The joint distribution over $\mathbb{P}(S)$ is denoted by μ. This defines a consideration set game $G^\mu = (N, S, u, \mu)$.

A pure strategy for player i is a mapping $f_i : \mathbb{P}(S_i) \to S_i$ with the property that $f_i(C_i) \in C_i$. Denote the space of all pure strategies for player i by F_i. Mixed strategies are defined in the usual way, as randomizations over the pure strategy mappings. The associated space of mixed strategies for player i is denoted by ΔF_i. A generic element of ΔF_i is denoted by τ_i, and a strategy profile of the consideration set game is denoted by τ. Given τ, let $\tau_{i|C_i}$ denote the conditional probability distribution over pure strategies in C_i when player i is of type C_i. Each profile τ induces a probability distribution τ^μ over the pure strategies of each player in the following way

$$\tau^\mu(s_i) = \sum_{\{C_i : s_i \in C_i\}} \tau_{i|C_i}(s_i)\mu_i(C_i)$$

This projection of τ onto the space of mixed strategies allows one to compute the payoffs to each player given a profile τ. For a profile τ, denote the induced mixed strategy profile by τ^μ.

A profile τ is said to be an equilibrium of the consideration-set game if for each player i and each $C_i \in \mathbb{P}(S_i)$,

$$u_i(\tau_{i|C_i}, \tau^{\mu}_{-i}) = max_{s_i \in C_i} u_i(s_i, \tau^{\mu}_{-i})$$

That is to say, if player i is of type C_i then player i chooses only best responses (from among those strategies contained in C_i) to the probability distribution induced by τ_{-i}. The equilibria are defined ex-ante. The following definitions will refer to subsets (referred to as "blocks") $T = \times_{i=1}^{N} T_i$ of the collection of pure strategy profiles. Each $T_i \subseteq S_i$ is a nonempty collection of pure strategies for each player.

Definition 5 *(Myerson and Weibull (2015)[48]) A strategy block $T \subseteq S$ is said to be coarsely tenable if $\exists\ \bar{\epsilon} \in (0,1)$ such that for all $\epsilon \in (0, \bar{\epsilon})$ and each type distribution μ such that $\mu(T) > 1 - \epsilon$,*

$$max_{s_i \in S_i} u_i(s_i, \tau^{\mu}_{-i}) = max_{t_i \in T_i} u_i(t_i, \tau^{\mu}_{-i})$$

for player i and each equilibrium τ of the incomplete information game induced by μ.

Coarse tenability requires that the block T be robust against all sufficiently small deviations from the convention T. The notion of *fine tenability* is a weaker concept in that it requires robustness against a specific class of deviations from convention. In particular, the deviations that are said to be $\epsilon-proper$. The definition of ϵ-properness for strategy blocks is quite similar to the one defined for strategy profiles defined by Myerson in his definition of *proper* equilibria (see Myerson (1978)[46]). Indeed, the definition imposes a requirement on the relative probability of trembles between "more" and "less" rational alternatives of strategy blocks. Here, rationality is used to describe the number of strategic alternatives that a player considers when playing as an unconventional type.

Definition 6 *(Myerson and Weibull (2015)[48]) Given a strategy block T and an $\epsilon \in (0,1)$, a type distribution μ is said to be ϵ-proper if, for every player i:*

- $\mu_i(T_i) > 1 - \epsilon$
- $\mu_i(C_i) > 0, \quad \forall C_i \subseteq S_i$
- $T_i \neq C_i \subset D_i \implies \mu_i(C_i) \leq \mu_i(D_i)\epsilon$

Where C_i and D_i are arbitrary subsets of S_i.

The first point in the definition requires that players are sufficiently likely to be of a conventional type. The second imposes a requirement that every possible type has a positive probability under μ. The third point reflects the requirement that players place greater probability on more rational deviations from convention.

I now introduce the final set-valued equilibrium concept of fine tenability.

Definition 7 *(Myerson and Weibull (2015)[48]) A block T is finely tenable if $\exists\ \bar{\epsilon} \in (0,1)$ such that for all $\epsilon \in (0, \bar{\epsilon})$ and for any type distribution μ that is $\epsilon - proper$,*

$$max_{t_i \in T_i} u_i(t_i, \tau^{\mu}_{-i}) = max_{s_i \in S_i} u_i(s_i, \tau^{\mu}_{-i})$$

for each player i and each equilibrium τ of the of the incomplete information game induced by μ.

As noted by Myerson and Weibull (2015), if a block T is coarsely tenable, then it also satisfies the weaker condition of fine tenability. It is important to note that as $\mu(T) \to 1$, the projections of equilibria of the consideration-set games G^{μ} converge to equilibria of the support restricted game $G^T = (N, T, u)$. This fact is given in Myerson and Weibull (2015)[48], and is used throughout the analysis in this paper.

3.3 Characterization Using Perturbations

In order to obtain a more concrete understanding of the structure of these block concepts and relate them to other solution concepts in the game theory literature, it is helpful to rephrase them in terms of the more familiar language of strategically perturbed games. A perturbation vector $\delta_i \in \mathbb{R}_+^{|S_i|}$ assigns to every strategy $s_i \in S_i$ a minimum probability $\delta_i(s_i)$ that this pure strategy must receive at any mixed strategy profile. These perturbation vectors must of course satisfy $\sum_{s_i \in S_i} \delta_i(s_i) \leq 1$. Given a tuple of perturbations $\{\delta_i\}_{i=1}^N$, the set of admissible mixed strategies for player i is defined as $\{\sigma_i \in \Delta S_i : \sigma_i(s_i) \geq \delta_i(s_i) \;\; \forall s_i \in S_i\}$. Let $Y_i^\delta(\sigma_i) = \{s_i \in S_i : \sigma_i(s_i) > \delta_i(s_i)\}$ denote the *δ-carrier* of the mixed strategy σ.

Definition 8 *An equilibrium of the perturbed game $G = (N, S, u, \delta)$ is a mixed strategy profile σ such that for each player i, $Y_i^\delta(\sigma_i) \subseteq BR_i(\sigma)$.*

With the appropriate definitions established, I proceed with relating projections of equilibria of consideration-set games to Nash equilibria of appropriately perturbed games. Keeping in mind that the final goal is to examine the properties of tenable strategy sets, I will suppose throughout this section that the analysis is given with an arbitrary, but fixed, strategy block T.

Lemma 1 *Fix a block T, an arbitrary distribution over types μ, and an equilibrium of the associated consideration-set game τ. Consider the induced probability distribution* *over pure strategies* τ^{μ}.

Claim: There is a probability distribution μ' *such that the marginal probability distributions* μ'_i *have the property that*

- $\mu'_i(T_i) = \mu_i(T_i)$
- $\mu'_i(C_i) = 0$ *for all* C_i *such that* $C_i \neq T_i$ *and* $|C_i| \neq 1$

(μ'_i *places positive probability only on* T_i *and individual strategies in* S_i*) and there is an equilibrium* τ' *of the consideration-set game induced by* μ' *with* $\tau^{\mu} = \tau'^{\mu}$*. That is to say,* τ' *induces the same probability distribution over pure strategies as* τ*.*

All proofs are in the appendix. The idea is that for a given equilibrium τ with projection τ^{μ}, reconstruct the distribution τ^{μ} directly using type distributions over individual strategies and T_i. Distributions of the form μ' described in the above lemma will be referred to as *simple* distributions throughout the rest of this paper. Note that when discussing *simple* distributions over types one requires the context of a specific T in order for the definition to make sense. The correct terminology would be "*simple with respect to* T." I omit this in future mentions of *simple* distributions with the understanding that the distributions are *simple* with respect to the block T in question.

Since deviations of the form described in the above lemma must be considered anyway, Lemma 1 says that there is no loss in generality in restricting attention to such simplified probability distributions. These distributions can be viewed in some sense as the familiar strategy perturbations used in describing various equilibrium concepts in the literature (*proper*, *(strictly) perfect*, etc.). The precise connection will soon become clear. For a mixed strategy profile σ, let

$BR_i^T(\sigma) = \{s_i \in T_i : u_i(s_i, \sigma_{-i}) \geq u_i(s_i', \sigma_{-i}) \;\; \forall s_i' \in T_i\}$ denote the set of best responses among strategies in T_i for player i against profile σ.

Definition 9 *For a fixed strategy block T and perturbation tuple $\delta = \{\delta_i\}_{i=1}^N$, a profile σ is a T-equilibrium of the perturbed game (N, S, u, δ) if for each player i, $Y_i^\delta(\sigma_i) \subseteq BR_i^T(\sigma)$.*

Note that *T-equilibria* need not coincide with Nash equilibria of the perturbed game. However, these are precisely the profiles which correspond to projections of equilibria of consideration-set games as $\mu(T) \to 1$ in a strong sense.

Lemma 2 *For a fixed block T in the game $G = (N, S, u)$ and for every simple type distribution μ with $\mu(T) > 0$, there is a perturbation tuple $\delta = \{\delta_i\}_{i=1}^N$ with $\sum_{s_i \in S_i} \delta_i(s_i) = 1 - \mu_i(T_i)$ for each player i and such that for every equilibrium τ of the consideration-set game G^μ, there is a T-equilibrium σ of the perturbed game (N, S, u, δ) such that $\tau^\mu = \sigma$.*

A variant of the reverse direction of Lemma 2 is also true.

Lemma 3 *For a fixed strategy block T, each perturbation tuple $\delta = \{\delta_i\}_{i=1}^N$ and every T-equilibrium σ of the perturbed game $G = (N, S, u, \delta)$, there is a simple type distribution μ such that $\mu_i(T_i) \geq 1 - \sum_{s_i \in S_i} \delta_i(s_i)$ and such that there is an equilibrium τ of the consideration-set game G^μ in which $\sigma = \tau^\mu$*

Lemmas 2 and 3 provide a simple way to express the relationship between consideration-set games and strategically perturbed games. This correspondence allows for the comparison of the set-valued equilibrium concepts found in Myerson and Weibull (2015)[48] to other solution concepts found in the game theory literature. In particular, I show that every *coarsely tenable* strategy block T contains a set that is *stable* in the sense of Kohlberg and Mertens (1986)[37] (henceforth *KM-stable*). Before proceeding, I provide a definition of *KM-stability* used in Jansen and Vermeulen (2001)[31] for the verification of KM-stable sets in bimatrix games.

Definition 10 *(Jansen and Vermeulen (2001)[31]) A closed set E of strategy pairs is called a KM-set if for each neighborhood V of E, there is $\eta > 0$ such that if the strategy perturbations satisfy $\|\delta\| < \eta$, then there is an equilibrium of the perturbed game lying in V. A set E that is minimal with respect to this property is referred to as (KM-)stable.* [1]

Note here that $\|\delta\|$ refers to the Euclidean norm of the full tuple of perturbations $\{\delta_i\}_{i=1}^{N}$ in $\times_{i=1}^{N}\mathbb{R}^{|S_i|}$.

Proposition 1 *In a finite game $G = (N, S, u)$, if a strategy block T is coarsely tenable, then T contains a set that is KM-stable.*

It is worth nothing that this result is known to be true for generic finite games (see Wikman (2019)[68]). I will return to the relationship between consideration-set games and perturbed games when discussing the verification of coarse tenability in the next section. I now move on to characterizing the relationship between projections of equilibria of consideration-set games with *ϵ-proper* type distributions and tuples (σ, δ) with specific structure.

Definition 11 *Consider a fixed strategy block T and $\epsilon > 0$. A tuple (σ, δ) consisting of a mixed strategy profile σ and a strategy perturbation δ is said to be $T(\epsilon)$-proper if the following conditions hold:*

[1] Note that Kohlberg and Mertens (1986)[37] required that the set E be composed only of Nash equilibria. Imposing minimality on E ensures that the sets considered in Jansen and Vermeulen (2001)[31] are indeed the sets defined as *stable* under the definition of Kohlberg and Mertens.

- $s_i \in argmax_{\{t_i \in T_i\}} u_i(t_i, \sigma_{-i}) \Rightarrow \sigma_i(s_i) \geq \delta_i(s_i)$
- $s_i \notin argmax_{\{t_i \in T_i\}} u_i(t_i, \sigma_{-i}) \Rightarrow \sigma_i(s_i) = \delta_i(s_i)$
- $\delta_i(s_i) > 0$
- $u_i(s_i, \sigma_{-i}) > u_i(s_j, \sigma_{-i}) \Rightarrow \delta_i(s_i)\epsilon \geq \delta_i(s_j)$
- $\sum_{\{s_i \in S_i\}} \delta_i(s_i) \leq \epsilon$

Note that the profile σ need not be $\epsilon-proper$, however the perturbations δ satisfy a properness criterion under the ordering over strategies induced by σ. The idea behind much of the work in the rest of this section is once again to consider all probability that arises from deviations from convention T as perturbations. The proof that for a fixed T the equilibria of consideration-set games with $\epsilon-$ *proper* type distributions admit tuples which are $T(\epsilon) -$ *proper* is almost exactly the same proof as Proposition 2 of Myerson and Weibull (2015)[48]. A proof is provided here for purposes of completeness and to note some subtle differences.

Lemma 4 *Fix a strategy block T. Suppose that μ is an ϵ^3-proper type distribution and that τ is an equilibrium of the consideration-set game G^μ. Then there is a tuple (σ, δ) that is $T(\epsilon)$-proper and such that $\tau^\mu = \sigma$.*

Now, I would like to show that any tuple that is $T(\epsilon)$-*proper* corresponds to an equilibrium of a consideration-set game with an ϵ-*proper* type distribution. Unfortunately, this in general requires type distributions which are proper according to $\epsilon^{\frac{1}{z}} > \epsilon$ for a given $T(\epsilon)$-*proper* tuple (σ, δ) and $z \in \mathbb{N}$. I will need to introduce some notation before continuing. Consider a mixed strategy profile σ. This profile induces a total preorder over the strategies of each player according to expected payoffs. Let $\succsim_i$ denote the total preorder of the form $s_i \succsim_i s_j$ if and only if $u_i(s_i, \sigma_{-i}) \geq u_i(s_j, \sigma_{-i})$ with $\sim$ denoting indifference. This total preorder partitions the strategy space of player i into indifference classes. Denote these indifference classes by $\zeta_1^i, \zeta_2^i, ..., \zeta_{r_i}^i$ with $r_i \leq |S_i|$. The classes are ordered such that $s \in \zeta_k^i$ and $s' \in \zeta_{k+1}^i$ implies $u_i(s, \sigma_{-i}) < u_i(s', \sigma_{-i})$.

Proposition 2 *For a fixed strategy block T, there exists $\epsilon^* > 0$ and $z \in \mathbb{N}$ such that for all $\epsilon \in (0, \epsilon^*)$ and any tuple (σ, δ) that is $T(\epsilon)$-proper, there is an $\epsilon^{\frac{1}{z}}$-proper type distribution and an equilibrium τ of the associated consideration-set game such that $\tau^\mu = \sigma$.*

Remark 1 *The proof of this proposition is essentially an application of the following observation. Fix $k \in \mathbb{N}$ and $M > 0$. There is an $\bar{\epsilon} \in (0, 1)$ and $z \in \mathbb{N}$ such that for any collection $\alpha_0, \alpha_1, ..., \alpha_k \in [0, M]$ and $\epsilon \in (0, \bar{\epsilon})$,*

$$\sum_{i=0}^{k} \frac{\alpha_i}{\epsilon^{\frac{1}{z}}} \leq \frac{1}{\epsilon}$$

Note that precisely how small ϵ^* in Proposition 2 needs to be is unimportant. Since fine tenability only considers behavior under small perturbations, it suffices to show that there is an equivalence between the two concepts in a neighborhood of the game where T is played with probability 1.

3.4 Verification of Coarsely Tenable Blocks

Given a strategy block T, a natural question is how to determine whether it is *coarsely tenable*. This section is devoted to determining how to answer this question in the case of two players. Throughout the remainder of this paper, I consider a bimatrix game in which player 1 has pure strategy set S_1 and player

2 has pure strategy set S_2. I suppose that $|S_1| = m$ and $|S_2| = n$. The associated mixed strategy sets are given by ΔS_1 and ΔS_2. The set of mixed strategy profiles is denoted by $\Sigma = \Delta S_1 \times \Delta S_2$. A generic element of ΔS_1 is denoted by p, while a generic element of ΔS_2 is denoted by q. Given a profile (p, q), payoffs are determined by the matrices A_{mxn} and B_{mxn} in the following way:

$$u_1(p, q) = pAq$$

$$u_2(p, q) = pBq$$

Nash equilibria of this game are the strategy pairs (p, q) such that

$$pAq \geq p'Aq \;\; \forall p' \in \Delta S^1$$

$$pBq \geq pBq' \;\; \forall q' \in \Delta S^2$$

Throughout this section, I assume that I am armed with a concise description of the Nash equilibria of the full bimatrix game. The set of all Nash equilibria in a bimatrix game is a finite union of convex polytopes using results of Jansen (1981)[29]. Furthermore, there exist methods of computing all Nash equilibria of a bimatrix game in finite time, regardless of assumptions on nondegeneracy (see Avis et al. (2010)[6]).

I now move on to the discussion of verifying whether a given block T satisfies the requirements imposed by *coarse tenability*. Much of this discussion is based on the work of Jansen and Vermeulen (2001)[31] who described a method of verifying whether a set is *stable* under the definition of Kohlberg and Mertens (1986)[37].

Consider a block T with the property that all equilibria of the block game G^T are also equilibria of the entire bimatrix game G. I am interested in determining whether it is coarsely tenable. This entails determining how the equilibria with support contained in the block T change as the distribution over types μ places small probability on blocks other than T. Lemma 1 implies that is suffices to consider deviations from convention T in which players consider individual pure strategies rather than larger strategy blocks. Lemmas 2 and 3 show that this problem can, with some care, be phrased as a problem about profiles in strategically perturbed games. Using the methodology of Jansen and Vermeulen (2001)[31], sub-blocks of T can be used to determine precisely how the equilibria of the perturbed game change, and thus how the equilibria of the consideration set game change as a result of altering the distribution over type. In what follows in Lemma 5 as well as throughout the rest of this paper, I refers to an indexed set of strategies in S_1. I sometimes use $i \in I$ to refer to a strategy $s_i \in I$ within this indexed set. I let $p_i = \sigma_i(s_i)$ for a strategy $s_i \in S_i$ and e_i denotes a basis vector corresponding to playing strategy s_i with probability 1. Similar notation is used for an indexed set $J \subseteq S_2$ and strategies for player 2.

Lemma 5 *Consider a fixed strategy block T. A profile (p, q) is a T-equilibrium for the perturbed (bimatrix game) $G = (N, S, u, \delta)$ if and only if there is a nonempty sub-block $I \times J$ with $I \subseteq T_1$ and $J \subseteq T_2$ such that $(p, \delta_1) \in S^1_{IJ}$ and $(q, \delta_2) \in S^2_{IJ}$, where*

$$S^1_{IJ} = \begin{cases} pBe_j - pBe_k \geq 0 & j \in J, k \in T_2 \\ p_i \geq \delta_1(s_i) & \forall i \in I \\ p_i = \delta_1(s_i) & \forall i \notin I \\ 0 \geq -\delta_1(s_i) & \forall i \\ \sum_i p_i = 1 & \end{cases}$$

$$
S^2_{IJ} = \begin{cases} e_i A q - e_k A q \geq 0 & i \in I, k \in T_1 \\ q_j \geq \delta_2(s_j) & \forall j \in J \\ q_j = \delta_2(s_j) & \forall j \notin J \\ 0 \geq -\delta_2(s_j) & \forall j \\ \sum_j q_j = 1 \end{cases}
$$

Since Lemmas 1-3 established a correspondence between *T-equilibria* and projections of equilibria of consideration-set games, every solution (p, q) to the above system given δ yields an equilibrium of a consideration-set game in which $\mu_i(T_i) \geq 1 - \sum_{s_i \in S_i} \delta_i(s_i)$ for each player i. For the case in which $|T_i| \geq 2$ for each player, the solutions yield equilibria in which $\mu_i(T_i) = 1 - \sum_{s_i \in S_i} \delta_i(s_i)$.

For a fixed sub-block $I \times J$, S^1_{IJ} describes a convex polytope in $\mathbb{R}^{2m}$, and S^2_{IJ} describes a convex polytope in $\mathbb{R}^{2n}$. To determine whether a given strategy block T is coarsely tenable, it suffices to enumerate only those sub-block $I \times J$ which possess a solution at $\delta = 0$. Indeed, the fact that equilibria of the incomplete information game induced by a type distribution μ converge to equilibria of the block game G^T as $\mu(T) \to 1$ implies that these are the only sub-blocks that are relevant to the problem. Any pair $I \times J$ which does not possess a solution tuple at $\delta = 0$ is such that there is no equilibrium of the block game in which each of the strategies in I and J are best responses. Then at least one of the associated polytopes is bounded away from the zero perturbation (full information) region. Since coarse tenability imposes restrictions only on perturbations sufficiently close to 0, such pairs $I \times J$ can safely be ignored.

Given a concise description of the equilibria of the full game and that T is such that each equilibria of the block game G^T are also equilibria of the full game, it is not difficult to enumerate the pairs $I \times J \subseteq T$ for which there are solutions at $\delta = 0$. Fix such a pair $I \times J$. In order to verify whether coarse tenability is violated, it suffices to determine whether there is a sequence of solutions $\{(p_k, q_k, \delta_k)\}_{k=1}^{\infty}$ with $\delta_k \to 0$ and such that for each (p_k, q_k), there is some player i and strategy $s_k \in S_i \backslash T_i$ such that s_k does strictly better than all strategies in T_i. The Bolzano-Weierstrass theorem implies that there exists a sequence in which the same strategy $s_k = s^*$ plays this role. Such a strategy, if it exists, must then be a (weak) best reply to a solution of the system when $\delta = 0$. Let us suppose for the moment that such a strategy s^* exists and without loss of generality that it belongs to player 2, and denote its index in S_2 by ω. Fix some strategy $s_j \in J$. Evidently the above sequence must lie entirely within the half-space $\{(p, \delta) : pBe_\omega - pBe_j > 0\}$. [2] Using the fact that the hyperplane $\{(p, \delta) : pBe_\omega - pBe_j = 0\}$ intersects S^1_{IJ} at $\delta = 0$ and the fact that a sequence of solutions in which s^* does strictly better than s_j exists, it must be that S^1_{IJ} possesses a vertex that lies in the half space $pBe_\omega - pBe_j > 0$. In fact, this is the only instance in which such a sequence may exist. If all vertices lie in the half space $pBe_\omega - pBe_j \leq 0$, then the hyperplane $pBe_\omega - pBe_j = 0$ is either tangent to S^1_{IJ} at $\delta = 0$, or aligns with one of its faces. In either case, s^* does not constitute a violation of coarse tenability.

Proposition 3 *Fix a strategy block T with the property that all equilibria of the block game G^T are equilibria of the entire game G. Enumerate all sub-blocks $I \times J$ of T for which S^1_{IJ} and S^2_{IJ} possess solutions when $\delta = 0$. For each such sub-block and for each player, enumerate all strategies outside of T that are (weak) best responses to an equilibrium with support contained within $I \times J$, denote the set of such strategies as*

[2] Note that implicitly this uses the fact that each player must be indifferent between all elements of I and J at any solution of the system.

$\{\Omega^1_{IJ}, \Omega^2_{IJ}\}$.

Claim: *T is coarsely tenable if and only if for each such sub-block* $I \times J$*:*

1. *For each strategy* $s_\omega \in \Omega^2_{IJ}$, S^1_{IJ} *does not possess a vertex that lies in the half-space* $\{(p, \delta) : pBe_\omega - pBe_j > 0\}$, *where* ω *denotes the index of* s_ω *in* S_2 *and* j *denotes the index of an arbitrary strategy in* J.
2. *For each strategy* $s_\omega \in \Omega^1_{IJ}$,$S^2_{IJ}$ *does not possess a vertex that lies in the half-space* $\{(q, \eta) : e_\omega Aq - e_i Aq > 0\}$, *where* ω *denotes the index of* s_ω *in* S_1 *and* i *denotes the index of an arbitrary strategy in* I.

This characterization of coarse tenability resembles the *undominated* characterization of perfect equilibria in bimatrix games due to van Damme (see Theorem 3.2.2 of van Damme (1983)[65]).

3.5 Verification of Finely Tenable Blocks

Recall the definition of fine tenability introduced in Section 2 before continuing. In order to determine whether a given block T is finely tenable, it often suffices to show that it is coarsely tenable. Indeed, Myerson and Weibull (2015)[48] show that the concepts of coarse and fine tenability agree for generic normal-form games. While this is a useful result, much of the usefulness of fine tenability as a solution concept stems from its bite in normal form representations of extensive form games. Furthermore, determining whether or not a game is of a given generic form, for instance the class of *nondegenerate* games or the class of *hyper-regular* games introduced in Myerson and Weibull (2015)[48], can be computationally expensive. With this in mind, a discussion on fine tenability seems appropriate. Fortunately, Proposition 2 has already supplied a useful characterization for checking the conditions of fine tenability. In order to verify whether a given block T is finely tenable, it is helpful to return to the sets S^1_{IJ} and S^2_{IJ}. It once again suffices to consider only those sub-blocks $I \times J$ of T that possess solutions when perturbations are set to zero. [3]

Fix such a sub-block $I \times J$. I am interested in determining whether there exists a sequence $\epsilon_k \to 0$ and a sequence of $T(\epsilon_k) - proper$ tuples (σ_k, δ_k) which solve the systems S^1_{IJ} and S^2_{IJ} and are such that there is some player i and $s_\omega \in S_i \backslash T_i$ with $u_i(s_\omega, \sigma_{-i,k}) > max_{t_i \in T_i} u_i(t_i, \sigma_{-i,k})$ for all k. Such a sequence $\{\epsilon_k, \sigma_k, \delta_k, s_\omega\}_{k=1}^{\infty}$ will be called an *$I \times J$-refutation*. If no such sub-block possesses a refutation, then T is finely tenable.

