ON THE STRUCTURE OF NONSMOOTH SINGULARITY MODELS OF RICCI FLOW

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The main goals of this work are to extend the structure theory of nonsmooth geometric limits of certain classes of Ricci flow solutions, and to apply this theory to the study of finite-time singularities of Ricci flow. We will separately consider Ricci flows in the settings of Type-I Scalar curvature, volume-noncollapsed Ricci flow with a lower Ricci curvature bound, and general solutions of Kähler-Ricci flow.

Given a Ricci flow satisfying a Type-I scalar curvature bound, it is known that any pointed Gromov-Hausdroff limit of appropriate rescalings is a singular shrinking gradient Ricci soliton with singularities of codimension 4. We prove that the entropy of a conjugate heat kernel based at the singular time converges to the soliton entropy of the singular soliton, and use this to characterize the singular set of the Ricci flow solution in terms of a heat kernel density function. This generalizes results previously only known with the stronger assumption of a Type-I curvature bound. We also show that in dimension 4, the singular Ricci soliton is smooth away from finitely many points, which are conical smooth orbifold singularities.

Next, we show that a simply-connected closed four-dimensional Ricci flow whose Ricci curvature is uniformly bounded below and whose volume does not approach zero must converge to a C^0 orbifold at any finite-time singularity, so has an extension through the singularity via orbifold Ricci flow. Moreover, a Type-I blowup of the flow based at any orbifold point converges to a flat cone in the Gromov-Hausdorff sense, without passing to a subsequence. In addition, we prove L^p bounds for the curvature tensor on time-slices for any $p < 2$. In higher dimensions, we show that every singular point of the flow is a Type-II point, and that any tangent flow at a singular point is a static flow corresponding to a Ricci flat cone.

Finally, in joint work with Wangjian Jian, we improve the description of F-limits of noncollapsed Ricci flows in the Kähler setting. In particular, the singular strata $\mathcal{S}^k$ of such metric flows satisfy $\mathcal{S}^{2j} = \mathcal{S}^{2j+1}$. We also prove an analogous result for quantitative strata, and show that any tangent flow with nonconstant potential function admits a nontrivial one-parameter action by isometries, which is locally free on the cone link in the static case. These results are established using parabolic regularizations of conjugate heat kernel potential functions based at almost-selfsimilar points, which may be of independent interest.

BIOGRAPHICAL SKETCH

Maximilien Hallgren was born in Fréjus, France, and grew up in the Finger Lakes region of upstate New York. He completed his Bachelor's degree in 2016 at Cornell University, and subsequently began the PhD program there.

This thesis is dedicated to my parents, Lisa and Morten.

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CHAPTER 1

INTRODUCTION

1.1 Historical Background

A solution of the Ricci flow on a closed manifold M is a one-parameter family of Riemannian metrics on M satisfying the nonlinear parabolic equation

$$\partial_t g_t = -2Rc(g_t),$$

where $Rc(g)$ denotes the Ricci tensor of a Riemannian metric g. It was shown by Hamilton in [H1] that for any initial Riemannian metric g_0 on M, a maximal smooth solution $(g_t)_{t\in[0,T)}$ exists, where $T \in (0,\infty]$. It was also shown that Ricci flow possesses certain regularization properties typical of nonlinear heat equations, and if $T < \infty$, then the curvature tensor must become unbounded as $t \to T$ at some definite rate; we then say the metric encounters a finite-time singularity at time T.

One of the central problems in the analysis of Ricci flow solutions is to determine the possible behavior at a finite-time singularity. It was shown in [Š] that the norm of the Ricci tensor must become unbounded as $t \nearrow T$, and it was later shown in [W] that if

$$Rc(g_t) \geq -Ag_t \quad \text{for all } t \in [0,T) \tag{1.1.1}$$

holds for some $A < \infty$, then the $L^{\frac{n+2}{2}}$-norm of the scalar curvature over spacetime must be infinite. Condition (1.1.1) by itself is insufficient to rule out finite-time singularities, but it is an open problem (see [CY]) whether a singularity can occur if we additionally assume that the volume does not approach zero at the singular time:

$$|M|_{g_t} \geq A^{-1} \quad \text{for all } t \in [0,T). \tag{1.1.2}$$

It has been shown that if in addition g_0 is Kähler [Z6] or $n \leq 3$ [CY], then such a singularity cannot occur. The question in general is the focus of Chapter 4 of this thesis.

Another open problem in this vein is whether a Ricci flow satisfying the uniform scalar curvature estimate

$$\sup_{M \times [0,T)} |R| \leq A \tag{1.1.3}$$

can develop a finite-time singularity. It has also been shown in this setting that if in addition g_0 is Kähler [Z5] or $n \leq 3$, then such a singularity cannot occur. In the general case, partial results were obtained in [BZ2, S3, CW2, CW4, B1].

So far, these statements all address a coarse description of singularities – they are of the form "at a finite-time singularity, this geometric quantity must be unbounded". For most applications, however, a finer description of singularity behavior is essential. One way to understand finer behavior of the flow is to find points $(x_i, t_i) \in M \times [0,T)$ with $t_i \nearrow T$, along with dilation factors $Q_i \nearrow \infty$ such that the parabolic dilations $g_t^i := Q_i g_{t_i + Q_i^{-1} t}$ have bounded geometry near x_i. Traditionally, x_i, t_i, Q_i are chosen so that the rescaled solutions obey bounds on the the full curvature tensor. Then local derivative estimates of Shi [S1] and Perelman's no-local collapsing theorem [P1] allow the application of Cheeger-Gromov-Hamilton's compactness theorem [H2] to extract a subsequential limit Ricci flow, called a (smooth) singularity model. These singularity models will be ancient solutions of Ricci flow (that is, they exist for times $t \in (-\infty, 0)$), and in certain cases, the singularity models are associated to special self-similar solutions. These are solutions of Ricci flow of the form $g_t = \sigma(t)\varphi_t g$, where $\sigma(t)$ is a scaling factor, and (φ_t) is a one-parameter family of diffeomorphisms. These self-similar solutions are equivalent to the data of a Riemannian manifold (M^n, g) equipped with a vector field $X \in C^\infty(M)$ satisfying the Ricci soliton equation:

$$Rc(g) + \mathcal{L}_X g = \frac{\lambda}{2} g.$$

If $X = \nabla f$ for some $f \in C^\infty(M)$, then (M, g, f) is called a gradient Ricci soliton (GRS). If $\lambda = 1$, the tuple (M, g, X) is called a shrinking Ricci soliton. It was shown by various methods [N1, EMT, MM] that if a Ricci flow satisfies the Type-I curvature bound

$$|Rm|(\cdot, t) \leq \frac{A}{T - t} \qquad \text{for all } t \in [0, T) \tag{1.1.4}$$

for some $A < \infty$, then for any singular point $x \in M$ of the flow, any singularity model of the Type-I rescaled pointed flows $(M, (\tau_i^{-1} g_{T+\tau_i t})_{t \in (-\tau_i^{-1}T, 0)}, x)$ for a given sequence $\tau_i \searrow 0$ must be a nontrivial shrinking GRS.

In dimension 3, it was shown by Perelman [P1] that singularity models are ancient flows with nonnegative curvature which are κ-noncollapsed at all scales; such solutions are called κ-solutions, and have since been entirely classified [Z3, B, BDS]. Moreover, Perelman's canonical neighborhood theorem [P1] showed that for any sequence $(x_i, t_i) \in M \times [0, T)$ with $t_i \nearrow T$ and $Q_i := R(x_i, t_i) \nearrow \infty$, the rescaled solutions $(M, (Q_i g_{t_i + Q_i^{-1} t})_{t \in (-Q_i t_i, 0]}, x_i)$ subsequentially converge to a κ-solution. This fact was essential for constructing Ricci flows with surgery and establishing the Poincare conjecture [P2, MT].

In higher dimensions, the options for obtaining well-understood singularity models are more limited, due to the failure of the Hamilton-Ivey pinching estimate in dimensions $n \geq 4$. In this case, there are rescaling methods for the Type-II case (that is, for solutions which do not satisfy (1.1.4)) which lead to non-flat eternal Ricci flow solutions (defined for all time). Though all such singularity models studied so far are steady Ricci solitons, it seems difficult to classify these singularity models unless strong additional assumptions are made. Moreover, the dilation factors Q_i in this case must correspond to $\sup_M |Rm|(\cdot, t_i)$ for some $t_i \nearrow T$, and the basepoints must be near the points where the spatial maxima are achieved. It thus becomes impossible to use this rescaling procedure to see what the flow looks like anywhere except very close to these spatial maxima.

It became apparent that, in order to get around this problem, it is necessary to consider a weaker notion of convergence and a more general notion of singularity model. Such a theory had previously been developed for Einstein manifolds (and more generally Riemannian manifolds with lower bounds on Ricci curvature) by Gromov, Cheeger, Colding, Tian, and Naber [CC1, C2, CC2, CCT, CN2, CN3]. A consequence of these works is that if (M_i^n, g_i, p_i) is a sequence of Riemannian manifolds satisfying the Einstein condition $Rc(g_i) = \lambda_i g_i$ for some $\lambda_i \in [-A, A]$ and a volume noncollapsing assumption $|B(p_i, 1)| \geq A^{-1} > 0$, then a subsequence will converge in the pointed Gromov-Hausdorff sense to a pointed metric length space (X, d, p) satisfying the following conditions:

- (i) An open dense subset $\mathcal{R} \subset X$ has the structure of a (not usually complete) Einstein manifold, and $X \setminus \mathcal{R}$ has Minkowski dimension at most $n - 4$
- (ii) There is a Riemannian metric g on $\mathcal{R}$ such that $d|(\mathcal{R} \times \mathcal{R})$ is the length metric d_g of $(\mathcal{R}, g)$
- (iii) There is an open precompact exhaustion (U_i) of $\mathcal{R}$ along with diffeomorphisms $\phi_i : U_i \to M_i$ such that $\phi_i \to \mathrm{id}_X$ with respect to the pointed Gromov-Hausdorff convergence, and $\phi_i^* g_i \to g$ in $C^\infty_{loc}(\mathcal{R})$.

That is, X is a smooth Einstein manifold away from a subset of codimension at least 4, the metric structure of (X, d) agrees with the Riemannian length structure of $(\mathcal{R}, g)$, and convergence is in the smooth Cheeger-Gromov sense away from the singular set. Moreover, the tangent cone at any point of X is a metric cone $C(Z)$ over some compact metric space Z, and there is a stratification of the singular set $\mathcal{S} := X \setminus \mathcal{R}$:

$$\mathcal{S}^0 \subset \mathcal{S}^1 \subset \cdots \subset \mathcal{S}^{n-4} = \mathcal{S}, \tag{1.1.5}$$

where the the Minkowski dimension of $\mathcal{S}^k$ is at most k, and any tangent cone of a point $x \in X \setminus \mathcal{S}^k$ is of the form $C(Z) \times \mathbb{R}^{k+1}$ for some metric cone $C(Z)$.

This produces one very specific type of weak compactness theorem for Ricci flows since the Ricci flow starting at any Einstein manifold evolves purely by rescaling, with each time slice also an Einstein manifold.

Much of the theory of Gromov-Hausdorff limits of Einstein manifolds goes through with the Einstein condition replaced by just $|Rc(g_i)| \le A$, though in this case g is only a $C^{1,\alpha}_{loc}$ Riemannian metric, and the convergence $\phi_i^* g_i \to g$ is only in $C^{1,\alpha}_{loc}(\mathcal{R})$. However, using Shi's estimates and an ϵ-regularity Theorem of Anderson, it can be shown (see [CY]) that if $(M_i, (g^i_t)_{t \in [-T,T]}, p_i)$ is a sequence of closed, pointed Ricci flows with $|Rc(g^i_t)| \le A$ for all $t \in [-T,T]$ and $|B_{g^i}(p_i,0,1)|_{g^i_0} \ge A^{-1}$, then after passing to a subsequence, we have pointed Gromov-Hausdorff convergence $(M_i, g^i_t, p_i) \to (X, d_t, p)$ for some continuous family of metric spaces (X, d_t), such that $\mathcal{R}$ admits a (possibly incomplete) Ricci flow $(\mathcal{R}, (g_t)_{t \in (-T,T]})$, satisfying property (ii) at every time $t \in (-T,T]$, and an analogous property to (iii). In particular, to find singularity models of a Ricci flow $(M, (g_t)_{t \in [0,T)})$, we only need to find basepoints $(x_i, t_i) \in M \times [0,T)$ and rescaling factors $Q_i \nearrow \infty$ such that the rescaled flows $g^i_t := Q_i g_{t_i + Q_i^{-1} t}$ have locally bounded Ricci curvature near x_i; the price to pay is that now the convergence is not smooth everywhere, and neither is the singularity model.

Unfortunately, this is still too restrictive for many Ricci flow settings in higher dimensions. An important example of this is Ricci flow starting at a Kähler metric in the canonical class of a Fano manifold. In this case, Perelman [ŠT] showed that a Type-I scalar curvature conditions holds along the flow:

$$|R|(\cdot,t) \le \frac{A}{T-t} \qquad \text{for all } t \in [0,T). \tag{1.1.6}$$

In particular, a Type-I rescaling procedure produces a sequence of Ricci flows with uniformly bounded scalar curvature, which is (a priori) substantially weaker than the Ricci curvature bound considered above. The Hamilton-Tian conjecture asks if, given a Fano Ricci flow (which only satisfies (1.1.6) instead of (1.1.4)), any sequence of Type-I recaled time slices $(M, (T-t_i) g_{t_i}, p)$ will converge in the pointed Gromov-Hausdorff sense to a Kähler-Ricci soliton, possibly with singularities of codimension four. This was answered in the affirmative in complex dimension two by Chen-Wang in [CW1], in complex dimension three by Tian-Zhang in [TZ], and in general dimensions by Chen-Wang in [CW3, CW4]. Soon afterwards, Bamler proved in [B2, B1] that given any closed Ricci flow satisfying (1.1.6), any Type-I rescaling has a subsequential limit which is a Ricci soliton with singularities of codimension four. Additional properties of this convergence is the focus of Chapter 3 of this thesis. The Hamilton-Tian conjecture was used to give a Kähler-Ricci flow proof of the Yau-Tian-Donaldson conjecture in [CDS]. Moreover, it is expected that in quite general settings, Kähler-Ricci flows automatically satisfy (1.1.6), while (1.1.4) generally fails. This is one of the primary motivations for studying Ricci flows with a scalar curvature bound.

To establish a structure theory for sequences of Ricci flows with bounded scalar curvature and an appropriate noncollapsing hypothesis (one sufficient hypothesis would be a lower bound on Perelman's ν-invariant, which holds automatically if the sequence of Ricci flows is a sequence of dilations of a fixed Ricci flow), Chen-Wang and Bamler establish a non-smooth version of Perelman's canonical neighborhood theorem. In particular, Bamler shows that if $(M, (g_t)_{t\in[-2,0]})$ is a closed Ricci flow with $\sup_{M\times[-2,0]} |R| \leq A$ and $\nu(g_{-2}) \geq -A$, and $\epsilon > 0$, then there is a scale $r_0 = r_0(\epsilon, A) > 0$ such that any metric ball $B_{g_t}(x,r) \subseteq M$ with $r < r_0$ is, after rescaling, ϵ-close in an appropriate sense to a nonsmooth singularity model. In the setting of [B1], the singularity models are Ricci flat spaces which satisfy many of the same properties as Gromov-Hausdorff limits of Einstein manifolds, including properties $(i) - (iii)$ considered above. Moreover, Bamler used the structure of nonsmooth singularity models to prove estimates for smooth Ricci flows with bounded scalar curvature. In particular, he established L^p estimates for the full curvature tensor and an ϵ-regularity theorem, both generalizing results [CN2, A1] known for Einstein manifolds.

Important elements of the above structure theory are distance distortion estimates and Gaussian lower bounds for the conjugate heat kernel which only depending on the scalar curvature bound. However, neither of these hold for general Ricci flow, though there is reason to expect that in an appropriate sense, a Type-I rescaling of a general Ricci flow should subsequentially converge to a singular shrinking GRS. In fact, it is known from

Ilmanen [I] that any Type-I rescaling at a singular point of a mean curvature flow converges in the varifold sense to a singular self-shrinker. Moreover, well-known examples of Type-II solutions of Ricci flow tend to have singularity models which converge (in appropriate senses) to singular shrinking GRS: the degenerate neckpinch of Angenent-Knopf [AIK] always has a nearby nondegenerate neckpinch singularity modeled on a shrinking cylinder, and a Type-I rescaling of Appleton's $U(2)$-invariant Ricci flow [A2] at the singular orbit converges in the pointed Gromov-Hausdorff sense to the flat cone $\mathbb{R}^4/\mathbb{Z}_2$ or a $\mathbb{Z}_2$-quotient of the shrinking cylinder.

In the series of papers [B4, B3, B5], Bamler defines a new notion of convergence, called $\mathbb{F}$-convergence, and proves a weak compactness theorem for solutions of Ricci flow with a lower bound on Nash entropy. He shows that Type-I dilations based at singular points of the Ricci flow $\mathbb{F}$-converge to singular shrinking solitons with singularities of codimension 4. This notion of convergence is not compatible with pointed Gromov-Hausdorff convergence in general, though we show in Chapter 5 that it implies pointed Gromov-Hausdorff convergence given assumptions (1.1.1) and (1.1.2).

Bamler's theory moreover provides a Ricci flow version of Cheeger-Colding theory. He proves that any sequence of noncollapsed (in the Nash entropy sense) Ricci flows subsequentially $\mathbb{F}$-converges to a metric flow $\mathcal{X}$, which has the structure of a smooth Ricci flow spacetime (in the sense of Kleiner-Lott [KL]) on an open dense subset $\mathcal{R} \subset \mathcal{X}$ of parabolic codimension four. He also shows that the infinitesimal structure of $\mathcal{X}$ is given by tangent flows rather than tangent cones, and any tangent flow is a singular shrinking GRS. Moreover, Bamler shows that the singular set $\mathcal{S} := \mathcal{X} \setminus \mathcal{R}$ admits a stratification as in (1.1.5). In Chapter 5, we show that if $\mathcal{X}$ is an $\mathbb{F}$-limit of Kähler-Ricci flows, then $\mathcal{S}^{2k+1} = \mathcal{S}^{2k}$, providing a Ricci flow analogue of a result known for Ricci limit spaces [CCT].

In [B5], Bamler finally used the structure theory he developed to prove new estimates for smooth noncollapsed Ricci flows. He established spacetime L^p estimates for the curvature tensor as well as a backwards pseudolocality theorem. Moreover, Bamler's structure theory is essential for many of the results in Chapter 4, and other authors have also found applications to problems for smooth Ricci flows [CMZ, CH].

1.2 Summary of the Main Results

We now summarize the main results of the thesis as well as their links to the historical developments outlined in the previous section. The precise statements of each of these results may be found in the first section of their respective chapter.

In Chapter 3, we extend Bamler's analysis of Ricci flows $(M^n, (g_t)_{t\in[0,T)})$ satisfying the Type-I scalar curvature bound (1.1.6). We prove Gaussian estimates for the conjugate heat kernel which allow us to define a density $\Theta : M \to (-\infty, 0]$ associated to conjugate heat kernels based at the singular time; this was previously known given a Type-I curvature assumption [MM]. This notion is somewhat analogous to the Gaussian density ratio for Mean curvature flow. Roughly speaking, $\Theta(x) = \lim_{\tau\searrow 0} \mathcal{W}(g_{T-\tau}, f_{T-\tau}, \tau)$, where $(4\pi(\tau))^{-\frac{n}{2}}e^{-f}$ is a conjugate heat kernel based at the singular time (the actually definition is slightly more technical dues to the nonuniquness of f).

Recall that in [B1], Bamler showed that any Gromov-Hausdorff limit resulting from a Type-I rescaling procedure of $(M, (g_t)_{t\in[0,T)})$ is a singular shinking gradient Ricci solion, with singularities of codimension 4. We show that any such singular soliton $(\mathcal{R}, g, f)$ is normalized in the sense that $\int_{\mathcal{R}} (4\pi)^{-\frac{n}{2}} e^{-f} dg = 1$, and has a well-defined entropy $\mathcal{W}(g, f)$. Moreover, we show the entropy only depends on the underlying (incomplete) Riemannian manifold $(\mathcal{R}, g)$ rather than the potential function f, generalizing a result of Naber. Next, we proceed to show that the heat kernel density coincides based at a point $x \in M$ coincides with the entropy of any singularity model: $\mathcal{W}(g, f) = \Theta(x)$. To do so, we use Bamler's volume estimates for high-curvature regions of Ricci flows with bounded scalar curvature to show that Perelman's entropy function does not accumulate near singularities of the rescaled flow. A consequence is that any two singularity models based at x have the same entropy, whereas (even with the stonger assumption (1.1.4)) uniqueness of the singularity models is still unknown. Moreover, we show that $\Theta(x) = 0$ if and only if $(\mathcal{R}, g, f)$ is the flat Euclidean space, so that $M \setminus \Theta^{-1}(0)$ is exactly the singular set of the Ricci flow.

Finally, we specialize to dimension 4, where Bamler's theory states that the singular soliton has singularities of codimension 4. We show that in this case, the singular shrinking GRS has only isolated conical singularities, each modeled on $\mathbb{R}^4/\Gamma$ for some finite subgroup $\Gamma \subseteq O(4, \mathbb{R})$ acting freely on $\mathbb{S}^3$. Moreover, we show the soliton admits the structure of a smooth Riemannian orbifold, and that there are only finitely many singular points. One technical difficulty is the a priori lack of volume comparison results on the singular soliton, which we address using Bamler's supercritical L^p estimates for the Ricci curvature, and an extension of elements of Cheeger-Colding theory developed in [TZ]. Another difficulty is the lack of local L^2 estimates for the curvature tensor. To handle this, we use the standard decomposition of the curvature tensor, estimating the trace-free Ricci component with an idea of [HM] and estimating the Weyl tensor using Chern-Simons invariants as in [DS].

In Chapter 4, we apply Bamler's more general weak compactness and partial regularity theory [B5] to answer the aforementioned question of X. Chen. Namely, can a closed Ricci flow satisfying the Ricci curvature lower bound (1.1.1) and volume lower bound (5.1.2) develop a finite-time singularity? Elementary arguments show that if $(M^n, (g_t)_{t\in[0,T)})$ is such a Ricci flow, then there is a compact metric space (X, d) such that

$$(M, d_{g_t}) \to (X, d)$$

in the Gromov-Hausdorff sense as $t \nearrow T$. In [CY], Chen-Yuan conjecture that (X, d) should have singularities of codimension four. We confirm this conjecture in the case $n = 4$, and show that in this case X has the structure of a C^0 Riemannian orbifold with finitely many conical orbifold singularities. This is surprisingly similar to Zhang-Bamler's description of finite-time singularities of Ricci flows with bounded scalar curvature when $n = 4$. However, our methods differ significantly, in part because we lack two-sided distortion estimates and bounds for the conjugate heat kernel. To establish this result, we first show that any tangent flow (in the sense of Bamler [B5]) is a static cone. A technical difficulty is that Bamler's notion of $\mathbb{F}$-convergence does not in general imply pointed Gromov-Hausdorff convergence. To deal with this, it suffices to prove a lower bound on the conjugate heat kernel on sets of almost-full measure near H_n-centers (roughly, centers of the conjugate heat kernel distribution). In dimension four, we can use the structure of the tangent flow to extract a good deal of information about the flow at the Type-I scale, eventually obtaining the orbifold structure of (X, d).

We also show that the L^p norm of the curvature is uniformly bounded along time slices of the flow for $p \in (0, 2)$:

$$\sup_{t \in [0,T)} \int_M |Rm|^p dg_t < \infty .$$

The coarse strategy is to decompose time-slices of the flow according to the size of the curvature relative to the Type-I scale. The main technical difficulty is to estimate the volume of the region with high curvature relative to the Type-I scale. Using Cheeger-Naber's quantitative stratification [CN2], it suffices to develop ϵ-regularity results akin to those in [CN3], which apply only at scales smaller than the Type-I scale. We establish this ϵ-regularity using contradiction-compactness arguments which combine techniques [CN3] from Cheeger-Colding theory with Ricci flow blowup analysis.

In the general (higher dimensional case), we show that any tangent flow of a closed Ricci flow satisfying (1.1.1),(5.1.2) must be a static, Ricci flat cone. We use this to show that any singular point of the original flow must be Type-II in the sense of [BM2]. We also show that the flows have singularities of codimension four below the Type-I scale, and use this to prove the L^p curvature estimate

$$\sup_{(x,t)\in M\times[\frac{T}{2},T)}\int_{B(x,t,\sqrt{T-t})}|Rm|^p dg_t \le C(A,n,T)(T-t)^{\frac{n}{2}-p}$$

for all $p\in(0,2)$.

In Chapter 5, we investigate $\mathbb{F}$-limits of Kähler-Ricci flows, in a joint work with Wangjian Jian. We establish results analogous to those previously shown for Gromov-Hausdorff limits of noncollapsed Kähler manifolds with lower Ricci curvature bounds [CCT, L1, LS2]. The singular strata $\mathcal{S}^k$ of such Gromov-Hausdorff limits X are known [CCT] to satisfy $\mathcal{S}^{2k+1}=\mathcal{S}^{2k}$. We show that this also holds for the singular strata of Bamler's metric flows $\mathcal{X}$ which are $\mathbb{F}$-limits of Kähler-Ricci flows. In fact, we show that an analogous statement holds for Bamler's quantitative stratification of Ricci flows. This relies on a construction of parabolic regularizations of functions in almost-selfsimilar regions of Ricci flow which behave like Ricci soliton potential functions. We also prove a Ricci flow version of the fact that Riemannian manifolds with lower Ricci curvature bounds which are Gromov-Hausdorff close to metric products admits harmonic almost-splitting functions (see [CC2]).

Moreover, we show that any tangent flow of a point in $\mathcal{X}$ whose potential function is not constant admits a nontrivial isometric S^1 action, generalizing the corresponding fact for tangent cones of noncollapsed Ricci limit spaces shown in [L1]. In the case where the tangent cone is static, we show that this action is locally free, again providing a Ricci flow analogue of a fact known [L1] for Ricci limit spaces. The proofs rely in part on the constructions mentioned in the previous paragraph. These results are one step towards showing that static tangent flows of Kähler-Ricci flows are homeomorphic to affine varieties (see the related statement [LS2]).

CHAPTER 2

PRELIMINARIES

2.1 Notation and Conventions

Throughout this paper, we use the following notation convention of Cheeger-Colding theory (c.f. [CC1]): we let $\Psi(a_1, ..., a_k | b_1, ..., b_\ell)$ denote a quantity depending on parameters $a_1, ..., a_k, b_1, ..., b_\ell$, which satisfies

$$\lim_{(a_1,...,a_k) \to (0,...,0)} \Psi(a_1, ..., a_k | b_1, ..., b_\ell) = 0$$

for any fixed $b_1, ..., b_\ell$. We also adhere to the following convention in [BK]: if we say that a proposition $P(\epsilon)$ depending on a parameter ϵ holds if $\epsilon \leq \overline{\epsilon}(b_1, ..., b_\ell)$, this means there exists a constant $\overline{\epsilon}$ depending on parameters $b_1, ..., b_\ell$ such that $P(\epsilon)$ holds whenever $\epsilon \in (0, \overline{\epsilon}]$. The notation $E \geq \underline{E}(b_1, ..., b_\ell)$ is defined analogously. We also let $\mathcal{P}(X)$ denote the space of Borel probability measures on a metric space X.

Given a Ricci flow $(M^n, (g_t)_{t\in[0,T)})$, along with $(x,t) \in M \times [0,T)$ and $r > 0$, we write

$$B_g(x,t,r) := \{y \in M; d_{g_t}(x,y) < r\}.$$

Let dg_t denote the Riemannian volume measure on M corresponding to g_t, and for a subset $A \subset M$, we let $|A|_{g_t}$ denote the volume of A with respect to g_t.

Given a metric space (X,d), along with $x \in X$ and $r > 0$, we denote by $B^X(x,r)$ the open metric ball, though we may write $B(x,r)$ when (X,d) is unambiguous.

2.2 Ricci Flow

2.2.1 Gradient Ricci Solitons

In this section we review a class of Riemannian manifolds which generalize Einstein manifolds, and whose Ricci flow solutions are self-similar.

Definition 1. *A gradient Ricci soliton (GRS) is a tuple (M^n, g, f) consisting of a Riemannian manifold (M, g) and a function $f \in C^\infty(M)$ satisfying*

$$Rc_g + \nabla^2 f = \frac{\lambda}{2} g$$

for some λ. If $\lambda > 0$, $\lambda = 0$, $\lambda < 0$, we call (M, g, f) a shrinking, steady, expanding soliton, respectively. We generally consider the normalized equation where $\lambda \in \{-1, 0, 1\}$, which can always be achieved by appropriate rescaling.

Next, we review some identities for shrinking GRS. Some analogous identities hold in the other cases, but we will not need them.

Proposition 1 (see Section 1.2 of [CCG+1]). *Suppose (M^n, g, f) is a connected shrinking GRS with $\lambda = 1$.*

(i) $R + \Delta f = \frac{n}{2}$.

(ii) $2Rc(\nabla f) = \nabla R$.

(iii) $R + |\nabla f|^2 - f$ *is constant.*

(iv) *If (M^n, g) is complete, then $R_g \geq 0$.*

From $(iii), (iv)$, one can show (see [CCG+4]) that if (M, g) is a complete Riemannian manifold, then the vector ∇f is complete. In this case, we can define a 1-parameter family of diffeomorphisms $\phi_t : M \to M$ with $\partial_t \phi_t(x) = \frac{1}{1-t} \nabla f(\phi_t(x))$ and $\phi_0 = \mathrm{id}_M$. If we set $g_t := (1-t)\phi_t^* g$, then $(M, (g_t)_{t\in(-\infty,1)})$ is a complete Ricci flow, and if we also define $f_t := f \circ \phi_t$, then (M, g_t, f_t) are all shrinking GRS:

$$Rc_{g_t} + \nabla^2 f_t = \frac{1}{2(1-t)} g_t.$$

If we view Ricci flow as a dynamical system in the space of Riemannian metrics on a smooth manifold modulo scaling and diffeomorphisms, then fixed points therefore correspond to Ricci solitons.

2.2.2 Ricci Flow Fundamentals

In this section, we denote by $(M^n, (g_t)_{t\in[0,T)})$ a closed Ricci flow.

The following foundational estimate roughly states that wherever a Ricci flow has bounded curvature, all of the derivatives of the curvature tensor are also bounded.

Theorem 1 (Shi's local derivative estimates, rescaled version [S1]). *There exist $C_k = C_k(n) < \infty$ such that for any Ricci flow $(M^n, (g_t)_{t\in[0,T]})$ and any $(x_0, t_0) \in M \times [0, T]$, $r > 0$ with $[t_0 - r^2, t_0] \subseteq [0, T]$ and $|Rm| \leq r^{-2}$ on $B(x_0, t_0, r) \times [t_0 - r^2, t_0]$, we have*

$$\sup_{B(x_0,t_0,\frac{r}{2})\times[t_0-\frac{1}{2}r^2,t_0]} |\nabla^k Rm| \leq C_k r^{-2-k}.$$

In general, knowing a curvature bound at a single point is much less useful than knowing a curvature bound at all nearby points, which motivates the following definitions of curvature scale.

Definition 2. *For $(x, t) \in M \times [0, T]$, we define*

$$r_{Rm}(x, t) := \sup\{r > 0; |Rm| \leq r^{-2} \text{ on } B(x, t, r) \times ([t - r^2, t + r^2] \cap [0, T))\},$$

$$\widetilde{r}_{Rm}(x, t) := \sup\{r > 0; |Rm| \leq r^{-2} \text{ on } B(x, t, r)\}.$$

Shi's estimates then imply that $|\nabla^k Rm|(x,t) \le C_k r_{Rm}^{-2-k}(x,t)$ whenever $[t - r_{Rm}^2, t] \subset [0,T]$. In Chapter 3, we only consider the curvature scale $\widetilde{r}_{Rm}$, so we will just write r_{Rm} instead of $\widetilde{r}_{Rm}$ throughout that chapter.

Let $(M^n, (g_t)_{t\in[0,T)})$ be a closed solution of Ricci flow. Standard theory (see, for example, Chapter 24, Section 2 of [CCG+3]) guarantees that, for any $(x,t) \in M \times [0,T)$, there exists a unique fundamental solution $K(x,t;\cdot,\cdot) : M \times [0,t) \to (0,\infty)$ of the conjugate heat equation based at (x,t). That is, $K(x,t;\cdot,\cdot)$ is the unique smooth function on $M \times [0,t)$ such that

$$(-\partial_s - \Delta_{g_s} + R_{g_s})K(y,s;\cdot,\cdot) = 0 \qquad \text{on } M \times [-2,t),$$

$$\int_M K(x,t;y,s)f(y)dg_s(y) \to f(x) \qquad \text{as } s \nearrow t$$

for any continuous $f : M \to \mathbb{R}$. Moreover, K is smooth on its domain, and if (y,s) are fixed, then $K(\cdot,\cdot;y,s)$ is the fundamental solution of the heat equation:

$$(\partial_t - \Delta_{g_t})K(\cdot,\cdot;y,s) = 0 \qquad \text{on } M \times (s,0),$$

$$\int_M K(x,t;y,s)f(x)dg_t(x) \to f(y) \qquad \text{as } t \searrow s.$$

2.2.3 Distance Distortion Estimates

The following is a well-known and elementary estimate for changing distances given uniform Ricci curvature bounds.

Proposition 2. *Suppose $(M^n, (g_t)_{t\in[0,T]})$ is any solution of Ricci flow satisfying $-A_1 g_t \le Rc(g_t) \le A_2 g_t$ for all $t \in [0,T]$. For any $x, y \in M$ and $0 \le t_1 \le t_2 \le T$, we then have*

$$e^{-A_1(t_2-t_1)}d_{g_{t_1}}(x,y) \le d_{g_{t_2}}(x,y) \le e^{A_2(t_2-t_1)}d_{g_{t_1}}(x,y).$$

Proof. For any $x \in M$ and $V \in T_xM$, we can compute

$$\frac{d}{dt}\log|V|_{g_t}^2 = -2\frac{Rc_{g_t}(V,V)}{g_t(V,V)} \in [-2A_2, 2A_1].$$

Upon integration, we have

$$\frac{|V|_{g_{t_2}}}{|V|_{g_{t_1}}} \in [e^{-A_2(t_2-t_1)}, e^{-A_1(t_2-t_1)}].$$

Now integrate along arbitrary smooth curves γ from x to y, and apply this estimate with $V = \dot{\gamma}$. □

The following is somewhat more involved, but is very useful to compare the distance between points which are far apart.

Proposition 3. *(Theorem 18.7 in [CCG⁺2]) Suppose $(M, (g_t)_{t\in[0,T)})$ is a complete solution of Ricci flow. Suppose $t_0 \in [0,T)$, $K \in [0,\infty)$, $r > 0$ and $x, y \in M$ are such that $Rc \leq (n-1)K$ on $B(x,t_0,r) \cup B(y,t_0,r)$ and $t \mapsto d_{g_t}(x,y)$ is differentiable at $t = t_0$. Then*

$$\left.\frac{\partial}{\partial t}\right|_{t=t_0} d_{g_t}(x,y) \geq -2(n-1)\left(Kr + \frac{1}{r}\right).$$

Because $t \mapsto d_{g_t}(x,y)$ is locally Lipschitz, this estimate can be integrated to obtain one-sided distance distortion bounds.

2.2.4 Nash Entropy and Perelman's $\mathcal{W}$-Functional

We first review the definition of several functional introduced by Perelman in [P1].

Definition 3. *Given a closed Riemannian manifold (M,g), a smooth function $f \in C^\infty(M)$, and $\tau > 0$, define*

$$\mathcal{W}(g,f,\tau) := (4\pi\tau)^{-\frac{n}{2}} \int_M \left(\tau(R + |\nabla f|^2) + f - n\right) e^{-f} dg.$$

For any Riemannian metric g on M and $\tau > 0$, we define

$$\mu[g,\tau] := \inf\left\{\mathcal{W}(g,f,\tau); (4\pi\tau)^{-\frac{n}{2}} \int_M e^{-f} dg = 1\right\},$$

$$\nu[g,\tau] := \inf_{s\in(0,\tau]} \mu[g,s].$$

The infimum defining $\mu[g,\tau]$ is always achieved on a closed Riemannian manifold (see Proposition 17.24 of [CCG+3]) by some $f \in C^\infty(M)$; in particular, $\mu[g,\tau] > -\infty$. Moreover, it is possible to show that $\lim_{\tau \searrow 0} \mu[g,\tau] = 0$, and $\tau \mapsto \nu[g,\tau]$ is continuous, so that $\nu[g,\tau] > -\infty$ as well. The important facts about these functionals which combine to make them extremely fruitful for singularity analysis are the following:

- (Lemma 1) scale-invariance,
- (Theorem 2) lower bounds on ν translate to upper and lower bounds for volume ratio relative to the curvature scale ,
- (Theorem 3) nondecreasing under Ricci flow, and constant only on shrinking Ricci solitons .

Lemma 1 (See Chapter 6 of [CCG+1]). *For any Riemannian metric g on M, and $f \in C^\infty(M)$, $\tau > 0$, the following hold:*

(i) $\mathcal{W}(\lambda g, f, \lambda\tau) = \mathcal{W}(g,\tau)$,

(ii) $\mu[\lambda g, \lambda\tau] = \mu[g,\tau]$,

(iii) $\nu[\lambda g, \lambda\tau] = \nu[g,\tau]$.

In the remainder of the section, we assume $(M, (g_t)_{t\in[0,T]})$ is a closed Ricci flow.

The following results are Perelman's no local collapsing theorem and Q. Zhang's no inflating result.

Theorem 2 ([P1],[Z2]). *Suppose $(M^n, (g_t)_{t\in[0,T)})$ is a closed Ricci flow satisfying $\nu[g_0, 2T] \geq -A$, and that $(x_0,t_0) \in M \times [0,T)$, $r_0 \in (0,\sqrt{t_0})$. Then the following hold, where $\kappa = \kappa(A) > 0$:*

(i) *If $R \leq r_0^{-2}$ on $B(x_0,t_0,r_0)$, then $|B(x_0,t_0,r_0)|_{g_{t_0}} \geq \kappa r_0^n$.*

(ii) *If $R \leq \frac{A}{t_0 - t}$ on $Q^-(x_0,t_0,r_0)$, then $|B(x_0,t_0,r_0)|_{g_{t_0}} \leq \kappa^{-1} r_0^n$.*

We now state Perelman's monotonicity formula.

Theorem 3 ([P1]). *Let $T' > T$, and suppose that*

$$u(x,t) = (4\pi(T' - t))^{-\frac{n}{2}} e^{-f_t(x)}$$

solves the conjugate heat equation $(-\partial_t - \Delta_{g_t} + R_{g_t})u = 0$. Then

$$\frac{d}{dt}\mathcal{W}(g_t, f_t, T - t) = 2(T' - t)\int_M \left| Rc_{g_t} + \nabla^2 f_t - \frac{1}{2(T' - t)} g_t \right|^2 u_t dg_t$$

for all $t \in [0,T]$.

From this formula, it is easy to conclude the monotonicity of Perelman's μ and ν functionals.

Corollary 1. *For $0 \le t_1 \le t_2 \le T$, we have the following:*

(i) $\mu[g_{t_1}, T' - t_1] \le \mu[g_{t_2}, T' - t_2]$,

(ii) $\nu[g_{t_1}, T' - T_1] \le \nu[g_{t_2}, T' - t_2]$.

Unfortunately, minimizers of the μ functional are not generally well-behaved along time-slices of a Ricci flow solution, since generally solution of elliptic equations along Ricci flow time slices do not satisfy uniform estimates. This makes these minimizers difficult to use for extracting a shrinking GRS structure from geometric limits, though it is possible in certain special cases [LS1].

An effective way of using Perelman's monotonicity formula to study singularity models is to consider the pointed entropy. We also define a related functional called the pointed Nash entropy.

Definition 4. *Suppose $K(x_0, t_0; y, s) = (4\pi(t_0 - s))^{-\frac{n}{2}} e^{-f_s}$ is a conjugate heat kernel. The pointed entropy at (x_0, t_0) is then*

$$\mathcal{W}_{x_0,t_0}(\tau) := \mathcal{W}(g_{t_0 - \tau}, f_{t_0 - \tau}, \tau)$$

for $\tau \in (0, t_0]$, while the Nash entropy is given by

$$\mathcal{N}_{x_0,t_0}(\tau) := \int_M \left(f_{t_0 - \tau} - \frac{n}{2}\right) K(x_0, t_0; \cdot, t_0 - \tau) dg_{t_0 - \tau}.$$

We now recall some standard facts about these functionals.

Proposition 4 (Proposition 5.2 of [B5]). *The Nash entropy* $[0,\tau] \cap (t_0 - I) \to \mathbb{R}$ *is continuous, and if* $R(\cdot, t_0 - \tau) \geq R_{min}$*, then for* $\tau > 0$*, the following hold:*

(i) $\frac{d}{d\tau}(\tau \mathcal{N}_{x_0,t_0}(\tau)) = \mathcal{W}_{x_0,t_0}(\tau) \leq 0$, so $\mathcal{N}_{x_0,t_0}(\tau) = \frac{1}{\tau}\int_0^\tau \mathcal{W}_{x_0,t_0}(s)ds \geq \mathcal{W}_{x_0,t_0}(\tau)$.

(ii) $\frac{d^2}{d\tau^2}(\tau \mathcal{N}_{x_0,t_0}(\tau)) = -2\tau \int_M \left| Rc_{t_0-\tau} + \nabla^2 f_{t_0-\tau} - \frac{1}{2\tau} g_{t_0-\tau}\right|^2 d\nu_{t_0-\tau} \leq 0$,

(iii) $-\left(\frac{n}{2\tau} + R_{min}\right) \leq \frac{d}{d\tau}\mathcal{N}_{x_0,t_0}(\tau) \leq 0$.

2.2.5 Estimates for the Heat and Conjugate Heat Kernels

Assume throughout this subsection that $(M^n, (g_t)_{t\in[0,T)})$ is a closed Ricci flow.

The following is a consequence of a gradient estimate due to Q. Zhang, which relies on no curvature assumptions.

Theorem 4 (Theorem 3.3 of [Z1]). *Suppose* $u \in C^\infty(M \times [0,T))$ *is a positive solution of the heat equation* $\partial_t u = \Delta_{g_t} u$*. If* $B := \sup_{M\times[0,T)} u$*, then for any* $x, y \in M$ *and* $t > 0$*, we have*

$$u(y,t) \leq B^{\frac{1}{2}} u^{\frac{1}{2}}(x,t) \exp\left(\frac{d^2_{g(t)}(x,y)}{4t}\right).$$

Now suppose $u(y,s) := K(x,t;y,s)$ is a conjugate heat kernel. Write $\tau(s) := t - s$ and $u = (4\pi\tau)^{-\frac{n}{2}} e^{-f}$. Perelman's differential Harnack quantity is then

$$w := \left(\tau(R + 2\Delta f - |\nabla f|^2) + f - n\right) u.$$

Theorem 5 (Perelman's differential Harnack estimate [P1]). $w \leq 0$.

This is referred to as a Harnack estimate because it can be integrated along curves in spacetime to obtain pointwise bounds on u. In order to state this estimate, we recall Perelman's definition [P1] of reduced length.

Definition 5. *The* $\mathcal{L}$*-length of a curve* $\gamma : [s,t] \to M$ *is*

$$\mathcal{L}(\gamma) := \int_s^t \sqrt{t-t'}\left(R(\gamma(t'),t') + |\dot{\gamma}(t')|^2_{g_{t'}}\right)dt'.$$

Given $(x,t) \in M \times [0,T)$*, the reduced length function* $\ell_{(x,t)} : M \times [0,t)$ *is given by*

$$\ell_{(x,t)}(y,s) := \frac{1}{2\sqrt{t-s}} \inf\{\mathcal{L}(\gamma); \gamma \text{ is a curve from } y \text{ to } x\}.$$

The following is then a consequence of Perelman's Harnack inequality.

Theorem 6 (Lower bound for conjugate heat kernels [P1])**.** *For* $x, y \in M$ *and* $0 \leq s < t < T$*, we have*

$$K(x,t;y,s) \geq \frac{1}{(4\pi(t-s))^{\frac{n}{2}}} e^{-\ell_{(x,t)}(y,s)}.$$

In [B4], Bamler established several new estimates for conjugate heat kernels which do not depend on any curvature assumption. In order to state these, we recall some basic definitions related to optimal transport and metric measure spaces.

Definition 6. *If* (X,d) *is a complete, locally compact metric space, and* μ_1, μ_2 *are Borel probability measures on* X*, then the* W_1*-distance between* μ_1, μ_2 *is*

$$d_{W_1}(\mu_1,\mu_2) := \sup\left\{\int_X f d\mu_1 - \int_X f d\mu_2; f : M \to \mathbb{R} \text{ bounded, } 1\text{-Lipschitz}\right\}.$$

A coupling of (μ_1,μ_2) *is a Borel probability measure* q *on* $X \times X$ *such that, if* $\pi_1, \pi_2 : X \times X \to X$ *are the projection maps, then* $(\pi_1)_* q = \mu_1$ *and* $(\pi_2)_* q = \mu_2$*.*

The following is a well-known alternate characterization of W_1-distance (c.f. [V]).

Proposition 5. *If* (X,d) *is a complete, locally compact metric space and* μ_1, μ_2 *are Borel probability measures on* X*, then*

$$d_{W_1}(\mu_1,\mu_2) = \inf\left\{\int_{X \times X} d(x,y)dq(x,y); q \text{ is a coupling of } (\mu_1,\mu_2)\right\}.$$

If $(M,(g_t)_{t\in I})$ is a Ricci flow, then for any $x \in M$ and $s,t \in I$ with $s<t$, define the corresponding conjugate heat kernel measure

$$\nu_{x,t;s} := K(x,t;\cdot,s)dg_s,$$

and set $\nu_{x,t;t} := \delta_x$.

The following results gives the monotonicity of W_1-distance between conjugate heat kernels.

Proposition 6 ([B2]). *If $(M^n,(g_t)_{t\in[0,T)})$ is a closed Ricci flow and $x_1,x_2 \in M$, then*

$$(0,t) \to [0,\infty), s \mapsto d_{W_1}^{g_s}(\nu_{x_1,t;s},\nu_{x_2,t;s})$$

is nondecreasing.

Definition 7. *The variance between Borel probability measures μ_1,μ_2 on a metric space (X,d) is*

$$\mathrm{Var}(\mu_1,\mu_2) := \int_X\int_X d^2(x_1,x_2)d\mu_1(x_1)d\mu_2(x_2).$$

We write $\mathrm{Var}(\mu) := \mathrm{Var}(\mu,\mu)$.

The following result bounds the variance between conjugate heat kernels.

Corollary 2. *For any $x_1,x_2 \in M$, $t_0 \in I$, and $t \leq t_0$ with $t \in I$, we have*

$$\mathrm{Var}_t(\nu_{x_1,t_0;t},\nu_{x_2,t_0;t}) \leq d_{t_0}^2(x_1,x_2)+H_n(t_0-t).$$

This result is even useful for a single conjugate heat kernel – it ensures the definition of points which are close to "most" of the conjugate heat kernel measure at a given time. This notion is essential for many of the results in Chapter 5.

Set $H_n := \frac{(n-1)\pi^2}{2}+4$.

Definition 8. *A point $(z,t) \in M \times I$ is an H_n-center of $(x_0,t_0) \in M \times I$ if $t \le t_0$ and $\mathrm{Var}_t(\delta_z, \nu_{x_0,t_0;t}) \le H_n(t_0 - t)$.*

Corollary 3. *For any $(x_0,t_0) \in M \times I$ and $t \le t_0$ with $t \in I$, there is at least one $z \in M$ such that (z,t) is an H_n-center of (x_0,t_0),*

The following is used frequently in Chapter 6. It was proved for $p = 2$ in [HN], and the general case was established in [B4].

Let $(M^n, (g_t)_{t \in I})$ be a closed Ricci flow.

Theorem 7. *If $(x_0,t_0) \in M \times I$ such that $[t_0 - \tau, t_0] \subset I$ for some $\tau > 0$, then for any $h \in C^1(M)$ and $p \in [1, \infty)$, $\nu := \nu_{x_0,t_0}$ satisfies*

$$\int_M h \, d\nu_{t_0 - \tau} = 0 \implies \int_M |h|^p d\nu_{t_0 - \tau} \le C(p)\tau^{\frac{p}{2}} \int_M |\nabla h|^p d\nu_{t_0 - \tau}.$$

Also, we can choose $C(1) = \sqrt{\pi}$ and $C(2) = 2$.

In [HN], an analogous log-Sobolev inequality was also proved, and used to prove the following, which will be essential for proving conjugate heat kernel estimates in Chapter 3.

Theorem 8 ([HN])**.** *Suppose $(M^n, (g_t)_{t \in [0,T)})$ is a closed Ricci flow. Fix $(x_0,t_0) \in M \times [0,T)$, and set $\nu_s := \nu_{x_0,t_0;s}$ for $s \in [0,t_0)$. Then, for any measurable subsets $A, B \subseteq M$, we have*

$$\nu_s(A)\nu_s(B) \le \exp\left(-\frac{\mathrm{dist}^2_{g_s}(A,B)}{8|s|} \right).$$

2.3 Ricci Limit Spaces

If $\mathrm{diam}(X) \le \pi$, we let $(C(X), d_{C(X)}, o_*)$ denote the metric cone over (X,d), the pointed metric space whose underlying set is $([0,\infty) \times X)/(\{0\} \times X)$, with vertex o_*, and with metric defined by

$$d^2_{C(X)}([r_1,x_1],[r_2,x_2]) = r_1^2 + r_2^2 - 2r_1r_2\cos(d_X(x_1,x_2))$$

for $[r_1, x_1], [r_2, x_2]$, where we let $[r, x] \in C(X)$ denote the equivalence class corresponding to (r, x).

For bounded subsets B_1, B_2 of a metric space (X, d), let $d_H^X(B_1, B_2)$ be the corresponding Hausdorff distance. For Borel probability measures μ_1, μ_2 on (X, d), let $d_{W_1}^X(\mu_1, \mu_2)$ denote their W_1-Wasserstein distance. For pointed metric spaces (X_1, d_1, x_1), (X_2, d_2, x_2), we denote by

$$d_{GH}((X_1, d_1), (X_2, d_2))$$

the Gromov-Hausdorff distance between the underlying metric spaces, and

$$d_{PGH}((X_1, d_1, x_1), (X_2, d_2, x_2))$$

the pointed Gromov-Hausdorff distance between the pointed metric spaces (for definitions and basic properties, see Chapters 7,8 of [BBI]). For convenience, when (X_i, d_i, x_i) are not bounded, we define

$$d_{PGH}((X_1, d_1, x_1), (X_2, d_2, x_2)) := \sum_{j=1}^{\infty} 2^{-j} \frac{d_{PGH}((B^{X_1}(x_1, j), d_1, x_1), (B^{X_2}(x_2, j), d_2, x_2))}{1 + d_{PGH}((B^{X_1}(x_1, j), d_1, x_1), (B^{X_2}(x_2, j), d_2, x_2))},$$

which metrizes the pointed Gromov-Hausdorff topology on the class of isometry classes of complete metric length spaces.

For $p \in M$ and $v \in T_pM$, $\gamma_v : \mathbb{R} \to M$ is the unique geodesic with initial velocity v, and $l_p(v) := \sup\{t \in [0, \infty); \gamma_v|[0, t]$ is a minimizing geodesic$\}$. $\mathcal{J}_p(v)$ is the Jacobian of the exponential map $\exp_p$ at v, so that if (x^i) are exponential coordinates centered at p, then $\exp_p^* g|_v = \mathcal{J}_p(v) dx^1 \wedge \cdots \wedge dx^n$. Let (r, ζ) be the associated polar geodesic coordinates, and let $\lambda = \lambda_p$ be the corresponding Jacobian, so that $\exp_p^* g|_{(r,\zeta)} = r^{n-1} \mathcal{J}_p(r\zeta) dr \wedge vol_{\mathbb{S}_p^{n-1}} = \lambda(r, \zeta) dr \wedge vol_{\mathbb{S}_p^{n-1}}$, hence $\lambda(r, \zeta) = r^{n-1} \mathcal{J}_p(r\zeta)$. However,

$$(\Delta r) vol = \mathcal{L}_{\nabla r} vol = \mathcal{L}_{\nabla r}(\lambda(r, \zeta) dr \wedge g_{\mathbb{S}_p^{n-1}}) = \partial_r(\log \lambda(r, \zeta)) vol,$$

hence $\Delta r = \partial_r \log \lambda$. Moreover, tracing the Riccati equation for the shape operator of geodesic spheres gives

$$\partial_r(\Delta r) = -|\nabla^2 r|^2 - Rc(\nabla r, \nabla r) \leq -(n-1)(\Delta r)^2 + (n-1)\kappa.$$

Riccati comparison leads to $\Delta r \leq (n-1)\sqrt{\kappa}\coth(\sqrt{\kappa} r)$. In particular, we get $\partial_r \log \lambda(r,\zeta) \leq \partial_r \log \sinh^{n-1}(r)$. By integrating in r, we therefore obtain the following directionally limited form of the area comparison theorem.

Theorem 9. *For any $\zeta \in S_p^{n-1}$ and $r_2 \geq r_1 \geq 0$, we have*

$$\frac{\lambda(r_2,\zeta)}{\lambda(r_1,\zeta)} \leq \left(\frac{\sinh(\sqrt{\kappa} r_2)}{\sinh(\sqrt{\kappa} r_1)}\right)^{n-1}.$$

Let S_κ^n be the simply connected Riemannian manifold of constant curvature κ, and let $v_\kappa(r) := |B(r)|$, $a_\kappa(r) := Area(\partial B(r))$ for any geodesic ball $B(r)$ of radius r in S_κ^n. Also let $A(p, r_1, r_2) := \{x \in M; r_1 < d(x,p) < r_2\}$, and let

$$v_\kappa(r_1, r_2) = \int_{r_1}^{r_2} a_\kappa(r) dr$$

be the volume of the corresponding annulus in the model space.

Theorem 10. *(Volume Comparison) If $r_1 \leq s_1$, $r_2 \leq s_2$, and $s_1 \leq s_2$, $r_1 \leq r_2$, then the following holds:*

$$\frac{|A(p, r_1, r_2)|}{v_\kappa(r_1, r_2)} \geq \frac{|A(p, s_1, s_2)|}{v_\kappa(s_1, s_2)}.$$

If $\kappa = 0$, then equality holds if and only if $A(p, r_1, r_2)$ is an annulus in a flat cone.

It is easy to use this volume comparison in combination with Gromov's compactness theorem to show that, given any sequence (M_i, g_i, p_i) of pointed complete Riemannian manifolds satisfying $Rc_{g_i} \geq -Ag_i$, we can pass to a subsequence to get pointed Gromov-Hausdorff convergence to some pointed metric space (X, d, p). By elementary metric geometry (see [BBI]), (X, d) is a complete metric length space. We will only consider the case where $|B(g_i, p_i)| \geq A^{-1}$ for some $A < \infty$. In this case, we refer to X as a noncollapsed Ricci limit space. We now consider the infinitesimal structure of X, by generalizing the notion of a tangent space.

Definition 9. *A tangent cone of a Ricci limit space X at $x \in X$ is any pointed Gromov-Hausdorff limit of (X, λ_i, x), where $\lambda_i \nearrow \infty$.*

By Gromov's compactness theorem and a diagonal argument, tangent cones at any point exist, but will not be unique in general (see [CC2] for an example). However, if X is a noncollapsed Ricci limit space, then we can say more about the structure of tangent cones using an almost-rigidity version of the rigidity part of Theorem 10, established in [CC2]. Roughly, it was shown that if $\kappa \geq 0$ and

$$r \mapsto \frac{|B(p,r)|}{v_{-\kappa}(r)},$$

is approximately constant on the interval $[1,2]$, then $B(p,r)$ is Gromov-Hausdorff close to some metric cone (relative to the scale r). Using this fact, it is not difficult to show that any tangent cone of a point in x is a metric cone $C(Z)$ for some compact metric space Z of diameter $\leq \pi$.

The notion of tangent space moreover gives a canonical partition of $X = \mathcal{R} \sqcup \mathcal{S}$, where $\mathcal{R}$ consists of points $x \in X$ all of whose tangent cones are $\mathbb{R}^n$. Moreover, the singular set admits a stratification $\mathcal{S}^0 \subset \mathcal{S}^1 \subset \cdots \subset \mathcal{S}^n$, where $M \setminus \mathcal{S}^k$ is defined to be the set of $x \in X$ such that some tangent cone at X is a metric product $C(Z) \times \mathbb{R}^{k+1}$ for some metric cone $C(Z)$. Roughly speaking, $\mathcal{S}^k$ is the set of $x \in X$ such that no tangent cone at x isometrically splits a factor of $\mathcal{R}^{k+1}$. It was shown in [CC2] that $\mathcal{S} = \mathcal{S}^{n-2}$ and that the Hausdorff dimension of $\mathcal{S}^k$ is at most k. If we also assume that the limiting sequence (M_i, g_i, p_i) satisfies $|Rc(g_i)| \leq A$, then it was shown in [CN3] that $\mathcal{S} = \mathcal{S}^{n-4}$.

In [CN2], the following notion of quantitative singular strata was established, and combined with the structure theory of Ricci limit spaces and an ϵ-regularity theorem for Einstein manifolds to get L^p curvature estimates.

Definition 10. *Let (X,d) be a noncollapsed Ricci limit space. Given $\eta, r > 0$ and $k \in \{0, ..., n\}$, we let $\mathcal{S}^k_{\eta,r}$ denote the set of $x \in X$ such that the following holds: for any $s \in [r, 1]$,* *there does not exist a metric cone $C(Z)$ with vertex o^* such that*

$$d_{PGH}\Big((B^X(x,s),x),(B^{C(Z)\times\mathbb{R}^{k+1}}((o^*,0^{k+1}),s)\Big)\leq\eta s.$$

Roughly speaking, $\mathcal{S}^k_{\eta,r}$ is the set of points which do not look (in the metric sense) like cones isometrically splitting a factor of $\mathbb{R}^{k+1}$ at any scale between r and 1. The singular strata $\mathcal{S}^k$ can be recovered from the quantitative strata:

$$\mathcal{S}^k=\bigcup_{\eta\in(0,1)}\bigcap_{r\in(0,1)}\mathcal{S}^k_{\eta,r}.$$

However, $\mathcal{S}^k_{\eta,r}$ are generally nonempty even on a smooth Riemannian manifold unlike $\mathcal{S}^k$. In [CN2], the following estimates were developed for the quantitative singular strata.

Theorem 11. *Suppose (M^n,g,p) is a pointed Riemannian manifold satisfying $Rc_g\geq -Ag$ and $|B(p,1)|\geq A^{-1}$. Then for any $\eta>0$, there exists $E=E(n,A,\eta)>0$ such that, for all $r\in(0,1]$, we have*

$$|\mathcal{S}^k_{\eta,r}\cap B(x,1))|\leq Er^{n-k-\eta}.$$

A consequence is that the Minkowski dimension of $\mathcal{S}^k$ is at most k. Though it is not very important for our results, we remark that the above estimate was improved in [CJN] so that the right hand side is of the form Er^{n-k}. The quantitative strata are useful for proving estimates on smooth Riemannian manifolds, using the following kind of ϵ-regularity theorem (of which there are many variations).

Theorem 12. [A1, CN3] *Suppose (M^n,g,p) is a pointed Riemannian manifold satisfying $Rc_g=\lambda g$ and $|B(p,1)|\geq A^{-1}$, where $|\lambda|\leq A$. Then there exists $\epsilon=\epsilon_0(n,A)>0$ such that if*

$$d_{PGH}\Big((B(p,r),d_g,p),(B^{C(Z)\times\mathbb{R}^{n-3}}((o^*,0^{k+1}),r)\Big)<\epsilon_0 r$$

for $r\in(0,1]$ and some metric cone $C(Z)$ with vertex o^*, then $\widetilde{r}_{Rm}(p)\geq\epsilon_0 r$.

By taking $\eta = \epsilon_0$ in Theorem, we may thus apply Theorem 12 to get the following estimate for the large-curvature region of $B(p,1)$:

$$|\{\tilde{r}_{Rm} < s\} \cap B(x,1)| \leq Es^4.$$

Such an estimate easily implies L^p bounds for $|Rm|$ for any $p \in (0,2)$. It was shown in [JN1] that the(sharp) L^2 estimate even holds.

The following consequence of volume element comparison will be important in Chapter 4. Given $x, y \in M$, we will let $\gamma_{x,y} : [0, l_{x,y}] \to M$ denote a unit-speed minimizing geodesic from x to y, which is unique for almost-every pair (x,y).

Proposition 7. *(Segment Inequality, see Theorem 7.1.10 of [P3]) Suppose (M^n, g, p) is a complete, pointed Riemannian manifold with $Rc \geq -(n-1)A$. Let $B_1, B_2 \subseteq M$ and a subset $W \subseteq M$ such that $\gamma_{x,y}([0, l_{x,y}]) \subseteq W$ for almost-every $(x,y) \in U_1 \times U_2$. Then, for any measurable $f : M \to [0, \infty)$ we have*

$$\int_{B_1}\int_{B_2}\int_0^{l_{x,y}} f(\gamma_{x,y}(t))dtddg(x)dg(y) \leq C(\mathrm{diam}(W), n, A)(|U_1| + |U_2|)\int_W f dg.$$

2.4 Singular Spaces

The notion of singular spaces which we will find useful was introduced by Bamler in [B2, B1], though a very similar notion was used in [CW3, CW4]. The definition encapsulates many of the properties of Gromov-Hausdorff limits of Einstein manifolds. It turns out that singularity models of noncollapsed Ricci flows with bounded scalar curvature [B1] and singular solitons obtained as $\mathbb{F}$-limits of general noncollapsed flows both have the structure of a singular space. A useful observation made in [B2] is that much of the analysis conducted on smooth Riemannian manifolds can be adapted (with sometimes nontrivial technical difficulties) to work on singular spaces. Depending on the setting, one may sometimes integrate by parts, use volume comparison, construct harmonic splitting functions, and so on.

Definition 11 (Bamler [B2, B5]). *A singular space is a tuple $\mathcal{X} = (X, d, \mathcal{R}, g)$, where (X, d) is a complete, locally compact metric length space, and $(\mathcal{R}, g)$ is a C^∞ Riemannian manifold satisfying the following:*

(i) *$d|(\mathcal{R} \times \mathcal{R})$ is the length metric of $(\mathcal{R}, g)$.*

(ii) *$\mathcal{R}$ is an open, dense subset of X.*

(iii) *for any compact subset $K \subset X$ and $D \in (0, \infty)$, there exists $\kappa = \kappa(K, D) > 0$ such that, for all $x \in K$ and $r \in (0, D)$, we have*

$$\kappa r^n \leq |B^X(x,r) \cap \mathcal{R}| \leq \kappa^{-1} r^n.$$

$\mathcal{X}$ is said to have singularities of codimension $p_0 > 0$ if, for all $p \in (0, p_0)$, $x \in X$ and $r_0 > 0$, there exists $E_{p,x,r} < \infty$ such that

$$|\{r_{Rm} < rs\} \cap B^X(x,r) \cap \mathcal{R}| \leq E_{p,x,r} r^n s^p$$

for all $r \in (0, r_0)$, $s \in (0,1)$. $\mathcal{X}$ is said to have mild singularities if, for any $p \in \mathcal{R}$, there exists a closed subset $\mathcal{Q}_p \subset \mathcal{R}$ of measure zero such that, for any $x \in \mathcal{Q}_p$, there exists a minimizing geodesic from p to x lying entirely in $\mathcal{R}$. $\mathcal{X}$ is Y-regular at scale a if, for any $x \in X$ and $r \in (0, a)$ satisfying $|B(x,r) \cap \mathcal{R}| > (\omega_n - Y^{-1}) r^n$, we have $r_{Rm}(x) > Y^{-1} r$.

We emphasize that any noncollapsed Gromov-Hausdorff limit of Einstein manifolds is a singular space, but a priori there could be many singular spaces which do not arise as such limits (even if they are Ricci-flat on their regular part $\mathcal{R}$).