Proposition 4 *A strategy block T is finely tenable if and only if every sub-block $I \times J$ of T does not possess an $I \times J$-refutation.*

Proposition 4 implies that the problem of verifying whether a block T is finely tenable reduces to the problem of finding *$I \times J$-refutations* among particular sub-blocks of T. However, this does not provide any indication of how to determine whether such refutations exist for a fixed T. Evidently, if an $I \times J$ refutation exists, then it is possible to construct a refutation with the desirable property that the total preorders $\succsim_{i,k}$ induced by σ_k are constant for all k. This suggests that a geometric approach to the problem of searching for refutations could be fruitful.

Fix a sub-block $I \times J$ of the strategy block T. For a given total preorder $\succsim_2$

[3] Note that while coarse tenability imposes the restriction that block equilibria are also equilibria of the entire game, fine tenability does not require this. See for instance Example 2 of Myerson and Weibull (2015)[48]

for player 2, the set of strategies of player 1 that induce this preorder is given by $p(\succsim_2)$. Note that in general $p(\succsim_2)$ is not closed, however the closure of this set is a polytope. Under the terminology of Jansen (1993)[30], the strategies in $p(\succsim_2)$ are said to be *order equivalent*. Indeed, the space of mixed strategies for each player can be partitioned into equivalence classes using *order equivalence* as an equivalence relation.

Now, suppose there is an *$I \times J$-refutation* $\{\epsilon_k, \sigma_k, \delta_k, s_\omega\}_{k=1}^{\infty}$ with constant total preorders $\succsim_1, \succsim_2$. Then $\sigma_{1,k} \in p(\succsim_2)$ and $\sigma_{2,k} \in q(\succsim_1)$ for all k. Furthermore, $\delta_{1,k}$ is ϵ_k*−proper* according to $\succsim_1$ for all k. Then it must be that $S^1_{IJ} \cap p(\succsim_2)$ and $S^2_{IJ} \cap q(\succsim_1)$ are nonempty and contain the perturbations $\{\delta_{1,k}\}_{k=1}^{\infty}$ and $\{\delta_{2,k}\}_{k=1}^{\infty}$ respectively.[4] The question is then: given a set $S^1_{IJ} \cap p(\succsim_2)$, according to what total preorders $\succsim_1$ does this set contain $\epsilon - proper$ perturbations δ_1 for all $\epsilon > 0$? Indeed, an answer to this question would provide a brute force method to the problem of verifying fine tenability. Namely, check all possible pairings of total preorders and determine whether the corresponding perturbations exist within S^1_{IJ} and S^2_{IJ} for all $\epsilon > 0$. The following results characterize necessary and sufficient conditions on $S^1_{IJ} \cap p(\succsim_2)$ to guarantee that it supports $\epsilon - proper$ perturbations of a given total preorder $\succsim_1$ for all $\epsilon > 0$. In what follows, the expression $v(\delta_1(s))$ denotes the value of $\delta_1(s)$ at a vertex v of a polytope.

Lemma 6 *Fix a sub-block $I \times J$ of a block T, $\succsim_1$ and $\succsim_2$. The set $S^1_{IJ} \cap p(\succsim_2)$ contains $T(\epsilon) - proper$ tuples (σ_1, δ_1) according to the order $\succsim_1$ for all $\epsilon > 0$ only if the following conditions hold*

- *There is a vertex of $cl(S^1_{IJ} \cap p(\succsim_2))$ that lies on the zero perturbation region (i.e.*

[4]Here, $p(\succsim_2)$ and $q(\succsim_1)$ are taken to be those elements (p, δ_1) and (q, δ_2) in which p induces $\succsim_2$ over S_2 and q induces $\succsim_1$ over S_1.

$v(\delta_1(s)) = 0$ for all $s \in S_1$).

- *For each strategy $s \in S_1$, there is a vertex of $cl(S^1_{IJ} \cap p(\succsim_2))$ in which $v(\delta_1(s')) = 0$ and $v(\delta_1(s)) > 0$ for all $s' \prec_1 s$.*

Before proving sufficiency, I will require the following lemma.

Lemma 7 *For a fixed sub-block $I \times J$ of a block T and total preorders $\{\succsim_1, \succsim_2\}$,*

$$relint(cl(S^1_{IJ} \cap p(\succsim_2))) \subseteq S^1_{IJ} \cap p(\succsim_2)$$

where relint($\cdot$) denotes the relative interior of a set (under the usual topology).

Lemma 8 *The conditions of Lemma 6 are sufficient to guarantee that for $\epsilon > 0$, there exists a tuple (σ_1, δ_1) that is $T(\epsilon) - proper$ according to $\succsim_1$ and that lies in the relative interior of $cl(S^1_{IJ} \cap p(\succsim_2))$.*

With these results about the geometry of $T(\epsilon) - proper$ tuples, I am able to provide an exact method for determining the presence of *$I \times J$-refutations* for any sub-block of T. Before describing the algorithm, I provide some basic definitions of the relevant objects used in the procedure.

Definition 12 *The hyperplane arrangements, in $\mathbb{R}^{2|S^1|}$ and $\mathbb{R}^{2|S^2|}$ respectively, associated with the block $I \times J$ are defined as*

$$H^1_{IJ} = \Big\{ \sum_i p_i = 1 \Big\} \cup \Big\{ \bigcup_{\{i,j \in S^2, i<j\}} \{pBe_i - pBe_j = 0\} \Big\} \cup \Big\{ \bigcup_{i \in S^1} \{\delta_i = 0\} \cup \{p_i = \delta_i\} \Big\}$$

$$H^2_{IJ} = \Big\{ \sum_j p_j = 1 \Big\} \cup \Big\{ \bigcup_{\{i,j \in S^1, i<j\}} \{e_iAq - e_jAq = 0\} \Big\} \cup \Big\{ \bigcup_{j \in S^2} \{\eta_j = 0\} \cup \{q_j = \eta_j\} \Big\}$$

The vertices of these hyperplane arrangements will be useful when computing vertices of the polytopes used in the procedure. Since it is unknown whether vertex enumeration can be done in polynomial time, it seems sensible to compute the vertices of this arrangement once and use them to quickly compute the vertices of one of the (possibly exponentially many) polytopes that are used in the procedure, rather than perform vertex enumeration at each step. Indeed, given a total preorder $\succsim_2$, the vertices of $cl(S^1_{IJ} \cap p(\succsim_2))$ will be a subset of the vertices of H^1_{IJ}.

Definition 13 *The zero perturbation regions of S^1_{IJ} and S^2_{IJ} projected onto the space of mixed strategies for each player are given by*

$$Z^1_{IJ} = \begin{cases} pBe_j - pBe_k \geq 0 & j \in J, k \in T^2 \\ p_i \geq 0 & \forall i \in I \\ p_i = 0 & \forall i \in S^1 \backslash I \\ \sum_i p_i = 1 & \end{cases}$$

$$Z^2_{IJ} = \begin{cases} e_iAq - e_kAq \geq 0 & i \in I, k \in T^1 \\ q_j \geq 0 & \forall j \in J \\ q_j = 0 & \forall j \in S^2 \backslash J \\ \sum_j q_j = 1 & \end{cases}$$

These sets are used to prune the number of total preorders used in the procedure. In particular, it suffices to only focus on those total preorders which are locally possible given Z^1_{IJ} and Z^2_{IJ}. For instance, if Z^1_{IJ} is a singleton then any strict preferences in the total preorder $\succsim_2$ induced at this point will be preserved as $\epsilon \to 0$. The algorithm is outlined on the next page.

Remark 2 *It is important to note what is meant by "associated inequalities" in step 6(a) below. The associated inequalities are those found by relaxing all strict inequalities imposed by the total preorders into weak inequalities. These result in the appropriate closures provided $S^1_{IJ} \cap p(\succsim_2)$ and $S^2_{IJ} \cap q(\succsim_1)$ are nonempty.*

Algorithm 1: Search for IJ-refutations

1. Compute the vertices V^i_H of H^i_{IJ} and V^i_Z of Z^i_{IJ} using a vertex enumeration algorithm for hyperplane arrangements, e.g. [5].
2. Set $j_1 \sim_2 j_2 \;\; \forall j_1, j_2 \in J$ and $i_1 \sim_1 i_2 \;\; \forall i_1, i_2 \in I$
3. For each $k_1, k_2 \in S_2 \backslash J, k_1 \neq k_2$:
   (a) If $V^1_Z \subset pBe_{k_1} - pBe_{k_2} > 0$, set $k_1 \succ_2 k_2$
   (b) Else if $V^1_Z \subset pBe_{k_1} - pBe_{k_2} < 0$, set $k_2 \succ_2 k_1$
   (c) Else place no restriction on k_1 and k_2 regarding $\succsim_2$
4. Repeat steps $3 - 4$ for restrictions on $\succsim_1$
5. For each pair $\succsim_1$ and $\succsim_2$ satisfying the restrictions from steps $2 - 4$ and such that there is some player i and strategy $s_\omega \in S_1(S_2)$ with $s_\omega \succ_i s$ for some $s \in I(J)$
   (a) Enumerate the vertices of $cl(S^1_{IJ} \cap p(\succsim_2))$ and $cl(S^2_{IJ} \cap q(\succsim_1))$ by checking which vertices of H^1_{IJ} and H^2_{IJ} satisfy the associated inequalities.
   (b) Check whether the vertices of $cl(S^1_{IJ} \cap p(\succsim_2))$ satisfy the conditions of Lemma 6 with respect to $\succsim_1$ to determine existence of $T(\epsilon) - proper$ perturbations for ϵ near zero. Repeat for $cl(S^2_{IJ} \cap q(\succsim_1))$ with respect to $\succsim_2$.
   (c) If the requisite conditions are satisfied for both $cl(S^1_{IJ} \cap p(\succsim_2))$ and $cl(S^2_{IJ} \cap q(\succsim_1))$ then a refutation has been found, and the procedure halts after printing the pairing of total preorders $\{\succsim_1, \succsim_2\}$.
   (d) Else there is no refutation for the total preorder pairing
6. If no total preorder pair yields a refutation, then the block $I \times J$ has no refutations.

Proposition 5 *Algorithm 1 correctly determines in finite time whether the sub-block $I \times J$ of T possesses a refutation.*

A number of comments on the method are in order. It is in principle possible to verify whether a pair of total preorders $\{\succsim_1, \succsim_2\}$ constitute a refutation by solving a pair of linear programs (see lemma 13). On the other hand, enumerating all total preorder pairs and performing this procedure very quickly becomes infeasible. The first 5 steps of the procedure will in general (at least in nondegenerate games) drastically reduce the number of pairs to check. It is of course possible to refine and improve upon this pruning of the total preorders by using the structure of the face lattice of the associated hyperplane arrangements alongside the restrictions imposed by properness. These improvements deserve some attention, and are a subject of ongoing research.

Algorithm 1 here is in some sense a formal generalization of the informal method of analytically determining whether a given equilibrium is proper given in Myerson (2001)[47] and fleshed out a bit more in Belhaiza (2012)[8]. The method here is adapted to a different solution concept, but the logic behind lemmas 6-8 and the procedure defined above are applicable to the problem of verifying proper equilibria. The formalization given here does provide a number of insights into the geometry of the problem, however.

The algorithm described here is exponential in the worst case, requiring the comparison of an exponential number of total preorder pairings. The next result shows that it is NP-hard to determine whether a block possesses a refutation, even when it consists of single strategies. The following reduction modifies the one provided by Hansen and Lund (2018)[27], who in turn modify a reduction given by Connitzer and Sandholm (2008)[17]. I construct a game in which a given block possesses a refutation if and only if an instance of a Boolean satisfiability problem in conjunctive normal form with exactly three literals per clause possesses a satisfying assignment.

Definition 14 *Let ϕ be a Boolean formula in conjunctive normal form with exactly three literals per clause. Let V denote the set of variables, with $|V| = n$. Let L denote the set of literals, with $+l_i$ and $-l_i$ denoting the positive and negative literals corresponding to variable x_i. Let the mapping $v : L \to V$ be defined by $v(+l_i) = v(-l_i) = x_i$, assigning to any literal its corresponding variable. Denote the set of clauses by C. Finally, each player will have three additional pure strategies denoted by f, g, and h. The symmetric, normal form game $G(\phi)$ that I consider is defined, with payoffs to the row player, in Table 1 below. The block to be considered is $T = \{g\} \times \{g\}$. In what follows, I will show that T is finely tenable if and only if the formula ϕ does not have a satisfying assignment.*

	l'	x'	h'	c'	f'	g'
l	$n-1$ if $l \neq -l'$ $n-4$ if $l = -l'$	$n-1$ if $v(l) \neq v$ $n-4$ if $v(l) = v$	$n-1$	$n-1$	$n-1$	$n-3$
x	$n-1$ if $v(l) \neq v$ n if $v(l) = v$	$n-1$	$n-1$	$n-1$	$n-1$	$n-2$
h	$n-1-\frac{1}{2n}$	$\frac{(n-1)^2+(n-4)}{n}$	$n-1$	$n-1$	$n-1$	$n-3$
c	1 if $l \notin c$ 0 if $l \in c$	$n-1$	n	$n-1$	$n-1$	$n-1$
f	$\frac{n-\frac{1}{2}}{n}$	$n-1$	$n-1$	$n-1$	$n-1$	$n-1$
g	$n-1$	$n-1$	$n-1$	n	0	$n-1$

Table 3.1: The game $G(\phi)$

Lemma 9 *Let (σ_1, σ_2) denote any $T(\epsilon) -$ proper profile for ϵ small. Then for each* *player i and each pair of variables x_j and x_k,*

$$\sigma_i(+l_j) + \sigma_i(-l_j) = \sigma_i(+l_k) + \sigma_i(-l_k)$$

and

$$\sigma_i(x_j) = \sigma_i(x_k)$$

Lemma 10 *If (σ_1, σ_2) is a $T(\epsilon) -$ proper profile with ϵ small, then there is a partition of the set of literals L into two sets L_1 and L_2 such that for each player*

$$\sigma_i(l_1) = \sigma_i(l'_1) \;\; \forall \; l_1, l'{}_1 \in L_1$$

$$\sigma_i(l_2) = \sigma_i(l'_2) \;\; \forall \; l_2, l'{}_2 \in L_2$$

$$l_j \in L_1 \;\; \Rightarrow \;\; -l_j \in L_2$$

$$l_1 \succsim_i l_2 \;\; \forall l_1 \in L_1, \;\; l_2 \in L_2$$

Lemma 11 *If ϕ has a satisfying assignment then $T = \{g\} \times \{g\}$ possesses a refutation.*

Lemma 12 *If $T = \{g\} \times \{g\}$ possesses a refutation, then ϕ has a satisfying assignment.*

Proposition 6 *It is NP-hard to verify whether a given sub-block $I \times J$ of a block T possesses an $I \times J$-refutation.*

Note that proposition 4 alongside lemmas 6-8 show that one can in fact view *$I \times J$-refutations* as a pair of total preorders over the pure strategies of each player. If one is provided with a pair of total preorders, one for each player, it is not difficult to determine whether it serves as an $I \times J$ refutation. This provides a proof for NP-completeness of finding refutations.

Lemma 13 *Given a block T, a sub-block $I \times J$, and a pair of total preorders $\succsim_1, \succsim_2$, one can determine whether $\{\succsim_1, \succsim_2\}$ serves as an $I \times J$ refutation for T by solving a polynomial number of linear programs, each of which contains only a polynomial number of contraints.*

Proposition 7 *The problem of verifying whether a given sub-block $I \times J$ of T possesses an $I \times J$-refutation is NP-complete.*

3.6 Application to Proper Equilibria

The results of Lemmas 6-8, used here to analyze fine tenability, can also be applied to proper equilibria. As I will soon show, it gives rise to an exact method of verifying whether a given equilibrium constitutes a proper equilibrium of a bimatrix game. Before proceeding, I introduce some basic definitions for the section.

Definition 15 *(Myerson (1978)[46]) A mixed strategy profile σ is said to be an ϵ-proper equilibrium if for each player i the following conditions hold:*

- $\sigma_i(s) > 0 \;\; \forall s \in S_i$
- $u_i(s, \sigma_{-i}) > u_i(s', \sigma_{-i}) \Rightarrow \sigma_i(s)\epsilon \geq \sigma_i(s') \;\; \forall s, s' \in S_i$

A Nash equilibrium σ is said to be a *proper equilibrium* if there exists a sequence $\{\epsilon_k, \sigma^k\}_{k=1}^{\infty}$ such that

- $\epsilon_k > 0$ and $\lim\limits_{k \to \infty} \epsilon_k = 0$

- σ^k is an ϵ_k*-proper equilibrium*

- $\lim_{k \to \infty} \sigma^k = \sigma$

The procedure below determines whether a given input Nash equilibrium $\sigma = (p^*, q^*)$ is a proper equilibrium. It is based on the logic of Lemmas 6-8 above that it is possible to determine whether a convergent sequence of ϵ*-proper* equilibria exists by analyzing the faces of a pair of hyperplane arrangements. Define the hyperplane arrangements:

$$\tilde{H}^1 = \left\{ \sum_i p_i = 1 \right\} \cup \left\{ \bigcup_{\{j,k \in S^2 : j < k\}} \{pBe_j - pBe_k = 0\} \right\} \cup \left\{ \bigcup_i \{p_i = 0\} \right\}$$

$$\tilde{H}^2 = \left\{ \sum_j q_j = 1 \right\} \cup \left\{ \bigcup_{\{i,k \in S^1 : i < k\}} \{e_i A q - e_k A q = 0\} \right\} \cup \left\{ \bigcup_j \{q_j = 0\} \right\}$$

Consider the following sets:

$$P^1(p^*, q^*) = \begin{cases} pBe_j - pBe_k \geq 0 & j \in Y(q^*), k \in S_2 \\ p_i \geq 0 & \forall i \in S^1 \\ \sum_i p_i = 1 \end{cases}$$

$$P^2(p^*, q^*) = \begin{cases} e_i A q - e_k A q \geq 0 & i \in Y(p^*), k \in S_1 \\ q_j \geq 0 & \forall j \in S^2 \\ \sum_j q_j = 1 \end{cases}$$

where $Y(\cdot)$ denotes the carrier of a mixed strategy. Each closed face (of positive dimension) f_1 of $\tilde{H}^1$ such that $p^* \in f_1$ and $f_1 \subseteq P^1(p^*, q^*)$ is associated with a total preorder $\succsim_2$, the preference ordering of player 2 over strategies in S^2 induced by mixed strategies that lie in the relative interior of f_1. A similar statement holds for faces f_2 of $\tilde{H}^2$. To determine whether (p^*, q^*) is a proper equilibrium, it suffices to check all pairings of such faces f_1 and f_2 for a set of conditions similar to those given in Lemma 6. In what follows, let $\succsim_{i,f_{-i}}$ denote the total preorder over strategies for player i induced by strategies on the relative interior of f_{-i}. For a given total preorder $\succsim_{i,f_{-i}}$, let ζ_k^i denote the k^{th} indifference class, with ζ_1^i denoting the best strategies for player i. Let r_i denote the total number of indifference classes in S_i according to the total preorder $\succsim_i$.

Lemma 14 *A Nash equilibrium $\sigma = (p^*, q^*)$ is a proper equilibrium if and only if there is a pair of faces f_1 (of $\tilde{H}^1$) and f_2 (of $\tilde{H}^2$) such that*

- *$p^* \in f_1$ and $q^* \in f_2$*
- *$f_1 \subseteq P^1(p^*, q^*)$ and $f_2 \subseteq P^2(p^*, q^*)$*
- *For each player i, each indifference class ζ_k^i and each strategy $s \in \zeta_k^i$, there is a vertex of f_i in which $p_s > 0$ and $p'_s = 0$ for all $s' \in \zeta_l^i$ with $l > k$.*

3.7 Discussion

The characterization in Section 3 provides a straightforward method of verifying coarse tenability of a given block T regardless of degeneracy assumptions on the game. A natural method of finding coarsely tenable blocks is then to determine all strategy blocks of the game which possess the property that all block equilibria are also equilibria of the full game, and then determining whether the block satisfies the additional conditions imposed by coarse tenability. Enumerating such blocks for nondegenerate games is straightforward, albeit computationally expensive. The relationship between projections of equilibria of consideration-set games and the usual notion of strategic perturbations has allowed me to relate coarse tenability to KM-stable sets. It would be interesting to explore the relationship between tenable strategy sets and other familiar solution concepts in the game theory literature.

The methods developed in Section 4 are based on simple geometric arguments about *proper* strategy profiles. Note that the procedures for verifying the conditions of fine tenability and proper equilibria, while exact, may require calculations involving extremely small numbers, and bounds for these values can be articulated using the geometry of bimatrix games; although, it be may computationally expensive to do so. This is the main drawback of the methods proposed here. On the other hand, it is quite useful to have bounds on the level of precision required for verifying these concepts. For instance, the results of this paper can be applied to the mixed 0-1 quadratic programming problem proposed in Belhaiza et al. (2012)[8] for the verification of proper equilibria.

There are a number of avenues in which the current paper can be improved. First, it would be very interesting to have a procedure which computes minimal strategy sets that satisfy the criteria of the two solution concepts discussed in this paper, even for bimatrix games. In particular, a procedure that is motivated by a population of agents collectively steering toward a stable convention seems the most promising. Note that, unlike sets which are closed under rational belief (CURB), the intersection of two coarsely (finely) tenable blocks need not be coarsely (finely) tenable in bimatrix games. This appears to preclude a procedure along the lines of Benisch et al. (2010)[9] in which a minimal block can be constructed by iteratively including additional strategies, as now the final product may not minimal under set-inclusion (this is due to the fact that the ordering of the addition of new strategies to the block is now important).

It is worth noting that complexity results for coarse tenability are also available, and NP-hardness of the problem of verifying coarse tenability can be proved by a straightforward modification of the reduction given in Connitzer and Sandholm (2008)[17]. A formal procedure for verifying coarse tenability is also a natural next step in the analysis. The procedure given here for fine tenability can also be improved upon a great deal. As mentioned in section 5, it is possible to use the properties of the face semi-lattice of the associated hyperplane arrangement to reduce the number of total preorders enumerated. The exact nature of this improvement is a subject of ongoing research.

3.8 Appendix of Proofs

Proof of Lemma 1

Proof Suppose that $|T_i| \geq 2$ for each player i. Construct μ' using the projection τ^{μ} as follows :

$$\mu_i'(T_i) = \mu_i(T_i) \ \ \forall i$$

$$\mu_i'(s_i) = \sum_{\{C_i : s_i \in C_i, C_i \neq T_i\}} \tau_{i|C_i}(s_i)\mu_i(C_i) \ \ \forall s_i \in S_i \text{ and } \forall i$$

$$|C_i| \neq 1 \ \wedge \ C_i \neq T_i \ \Rightarrow \mu_i'(C_i) = 0$$

Now τ' has essentially been constructed for us. Given the restrictions imposed by μ', τ_i' is pinned down for all sets other than T_i for each player i. Finally, set $\tau'_{i|T_i} = \tau_{i|T_i}$. To see that the proof is complete, note that by construction $\tau^{\mu} = \tau'^{\mu}$, the equilibria induce the same distribution over pure strategies for each player. Therefore since $\tau_{i|T_i}$ was part of an equilibrium, it must also be that $\tau'_{i|T_i}$ is as well.

For the case in which $T_i = s_i$ for some player i, define $\mu_i^{'}(T_i) = \tau^{\mu}(s_i)$.

Proof of Lemma 2

Proof Fix a *simple* type distribution μ and suppose for now that $|T_i| \geq 2$ for each player i. Let τ be an equilibrium of the consideration-set game G^{μ} with associated projection τ^{μ}. Define $\delta_i(s_i) = \mu_i(s_i)$ for each strategy $s_i \in S_i$ and each player i. Define $\sigma_i(s_i) = \tau^{\mu}(s_i)$. To see that σ is a *T-equilibrium*, note that if $s_i \notin BR_i^T(\sigma)$ then the fact that τ is an equilibrium implies that $\tau_{i|T_i}(s_i) = 0$ and hence $\sigma_i(s_i) = \mu_i(s_i) = \delta_i(s_i)$ which implies that $s_i \notin Y_i^{\delta}(\sigma_i)$.

For the case in which $|T_i| = 1$ for some player i, define $\delta_i(s_i) = 0$ for such players.

Proof of Lemma 3

Proof Suppose that $|T_i| \geq 2$ for each player i. Given δ, set $\mu_i(s_i) = \delta_i(s_i)$ for each $s_i \in S_i$ and each player i. Set $\mu_i(T_i) = 1 - \sum_{s_i \in S_i} \delta_i(s_i)$ for each player i. Define

$$\tau_{i|T_i}(s_i) = \frac{\sigma_i(s_i) - \delta_i(s_i)}{\mu_i(T_i)} \quad \forall s_i \in T_i$$

for each player i. The fact that μ is assumed to be a *simple* type distribution implies that τ is pinned down for all blocks aside from T. Thus by construction

$$\tau_i^{\mu}(s_i) = \sigma_i(s_i)$$

and furthermore the fact that σ is a *T-equilibrium* implies that all strategies $s_i \in T_i$ such that $\tau_{i|T_i}(s_i) > 0$ are an element of $BR_i^T(\tau^{\mu})$.

For the case in which $|T_i| = 1$ for some player i, set $\mu_i(T_i) = 1 - \sum_{s_i \notin T_i} \delta_i(s_i)$. Note that τ_i is automatically constructed for these players since μ is *simple*.

Proof of Proposition 1

Proof I will show that T contains a KM-set, and hence must contain a minimal KM-set. Let E denote the set of equilibria of the block game G^T (which are also equilibria of the entire game under the assumption of coarse tenability). Since E is the entire set of equilibria of a block game, it is closed.

Suppose, seeking contradiction, that this set E is not a KM-set. Then there is a neighborhood V of E in the space of mixed strategies such that for any $\eta > 0$, there is a perturbation δ with $\|\delta\| < \eta$ and such that all equilibria of the game perturbed under δ lie outside of V. Thus there exists a sequence of perturbations $\{\delta_k\}_{k=1}^{\infty}$ such that $\|\delta_k\| > 0$ for each k and $\|\delta_k\| \to 0$, and such that for each k, the perturbed game game (N, S, u, δ) possesses no equilibria within V.

Suppose that $|T_i| \geq 2$ for each player i. Interpret the perturbations $\{\delta_{i,k}\}_{i=1}^{N}$ as a *simple* type distribution μ_k with respect to T for a moment, with $\mu_{i,k}(T_i) = 1 - \sum_{s_i \in S_i} \delta_{i,k}(s_i)$ and $\mu_{i,k}(s_i) = \delta_{i,k}(s_i)$. Recall that the projections of equilibria of the associated consideration-set games converge to equilibria of the block game G^T as $\mu(T) \to 1$ (i.e. $k \to \infty$). Hence there is a $K_1 > 0$ such that for all $k \geq K_1$, there is an equilibrium of the consideration-set game induced by the *simple* distribution μ_k that lies in the neighborhood of V. Now, if I can show that the projections of these equilibria of the consideration-set game induced by μ_k are (eventually) also equilibria of the perturbed game (N, S, u, δ), the proof will be complete. This is precisely where the assumption of coarse tenability has bite. Indeed, there is $\bar{\epsilon} > 0$ such that if $\mu(T) > 1 - \bar{\epsilon}$, then all projections of equilibria of the consideration-set game are also equilibria of the associated perturbed game. To see why, recall that the profile σ is an equilibrium of the perturbed game

(N, S, u, δ) if and only if σ satisfies the minimum probability constraints imposed by δ and for each player i:

$$Y_i^\delta(\sigma_i) \subseteq BR_i(\sigma)$$

Let $K_2 > 0$ be such that for all $k \geq K_2$, $\mu_k(T) > 1 - \bar{\epsilon}$, and let $K^* = max\{K_1, K_2\}$. Consider any equilibrium τ of a consideration-set game μ_k for $k \geq K^*$. Then

$$\tau_{i|T_i}(s_i) > 0 \Rightarrow s_i \in BR_i(\tau^{\mu_k})$$

since T is coarsely tenable. This then implies that

$$\tau_i^{\mu_k}(s_i) > \mu_k(s_i) = \delta_{i,k}(s_i) \Rightarrow s_i \in BR_i(\tau^{\mu_k})$$

Thus $\tau^{\mu_k} = \sigma$ constitutes an equilibrium of the perturbed game (N, S, u, δ) that lies within the set V, a contradiction.

For the case in which $|T_i| = 1$ for some player i, define $\mu_{i,k}(T_i) = 1 - \sum_{s_i \notin T_i} \delta_{i,k}(s_i)$ for such players. The rest of the argument is the same.

Proof of Lemma 4

Proof Let $|T_i| \geq 2$ for each player. For a player i and given strategy $s_i \in S_i$, let $\beta_i(s_i) \subseteq \mathbb{P}(S_i)$ denote the collection of sets $C_i \in \mathbb{P}(S_i)$ such that

$$s_i \in argmax_{t_i \in C_i} u_i(t_i, \tau_{-i}^\mu)$$

It is important to distinguish between two cases:

- Case 1: There exist two strategies $r_i, t_i \in S_i$ such that $u_i(r_i, \tau_{-i}^\mu) < u_i(t_i, \tau_{-i}^\mu)$ and such that there is $C_i^* \in \beta_i(r_i)$ with $T_i = C_i^* \cup \{t_i\}$.

In this case, note that t_i is the unique best response in T_i. Define

$$\delta_i(s_i) = \sum_{C_i \neq T_i} \tau_{i|C_i}(s_i)\mu_i(C_i)$$

for all $s_i \neq t_i$. Define

$$\delta_i(t_i) = \sum_{C_i \neq T_i} \tau_{i|C_i}(t_i)\mu_i(C_i) + \frac{1}{\epsilon}\mu_i(C_i^*)$$

Define $\sigma_i = \tau_i^{\mu}$. Then

$$\delta_i(r_i) \leq \sum_{\{C_i \in \beta_i(r_i)\}} \mu_i(C_i) \leq \epsilon^3 \sum_{\{C_i \in \beta_i(r_i), C_i \neq C_i^*\}} \mu_i(C_i \cup \{t_i\}) + \mu_i(C_i^*) \leq \epsilon\delta_i(t_i)$$

Now let s_i be such that $u_i(s_i, \tau_{-i}^{\mu}) > u_i(t_i, \tau_{-i}^{\mu})$. Then

$$\delta_i(t_i) \leq \sum_{\{C_i \in \beta_i(t_i), C_i \neq T_i\}} \mu_i(C_i) + \frac{1}{\epsilon}\mu_i(C_i^*) \leq \epsilon^3 \sum_{\{C_i \in \beta_i(t_i), C_i \neq T_i\}} \mu_i(C_i \cup \{s_i\}) + \epsilon\mu_i(T_i \cup s_i) \leq \epsilon\delta_i(s_i)$$

Verifying all other pairings uses the same logic, and is more simple due to the fact that T_i does not appear in the calculations. See case 2 for how this is done explicitly. Note that

$$\sum_{s_i \in S_i} \delta_i(s_i) = \sum_{C_i \neq T_i} \mu_i(C_i) + \frac{1}{\epsilon}\mu_i(C_i^*) \leq \epsilon^3 + \epsilon^2\mu_i(T_i) \leq \epsilon$$

- Case 2: There do not exist two strategies $r_i, t_i \in S_i$ such that $u_i(r_i, \tau_{-i}^{\mu}) < u_i(t_i, \tau_{-i}^{\mu})$ and such that there is $C_i^* \in \beta_i(r_i)$ with $T_i = C_i^* \cup \{t_i\}$.

  In this case, it suffices to set

$$\delta_i(s_i) = \sum_{C_i \neq T_i} \tau_{i|C_i}(s_i)\mu_i(C_i)$$

  Then let $r_i, t_i \in S_i$ such that $u_i(r_i, \tau_{-i}^{\mu}) < u_i(t_i, \tau_{-i}^{\mu})$. Then

$$\delta_i(r_i) \leq \sum_{\{C_i \in \beta_i(r_i), C_i \neq T_i\}} \mu_i(C_i) \leq \epsilon^3 \sum_{\{C_i \in \beta_i(r_i), C_i \neq T_i\}} \mu_i(C_i \cup \{t_i\}) \leq \epsilon\delta_i(t_i)$$

  The case in which $|T_i| = 1$ for some player i only requires slight modifications, and only in the case in which the strategy $s_i = T_i$ is among the *worst* strategies in S_i. In such a case, choose $\delta_i(s_i)$ to be a sufficiently small positive number, and construct δ_i the same way as above for all other strategies.

Proof of Proposition 2

Proof Suppose without loss of generality that $m = |S_1| = max_i\{|S_i|\}$. Fix $\epsilon^* = \frac{1}{m^2 2^{2m}}$ and $z = 2m$. Let (σ, δ) be a $T(\epsilon) - proper$ tuple for $\epsilon < \epsilon^*$. I will construct a type distribution μ that is $\epsilon^{\frac{1}{z}} - proper$ as well as an equilibrium τ of the associated consideration-set game directly. Enumerate the indifference classes $\zeta_1^i, \zeta_2^i, ..., \zeta_{r_i}^i$ induced by σ for each player i. For a given strategy s, let $\zeta(s)$ denote the indifference class that strategy s is an element of. The construction below is for player 1, without any loss of generality.

I will distinguish between two cases.

- Case 1: $T_1 \neq \bigcup_{l \leq k} \zeta_l^1$ for any k.

  Define the term:

$$R(s) = \sum_{\{C \subset \bigcup_{\{s' \precsim s\}} s',\ C \neq T_1\ :\ s \in C\}} \frac{1}{\epsilon^{\frac{|C|-1}{z}} |C \cap \zeta(s)|}$$

  Note that for strategies in the same indifference class, R can differ only if one is an element of T and the other is not. If $|\zeta_1^1| = 1$, let s_1 denote its element and define $\mu_1(s) = \delta_1(s) := \gamma_1^1$ for all $s \in S^1$. Otherwise let

$$\mu_1(s) = min \left[\frac{min_{\{s \in \zeta_1^1\}} \delta_1(s)}{max_{\{s \in \zeta_1^1\}} R(s)}, \frac{\sum_{s \in \zeta_1^1} \delta_1(s)}{\sum_{s \in \zeta_1^1} R(s) + \frac{1}{\epsilon^{\frac{|\zeta_1^1|-1}{z}}}} \right] := \gamma_1^1$$

for all $s \in S^1$. Set $\mu_1(C) = \frac{\mu_1(s)}{\epsilon^{\frac{|C|-1}{z}}}$ for all C such that $|C| < |\zeta_1^1|$ and $C \neq T_1$. Define

$$\mu_1(C) = \sum_{s \in \zeta_1^1} \delta_1(s) - \gamma_1^1 \sum_{s \in \zeta_1^1} R(s) := \gamma_2^1$$

for all $C \neq T_1$ such that $|C| = |\zeta_1^1|$. For all subsets C such that $|\zeta_1^1| < |C| < |\zeta_1^1| + |\zeta_2^1|$, set $\mu_1(C) = \frac{\gamma_2^1}{\epsilon^{\frac{|C|-|\zeta_1^1|}{z}}}$. For any strategy s, define

$$\bar{R}(s) = \sum_{\{C \subset \bigcup_{\{s' \precsim s\}} s',\ C \neq T_1\ :\ s \in C\}} \frac{\mu_1(C)}{|C \cap \zeta(s)|}$$

Here, $\bar{R}(s)$ can be interpreted as the probability that a strategy s receives from all blocks in which it is among the best alternatives, aside from the largest such block. This is of course under the assumption that the conditional distribution over the best alternatives in a block is uniform. When moving to the largest block in which a given strategy is a best alternative, the conditional distribution will no longer be uniform. This will be shown in the construction of τ_1 later on.

Let

$$\mu_1(C) = \sum_{s \in \zeta_2^1} \delta_1(s) - \sum_{s \in \zeta_2^1} \bar{R}(s) := \gamma_3^1$$

for all sets $C \neq T_1$ such that $|C| = |\zeta_1^1| + |\zeta_2^1|$. Now it is possible to give an iterative construction of μ_1 for all subsets. For all sets $C \neq T_1$ such that $\sum_{l=1}^{k} |\zeta_l^1| < C < \sum_{l=1}^{k+1} |\zeta_l^1|$ for some $k < r_1$, define

$$\mu_1(C) = \frac{\gamma_{k+1}^1}{\epsilon^{\frac{|C| - \sum_{l=1}^{k} |\zeta_l^1|}{z}}}$$

where

$$\gamma_k^1 = \sum_{s \in \zeta_{k-1}^1} \delta_1(s) - \sum_{s \in \zeta_{k-1}^1} \bar{R}(s), \quad 2 \leq k \leq r_1 + 1$$

and $\mu_1(C) = \gamma_{k+1}^1$ if $C \neq T_1$ and $|C| = \sum_{l=1}^{k} |\zeta_l^1|$. This completes μ_1.

Now we must construct τ_1. Fix a strategy $s \in S^1$. For each set $C \neq T_1$ such that $s \in C$, $s \succsim s'$ for all $s' \in C$ and $C \neq \bigcup_{s' \precsim s} s'$, define

$$\tau_{1|C}(s) = \frac{1}{|\zeta(s) \cap C|}$$

For the set $C = \bigcup_{s' \precsim s} s'$, define

$$\tau_{1|C}(s) = \frac{\delta_1(s) - \bar{R}(s)}{\mu_1(C)}$$

Finally, set

$$\tau_{1|T}(s) = \frac{\sigma_1(s) - \delta_1(s)}{\mu_1(T)}$$

where $\mu_1(T) = 1 - \sum_{C \neq T_1} \mu_1(C)$. Note that in order for this construction to work, it must be that

$$\bar{R}(s) \leq \delta_1(s)$$

for each strategy s. Suppose $s \in \zeta_1^1$, then by the definition of γ_1^1 one has that

$$\bar{R}(s) = \gamma_1^1 R(s) \leq \frac{\delta_1(s)}{R(s)} R(s) = \delta_1(s)$$

If $s \in \zeta_k^1$ for $k \geq 2$, then

$$\bar{R}(s) = \sum_{\{C \subset \bigcup_{\{s' \precsim s\}} s',\ C \neq T_1 \ : \ s \in C\}} \frac{\mu_1(C)}{|C \cap \zeta(s)|} \leq \frac{\gamma_k^1}{\epsilon^{\frac{|\zeta_k^1|-1}{z}}} 2^{|S^1|} \leq \frac{\sum_{s' \in \zeta_{k-1}^1} \delta_1(s')}{\epsilon^{\frac{|\zeta_k^1|-1}{z}}} 2^{|S_1|}$$

Using the fact that for all $s \in \zeta_k^1$ and $s' \in \zeta_{k-1}^1$,

$$\delta_1(s)\epsilon \geq \delta_1(s')$$

it must be that

$$\bar{R}(s) \leq \frac{m\delta_1(s)\epsilon}{\epsilon^{\frac{m}{z}}} 2^{|S_1|} = m\delta_1(s) 2^{|S_1|} \epsilon^{\frac{z-m}{z}} = m\delta_1(s) 2^{|S_1|} \epsilon^{\frac{1}{2}} \leq m\delta_1(s) 2^{|S_1|} \epsilon^{*\frac{1}{2}} = \frac{m\delta_1(s) 2^{|S_1|}}{m 2^m} \leq \delta_1(s)$$

There are still some details to verify. In particular, to show that $\tau_1^{\mu_1} = \sigma_1$, and that μ_1 is $\epsilon^{\frac{1}{z}} - proper$. For a strategy s, its probability under this specification is given by

$$\tau_1^{\mu_1}(s) = \bar{R}(s) + \delta_1(s) - \bar{R}(s) + \sigma_1(s) - \delta_1(s) = \sigma_1(s)$$

To verify properness, it suffices to show that for all $k \geq 2$

$$\gamma_{k+1}^1 \epsilon^{\frac{1}{z}} \geq \frac{\gamma_k^1}{\epsilon^{\frac{|\zeta_k^1|-1}{z}}}$$

or

$$\gamma_{k+1}^1 \geq \frac{\gamma_k^1}{\epsilon^{\frac{|\zeta_k^1|}{z}}}$$

One has that

$$\gamma_{k+1}^1 = \sum_{s \in \zeta_k^1} (\delta_1(s) - \bar{R}(s)) \geq \sum_{s \in \zeta_k^1} \delta_1(s) - \frac{2^{|S^1|} - 1}{\epsilon^{\frac{|\zeta_k^1|}{z}}} \sum_{s' \in \zeta_{k-1}^1} \delta_1(s')$$

Furthermore

$$\frac{\gamma_k^1}{\epsilon^{\frac{|\zeta_k^1|}{z}}} \leq \frac{\sum_{s' \in \zeta_{k-1}^1} \delta_1(s')}{\epsilon^{\frac{|\zeta_k^1|}{z}}}$$

Finally,

$$\sum_{s \in \zeta_k^1} \delta_1(s) \geq \frac{max_{s' \in \zeta_{k-1}^1} \delta_1(s')}{\epsilon^{\frac{z-|\zeta_k^1|}{z}} \epsilon^{\frac{|\zeta_k^1|}{z}}} \geq \frac{max_{s' \in \zeta_{k-1}^1} \delta_1(s')}{\epsilon^{\frac{|\zeta_k^1|}{z}}} \frac{1}{\epsilon^{*\frac{1}{2}}} \geq \frac{max_{s' \in \zeta_{k-1}^1} \delta_1(s')}{\epsilon^{\frac{|\zeta_k^1|}{z}}} m 2^{|S_1|} \geq \frac{\sum_{s' \in \zeta_{k-1}^1} \delta_1(s')}{\epsilon^{\frac{|\zeta_k^1|}{z}}} 2^{|S_1|}$$

as desired. As a final note, γ_1^1 was constructed so that properness holds for $k = 1$.

- Case 2: $T = \bigcup_{l \leq k} \zeta_l^1$ for some k. It will be helpful to identify $k_{T_1} = max\{k : T_1 \cap \zeta_k^1 \neq \varnothing\}$. For all $k < k_{T_1} - 1$, the construction is identical to Case 1. If $|\zeta_{k_{T_1}}^1| \geq 2$, then $k_{T_1} - 1$ can also be constructed in the same way as

Case 1. To handle instances in which $|\zeta^1_{k_{T_1}}| \geq 2$, one needs only to modify the construction from Case 1 for the sets

$$\mathbf{C}_{k_{T_1}} = \{C : C \subset \bigcup_{l \leq k_{T_1}} \zeta^1_l,\ |C| = -1 + \sum_{l \leq k_{T_1}} |\zeta^1_k|\}$$

This modification is due to the fact that it is no longer possible to use the set

$$\bigcup_{l \leq k_{T_1}} \zeta^1_k = T$$

to manage residual probabilities. Instead, these remaining probabilities must be distributed among the blocks in $\mathbf{C}_{k_{T_1}}$. To formalize this idea, define a modified version of $\bar{R}$ for strategies in $\zeta^1_{T_1}$. For $s \in \zeta^1_{k_{T_1}}$, let

$$\bar{R}_{k_{T_1}}(s) = \sum_{\{C \subset \cup_{s' \precsim s} s',\ |C| \leq -2 + \sum_{l \leq k_{T_1}} |\zeta^1_l|,\ s \in C\}} \frac{\mu_1(C)}{|C \cap \zeta^1_{k_{T_1}}|}$$

where $\mu_1(C)$ is constructed in the same way as Case 1 above for all sets $|C| \leq -2 + \sum_{l \leq k_{T_1}} |\zeta^1_l|$ by using the γ^1_l. Now partition $\mathbf{C}_{k_{T_1}}$ into two separate subsets.

$$\mathbf{C}^*_{k_{T_1}} = \{C : C \in \mathbf{C}_{k_{T_1}},\ \zeta^1_{k_{T_1}} \subseteq C\}$$

$$\mathbf{C}^{**}_{k_{T_1}} = \{C : C \in \mathbf{C}_{k_{T_1}},\ \zeta^1_{k_{T_1}} \nsubseteq C\}$$

Note that each element of $\mathbf{C}^{**}_{k_{T_1}}$ contains all strategies in $\zeta^1_{k_{T_1}}$ except for one. The idea is to use the smallest possible probability, while maintaining properness, for elements in $\mathbf{C}^*_{k_{T_1}}$. Then use $\mathbf{C}^{**}_{k_{T_1}}$ to sort the rest of the remaining probabilities out. In particular, let

$$\mu_1(C) = \frac{\gamma^1_{k_{T_1}}}{\epsilon^{\frac{|\zeta^1_{k_{T_1}}|-1}{z}}}$$

$$\tau_{1|C}(s) = \frac{1}{|\zeta^1_{k_{T_1}}|}$$

for all $C \in \mathbf{C}^{*}_{k_{T_1}}$ and $s \in \zeta^1_{k_{T_1}}$. For elements of $\mathbf{C}^{**}_{k_{T_1}}$, there is always one strategy from $\zeta^1_{k_{T_1}}$ that is excluded. Enumerate the elements of $\zeta^1_{k_{T_1}}$ as $s_1, s_2, ..., s_{|\zeta^1_{k_{T_1}}|}$. Given C, suppose that $s_i \notin C$, then define

$$\mu_1(C) = \delta_1(s_{i+1}) - \bar{R}_{k_{T_1}}(s_{i+1}) - \frac{\gamma^1_{k_{T_1}}}{\epsilon^{\frac{|\zeta^1_{k_{T_1}}|-1}{z}}|\zeta^1_{k_{T_1}}|}\sum_{l\leq k_{T_1}-1}|\zeta^1_l|$$

$$\tau_{1|C}(s_{i+1}) = 1$$

where with an abuse of notation $s_{|\zeta^1_{k_{T_1}}|+1} = s_1$. Now to verify that

$$\mu_1(C) \geq \frac{\gamma^1_{k_{T_1}}}{\epsilon^{\frac{|\zeta^1_{k_{T_1}}|-1}{z}}}$$

for all $C \in \mathbf{C}^{**}_{k_{T_1}}$ in order to maintain properness. This is of course verified in the same way as Case 1. For any $s \in \zeta^1_{k_{T_1}}$,

$$\delta_1(s) - \bar{R}_{k_{T_1}}(s) - \frac{\gamma^1_{k_{T_1}}}{\epsilon^{\frac{|\zeta^1_{k_{T_1}}|-1}{z}}|\zeta^1_{k_{T_1}}|}\sum_{l\leq k_{T_1}-1}|\zeta^1_l| \geq \delta_1(s) - \frac{\gamma^1_{k_{T_1}}}{\epsilon^{\frac{|\zeta^1_{k_{T_1}}|-1}{z}}}(2^{|S_1|}-1)$$

Thus I would like to say that

$$\delta_1(s) \geq \frac{\gamma^1_{k_{T_1}}}{\epsilon^{\frac{|\zeta^1_{k_{T_1}}|-1}{z}}}2^{|S_1|}$$

To conclude,

$$\frac{\gamma^1_{k_{T_1}}}{\epsilon^{\frac{|\zeta^1_{k_{T_1}}|-1}{z}}}2^{|S_1|} \leq \frac{\sum_{s'\in\zeta^1_{k_{T_1}-1}}\delta_1(s')}{\epsilon^{\frac{m}{z}}}2^{|S_1|} \leq \frac{m\delta_1(s)\epsilon^*}{\epsilon^{*\frac{1}{2}}}2^{|S_1|} = \delta_1(s)$$

Thus the distribution up to this point is $\epsilon^{\frac{1}{z}} - proper$. Define

$$\gamma^1_{k_{T_1}+1} = max_{s\in\zeta^1_{k_{T_1}}}\left[\delta_1(s) - \bar{R}_{k_{T_1}}(s) - \frac{\gamma^1_{k_{T_1}}}{\epsilon^{\frac{|\zeta^1_{k_{T_1}}|-1}{z}}|\zeta^1_{k_{T_1}}|}\sum_{l\leq k_{T_1}-1}|\zeta^1_l|\right]\frac{1}{\epsilon^{\frac{1}{z}}}$$

With $\gamma^1_{k_{T_1}+1}$ defined, the rest of the construction is the same as Case 1, with the special note that

$$\mu_1(C) = \gamma^1_{k_{T_1}+1}\epsilon^{\frac{1}{z}}$$

for all C such that $C \notin \mathbf{C}_{k_{T_1}}$ with $|C| = |T| - 1$.

The sub-case in which $|\zeta^1_{k_{T_1}}| = 1$ requires only slight modifications to the above argument.

It is immediate that $\tau^{\mu_i}(s) = \sigma_i(s)$ for each player i and each strategy $s \in S^i$. Furthermore, since conditional distributions $\tau_{i|C}$ are such that only best responses in C receive positive probability, the profile τ constructed above is indeed an equilibrium of the consideration-set game.

Proof of Lemma 5

Proof To show necessity, suppose that (p, q) is a *T-equilibrium* under perturbations $\{\delta_1, \delta_2\}$. Let $I = Y_1^{\delta}(p) \subseteq T_1$ and $J = Y_2^{\delta}(q) \subseteq T_2$. Note that for all strategies outside of I (J), it must be that $p_i = \delta_1(s_i)$ $(q_j = \delta_2(s_j))$. Finally note that $I \subseteq BR_1^T(p, q)$ and $J \subseteq BR_2^T(p, q)$. Hence $(p, \delta_1) \in S_{IJ}^1$ and $(q, \delta_2) \in S_{IJ}^2$.

To show sufficiency, suppose that $(p, \delta_1) \in S_{IJ}^1$ and $(q, \delta_2) \in S_{IJ}^2$ for some strategy sub-block $I \times J$ of T. Clearly then $Y_i^{\delta}(p, q) \subseteq BR_i^T(p, q)$ for each player i, and hence the tuple (p, q) is a *T-equilibrium* since the other constraints imposed by S_{IJ}^1 and S_{IJ}^2 ensure that these mixed strategies lie in the set of admissible strategies given the perturbation δ.