Definition 12 (Bamler [B2]). *If (M_i, g_i, q_i) is a sequence of complete, pointed Riemannian manifolds and $(\mathcal{X}, q_\infty) = (X, d, \mathcal{R}, g, q_\infty)$ is a pointed singular space, a convergence scheme (U_i, V_i, ϕ_i) for the convergence $(M_i, g_i, q_i) \to (\mathcal{X}, q_\infty)$ consists of open subsets $V_i \subset M_i$, $U_i \subset \mathcal{R}$, and diffeomorphisms $\phi_i : U_i \to V_i$ such that the following hold:*

(i) *(U_i) is an increasing sequence with $\bigcup_i U_i = \mathcal{R}$,*

(ii) *$\phi_i^* g_i \to g$ in $C^\infty_{loc}(\mathcal{R})$,*

(iii) there exist $q_i' \in U_i$ with $q_i' \to q_\infty$ and $d_{M_i}(q_i, \phi_i(q_i')) \to 0$,

(iv) for any $D < \infty$ and $\epsilon > 0$, there exists $i_0 = i_0(D, \epsilon) \in \mathbb{N}$ such that for all $i \geq i_0$ and $x_1, x_2 \in B^X(q_\infty, D) \cap U_i$, we have

$$|d_X(x_1, x_2) - d_{M_i}(\phi_i(x_1), \phi_i(x_2))| < \epsilon,$$

and such that, for any $y \in B^{M_i}(q_i, D)$, there exists $x \in U_i$ such that $d_{M_i}(\phi_i(x), y) < \epsilon$.

Note that conditions (iii),(iv) imply that ϕ_i are Gromov-Hausdorff approximations. If a convergence scheme exists, we say that (M_i, g_i, q_i) converges to (X, q_∞).

2.5 Ricci Flow with Bounded Scalar Curvature

We now state Bamler's main weak compactness results for noncollapsed Ricci flows with bounded scalar curvature. A similar result was shown in the Kähler setting in [CW3, CW4]

Theorem 13. *(Theorems 1.4, 1.7 in [B1]) Suppose $(M_i^n, (g_t^i)_{t \in [-2,0]}, q_i)$ is a sequence of closed solutions of Ricci flow satisfying the following:*

i. $\nu[g^i_{-2}, 4] \geq -A$,

ii. $|R_{g^i}| \leq \rho_i \leq A$ on $M_i \times [-2, 0]$,

where $A < \infty$. Then some subsequence of (M_i, g_0^i, x_i) converges to a pointed singular space (X, q_∞) with singularities of codimension 4, which is Y-regular at the scale 1, where $Y = Y(n, A) < \infty$. Also, for any $\epsilon > 0$, there exists $C = C(A, \epsilon, n) < \infty$ such that

$$\int_{B(x,t,r)} (\widetilde{r}_{Rm}(\cdot, t))^{-4+2\epsilon} dg_t^i < Cr^{n-4+2\epsilon}.$$

for any $r \in (0, 1]$. If in addition $\rho_i \to 0$, then X is Ricci flat and has mild singularities.

The following estimate is a consequence of Theorem 2 and a basic covering argument (Lemma 2.1 of [BZ1]).

Proposition 8. *For any $A < \infty$, there exists $C = C(A) < \infty$ such that, for any closed solution $(M^n, (g_t)_{t\in[-2,0)})$ of Ricci flow satisfying:*

(i) $\nu[g_{-2}, 4] \ge -A$,

(ii) $|R| \le A$ on $M \times [-2, 0)$,

then for any $(x,t) \in M \times [-1, 0]$, $r > 0$, we have

$$C^{-1}(\min\{1, r\})^n \le |B(x,t,r)|_t \le C r^n e^{Cr}.$$

If instead of (ii) we have $|R| \le A|t|^{-1}$ on M for all $t \in [-2, 0)$, then

$$|B(x,t,r)|_t \le C r^n$$

for all $r \in (0, 1]$.

We will also need the following distortion estimate for Ricci flows with bounded scalar curvature.

Theorem 14 (Theorem 1.1 in [BZ1])**.** *Given $A < \infty$, there exists $B = B(A,n) < \infty$ such that the following holds. Suppose $(M^n, (g_t)_{t\in[-2,0]})$ is a closed Ricci flow satisfying:*

(i) $\nu[g_{-2}, 4] \ge -A$,

(ii) $|R| \le A$ on $M \times [-2, 0]$,

then for all $x, y \in M$ and $s, t \in [-1, 0]$, we have

$$\frac{1}{B} d_s(x,y) - B\sqrt{|t-s|} \le d_t(x,y) \le B d_s(x,y) + B\sqrt{|t-s|}.$$

Given $(q,t) \in M \times (-2, 0)$, let $u_{q,t} : M \times [-2, t) \to (0, \infty)$ be the conjugate heat kernel based at (q,t), and write $u_{q,t}(y,s) = (4\pi(t-s))^{-\frac{n}{2}} e^{-f_{q,t}(y,s)}$. The corresponding pointed entropy is defined to be $\mathcal{W}_{q,t}(\tau) := \mathcal{W}(g_{t-\tau}, f_{q,t}(t-\tau), \tau)$. Note that, if $(M^n, g_t) = (\mathbb{R}^n, g_{euc})$ is the static, flat Euclidean space, then $u_{x,t}(y,s) = (4\pi(t-s))^{-\frac{n}{2}} e^{-\frac{|x-y|^2}{4(t-s)}}$, and $\mathcal{W}_{x,t}(\tau) = 0$ for all $\tau > 0$. Perelman's differential Harnack inequality guarantees that $\mathcal{W}_{x,t}(\tau) \le 0$ in general, and the following ϵ-regularity theorem demonstrates that, wherever the pointed entropy is almost-Euclidean, the space-time geometry nearby is almost-Euclidean as well.

Theorem 15. *(Theorem 1.16 of [HN]) For any $A < \infty$, there exists $\epsilon = \epsilon(n, A) > 0$ such that the following holds. Let $(M^n, (g_t)_{t\in[-2,0)})$ be a closed Ricci flow satisfying $\nu[g_{-2}, 4] \geq -A$ and $|Rm|(\cdot, t) \leq A|t|^{-1}$ on M for all $t \in [-2, 0)$. If $(q, t) \in M \times [-1, 0)$ satisfies $\mathcal{W}_{q,t}(\tau) \geq -\epsilon$, then $(r_{Rm}(q, t))^2 \geq \epsilon\tau$.*

The hypotheses were later significantly weakened in [B5].

2.6 Metric Flows and $\mathbb{F}$-Convergence

The parabolic analogue of a metric space, defined in [B3], is that of a metric flow; metric flow pairs play the role of pointed metric spaces. The definition of a metric flow relies on an auxiliary function $\Phi(x) := \int_{-\infty}^{x} \frac{1}{\sqrt{4\pi}} e^{-\frac{y^2}{4}} dy$.

Definition 13 (Metric Flow Pairs, Definitions 3.2, 5.1 in [B3])**.** *A metric flow over $I \subseteq \mathbb{R}$ is a tuple $(\mathcal{X}, \mathfrak{t}, (d_t)_{t\in I}, (\nu_{x;s})_{x\in\mathcal{X}, s\in I\cap(-\infty,\mathfrak{t}(x)]})$, where $\mathcal{X}$ is a set, $\mathfrak{t} : \mathcal{X} \to I$ is a function, d_t are metrics on the level sets $\mathcal{X}_t := \mathfrak{t}^{-1}(t)$, and $\nu_{x;s} \in \mathcal{P}(\mathcal{X}_s)$, $s \leq \mathfrak{t}(x)$ are such that $\nu_{x;\mathfrak{t}(x)} = \delta_x$ and the following hold:*

(i) *(Gradient estimate for heat flows) For $s, t \in I$, $s < t$, $T \geq 0$, if $u_s : \mathcal{X}_s \to [0, 1]$ is such that $\Phi^{-1} \circ u_s$ is $T^{-\frac{1}{2}}$-Lipschitz (or just measurable if $T = 0$), then either $u_t : \mathcal{X}_t \to [0, 1]$, $x \mapsto \int_{\mathcal{X}_s} u_s d\nu_{x;s}$, is constant or $\Phi^{-1} \circ u_t$ is $(T + t - s)^{-\frac{1}{2}}$-Lipschitz,*

(ii) *(Reproduction formula) For $t_1 \leq t_2 \leq t_3$ in I, $\nu_{x;t_1}(E) = \int_{\mathcal{X}_{t_2}} \nu_{y;t_1}(E) d\nu_{x;t_2}(y)$ for $x \in \mathcal{X}_{t_3}$ and all Borel sets $E \subseteq \mathcal{X}_{t_1}$.*

A conjugate heat flow on $\mathcal{X}$ is a family $\mu_t \in \mathcal{P}(\mathcal{X}_t)$, $t \in I'$, such that for $s \leq t$ in I', we have $\mu_s(E) = \int_{\mathcal{X}_t} \nu_{x;s}(E) d\mu_t(x)$ for any Borel subset $E \subseteq \mathcal{X}_s$. A metric flow pair $(\mathcal{X}, (\mu_t)_{t\in I'})$ consists of a metric flow $\mathcal{X}$, along with a conjugate heat flow $(\mu_t)_{t\in I'}$ such that supp$(\mu_t) = \mathcal{X}_t$ and $|I \setminus I'| = 0$.

The parabolic analogue of pointed Gromov-Hausdorff convergence is replaced with $\mathbb{F}$-

convergence, also introduced in [B3].

Definition 14 (Correspondences and $\mathbb{F}$-Distance, Definitions 5.4, 5.6 in [B3]). *Given metric flows $(\mathcal{X}^i)_{i\in I}$ defined over $I^{0,i}$, a correspondence over $I'' \subset \mathbb{R}$ is a pair*

$$\mathfrak{C} = \left((Z_t, d_t)_{t\in I''}, (\varphi_t^i)_{t\in I''^{,i}, i\in I}\right)$$

where (Z_t, d_t^Z) are metric spaces, $I''^{,i} \subset I^{0,i} \cap I''$, and $\varphi_t^i : (\mathcal{X}_t^i, d_t^i) \to (Z_t, d_t^Z)$ are isometric embeddings. The $\mathbb{F}$-distance between metric flow pairs $(\mathcal{X}^j, (\mu_t^j)_{t\in I^{0,j}})$, $j = 1, 2$, within $\mathfrak{C}$ is the infimum of $r > 0$ such that there exists a measurable set $E \subset I''$ such that $I'' \setminus E \subset I''^{,1} \cap I''^{,2}$, $|E| \le r^2$, and there exist couplings q_t of (μ_t^1, μ_t^2), $t \in I'' \setminus E$, such that for all $s, t \in I'' \setminus E$ with $s \le t$, we have

$$\int_{\mathcal{X}_t^1 \times \mathcal{X}_t^2} d_{W_1}^{Z_s}\left((\varphi_s^1)_* \nu_{x^1;s}^1, (\varphi_s^2)_* \nu_{x^2;s}^2\right) dq_t(x^1, x^2) \le r.$$

The $\mathbb{F}$-distance between metric flow pairs is the infimum of $\mathbb{F}$-distances within a correspondence $\mathfrak{C}$, where $\mathfrak{C}$ is varied among all correspondences.

For the next definition, we suppose $(\mathcal{X}^i, (\mu_t^i)_{t\in I^{0,i}})$ $\mathbb{F}$-converge to $(\mathcal{X}^\infty, (\mu_t^\infty)_{t\in I^{0,\infty}})$ within the correspondence $\mathfrak{C}$.

Definition 15 (Convergence within a correspondence, Definition 6.18 in [B3]). *Given $\mu^i \in \mathcal{P}(\mathcal{X}_{t_i}^i)$ and $\mu^\infty \in \mathcal{P}(\mathcal{X}_{t_\infty}^\infty)$, we write $\mu^i \xrightarrow[i\to\infty]{\mathfrak{C}} \mu^\infty$ if $t_i \to t_\infty$ and there exist $E_i \subset I''$ such that $|I'' \setminus E_i| \to 0$, $E_i \subset I''$ and*

$$\lim_{i\to\infty} \sup_{t\in I''\setminus E} d_W^{Z_t}\left((\varphi_t^i)_* \mu_t^i, (\varphi_t^\infty)_* \mu_t^\infty\right) = 0,$$

where μ_t^i is the conjugate heat flow on $\mathcal{X}^i$ with $\mu_{t_i}^i = \mu^i$, for $i \in \mathbb{N} \cup \{\infty\}$. We write $x_i \xrightarrow[i\to\infty]{\mathfrak{C}} x_\infty$ if $\delta_{x_i} \xrightarrow[i\to\infty]{\mathfrak{C}} \delta_{x_\infty}$.

2.7 Partial Regularity Theory of Ricci Flows

In [KL], it was shown that given a sequence of 3-dimensional Ricci flows with surgery with fixed (or more generally, convergent) initial data, the Ricci flows converge in a certain sense to a smooth 4-manifold which locally looks like a Ricci flow on a fixed manifold. This limiting object is called a Ricci flow spacetime, and it turns out that such spacetimes occur naturally as the regular part of Bamler's $\mathbb{F}$-limits of noncollapsed Ricci flows.

Definition 16 (Ricci Flow Spacetime, Definition 1.2 in [KL])**.** *A Ricci flow spacetime is a tuple* $(\mathcal{M}, \mathfrak{t}, \partial_{\mathfrak{t}}, g)$ *consisting of a manifold* $\mathcal{M}$*, a time function* $\mathfrak{t} : \mathcal{M} \to \mathbb{R}$*, a "time-like" vector field* $\partial_{\mathfrak{t}} \in \mathfrak{X}(\mathcal{M})$ *with* $\partial_{\mathfrak{t}}\mathfrak{t} = 1$*, and a bundle metric* g *on the subbundle* $\ker(d\mathfrak{t}) \subseteq T\mathcal{M}$ *satisfying* $\mathcal{L}_{\partial_{\mathfrak{t}}} g = -2Rc(g)$*, where* $Rc(g)|\mathfrak{t}^{-1}(t)$ *is defined to be the Ricci curvature of* $g|\mathfrak{t}^{-1}(t)$*. We write* $\mathcal{M}_t := \mathfrak{t}^{-1}(t)$ *and* $g_t := g|\mathcal{M}_t$*.*

Given a Ricci flow $(M, (g_t)_{t\in I})$ and some $(x,t) \in M \times I$, we let $K(x,t;\cdot,\cdot) : M \times (I \cap (-\infty, t)) \to (0, \infty)$ denote the conjugate heat kernel based at (x,t), and define $d\nu_{x,t;s} := K(x,t;\cdot,s)dg_s \in \mathcal{P}(M)$. We now summarize some of the main points of Bamler's weak compactness and partial regularity theory.

Theorem 16 (c.f. Theorems 7.6, 9.12, 9.31 in [B3])**.** *Suppose* $(M_i^n, (g_{i,t})_{t\in(-T_i,0]}, (x_i, 0))$ *is a sequence of pointed Ricci flows satisfying* $\mathcal{N}_{x_i,0}(1) \geq -Y$ *for some* $Y < \infty$*. Then we can pass to a subsequence to obtain a future-continuous metric flow pair* $(\mathcal{X}, (\nu_{x_\infty,t})_{t\in(-T,0]})$ *along with a correspondence* $\mathfrak{C}$ *such that we have the following* $\mathbb{F}$*-convergence within the correspondence on compact time intervals:*

$$(M_i^n, (g_{i,t})_{t\in(-T_i,0]}, (\nu_{x_i,0;t})_{t\in(-T_i,0]}) \xrightarrow[i\to\infty]{\mathbb{F},\mathfrak{C}} (\mathcal{X}, (\nu_{x_\infty,t})_{t\in(-T,0]}).$$

Moreover, there is an open, dense subset $\mathcal{R} \subseteq \mathcal{X}$ *(with respect to the natural topology defined in Section 3 of [B3]) which admits the structure of a Ricci flow spacetime* $(\mathcal{R}, \mathfrak{t}, \partial_{\mathfrak{t}}, g)$*, where* $\mathfrak{t}$ *is the restricted function from the metric flow structure, and each* $(\mathcal{X}_t, d_t)$ *is the completion of* *the Riemannian length metric on $(\mathcal{R}_t, d_{g_t})$. In addition, the subbundle $\ker(d\mathfrak{t}) \subseteq T\mathcal{R}$ admits an endomorphism J satisfying $\mathcal{L}_{\partial_{\mathfrak{t}}} J = 0$ and restricting to an almost-complex structure J_t on each $\mathcal{R}_t$ such that each $(\mathcal{R}_t, g_t, J_t)$ a Kähler manifold, and there is an increasing exhaustion (U_i) of $\mathcal{R}$ by precompact open sets along with time-preserving diffeomorphisms $\psi_i : U_i \to M_i$ such that the following hold:*

(i) $\psi_i^* g_i \to g$ *in* $C^\infty_{loc}(\mathcal{R})$*,*

(ii) $(\psi_i^{-1})_* \partial_t \to \partial_{\mathfrak{t}}$ *in* $C^\infty_{loc}(\mathcal{R})$*,*

(iii) If $J_i \in \mathrm{End}(TM_i)$ *denote the given complex structures, then* $\psi_i^* J_i \to J$ *in* $C^\infty_{loc}(\mathcal{R})$*,*

(iv) If we write $d\nu_{x_\infty,t} = v_t dg_t$ *on* $\mathcal{R}$*, then* $\psi_i^* K(x_i, 0; \cdot, \cdot) \to v$ *in* $C^\infty_{loc}(\mathcal{R})$*.*

Proof. By the mentioned theorems in [B3, B5], it suffices to verify the claims concerning the complex structures. Because $|J_i|_{g_{i,t}} = \sqrt{2n}$ and $\nabla J_i = 0$, the Arzela-Ascoli theorem lets us pass to a subsequence so that $\psi_i^* J_i \to J$, where J restricts to an almost-complex structure on $T\mathcal{R}_t$ for each $t \in (-T, 0]$. Moreover, if $\omega_{i,t} \in \Omega^2(M_i)$ denote the Kähler forms of $(M_i, g_{i,t})$, then $\psi_i^* \omega_{i,t} \to \omega_t$, where $\omega_t(\cdot, \cdot) := g_t(J\cdot, \cdot)$. Then $d\omega_{i,t} = 0$ and $\partial_t J_i = 0$ pass to the limit to give $d\omega_t = 0$ and $\mathcal{L}_{\partial_{\mathfrak{t}}} J = 0$, so $(\mathcal{R}_t, g_t, J_t)$ is Kähler, where $J_t := J|T\mathcal{R}_t$. □

An important application of this compactness theorem is (via a diagonal argument) the existence of tangent flows based at any point of the aforementioned $\mathbb{F}$-limits.

Theorem 17 (Theorems 2.6, 2.16, 2.18 in [B5])**.** *If $\mathcal{X}$ is a metric flow obtained as in Theorem 16, and $y \in \mathcal{X}$, $t_0 := \mathfrak{t}(x)$, then for any sequence $\lambda_k \nearrow \infty$, we can pass to a further subsequence so that the time shifted and parabolically rescaled metric flows $(\mathcal{X}^{t_0,\lambda_k}, (\nu^{t_0,\lambda_k}_{y;t})_{t \in (-\lambda_k^2(t_0 - T), 0]})$ $\mathbb{F}$-converge to a metric flow pair $(\mathcal{Y}, (\nu_{y_\infty;t})_{t \in (-\infty, 0]})$, where $(\mathcal{Y}_{<0}, (\nu_{y_\infty;t_\infty})_{t \in (-\infty, 0)})$ is a metric soliton modeled on a singular shrinking Kähler GRS $(Y, d, \mathcal{R}_Y, g_Y, J_Y, f_Y)$. Also, there are diffeomorphisms as in Theorem 16 realizing smooth convergence on the regular part of $\mathcal{Y}$. There is an identification $\mathcal{Y}_{<0} = Y \times (-\infty, 0)$ restricting to isometries $(\mathcal{Y}_t, d_t) = (Y, \sqrt{|t|}d)$, and also identifying the spacetime $\mathcal{R} \subseteq \mathcal{Y}$ with*

$(\mathbb{R}_Y \times (-\infty,0), t, \partial_t - \nabla f_Y, |t| g_Y)$. Writing $d\nu_{y_\infty;t} = (4\pi\tau)^{-\frac{n}{2}} e^{-f} dg_t$, we have that $f(\cdot,t)$ corresponds to f_Y for all $t<0$ with respect to this identification.

If $Rc(g_Y)=0$, then $\mathcal{Y}_{<0}$ is a static metric flow modeled on the metric cone (Y,d) with vertex o. Moreover, in this case, there is an identification $\mathcal{Y}_{<0} = Y \times (-\infty,0)$ restricting to isometries $(\mathcal{Y}_t, d_t) = (Y,d)$, and identifying the spacetime $\mathcal{R}$ with $(\mathcal{R}_Y \times (-\infty,0), t, \partial_t, g_Y)$; $f(\cdot,t)$ then corresponds to $\frac{1}{4|t|}d^2(o,\cdot)+W_\infty$ for each $t<0$. Moreover, $\mathcal{R}_Y \setminus \partial B(o,1)$ equipped with the restricted Riemannian metric is a Sasaki-Einstein manifold.

Proof. By the mentioned theorems in [B5], and by Theorem 16, it suffices to recall that a gradient Ricci soliton structure on a Kähler manifold is automatically a Kähler-Ricci soliton (see Section 2.2 of [FIK]). □

Bamler used these tangent flows to define a stratification of the singular set analogous to that in Cheeger-Colding theory. For the definition, we must review the notions of almost-split, almost-static, and almost-selfsimilar.

Definition 17 (Definitions 5.1, 5.5, 5.6, 5.7 in [B5]). *Suppose $(M^n,(g_t)_{t\in I})$ is a closed Ricci flow, $(x_0,t_0)\in M\times I$, $r>0$, $\epsilon\in(0,1)$, and write $d\nu_{x_0,t_0} = (4\pi\tau)^{-\frac{n}{2}}e^{-f}dg$.*

(i) (x_0,t_0) is (ϵ,r)-selfsimilar if $[t_0-\epsilon^{-1}r^2,t_0]\subseteq I$ and the following hold for $W:=\mathcal{N}_{x,t}(r^2)$:

$$\int_{t_0-\epsilon^{-1}r^2}^{t_0-\epsilon r^2}\int_M \tau\left|Rc+\nabla^2 f-\frac{1}{2\tau}g\right|^2 d\nu_{x_0,t_0;t}dt \le \epsilon,$$

$$\sup_{t\in[t_0-\epsilon^{-1}r^2,t_0-\epsilon r^2]}\int_M \left|\tau(R+2\Delta f-|\nabla f|^2)+f-n-W\right| d\nu_{x_0,t_0;t} \le \epsilon,$$

$$\inf_{M\times[t_0-\epsilon^{-1}r^2,t_0-\epsilon r^2]} r^2 R \ge -\epsilon.$$

(ii) (x_0,t_0) is (ϵ,r)-static if $[t_0-\epsilon^{-1}r^2,t_0]\subseteq I$ and the following hold:

$$r^2\int_{t_0-\epsilon^{-1}r^2}^{t_0-\epsilon r^2}\int_M |Rc|^2 d\nu_{x_0,t_0;t}dt \le \epsilon,$$

$$\sup_{t\in[t_0-\epsilon^{-1}r^2,t_0-\epsilon r^2]}\int_M R\,d\nu_{x_0,t_0;t}\le\epsilon,$$
$$\inf_{M\times[t_0-\epsilon^{-1}r^2,t_0-\epsilon r^2]} r^2R\ge-\epsilon.$$

(iii) (x_0,t_0) *is weakly* (k,ϵ,r)*-split if* $[t_0-\epsilon^{-1}r^2,t_0]\subset I$ *and there is a map* $y=(y_1,...,y_k):M\times[t_0-\epsilon^{-1}r^2,t_0-\epsilon r^2]\to\mathbb{R}^k$*, called a weak* (k,ϵ,r)*-splitting map, satisfying the following:*

$$r^{-1}\int_{t_0-\epsilon^{-1}r^2}^{t_0-\epsilon r^2}\int_M|\square y_i|\,d\nu_{x_0,t_0;t}dt\le\epsilon,$$
$$r^{-2}\int_{t_0-\epsilon^{-1}r^2}^{t_0-\epsilon r^2}\int_M|\langle\nabla y_i,\nabla y_j\rangle-\delta_{ij}|\,d\nu_{x_0,t_0;t}dt\le\epsilon.$$

(iv) (x_0,t_0) *is strongly* (k,ϵ,r)*-split if* $[t_0-\epsilon^{-1}r^2,t_0]\subset I$ *and there is a map* $y:M\times[t_0-\epsilon^{-1}r^2,t_0-\epsilon r^2]\to\mathbb{R}^k$*, called a strong* (k,ϵ,r)*-splitting map, satisfying the following:*

$$\square y_i=0 \text{ on } M\times[t_0-\epsilon^{-1}r^2,t_0-\epsilon r^2],$$
$$r^{-2}\int_{t_0-\epsilon^{-1}r^2}^{t_0-\epsilon r^2}\int_M|\langle\nabla y_i,\nabla y_j\rangle-\delta_{ij}|\,d\nu_{x_0,t_0;t}dt\le\epsilon,$$
$$\int_M y_i\,d\nu_{x_0,t_0;t}=0 \text{ for all } t\in[t_0-\epsilon^{-1}r^2,t_0-\epsilon r^2].$$

We now review the quantitative stratification of metric flows introduced by Bamler. We will use a slightly stricter definition of (k,ϵ,r)-symmetric points than that in [B5]. Let $(\mu_t^{\mathbb{R}^k})_{t<0}$ be the Euclidean backwards heat kernel based at $0^k\in\mathbb{R}^k$.

Definition 18 ((k,ϵ,r)-symmetric points, c.f. Definition 2.21 in [B3])**.** *Given a metric flow* $\mathcal{X}$ *over* I*, a point* $x_0\in\mathcal{X}_{t_0}$ *is* (k,ϵ,r)*-symmetric if* $[t_0-\epsilon^{-1}r^2,t_0]\subset I$ *and there is a metric flow pair* $(\mathcal{X}',(\mu'_t)_{t\le0})$ *over* $(-\infty,0]$ *which is an* $\mathbb{F}$*-limit of noncollapsed Ricci flows as in Theorem 16, and which satisfies one of the following:*

(b1) $(\mathcal{X}'_{<0},(\mu'_t)_{t\in(-\infty,0)})=(\mathcal{X}''\times\mathbb{R}^k,(\mu''_t\otimes\mu^{\mathbb{R}^k}_t)_{t\in(-\infty,0)})$ *as metric flow pairs for some metric soliton* $(\mathcal{X}'',(\mu''_t)_{t\in(-\infty,0)})$*, and this identification restricts to an isometry of Ricci flow spacetimes* $\mathcal{R}'=\mathcal{R}''\times\mathbb{R}^k$*,* (b2) $(\mathcal{X}'_{<0}, (\mu'_t)_{t\in(-\infty,0)}) = (\mathcal{X}'' \times \mathbb{R}^{k-2}, (\nu'_{x';t} \otimes \mu_{\mathbb{R}^{k-2}})_{t\in(-\infty,0)})$ *as metric flow pairs for some static cone $\mathcal{X}''$ with vertex x', and this identification restricts to an isometry of Ricci flow spacetimes $\mathcal{R}' = \mathcal{R}'' \times \mathbb{R}^{k-2}$.*

In addition, $\mathcal{X}', \mathcal{X}''$ must satisfy the following:

(c) *Writing $d\mu'_t = (4\pi\tau)^{-\frac{n}{2}} e^{-f'} dg$ on $\mathcal{R}'$ and $d\mu''_t = (4\pi\tau)^{-\frac{n}{2}} e^{-f''} dg''$ on $\mathcal{R}'$, we have $Rc(g') + \nabla^2 f' = \frac{1}{2\tau} g'$ on $\mathcal{R}'$, $Rc(g'') + \nabla^2 f'' = \frac{1}{2\tau} g''$ on $\mathcal{R}''$, and $f' = f'' + \frac{1}{4\tau}|x|^2$ on $\mathcal{R}'$. In case (b2), we also have $Rc(g') = 0$ and $Rc(g'') = 0$,*

(d) *$\mathcal{N}_{(\mu'_t)}(\tau) = W$ for all $\tau > 0$, where $W \in [-Y, 0]$.*

Finally, we require that $|\mathcal{N}_{x_0}(r^2) - W| < \epsilon$ and

$$d_{\mathbb{F}}\left((\mathcal{X}^{t_0,r^{-1}}_{[-\epsilon^{-1},0]}, (\nu^{t_0,r^{-1}}_{x_0;t})_{t\in[-\epsilon^{-1},0]}), (\mathcal{X}'_{[-\epsilon^{-1},0]}, (\mu'_t)_{t\in[-\epsilon^{-1},0]})\right) < \epsilon.$$

The main difference between Definition 2.21 in [B5] and Definition 18 is the added assumptions on the Nash entropy.

There is another notion of almost-symmetric points of a metric flow, defined in terms of smooth Ricci flow approximants.

Definition 19 (Weakly (k,ϵ,r)-symmetric points, Definition 20.1 in [B5])**.** *Given a metric flow $\mathcal{X}$ over I, a point $x \in \mathcal{X}_{t_0}$ is weakly (k,ϵ,r)-symmetric if $[t_0 - \epsilon^{-1} r^2, t_0] \subset I$ and there is a closed, pointed Ricci flow $(M', (g'_t)_{t\in[-\epsilon^{-1},0]}, x')$ such that $(x',0)$ is $(\epsilon,1)$-selfsimilar and either strongly $(k,\epsilon,1)$-split or both $(\epsilon,1)$-static and strongly $(k-2,\epsilon,1)$-split, which satisfies $|\mathcal{N}_{x',0}(1) - \mathcal{N}_{x_0}(r^2)| \le \epsilon$ and*

$$d_{\mathbb{F}}\left((\mathcal{X}^{-t_0,r}, (\nu^{t_0,r}_{x_0;t})_{t\in[-\epsilon^{-1},0]}), (M', (g'_t)_{t\in[-\epsilon^{-1},0]}, (\nu_{x',0;t})_{t\in[-\epsilon^{-1},0]})\right) < \epsilon.$$

Definition 20 (Strata and Quantitative strata of a metric flow, Definitions 2.21, 20.2 in [B5])**.** *If $\mathcal{X}$ is a metric flow over I and $0 \le r_1 < r_2 \le \infty$, $\epsilon > 0$, then $\mathcal{S}^{\epsilon,k}_{r_1,r_2}$ consists of* *the points $x \in X_t$ such that $[t - \epsilon^{-1} r_2^2, t] \subseteq I$ and x is not $(k+1, \epsilon, r')$-symmetric for any $r' \in (r_1, r_2)$. Analogously, $x \in \widehat{S}^{\epsilon,k}_{r_1,r_2}$ if $[t - \epsilon^{-1} r_2^2, t] \subseteq I$ and x is not weakly $(k+1, \epsilon, r')$-symmetric for any $r' \in (r_1, r_2)$. Define*

$$S^k := \bigcup_{\epsilon \in (0,1)} \widehat{S}^{\epsilon,k}_{0,\epsilon}.$$

Note that, because $\epsilon \mapsto \widehat{S}^{\epsilon,k}_{0,r}$ and $r \mapsto \widehat{S}^{\epsilon,k}_{0,r}$ are both decreasing, we can write

$$S^k = \bigcup_{\epsilon \in (0,1)} \bigcup_{r \in (0,1)} \widehat{S}^{\epsilon,k}_{0,r}.$$

Bamler showed that, roughly speaking, the weak quantitative strata are qualitatively at least as large as the quantitative strata.

Lemma 2 (c.f. Lemma 20.3 in [B5])**.** *Suppose X is an $\mathbb{F}$-limit of closed noncollapsed Ricci flows as in Theorem 16. Given $Y < \infty$, $\epsilon > 0$ there exists $\epsilon'(Y, \epsilon) > 0$ such that for all $0 \leq r_1 < r_2 < \infty$, we have*

$$\{x \in X ; \mathcal{N}_x(r_2^2) \geq -Y\} \cap S^{\epsilon,k}_{r_1,r_2} \subseteq \widehat{S}^{\epsilon'(Y,\epsilon),k}_{r_1,r_2}.$$

As a consequence, we have

$$\bigcup_{\epsilon \in (0,1)} S^{\epsilon,k}_{0,\epsilon} \subseteq S^k.$$

Proof. Suppose $x \in X \setminus \widehat{S}^{\epsilon'(Y,\epsilon),k}_{r_1,r_2}$ and $\mathcal{N}_x(r_2^2) \geq -Y$. By definition, there exists $r \in (r_1, r_2)$ such that x is weakly $(k, \epsilon'(Y, \epsilon), r)$-symmetric. Lemma 20.3 of [B5] then states that x is (k, ϵ, r)-symmetric, hence $x \notin S^{\epsilon,k}_{r_1,r_2}$. We note that this Lemma holds even with our stricter definition of (k, ϵ', r)-symmetric points, by Nash entropy convergence (Theorem 15.45 of [B5]) and Proposition 7.1 of [B5].

Taking $r_1 = 0$, $r_2 = \epsilon$, and taking the union over $\epsilon \in (0,1)$ gives

$$\{x \in X ; \mathcal{N}_x(1) \geq -Y\} \cap \left(\bigcup_{\epsilon \in (0,1)} S^{\epsilon,k}_{0,\epsilon} \right) \subseteq S^k,$$

so the remaining claim follows by taking $Y \nearrow \infty$. □

Remark 1. *We will later show that* $\bigcup_{\epsilon\in(0,1)}\mathcal{S}^{\epsilon,k}_{0,\epsilon}=\mathcal{S}^k$.

We now recall a result from [B5] asserting the existence of good cutoff functions vanishing near the singular set of an $\mathbb{F}$-limit of Ricci flows, which will be useful in Section 6. Assume $\mathcal{X}$ is a metric flow as in Theorem 16.

Lemma 3 (Lemma 15.27 in [B5]). *There is a family of smooth functions* $\eta_r\in C^\infty(\mathcal{R},[0,1])$ *satisfying the following:*

(i) $r_{Rm}\geq r$ *on* $\{\eta_r>0\}$,

(ii) $\eta_r|_{\{r_{Rm}\geq 2r\}}\equiv 1$,

(iii) $|\nabla\eta_r|\leq C_0 r^{-1}$, $|\partial_{\mathfrak{t}}\eta_r|\leq C_0 r^{-2}$, *where* $C_0<\infty$ *is universal*

(iv) *for any* $x\in\mathcal{X}_t$, $A<\infty$, *and* $r>0$, *the set* $\{\eta_r>0\}\cap P^{*}(x;A,-A^2)\cap\mathcal{R}_t$ *is relatively compact in* $\mathcal{R}_t$,

(v) *for any maximal integral curve* $\gamma: I\to\mathcal{R}$ *of* $\partial_{\mathfrak{t}}$ *(assume* $\mathfrak{t}(\gamma(t))=t$ *by a constant reparametrization), we have either* $\eta_r(\gamma(t))=0$ *for* t *near* $t_{\min}:=\inf(I)$, *or else* $t_{\min}=-T$.

The following estimate gives rough L^p bounds on various geometric quantities in almost-selfsimilar regions, which we will use frequently in Chapter 5.

Proposition 9 (c.f. Proposition 6.2 in [B5]). *Given* $\epsilon>0$, *if* $\alpha\leq\overline{\alpha}$ *and* $\delta\leq\overline{\delta}(\epsilon)$, *then the following holds. Suppose* $(M^n,(g_t)_{t\in I})$ *is a Ricci flow,* $r>0$, $(x_0,t_0)\in M\times I$ *is* (δ,r)-*selfsimilar,* $W:=\mathcal{N}_{x_0,t_0}(1)\geq -Y$, *and write* $(4\pi\tau)^{-\frac{n}{2}}e^{-f}dg:=d\nu:=d\nu_{x_0,t_0}$. *Then*

$$\int_{t_0-\epsilon^{-1}r^2}^{t_0-\epsilon r^2}\int_M\left(\tau|Rc|^2+\tau|\nabla^2 f|^2+|\nabla f|^2+\tau|\nabla f|^4+\tau^{-1}e^{\alpha f}\right)e^{2\alpha f}\,d\nu_t\,dt\leq C(Y,\epsilon),\tag{2.7.1}$$

$$\sup_{t\in[t_0-\epsilon^{-1}r^2,\,t_0-\epsilon r^2]}\int_M\left(\tau|R|+\tau|\Delta f|+\tau|\nabla f|^2+e^{\alpha f}\right)e^{2\alpha f}\,d\nu_t\leq C(Y,\epsilon).\tag{2.7.2}$$

Proof. By Proposition 7.1 of [B5], we have $\mathcal{N}_{x_0,t_0}(\tau)\geq W-\Psi(\delta|Y,\epsilon)$. We can therefore apply Proposition 6.2 of [B5] for any $r\in[\epsilon^{\frac{1}{2}},\epsilon^{-\frac{1}{2}}]$, taking $\theta:=\epsilon^2$, to obtain (2.7.1), (2.7.2). $\square$

2.8 Chern-Simons Invariants

A Chern-Simons invariant associated to a $(2n-1)$-manifold equipped with a principal G-bundle $\pi : P \to M$ and a principal connection $\theta \in \Omega^1(P;\mathfrak{g})$ is an $\mathbb{R}/\mathbb{Z}$-valued gauge invariant constructed by Cheeger-Simons in [CS2]. These are associated to G-invariant polynomials on $\mathfrak{g}$, and hence to certain characteristic classes, but are defined on odd-dimensional manifolds, and generally depend on the connection; they are thus an example of "secondary characteristic classes". These will be useful in Chapter 3, so we review enough of the theory for our desired application. We first discuss some preliminaries necessary for the construction of these invariants.

Definition 21. *(n-Classifying bundles with connection) Given a compact Lie group G with Lie algebra $\mathfrak{g}$, let $\varepsilon(G)$ denote the category of triples $\alpha = (P, M, \theta)$, where $\pi : E \to M$ is a principal G-bundle and $\theta \in \Omega^1(E;\mathfrak{g})$ is a principal connection; morphisms $\varphi : \alpha \to \widehat{\alpha}$ are bundle maps $\varphi : P \to \widehat{P}$ compatible with the chosen connections: $\varphi^*\widehat{\theta} = \theta$. An object $A \in \varepsilon(G)$ is n-classifying if the following hold:*

(i) *for every $\alpha \in \varepsilon(G)$ with $\dim(M) \leq n$, there is a morphism $\alpha \to A$,*

(ii) *any two such morphisms are homotopic through bundle maps: given morphisms $\varphi_0, \varphi_1 : \alpha \to A$, there exists a 1-parameter family of bundle maps $\varphi_t : \alpha \to A$ (which are not necessarily compatible with the connections of α, A) from φ_0 to φ_1.*

Theorem 18. *(Narasimhan-Ramanan [NR]) For each $n \in \mathbb{N}$, there exists an n-classifying $A \in \varepsilon(G)$.*

Remark 2. *It is convenient to use the definition of n-classifying bundle in order to avoid discussing connections on infinite-dimensional principal bundles (note that classifying bundles in the usual topological sense will be infinite-dimensional manifolds).*

Definition 22. *(de Rahm cohomology with representatives) Let $Z^k_{\mathbb{Z}}(M) \subset \Omega^k(M)$ denote the* *set of closed differential forms $\omega \in \Omega^k(M)$ with $\int_\sigma \omega \in \mathbb{Z}$ for all all smooth singular cycles $\sigma : \Delta_k \to M$. Given a smooth manifold M^n, we define*

$$R^k(M,\mathbb{Z}) := \{(\omega, u) \in Z^k_{\mathbb{Z}}(M) \times H^k(M;\mathbb{Z}); i_* u = [\omega]\},$$

where $[\omega]$ denotes the de Rahm cohomology class of ω, and $i_* : H^k(M,\mathbb{Z}) \to H^k(M,\mathbb{R}) = H^k_{dR}(M)$ is induced by the inclusion $\mathbb{Z} \hookrightarrow \mathbb{R}$, composed with the de Rahm isomorphism.

In order to construct the Chern-Simons invariants, it is necessary to first construct objects with more information, called differential characters.

Definition 23. *A differential character of degree k is a group homomorphism $f : Z_k(M) \to \mathbb{R}/\mathbb{Z}$ such that there exists $\omega \in \Omega^{k+1}(M)$ with*

$$f(\partial\sigma) = \widetilde{\int_\sigma \omega}$$

for all $\sigma \in C_{k+1}(M)$, where $C_k(M)$ is the abelian group of smooth singular chains in M with coefficients in $\mathbb{Z}$, $Z_k(M) \subset C_k(M)$ is the subgroup of cycles, and $\widetilde{x} \in \mathbb{R}/\mathbb{Z}$ denotes the equivalence class of $x \in \mathbb{R}$. The group of such differential characters is denoted $\widehat{H}^k(M,\mathbb{R}/\mathbb{Z})$.

Theorem 19. [CS2] *There are natural short exact sequences*

(a) $0 \to H^k(M;\mathbb{R}/\mathbb{Z}) \to \widehat{H}^k(M,\mathbb{R}/\mathbb{Z}) \xrightarrow{\delta_1} Z^{k+1}_{\mathbb{Z}}(M) \to 0$,

(b) $0 \to \Omega^k(M)/Z^k_{\mathbb{Z}}(M) \to \widehat{H}^k(M,\mathbb{R}/\mathbb{Z}) \xrightarrow{\delta_2} H^{k+1}(M;\mathbb{Z}) \to 0$,

(c) $0 \to H^k(M;\mathbb{R})/i_*(H^k(M;\mathbb{Z})) \to \widehat{H}^k(M;\mathbb{R}/\mathbb{Z}) \xrightarrow{(\delta_1,\delta_2)} R^{k+1}(M;\mathbb{Z}) \to 0$.

The maps δ_1, δ_2 can be described as follows: given $f \in \widehat{H}^k(M,\mathbb{R}/\mathbb{Z})$, choose any group homomorphism $\widehat{f} : C_k(M) \to \mathbb{R}/\mathbb{Z}$ extending f; choosing $\delta_1(f) := \omega \in \Omega^{k+1}(M)$ as in the previous definition, it can be shown that $c := \omega - \delta\widehat{f} \in C^{k+1}(M)$ is closed, hence defines an integral cohomology class $\delta_2(f) := [c]$, that $\omega \in Z^{k+1}_{\mathbb{Z}}(M)$, and that $\omega, [c]$ are independent of any choices made. The inclusion $H^k(M;\mathbb{R}/\mathbb{Z}) \hookrightarrow \widehat{H}^k(M,\mathbb{R}/\mathbb{Z})$ is defined by taking any $[u] \in$

$H^k(M;\mathbb{R}/\mathbb{Z})$ to the restriction $u|Z_k(M)$, while the inclusion $\Omega^k(M)/Z^k_{\mathbb{Z}}(M) \hookrightarrow \widehat{H}^k(M,\mathbb{R}/\mathbb{Z})$ is induced by the map $\Omega^k(M) \to \widehat{H}^k(M,\mathbb{R}/\mathbb{Z})$ taking $\alpha \in \Omega^k(M)$ to $Z_k(M) \to \mathbb{R}/\mathbb{Z}, \sigma \mapsto \widetilde{\int_\sigma \alpha}$.

Remark 3. (a) *implies that the usual cohomology group $H^k(M;\mathbb{R}/\mathbb{Z})$ can be seen as a subgroup of the differential characters.* (c) *indicates that $\widehat{H}^k(M;\mathbb{R}/\mathbb{Z})$ is an extension of $R^{k+1}(M;\mathbb{Z})$ by a finite-dimensional torus. The upshot is that any $f \in \widehat{H}^k(M;\mathbb{R}/\mathbb{Z})$ uniquely determines a closed $(k+1)$-form ω and an integral cohomology class $[c] \in H^{k+1}(M;\mathbb{Z})$.*

Example 1. *Let $\pi : E \to M^n$ be a principle $SO(2,\mathbb{R})$-bundle over M with connection $\theta \in \Omega^1(P,i\mathbb{R})$, and corresponding curvature $F = F_\theta \in \Omega^2(M,i\mathbb{R})$, so that $\frac{1}{2\pi i}F \in Z^2_{\mathbb{Z}}(M)$ is a representative of the Euler class of E. Given a closed curve $\gamma : \mathbb{S}^1 \to M$, let $H(\gamma) \in SO(2,\mathbb{R})$ be the holonomy along γ, and define $\widehat{\chi}(\gamma) \in \mathbb{R}/\mathbb{Z}$ by $H(\gamma) = e^{2\pi i \widehat{\chi}(\gamma)}$. Extend $\widehat{\chi}$ to all 1-cycles as follows: given $x \in Z_1(M)$, we can choose a closed curve $\gamma : \mathbb{S}^1 \to M$ and a 2-chain $y \in C_2(M)$ such that $x = \gamma + \partial y$. Define*

$$\widehat{\chi}(x) := \widehat{\chi}(\gamma) + \frac{1}{2\pi i}\widetilde{\int_y F}.$$

To check that this is well-defined, we note that if $\gamma : \mathbb{S}^1 \to M$ is a closed curve with $\gamma = \partial y$ for some $y \in C_2(M)$, then $e^{2\pi i \widehat{\chi}(\gamma)} = H(\gamma) = \exp\left(\int_y F\right)$. We have $\widehat{\chi} \in \widehat{H}^1(M;\mathbb{R}/\mathbb{Z})$, $\delta_1(\widehat{\chi}) = \frac{1}{2\pi i}F$, and $\delta_2(\widehat{\chi})$ is the integral Euler class.

Proposition 10. [CS2] *Let $B : H^k(M;\mathbb{R}/\mathbb{Z}) \to H^{k+1}(M;\mathbb{Z})$ be the Bockstein homomorphism associated to the exact sequence $0 \to \mathbb{Z} \to \mathbb{R} \to \mathbb{R}/\mathbb{Z} \to 0$.*

(i) $\delta_1 | (\Omega^k(M)/Z^k_{\mathbb{Z}}(M)) = d$.

(ii) $\delta_2 | H^k(M;\mathbb{R}/\mathbb{Z}) = -B$.

Now let B_G be the (infinite-dimensional) classifying space of G, and let $I^*(G)$ be the ring of invariant polynomials on $\mathfrak{g}$ with respect to the Ad representation of G on $\mathfrak{g}$. Then there is a ring homomorphism $w : I^k(G) \to H^{2k}(B_G,\mathbb{R})$, which can be constructed from the usual Chern-Weil homomorphism using the simplicial de Rahm complex on B_G (see [D]). Given $\alpha = (P, M, \theta) \in \varepsilon(G)$, the Chern-Weil homomorphism gives a ring homomorphism $W : I^k(G) \to \Omega^{2k}(M)$, and W is a natural transformation in the sense that for any morphism $\varphi : \alpha \to \widehat{\alpha}$, we have $\varphi^* \widehat{W}(p) = W(p)$, where $\widehat{W} : I^k(G) \to \Omega^{2k}(\widehat{M})$ is the Chern-Weil homomorphism associated to $\widehat{\alpha}$. Explicitly $W(p) = p(F_\theta)$, where $F_\theta \in \Omega^2(P, \mathfrak{g})$ is the curvature of θ, and we view $p(F_\theta)$ as living on M via pullback by arbitrary local sections (this is well-defined by the Ad-invariance of $p \in I^k(G)$). Let $C_{\mathbb{Z}} : H^*(B_G; \mathbb{Z}) \to H^*(M; \mathbb{Z})$ and $C_{\mathbb{R}} : H^*(B_G; \mathbb{R}) \to H^*(M; \mathbb{R})$ be the characteristic class maps associated to $P \to M$, which are defined by $C_{\mathbb{Z}}(u) := f^* u$, where $f : M \to B_G$ is any continuous map satisfying $f^* E_G = P$ as topological principal G-bundles; $C_{\mathbb{R}}$ is defined similarly.

$$K^{2k}(G; \Lambda) := \left\{ (f, u) \in I^k(G) \times H^{2k}(B_G; \mathbb{Z}); w(f) = r(u) \right\},$$

where $r : H^*(B_G; \mathbb{Z}) \to H^*(B_G; \mathbb{R})$ is the map on cohomology induced by inclusion of $\mathbb{Z}$-valued cochains into $\mathbb{R}$-valued cochains.

Theorem 20. *(Theorem 2.2 in [CS2]) For any $\alpha \in \varepsilon(G)$ and $(P, u) \in K^{2k}(G; \Lambda)$, there exists a unique $S_{P,u}(\alpha) \in \widehat{H}^{2k-1}(M; \mathbb{R}/\Lambda)$ satisfying the following:*

(i) $\delta_1(S_{P,u}(\alpha)) = P(\Theta)$,

(ii) $\delta_2(S_{P,u}(\alpha)) = u(\alpha)$,

(iii) if $\beta \in \varepsilon(G)$ and $\phi : \alpha \to \beta$ is a morphism, then $\phi^*(S_{P,u}(\beta)) = S_{P,u}(\alpha)$.

Remark 4. *That is, the following diagram commutes:*

$$\begin{array}{ccc} & & \widehat{H}^*(M, \mathbb{R}/\mathbb{Z}) \\ & \overset{S}{\nearrow} & \downarrow {\scriptstyle (\delta_1, \delta_2)} \\ K^*(G, \mathbb{Z}) & \xrightarrow[(W, C_{\mathbb{Z}})]{} & R^*(M, \mathbb{Z}) \end{array}$$

and the map S is natural. Moreover, S is a ring homomorphism.

Proof Sketch: By Theorem 18, there exists $\beta_N = (P_N, A_N, \theta_N) \in \varepsilon(G)$ which is N-classifying. It is known that then $H^{2k-1}(A_N; \mathbb{R}) = 0$ for $N \in \mathbb{N}$ sufficiently large, so $(\delta_1, \delta_2) : \widehat{H}^{2k-1}(A_N, \mathbb{R}/\mathbb{Z}) \to R^{2k}(A_N)$ is an isomorphism by Theorem 19, and the theorem follows for N-classifying spaces by taking $S_{P,u}(\beta_N) := (\delta_1, \delta_2)^{-1}(P(F_N), C_{\mathbb{Z}}(u))$, where $F_N \in \Omega^2(E_N; \mathfrak{g})$ is the curvature of θ_N. Given arbitrary $\alpha = (P, M, \theta)$, we would like to define

$$S_{P,u}(\alpha) := f^* \beta_N,$$

where β_N is N-classifying, and $f : M \to A_N$ is the smooth map of base spaces induced by a morphism $\alpha \to \beta_N$ (which exists for $N \in \mathbb{N}$ sufficiently large by Theorem 18). To check that this is well-defined, we use property (ii) of Definition 21. □

Corollary 4. *If $\alpha = (E, G, \theta) \in \varepsilon(G)$ and $P(F) = 0$, then $S_{P,u}(\alpha) \in H^{2k-1}(M, \mathbb{R}/\mathbb{Z})$ and $B(S_{P,u}(\alpha)) = -u(\alpha)$.*

Definition 24. *If M has dimension $2n-1$ and P has degree n, then $P(F) = 0$, so we may evaluate $S_{P,u}(\alpha) \in H^{2n-1}(M, \mathbb{R}/\mathbb{Z})$ on the fundamental class of M to obtain*

$$S_{P,u}(\alpha)[M] \in \mathbb{R}/\mathbb{Z},$$

which we call the Chern-Simons invariant associated to P, u, α.

Remark 5. *If in addition $M = \partial \overline{M}^{2n}$ and E is the restriction of a principal G-bundle $\overline{E} \to \overline{M}$, and if $\overline{\theta}$ is any extension of θ to a connection on $\overline{E}$, then we have a morphism from α to $\overline{\alpha} := (\overline{E}, \overline{M}, \overline{\theta})$ whose map on base spaces is the inclusion $M \hookrightarrow \overline{M}$, hence $S_{P,u}(\overline{\alpha})|M^{2n-1} = S_{P,u}(\alpha)$. Because $\delta_1(S_{P,u}(\overline{\alpha})) = P(\overline{F})$, it follows that*

$$S_{P,u}(\alpha)[M^{2n-1}] = \int_{\overline{M}} \wedge P(\overline{F}).$$

To compute Chern-Simons invariants when M is not a boundary, we consider the transgression forms studied in [CS3].

Definition 25. *Given $\alpha = (E, M, \theta)$ and $P \in I^k(G)$, the transgression form $TP(\theta)$ is*

$$TP(\theta) := k \int_0^1 P(\theta \wedge F_t^{k-1}) dt \in \Omega^{2k-1}(E)$$

where $F_t := tF + \frac{1}{2}(t^2 - t)[\theta, \theta]$ and F is the curvature of θ. We also consider the relative transgression form $TP(\theta, \theta')$ associated to a pair of connections on $E \to M$, given by

$$TP(\theta, \theta') = k \int_0^1 P\left((\theta - \theta') \wedge F_t(\theta, \theta')^{k-1}\right) dt \in \Omega^{2k-1}(E),$$

where $F_t(\theta, \theta') = d\theta_t + \frac{1}{2}[\theta_t, \theta_t]$ is the curvature of $\theta_t := (1 - t)\theta + t\theta' \in \Omega^1(E; \mathfrak{g})$.

Remark 6. *An elementary but tedious computation shows that $TP(\theta)$ can equivalently be written as*

$$TP(\theta) = \sum_{j=0}^{k-1} (-1)^j \frac{(k-1)!k!}{2^j (k-j-1)!(j+k)!} P(\theta \wedge [\theta, \theta]^j \wedge F^{k-1-j}).$$

The forms $TP(\theta)$ are useful because they are relatively easy to compute, and because of their relation to the Chern-Weil homomorphism.

Lemma 4. (i) $d(TP(\theta)) = P(F)$*.*

(ii) $d(TP(\theta, \theta')) = P(F) - P(F')$*,*

(iii) *If $\varphi : (E, M, \theta) \to (\overline{E}, \overline{M}, \overline{\theta})$, then $\varphi^* TP(\overline{\theta}) = TP(\theta)$.*

Proof. Direct computation, using the infinitesimal invariance of P under the ad representation of $\mathfrak{g}$. □

Proposition 11. [CS2] *Suppose $\alpha = (E, M, \theta) \in \varepsilon(G)$, and let $\pi : E \to M$ be the projection map. Then*

$$\pi^* S_{P,u}(\alpha) = \widehat{TP}(\theta)|Z_{2k-1}(E).$$

If in addition $\dim(M) = 2k - 1$ and $E \to M$ is trivial, with $s : M \to E$ a global section, then

$$S_{P,u}(\alpha)[M] = \int_M s^* \widehat{TP}(\theta).$$

Proof. First consider the case where $\alpha_N = (E_N, A_N, \theta_N)$ is N-classifying for large $N \in \mathbb{N}$, and let $\pi_N : E_N \to A_N$ be the bundle projection map. Because $\pi_N^* E_N$ is trivial, we know that $\delta_2(\pi_N^* S_{P,u}(\alpha_N)) = 0$, so that $\pi_N^* S_{P,u}(\alpha_N) \in \Omega^k(E_N)/\Omega^k_{\mathbb{Z}}(E_N)$; choosing a representative $\beta \in \Omega^{2k-1}(E_N)/Z^{2k-1}_{\mathbb{Z}}(E_N)$, we have $\widetilde{\beta}|Z_{2k-1}(E_N) = \pi_N^* S_{P,u}(\alpha_N)$

$$d(TP(\theta_N)) = P(F_N) = \delta_1(\pi^* S_{P,u}(\alpha)) = d\beta.$$

Because $H^{2k-1}(E_N, \mathbb{R}) = 0$ for sufficiently large $N \in \mathbb{N}$, we know $TP(\theta_N) - \beta = d\gamma$ for some $\gamma \in \Omega^{2k-2}(E_N)$.

In general, suppose $\varphi : \alpha \to \alpha_N$ is a morphism, and observe that

$$d(\varphi^* \gamma) = \varphi^* TP(\theta_N) - \varphi^* \beta = TP(\theta) - \varphi^* \beta,$$

where

$$\widetilde{\varphi^* \beta}|Z_{2k-1}(E) = \varphi^* (\pi_N^* S_{P,u}(\alpha_N)) = \pi^* (f^* S_{P,u}(\alpha_N)) = \pi^* (S_{P,u}(\alpha)),$$

and $f : M \to A_N$ is the map of base spaces corresponding to φ. Because $d(\varphi^* \gamma)$ vanishes on $Z_{2k-2}(E)$, we conclude that

$$\widehat{TP}(\theta)|Z_{2k-1}(E) = \widetilde{\varphi^* \beta}|Z_{2k-1}(E) = \pi^* (S_{P,u}(\alpha)).$$

If $\dim(M) = 2k-1$, then $S_{P,u}(\alpha) \in H^{2k-1}(M, \mathbb{R}/\mathbb{Z})$, so we can evaluate both sides on $s(M)$ to obtain

$$\int_M s^* \widehat{TP}(\theta) = \int_{s(M)} \widehat{TP}(\theta) = \pi^* (S_{P,u}(\alpha))[s(M)] = S_{P,u}(\alpha)[M].$$

□

Example 2. *Consider the special case where* $\dim(M) = 3$, $G = SO(m, \mathbb{R})$, *and* $P(X) = \frac{1}{8\pi^2} tr(X^2) \in I^2(SO(m, \mathbb{R}))$, *which corresponds via the Chern-Weil homomorphism to the* *rst Pontryagin class of E. Given $\alpha = (E, M, \theta)$, we have*

$$
\begin{aligned}
TP(\theta) =& \sum_{j=0}^{1} (-1)^j \frac{2}{2^j(1-j)!(j+2)!} P(\theta \wedge [\theta,\theta]^j \wedge F^{1-j}) \\
=& P(\theta \wedge F) - \frac{1}{6} P(\theta \wedge [\theta,\theta]) \\
=& -\frac{1}{8\pi^2}\left(tr(\theta \wedge d\theta) + \frac{2}{3} tr(\theta \wedge \theta \wedge \theta) \right).
\end{aligned}
$$

If $V \to M$ is a vector bundle of rank m, and we choose a bundle metric, along with a linear metric connection ∇, then this connection lifts to a connection θ on the orthonormal frame bundle E of V. If V is a trivial bundle, and with respect to a given trivialization we have $\nabla = d + A$ for some $A \in \Omega^1(M, \mathfrak{o}(m, \mathbb{R}))$, we can de ne the Chern-Simons invariant $CS(V, M, \nabla)$ of (V, M, ∇) to be the Chern-Simons invariant of (E, M, θ), which is

$$
CS(V, M, \nabla) = -\frac{1}{8\pi^2} \int_M tr\left(A \wedge dA + \frac{2}{3} A \wedge A \wedge A \right).
$$

Suppose $\theta_0, \theta_1 \in \Omega^1(E; \mathfrak{g})$ are principal connections, and set $\alpha_i := (E, M, \theta_i)$. Because

$$
\delta_2 \left(S_{P,u}(\alpha_1) - S_{P,u}(\alpha_2) \right) = u(\alpha_1) - u(\alpha_2) = 0,
$$

our knowledge of $\ker(\delta_2)$ implies that there exists $\beta \in \Omega^{2k-1}(M)$ such that

$$
S_{P,u}(\alpha_1) - S_{P,u}(\alpha_2) = \widetilde{\beta}|Z_{2k-1}(M).
$$

Let $(\theta_t)_{t \in [0,1]}$ be a smooth curve of connections joining θ_0, θ_1, let $\Theta_t := d\theta_t + \frac{1}{2}[\theta_t, \theta_t] \in \Omega^2(E; \mathfrak{g})$ be the corresponding curvature, and set $\dot{\theta}_t := \frac{d}{dt}\theta_t \in \Omega^1(E; \mathfrak{g})$.

Proposition 12. [CS2] *$S_{P,u}(\alpha_1) - S_{P,u}(\alpha_2) = k\int_0^1 P(\dot{\theta}_t \wedge F_t^{k-1}) dt|Z_{2k-1}(M)$.*

Proof. By Proposition 11, we have

$$
\pi^* \left(S_{P,u}(\alpha_1) - S_{P,u}(\alpha_2) \right) = \left(TP(\theta_1) \wedge TP(\theta_2) \right) |Z_{2k-1}(E).
$$

On the other hand, we know that

$$
d\left(TP(\theta_1) - TP(\theta_0) \right) = d\left(k \int_0^1 P(\dot{\theta}_t \wedge F_t^{k-1}) dt \right),
$$

so by arguing on an N-classifying bundle and then pulling back as in Proposition 11, we have

$$k\int_0^1 P(\dot{\theta}_t \wedge F_t^{k-1})dt|Z_{2k-1}(E) = \pi^*(S_{P,u}(\alpha_1) - S_{P,u}(\alpha_2)).$$

Because $\dot{\theta}_t, F_t$ are both horizontal forms, $P(\dot{\theta}_t \wedge F_t^{k-1})$ is as well. Because P is $\mathrm{Ad}(G)$-invariant, we moreover have that $s^* P(\dot{\theta}_t \wedge F_t^{k-1})$ is independent of the choice of local section $s : M \supset U \to P$. For any $\tau \in Z_{2k-1}(M)$, with $\mathrm{im}(\tau) \subset U$ for some open subset $U \subset M$ with $E|U$ trivial, we can thus find a local section $s : U \to P$, and $s \circ \tau \in Z_{2k-1}(E)$, so that

$$\begin{aligned}(S_{P,u}(\alpha_1) - S_{P,u}(\alpha_2))(\tau) =& \pi^*(S_{P,u}(\alpha_1) - S_{P,u}(\alpha_2))(s\circ\tau) = \left(k\int_0^1 P(\dot{\theta}_t \wedge \Theta_t^{k-1})dt + d\gamma\right)(s \circ \tau)\\ =& \int_\tau k\left(\int_0^1 P(\dot{\theta}_t \wedge \Theta_t^{k-1})dt\right).\end{aligned}$$

For general $\tau \in Z_{2k-1}(M)$ we can (by barycentric subdivision) find $\sigma \in Z_{2k}(M)$ such that $\tau - \partial\sigma$ is a finite sum of cycles each contained in some set $U \subset M$ with $E|U$ trivial. Applying the above identity and summing then gives

$$\begin{aligned}(S_{P,u}(\alpha_1) - S_{P,u}(\alpha_2))(\tau) - \int_\tau k\left(\int_0^1 P(\dot{\theta}_t \wedge F_t^{k-1})dt\right)&\\ =(S_{P,u}(\alpha_1) - S_{P,u}(\alpha_2))(\partial\sigma) - \int_{\partial\sigma} k\left(\int_0^1 P(\dot{\theta}_t \wedge F_t^{k-1})dt\right)&\\ =\int_\sigma (\delta_1(S_{P,u}(\alpha_1)) - \delta_1(S_{P,u}(\alpha_2))) - \int_\sigma (P(F_1) - P(F_0))&\\ =0.&\end{aligned}$$

□

Example 3. *Returning to the example where $P(A) = -\frac{1}{8\pi^2}tr(A^2)$, if we are given metric connections $\nabla, \overline{\nabla}$ on $V \to M$ with $\nabla = \overline{\nabla} + A$, we can compute (even when $V \to M$ is not* *a trivial bundle)*

$$\begin{aligned} 8\pi^2 \int_0^1 P(\dot{\theta}_t \wedge F_t^{k-1})dt &= \int_0^1 \mathrm{tr}(A \wedge F_{\overline{\nabla}+tA})dt \\ &= \int_0^1 \mathrm{tr}(A \wedge F_{\overline{\nabla}} + tA \wedge d_{\overline{\nabla}}A + t^2 A \wedge A \wedge A)dt \\ &= \mathrm{tr}(A \wedge F_{\overline{\nabla}}) + \frac{1}{2}\mathrm{tr}(A \wedge d_{\overline{\nabla}}A) + \frac{1}{3}\mathrm{tr}(A \wedge A \wedge A), \end{aligned}$$

where $d_{\overline{\nabla}}$ is the exterior derivative extended to act on $\Omega^2(M, \mathrm{Ad}(E))$ using the connection $\overline{\nabla}$. Thus

$$CS(V, \nabla, M) - CS(V, \overline{\nabla}, M) = -\frac{1}{8\pi^2}\widetilde{\int_M} \left(2\mathrm{tr}(A \wedge F_{\overline{\nabla}}) + A \wedge d_{\overline{\nabla}}A + \frac{2}{3}\mathrm{tr}(A \wedge A \wedge A)\right).$$

We now expand on the particular setting which will be useful to us in Chapter 3.