Proof of Proposition 3

Proof I begin with the "only if" direction. Suppose that T is coarsely tenable. Suppose, seeking contradiction, there is some sub-block $I \times J$ that possesses a solution to the above systems at $\delta = 0$, but fails at least one of (1) or (2). Fix this sub-block and suppose without loss of generality that (1) fails. Then Ω_{IJ}^2 is nonempty, and there is some $s_\omega \in \Omega_{IJ}^2$ such that S_{IJ}^1 possesses a vertex within the half-space $\{(p, \delta) : pBe_\omega - pBe_j > 0\}$, call this vertex (p^*, δ^*). Furthermore, since $s_\omega \in \Omega_{IJ}^2$, the hyperplane $\{(p, \delta) : pBe_\omega - pBe_j\}$ intersects S_{IJ}^1 at a point where $\delta_1 = 0$. Hence there is some point $(\hat{p}, 0) \in S_{IJ}^1$ and satisfies $pBe_\omega = pBe_j$. Using the fact that S_{IJ}^1 is a convex set, it holds that $\lambda(p^*, \delta*) + (1 - \lambda)(\hat{p}, 0) \in S_{IJ}^1$ for all $\lambda \in [0, 1]$. Note also that

$$(\lambda p^* + (1 - \lambda)\hat{p})Be_\omega - (\lambda p^* + (1 - \lambda)\hat{p})Be_j > 0 \ \ \forall \lambda \in (0, 1]$$

Therefore, for any $\lambda \in (0,1)$, there is a tuple $(\lambda p^* + (1-\lambda)\hat{p}, \lambda\delta^*, q, 0)$ that solves the above system, and in which s_ω is a strictly better reply to $\lambda p^* + (1-\lambda)\hat{p}$ than all elements of T_2. Sending $\lambda \to 0$ and invoking Lemma 3 yields a contradiction.

Proceeding with the "if" direction, suppose that all sub-blocks outlined in the proposition satisfy (1) and (2). Suppose, seeking contradiction, that T is not coarsely tenable. Then there exists a sequence $\epsilon_k \to 0$ and a sequence of (without loss of generality *simple*) distributions μ_k with $\mu_k(T) > 1 - \epsilon_k$ for each k such that there is an equilibrium τ_k of the consideration-set game induced by μ_k where some player, without loss of generality player 2, possesses a strategy $s_\omega \in S_2 \backslash T_2$ such that

$$u_2(s_\omega, \tau^\mu_{1,k}) > max_{t_2 \in T_2} u_2(t_2, \tau^\mu_{1,k})$$

Consider the sequences $\{\tau_{1|T_1}\}_k$ and $\{\tau_{2|T_2}\}_k$ of conditional distributions over pure strategies in T at equilibrium. The support of $\tau_{i|T_{ik}}$ is a finite subset of T_i for each k. The Bolzano-Weierstrass theorem then implies that the sequence of supports possesses a convergent (constant) subsequence. Hence one can assume that the above sequence is constructed such that the supports of $\{\tau_{i|T_i}\}_k$ remain constant for each i. Denote the support of $\{\tau_{1|T_1}\}_k$ by I, and the support of $\{\tau_{2|T_2}\}_k$ by J. Lemmas 2 and 5 imply that τ_k corresponds to some $(p, \delta_1) \in S^1_{IJ}$ and $(q, \delta_2) \in S^2_{IJ}$ under the chosen sub-block of supports for each k. The fact that the equilibria of the consideration-set games converge to equilibria of the block game G^T as $\mu(T) \to 1$ implies that the above sequence possesses a subsequence which converges to an equilibrium τ^* of the block game. I suppose without loss of generality that the entire sequence has this property. This implies that S^1_{IJ} and S^2_{IJ} have solutions when $\delta_i = 0$. The fact that

$$u_2(s_\omega, \tau^\mu_{1,k}) > max_{t_2 \in T_2} u_2(t_2, \tau^\mu_{1,k})$$

for each k implies that the associated solution to the system is such that, for each k,

$$p_k Be_\omega - p_k Be_j > 0$$

where $p_k = \tau^{\mu}_{1,k}$ is the distribution over pure strategies for player 1 induced by τ_k. Furthermore, sending $\epsilon_k \to 0$ ($\mu_k(T) \to 1$) implies that the hyperplane $\{(p, \delta) : pBe_\omega - pBe_j = 0\}$ intersects the set of solutions to the system at a point in which $\delta_1 = 0$. Therefore, S^1_{IJ} has a vertex that lies within the half-space $\{(p, \delta) : pBe_\omega - pBe_j > 0\}$. This is a contradiction. Thus the result is proved.

Proof of Proposition 4

Proof It suffices only to check sub-blocks $I \times J$ in which both S^1_{IJ} and S^2_{IJ} possess solutions on the zero perturbation region. All other sub-blocks will of course lack $I \times J$*-refutations* since solutions of the corresponding system are bounded away from the zero perturbation region.

Suppose that no sub-block $I \times J$ of T possesses an $I \times J$*-refutation*. Suppose, seeking contradiction, that T is not finely tenable. Then there exists a sequence $\{\epsilon_k, \mu_k, \tau_k\}_{k=1}^{\infty}$ such that $\epsilon_k \to 0$, μ_k is an $\epsilon_k - proper$ type distribution with respect to T and τ_k is an equilibrium of the consideration-set game induced by μ_k with the property that for each k, there is some player i and strategy $s_\omega \in S_i \backslash T_i$ such that

$$u_i(s_\omega, \tau_{-i,k},) > max_{t_i \in T_i} u_i(t_i, \tau_{-i,k})$$

The sequence can be chosen such that the above inequality holds for the same player and the same strategy s_ω for each k. Furthermore, it is safe to assume that $supp_i(\tau_{i|T_i})_k$ is constant for each player i and all k. These follow from application of Bolzano-Weierstrass. Let $I = supp_1(\tau_{1|T_1})_k$ and $J = supp_2(\tau_{2|T_2})_k$. For each k, Lemma 4 implies that there is a $T(\epsilon_k^{\frac{1}{3}}) - proper$ tuple (σ_k, δ_k) that solves the systems S^1_{IJ} and S^2_{IJ}. This implies that the sequence of tuples $\{\epsilon_k^{\frac{1}{3}}, \sigma_k, \delta_k\}_{k=1}^{\infty}$ obtained from the projections of $\tau_k^{\mu_k}$ constitutes an $I \times J$*-refutation*, a contradiction.

Proceeding with the "only if" direction, suppose that T is finely tenable. Suppose, seeking contradiction, that there is some sub-block $I \times J$ of T which possesses an $I \times J$*-refutation*. Fix this sub-block and let $\{\epsilon_k, \sigma_k, \delta_k, s_\omega\}_{k=1}^{\infty}$ denote a refutation. Suppose without loss of generality that $m = |S_1| = max\{|S_1|, |S_2|\}$. Since $\epsilon_k \to 0$, there is $K \in \mathbb{N}$ such that for all $k \geq K$, $\epsilon_k \leq \frac{1}{m^2 2^{2m}} = \epsilon^*$. Proposition 2 then implies that for all $k \geq K$, there is a type distribution μ_k that is $\epsilon^{\frac{1}{2m}} - proper$ and τ_k such that $\tau_k^{\mu_k} = \sigma_k$. This immediately provides a sequence $\{(\epsilon_k)^{\frac{1}{2m}}, \mu_k, \tau_k,\}_{k \geq K}$ that constitutes a direct violation of fine tenability, a contradiction.

Proof of Lemma 6

Proof The first condition is clearly necessary, as $T(\epsilon) - proper$ tuples tend to zero perturbation at $\epsilon \to 0$.

Suppose, seeking contradiction, that $I \times J$ contains $T(\epsilon) - proper$ tuples (σ_1, δ_1) according to the order $\succsim_1$ for all $\epsilon > 0$, but that the second condition above fails. Then there is some strategy $s^* \in S_1$ such that for every vertex v of $cl(S^1_{IJ} \cap p(\succsim_2))$ of the form

$$\{v : v(\delta_1(s^*)) > 0\} := Pos(s^*)$$

there is some strategy $s' \in S_1$ in which $s > s'$ and $v(\delta_1(s')) > 0$. Let

$$\epsilon^* = \frac{min_{v \in Pos(s^*)} min_{\{s: v(\delta_1(s)) > 0\}} v(\delta_1(s))}{|Pos(s^*)| max_{\{v \in Pos(s^*)\}} v(\delta_1(s^*))}$$

I claim that there is no tuple $(\sigma_1, \delta_1) \in cl(S^1_{IJ} \cap p(\succsim_2))$ that is $T(\frac{\epsilon^*}{2}) - proper$ according to the order $\succsim_1$. Let,

$$\{v : v(\delta_1(s^*)) = 0\} := Zero(s^*)$$

and consider any element

$$(\sigma_1, \delta_1) = \sum_{v \in Pos(s^*)} \lambda_v v + \sum_{v \in Zero(s^*)} \lambda_v v$$

within $cl(S^1_{IJ} \cap p(\succsim_2))$. In order for properness to hold for any ϵ, it must be that $\delta_1(s) > 0$ for all $s \in S_1$. It must then be that there is $v \in Pos(s^*)$ with $\lambda_v > 0$. Let $\lambda^*_{Pos(s^*)} = max_{\{v \in Pos(s^*)\}} \lambda_v$ and let v^* be a maximizing vertex. Choose any $s' \prec_1 s$ such that $v^*(\delta_1(s')) > 0$. Then,

$$\frac{\delta_1(s')}{\delta_1(s^*)} = \frac{\sum_{v \in Pos(s^*)} \lambda_v v(\delta_1(s')) + \sum_{v \in Zero(s^*)} \lambda_v v(\delta_1(s'))}{\sum_{v \in Pos(s^*)} \lambda_v v(\delta_1(s^*))} \geq \frac{\sum_{v \in Pos(s^*)} \lambda_v v(\delta_1(s'))}{\sum_{v \in Pos(s^*)} \lambda_v v(\delta_1(s^*))}$$

$$\geq \frac{\lambda^*_{Pos(s^*)} min_{v \in Pos(s^*)} min_{\{s : v(\delta_1(s)) > 0\}} v(\delta_1(s))}{\lambda^*_{Pos(s^*)} |Pos(s^*)| max_{\{v \in Pos(s^*)\}} v(\delta_1(s^*))} = \epsilon^*$$

Hence there are no $T(\frac{\epsilon^*}{2}) - proper$ tuples in $cl(S^1_{IJ} \cap p(\succsim_2))$, a contradiction.

Proof of Lemma 7

Proof The proof is a straightforward result of the fact that S^1_{IJ} and $p(\succsim_2)$ are convex. This implies that their intersection is convex, and hence

$$relint(cl(S^1_{IJ}) \cap p(\succsim_2)) = relint(S^1_{IJ} \cap p(\succsim_2)) \subseteq S^1_{IJ} \cap p(\succsim_2)$$

where the first equality follows from known results about convex sets. (See Proposition 1.4.3 of Bertsekas et al. (2003)[10]).

Proof of Lemma 8

Proof Given $\epsilon \in (0,1)$, I seek to construct a tuple (σ_1, δ_1) that satisfies the properness criteria for ϵ and such that

$$(\sigma_1, \delta_1) = \sum_{\{v \in cl(S^1_{IJ} \cap p(\succsim_2))\}} \lambda_v v$$

with $\lambda_v > 0$ for all vertices v. Fix the following values:

$$\bar{\delta} = max_{\{v \in cl(S^1_{IJ} \cap p(\succsim_2))} \; max_{\{s: v(\delta_1(s))>0\}} v(\delta_1(s))$$

$$\underline{\delta} = min_{\{v \in cl(S^1_{IJ} \cap p(\succsim_2))} \; min_{\{s: v(\delta_1(s))>0\}} v(\delta_1(s))$$

Partition S^1 into the indifference classes $\zeta^1_1, \zeta^1_2, ..., \zeta^1_{r_1}$ according to $\succsim_1$, but in decreasing order this time (i.e. ζ^1_1 contains the best strategies according to $\succsim_1$). For each ζ^1_k and each $s \in \zeta^1_k$, the conditions of Lemma 6 imply that there is a corresponding vertex v_s such that $v_s(\delta_1(s)) > 0$ and for each ζ^1_j with $j > k$,

$$v_s(\delta_1(s')) = 0 \;\; \forall s' \in \zeta^1_j$$

Note that multiple strategies within the same indifference class may possess the same corresponding vertex. For each indifference class ζ^1_k, choose a collection of vertices V_k such that for each strategy $s \in \zeta^1_k$, there is a vertex $v_s \in V_k$ such that $v(\delta_1(s)) > 0$ and for each ζ^1_j with $j > k$,

$$v_s(\delta_1(s')) = 0 \;\; \forall s' \in \zeta^1_j$$

Note that $V_k \cap V_j = \varnothing$ for $k \neq j$. For $v \in V_k$, let

$$\lambda_v = \frac{(\epsilon \underline{\delta})^k}{(|S^1|d)^k}$$

where d is the number of vertices of $cl(S^1_{IJ} \cap p(\succsim_2))$. For all vertices $v \notin \bigcup_k V_k$ that do not lie on the zero perturbation region, let

$$\lambda_v = \frac{(\epsilon\underline{\delta})^{r_1}}{(|S^1|d)^{r_1}}$$

For vertices on the zero perturbation region (of which there is at least one by assumption), allocate all remaining probability uniformly. Note that under this construction all λ_v can be scaled down in order to construct a valid probability distribution while maintaining the requisite properness criteria. Let us now verify that the profile

$$(\sigma_1, \delta_1) = \sum_{\{v \in cl(S^1_{IJ} \cap p(\succsim_2))\}} \lambda_v v$$

is $T(\epsilon) - proper$. The fact that $(\sigma_1, \delta_1) \in S^1_{IJ}$ (Lemma 7) implies that the first two conditions are satisfied. The third condition is clearly satisfied. To verify the fourth condition, let $s \in \zeta^1_k$ and $s' \in \zeta^1_{k+1}$ for $k \leq r_1 - 1$. Then

$$\delta_1(s) \geq \frac{(\epsilon\underline{\delta})^k}{(|S^1|d)^k}$$

$$\delta_1(s') \leq \bar{\delta}(d - \sum_{i=1}^{r_1} |V_i|) \frac{(\epsilon\underline{\delta})^{r_1}}{(|S^1|d)^{r_1}} + \sum_{i=k+1}^{r_1} |V_i| \frac{(\epsilon\underline{\delta})^i \bar{\delta}}{(|S^1|d)^i}$$

then

$$\frac{\delta_1(s')}{\delta_1(s)} \leq \frac{\bar{\delta}(d - \sum_{i=1}^{r_1} |V_i|) \frac{(\epsilon\underline{\delta})^{r_1}}{(|S^1|d)^{r_1}} + \sum_{i=k+1}^{r_1} |V_i| \frac{(\epsilon\underline{\delta})^i \bar{\delta}}{(|S^1|d)^i}}{\frac{(\epsilon\underline{\delta})^k}{(|S^1|d)^k}}$$

$$\leq \frac{d\bar{\delta}(\epsilon\underline{\delta})^{k+1}}{(\epsilon\underline{\delta})^k} \frac{(|S^1|d)^k}{(|S^1|d)^{k+1}} = \frac{\bar{\delta}\underline{\delta}\epsilon}{|S^1|} \leq \epsilon$$

The final condition for (σ_1, δ_1) to be $T(\epsilon) - proper$ can be made to hold due to the fact that all λ_v that do not lie on the zero perturbation region may be scaled down while maintaining the first four conditions. Hence it is possible to extract a profile $(\sigma_1, \delta_1) \in relint(cl(S^1_{IJ}) \cap p(\succsim_2))$ which is $T(\epsilon) - proper$ when the conditions of Lemma 6 are satisfied.

Proof of Proposition 5

Proof Suppose that $I \times J$ possesses a refutation $\{\epsilon_k, \sigma_k, \delta_k, s_\omega\}_{k=1}^{\infty}$ with constant total preorders $\{\succsim_1, \succsim_2\}$. One must show that

1. The vertices of $cl(S^1_{IJ} \cap p(\succsim_2))$ and $cl(S^2_{IJ} \cap q(\succsim_1))$ are also vertices of H^1_{IJ} and H^2_{IJ} respectively.
2. The total preorders $\succsim_1$ and $\succsim_2$ satisfy the restrictions imposed by steps 2−5 of Algorithm 1.
3. The total preorders $\succsim_1$ and $\succsim_2$ satisfy the conditions of Lemma 6.

To verify (1), note that $cl(S^1_{IJ} \cap p(\succsim_2))$ corresponds to a face of the arrangement H^1_{IJ}. Similarly $cl(S^2_{IJ} \cap q(\succsim_1))$ is a face of H^2_{IJ}.

Condition (2) must clearly be satisfied as a refutation profile σ_k with constant total preorders for all k has a subsequence which converges to some point on $Z^1_{IJ} \times Z^2_{IJ} = Conv(V^1_Z) \times Conv(V^2_Z)$ as $\epsilon_k \to 0$.

Lemmas 6-8 guarantee that if such (σ_k, δ_k) exists then the sets $cl(S^1_{IJ} \cap p(\succsim_2))$ and $cl(S^2_{IJ} \cap q(\succsim_1))$ satisfy the conditions of Lemma 6. Thus Algorithm 1 detects the refutation $\{\epsilon_k, \sigma_k, \delta_k, s_\omega\}_{k=1}^{\infty}$ provided it compares the preorders $\{\succsim_1, \succsim_2\}$ associated with the sequence σ_k.

On the other hand, if the algorithm encounters two total preorders $\succsim_1$ and $\succsim_2$ such that $cl(S^1_{IJ} \cap p(\succsim_2))$ and $cl(S^2_{IJ} \cap q(\succsim_1))$ satisfy the conditions of Lemma 6, then Lemma 6 implies that it is possible to construct an $I \times J$-*refutation*.

Proof of Lemma 9

Proof Begin by proving the first claim of the lemma. Suppose seeking contradiction and w.l.o.g. that player one possesses a pair of variables x_j and x_k such that

$$\sigma_1(+l_j) + \sigma_1(-l_j) > \sigma_1(+l_k) + \sigma_1(-l_k)$$

Then it must be that

$$u_2(\sigma_1, x_j) > u_2(\sigma_1, x_k)$$

Thus $x_j \succ_2 x_k$ and $\sigma_2(x_j)\epsilon \geq \sigma_2(x_k)$. Note that as $\epsilon \to 0$ one has that

$$x \succ_2 l \;\; \forall x \in V, l \in L$$

and so

$$\sigma_2(x_j)\epsilon^2 \geq \sigma_2(x_k)\epsilon \geq \sigma_2(l) \;\; \forall l \in L$$

As $\epsilon \to 0$, it must then be that

$$u_1(\pm l_j, \sigma_2) < u_1(\pm l_k, \sigma_2)$$

and thus σ_1 does not satisfy criteria for properness.

Similarly, suppose that $\sigma_1(x_j) > \sigma_1(x_k)$. Then $\pm l_k \succ_2 \pm l_j$, and hence player two places much less probability on $\{+l_j, -l_j\}$ than on $\{+l_k, -l_k\}$ as $\epsilon \to 0$. This implies that $u_1(x_j, \sigma_2) < u_1(x_k, \sigma_2)$, and hence σ_1 cannot be proper.

Proof of Lemma 10

Proof Suppose without loss of generality and seeking contradiction that σ_1 is such that there exists two variables x_i and x_j with $max\{\sigma_1(+l_i), \sigma_1(-l_i)\} >$

$max\{\sigma_1(+l_j), \sigma_1(-l_j)\}$. Lemma 9 of course implies that $min\{\sigma_1(+l_i), \sigma_1(-l_i)\} < min\{\sigma_1(+l_j), \sigma_1(-l_j)\}$. This implies that

$$max\{u_2(\sigma_1, +l_i), u_2(\sigma_1, -l_i)\} > max\{u_2(\sigma_1, +l_j), u_2(\sigma_1, -l_j)\} \geq min\{u_2(\sigma_1, +l_j), u_2(\sigma_1, -l_j)\}$$

and hence $\sigma_2(+l_i) + \sigma_2(-l_i) > \sigma_2(+l_j) + \sigma_2(-l_j)$ in order to maintain properness. But this violates Lemma 9, a contradiction. Thus for every pair of variables x_i and x_j

$$max\{\sigma_i(+l_i), \sigma_i(-l_i)\} = max\{\sigma_i(+l_j), \sigma_i(-l_j)\}$$

$$min\{\sigma_i(+l_i), \sigma_i(-l_i)\} = min\{\sigma_i(+l_j), \sigma_i(-l_j)\}$$

for each player i. Thus a partition L_1^i and L_2^i exists for individual players. To show that $L_1^1 = L_1^2$, note that if $\sigma_i(+l_j) > \sigma_i(-l_j)$ then $u_{-i}(\sigma_{-i}, +l_j) > u_{-i}(\sigma_{-i}, -l_j)$ and hence $\sigma_{-i}(+l_j) > \sigma_{-i}(-l_j)$. Similarly it is easy to prove that if $\sigma_i(+l_j) = \sigma_i(-l_j)$, it must be that $\sigma_{-i}(+l_j) = \sigma_{-i}(-l_j)$. Thus it must be that $L_1^1 = L_1^2$ and $L_2^1 = L_2^2$.

Proof of Lemma 11

Proof Let $A = \{l_1, l_2, ..., l_n\}$ denote a satisfying assignment of ϕ. Let $C_1 = \{c \in C : c \cap A = 1\}$, $C_2 = \{c \in C : c \cap A = 2\}$ and $C_3 = \{c \in C : c \cap A = 3\}$. Construct a refutation as follows:

$$\delta_i(f) = \sigma_i(f) = \epsilon$$

$$\delta_i(c) = \sigma_i(c) = \epsilon^2 \quad \forall c \in C_1$$

$$\delta_i(c) = \sigma_i(c) = \epsilon^3 \quad \forall c \in C_2$$

$$\delta_i(c) = \sigma_i(c) = \epsilon^4 \quad \forall c \in C_3$$

$$\delta_i(g) = \epsilon^5$$

$$\delta_i(x) = \sigma_i(x) = \epsilon^6 \quad \forall x \in V$$

$$\delta_i(l) = \sigma_i(l) = \epsilon^7 \quad \forall l \in A$$

$$\delta_i(h) = \sigma_i(h) = \epsilon^8$$

$$\delta_i(l) = \sigma_i(l) = \epsilon^9 \quad \forall l \notin A$$

$$\sigma_i(g) = 1 - \sum_{s \neq g} \sigma_i(s)$$

I have omitted the normalizing factor that would ensure that $\sigma_i(g) > 1 - \epsilon$, but this does not present an issue to the proof. All probabilities aside from $\sigma_i(g)$ could be scaled down by for instance a factor of $\frac{1}{2}$ to ensure this holds. Now to show that it is indeed the case that

$$f \succ_i c \succ_i g \succ x \succ_i l(\in A) \succ_i h \succ_i l(\notin A)$$

for each player i. Note that since $\sigma_i(g) \to 1$ as $\epsilon \to 0$, it must be that $f, g, c \succ x \succ l, h$ for small ϵ. Computing expected payoffs:

$$u_i(\sigma_{-i}, f) = (\frac{n - \frac{1}{2}}{n})(n\epsilon^7 + n\epsilon^9) + (1 - n\epsilon^7 - n\epsilon^9)(n - 1)$$

$$u_i(\sigma_{-i}, c) = (n-1)\epsilon^7 + (n-2)\epsilon^9 + (1 - n\epsilon^7 - n\epsilon^9)(n-1) \quad \forall c \in C_1$$

$$u_i(\sigma_{-i}, c) = (n-2)\epsilon^7 + (n-1)\epsilon^9 + (1 - n\epsilon^7 - n\epsilon^9)(n-1) \quad \forall c \in C_2$$

$$u_i(\sigma_{-i}, c) = (n-3)\epsilon^7 + (n)\epsilon^9 + (1 - n\epsilon^7 - n\epsilon^9)(n-1) \quad \forall c \in C_3$$

$$u_i(\sigma_{-i}, g) = n(|C_1|\epsilon^2 + |C_2|\epsilon^3 + |C_3|\epsilon^4) + (1 - \epsilon - |C_1|\epsilon^2 - |C_2|\epsilon^3 - |C_3|\epsilon^4)(n-1)$$

$$u_i(\sigma_{-i}, l) = (\epsilon + |C_1|\epsilon^2 + |C_2|\epsilon^3 + |C_3|\epsilon^4 + \epsilon^8)(n-1) + (n-1)^2\epsilon^6 + (n-4)\epsilon^6 +$$

$$n(n-1)\epsilon^7 + (n-1)^2\epsilon^9 + (n-4)\epsilon^9 + (n-3)\sigma_i(g) \quad \forall l \in A$$

$$u_i(\sigma_{-i}, h) = (\epsilon + |C_1|\epsilon^2 + |C_2|\epsilon^3 + |C_3|\epsilon^4 + \epsilon^8)(n-1) + (n-1)^2\epsilon^6 + (n-4)\epsilon^6 +$$

$$n(n - 1 - \frac{1}{2n})(\epsilon^7 + \epsilon^9) + (n-3)\sigma_i(g)$$

$$u_i(\sigma_{-i}, l) = (\epsilon + |C_1|\epsilon^2 + |C_2|\epsilon^3 + |C_3|\epsilon^4 + \epsilon^8)(n-1) + (n-1)^2\epsilon^6 + (n-4)\epsilon^6 +$$

$$(n-1)^2\epsilon^7 + (n-4)\epsilon^7 + n(n-1)\epsilon^9 + (n-3)\sigma_i(g) \ \ \forall l \notin A$$

Using these calculated payoffs, it is possible to verify that the appropriate ordering holds for $\succsim_i$, and hence this profile serves as a refutation.