Corollary 5. *Define $A_j := B(0^4, 2^{-j+1}) \setminus B(0^4, 2^{-j-1}) \subset \mathbb{R}^4$, $S_j := \partial B(0^4, 2^{-j}) = \mathbb{S}^3$, and suppose g is a Riemannian metric on $B(0^4, 2) \setminus \{0^4\}$ such that*

$$\lim_{j \to \infty} (2^{-j})^{-2-k} \|\overline{\nabla}^k (g - \overline{g})\|_{C^0(A_j, \overline{g})} = 0$$

for all $k \in \mathbb{N}$, where $\overline{g}$ is the standard Euclidean metric on A_0, and $\overline{\nabla}$ is the standard flat connection. If $\Lambda^2_+(M, g)$ denotes the bundle of self-dual 2-forms on a Riemannian 4-manifold (M, g), then we have

$$\lim_{j \to \infty} CS(\Lambda^2_+(A_j, g)|S_j, \nabla^g, S_j) = CS(\Lambda^2_+(A_0)|S_0, \nabla^{\overline{g}}, S_0) = 0 \in \mathbb{R}/\mathbb{Z},$$

where ∇^g denotes the connection on $\Lambda^2_+(A_j, g)$ induced by the Levi-Civita connection of g, and $\Lambda^2_+(A_j, g)|S_j$ is the restricted bundle equipped with the pullback connection by the inclusion $S_j \hookrightarrow A_j$.

Proof. Letting $\lambda_j(x) := 2^{-j}x$ be the dilation map, and letting (x^i) be the standard coordinates on $B(0^n, 1) \setminus \{0^n\}$, we have

$$|(2^{-j})^{-2}\lambda_j^* g - \overline{g}|_{\overline{g}}(x) = (2^{-j})^{-2}|\lambda_j^*(g - \overline{g})|_{\overline{g}}(x) = |g - \overline{g}|_{\overline{g}}(2^{-j}x),$$

hence

$$\|g-\bar{g}\|_{C^0(A_j,\bar{g})} = \|(2^{-j})^{-2}\lambda_j^* g-\bar{g}\|_{C^0(A_0,\bar{g})},$$

and similarly for higher derivatives, so that $g_j := 2^{2j}\lambda_j^* g \to \bar{g}$ in $C^\infty_{\mathrm{loc}}(A_0)$. We also have $\lambda_j^*\Lambda^2_+(A_j,g) = \Lambda^2_+(A_0,g_j)$, and the pullback connection $\lambda_j^*\nabla^g$ corresponds to ∇^{g_j} under this bundle isomorphism. By the naturality of Chern-Simons invariants, we have

$$\mathrm{CS}(\Lambda^2_+(A_j,g)|S_j,\nabla^g,S_j) = \mathrm{CS}(\Lambda^2_+(A_0,g_j)|S_0,\nabla^{g_j},S_0).$$

Because $g_j \to g$ in $C^\infty_{loc}(A_0)$, it follows that the composition (where the second map is the projection $\alpha \mapsto \frac{1}{2}(\alpha + *_{\bar{g}}\alpha)$)

$$\Lambda^2_+(A_0,g_j) \hookrightarrow \Lambda^2T^*A_0 \to \Lambda^2_+(A_0,\bar{g})$$

restricted to S_0 is an isomorphism for large $j \in \mathbb{N}$. Regarding this isomorphsm as a bundle endomorphism T_j of $\Lambda^2T^*A_0|S_0$ which is zero on $\Lambda^2_-(A_0,g_j)$, we see that $T_j \to 0$ in $C^\infty(S_0,\mathrm{End}(\Lambda^2T^*A_0))$, hence under this identification, $B_j := \nabla^{\lambda_j^* g} - \nabla^{\bar{g}} \in \Omega^1(S_0,\mathrm{Ad}(\Lambda^2_+(A_0)))$ converges to 0 in $C^\infty_{loc}(S_0)$. Because $\nabla^{\bar{g}}$ is flat, Proposition 12 gives

$$\mathrm{CS}(\Lambda^2_+(A_j)|S_j,\nabla^g,S_j) - \mathrm{CS}(\Lambda^2_+(A_0)|S_0,\nabla^{\bar{g}},S_0) = \frac{1}{8\pi^2}\int_M \left(B_j \wedge d_{\nabla^{\bar{g}}}B_j + \frac{2}{3}\mathrm{tr}(B_j \wedge B_j \wedge B_j)\right) \to 0$$

in $\mathbb{R}/\mathbb{Z}$ as $j \to \infty$. Observing that $\Lambda^2_+(A_0,\bar{g})$ has a canonical parallel orthonormal frame with connection 1-form equal to zero, Proposition 11 then gives

$$\mathrm{CS}(\Lambda^2_+(A_0)|S_0,\nabla^{\bar{g}},S_0) = 0.$$

□

CHAPTER 3

TYPE-I SCALAR CURVATURE

3.1 Statement of Results

This chapter is concerned with the finite-time singularities of solutions $(M^n, (g_t)_{t\in[0,T)})$ to the Ricci flow

$$\frac{\partial}{\partial t} g_t = -2Rc(g_t)$$

on a closed manifold which satisfy the Type-I scalar curvature condition

$$\limsup_{t\to T} \max_M R(\cdot, t)(T-t) < \infty, \tag{3.1.1}$$

where $T < \infty$ is the maximal existence time. In particular, we generalize some of the theory of Ricci flow solutions which satisfy the more stringent Type-I curvature assumption

$$\limsup_{t\to T} \max_M |Rm|(\cdot, t)(T-t) < \infty. \tag{3.1.2}$$

Ricci flow solutions satisfying (3.1.2) have been studied in [EMT, MM, N1], where it was shown that, for any fixed $q \in M$ and sequence of times $t_i \nearrow T$, a subsequence of $(M^n, (T-t_i)^{-1} g_{t_i}, q)$ converges in the pointed Cheeger-Gromov sense to a complete Riemannian manifold $(M_\infty, g_\infty, q_\infty)$ equipped with a function $f_\infty \in C^\infty(M_\infty)$ which satisfies the shrinking gradient Ricci soliton (GRS) equation

$$Rc_{g_\infty} + \nabla^2 f_\infty = \frac{1}{2} g_\infty.$$

While it is unknown whether this limiting soliton is uniquely determined by the base-point q, in [MM] it is shown that all such solitons share a numerical invariant, called the shrinker entropy $\mathcal{W}(g_\infty, f_\infty)$, (see Section 4), which is determined by q. They also show that $\mathcal{W}(g_\infty, f_\infty) = 0$ if and only if $(M_\infty, g_\infty, f_\infty)$ is the Gaussian shrinking soliton on flat Euclidean space.

While interesting questions about solutions satisfying (3.1.2) condition remain open, this condition is often too restrictive for useful applications of Ricci flow to geometry and topology. Condition (3.1.1), on the other hand, is satisfied by Kähler-Ricci flow on a Fano manifold with initial metric in the canonical Kähler class by the work of Perelman (see [ŠT]), and is conjectured to be satisfied for a much larger class of Kähler-Ricci flow solutions (Conjecture 7.7 of [SW]). One of the main technical difficulties that arises when studying Ricci flows satisfying (3.1.1) is that one cannot expect Type-I blowups to result in a smooth limiting space. In fact, most results about Ricci flows satisfying (3.1.2), including [CZ1, EMT, N1, MM], depend crucially on applying the Cheeger-Gromov compactness theorems to rescaled solutions. However, in [B2],[B1], Bamler develops an extensive theory for taking weak limits of Ricci flows with uniformly bounded scalar curvature, modeled on the Cheeger-Colding-Naber-Tian theory of noncollapsed Riemannian manifolds with bounded Ricci curvature. In particular, Theorem 1.2 of [B1] shows that any Ricci flow satisfying (3.1.1) has a dilation limit which is a singular space (see section 2), and which possesses the structure of a smooth but incomplete shrinking Ricci soliton outside of a subset of Minkowski codimension 4.

The main goal of this chapter is to extend Bamler's analysis of the singular limits of dilated Ricci flows satisfying (3.1.1), and to relate some of their properties to the original Ricci flow. The first main theorem generalizes the aforementioned results in [MM].

In order to state this theorem, we first recall a result in [B1]. Assume $(M^n, (g_t)_{t\in[0,T)})$ is a closed, pointed solution of Ricci flow satisfying (3.1.1), and fix any sequence $t_i \nearrow T$. According to Theorem 1.2 of [B1], we can pass to a subsequence to get pointed Gromov-Hausdorff convergence of $(M^n, (T-t_i)^{-1}g_{t_i}, q)$ to a pointed singular space $(\mathcal{X}, q_\infty) = (X, d, \mathcal{R}, g_\infty, q_\infty)$. Moreover, there exists $f_\infty \in C^\infty(\mathcal{R})$ obtained as a limit of rescalings of a conjugate heat kernel based at the singular time, which satisfies the Ricci soliton equation on $\mathcal{R}$. The Ricci soliton $(\mathcal{R}, g_\infty, f_\infty)$ has a well-defined entropy $\mathcal{W}(g_\infty, f_\infty)$, defined in Section 4, and there is a heat kernel density function (defined in Section 3) $\Theta(q) \in (-\infty, 0]$ associated to the basepoint q.

Theorem 21. $\Theta(q) = \mathcal{W}(g_\infty, f_\infty)$, *with* $\Theta(q) = 0$ *if and only if* $(\mathcal{R}, g_\infty, f_\infty)$ *is the Gaussian shrinker on flat* $\mathbb{R}^n$, *in which case there is a neighborhood* U *of* q *in* M *such that* $\sup_{U \times [-2,0)} |Rm| < \infty$.

In particular, all singular shrinking GRS which arise as Type-I dilation limits at a fixed point in M possess the same shrinker entropy. We recall the definition of the singular set of $(M, (g_t)_{t \in [0,T)})$, defined in [EMT] as

$$\Sigma := \left\{ x \in M; \sup_{U \times [0,T)} |Rm| = \infty \text{ for every neighborhood } U \text{ of } q \text{ in } M \right\}.$$

In the general Riemannian case, little is known about the regularity or structure of Σ. In the case where (M, g_0) is Kähler, it is known that Σ is actually an analytic subvariety of M, even without the Type-I assumption (see [CT]). With a Type-I curvature assumption, it was shown in [MM] that Σ is characterized by the density function: $\Sigma = \Theta^{-1}(0)$. We are able to generalize this result to the case of Type-I scalar curvature bounds.

Corollary 6. *Suppose* $(M^n, (g_t)_{t \in [0,T)}, q)$ *is a closed, pointed solution of Ricci flow satisfying* (3.1.1). *Then* $\Sigma = \Theta^{-1}(0)$.

Finally, in dimension 4, we extend Bamler's results on the structure of singular shrinking GRS by giving a more precise description of the singular part of the shrinking soliton. We let $(X, d, \mathcal{R}, g_\infty)$ be the singular space of Theorem 21 and the discussion preceding it.

Theorem 22. *If* $n = 4$, *then* $X \setminus \mathcal{R}$ *consists of finitely many points, and* X *has the structure of a* C^∞ *Riemannian orbifold.*

In particular, if $\mathcal{X} = (X, d, \mathcal{R}, g)$ is the singular space in Theorem 22, then there exists $f \in C^\infty(\mathcal{R})$ such that $(\mathcal{R}, g, f)$ is an incomplete but smooth shrinking GRS, and each $x \in X \setminus \mathcal{R}$ admits a finite group $\Gamma_x \subset \mathbb{R}^4$ acting linearly and freely away from the origin, along with a homeomorphism $\varphi_x : \mathbb{R}^4/\Gamma_x \supseteq B(0^4, r_0) \to B^X(x, r_0)$ such that, if $\pi_x : \mathbb{R}^4 \to \mathbb{R}^4/\Gamma_x$ is the quotient map, then $\varphi_x \circ \pi_x$ is a smooth map on $B(0^4, r_0) \setminus \{0\}$, and $(\varphi_x \circ \pi_x)^* g$, $(\varphi_x \circ \pi_x)^* f$ extend smoothly to a Riemannian metric and function on $B(0^4, r_0)$.

Theorem 22 was proved in the setting of Fano Kähler-Ricci flow in [CW1], where it was essential that the L^2 norm of the curvature tensor is uniformly bounded along the flow. This fails in the general Riemannian setting (even if we assume (3.1.2)), so our proof must rely on different arguments.

In Section 3.2, we collect definitions and results related to Ricci flows satisfying certain scalar curvature bounds. In Section 3.3, we establish Gaussian-type estimates for conjugate heat kernels based at the singular time, largely along the lines of Bamler and Zhang's heat kernel estimates. In Section 3.4, we define shrinker entropy, and prove an important integration-by-parts lemma for singular shrinking GRS. In Section 3.5, we prove the convergence of entropy and the heat kernel measure. In Section 3.6, we show that the shrinker entropy of a normalized singular GRS only depends on the underlying manifold, and use this to complete the proof of Theorem 21 and Corollary 6. Finally, in Section 3.7, we specialize to the case of dimension 4, and prove Theorem 22.

3.2 Gaussian Estimates for Conjugate Heat Kernels Based at the Singular Time

The following lemma is mostly a combination of the proofs of Theorem 1.2 in [B1] and Theorem 1.4 in [BZ2].

Lemma 5. *For any $A < \infty$, there exists $C = C(A, n) < \infty$ such that the following holds. Let $(M^n, (g_t)_{t \in [-2,0)})$ is a closed solution of Ricci flow satisfying $\nu[g_{-2}, 4] \geq -A$ and*

$|R|(x,t) \le A|t|^{-1}$ *for all* $t \in [-2,0)$. *Then, for any* $x,y \in M$ *and* $-\frac{1}{2} \le s < t < 0$, *we have*

$$\frac{1}{C(t-s)^{\frac{n}{2}}}\exp\left(-\frac{C\, d_s^2(x,y)}{t-s}\right) \le K(x,t;y,s) \le \frac{C}{(t-s)^{\frac{n}{2}}}\exp\left(-\frac{d_s^2(x,y)}{C(t-s)}\right).$$

Proof. First note the reduced distance bound

$$\ell_{(x,t)}(x,s) \le \frac{1}{2\sqrt{t-s}}\int_0^{t-s}\sqrt{\tau}\frac{A}{|t|+\tau}d\tau \le \frac{1}{2\sqrt{t-s}}\int_0^{t-s}\frac{A}{\sqrt{\tau}}d\tau = A,$$

so by Perelman's differential Harnack inequality, $K(x,t;x,s) \ge (4\pi(t-s))^{-\frac{n}{2}}e^{-A}$ for all $x \in M$ and $-2 \le s < t < 0$.

Claim: There exists $C' = C'(A,n) < \infty$ such that, for $-\frac{1}{2} \le s < 0$ and $t \in (s, \frac{1}{2}s]$, $x,y \in M$, we have

$$\frac{1}{C'(t-s)^{\frac{n}{2}}}\exp\left(-\frac{C' d_\tau^2(x,y)}{t-s}\right) \le K(x,t;y,s) \le \frac{C'}{(t-s)^{\frac{n}{2}}}\exp\left(-\frac{d_\tau^2(x,y)}{C'(t-s)}\right), \tag{3.2.1}$$

where $\tau \in \{s,t\}$.

This will just follow from an appropriate rescaling and the corresponding Gaussian bounds for Ricci flow with bounded scalar curvature. In fact, consider the rescaled flow $\widetilde{g}_r := |t|^{-1}g_{t+|t|r}$, $r \in [-|t|^{-1}(2+t),0]$. Then $|\widetilde{R}| \le A$ on $M \times [-2,0]$, and

$$\nu[\widetilde{g}_{-2},4] = \nu[g_{3t},4|t|] \ge \nu[g_{-2},4|t|+(2-3|t|)] \ge \nu[g_{-2},2+|t|] \ge -A,$$

so the Bamler-Zhang heat kernel estimates [BZ2] give $C' = C'(A,n) < \infty$ such that for all $x,y \in M$ and $r \in [-1,0)$ we have

$$\frac{1}{C'|r|^{\frac{n}{2}}}\exp\left(-\frac{C'\widetilde{d}_\tau^2(x,y)}{|r|}\right) \le \widetilde{K}(x,0;y,r) \le \frac{C'}{|r|^{\frac{n}{2}}}\exp\left(-\frac{\widetilde{d}_\tau^2(x,y)}{C'|r|}\right),$$

where $\tau \in \{0,r\}$. Also, we know the behavior of the heat kernel under rescaling: $\widetilde{K}(x,0;y,r) = |t|^{\frac{n}{2}}K(x,t;y,t+|t|r)$, so taking $r := -|t|^{-1}(t-s)$ gives $K(x,t;y,s) = |t|^{-\frac{n}{2}}\widetilde{K}(x,0;y,-|t|^{-1}(t-s))$. Note that $|t|^{-1}(t-s) \le (|s|/2)^{-1}(s/2-s) = 1$ because $t \in (s,s/2]$, so the claim follows.

Now consider the case where $s/2 < t < 0$. A special case of the reproduction formula for the heat kernel is

$$K(x,t;y,s) = \int_M K(x,t;z,\frac{1}{2}(t+s))K(z,\frac{1}{2}(t+s);y,s)dg_{\frac{1}{2}(t+s)}(z)$$

for all $x, y \in M$ and $-2 \le s < 0$ and $t \in (s/2, 0)$. Also, the above claim implies

$$K(z,\frac{1}{2}(t+s);y,s) \le \frac{2^{\frac{n}{2}}C'}{(t-s)^{\frac{n}{2}}},$$

so combining this with the reproduction formula gives

$$K(x,t;y,s) \le \frac{2^{\frac{n}{2}}C'}{(t-s)^{\frac{n}{2}}}\int_M K(x,t;z,\frac{1}{2}(t+s))dg_{\frac{1}{2}(t+s)}(z) = \frac{2^{\frac{n}{2}}C'}{(t-s)^{\frac{n}{2}}}.$$

Set $c_0 := (4\pi)^{-\frac{n}{2}}e^{-A}$, so that the above claim gives $D = D(A,n) < \infty$ such that, for any $x, y \in M$ with $d_{\frac{1}{2}(t+s)}(x,y) \le D\sqrt{t-s}$, we have

$$K(x,\frac{1}{2}(t+s);y,s) \ge \frac{c_0}{2(t-s)^{\frac{n}{2}}}.$$

Thus, for any $x \in M$, we have

$$\begin{aligned}\frac{c_0}{(t-s)^{\frac{n}{2}}} \le K(x,t;x,s) &= \int_M K(x,t;y,\frac{1}{2}(t+s))K(y,\frac{1}{2}(t+s);x,s)dg_{\frac{1}{2}(t+s)}(y) \\ &\le \frac{2^{\frac{n}{2}}C'}{(t-s)^{\frac{n}{2}}}\int_{B(x,\frac{1}{2}(t+s),D\sqrt{t-s})} K(x,t;y,\frac{1}{2}(t+s))dg_{\frac{1}{2}(t+s)}(y) + \frac{c_0}{2(t-s)^{\frac{n}{2}}},\end{aligned}$$

so applying the upper bound of (3.2.1) with $\tau = \frac{1}{2}(t+s)$ gives

$$\int_{B(x,\frac{1}{2}(t+s),D\sqrt{t-s})} K(x,t;y,\frac{1}{2}(t+s))dg_{\frac{1}{2}(t+s)}(y) \ge c_0 2^{-\frac{(n+2)}{2}}(C')^{-1} =: c_0'. \tag{3.2.2}$$

Now consider the rescaled flow $\widehat{g}_r := (t-s)^{-1}g_{(t-s)r+\frac{1}{2}(s+t)}$, which satisfies

$$|\widehat{R}| \le \frac{2A}{|s+t|}(t-s) \le 2A$$

for $r \in [-2, 0]$, and

$$\nu[\widehat{g}_{-2},4] = \nu[g_{\frac{1}{2}(s+t)-2(t-s)},4(t-s)] \ge \nu[g_{-2},4(t-s)+\frac{1}{2}(s+t)-2(t-s)+2] \ge \nu[g_{-2},4]$$

since $-\frac{1}{2} \le s \le t < 0$. We can thus apply Theorem 14 to obtain $B = B(A,n) < \infty$ such that

$$\frac{1}{B}\widehat{d}_{r_1}(x_1,x_2) - B\sqrt{|r_2 - r_1|} \le \widehat{d}_{r_2}(x_1,x_2) \le B\widehat{d}_{r_2}(x_1,x_2) + B\sqrt{|r_2 - r_1|}$$

for all $r_1, r_2 \in [-\frac{1}{2}, 0]$ and $x_1, x_2 \in M$. In terms of the unrescaled flow, this gives

$$\frac{1}{B}d_{t_1}(x_1,x_2) - B\sqrt{|t_2 - t_1|} \le d_{t_2}(x_1,x_2) \le Bd_{t_1}(x_1,x_2) + B\sqrt{|t_2 - t_1|} \tag{3.2.3}$$

for all $t_1, t_2 \in [s, \frac{1}{2}(s+t)]$ and $x_1, x_2 \in M$. For any $x, y \in M$, we combine (3.2.1),(3.2.2),(3.2.3) to obtain

$$\begin{aligned}
K(x,t;y,s) &\ge \int_{B(x,\frac{1}{2}(t+s),D\sqrt{t-s})} K(x,t;z,\tfrac{1}{2}(t+s))K(z,\tfrac{1}{2}(t+s);y,s)\,dg_{\frac{1}{2}(t+s)}(z) \\
&\ge \int_{B(x,\frac{1}{2}(t+s),D\sqrt{t-s})} K(x,t;z,\tfrac{1}{2}(t+s)) \\
&\qquad \cdot \frac{1}{C'(t-s)^{\frac{n}{2}}}\exp\left(-\frac{2C'}{t-s}(d_{\frac{1}{2}(s+t)}(x,y) + D\sqrt{t-s})^2\right) dg_{\frac{1}{2}(t+s)}(z) \\
&\ge \frac{c_0'}{C'(t-s)^{\frac{n}{2}}}e^{-4C'D^2}\exp\left(-\frac{4C'}{t-s}\left(Bd_s(x,y) + B\sqrt{|t-s|}\right)^2\right) \\
&\ge \frac{c_0'}{C'(t-s)^{\frac{n}{2}}}e^{-4C'D^2-8C'B^2}\exp\left(-\frac{8C'B^2d_s^2(x,y)}{t-s}\right)
\end{aligned}$$

This and (3.2.1) give $C\ (A,n) < 0$ such that, for all $x, y \in M$ and $-\frac{1}{2} \le s < t < 0$, we have

$$K(x,t;y,s) \ge \frac{1}{C(t-s)^{\frac{n}{2}}}\exp\left(-\frac{C d_s^2(x,y)}{t-s}\right).$$

In particular, for any $r \in [s, \frac{1}{2}(s+t)]$, we have

$$\int_{B(x,r,\sqrt{t-r})} K(x,t;y,r)\,dg_r(y) \ge \frac{e^{-C}}{C(t-s)^{\frac{n}{2}}}|B(x,r,\sqrt{t-r})|_r.$$

Applying Theorem 8 to the rescaled flow gives

$$|\widehat{B}(y,r,\tfrac{1}{\sqrt{2}})|_{\widetilde{g}_r} \ge b \quad \text{for all} \quad r \in [-\frac{1}{2}, 0],$$

where $b = b(A,n) > 0$. Thus, for any $r \in [s, \frac{1}{2}(s+t)]$, we have

$$\begin{aligned}
|B(x,r,\sqrt{t-r})|_{g_r} &\ge |B(x,r,\tfrac{1}{\sqrt{2}}\sqrt{t-s})|_{g_r} = (t-s)^{\frac{n}{2}}|\widehat{B}(y,\tau,\tfrac{1}{\sqrt{2}})|_{\widetilde{g}_r} \\
&\ge b(t-s)^{\frac{n}{2}},
\end{aligned}$$

where τ is defined by $\frac{1}{2}(s+t)+(t-s)\tau = r$, so that $\tau \in [-\frac{1}{2}, 0]$. Combining estimates gives

$$\int_{B(x,r,\sqrt{t-r})} K(x,t;y,r)dg_r(y) \geq b(C)^{-1}e^{-C} =: c \tag{3.2.4}$$

for all $r \in [s, \frac{1}{2}(s+t)]$.

Case 1: For any $y \in M$ with $d_s(x,y) \geq 4B^2\sqrt{t-s}$, the distortion estimate (3.2.3) gives

$$d_r(x,y) \geq \frac{1}{B}d_s(x,y) - B\sqrt{r-s} \geq \frac{1}{2B}d_s(x,y)$$

for all $r \in [s, \frac{1}{2}(s+t)]$. Then the Hein-Naber concentration inequality (Theorem 1.30 of [HN]) gives

$$\begin{aligned}\left(\int_{B(x,r,\sqrt{t-r})} K(x,t;z,r)dg_r(z)\right)&\left(\int_{B(y,r,\sqrt{t-r})} K(x,t;z,r)dg_r(z)\right) \\ &\leq \exp\left(-\frac{(d_r(x,y)-2\sqrt{t-r})^2}{8(t-r)}\right) \\ &\leq \exp\left(-\frac{1}{8(t-r)}\left(\frac{1}{B}d_s(x,y)-2\sqrt{t-s}\right)^2\right) \\ &\leq \exp\left(-\frac{d_s^2(x,y)}{32B^2(t-s)}\right).\end{aligned}$$

Combining this with 3.2.4 gives

$$\int_{B(y,r,\sqrt{t-r})} K(x,t;z,r)dg_r(z) \leq c^{-1}\exp\left(-\frac{d_s^2(x,y)}{32B^2(t-s)}\right)$$

We integrate from $r=s$ to $r=\frac{1}{2}(t+s)$ to get

$$\int_{Q^+(y,s,\sqrt{\frac{1}{2}(t-s)})} K(x,t;z,r)dg_r(z)dr \leq c^{-1}(t-s)\exp\left(-\frac{d_s^2(x,y)}{32B^2(t-s)}\right).$$

We now combine this with the on-diagonal upper bound, obtaining $\overline{C}=\overline{C}(A,n)<\infty$ such that

$$\int_{Q^+(y,s,\sqrt{\frac{1}{2}(t-s)})} K^2(x,t;z,r)dg_r(z)dr \leq \frac{\overline{C}}{(t-s)^{\frac{n}{2}-1}}\exp\left(-\frac{d_s^2(x,y)}{\overline{C}(t-s)}\right).$$

In terms of the rescaled flow, this is

$$\int_{\widehat{Q}^+(y,-\frac{1}{2},\frac{1}{\sqrt{2}})} \widehat{K}^2(x,1/2;z,r)d\widehat{g}_r(z)dr \leq \overline{C}\exp\left(-\frac{d_s^2(x,y)}{\overline{C}(t-s)}\right),$$

The Bamler-Zhang parabolic mean value inequality for solutions to the conjugate heat equation (Lemma 4.2 in [BZ2]) applied to the rescaled flow (on the time interval $[-\frac{1}{2},0]$) gives

$$\widehat{K}^2(x,\frac{1}{2};y,-\frac{1}{2}) \leq C''' \int_{\widehat{Q}^+(y,-\frac{1}{2},\sqrt{\frac{1}{2}})} \widehat{K}^2(x,\frac{1}{2};z,r)\,d\widehat{g}_u(z)\,du \leq C'''\overline{C}\exp\left(-\frac{d_s^2(x,y)}{\overline{C}(t-s)}\right),$$

for some $C''' = C'''(A,n) < \infty$, so rescaling back gives

$$K^2(x,t;y,s) \leq \frac{C'''\overline{C}}{(t-s)^n}\exp\left(-\frac{d_s^2(x,y)}{\overline{C}(t-s)}\right).$$

Case 2: If instead $d_s(x,y) \geq 4B^2\sqrt{t-s}$, then

$$K(x,t;y,s) \leq \frac{2^{\frac{n}{2}}C'}{(t-s)^{\frac{n}{2}}} \leq \frac{e2^{\frac{n}{2}}C'}{(t-s)^{\frac{n}{2}}}\exp\left(-\frac{d_s^2(x,y)}{16B^2(t-s)}\right).$$

□

Throughout this section, let $u_{q,t}$ be the conjugate heat kernel based at (q,t), and write $u_{q,t}(x,s) = (4\pi(t-s))^{-\frac{n}{2}}e^{-f_{q,t}(x,s)}$. The following lemma is essentially obtained by passing Lemma 5 to the limit as $t \nearrow 0$, and extends Propositions 2.7 and 2.8 of [MM].

Lemma 6. *Let $(M^n,(g_t)_{t\in[-2,0)},q)$ be a closed, pointed Ricci flow with $\nu[g_{-2},4] \geq -A$ and $|R(x,t)| \leq A|t|^{-1}$ for all $(x,t) \in M\times[-2,0)$. Also suppose $t_i \nearrow 0$ and $q_i \to q$ in M. Then there is some subsequence of $(u_{q_i,t_i})_{i\in\mathbb{N}}$ which converges in $C^\infty_{loc}(M\times(-1,0))$ to some $u_{q,0} \in C^\infty(M\times[-1,0))$ satisfying $\int_M u_{q,0}(x,t)dg_t(x) = 1$ for $t \in [-1,0)$, as well as the conjugate heat equation $(-\partial_t - \Delta + R)u_{q,0} = 0$. In addition, there exists $C = C(A,n) < \infty$ such that*

$$\frac{1}{C|s|^{\frac{n}{2}}}\exp\left(-\frac{Cd_s^2(y,q)}{|s|}\right) \leq u_{q,0}(y,s) \leq \frac{C}{|s|^{\frac{n}{2}}}\exp\left(-\frac{d_s^2(y,q)}{C|s|}\right)$$

for all $(y,s) \in M\times[-1,0)$.

Proof. For any closed solution of Ricci flow, a subsequence of u_{q_i,t_i} must converge in $C^\infty_{loc}(M\times[-2,0))$ to some $u_{q,0}$ solving the conjugate heat equation on $M\times[-2,0)$, as shown in [MM].

Since M is closed, $\int_M u_{q,0}(x,t)dg_t(x) = 1$ is immediate, so it suffices to prove the Gaussian bounds for any limit $u_{q,0}$. Fix $\alpha \in (0,1]$, and let $i_0 \in \mathbb{N}$ be sufficiently large so that $t_i \geq -\frac{1}{2}\alpha$ for all $i \geq i_0$. By the previously established heat kernel bounds, there exists $C^* = C^*(A,n) < \infty$ such that, for all $(y,s) \in M \times [-1,-\alpha]$ and $i \geq i_0$, we have

$$\begin{aligned} u_{q_i,t_i}(y,s) &\geq \frac{1}{C^*(t_i-s)^{\frac{n}{2}}} \exp\left(-\frac{2C^*(d_s^2(q_i,q)+d_s^2(q,y))}{t_i-s}\right) \\ &\geq \frac{1}{C^*|s|^{\frac{n}{2}}} \exp\left(-\frac{2C^* d_s^2(q_i,q)}{\frac{1}{2}\alpha}\right) \exp\left(-\frac{2C^* d_s^2(q,y)}{\frac{1}{2}|s|}\right). \end{aligned}$$

Note that $d_s(q_i,q) \to 0$ uniformly in $s \in [-1,-\alpha]$ as $i \to \infty$, so for any $(y,s) \in M \times [-1,-\alpha]$,

$$u_{q,0}(y,s) = \lim_{i \to \infty} u_{q_i,t_i}(y,s) \geq \frac{1}{C^*|s|^{\frac{n}{2}}} \exp\left(-\frac{4C^* d_s^2(q,y)}{|s|}\right).$$

Similarly, for any $(y,s) \in M \times [-1,-\alpha]$ and $i \geq i_0$, we have (since $(a-b)_+^2 \geq \frac{1}{2}a^2 - b^2$ for $a,b > 0$)

$$\begin{aligned} u_{q_i,t_i}(y,s) &\leq \frac{C^*}{(t_i-s)^{\frac{n}{2}}} \exp\left(\frac{\frac{1}{2}d_s^2(q,y)+d_s^2(q,q_i)}{C^*(t_i-s)}\right) \\ &\leq \frac{2^{\frac{n}{2}}C^*}{|s|^{\frac{n}{2}}} \exp\left(\frac{d_s^2(q,q_i)}{\frac{1}{2}C^*|s|}\right) \exp\left(-\frac{d_s^2(q,y)}{2C^*|s|}\right), \end{aligned}$$

so the claim follows as for the lower bound. □

Definition 26. *Any limit $u_{q,0}$ as in the statement of Lemma 10 is called a conjugate heat kernel based at the singular time. The set of such functions $u_{q,0}$ is denoted $\mathcal{U}_q$, as in* [MM].

Note that we are not able to establish the uniqueness of $u_{q,0}$ given a point $q \in M$ (in fact, this is not even known under assumption (3.1.2)), but the collection of such functions satisfies strong compactness properties. By the uniform Gaussian estimates and parabolic regularity on compact subsets of $M \times (-1,0)$, $\mathcal{U}_q$ is compact in C^∞_{loc}. Let $\mathcal{F}_q$ be the set of $f_{q,0} \in C^\infty(M \times (-1,0))$, where $u_{q,0}(x,t) = (4\pi|t|)^{-\frac{n}{2}}e^{-f_{q,0}(x,t)}$. By the locally uniform bounds on $u_q \in \mathcal{U}_q$ and their derivatives, we observe that $\mathcal{F}_q$ is also compact in C^∞_{loc}. Thus Perelman's differential Harnack inequality passes to the limit to give

$$\tau(R + 2\Delta f_q - |\nabla f_q|^2) + f_q - n \leq 0 \quad \text{on} \quad M \times (-1,0)$$

for any $f_q \in \mathcal{F}_q$, where $\tau := |t|$. As in [MM], we also define

$$\theta_q(t) := \inf_{f \in \mathcal{F}_q} \mathcal{W}(g_t, f(t), \tau(t)).$$

Because $\mathcal{F}_q$ is compact in C^{∞}_{loc}, and because $\theta_q(t) \geq \mu[g_t, \tau(t)] > -\infty$, this infimum is actually achieved at any $t \in (-\infty, 0)$ by some $f_t \in \mathcal{F}_q$. For $-\infty < s < t < 0$, Perelman's entropy monotonicity gives

$$\theta_q(s) \leq \mathcal{W}(g_s, f_t(s), \tau(s)) \leq \mathcal{W}(g_t, f_t(t), \tau(t)) = \theta_q(t),$$

so θ_q is nondecreasing. Similar reasoning gives

$$0 \leq \theta_q(t) - \theta_q(s) \leq \int_s^t 2|r| \int_M \left| Rc_{g_r} + \nabla^2 f_s(r) - \frac{g_r}{2|r|} \right|^2 \frac{e^{-f_s(r)}}{(4\pi|r|)^{\frac{n}{2}}} dg_r dr,$$

but the integrand is bounded on any compact subset of $M \times (-\infty, 0)$, by the uniform estimates for $f \in \mathcal{F}_q$. Thus θ_q is locally Lipschitz. Moreover, $\theta_q(t) \leq 0$ for all $t \in (-\infty, 0)$ by Perelman's Harnack inequality, so we can define the heat kernel density function

$$\Theta(q) := \lim_{t \nearrow 0} \theta_q(t).$$

Fix a sequence $t_i \nearrow 0$, and consider the rescaled flows $\widetilde{g}^i_t := |t_i|^{-1} g_{t_i + |t_i| t}$, $t \in [-2, 0]$, and $\widetilde{f}_i(t) := f_{t_i}(t_i + |t_i| t)$. By the monotonicity of θ_q, we have $\lim_{i \to \infty} (\theta_q(t_i) - \theta_q(t_i - \rho |t_i|)) = 0$ for any fixed $\rho > 0$. Since

$$\begin{aligned}
0 &\leq \mathcal{W}(\widetilde{g}^i_0, \widetilde{f}_i(0), 1) - \mathcal{W}(\widetilde{g}^i_{-\rho}, \widetilde{f}_i(-\rho), \tau(-\rho)) \\
&= \mathcal{W}(g_{t_i}, f_{t_i}(t_i), |t_i|) - \mathcal{W}(g_{t_i - \rho|t_i|}, f_{t_i}(t_i - \rho|t_i|), |t_i| + \rho|t_i|) \\
&\leq \theta_q(t_i) - \theta_q(t_0 - \rho|t_i|),
\end{aligned}$$

we may conclude that

$$0 = \lim_{i \to \infty} \mathcal{W}(\widetilde{g}^i_0, \widetilde{f}_i(0), 1) - \mathcal{W}(\widetilde{g}^i_{-\rho}, \widetilde{f}_i(-\rho), 1 + \rho) \tag{3.2.5}$$

$$= \lim_{i \to \infty} \int_{-\rho}^0 2(1 + |t|) \int_M \left| Rc_{\widetilde{g}^i_t} + \nabla^2 \widetilde{f}_i(t) - \frac{\widetilde{g}^i_t}{2(1 + |t|)} \right|^2 \frac{e^{-\widetilde{g}^i_t}}{(4\pi(1 + |t|))^{-\frac{n}{2}}} d\widetilde{g}^i_t dt. \tag{3.2.6}$$

By the argument of Theorem 1.2 of [B1], we know that, after passing to a subsequence, $(M, \widetilde{g}^i_0, q)$ converge to a singular shrinking GRS, and $\widetilde{f}_i(0)$ converge to the corresponding potential function. The only difference is that, in [B1], the soliton potential function is obtained from limiting a fixed conjugate heat kernel based at the singular time, whereas we are obtaining a soliton potential function from a sequence in $\mathcal{F}_q$. The proof is almost exactly the same, since the estimates for elements of $\mathcal{F}_q$ are uniform, but we rewrite the relevant parts of the argument in [B1] here for completeness, and because we would like to pass the heat kernel bounds of Lemma 6 to the limit.

By Bamler's compactness theorem (Theorem 13), we can pass to a subsequence so that $(M, \widetilde{g}^i_0, q)$ converge to a pointed singular space $(\mathcal{X}, q_\infty) = (X, d, \mathcal{R}, g, q_\infty)$, with associated convergence scheme $\Phi_i : U_i \to V_i$. For any $x \in \mathcal{R}$, we have $r := r^X_{Rm}(x) > 0$, so by Proposition 4.1 in [B1], $r^{\widetilde{g}^i}_{Rm}(\Phi_i(x), 0) > \frac{r}{2}$ for sufficiently large $i \in \mathbb{N}$. Because $|R_{\widetilde{g}^i_0}| \le A$ and $\nu[\widetilde{g}^i_{-2}, 4] \ge -A$, backwards pseudolocality (Theorem 1.5 in [BZ2]) gives $\alpha = c(n, A) > 0$ such that $r_{Rm}(y, s) > \alpha r$ for all $(y, s) \in B_{\widetilde{g}^i}(\Phi_i(x), 0, \alpha r) \times [-\alpha^2 r^2, 0]$ for all $i \in \mathbb{N}$. By 59 6, we have the uniform bounds

$$\frac{1}{C}\exp\left(-\frac{Cd_s^2(y,q)}{|s|}\right) \le e^{-f_{t_i}(y,s)} \le C\exp\left(-\frac{d_s^2(y,q)}{C|s|}\right)$$

for all $(y, s) \in [-1, 0)$. This implies

$$-\log C + \frac{\widetilde{d}_t^2(y,q)}{C(1-t)} \le \widetilde{f}_i(t) \le \log C + C\frac{\widetilde{d}_t^2(y,q)}{(1-t)}$$

for all $t \in [-2, 0]$, hence (because Φ_i is a ϵ_i-Gromov-Hausdorff map for some sequence $\epsilon_i \to 0$)

$$-\log C' + \frac{d^2(q_\infty, x)}{C'} \le \widetilde{f}_i(\Phi_i(x), 0) \le \log C' + C' d^2(q_\infty, x)$$

for all $x \in U_i \setminus B^X(q_\infty, D_i)$, where $D_i \to \infty$. By parabolic regularity theory applied to $\widetilde{u}_i(t) := (4\pi(1-t))^{-\frac{n}{2}} e^{-\widetilde{f}_i(t)}$ on $B_{\widetilde{g}^i}(\Phi_i(x), 0, \alpha r) \times [-\alpha^2 r^2, 0]$, we find that

$$\limsup_{i\to\infty} \sup_{B_{\widetilde{g}^i}(\Phi_i(x),0,\frac{1}{2}\alpha r)\times[-\frac{1}{2}\alpha^2 r^2, 0]} |\nabla^k \widetilde{u}_i|_{\widetilde{g}^i} > 0$$

for all k 2 N. Along with the locally uniform upper bound for $\widetilde{f}_i$, we get similar bounds for $\widetilde{f}_i$, so that we can pass to a subsequence such that $\widetilde{f}_i(0)$ converges in C^{1}_{loc} to some f_1 2 $C^1(R)$. Suppose by way of contradiction that there exists x 2 R such that

$$\left| Rc_{g_1} + r^2 f_1 \quad \frac{g_1}{2} \right|^2 (x\ ) \quad c_0 > 0.$$

Then this quantity is at least $\frac{1}{2}c_0$ on some ball $B^X(x\ , r)$ $\quad R$, so for x 2 $B_{\widetilde{g}^i_0}(\Phi_i(x\ ), \frac{1}{2}r)$ and sufficiently large i, we have

$$\left| Rc_{\widetilde{g}^i_0} + r^2 \widetilde{f}_i(0) \quad \frac{\widetilde{g}^i_0}{2} \right|^2 (x) \quad \frac{c_0}{4}.$$

However, this along with backwards pseudolocality and parabolic regularity give

$$\left| Rc_{\widetilde{g}^i_t} + r^2 \widetilde{f}_i(t) \quad \frac{\widetilde{g}^i_t}{2(1 + jtj)} \right|^2 (x) \quad \delta$$

for (x,t) 2 $B_{\widetilde{g}^i}(\phi_i(x\ ), 0, \delta)$ $\quad [\ \ \delta^2, 0]$, where $\delta > 0$ is small, (depending on x but not on i) contradicting (3.2.5). The estimate (3.2) passes to the limit to give

$$\log C \ + \frac{d^2(q_1, x)}{C} \quad f_1(x) \quad \log C \ + C\ d^2(q_1, x) \tag{3.2.7}$$

for all x 2 R.

3.3 Integration by Parts on the Singular Ricci Soliton

Now let (X, d, R, g_1, f_1) be a singular shrinking GRS as obtained in the previous section.

Lemma 7. *There exists $T = T(A,n) <$ 1 such that, for all $r > 0$, we have*

$$jB^X(q_1, r) \setminus Rj \quad Tr^n.$$

Proof. By Proposition 6, we obtain $C = C(A) <$ 1 such that $jB(x,t,r)j_t$ $\quad Cr^n$ for all r 2 $(0,1]$ and (x,t) 2 M $\quad [\ \ 1, 0)$. For the rescaled flows $\widetilde{g}^i_t := jt_ij\ \ ^1 g_{t_i + jt_ijt}$, this means that

$|B_{\widetilde{g}^i}(x,t,r)|_{\widetilde{g}^i_t} \leq Cr^n$ for all $r \in (0,|t_i|^{-\frac{1}{2}})$ and $(x,t) \in M \times [-2|t_i|^{-1},0)$. Now let (U_i,V_i,Φ_i) be a convergence scheme for the convergence $(M,\widetilde{g}^i_0,q) \to (\mathcal{X},q_\infty)$. Let K be any compact subset of $B^X(q_\infty,r) \cap \mathcal{R}$. Then, for sufficiently large $i \in \mathbb{N}$, we have $K \subseteq U_i$ and

$$|K| \leq 2|\varphi_i(K)|_{\widetilde{g}^i_0} \leq 2|B_{\widetilde{g}^i}(q,0,2r)|_{\widetilde{g}^i_0} \leq 2^{n+1}Cr^n.$$

Since K was arbitrary, this means $|B^X(q_\infty,r) \cap \mathcal{R}| \leq 2^{n+1}Cr^n$. □

Definition 27. *The shrinker entropy of the singular shrinking GRS $(X,d,\mathcal{R},g_\infty,f_\infty)$ is*

$$\mathcal{W}(g_\infty,f_\infty) := \int_{\mathcal{R}} (R_{g_\infty} + |\nabla f_\infty|^2 + f_\infty - n)(4\pi)^{-\frac{n}{2}} e^{-f_\infty}\, dg_\infty.$$

This integral is finite by the previous lemma, since $|R_\infty|$ is bounded, f_∞ has quadratic growth, and $|\nabla f_\infty|^2 \leq R + |\nabla f_\infty|^2 = f_\infty - C$ for some constant $C \in \mathbb{R}$.

In order to prove convergence of entropy, it is essential to use Perelman's differential Harnack inequality, so that the entropy can be rewritten as the integral of a nonpositive quantity. However, it is then necessary to prove that the integration by parts formula

$$\int_{\mathcal{R}} \Delta f_\infty e^{-f_\infty}\, dg_\infty = \int_{\mathcal{R}} |\nabla f_\infty|^2 e^{-f_\infty}\, dg_\infty$$

holds in the singular case. This is equivalent to showing that

$$\int_{\mathcal{R}} \mathrm{div}(\nabla e^{-f_\infty}) dg_\infty = 0.$$

To this end, we recall the following integration by parts formula

Lemma 8. *([B2] Prop 5.2) Let $\mathcal{X} = (X,d,\mathcal{R},g)$ be a singular space with singularities of codimension $p_0 > 2$, and Z a C^1 vector field on $\mathcal{R}$ that vanishes on $\mathcal{R} \setminus B(x,r)$ for some large $r > 0$. Assume there is a constant $C < \infty$ such that*

$$|Z| < Cr_{Rm}^{-1} \quad \text{and} \quad |\mathrm{div}(Z)| < Cr_{Rm}^{-2} \quad \text{on} \quad B(x,r) \cap \mathcal{R}.$$

Then

$$\int_{\mathcal{R}} (\mathrm{div}\, Z) dg = 0.$$

The hypotheses of this lemma will follow from various identities for soliton potential functions.

Lemma 9. $\int_R \Delta f_\infty e^{-f_\infty} dg_\infty = \int_R |\nabla f_\infty|^2_{g_\infty} e^{-f_\infty} dg_\infty$.

Proof. Now fix $r > 0$, and let $\phi \in C^\infty(R)$ be a smoothing of a radial function, chosen such that $\phi|B(q_\infty, r) = 1$, $0 \leq \phi \leq 1$, $|\nabla\phi| \leq 4$, and $\text{supp}(\phi) \subseteq B^X(q_\infty, r+1)$. We want to apply the previous lemma to $Z := \phi\nabla e^{-f_\infty}$. Note that $R_{g_\infty} \geq 0$ since R_{g_t} is uniformly bounded below. Also, the bound $|R|(x,t) \leq A|t|^{-1}$ passes to the limit to give $R_{g_\infty} \leq A$. We know that $R_{g_\infty} + |\nabla f_\infty|^2 - f_\infty = C$ for some constant $C \in \mathbb{R}$. For the purpose of this section, we may assume that $C = 0$, so that $f_\infty \geq 0$ and $|\nabla f_\infty|^2 \leq f_\infty$. Also, $R_{g_\infty} + \Delta f_\infty = \frac{n}{2}$ implies that $|\Delta f_\infty| \leq A + \frac{n}{2}$. The quadratic growth estimates (3.2.7) give

$$|Z(x)| \leq |\nabla f_\infty| e^{-f_\infty} \leq (\log C + C d^2(x, q_\infty))^{\frac{1}{2}} \exp\left(\log C - \frac{1}{C} d^2(q_\infty, x)\right),$$

$$\begin{aligned}|\text{div}(Z(x))| &\leq 2(|\nabla f_\infty| + |\nabla f_\infty|^2 + |\Delta f_\infty|)e^{-f_\infty} \\ &\leq 4(\log C + C d^2(x, q_\infty) + A + \frac{n}{2}) \exp\left(\log C - \frac{1}{C} d^2(q_\infty, x)\right),\end{aligned}$$

for $x \in R$. Both of these terms are locally bounded on R, so we may apply the previous lemma to Z to obtain $0 = \int_R \text{div}(\phi\nabla e^{-f_\infty}) dg_\infty$. Using the volume upper bound, we can conclude

$$\begin{aligned}\int_R |\nabla\phi| |\nabla f_\infty| e^{-f_\infty} dg_\infty &\leq C(n) \int_{R \cap (B^X(q_\infty, r+1) \setminus B^X(q_\infty, r))} |\nabla f_\infty| e^{-f_\infty} dg_\infty \\ &\leq C(n) \int_{R \cap (B^X(q_\infty, r+1) \setminus B^X(q_\infty, r))} e^{-\frac{1}{2} f_\infty} dg_\infty \\ &\leq C(n, A) r^n \exp\left(-\frac{r^2}{C(n, A)}\right).\end{aligned}$$

The claim then follows by taking $r \to \infty$, and using the dominated convergence theorem. □

Corollary 7. *The soliton entropy can also be expressed as*

$$W(g_\infty, f_\infty) = (4\pi)^{-\frac{n}{2}} \int_R (R_{g_\infty} + 2\Delta f_\infty - |\nabla f_\infty|^2 + f_\infty - n) dg_\infty,$$

which has nonpositive integrand by passing Perelman's differential Harnack inequality to the limit.

3.4 Proof of Entropy Convergence

Theorem 23. *Suppose* $(M^n, (g_t)_{t\in[-2,0)}, q)$ *is a closed, pointed solution of Ricci flow satisfying* $\nu[g_{-2}, 4] \geq -A$ *and*

$$|R(\cdot, t)| \leq \frac{A}{|t|}$$

for all $t \in [-2, 0)$*. Let* $(\mathcal{X}, q_\infty) = (X, d, \mathcal{R}, g_\infty, q_\infty)$ *be a singular space obtained as a pointed limit of* $(M, \widetilde{g}^i_0, q)$*, where* $t_i \nearrow 0$*, and* $\widetilde{g}^i_t := |t_i|^{-1} g_{t_i+|t_i|t}$*. Also assume* $f_\infty \in C^\infty(\mathcal{R})$ *is obtained by limiting* $\widetilde{f}_i(0)$ *as in Section 3, where* $f_i(t) := f_{t_i}(t_i + |t_i|t)$*, and* $f_{t_i} \in \mathcal{F}_q$ *satisfy* $\theta_q(t_i) = \mathcal{W}(g_{t_i}, f_{t_i}, |t_i|)$*. Then*

$$\Theta(q) = \lim_{i\to\infty} \mathcal{W}(\widetilde{g}^i_0, \widetilde{f}_i(0), 1) = \mathcal{W}(g_\infty, f_\infty).$$

Proof. The first equality is by definition. Let (U_i, V_i, Φ_i) be the convergence scheme for $(M, \widetilde{g}^i_0, q) \to (\mathcal{X}, q_\infty)$. Then, for any compact subset $K \subseteq \mathcal{R}$, we have for large enough $i \in \mathbb{N}$ that

$$\begin{aligned}\int_K (R_{g_\infty} + 2\Delta f_\infty - |\nabla f_\infty|^2 + f_\infty - n)(4\pi)^{-\frac{n}{2}} dg_\infty \\ = \lim_{i\to\infty} \int_{\Phi_i(K)} (R_{\widetilde{g}^i_0} + 2\Delta \widetilde{f}_i(0) - |\nabla \widetilde{f}_i(0)|^2 + \widetilde{f}_i(0) - n)(4\pi)^{-\frac{n}{2}} d\widetilde{g}^i_0 \\ \leq \limsup_{i\to\infty} \mathcal{W}(\widetilde{g}^i_0, \widetilde{f}_i(0), 1).\end{aligned}$$

Taking the infimum over all compact subsets $K \subseteq \mathcal{R}$ gives $\mathcal{W}(g_\infty, f_\infty) \leq \Theta(q)$. Now fix $\epsilon > 0$, and choose $K \subseteq \mathcal{R}$ compact such that

$$(4\pi)^{-\frac{n}{2}} \int_{\mathcal{R}\setminus K} |R_{g_\infty} + |\nabla f_\infty|^2 + f_\infty - n| e^{-f_\infty} dg_\infty < \epsilon.$$

Then, for any $K' \subset \mathcal{R}$ compact with $K \subset K'$, we have

$$\begin{aligned} \mathcal{W}(g_\infty, f_\infty) &\le \int_{K'} (R_{g_\infty} + |\nabla f_\infty|^2 + f_\infty - n)(4\pi)^{-\frac{n}{2}} e^{-f_\infty} dg_\infty + \epsilon \\ &= \lim_{i \to \infty} \int_{\Phi_i(K')} (R_{\widetilde{g}^i_0} + |\nabla \widetilde{f}_i(0)|^2 + \widetilde{f}_i(0) - n)(4\pi)^{-\frac{n}{2}} e^{-\widetilde{f}_i(0)} d\widetilde{g}^i_0 + \epsilon. \end{aligned}$$

In order to show $\mathcal{W}(g_\infty, f_\infty) \le \Theta(q)$, it therefore suffices to find some $K' \subset \mathcal{R}$ compact (possibly depending on ϵ) with $K \subset K'$ and

$$\liminf_{i \to \infty} \int_{M \setminus \Phi_i(K')} (R_{\widetilde{g}^i_0} + |\nabla \widetilde{f}_i(0)|^2 + \widetilde{f}_i(0) - n)(4\pi)^{-\frac{n}{2}} e^{-\widetilde{f}_i(0)} d\widetilde{g}^i_0 > -\epsilon.$$

Since $\widetilde{f}_i(0)$ have uniform quadratic growth, and because $|R_{\widetilde{g}^i_0}| \le A$, we can find $D = D(A,n) < \infty$ uniform such that

$$R_{\widetilde{g}^i_0} + |\nabla \widetilde{f}_i(0)|^2 + \widetilde{f}_i(0) - n \ge 0 \quad \text{on} \quad M \setminus B_{\widetilde{g}^i}(q,0,D)$$

for all $i \in \mathbb{N}$. Moreover, Bamler's upper bound (Theorem 13) on the size of the quantitative singular set gives us $E = E(A,n) < \infty$ such that

$$|\{r^{\widetilde{g}^i}_{Rm}(\cdot,0) > s\} \cap B_{\widetilde{g}^i_0}(q,2D)|_{g^i_0} \le E s^3$$

for all $s \in (0,1]$.

We also know that the entropy integrand is bounded uniformly from below on $B_{\widetilde{g}^i}(q,0,D)$, and that for $i \in \mathbb{N}$ sufficiently large we have

$$\{r^{\widetilde{g}^i}_{Rm}(\cdot,0) \ge s\} \cap B_{\widetilde{g}^i}(q,0,2D) \subset V_i.$$

Thus we can choose $s = s(A,n,\epsilon) > 0$ sufficiently small so that

$$\int_{\{r^{\widetilde{g}^i}_{Rm}(\cdot,0)<s\} \cap B_{\widetilde{g}^i}(q,0,2D)} (R_{\widetilde{g}^i_0} + |\nabla \widetilde{f}_i(0)|^2 + \widetilde{f}_i(0) - n)(4\pi)^{-\frac{n}{2}} e^{-\widetilde{f}_i(0)} d\mu_{\widetilde{g}^i_0} > -\epsilon.$$

Finally, by the definition of a convergence scheme, we can choose $K' \subset \mathcal{R}$ such that $K \subset K'$ and $\Phi_i(K) \supset \{r^{\widetilde{g}^i}_{Rm}(\cdot,0) \ge s\} \cap B_{\widetilde{g}^i}(q,0,2D)$ (in fact, this will follow by taking $K' = \overline{U}_i$ for some large $i \in \mathbb{N}$). □

Definition 28. *A singular GRS $(\mathcal{R}, g, f)$ is called normalized if*

$$\int_{\mathcal{R}} (4\pi)^{-\frac{n}{2}} e^{-f} dg = 1$$

.

Recall that $R + |\nabla f|^2 - f$ is some constant $c \in \mathbb{R}$ and $R + \Delta f = \frac{n}{2}$, we can write

$$\begin{aligned} \mathcal{W}(g, f) =& (4\pi)^{-\frac{n}{2}} \int_{\mathcal{R}} (R + 2\Delta f - |\nabla f|^2 + f - n) e^{-f} dg \\ =& (4\pi)^{-\frac{n}{2}} \int_{\mathcal{R}} (-R - |\nabla f|^2 + f) e^{-f} dg = -c \int_{\mathcal{R}} (4\pi)^{-\frac{n}{2}} e^{-f} dg. \end{aligned}$$

That is, for a normalized soliton, we know $R + |\nabla f|^2 = f - \mathcal{W}(g, f)$.

Proposition 13. *The singular shrinking GRS $(\mathcal{R}, g_\infty, f_\infty)$ of Theorem 23 is normalized:*

$$\int_{\mathcal{R}} (4\pi)^{-\frac{n}{2}} e^{-f_\infty} dg_\infty = 1.$$

Proof. For any compact subset $K \subset \mathcal{R}$, we have

$$\int_K e^{-f_\infty} dg_\infty = \lim_{i \to \infty} \int_{\varphi_i(K)} e^{-\tilde{f}_i(0)} d\tilde{g}^i_0 \le (4\pi)^{\frac{n}{2}},$$

so it suffices to prove that $\int_{\mathcal{R}} (4\pi)^{-\frac{n}{2}} e^{-f_\infty} dg_\infty \ge 1$. In fact, fix $\epsilon > 0$. By the uniform volume upper bound (Proposition 8) and heat kernel lower bound (Lemma 6), we have some $D = D(\epsilon) < \infty$ such that

$$\begin{aligned} \int_{M \setminus B_{\tilde{g}^i}(q,0,D)} (4\pi)^{-\frac{n}{2}} e^{-\tilde{f}_i(0)} d\tilde{g}^i_0 \le & C(n, A) \int_{M \setminus B_{\tilde{g}^i}(q,0,D)} \exp\left(-\frac{1}{C} \tilde{d}^2_{\tilde{g}^i_0}(q, x)\right) d\tilde{g}^i_0 \\ =& C(n, A) \int_D^\infty \mathrm{Area}_{\tilde{g}^i}(\partial B_{\tilde{g}^i}(q, 0, r)) e^{-\frac{r^2}{C}} dr \\ =& C(n, A) \int_D^\infty e^{-\frac{r^2}{C}} \frac{d}{dr} |B_{\tilde{g}^i}(q, 0, r)|_{\tilde{g}^i_0} dr \\ \le & C(n, A) \int_D^\infty r |B_{\tilde{g}^i}(q, 0, r)|_{\tilde{g}^i_0} e^{-\frac{r^2}{C}} dr \\ \le & C(n, A) \int_D^\infty r^{n+1} e^{-\frac{r^2}{C}} dr < \frac{\epsilon}{2} \end{aligned}$$

for all i 2 N. Moreover, since $e^{\ f_1}$ is uniformly bounded on $B_{\widetilde{g}^i}(q,0,2D)$ (independently of i) we also have

$$\int_{B_{\widetilde{g}^i}(q,0,2D) n V_i} (4\pi)^{\ \frac{n}{2}} e^{\ \widetilde{f}_i(0)} d\mu_{\widetilde{g}^i_0} \quad C \left| f r^{\widetilde{g}^i}_{Rm}(\ ,0) > s g \ \backslash \ B_{\widetilde{g}^i}(q,0,2D) \right|_{\widetilde{g}^i_0}$$

for any $s > 0$, when sufficiently large i. By taking $s > 0$ sufficiently small, the upper bound on the size of the quantitative singular set (as in the previous section) tells us that the right hand side is less than $\frac{1}{2}\epsilon$. This means that

$$\int_{B_{\widetilde{g}^i}(q,0,2D) \backslash V_i} (4\pi)^{\ \frac{n}{2}} e^{\ \widetilde{f}_i(0)} d\mu_{\widetilde{g}^i_0} \quad 1 \quad \epsilon$$

for i sufficiently large, hence

$$\int_{R \backslash B(q,2D)} (4\pi)^{\ \frac{n}{2}} e^{\ f_1} d\mu_{g_1} \quad 1 \quad 2\epsilon,$$

and the claim follows. □

Remark 7. *As in the Type-I curvature case* [MM]*, we note that Proposition 13 and the entropy convergence part of Theorem 23 also hold if the sequence $\widetilde{f}_i$ is replaced by $f(\ , t_i + j t_i j t)$ for some xed f 2 F_q. The equality $W(g_1, f_1) = \Theta(q)$ could fail a priori in that setting, though equality will follow from the results of Section 6.*

3.5 Entropy Rigidity of the Gaussian Soliton

The following result extends Lemma 2.1 of [N1] to the setting of singular shrinking GRS. The proof of that lemma used essentially the fact that the underlying Riemannian manifold is complete, which in our setting is only true if the singular set X n R is empty. However, we will see that the proof can be modified to work when X n R has singularities of codimension strictly greater than 3, using the arguments of Claim 2.32 of [CW3]. In fact, the part of the following proof establishing the flow properties of a function f 2 $C^1(R)$ with $r^2 f = 0$ is taken from this claim, but since the setting of [CW3] is somewhat different, we rewrite the part of this claim we need.

Proposition 14. *Suppose $\mathcal{X} = (X, d, \mathcal{R}, g, f_i)$, $i = 1, 2$ are normalized singular shrinking GRS with singularities of codimension 4. Then*

$$\mathcal{W}(g, f_1) = \mathcal{W}(g, f_2).$$

Proof. We can assume that $f_1 - f_2$ is not constant, otherwise the normalization condition gives the claim. Set $f := |\nabla(f_1 - f_2)|^{-1}(f_1 - f_2)$, so that $|\nabla f| = 1$ and $\nabla^2 f = 0$ on $\mathcal{R}$. Let $\varphi_t(x)$ be the flow of ∇f starting at $x \in \mathcal{R}$ for $t \in \mathbb{R}$ such that this is defined. Fix $p \in (2, 4)$, $s \in (0, 1]$. We first show that, for any $q \in X$, $s \in (0, 1]$, and $D < \infty$, the set

$$S_{D,s} := \{x \in \mathcal{R} \cap B^X(q, D)\, ;\, r^X_{Rm}(\varphi_t(x)) < \frac{1}{2}s \text{ for some } t \in [-D, D]\}$$

has Minkowski codimension at least $p - 1$. We denote by $\mathcal{H}^{n-1}$ the $(n-1)$-dimensional Hausdorff measure on $\mathcal{R}$, which coincides with the Lebesgue measure on any hypersurface.

Because r^X_{Rm} is 1-Lipschitz, we can find $h \in C^\infty(\mathcal{R})$ such that $|\nabla h| \leq 2$ and

$$\frac{1}{2}r^X_{Rm} < h < 2r^X_{Rm} \qquad \text{on } \mathcal{R}.$$

Using the coarea formula and the fact that singularities are of codimension 4, we have

$$\begin{aligned}\int_s^{2s} \mathcal{H}^{n-1}(h^{-1}(t) \cap B^X(q, 3D))dt &= \int_{\{s \leq h \leq 2s\} \cap B^X(q,3D)} |\nabla h| dg \\ &\leq 2|\{r^X_{Rm} \leq 4s\} \cap B^X(q, 3D) \cap \mathcal{R}| \leq 2 \cdot 4^p E_{p,3D,q} s^p.\end{aligned}$$

By Sard's theorem, we may therefore find $t = t(s) \in (s, 2s)$ such that $\Sigma_s := h^{-1}(t) \cap B^X(q, 3D)$ is smooth and satisfies $\mathcal{H}^{n-1}(\Sigma_s) \leq 2^{2p+1}E_{p,3D,q}s^{p-1}$. Next, write $S_{D,s} = I_s \cup II_s$, where

$$I_s := \{x \in S_{D,s}\, ;\, r^X_{Rm}(x) \leq 4s\},$$

$$II_s := \{x \in S_{D,s}\, ;\, r^X_{Rm}(x) > 4s\}.$$

Since the singularities of X are codimension 4, we have

$$|I_s| \le |\{r^X_{Rm} \le 4s\} \cap B^X(q,D) \cap \mathcal{R}| \le 4^p E_{p,3D,q} s^p.$$

For any $y \in II_s$, there exists $t \in (-D,D)$ such that $\varphi_t(y) \in \Sigma_s$. Moreover, $|\nabla f| = 1$ implies $d(\varphi_t(y),q) \le 2D < 3D$. Now set

$$\Omega_s := \{(t,x) \in (-D,D) \times \Sigma_s \,;\, \varphi_t(x) \text{ is well defined}\},$$

which is open in $(-D,D) \times \Sigma_s$.

Claim 1: The Jacobian of $\eta : (\Omega_s, dt^2 + g_{\Sigma_s}) \to (\mathcal{R}, g), (t,x) \mapsto \varphi_t(x)$ is ≤ 1 everywhere.

In fact, since each φ_t is a local isometry, we have that

$$d\eta_{(t,x)}|_{T_x\Sigma_s} : T_x\Sigma_s \to T_{\varphi_t(x)}(\varphi_t(\Sigma_s))$$

is a linear isometric embedding for all $(t,x) \in (-D,D) \times \Sigma_s$. Moreover, $d\eta_{(t,x)}(\partial/\partial t) = -\nabla f(\varphi_t(x))$, and so the Jacobian of η at $(t,x) \in \Sigma_s \times (-D,D)$ is $|(-\nabla f(\varphi_t(x)))^{\perp}| \le 1$, where $\perp$ denotes the projection $T_{\varphi_t(x)}\mathcal{R} \to (T_{\varphi_t(x)}\varphi_t(\Sigma_s))^{\perp}$.

Note that $II_s \subset \eta(\Omega_s)$ and the claim give $|\eta(\Omega_s)| \le \mathcal{H}^{n-1}(\Sigma_s) \cdot 2D \le 4^{p+2}E_{q,3D,p}Ds^{p-1}$, so we may conclude

$$|S_{D,s}| = |I_s| + |II_s| \le 4^{p+3}E_{q,3D,p}s^{p-1}.$$

Claim 2: $\{x \in \mathcal{R} \,;\, d(x,S_{D,s}) < s\} \subset S_{2D,4s}$.

In fact, suppose $x \in \mathcal{R}$ satisfies $d(x,S_{D,s}) < s$. If $r^X_{Rm}(x) < 2s$, then $x \in S_{2D,4s}$ by definition. If $r^X_{Rm}(x) > 2s$, then there is a minimal geodesic from x to some point in $S_{2D,s}$, and this geodesic lies entirely in $\{r^X_{Rm} > s\} \subset \mathcal{R}$. By construction, there is some $t \in (-D,D)$ such that $\varphi_t(\gamma) \cap \{r^X_{Rm} \le \tfrac{1}{2}\} \ne \emptyset$. Let $t_0 \in (-D,D)$ be such that $|t_0|$ is minimal among such t. We can assume, by replacing f with $-f$, that $t_0 > 0$. Then, since φ_{t_0} is a local isometry, and $r^X_{Rm} \ge \tfrac{1}{2}s$ along $\varphi_t(\gamma)$ for $0 \le t \le t_0$, we know that φ_{t_0} is defined on γ and that

$L_g(\varphi_{t_0}(\gamma)) = L_g(\gamma) < s$. Also, by construction we have $\varphi_{t_0}(\gamma) \cap \{r^X_{Rm} = \frac{1}{2}s\} \neq \emptyset$. Since r^X_{Rm} is 1-Lipschitz, this implies $r^X_{Rm}(\varphi_{t_0}(x)) \geq 3/2$, hence $x \in S_{2D,4s}$.

This along with $|S_{2D,2s}| \leq 4^{p+10}E_{q,6D,p}s^{p-1}$ implies the Minkowski dimension claim. In particular, the set S of $x \in R$ such that $\varphi_t(x)$ does not exist for all time satisfies $|S| = 0$ and $\mathcal{H}^{n-1}(S \cap f^{-1}(0)) = 0$. Define $N := f^{-1}(0) \cap R$, and let $U \subset \mathbb{R} \times N$ be the (open) maximal subset where $\psi(t,x) := \varphi_t(x)$ is defined. Then $R \setminus S \subset \psi(U)$, since for any $x \in R \setminus S$, we have $x = \psi(f(x), \varphi_{-f(x)}(x))$. In particular, $|R \setminus \psi(U)| = 0$. By an computation similar to that in Claim 1, and noting that now $(\nabla f(\varphi_t(x)))^{\perp} = \nabla f(\varphi_t(x))$, where $\perp$ denotes the projection $T_{\varphi_t(x)}R \to (T_{\varphi_t(x)}\varphi_t(N))^{\perp}$, we get that $\psi(U)$ is a Riemannian isometry $(U, dt^2 + \widetilde{g}) \to (\psi(U), g)$, where $\widetilde{g}$ is the Riemannian metric $\widetilde{g}$ on $N := f^{-1}(0) \cap R$ induced from g. In particular, $\widetilde{f}_i := f_i \circ \psi \in C^{\infty}(U)$ are soliton functions, and $(f \circ \psi)(t,x) = t$.

Claim 3: There are $a_i \in \mathbb{R}$ such that

$$\widetilde{f}_i(t,x) = \widetilde{f}_i(0,x) + a_i t + \frac{1}{4}t^2.$$

for all $(t,x) \in U$.

In fact, the pulled back soliton equation gives $\partial_t^2 \widetilde{f}_i = \frac{1}{2}$ everywhere, so

$$\widetilde{f}_i(t,x) = \widetilde{f}_i(0,x) + \partial_t \widetilde{f}_i(0,x)t + \frac{1}{4}t^2$$

for $(t,x) \in U$. Moreover, for any $X \in \mathfrak{X}(N)$, we have $\nabla_X \partial_t = 0$, so the Riemannian product structure and the soliton equation give

$$X(\partial_t \widetilde{f}_i) = \nabla^2 \widetilde{f}_i(\partial_t, X) = \frac{1}{2}g(\partial_t, X) - Rc(\partial_t, X) = 0.$$

This means that $\nabla(\partial_t \widetilde{f}_i - \frac{1}{2}t) = 0$ on U, hence $\nabla(\langle \nabla f, \nabla f_i \rangle - \frac{1}{2}f) = 0$ on the dense open subset $\psi(U)$ of R. Because f, f_i are smooth and R is connected, we get that $\langle \nabla f, \nabla f_i \rangle - \frac{1}{2}f$

is constant on $\psi(U)$, hence $\partial_t \widetilde{f}_i - \frac{1}{2}t$ is constant on U. In particular, $\partial_t \widetilde{f}_i$ is constant on $\{0\} \times N$, and the claim follows.

Now we use the normalization conditions on $\widetilde{f}_i$. Since $|\mathbb{R} \setminus \psi(U)| = 0$ and $\partial_t(\widetilde{f}_1 - \widetilde{f}_2) = 1$, we have

$$(4\pi)^{-\frac{n}{2}} \left(\int_N e^{-\widetilde{f}_2(0,x)} d\widetilde{g}(x) \right) \left(\int_{\mathbb{R}} e^{-\frac{1}{4}t^2 - a_2 t} dt \right) = 1,$$
$$(4\pi)^{-\frac{n}{2}} \left(\int_N e^{-\widetilde{f}_2(0,x)} d\widetilde{g}(x) \right) \left(\int_{\mathbb{R}} e^{-\frac{1}{4}t^2 - a_1 t} dt \right) = 1,$$

since $\widetilde{f}_1 = \widetilde{f}_2$ on $\{0\} \times N$. Thus

$$e^{a_2^2} \int_{\mathbb{R}} e^{-\frac{1}{4}t^2} dt = e^{a_2^2} \int_{\mathbb{R}} e^{-\frac{1}{4}(t-2a_2)^2} dt = e^{a_1^2} \int_{\mathbb{R}} e^{-\frac{1}{4}(t-2a_1)^2} dt = e^{a_1^2} \int_{\mathbb{R}} e^{-\frac{1}{4}t^2} dt,$$

which implies $a_1^2 = a_2^2$. Noting that $\nabla_X \widetilde{f}_1(0,x) = \nabla_X \widetilde{f}_2(0,x)$ for all $x \in N$ and $X \in T_x N$, we thus have

$$|\nabla \widetilde{f}_1(0,x)|^2 - |\nabla \widetilde{f}_2(0,x)|^2 = a_1^2 - a_2^2 = 0.$$

In particular, on $\{0\} \times N$,

$$R + |\nabla \widetilde{f}_1|^2 - \widetilde{f}_1 = R + |\nabla \widetilde{f}_2|^2 - \widetilde{f}_2.$$

Since f_i are normalized, we have $R + |\nabla f_i|^2 - f_i = -\mathcal{W}(g, f_i)$, so the proposition follows. $\square$

Next, we address the rigidity statement of Theorem 21.

Proposition 15. *Suppose $(M, (g_t)_{t \in [-2,0)}, q)$ is a closed, pointed solution of Ricci flow with*

$$\sup_{t \in [-2,0)} |R(\cdot, t)|(T - t) < \infty,$$

and let (X_∞, q_∞) be a singular shrinking GRS obtained as a Type-I limit. If $\mathcal{W}(g_\infty, f_\infty) = 0$, then X_∞ is the Gaussian shrinker. If this occurs, there is a neighborhood U of q in M such that

$$\sup_{U \times [-2,0)} |Rm| < \infty.$$

Proof. Fix $x \in \mathcal{R}$, and let (U_i, V_i, Φ_i) be the convergence scheme for $(M, \widetilde{g}_0^i, q) \to (\mathcal{X}, q_\infty)$. Note that

$$d^X(x, q_\infty) = \lim_{i \to \infty} d_{\widetilde{g}_0^i}(\Phi_i(x), q) = \lim_{i \to \infty} |t_i|^{-\frac{1}{2}} d_{g_{t_i}}(\phi_i(x), q),$$

so $\phi_i(x) \to q$ in M. Thus, after passing to a subsequence, $u_{\phi_i(x),t_i}$ converges to a conjugate heat kernel at the singular time $u \in \mathcal{U}_q$ in $C^\infty_{loc}(M \times (-\infty, 0))$. Writing $u(y,s) = (4\pi|s|)^{-\frac{n}{2}} e^{-f(y,s)}$, we know from previous sections that, if $f_i(s) := f(t_i + |t_i|s)$, then $f_i(0) \circ \Phi_i$ converges in $C^\infty_{loc}(\mathcal{R})$ to a normalized soliton function $\overline{f}_\infty$, which must satisfy $\mathcal{W}(g_\infty, \overline{f}_\infty) = \mathcal{W}(g_\infty, f_\infty) = 0$ by Remark 7 and Proposition 14, hence (again using Remark 7)

$$\begin{aligned} 0 &= \lim_{i \to \infty} \mathcal{W}(g_0^i, f_i(0), 1) = \lim_{i \to \infty} \mathcal{W}(|t_i|^{-1} g_{t_i}, f(t_i), 1) \\ &= \lim_{i \to \infty} \mathcal{W}(g_{t_i}, f(t_i), |t_i|) = \lim_{t \nearrow 0} \mathcal{W}(g_t, f(t), |t|) \end{aligned}$$

by Theorem 23. Now let $\epsilon = \epsilon(n, C) > 0$ be the constant from Theorem 15. Then there exists $\delta > 0$ such that $\mathcal{W}(g_t, f(t), |t|) \geq -\frac{1}{2}\epsilon$ for all $t \in [-\delta, 0)$. Because $f_{\phi_i(x),t_i} \to f$ in $C^\infty_{loc}(M \times (-\infty, 0))$, we know that for any fixed $t \in (-\infty, 0)$, we have

$$\mathcal{W}(g_t, f(t), |t|) = \lim_{i \to \infty} \mathcal{W}(g_{t+t_i}, f_{\phi_i(x),t_i}(t + t_i), |t|).$$

In particular, $\mathcal{W}(g_{-\delta+t_i}, f_{\phi_i(x),t_i}(-\delta + t_i), \delta) \geq -\epsilon$ for sufficiently large $i \in \mathbb{N}$. By Theorem 15, we conclude $(r^g_{Rm}(\Phi_i(x), t_i))^2 \geq \epsilon\delta$. This means $\left(r^{\widetilde{g}^i}_{Rm}(\Phi_i(x), 0)\right)^2 > \epsilon\delta|t_i|^{-1}$, so by backwards Pseudolocality, it follows that $(M, |t_i|^{-1} g_{t_i}, q)$ actually converges in the C^∞ Cheeger-Gromov sense to the Gaussian shrinker on flat $\mathbb{R}^n$.

Now apply a version of Perelman's pseudolocality theorem (Theorem 1.2 of [L2]) to the ball $B(q, t_i, D\sqrt{|t_i|})$, with $D < \infty$ and $i \in \mathbb{N}$ sufficiently large, to conclude that $|Rm|(x,t) \leq C$ for all $x \in B(q, t_i, \sqrt{|t_i|})$, $t \in (t_i, 0)$, (see also Lemma 2.4 of [EMT]). □

Proof of Theorem 1. By Section 3, we can pass to a further subsequence in order to assume that $\widetilde{f}_i(0)$ converge to another smooth soliton potential function $f'_\infty \in C^\infty(\mathcal{R})$, which satisfies

$\mathcal{W}(g_1, f_1^0) = \Theta(q)$. By Proposition 14, we have $\mathcal{W}(g_1, f_1^0) = \mathcal{W}(g_1, f_1)$. The remaining claim is Proposition 15.

3.6 Removable Singularities

In this section, we specialize to the four-dimensional case, where we first sharpen Bamler's Minkowski dimension estimates for the singular set, obtaining that the limiting singular GRS is actually smooth outside of a discrete set of points. Using this, we are able to show the singularities are conical C^0 orbifold singularities, without knowing that the global L^2 norm of the curvature tensor on the regular set is finite (this is not true in general, even if we assume (3.1.2) so that X is smooth). In fact, it is not clear how one can prove local L^2 estimates for the curvature on the rescaled Ricci flow. This is because the L^2 curvature bound in dimension 4 is usually proved using the Chern-Gauss-Bonnet formula, but the argument relies crucially on the (rescaled) flow having uniformly bounded diameter. Moreover, it is not clear how to effectively localize the Chern-Gauss-Bonnet formula in this situation: applying the formula on a subdomain results in boundary terms which depend on the principal curvatures of the boundary. In [HM], this difficulty was overcome by using properties of level sets of a shrinking GRS, which suggests that it may be easier to prove the L^2 curvature estimate on the limiting singular space rather than on the Ricci flow itself.

Therefore, we aim to prove a local L^2 bound for $|Rm|$ near the singular points of X, and then apply the removable singularity techniques of [T],[CS1], [U]. We achieve this by estimating separately the traceless Ricci and the Weyl parts of the curvature tensor, using ideas of Haslhoffer-Muller [HM] and Donaldson-Sun [DS], respectively. After overcoming this difficulty, the proof is fairly standard, and Uhlenbeck's theory [U] of removable singularities along with the ϵ-regularity theorem proved in [H4], and later [JN2], let us conclude that in fact the singular GRS has a C^1 orbifold structure.

Throughout this section, we suppose that $(M^4,(g_t)_{t\in[-2,0)})$ is a closed solution of Ricci Flow satisfying $\nu[g_{-2},4]\geq -A$ and

$$|R(x,t)|\leq \frac{A}{|t|}$$

for all $(x,t)\in M\times[-2,0)$. Fix a basepoint $q\in M$ and a sequence of times $t_i\nearrow 0$. Define the rescaled sequence $\widetilde{g}^i_t := |t_i|^{-1}g_{t_i+|t_i|t}$ for $t\in[-2,0)$. Then the rescaled solutions satisfy $\sup_{M\times[-2,0]}|R_{\widetilde{g}^i}|\leq A$ and $\nu[\widetilde{g}^i_{-2},4]\geq -A$ for all $i\in\mathbb{N}$. By Theorem 1.2 of [B1], we may pass to a subsequence so that $(M,\widetilde{g}^i_0,q)$ converges to a pointed singular space $(\mathcal{X},q_\infty)=(X,d,\mathcal{R},g,q_\infty)$ with singularities of codimension 4, that is Y-regular at all scales, for some $Y=Y(A)<\infty$, and satisfies the shrinking soliton equation $Rc+\nabla^2 f=\frac{1}{2}g$ on the regular part $\mathcal{R}$, where $f\in C^\infty(\mathcal{R})$ is the obtained from a sequence of rescaled conjugate heat kernels based at the singular time. We recall that $|R|\leq A$ on $\mathcal{R}$, and that f satisfies quadratic growth estimates (3.2.7), which combine with the equation $R+|\nabla f|^2=f-\mathcal{W}(g,f)$ to give a locally uniform gradient estimate for f.

Lemma 10. *$X\setminus\mathcal{R}$ is discrete, and every tangent cone at $x\in X\setminus\mathcal{R}$ is isometric to $\mathbb{R}^4/\Gamma$ for some finite subgroup $\Gamma\subseteq O(4,\mathbb{R})$ (which may depend on x and the choice of rescalings). Moreover, there exists $N=N(A)>0$ such that $|\Gamma|\leq N$.*

Proof. Fix $x_0\in X\setminus\mathcal{R}$, and let (Z,d_Z,c_Y) be a tangent cone at x_0, with $\lambda_i\to\infty$ such that $(X,\lambda_i d_X,x_0)\to(Z,d_Z,c_Z)$ in the pointed Gromov-Hausdorff sense. By Corollary 1.5 of [B1], Z is a metric cone. Choose $x_i\in M$ such that $x_i\to x_0$ as $i\to\infty$. By definition of the convergence $(M,g^i_0,q)\to(\mathcal{X},q_\infty)$, for each $i\in\mathbb{N}$, we can choose $j=j(i)\geq i$ such that $(M,\lambda_i^2 g^{j(i)}_0,x_{j(i)})$ is $\lambda_i i^{-1}$-close in the pointed Gromov-Hausdorff topology to $(X,\lambda_i d_X,x_0)$. Setting $\widetilde{g}^i_t:=\lambda_i^2 g^{j(i)}_{\lambda_i^{-2}t}$, we get that $(M,(\widetilde{g}^i_t)_{t\in[-2,0]},x_{j(i)})_{i\in\mathbb{N}}$ is a sequence of pointed Ricci flows with $\sup_{M\times[-2,0]}|R_{\widetilde{g}^i}|\to 0$ and $\nu[\widetilde{g}^i_{-2},4]\geq -A$, which converges in the pointed Gromov-Hausdorff sense to (Z,d_Z,c_Z). In particular, (Z,d_Z,c_Z) has the structure of a singular space $\mathcal{Z}=(Z,d_Z,\mathcal{R}_Z,g_Z,c_Z)$ with mild singularities of codimension 4, such that $Rc_{g_Z}=0$ on

$\mathcal{R}_Z$. However, $Z = C(\Sigma)$ is a metric cone, so the link Σ of Z is a smooth 3-dimensional Riemannian manifold. That is, $Z \setminus \{c_Z\}$ is a smooth metric cone $g_Z = dr^2 + r^2 g_\Sigma$ for some smooth Riemannain metric g_Σ on Σ. However, $Rc_{g_Z} = 0$ implies $Rc_{g_\Sigma} = (n-1)g_\Sigma$, and since $\dim(\Sigma) = 3$, (Σ, g_Σ) must be a disjoint union of spherical space forms. Because $\mathcal{R}_Z = Z \setminus \{c_Z\}$ is connected, Σ must be connected. Thus, $Z = C(\mathbb{S}^3/\Gamma) = \mathbb{R}^4/\Gamma$ for some finite subgroup $\Gamma \subset O(4,\mathbb{R})$. Moreover, because Z is Y-tame for some $Y = Y(A) < \infty$ (by Proposition 4.2 of [B2]), we have

$$c(A) < |B^Z(c_Z, 1) \setminus \{c_Z\}|_{g_Z} = \omega_n/|\Gamma|.$$

It remains to show that x_0 is an isolated point of $X \setminus \mathcal{R}$. Suppose by way of contradiction that there exist $y_i \in X \setminus (\mathcal{R} \cup \{x_0\})$ such that $y_i \to x_0$. Set $\lambda_i := 1/d(x_0, y_i)$. By passing to a subsequence, we can assume $(X, \lambda_i d_X, x_0)$ converges in the pointed Gromov-Hausdorff sense to a tangent cone (Z, d_Z, c_Z) as above. For any $\alpha \in (0,1)$, we can pass to a further subsequence so that $(B^X(y_i, \alpha\lambda_i^{-1}), \lambda_i d, y_i)$ converges in the pointed Gromov-Hausdorff sense to $(B^Z(y_\infty, \alpha), d_Z, y_\infty)$ for some $y_\infty \in Z$ with $d(c_Z, y_\infty) = 1$. By possibly shrinking $\alpha > 0$, we can assume that $B^Z(y_\infty, \alpha)$ is isometric to a ball in $\mathbb{R}^n$. Applying Theorem 2.37 of [TZ] (see the appendix of this paper), we have

$$|B^X(y_i, \alpha\lambda_i^{-1}) \cap \mathcal{R}|_g \geq (\omega_n - \epsilon_i)(\alpha\lambda_i^{-1})^4$$

for some sequence $\epsilon_i \to 0$. However, the $Y(A)$-regularity of X then implies $r_{Rm}(y_i) > 0$, contradicting $y_i \in X \setminus \mathcal{R}$. □

Theorem 24. *X has the structure of a C^∞ Riemannian orbifold with finitely many conical orbifold singularities, such that in orbifold charts around the singular points, f extends smoothly across the singular points, and satisfies the gradient Ricci soliton equation everywhere.*

Proof. Fix $x \in X \setminus \mathcal{R}$. Suppose by way of contradiction that there exists a sequence $x_i \to x$ such that $\liminf_{i\to\infty} |Rm(x_i)| d_X^2(x_i, x) > 0$ (since x is an isolated point of $X \setminus \mathcal{R}$,we can assume $x_i \in R$). Set $r_i := d_X(x_i, x_\infty)$, so that by passing to a subsequence, we may assume that $(X, r_i^{-1} d_X, x_\infty)$ converge in the pointed Gromov-Hausdorff sense to $(C(\mathsf{S}^3/\Gamma), d_{C(\mathsf{S}^3/\Gamma)}, c_0)$ for some finite subgroup $\Gamma \subset O(4, \mathsf{R})$, where $c_0 \in C(\mathsf{S}^3/\Gamma)$ is the cone point.

We claim that there are $\epsilon_i \to 0$ such that, for all $x \in B^X(x_\infty, 2r_i) \setminus \overline{B}^X(x_\infty, \frac{1}{2}r_i)$, we have the Gromov-Hausdorff distance estimate

$$d_{GH}\Big((B^X(x, \alpha r_i), r_i^{-1} d_X, x), (B^{C(\mathsf{S}^3/\Gamma)}(\overline{x}, \alpha), d_{C(\mathsf{S}^3/\Gamma)}, \overline{x})\Big) < \epsilon_i$$

for some $\overline{x} \in C(\mathsf{S}^3/\Gamma)$ with $\frac{1}{4} \leq d(c_0, \overline{x}) \leq 4$. Suppose by way of contradiction that there exist $\epsilon > 0$ and $y_i \in B^X(x_\infty, 2r_i) \setminus \overline{B}^X(x_\infty, \frac{1}{2}r_i)$ where

$$d_{GH}((B^X(y_i, \alpha r_i), r_i^{-1} d_X, y_i), (B^{C(\mathsf{S}^3/\Gamma)}(\overline{x}, \alpha), d_{C(\mathsf{S}^3/\Gamma)}, \overline{x})) > \epsilon$$

for every $\overline{x} \in C(\mathsf{S}^3/\Gamma)$ with $\frac{1}{4} \leq d(c_0, \overline{x}) \leq 4$. Let $\psi_i : B^X(x_\infty, 100r_i) \to C(\mathsf{S}^3/\Gamma)$ be δ_i-Gromov-Hausdorff approximations

$$(B^X(x_\infty, 100r_i), r_i^{-1} d_X, x_\infty) \to (B^{C(\mathsf{S}^3/\Gamma)}(c_0, 100), d_{C(\mathsf{S}^3/\Gamma)}, c_0),$$

where $\delta_i \to 0$. Then $\psi_i|B^X(y_i, \alpha r_i)$ is a $\frac{1}{2}\delta_i$-Gromov-Hausdorff approximation from

$$(B^X(y_i, \alpha r_i), r_i^{-1} d_X, y_i) \to (B^{C(\mathsf{S}^3/\Gamma)}(\psi_i(y_i), \alpha), d_{C(\mathsf{S}^3/\Gamma)}, \psi_i(y_i)),$$

where $\frac{1}{3} \leq d(c_0, \psi_i(y_i)) \leq 3$. Passing to a subsequence, we may assume that $\psi_i(y_i)$ converges in to some $y_\infty \in C(\mathsf{S}^3/\Gamma)$ with $\frac{1}{3} \leq d(c_0, y_\infty) \leq 3$. Then $(B^X(y_i, \alpha r_i), r_i^{-1} d_X, y_i)$ converges in the pointed Gromov-Hausdorff sense to $(B^{C(\mathsf{S}^3/\Gamma)}(y_\infty, \alpha), d_{C(\mathsf{S}^3/\Gamma)}, y_\infty)$, a contradiction.

Because $|\Gamma| \leq N(A)$, we can choose $\alpha = \alpha(A) > 0$ sufficiently small so that $|B^{C(\mathsf{S}^3/\Gamma)}(y, \alpha)| = \omega_n \alpha^n$ for all $y \in C(\mathsf{S}^3/\Gamma)$ with $\frac{1}{4} \leq d(y, c_0) \leq 4$. Another application of Theorem 2.37 of [TZ] tells us that the volume ratios $(\alpha r_i)^{-n}|B^X(x, \alpha r_i)|$ converge to the Euclidean volume ratio, locally uniformly for $x \in X$ with $2r_i > d_X(x, x_\infty) > \frac{1}{2}r_i$. Note that we could also perform a conformal change using the potential function f as in [Z4], and appeal to the volume convergence theorem for manifolds with Ricci curvature bounded below. By the $Y(A)$-regularity of the singular space X, we may conclude that $r^X_{Rm}(x_i) > Y^{-1} r_i$ for some $Y = Y(A) < \infty$. In particular, also using the lower volume bound, we may assume by passing to a subsequence and applying the Cheeger-Gromov compactness theorem, that the convergence of the rescaled metrics $(B^X(x_i, \frac{1}{2}\alpha_i r_i), r_i^{-2} g)$ to a ball in $B^{C(\mathsf{S}^3/\Gamma)}(c_0, 4) \setminus B^{C(\mathsf{S}^3/\Gamma)}(c_0, \frac{1}{4})$ is smooth. However, $C(\mathsf{S}^3/\Gamma)$ is flat away from the cone point, so actually $|Rm(x_i)| d_X^2(x_i, x_\star) \to 0$ as $i \to \infty$, a contradiction. We may therefore conclude that $|Rm|(x) d_X^2(x, x_\star) \leq \epsilon(x)$ for $x \in B^X(x_\star, \delta) \setminus \{x_\star\}$, where $\lim_{x \to x_\star} \epsilon(x) = 0$. By local estimates for shrinking GRS (c.f. the proof of Theorem 2.5 in [HM]), we even have $|\nabla^k Rm|(x) d_X^{2+k}(x, x_\star) \leq \epsilon(x)$ for $x \in B^X(x_\star, \delta)$, any $k \geq 0$.

Claim: $C(\mathsf{S}^3/\Gamma)$ is the unique tangent cone of X at $x_\star$.

We proceed as in the proof of Lemma 5.13 in [BKN]. The above arguments give that, for any tangent cone $C(\mathsf{S}/\Gamma')$ of X at $x_\star$, there is a sequence $s_i \searrow 0$ such that $B^X(x_\star, 2s_i) \setminus B^X(x_\star, s_i)$ is diffeomorphic to $(1,2) \times \mathsf{S}^3/\Gamma'$. It therefore suffices to find $\overline{r} > 0$ such that $B^X(x_\star, 2r) \setminus \overline{B}^X(x_\star, r)$ and $B^X(x_\star, 2s) \setminus \overline{B}^X(x_\star, s)$ are diffeomorphic for all $s, r \in (0, \overline{r})$. Set

$$\theta(r) := \sup\{\angle(\dot{\gamma}(r), \dot{\eta}(r)); \gamma, \eta : [0, r] \to B^X(x, r)$$
$$\text{are unit-speed minimizing geodesics from } x_\star \text{ to some } x \in \partial B^X(x, r)\}.$$

We claim that $\lim_{r \to 0} \theta(r) = 0$. Otherwise, there exists $\tau > 0$ and a sequence of unit-speed geodesics $\gamma_i, \eta_i : [0, r_i] \to X$ from x to some $x_i \in \partial B^X(x_\star, r_i)$, such that $\angle(\dot{\gamma}(r_i), \dot{\eta}_i(r_i)) \geq \tau$. After passing to a subsequence, we have pointed Cheeger-Gromov convergence

$$(B^X(x_\star, \alpha r_i), r_i^{-2} g, x_i) \to (B^{C(\mathsf{S}^3/\Gamma')}(x_\infty, \alpha), g_{C(\mathsf{S}^3/\Gamma')}, x_\infty)$$

for some $x_\infty \in \partial B^{C(\mathsf{S}^3/\Gamma')}(c_0, 1)$,

$$r_i^{-1} \dot{\gamma}_i(r_i) \to v \in T_{x_\infty} C(\mathsf{S}^3/\Gamma'),$$

$$r_i^{-1} \dot{\eta}_i(r_i) \to w \in T_{x_\infty} C(\mathsf{S}^3/\Gamma'),$$

pointed Gromov-Hausdorff convergence

$$(B^X(x_\infty,\delta), r_i^{-1}d_X, x_\infty) \to (C(S^3/\Gamma'), g_{C(S^3/\Gamma')}, c_0)$$

and (after constant-speed reparametrization) γ_i, η_i converge (smoothly on $(0,1]$) to unit-speed geodesics $\gamma_\infty, \eta_\infty : [0,1] \to X$ from c_0 to x_∞ with

$$\angle(\dot\gamma_\infty(1), \dot\eta_\infty(1)) = \angle(v,w) \geq \tau,$$

contradicting the fact that there is a unique minimizing geodesic from c_0 to any $x \in C(S^3/\Gamma')$. Therefore $\lim_{r\to 0}\theta(r) = 0$, so for $r_0 > 0$ sufficiently small, we can construct a smooth vector field whose flow can be used to construct a homeomorphism (see Proposition 12.1.2 and Lemma 12.1.3 of [P3]) between $B^X(x_\infty, 2s_i) \setminus B^X(x_\infty, s_i)$, $i = 1,2$, for any $s_1, s_2 \in (0, r_0]$. However, we know $B^X(x_\infty, 2s_i) \setminus B^X(x_\infty, s_i)$ is homeomorphic to $(1,2) \times S^3/\Gamma_i$ for some finite subgroups, so we must have $S^3/\Gamma_1 = S^3/\Gamma_2$.

We can now apply the argument in Step 1 of [DS] (see also [BZ2], [BKN],[T]) verbatim to our situation to conclude that there is a diffeomorphism $F : (B(0^4, r_0) \setminus \{0^4\})/\Gamma \to B^X(x_\infty, r_0) \setminus \{x_\infty\}$ such that $(F \circ \pi)^* g$ extends to a C^0 Riemannian metric on $B(0^4, r_0)$, where $\pi : \mathbb{R}^4 \to \mathbb{R}^4/\Gamma$ is the quotient map. By replacing g with $(F\circ\pi)^*g$, we may as well assume Γ is trivial. Define $A(r_1, r_2) := B^X(x_\infty, r_2) \setminus \overline{B}^X(x_\infty, r_1)$ and $B^* := B^X(x_\infty, r_0) \setminus \{x_\infty\}$.

Claim: $\int_{B^*} |Rm|^2 dg < \infty$.

In four dimensions, the curvature tensor Rm admits the orthogonal decomposition

$$Rm = \frac{R}{24} g \odot g + \frac{1}{2}\left(Rc - \frac{R}{4}g\right) \odot g + W. \tag{3.6.1}$$

Because $|R| \leq A$ on $\mathcal{R}$, the first term of (3.6.1) is bounded pointwise. We use the method of [HM] to estimate the second term of (3.6.1). Fix $\beta \in (0,1)$,and let $\phi \in C_c^\infty(A(\beta r_0, r_0))$ be a cutoff function with $|\nabla\phi| \leq C(n)(\beta r_0)^{-1}$ on $A(\beta r_0, 2\beta r_0)$, $|\nabla \phi| \leq C(n) r_0^{-1}$ on $A(\frac{1}{2}r_0, 2r_0)$, and $\phi = 1$ on $A(2\beta r_0, \frac{1}{2}r_0)$. Then, setting $E := \sup_B(e^{-f} + |\nabla f|)$, we get

$$\begin{aligned}
\int_B |Rc|^2\phi^2 e^{-f}dg &= \int_B \left\langle \frac{1}{2}g - \nabla^2 f, Rc \right\rangle \phi^2 e^{-f}dg \\
&= C(A,E) + \int_B \langle \nabla f, \mathrm{div}_f(\phi^2 Rc)\rangle e^{-f}dg \\
&\leq C(A,E) + \int_B 2|\nabla f||\nabla\phi|\phi|Rc|e^{-f}dg \\
&\leq C(A,E) + \frac{1}{2}\int_B |Rc|^2\phi^2 e^{-f}dg + 2\int_B |\nabla f|^2|\nabla\phi|^2 e^{-f}dg,
\end{aligned}$$

since $\mathrm{div}_f Rc = 0$. Rearranging, we conclude

$$\begin{aligned}
\int_B |Rc|^2\phi^2 e^{-f}dg &\leq C(A,E) + C(A,E,r_0)\beta^{-2}\mathrm{Vol}_g(A(0,2\beta r_0)) \\
&\leq C(A,E) + C(A,E,r_0)\beta^2.
\end{aligned}$$

Taking $\beta \to 0$, and recalling that f is locally bounded above, we obtain $\int_B |Rc|^2 dg < \infty$. Finally, to estimate the third term of (3.6.1), we further decompose W into the self-dual and anti-self-dual parts $W_\pm$, and then employ the strategy of [DS]. Let A_+ be the connection on the bundle Λ_+ of self-dual forms on $\mathcal{R}$ induced by the Levi-Civita connection of $(\mathcal{R}, g)$ (see section 6.D of [B6] for definitions). Then, because W_+ is self-dual, $(Rc - \frac{R}{4}g) \odot g$ is anti-self-dual, and Λ_+, Λ_- are orthogonal, we have

$$\int_{A(s,r)} \left(|W_+|^2 + \frac{R^2}{12} - \left|Rc - \frac{R}{4}g\right|^2\right) dg = \int_{A(s,r)} tr(F_{A_+} \wedge F_{A_+}),$$

but Remark 5 gives

$$\int_{A(s,r)} tr(F_{A_+} \wedge F_{A_+}) = CS(A_+, \partial B(x, r)) - CS(A_+, \partial B(x, s)) \quad (\mathrm{mod}\ \mathbb{Z}),$$

where (noting the difference in sign convention and notation from Chapter 2)

$$CS(B,\Sigma) := \frac{1}{8\pi^2}\int tr\left(dB \wedge B + \frac{2}{3}B \wedge B \wedge B\right) \in \mathbb{R}/\mathbb{Z}.$$

is the Chern-Simons invariant (associated to the first Pontryagin class) of a connection $\nabla = d + B$ on a trivial bundle over a 3-manifold Σ, once we have chosen an arbitrary global section of the bundle. However, by the Cheeger-Gromov convergence of $r^{-2}g|\partial B(x_\infty, r)$ to a flat bundle metric on $(T\mathbb{R}^4)|\mathbb{S}^3$ as $r \to \infty$, Corollary 5 gives

$$\lim_{r \searrow 0} CS(A_+, \partial B(x_\infty, r)) = 0$$

as $r \searrow 0$, so we can choose $r \in (0, r_0]$ sufficiently small such that $|CS(A_+, \partial B(x_\infty, s))| \leq \frac{1}{8}$ mod $\mathbb{Z}$ for all $s \in [0, r]$. Because

$$s \mapsto \int_{A(s,r)} tr(F_{A_+} \wedge F_{A_+}) \in [-\frac{1}{4}, \frac{1}{4}] \quad (\mathrm{mod}\ \mathbb{Z})$$

is continuous, we conclude that the integral is bounded uniformly (in $\mathbb{R}$) for all $s < r$. In particular, we can take $s \searrow 0$ to obtain $\int_{B_\infty} |W_+|^2 dg < \infty$, and the proof of $\int_{B_\infty} |W_-|^2 dg < \infty$ is similar.

We can now argue as in [CS1], [T], to conclude that in fact B_∞ has the structure of a C^∞ orbifold at x_∞. Note that, because we have bounds on $f, |\nabla f|$ on B_∞, the only difference in our setting is that we must use the ϵ-regularity theorem that is Theorem 1.1 of [JN2] or Theorem 1.2 of [JN2] (note that the completeness condition can be replaced with the condition that a larger geodesic ball is locally compact). Also, $R + |\nabla f|^2 = f - W(g, f)$, $|R| \leq A$, and the quadratic growth of f imply that all critical points of f must occur in some bounded set. On the other hand, any orbifold point of X must be a critical point: if $\varphi : \mathbb{R}^4/\Gamma \supseteq U \to B^X(x, \delta)$ is an orbifold chart, and $\pi : \mathbb{R}^4 \to \mathbb{R}^4/\Gamma$ is the quotient map, then $\nabla^{(\pi \circ \varphi)^* g}(\pi \circ \varphi)^* f$ must be fixed by all of Γ, so must be the zero vector. Since $X \setminus \mathcal{R}$ is discrete and bounded, it must be finite. □

Proof of Theorem 3. This is immediate from Theorem 24.

3.7 Appendix: L^p Ricci Curvature bounds and volume convergence

In this section, we give further details for the claim in Lemma 10 that

$$|B^X(y_i, \alpha\lambda_i^{-1}) \cap \mathcal{R}|_g \geq (\omega_n - \epsilon_i)(\alpha\lambda_i^{-1})^4$$

for some sequence $\epsilon_i \to 0$. The main idea is to use the fact that, for $i \in \mathbb{N}$ sufficiently large, $(B^X(y_i, \alpha\lambda_i^{-1}), \lambda_i d, y_i)$ is arbitrarily close to a Euclidean ball in the pointed Gromov-Hausdroff sense, and to then appeal to a volume convergence theorem for Riemannian manifolds with integral Ricci lower bounds.

Observe that, by Lemma 6.1 of [BZ2], we have

$$|Rc|_{\tilde{g}^i}(\cdot, 0) \leq C(A)(r^{\tilde{g}^i}_{Rm})^{-1}(\cdot, 0),$$

so combining this with the integral estimate for the curvature scale (Theorem 1.7 of [B1]) gives

$$\int_{B_{\tilde{g}^i}(x_i,0,1)} |Rc|^3(\cdot, 0) d\tilde{g}^i_0 \leq \int_{B_{\tilde{g}^i}(x_i,0,1)} (r^{g^i}_{Rm}(\cdot, 0))^{-3} d\tilde{g}^i_0 \leq C(A).$$

Note that we actually have a local L^p bound for Rc for any $p < 4$, and the following arguments will work for any $p \in (2, 4)$, but we choose $p = 3$ for convenience.

Let $\mathcal{H}^4_d = \mathcal{H}^4$ be the 4-dimensional Hausdorff measure on the metric space (X, d). Because $\mathcal{H}^4(X \setminus \mathcal{R}) = 0$, and because $\mathcal{H}^4$ agrees with the Riemannian volume measure on any 4-dimensional Riemannian manifold (in particular, on $\mathcal{R}$), we have $\mathcal{H}^4(S) = |S \cap \mathcal{R}|$ for any subset $S \subset X$. Thus

$$\mathcal{H}^4_{\lambda_i d}(B^X(y_i, \alpha\lambda_i^{-1})) = \lambda_i^4 |B^X(y_i, \alpha\lambda_i^{-1}) \cap \mathcal{R}|_g,$$

$$\mathcal{H}^4_{d_Z}(B^Z(y_\infty, \alpha)) = \omega_n \alpha^n.$$

We now restate the modification of Theorem 2.37 of [TZ] that we will be using. Denote by $|Rc_-|(x)$ the absolute value of the smallest negative eigenvalue of $Rc(x)$ (if $Rc(x) \geq 0$, then $|Rc_-| = 0$).

Lemma 11. *For any $\kappa > 0$, $\Lambda < \infty$, $n \in \mathbb{N}$, and $p > n$, there exist $r_0 = r_0(n, p, \kappa, \Lambda, \epsilon) > 0$ such that the following holds. Suppose (M_i^n, g_i, x_i) is a sequence of complete Riemannian manifolds satisfying:*

(i) $\int_{B(x,1)} |Rc_-|^p dg \leq \Lambda$ *for all* $x \in M_i$,

(ii) $|B(x,r)| \geq \kappa r^n$ *for all* $r \in (0,1]$, $x \in M$.

Assume that (M_i^n, g_i, x_i) converge in the pointed Gromov-Hausdorff sense to the complete metric length space (X, d, p). Then, for any $r \in (0, r_0]$, we have

$$\mathcal{H}_d^n(B(x,r)) = \lim_{i \to \infty} |B(x_i, r_i)|.$$

The difference between this lemma and Theorem 2.37 of [TZ] is that we only require a local integral Ricci bound (i) rather than the global bound

$$\int_M |Rc_-|^p dg \leq \Lambda$$

assumed in [TZ]. However, in [TZ], the objects under consideration are time slices of a normalized Ricci flow on a Fano threefold, which have uniformly bounded diameter. The proof of Theorem 2.37 is stated to be a modification of volume convergence for noncollapsed Riemannian manifolds with Ricci curvature bounded below, given in [C2, CC1]. A careful examination of the proof shows that only the conditions $(i), (ii)$ are used, essentially due to the fact that the involved arguments are all local.

The following elementary lemma is essentially a consequence of Lemma 22 and a diagonal argument.

Lemma 12. *Let (X_k, d_k, p_k) be a sequence of limit spaces as in Lemma 22, converging in the pointed Gromov-Hausdorff sense to (X,d,p), and suppose $r \leq r_0(n, p, \kappa, \Lambda)$. Then*

$$\mathcal{H}^n(B(p_k, r)) \to \mathcal{H}^n(B(p, r)).$$

Proof. For each $k \in \mathbb{N}$, let $(M_{k,i}, g_{k,i}, x_{k,i})$ be a sequence of complete, pointed Riemannian manifolds satisfying $(i), (ii)$ of Lemma 11, which converge in the pointed Gromov-Hausdorff sense to (X_k, d_k, x_k) as $i \to \infty$. Also let (M_i, g_i, x_i) be a sequence of such manifolds converging to (X, d, p) in the pointed Gromov-Hausdorff sense. By Lemma 11, we know that

$$\lim_{i \to \infty} |B(x_{k,i}, r)|_{g_{k,i}} = \mathcal{H}^n(B(x_k, r))$$

for each $k \in \mathbb{N}$. Thus, for each $k \in \mathbb{N}$, we can find $i(k) \in \mathbb{N}$ such that

$$\left| |B(x_{k,i(k)}, r)|_{g_{k,i(k)}} - \mathcal{H}^n(B(x_k, r)) \right| \leq 2^{-k},$$

$$d_{GH}\Big((B(x_{k,i(k)}, \alpha_k r), d_{g_{k,i(k)}}, x_{k,i(k)}), (B(x_k, \alpha_k r), d_k, x_k)\Big) \leq r2^{-k},$$

where $\alpha_k \to \infty$. In particular, $(M_{k,i(k)}, g_{k,i(k)}, x_{k,i(k)})$ converge in the pointed Gromov-Hausdorff sense to (X, d, p), so

$$\left| |B(x_{k,i(k)}, r)|_{g_{k,i(k)}} - \mathcal{H}^n(B(x, r)) \right| \to 0.$$

Combining expressions gives the claim. □

After possibly shrinking α so that $\alpha < r_0$, we can apply the previous lemma to

$$(B^X(y_i, \alpha\lambda_i^{-1}), \lambda_i d, y_i) \to (B^Z(y_\infty, \alpha), d_Z, y_\infty),$$

we conclude that

$$\mathcal{H}^4_{\lambda_i d}(B^X(y_i, \alpha\lambda_i^{-1}) \to \mathcal{H}^4(B^Z(y_\infty, \alpha))$$

as $i \to \infty$. That is,

$$\lim_{i \to \infty} \lambda_i^4 |B^X(y_i, \alpha\lambda_i^{-1}) \cap \mathcal{R}| = \omega_n \alpha^4,$$

so the claim follows.

CHAPTER 4

VOLUME-NONCOLLAPSED RICCI FLOW WITH RICCI CURVATURE BOUNDED BELOW

4.1 Statement of Results

In this paper, we study closed solutions $(M^n, (g_t)_{t\in[0,T)})$ of Ricci flow satisfying the following assumptions for all $t \in [0,T)$:

$$Rc(g_t) \geq -Ag_t, \tag{4.1.1}$$

$$|M|_{g_t} \geq A^{-1}, \tag{4.1.2}$$

where $A < \infty$ is constant. One motiviation for studying such flows is to get a clearer picture of how the curvature of a Ricci flow fails to be controlled near finite-time singularities. N. Šešum showed in [Š] that any Ricci flow satisfying a two-sided curvature bound

$$-Ag_t \leq Rc(g_t) \leq Ag_t$$

cannot develop a finite-time singularity. B. Wang showed in [W] that a Ricci flow satisfying (4.1.1) as well as the spacetime integral estimate for scalar curvature

$$\int_0^T \int_M |R|^{\frac{n+2}{2}} dg_t dt < \infty$$

cannot develop a singularity either. Even when $n = 4$, it is still an open problem whether a finite-time singularity can occur for a Ricci flow with bounded scalar curvature:

$$\sup_{M\times[0,T)} R < \infty, \tag{4.1.3}$$

(in which case assumption (4.1.2) holds automatically) though considerable progress has been made [BZ2, BZ1, B1, CW1, CW3, CW4, S3]. In particular, Bamler-Zhang [BZ2] proved, and it was shown independently by M. Simon [S3], that in four dimensions, a Ricci flow satisfying (4.1.3) must converge in the Gromov-Hausdorff sense as $t \nearrow T$ to a C^0 Riemannian orbifold, with smooth convergence away from the orbifold points. Simon also showed that such a flow can be continued through the singularity via orbifold Ricci flow (Theorem 9.1 of [S3]).

Ricci flow solutions satisfying (4.1.1),(4.1.2) were studied by X. Chen and F. Yuan in [CY], where they asked whether such a flow can develop a singularity in finite time. They found that this cannot occur in dimension three. This had previously been shown by Z. Zhang (Theorem 1.1 of [Z6]) in all dimensions if we assume g_0 is Kähler. Assumption (4.1.1) by itself is clearly insufficient to rule out singularities (the round shrinking sphere is a counterexample), and in general the behavior of Ricci flows satisfying (4.1.1) and $\lim_{t \to T} |M|_{g_t} = 0$ is more complicated than those satisfying (4.1.1),(4.1.2).

Another motivation for considering conditions (4.1.1),(4.1.2) is the extensive compactness and partial regularity theory developed by Cheeger-Colding-Naber-Tian-Jiang [C2, CC1, CC2, CC3, CCT, CN2, CN3, CJN] for sequences of Riemannian manifolds (M_i, g_i) satisfying assumptions

$$Rc(g_i) \geq -Ag_i, \tag{4.1.4}$$

$$|M_i|_{g_i} \geq A^{-1}, \tag{4.1.5}$$

$$\text{diam}_{g_i}(M_i) \leq A. \tag{4.1.6}$$

We refer to Gromov-Hausdorff limits of such (M_i, g_i) (or pointed Gromov-Hausdorff limits if we drop assumption (4.1.6)) as noncollapsed Ricci limit spaces. The aforementioned works showed that any sequence satisfying (4.1.4),(4.1.5),(4.1.6) must subsequentially converge to a compact metric length space (X, d) which is bi-Hölder homeomorphic to a smooth

Riemannian manifold on a subset $\mathcal{R}$ whose complement has Hausdorff codimension two [CJN]. Also, if dg_i is the Riemannian volume measure of (M_i, g_i), then

$$(M_i, g_i, dg_i) \to (X, d, \mathcal{H}^n)$$

in the measured Gromov-Hausdorff sense, where $\mathcal{H}^n$ is the n-dimensional Hausdorff measure of X (this is known as Colding's Volume Convergence Theorem [C2]). Theorem 1 of [CY] states that if $(M_i, g_i) = (M, g_{t_i})$ for some sequence of times $t_i \nearrow T$, where $(M, (g_t)_{t\in[0,T)})$ is a closed Ricci flow satisfying (4.1.1),(4.1.2), then $\mathcal{R}$ is open and actually has the structure of a smooth Riemannian manifold. The intuition espoused in [CY] is that the Ricci flow equation should imply that in some respects, $(M, g_{t_i})_{i\in\mathbb{N}}$ behaves like a sequence of Riemannian manifolds satisfying (4.1.5),(4.1.6), and the two-sided Ricci bound

$$-Ag_i \leq Rc(g_i) \leq Ag_i, \tag{4.1.7}$$

whose limit spaces have singularities of codimension four [CN3]. Some evidence for this idea is that limits of $(M, g_{t_i})_{i\in\mathbb{N}}$ must satisfy Anderson's ϵ-regularity theorem for the volume ratio, which usually only holds for Ricci limit spaces corresponding to sequences satisfying (4.1.7) (Theorem 3.2 and Remark 3.3 of [A1]).

In this paper, we address the question posed in [CY], with our strongest results holding only in the special case of dimension four. The structure of the limit space in four dimensions is summarized in the following theorem.

Theorem 25. *Suppose $(M^4, (g_t)_{t\in[0,T)})$ is a simply connected four-dimensional closed Ricci flow satisfying $Rc(g_t) \geq -Ag_t$ and $\inf_{t\in[0,T)} |M|_{g_t} \geq A^{-1} > 0$ for some constant $A < \infty$. Then (M, d_{g_t}) converge in the Gromov-Hausdorff sense as $t \nearrow T$ to a Ricci limit space (X, d) with the structure of a C^0 Riemannian orbifold with finite singular set. Moreover, the convergence is smooth away from the singular points of X, and each singular point $\overline{x} \in X$ has unique tangent cone equal to $C(\mathbb{S}^3/\Gamma_{\overline{x}})$ for some finite subgroup $\Gamma_{\overline{x}} \subseteq O(4,\mathbb{R})$.*

Remark 8. *The assumption of simple connectedness is only used to imply orientability and the generalized Jordan-Brouwer Separation Theorem, so may be replaced with the weaker condition $H_1(M, \mathbb{Z}/2\mathbb{Z}) = \{0\}$. Even this condition may not be necessary, though it is convenient for our proof of Theorem 25.*

In addition, $X = M/\sim$ is a topological quotient of the original space M (see Section 3, and also Corollary 1.3 of [BZ1]). The orbifold structure of X mirrors the corresponding result for four-dimensional Gromov-Hausdorff limits of sequences of pointed Riemannian manifolds satisfying (4.1.5),(4.1.6),(4.1.7). In fact, [BKN, CN3] imply that such limits must be C^0 Riemannian orbifolds. We also note that the same result was proved for time-slices of Ricci flows by [BZ2, S3] with the hypotheses (4.1.1),(4.1.2) replaced by (4.1.3). Using Theorem 25, we can apply Theorem 9.1 of [S3] to conclude that there is a flow through the singularity at time T.

Corollary 8. *With the same hypotheses of Theorem 25, there exists $\delta > 0$ and a solution $(\widetilde{M}^4, (\widetilde{g}_t)_{t \in [T, T+\delta)})$ of the orbifold Ricci flow such that*

$$\lim_{t \searrow T} d_{GH}\left((\widetilde{M}, d_{\widetilde{g}_t}), (X, d) \right) = 0.$$

One major difference between our setting and that of Ricci flows satisfying (4.1.3) is that in [BZ2], it is shown that the 'deepest bubbles' of $(M, (g_t)_{t \in [0,T)})$ satisfying (4.1.3) (that is, the dilation limits arising from rescaling so that the maximum value of $|Rm|_{g_t}$ is 1) are Ricci-flat ALE spaces (Corollary 1.9 of [BZ2]), so that orbifold singularities are precluded by a simple topological condition (Corollary 1.10 of [BZ2]). We are unable to prove such results in our setting at present, but we are still able to give a fairly complete description of the behavior of the flow at the Type-I curvature scale. Both the statement and the proof of this behavior rely heavily on the recent convergence and partial regularity theory for Ricci flows produced by R. Bamler [B4, B3, B5], which can be seen as the parabolic analogue of Cheeger-Colding-Naber-Tian-Jiang's theory. In [B3], Bamler defines the parabolic notion of a metric space, called a metric flow (Definition 3.2 of [B3]), and the parabolic analogue of pointed Gromov-Hausdorff convergence of pointed metric spaces, termed $\mathbb{F}$-convergence of metric flow pairs (Definition 5.8 [B3]). Roughly speaking, he shows that any sequence of Ricci flows $(M_i, (g_t^i)_{t\in I^i}, (\nu^i_{x_i,t_i;t})_{t\in I^i\cap(-\infty,t_i)})$, where $\nu^i_{x_i,t_i;t} = K^i(x_i,t_i;\cdot,t)dg_t$, and $K^i(x_i,t_i;\cdot,\cdot)$ is the conjugate heat kernel of $(M_i,(g_t^i)_{t\in I^i})$ based at (x_i,t_i), $\mathbb{F}$-converge (after passing to a subsequence) to some metric flow, which is a smooth Ricci flow spacetime (see Definition 9.1 of [B3]) outside of a set of parabolic codimension four. If $(M_i,(g_t^i)_{t\in I^i})$ are Type-I rescalings of any fixed Ricci flow with fixed basepoint, Bamler proves that any $\mathbb{F}$-limit (called a tangent flow) must be a singular shrinking gradient Ricci soliton with singularities of Minkowski codimension four (Theorem 1.19 of [B5]). We show that under assumptions (4.1.1),(4.1.2), such a tangent flow must be a static flow modeled on $C(\mathbb{S}^3/\Gamma)$ for some finite subgroup $\Gamma\subset O(4,\mathbb{R})$, and show that convergence also occurs in the Gromov-Hausdorff sense on each time slice.

Theorem 26. *Given the notation and hypotheses of Theorem 25, let $x\in M$ correspond to a singular point $\overline{x}\in X = M/\sim$, and let $C(\mathbb{S}^3/\Gamma_{\overline{x}})$ be the corresponding tangent cone as in Theorem 25.*

(i) Let $(\nu_{x,T;t})_{t\in(0,T)}$ be a conjugate heat kernel at the singular time based at x (see the discussion at the beginning of Section 3). Then every corresponding tangent flow (c.f. Theorem 1.38 of [B5]) is a static metric flow corresponding to the singular space $C(\mathbb{S}^3/\Gamma_{\overline{x}})$.

(ii) If o_* is the vertex of $C(\mathbb{S}^3/\Gamma_{\overline{x}})$, then

$$(M,(T-t)^{-\frac{1}{2}}d_{g_t},x) \to (C(\mathbb{S}^3/\Gamma_{\overline{x}}), d_{C(\mathbb{S}^3/\Gamma_{\overline{x}})}, o_*)$$

in the pointed Gromov-Hausdorff sense as $t\nearrow T$.

It was shown in [CN2] that any Riemannian manifold (M,g) satisfying (4.1.5),(4.1.6),(4.1.7) also satisfies an estimate of the form

$$\int_M r_h{}^{-p}(x)dg \le C(A,p)$$

for any $p \in (0,4)$, where r_h is the $C^{1,\alpha}$ harmonic radius. If in addition (M,g) is Einstein, then r_h can be replaced by the curvature scale $\widetilde{r}_{Rm}$ (see Section 2 for the definition). Our next theorem states that such an estimate also holds for the time slices of a Ricci flow satisfying (4.1.1),(4.1.2) when $n=4$.

Theorem 27. *Given the hypotheses of Theorem 25, we have*

$$\sup_{t\in[0,T)}\int_M |Rm|^{\frac{p}{2}}(x,t)dg_t(x) \le \sup_{t\in[0,T)}\int_M \widetilde{r}_{Rm}^{\,-p}(x,t)dg_t(x) < \infty$$

for any $p \in (0,4)$. Moreover, there exists $E<\infty$ such that for all $s \in (0,1]$ and $t \in [0,T)$, we have

$$|\{\widetilde{r}_{Rm}(\cdot,t) < s\}|_{g_t} \le Es^4.$$

For $p \in [2,4)$, we are unable to get a bound which is uniform in the parameter A, even if we only take the supremum over $t \in [\frac{T}{2},T)$ and include an upper bound on diameter or only integrate over geodesic balls of fixed radius. It is unclear whether or not the estimate of Theorem 27 can be made uniform in the parameters A,T,p. Another remaining question is whether this estimate holds when $p=4$.

We now consider the higher-dimensional case.

Theorem 28. *Let $(M^n,(g_t)_{t\in[0,T)})$ be a closed Ricci flow satisfying (4.1.1),(4.1.2), and let $(\nu_{x,T;t})_{t\in(0,T)}$ be a conjugate heat kernel based at the singular time at $x \in M$, corresponding to a singular point $\overline{x}$ in the Gromov-Hausdorff limit $\lim_{t\nearrow T}(M,d_{g_t})$. Then every tangent flow at $\nu_{x,T}$ is a static metric flow modeled on a metric cone $(C(Z),o_*)$ which is Ricci-flat on its regular part, but $C(Z) \neq \mathbb{R}^n$. Moreover, if $(g_t^i)_{t\in[-\tau_i^{-1}T,0)}$ is a sequence of Type-I rescalings $\mathbb{F}$-converging to a given tangent flow, then we can pass to a subsequence so that for every*

$t \in (-\infty, 0)$, *there exist* $x_t^i \in M$ *such that*

$$(M, d_{g_t^i}, x_t^i) \to (C(Z), d_{C(Z)}, o_*)$$

in the pointed Gromov-Hausdorff sense as $i \to \infty$.

We also prove a local version of the fact that a Ricci flow satisfying (4.1.1),(4.1.2) cannot develop a Type-I singularity.

Theorem 29. *Let* $(M^n, (g_t)_{t\in[0,T)})$ *be a closed Ricci flow satisfying* (4.1.1), (4.1.2)*. Again let* $(X, d) = \lim_{t\nearrow T}(M, d_{g_t})$ *be the corresponding Ricci limit space. Then, for any* $x \in M$ *corresponding to a singular point* $\overline{x} \in X$*, we have*

$$\limsup_{t\nearrow T}(T-t)r_{Rm}^{-2}(x,t) = \infty.$$

Observe that this implies there is a sequence $(x_i, t_i) \in M \times [0,T)$ with $t_i \nearrow T$, $d_{g_{t_i}}(x_i, x) = o(\sqrt{T-t_i})$ and

$$\lim_{i\to\infty} |Rm|(x_i,t_i)(T-t_i) = \infty.$$

In the notation and terminology of Definition 1.1 of [BM2], this means that every singular point $x \in M$ is a Type-II point, so $\Sigma = \Sigma_{II}$.

Finally, though we are so far unable to prove L^p curvature estimates for $p \in [1,2)$ in higher dimensions, we do get L^p bounds on the (time-dependent) set where the curvature scale is smaller than $\sqrt{T-t}$. This is made precise in the following Theorem.

Theorem 30. *For any* $A < \infty$, $\underline{T} > 0$, *and* $p \in (0,4)$, *there exist* $r_0 = r_0(p, A, \underline{T}) > 0$, $E = E(A, \underline{T}) < \infty$, *such that the following hold. Let* $(M^n, (g_t)_{t\in[0,T)})$ *be a closed Ricci flow satisfying* $Rc(g_t) \geq -Ag_t$ *and* $|B(x,t,r)|_{g_t} \geq A^{-1}r^n$ *for all* $(x,t) \in M \times [0,T)$ *and* $r \in (0,1]$*, where* $T \geq \underline{T}$.

(i) *For any* $(x,t) \in M \times [\frac{T}{2}, T)$, $r \in (0, r_0\sqrt{T-t}]$, *and* $s \in (0,1]$, *we have*

$$|\{\widetilde{r}_{Rm}(\cdot,t) < sr\} \cap B(x,t,r)|_{g_t} \leq Es^4 r^n.$$

(ii) *For any* $\alpha \in [1, \infty)$*, there exists* $C_\alpha := C_\alpha(p, A, \underline{T}) < \infty$ *such that*

$$\int_{B(x,t,\alpha\sqrt{T-t})} |Rm|^{\frac{p}{2}} dg_t \leq \int_{B(x,t,\alpha\sqrt{T-t})} \widetilde{r}_{Rm}^{\,-p}(\cdot, t) dg_t \leq C_\alpha (T-t)^{\frac{n-p}{2}}.$$

for all $(x,t) \in M \times [\frac{T}{2}, T)$*.*

We now give an outline of the paper and the proofs of the main theorems.

In Section 2, we recall and develop some necessary facts relevant to Ricci flow and Ricci limit spaces.

In Section 3, we prove Theorems 28,29, and part (i) of Theorem 26. The rough idea of the proof comes from observing that any singularity model which is a shrinking gradient Ricci soliton (GRS) with nonnegative Ricci curvature and maximal volume growth must be flat (Corollary 1.1 in [CN1]). If the flow $(M, (g_t)_{t \in [0,T)})$ were Type-I, then any Type-I singularity model at any point would therefore be flat, so that no singularity occurs (for example, using Theorem 1.2 of [EMT]). To make this work for Type-II flows, we need to apply the compactness and partial regularity theory of [B4, B3, B5]. In particular, we fix $x \in M$ corresponding to a singular point $\overline{x} \in X$, a sequence $\tau_i \searrow 0$, and consider the Type-I rescaled solutions $g_t^i := \tau_i^{-1} g_{T+\tau_i t}$ and their corresponding conjugate heat kernels K^i based at x at the singular time. Seen as metric flow pairs, we can pass to a subsequence to obtain $\mathbb{F}$-convergence (see Section 5 of [B3]) to a metric soliton modeled on a singular shrinking GRS $(Y, d, \mathcal{R}_Y, g_Y, f)$ with nonnegative Ricci curvature on $\mathcal{R}_Y$ (see Section 2 for relevent definitions). This implies in particular that there are diffeomorphisms $\psi_i : U_i \to M$, where (U_i) is an exhaustion of $\mathcal{R}_Y$, such that $\psi_i^* g^i \to g^\infty$ in $C^\infty_{loc}(\mathcal{R}_Y)$.

One technical hurdle for proceeding as in the Type-I case is that in general volume convergence does not follow from $\mathbb{F}$-convergence, which makes it difficult to show that the limiting GRS has maximal volume growth. To address this, we prove an estimate (Lemma 15) for the flows g^i showing that, for some $\sigma > 0$, the set of points near an H_n-center of

x which satisfy $K^i(\;,\; 1) \quad \sigma$ has almost-full measure. This relies on several results from Cheeger-Colding theory, including Cheeger-Colding's segment inequality and an estimate on the size of the quantitative singular strata on each time slice. From this inequality, we can extract pointed Gromov-Hausdorff convergence of time slices from W_1-Gromov-Wasserstein convergence, and then apply Colding's volume convergence theorem. We use this to establish Theorem 28. A modification of a proof of Ni's (Proposition 1.1 in [N2]) shows that Corollary 1.1 of [CN1] holds for singular shrinking solitons in our setting, which we use to prove Theorem 29 and part (i) of Theorem 26.

In Section 4, we specialize to dimension four, and prove part (ii) of Theorem 26 by showing that for each time slice of the Type-I rescaled solutions, x is near the part of M corresponding to the vertex $o \;\; 2 \; C(\mathsf{S}^3/\Gamma)$. The idea is that the (rescaled) conjugate heat kernels based at (x, T) converge in the Cheeger-Gromov sense to the heat kernel of $C(\mathsf{S}^3/\Gamma)$, which gives a lower bound on K^i at bounded distance from points corresponding to the vertex. On the other hand, Bamler's Gaussian upper bound on the conjugate heat kernel in terms of distance from an H_4-center contradicts this lower bound if an H_4 center is too far away from points corresponding to the vertex. We combine this with a Gaussian heat kernel upper bound (see the Appendix) to show that x is in the "inner" component of M separated by some hypersurfaces $\psi_i(\partial B(o \;, \alpha_i))$, where $\alpha_i \;\, \& \;\, 0$, and then show that the diameter of this component goes to zero. We prove Theorem 25 by applying Perelman's pseudolocality theorem on regions of (M, g_t^i) corresponding to annuli centered at the vertex of $C(\mathsf{S}^3/\Gamma)$, which (as in [EMT]) implies that the curvature bound extends forwards in time all the way to the singular time, and is uniform on annuli centered at x. We can extract from this that $\overline{x}$ is an isolated singular point of X, and that the curvature bound blows up no faster than the rate $\frac{1}{d^2(\;,\overline{x})}$ on X. Then a maximum principle argument for the Ricci tensor rules out nonflat tangent cones, and implies that the curvature blows up along X at a rate strictly slower than the rate $\frac{1}{d^2(\;,\overline{x})}$. Standard arguments then show that X must be a C^0 orbifold, and Simon's construction allows one to flow through the singularity.