Proof of Lemma 12

Proof Suppose that T has a refutation. Then the total preorder must satisfy $f \succsim_i c$ for all clauses c for ϵ sufficiently small. Indeed, if $c^* \succ_i f$ for some clause c^*, then $g \succ_i c^*$ and hence the preorder can't serve as a refutation for all ϵ as g would then be the most preferred strategy. Lemmas 9 and 10 also imply that, for ϵ small, $f \sim_i c$ cannot occur. This is due to the fact that the payoffs from f and c differ only on literals and h. In particular, let L_1 and L_2 be a partition from Lemma 10 and suppose that $L_1 \succ_i L_2$ (this implies that $L_1 \succ_i h$). Noting that each literal in L_1 must be given the same probability at $T(\epsilon)$ proper profiles, we see that for all sufficiently small ϵ, either $c \succ_i f$ or $f \succ_i c$. If it is the case that $L_1 \sim_i L_2$, then $h \succ_i L_1, L_2$ so that $c \succ_i f$ for all clauses c. Hence there cannot be a refutation due to the fact that $g \succ_i c$. Therefore, if a refutation exists, the total preorder must satisfy $f \succ_i c$ for all clauses c, and $L_1 \succ L_2$ in the partition from Lemma 10. Note that if $L_1 \succ_i L_2$, it must be the case that h lies between them so that $L_1 \succ_i h \succ_i L_2$. Given this, if $f \succ_i c$ for all clauses, then every clause c possesses a literal from the set L_1. To see why, if some clause c^* is such that every literal from L_1 is not in $c*$, then it must be the case that $c \succ_i f$, and hence $g \succ_i f$. Thus if a refutation exists, then it must be the case that there is an assignment of literals L_1 such that each clause possesses a literal from L_1. Hence L_1 serves as a satisfying assignment.

Proof of Lemma 13

Proof Suppose that one is given a sub-block $I \times J$ of T and a pair of total pre-orders $\{\succsim_1, \succsim_2\}$. To verify whether they serve as a refutation, it suffices to show that $p(\succsim_2)$ contains profiles which are $T(\epsilon)$*-proper* in its relative interior, with a similar statement holding for $q(\succsim_1)$ with ϵ sufficiently small. Lemmas 6-8 imply that one need only determine whether there are a collection of vertices of $p(\succsim_2)$ and $q(\succsim_1)$ that satisfy the conditions of lemma 6. Consider w.l.o.g. the total pre-order $\succsim_1$ and enumerate the indifference classes $\zeta_1^1, ..., \zeta_r^1$ in decreasing order, so that strategies in ζ_i^1 are strictly preferred to all $s' \in \zeta_j^1$ with $j > i$. To verify the conditions of lemma 6 for a given strategy $s \in \zeta_i^1$, it suffices to solve

$$\max \delta_1(s)$$

$$s.t.\ \ \delta_1(s') = 0 \ \ \forall s' \in \zeta_j^1, j > i$$

$$\delta_1(s) \in cl(S^1_{IJ} \cap p(\succsim_2))$$

and verify whether there is a basic feasible solution with $\delta_1(s) > 0$. Note that the constraint set for this problem is defined by a number of linear inequalities that is polynomial in $|S_1|$ and $|S_2|$. If the desired solution exists for all $s \in S_1$, then lemma 8 implies that there are $T(\epsilon)$*-proper* profiles according to $\succsim_1$ that lie within $p(\succsim_2)$. A similar process can be performed for strategies in $\succsim_2$.

Proof of Proposition 7

Proof By proposition 4, the problem of verifying whether a given block T is finely tenable is equivalent to determining whether there is a sub-block $I \times J$ that possesses an $I \times J$*- refutation*. Lemmas 6-8 imply that any $I \times J$*refutation* can be equivalently expressed as a pair of total preorders $\{\succsim_1, \succsim_2\}$. Thus the problem of verifying fine tenability of a block T is equivalent to finding $\{I, J, \succsim_1, \succsim_2\}$ with $I \times J$ a sub-block of T and $\succsim_1, \succsim_2$ satisfying the conditions outlined in lemma 6. Given such a tuple $\{I, J, \succsim_1, \succsim_2\}$, lemma 13 implies that it can be verified in polynomial time. This, combined with proposition 7, implies that the problem is NP-complete.

Proof of Lemma 14

Proof To prove necessity, suppose that $\sigma = (p^*, q^*)$ is indeed a proper equilibrium. Then there is a sequence $\{\epsilon_k, \sigma_k\}_{k=1}^{\infty}$ of ϵ_k-*proper* equilibria converging to σ with $\epsilon_k \to 0$. The total preorders over strategies induced by $\sigma_{1,k}$ and $\sigma_{2,k}$ can be taken to be constant in k. Hence the sequence $\sigma_{1,k}$ and $\sigma_{2,k}$ lie on single closed faces f_1 and f_2 respectively. Since these faces are closed and the sequences converge to p^* and q^*, it must be that $p^* \in f_1$ and $q^* \in f_2$. The fact that σ_k are ϵ_k-*proper* with $\epsilon_k \to 0$ implies that $f_i \subseteq P^i(p^*, q^*)$. Showing the final condition is almost identical to the logic of Lemma 6.

To show sufficiency, it suffices to construct a convergent sequence of ϵ_k-*proper* equilibria directly. Since $p^* \in f_1$, it follows that for every strategy $s \in Y_1(p^*)$, there is a vertex v of f_1 in which $v(p_s) > 0$ and in which $v(p_{s'}) = 0$ for all $s' \notin Y_1(p^*)$. Furthermore, p^* can be realized as a convex combination of such vertices. Let V_{p^*} denote the largest collection of vertices of f_1 such that $v(p_s) > 0$ for some $s \in Y_1(p^*)$ and $v_(p'_s) = 0$ for all $s' \notin Y_1(p^*)$, and let $\{\lambda^*_v\}_{v \in V_{p^*}}$ denote a convex combination such that

$$\sum_{v \in V_{p^*}} \lambda^*_v v = p^*$$

Let $Z(V_{p^*}) = \{v \in V_{p^*} : \lambda_v^* = 0\}$. Fix the following values:

$$\bar{p} = max_{v \in f_1} max_{\{s: v(p_s) > 0\}} v(p_s)$$

$$\underline{p} = min_{v \in f_1} min_{\{s: v(p_s) > 0)\}} v(p_s)$$

Let $\succsim_1$ denote the total preorder induced over S_1 induced by strategies on the relative interior of f_2. Let V_1 denote the collection of vertices of f_1 such that $v(p_s) > 0$ for some $s \in \zeta_1^1 \backslash Y_1(p^*)$ and $v(p_{s'}) = 0$ for all $s' \in \zeta_k^1$, $k > 1$. Similarly, let V_k, $k \geq 2$, denote the collection of vertices of f_1 such that $v(p_s) > 0$ for some $s \in \zeta_k^1$ and $v(p_{s'}) = 0$ for all $s' \in \zeta_l^1$, with $l > k$. Note that $V_k \cap V_j = \emptyset$ for $j \neq k$. Let $|V(f_1)|$ denote the total number of vertices of the closed face f_1. For a given, small $\epsilon > 0$, I will construct a profile p that lies within the relative interior of f_1 and is *ϵ-proper* according to $\succsim_1$. For $v \in V_k$, define

$$\lambda_v = \frac{(\epsilon \underline{p})^k}{(|S_1||V(f_1)|)^k}$$

For $v \notin \{\cup_k V_k\} \cup V_{p^*}$, define

$$\lambda_v = \frac{(\epsilon \underline{p})^{r_1}}{(|S_1||V(f_1)|)^{r_1}}$$

For $v \in Z(V_{p^*})$, define

$$\lambda_v = \frac{(\epsilon \underline{p})^{r_1}}{(|S_1||V(f_1)|)^{r_1}}$$

For $v \in V_{p^*} \backslash Z(V_{p^*})$, define

$$\lambda_v = (1 - \sum_{\{v' \notin V_{p^*} \backslash Z(V_{p^*})\}} \lambda_{v'}) \lambda_v^*$$

Note that as $\epsilon \to 0$,

$$\sum_{v' \notin V_{p^*} \backslash Z(V_{p^*})} \lambda_{v'} \to 0$$

Therefore

$$p_\epsilon = \sum_{v \in f_1} \lambda_v v \to p^*$$

as $\epsilon \to 0$ since $p^* = \sum_{v \in V_{p^*}} \lambda_v^* v$. Note also that $p_\epsilon(s) > 0$ for all $s \in S_1$ and all $\epsilon > 0$. All that is left to show is that p_ϵ is indeed *ϵ-proper* according to $\precsim_1$ for small enough ϵ. For any $s \in \zeta_k^1 \backslash Y_1(p^*)$, with $1 \le k \le r_1 - 1$,

$$p_\epsilon(s) \ge \frac{(\epsilon \underline{p})^k}{(|S^1||V(f_1)|)^k}$$

and for any $s' \in \zeta_{k+1}^1$

$$p_\epsilon(s') \le \sum_{i=k+1}^{r_1} |V_i| \bar{p} \frac{(\epsilon \underline{p})^i}{(|S_1||V(f_1)|)^i} + |Z(V_{p^*})| \bar{p} \frac{(\epsilon \underline{p})^{r_1}}{(|S_1||V(f_1)|)^{r_1}}$$

and hence

$$\frac{p_\epsilon(s')}{p_\epsilon(s)} \le \frac{\sum_{i=k+1}^{r_1} |V_i| \bar{p} \frac{(\epsilon \underline{p})^i}{(|S_1||V(f_1)|)^i} + |Z(V_{p^*})| \bar{p} \frac{(\epsilon \underline{p})^{r_1}}{(|S_1||V(f_1)|)^{r_1}}}{\frac{(\epsilon \underline{p})^k}{(|S^1||V(f_1)|)^k}}$$

$$\le \frac{|V(f_1)| \bar{p} (\epsilon \underline{p})^{k+1}}{(|S_1||V(f_1)|)^{k+1}} \frac{(|S_1||V(f_1)|)^k}{(\epsilon \underline{p})^k} = \frac{\bar{p}(\epsilon \underline{p})}{|S_1|} \le \epsilon$$

Now let $s \in Y_1(p^*)$ and $s' \in \zeta_2^1$. Then

$$p(s) \ge \underline{p} min_{\{v : \lambda_v^* > 0\}} \lambda_v^* (1 - \sum_{\{v' \notin V_{p^*} \backslash Z(V_{p^*})\}} \lambda_v')$$

$$p(s') \le \sum_{i=2}^{r_1} |V_i| \bar{p} \frac{(\epsilon \underline{p})^i}{(|S_1||V(f_1)|)^i} + |Z(V_{p^*})| \bar{p} \frac{(\epsilon \underline{p})^{r_1}}{(|S_1||V(f_1)|)^{r_1}}$$

Then

$$\frac{p(s')}{p(s)} \le \frac{\bar{p} \underline{p} \epsilon^2}{|S_1|^2 |V(f_1)| min_{\{v : \lambda_v^* > 0\}} \lambda_v^* (1 - \sum_{\{v' \notin V_{p^*} \backslash Z(V_{p^*})\}} \lambda_v')} \le \epsilon$$

for ϵ small enough. Thus p_ϵ constructed here is a sequence of *ϵ-proper* equilibria which converge to p^* as $\epsilon \to 0$. Note that since p_ϵ are all in the relative interior of f_1, they induce the same total preorder over S_2. A mirrored construction allows one to construct q_ϵ which are *ϵ-proper* for ϵ small enough and which lies in the relative interior of f_2, and hence induce the same total preorder over strategies in S_1. This provides a pair (p_ϵ, q_ϵ) satisfying the desired properties for ϵ small and converging to (p^*, q^*) as $\epsilon \to 0$.

CHAPTER 4

CONTRACTING FOR ATTENTION INTERMEDIARIES

4.1 Introduction

The internet has developed into a mainstay of everyday modern life. This has changed not only the way in which business is conducted, but also how individuals find entertainment. As of writing this, the popular video sharing platform Youtube accounts for nearly fifteen percent of all consumer broadband traffic in the U.S. Twitch.tv, a streaming platform owned by Amazon, commands nearly 2 percent at peak hours.[1]

Youtube and Twitch belong to a category of media platforms which rely on value generation by individuals producing content. The platforms themselves offer little to no value to a viewer without the vast library of streams and videos that it hosts. Thus, like all platforms appearing in the economics literature, media platforms that feature content produced by third-parties offer a service which eliminates the frictions associated with viewers digging through independent websites searching for entertainment. Now, more than ever, people are sharing and streaming their hobbies, daily life, and major events through media platforms, making their vocation a vacation, as Mark Twain may have put it.

Media platforms are monetized through several different channels. The most prominent channel is by hosting advertisements and charging advertisers per impression, or by receiving a cut of profits from all purchases generated

[1]Some notable public figures, such as U.S. congresswoman Alexandria Ocasio-Cortez, have even sought to use streaming platforms as a method of reaching constituents and gaining popularity.

by clicks on advertisements hosted on the platform. A number of platforms also feature a, sometimes optional, subscription fee for consumers that enhances the viewing experience by, for instance, providing additional features or reducing advertising for subscription holders. In this paper, I focus only on advertising and abstract away from all other monetization policies featured on media platforms.

While content on these platforms is produced by third-parties that oftentimes receive only modest popularity in return, a number of individuals pursue content creation as a primary source of income. Those that are able to do so command large viewership hours and, consequently, prove to be an excellent matching medium between advertisers and consumers. The quality and quantity of this content, indeed the average value of a platform for a viewer, is determined by the incentives provided to the individuals producing it. When those producing content on these platforms receive a greater share of the revenue that they generate, this leads to greater incentives to not only produce content for that platform, but also to produce content of a higher quality.

In this paper, I consider the provision of incentives by advertising-funded media platforms such as Youtube, Twitch and Facebook in a simple competitive framework. These platforms offer simple limited-liability contracts to content creators that pay contingent on the realization of a high quality of content. Consumers are assumed to single-home in this model, and receive value from joining a platform by allocating scarce attention to high quality content available on that platform. Consumers are not charged monetary fees for joining the platform, and instead pay with their attention by watching informative advertisements. Advertisers are charged per-impression by platforms.

Of particular note in the analysis that follows is the focus on exclusivity versus multi-homing among those producing content for these sites. This has been a recent point of contention in the market for online entertainment. Youtube, Twitch and Facebook's streaming service have a strict "no dual-screening" policy which prevents individuals from streaming to two platforms separately.[2] The most popular content creators on these platforms are sometimes given explicit exclusivity clauses in their contracts.[3] Twitch, in response to the entry of competing live-streaming platform Mixer and Youtube's live-streaming service, began imposing exclusivity clauses in several Partnership contracts in 2015. Partnership is a very common form of incentive scheme offered on media platforms wherein the top-performing agents receive a greater share of the revenue that they generate. The vast majority of viewership on media platforms is received by partnered channels. A natural question is then whether exclusivity is good for consumers in this market. While exclusivity of course reduces access to high-quality content for consumers, platforms are much more likely to provide greater incentives to exclusive agents due to the fact that these incentives can only raise the value of the platform relative to competition. In sections 3 and 4 below, I consider this tradeoff in a very simple model.

4.1.1 Related Literature

This paper belongs to the literature in industrial organization on platforms and multi-sided markets. Since the major contributions of Rochet and Tirole

[2] Additionally, clips and stream recordings from Twitch cannot be uploaded elsewhere for a minimum of 24 hours after creation.

[3] A notable collection of examples are in Mixer's (now merged with Facebook's live-streaming service) poaching of streaming giants "Ninja" and "Shroud." Exclusivity clauses are also common in media platforms hosting music, such as Apple Music's former agreement with pop star Taylor Swift.

(2003)[56] and Armstrong (2006)[4], the literature on platforms has grown far too large to summarize here.[4] Perhaps the most related papers are those within this literature that explicitly attempt to model attention intermediaries and media platforms. Anderson and Coate (2005)[2] consider the market for radio broadcasting and provide conditions under which the advertising levels in the market equilibrium are below (above) the socially efficient level.[5] As in most papers on media markets, see for instance Gabszewicz et al. (2004)[32], Anderson and Coate (2005)[2], and Peitz and Valetti (2008)[51], this paper assumes that consumers are not charged for access to content, and instead "pay" for access by watching advertisements. Within this literature, the most related paper to this one is Anderson and Peitz (2020)[3] who use the an aggregative games framework to analyze the effects of entry and mergers on consumer and advertiser surplus. The novelty of this paper within the literature on platforms and multi-sided markets is its departure from Hotelling lines and Salop circles, which generally feature linear transportation costs, to a microfounded model of Logit demand. The primary benefit of this framework is that it yields tractability even when one moves beyond two competing platforms. In this paper, I build upon this approach by considering Logit membership decisions by both content creators and consumers. This extension does not come without some sacrifice, however. In particular, I consider a model with identical advertisers and impose specific parametric forms on the demand and cost functions.

This paper differs from much of the literature on media platforms in that consumer values for content and advertising are endogenized using an attention model of consumer demand. While endogenous network effects appear

[4] For excellent reviews, see Rysman (2009)[58] and Sanchez-Cartas and Leon (2021)[60]

[5] One of the best contributions of this paper is its novel model of an advertising market. In particular, I use a simplified version of their advertising setup in this paper.

frequently in the literature on buyer-seller platforms,[6] the attention model used in this paper, which is adapted from Anderson et al. (2011)[61], yields a distortionary form for advertising nuisance that differs from the additive form appearing in much of the literature on media markets.

A number of papers consider the effects of exclusivity and multi-homing within the framework of platform markets, and specifically within the context of media markets. Nicita and Ramello (2005)[50] argue against exclusivity in the European Pay-TV market, claiming that exclusivity serves as a barrier to entry and dampens the prospect for innovation by precluding the development of alternative platforms. Stennek (2014)[64] demonstrates in a short paper that exclusive broadcasting rights may encourage a higher investment in quality among content providers. Carroni et al. (2018)[20] consider the welfare effects of imposing exclusivity on an atomistic superstar in a two-sided market for content. This paper differs from this literature in that it considers explicitly the provision of incentives by attention intermediaries in a moral hazard framework. I show that, when the extent of multi-homing is exogenously given,[7] consumer surplus is increasing in the extent of multi-homing at the symmetric equilibrium that arises.

Finally, there are some papers which seek to model precisely the kind of content-creation and streaming platforms discussed in the introduction, though these are primarily in a monopolistic setting. Carroni and Paolini (2019)[12] consider the problem of whether a streaming platform, such as Spotify, should offer premium (subscription) services for consumers. Casner (2018)[13] develops a

[6]See Hagiu (2009)[25] for an excellent analysis of endogenous network effects and consumer love of variety.

[7]This is a common assumption in two-sided markets with multi-homing. See, e.g., Choi (2010)[16]

model of multi-sided markets with third-party content creation, with Youtube as an explicit motivation, and considers the welfare effects of introducing a subscription that eliminates advertising for consumers. Alaei et al. (2020)[59] also consider an advertising-funded media platform, motivated by music streaming. Their focus is on the merits of different revenue allocation rules for artists, however. Finally, Galeotti and Fainmesser (2021)[22] consider the trade-off between sponsored and organic content faced by those endorsing products on social media platforms such as Instagram.

4.2 Model

This paper presents a stylized, parametric model of of competition among attention intermediaries that incentivize content creation through the design of simple limited liability contracts. The model features four kinds of players: attention intermediaries (principals), content creators (agents), consumers, and advertisers. There are assumed to be m attention intermediaries, indexed by $i \in \{1, 2, ..., m\}$ who each operate separate media platforms. These attention intermediaries incentivize content creators to create value for consumers by producing high-quality content on the intermediary's platform. The cost to each intermediary in this model is described simply by the transfers detailed in a simple limited liability contract. Thus I am assuming that there is no variable cost to an intermediary that depends on the volume of content or viewers present on the platform. Consumers value high-quality content, and have a tendency to join platforms which feature greater levels of high-quality content, all else equal. Attention intermediaries are assumed to not price consumers directly via, for instance, subscriptions or membership fees. Instead, each intermediary accrues revenue through a simple advertising channel by charging a per-viewer price to advertisers. Each intermediary's revenue is extracted solely from the advertising channel. Advertising is assumed to deteriorate the value for consumers associated with watching content on a platform so that each intermediary faces a trade-off between greater per-consumer revenues versus reduced demand associated with a higher level of advertising.

There is a unit mass of identical consumers that single-home on either a platform or an outside option. Each of these consumers is endowed with a unit budget of attention that they allocate across content available on a platform, should they choose to join one. Content in this model is assumed to take one of two types, *high* or *low*. *Low* quality content is not valued by consumers and receives no attention. *High* quality content is assumed to be the only source of value for consumers from allocating attention to content on a platform. Implicit in this framework are a number of simplifying, and restrictive, assumptions. First, all consumers in the market agree about the value of any given piece of content. Second, the assumption that *low* quality content receives no attention from consumers implicitly assumes that consumers spend no time sifting through content to separate the high and low.

Content creators, the *agents* to the intermediary's *principal* in this framework, are assumed to be non-atomic, with a unit mass of them present in the market. These content creators are assumed to be ex-ante identical from the perspective of all players in the model. These content creators exert effort $e \in [0, 1]$ in order to increase the probability of a *high* realization of content quality. For the sake of simplicity, the probability of a high realization of content quality conditional on effort e is assumed to be $P(H, e) = e$. That is, the probability of a high realization is equal to the agent's effort level e.[8] Furthermore, I suppose that the level of effort e exerted by any given agent is known only to the exerting agent, and in particular is not observable, nor verifiable, to any other player.

Let n_{H_i} denote the mass of high-quality content featured on platform i, and let κ_i denote the level of advertising on platform i. The mass of consumers that choose to join platform i, here denoted by a_{P_i}, takes a logit form in this model[9,10]:

$$a_{P_i} = \frac{n_{H_i}(1-\kappa_i)^{\frac{\alpha}{1-\alpha}}}{\sum_{j=1}^{m} n_{H_j}(1-\kappa_j)^{\frac{\alpha}{1-\alpha}} + n_0} \tag{4.1}$$

where $\alpha \in (0,1)$ is a parameter measuring the diminishing returns to attention for consumers, and n_0 is the (fixed) mass of high-quality content available outside of the market. Note in particular that the relative weight given to platform i is linear in the mass of high-quality content on platform i. This linear form is common in models of two-sided markets, and requires some additional assumptions to generate in this case. The weight $n_{H_i}(1-\kappa_i)^{\frac{\alpha}{1-\alpha}}$ is sometimes referred to as a *utility intensity*,[11] but is not the utility for a consumer choosing to join platform i in this model. Instead, this utility is given by the following form:

$$U_i = ln(v_i) + \epsilon_i$$

where $ln(v_i)$ is a deterministic value for joining platform i, and ϵ_i is a random variable representing a given consumer's idiosyncratic value for platform i. Each ϵ_i is drawn independently and according to the same distribution for each platform. The deterministic component $ln(v_i)$ for utility achieved from joining

[8] This is a common form for stochastic output in models of contracting, and appears in nearly any textbook on the principal-agent problem. See, for instance, Laffont and Martimort (2002)[38].

[9] Logit models of discrete choice are common in applied models. In particular, the form used in this paper is taken from Anderson and Peitz (2020) [3] .

[10] This form for platform demand can alternatively be obtained using a representative consumer that allocates attention across all of the content available in the market. However, this connection breaks down when one moves to a regime with multi-homing content creators.

[11] See, e.g. Chambers and Echenique (2016)[14].

platform i is derived from solving a simple attention allocation problem for a consumer present on platform i. To make this precise, suppose that high-quality content on platform i is indexed by $s \in [0, n_{H_i}]$. A consumer patronizing platform i is assumed to solve the following trivial optimal control problem:

$$\max_{a_s} \; ln\Big(\int_0^{n_{H_i}} (a_s(1-\kappa_i))^{\alpha} ds\Big)^{\frac{1}{1-\alpha}}$$
$$s.t. \quad \int_0^{n_{H_i}} a_s = 1$$
$$a_s \geq 0 \quad \forall s \in [0, n_{H_i}]$$

Given that $\alpha \in (0,1)$, the solution is given by $a_s = \frac{1}{n_{H_i}}$ for all $s \in [0, n_{H_i}]$. There are a number of comments in order for this problem. First, observe that advertising is distortionary in this model. When a consumer allocates attention a_s to a piece of content s when the advertising level is κ_i, they spend time $a_s\kappa_i$ watching advertisements and $a_s(1-\kappa_i)$ viewing actual content. The integrand here also demonstrates precisely how α captures the diminishing returns to attention, as briefly mentioned earlier. Finally, the transformation $ln(\cdot)^{\frac{1}{1-\alpha}}$ is imposed in order to ensure that demand takes the stochastic form given in (1), and also that the utility intensity associated with platform i is linear in the amount of high-quality content on platform i. Note that this transformation does not alter the optimal attention allocation of a consumer on platform i, provided one exists, even when content takes a richer form, such as a continuum of qualities. The value of this problem at an optimal solution defines the deterministic component $ln(v_i)$ of a consumer's utility from joining platform i.