In Section 5, we prove a codimension two ϵ-regularity result, which roughly says that, for a closed Ricci flow satisfying (4.1.1),(4.1.2), Gromov-Hausdorff closeness to a cone which splits $\mathbb{R}^{n-2}$ implies a curvature bound, at scales $<< \sqrt{T-t}$. This is done by a contradiction-compactness argument with careful point-picking, where we first rescale and obtain points which converge in the Gromov-Hausdorff sense to $\mathbb{R}^{n-2} \times Z$ for some metric space Z, but whose curvature blows up. We use Cheeger-Naber's slicing theorem as in the proof of Theorem 5.2 in [CN3] to change the blowup points (without changing the time slices) and further rescale to get smooth Cheeger-Gromov convergence to $\mathbb{R}^{n-2} \times S$, where S is an immortal 2-dimensional Ricci flow with nonnegative curvature. Using the classification of 2-dimensional steady and expanding solitons, we can change basepoints and scales once again to get smooth Cheeger-Gromov convergence to $\mathbb{R}^{n-2} \times E$, where E is an expanding Ricci soliton asymptotic at infinity to $C(\mathsf{S}^1_\beta)$, where S^1_β denotes the circle with circumference $\beta \in (0, 2\pi)$. Because this is still a blowup of the original flow, we can use Bamler's compactness theory for $\mathbb{F}$-convergence to obtain an ancient metric flow coinciding with $\mathbb{R}^{n-2} \times E$ for time $t \in (0, 1)$. However, at its initial time $t = 0$, the soliton E converges smoothly to the flat cone $C(\mathsf{S}^1_\beta)$ away from its vertex, so we can show that the metric flow is static for all times $t \in (-\infty, 0]$, and in fact coincides with $\mathbb{R}^{n-2} \times C(\mathsf{S}^1_\beta)$ on each time slice, contradicting the Minkowski dimension estimates for the singular set of static metric flows.

In Section 6, we prove a codimension three ϵ-regularity result via another contradiction argument, where now the Gromov-Hausdorff limit is $\mathbb{R}^{n-3} \times C(Z)$ for some two dimensional metric space Z. To show Z is smooth, we use the codimension two ϵ-regularity theorem to estimate size of the quantitative singular set at scales $<< \sqrt{T-t}$. This rules out codimension 2 singularities for $\mathbb{R}^{n-3} \times C(Z)$, showing that Z is smooth. Then a maximum principle argument for the Ricci curvature on the regular set of $\mathbb{R}^{n-3} \times C(Z)$ guarantees that the cone is flat, and because M is orientable, it follows that $\mathbb{R}^{n-3} \times C(Z)$ is the Euclidean cone. From this, we obtain Theorem 30.

In Section 7, we prove Theorem 27 using a time-dependent decomposition of time slices to estimate the size of the set $\{\widetilde{r}_{Rm}(\cdot,t) < s\}$. When $s << \sqrt{T-t}$, this is achieved by combining codimension three ϵ-regularity with Cheeger-Jiang-Naber's volume estimates for quantitative singular strata. When $s \sim \sqrt{T-t}$, our estimate uses our knowledge of the flow at the Type-I scale. When $s >> \sqrt{T-t}$, we use pseudolocality to estimate this region using the corresponding region of the limit space X.

Finally, in the Appendix, we indicate a modification of the proof of Theorem 3.1 in [CZ1], which leads to a Gaussian upper bound for the heat kernel in the presence of a negative Ricci curvature lower bound. This result was used in Section 4.

4.2 Preliminary Estimates

Throughout this chapter, we usually omit the dependence of constants on the dimension n.

We now establish several basic geometric estimates that follow from assumptions (4.1.1),(4.1.2).

Lemma 13. *There exists $\overline{A} = \overline{A}(n, A, T, \mathrm{diam}_{g_0}(M)) \in (1, \infty)$ such that the following hold for any Ricci flow satisfying (4.1.1),(4.1.2):*

(i) $\overline{A}^{-1} \leq \mathrm{diam}_{g_t}(M) \leq \overline{A}$ for all $t \in [0,T)$,

(ii) For every $r \in (0, \max\{1, \sqrt{T}\}]$ and $(x,t) \in M \times [0,T)$, $\overline{A}^{-1} r^n \leq |B(x,t,r)|_{g_t} \leq \overline{A} r^n$,

(iii) $\mathcal{N}_{x,t}(\tau) \geq -\overline{A}$ for all $(x,t) \in M \times [0,T)$ and $\tau \in (0,t]$.

Proof. (i) Since $[0,T) \to (0,\infty)$, $t \mapsto \mathrm{diam}_{g(t)}(M)$ is locally Lipschitz, it suffices to note that, for any $x \in M$ and $V \in T_xM$, we have

$$\log\left(\frac{g_t(V,V)}{g_0(V,V)}\right) = \int_0^t \partial_s \log g_s(V,V)ds = -2\int_0^t \frac{Rc_{g_s}(V,V)}{g_s(V,V)}ds \leq 2At,$$

so that $\text{diam}_{g_t}(M) \leq e^{AT}\text{diam}_{g_0}(M)$. The lower bound follows by combining $\inf_{t\in[0,T)}|M|_{g_t} \geq A^{-1}$ with $|M|_{g_t} \leq v_{-A}(\text{diam}_{g_t}(M))$.

(ii) The upper bound follows from Bishop volume comparison. For the lower bound, fix $(x,t) \in [0,T)$, $r \in (0, \max\{1,\sqrt{T}\}]$, and let $\overline{A} = \overline{A}(n, \text{diam}_{g_0}(M), T, A)$ be the constant from (i). Then

$$
\begin{aligned}
|B(x,t,r)|_{g_t} &= \frac{|B(x,t,r)|_{g_t}}{|B(x,t,\overline{A}+\max\{1,\sqrt{T}\})|_{g_t}}|M|_{g_t} \\
&\geq \frac{v_{-A}(r)}{v_{-A}(\overline{A}+\max\{1,\sqrt{T}\})}A^{-1} \\
&\geq C(n,\text{diam}_{g_0}(M),T,A)r^n,
\end{aligned}
$$

so (ii) follows after possibly modifying $\overline{A}$.

(iii) Because $R \geq -nA$, we can apply Theorem 8.1 of [B4] with $r = \sqrt{\tau}$ and use (ii) to estimate

$$
\overline{A}^{-1}\tau^{\frac{n}{2}} \leq \left|B\left(x,t,\sqrt{\tau}\right)\right|_{g_t} \leq C(n,A,T)\exp(\mathcal{N}_{x,t}(\tau))\tau^{\frac{n}{2}}.
$$

□

We recall the following notions of curvature scale:

Definition 29. *For* $(x,t) \in M \times [0,T)$, *we define*

$$
r_{Rm}(x,t) := \sup\{r>0; |Rm| \leq r^{-2} \text{ on } B(x,t,r) \times ([t-r^2,t+r^2] \cap [0,T))\},
$$

$$
\widetilde{r}_{Rm}(x,t) := \sup\{r>0; |Rm| \leq r^{-2} \text{ on } B(x,t,r)\}.
$$

It is important that these two notions are comparable, which follows from combining Perelman's pseudolocality theorem [P1] *(we use the version stated and proved in* [L2]*) with the backwards pseudolocality theorem, proved for Ricci flows with a Ricci lower bound by Chen-Yuan (Theorem 3 of* [CY]*), and later in the general setting by Bamler (Theorem 1.48 of* [B5]*).*

Theorem 31. *(Forwards and Backwards Pseudolocality) For any $A < \infty$, there exists $\epsilon_P = \epsilon_P(A) > 0$ such that the following hold. Suppose $(M^n, (g_t)_{t\in[-2,T]})$ is a closed, pointed Ricci flow (where $T \geq 0$) satisfying $Rc(g_t) \geq -Ag_t$ and $|B(x,t,r)|_{g_t} \geq A^{-1}r^n$ for all $(x,t) \in M \times [-1,T]$ and $r \in (0,1]$. If $x \in M$ and $r \in (0,1]$ are such that $|Rm|(\cdot,0) \leq r^{-2}$ on $B(x,0,r)$, then $|Rm| \leq (\epsilon_P r)^{-2}$ on $B(x,0,\epsilon_P r) \times [-(\epsilon_P r)^2, \min((\epsilon_P r)^2, T)]$. In particular, we have*

$$r_{Rm}(y,t) \geq \widetilde{r}_{Rm}(y,t) \geq \epsilon_P^{-1}(A) r_{Rm}(y,t)$$

for all $(y,t) \in M \times [-1,T]$.

Proof. The bound $|Rm| \leq (\epsilon_0 r)^{-2}$ on $B(x,0,\epsilon_0 r) \times [0, \min((\epsilon_0 r)^2, T)]$ for some $\epsilon_0 = \epsilon_0(A) > 0$ is Proposition 2.1 of [L2], while the analogous bound on $B(x,0,\epsilon_1 r) \times [-(\epsilon_1 r)^2, 0]$ for some $\epsilon_1 = \epsilon_1(A) > 0$ is Theorem 2.47 of [B5]. Take $\epsilon_P(A) := \min(\epsilon_0(A), \epsilon_1(A))$. For the remaining claim, we suppose $(y,t) \in M \times [-1,T]$ satisfies $\widetilde{r}_{Rm}(y,t) = r$. Then $|Rm|(\cdot,t) \leq r^{-2}$, so applying the first claim to the time-translated flow gives $r_{Rm}(y,t) \geq \epsilon_P(A) r$. □

If (M,g) is a (possibly incomplete) Riemannian manifold, we let $\widetilde{r}^{(M,g)}_{Rm}(x)$ be the supremum of all $r > 0$ such that $B_g(x,r)$ has compact closure in M and $|Rm| \leq r^{-2}$ on $B_g(x,r)$. If (X,d) is a noncollapsed Ricci limit space whose regular part $\mathcal{R}$ is open and has the structure of a smooth Riemannian manifold $(\mathcal{R}, g)$, we let $\widetilde{r}^X_{Rm}(x)$ be the supremum of $r > 0$ such that $B(x,r) \subseteq \mathcal{R}$ and $|Rm| \leq r^{-2}$ on $B(x,r)$. Equivalently, $\widetilde{r}^X_{Rm} = \widetilde{r}^{(\mathcal{R},g)}_{Rm}$ on $\mathcal{R}$, while $\widetilde{r}^X_{Rm} = 0$ on $X \setminus \mathcal{R}$.

We also make use of the following theorem from [CY], which is essentially a combination of backwards pseudolocality and the point-picking procedure from Anderson's ϵ-regularity theorem. We let $v_{-\rho}(r)$ denote the volume of a geodesic ball of radius r in the simply connected n-dimensional space form with curvature $-\frac{\rho}{n-1}$, and let ω_n be the volume of the unit ball in $\mathbb{R}^n$.

Theorem 32. *(Chen-Yuan ϵ-Regularity) For any $A \in [0,\infty)$ and $\lambda \in (0,1]$, there exists $\delta = \delta(A) > 0$ such that the following hold. If $(M^n,(g_t)_{t\in[-\lambda^{-1},0]})$ is a closed, pointed Ricci flow satisfying $Rc(g_t) \geq -\lambda A g_t$ and $|B(x,t,r)|_{g_t} \geq A^{-1}r^n$ for all $(x,t) \in M \times [-\lambda^{-1},0]$ and $r \in (0,\lambda^{-\frac{1}{2}}]$, then for any $x \in M$ and $r \in (0,\lambda^{-\frac{1}{2}}]$ with $|B(x,0,r)|_{g_0} > (1-\delta)v_{-\lambda A}(r)$, we have $\widetilde{r}_{Rm}(x,0) \geq \delta r$.*

Remark 9. *We will often consider a blowup sequence $(g_t^i)_{t\in[-\lambda_i,0]}$ of Ricci flows satisfying ([4.1.1](#)),([4.1.2](#)), in which case we will apply this theorem with parameters $\lambda_i \searrow 0$. Then for any $r \in (0,D]$, we can replace the almost-maximal volume ratio with the assumption*

$$|B_{g^i}(x,0,r)|_{g_0^i} > (1-\delta)\omega_n r^n$$

when $i = i(D) \in \mathbb{N}$ is sufficiently large, after possibly adjusting $\delta > 0$, since

$$\inf_{r\in(0,D]} \frac{\omega_n r^n}{v_{-\lambda_i A}(r)} = \frac{\int_0^D s^{n-1}ds}{\int_0^D \left(\sqrt{\frac{n-1}{\lambda_i A}}\sinh\left(\sqrt{\frac{\lambda_i A}{n-1}}s\right)\right)^{n-1}ds} \to 1$$

as $i \to \infty$.

The proof of Theorem [32](#) given in [[CY](#)] contains a small gap (their point-picking does not lead to a limit with Euclidean volume growth), so we indicate the changes needed to give a correct proof of (a modification of) their statement.

Proof. We first prove the theorem when $\lambda = 1$. As in [[CY](#)], proceed by contradiction and choose (after parabolic rescaling) a sequence $(M_i,(g_t^i)_{t\in[-1,0]})$ along with $x_i \in M_i$, $\delta_i \searrow 0$, $A_i \in [0,A]$, such that $Rc(g_t^i) \geq -A_i g_t^i$, $|B_{g^i}(x,t,r)|_{g_t^i} \geq A^{-1}r^n$ for all $(x,t) \in M \times [-1,0]$ and $r \in (0,1]$, $|B_{g^i}(x_i,0,1)|_{g_0^i} > (\omega_n - \delta_i)v_{-A_i}(1)$, but also $\widetilde{r}^{g^i}_{Rm}(x_i,0) < \delta_i$. Choose $s_i \searrow 0$ such that $\lim_{i\to\infty} \frac{\widetilde{r}^{g^i}_{Rm}(x_i,0)}{s_i} = 0$, and then choose $y_i \in B_{g^i}(x_i,0,s_i)$ minimizing

$$\epsilon_i(y) := \frac{\widetilde{r}^{g^i}_{Rm}(y,0)}{d_{g_0^i}(y,\partial B_{g^i}(x_i,0,s_i))},$$

so that

$$\epsilon_i(y_i) \leq \epsilon_i(x_i) = \frac{\widetilde{r}^{g^i}_{Rm}(x_i,0)}{s_i} \to 0$$

as $i \to \infty$. Set $r_i := \widetilde{r}^{g^i}_{Rm}(y_i, 0)$, $\widetilde{g}^i_t := r_i^{-2} g^i_{r_i^2 t}$ for $t \in [-1, 0]$, so that $\widetilde{r}^{\widetilde{g}^i}_{Rm}(y_i, 0) = 1$. For $y \in M_i$ with $d_{\widetilde{g}^i_0}(y_i, y) \le \frac{1}{2} d_{\widetilde{g}^i_0}(y_i, \partial B_{g^i_0}(x_i, 0, s_i))$, we have

$$\widetilde{r}^{\widetilde{g}^i}_{Rm}(y, 0) = \frac{\widetilde{r}^{\widetilde{g}^i}_{Rm}(y, 0)}{\widetilde{r}^{\widetilde{g}^i}_{Rm}(y_i, 0)} \ge \frac{d_{\widetilde{g}^i_0}(y, \partial B_{g^i_0}(x_i, 0, s_i))}{d_{\widetilde{g}^i_0}(y_i, \partial B_{g^i_0}(x_i, 0, s_i))} \ge \frac{1}{2}.$$

However, we also know that

$$d_{\widetilde{g}^i_0}(y_i, \partial B_{g^i_0}(x_i, 0, s_i)) = \frac{d_{g^i_0}(y_i, \partial B_{g^i_0}(x_i, 0, s_i))}{\widetilde{r}^{g^i_0}_{Rm}(y_i, 0)} = \epsilon_i(y_i)^{-1} \to \infty,$$

so, for any $D < \infty$, we have $\widetilde{r}^{\widetilde{g}^i}_{Rm}(\cdot, 0) \ge \frac{1}{2}$ on $B_{\widetilde{g}^i}(y_i, 0, D)$ for $i = i(D) \in \mathbb{N}$ sufficiently large. By Theorem 31, Shi's estimates, and the assumed lower bound on volume of geodesic balls, we can pass to a subsequence to get C^∞ Cheeger-Gromov convergence $(M_i, \widetilde{g}^i_0, y_i) \to (M_\infty, \widetilde{g}^\infty, y_\infty)$ for some complete Riemannian manifold with bounded curvature, such that $\widetilde{r}^{\widetilde{g}^\infty}_{Rm}(y_\infty) = 1$ and $Rc(\widetilde{g}^\infty) \ge 0$ everywhere.

Claim: $|B_{\widetilde{g}^\infty}(y_\infty, r)|_{\widetilde{g}^\infty} = \omega_n r^n$ for all $r \in (0, \infty)$.

Since $Rc(g^i_0) \ge -A_i g^i_0$, $|B_{g^i}(y_i, 0, 1)|_{g^i_0} > (1 - \delta_i) v_{-A_i}(1)$, and $d_i := d_{g^i_0}(x_i, y_i) \to 0$, we can use Bishop volume comparison to conclude

$$\begin{aligned} |B_{g^i}(y_i, 0, 1)|_{g^i_0} &\ge \frac{v_{-A_i}(1)}{v_{-A_i}(1 + d_i)} |B_{g^i}(y_i, 0, 1 + d_i)|_{g^i_0} \\ &> (1 - \delta_i) \frac{v_{-A_i}(1)}{v_{-A_i}(1 + d_i)} v_{-A_i}(1) =: \tau_i v_{-A_i}(1), \end{aligned}$$

where $\tau_i \to 1$ as $i \to \infty$. Now rescale and pass to the limit to get, for any fixed $r \in (0, \infty)$,

$$\begin{aligned} |B_{\widetilde{g}^\infty}(y_\infty, r)|_{\widetilde{g}^\infty} &= \lim_{i \to \infty} |B_{\widetilde{g}^i}(y_i, 0, r)|_{\widetilde{g}^i_0} = \lim_{i \to \infty} r_i^{-n} |B_{g^i}(y_i, 0, r_i r)|_{g^i_0} \\ &\ge \limsup_{i \to \infty} r_i^{-n} \tau_i v_{-A_i}(r_i r) = \limsup_{i \to \infty} v_{-r_i^2 A_i}(r) = \omega_n r^n. \end{aligned}$$

The only complete Riemannian manifold with nonnegative Ricci curvature and Eulidean volume growth is flat $\mathbb{R}^n$, so $(M_\infty, \widetilde{g}^\infty)$ is flat $\mathbb{R}^n$, contradicting $\widetilde{r}^{g^\infty}_{Rm}(y_\infty) = 1$.

Finally, consider the case where $\lambda \in (0,1)$. Define $\overline{g}_t := \lambda g_{\lambda^{-1}t}$ for $t \in [-1,0]$, so that $Rc(\overline{g}_t) \geq -A\overline{g}_t$, $|B_{\overline{g}}(y,t,s)|_{\overline{g}_t} \geq A^{-1}s^n$ for all $(y,t) \in M \times [-1,0]$, $s \in (0,1]$, and also

$$|B_{\overline{g}}(x,0,\lambda^{\frac{1}{2}}r)|_{\overline{g}} \geq (1-\delta)\lambda^{\frac{n}{2}} v_{-\lambda A}(r) = (1-\delta) v_{-A}(\lambda^{\frac{1}{2}}r),$$

where $\lambda^{\frac{1}{2}}r \in (0,1]$. Then we can apply the $\lambda = 1$ case, replacing r with $\lambda^{\frac{1}{2}}r$, to obtain $\widetilde{r}^{\overline{g}}_{Rm}(x,0) \geq \delta\lambda^{\frac{1}{2}}r$, and the claim follows. □

Using Theorem 32 and volume comparison, Chen-Yuan prove that the regular set of a noncollapsed Ricci limit space is open and equipped with a smooth Riemannian metric if it is the Gromov-Hausdorff limit of time-slices of Ricci flows satisfying (4.1.1),(4.1.2). This is made precise in the following.

Theorem 33. *(Theorem 1 of [CY]) Given $A < \infty$ and $\underline{T} > 0$, suppose $(M_i, (g^i_t)_{t \in [-T_i,0]})$ is a sequence of Ricci flows satisfying $Rc(g^i_t) \geq -Ag^i_t$ and $|B_{g^i}(x,t,r)|_{g^i_t} \geq A^{-1}r^n$ for all $(x,t) \in M \times [-T_i,0]$ and $r \in (0,1]$, where $T_i \geq \underline{T}$. If (X,d,x) is a pointed, complete metric length space such that*

$$(M, d_{g^i_0}, x_i) \to (X,d,x)$$

in the pointed Gromov-Hausdorff sense for some $x_i \in M_i$, then (X,d) is a noncollapsed Ricci limit space with open regular part $\mathcal{R}$, and $\mathcal{R}$ admits the structure of a smooth Riemannian manifold $(\mathcal{R}, g)$ such that $d|(\mathcal{R} \times \mathcal{R})$ is the length metric d_g of $(\mathcal{R}, g)$. Moreover, there is an increasing exhaustion (U_i) of $\mathcal{R}$ by precompact open sets with diffeomorphisms $\psi_i : U_i \to M$ such that $\psi_i(y) \to y$ locally uniformly in $\mathcal{R}$ (with respect to the Gromov-Hausdorff convergence) and $\psi_i^* g^i_0 \to g$ in $C^\infty_{loc}(\mathcal{R})$.

The following proposition gives a sufficient criterion for Gromov-Wasserstein convergence to imply Gromov-Hausdorff convergence.

Proposition 16. *Suppose (X_i, d_i, μ_i) is a sequence of complete metric measure spaces converging in the W_1-Gromov-Wasserstein sense to a complete, locally compact metric measure* *space* $(X_\infty, d_\infty, \mu_\infty)$ *with supp*$(\mu_\infty) = X_\infty$, *such that* $(X_\infty, d_\infty), (X_i, d_i)$ *are all length spaces. Suppose* $x_i \in X$ *are such that, for any* $D < \infty$ *and* $r > 0$, *there exists* $c = c(r, D) > 0$ *with* $\mu_i(B^{X_i}(x, r)) \geq c$ *for all* $x \in B^{X_i}(x_i, D)$. *Finally, assume Var*$(\mu_\infty) \leq H < \infty$, *and choose* $x_\infty \in X_\infty$ *such that Var*$(\mu_\infty, \delta_{x_\infty}) \leq H$. *Then there exist* $x_i' \in X_i$ *with* $d_i(x_i, x_i') \leq C(H, c(\frac{1}{2}, 1))$ *such that*

$$(X_i, d_i, x_i') \to (X_\infty, d_\infty, x_\infty)$$

in the pointed Gromov-Hausdorff sense. In fact, if $\phi_i : (X_i, d_i) \to (Z, d^Z)$ *and* $\phi_\infty : (X_\infty, d_\infty) \to (Z, d^Z)$ *are isometric embeddings into a common metric space such that*

$$\lim_{i \to \infty} d^Z_{W_1}((\phi_i)_* \mu_i, (\phi_\infty)_* \mu_\infty) = 0, \tag{4.2.1}$$

then we also have

$$\lim_{i \to \infty} d^Z_H \left(B^Z(\phi_i(x_i'), r), B^Z(\phi_\infty(x_\infty), r) \right) = 0$$

for each fixed $r \in (0, \infty)$.

By passing to a subsequence, we can therefore find $x_\infty \in X_\infty$ *such that* $(X_i, d_i, x_i) \to (X_\infty, d_\infty, x_\infty)$ *in the pointed Gromov-Hausdorff sense.*

Proof. Given that $(X_i, d_i, \mu_i) \to (X_\infty, d_\infty, \mu_\infty)$ in the W_1-Gromov-Wasserstein sense, a direct limit construction gives $(Z, d^Z), \phi_i, \phi_\infty$ such that (4.2.1) is satisfied. Let q_i be couplings of μ_i, μ_∞ realizing this convergence, so that

$$\lim_{i \to \infty} \int_{X_i \times X_\infty} d^Z(\phi_i(x), \phi_\infty(y)) dq_i(x, y) = 0.$$

Claim 1: For each $D < \infty$, we have

$$\lim_{i \to \infty} \sup_{x \in B^{X_i}(x_i, D)} d^Z(\phi_i(x), \phi_\infty(X_\infty)) = 0.$$

Otherwise, we can pass to a subsequence to obtain $\epsilon > 0$ and $y_i \in B^{X_i}(x_i, D)$ such that $d^Z(\phi_i(y_i), \phi_\infty(X_\infty)) \geq 2\epsilon$ for all $i \in \mathbb{N}$. Then $d^Z(\phi_i(x), \phi_\infty(X_\infty)) \geq \epsilon$ for all $x \in B^{X_i}(y_i, \epsilon)$,

hence

$$\epsilon\mu_i(B^{X_i}(y_i,\epsilon)) \le \int_{B^{X_i}(y_i,\epsilon)\times X_\infty} d^Z(\phi_i(x),\phi_\infty(y))dq_i(x,y) \to 0,$$

contradicting $\liminf_{i\to\infty}\mu_i(B^{X_i}(y_i,\epsilon)) \ge c(\epsilon,D)>0$.

Now choose $x_i' \in X_\infty$ such that $d^Z(\phi_\infty(x_i'),\phi_i(x_i)) \to 0$.

Claim 2: $\liminf_{i\to\infty}\mu_\infty(B^{X_\infty}(x_i',1)) \ge c(1/2,1)$.

When $i\in\mathbb{N}$ is sufficiently large, we have

$$d^Z(\phi_i(x),\phi_\infty(y)) \ge d^Z(\phi_\infty(y),\phi_\infty(x_i')) - d^Z(\phi_\infty(x_i'),\phi_i(x_i)) - d^Z(\phi_i(x_i),\phi_i(x)) \ge \frac{1}{4}$$

for all $x\in B^{X_i}(x_i,\frac12)$ and $y\in X_\infty\setminus B^{X_\infty}(x_i',1)$ so

$$\frac{1}{4}q_i\left(B^{X_i}(x_i,\frac{1}{2})\times\left(X_\infty\setminus B^{X_\infty}(x_i',1)\right)\right) \le \int_{B^{X_i}(x_i,\frac12)\times\left(X_\infty\setminus B^{X_\infty}(x_i',1)\right)} d^Z(\phi_i(x),\phi_\infty(y))dq_i(x,y),$$

but

$$q_i\left(B^{X_i}(x_i,\frac{1}{2})\times B^{X_\infty}(x_i',1)\right) \le \mu_\infty(B^{X_\infty}(x_i',1)),$$

so combining estimates gives

$$\begin{aligned}c(1/2,1) &\le \liminf_{i\to\infty}\mu_i\left(B^{X_i}(x_i,\frac{1}{2})\right) = \liminf_{i\to\infty} q_i\left(B^{X_i}(x_i,\frac{1}{2})\times X_\infty\right)\\ &\le \liminf_{i\to\infty}\mu_\infty(B^{X_\infty}(x_i',1)),\end{aligned}$$

If $d(x_i',x_\infty) > \sqrt{\frac{H}{c(1/2,1)}}+1$, then

$$H \ge \int_{B(x_i',1)} d^2(x_\infty,x)d\mu_\infty(x) > \frac{H}{c(1/2,1)}\mu_\infty(B(x_i',1)) \ge H,$$

a contradiction. We therefore have $\Lambda=\Lambda(H,c(1/2,1))<\infty$ such that $x_i'\in B(x_\infty,\frac12\Lambda)$ for all $i\in\mathbb{N}$ sufficiently large.

Claim 3: For each $D<\infty$, we have

$$\lim_{i\to\infty}\sup_{x\in B^{X_\infty}(x_\infty,D)} d^Z(\phi_i(X_i),\phi_\infty(x)) = 0.$$

Otherwise, we can pass to a subsequence to obtain $\epsilon > 0$ and $y_i \in B^{X_\infty}(x_\infty, D)$ such that $d^Z(\phi_\infty(y_i), \phi_i(X_i)) \geq 3\epsilon$ for all $i \in \mathbb{N}$. We can pass to a further subsequence so that $y_i \to y_\infty \in \overline{B}^{X_\infty}(x_\infty, D)$. Then

$$d^Z(\phi_\infty(y_\infty), \phi_i(X_i)) \geq 3\epsilon - d^Z(\phi_\infty(y_i), \phi_\infty(y_\infty)) \geq 2\epsilon$$

for sufficiently large $i \in \mathbb{N}$, hence

$$\epsilon\mu_\infty\left(B^{X_\infty}(y_\infty, \epsilon)\right) \leq \int_{X_i \times B^{X_\infty}(y_\infty,\epsilon)} d^Z(\phi_i(x), \phi_\infty(y)) dq_i(x, y) \to 0$$

as $i \to \infty$, contradicting the fact that μ_∞ has full support.

In particular, we can find $x_i \in X_i$ such that $d^Z(\phi_i(x_i), \phi_\infty(x_\infty)) \to 0$, which implies

$$d_i(x_i, x_i) \leq d^Z(\phi_i(x_i), \phi_\infty(x_i')) + d_\infty(x_i', x_\infty) + d^Z(\phi_i(x_i), \phi_\infty(x_\infty)) \leq \Lambda$$

for sufficiently large $i \in \mathbb{N}$. This implies the following for $i \in \mathbb{N}$ sufficiently large: for any $D < \infty$, $r > 0$, and $y \in B^{X_i}(x_i, r)$, we have

$$\mu_i(B^{X_i}(y, r)) \geq c(r, D + \Lambda).$$

Arguing as in Claim 1 thus gives

$$\lim_{i \to \infty} \sup_{x \in B^{X_i}(x_i, D)} d^Z(\phi_i(x), \phi_\infty(X_\infty)) = 0$$

for any $D < \infty$.

Now fix $D \in (0, \infty)$ and $\epsilon \in (0, D)$. For any $x \in B^{X_i}(x_i, D)$, because (X_i, d_i) is a length space, we can find $x' \in B^{X_i}(x_i, D - \epsilon/2)$ such that $d_i(x, x') < \epsilon/2$. We have shown that there exists $x_i' \in X_\infty$ such that

$$d^Z(\phi_i(x'), \phi_\infty(y')) < \frac{\epsilon}{4}$$

when $i = i(\epsilon, D)$ is sufficiently large. Thus, for $i = i(\epsilon, D) \in \mathbb{N}$ large, we have

$$d_\infty(y', x_\infty) \leq d^Z(\phi_\infty(y'), \phi_i(x')) + d_i(x', x_i) + d^Z(\phi_i(x_i), \phi_\infty(x_\infty)) < D,$$

hence

$$\lim_{i\to\infty}\sup_{x\in B^{X_i}(x_i,D)} d^Z\left(\phi_i(x),\phi_\infty(B^{X_\infty}(x_\infty,D))\right)=0.$$

Similarly, because (X_∞, d_∞) is a length space, we have

$$\lim_{i\to\infty}\sup_{x\in B^{X_\infty}(x_\infty,D)} d^Z\left(\phi_\infty(x),\phi_\infty(B^{X_i}(x_i,D))\right)=0.$$

Together, these facts give

$$\lim_{i\to\infty} d_H\left(\phi_i(B^{X_i}(x_i,D)),\phi_\infty(B^{X_\infty}(x_\infty,D))\right)=0.$$

The remaining claim follows from the distance bound $d_i(x_i, x_i) \leq \Lambda$. □

Definition 30. *We recall the definition of the quantitative singular strata of a Riemannian manifold (M^n, g): $S^k_{\eta,r}$ is the set of $x \in M$ such that there does not exist $s \in [r,2]$ and a metric cone $C(Z)$ such that*

$$d_{PGH}\left((B(x,s),d_g,x),\left(B\left((0^{k+1},z^*),s\right),d_{\mathbb{R}^{k+1}\times C(Z)},(0^{k+1},z^*)\right)\right)<\eta s,$$

where z^* is the vertex of the cone $C(Z)$. Given a noncollapsed Ricci limit space (X,d), we let $S^k(X)$ be the set of $x \in X$ such that no tangent cone of (X,d) at x isometrically splits a factor of $\mathbb{R}^{k+1}$, and let $S(X)$ be the singular set, which consists of $x \in X$ such that no tangent cone of (X,d) at x is $\mathbb{R}^n$. The regular set is $R(X) := X \setminus S(X)$.

The following obtained from Theorem 1.7 in [CJN] by rescaling.

Theorem 34. *(Estimating the Size of Quantitative Singular Strata) For any $\eta > 0$ and $A < \infty$, there exists $C = C(A,\eta) > 0$ such that for any Riemannian manifold (M^n, g) satisfying $Rc(g) \geq -Ag$ and $|B_g(y,1)|_g \geq A^{-1}$ for all $y \in M$, we have*

$$|S^k_{\eta,sr}\cap B_g(y,r)|_g \leq C s^{n-k} r^n$$

for any $y \in M$ and $r, s \in (0,1]$.

Remark 10. *In fact, if* $\widetilde{g} := r^{-2}g$ *and* $\widetilde{\mathcal{S}}^k_{\eta,s}$ *denotes the quantitative singular set of the rescaled manifold* $(M,\widetilde{g})$, *we just note that* $\mathcal{S}^k_{\eta,sr} \subseteq \widetilde{\mathcal{S}}^k_{\eta,s}$.

We will combine this with various Gromov-Hausdorff ϵ-regularity theorems as in [CN2] to get L^p estimates for $\widetilde{r}^{\,-2}_{Rm}$.

Proposition 17. *(Codimension one* ϵ*-Regularity) For any* $\underline{T} > 0$, *and* $A < \infty$, *there exists* $\epsilon_0 = \epsilon_0(\underline{T}, A) > 0$ *such that the following holds for any closed Ricci flow* $(M^n, (g_t)_{t\in[0,T)})$ *with* $T \geq \underline{T}$ *satisfying* $Rc(g_t) \geq -Ag_t$ *and* $|B(x,t,r)|_{g_t} \geq A^{-1}r^n$, *for all* $(x,t) \in M \times [0,T)$ *and* $r \in (0,1]$. *For any* $(x,t) \in M \times [\frac{T}{2}, T)$ *and* $r \in (0,1]$, *if*

$$d_{PGH}\left(\left(B(x,t,\epsilon_0^{-1}r), d_{g_t}, x\right), \left(B\left((0^{n-1}, z^*), \epsilon_0^{-1}r\right), d_{\mathbb{R}^{n-1}\times C(Z)}, (0^{n-1}, z^*)\right)\right) < \epsilon_0 r$$

for some metric cone $C(Z)$, *then* $\widetilde{r}_{Rm}(x,t) \geq \epsilon_0 r$.

Proof. Suppose the claim is false. Then, after rescaling and time translating, we can find closed Ricci flows $(M_i^n, (g_t^i)_{t\in[-\frac{T_i}{2},0]})$ with $T_i \geq \underline{T}$ satisfying $Rc(g_t^i) \geq -Ag_t^i$ and $|B_{g^i}(x,t,r)|_{g_t^i} \geq A^{-1}r^n$ for all $(x,t) \in M \times [-\frac{T_i}{2}, 0]$ and $r \in (0,1]$, and also $\epsilon_i \searrow 0$, $x_i \in M_i$ such that

$$d_{PGH}\left(\left(B_{g^i}(x_i,0,\epsilon_i^{-1}), d_{g_t^i}, x_i\right), \left(B\left((0^{n-1}, z^i), \epsilon_i^{-1}\right), d_{\mathbb{R}^{n-1}\times C(Z_i)}, (0^{n-1}, z^i)\right)\right) < \epsilon_i$$

for some metric spaces (Z_i, d_i), yet $\widetilde{r}^{\,g^i}_{Rm}(x_i, 0) < \epsilon_i$. We can pass to a subsequence to assume that $(M_i, d_{g_0^i}, x_i)$ converges in the pointed Gromov-Hausdorff sense to some metric cone $(\mathbb{R}^{n-1} \times C(Z_\infty), d_{\mathbb{R}^{n-1}\times C(Z_\infty)}, (0^{n-1}, z^\infty))$. By Theorem 6.1 of [CC2], $\mathcal{S}(\mathbb{R}^{n-1} \times C(Z_\infty))$ has Hausdorff dimension $\leq n-2$, so $\mathcal{S}(C(Z_\infty)) = \emptyset$, hence $C(Z_\infty) = \mathbb{R}$. By Theorem 33, the convergence is actually smooth everywhere, so $\lim_{i\to\infty} \widetilde{r}_{Rm}(x_i, 0) = \infty$, a contradiction. □

Remark 11. *Suppose that we replace the Gromov-Hausdorff closeness assumption of Proposition 17 by*

$$d_{PGH}\left((B_g(x,t_0,r), d_{g_t}, x), \left(B\left((0^{n-1}, z^*), r\right), d_{\mathbb{R}^{n-1}\times C(Z)}, (0^{n-1}, z^*)\right)\right) < \epsilon_0^2 r$$

for some $r \in (0,1]$. Then we can apply Proposition 17 with r replaced by $\epsilon_0 r$, to get

$$\widetilde{r}^{g}_{Rm}(x,t_0) \geq \epsilon_0^2 r.$$

Thus the conclusion of 17 holds (after replacing ϵ_0 with ϵ_0^2) if we only ask for a Gromov-Hausdorff closeness condition for $B_g(x,t,1)$ rather than $B_g(x,t,\epsilon_0^{-1})$.

Corollary 9. *(Estimating the Size of High-Curvature Regions) For any $p \in (0,2)$, $\underline{T} > 0$, $D < \infty$, and $A < \infty$, there exist $E = E(D,\underline{T},A) < \infty$ and $C = C(\underline{T},A,p,D) < \infty$ such that the following hold for any closed Ricci flow $(M^n,(g_t)_{t\in[0,T)})$ with $T \geq \underline{T}$ satisfying $Rc(g_t) \geq -Ag_t$ and $|B(x,t,r)|_{g_t} \geq A^{-1}r^n$, for all $(x,t) \in M \times [0,T)$ and $r \in (0,1]$.*

(i) *For all $(x,t) \in M \times [\frac{T}{2},T)$ and $s \in (0,1]$,*

$$|\{\widetilde{r}_{Rm}(\cdot,t) < s\} \cap B_g(x,t,D)|_{g_t} \leq Es^2.$$

(ii) *For all $(x,t) \in M \times [\frac{T}{2},T)$,*

$$\int_{B(x,t,D)} |Rm|^{\frac{p}{2}}(\cdot,t)dg_t \leq \int_{B(x,t,D)} \widetilde{r}^{\,-p}_{Rm}(\cdot,t)dg_t \leq C.$$

Proof. (i) We first assume $D = 1$. Then, for any $y \in B_g(x,t,1)$ with $\widetilde{r}_{Rm}(y,t) < \epsilon_0 s$, Proposition 17 and Remark 11 give

$$d_{PGH}\left((B(x,t,r'),d_g,x),\left(B\left((0^{n-1},z^*),r'\right),d_{\mathbb{R}^{n-1}\times C(Z)},(0^{n-1},z^*)\right)\right) > \epsilon_0 r'$$

for all metric cones $C(Z)$ and all $r' \in [s,1]$. In other words, $y \in \mathcal{S}^{n-2}_{\epsilon_0,s}$, so Theorem 9 with $\eta := \epsilon_0$ gives

$$|\{\widetilde{r}_{Rm}(\cdot,t) < s\} \cap B_g(x,t,1)|_{g_t} \leq |\mathcal{S}^{n-2}_{\epsilon_0,\epsilon_0^{-1}s} \cap B(x,t,1)|_{g_t} \leq C(A,\underline{T})s^2$$

for $s \in (0,\epsilon_0]$. For $s \in (\epsilon_0,1]$, we estimate

$$|\{\widetilde{r}_{Rm}(\cdot,t) < s\} \cap B_g(x,t,1)|_g \leq C(A) \leq C(A,\underline{T})s^2.$$

For arbitrary $D < \infty$, we can combine the $D = 1$ case with a standard covering argument to get

$$|\{\widetilde{r}_{Rm}(\cdot, t) < s\} \cap B_g(x,t,D)|_{g_t} \leq C(A, \underline{T}, D)s^2$$

for all $s \in (0, 1]$.

(ii) For the remaining claim, we apply the first estimate to get

$$\begin{aligned}
\int_{B(x,t,D)} |Rm|^{\frac{p}{2}}(\cdot, t)dg_t &\leq \int_{B(x,t,D)} \widetilde{r}_{Rm}^{\,-p}(\cdot,t)dg_t \\
&\leq |B(x,t,D)|_{g_t} + \frac{1}{p}\int_1^{\infty} s^{p-1}|B(x,t,D) \cap \{\widetilde{r}_{Rm}^{\,-1}(\cdot,t) > s\}|_{g_t} ds \\
&\leq C(\underline{T}, A, p, D)\left(1 + \int_1^{\infty} s^{p-1} s^{-2} ds\right) \\
&\leq C(\underline{T}, A, p, D).
\end{aligned}$$

□

Remark 12. *After possibly modifying the constants E_p, C, Corollary 9 also holds with $\widetilde{r}_{Rm}$ replaced with r_{Rm}, using the proof of part (iii) of Lemma 13.*

Remark 13. *The proof of Corollary 9 can be trivially modified to produce stronger estimates given stronger ϵ-regularity theorems.*

By [B3, B5], a metric flow arising as an $\mathbb{F}$-limit of noncollapsed Ricci flow is characterized by its regular set, which is endowed with the structure of a Ricci flow spacetime. We now introduce some definitions and notation relevant to this notion.

Definition 31. *(Definition 1.2 in [KL]) A Ricci flow spacetime is a tuple $(\mathcal{R}, g, \mathfrak{t}, \partial_{\mathfrak{t}})$, where $\mathcal{R}$ is a smooth $(n+1)$-dimensional manifold, $\mathfrak{t} : \mathcal{R} \to \mathbb{R}$ is a smooth submersion, $\partial_{\mathfrak{t}} \in \mathfrak{X}(\mathcal{R})$ satisfies $\partial_{\mathfrak{t}}\mathfrak{t} = 1$, and g is a bundle metric on $\ker(d\mathfrak{t})$ satisfying $\mathcal{L}_{\partial_{\mathfrak{t}}} g = -2Rc(g_t)$, where g_t is Riemannian metric on $\mathcal{R}_t := \mathfrak{t}^{-1}(t)$ obtained by restricting g. Given a point $x \in \mathcal{R}$, let $\gamma : I_x \to \mathcal{R}$ denote the maximally defined integral curve of $\partial_{\mathfrak{t}}$ satisfying $\gamma(\mathfrak{t}(x)) = x$. If*

$t \in (\alpha, \beta)$, *then we say* x *survives until time* t, *and we write* $x(t) := \gamma(t)$. *For any subset* $S \subset \mathcal{R}$, *write* $S_t := S \cap \mathcal{R}_t$, *and if* $S \subset \mathcal{R}_t$, *define*

$$S(t') := \{y(t'); y \in S, t' \in I_y\}.$$

Also define the parabolic neighborhood

$$P(x; A, -T^-, T^+) := \bigcup_{t \in [\mathsf{t}(x) - T^-, \mathsf{t}(x) + T^+]} \left(B_{g_{\mathsf{t}(x)}}(x, A) \right)(t).$$

We call $P(x; A, -T^-, T^+)$ *unscathed if* $B_{g_{\mathsf{t}(x)}}(x, A)$ *has compact closure in* $\mathcal{R}_t$ *and* $[\mathsf{t}(x) - T^-, \mathsf{t}(x) + T^+] \subset I_y$ *for all* $y \in B_{g_{\mathsf{t}(x)}}(x, A)$.

According to [B5], special limits of noncollapsed Ricci flows (in particular, static flows and metric solitons) are actually determined by the restriction of the metric flow to a single time slice, where the corresponding metric space has the structure of singular space. We will thus often restrict our attention to studying the properties of these singular spaces, so it is important to review the following definitions.

Definition 32. *(Definition 1.7 in* [B2]*) A singular space is a tuple* $(X, d, \mathcal{R}_X, g_X)$, *where* (X, d) *is a complete metric length space and* $\mathcal{R} \subset X$ *is a dense open subset equipped with the structure of an* n*-dimensional Riemannian manifold* $(\mathcal{R}_X, g_X)$ *such that* $d|(\mathcal{R}_X \times \mathcal{R}_X)$ *is the length metric* d_{g_X}, *and for any compact subset* $K \subset X$ *and* $D < \infty$, *there exist* $0 < \kappa_1(K, D) < \kappa_2(K, D) < \infty$ *such that*

$$\kappa_1 r^n < |B^X(x, r) \cap \mathcal{R}_X|_{g_X} < \kappa_2 r^n$$

for all $x \in K$ *and* $r \in (0, D)$.

A singular shrinking soliton is a tuple $(X, d, \mathcal{R}_X, g_X, f_X)$, *where* $(X, d, \mathcal{R}_X, g_X)$ *is a singular space and* $f_X \in C^\infty(\mathcal{R}_X)$ *satisfies the Ricci soliton equation*

$$Rc(g_X) + \nabla^2 f_X = \frac{1}{2} g_X$$

on $\mathcal{R}_X$.

4.3 Tangent Flows are Ricci-Flat Cones

We first recall the notion of a conjugate heat kernel based at the singular time. Suppose $(M^n, (g_t)_{t\in[0,T)}, p)$ is any closed, pointed Ricci flow. Let $K(x,t;\cdot,\cdot)$ denote the conjugate heat kernel based at $(x,t)\in M\times(0,T)$. By Lemma 2.2 of [MM], for any sequence of times $t_i\nearrow T$, we can pass to a subsequence so that $K(x,t_i;\cdot,\cdot)$ converge in $C^\infty_{loc}(M\times[0,T))$ to a solution $K(x,T;\cdot,\cdot)$ of the conjugate heat equation on $M\times[0,T)$ satisfying $\int_M K(x,T;y,t)dg_t(y)=1$ for all $t\in[0,T)$. In particular, $d\nu_{x,T;t}:=K(x,T;\cdot,t)dg_t$ is a probability measure. By a slight abuse of language, we refer to both $K(x,T;\cdot,\cdot)$ and $(\nu_{x,T;t})_{t\in[0,T)}$ as a conjugate heat kernel at the singular time based at x. Note that $K(x,T;\cdot,\cdot)$ is not unique, and may depend on the sequence (t_i).

Lemma 14. (i) $\lim_{i\to\infty} d^{g_s}_{W_1}(\nu_{x,t_i;s},\nu_{x,T;s})$ *for each* $s\in[0,T)$.

(ii) *Var*$(\nu_{x,T;s})\leq H_n(T-s)$ *for all* $s\in[0,T)$.

Proof. (i) In fact, for any 1-Lipschitz function $f:(M,d_{g_s})\to\mathbb{R}$, and any fixed point $y_0\in M$,

$$\begin{aligned}\int_M f d\nu_{x,t_i;s}-\int_M f d\nu_{x,T;s} &= \int_M f(y)\left(K(x,t_i;y,s)-K(x,T;y,s)\right)dg_s(y)\\ &= \int_M (f(y)-f(y_0))\left(K(x,t_i;y,s)-K(x,T;y,s)\right)dg_s(y)\\ &\leq \left(\sup_{t\in[0,T)}\mathrm{diam}_{g_t}(M)\right)\int_M |K(x,t_i;y,s)-K(x,T;y,s)|dg_s(y),\end{aligned}$$

but the right hand side approaches 0 as $i\to\infty$ by the dominated convergence theorem.

(ii) Corollary 3.8 of [B4] gives $\mathrm{Var}(\nu_{x,t_i;s})\leq H_n(t_i-s)$, so the claim follows from the lower semicontinuity of Var under W_1-convergence. □

Remark 14. *In Section 1.7 of* [B5], *Bamler defines* $\nu_{x,T;s}$ *as the limit in* W_1 *of* $\nu_{x,t_i;s}$ *as* $t_i\nearrow T$. *We choose to define* $\nu_{x,T;s}$ *via the smooth convergence of* $K(x,t_i;\cdot,\cdot)$ *because we will need to pass the Gaussian heat kernel estimate stated in Proposition 25 to the limit as* $t_i\nearrow T$ *in Section 4, and it is not immediate to us how to justify this using only* W_1*-convergence.*

By assertion (ii) of Lemma 14, $(\nu_{x,T;t})_{t\in[0,T)}$ admits H_n-centers: for any $t \in [0,T)$, there exists $y \in M$ such that $\mathrm{Var}(\nu_{x,T;t},\delta_y) \le H_n(T-t)$. The following lemma shows that near such an H_n-center y, there is a pointwise lower bound for $K(x,T;\cdot,t)$ on a set of almost full measure in $B(y,t,1)$.

Lemma 15. *Let $(M,(g_t)_{t\in[-4,0)},(\nu_t)_{t\in[-4,0)})$ be a closed Ricci flow satisfying $Rc(g_t) \le Ag_t$ and $A^{-1}r^n < |B(x,t,r)|_{g_t}$ for all $(x,t) \in M \times [-4,0)$ and $r \in (0,1]$, where ν_t is a conjugate heat kernel based at the singular time. For any $D < \infty$ and $\delta > 0$, there exists $\sigma = \sigma(A,D,\delta) > 0$ such that the following holds. For any H_n-center $(y,t) \in M \times [-2,-1]$ of ν, there is a subset $S \subseteq B(y,t,D)$ such that $|B(y,t,D)\setminus S|_{g_t} < \delta$ and $K(\cdot,t) > \sigma$ on S, where $d\nu_t = K(\cdot,t)dg_t$.*

Proof. **Step 1:** Find a small set in a future time-slice with controlled curvature and conjugate heat kernel.

Fix an H_n-center $(y,t) \in M \times [-2,-1]$ of ν. Fix $\tau_0 \in (0,\frac{1}{2})$ to be determined, and let $(z,t+\tau_0)$ be an H_n-center of ν, so that

$$\nu_{t+\tau_0}(B(z,t+\tau_0,\sqrt{4H_n})) \ge \frac{1}{2}.$$

By assertion (iii) of Lemma 2.6, for any $(x,t) \in M \times [-2,0]$ and $\tau \in (0,2]$, we can estimate $\mathcal{N}_{x,t}(\tau) \ge -Y$ for some $Y = Y(A) < \infty$. Thus Bamler's on-diagonal heat-kernel upper bound (Theorem 7.1 of [B4]) gives $K \le C_1(A)$ on $M \times [-2,-\frac{1}{2}]$. Because (by volume comparison)

$$|B(z,t+\tau_0,\sqrt{4H_n})|_{g_{t+\tau_0}} \le C_2(n,A),$$

the subset $\widetilde{S}$ consisting of points $z' \in B(z,t+\tau_0,\sqrt{4H_n})$ with $K(z',t+\tau_0) > \frac{1}{4C_2}$ satisfies

$$\begin{aligned}\frac{1}{2} &\le \int_{B(z,t+\tau_0,\sqrt{4H_n})} K(\cdot,t+\tau_0)dg_{t+\tau_0} \\ &\le \int_{\widetilde{S}} K(\cdot,t+\tau_0)dg_{t+\tau_0} + \frac{1}{4C_2}|B(z,t+\tau_0,\sqrt{4H_n})\setminus\widetilde{S}|_{g_{t+\tau_0}} \\ &\le C_1|\widetilde{S}|_{g_{t+\tau_0}} + \frac{1}{4},\end{aligned}$$

and in particular, $|\widetilde{S}|_{g^i_{t+\tau_0}} \geq c_3(A) > 0$. By Corollary 9, there exists $E_0 := E(\sqrt{4H_n}, 1, A) < \infty$ such that

$$|\{\widetilde{r}_{Rm}(\cdot, t+\tau_0) < s\} \cap B(z, t+\tau_0, \sqrt{4H_n})|_{g_{t+\tau_0}} \leq E_0 s^2$$

for all $s \in (0,1]$. In particular, we can choose $s_0 = s_0(A) > 0$ such that

$$\widehat{S} := \{z' \in \widetilde{S}; \widetilde{r}_{Rm}(z', t+\tau_0) \geq s_0\}$$

satisfies $|\widehat{S}|_{g_{t+\tau_0}} \geq \frac{1}{2}c_3$. By Theorem 31, we can then modify $\tau_0 = \tau_0(A) > 0$ so that $\widetilde{r}_{Rm}(z', t') \geq \tau_0$ for all $(z', t') \in \widehat{S} \times [t, t+2\tau_0]$. Now fix $z' \in \widehat{S}$, and apply Lemma 9.15 of [B3] to get

$$|\partial_t K| + |\nabla K| \leq C_4(A)$$

on $B(z', t+\tau_0, \frac{\tau_0}{2}) \times [t, t+\tau_0]$. By again modifying $\tau_0(A)$, and by standard distortion estimates, we can integrate along geodesics in $B(z', t+\tau_0, \frac{\tau_0}{2})$ emanating from z', and then integrate backwards in time to conclude that $K(z'', s) > (8C_2)^{-1}$ for all $s \in [t, t+\tau_0]$ and $z'' \in B(z', t+\tau_0, \alpha)$, where $\alpha = \alpha(A) > 0$.

Applying the volume lower bound once again, we get

$$\nu_t(B(z', t, \alpha)) \geq \frac{A^{-1}\alpha^{\frac{n}{2}}}{4C_2} =: c_5,$$

where $c_5 = c_5(A) > 0$. On the other hand, we know

$$\nu_t\left(B\left(y, t, \sqrt{4H_n c_5^{-1}}\right)\right) \geq 1 - \frac{c_5}{2},$$

so that

$$B\left(y, t, \sqrt{4H_n c_5^{-1}}\right) \cap B(z', t, \alpha) \neq \emptyset,$$

hence $z' \in B(y, t, C_6)$ for some $C_6 = C_6(A) < \infty$.

Step 2: Combine weak L^p curvature scale estimates with Colding's segment inequality to construct curves with controlled $\mathcal{L}$-length.

Now apply Corollary 9 to obtain $E := E(D', 1, A) < \infty$ such that

$$|\{\widetilde{r}_{Rm}(\cdot, s) < \theta\} \cap B(y, s, D')|_{g_s} \leq E\theta^2$$

for any $s \in [t, t+\tau_0]$, $D' < \infty$, and $\theta \in (0,1]$. Fix $\tau_1 \in (0, \frac{1}{2}\tau_0)$ to be determined, and set $D' := D'(A, D) := 2C_6 + 8(D+\alpha)$.

Let $\gamma_{y_1,y_2} : [0, l_{y_1,y_2}] \to M$ denote a unit-speed minimizing geodesic from $y_1 \in B(z', t, \alpha)$ to $y_2 \in B(y, t, D)$ with respect to $g_{t+\tau_1}$. To deal with nonuniqueness, we will only integrate over a set of (y_1, y_2) with full $dg_{t+\tau_1} \times dg_{t+\tau_1}$-measure such that there is a unique such γ_{y_1,y_2}, and so that $(y_1, y_2) \mapsto l_{y_1,y_2}$ and $(y_1, y_2, u) \mapsto \gamma_{y_1,y_2}(u)$ are smooth. For any such geodesic γ_{y_1,y_2}, and any $u \in [0, l_{y_1,y_2}]$, we can estimate

$$\begin{aligned} d_{g_{t+\tau_1}}(\gamma_{y_1,y_2}(u), y) &\leq d_{g_{t+\tau_1}}(y_1, y_2) + d_{g_{t+\tau_1}}(y_2, y) \\ &\leq 4(D+\alpha) + d_{g_{t+\tau_1}}(z', y) + 2D \leq D', \end{aligned}$$

We apply the Cheeger-Colding segment inequality (Theorem 2.11 of [CC1]) with respect to the time slice $g_{t+\tau_1}$, the sets

$$B(z', t, \alpha) \subseteq B(z', t+\tau_1, 2\alpha),$$

$$B(y, t, D) \subseteq B(y, t+\tau_1, 2D),$$

and with the function χ_W, where

$$W := \{w \in B(y, t+\tau_1, D'); \widetilde{r}_{Rm}(w, t+\tau_1) < \theta\},$$

and $\theta \in (0,1]$ is to be determined, to get (since $d_{g_{t+\tau_1}}(y, z') \leq 2C_6$)

$$\begin{aligned} &\int_{B(z',t,\alpha) \times B(y,t,D)} |\{u \in [0, l_{y_1,y_2}]; \widetilde{r}_{Rm}(\gamma_{y_1,y_2}(u), t+\tau_1) < \theta\}| dg_{t+\tau_1}(y_1) dg_{t+\tau_1}(y_2) \\ &\leq C_7(A, D') \left(|B(z', t, \alpha)|_{g_{t+\tau_1}} + |B(y, t, D)|_{g_{t+\tau_1}}\right) \int_{B(y, t+\tau_1, D')} \chi_W dg_{t+\tau_1} \\ &\leq C_7(A, D') \left(|B(z', t+\tau_1, 2\alpha)|_{g_{t+\tau_1}} + |B(y, t+\tau_1, D)|_{g_{t+\tau_1}}\right) |W|_{g_{t+\tau_1}} \\ &\leq C_8(A, D)\theta^2. \end{aligned}$$

We now define

$$S := \left\{ y_2 \in B(y,t,D); \int_{B(z',t,\alpha)} |\{u \in [0, l_{y_1,y_2}]; \widetilde{r}_{Rm}(\gamma_{y_1,y_2}(u), t+\tau_1) < \theta\}| dg_{t+\tau_1}(y_1) \leq \theta^{\frac{3}{2}} \right\}.$$

Then

$$\begin{aligned} &\theta^{\frac{3}{2}} |B(y,t,D) \setminus S|_{g_{t+\tau_1}} \\ &\leq \int_{B(z',t,\alpha) \times (B(y,t,D)\setminus S)} |\{u \in [0, l_{y_1,y_2}]; \widetilde{r}_{Rm}(\gamma_{y_1,y_2}(u), t+\tau_1) < \theta\}| dg_{t+\tau_1}(y_1) dg_{t+\tau_1}(y_2) \\ &\leq C_8 \theta^2 \end{aligned}$$

implies $|B(y,t,D) \setminus S|_{g_{t+\tau_1}} \leq C_8\theta^{\frac{1}{2}}$. For any $y_2 \in S$, the subset S_{y_2} consisting of $y_1 \in B_{g^i}(z',t,\alpha)$ such that

$$|\{u \in [0, l_{y_1,y_2}]; \widetilde{r}_{Rm}(\gamma_{y_1,y_2}(u), t+\tau_1) < \theta\}| \geq \frac{1}{4}\theta$$

must satisfy

$$\frac{1}{4}\theta |B(z',t,\alpha) \setminus S_{y_2}|_{g_{t+\tau_1}} \leq \int_{B(z',t,\alpha)} |\{u \in [0, l_{y_1,y_2}]; \widetilde{r}_{Rm}(\gamma_{y_1,y_2}(u), t+\tau_1) < \theta\}| dg_{t+\tau_1}(y) \leq \theta^{\frac{3}{2}},$$

or equivalently, $|B(z',t,\alpha) \setminus S_{y_2}|_{g_{t+\tau_1}} \leq 4\theta^{\frac{1}{2}}$. For any $y_2 \in S$ and $y_1 \in S_{y_2} \setminus B(y_2, t+\tau_1, \theta)$, if $\widetilde{r}_{Rm}(\gamma_{y_1,y_2}(u), t+\tau_1) < \frac{1}{2}\theta$ for some $u \in [0, l_{y_1,y_2}]$, then because $\widetilde{r}_{Rm}(\cdot, t+\tau_1)$ is 1-Lipschitz with respect to $g_{t+\tau_1}$, we get

$$|\{u \in [0, l_{y_1,y_2}]; \widetilde{r}_{Rm}(\gamma_{y_1,y_2}(u), t+\tau_1) < \theta\}| > \frac{1}{2}\theta,$$

a contradiction. Assume we have chosen $\theta < \min\{\alpha, \tau_0\}$. Then, because $\widetilde{r}_{Rm}(\cdot, t+\tau_1) \geq \tau_0$ on $B(z', t+\tau_1, \alpha)$, we know that if $y_1 \in S_{y_2} \cap B(y_2, t+\tau_1, \theta)$, we must have

$$\gamma_{y_1,y_2}([0, l_{y_1,y_2}]) \subseteq B(z', t+\tau_1, \alpha) \subseteq \{\widetilde{r}_{Rm}(\cdot, t+\tau_1) \geq \theta\}.$$

In either case, we conclude that $\widetilde{r}_{Rm}(\cdot, t+\tau_1) > \frac{1}{2}\theta$ along γ_{y_1,y_2} for any $y_2 \in S_1$, $y_1 \in S_{y_2}$. Let $\epsilon_P = \epsilon_P(A) > 0$ be as in Theorem 31, and assume we have chosen $\tau_1 = \tau_1(A, D, \theta) \in (0, \epsilon_P^2\theta^4)$. Then, for any $y_2 \in S$, $y_1 \in S_{y_2}$, $u \in [0, l_{y_1,y_2}]$, and $s \in [t, t+\tau_1]$, we have

$$|Rm|(\gamma_{y_1,y_2}(u), s) \leq {\epsilon_P}^{2}\theta^{-2}.$$

In particular, we can estimate

$$\left|\partial_s \log|\dot\gamma_{y_1,y_2}(u)|^2_{g_s}\right| \le 2n\epsilon_P{}^2\theta^{-2},$$

so that integration in time and $|\dot\gamma_{y_1,y_2}(u)|^2_{g_{t+\tau_1}} = 1$ give $|\dot\gamma_{y_1,y_2}(u)|_{g_s} \le e^{n\epsilon_P{}^2}$ for all $s \in [t, t+\tau_1]$.

For the moment, fix $y_2 \in S$ and $y_1 \in S_{y_2}$, and set $\eta(r) := \gamma_{y_1,y_2}(\tau_1^{-1} l_{y_1,y_2} r)$ for $t \in [0,\tau_1]$. Because $l_{y_1,y_2} \le D'$, we have the reduced length estimate

$$\begin{aligned}\ell_{(y_1,t+\tau_1)}(y_2,t) &\le \frac{1}{2\sqrt{\tau_1}}\int_0^{\tau_1}\sqrt{r}\left(R(\eta(r),t+\tau_1-r)+|\dot\eta(r)|^2_{g_{t+\tau_1-r}}\right)dr\\ &\le C(n)\epsilon_P{}^2\theta^{-2}+e^{2n\epsilon_P{}^2}\int_0^{\tau_1}\frac{l^2_{y_1,y_2}}{\tau_1^2}dr \le C_9(A,D,\theta,\tau_1).\end{aligned}$$

Because $\widetilde{r}_{Rm}(z'', s) \ge \tau_0$ for all $z'' \in B(z', t, \alpha)$, we can integrate $\partial_s dg_s|_{z''} \ge -c(n)\tau_0{}^{-2} dg_s|_{z''}$ from t to $t+\tau_1$ to get

$$|B(z',t,\alpha)|_{g_{t+\tau_1}} \ge c_{10}(A).$$

Then, for $y_2 \in S$ fixed, we integrate over $y_1 \in S_{y_2}$ and use the integrated form of Perelman's differential Harnack inequality (see Proposition 16.54 of [CCG+2]) to get a lower bound for K at (y_2, t):

$$\begin{aligned}K(y_2,t) &\ge (4\pi\tau_1)^{-\frac{n}{2}}e^{-C_9}\int_{S_{y_2}}K(y',t+\tau_1)dg_{t+\tau_1}(y')\\ &\ge c_{11}(A,D,\theta,\tau_1)\left(|B(z',t,\alpha)|_{g_{t+\tau_1}} - |B(z',t,\alpha)\setminus S_{y_2}|_{g_{t+\tau_1}}\right)\\ &\ge c_{11}(A,D,\theta,\tau_1)\left(c_{10}(A) - 4\theta^{\frac{1}{2}}\right) =: \sigma > 0.\end{aligned}$$

assuming we have chosen $\theta \le (\frac{1}{8}c_{10}(n,A))^2$.

Finally, we have $\partial_s dg_s|_{z''} \le \frac{c(n)}{\tau_1} dg_s|_{z''}$ for all $z'' \in M$, $s \in [t, t+\tau_1]$ with $\widetilde{r}_{Rm}(z'', t) \ge \epsilon_P{}^{-1}\sqrt{\tau_1}$, so we can integrate from t to $t+\tau_1$ to obtain $dg_{t+\tau_1} \ge e^{-c(n)} dg_t$ for such z'', hence

$$\begin{aligned}|B(y,t,D)\setminus S|_{g_t} &\le \left|(B(y,t,D)\setminus S)\cap\{\widetilde{r}_{Rm}(\cdot,t)\ge\epsilon_P{}^{-1}\sqrt{\tau_1}\}\right|_{g_t}\\ &\quad + |B(y,t,D)\cap\{\widetilde{r}_{Rm}(\cdot,t)<\epsilon_P{}^{-1}\sqrt{\tau_1}\}|_{g_t}\\ &\le e^{c(n)}|B(y,t,D)\setminus S|_{g_{t+\tau_1}} + E(\epsilon_P{}^{2}\sqrt{\tau_1})^2\\ &\le C_8 e^{c(n)}\theta^{\frac{1}{4}} + C(A,D)\tau_1.\end{aligned}$$

The claim follows by taking $\theta = \theta(n, A, D, \delta) > 0$, then (since τ_1 depends on θ) $\tau_1 = \tau_1(A, D, \delta) > 0$ sufficiently small. $\square$

Now let $(M_i, (g_t^i)_{t\in[-4,0)}, (\nu_t^i)_{t\in[-4,0)})$ be a sequence of Ricci flow solutions satisfying $Rc(g_t^i) \geq -Ag_t^i$ and $|B_{g^i}(x,t,r)|_{g_t^i} > A^{-1}r^n$ for all $(x,t) \in M_i \times [-4,0)$ and $r \in (0,1]$, where $\nu_t^i = K^i(\cdot,t)dg_t^i$ are conjugate heat kernels based at the singular time (if $(g_t^i)_{t\in[-4,0)}$ does not develop a singularity at time $t=0$, these are just the usual conjugate heat kernel based at some points $x_i \in M_i$). By Theorem 1.38 of [B5], we can pass to a subsequence to obtain $\mathbb{F}$-convergence within some correspondence $\mathfrak{C}$:

$$(M, (g_t^i)_{t\in(-4,0)}, (\nu_t^i)_{t\in(-4,0)}) \xrightarrow[i\to\infty]{\mathbb{F},\mathfrak{C}} (\mathcal{X}, (\mu_t^\infty)_{t\in(-4,0)}),$$

where $\mathcal{X}$ is an H_n-concentrated, future-continuous metric flow of full support.

Lemma 16. *Let (y_i, t) be H_n-centers of ν^i for some fixed time $t \in [-2,-1]$ where the $\mathbb{F}$-convergence is timewise. Then there exist $\Lambda < \infty$, $y_i' \in B_{g^i}(y_i, t, \Lambda)$, and $y_\infty' \in \mathcal{X}_t$ such that, after passing to a subsequence,*

$$(M_i, d_{g_t^i}, y_i') \to (\mathcal{X}_t, d_{g_t^\infty}, y_\infty')$$

in the pointed Gromov-Hausdorff sense. By passing to a subsequence, we may also find $y_\infty \in \mathcal{X}_t$ such that

$$(M_i, d_{g_t^i}, y_i) \to (\mathcal{X}_t, d_{g_t^\infty}, y_\infty)$$

in the pointed Gromov-Hausdorff sense.

Proof. Fix $D < \infty$ and $r \in (0,1]$. Fix $\delta > 0$ to be determined, and apply Lemma 15 to obtain subsets $S_i \subseteq B_{g^i}(y_i, t, D+1)$ such that

$$|B_{g^i}(y_i, t, D+1) \setminus S_i|_{g_t^i} < \delta$$

and $\sigma = \sigma(A, D, \delta) > 0$ such that $K^i(\cdot, t) > \sigma$ on S_i. For any $z \in B_{g^i}(y_i, t, D+1)$ and $r \in (0, 1]$, we estimate

$$\begin{aligned} |B_{g^i}(z,t,r) \cap S_i|_{g_t^i} &\geq |B_{g^i}(z,t,r)|_{g_t^i} - |B_{g^i}(y_i,t,D+1) \setminus S_i|_{g_t^i} \\ &\geq A^{-1} r^n - \delta, \end{aligned}$$

so if we choose $\delta := \frac{1}{2} A^{-1} r^n$, then

$$\nu^i(B_{g^i}(z,t,r)) = \int_{B_{g^i}(z,t,r)} K^i(\cdot, t) dg_t^i \geq \sigma |B_{g^i}(z,t,r) \cap S_i|_{g_t^i} \geq \sigma',$$

where $\sigma' = \sigma'(A, D, r) > 0$. We may therefore apply Proposition 16 along with the fact that time-wise convergence at time t implies

$$(M_i, d_{g_t^i}, \nu_t^i) \to (Y, d_{g_t^\infty}, \nu_t^\infty)$$

in the W_1-Gromov-Wasserstein sense. □

Lemma 17. *Suppose that $(X, d, \mathcal{R}, g, f)$ is a singular shrinking GRS with $Rc(g) \geq 0$ on $\mathcal{R}$, corresponding to a tangent flow of a smooth, closed Ricci flow at the singular time. Also assume that there are closed Ricci flows $(M_i, (g_t^i)_{t \in [-4,0]}, p_i)$ satisfying $Rc(g_t^i) \leq A g_t^i$ and $|B_{g^i}(x,t,r)|_{g_t^i} \geq A^{-1} r^n$ for all $(x,t) \in M_i \times [-4, 0]$ and $r \in (0,1]$, such that $(M_i, d_{g^i_{t'}}, p_i) \to (X, d, p)$ in the pointed Gromov-Hausdorff sense for some $p \in \mathcal{R}$ and $t' \in [-2, -1]$.*

If $(\mathcal{R}, g)$ is not Ricci flat, then $\inf_{\mathcal{R}} R(g) > 0$.

Remark 15. *Both the statement and the proof of this are modifications of Proposition 1.1 in [N2]. The main technical difficulty is showing that the integral curve of ∇f starting at $x \in \mathcal{R}$ is complete for almost-every $x \in \mathcal{R}$. To establish this, we argue similarly to Claim 2.32 of [CW3]. The argument of Claim 2.32 used $\nabla^2 f = 0$ to establish estimates for the distortion of the volume form along the gradient flow of ∇f. We no longer have this estimate, but $Rc \geq 0$ along with $R \leq f$ and a locally uniform upper bound for f tell us that $|Rc|$ is locally bounded on $\mathcal{R}$, and it turns out this is enough to make the argument work.*

We observe that the proof is a trivial modification of Ni's when $n = 4$ since the regular set $\mathcal{R}$ is convex and all orbifold points are critical points for f, hence the gradient flow is complete.

Proof. Suppose $(\mathcal{R}, g)$ is not Ricci flat, so that $R > 0$ on $\mathcal{R}$ by Theorem 1.19 of [B5]. Because $\mathcal{R}$ is connected, we can add a constant to f to assume that $R + |\nabla f|^2 = f$. Write $r := d(\cdot, p)$. Integrating $|\nabla \sqrt{f}| \leq \frac{1}{2}$ along almost-minimizing curves in $\mathcal{R}$ from p to $x \in \mathcal{R}$ gives $\sqrt{f(x)} \leq \sqrt{f(p)} + \frac{1}{2} r(x)$, so that

$$f(x) \leq \frac{1}{2} r^2(x) + 2f(p)$$

for all $x \in \mathcal{R}$. By Theorem 3.7 of [CC3] (here we are also using that $\mathcal{R}$ is open by Theorem 33) and the proof of Proposition 2.3(c) in [B2], there is an open subset $\mathcal{G} \subset \mathcal{R}$ of full measure such that, for any $x \in \mathcal{G}$, there is a unique minimizing geodesic of (X, d) from p to x that lies entirely in $\mathcal{R}$. Given $x \in \mathcal{G}$, let $\gamma : [0, l] \to \mathcal{R}$ be such a unit-speed arclength minimizing geodesic, where $l := r(x)$. Let $(E_i)_{i=1}^n$ be an orthonormal frame at p with $E_n := \dot{\gamma}(0)$, and let $E_i \in \mathcal{X}(\gamma)$ be the corresponding parallel translations along γ. If $l \geq 2$, then for any $r_0 \in [0, 1]$, we define

$$Y_i(s) := \begin{cases} sE_i(s), & s \in [0,1] \\ E_i(s), & s \in [1, l - r_0] \\ \frac{l-s}{r_0} E_i(s) & s \in [l - r_0, l] \end{cases}$$

for $i = 1, ..., n-1$. Because γ is minimizing, hence stable, we have

$$\begin{aligned} 0 &\leq \int_0^l \left(|\nabla_{\dot{\gamma}} Y_i|^2 - R(Y_i, \dot{\gamma}, \dot{\gamma}, Y_i) \right) ds \\ &= \int_0^1 \left(1 - s^2 R(E_i, \dot{\gamma}, \dot{\gamma}, E_i) \right) ds - \int_1^{l - r_0} R(E_i, \dot{\gamma}, \dot{\gamma}, E_i) ds \\ &\quad + \int_{l - r_0}^l \left(\frac{1}{r_0^2} - \left(\frac{l-s}{r_0} \right)^2 R(E_i, \dot{\gamma}, \dot{\gamma}, E_i) \right) ds, \end{aligned}$$

so we can sum to obtain (using $Rc \geq 0$)

$$0 \leq n - 1 + \frac{n-1}{r_0} + \int_0^1 (1-s^2) Rc(\dot\gamma,\dot\gamma)ds - \int_0^{l-r_0} Rc(\dot\gamma,\dot\gamma)ds - \int_{l-r_0}^{l} \left(\frac{l-s}{r_0}\right)^2 Rc(\dot\gamma,\dot\gamma)ds$$
$$\leq C\left(n, \sup_{B(p,1)\cap R} |Rc|\right) - \int_0^{l-r_0} Rc(\dot\gamma,\dot\gamma)ds + \frac{n-1}{r_0}.$$

Moreover, we know $Rc \geq 0$ and $R \leq f \leq \frac{1}{2}r^2(x) + 2f(p)$ on R, so combining these gives

$$\sup_{B(p,1)\cap R} |Rc| \leq \sup_{B(p,1)\cap R} R \leq \frac{1}{2} + 2f(p) < \infty$$

Claim 1: There exist $C < \infty$ and $\Lambda < \infty$ such that if $R(x) \geq 1$ and $l \geq \Lambda$, then

$$\int_0^l Rc(\dot\gamma,\dot\gamma)ds \leq \frac{l}{4} + C.$$

Choose $\Lambda > 1$ such that $f(x) \leq r^2(x)$ for $x \in G \setminus B(p,\Lambda)$. Since $R + |\nabla f|^2 = f$, this implies $|\nabla f(x)| \leq r(x)$ for $x \in G \setminus B(p,\Lambda)$. Thus

$$|\nabla R|^2 = 4|Rc(\nabla f)|^2 \leq 4R^2|\nabla f|^2.$$

Combining estimates gives $|\nabla \log R| \leq 2|\nabla f| \leq 2r$ on $G \setminus B(p,\Lambda)$. Set $r_0 := \min\{\frac{4(n-1)}{l}, 1\}$, so that if $l \geq 2\Lambda$, then for $s \in [l - r_0, l]$,

$$\log\left(\frac{R(\gamma(s))}{R(x)}\right) \leq 2lr_0 \leq 2(n-1),$$

hence $R(\gamma(s)) \leq e^{2(n-1)}R(x)$ for $s \in [l - r_0, l]$. Combining estimates gives

$$\int_0^l Rc(\dot\gamma,\dot\gamma)ds \leq \int_0^{l-r_0} Rc(\dot\gamma,\dot\gamma)ds + \int_{l-r_0}^{l} R(\gamma(s))ds$$
$$\leq C(g,p) + \frac{n-1}{r_0} + e^{n-1}r_0R(x) \leq C(g,p) + \frac{l}{4}.$$

Whenever $r(x) = l \geq \Lambda$ and $R(x) \geq 1$, we can thus estimate

$$\langle \nabla f(x), \dot\gamma(l)\rangle - \langle \nabla f(p), \dot\gamma(0)\rangle = \int_0^l \nabla^2 f(\dot\gamma,\dot\gamma)ds = \int_0^l \left(\frac{1}{2} - Rc(\dot\gamma,\dot\gamma)\right)ds$$
$$\geq \frac{l}{2} - C - \frac{l}{4} \geq \frac{l}{8},$$

so that,

$$\langle \nabla f(x), \dot{\gamma}(l) \rangle \geq \left(\frac{1}{8} r(x) - |\nabla f(p)| \right),$$

which implies that $\nabla f(x) \neq 0$ if in addition $r(x) > 8|\nabla f(p)|$.

By Theorem 1.19 of [B5], $\mathcal{S}$ has Minkowski dimension 4, which implies that for any $D < \infty$, there exists $E = E(A, D) < \infty$ such that

$$|\{d(\cdot, \mathcal{S}) < s\} \cap B(p, D) \cap \mathcal{R}|_g \leq E s^{\frac{7}{2}} \tag{4.3.1}$$

for all $s \in (0, 1]$.

Claim 2: There is a Borel subset $\mathcal{G}' \subseteq \mathcal{G}$ of full measure such that for any $x \in \mathcal{G}'$, the integral curve of ∇f through x exists for all time.

Let (φ_t) be the (partially defined) flow corresponding to ∇f. We first observe that because $|\nabla f|$ is locally bounded, the escape lemma for ODEs guarantees that $\varphi_t(x)$ exists for all time unless $t \mapsto \varphi_t(x)$ has a limit point in $\mathcal{S}$. For each $D < \infty$ and $s \in (0, 1]$, define

$$S_{D,s} := \left\{ x \in \mathcal{R} \cap B(p, D); d(\varphi_t(x), \mathcal{S}) < \frac{1}{2} s \text{ for some } t \in [-D, D] \right\}.$$

Because $d(\cdot, \mathcal{S})$ is 1-Lipschitz, we can find $h \in C^\infty(\mathcal{R})$ such that $|\nabla h| \leq 2$ and $\frac{1}{2} d(\cdot, \mathcal{S}) < h < 2d(\cdot, \mathcal{S})$ on $\mathcal{R}$. For any $D' < \infty$, (4.3.1) gives $E = E(A, D') < \infty$ such that

$$|\{h < 2s\} \cap B(p, D') \cap \mathcal{R}|_g \leq E s^{\frac{7}{2}}$$

for all $s \in (0, 1]$. Thus, by the coarea formula,

$$\begin{aligned} \int_s^{2s} \mathcal{H}^{n-1}(h^{-1}(t) \cap B(p, D')) dt &= \int_{\{s \leq h \leq 2s\} \cap B(p, D') \cap \mathcal{R}} |\nabla h| dg \\ &\leq 2|\{h < 2s\} \cap B(p, D') \cap \mathcal{R}|_g \leq 2E s^{\frac{7}{2}}. \end{aligned}$$

By Sard's theorem, for any $s \in (0, 1]$, we can find $t = t(s) \in (s, 2s)$ such that $\Sigma_s := h^{-1}(t) \cap B(p, D')$ is smooth and $\mathcal{H}^{n-1}(\Sigma_s) \leq 4E s^{\frac{5}{2}}$. Write $S_{D,s} := I_s \cup II_s$, where

$$I_s := \{x \in S_{D,s}; d(x, \mathcal{S}) \leq 4s\},$$

$$II_s := \{x \in S_{D,s}; d(x,\mathcal{S}) > 4s\},$$

so that $|II_s|_g \le Es^{\frac{7}{2}}$. For any $x \in II_s$, there exists $t \in (-D,D)$ such that $\varphi_t(x) \in \Sigma_s$. Because $|\nabla f| \le \sqrt{f} \le C(r+1)$, we have

$$\left|\frac{d}{dt} r(\varphi_t(x))\right| = |\langle \nabla r, \nabla f\rangle|(\varphi_t(x)) \le C\left(r(\varphi_t(x)) + 1\right)$$

for almost every t, so we can find $D' = D'(D) < \infty$ such that $\varphi_t(x) \in B(p, D')$ for all $x \in II_s$ and $t \in (-D,D)$ such that $\varphi_t(x)$ exists. Set

$$\Omega_s := \{(t,x) \in (-D',D') \times \Sigma_s; \varphi_t(x) \text{ is well defined}\},$$

which is open in $(-D,D) \times \Sigma_s$, and define $\eta : \Omega_s \to R, (t,x) \mapsto \varphi_t(x)$. For any $x \in \Sigma_s$ and $v \in T_x\Sigma_s$, we have (since $|Rc| \le C(D)$ on $\eta(\Omega_s)$)

$$\begin{aligned}\left|\frac{d}{dt}(\eta^* g)_{(t,x)}(v,v)\right| &= \left|\frac{d}{dt}(\varphi_t^* g)_x(v,v)\right| \\ &= |(\mathcal{L}_{\nabla f} g)(d(\varphi_t)_x v, d(\varphi_t)_x v)| \\ &= \left|2\nabla^2 f_{\varphi_t(x)}(d(\varphi_t)_x v, d(\varphi_t)_x v)\right| \\ &\le 2|\nabla^2 f|(\varphi_t(x))|g_{\varphi_t(x)}(d(\varphi_t)_x v, d(\varphi_t)_x v)| \\ &\le C(D)\left|(\eta^* g)_{(t,x)}(v,v)\right|\end{aligned}$$

which we can integrate in t to obtain $(\eta^* g)_{(t,x)}(v,v) \le C(D)$ for $|v|_{g_{\Sigma_s}} = 1$, where g_{Σ_s} is the restriction of g to Σ_s. Also,

$$(\eta^* g)_{(t,x)}(\partial_t, \partial_t) = |\nabla f|^2_{\varphi_t(x)} \le C(D)$$

so the Jacobian $J = \frac{\eta^* dg}{dt \wedge dg_{\Sigma_s}}$ satisfies $|J| \le C(D)$. Because $II_s \subseteq \eta(\Omega_s)$, we thus obtain

$$|II_s|_g \le \int_{\Sigma_s \times [-D,D]} d(\eta^* g) \le \int_{-D}^{D}\int_{\Sigma_s} C(D)dg_{\Sigma_s}dt \le C(D)\mathcal{H}^{n-1}(\Sigma_s) \le C(A,D)s^{\frac{5}{2}}.$$

Combine estimates to get $|S_{D,s}| \le C(A,D)s^{\frac{5}{2}}$. By taking $s \searrow 0$, we get that the set S_D of $x \in B(p,D)$ such that $\varphi_t(x)$ is undefined for some $t \in (-D,D)$ has measure zero. Now taking $D \nearrow \infty$, we see that the set of $x \in \mathcal{R}$ such that $\varphi_t(x)$ is undefined for some $t \in \mathbb{R}$ has measure zero.

Next, we observe that $\mathcal{G}' \subset \mathcal{R}$ is a set of full measure which is preserved by the flow (φ_t). If $D \subset \mathbb{R} \times \mathcal{R}$ is the (open) maximal flow domain, then

$$\xi : D \to \mathbb{R} \times \mathcal{R}, \quad (t,x) \mapsto (t, \varphi_t(x))$$

is a diffeomorphism onto its (open) image. Note that $\mathbb{R} \times (\mathcal{R} \setminus \mathcal{G})$ has measure zero in $\mathbb{R} \times \mathcal{R}$, hence

$$\xi^{-1}(\mathbb{R} \times (\mathcal{R} \setminus \mathcal{G})) = \{(t,x) \in D; \varphi_t(x) \in \mathcal{R} \setminus \mathcal{G}\}$$

has measure zero in D. In particular,

$$\{(t,x) \in \mathbb{R} \times \mathcal{G}'; \varphi_t(x) \in \mathcal{R} \setminus \mathcal{G}\} \subset \xi^{-1}(\mathbb{R} \times (\mathcal{R} \setminus \mathcal{G}'))$$

has measure zero in $\mathbb{R} \times \mathcal{G}'$. By Fubini's theorem, we may conclude that the set

$$\mathcal{G}'' := \{x \in \mathcal{G}'; |\{t \in \mathbb{R}; \varphi_t(x) \in \mathcal{R} \setminus \mathcal{G}\}| = 0\}$$

has full measure in $\mathcal{G}'$, hence in $\mathcal{R}$.

Suppose $x \in \mathcal{G}'' \setminus B(p, \Lambda)$ satisfies $R(x) \geq 1$, and let $\sigma : \mathbb{R} \to \mathcal{R}$ be the integral curve of ∇f with $\sigma(0) = x$. Then

$$\frac{d}{dt} R(\sigma(t)) = \langle \nabla R(\sigma(t)), \nabla f(\sigma(t)) \rangle = 2Rc(\nabla f, \nabla f)(\sigma(t)) \geq 0,$$

which implies $R(\sigma(t)) \geq R(x)$ for $t \geq 0$. Recall that r in smooth on $\mathcal{G}$ (see Proposition 2.3 of [B2]), so for any $t \in \mathbb{R}$ such that $r(\sigma(t)) \geq \Lambda$ and $\sigma(t) \in \mathcal{G}$, we have

$$\frac{d}{dt} r(\sigma(t)) = \langle \nabla r(\sigma(t)), \nabla f(\sigma(t)) \rangle \geq \frac{1}{8} r(\sigma(t)) - |\nabla f(p)|,$$

so that

$$\frac{d}{dt} \log\left(\frac{1}{8} r(\sigma(t)) - |\nabla f(p)|\right) \geq \frac{1}{8}.$$

Because $t \mapsto \log(\frac{1}{8}r(\sigma(t)) - |\nabla f(p)|)$ is locally Lipschitz, with derivative almost-everywhere $\geq \frac{1}{8}$, we can thus integrate to obtain that σ moves x into $B(p,\Lambda) \cap \mathcal{R}$ for sufficiently negative $t < 0$, but R decreases as t decreases, so

$$R(x) \geq \min\left\{ \inf_{B(p,\Lambda)\cap\mathcal{R}} R, 1 \right\}$$

for all $x \in \mathcal{G}''$. Next, recall that we have the estimate $|\nabla R| = 2|Rc(\nabla f)| \leq CR\sqrt{f} \leq C(\Lambda)R$ on $B(p,2\Lambda) \cap \mathcal{R}$, or equivalently $|\nabla \log R| \leq C(\Lambda)$. Fix $\eta > 0$, and suppose $R(y) < \eta$ for some $y \in B(p,\Lambda) \cap \mathcal{R}$. Integrating $|\nabla \log R| \leq C(\Lambda)$ along a curve from p to y within $B(p,2\Lambda) \cap \mathcal{R}$ gives $R(p) \leq C(\Lambda)\eta$. By choosing $\eta < c(\Lambda, R(p))$ sufficiently small, we get a contradiction, so we conclude that $\inf_{B(p,\Lambda)\cap\mathcal{R}} R > 0$. Because $R|\mathcal{R}$ is continuous and $\mathcal{G}''$ is dense in $\mathcal{R}$, the claim follows. □

We now restrict our attention to tangent flows of a fixed Ricci flow.

For the remainder of this section, we assume that $(M^n,(g_t)_{t\in[0,T)})$ is a closed Ricci flow satisfying $Rc(g_t) \geq -Ag_t$ for all $t \in [0,T)$, as well as $A^{-1}r^n \leq |B(x,t,r)|_{g_t} \leq Ar^n$ for all $(x,t) \in M \times [0,T)$ and $r \in (0,1]$. Also assume $\text{diam}_{g_t}(M) \leq A$ for all $t \in [0,T)$, and that $T \leq A$. By Lemma 13, all of these assumptions hold with A replaced by some $\overline{A}(n,A,T,\text{diam}_{g_0}(M)) < \infty$ for a closed Ricci flow satisfying (4.1.1),(4.1.2).

The Ricci lower bound implies that $t \mapsto e^{-At}d_{g_t}$ are pointwise nonincreasing as functions on $M \times M$, so in particular

$$d_{g_T}(x,y) := \lim_{t\to\infty} d_{g_t}(x,y) = e^{AT}\lim_{t\to\infty} e^{-At}d_{g_t}(x,y) \in [0,\infty)$$

is a well-defined pseudometric on M. We can form the metric space whose underlying set is $X = M/\sim$, where $x \sim y$ if and only if $\lim_{t\nearrow T} d_t(x,y) = 0$, and equip X with the induced metric d_X from passing d_{g_T} to the quotient.

Lemma 18. $\lim_{t\nearrow T} d_{GH}((M,d_{g_t}),(X,d_X)) = 0.$

Proof. It suffices to show that for any $\epsilon > 0$, there exists $\delta > 0$ such that the quotient map $\pi : (M, d_{g_t}) \to (X, d_X)$ is an ϵ-Gromov-Hausdorff approximation for all $t \in (T - \delta, T)$. Let $\{x_1, ..., x_N\}$ be an $\frac{1}{5}e^{-AT}\epsilon$-dense subset of (M, d_{g_0}), which is thus $\frac{\epsilon}{5}$-dense in each (M, d_{g_t}). Next, choose $\delta > 0$ such that $|d_{g_t}(x_i, x_j) - d_{g_T}(x_i, x_j)| < \frac{\epsilon}{5}$ for all $i, j \in \{1, ..., N\}$ whenever $t \in (T - \delta, T)$. Let $x, y \in M$ and $t \in (T - \delta, T)$ be arbitrary, and choose $i, j \in \{1, ..., N\}$ such that $d_{g_t}(x, x_i), d_{g_t}(y, x_j) < \frac{1}{5}e^{-AT}\epsilon$, hence $d_{g_T}(x, x_i), d_{g_T}(y, x_j) < \frac{\epsilon}{5}$. Then we can estimate

$$\begin{aligned} |d_{g_t}(x,y) - d_{g_T}(x,y)| &\leq d_{g_t}(x, x_i) + d_{g_t}(y, x_j) + d_{g_T}(x, x_i) + d_{g_T}(y, x_j) \\ &\quad + |d_{g_t}(x_i, x_j) - d_{g_T}(x_i, x_j)| \\ &< \epsilon. \end{aligned}$$

Because $d_{g_T}(x, y) = d_X(\pi(x), \pi(y))$ and π is surjective (hence ϵ-dense), the claim follows. $\square$

In particular, (X, d_X) is a noncollapsed Ricci limit space, so by Theorem 33, there is an open dense subset $\mathcal{R}_X$ defined as the set of points $x \in X$ which have a tangent cone isometric to the standard Euclidean space $\mathbb{R}^n$. Moreover, (X, d_X) is a compact metric length space, and it follows from Theorem 3.7 of [CC3] that (X, d_X) is the completion of $\mathcal{R}_X$ equipped with the length metric corresponding to $d_X|(\mathcal{R}_X \times \mathcal{R}_X)$. We also set $\mathcal{S}(X) := X \setminus \mathcal{R}_X$, and for any $x \in M$, we denote by $\overline{x} \in X$ the corresponding equivalence class.

Lemma 19. *π restricts to a homeomorphism from the open dense subset*

$$M \setminus \Sigma := \left\{ x \in M; \sup_{U \times [0,T)} |Rm| < \infty \text{ for some neighborhood } U \text{ of } x \text{ in } M \right\}$$

of M to its image $(\mathcal{R}_X, g_X)$.

Proof. If $x \in M \setminus \Sigma$, then standard distortion estimates imply $d_{g_T}(x, y) \neq 0$ for all $y \in M \setminus \{x\}$, so $\pi|(M \setminus \Sigma)$ is injective. Also, $\pi : (M \setminus \Sigma, d_{g_T}) \hookrightarrow (\mathcal{R}, d)$ is an isometry, hence a homeomorphism onto its image. Now suppose we are given $\overline{x} \in \mathcal{R}_X$. Then for any $\sigma > 0$, there exists $r_0 = r_0(\sigma) > 0$ such that

$$d_{PGH}\left((B^X(\overline{x}, r), d_X, \overline{x}), (B^{\mathbb{R}^n}(0^n, r)), d_{\mathbb{R}^n}, 0^n)\right) < \sigma r$$

for all $r \in (0, r_0]$.

Claim: $\lim_{t\nearrow T} d_{PGH}\big((B^X(\overline{x}, r), d_X, \overline{x}), (B_g(x,t,r), d_{g_t}, x)\big) = 0$ for any fixed $r \in (0, r_0]$.

Fix $\epsilon > 0$. By the proof of the previous lemma, there exists $\delta = \delta(\epsilon) > 0$ such that π satisfies

$$|d_X(\overline{y}_1, \overline{y}_2) - d_{g_t}(y_1, y_2)| < \epsilon$$

for all $t \in (T - \delta, T)$ and $y_1, y_2 \in B_g(x,t,r)$. It thus remains to show that $\pi(B_g(x,t,r))$ is ϵ-dense in $B^X(\overline{x}, r)$. To see this, choose an $\frac{\epsilon}{2}$-dense subset $\{\overline{y}_1, ..., \overline{y}_N\}$ of $B^X(\overline{x}, r)$. Then, for any choice of representatives $y_i \in \overline{y}_i$, we can modify δ to assume that $d_t(y_i, x) < r$ for all $i \in \{1, ..., N\}$ whenever $t \in (T - \delta, T)$.

We may therefore find $\tau_r = \tau_r(\sigma) > 0$ such that for any $t \in (T - \tau_r, T)$, we have

$$d_{PGH}\big((B^X(\overline{x}, r), d_X, \overline{x}), (B^{\mathbb{R}^n}(0, r), d_{\mathbb{R}^n}, 0^n)\big) < \sigma r.$$

By Proposition 32, Remark 13, and choosing $\sigma = \sigma(A) > 0$ sufficiently small, we have $\widetilde{r}_{Rm}(x,t) > \sigma r$ for all $t \in (T - \tau_{r_0}, T)$. By choosing $t = t(r')$ sufficiently close to t, we get a curvature bound $|Rm| \leq (r')^{-2}$ on $B_g(x,t,r') \times [t,T]$, so $x \in M \setminus \Sigma$. $\square$

In particular, $\mathcal{R}_X$ has the structure of a smooth Riemannian manifold, equipped with the metric g_X using the homeomorphism $\pi|(M \setminus \Sigma) \to \mathcal{R}_X$. Moreover, it is standard that $g_t \to g_T$ in $C^\infty_{loc}(M \setminus \Sigma)$ as $t \nearrow T$, hence $(M \setminus \Sigma, g_t) \to (\mathcal{R}_X, g_X)$ in the C^∞ Cheeger-Gromov sense, with canonical diffeomorphism $\pi|M \setminus \Sigma$ and Gromov-Hausdorff approximations π. Moreover, we know that $\mathcal{S}(X)$ has Hausdorff dimension $\leq n - 2$ by [CC2].

Now fix $x \in M$ corresponding to a singular point $\overline{x} \in \mathcal{S}(X)$. Choose $t_i \nearrow T$ such that the conjugate heat kernels $K(x, t_i; \cdot, \cdot)$ converge in $C^\infty_{loc}(M \times [0,T))$ to some conjugate heat kernel $K \in C^\infty(M \times (0,T))$ at the singular time based at x. Write $d\nu_s := K(\cdot, s)dg_s$. Fix any sequence $\tau_i \searrow 0$, and define the rescaled flows $g^i_t := \tau_i^{-1} g_{T+\tau_i t}$, as well as the correspondingly rescaled conjugate heat kernels $K^i(y,t;z,s) := \tau_i^{\frac{n}{2}} K(y,T+\tau_i t; z, T+\tau_i s)$. Also set $K^i(y,t) := \tau_i^{\frac{n}{2}} K(y, T+\tau_i t)$ and $d\nu^i_t := d\nu_{x,T;T+\tau_i t} = K^i(\cdot,t)dg^i_t$. By Theorem 1.38 of [B5], we can pass to a subsequence to obtain uniform $\mathbb{F}$-convergence within some correspondence $\mathfrak{C}$ on compact time intervals:

$$(M,(g^i_t)_{t\in[-\tau_i^{-1}T,0)},(\nu^i_t)_{t\in[-\tau_i^{-1}T,0)}) \xrightarrow[i\to\infty]{\mathbb{F},\mathfrak{C}} (\mathcal{Y},(\mu^\infty_t)_{t<0}),$$

where $(\mathcal{Y},(\mu^\infty_t)_{t\in(-\infty,0)})$ is a metric soliton with Nash entropy $\mathcal{N}_{(\mu^\infty_t)}(\tau) < 0$ (in particuar, the soliton is not flat Euclidean space). Let $\mathcal{R}$ be the regular set of $\mathcal{Y}$, equipped with a Ricci flow spacetime $(\mathcal{R},\mathfrak{t},\partial_{\mathfrak{t}},(g^\infty_t)_{t\in(-\infty,0)})$. By Theorem 1.19 of [B5], there is an n-dimensional singular space $(Y,d_Y,\mathcal{R}_Y,g_Y)$, a probability measure μ on Y, a smooth function $f_Y \in C^\infty(\mathcal{R}_Y)$, and an identification $\mathcal{Y} = Y \times (-\infty,0)$ such that the following hold for all $t\in(-\infty,0)$:

(a) $(\mathcal{Y}_t,d_t,\mu^\infty_t) = (Y\times\{t\},|t|^{\frac{1}{2}}d,\mu)$,

(b) $\mathcal{R} = \mathcal{R}_Y \times (-\infty,0)$, and $\partial_{\mathfrak{t}} - \nabla f$ corresponds to the standard vector field on the second factor,

(c) $(\mathcal{R}_t, g^\infty_t, f(\cdot,t)) = (\mathcal{R}_Y \times\{t\}, |t|g_Y, f_Y)$, where $d\mu^\infty_t = K^\infty(\cdot,t)dg^\infty_t =: (4\pi|t|)^{-\frac{n}{2}}e^{-f}dg^\infty_t$ on $\mathcal{R}$

(d) $Rc + \nabla^2 f = \frac{1}{2\tau}g$ on $\mathcal{R}$, hence $Rc(g_Y) + \nabla^2 f_Y = \frac{1}{2}g_Y$ on $\mathcal{R}_Y$

Using this identification, we can apply Theorem 9.31 of [B3] to obtain an exhaustion $(U_i)_{i\in\mathbb{N}}$ of $\mathcal{R}_Y$ by precompact open sets, a sequence $\alpha_i \searrow 0$, along with embeddings $\psi_i : U_i \times (-\alpha_i^{-1},-\alpha_i) \to M \times (-\alpha_i^{-1},-\alpha_i)$ satisfying

$$\begin{aligned} \|\partial_t - (\psi_i^{-1})_*\partial^i_t\|_{C^i(U_i\times(-\alpha_i^{-1},-\alpha_i))} &< \alpha_i, \\ \|\psi_i^* g^i - g^\infty\|_{C^i(U_i\times(-\alpha_i^{-1},-\alpha_i))} &< \alpha_i, \\ \|\psi_i^* K^i - K^\infty\|_{C^i(U_i\times(-\alpha_i^{-1},-\alpha_i))} &< \alpha_i, \end{aligned} \tag{4.3.2}$$

for all $i\in\mathbb{N}$, where the C^i norms are with respect to the metric $|t|g_Y$. Set $\psi_{i,t}|U_i\times\{t\} : U_i \to$

M. Because the $\mathcal{Y}$ is a continuous metric flow on $(-\infty, 0)$, we can pass to a subsequence so that the $\mathbb{F}$-convergence is timewise for all $t \in (-\infty, 0)$.

Proposition 18. *For any $y_\infty \in \mathcal{R}_Y$ and $r > 0$, we have*

$$\mathcal{H}^n(B^Y(y_\infty, r)) = |B^Y(y_\infty, r) \cap \mathcal{R}_Y|_{g_Y} \geq A^{-1} r^n.$$

Moreover, the volume ratio

$$r \mapsto \frac{\mathcal{H}^n(B^Y(y_\infty, r))}{r^n}$$

is nonincreasing.

Proof. The first equality follows from $\mathcal{H}^n(Y \setminus \mathcal{R}_Y) = 0$ along with the fact that for any connected Riemannian manifold (N^n, h), the n-dimensional Lebesgue measure induced by h coincides with the n-dimensional Hausdorff measure induced by d_h. Fix any sequence $(z_i, -1)$ of H_n-centers of ν^i. By Lemma 16, we have pointed Gromov-Hausdorff convergence

$$(M, d_{g^i_{-1}}, z_i) \to (Y, d, z_\infty)$$

for some $z_\infty \in Y$, so we can find $y_i \in M$ converging to y_∞ with respect to this Gromov-Hausdorff convergence. By Colding's volume convergence theorem, we get

$$\mathcal{H}^n(B^Y(y_\infty, r)) = \lim_{i \to \infty} |B_{g^i}(y_i, -1, r)|_{g^i_{-1}} \geq A^{-1} r^n.$$

For any $\kappa > 0$, we know that $Rc(g^i) \geq -\kappa g^i$ for sufficiently large $i \in \mathbb{N}$, so that Bishop volume comparison and volume convergence tell us that

$$r \mapsto \frac{\mathcal{H}^n(B^Y(y_\infty, r))}{v_{-\kappa}(r)}$$

is nonincreasing. Taking $\kappa \searrow 0$ thus gives the remaining claim. □

Proposition 19. *If $(Y, d, \mathcal{R}_Y, g_Y)$ is the metric soliton corresponding to the tangent flow of*

$$(M, (g^i_t)_{t \in [-\tau_i^{-1} T, 0)}, (\nu^i_t)_{t \in [-\tau_i^{-1} T, 0)})$$

as described above, then $(\mathcal{R}_Y, g_Y)$ is Ricci flat.

Proof. Fix any sequence $\lambda_j \nearrow \infty$ and basepoint $y_0 \in \mathcal{R}_Y$. By Proposition 18, we know $\mathcal{H}^n(B(y_0,r)) \geq A^{-1}r^n$ for all $r>0$, and that the volume ratios monotonically decrease to a positive number as $r \nearrow \infty$. Because Y is a Gromov-Hausdorff limit of some noncollapsed $(M_i, d_{g^i_{-1}}, x_i)$, where $(M_i,(g^i_t)_{t\in[-4,0]})$ are closed Ricci flows satisfying $Rc(g^i_t) \geq -\tau_i A g^i_t$ where $\tau_i \searrow 0$, we may use a diagonal argument to get that the rescaled spaces $(Y, \lambda_j^{-1}d_Y, y_0)$ converge in the pointed Gromov-Hausdorff sense to another noncollapsed Ricci limit space (Z, d_Z, z_0), which is moreover a metric cone. In particular, we can choose $z' \in Z \setminus \{z_0\}$ in the regular set of Z, which corresponds under the Gromov-Hausdorff convergence to a sequence $y_j \in Y$ with

$$\lim_{j\to\infty} \lambda_j^{-1} d_Y(y_j,y_0) = d_Z(z_0,z') \in (0,\infty).$$

Recall that Colding's volume convergence theorem also holds for sequences of noncollapsed Ricci limit spaces (Theorem 10.15 of [C1]), so for any $\delta>0$, there exists $r=r(\delta)>0$ such that

$$\lambda_j^{-n}\mathcal{H}^n(B^Y(y_j,\lambda_j r)) \geq (\omega_n - \delta/2)r^n$$

for $j=j(\delta)\in\mathbb{N}$ sufficiently large. On the other hand, we know that $(M, d_{g^i_{-1}}, x_i) \to (Y,d,y_0)$ in the pointed Gromov-Hausdorff sense for some sequence $x_i \in M$, so that for any fixed $j\in\mathbb{N}$, there is a sequence $y_{i,j}\in M$ with

$$(B_{g^i}(y_{i,j},-1,\lambda_j r), d_{g^i_{-1}}, y_{i,j}) \to (B^Y(y_j,\lambda_j r), d_{g^i_{-1}}, y_{i,j}))$$

in the pointed Gromov-Hausdorff sense as $i\to\infty$. By Colding's volume convergence theorem, for any $j=j(\delta)\in\mathbb{N}$ large, we have the following for all $i=i(\delta,j)\in\mathbb{N}$ sufficiently large:

$$|B_{g^i}(y_{i,j},-1,\lambda_j r)|_{g^i_{-1}} \geq (\omega_n-\delta)(\lambda_j r)^n.$$

Moreover, we have $Rc(g^i_{-1}) \geq -\tau_i A g^i_{-1}$, but $\tau_i \searrow 0$, so we can choose $\delta=\delta(A)>0$ according to Theorem 32 and Remark 9, so that

$$\widetilde{r}^{g^i}_{Rm}(y_{i,j},\lambda_j r) \geq c(A)(\lambda_j r)$$

for $i = i(j) \in \mathbb{N}$ large. Taking $i \to \infty$, we get that $y_j \in \mathcal{R}_Y$ and $\widetilde{r}^{g_Y}_{Rm}(y_j) \geq c(A)\lambda_j r$. Since $\lambda_j \nearrow \infty$, we have $\inf_{\mathcal{R}_Y} R = 0$, hence $(\mathcal{R}_Y, g_Y)$ is Ricci flat by Lemma 17. □

Proof of Theorem 28. Let $\mathcal{Y}$ be a tangent flow based at $\nu_{x,T}$, which is modeled on a singular shrinking soliton $(Y, d_Y, \mathcal{R}_Y, g_Y)$. We first pass to a subsequence so that the $\mathbb{F}$-convergence is timewise at almost every time. We can apply Lemma 16 to the sequence $(M,(g^i_t))$ or a rescaling $\widetilde{g}^i_t := \lambda g^i_{\lambda^{-1}t}$ to obtain $y'_{i,t} \in M$ and $y'_{\infty,t} \in \mathcal{Y}_t$ such that $(M, d_{g^i_t}, y'_{i,t}) \to (\mathcal{Y}_t, d_t, y'_{\infty,t})$ for almost every $t \in (-\infty, 0)$. By Proposition 19, $Rc(g_Y) = 0$ on $\mathcal{R}_Y$, so Y is a metric cone by Theorem 2.18 of [B5]. □

Proof of Theorem 29. Suppose by way of contradiction that $L := \limsup_{t \nearrow T}(T - t)r^{-2}_{Rm}(x,t) < \infty$. Since $\overline{x} \notin S(X)$, we know $L > 0$. Because $r_{Rm}(x,t) \geq \frac{1}{2\sqrt{L}}\sqrt{T-t}$ for all $t \in (0,T)$ sufficiently close to T, we have

$$\frac{1}{\sqrt{T-t}}\int_t^T \sqrt{T-s}\,R(x,s)ds \leq \frac{1}{\sqrt{T-t}}\int_t^T \frac{4L}{\sqrt{T-s}}ds = 8L.$$

Fix $t_i \nearrow T$ such that $K(x,t_i;\cdot,\cdot)$ converge in $C^\infty_{loc}(M \times [0,T))$ to a conjugate heat kernel at the singular time based at $\overline{x}$. Then we can apply the heat kernel lower bound of [Z2] to get

$$K(y,t) \geq \frac{ce^{-8L}}{(T-t)^{\frac{n}{2}}}\exp\left(-\frac{d^2_{g_t}(x,y)}{c(T-t)}\right),$$

for all $y \in M$, (in fact, we even have the stronger bound where we replace $d_{g_t}(x,y)$ with $d_{g_T}(x,y)$) where $c = c(g_0) > 0$. Now fix $\tau_i \searrow 0$, and consider the $\mathbb{F}$-convergence within a correspondence

$$(M,(g^i_t)_{t\in[-\tau_i^{-1}T,0)},(\nu^i)_{t\in[-\tau_i^{-1}T,0)}) \xrightarrow[i\to\infty]{\mathbb{F},\mathfrak{C}} (\mathcal{Y},(\mu^\infty_t)_{t<0})$$

in the discussion preceding Proposition 18, where $\mathcal{Y}$ is a metric soliton corresponding to the singular shrinking GRS $(Y, d, \mathcal{R}_Y, g_Y, f)$. The rescaled metrics satisfy

$$\frac{1}{C}\exp\left(-C\,d^2_{g^i_{-1}}(x,y)\right) \leq K^i(y,t)$$

for all $(y,t) \in M \times [-2,0)$ and $i \in \mathbb{N}$, where $C = C(n,A,T,g_0,L) < \infty$. In particular, for any $D < \infty$, $r > 0$, and $y \in B_{g^i}(x,-1,D)$, we have

$$\int_{B_{g^i}(y,-1,r)} K^i(y,t) \geq \frac{1}{C}\exp\left(-CD^2\right)|B_{g^i}(y,-1,r)|_{g^i} \geq \frac{e^{-CD^2}}{C}A^{-1}r^n,$$

hence the hypotheses of Proposition 16 are satisfied with $x_i = x$.

Because $\mathcal{Y}$ is continuous on $[-2,0)$, the $\mathbb{F}$-convergence is timewise at $t = -1$. We let $\mathfrak{C} = ((Z_t, d_t^Z)_{t\in I'}, (\phi_t^i)_{t\in I'^{,i}, i\in\mathbb{N}\cup\{\infty\}})$ be a correspondence such that

$$\lim_{i\to\infty} d_{W_1}^{Z_t}\left((\phi^i_{-1})_*\nu^i_{-1}, (\phi^\infty_{-1})_*\mu^\infty_{-1}\right) = 0.$$

Let U_i, ψ_i be as in (4.3.2). By Proposition 16, we can pass to a subsequence to find $x_\infty \in Y$ such that

$$\lim_{i\to\infty} d_t^Z((\phi^i_{-1})(x), (\phi^\infty_{-1})(x_\infty)) = 0$$

and

$$(M, d_{g^i_{-1}}, x) \to (Y, d, x_\infty)$$

in the pointed Gromov-Hausdorff sense. Because $\liminf_{i\to\infty} r_{Rm}^{g^i}(x,-1) \geq \frac{1}{2\sqrt{L}} > 0$, we moreover have $x_\infty \in \mathcal{R}$.

By Proposition 19, $(\mathcal{R}_Y, g_Y)$ must be Ricci flat. By Theorems 1.17, 1.19 of [B5], we know $\mathcal{Y}$ is a static flow, and (Y,d) is a metric cone $C(Z)$ over some compact metric space (Z,d_Z). Choose $r_0 \in (0,1]$ such that $B(x_\infty, 2r_0) \subseteq \mathcal{R}_{-1}$.

Let $\gamma : [0,1] \to M$ be a minimizing curve from $\psi_i(x_\infty)$ to x with respect to g^i_{-1}. Suppose by way of contradiction that $\gamma([0,1]) \not\subseteq \psi_i(U_i)$, and define

$$u_* := \inf\{u \in [0,1]; \gamma(u) \in \psi_i(\partial B(x_\infty, r_0))\}.$$

Because $B(x_\infty, r_0) \subseteq U_i \times \{-1\}$ for sufficiently large $i = i(r_0, x_\infty) \in \mathbb{N}$, we must have

$$d_{g^i_{-1}}(\psi_i(x_\infty), x) \geq \mathrm{length}_{g^i_{-1}}(\gamma|[0,u_*]) \geq \frac{1}{2}\mathrm{length}_{g^\infty_{-1}}(\psi_i^{-1}\circ\gamma|[0,u_*]) \geq \frac{1}{2}r_0.$$

However,

$$d_{g^i_{-1}}(\psi_i(x_\infty),x) = d^Z_{-1}\big((\phi^i_{-1}\circ\psi_i)(x_\infty),\phi^i_{-1}(x)\big)$$
$$\le d^Z_{-1}(\phi^i_{-1}(x),\phi^\infty_{-1}(x_\infty)) + d^Z_{-1}\big((\phi^i_{-1}\circ\psi_i)(x_\infty),\phi^\infty_{-1}(x_\infty)\big) \to 0$$

as $i\to\infty$ by the choice of x_∞ and Theorem 9.31(d) in [B3], a contradiction. We therefore have $x_i := \psi_i^{-1}(x) \in U_i$ for sufficiently large $i\in\mathbb{N}$, and

$$d_{g^\infty_{-1}}(x_i,x_\infty) \le \text{length}_{g^\infty_{-1}}(\psi_i^{-1}\circ\gamma) \le 2\text{length}_{g^i_{-1}}(\gamma) \le 2d_{g^i_{-1}}(\psi_i(x_\infty),x) \to 0$$

as $i\to\infty$.

Claim: For any $\tau>0$, we have $\widetilde{r}^{g^i}_{Rm}(x,-\tau) \ge \frac{1}{2}r_0$ for $i=i(\tau)\in\mathbb{N}$ sufficiently large.

Because $x_\infty \in \mathcal{R}_{-1}$ and $\mathcal{Y}$ is static, we have $x_\infty(t)\in\mathcal{R}$ for all $t\in[-1,0)$. Moreover, for any $\tau>0$, $\{x_\infty(t); t\in[-1,-\tau]\}$ is a compact subset of $\mathcal{R}$, so admits a neighborhood $V \Subset \mathcal{R}$. Because $(\psi_i^{-1})_*\partial_t \to \partial_{\mathfrak{t}}$ in $C^\infty_{loc}(\mathcal{R})$, a standard statement about continuous dependence of ODE solutions on parameters (see Chapter 5 of [H3]) implies that for large $i\in\mathbb{N}$, the integral curve γ_i of $(\psi_i^{-1})_*\partial_t$ starting at x_i is well-defined for all $t\in[-1,-\tau]$, and $\gamma_i(t)\to x_\infty(t)$ uniformly in $t\in[-1,-\tau]$ as $i\to\infty$, so $\gamma_i([-1,-\tau]) \subseteq V \quad U_i \quad \{-\tau\}$ for sufficiently large $i\in\mathbb{N}$. In particular, $B(\gamma_i(\tau),\frac{3}{2}r_0) \Subset \mathcal{R}$ when $i=i(\tau)\in\mathbb{N}$ is large, hence $B(\gamma_i(\tau),\frac{3}{2}r_0)\subseteq U_i$ and

$$B_{g^i}(x,-\tau,r_0) \subseteq \psi_i\left(B(\gamma_i(\tau),\frac{3}{2}r_0)\right).$$

However, because $\mathcal{Y}$ is static, we know $|Rm|_{g^\infty}(y(t),t) \le \overline{C}$ for all $y\in B(x_\infty,\frac{3}{2}r_0)$ and $t\in[-1,0)$, which implies that $|Rm|_{g^i}(\cdot,-\tau)\le 2\overline{C}$ on $B_{g^i}(x,-\tau,r_0)$ for $i=i(\tau)\in\mathbb{N}$ large.

Now apply Theorem 31 on the ball $B_{g^i}(x,-\tau,\min\{r_0,\overline{C}^{-2}\})$ for some $\tau=\tau(r_0,\overline{C},A)\in(-1,0)$ sufficiently small in order to contradict the fact that x is a singular point. $\square$

4.4 C^0 Orbifold Structure in Dimension 4

Throughout this section, we assume that $(M^4,(g_t)_{t\in[0,T)})$ is a closed, simply connected four-dimensional Ricci flow satisfying $Rc(g_t) \geq -Ag_t$ for all $t \in [0,T)$, as well as $A^{-1}r^4 \leq |B(x,t,r)|_{g_t} \leq Ar^4$ for all $(x,t) \in M \times [0,T)$ and $r \in (0,1]$.

As before, fix a sequence $\tau_i \searrow 0$, a singular point $\overline{x} \in X$, let $K \in C^\infty(M \times [0,T))$ be a conjugate heat kernel at the singular time based at x, and let $g_t^i := \tau_i^{-1}g_{T+\tau_i t}$, $K^i(\cdot,t) := \tau_i^{\frac{n}{2}}K(\cdot,T+\tau_i t)$, $d\nu_t^i := K^i(\cdot,t)dg_t^i$. By Proposition 19 and Theorems 1.17, 1.38, 1.47 of [B5], we can pass to a subsequence to obtain uniform $\mathbb{F}$-convergence within some correspondence $\mathfrak{C}$ on compact time intervals:

$$(M,(g_t^i)_{t\in[-\tau_i^{-1}T,0)},(\nu_t^i)_{t\in[-\tau_i^{-1}T,0)}) \xrightarrow[i\to\infty]{\mathbb{F},\mathfrak{C}} (\mathcal{Y},(\mu_t^\infty)_{t\in(-\infty,0)}),$$

where $(\mathcal{Y},(\mu_t^\infty)_{t\in(-\infty,0)})$ is a static, H_4-concentrated metric soliton. Let $(\mathcal{R},g^\infty,\mathfrak{t},\partial_\mathfrak{t})$ be the spacetime structure on the regular set of $\mathcal{Y}$ as before, so that by Theorem 1.47 of [B5], there is a finite subgroup $\Gamma \subseteq O(4,\mathbb{R})$ and an identification $\mathcal{Y} = C(\mathbb{S}^3/\Gamma) \times (-\infty,0)$ such that the following hold for all $t \in (-\infty,0)$:

(a) $(\mathcal{Y}_t,d_t) = (C(\mathbb{S}^3/\Gamma) \times \{t\},d)$, where d is the cone metric on $C(\mathbb{S}^3/\Gamma)$,

(b) $\mathcal{R} = (C(\mathbb{S}^3/\Gamma) \setminus \{o_*\}) \times (-\infty,0)$, where o_* is the vertex of $C(\mathbb{S}^3/\Gamma)$, and $\partial_\mathfrak{t}$ corresponds to the standard vector field on the second factor,

(c) $(\mathcal{R}_t,g_t^\infty) = ((C(\mathbb{S}^3/\Gamma) \setminus \{o_*\}) \times \{t\},g_Y)$, where $g_Y = dr^2 + r^2 g_{\mathbb{S}^3/\Gamma}$ is the smooth cone metric over $\mathbb{S}^3/\Gamma$,

(d) $Rc = 0$ and $d\mu_t^\infty = K^\infty(\cdot,t)dg_t^\infty = \frac{1}{|\Gamma|}(4\pi|t|)^{-2}\exp\left(-\frac{d^2(o_*,\cdot)}{4|t|}\right)dg_t^\infty$ on $\mathcal{R}$.

For ease of notation, we also write $B(o_*,r_2) := B^{C(\mathbb{S}^3/\Gamma)}(o_*,r_2)$, $A(o_*,r_1,r_2) := B(o_*,r_2) \setminus \overline{B}(o_*,r_1)$ for all $r_2 > r_1 > 0$. There is a sequence $\alpha_i \searrow 0$ such that if $A_i := A(o_*,\alpha_i,\alpha_i^{-1})$, then there are embeddings $\psi_i : A_i \times (-2,0) \to M$ satisfying

$$\|\partial_t - (\psi_i^{-1})_* \partial_t\|_{C^i(U_i \times (-\alpha_i^{-1}, -\alpha_i))} < \alpha_i,$$

$$\|\psi_i^* g^i - g^\infty\|_{C^i(U_i \times (-\alpha_i^{-1}, -\alpha_i))} < \alpha_i,$$

$$\|\psi_i^* K^i - K^\infty\|_{C^i(U_i \times (-\alpha_i^{-1}, -\alpha_i))} < \alpha_i$$

for all $i \in \mathbb{N}$, where the C^i norms are with respect to the metric g_Y. By abuse of notation, we write ψ_i for the restriction $\psi_i|(A_i \times \{-1\}) : A_i \to M$. In the remainder of this section, all geometric quantities (H_4-centers, for example) correspond to the rescaled flows g^i.

Proposition 20. $(M, d_{g^i_{-1}}, x)$ *converge in the pointed Gromov-Hausdorff sense to* $(C(\mathbb{S}^3/\Gamma), d, o_*)$*, with smooth convergence on* $C(\mathbb{S}^3/\Gamma) \setminus \{o_*\}$*.*

Proof. Suppose that $(x_i, -1)$ are H_4-centers of ν^i. Bamler's Gaussian upper bounds for the conjugate heat kernel (Theorem 7.2 of [B4]) then give

$$K^i(y, -1) \le C \exp\left(-\frac{1}{C} d^2_{g^i_{-1}}(x_i, y)\right)$$

for all $y \in M$, where $C = C(A) < \infty$. On the other hand, we know

$$K^\infty(v, -1) = \frac{1}{|\Gamma|(4\pi)^2} \exp\left(-\frac{1}{4} d^2(o_*, v)\right),$$

for all $v \in C(\mathbb{S}^3/\Gamma) \setminus \{o_*\}$, where $|\Gamma| \le C(A)$. Thus $\psi_i^* K^i \to K^\infty$ in $C^\infty_{loc}(A_i)$ implies that (after possibly reindexing)

$$K^i(y, -1) \ge \frac{1}{|\Gamma|(8\pi)^2} \exp\left(-\frac{1}{4} d^2(o_*, \psi_i^{-1}(y))\right)$$

for any $y \in \psi_i(A_i)$, when $i \in \mathbb{N}$ is sufficiently large. Combining estimates, we see that

$$d^2(o_*, \psi_i^{-1}(y)) \ge \frac{1}{C} d^2_{g^i_{-1}}(x_i, y) - C$$

for all $y \in \psi_i(A_i)$ when $i \in \mathbb{N}$ is sufficiently large, where $C = C(A) < \infty$. This gives an upper bound of the form $d_{g^i_{-1}}(x_i, y) \le D$ for all $y \in \psi_i(\partial B(o_*, 1))$, where $D = D(A) < \infty$.

Because M is simply connected, for any $r \in (\alpha_i, \alpha_i^{-1})$, $\psi_i(\partial B(o, r))$ disconnects M into two pieces $M'_{i,r}, M''_{i,r}$ by the generalized Jordan-Brouwer theorem, where $\psi_i(A(o, \alpha_i, r)) \subset M'_{i,r}$ and $\psi_i(A(o, r, \alpha_i^{-1})) \subset M''_{i,r}$.

Claim 1: When $i \in \mathbb{N}$ is sufficiently large, we have $x_i \in M'_{i,4D}$.

Suppose instead that $x_i \in M''_{i,4D}$, and let $\gamma : [0,1] \to M$ be a g^i_{-1}-minimizing geodesic from x_i to some $y \in \psi_i(\partial B(o, 1))$. Because M is simply connected, we know $\psi_i(\partial B(o, 4D))$, $\psi_i(\partial B(o, 1))$ each separate M into two components, so we can find $0 \leq s_1 < s_2 \leq 1$ such that $\gamma(s_1) \in \psi_i(\partial B(o, 1))$, $\gamma(s_2) \in \psi_i(\partial B(o, 4D))$, and $\gamma|[s_1, s_2] \subset \psi_i(A(o, 1, 4D))$. Then, for $i \in \mathbb{N}$ sufficiently large,

$$3D \leq \text{length}_{g_Y}(\psi_i^{-1} \circ \gamma|[s_1, s_2]) \leq 2\text{length}_{g^i_{-1}}(\gamma|[s_1, s_2]) \leq 2d_{g^i_{-1}}(x_i, y) \leq 2D,$$

a contradiction.

For any $y \in \psi_i(A(o, D, \frac{1}{2}\alpha_i^{-1}))$ and $z \in B_{g^i}(y, -1, \frac{D}{4})$, let $\gamma : [0,1] \to M$ be a minimizing g^i_{-1}-geodesic from y to z. If γ leaves $\psi_i(A(o, \alpha_i, \alpha_i^{-1}))$, we can argue as in Claim 1 to obtain $\text{length}_{g^i_{-1}}(\gamma) \geq \frac{1}{2}D$ for $i \in \mathbb{N}$ sufficiently large, a contradiction. Thus, for i large, we have $|Rm|_{g^i}(\cdot, -1) \leq \epsilon_i$ on $B_{g^i}(y, -1, \frac{D}{4})$ for all $y \in \psi_i(A(o, D, \frac{1}{2}\alpha_i^{-1}))$, where $\epsilon_i \searrow 0$. We can thus apply Theorem 31 to the ball $B(y, -1, \frac{D}{4})$ to get (after possibly increasing $D = D(A) < \infty$) $|Rm|_{g^i} \leq C(A)$ on $\psi_i(A(o, D, \frac{1}{2}\alpha_i^{-1})) \times (-2, 0)$ when $i = i(A) \in \mathbb{N}$ is large.

Claim 2: For $i \in \mathbb{N}$ sufficiently large, we have $x \in M'_{i,16D}$.

Suppose instead that $x \in M''_{i,16D}$. Let $\gamma : [0,1] \to M$ be a g^i_t-minimizing curve from x to any $y \in M'_{i,8D}$, where $t \in [-1, 0)$ is arbitrary. Arguing as in Claim 1, we find $0 \leq s_1 < s_2 \leq 1$ such that $\gamma(s_1) \in \psi_i(\partial B(o, 4D))$ and $\gamma(s_2) \in \psi_i(\partial B(o, 8D))$, and $\gamma|[s_1, s_2] \in \psi_i(A(o, 8D, 16D))$. From the curvature bound $|Rm|_{g^i} \leq C(A)$ on $\psi_i(A(o, 8D, 16D)) \times (-1, 0)$, we can estimate

$$d_{g^i_t}(x, y) \geq \text{length}_{g^i_t}(\gamma|[s_1, s_2]) \geq \frac{1}{C(A)}\text{length}_{g^i_{-1}}(\gamma|[s_1, s_2]) \geq cD$$

for all $t \in [-1,0)$, where $c = c(A) > 0$. However, Proposition 25 applied to the original flow $(M,(g_t)_{t\in[0,T)})$ gives the following after rescaling:

$$K^i(y,-1) \le C' \exp\left(-\frac{1}{C'}\lim_{t\searrow 0} d^2_{g^i_t}(y,x)\right)$$

for all $y \in M$, where $C' = C'(A)$. In particular, for $y \in M'_{i,8D}$, we have

$$K^i(y,-1) \le C' \exp\left(-\frac{1}{C'}c^2D^2\right).$$

Now we integrate on $B(x_i,-1,\sqrt{2H_4}) \subseteq M'_{i,8D}$, using the volume upper bound, and the concentration estimate near H_4-centers (Proposition 3.13 of [B4]) to get

$$\frac{1}{2} \le \int_{B(x_i,-1,\sqrt{2H_4})} K^i(y,-1)\,dg^i_{-1}(y) \le C(A)\exp\left(-\frac{c^2D^2}{C}\right),$$

so that $D \le C(A)$, a contradiction after adjusting $D = D(A)$.

For any $\gamma \in (0,1]$, we note that $\tilde{r}^{g^i}_{Rm}(\cdot,-1) \ge c(\gamma)$ on $\psi_i(A(o_\infty,\gamma,\gamma^{-1}))$ $i = i(\gamma) \in \mathbb{N}$ is sufficiently large.

Claim 3: For any $\gamma \in (0,1]$, we have $x \in M'_{i,\gamma}$ when $i = i(\gamma,A) \in \mathbb{N}$ is large.

Suppose not, so that after passing to a subsequence we have $x \in M'_{i,16D} \setminus M''_{i,\gamma}$ for all $i \in \mathbb{N}$. Then we can pass to a subsequence to get $\psi_i^{-1}(x) \to x_\infty \in A(o_\infty,\frac{\gamma}{2},32D)$. Because $\mathcal{Y}$ is static, we can argue as in the proof of Theorem 29 to get $\psi_{i,t}^{-1}(x) \to x_\infty(t)$ as $i \to \infty$, uniformly in $t \in [-1,-\beta]$ for $\beta \in (0,1)$ fixed. In particular, we can find $c(\gamma) > 0$ such that $\tilde{r}^{g^i}_{Rm}(x,-\beta) \ge c(\gamma)$, so by taking $\beta = \beta(\gamma,A)$ sufficiently small, applying Theorem 31 gives a contradiction.

Claim 4: For any $\delta > 0$, there exists $\gamma = \gamma(A,\delta) > 0$ such that

$$\limsup_{i\to\infty} \mathrm{diam}_{g^i_{-1}}(M'_{i,\gamma}) < \delta.$$

Set $\mathcal{B}_i := M'_{i,\gamma}$, so that $\mathrm{Area}_{g^i_{-1}}(\partial \mathcal{B}_i) \le 2\mathrm{Area}_{g^i_{-1}}(\partial B(o_\infty,\gamma)) \le C(n)\gamma$ as $i \to \infty$. Thus we can apply Croke's isoperimetric inequality (Theorem 13 of [C3]) to the original (unrescaled)

Riemannian manifold $(M, g_{T-\tau_i})$ and then rescale to obtain

$$\mathrm{Area}_{g^i_{-1}}(\partial B_i) \geq c(A)\left(\min\{|B_i|_{g^i_{-1}}, |M \setminus B_i|_{g^i_{-1}}\}\right)^{\frac{3}{4}}.$$

On the other hand, for $i \in \mathbb{N}$ large, we know

$$|M \setminus B_i|_{g^i_{-1}} \geq |\psi_{i,-1}(A(o_*, 1, 2))|_{g^i_{-1}} \geq c(\Gamma),$$

hence $|B_i|_{g^i_{-1}} \leq C\gamma^{\frac{4}{3}}$. For any $r > 2C^{\frac{1}{4}}A^{\frac{1}{4}}\gamma^{\frac{1}{3}}$, when $i = i(r) \in \mathbb{N}$ is sufficiently large, we have $B_{g^i}(y, -1, r) \cap \partial B_i \neq \emptyset$ for all $y \in B_i$, since otherwise $B_{g^i}(y, -1, r) \subseteq B_i$, in which case $A^{-1}r^4 \leq |B_i|_{g^i_{-1}} < C\gamma^{\frac{4}{3}}$,a contradiction. Thus

$$\mathrm{diam}_{g^i_{-1}}(B_i) \leq C(A)\gamma^{\frac{1}{3}} + \mathrm{diam}_{g^i_{-1}}(\partial B_i) \leq 2C(A)\gamma^{\frac{1}{3}}$$

for sufficiently large $i \in \mathbb{N}$.