Now, in order to generate the demand equation (1) using this microfoundation, I assume that each ϵ_i is drawn independently according to a Type-1 Extreme Value Distribution (MacFadden (1973)[39]). Under these assumptions, the probability of the event $(U_i \geq \max_{j \neq i} U_j)$ is given by the logit form in the right-hand side of equation (1). Given that all consumers are assumed to draw independent idiosyncratic values, a strong law of large numbers implies that the mass of consumers that choose platform i is given as in (1). Under these assumptions, and using results of Small and Rosen (1981)[62], consumer surplus in this setting takes the following log-sum form (up to an additive constant):

$$CS = \mu ln\Big(\sum_{i=1}^{m} e^{ln(v_i)} + e^{ln(v_0)}\Big) = \mu ln\Big(\sum_{i=1}^{m} n_{H_i}(1-\kappa_i)^{\frac{\alpha}{1-\alpha}} + n_0\Big)$$

where μ is the standard deviation of ϵ_i. Now, it is important to note that this expression is a monotone transformation of the aggregate term appearing in the denominator of the demand function (1):

$$A = \sum_{i=1}^{m} n_{H_i}(1-\kappa_i)^{\frac{\alpha}{1-\alpha}} + n_0$$

For the purposes of the results in this paper, I use the aggregate A for all comparisons of consumer surplus.

Advertising in this model serves an informative role for consumers and is a simplified variant of the advertising model appearing in Anderson and Coate (2005)[2]. Each advertiser has a unique product, which they are the sole, monopolistic producer of. These products are assumed to be produced at a constant marginal cost, set at zero without loss of generality. When a consumer observes an advertisement, they are informed about the existence, price, and their willingness to pay of an advertiser's product. This willingness to pay is either $\omega > 0$ or 0, and each consumer only desires at most one of any advertiser's product. The probability of a positive match value, $P(\omega)$, between an advertiser and consumer is assumed to be constant, and independent across all advertisers. Given the form for a consumer's willingness to pay, each advertiser sets a price ω for their product. This implies that consumers receive no benefit from the informative advertising.

Letting $P(\omega)\omega \equiv \phi$ denote the expected value to an advertiser of reaching a consumer with an advertisement, each advertiser is willing to accept a per-viewer price of $p \leq \phi$ for showing their advertisement. Hence each platform sets a per-viewer price of ϕ to advertisers, so that advertisers receive a payoff of 0 at any equilibrium. In this model, surplus generated through advertising is transferred entirely to the attention intermediaries. I assume that platforms are able to exclude any additional unwanted advertising, so that they are able to choose an advertising level κ_i directly. When a platform hosts κ_i advertisements, each consumer on the platform is informed of the price and their willingness to pay for the products of κ_i advertisers. Note here that the assumption of consumer single-homing is critical to this framework, as each consumer never encounters the same advertisement multiple times. When a platform i achieves a consumer membership of a_{P_i} at advertising level κ_i, they accrue revenue $\phi\kappa_i a_{P_i}$, as each of the κ_i advertisers is charged ϕ per consumer.

As discussed above, the value to consumers associated with joining a given platform is determined by the mass of high-quality content available on it. This high-quality content produced by a unit mass of content creators present in the market that exert costly effort $e \in [0, 1]$ to improve the probability that their content realization is high. In particular, the probability of a high content realization given effort e is assumed to be $P(H, e) = e$. The cost of exerting effort e is described by the thrice-continuously differentiable, strictly increasing, and strictly convex cost function C. Additionally, C satisfies

$$\lim_{e \to 0} C(e) = \lim_{e \to 0} C'(e) = 0$$

so that attention intermediaries will incentivize a positive level of effort. Wherever necessary, I impose that C is sufficiently convex to guarantee interiority of solutions. I assume that agents exert effort only *after* choosing a platform, and hence only evaluate their decision to join a platform based on expected payoffs. Suppose now that an agent is faced with a tuple $(t(H), t(L))$ of value contingent transfers offered by an attention intermediary corresponding to the platform that they have chosen to join, the agent solves:

$$\max_{e} \quad et(H) + (1-e)t(L) - C(e)$$
$$s.t. \quad e \in [0,1]$$

Note that this formulation assumes not only risk neutrality, but quasilinearity of content creator preferences. This problem is solved by the first order condition:

$$t(H) - t(L) = C'(e)$$

Given that output is binary, one can impose without loss of generality that contracts take the form $(t(H), t)$ where $t(H)$ is a transfer contingent on a high realization of output and t is a flat rent for joining the platform. Given the tuple $(t(H), t)$, define $r(t(H), t)$ denote the expected rent from accepting the contract and exerting the optimal effort.

Note that this model departs from much of the literature on platforms and two-sided markets in that content creators here only care about transfers from attention intermediaries rather than indirect network effects generated by the mass of agents on the other side of the market. In reality, content creators may be motivated by factors such as popularity and peer recognition. Such effects would in general lead to greater effort exertion at any tuple of transfers and relax, at least to some extent, the distortions imposed by limited liability. In this paper, I abstract away from such indirect network effects and focus solely on the incentives provided by the attention intermediaries. This simplifying assumption allows for considerable flexibility in the timing of equilibria in the game that follows.

Each attention intermediary offers a tuple $(t(H), t)$, where $t(H)$ is a transfer contingent on a high realization of content value, and t is a flat transfer to an agent for participating on the platform. I assume that the effort exerted by each content creator is unobservable to both attention intermediaries so that, coupled with stochastic output, the contracting environment features a very standard form of moral hazard. This is motivated by the fact that whether or not content becomes successful oftentimes depends on a number of factors such as whether or not it goes viral as well as how it relates to the objectives of a recommendation algorithm. Given that output is discrete and each platform will in general possess a positive mass of agents accepting the associated contract and, at equilibrium, exerting the same level of effort, a simple strong law of large numbers implies that the quantities of high and low quality content on the platform are deterministic. Thus in this framework the risk preferences of the intermediaries do not affect the form of incentives. I assume that attention intermediaries are also constrained by limited liability in transfers, so that $t(H), t \geq 0$. This is imposed as a realistic assumption. Content creators are frequently individuals seeking to use media as a form of income, and as such do not have deep pockets. Additionally, the possibility of being in debt to the platform may deter agents from choosing to enter the market.

4.3 Agent Single-Homing

In this section, I consider the case in which attention intermediaries offer contracts that impose exclusivity on any agent that chooses to be active on their platform. Content creator participation in this setting is described by a simple stochastic form similar to consumer demand. In particular, let $(t_i(H), t_i)$ denote the transfers offered by platform i, and let $r(t_i(H), t_i)$ denote the expected rent associated with the contract. The mass of agents that choose to join platform i and accept the terms of the associated contract is assumed to be:

$$n_i = \frac{(r(t_i(H), t_i))^\rho}{\sum_{j=1}^{m} (r(t_j(H), t_j))^\rho} \tag{4.2}$$

This simple form for agent participation is assumed in order to avoid discontinuities in platform membership generated by a model in which content creators simply choose the contract with the largest expected rents, which would preclude existence of equilibria in general. The parameter $\rho \in (0, 1]$ in this expression is a measure of the responsiveness of content creators to changes in expected rents.[12] As ρ tends to 0, content creators become increasingly unresponsive to expected rents. In the limit, content creators are split uniformly across all platforms offering positive expected rents. Under the lens of the microfoundation for consumers, equation (2) implies that each content creator evaluates their decision to join a platform according to a deterministic component, $ln(r)$, and an idiosyncratic value ν_i drawn independently for each platform according to a Type-1 extreme value distribution.

Each attention intermediary chooses a tuple $(\kappa_i, t_i(H), t_i)$ consisting of an advertising level $\kappa_i \geq 0$, a transfer to high-value content creators $t_i(H) \geq 0$, and a flat transfer t_i for joining the platform. Attention intermediary i solves:

$$\max a_{P_i} \phi \kappa_i - n_i(e_i t_i(H) + t_i)$$
$$s.t. \quad \kappa_i, t_i(H), t_i \geq 0$$

where e_i is the optimal level of effort contingent on accepting platform i's incentive scheme. It's helpful to note that when each agent responds only to the

[12]In principle ρ can be much larger than 1. The restrictions imposed here on ρ are sufficient to guarantee uniqueness of conditional best responses for platforms in this setting.

incentives of a single platform, the first-order condition of the agent's problem gives

$$t_i(H) = C'(e_i)$$

and

$$r(t_i(H), t_i) = e_i C'(e_i) - C(e_i) + t_i$$

Hence one can write the intermediary's problem as choosing an effort level e_i directly and without appealing to the transfer $t_i(H)$. Rewritten, one has:

$$\max a_{P_i} \phi \kappa_i - n_i(e_i C'(e_i) + t_i)$$

$$s.t. \quad t_i \geq 0$$

$$e_i, \kappa_i \in [0, 1]$$

where a_{P_i} is given by (1) and n_i is given by (2) above. I'm now in a position to define an equilibrium with exclusivity in this setting. In particular, an equilibrium of the exclusivity regime is a collection of tuples $\{(\kappa_i, t_i(H), t_i)\}_{i=1}^m$, $\{(n_i, e_i)\}_{i=1}^m$, $\{a_{P_i}\}_{i=1}^m$ such that

1. Each e_i is an optimal effort level given $(t_i(H), t_i)$.
2. Each n_i is given as in (2).
3. Demand for each platform a_{P_i} is given as in (1) for each i.
4. For each i, given $\{(\kappa_j, t_j(H), t_j)\}_{j \neq i}$, the advertising and contract tuple $(\kappa_i, t_i(H), t_i)$ is optimal for intermediary i.

In order to provide concrete comparisons between exclusivity and multi-homing, it's helpful to provide a parametric form for the cost of effort C. Suppose that C takes the common form:

$$C(e) = \gamma e^{\delta}$$

where $\gamma, \delta > 0$ are sufficiently large relative to ϕ. Given the parametric form for costs, the symmetric equilibrium of this game can be pinned down in a simple form.

Proposition 4.3.1 *Suppose that $C(e) = \gamma e^{\delta}$ with γ and δ sufficiently large. There is ρ^* such that for all $\rho < \rho^*$, the unique symmetric equilibrium is given by the implicit solutions to the following system for each platform i:*

$$(1 - a_{P_i}) = \frac{1 - \kappa_i}{\kappa_i} \frac{1 - \alpha}{\alpha} \tag{4.3}$$

$$\phi \kappa_i a_{P_i} = \frac{\gamma \delta e_i^{\delta}}{m} \frac{[\rho \delta \frac{m-1}{m} + \delta]}{[\rho \delta \frac{m-1}{m} + (1 - a_{P_i})]} \tag{4.4}$$

additionally, $t_i = 0$ for each platform i.

All proofs are in the appendix. Note that I am not claiming that this is the unique equilibrium of the game, only that it is the unique symmetric equilibrium.

The equations of Proposition 1 provide unique, implicit solutions for the equilibrium effort levels and advertising. It is useful to note that equation (3) determines a one-to-one relationship between the level of advertising at the symmetric equilibrium and the demand for any given platform i. At the symmetric equilibrium, a larger demand a_{P_i} implies a higher level of advertising κ_i. Given that all platforms in this model are identical and I am using symmetric equilibria as the focus of this discussion, the consumer surplus can be characterized solely by the amount of demand for any given platform in the market, provided m is fixed. When the platforms in the market receive a larger share of the consumers in the market, consumer surplus must necessarily be higher.

Equation (3) then implies that the level of advertising serves as a simple way of determining changes in consumer surplus for a fixed market size m.

It's useful to compare equations (3) and (4) to the solution obtained by a monopoly intermediary that hosts all of the content creators in the market. In this case, the solution is given by the system:

$$(1 - a_{P_i}) = \frac{1 - \kappa_i}{\kappa_i} \frac{1 - \alpha}{\alpha}$$

$$\phi \kappa_i a_{P_i} = \gamma \delta e_i^{\delta} \frac{\delta}{1 - a_{P_i}}$$

In each case, the terms $\gamma \delta e_i^{\delta}$ and $\frac{\gamma \delta e_i^{\delta}}{m}$ represent the total costs to the intermediary in the monopoly and oligopoly settings respectively. The ratios attached to these terms in the equations given above show that total costs at equilibrium are proportional total revenues. Additionally, equation (4) demonstrates that as $\rho \to 0$, this proportion in the oligopoly setting tends to that of the monopoly setting.

In the above equilibrium equations, the parameter capturing responsiveness of content creators to changes in rents, ρ, determines the extent to which limited liability affects the level of effort implemented by the attention intermediaries. As content creators become more responsive, platforms compete more fiercely for their membership. This implies a higher level of effort implemented by the attention intermediaries, and a greater level of consumer surplus. It is also important to note that profits are decreasing in ρ, so that platforms prefer as little competition in the market for content creators as possible, assuming symmetric equilibria as a solution.

The equations outlined in Proposition 1 provide straightforward comparative statics for the symmetric equilibrium in this model via the implicit function theorem.

Proposition 4.3.2 *The level of effort implemented and the level of advertising at the symmetric equilibrium are increasing in ρ, ϕ and decreasing in α, n_0 and γ.*

For the model in this section, the level of advertising κ_i and effort e_i tend to move together. As noted before, as ρ rises, attention intermediaries tend to provide greater incentives to content creators due to a greater responsiveness. Of particular note is the fact that effort and advertising are decreasing in α. Under the assumption that output is binary and n_0 is the measure of high quality content outside of the market, the parameter α in the consumer's problem takes on the interpretation of an aversion to advertising. When α is larger, the losses to the consumer from any given level of advertising become greater. In this way, platforms will tend to advertise less and, as a result, provide weaker incentives to content creators.

4.4 Agent Multi-Homing

The goal of this section is to compare the symmetric equilibria that result when attention intermediaries are able to screen for exclusivity against those that arise when intermediaries are restricted to write contracts contingent only on the quality of content without any regard to the number of platforms on which the content appears. In order to proceed with a tractable equilibrium analysis, I adopt a common assumption within the literature on platforms and suppose that an exogenous fraction $\lambda \in [0, 1]$ of content creators multi-home on every platform, and that this fraction is independent of the incentive schemes offered by platforms in the market. I view this as the first natural step toward an analysis that allows for endogenous multi-homing, as this in general requires knowledge of the structure of equilibria with fixed multi-homers. The focus of this section is on how the parameter λ representing the mass of multi-homing agents affects incentives, as well as consumer surplus and platform profits. Note that, for fixed incentives, this model has the feature that multi-homing has the effect of increasing the mass of high-quality content in the market when consumers single-home. While this has an immediate positive effect on consumer surplus, the fact that multi-homing helps a platform's competitors has a dampening effect on incentives. A natural question is then whether or not consumer surplus rises as a result of multi-homing when one endogenizes incentives, as in this model. The main result of this section is that as the fraction of exogenous multi-homing content creators rises (λ increases), consumer surplus at the corresponding symmetric equilibrium also increases. Thus, exclusivity tends to harm consumers in this model.

The model of this section is similar to that of the single-homing regime. The mass $1 - \lambda$ of single-homing agents choose their platform based on the expected rents of the associated contract according to equation (2) in section 3. The mass λ of multi-homing agents choose to join every platform and respond to the incentives offered by all contracts in the market. Given that attention intermediaries are assumed to be unable to write contracts contingent on exclusivity, the action of intermediary i is a tuple $\{t_i(H), t_i, \kappa_i\}$ which specifies a transfer $t_i(H) \geq 0$ upon a high realization of content quality as well as a flat transfer $t_i \geq 0$ for agents that choose to join. The advertising level $\kappa_i \in [0, 1]$ is also assumed to be uniform across both single-homing and multi-homing agents. I view this as the most incomplete variant of a contracting regime for principals in this model. It imposes total non-discrimination across agents. Now, given that multi-homing agents respond to the incentives of all platforms in the market, it is in general no longer possible to write the problem of each intermediary as that of implementing direct levels of effort. Consider now the problem of a multi-homing agent given the incentives offered by each platform. They solve the following simple optimization problem:

$$\max_{e_M} \; e_M \sum_{i=1}^{m} t_i(H) - C(e_M)$$

$$s.t. \quad e_M \in [0, 1]$$

Provided C is sufficiently convex, the solution is given by the first order condition:

$$\sum_{i=1}^{m} t_i(H) = C'(e_M)$$

A straightforward application of the implicit function theorem yields the following expression for how multi-homers respond to a change in incentives offered by platform i:

$$\frac{\partial e_M}{\partial t_i(H)} = \frac{1}{C''(e_M)}$$

As noted before, since multi-homers respond to the contracts offered by all intermediaries and produce content for each of a given intermediary's competitors, incentives offered at any symmetric equilibrium are weaker than those under exclusivity. The focus of the results that follow are whether or not this reduction in incentives, when taken together with the fact that a single-homing consumer now has access to a greater amount of content on every platform, leads to greater consumer surplus than under the single-homing regime. Consumer surplus in this setting is identical to the single-homing regime, and I will continue to use the aggregate

$$\sum_{i=1}^{m} n_{H_i}(1-\kappa_i)^{\frac{\alpha}{1-\alpha}} + n_0$$

where n_{H_i} denotes the mass of high-quality content available on platform i, in comparisons of consumer surplus. Note that $\lambda = 0$ implicitly defines the single-homing regime, whereas $\lambda = 1$ defines the familiar competitive bottleneck setting from the literature on two-sided markets. This is perhaps a good place to mention that the interpretation of each consumer's decision to join a particular platform as stemming in part from an idiosyncratic value becomes somewhat more difficult to justify as λ tends to 1. While there is little discussion of this in models of competitive bottlenecks, whether characterized as idiosyncratic values or transportation costs, one potential source for horizontal differentiation in this model is the tendency of platforms to offer additional features to consumers that patronize partnered platforms in alternative markets. For instance, Twitch offers benefits (free subscriptions, specialized content) to viewers that also have an Amazon Prime subscription.

In pursuance of an equilibrium analysis with multi-homing, it helps to explicitly define the objectives of intermediaries and the equilibrium concept for this section. Given the actions $\{t_j(H), t_j, \kappa_j\}_{j\neq i}$ of all intermediaries other than i, the problem of intermediary i is given by:

$$\begin{aligned} \max \;\; & a_{P_i}\phi\kappa_i - n_{i,SH}(e_i t_i(H) + t_i) - \lambda(e_M t_i(H) + t_i) \\ s.t. \;\; & t_i(H), t_i \geq 0 \\ & \kappa_i \in [0,1] \end{aligned}$$

where $n_{i,SH}$, the mass of single-homing agents on platform i, is given by $(1-\lambda)n_i$ with n_i given as in equation (2) of the single-homing regime. The effort level

e_i is the optimal level of effort for single-homing agents on platform i, and the effort level e_M is the optimal level of effort for multi-homing agents. Demand a_{P_i} for platform i is given by equation (1) in the preliminaries. Given the profit maximization problem of each platform, an equilibrium of the multi-homing regime is a collection of tuples $\{(t_i(H), t_i, \kappa_i)\}_{i=1}^m$, $\{e_i\}_{i=1}^m, e_M$, $\{n_i\}_{i=1}^m$, $\{a_{P_i}\}_{i=1}^m$ such that:

1. Each e_i is an optimal effort level given the incentive scheme $(t_i(H), t_i)$ offered by platform i.
2. The multi-homing effort level e_M is optimal given the incentive schemes $\{(t_i(H), t_i)\}_{i=1}^m$.
3. Single-homing membership is given by $(1 - \lambda)n_i$, where n_i is determined by equation (2) for each i.
4. Demand for each platform a_{P_i} is given as in (1) for each platform i.
5. For each i, given $\{(t_j(H), t_j, \kappa_j)\}_{j\neq i}$, the advertising and contract tuple $(t_i(H), t_i, \kappa_i)$ is optimal for intermediary i.

It is important to note that, as λ tends to 1, intermediaries have less incentive to provide flat rents to agents that choose to join their platform. Indeed, given that intermediaries are constrained to offer the same contract to all agents producing content on their platform, the value of these flat rents (incentivizing additional agents to join the intermediary's platform) vanish as λ tends to 1. From a formal standpoint, the threshold ρ^* below which platforms provide no flat transfers t_i is increasing in λ. In the competitive bottleneck setting ($\lambda = 1$), the responsiveness parameter ρ becomes entirely irrelevant. Throughout the remainder of this section, I will focus on the case in which ρ is sufficiently small so that flat transfers are not provided at the symmetric equilibrium. Additionally, I suppose that costs take the parametric form $C(e) = \gamma e^{\delta}$ with γ sufficiently large relative to ϕ and $\delta > 0$ is sufficiently large to guarantee interior solutions of effort at the symmetric equilibrium.

Proposition 4.4.1 *For each $\lambda \in [0,1]$, there is $\rho^*(\lambda)$ such that for all $\rho < \rho^*(\lambda)$, the unique symmetric equilibrium of the game with multi-homing with m platforms is given by the implicit solutions to the following system for each i:*

$$(1 - a_{P_i}) = \frac{1-\kappa_i}{\kappa_i}\frac{1-\alpha}{\alpha}$$

$$\phi\kappa_i a_{P_i} = \Big[\frac{1-\lambda}{m} + \lambda m^{\frac{1}{\delta-1}}\Big]\gamma\delta e_i^{\delta}\frac{\Big[\rho\delta\frac{1-\lambda}{m}\frac{(m-1)(1-\lambda)}{m} + \delta\frac{1-\lambda}{m} + \lambda m^{\frac{1}{\delta-1}}(\frac{1}{m} + \delta - 1)\Big]}{\Big[\rho\delta\frac{1-\lambda}{m}\frac{(m-1)(1-\lambda)}{m} + \frac{1-\lambda}{m}(1 - a_{P_i}) + \frac{\lambda m^{\frac{1}{\delta-1}}(1-ma_{P_i})}{m}\Big]}$$

and in which $t_i = 0$ for all i.

The first equation above is the familiar advertising equation from the single-homing section. The second condition may at first seem unwieldy, but it helps to consider the terms that appear in the expression. To begin with, $\frac{(1-\lambda)}{m}$ is the mass of single-homing content creators on any given platform at the symmetric equilibrium, so that $\frac{(m-1)(1-\lambda)}{m}$ is the mass of single-homing content creators on the other $m-1$ platforms. Furthermore, given that all platforms offer the same terms at a symmetric equilibrium, it's possible to relate the effort exerted by single-homing and multi-homing agents at the equilibrium as:

$$\frac{e_M}{e_i} = m^{\frac{1}{\delta-1}}$$

Thus the expression $e_i\Big[\frac{1-\lambda}{m} + \lambda m^{\frac{1}{\delta-1}}\Big] = n_{H_i}$ represents the total mass of high quality content on platform i at a symmetric equilibrium. Additionally, given the parametric form for costs, $\gamma\delta e_i^{\delta-1} = C'(e_i) = t_i(H)$ are the transfers provided to high-quality content producers by platform i at the symmetric equilibrium. Hence the product of these terms represents total cost. Comparing this expression to equation (4) in the single-homing section, note that when $\lambda = 0$ the solutions coincide. Additionally, total revenues are once again related to total costs by a ratio term that is increasing in ρ.

Given that the symmetric equilibrium solution in the multi-homing regime is well-defined for each $\lambda \in [0, 1]$, it's possible to show the following welfare result:

Proposition 4.4.2 *Incentives $t_i(H)$ at the symmetric equilibrium are decreasing in the amount of multi-homers λ. Moreover, for ρ sufficiently small, advertising and consumer surplus at the symmetric equilibrium are increasing in λ.*

The fact that incentives are decreasing in the extent of multi-homers is not surprising. As the number of multi-homers increases, the value of providing additional incentives for any given platform is, ceterus paribus, attenuated. Despite this, the loss in consumer surplus due to reduced incentives at the symmetric equilibrium is outweighed by the fact that single-homing consumers now have access to a greater amount of total content on each platform. At the equilibrium, the total amount of high-quality content on each individual platform rises in λ, provided ρ is sufficiently small.[13] This result suggests that consumers and platforms may be better off when platforms are unable to impose exclusivity restrictions on content creators. A more complete answer to the question of whether multi-homing offers welfare improvements for consumers and platforms requires an analysis that endogenizes multi-homing among content creators. This offers a clear next step for future research with this model.

[13]Simulations indicate that this holds for ρ in the entirety of $(0, 1]$.

4.5 Discussion

There are numerous avenues in which the current model can be extended.

4.5.1 Endogenous Multi-homing

The discussion in section 3 above relies on an exogenous formulation of multi-homing to serve as a tractable starting point for an analysis of consumer surplus. Allowing for endogenous multi-homing requires a number of additional assumptions which may at first be difficult to lay down in a meaningful way. In particular, this requires two primary ingredients. The first is to determine the completeness of the contracts that can be offered by attention intermediaries. One could continue with the non-discrimination regime of section 3 above, so that each platform offers a tuple $(t_i(H), t_i)$ to content creators that post content on that platform. On the other hand, there are a number of other regimes that appear reasonable, and perhaps more tractable. The most complete environment would allow for each platform to post a separate incentive scheme for every element of $\mathbb{P}(P)$ which includes that platform. Regardless of the contracting environment, let $\mathbb{I}$ denote the set of possible incentive schemes that can be offered by a platform. The second ingredient for an endogenous multi-homing setting is a matching function:

$$G : \mathbb{I}^m \times \mathbb{P}(P) \to [0, 1]$$

which assigns to every nonempty subset of platforms a mass of agents that produce content for exactly that subset, given the incentive schemes offered by all intermediaries. Given the generality of $\mathbb{I}$, it's difficult ex-ante to place meaningful assumptions on the mapping G. One potential assumption that stands out is that agents will only multi-home when the expected rents associated with it are at least as large as those associated with the worst single-homing contract. Such an assumption, coupled with a rich contracting environment for intermediaries, seems likely to produce an equilibrium with exclusivity as in section 2.