Claims 1-4 imply that $\sup_{i\in\mathbb{N}} d_{g^i_{-1}}(x, x_i) < \infty$ for each $t \in [-1, 0)$. Thus, given any subsequence of $(M, d_{g^i_{-1}}, x)$, we can pass to a further subsequence so that

$$\lim_{i\to\infty} d^Z_{-1}(\phi^i_{-1}(x), \phi^\infty_{-1}(x_\infty)) = 0,$$

for some $x_\infty \in C(\mathbb{S}^3/\Gamma)$, hence

$$(M, d_{g^i_{-1}}, x) \to (C(\mathbb{S}^3/\Gamma), d_{C(\mathbb{S}^3/\Gamma)}, x_\infty)$$

in the pointed Gromov-Hausdorff sense. Claims 3,4 imply that for any $y \in \partial B(o_*, 2\gamma)$,

$$\begin{aligned}
d(x_\infty, y) =& d^Z_{-1}(\phi^\infty_{-1}(x_\infty), \phi^\infty_{-1}(y)) \\
\leq& d^Z_{-1}\left(\phi^i_{-1}(x), \phi^i_{-1}(\psi_i(y))\right) + d^Z_{-1}\left(\phi^i_{-1}(\psi_i(y)), \phi^\infty_{-1}(y)\right) + d^Z_{-1}\left(\phi^\infty_{-1}(x_\infty), \phi^i_{-1}(x)\right) \\
\leq& \mathrm{diam}_{g^i_{-1}}(M^0_{i,2\gamma}) + \sup_{z\in U_i \cap A(o_*,\gamma,4\gamma)} d^Z_{-1}\left(\phi^i_{-1}(\psi_i(z)), \phi^\infty_{-1}(z)\right) + d^Z_{-1}\left(\phi^\infty_{-1}(x_\infty), \phi^i_{-1}(x)\right).
\end{aligned}$$

Taking $i \to \infty$ and appealing to Theorem 9.31(d) of [B3], we have

$$d(x_\infty, y) \leq \liminf_{i\to\infty} \mathrm{diam}_{g^i_{-1}}(M^0_{i,2\gamma}) \leq C(A)\gamma^{\frac{1}{3}}.$$

Thus $d(o_*, x_\infty) \leq 2C(A)\gamma^{\frac{1}{3}}$, but $\gamma > 0$ was arbitrary, so $x_\infty = o_*$. □

Lemma 20. *For any $x \in M$ corresponding to a singular point $\overline{x} \in S$, there exists a finite subgroup $\Gamma \subseteq O(4,\mathbb{R})$ only depending on x such that*

$$\lim_{t \nearrow T} d_{PGH}\Big((M,(T-t)^{-\frac{1}{2}} d_{g_t}, x), (C(\mathbb{S}^3/\Gamma), d_{C(\mathbb{S}^3/\Gamma)}, o_*)\Big) = 0,$$

where o_* is the vertex of $C(\mathbb{S}^3/\Gamma)$.

Proof. We first show that, for any $\epsilon > 0$, there exists $\delta > 0$ such that for any $t \in (T-\delta, T)$, there exists a finite subgroup Γ_t (possibly depending on t) such that

$$d_{PGH}\Big((M,(T-t)^{-\frac{1}{2}} d_{g_t}, x), (C(\mathbb{S}^3/\Gamma_t), d_{C(\mathbb{S}^3/\Gamma_t)}, o_*)\Big) < \epsilon,$$

and where $C(\mathbb{S}^3/\Gamma_t)$ models a tangent flow of $(M,(g_t)_{t\in[0,T)})$ at (x,T). Suppose this does not hold, so that there are $t_i \nearrow T$ such that

$$d_{PGH}\Big((M,(T-t_i)^{-\frac{1}{2}} d_{g_{t_i}}, x), (C(\mathbb{S}^3/\Gamma), d_{C(\mathbb{S}^3/\Gamma)}, o_*)\Big) \geq \epsilon$$

for all $i \in \mathbb{N}$ and all finite subgroups $\Gamma \subseteq O(4,\mathbb{R})$. Let $K \in C^\infty(M \times (0,T))$ be a conjugate heat kernel at x based at the singular time T, and let $K^i \in C^\infty(M \times (0,T))$ be the conjugate heat kernels corresponding to the rescaled solutions $g_t^i := (T-t_i)^{-1} g_{T+(T-t_i)t}$. Write $\nu_t^i := K^i(\cdot,t)dg_t$. After passing to a subsequence, we have $\mathbb{F}$-convergence

$$(M,(g_t^i)_{t\in(-2,0)},(\nu_t^i)_{t\in(-2,0)}) \xrightarrow[i\to\infty]{\mathbb{F},\mathfrak{C}} (\mathcal{X},(\mu_t^\infty)_{t\in(-2,0)})$$

where $\mathcal{X}$ is the metric flow corresponding to the static flow on $C(\mathbb{S}^3/\Gamma)$ for some finite subgroup $\Gamma \subseteq O(4,\mathbb{R})$. From the previous section, we know this implies

$$(M, d_{g^i_{-1}}, x) \to (C(\mathbb{S}^3/\Gamma), d_{C(\mathbb{S}^3/\Gamma)}, o_*)$$

in the pointed Gromov-Hausdorff sense, a contradiction.

Now, recall that if $C(\mathbb{S}^3/\Gamma)$ models a tangent flow at (x,T), and if $\mathcal{N}(\tau)$ is the pointed Nash entropy corresponding to K, then

$$\log\left(\frac{1}{|\Gamma|}\right) = \lim_{\tau\searrow 0} \mathcal{N}(\tau),$$

so if we take $N(x)$ to be any integer greater than $\exp(-\lim_{\tau\searrow 0} \mathcal{N}(\tau))$, then we obtain

$$\lim_{t\nearrow T} \inf_{|\Gamma|\le N} d_{PGH}\Big((M,(T-t)^{-\frac{1}{2}}d_{g_t},x),(C(\mathsf{S}^3/\Gamma),d_{C(\mathsf{S}^3/\Gamma)},o_\Gamma)\Big)=0,$$

where the infimum is taken over all finite subgroups $\Gamma\subseteq O(4,\mathsf{R})$ with $|\Gamma|\le N$. Finally, we note that there are only finitely many finite subgroups of $O(4,\mathsf{R})$ with $|\Gamma|\le N$ up to conjugation (in fact, embeddings of a finite group $G\hookrightarrow O(4,\mathsf{R})$ correspond to real representations $G\curvearrowright \mathsf{R}^4$, of which there are finitely many up to isomorphism, and if two representations of G are isomorphic, then the corresponding subgroups of $O(4,\mathsf{R})$ are conjugate). Also, $C(\mathsf{S}^3/\Gamma)$, $C(\mathsf{S}^3/\Gamma')$ are isometric whenever Γ,Γ' are conjugate in $O(4,\mathsf{R})$, so there is a fixed distance $d>0$ such that any non-isometric $(C(\mathsf{S}^3/\Gamma),d_{C(\mathsf{S}^3/\Gamma)},o_\Gamma),(C(\mathsf{S}^3/\Gamma'),d_{C(\mathsf{S}^3/\Gamma')},o'_\Gamma)$ with $|\Gamma|,|\Gamma'|\le N$ satisfy

$$d_{PGH}\big((B(o_\Gamma,1),d_{C(\mathsf{S}^3/\Gamma)},o_\Gamma),(B(o'_\Gamma,1),d_{C(\mathsf{S}^3/\Gamma')},o'_\Gamma)\big)\ge d.$$

However, $t\mapsto (B_g(x,t,1),(T-t)^{-\frac{1}{2}}d_{g_t},x)$ is continuous in the pointed Gromov-Hausdorff topology, so the conjugacy class of Γ_t, hence the isometry class of $C(\mathsf{S}^3/\Gamma_t)$, must be constant for $t\in(T-\delta,T)$ for sufficiently small $\delta>0$. □

Remark 16. *A similar idea to the above argument for showing that Γ does not change along the subsequence was used in Proposition 2.2 of* [BCD⁺].

Proof of Theorem 26. Part (ii) is exactly the statement of Lemma 20. For part (i), Proposition 19 tells us that any tangent flow is a static flow corresponding to $C(\mathsf{S}^3/\Gamma_x)$ for some finite subgroup $\Gamma_x\subseteq O(4,\mathsf{R})$. However, Proposition 20 implies that the time -1 time-slices of the rescaled flows also Gromov-Hausdorff converge to $C(\mathsf{S}^3/\Gamma_x)$, so S^3/Γ and S^3/Γ_x are isometric, hence we can assume $\Gamma=\Gamma_x$. □

Proof of Theorem 25. Let $x\in M$ correspond to a singular point $\overline{x}\in \mathcal{S}$. By Lemma 20, there exists $\delta=\delta(x)>0$ and a subgroup $\Gamma\subseteq O(4,\mathsf{R})$ such that

$$\lim_{t\nearrow T} d_{PGH}\Big((M,(T-t)^{-\frac{1}{2}}d_{g_t},x),(C(\mathsf{S}^3/\Gamma),d_{C(\mathsf{S}^3/\Gamma)},o_\Gamma)\Big),$$

where o_* is the vertex of $C(\mathbb{S}^3/\Gamma)$.

Claim 1: For any $\beta \in (0,1)$, there exists $\delta = \delta(x,\beta) > 0$ such that $\widetilde{r}^g_{Rm}(y,t) \geq \frac{1}{6} d_{g_t}(y,x)$ for all $t \in (T-\delta, T)$ and $y \in B_g(x,t,\beta^{-1}\sqrt{T-t}) \setminus \overline{B}_g(x,t,\beta\sqrt{T-t})$.

Suppose by way of contradiction that $t_i \nearrow T$ and

$$y_i \in B_g(x,t_i,\beta^{-1}\sqrt{T-t_i}) \setminus \overline{B}_g(x,t_i,\beta\sqrt{T-t_i})$$

satisfy

$$\frac{\widetilde{r}^g_{Rm}(y_i,t_i)}{d_{g_{t_i}}(y_i,x)} \leq \frac{1}{6}.$$

and let $\phi_i : (B(o_*,\alpha_i^{-1}), d, o_*) \to (B_g(x,t_i,\alpha_i^{-1}(T-t_i)^{\frac{1}{2}}), (T-t_i)^{-\frac{1}{2}} d_{g_{t_i}}, x)$ be α_i-Gromov-Hausdorff maps for some sequence $\alpha_i \searrow 0$. By hypothesis, we have $(T-t_i)^{-\frac{1}{2}} d_{g_{t_i}}(x,y_i) \in (\beta, \beta^{-1})$, so we can pass to a subsequence so that

$$D := \lim_{i\to\infty} (T-t_i)^{-\frac{1}{2}} d_{g_{t_i}}(x,y_i) \in [\beta,\beta^{-1}]$$

exists. Also choose $y_i' \in C(\mathbb{S}^3/\Gamma)$ with $(T-t_i)^{-\frac{1}{2}} d_{g_{t_i}}(\phi_i(y_i'), y_i) \to 0$, so that $d(y_i', o_*) \to D$ as $i \to \infty$. Then $B(y_i', \frac{D}{2}) \subseteq C(\mathbb{S}^3/\Gamma) \setminus \{o_*\}$, and

$$B_g(y_i,t_i,\frac{D}{3}\sqrt{T-t_i}) \subseteq \psi_i\left(B(y_i', D/2)\right)$$

for large $i \in \mathbb{N}$, so because $C(\mathbb{S}^3/\Gamma) \setminus \{o_*\}$ is flat and (by Theorem 33) we have locally uniform smooth convergence on $C(\mathbb{S}^3/\Gamma) \setminus \{o_*\}$,

$$\lim_{i\to\infty} \sup_{B_g(y_i,t_i,D/3)} |Rm|_g(\cdot,t_i)(T-t_i) = 0.$$

This means $\widetilde{r}^g_{Rm}(y_i,t_i) \geq \frac{D}{4}(T-t_i)^{\frac{1}{2}}$ for sufficiently large, hence

$$\frac{1}{6} \geq \frac{\widetilde{r}^g_{Rm}(y_i,t_i)}{d_{g_{t_i}}(y_i,x)} \geq \frac{\widetilde{r}^g_{Rm}(y_i,t_i)}{\sqrt{T-t_i}} \frac{\sqrt{T-t_i}}{d_{g_{t_i}}(y_i,x)} \geq \frac{D}{4}\frac{4}{5D},$$

for large $i \in \mathbb{N}$, a contradiction.

Now set $r := r(A) := 8\max\{\epsilon_P^{-1}, e^4\} < \infty$, where $\epsilon_P(A) > 0$ is as in Theorem 31. Also fix $E \geq 2$, so that for $t \in (T-\delta, T)$ and $y \in B_g(x,t,Er\sqrt{T-t}) \setminus \overline{B}_g(x,t,\frac{1}{2}r\sqrt{T-t})$, we have

$$|Rm|_g(y,s) \leq \frac{1}{T-t}.$$

for all $s \in (t,T)$. Here we have fixed $\delta = \delta(x, r^{-1}E^{-1}) > 0$ as in Claim 1. In particular, $y \in M \setminus \Sigma$, and $(g_t)_{t\in[0,T)}$ extends smoothly to the Riemannian metric g_T on

$$B_g(x,t,Er\sqrt{T-t}) \setminus \overline{B}_g(x,t,\frac{1}{2}r\sqrt{T-t}) \subseteq M \setminus \Sigma$$

whenever $t \in (T-\delta, T)$. Moreover, we have

$$|Rm|_{g_T} \leq \frac{1}{T-t}$$

on this subset. Now suppose $y \in \partial B_g(x,t,r\sqrt{T-t})$, where $t \in (T-\delta,T)$, and fix $z \in B_{g_T}(y,\sqrt{T-t})$. Let $\gamma : [0,1] \to M \setminus \Sigma$ be a curve from y to z with $\text{length}_{g_T}(\gamma) \leq 2\sqrt{T-t}$. If $\gamma([0,1]) \subseteq B_g(y,t,\frac{r}{2}\sqrt{T-t})$, then $|Rm|_g(\gamma(u),s) \leq \frac{1}{T-t}$ for all $u \in [0,1]$ and $s \in [t,T]$, so standard distortion estimates give

$$d_{g_t}(y,z) \leq \text{length}_{g_t}(\gamma) \leq e^4\text{length}_{g_T}(\gamma) \leq 2e^4\sqrt{T-t}.$$

If $\gamma([0,1]) \not\subseteq B_g(y,t,\frac{r}{2}\sqrt{T-t})$, then set

$$u^* := \inf\{u \in [0,1]; d_{g_t}(y,\gamma(u)) = \frac{r}{2}\sqrt{T-t}\},$$

and again apply distortion estimates, this time concluding

$$\frac{r}{2}\sqrt{T-t} = d_{g_t}(y,\gamma(u)) \leq e^4\text{length}_{g_T}(\gamma|[0,u^*]) \leq 2e^4\sqrt{T-t},$$

contradicting our choice of r. We thus conclude that $\widetilde{r}_{Rm}^{(M\setminus\Sigma, g_T)}(y) \geq \sqrt{T-t}$.

Now let $y \in B_g(x,T-\delta/2, r\sqrt{\delta/2})$ satisfy $\overline{y} \neq \overline{x}$, and define

$$D(t) := \frac{d_{g_t}(x,y)}{\sqrt{T-t}}.$$

Then $\lim_{t\nearrow T} D(t) = \infty$ and $D(T-\delta/2) < r$, so there exists $t = t(y) \in (T-\delta/2, T)$ such that $D(t) = r$. Because $2d_{g_t}(x,y) \geq d_X(\overline{x},\overline{y})$ for all $t \in (T-\delta, T)$,

$$r^X_{Rm}(\overline{y}) = r^{(M\setminus\ ,g_T)}_{Rm}(y) \geq \sqrt{T-t(y)} = r^{-1} d_{g_{t(y)}}(x,y) \geq c(A) d_X(\overline{x},\overline{y}).$$

Claim 2: If $y \in M \setminus B_g(x,t,r\sqrt{T-t})$ for some $t \in (T-\delta/2, T)$, then $\overline{y} \neq \overline{x}$.

Fix $\epsilon > 0$, and let $\gamma : [0,1] \to M$ be any curve from x to y. Set

$$u_- := \sup\left\{u \in [0,1]; d_{g_t}(\gamma(u),x) = \frac{r}{2}\sqrt{T-t}\right\},$$

$$u_+ := \inf\left\{u \in [0,1]; d_{g_t}(\gamma(u),x) = r\sqrt{T-t}\right\},$$

so that $\text{length}_{g_t}(\gamma|[u_-,u_+]) \geq \frac{r}{2}\sqrt{T-t}$, and $|Rm|(\gamma(u),s) \leq \frac{1}{T-t}$ for all $u \in [u_-,u_+]$ and $s \in [t,T)$. In particular, we can estimate

$$\text{length}_{g_s}(\gamma) \geq \text{length}_{g_s}(\gamma|[s_-,s_+]) \geq \frac{r}{2}e^{-4}\sqrt{T-t},$$

so taking the infimum over all curves γ gives $d_{g_s}(x,y) \geq c(x,A)$, and taking $s \nearrow T$ gives $d_X(\overline{x},\overline{y}) \geq c(x,t,A) > 0$.

By Claim 2, $B_g(x,T-\delta/2,\sqrt{\delta/2}r)$ is a saturated open set with respect to the equivalence relation $\sim$, so if $\pi : M \to X$ is the quotient map, then

$$U := \pi\left(B_g(x,T-\delta/2,\sqrt{\delta/2}r)\right)$$

is a neighborhood of $\overline{x}$ in X such that $U \cap \mathcal{S}(X) = \{\overline{x}\}$. Moreover, we have $\widetilde{r}^X_{Rm}(\overline{y}) \geq c(A)d(\overline{x},\overline{y})$ for all $\overline{y} \in U \setminus \{\overline{x}\}$, so by applying Theorem 31 and then Shi's local derivative estimates on

$$B_{g_T}(y, c(A)\epsilon_P d(\overline{x},\overline{y})) \times [T-(c(A)\epsilon_P d(\overline{x},\overline{y}))^2, T],$$

we obtain $C_k = C_k(A) < \infty$ for each $k \in \mathbb{N}$ such that

$$|\nabla^k Rm|_g(\overline{y}) \leq C_k d^{-2-k}(\overline{x}, \overline{y})$$

for all $\overline{y} \in U \setminus \{\overline{x}\}$. Thus, for any tangent cone $(C(Y), c_*)$ of (X, d) at $\overline{x}$, convergence is smooth away from the vertex c_*, and in particular $C(Y)$ must have smooth link, so $C(Y)$ cannot split any factor of $\mathbb{R}$ (otherwise the vertex would produce a line of nonsmooth points, a contradiction). In particular, $\mathcal{S}(X) = \mathcal{S}^0(X)$. Moreover, we showed that $\mathcal{S}(X)$ is discrete, so by the compactness of X, $\mathcal{S}(X)$ must be finite.

Claim 3: $C(Y) \setminus \{c_*\}$ is flat.

Fix $x_0 = [r, y_0] \in (C(Y) \setminus \{c_*\})$ arbitrary (see Section 2 for notation), and set $v := \partial_r|_{x_0}$ (where ∂_r is the radial vector field corresponding to c_*) which is a zero eigenvector for $Rc(g_{C(Y)})|_{x_0}$. By a standard diagonal argument, $C(Y)$ is itself a Gromov-Hausdorff limit of appropriate dilations of $(M, d_{g_{t_i}}, x_i)$ for some $(x_i, t_i) \in M \times [0, T)$, so by Theorems 31, 33, Shi's estimates, and the Hamilton-Cheeger-Gromov compactness theorem, there is a neighborhood U_0 of x_0 in $C(Y) \setminus \{c_*\}$, and some $\delta_0 > 0$ such that we can extract a Ricci flow $(U_0, (\overline{g}_t)_{t \in (-\delta_0, 0]})$ satisfying $\overline{g}_0 = g_{C(Y)}$ on U_0 as well as $Rc(\overline{g}_t) \geq 0$ for all $t \in (-\delta, 0]$. By possibly shrinking U_0, we can extend v to a unit-length vector field V on $(U_0, \overline{g}_0)$ by parallel translation along radial geodesics emanating from x_0, and then extend to a vector field on $U_0 \times (-\delta_0, 0]$ by letting V be constant in time. Then $\phi := Rc_{\overline{g}_0}(V, V) \in C^\infty(U_0 \times (-\delta_0, 0])$ satisfies $\phi \geq 0$ on $U \times (-\delta, 0]$, with $\phi(x_0, t_0) = 0$. We observe that $Rc_{\overline{g}_0}(\partial_r) = 0$ and $Rm_{\overline{g}_0}(\partial_r, \cdot, \cdot, \cdot) = 0$, hence at $(x_0, 0)$, we have (where $(\partial_r, e_1, e_2, e_3)$ is an orthonormal basis of $T_{x_0}(C(Y) \setminus \{c_*\})$)

$$\begin{aligned}(\partial_t - \Delta)\phi(x_0, 0) =& (\partial_t Rc_{\overline{g}})(v, v) - (\Delta Rc_{\overline{g}})(v, v) \\ =& 2\sum_{j,k=1}^{3} Rm_{\overline{g}_0}(e_j, \partial_r, \partial_r, e_k) Rc_{\overline{g}_0}(e_j, e_k) - 2\sum_{j=1}^{3} Rc_{\overline{g}_0}(e_j, \partial_r) Rc_{\overline{g}_0}(e_j, \partial_r) = 0.\end{aligned}$$

Because $\partial_t \phi(x_0, 0) \leq 0$ and $\Delta\phi(x_0, 0) \geq 0$, we have (abbreviating $Rc = Rc(g_{C(Y)})$)

$$0 = \Delta\phi(x_0, 0) = (\Delta Rc)_{x_0}(\partial_r, \partial_r).$$

Since $x_0 \in C(Y) \setminus \{c\}$ was arbitrary, we obtain $(\Delta Rc)(\partial_r, \partial_r) = 0$ everywhere. Now once again fix $x_0 \in C(Y) \setminus \{c\}$, and choose an orthonormal frame $(E_i)_{i=0}^3$ for $g_{C(Y)}$ in a neighborhood U_0 of x_0 with $E_0 = \partial_r$. For any $W \in \mathfrak{X}(U_0)$, we compute

$$(\nabla_W Rc)(\partial_r, \partial_r) = -2Rc(\nabla_W \partial_r, \partial_r) = 0$$

on U_0 since $Rc(\partial_r) = 0$. Because $\nabla_{E_i}\partial_r = \frac{1}{r}E_i$ for $i = 1, 2, 3$, we have (at x_0)

$$\begin{aligned}
0 &= (\nabla_{\partial_r}(\nabla_{\partial_r} Rc))(\partial_r, \partial_r) + \sum_{i=1}^{3} (\nabla_{E_i}(\nabla_{E_i} Rc))(\partial_r, \partial_r) - \sum_{i=1}^{3} (\nabla_{\nabla_{E_i} E_i} Rc)(\partial_r, \partial_r) \\
&= -2\sum_{i=1}^{3} (\nabla_{E_i} Rc)(\nabla_{E_i}\partial_r, \partial_r) = 2\sum_{i=1}^{3} Rc(\nabla_{E_i}\partial_r, \nabla_{E_i}\partial_r) \\
&= \frac{2}{r^2}\sum_{i=1}^{3} Rc(E_i, E_i).
\end{aligned}$$

Because $Rc \geq 0$ on $C(Y) \setminus \{c\}$, we conclude that $C(Y) \setminus \{c\}$ is Ricci-flat, hence $Rc(g_Y) = 2g_Y$. Because $\dim(Y) = 3$, this is only possible if (Y, g_Y) has constant sectional curvature 1, so that $C(Y) \setminus \{c\}$ is flat.

Because the Gromov-Hausdorff convergence of dilations of $(X, d_X, \overline{x})$ to $(C(Y), d_{C(Y)}, c)$ is smooth away from the vertex, and because $C(Y)$ was an arbitrary tangent cone at $\overline{x}$, we conclude that

$$\lim_{\overline{y} \to \overline{x}} d^{2+k}(\overline{x}, \overline{y})|\nabla^k Rm|(\overline{y}) = 0.$$

Now one can show (see, for example, Chapter 3) that there exists a finite subgroup $\Gamma' \subset O(4, \mathbb{R})$ such that any tangent cone of (X, d) at $\overline{x}$ is $C(\mathbb{S}^3/\Gamma')$. Gluing together Cheeger-Gromov diffeomorphisms on dyadic annuli as in the proof of Proposition 3.2 of [T], we get that X has the structure of a C^0 orbifold (which is C^∞ away from the singular points) with finitely many conical orbifold singularities. In fact, since we know that every tangent cone at $\overline{x}$ is $C(\mathbb{S}^3/\Gamma')$, with smooth convergence away from the vertex, we can appeal to Step 1 of Theorem 5.7 in [DS] verbatim. It remains only to establish the following claim.

Claim 4: $C(\Gamma) = C(\Gamma')$.

In fact, for any fixed $\beta \in (0,1)$, we have (writing $t = T - \beta\tau$)

$$
\begin{aligned}
&d_{PGH}\Big((B_g(x,T-\beta\tau,\tau^{\frac{1}{2}}),\tau^{-\frac{1}{2}}d_{g_{T-\beta\tau}},x),(B(o_\Gamma,1),d_{C(S^3/\Gamma)},o_\Gamma)\Big)\\
&\quad = d_{PGH}\Big((B_g(x,T-\beta\tau,\beta^{-\frac{1}{2}}(\beta\tau)^{\frac{1}{2}}),\beta^{\frac{1}{2}}(\beta\tau)^{-\frac{1}{2}}d_{g_{T-\beta\tau}},x),(B(o_\Gamma,\beta^{-\frac{1}{2}}),\beta^{\frac{1}{2}}d_{C(S^3/\Gamma)},o_\Gamma)\Big)\\
&\quad = \beta^{\frac{1}{2}}d_{PGH}\Big((B_g(x,t,\beta^{-\frac{1}{2}}\sqrt{T-t}),(T-t)^{-\frac{1}{2}}d_{g_t},x),(B(o_\Gamma,\beta^{-\frac{1}{2}}),d_{C(S^3/\Gamma)},o_\Gamma)\Big) \to 0
\end{aligned}
$$

as $\tau \searrow 0$ (or equivalently as $t \nearrow T$). By a diagonal argument, we can therefore find sequences $\beta_j \searrow 0$ and $\tau_j \searrow 0$ such that

$$
\lim_{j\to\infty} d_{PGH}\Big((B_g(x,T-\beta_j\tau_j,\tau_j^{\frac{1}{2}}),\tau_j^{-\frac{1}{2}}d_{g_{T-\beta_j\tau_j}},x),(B(o_\Gamma,1),d_{C(S^3/\Gamma)},o_\Gamma)\Big)=0,
$$

$$
\lim_{j\to\infty} d_{PGH}\Big((B^X(\overline{x},\tau_j^{\frac{1}{2}}),\tau_j^{-\frac{1}{2}}d,\overline{x}),(B(o',1),d_{C(S^3/\Gamma')},o')\Big)=0,
$$

so it suffices to prove that if $g^j := \tau_j^{-1}g_{T-\beta_j\tau_j}$ and $d_j := \tau_j^{-\frac{1}{2}}d$, then

$$
\lim_{j\to\infty} d_{PGH}\big((B_{g^j}(x,1),d_{g^j},x),(B^{(X,d_j)}(\overline{x},1),d_j,\overline{x})\big)=0.
$$

Recall that $d_j|(\mathcal{R}_X\times\mathcal{R}_X)$ is equal to the length metric of $(M\setminus\Sigma,\tau_j^{-1}g_T)$, and $\pi : M \to X = M/\Gamma$ is given by $\pi(x)=\overline{x}$. For all $y,z\in B_{g^j}(x,1)$, we have

$$
\begin{aligned}
d_j(\pi(y),\pi(z)) - d_{g^j}(y,z) &\leq \tau_j^{-\frac{1}{2}}(e^{A\beta_j\tau_j}-1)d_{g_{T-\beta_j\tau_j}}(y,z)\\
&\leq 2\tau_j^{-1}A\beta_j d_{g^j}(y,z) \leq \beta_j.
\end{aligned}
$$

Because $(B_{g^j}(x,2),d_{g^j},x) \to (B(o_\Gamma,2),d_{C(S^3/\Gamma)},o_\Gamma)$ in the pointed Gromov-Hausdorff sense, with smooth convergence away from the vertex (by Theorem 33), we have the following: for any $\epsilon>0$, there exists $\sigma=\sigma(\epsilon)>0$ such that for $j=j(\epsilon)\in\mathbb{N}$ sufficiently large, we have $\widetilde{r}^{g^j}_{Rm}(y)>\sigma$ for all $y\in B_{g^j}(x,2)\setminus B_{g^j}(x,\epsilon)$. Now fix $y_1,y_2\in B_{g^j}(x,2)\setminus B_{g^j}(x,\epsilon)$. For $j=j(\sigma)\in\mathbb{N}$ sufficiently large, we can apply Theorem 31 on $B_{g^j}(y_k,\sigma)$ for each $u\in[0,1]$ to get

$$
\sup_{B_g(y_k,T-\beta_j\tau_j,\epsilon_P\sigma\tau_j^{\frac{1}{2}})\times[T-\beta_j\tau_j,T)} |Rm| \leq \tau_j^{-1}(\epsilon_P\sigma)^{-2}
$$

for $k = 1, 2$ and all sufficiently large $j = j(\sigma) \in \mathbb{N}$. By an easy distortion estimate,

$$\sup_{s\in[T-\beta_j\tau_j,T)} \sup_{B_g(y_k,s,e^{-4}\epsilon_P\sigma\tau_j^{\frac{1}{2}})} |Rm| \le \tau_j^{-1}(\epsilon_P\sigma)^{-2}$$

for $k = 1, 2$ and large $j = j(\sigma) \in \mathbb{N}$. We can thus apply the standard lower bound for distance distortion (Theorem 18.7 in [CCG$^+$3]) with $r_0 = e^{-4}\epsilon_P\sigma\tau_j^{\frac{1}{2}}$ and $K = \tau_j^{-1}(\epsilon_P\sigma)^{-2}$ to get

$$\partial_s d_{g_s}(y_1, y_2) \ge -6\left(Kr_0 + \frac{1}{r_0}\right) \ge -c(\sigma, A)\tau_j^{-\frac{1}{2}} \qquad (4.4.1)$$

for all $s \in [T - \beta_j\tau_j, T)$. Integrating from $s = T - \beta_j\tau_j$ to T gives

$$d_{g_T}(y_1, y_2) \ge d_{T-\beta_j\tau_j}(y_1, y_2) - c(\sigma, A)\beta_j\tau_j^{\frac{1}{2}},$$

or equivalently

$$d_j(\pi(y_1), \pi(y_2)) \ge d_{g^j}(y_1, y_2) - c(\sigma, A)\beta_j.$$

Now fix $\epsilon > 0$, and suppose $\overline{y} \in B^{(X,d_j)}(\overline{x}, 1 - 3\epsilon)$. Since $d_j(\overline{x}, \overline{y}) = \lim_{t\nearrow T} \tau_j^{-\frac{1}{2}} d_{g_t}(x, y)$, for any $j \in \mathbb{N}$, we can find $t_j \in (T - \beta_j\tau_j, T)$ satisfying $\tau_j^{-\frac{1}{2}} d_{g_{t_j}}(x, y) < 1 - 3\epsilon$. Let let $\gamma : [0, 1] \to M$ be any g_{t_j}-minimizing geodesic from x to y. Suppose by way of contradiction that $\text{length}_{g^j}(\gamma) \ge 1$, and set

$$u_- := \sup\left\{u \in [0, 1]; d_{g^j}(\gamma(u), x) = \epsilon\right\},$$

$$u_+ := \inf\left\{u \in [0, 1]; d_{g^j}(\gamma(u), x) = 1 - \epsilon\right\}.$$

Also let $\sigma = \sigma(\epsilon) > 0$ be as above, so that integrating (4.4.1) again gives

$$\tau_j^{-\frac{1}{2}} d_{g_{t_j}}(x, y) \ge \tau_j^{-\frac{1}{2}} d_{g_{t_j}}(\gamma(u_1), \gamma(u_2)) \ge d_{g^j}(\gamma(u_1), \gamma(u_2)) - c(\sigma, A)\beta_j$$
$$\ge 1 - 2\epsilon - c(\sigma, A)\beta_j,$$

a contradiction for sufficiently large $j = j(\epsilon, A) \in \mathbb{N}$ (independent of y and γ). In particular, $y \in B_{g^j}(x, 1)$ for $j = j(\epsilon, A) \in \mathbb{N}$ sufficiently large, so that

$$\pi(B_{g^j}(x, 1)) \supseteq B^{(X,d_j)}(\overline{x}, 1 - 3\epsilon).$$

For any fixed $\epsilon > 0$, π is thus a 3ϵ-Gromov-Hausdorff map

$$B_{g^j}(x,1) \to B^{(X,d_j)}(\overline{x},1)$$

for $j = j(\epsilon) \in \mathbb{N}$ sufficiently large. Since $\epsilon > 0$ was arbitrary, the claim follows. □

Remark 17. *With small modifications, the proof of Claim 4 can be used to prove Theorem 1.1 using Lemma 20 (without Claims 1-3). However, Claims 1-3 or their proofs will be referenced in later sections.*

4.5 Codimension Two ϵ-Regularity

Proposition 21. *(Codimension two ϵ-Regularity) For any $A < \infty$ and $\underline{T} \geq 0$, there exists $\epsilon_0 = \epsilon_0(A, \underline{T}) > 0$ such that the following holds. Suppose $(M^n, (g_t)_{t\in[0,T)})$ is a closed Ricci flow satisfying $Rc(g_t) \geq -Ag_t$ and $|B(x,t,r)|_{g_t} \geq A^{-1}r^n$, for all $(x,t) \in M \times [0,T)$ and $r \in (0,1]$, where $T \geq \underline{T}$. Then for any $(x,t) \in M \times [\frac{T}{2}, T)$ and $r \in (0, \epsilon_0\sqrt{T-t}]$, if*

$$d_{PGH}\left(\left(B(x,t,\epsilon_0^{-1}r), d_{g_t}, x\right), \left(B\left((0^{n-2},z),\epsilon_0^{-1}r\right), d_{\mathbb{R}^{n-2}\times Z}, (0^{n-2},z)\right)\right) < \epsilon_0 r$$

for some pointed metric space (Z, d_Z, z), then $\widetilde{r}_{Rm}(x,t) \geq \epsilon_0 r$.

Remark 18. *The first part of the proof is a modification of Theorem 5.2 in* [CN3] *and Lemma 13.5 in* [B2].

Proof. Suppose the claim is false. Then we can find closed Ricci flows $(M_i^n, (g_t^i)_{t\in[0,T_i)})$ satisfying $Rc(g_t^i) \geq -Ag_t^i$, $|B_{g^i}(x,t,r)|_{g_t^i} \geq A^{-1}r^n$ for all $(x,t) \in M_i \times [0,T_i)$ and $r \in (0,1]$, where $T_i \geq \underline{T}$, along with points $(x_i,t_i) \in M \times [\frac{T_i}{2}, T_i)$, a sequence $\epsilon_i \searrow 0$, and scales $r_i \in (0, \epsilon_i\sqrt{T_i - t_i})$, such that

$$d_{PGH}\left(\left(B_{g^i}(x_i,t_i,\epsilon_i^{-1}r_i), d_{g_t^i}, x_i\right), \left(B_{g^i}\left((0^{n-2},z_i),\epsilon_i^{-1}r_i\right), d, (0^{n-2},z_i)\right)\right) < \epsilon_i r_i$$

for some pointed metric spaces (Z_i, d_i, z_i), yet $\widetilde{r}^{g^i}_{Rm}(x_i, t_i) < \epsilon_i r_i$. Define $\widetilde{g}^i_t := r_i^{-2} g^i_{t_i + r_i^2 t}$ for $t \in [-r_i^{-2} t_i, r_i^{-2}(T - t_i))$, so that we can pass to a subsequence to assume that $(M, d_{\widetilde{g}^i_0}, x_i)$ converges in the pointed Gromov-Hausdorff sense to some metric product

$$(\mathbb{R}^{n-2} \times Z_\infty, d_{\mathbb{R}^{n-2} \times Z_\infty}, (0^{n-2}, z_\infty))$$

Because $|B_{\widetilde{g}^i}(x_i, 0, 1)|_{\widetilde{g}^i_0} \geq A^{-1}$ and $Rc(\widetilde{g}^i_0) \geq -(n-1) r_i^2 \to 0$ as $i \to \infty$, we can apply the "almost metric product implies existence of splitting maps" theorem (Theorem 9.29 of [C1]) to obtain δ_i-splitting maps (see Definition 1.20 of [CN3])

$$u_i : B_{\widetilde{g}^i}(x_i, 0, 2) \to \mathbb{R}^{n-2},$$

where $\lim_{i \to \infty} \delta_i = 0$. Now fix a sequence $\delta_i' \searrow 0$ such that we can apply Cheeger-Naber's Slicing theorem (Theorem 1.23 of [CN3]) with parameter δ_i' given δ_i-splitting maps. That is, there are subsets $G_{\delta_i'} \subseteq B^{\mathbb{R}^{n-2}}(0^{n-2}, 1)$ such that the following hold:

(1) $|G_{\delta_i'}|_{g_{\mathbb{R}^{n-2}}} > \omega_{n-2} - \delta_i'$,

(2) If $s \in G_{\delta_i'}$, then $u_i^{-1}(s) \cap B_{\widetilde{g}^i}(x_i, 0, 1) \neq \emptyset$,

(3) For each $z \in u_i^{-1}(G_{\delta_i'})$, and $r \in (0, 1]$, there is a lower triangular matrix $T \in GL(n-2, \mathbb{R})$ with positive diagonal entries such that $T \circ u_i : B_{\widetilde{g}^i}(z, 0, r) \to \mathbb{R}^{n-2}$ is a δ_i'-splitting map.

Observe that, by applying this theorem on $B_{\widetilde{g}^i}(x_i, 0, \gamma)$ for some fixed $\gamma \in (0, 1)$, and using the fact that for any $\epsilon > 0$, $u_i|B_{\widetilde{g}^i}(x_i, 0, \gamma)$ is an ϵ-splitting map for sufficiently large $i \in \mathbb{N}$, a diagonal argument gives regular values $s_i \in \mathbb{R}^{n-2}$ of u_i such that (3) holds, and

$$u_i^{-1}(s_i) \cap B_{\widetilde{g}^i}(x_i, 0, \gamma_i) \neq \emptyset,$$

where $\gamma_i \searrow 0$. Any $w_i \in u_i^{-1}(s_i) \cap B_{\widetilde{g}^i}(x_i, 0, \gamma_i)$ satisfy $\widetilde{r}^{\widetilde{g}^i}_{Rm}(w_i, 0) \leq \widetilde{r}_{Rm}(x_i, 0) + \gamma_i$, so that

$$\min_{y \in u_i^{-1}(s_i) \cap B_{\widetilde{g}^i}(x_i, 0, 1)} \frac{\widetilde{r}^{\widetilde{g}^i}_{Rm}(y, 0)}{d_{\widetilde{g}^i_0}\left(y, \partial B_{\widetilde{g}^i}(x_i, 0, 1)\right)} \to 0$$

Choose $y_i \in u_i^{-1}(s_i) \cap B_{\widetilde{g}^i}(x_i,0,1)$ achieving the minima, set $\rho_i := \widetilde{r}^{\widetilde{g}^i}_{Rm}(y_i,0)$, and define $\widehat{g}^i_t := \rho_i^{-2}\widetilde{g}^i_{\rho_i^2 t}$, $t \in [-(r_i\rho_i)^{-2}t_i, (r_i\rho_i)^{-2}(T-t_i))$. We know $\widetilde{r}^{\widehat{g}^i}_{Rm}(y_i,0) = 1$, $\lim_{i\to\infty} d_{\widehat{g}^i_0}(y_i,\partial B_{\widetilde{g}^i}(x_i,0,1)) = \infty$, and for any $z \in u_i^{-1}(s_i)$ with $d_{\widehat{g}^i_0}(y,z) \leq \frac{1}{2}d_{\widehat{g}^i_0}(y_i,\partial B_{\widetilde{g}^i}(x_i,0,1))$, we have

$$\widetilde{r}^{\widehat{g}^i}_{Rm}(z,0) = \frac{\widetilde{r}^{\widetilde{g}^i}_{Rm}(z,0)}{\widetilde{r}^{\widetilde{g}^i}_{Rm}(y_i,0)} \geq \frac{d_{\widehat{g}^i_0}\left(z,\partial B_{\widetilde{g}^i}(x_i,0,1)\right)}{d_{\widehat{g}^i_0}\left(y,\partial B_{\widetilde{g}^i}(x_i,0,1)\right)} \geq \frac{1}{2}.$$

That is, for any $D < \infty$, we have $\widetilde{r}^{\widehat{g}^i}_{Rm}(\cdot,0) \geq \frac{1}{2}$ on $B_{\widehat{g}^i_0}(y_i,0,D) \cap u_i^{-1}(s_i)$ for sufficiently large $i \in \mathbb{N}$. By (3) and Theorem 9.29 of [C1], we can find δ_i''-splitting maps (with respect to $\widehat{g}^i_0$) $v_i := T_i \circ (u_i - s_i) : B_{\widehat{g}^i_0}(z_i,0,1) \to \mathbb{R}^{n-2}$, where $T_i \in GL(n-2,\mathbb{R})$ and $\delta_i'' \searrow 0$.

After passing to a subsequence, we can assume $(M, d_{\widehat{g}^i_0}, y_i)$ converge in the pointed Gromov-Hausdorff sense to some noncollapsed Ricci limit space (W, d_W, w). Also, $|B_{\widehat{g}^i_0}(y,0,r)|_{\widehat{g}^i_0} \geq A^{-1}r^n$ for all $y \in M$ and $r > 0$ when $i = i(r) \in \mathbb{N}$ is sufficiently large, so $\mathcal{H}^n(B^W(w,r))) \geq A^{-1}r^n$ for all $r \in (0,\infty)$ by Colding's volume convergence theorem. It follows from the proof of the Transformation theorem (Theorem 1.32 of [CN3]) that in fact $v_i : B_{\widehat{g}^i}(y_i,0,r) \to \mathbb{R}^{n-2}$ are $C(r)\delta_i''$-splittings for any $r > 0$ when $i = i(r) \in \mathbb{N}$ is sufficiently large. This is essentially because we know $T_{i,r}v_i : B_{\widehat{g}^i}(y_i,0,r) \to \mathbb{R}^{n-2}$ are δ_i''-splittings for some $T_{i,r} \in GL(n-2,\mathbb{R})$, and we can then compare T_i with $T_{i,r}$, using the fact that they are both lower triangular with positive diagonal entries, to conclude they are actually close (see the proof of Theorem 1.32 in [CN3] for details). We can conclude from the "splitting maps imply metric almost-splitting" theorem (Theorem 3.6 of [CC1]) that W splits off $\mathbb{R}^{n-2}$ isometrically, and v_i converge locally uniformly (with respect to the Gromov-Hausdorff convergence) to a 1-Lipschitz function $v : W \to \mathbb{R}^{n-2}$, such that $v : W = \mathbb{R}^{n-2} \times S \to \mathbb{R}^{n-2}$ is the projection map.

Claim 1: S is smooth.

We follow Lemma 13.5 of [B2]. Because $\widetilde{r}^{\widehat{g}^i}_{Rm}(y_i,0) = 1$, we know w is in the (smooth and open) regular set of W. Suppose $z \in W$ is also in the regular set, and choose a sequence $z_i \in M_i$ converging to z with respect to the Gromov-Hausdorff convergence. By smooth convergence on the regular set, we have $\sigma := \liminf_{i\to\infty} \widetilde{r}^{\widehat{g}^i}_{Rm}(z_i,0) > 0$. Let $f_i : B^W(z,\frac{1}{2}\sigma) \to B_{\widehat{g}^i}(y_i,0,\sigma)$ be diffeomorphisms with $f_i^*\widehat{g}^i_0 \to g^W$, and such that $f_i \to \mathrm{id}_W$ locally uniformly with respect to the Gromov-Hausdorff convergence, where g^W is the metric on the regular set of W. Because v_i are harmonic and converge locally uniformly to v, local elliptic regularity and Arzela-Ascoli's theorem lets us pass to a subsequence such that $f_i^* v_i \to v$ in $C^\infty_{loc}(B^W(z,\frac{1}{2}))$. The C^2 convergence allows us to apply a quantitative version of the implicit function theorem to find $z_i' \in u_i^{-1}(0)$ with $d_{\widehat{g}^i_0}(z_i',z_i) \to 0$. This implies $z_i' \to z$, and $\widetilde{r}^{\widehat{g}^i}_{Rm}(z_i',0) \geq \frac{1}{2}$ then implies $r^W_{Rm}(z) \geq \frac{1}{2}$ (using backwards pseudolocality and Shi's local derivative estimates). Because S is connected, this implies $r^W_{Rm} \geq \frac{1}{2}$ on all of S.

Since $W = \mathbb{R}^{n-2} \times S$ is a metric product and S is C^∞, we can conclude that W is C^∞, and (by Theorem 33) $(M,\widehat{g}^i_0,y_i)$ converges to (W,g^W,w) in the C^∞ Cheeger-Gromov sense, where g^W is a C^∞ Riemannian metric on W whose length metric coincides with d_W. We can also conclude that $Rc(g_W) \geq 0$ everywhere, and $g^W = g_{\mathbb{R}^{n-2}} + g^S$, where g^S is a C^∞ Riemannian metric on S with nonnegative curvature. In fact, we can conclude by Theorem 31 and Shi's derivative estimates that $(M,(\widehat{g}^i_t)_{t\in[-\epsilon,\epsilon]},y_i)$ converges in the C^∞ Cheeger-Gromov sense to a complete, pointed Ricci flow $(\mathbb{R}^{n-2} \times S, g_{\mathbb{R}^{n-2}} + g^S_t, (0^{n-2},w'))_{t\in[-\epsilon,\epsilon]}$ with $g^S_0 = g^S$, $w = (0^{n-2},w')$, for some $\epsilon > 0$, which has uniformly bounded curvature, and satisfies $Rc(g^S_t) \geq 0$ everywhere. Set $g_t^\infty := g_{\mathbb{R}^{n-2}} + g^S_t$. If (S,g^S) has vanishing curvature somewhere, then $R(g_t^\infty) = 0$ somewhere, and because $R(g_t^\infty) \geq 0$ everywhere, the strong maximum principle for $R(g^\infty)$ and uniqueness for complete Ricci flow solutions with bounded curvature (Theorem 1.1 of [CZ2]) implies that $R(g_t^\infty) = 0$ everywhere. This is only possible if g_t^∞ is flat, contradicting $\widetilde{r}^{g^\infty}_{Rm}(w,0) = 1$. We may therefore assume (S,g^S) has strictly positive curvature everywhere. In particular, S is diffeomorphic to $\mathbb{R}^2$. The Cohn-Vossen inequality thus gives $\int_S R(g_0^S)dg_0^S \leq 4\pi$, so that for any $x \in M$ and $r > 0$, we have

$$\frac{1}{|B_{g^S}(x,0,r)|_{g_0^S}} \int_{B_{g^S}(x,0,r)} R(g_0^S)dg_0^S \leq \frac{\omega_{n-2}r^{n-2}}{|B_{g^S}(x,0,r) \times B^{\mathbb{R}^{n-2}}(0^{n-2},r)|_{g_0^\infty}} 4\pi \leq \frac{4\pi A\omega_{n-2}}{r^2}.$$

By a criterion of Shi (Theorem 9.2 in [CLN]), the flow $(g^S_t)_{t\in(-\epsilon,\epsilon)}$ can be extended to a complete, immortal Ricci flow $(g^S_t)_{t\in(-\epsilon,\infty)}$ with bounded curvature on compact time intervals.

Claim 2: After passing to a further subsequence, we can assume $(M_i,(\widehat{g}^i_t)_{t\in(-\epsilon,(r_i\rho_i)^{-2}(T-t_i))},y_i)$ converge in the C^∞ pointed Cheeger-Gromov-Hamilton sense to $(W,(g^\infty_t)_{t\in(-\epsilon,\infty)},w)$.

By $\lim_{i\to\infty} r_i(T-t_i)^{-\frac{1}{2}}=0$, for any $\tau<\infty$, $\widehat{g}^i_t$ is defined for $t\in(-\epsilon,\tau]$ whenever $i=i(\tau)\in\mathbb{N}$ is sufficiently large. Let $t^*\in[0,\infty]$ be supremal such that there exists $C<\infty$ such that for any $D<\infty$ and $\delta>0$, we have

$$\sup_{B_{\widehat{g}^i}(y_i,t,D)\times[0,t^*-\delta]}|Rm|_{\widehat{g}^i}\le C$$

for all $i=i(D,\delta,t^*)\in\mathbb{N}$ sufficiently large. Suppose by way of contradiction that $t^*<\infty$. Then we can pass to a subsequence to get pointed Cheeger-Gromov-Hamilton convergence

$$(M_i,(\widehat{g}^i_t)_{t\in(-\epsilon,t^*)},y_i)\to(\overline{W},(\overline{g}^\infty_t)_{t\in(-\epsilon,t^*)},\overline{w})$$

to some complete Ricci flow with bounded curvature on compact time intervals. By the uniqueness [CZ2] of complete Ricci flows with bounded curvature, and because $(M_i,\widehat{g}^i_0,y_i)\to(W,g^\infty,w)$ in the pointed Cheeger-Gromov sense, we have $(\overline{W},(\overline{g}^\infty_t)_{t\in(-\epsilon,t^*)},\overline{w})=(W,(g^\infty_t)_{t\in(-\epsilon,t^*)},w)$.

Because $(W,(g^\infty_t)_{t\in(-\epsilon,\infty)})$ is immortal, with bounded curvature on compact time intervals, there exists $C=C(t^*)<\infty$ such that for any $D'<\infty$ and $\delta>0$,

$$\sup_{t\in[0,t^*-\delta]}\sup_{B_{\widehat{g}^i}(y_i,t,D')}|Rm|\le C$$

for $i=i(D',\delta)\in\mathbb{N}$ sufficiently large. For any $D<\infty$, we can thus take $\delta=\delta(t^*)>0$ small and $D'=D'(D,t^*)<\infty$ large and apply Theorem 31 to obtain

$$\sup_{t\in[0,t^*+\delta)}\sup_{B_{\widehat{g}^i}(y_i,t,D)}|Rm|\le\overline{C}(t^*),$$

for $i=i(D,t^*)\in\mathbb{N}$ sufficiently large. This contradicts the definition of t^*, so we must have $t^*=\infty$, and the claim follows from the Cheeger-Gromov-Hamilton compactness theorem and the aforementioned uniqueness result [CZ2].

If $(S,(g^S_t)_{t\in(-\epsilon,\infty)})$ is a Type-IIb immortal solution:

$$\lim_{t\to\infty}\sup_{x\in S} t|Rm|_{g^S}(x,t)=\infty,$$

then a Type-IIb rescaling (as in Proposition 8.20 of [CLN]) procedure produces a singularity model which is a the Cigar soliton (Theorem 9.4 of [CLN]). However, because the limit of a limit is a limit (see Lemma 8.26 of [CLN]), we obtain the Cigar soliton (times flat $\mathbb{R}^{n-2}$) as a smooth Cheeger-Gromov-Hamilton limit of some Ricci flows $(M_i,(\overline{g}^i_t)_{t\in(a_i,b_i)},z_i)$ satisfying $Rc(\overline{g}^i_t)\geq -A\overline{g}^i_t$ and $|B_{\overline{g}^i}(z_i,0,r)|_{\overline{g}^i_0}\geq A^{-1}r^n$ for all $r\in(0,\infty)$, when $i=i(r)\in\mathbb{N}$ is sufficiently large. We thus obtain that the Cigar soliton (times flat $\mathbb{R}^{n-2}$) has maximal volume growth, which is a contradiction.

We may therefore assume the immortal solution is Type-III, so that

$$\Lambda:=\limsup_{t\to\infty}\sup_{x\in S} tR_{g^S}(x,t)<\infty.$$

Hamilton's Harnack inequality implies $\partial_t((t+\epsilon)R_{g^S}(w,t))\geq 0$, so $tR_{g^S}(w,t)\geq\frac{1}{2}\epsilon R_{g^S}(w,0)$ for all $t\in[1,\infty)$, hence $\Lambda>0$. Choose $(w_i,\tau_i)\in S\times(0,\infty)$ such that $\tau_i\nearrow\infty$ and $\lim_{i\to\infty}\tau_i R_{g^S}(w_i,\tau_i)=\Lambda$. Define $\overline{Q}_i:=R_{g^S}(w_i,\tau_i)\searrow 0$ and $\overline{g}^i_t:=\overline{Q}_i g^S_{\overline{Q}_i^{-1}t}$ for $t\in[0,\infty)$, so that

$$R_{\overline{g}^i}\left(w_i,\tau_i\overline{Q}_i\right)=\overline{Q}_i^{-1}R_{g^S}(w_i,\tau_i)=1,$$

and for $\sigma\in(0,1)$, we have the following for all $(x,t)\in S\times[\sigma,\sigma^{-1}]$ when $i=i(\sigma)\in\mathbb{N}$ is sufficiently large:

$$tR_{\overline{g}^i}(x,t)=t\overline{Q}_i^{-1}R_{g^S}(x,\overline{Q}_i t)\leq\Lambda+\sigma.$$

After passing to a subsequence, $(S,(\tau_i^{-1}g^S_{\tau_i t})_{t\in(0,\infty)},w)$ thus converges in the C^∞ pointed Cheeger-Gromov sense as $i\to\infty$ to a complete noncompact immortal Ricci flow $(E,(g^E_t)_{t\in(0,\infty)},e_\infty)$ with positive curvature satisfying $tR_{g^E}(x,t)\leq\Lambda$ for all $t>0$, with equality at $t=\Lambda$, since $\tau_i\overline{Q}_i\to\Lambda$. By Theorem 1.4 of [C], $(E,(g^E_t)_{t\in(0,\infty)})$ is a nonflat expanding gradient Ricci soliton $(E,(g^E_t)_{t\in(0,\infty)},e_\infty)$ by Theorem 1.4 of [C]. Reasoning as in the Type-IIb case, (E,g^E_1) has maximal volume growth. By the classification of 2-dimensional gradient Ricci solitons in [BM1], g^E must be an asymptotically conical expanding Ricci soliton asymptotic at infinity to $C(\mathsf{S}^1_\beta)$ for some $\beta \in (0, 2\pi)$, constructed in Appendix A of [GHMS] (see also section 4.5 of [CLN]). Here, $e_* \in E$ corresponds to the point fixed by the isometric S^1 action, which is also the global maximum of the soliton potential function.

Again using the diagonal argument from Lemma 8.26 in [CLN], we can find $(\check{y}_i, \check{t}_i) \in M_i \times [\frac{T_i}{2}, T_i)$ and some $\check{r}_i \searrow 0$ such that $\check{r}_i^{-2}\check{t}_i \to \infty$, $(T - \check{t}_i)\check{r}_i^{-2} \to \infty$ and the rescaled flows $\check{g}^i_t := \check{r}_i^{-2} g_{\check{r}_i^2 t + t_i}$ are such that $(M_i, (\check{g}^i_t)_{t\in(0,(T-t_i)r_i^{-2})}, \check{y}_i)$ converge in the pointed Cheeger-Gromov sense to the expanding soliton $(\mathbb{R}^{n-2} \times E, (g_{\mathbb{R}^{n-2}} + g^E_t)_{t\in(0,\infty)}, (0^{n-2}, e_*))$. Moreover, if we let $(\check{\nu}^i_t)_{t\in(-r_i^{-2}t_i,1)}$ be the conjugate heat kernels of the rescaled flows based at $(\check{y}_i, 1)$, then we can pass to a subsequence to obtain a metric flow pair $(\check{\mathcal{X}}, (\check{\mu}^\infty_t)_{t<0})$ such that

$$(M_i, (\check{g}^i_t)_{t\in[-r_i^{-2}t_i,1]}, (\check{\nu}^i_t)_{t\in[-r_i^{-2}t_i,1[}) \xrightarrow[i\to\infty]{\mathbb{F},\mathfrak{C}} (\check{\mathcal{X}}, (\check{\mu}^\infty_t)_{t\in(-\infty,1[})$$

on compact time intervals, where $\mathfrak{C}$ is a fixed correspondence, and

$$\check{\mathcal{X}} = \left((\check{\mathcal{X}}_t)_{t\in(-\infty,1]}, \check{\mathfrak{t}}, (\check{d}_t)_{t\in(-\infty,1]}, (\check{\nu}_{x;s})_{x\in\mathcal{X}, s\in(-\infty,\mathfrak{t}(x))}\right)$$

is an H_n-concentrated future-continuous metric flow of full support. In fact, this follows from Section 2 of [B5], where the required lower bound on Nash entropy follows as in Lemma 13, using Theorem 8.1 of [B4] to estimate

$$A^{-1} < |B_{g^i}(\check{y}_i, 0, 1)|_{g^i_0} \leq C(A, n)\exp\left(\mathcal{N}^{g^i}_{y_i,0}(1)\right).$$

Let $(\check{\mathcal{R}}, \check{\mathfrak{t}}, \partial_{\mathfrak{t}}, \check{g}^\infty)$ denote the corresponding Ricci flow spacetime structure on the regular set of $\check{\mathcal{X}}$. Let $(\check{U}_i)$ be an increasing compact exhaustion of $\check{\mathcal{R}}$, and $\check{\psi}_i : \check{U}_i \to M_i$ be diffeomorphisms such that

$$\|\check{\psi}_i^* \check{K}^i - \check{K}^\infty\|_{C^i(U_i)} + \|\check{\psi}_i^* \check{g}^i - \check{g}^\infty\|_{C^i(U_i)} + \|(\check{\psi}_i^{-1})_* \partial_t - \partial_{\mathfrak{t}}\|_{C^i(U_i)} \leq \alpha_i$$

for some sequence $\alpha_i \searrow 0$, where $d\check{\mu}^\infty_t = \check{K}^\infty(\cdot, t)d\check{g}^\infty_t$ on $\check{\mathcal{R}}$ and $d\check{\nu}^i_t = \check{K}^i(\cdot, t)d\check{g}^i_t$.

Claim 3: $\check{\mathcal{X}}_{(0,1)}$ is isometric as a metric flow to the smooth Ricci flow $(\mathbb{R}^{n-2} \times E, (g_{\mathbb{R}^{n-2}} + g^E_t)_{t\in(0,1)})$.

From the smooth Cheeger-Gromov-Hamilton convergence

$$(M_i, (\check{g}^i_t)_{t\in(0,2)}, \check{y}_i) \to (\mathbb{R}^{n-2} \times E, (g_{\mathbb{R}^{n-2}} + g^E_t)_{t\in(0,2)}, (0^{n-2}, e_\infty)),$$

we know that

$$\limsup_{i\to\infty} \sup_{B_{g^i}(y_i,0,D)\times(-\kappa,1]} |Rm|_{g^i} \leq C(\kappa) < \infty$$

for any fixed $D < \infty$ and $\kappa \in (0,1)$. Thus, for any fixed $\kappa \in (0,1)$, we can apply Proposition 9.5 of [B2] to get

$$\limsup_{i\to\infty} d^{g^i_{-\kappa}}_{W_1}(\check{\nu}^i_{1-\kappa}, \delta_{y_i}) \leq C$$

for some $C = C(\kappa) < \infty$. Fix $\overline{y}_\infty \in \check{\mathcal{R}}_{-\kappa}$, and set $\overline{y}_i := \check{\psi}_i(\overline{y}_\infty)$, so that $\overline{y}_i \xrightarrow[i\to\infty]{C} \overline{y}_\infty$ by Theorem 9.31(c) of [B3]. Because $\check{\nu}^\infty_{1-\kappa}(B(\overline{y}_\infty,1)) > 0$ and $\lim_{i\to\infty} ||\check{\psi}_i \check{K}^i - \check{K}^\infty||_{C^0(U_i)} = 0$, we have

$$\liminf_{i\to\infty} \check{\nu}^i(B_{g^i}(\overline{y}_i, 1-\kappa, 2)) \geq \check{\nu}^\infty_{1-\kappa}(B(\overline{y}_\infty,1)) > 0.$$

Thus, for sufficiently large $i \in \mathbb{N}$,

$$\frac{1}{2}\check{\nu}^\infty_{1-\kappa}(B(\overline{y}_\infty,1))\left(d_{g^i_{1-\kappa}}(\check{y}_i,\overline{y}_i) - 1\right) < \int_{M_i} d_{g^i_{1-\kappa}}(\check{y}_i, y)\, d\check{\nu}^i_{1-\kappa}(y) \leq C,$$

and in particular,

$$\limsup_{i\to\infty} d_{g^i_{1-\kappa}}(\check{y}_i, \overline{y}_i) < \infty.$$

We can now apply Theorem 9.58 of [B5] to conclude that the parabolic cylinders $P(\overline{y}_\infty; D, -1+2\kappa, 0) \subseteq \check{\mathcal{R}}$ are unscathed for any $\kappa \in (0,\frac{1}{4})$ and $D < \infty$. Because κ, D were arbitrary, it follows that $\check{\mathcal{R}}_{(-\infty,0)} = \check{\mathcal{X}}_{(-\infty,0)}$, and we can identify $\check{\mathcal{R}}_{(-\infty,0)}$ with the canonical spacetime of a smooth Ricci flow. Moreover, we have Cheeger-Gromov-Hamilton convergence $(M, (\check{g}^i_t)_{t\in(0,1)}, \overline{y}_i) \to (\check{\mathcal{R}}_t, (\check{g}^\infty_t)_{t\in(0,1)}, \overline{y}_\infty)$, hence the Ricci flow spacetime is isometric to $(\mathbb{R}^{n-2} \times E, (g_{\mathbb{R}^{n-2}} + g^E_t)_{t\in(0,1)})$.