4.5.2 Continuous Quality

The assumption of a binary quality in the model is clearly restrictive. A natural extension is to allow for content quality to be continuous. Doing so requires only slight modifications to the consumers' optimal control problem for attention allocation. Let us maintain the assumption that qualities are in $[0, 1]$. For any level of effort $e \in [0, 1]$, let $\pi(v, e)$ denote the probability of realizing quality v given effort e. Supposing that, for instance, there is a unit mass of agents which all exert the same effort e^* on a platform, the density of agents with value $v \in [0, 1]$ is given by $\pi(v, e^*)$[14]. The problem of a consumer is then to choose a continuous mapping

$$A : [0, 1] \to \mathbb{R}_+$$

to solve:

$$\max_{A(v)} \int_0^1 v(A(v))^\alpha \pi(v, e^*)dv$$
$$s.t. \quad \int_0^1 A(v)\pi(v, e^*)dv = 1$$

The constraint reflects the fact that a consumer has a unit budget of attention, and the parameter $\alpha \in (0, 1)$ captures diminishing returns to attention. Note that

[14] This relies on a particularly restrictive form of a strong law of large numbers. See Judd (1985)[35] for a discussion.

this formulation assumes a linear value in quality v for the consumer. This is an isoperimetric optimal control problem. In particular writing:

$$y(x) = \int_0^x A(v)\pi(v, e^*)dv$$

the fundamental theorem of calculus implies that:

$$y'(x) = A(x)\pi(x, e^*)$$

The consumer problem can then be written as:

$$\max_{y(v)} \int_0^1 v\Big(\frac{y'(v)}{\pi(v, e^*)}\Big)^\alpha \pi(v, e^*)dv$$
$$s.t. \quad y(0) = 0, \quad y(1) = 1$$

The Euler-Lagrange equation is given by:

$$\frac{d}{dv}\Big(v\alpha\Big(\frac{y'(v)}{\pi(v, e^*)}\Big)^{\alpha-1}\Big) = \frac{d}{dv}\Big(v\alpha(A(v))^{\alpha-1}\Big) = 0$$

This can be written as:

$$\frac{1}{1-\alpha}A(v) = vA'(v)$$

with solution given by:

$$A(v) = cv^{\frac{1}{1-\alpha}}$$

with the constant c obtained using the consumer's attention constraint.

The consumer's problem can be extended to allow for continuous output without much difficulty. The most sensitive part of an analysis of continuous output is the assumptions to be placed on $\pi(v, e)$. Fortunately, Poblete and Spulber (2012)[52] have already characterized the optimal incentive contracts under moral hazard and limited liability with risk neutrality. The most relevant part of their analysis lies in the conditions that they provide in order to guarantee that the optimal incentive contract takes the form of debt, which I view as the most realistic form of incentives given that this is a static model. Platforms frequently offer increasing incentives once agents reach certain viewership benchmarks, for instance in the form of a partnership.

4.5.3 Network Effects for Agents

The above model assumes that content creators respond only to the incentives offered by attention intermediaries and receive no intrinsic benefit from viewership. This is certainly an unrealistic assumption. Content creators frequently receive additional benefits in the form of donations and peer recognition which platforms in general cannot influence directly. Allowing for such indirect network effects can be difficult from a modeling standpoint. In particular, the timing of membership decisions now plays a crucial role in the structure of equilibria. Tipping situations may arise in which a single platform receives nearly, if not all, of the agents and consumers in the market. The existence of such equilibria depends on how agents value network effects and rents.

4.5.4 Conclusion

In this short paper, I have developed a tractable, parametric model of media platforms with the aim of capturing relevant features of modern streaming and video-sharing sites. I have shown that under the assumption of exogenous multi-homing consumer surplus rises in the extent of such multi-homing, despite reductions in incentives provided by attention intermediaries. Future work will focus on relaxing the many restrictive assumptions in the model in order to determine the robustness of these results.

4.6 Appendix

Proof of Proposition 1 Consider the problem of intermediary i given the actions of all other platforms:

$$\begin{aligned} \max a_{P_i}\phi\kappa_i - n_i(e_iC'(e_i) + t_i) \\ s.t. \quad \kappa_i, e_i, t_i \geq 0 \\ e_i, \kappa_i \leq 1 \end{aligned}$$

Note that if $\gamma > \phi$ and $\delta > 0$ is sufficiently large, the intermediary would never choose to implement an effort of 1. Similarly, if $e_i = 0$ then the platform receives no profit. Finally, if κ_i is 0 or 1, then the profit is also 0. Provided there is an interior solution with positive profit, these boundary cases can of course be ruled out. I will proceed by supposing that the $k_i, e_i \in (0, 1)$ and demonstrate that profit must be positive at the symmetric equilibrium. The solution is found by writing down the KKT conditions for an arbitrary intermediary i, assuming symmetry, and showing a single-crossing condition in the appropriate region for the choice variables given the implicit equations defining a solution. The bordered Hessian for this problem is far too complex to reason about, and in general requires explicit values for the actions of other players to show strict concavity. Instead, I reason using uniqueness of implicit solutions to a reduced system. Consider the first order conditions associated with an interior solution for e_i and κ_i:

$$\{e_i\} : \frac{\partial a_{P_i}}{\partial e_i}\phi\kappa_i - \frac{\partial n_i}{\partial e_i}[e_iC'(e_i) + t_i] - n_i(C'(e_i) + e_iC''(e_i)) = 0$$

Written explicitly:

$$\frac{\partial a_{P_i}}{\partial e_i} = \frac{[\frac{\partial n_i}{\partial e_i}e_i(1-\kappa_i)^{\frac{\alpha}{1-\alpha}} + n_i(1-\kappa_i)^{\frac{\alpha}{1-\alpha}}]A - v_i[\frac{\partial n_i}{\partial e_i}e_i(1-\kappa_i)^{\frac{\alpha}{1-\alpha}} + n_i(1-\kappa_i)^{\frac{\alpha}{1-\alpha}} + \sum_{j\neq i}\frac{\partial n_j}{\partial e_i}e_j(1-\kappa_j)^{\frac{\alpha}{1-\alpha}}]}{A^2}$$

where

$$A = \sum_{j=1}^{m} n_j e_j (1-\kappa_j)^{\frac{\alpha}{1-\alpha}} + n_0$$

is an aggregate term and

$$v_i = n_i e_i (1-\kappa_i)^{\frac{\alpha}{1-\alpha}}$$

is the utility intensity associated with platform i. Now, at a symmetric equilibrium, $e_j = e_i$ and $\kappa_j = \kappa_i$ for all $j \neq i$. Furthermore, the fact that the size of the content-creator side of the market is fixed implies that

$$\frac{\partial n_i}{\partial e_i} = -\sum_{j \neq i} \frac{\partial n_j}{\partial e_i}$$

so that $\frac{\partial a_{P_i}}{\partial e_i}$ can be rewritten as:

$$\frac{\partial a_{P_i}}{\partial e_i} = \frac{[\frac{\partial n_i}{\partial e_i} e_i (1-\kappa_i)^{\frac{\alpha}{1-\alpha}} + n_i (1-\kappa_i)^{\frac{\alpha}{1-\alpha}}]A - (n_i e_i (1-\kappa_i)^{\frac{\alpha}{1-\alpha}}[n_i (1-\kappa_i)^{\frac{\alpha}{1-\alpha}}]}{A^2}$$

straightforward simplification and the assumption of symmetry yields:

$$\frac{\partial a_{P_i}}{\partial e_i} = \frac{\frac{\partial n_i}{\partial e_i} e_i (1-\kappa_i)^{\frac{\alpha}{1-\alpha}}}{A} + \frac{n_i (1-\kappa_i)^{\frac{\alpha}{1-\alpha}} (1-a_{P_i})}{A}$$

To continue simplifying, consider

$$\frac{\partial n_i}{\partial e_i} = \frac{\rho r_i^{\rho-1} \frac{\partial r_i}{\partial e_i} (\sum_{j \neq i} r_j^{\rho})}{(\sum_{j=1}^{m} r_j^{\rho})^2}$$

where $r_i = e_i C'(e_i) - C(e_i) + t_i$ is the expected rents associated with the contract implementing effort e_i with flat transfer t_i. Symmetry yields:

$$\frac{\partial n_i}{\partial e_i} = \rho \frac{\frac{\partial r_i}{\partial e_i}}{r_i} n_i (1-n_i) = \rho \frac{\frac{\partial r_i}{\partial e_i}}{r_i} \frac{m-1}{m^2}$$

Re-writing the full first-order condition, one has:

$$\phi \kappa_i a_{P_i} \Big(\rho \frac{\frac{\partial r_i}{\partial e_i}}{r_i} \frac{m-1}{m} + \frac{(1-a_{P_i})}{e_i} \Big) = \rho \frac{\frac{\partial r_i}{\partial e_i}}{r_i} \frac{m-1}{m^2} [e_i C'(e_i) + t_i] + \frac{C'(e_i) + e_i C''(e_i)}{m}$$

Plugging in the explicit parametric form for costs in this model, the final FOC can be written, assuming a symmetric equilibrium, as:

$$\{e_i\} : \phi\kappa_i a_{P_i}\Big(\rho\frac{\gamma\delta(\delta-1)e_i^{\delta-1}}{\gamma(\delta-1)e_i^{\delta}+t_i}\frac{m-1}{m}+\frac{(1-a_{P_i})}{e_i}\Big)=\rho\frac{\gamma\delta(\delta-1)e_i^{\delta-1}}{\gamma(\delta-1)e_i^{\delta}+t_i}\frac{m-1}{m^2}[\gamma\delta e_i^{\delta}+t_i]+\frac{\gamma\delta^2 e_i^{\delta-1}}{m}$$

I will return to this first-order condition later. For now, notice that $t_i = 0$ simplifies several terms in this expression. Consider now the first-order condition on t_i. This is written as:

$$\{t_i\} : \frac{\partial a_{P_i}}{\partial t_i}\phi\kappa_i - \frac{\partial n_i}{\partial t_i}[e_i C'(e_i) + t_i] - n_i = \lambda_t$$

where $\lambda_t \leq 0$ is the Lagrange multiplier associated with the non-negativity constraint on t_i. After some simplification and assuming symmetry, one has that:

$$\frac{\partial a_{P_i}}{\partial t_i} = \frac{\rho}{r_i}\frac{m-1}{m}a_{P_i}(1-a_{P_i})$$

Re-writing the expression for the first-order condition:

$$\{t_i\} : \phi\kappa_i\frac{\rho}{r_i}\frac{m-1}{m}a_{P_i}(1-a_{P_i}) - \frac{\rho}{r_i}\frac{m-1}{m^2}[\gamma\delta e_i^{\delta}+t_i] - \frac{1}{m} = \lambda_t$$

Before analyzing t_i, we will need conditions on advertising. The first-order condition for κ_i simplifies to:

$$\{\kappa_i\} : (1-a_{P_i}) = \frac{1-\kappa_i}{\kappa_i}\frac{1-\alpha}{\alpha}$$

Now, notice that at a symmetric equilibrium with $m \geq 2$, $a_{P_i} \leq \frac{1}{2}$ for each i, so that advertising must be bounded from below at any equilibrium. Now, begin by sending $\rho \to 0$. Intuitively, when agents become increasingly unresponsive to expected rents, no intermediary would ever choose to provide flat transfers, as this would add a vanishingly small amount of value to their platform (additional agents) while the cost of an increase in transfers is roughly constant. To be precise, as $\rho \to 0$, the first-order condition on e_i approaches the following system:

$$\phi\kappa_i a_{P_i}(1 - a_{P_i}) = \frac{\gamma\delta^2 e_i^\delta}{m}$$

As a note for this claim, it helps to analyze the powers of e_i in each of the terms with a ρ appearing in it (so the claim holds even if $e_i \to 0$ as $\rho \to 0$, which turns out to not even be true). Plugging this expression into the first-order condition for t_i, one obtains:

$$\frac{m-1}{m^2}\rho\Big[\frac{\gamma\delta(\delta-1)e_i^\delta - t_i}{\gamma(\delta-1)e_i^\delta}\Big] - \frac{1}{m} = \lambda_t$$

Now, for any $t \geq 0$, the left-hand side is strictly negative for ρ sufficiently close to 0. This immediately implies that $t_i = 0$ for ρ sufficiently small. Writing $t_i = 0$ in the first order condition for e_i, the symmetric equilibrium solution is defined by (3) and (4). Supposing that ρ is indeed small enough, one has left to show that profits are positive and that the solution is unique in the desired region. If profits are indeed positive and the solution is unique, then it must necessarily be a maximum of each intermediary's problem. Note that since

$$\frac{\rho\delta\frac{m-1}{m} + \delta}{\rho\delta\frac{m-1}{m} + (1 - a_{P_i})} > 1$$

profit must be positive for each platform at the solution. To show single-crossing, it helps to write the system as an implicit equation with a single variable. Starting with equation (3), one can write:

$$e = n_0 \frac{1 - \frac{1-\kappa}{\kappa}\frac{1-\alpha}{\alpha}}{(1-\kappa)^{\frac{\alpha}{1-\alpha}}\Big[\frac{1}{m} - 1 + \frac{1-\kappa}{\kappa}\frac{1-\alpha}{\alpha}\Big]}$$

Note that the expression on the right is increasing in κ. Substituting this into (4), one has

$$\phi\kappa(1-\kappa)^{\delta\frac{\alpha}{1-\alpha}}\Big[\frac{1}{m} - 1 + \frac{1-\kappa}{\kappa}\frac{1-\alpha}{\alpha}\Big]^\delta\Big[\rho\delta\frac{m-1}{m} + \frac{1-\kappa}{\kappa}\frac{1-\alpha}{\alpha}\Big] = \gamma\delta(n_0)^\delta\Big[1 - \frac{1-\kappa}{\kappa}\frac{1-\alpha}{\alpha}\Big]^{\delta-1}\Big[\rho\delta\frac{m-1}{m} + \delta\Big]$$

The right-hand side of this equation is clearly increasing in κ. The left-hand side is decreasing in κ since

$$\frac{\partial}{\partial\kappa}\Big[\kappa(1-\kappa)^{\delta\frac{\alpha}{1-\alpha}}\Big] = (1-\kappa)^{\delta\frac{\alpha}{1-\alpha}}\Big[1-\delta\frac{\kappa}{1-\kappa}\frac{\alpha}{1-\alpha}\Big] < 0 \tag{4.5}$$

since equation (3) imposes

$$\frac{\kappa}{1-\kappa}\frac{\alpha}{1-\alpha} = \frac{1}{1-a_{P_i}} > 1$$

Hence there is a unique κ^* that defines the symmetric equilibrium.

Now, I have not yet shown that the best response of each intermediary is uniquely defined. Showing this requires some care. I will use a method similar to showing single-crossing in the case of equilibria. It is important to start by noting that the advertising equation:

$$(1-a_{P_i}) = \frac{1-\kappa_i}{\kappa_i}\frac{1-\alpha}{\alpha}$$

is derived without any assumptions of symmetry. Hence this must hold for each intermediary at any best response under the appropriate boundary conditions. Now, let us begin by supposing that e^* and κ^* are the corresponding equilibrium values at the symmetric equilibrium claimed above. Suppose that all players other than some intermediary i are implementing e^* and advertising according to κ^*. Note that the advertising equation for intermediary i gives a relation between effort and advertising as:

$$((m-1)e^*(1-\kappa^*)^{\frac{\alpha}{1-\alpha}} + n_0)\frac{1-\frac{1-\kappa_i}{\kappa_i}\frac{1-\alpha}{\alpha}}{\frac{1-\kappa_i}{\kappa_i}\frac{1-\alpha}{\alpha}(1-\kappa_i)^{\frac{\alpha}{1-\alpha}}} = e_i \tag{4.6}$$

Now, the first-order condition on effort e_i without assuming symmetry, but assuming $t_i = 0$, is given, after some simplification and using the above best-response relation between effort and advertising, by:

$$\phi\kappa_i a_{P_i}(1-a_{P_i})\Big[(1-n_i)\rho\gamma\delta\Big[1+\frac{(1-\kappa^*)^{\frac{\alpha}{1-\alpha}}e^*}{(m-1)(1-\kappa^*)^{\frac{\alpha}{1-\alpha}}e^*+n_0}\Big]+1\Big] = n_i\gamma\delta e_i^{\delta}\Big[\delta+\rho\gamma\delta(1-n_i)\Big]$$

Note that when flat transfers $t_i = 0$, the expression for n_i is given by: $n_i = \frac{e_i^{\rho\delta}}{(m-1)(e^*)^{\rho\delta}+e_i^{\rho\delta}}$. Substituting this into the first-order condition and using the advertising equation, one obtains:

$$\phi\kappa_i(1-\frac{1-\kappa_i}{\kappa_i}\frac{1-\alpha}{\alpha})(\frac{1-\kappa_i}{\kappa_i}\frac{1-\alpha}{\alpha})\Big[\frac{(m-1)(e^*)^{\rho\delta}}{(m-1)(e^*)^{\rho\delta}+e_i^{\rho\delta}}\rho\delta\Big[1+\frac{(1-\kappa^*)^{\frac{\alpha}{1-\alpha}}e^*}{(m-1)(1-\kappa^*)^{\frac{\alpha}{1-\alpha}}e^*+n_0}\Big]+1\Big]$$

$$=\frac{e_i^{\rho\delta}}{(m-1)(e^*)^{\rho\delta}+e_i^{\rho\delta}}\gamma\delta e_i^{\delta}\Big[\delta+\rho\delta\frac{(m-1)(e^*)^{\rho\delta}}{(m-1)(e^*)^{\rho\delta}+e_i^{\rho\delta}}\Big]$$

After substituting for advertising, the above equation has a unique solution for κ_i, which dictates platform i's best response whenever ρ is not *too* large. Indeed, after substitution and simplification, one obtains uniqueness of conditional best responses whenever $\rho \leq \frac{\delta}{\delta+1}$. This is a strong sufficient condition for uniqueness of conditional best responses, in principle ρ can be much larger. To finalize this argument, recall equation (6) and note that the above system can be simplified to:

$$\phi(1-\kappa_i(\frac{1-\alpha}{\alpha}))\Big[\frac{(m-1)(e^*)^{\rho\delta}}{(m-1)(e^*)^{\rho\delta}+e_i^{\rho\delta}}\rho\delta\Big[1+\frac{(1-\kappa^*)^{\frac{\alpha}{1-\alpha}}e^*}{(m-1)(1-\kappa^*)^{\frac{\alpha}{1-\alpha}}e^*+n_0}\Big]+1\Big]$$

$$=\frac{e_i^{\rho\delta}}{(m-1)(e^*)^{\rho\delta}+e_i^{\rho\delta}}\gamma\delta e_i^{\delta-1}\Big[\delta+\rho\delta\frac{(m-1)(e^*)^{\rho\delta}}{(m-1)(e^*)^{\rho\delta}+e_i^{\rho\delta}}\Big]\frac{((m-1)e^*(1-\kappa^*)^{\frac{\alpha}{1-\alpha}}+n_0)}{\frac{1-\kappa_i}{\kappa_i}\frac{1-\alpha}{\alpha}(1-\kappa_i)^{\frac{\alpha}{1-\alpha}}}$$

Now, since e_i can be written as an increasing function of κ_i according to (6), the left-hand side of this system is decreasing in κ_i. Furthermore, if $\rho \leq \frac{\delta}{\delta+1}$, the right-hand side is increasing in κ_i. A solution to this system is then unique whenever it exists, and provides the optimal best response κ_i for player i, which then gives the optimal effort implemented by equation (6). This completes the proof.

Proof of Proposition 2

The proof is a straightforward application of the implicit function theorem. Suppose that $\rho < \rho^*$ as in Proposition 1, and consider the symmetric equilibrium solutions described by equations (3) and (4). Let η denote an arbitrary parameter. Write:

$$F^1 = \phi\kappa\Big(1 - \frac{1-\kappa}{\kappa}\frac{1-\alpha}{\alpha}\Big)\Big[\rho\delta\frac{m-1}{m} + \frac{1-\kappa}{\kappa}\frac{1-\alpha}{\alpha}\Big] - \frac{\gamma\delta e^{\delta}}{m}\Big[\rho\delta\frac{m-1}{m} + \delta\Big]$$

$$F^2 = \Big(\frac{m-1}{m}e(1-\kappa)^{\frac{\alpha}{1-\alpha}} + n_0\Big) - \Big(e(1-\kappa)^{\frac{\alpha}{1-\alpha}} + n_0\Big)\frac{1-\kappa}{\kappa}\frac{1-\alpha}{\alpha}$$

Applying the implicit function theorem, one has that:

$$\begin{bmatrix} \frac{\partial\kappa}{\partial\eta} \\ \frac{\partial e}{\partial\eta} \end{bmatrix} = \frac{-1}{det(J)}\begin{bmatrix} J_{22} & -J_{12} \\ -J_{21} & J_{11} \end{bmatrix}\begin{bmatrix} \frac{\partial F^1}{\partial\eta} \\ \frac{\partial F^2}{\partial\eta} \end{bmatrix} \tag{4.7}$$

where

$$J_{11} = \frac{\partial F^1}{\partial\kappa}$$
$$J_{12} = \frac{\partial F^1}{\partial e}$$
$$J_{21} = \frac{\partial F^2}{\partial\kappa}$$
$$J_{22} = \frac{\partial F^2}{\partial e}$$

assuming that the Jacobian J is invertible. I claim that $det(J) > 0$. To show this, it helps to write out each element of the Jacobian directly and then use equations (3) and (4) to determine its sign. One has:

$$J_{11} = \phi\Big(1 - \frac{1-\kappa}{\kappa}\frac{1-\alpha}{\alpha}\Big)\Big[\rho\delta\frac{m-1}{m} + \frac{1-\kappa}{\kappa}\frac{1-\alpha}{\alpha}\Big] + \frac{\phi}{\kappa}\frac{1-\alpha}{\alpha}\Big[\rho\delta\frac{m-1}{m} + \frac{1-\kappa}{\kappa}\frac{1-\alpha}{\alpha}\Big] - \frac{\phi}{\kappa}\frac{1-\alpha}{\alpha}\Big(1 - \frac{1-\kappa}{\kappa}\frac{1-\alpha}{\alpha}\Big)$$

$$J_{12} = -\frac{\gamma\delta^2 e^{\delta-1}}{m}\Big[\rho\delta\frac{m-1}{m} + \delta\Big]$$

$$J_{21} = -\frac{\alpha}{1-\alpha}\frac{m-1}{m}e(1-\kappa)^{\frac{2\alpha-1}{1-\alpha}} + \frac{\alpha}{1-\alpha}e(1-\kappa)^{\frac{2\alpha-1}{1-\alpha}}\Big(\frac{1-\kappa}{\kappa}\frac{1-\alpha}{\alpha}\Big) + \Big(e(1-\kappa)^{\frac{\alpha}{1-\alpha}} + n_0\Big)\frac{1-\alpha}{\alpha}\frac{1}{\kappa^2}$$

$$J_{22} = \frac{m-1}{m}(1-\kappa)^{\frac{\alpha}{1-\alpha}} - (1-\kappa)^{\frac{\alpha}{1-\alpha}}\frac{1-\kappa}{\kappa}\frac{1-\alpha}{\alpha}$$

Now, we want to say that

$$J_{11}J_{22} - J_{12}J_{21} > 0$$

Using the fact that the advertising equation implies

$$e(1-\kappa)^{\frac{\alpha}{1-\alpha}}\Big[\frac{m-1}{m} - \frac{1-\kappa}{\kappa}\frac{1-\alpha}{\alpha}\Big] = n_0\Big[\frac{1-\kappa}{\kappa}\frac{1-\alpha}{\alpha} - 1\Big]$$

at any solution, some simplification yields that this condition on the Jacobian is equivalent to:

$$n_0\Big[\Big(1 - \frac{1-\kappa}{\kappa}\frac{1-\alpha}{\alpha}\Big)\Big[\rho\delta\frac{m-1}{m} - \frac{1-\alpha}{\alpha}\Big] + \frac{1-\alpha}{\alpha}\frac{1}{\kappa}\Big[\rho\delta\frac{m-1}{m} + \frac{1-\kappa}{\kappa}\frac{1-\alpha}{\alpha}\Big]\Big]$$

$$< \delta\frac{\kappa}{1-\kappa}\Big(\rho\delta\frac{m-1}{m} + \frac{1-\kappa}{\kappa}\frac{1-\alpha}{\alpha}\Big)\Big[n_0\frac{\alpha}{1-\alpha}\Big(1 - \frac{1-\kappa}{\kappa}\frac{1-\alpha}{\alpha}\Big) + (e(1-\kappa)^{\frac{\alpha}{1-\alpha}} + n_0)\frac{1-\alpha}{\alpha}\frac{1-\kappa}{\kappa}\frac{1}{\kappa}\Big]$$

which can be verified piecewise as:

$$n_0\Big(1 - \frac{1-\kappa}{\kappa}\frac{1-\alpha}{\alpha}\Big)\Big[\rho\delta\frac{m-1}{m} - \frac{1-\alpha}{\alpha}\Big] < n_0\Big(1 - \frac{1-\kappa}{\kappa}\frac{1-\alpha}{\alpha}\Big)\frac{\delta}{(1-a_{P_i})}\Big[\rho\delta\frac{m-1}{m} + \frac{1-\kappa}{\kappa}\frac{1-\alpha}{\alpha}\Big]$$

$$n_0\frac{1-\alpha}{\alpha}\frac{1}{\kappa}\Big[\rho\delta\frac{m-1}{m} + \frac{1-\kappa}{\kappa}\frac{1-\alpha}{\alpha}\Big] < (e(1-\kappa)^{\frac{\alpha}{1-\alpha}} + n_0)\delta\frac{1-\alpha}{\alpha}\frac{1}{\kappa}\Big[\rho\delta\frac{m-1}{m} + \frac{1-\kappa}{\kappa}\frac{1-\alpha}{\alpha}\Big]$$

hence $det(J) > 0$. Now, we also see that $J_{22}, J_{12} < 0$ and $J_{21}, J_{11} > 0$. Armed with this, one can now check the following:

$$\frac{\partial F^1}{\partial \phi} = \kappa\Big(1 - \frac{1-\kappa}{\kappa}\frac{1-\alpha}{\alpha}\Big)\Big[\rho\delta\frac{m-1}{m} + \frac{1-\kappa}{\kappa}\frac{1-\alpha}{\alpha}\Big] > 0$$

$$\frac{\partial F^2}{\partial \phi} = 0$$

$$\frac{\partial F^1}{\partial \gamma} = -\frac{\delta e^{\delta}}{m}\Big[\rho\delta\frac{m-1}{m} + \delta\Big] < 0$$

$$\frac{\partial F^2}{\partial \gamma} = 0$$

$$\frac{\partial F^1}{\partial \alpha} = \phi\kappa\frac{1-\kappa}{\kappa}\Big(\frac{(1-\alpha)}{\alpha}\Big)^2\Big[\rho\delta\frac{m-1}{m} + 2\frac{1-\kappa}{\kappa}\frac{1-\alpha}{\alpha} - 1\Big]\frac{1}{(1-\alpha)^2} > 0$$

since $\frac{1-\kappa}{\kappa}\frac{1-\alpha}{\alpha} = (1-a_{P_i}) > \frac{1}{2}$.

$$\frac{\partial F^2}{\partial \alpha} = \frac{1}{(1-\alpha)^2}\Big[ln(1-\kappa)e(1-\kappa)^{\frac{\alpha}{1-\alpha}}\Big(\frac{m-1}{m} - \frac{1-\kappa}{\kappa}\frac{1-\alpha}{\alpha}\Big) + (e(1-\kappa)^{\frac{\alpha}{1-\alpha}} + n_0)\frac{1-\kappa}{\kappa}\Big(\frac{\alpha}{1-\alpha}\Big)^2\Big] > 0$$

since $ln(1-\kappa) < 0$ and $\frac{1-\kappa}{\kappa}\frac{1-\alpha}{\alpha} = (1-a_{P_i}) > \frac{m-1}{m}$ at a symmetric equilibrium.

$$\frac{\partial F^1}{\partial n_0} = 0$$

$$\frac{\partial F^2}{\partial n_0} = 1 - \frac{1-\kappa}{\kappa}\frac{1-\alpha}{\alpha} = a_{P_i} > 0$$

$$\frac{\partial F^1}{\partial \rho} = \phi\kappa\Big(1 - \frac{1-\kappa}{\kappa}\frac{1-\alpha}{\alpha}\Big) - \frac{\gamma\delta e^{\delta}}{m} > 0$$

since profit is positive.

$$\frac{\partial F^2}{\partial \rho} = 0$$

Using these calculated derivatives, it's straightforward to verify the comparative statics using the associated signs and the form given in (7).

Proof of Proposition 3

Similar to the single-homing regime, I will use the KKT conditions and the assumption of symmetry to derive the equilibrium conditions, then show single-crossing in much the same way. To begin, consider the problem of platform i:

$$\begin{aligned} \max\ \ a_{P_i}\phi\kappa_i - n_{i,SH}(e_i t_i(H) + t_i) - \lambda(e_M t_i(H) + t_i) \\ s.t. \quad t_i(H), t_i \geq 0 \\ \kappa_i \in [0,1] \end{aligned}$$

Given that advertising is assumed to be uniform across single-homing and multi-homing content, one can factor our $(1-\kappa)^{\frac{\alpha}{1-\alpha}}$ from a_{P_i} and obtain the exact same advertising equation:

$$(1 - a_{P_i}) = \frac{1-\kappa_i}{\kappa_i}\frac{1-\alpha}{\alpha}$$

Now let us consider incentives.

Differentiating profit with respect to $t_i(H)$ and assuming symmetry yields:

$$\phi\kappa_i \frac{\partial a_{P_i}}{\partial t_i(H)} = \frac{\partial n_{i,SH}}{\partial t_i(H)}(e_i t_i(H) + t_i) - \frac{(1-\lambda)}{m}\Big(\frac{\partial e_i}{\partial t_i(H)} t_i(H) - e_i\Big) - \lambda\Big(\frac{\partial e_M}{\partial t_i(H)} t_i(H) + e_M\Big) \tag{4.8}$$

Now, symmetry and the explicit form for content creator membership yields, assuming $t_i = 0$:

$$\frac{\partial n_{i,SH}}{\partial t_i(H)} = \frac{\rho \frac{\partial r_i}{\partial t_i(H)}}{r_i}\frac{(1-\lambda)}{m}\frac{(m-1)(1-\lambda)}{m} = \frac{\rho e_i}{r_i}\frac{(1-\lambda)}{m}\frac{(m-1)(1-\lambda)}{m}$$

Since

$$\frac{\partial r_i}{\partial t_i(H)} = \Big(\frac{t_i(H)}{\gamma\delta}\Big)^{\frac{1}{\delta-1}} = e_i$$

Using this, equation (8) can be written as:

$$\phi\kappa_i \frac{\partial a_{P_i}}{\partial t_i(H)} = e_i\Big[\frac{\delta}{\delta-1}\rho\frac{(1-\lambda)}{m}\frac{(m-1)(1-\lambda)}{m} + \frac{\delta}{\delta-1}\frac{1-\lambda}{m} + \lambda m^{\frac{1}{\delta-1}}\Big(\frac{1}{m(\delta-1)} + 1\Big)\Big]$$

with the caveat that at a symmetric equilibrium:

$$e_M = e_i m^{\frac{1}{\delta-1}}$$

Now, let us consider the left-hand side. We have that:

$$\frac{\partial a_{P_i}}{\partial t_i(H)} = \frac{\Big[\frac{\partial e_i}{\partial t_i(H)} n_{i,SH}(1-\kappa_i)^{\frac{\alpha}{1-\alpha}} + \frac{\partial n_{i,SH}}{\partial t_i(H)} e_i (1-\kappa_i)^{\frac{\alpha}{1-\alpha}} + \frac{\partial e_M}{\partial t_i(H)}\lambda(1-\kappa_i)^{\frac{\alpha}{1-\alpha}}\Big]A}{A^2}$$

$$-\frac{v_i\Big[\frac{\partial n_{i,SH}}{\partial t_i(H)}e_i(1-\kappa_i)^{\frac{\alpha}{1-\alpha}}+\frac{\partial e_i}{\partial t_i(H)}n_{i,SH}(1-\kappa_i)^{\frac{\alpha}{1-\alpha}}+\lambda\frac{\partial e_M}{\partial t_i(H)}(1-\kappa_i)^{\frac{\alpha}{1-\alpha}}+\sum_{j\neq i}\Big(\frac{\partial n_{j,SH}}{\partial t_i(H)}e_j(1-\kappa_j)^{\frac{\alpha}{1-\alpha}}++\lambda\frac{\partial e_M}{\partial t_i(H)}(1-\kappa_j)^{\frac{\alpha}{1-\alpha}}\Big)}{A^2}$$

where $v_i = (1-\kappa_i)^{\frac{\alpha}{1-\alpha}}(n_{i,SH}e_i + \lambda e_M)$. Assuming symmetry and simplifying, this reduces to:

$$\frac{\partial a_{P_i}}{\partial t_i(H)} = \frac{\rho\frac{1-\lambda}{m}\frac{(m-1)(1-\lambda)}{m}(1-\kappa_i)^{\frac{\alpha}{1-\alpha}}}{A}\frac{1}{\gamma(\delta-1)e_i^{\delta-2}} + (1-a_{P_i})\frac{\frac{1-\lambda}{m}(1-\kappa_i)^{\frac{\alpha}{1-\alpha}}}{A}\frac{1}{\gamma\delta(\delta-1)e_i^{\delta-2}}$$

$$+\frac{\lambda(1-\kappa_i)^{\frac{\alpha}{1-\alpha}}}{A}(1-ma_{P_i})\frac{1}{\gamma\delta(\delta-1)e_M^{\delta-2}}$$

Now, using the fact that at a symmetric equilibrium:

$$a_{P_i} = \frac{e_i(1-\kappa_i)^{\frac{\alpha}{1-\alpha}}\Big[\frac{1-\lambda}{m}+\lambda m^{\frac{1}{\delta-1}}\Big]}{A}$$

and the above relation between e_i and e_M, this expression simplifies to:

$$\frac{\partial a_{P_i}}{\partial t_i(H)} = \frac{1}{\frac{1-\lambda}{m}+\lambda m^{\frac{1}{\delta-1}}}\frac{a_{P_i}}{\gamma(\delta-1)e_i^{\delta-1}}\Big[\rho\frac{1-\lambda}{m}\frac{(m-1)(1-\lambda)}{m}+\frac{\frac{1-\lambda}{m}(1-a_{P_i})}{\delta}+\frac{\lambda(1-ma_{P_i})}{\delta m^{\frac{\delta-2}{\delta-1}}}\Big]$$

Combining the expressions, we have the equilibrium condition:

$$\phi\kappa_i a_{P_i} = \Big[\frac{1-\lambda}{m}+\lambda m^{\frac{1}{\delta-1}}\Big]\gamma\delta e_i^{\delta}\frac{\Big[\rho\delta\frac{1-\lambda}{m}\frac{(m-1)(1-\lambda)}{m}+\delta\frac{1-\lambda}{m}+\lambda m^{\frac{1}{\delta-1}}(\frac{1}{m}+\delta-1)\Big]}{\Big[\rho\delta\frac{1-\lambda}{m}\frac{(m-1)(1-\lambda)}{m}+\frac{1-\lambda}{m}(1-a_{P_i})+\frac{\lambda m^{\frac{1}{\delta-1}}(1-ma_{P_i})}{m}\Big]}$$

To show single-crossing, note that the advertising equation implies that:

$$e = \frac{\Big[1-\frac{1-\kappa}{\kappa}\frac{1-\alpha}{\alpha}\Big]n_0}{(1-ma_{P_i})\Big[\frac{1-\lambda}{m}+\lambda m^{\frac{1}{\delta-1}}\Big]}$$

Plugging this into the equilibrium condition for effort and rearranging, one has that:

$$\phi\kappa(1-\kappa)^{\delta\frac{\alpha}{1-\alpha}}(1-ma_{P_i})^{\delta}\frac{1}{(n_0)^{\delta}\Big[1-\frac{1-\kappa}{\kappa}\frac{1-\alpha}{\alpha}\Big]^{\delta-1}} = \frac{\gamma\delta}{\Big[\frac{1-\lambda}{m}+\lambda m^{\frac{1}{\delta-1}}\Big]^{\delta-1}}\frac{\Big[\rho\delta\frac{1-\lambda}{m}\frac{(m-1)(1-\lambda)}{m}+\delta\frac{1-\lambda}{m}+\lambda m^{\frac{1}{\delta-1}}(\frac{1}{m}+\delta-1)\Big]}{\Big[\rho\delta\frac{1-\lambda}{m}\frac{(m-1)(1-\lambda)}{m}+\frac{1-\lambda}{m}(1-a_{P_i})+\frac{\lambda m^{\frac{1}{\delta-1}}(1-ma_{P_i})}{m}\Big]}$$

Once again, the left-hand side is decreasing in κ since $a_{P_i} = 1-\frac{1-\kappa}{\kappa}\frac{1-\alpha}{\alpha}$ and using (5). The right-hand side is then easily seen to be increasing in κ. To demonstrate that $t_i = 0$ for ρ sufficiently small, the method is exactly the same as in Proposition 1. In particular, $t_i = 0$ at the symmetric equilibrium for all $\rho < \rho^*(0)$ from Proposition 1.

To show that $t_i = 0$ at a symmetric equilibrium with ρ small, note that:

$$\frac{\partial a_{P_i}}{\partial t_i}\phi\kappa_i - \frac{\partial n_{i,SH}}{\partial t_i}(e_i t_i(H) + t_i) - n_{i,SH} - \lambda = \mu_{t_i}$$

where $\mu_{t_i} \leq 0$. Now, for ρ near 0, $\frac{\partial a_{P_i}}{\partial t_i} \approx 0$ so that profit cannot be increasing in flat transfers for any $\lambda > 0$ when ρ is sufficiently close to 0.

Proof of Proposition 4 In order to show that consumer welfare is increasing in λ, it suffices to use the advertising equation. In particular, if advertising is increasing at the symmetric equilibrium, it must then imply that demand a_{P_i} is increasing for each platform i. This then implies that consumer surplus also necessarily rises, as the value in the content market is larger relative to that of the outside option, which of course remains constant. In order to show this, I will use the implicit function theorem. To begin, return to the equation from Proposition 3 for single-crossing. Define the function:

$$F^1 = \frac{\phi}{\gamma\delta}\kappa(1-\kappa)^{\delta\frac{\alpha}{1-\alpha}}(1-ma_{P_i})^{\delta}\frac{1}{(n_0)^{\delta}\left[1-\frac{1-\kappa}{\kappa}\frac{1-\alpha}{\alpha}\right]^{\delta-1}} - \frac{1}{\left[\frac{1-\lambda}{m}+\lambda m^{\frac{1}{\delta-1}}\right]^{\delta-1}}\frac{\left[\rho\delta\frac{1-\lambda}{m}\frac{(m-1)(1-\lambda)}{m}+\delta\frac{1-\lambda}{m}+\lambda m^{\frac{1}{\delta-1}}(\frac{1}{m}+\delta-1)\right]}{\left[\rho\delta\frac{1-\lambda}{m}\frac{(m-1)(1-\lambda)}{m}+\frac{1-\lambda}{m}(1-a_{P_i})+\frac{\lambda m^{\frac{1}{\delta-1}}(1-ma_{P_i})}{m}\right]} = 0 \tag{4.9}$$

As established in Proposition 3, $\frac{\partial F^1}{\partial\kappa} < 0$. Applying the implicit function theorem, we have

$$\frac{\partial\kappa}{\partial\lambda} = -\frac{\frac{\partial F^1}{\partial\lambda}}{\frac{\partial F^1}{\partial\kappa}}$$

Thus we would like to show that $\frac{\partial F^1}{\partial\lambda} > 0$. Let us begin by sending $\rho \to 0$[15] and demonstrate a strict inequality. The fact that F^1 is a smooth function of ρ

[15]Note that the $\frac{\partial F^1}{\partial\kappa} < 0$ for any value of ρ.

implies that a positivity theorem yields the result for ρ sufficiently small. After some differentiation and careful simplification, one has that:

$$\frac{\partial F^1}{\partial \lambda} = (\delta - 1)\frac{m^{\frac{1}{\delta-1}} - \frac{1}{m}}{\frac{1-\lambda}{m} + \lambda m^{\frac{1}{\delta-1}}}\Big[\delta\frac{1-\lambda}{m} + \lambda m^{\frac{1}{\delta-1}}(\frac{1}{m} + \delta - 1)\Big] - m^{\frac{1}{\delta-1}}\frac{(\frac{1}{m} + \delta - 1)(\frac{1-\kappa}{\kappa}\frac{1-\alpha}{\alpha})\frac{1}{m} - \frac{(1-ma_{P_i})}{m}\frac{\delta}{m}}{\frac{1-\lambda}{m}(1 - a_{P_i}) + \frac{\lambda}{m}m^{\frac{1}{\delta-1}}(1 - ma_{P_i})}$$

This is at least as large as:

$$\frac{1}{\frac{1-\lambda}{m} + \lambda m^{\frac{1}{\delta-1}}}\Big[(\delta-1)(m^{\frac{1}{\delta-1}} - \frac{1}{m})(\delta\frac{1-\lambda}{m} + \lambda m^{\frac{1}{\delta-1}}(\frac{1}{m} + \delta - 1)) - m^{\frac{1}{\delta-1}}(\frac{1}{m} + \delta - 1)(\frac{1-\kappa}{\kappa}\frac{1-\alpha}{\alpha})\frac{1}{m}\Big] \tag{4.10}$$

Now, I want to show that (10) is positive for all $\lambda \in [0, 1]$. Note that

$$m^{\frac{\delta}{\delta-1}} > \frac{\delta - 1}{\delta - 2}$$

provided δ is sufficiently large. This implies that:

$$(\delta - 1)(m^{\frac{1}{\delta-1}} - \frac{1}{m}) > m^{\frac{1}{\delta-1}}$$

so that (10) is greater than 0 at $\lambda = 0$. Now, differentiating (10) with respect to λ, we see that this is:

$$(\delta - 1)(m^{\frac{1}{\delta-1}} - \frac{1}{m})\Big[-\frac{\delta}{m} + m^{\frac{1}{\delta-1}}(\frac{1}{m} + \delta - 1)\Big] > 0$$

Hence, for ρ sufficiently small, advertising is increasing in λ for all $\lambda \in (0, 1)$. The advertising equation then implies that consumer surplus must also be increasing in λ.

To show that incentives are decreasing in λ, I will once again use the implicit function theorem. Using the equilibrium conditions, define the following functions:

$$F^1 = \phi\kappa\Big(1 - \frac{1-\kappa}{\kappa}\frac{1-\alpha}{\alpha}\Big)\Big[\rho\delta\frac{1-\lambda}{m}\frac{(m-1)(1-\lambda)}{m} + \frac{1-\lambda}{m}\frac{1-\kappa}{\kappa}\frac{1-\alpha}{\alpha} + \lambda m^{\frac{2-\delta}{\delta-1}}(1 - m(1 - \frac{1-\kappa}{\kappa}\frac{1-\alpha}{\alpha}))\Big]$$

$$-\Big[\frac{1-\lambda}{m} + \lambda m^{\frac{1}{\delta-1}}\Big]\gamma\delta e^{\delta}\Big[\rho\delta\frac{1-\lambda}{m}\frac{(m-1)(1-\lambda)}{m} + \delta\frac{1-\lambda}{m} + \lambda m^{\frac{1}{\delta-1}}(\frac{1}{m} + \delta - 1)\Big] = 0$$

$$F^2 = \Big[1 - \frac{1-\kappa}{\kappa}\frac{1-\alpha}{\alpha}\Big]\Big(\Big[1-\lambda+\lambda m^{\frac{\delta}{\delta-1}}\Big]e + n_0\Big) - \Big(\frac{1-\lambda}{m} + \lambda m^{\frac{1}{\delta-1}}\Big)e = 0$$

sending $\rho \to 0$ and writing:

$$J = \begin{bmatrix} J_{11} & J_{12} \\ J_{21} & J_{22} \end{bmatrix}$$

with

$$J_{11} = \phi\Big(1 - \frac{1-\kappa}{\kappa}\frac{1-\alpha}{\alpha}\Big)\Big[\frac{1-\lambda}{m}\frac{1-\kappa}{\kappa}\frac{1-\alpha}{\alpha} + \lambda m^{\frac{2-\delta}{\delta-1}}\Big(1 - m(1 - \frac{1-\kappa}{\kappa}\frac{1-\alpha}{\alpha}))\Big]$$
$$+\frac{\phi}{\kappa}\Big[\lambda m^{\frac{2-\delta}{\delta-1}}(1 - m(1 - \frac{1-\kappa}{\kappa}\frac{1-\alpha}{\alpha})) + \frac{1-\lambda}{m}\frac{1-\alpha}{\alpha} + \lambda m^{\frac{2-\delta}{\delta-1}} - \frac{1-\kappa}{\kappa}\frac{1-\alpha}{\alpha}\lambda m^{\frac{2-\delta}{\delta-1}}\Big] > 0$$

$$J_{12} = -\Big[\frac{1-\lambda}{m} + \lambda m^{\frac{1}{\delta-1}}\Big]\gamma\delta^2 e^{\delta-1}\Big[\delta\frac{1-\lambda}{m} + \lambda m^{\frac{1}{\delta-1}}\Big(\frac{1}{m} + \delta - 1\Big)\Big] < 0$$

$$J_{21} = \frac{1}{\kappa^2}\frac{1-\alpha}{\alpha}\Big([1-\lambda+\lambda m^{\frac{\delta}{\delta-1}}]e + n_0\Big) > 0$$

$$J_{22} = \Big(1 - \frac{1-\kappa}{\kappa}\frac{1-\alpha}{\alpha}\Big)\Big[1-\lambda+\lambda m^{\frac{\delta}{\delta-1}}\Big] - \Big(\frac{1-\lambda}{m} + \lambda m^{\frac{1}{\delta-1}}\Big)$$

Similar to the proof of proposition 2, we see that:

$$det(J) > 0$$

using the fact that:

$$J_{12} = -\frac{\delta}{e}\phi\kappa a_P\Big[\frac{1-\lambda}{m}(1-a_P) + \lambda m^{\frac{2-\delta}{\delta-1}}(1 - m a_P)\Big]$$

Hence in order to show that $\frac{\partial e}{\partial \lambda} < 0$, we would like to show that:

$$-J_{21}\frac{\partial F^1}{\partial \lambda} + J_{11}\frac{\partial F^2}{\partial \lambda} > 0 \tag{4.11}$$

we have:

$$\frac{\partial F^1}{\partial \lambda} = -\phi\kappa a_P \frac{1}{m}(1-a_P) + \phi\kappa a_P m^{\frac{2-\delta}{\delta-1}}(1-ma_P) - \Big[\frac{-1}{m} + m^{\frac{1}{\delta-1}}\Big]\gamma\delta e^{\delta}\Big[\delta\frac{1-\lambda}{m} + \lambda m^{\frac{1}{\delta-1}}\Big(\frac{1}{m}+\delta-1\Big)\Big]$$
$$-\Big[\frac{1-\lambda}{m} + \lambda m^{\frac{1}{\delta-1}}\Big]\gamma\delta e^{\delta}\Big[-\frac{\delta}{m} + m^{\frac{1}{\delta-1}}\Big(\frac{1}{m}+\delta-1\Big)\Big] < 0$$

$$\frac{\partial F^2}{\partial \lambda} = e\Big(a_P\Big[m^{\frac{\delta}{\delta-1}} - 1\Big] - \Big(-\frac{1}{m} + m^{\frac{1}{\delta-1}}\Big)\Big) < 0$$

Now, the condition (11) after substitution and simplification reduces to:

$$\phi a_P e(1-ma_P)\Big(m^{\frac{1}{\delta-1}} - \frac{1}{m}\Big)\Big[a_P\frac{1-\lambda}{m}(1-a_P) + a_P\lambda m^{\frac{2-\delta}{\delta-1}}(1-ma_P) + \lambda m^{\frac{2-\delta}{\delta-1}}\frac{1-ma_P}{\kappa} + \frac{1-a_P}{1-\kappa}\frac{1-\lambda}{m} + \frac{\lambda}{\kappa}m^{\frac{2-\delta}{\delta-1}}a_P\Big]$$
$$< \frac{(1-a_P)}{\kappa(1-\kappa)}\Big([1-\lambda+\lambda m^{\frac{\delta}{\delta-1}}]e + n_0\Big)\Big[\phi\kappa\frac{a_P}{m}(1-a_P) - m^{\frac{2-\delta}{\delta-1}}(1-ma_P)\phi\kappa a_P + \gamma\delta e^{\delta}\Big[\frac{1-\lambda}{m}(2\delta-1) + \lambda m^{\frac{1}{\delta-1}}(\frac{1}{m}+2\delta-1)\Big]$$

Using

$$\gamma\delta e^{\delta} = \phi\kappa a_P \frac{\frac{1-\lambda}{m}(1-a_P) + \lambda m^{\frac{2-\delta}{\delta-1}}(1-ma_P)}{\delta\frac{1-\lambda}{m} + \lambda m^{\frac{1}{\delta-1}}(\frac{1}{m}+\delta-1)} \frac{1}{\frac{1-\lambda}{m} + \lambda m^{\frac{1}{\delta-1}}}$$

for δ sufficiently large, the above inequality is true if:

$$e(1-ma_P)(m^{\frac{1}{\delta-1}} - \frac{1}{m}) < \Big([1-\lambda+\lambda m^{\frac{\delta}{\delta-1}}]e + n_0\Big)\Big[\frac{1-a_P}{m} - m^{\frac{2-\delta}{\delta-1}}(1-ma_P) + m(1-a_P)\Big]$$

which is true for δ sufficiently large.

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