Suppose $(U_i)_{i\in\mathbb{N}}$ is an exhaustion of $\mathbb{R}^{n-2} \times E$ and $\zeta_i : U_i \to M_i$ are embeddings such that $\zeta_i(\overline{y}_\infty) = \overline{y}_i$ and $\zeta_i \check{g}^i \to \check{g}^\infty$ as $i \to \infty$ in $C^\infty_{loc}((\mathbb{R}^{n-2} \times E) \times (0,1))$. Direct computation (see

Section 5 of [CLN]) shows that

$$\lim_{d_{g_1^E}(e_*,y)\to\infty} |Rm|_{g^E}(y,1)\, d^2_{g_1^E}(e_*,y) = 0.$$

Now fix $r_0 \in (0,\infty)$ such that

$$|Rm|_{g^E}(y,1)\, d^2_{g_1^E}(e_*,y) \leq \frac{1}{4}$$

for all $y \in E$ with $d_{g_1^E}(y,e_*) \geq r_0$. Suppose $y \in E$ satisfies $r_y := d_{g_1^E}(y,e_*) \geq r_0$, so that

$$\sup_{B_{g^E}(y,1,\frac{1}{2}r_y)} |Rm|_{g^\infty}(\cdot,1) \leq \frac{1}{r_y^2},$$

hence

$$\sup_{B_{g^i}(\zeta_i(0^{n-2},y),1,\frac{1}{4}r_y)} |Rm|_{g^i}(\cdot,1) \leq \frac{2}{r_y^2}$$

for sufficiently large $i \in \mathbb{N}$. By Theorem 31 and Shi's estimates on the backwards cylinder $B_{g^i}(\zeta_i(0^{n-2},y),1,\frac{\epsilon_P}{4}r_y) \times [1-\frac{1}{16}(\epsilon_P r_y)^2, 1]$ when $i \in \mathbb{N}$ is large, we get

$$|\nabla^k Rm|_{g^i}(\zeta_i(0^{n-2},y),1) \leq C(k,A) r_y^{-2-k},$$

hence taking $i \to \infty$ gives $|\nabla^k Rm|_{g^\infty}(0^{n-2},y) \leq C(k,A) r_y^{-2-k}$, hence $|\nabla^k Rm|_{g_E}(e_*,y) \leq C(k,A) r_y^{-2-k}$. We have thus shown

$$\sup_{y\in E} |\nabla^k Rm|_{g_E}(y)\, d^{2+k}_{g_1^E}(e_*,y) \leq C(k,A) < \infty.$$

This is the condition needed to apply Theorem 4.3.1 of [S2] to guarantee that the expanding soliton $(E,(g_t^E)_{t\in(-1,\infty)})$ converges smoothly to $C(\mathbb{S}^1_\beta)$ away from the vertex:

$$\lim_{t\searrow 0} \sup_K |(\nabla^{g_{C(\mathbb{S}^1_\beta)}})^k (g_{C(\mathbb{S}^1_\beta)} - g_t^E)|_{g_{C(\mathbb{S}^1_\beta)}} = 0$$

for any compact subset $K \subseteq C(\mathbb{S}^1_{\beta''}) \setminus \{o\} = E \setminus \{e_*\}$ and integer $k \in \mathbb{N}$. In particular, we have

$$\lim_{t\searrow 0} \sup_K |Rm|_{g^E}(\cdot,t) = 0.$$

Now let $y' \in ((\mathbb{R}^{n-2} \times E) \setminus (\mathbb{R}^{n-2} \times \{e_*\})) \times \{1\} \subseteq \check{X}_1$ be arbitrary. Because $\lim_{t\searrow 0} \sup |Rm|_{g^E}(\cdot,t) \to 0$, Theorem 31 gives uniform curvature bounds on

$B_{\widehat{g}^i}(\zeta_i(y'), t', 2r') \subset [t' - 2(r')^2, t' + 2(r')^2]$ for all $t' \in (0,1]$, where $r' = r'(y') > 0$ is independent of t'. We can again apply Theorem 9.58, now concluding that $P(y'(t); r', -(r')^2, 0) \subset \check{\mathcal{R}}_{(t-(r')^2,\delta)}$ is unscathed for all $t \in (0,1]$. Choosing $t' \in (0, \frac{1}{4}(r')^2)$, we know that $P(y'; r', -(r')^2, 0) \cap \check{\mathcal{X}}_{(0,t)}$ corresponds to the flow $(g_{\mathbb{R}^{n-2}} + g_t^E)_{t \in (0,t')}$ on some open subset $U \subset (\mathbb{R}^{n-2} \times E) \setminus (\mathbb{R}^{n-2} \times \{e\})$, so that

$$(P(y'; r', -(r')^2, 0) \cap \check{\mathcal{X}}_0, \check{g}_0^\infty)$$

is isometric to an open subset of the flat Riemannian cone $\mathcal{C} := \mathbb{R}^{n-2} \times (C(\mathbb{S}^1_\beta) \setminus \{o\})$ and in particular, $Rm(\check{g}_0^\infty) = 0$ on $P(y'; r', -(r')^2, 0) \cap \check{\mathcal{X}}_0$. However, we know that $R_{g^\infty} \geq 0$ everywhere on $\check{\mathcal{R}}$, so the strong maximum principle for R_{g^∞} (Corollary 12.43 of [CCG+2]) gives $Rc(\check{g}^\infty) = 0$ on $\check{\mathcal{R}}_{<0}$. Because y' was arbitrary, we see that the flow of $\partial_{\mathfrak{t}}$ produces an open embedding

$$\iota : \mathcal{C} = (\mathbb{R}^{n-2} \times E) \setminus (\mathbb{R}^{n-2} \times \{e\}) \times \{1\} \hookrightarrow \check{\mathcal{R}}_0,$$

and that the image of ι is Riemannian isometric to $\mathcal{C}$ equipped with the flat cone metric $g_{\mathcal{C}}$.

Suppose by way of contradiction that $z \in \check{\mathcal{R}}_0 \setminus \iota(\mathcal{C})$, and let $\epsilon > 0$ be such that the integral curve $\gamma : (-2\epsilon, 2\epsilon) \to \check{\mathcal{R}}$ of $\partial_{\mathfrak{t}}$ with $\gamma(0) = z$ is well-defined. Then $\gamma(\epsilon)$ corresponds to a point in $\mathbb{R}^{n-2} \times E$ not in the domain of ι; that is, $\gamma(\epsilon) \in (\mathbb{R}^{n-2} \times \{e\}) \times \{\epsilon\}$. However, $\lim_{t \searrow -1} |Rm|_{g^\infty}(\gamma(t)) = \infty$, contradicting $r^{\mathcal{X}}_{Rm}(z) > 0$. Thus ι is surjective. Because $\check{\mathcal{X}}_{-1}$ is the metric completion of $\check{\mathcal{R}}_{-1}$ equipped with its length metric, it follows that ι extends to a metric isometry $\check{\mathcal{X}}_0 \to \mathbb{R}^{n-2} \times C(\mathbb{S}^1_\beta)$.

Because $Rc(\check{g}^\infty) = 0$ on $\check{\mathcal{R}}_{\leq 0}$, we know that $\check{\mathcal{X}}_{<0}$ is a static metric flow by Theorem 1.17 of [B5], corresponding to some Ricci flat singular space $(\check{X}', \check{d}', \check{\mathcal{R}}', \check{g}')$.

Claim 4: $(\check{X}', \check{d}')$ is isometric to $\mathbb{R}^{n-2} \times C(\mathbb{S}^1_\beta)$.

For any $y' \in \check{\mathcal{R}}_0$, there exists $\epsilon > 0$ such that the integral curve $\gamma : (-\epsilon, \epsilon) \to \check{\mathcal{R}}$ of $\partial_{\mathfrak{t}}$ with $\gamma(0) = y'$ is well-defined. Because $\check{\mathcal{X}}_{<0}$ is static and $\check{\mathcal{X}}_{(0,1)} = \check{\mathcal{R}}_{(0,1)}$, we conclude that

γ extends to an integral curve of ∂_{t} defined on $(-\infty, 1)$. Flowing along ∂_{t} therefore gives a smooth embedding $\iota' : \check{R}_0 \hookrightarrow \check{R}_1 = \check{R}'$, or equivalently $C \hookrightarrow \check{X}'$. Moreover, because $Rc(\breve{g}^{\infty}) = 0$ on $\check{R}_{(-\infty,0]}$, we have $\check{d}_{-1}(\iota'(z_1), \iota'(z_2)) \leq \check{d}_0(z_1, z_2)$ for all $z_1, z_2 \in \check{R}_0$.

Suppose by way of contradiction that there exist $\epsilon > 0$, $z', z'' \in \check{R}_0$, and a curve $\gamma : [0,1] \to \check{R}_{-1}$ from $\iota'(z')$ to $\iota'(z'')$ such that $\mathrm{length}_{g^{\infty}_{-1}}(\gamma) < \check{d}_0(z', z'') - \epsilon$. For sufficiently large $i \in \mathbb{N}$, we know that $\gamma([0,1]) \subset \check{U}_i$, so we can define $\gamma_i := \check{\psi}_i \circ \gamma$. Then, for large $i \in \mathbb{N}$, γ_i is a curve from $z_i' := (\check{\psi}_i \circ \iota')(z')$ to $z_i'' := (\check{\psi}_i \circ \iota')(z'')$ in M with

$$\mathrm{length}_{g^i_{-1}}(\gamma_i) < \check{d}_0(z', z'') - \frac{\epsilon}{2}.$$

Because $Rc(\breve{g}^i_t) \geq -\check{r}_i^2 A$, we obtain

$$\mathrm{length}_{g^i_{\kappa}}(\gamma_i) < \check{d}_0(z', z'') - \frac{\epsilon}{4}$$

for all $\kappa \in (0,1)$ when $i \in \mathbb{N}$ is sufficiently large. Now let $\eta_{z'}, \eta_{z''} : [-1, \frac{1}{2}] \to \check{R}$ be the integral curves of ∂_{t} with $\eta_{z'}(0) = z'$ and $\eta_{z''}(0) = z''$, so that $\eta_{z'}(-1) = \iota'(z')$, $\eta_{z''}(-1) = \iota'(z'')$, and $\eta_{z'}([-1, \frac{1}{2}]), \eta_{z''}([-1, \frac{1}{2}]) \subset V \subset \check{R}$, where V is some open product domain $V' \times [-1, \frac{1}{2}]$. Because

$$\|(\check{\psi}_i^{-1})_* \partial_t - \partial_{\mathsf{t}}\|_{C^0(U_i)} < \alpha_i,$$

a standard result about continuous dependence of ODE solutions on parameters (see Chapter 5 of [H3]) tells us that the integral curves $\eta^i_{z'}, \eta^i_{z''} : [-1, \frac{1}{2}] \to \check{R}$ of $(\check{\psi}_i^{-1})_* \partial_t$ satisfying $\eta^i_{z'}(-1) = \iota'(z')$, $\eta^i_{z''}(-1) = \iota'(z'')$ are well-defined (do not escape $V' \times [-1, \frac{1}{2}]$), and satisfy $\lim_{i \to \infty} \eta^i_{z'}(s) = \eta_{z'}(s)$, $\lim_{i \to \infty} \eta^i_{z''}(s) = \eta_{z''}(s)$ for any $s \in [-1, \frac{1}{2}]$. In particular, we have $\gamma_i(0) \times [-1, \frac{1}{2}] \in \check{\psi}_i(\check{U}_i)$ for sufficiently large $i \in \mathbb{N}$. Using the locally uniform curvature estimates at times $t \in [\frac{\kappa}{2}, \kappa]$, Theorem 9.58 of [B3] guarantees that $\gamma_i([0,1]) \times \{\kappa\} \subset \check{\psi}_i(\check{U}_i)$ for sufficienly large $i \in \mathbb{N}$. Because $\breve{g}^i \to \breve{g}^{\infty}$ in $C^0_{loc}(\check{R})$, we obtain

$$\check{d}_{\kappa}(\check{\psi}_i^{-1}(z_i'), \check{\psi}_i^{-1}(z_i'')) \leq \mathrm{length}_{g^{\infty}_{\kappa}}(\psi_{i,\kappa}^{-1} \circ \gamma_i) < \check{d}_0(z', z'') - \frac{\epsilon}{8}.$$

Moreover, we have

$$\check{d}_\kappa(\check{\psi}_i^{-1}(z_i'), z'(\kappa)) = \check{d}_\kappa(\eta^i_{z'}(\kappa), \eta_{z'}(\kappa)) \to 0$$

as $i \to \infty$, and similarly for z''. Combining estimates gives

$$\check{d}_\kappa(z'(\kappa), z''(\kappa)) \leq \check{d}_0(z', z'') - \frac{\epsilon}{16}$$

for all $\kappa \in (0, \frac{1}{2}]$. For any $\kappa \in (0, \frac{1}{2}]$, we can choose a curve $\zeta : [0,1] \to \check{R}_{-1}$ from $z'(\kappa)$ to $z''(\kappa)$ with

$$\text{length}_{g^1_\kappa}(\zeta) < \check{d}_\kappa(z'(\kappa), z''(\kappa)) + \frac{\epsilon}{32},$$

and $\zeta([0,1]) \cap (\mathbb{R}^{n-2} \times B(e_\infty, c\epsilon)) = \emptyset$ (this is possible because $\mathbb{R}^{n-2} \times \{e_\infty\}$ has codimension 2 in $\mathbb{R}^{n-2} \times E$). However, we know that

$$\sup_{\mathbb{R}^{n-2} \times (E \setminus B(e_\infty, c\epsilon))} |Rc|(\cdot, t) \leq 1$$

for $t \in [0, \tau(\epsilon)]$. Defining $\zeta_t : [0,1] \to R_t$ by $\zeta_t(s) := (\zeta(s))(t)$, we have

$$\frac{d}{dt}\left(e^t \text{length}_{g^1_t}(\zeta_t)\right) = e^t \left(1 - \int_0^1 \frac{Rc(\dot{\zeta}_t(s), \dot{\zeta}_t(s))}{|\dot{\zeta}_t(s)|} ds\right) \text{length}_{g^1_t}(\zeta_t) \geq 0$$

for any $t \in [0, \tau(\epsilon)]$, so that

$$\limsup_{\kappa \searrow 0} \check{d}_\kappa(z'(\kappa), z''(\kappa)) \geq \limsup_{\kappa \searrow 0} e^{-\kappa} \text{length}_{g^1_\kappa}(\zeta_0) - \frac{\epsilon}{32} \geq \check{d}_0(z', z'') - \frac{\epsilon}{32}.$$

Combining estimates, we therefore arrive at

$$\check{d}_0(z', z'') - \frac{\epsilon}{16} \geq \limsup_{\kappa \searrow 0} \check{d}_\kappa(z'(\kappa), z''(\kappa)) \geq \check{d}_0(z', z'') - \frac{\epsilon}{32},$$

a contradiction.

Therefore, $\iota' : (\check{R}_0, \check{d}_0) \hookrightarrow (\check{R}_{-1}, \check{d}_{-1})$ is an isometric embedding, which extends to an isometric embedding $(\check{X}_0, \check{d}_0) \hookrightarrow (\check{X}_{-1}, \check{d}_{-1})$. Suppose by way of contradiction that $\iota'(\check{R}_0) \neq \check{R}_{-1}$. Because $\check{R}_{-1}$ is connected, we can find a curve $\gamma : [0,1] \to \check{R}_{-1}$ from a point $v_1 \in \iota'(\check{R}_0)$ to some $v_2 \in \check{R}_{-1} \setminus \iota'(\check{R}_0)$, and such that $\widetilde{r}^{\mathcal{X}_t}_{Rm}((\gamma(u))(t)) \geq \sigma$ for all $u \in [0,1]$ and $t \in [-1, 0)$, where $\sigma > 0$ is an appropriately chosen constant. By truncating γ, we can assume

$\gamma([0,1)) \subset \iota'(\check{R}_0)$ and $\gamma(1) \in \iota'(\check{X}_0 \setminus \check{R}_0)$. For any fixed $t \in [-1,0)$, we have $(\gamma([0,1]))(t) \subset \check{U}_i$ for $i = i(t) \in \mathbb{N}$ sufficiently large, hence $\widetilde{r}^{g^i}_{Rm}(\psi_i(\gamma(u)(t)),t) \geq c(A,\sigma)$ for all $u \in [0,1]$ for large $i = i(t) \in \mathbb{N}$. By choosing $t \in [-1,0)$ sufficiently close to 0, we can apply Theorem 31 to conclude $r^{g^i}_{Rm}(\psi_i(\gamma(u)(0)),0) \geq c(A,\sigma)$ for all $u \in [0,1)$ when $i = i(u) \in \mathbb{N}$ is sufficiently large, since by hypothesis $\gamma(u)(0) \in \check{R}_0$ for all $u \in [0,1)$. For $u \in [0,1)$ sufficiently close to 1, Theorem 9.58 of [B3] gives

$$r^{g^\infty}_{Rm}((\gamma(u))(0),0) \leq 2\check{d}_0((\gamma(u))(0), \mathbb{R}^{n-2} \times \{o\})$$

since $(\iota')^{-1}(\gamma(1)) \notin \check{R}_0$, a contradiction. Thus $\iota'(\check{R}_0) = \check{R}_{-1}$, but because $\check{X}_{-1}$ is the metric completion of $\check{R}_{-1}$, it must be isometric to $\mathbb{R}^{n-2} \times C(S^1_\beta)$.

However, Theorem 1.17 of [B5] then tells us that the singular set $\mathbb{R}^{n-2} \times \{o\}$ of $\mathbb{R}^{n-2} \times C(S^1_{\beta''})$ has Minkowski dimension at most $n-4$, which is false. □

4.6 Codimension Three ϵ-Regularity

We can use codimension two ϵ-regularity to estimate the size of the large curvature region at curvature scales $\widetilde{r}_{Rm} \lesssim \sqrt{T-t}$.

Proposition 22. *For any $A < \infty$ and $\underline{T} > 0$, there exist $r_0 = r_0(A,\underline{T}) > 0$ and $E = E(A,\underline{T}) < \infty$ such that the following holds. Let $(M^n,(g_t)_{t \in [0,T)})$ be a closed Ricci flow satisfying $Rc(g_t) \geq -Ag_t$ and $|B(x,t,r)|_{g_t} \geq A^{-1}r^n$ for all $(x,t) \in M \times [0,T)$ and $r \in (0,1]$, where $T \geq \underline{T}$. Then, for all $(x,t) \in M \times [\frac{T}{2},T)$, $r \in (0, r_0\sqrt{T-t}]$, and $s \in (0,1]$ we have*

$$|\{\widetilde{r}^g_{Rm}(\cdot,t) < sr\} \cap B_g(x,t,r)|_{g_t} \leq Es^3r^n.$$

Proof. Let $\epsilon_0 > 0$ be as in Theorem 21. Define the rescaled metric $\widetilde{g} := (T-t)^{-1}g_t$, and suppose $x \in M$ satisfies

$$d_{PGH}\left((B_{\widetilde{g}}(x,r'),d_{\widetilde{g}},x),\left(B\left((0^{n-2},z_*),r'\right),d_{\mathbb{R}^{n-2}\times C(Z)},(0^{n-2},z_*)\right)\right) < \epsilon_0^2 r'$$

for some $r' \in [s,1]$ and some metric cone $C(Z)$. Then

$$d_{PGH}\left(\left(B_g(x,t,\sqrt{T-t}\,r'),(T-t)^{-\frac{1}{2}}d_{g_t},x\right),\left(B\left((0^{n-2},z^*),r'\right),d_{\mathbb{R}^{n-2}\times C(Z)},(0^{n-2},z^*)\right)\right)<\epsilon_0^2 r',$$

so we can apply Theorem 21 with $r=\epsilon_0\sqrt{T-t}\,r' < \epsilon_0\sqrt{T-t}$ to get $\widetilde{r}^g_{Rm}(x,t) \geq \epsilon_0^2\sqrt{T-t}\,r'$, or equivalently $\widetilde{r}^{\widetilde{g}}_{Rm}(x) \geq \epsilon_0^2 r' \geq \epsilon_0^2 s$. Let $\widetilde{\mathcal{S}}^k_{\eta,r}$ denote the quantitative singular strata of the rescaled metric $\widetilde{g}$. Then

$$\{\widetilde{r}^{\widetilde{g}}_{Rm}<s\} \subseteq \widetilde{\mathcal{S}}^{n-3}_{\epsilon_0^2,\epsilon_0^{-2}s}$$

for all $s\in(0,\epsilon_0^2]$. We can thus apply Theorem 34 with $\eta:=\epsilon_0^2$ to obtain

$$|\{\widetilde{r}^{\widetilde{g}}_{Rm}<sr\}\cap B_{\widetilde{g}}(x,r)|_{\widetilde{g}} \leq |\widetilde{\mathcal{S}}^{n-3}_{\epsilon_0^2,\epsilon_0^{-2}sr}\cap B_{\widetilde{g}}(x,r)|_{\widetilde{g}} \leq Cs^3r^n$$

for all $s\in(0,1]$ and $r\in(0,\epsilon_0^2]$, where $C=C(A,\underline{T})<\infty$. In terms of the unrescaled metric, this is

$$|\{\widetilde{r}^g_{Rm}(\cdot,t)<sr\}\cap B_g(x,t,r)|_{g_t} \leq Cs^3r^n$$

for all $s\in(0,1]$ and $r\in(0,\epsilon_0^2\sqrt{T-t}]$. □

Proposition 23. *(Codimension 3 ϵ-Regularity) For any $A<\infty$ and $\underline{T}\geq 0$, there exists $\epsilon_0=\epsilon_0(A,\underline{T})>0$ such that the following holds. Suppose $(M^n,(g_t)_{t\in[0,T)})$ is a closed Ricci flow satisfying $Rc(g_t)\geq -Ag_t$ and $|B(x,t,r)|_{g_t}\geq A^{-1}r^n$, for all $(x,t)\in M\times[0,T)$ and $r\in(0,1]$, where $T\geq\underline{T}$. For any $(x,t)\in M\times[\frac{T}{2},T)$ and $r\in(0,\epsilon_0\sqrt{T-t}]$, if*

$$d_{PGH}\left((B(x,t,r),d_{g_t},x),\left(B\left((0^{n-3},z^*),r\right),d_{\mathbb{R}^{n-3}\times C(Z)},(0^{n-3},z^*)\right)\right)<\epsilon_0 r$$

for some metric cone $C(Z)$, then $\widetilde{r}_{Rm}(x,t)\geq\epsilon_0 r$.

Proof. Suppose the claim is false. Then we can find closed Ricci flows $(M_i^n,(g_t^i)_{t\in[0,T_i)})$ satisfying $Rc(g_t^i)\geq -Ag_t^i$, $|B_{g^i}(x,t,r)|_{g_t^i}\geq A^{-1}r^n$ for all $(x,t)\in M_i\times[0,T_i)$ and $r\in(0,1]$, where $T_i\geq\underline{T}$, along with points $(x_i,t_i)\in M\times[\frac{T_i}{2},T_i)$, a sequence $\epsilon_i\searrow 0$, and scales $r_i\in(0,\epsilon_i\sqrt{T_i-t_i})$, such that

$$d_{PGH}\left((B_{g^i}(x_i,t_i,\epsilon_i^{-1}r_i),d_{g_t^i},x_i),\left(B\left((0^{n-3},z^i),\epsilon_i^{-1}r_i\right),d_{\mathbb{R}^{n-3}\times C(Z_i)},(0^{n-3},z^i)\right)\right)<\epsilon_i r_i$$

for some metric spaces (Z_i, d_i), yet $\widetilde{r}^{g^i}_{Rm}(x_i, t_i) < \epsilon_i r_i$. Define $\widetilde{g}^i_t := r_i^{-2} g^i_{t_i + r_i^2 t}$ for $t \in [-r_i^{-2} t_i, r_i^{-2}(T - t_i))$, so that we can pass to a subsequence to assume that $(M, d_{\widetilde{g}^i_0}, x_i)$ converges in the pointed Gromov-Hausdorff sense to some metric cone

$$(\mathbb{R}^{n-3} \times C(Z_\infty), d_{\mathbb{R}^{n-3} \times C(Z_\infty)}, (0^{n-3}, z_\infty))$$

based at its vertex, where $\mathrm{diam}(Z_\infty) \leq \pi$. For brevity, we write $(W, d_W, w_\infty) := (\mathbb{R}^{n-3} \times C(Z_\infty), d_{\mathbb{R}^{n-3} \times C(Z_\infty)}, (0^{n-3}, z_\infty))$ and $(\mathcal{R}_W, g_W)$ for the smooth Riemannian structure on $\mathcal{R}(W)$.

By Proposition 22, there exist $E = E(A, \underline{T}) < \infty$ and $r_0 = r_0(A, \underline{T}) > 0$ such that

$$|\{\widetilde{r}^{g^i}_{Rm}(\cdot, t) < sr\} \cap B_g(x, t, r)|_{g_t} \leq E s^3 r^n$$

for all $(x, t) \in M_i \times [\frac{T_i}{2}, T_i)$, $r \in (0, r_0 \sqrt{T - t})$ $s \in (0, 1]$. In terms of the rescaled metrics, this is

$$\begin{aligned} |\{\widetilde{r}^{\widetilde{g}^i}_{Rm}(\cdot, 0) < sr\} \cap B_{\widetilde{g}^i}(x, 0, r)|_{\widetilde{g}^i_0} &= r_i^{-n} |\{\widetilde{r}^{g^i}_{Rm}(\cdot, t_i) < s r r_i\} \cap B_{g^i}(x, t_i, r r_i)|_{g^i_{t_i}} \\ &\leq r_i^{-n} E s^3 (r r_i)^n = E s^3 r^n \end{aligned} \tag{4.6.1}$$

for all $s \in (0, 1]$ and $r \in (0, r_0 r_i^{-1} \sqrt{T - t_i}]$. Since $r_i^{-1} \sqrt{T - t_i} \to \infty$, (4.6.1) holds for all $s, r \in (0, 1]$ when $i \in \mathbb{N}$ is sufficiently large.

Claim 1: For any $y \in W$ and $s \in (0, \frac{1}{2}]$, we have

$$|\{\widetilde{r}^W_{Rm} < s\} \cap B^W(y, 1) \cap \mathcal{R}_W|_{g_W} \leq 2^{10n} E s^3.$$

Suppose by way of contradiction there are $y \in W$, $r, s \in (0, \frac{1}{2}]$ such that the claim fails. Because $\mathcal{R}_W \subseteq W$ is dense, we can assume $y \in \mathcal{R}_W$. Then we can find $\sigma > 0$ such that the set

$$S := \{\sigma < \widetilde{r}^W_{Rm} < s\} \cap B^W(y, 1)$$

satisfies $|S|_{g_W} > 2^{10n} E s^3$. By Theorem 33, there is an exhaustion (U_i) of $\mathcal{R}_W$ along with diffeomorphisms $\psi_i : U_i \to M$ such that $\psi_i^* \widetilde{g}^i_0 \to g_W$ in $C^\infty_{loc}(\mathcal{R}_W)$ as well as

$$\sup_{K \cap U_i} |d_{\widetilde{g}^i_0}(\psi_i(y_1), \psi_i(y_2)) - d_W(y_1, y_2)| \le \eta_i(K) \tag{4.6.2}$$

for each compact subset $K \subset \mathcal{R}_W$, where $\lim_{i \to \infty} \eta_i(K) = 0$. Because $S \subset \mathcal{R}_W$, we have $S \subset U_i$ for sufficiently large $i \in \mathbb{N}$. Moreover, (4.6.2) implies that $\psi_i(S) \subset B_{\widetilde{g}^i_0}(\psi_i(y), 0, 2)$ for sufficiently large $i \in \mathbb{N}$.

Subclaim: $\widetilde{r}^{\widetilde{g}^i}_{Rm}(\psi_i(x), 0) < 4s$ for all $x \in S$ when $i \in \mathbb{N}$ is sufficiently large.

Otherwise, there are $x_i \in S$ such that

$$\sup_{B_{\widetilde{g}^i}(\psi_i(x_i),0,4s)} |Rm|_{\widetilde{g}^i_0} \ge \frac{1}{16} s^{-2}.$$

Pass to a subsequence so that

$$x_i \to x \in \{\sigma \le \widetilde{r}^W_{Rm} \le s\} \cap \overline{B}^W(y, 1)$$

with respect to the Gromov-Hausdorff convergence. Then

$$(B_{\widetilde{g}^i}(\psi_i(x_i), 0, 4s), d_{\widetilde{g}^i_0}, x_i) \to (B^W(x, 4s), d, x)$$

in the pointed Gromov-Hausdorff sense, but the Cheeger-Gromov compactness theorem tells us that

$$(B_{\widetilde{g}^i}(\psi_i(x_i), 0, 4s), \widetilde{g}^i_0, x_i) \to (\widehat{B}, \widehat{g}, \widehat{x})$$

in the C^∞ Cheeger-Gromov sense for some smooth (incomplete) Riemannian manifold $\widehat{B}$ with curvature tensor satisfying $|Rm|_{\widehat{g}} \le \frac{1}{16} s^{-2}$. Moreover,

$$(B_{\widetilde{g}^i}(\psi_i(x_i), 0, 2s), d_{\widetilde{g}^i_0}, x_i) \to (B_{\widehat{g}}(\widehat{x}, 2s), d_{\widehat{g}}, \widehat{x}),$$

so $B^W(x, 2s)$ is isometric to a ball in a smooth Riemannian manifold equipped with its length metric, and with $|Rm| \le \frac{1}{16} s^{-2}$. In particular, $B^W(x, 2s) \subset \mathcal{R}$ and $|Rm|_g \le \frac{1}{4} s^{-2}$ on $B^W(x, 2s)$, so that $\widetilde{r}^W_{Rm}(x) \ge 2s$, a contradiction.

From the subclaim, we get

$$\psi_i(S) \subset \{\widetilde{r}^{\widetilde{g}^i}_{Rm}(\cdot,0) < 4s\} \cap B_{\widetilde{g}^i}(x,0,2),$$

so that (4.6.1) and $\psi_i^* \widetilde{g}^i_0 \to g_W$ in $C^\infty_{loc}(\mathcal{R}_W)$ imply

$$\begin{aligned} 2^{10n}Es^3 < |S|_{g_W} &= \lim_{i\to\infty} |\psi_i(S)|_{\widetilde{g}^i_0} \le \liminf_{i\to\infty} |\{\widetilde{r}^{\widetilde{g}^i}_{Rm}(\cdot,0) < 4s\} \cap B_{\widetilde{g}^i}(x,0,2)|_{\widetilde{g}^i_0} \\ &\le E(4s)^3 2^n < 2^{8n}Es^3, \end{aligned}$$

a contradiction.

Claim 2: Z_∞ is the length space corresponding to a smooth Riemannian manifold.

If Z_∞ is not smooth, then $\mathcal{S}(W)$ contains a subset isometric to $\mathbb{R}^{n-3} \times [0,\infty)$, so that the Hausdorff dimension of $\mathcal{S}(W)$ is at least $n-2$. To get a contradiction, it therefore suffices to prove that the Hausdorff dimension of

$$S := \mathcal{S}(W) \cap B^W(y,1)$$

is at most $n-3$ for any $y \in W$. The proof of this given Claim 1 is standard, but we include it for completeness.

Given $s \in (0,1]$, let $\{y_1,...,y_N\}$ be a maximal subset of S such that $d_W(y_i,y_j) \ge \frac{s}{2}$ for any distinct $i,j \in \{1,...,N\}$. Then $S \subset \cup_{j=1}^N B^W(y_j,s)$, and

$$\cup_{j=1}^N B^W(y_j,s) \subset \{\widetilde{r}^W_{Rm} < s\} \cap B^W(y,2)$$

since $\widetilde{r}^W_{Rm}$ is 1-Lipschitz. Thus

$$\begin{aligned} NA^{-1}s^n &\le \sum_{j=1}^N \mathcal{H}^n(B^W(y_j,s)) \le \mathcal{H}^n(\{\widetilde{r}^W_{Rm} < s\} \cap B^W(y,2)) \\ &= |\{\widetilde{r}^W_{Rm} < s\} \cap B^W(y,2) \cap \mathcal{R}_W|_{g_W} \le 2^{14n}Es^3, \end{aligned}$$

which implies that $N \le 2^{14}AEs^{3-n}$, hence

$$\mathcal{H}^{n-3}_s(S) \le CNs^{n-3} \le 2^{14n}CAE.$$

Taking $s \searrow 0$ gives $\mathcal{H}^{n-3}(S) < \infty$.

In particular, $\mathbb{R}^{n-3} \times (C(Z_\infty) \setminus \{z_\infty\})$ is a smooth 3-dimensional Riemannian cone with nonnegative Ricci curvature. By the same argument as Claim 3 of Theorem 1, we know that Z_∞ has constant curvature 1, hence that Z_∞ is the round sphere or the round $\mathbb{R}P^2$, but because the pointed Gromov-Hausdorff convergence

$$(M, d_{\widetilde{g}_0^i}, x_i) \to (\mathbb{R}^{n-3} \times C(Z_\infty), d_{\mathbb{R}^{n-3} \times C(Z_\infty)}, (0^{n-3}, z_\infty))$$

is smooth away from $\mathbb{R}^{n-3} \times \{z_\infty\}$, and because M is orientable, we must have $Z_\infty = \mathbb{S}^2$ (otherwise, there is an embedding $B(0^{n-3}, 1) \times \mathbb{R}P^2 \times (1, 2) \hookrightarrow M$, which is impossible). That is, $(M, d_{\widetilde{g}_0^i}, x_i)$ converges to flat $\mathbb{R}^n$, so $r_{Rm}^{\widetilde{g}_0^i}(x_i, 0) \to \infty$, contradicting $r_{Rm}^{\widetilde{g}_0^i}(x_i, 0) < \epsilon_i \to 0$. □

Proof of Theorem 30. The proof of (i) is a trivial modification of the proof of Proposition 22, where we use codimension three ϵ-regularity (Proposition 23) instead of codimension two (that is, we replace $\mathcal{S}^{n-3}_{\epsilon,r}$ with $\mathcal{S}^{n-4}_{\epsilon,r}$).

(ii) Replacing r with $r_0\sqrt{T-t}$ and s with $\frac{s^{-1}}{r_0\sqrt{T-t}}$ in (i), we can estimate

$$\begin{aligned}
\int_{B(x,t,r_0\sqrt{T-t})} \widetilde{r}_{Rm}^{-p}(y,t)\,dg_t(y) &\leq p\int_{r_0^{-\frac{1}{2}}(T-t)^{-\frac{1}{2}}}^{\infty} s^{p-1}|\{\widetilde{r}_{Rm}^{-1}(\cdot,t) > s\} \cap B(x,t,r_0\sqrt{T-t})|_{g_t}\,ds \\
&\quad + \left(r_0^{-\frac{1}{2}}(T-t)^{-\frac{1}{2}}\right)^p |B(x,t,r_0\sqrt{T-t})|_{g_t} \\
&\leq pE\int_{r_0^{-\frac{1}{2}}(T-t)^{-\frac{1}{2}}}^{\infty} s^{p-1}\left(\frac{s^{-1}}{r_0\sqrt{T-t}}\right)^4 \left(r_0\sqrt{T-t}\right)^n ds \\
&\quad + C(A,\underline{T})(T-t)^{\frac{n-p}{2}} \\
&\leq C(A,\underline{T},p)(T-t)^{\frac{n-p}{2}}.
\end{aligned}$$

The claim follows from a standard covering argument. □

4.7 Curvature Scale Decomposition in Dimension 4

In this section, we again specialize to dimension four, where we decompose each time slice of a Ricci flow satisfying (4.1.1),(4.1.2) according its curvature scale relative to the Type-I scale. The region where $\widetilde{r}_{Rm}(\cdot,t) << \sqrt{T-t}$ was estimated using codimension three ϵ-regularity. The region where $\widetilde{r}_{Rm}(\cdot,t) >> \sqrt{T-t}$ is estimated using the following proposition, and the intermediate region where $\widetilde{r}_{Rm}(\cdot,t) \sim \sqrt{T-t}$ is dealt with in the proof of Theorem 27.

Proposition 24. *Suppose $(M^4,(g_t)_{t\in[0,T)})$ is a closed, simply connected Ricci flow satisfying (4.1.1),(4.1.2). Then there exist $C=C(A,T)<\infty$, and $E'=E'(X)<\infty$ such that the following hold:*

(i) *For any $(x,t)\in M\times[\frac{T}{2},T)$ and $s\in[\epsilon_P^{-1}\sqrt{T-t},1]$, we have*

$$|\{\epsilon_P^{-1}\sqrt{T-t}\le \widetilde{r}^g_{Rm}(\cdot,t)<s\}|_{g_t}\le C(A,T)|\{\widetilde{r}^X_{Rm}<\epsilon_P^{-1}s\}\cap \mathcal{R}_X|_{g_X},$$

(ii) *For any $s\in(0,1]$, we have*

$$|\{\widetilde{r}^X_{Rm}<s\}\cap \mathcal{R}_X|_{g_X}\le E's^4.$$

Proof. (i) By Theorem 31,

$$\{\epsilon_P^{-1}\sqrt{T-t}<\widetilde{r}^g_{Rm}(\cdot,t)<s\}\subseteq\{x\in M\setminus\Sigma;\widetilde{r}^g_{Rm}(x,T)<\epsilon_P^{-1}s\}.$$

Moreover, for any $x\in M$ with $\widetilde{r}^g_{Rm}(x,t)\ge\epsilon_P^{-1}(T-t)$, we have $|Rm|(x,\tau)\le\frac{1}{T-\tau}$ for all $\tau\in[t,T]$, so we can integrate $\partial_\tau dg_\tau|_x\le c(T-t)^{-1}dg_\tau|_x$ from $\tau=t$ to $\tau=T$ to obtain $dg_T|_x\le c\,dg_t|_x$. Thus

$$|\{\epsilon_P^{-1}\sqrt{T-t}<\widetilde{r}^g_{Rm}(\cdot,t)<s\}|_{g_t}\le C|\{x\in M\setminus\Sigma;\widetilde{r}^g_{Rm}(x,T)<\epsilon_P^{-1}s\}|_{g_T}$$
$$=C|\{\overline{x}\in\mathcal{R}_X;\widetilde{r}^X_{Rm}(\overline{x})<\epsilon_P^{-1}s\}|_{g_X}.$$

(ii) Let $\overline{x}_1,...,\overline{x}_N\in X$ be the singular points. Fix $r_0>0$ such that $d(\overline{x}_i,\overline{x}_j)>2r_0$ for distinct $i,j\in\{1,...,N\}$. Because $X\setminus\{\overline{x}_1,...,\overline{x}_N\}$ is smooth, we can find $\sigma_0>0$ such that

$\{\widetilde{r}^X_{Rm} < \sigma_0\} \subset \bigcup_{j=1}^N B^X(\overline{x}_j, r_0)$. Moreover, by possibly shrinking r_0 (and σ_0 accordingly), we can assume $\widetilde{r}^X_{Rm} \geq c_0 d(\overline{x}_j, \cdot)$ on $B^X(\overline{x}_j, r_0)$, for $j = 1, ..., N$, and some constant $c_0 > 0$. When $s \geq \sigma_0$, we can estimate

$$|\{\widetilde{r}^X_{Rm} < s\} \cap R_X|_{g_X} \leq |R_X|_{g_X} \leq C\sigma^{-4} s^4,$$

so it suffices to consider the case where $s < \sigma_0$. Then

$$\begin{aligned} |\{\widetilde{r}^X_{Rm} < s\} \cap R_X|_{g_X} &\leq \sum_{j=1}^{N} |B^X(\overline{x}_j, c_0^{-1} s) \cap R_X|_{g_X} \\ &\leq NC(A,T) c_0^{-4} s^4. \end{aligned}$$

□

Proof of Theorem 27. Since $(M^4, (g_t)_{t \in [0, T-\tau]})$ is smooth, it suffices to prove this for $t \in [T-\tau, T)$. Let $\overline{x}_1, ..., \overline{x}_N \in X$ be the singular points of X, and choose representative points $x_1, ..., x_N \in M$.

Claim 1: For any $\alpha \in [1, \infty)$, there exists $\delta = \delta(\alpha) > 0$ such that

$$\{\widetilde{r}^g_{Rm}(\cdot, t) < \alpha\sqrt{T-t}\} \subset \cup_{j=1}^N B_g(x_j, t, \delta^{-1}\sqrt{T-t})$$

for all $t \in [T-\delta, T)$.

Assume not, so that we can find sequences $\delta_i \searrow 0$, $t_i \in [T-\delta_i, T)$, and points

$$y_i \in M \setminus \cup_{j=1}^N B_g(x_j, t, \delta_i^{-1}\sqrt{T-t})$$

satisfying

$$\frac{\widetilde{r}^g_{Rm}(y_i, t_i)}{\sqrt{T-t_i}} < \alpha.$$

Pass to a subsequence so that $y_i \to y$ in M. Because $\widetilde{r}^g_{Rm}(y_i, t_i) \to 0$, we must have $y \in \Sigma$, hence $\overline{y} \in \{\overline{x}_1, ..., \overline{x}_N\}$. By Claim 2 in the proof of Theorem 25, we get

$\limsup_{t\nearrow T} d_{g_t}(y,x)(T-t)^{-\frac{1}{2}} < \infty$ for some $x \in \{x_1,...,x_N\}$. By the choice of y_i, this implies

$$\liminf_{i\to\infty}\frac{d_{g_{t_i}}(y,y_i)}{\sqrt{T-t_i}} \geq \liminf_{i\to\infty}\left(\frac{d_{g_{t_i}}(x,y_i)-d_{g_{t_i}}(x,y)}{\sqrt{T-t_i}}\right) = \infty.$$

Let $E \geq 2$ be arbitrary. Let $\gamma : [0,1] \to M$ be any curve from y to y_i. By Claim 1 of Theorem 25, there exist $\delta = \delta(A,E) > 0$ and $r = r(A) < \infty$ such that for $t \in (T-\delta,T)$ and $y' \in B_g(y,t,2Er\sqrt{T-t}) \setminus \overline{B}_g(y,t,\frac{1}{2}r\sqrt{T-t})$, we have

$$|Rm|(y',s) \leq \frac{1}{T-t}$$

for all $s \in (T-\delta,T)$. Similarly to the proof of Theorem 25, we define

$$u^i_- := \sup\left\{u \in [0,1]; d_{g_{t_i}}(\gamma(u),y) = r\sqrt{T-t_i}\right\},$$
$$u^i_+ := \inf\left\{u \in [0,1]; d_{g_{t_i}}(\gamma(u),y) = (E+1)r\sqrt{T-t_i}\right\},$$

so that $\text{length}_{g_{t_i}}(\gamma|[u^i_-,u^i_+]) \geq Er\sqrt{T-t_i}$ for $i \in \mathbb{N}$ sufficiently large (independently of γ). Because $|Rm|_g(\gamma(u),s) \leq \frac{1}{T-t_i}$ for all $u \in [u^i_-,u^i_+]$ and $s \in [T-t_i,T)$, we get

$$\text{length}_{g_s}(\gamma|[u^i_-,u^i_+]) \geq \frac{r}{e^4}E\sqrt{T-t_i},$$

so letting $s \nearrow T$, then taking the infimum over all curves γ gives

$$d_X(\overline{y},\overline{y}_i) \geq \frac{r}{e^4}E\sqrt{T-t_i},$$

hence

$$\liminf_{i\to\infty}(T-t_i)^{-\frac{1}{2}}d_X(\overline{y},\overline{y}_i) \geq \frac{r}{e^4}E.$$

However, $E < \infty$ was arbitrary, so we have

$$\lim_{i\to\infty}(T-t_i)^{-\frac{1}{2}}d_X(\overline{y},\overline{y}_i) = \infty.$$

In particular, we can use $\liminf_{\overline{z}\to\overline{y}} d_X^{-1}(\overline{z},\overline{y})\widetilde{r}^X_{Rm}(\overline{z}) > 0$ to get

$$(T-t_i)^{-\frac{1}{2}}\widetilde{r}^X_{Rm}(\overline{y}_i) = \frac{d_X(\overline{y},\overline{y}_i)}{\sqrt{T-t_i}}\frac{\widetilde{r}^X_{Rm}(\overline{y}_i)}{d_X(\overline{y},\overline{y}_i)} \to \infty$$

as $i \to \infty$. By Theorem 31, this implies

$$\lim_{i \to \infty} \frac{\widetilde{r}^{g}_{Rm}(y_i, t_i)}{\sqrt{T - t_i}} = \infty,$$

a contradiction.

We now apply Claim 1 with $\alpha = \epsilon_P^{-1}$ as in Theorem 31. Then a standard covering argument on each ball $B(x_j, t, \delta^{-1}\sqrt{T-t})$ using volume doubling gives, for any $t \in (T-\delta, T)$, some $N_1 = N_1((g_t)_{t\in[0,T)}) \in \mathbb{N}$ (independent of t) along with points $z_t^1, ..., z_t^{N_1}$ such that

$$\{\widetilde{r}^{g}_{Rm}(\cdot, t) < \epsilon_P^{-1}\sqrt{T-t}\} \subset \bigcup_{j=1}^{N_1} B_g(z_t^j, t, r_0\sqrt{T-t}),$$

where r_0 is as in Theorem 30.

For any $s \in (0, r_0\sqrt{T-t}]$, we may thus apply Theorem 30 to get

$$\begin{aligned}
|\{\widetilde{r}^{g}_{Rm}(\cdot, t) < s\}|_{g_t} &\le \sum_{j=1}^{N_1} |\{\widetilde{r}^{g}_{Rm}(\cdot, t) < s\} \cap B_g(z_t^j, t, r_0\sqrt{T-t})|_{g_t} \\
&\le N_1 E \left(\frac{s}{r_0\sqrt{T-t}}\right)^4 \left(r_0\sqrt{T-t}\right)^4 \\
&\le N_1 E s^4.
\end{aligned}$$

If $s \in [r_0\sqrt{T-t}, \epsilon_P^{-1}\sqrt{T-t}]$, then Claim 1 gives

$$\begin{aligned}
|\{\widetilde{r}^{g}_{Rm}(\cdot, t) < s\}|_{g_t} &\le \sum_{j=1}^{N} |B_g(x_j, t, \delta^{-1}\sqrt{T-t})|_{g_t} \\
&\le C(A)\delta^{-4} N (T-t)^2 \le C(A)\delta^{-4} N r_0^{-4} s^4.
\end{aligned}$$

If $s \in [\epsilon_P^{-1}\sqrt{T-t}, \infty)$, use Proposition 24 and Claim 1 to get

$$\begin{aligned}
|\{\widetilde{r}^{g}_{Rm}(\cdot, t) < s\}|_{g_t} &\le |\{\epsilon_P^{-1}\sqrt{T-t} \le \widetilde{r}^{g}_{Rm}(\cdot, t) < s\}|_{g_t} \\
&\quad + |\{\widetilde{r}^{g}_{Rm}(\cdot, t) < \epsilon_P^{-1}\sqrt{T-t}\}|_{g_t} \\
&\le C(A)\delta^{-4} N r_0^{-4} \epsilon_P^{-4} (T-t)^2 + E' s^4 \\
&\le C((g_t)_{t\in[0,T)}) s^4.
\end{aligned}$$

□

4.8 Appendix: Gaussian Upper Bound for the Heat Kernel

The following is a modification of the proof of Theorem 3.1 in [CZ1], which now applies to manifolds with possibly negative lower Ricci curvature bounds. This estimate is similar to Theorem 1.4 of [Z7], but importantly does not depend on a bound for $\int_0^T \sup_M |Rc|(\cdot, t)dt$.

Proposition 25. *Let $(M^n, (g_t)_{t\in[0,T)})$ be a solution to the Ricci flow satisfying $Rc(g_t) \geq -Ag_t$ and $|B(x,t,r)|_{g_t} \geq A^{-1}r^n$ for all $(x,t) \in M \times [0,T)$ and $r \in (0,1]$. Then there exists $C = C(n, A, T, \mathrm{diam}_{g_0}(M)) < \infty$ such that*

$$K(x,t;y,s) \leq \frac{C}{(t-s)^{\frac{n}{2}}} \exp\left(-\frac{d^2_{g_t}(x,y)}{C(t-s)}\right)$$

for all $x, y \in M$ and $0 \leq s < t < T$.

Proof. First suppose $u \in C^\infty(M \times [t_2, t_1])$ is a positive solution of $\partial_t u = \Delta_{g_t} u$. Fix $\xi \in C^\infty(M \times [t_2, t_1])$ to be determined. Then

$$\begin{aligned} \frac{d}{dt}\int_M u^2 e^\xi dg_t &= \int_M (2u\Delta_{g_t} u + u^2 \partial_t \xi - Ru^2)e^\xi dg_t \\ &= \int_M \left(-2|\nabla u|^2_{g(t)} - 2u\langle \nabla u, \nabla \xi\rangle + u^2\partial_t \xi - Ru^2\right) e^\xi dg_t \\ &\leq \int_M \left(\frac{1}{2}|\nabla \xi|^2_{g(t)} + \partial_t \xi \right) u^2 e^\xi dg_t + nA\int_M u^2 e^\xi dg_t. \end{aligned}$$

Fix $x \in M$, and define

$$I_r(t) := \int_{M\setminus B(x,t,r)} u^2(y,t)dg_t(y).$$

Fix $t_0 \in (t_1, T)$, and define

$$\xi(y,t) := -\frac{(r - d_{g_t}(x,y))_+^2}{8(t_0 - t)},$$

for $(y,t)\in M\times[t_2,t_0)$. For any $t\in[t_2,t_1]$ where $t\mapsto d_{g_t}(x,y)$ is differentiable,

$$\begin{aligned}\partial_t d_{g_t}(x,y) =& \frac{d}{dt}\left(\inf_\gamma \int_0^l |\dot\gamma(\tau)|_{g_t} d\tau\right) = \inf_\gamma\left(-\int_0^l \frac{Rc(\dot\gamma(\tau),\dot\gamma(\tau))}{|\dot\gamma(\tau)|} d\tau\right)\\ \le& \inf_\gamma\left(A\int_0^l |\dot\gamma(\tau)|_{g_t} d\tau\right) = A d_{g_t}(x,y),\end{aligned}$$

where the infimum is taken over all g_t-minimizing geodesics $\gamma:[0,l]\to M$ from x to y. We can thus estimate (when $d_{g_t}(x,y)\le r$, otherwise everything is zero)

$$\begin{aligned}\partial_t\xi+\frac{1}{2}|\nabla\xi|^2_{g_t} =& -\frac{(r-d_{g_t}(x,y))^2}{8(t_0-t)^2}+\frac{(r-d_{g_t}(x,y))}{4(t_0-t)}\partial_t d_{g_t}(x,y)+\frac{(r-d_{g_t}(x,y))^2}{32(t_0-t)^2}\\ \le& -\frac{(r-d_{g_t}(x,y))^2}{16(t_0-t)^2}+Ar\frac{(r-d_{g_t}(x,y))}{4(t_0-t)}\\ \le& A^2r^2.\end{aligned}$$

This implies

$$\frac{d}{dt}\left(e^{-(nA+A^2r^2)t}\int_M u^2e^\xi dg_t\right)\le 0.$$

For $r>\rho>0$ and $0<t_2<t_1<t_0$, we can integrate to obtain

$$\begin{aligned}I_r(t_1) \le& \int_M u^2(y,t_1)e^{\xi(y,t_1)}dg_{t_1}(y)\\ \le& e^{(nA+A^2r^2)(t_1-t_2)}\int_M u^2(y,t_2)e^{\xi(y,t_2)}dg_{t_2}(y)\\ \le& e^{(nA+A^2r^2)(t_1-t_2)}\left(I_\rho(t_2)+\int_{B(x,t_2,\rho)}u^2(y,t_2)e^{\xi(y,t_2)}dg_{t_2}(y)\right)\\ \le& e^{(nA+A^2r^2)(t_1-t_2)}\left(I_\rho(t_2)+\exp\left(-\frac{(r-\rho)^2}{8(t_0-t_2)}\right)\int_{B(x,t_2,\rho)}u^2(y,t_2)dg_{t_2}(y)\right).\end{aligned}$$

By Lemma 13, Theorem 7.1 of [B4] gives $B = B(A,T,D)<\infty$ such that

$$K(y,t;y,s)\le\frac{B}{(t-s)^{\frac{n}{2}}}$$

for all $y\in M$ and $0\le s<t<T$, where $D:=\mathrm{diam}_{g_0}(M)$. From

$$\frac{d}{dt}\int_M u\,dg_t = -\int_M Ru\,dg_t \le nA\int_M u\,dg_t,$$

we get $\int_M u\,dg_t \le e^{nAT}\int_M u\,dg_s$, so by taking $u(y,t) := K(y,t;x,s)$, we obtain

$$\int_{B(x,t_2,\rho)} K^2(y,t_2;x,s)\,dg_{t_2}(y) \le \frac{B}{(t_2-s)^{\frac{n}{2}}}\int_{B(x,t_2,\rho)} u(y,t_2)\,dg_{t_2}(y) \le \frac{B}{(t_2-s)^{\frac{n}{2}}},$$

where $B = B(A,T,D) < \infty$. Combining estimates, and then taking $t_0 \searrow t_1$ gives

$$I_r(t_1) \le e^{(nA+A^2r^2)(t_1-t_2)}\left(I_\rho(t_2) + \frac{B}{(t_2-s)^{\frac{n}{2}}}\exp\left(-\frac{(r-\rho)^2}{8(t_1-t_2)}\right)\right).$$

Now fix $r>0$, $0\le s<t\le T$ and suppose $(r_k),(t_k)$ are decreasing sequences with $r_0=r$, $t_0=t$, $t_k \searrow s$. Then

$$I_{r_k}(t_k) \le e^{(nA+A^2r_k^2)(t_k-t_{k+1})}\left(I_{r_{k+1}}(t_{k+1}) + \frac{B}{(t_{k+1}-s)^{\frac{n}{2}}}\exp\left(-\frac{(r_k-r_{k+1})^2}{8(t_k-t_{k+1})}\right)\right),$$

so iterating and using $I_{r_k}(t_k) \to 0$ (since $K(\cdot,t;x,s) \to 0$ locally uniformly on $M\setminus\{x\}$ as $t\searrow s$) gives

$$\begin{aligned}
I_r(t) = I_{r_0}(t_0) &\le \sum_{k=0}^{\infty}\exp\left(\sum_{j=0}^{k}(nA+A^2r_j^2)(t_j-t_{j+1})\right)\frac{B}{(t_{k+1}-s)^{\frac{n}{2}}}\exp\left(-\frac{(r_k-r_{k+1})^2}{8(t_k-t_{k+1})}\right)\\
&\le c(A,T,D)e^{A^2Tr^2}\sum_{k=0}^{\infty}\frac{1}{(t_{k+1}-s)^{\frac{n}{2}}}\exp\left(-\frac{(r_k-r_{k+1})^2}{8(t_k-t_{k+1})}\right).
\end{aligned}$$

We now choose the sequences of radii and times:

$$r_k := \left(\frac{1}{2}+\frac{1}{k+2}\right)r,\ t_k := s+2^{-k}(t-s),$$

so that

$$r_k - r_{k+1} \ge \frac{r}{(k+3)^2}, \qquad t_k - t_{k+1} = 2^{-k-1}(t-s),$$

hence we have

$$\frac{(r_k-r_{k+1})^2}{8(t_k-t_{k+1})} \ge \frac{2^{k-2}}{(k+3)^4}\frac{r^2}{t-s} \ge \frac{\gamma 2^{\frac{k}{2}+1}r^2}{t-s}$$

for all $k\in\mathbb{N}$, where $\gamma>0$ is universal. Assuming $r^2 \ge t-s$, we therefore have

$$\begin{aligned}
I_r(t) &\le \frac{C(A,T,D)}{(t-s)^{\frac{n}{2}}}\exp\left(A^2Tr^2-\frac{\gamma r^2}{t-s}\right)\sum_{k=0}^{\infty}2^{nk}\exp\left(-\gamma 2^{\frac{k}{2}}\right)\\
&\le \frac{C(A,T,D)}{(t-s)^{\frac{n}{2}}}\exp\left(A^2Tr^2-\frac{\gamma r^2}{t-s}\right).
\end{aligned}$$

Now set $\eta := \frac{\gamma}{2A^2T}$. If $|t - s| < \eta$, and if $r := d_{g_t}(x,y)$ satisfies $r^2 \geq 4(t-s)$, then

$$\begin{aligned}\int_{B(y,t,\sqrt{t-s})} K^2(z,t;x,s)dg_t(z) &\leq \int_{M\setminus B(x,t,\frac{r}{2})} K^2(z,t;x,s)dg_t(z)\\ &\leq \frac{C(A,T,D)}{(t-s)^{\frac{n}{2}}}\exp\left(-\frac{\gamma}{8}\frac{d_{g_t}^2(x,y)}{t-s}\right),\end{aligned}$$

so there exists $z_0 \in B(y,t,\sqrt{t-s})$ such that

$$K^2(z_0,t;x,s)|B(y,t,\sqrt{t-s})|_{g_t} \leq \frac{C(A,T,D)}{(t-s)^{\frac{n}{2}}}\exp\left(-\frac{\gamma}{8}\frac{d_{g_t}^2(x,y)}{t-s}\right),$$

By the volume lower bound and Q. Zhang's gradient estimate (Theorem 3.3 of [Z1]) we therefore conclude

$$K(y,t;x,s) \leq \frac{C(A,T,D)}{(t-s)^{\frac{n}{2}}}\exp\left(-\frac{\gamma}{16}\frac{d_{g_t}^2(x,y)}{t-s}\right)$$

for all $x,y \in M$ and $0 \leq s < t \leq T$ with $|t-s| < \eta$ and $d_{g_t}^2(x,y) \geq 4(t-s)$. If instead $d_{g_t}^2(x,y) \leq 4(t-s)$, then the on-diagonal upper bound gives

$$K(y,t;x,s) \leq \frac{C(A,T,D)}{(t-s)^{\frac{n}{2}}} \leq \frac{C(A,T,D)}{(t-s)^{\frac{n}{2}}}\exp\left(-\frac{\gamma}{4}\frac{d_{g_t}^2(x,y)}{t-s}\right),$$

so the estimate holds in this case as well. Finally, from $\mathrm{diam}_{g_t}(M) \leq C(A,T,D)$, if $|t-s| \geq \eta$,then

$$\begin{aligned}K(y,t;x,s) &\leq \frac{C(A,T,D)}{(t-s)^{\frac{n}{2}}} \leq \frac{C(A,T,D)}{(t-s)^{\frac{n}{2}}}\exp\left(-\frac{\gamma}{4}\frac{d_{g_t}^2(x,y)}{t-s}\right)\exp\left(\frac{\gamma}{4}\frac{D^2}{\eta}\right)\\ &\leq \frac{C(A,T,D)}{(t-s)^{\frac{n}{2}}}\exp\left(-\frac{\gamma}{4}\frac{d_{g_t}^2(x,y)}{t-s}\right).\end{aligned}$$

□

CHAPTER 5

TANGENT FLOWS OF KÄHLER METRIC FLOWS

5.1 Statement of Results

Suppose (M_i^{2n}, g_i, p_i) is a sequence of pointed, complete Riemannian manifolds satisfying

$$Rc(g_i) \geq -(n-1)g_i, \tag{5.1.1}$$

$$\mathrm{Vol}(B(p_i,1)) \geq \nu, \tag{5.1.2}$$

where $\nu > 0$. Assume moreover that the sequence (M_i, d_{g_i}, p_i) converges in the pointed Gromov-Hausdorff sense to a metric space (X, d, p_∞). It was shown in [CC2] that any tangent cone based at a point in X is a metric cone. The singular strata $\mathcal{S}^k$ were defined to be set of $x \in X$ such that no tangent cone based at x is isometric to $C(Z) \times \mathbb{R}^{k+1}$, where Z is any compact metric space with diameter at most π. Moreover, the Hausdorff dimension estimates $\dim_{\mathcal{H}}(\mathcal{S}^k) \leq k$ were established.

It is natural to ask what additional properties X satisfies if (M_i, g_i) are assumed to be Kähler. Theorem 9.1 of [CCT] showed that in this case $\mathcal{S}^{2j} = \mathcal{S}^{2j+1}$ for $j = 0, ..., n-1$; roughly speaking, if a tangent cone $C(Y)$ of some $x \in X$ splits a factor of $\mathbb{R}^{2j+1}$, then it actually splits a factor of $\mathbb{R}^{2j+2}$. It was also shown in [L1] that any tangent cone $C(Y)$ of X admits a 1-parameter action by isometries, which extends to an effective isometric action by a torus. This action is used in [LS2] to obtain an embedding $C(Y) \hookrightarrow \mathbb{C}^N$ whose image is a normal affine algebraic variety, such that the action on $C(Y)$ is the restriction of a linear torus action on $\mathbb{C}^N$.

The goal of this chapter is to prove analogous results in the setting of Ricci flow. A Ricci flow analogue of the singular stratification was first introduced for Ricci flows satisfying a Type-I curvature assumption in [G1]. A version defined for general Ricci flows was later defined and studied in [B5]. To state our results more precisely, we let $(M_i^{2n}, (g_{i,t})_{t\in(-T_i,0]})$ be any sequence of Ricci flows equipped with conjugate heat kernel measures $(\nu_{x_i,0;t})_{t\in(-T_i,0]}$ based at $(x_i, 0)$, where $T_\infty := \lim_{i\to\infty} T_i \in (0, \infty]$ and $\mathcal{N}_{x_i,0}(1) \geq -Y$ for some $Y < \infty$. In [B4, B3, B5], Bamler establishes a Ricci flow version of Gromov's compactness theorem and Cheeger-Colding theory, where $\mathbb{F}$-convergence takes the role of pointed Gromov-Hausdorff, and the volume noncollapsing assumption is replaced by the assumed lower bound for Nash entropy. We will review related definitions in Section 2. After passing to a subsequence, we can suppose that

$$(M_i^{2n}, (g_{i,t})_{t\in(-T_i,0]}, (\nu_{x_i,0;t})_{t\in(-T_i,0]}) \xrightarrow[i\to\infty]{\mathbb{F}} (\mathcal{X}, (\nu_{x_\infty;t})_{t\in(-T_\infty,0]})$$

uniformly on compact time intervals, for some future-continuous metric flow $\mathcal{X}$ of full support over $(-T_\infty, 0]$. There is stratification $\mathcal{S}^0 \subset \cdots \subset \mathcal{S}^{2n-2} = \mathcal{S}$ of the singular set $\mathcal{S}$ of $\mathcal{X}$ analogous to that of Ricci limit spaces (see Section 2 for details). Our first main theorem can then be stated as follows.

Theorem 35. *If $(M_i, (g_{i,t})_{t\in(-T_i,0]})$ are Kähler-Ricci flows, then $\mathcal{S}^{2j+1} = \mathcal{S}^{2j}$ for $j = 0, ..., n-1$.*

For applications to smooth Ricci flows, it is often useful to study the quantitative stratification. A similar stratification was studied for Riemannian manifolds satisfying (5.1.1),(5.1.2) in [CN2], where it was used to prove L^p estimates for the Riemannian curvature tensor of Einstein manifolds. The Ricci flow version was again first studied for Type-I Ricci flows [G2], while two different but related definitions for general Ricci flows were used in [B5]. These are the quantitative strata $\mathcal{S}^{\epsilon,k}_{r_1,r_2}$ and the weak quantitative strata $\widehat{\mathcal{S}}^{\epsilon,k}_{r_1,r_2}$. Our next result is a quantitative form of Theorem 35, from which Theorem 35 is easily derived. We note that our definition of quantitative strata $\mathcal{S}^{\epsilon,k}_{r_1,r_2}$ is slightly more restrictive than the definition given in [B5] (see Section 2).

Theorem 36. *For any $\epsilon > 0$ and $Y, A < \infty$, there exists $\delta = \delta(\epsilon, Y, A) > 0$ such that for all*

$r_2 > r_1 \geq 0$ *and* $j \in \{0, ..., n-1\}$*, we have*

$$\mathcal{S}^{\epsilon,2j+1}_{r_1,r_2} \cap P^*(x_\infty; A, -A^2) \subseteq \mathcal{S}^{\delta,2j}_{r_1,r_2} \cap P^*(x_\infty; A, -A^2),$$

$$\widehat{\mathcal{S}}^{\epsilon,2j+1}_{r_1,r_2} \cap P^*(x_\infty; A, -A^2) \subseteq \widehat{\mathcal{S}}^{\delta,2j}_{r_1,r_2} \cap P^*(x_\infty; A, -A^2).$$

We observe that unlike Theorem 35, Theorem 36 applies directly to smooth Kähler-Ricci flows, since the quantitative strata are generally nonempty even for smooth flows.

Next, we consider a fixed point $x_0 \in \mathcal{X}_{t_0}$ with $t_0 < 0$, and consider a sequence of parabolic rescalings

$$(\mathcal{X}^{-t_0,\lambda_k}, (\nu^{-t_0,\lambda_k}_{x_0;t})_{t\in[-\lambda_k^2(t_0-T_\infty),0]}),$$

where $\lambda_k \nearrow \infty$. After passing to a subsequence, we can assume $\mathbb{F}$-convergence

$$(\mathcal{X}^{-t_0,\lambda_k}, (\nu^{-t_0,\lambda_k}_{x_0;t})_{t\in[-\lambda_k^2(t_0-T_\infty),0]}) \xrightarrow[i\to\infty]{\mathbb{F}} (\mathcal{Y}, (\nu_{y_\infty;t})_{t\in(-\infty,0]}),$$

where $\mathcal{X}^{-t_0,\lambda_k}$ is the metric flow $\mathcal{X}$ with a time translation by $-t_0$ and a parabolic rescaling by λ_k, and $\mathcal{Y}$ is a metric soliton modeled on a singular shrinking Kähler-Ricci soliton $(Y, d_Y, \mathcal{R}_Y, g_Y, f)$ with singularities of codimension four (see Section 2). We now give an analogue of Liu's construction [L1] of a 1-parameter isometric action in the setting of these singular solitons.

Theorem 37. *If* f *is not constant, then* (Y, d_Y) *admits a nontrivial 1-parameter action by isometries* $(\sigma_s)_{s\in\mathbb{R}}$ *which preserves* $\mathcal{R}_Y$*. Moreover, the infinitesimal generator of the restriction to* $\mathcal{R}_Y$ *is* $J\nabla f$*. If in addition* $Rc(g_Y) = 0$*, then* Y *is a metric cone (by* [B5]*), and the action restricted to the cone link is locally free.*

Taking the closure of this 1-parameter subgroup would induce a faithful action of a torus on Y, if we knew that the isometry group of (Y, d_Y) is a Lie group – this holds for Ricci limit spaces by [CC3, CN4], and it is likely the arguments can be extended to certain $\mathbb{F}$-limits of Ricci flows (c.f. Remark 2.7 of [B5]). However, it is currently uncertain whether the arguments of [LS2] can be adapted to show that every tangent cone of a Kähler-Ricci flow is an affine variety. This is because the proof of Lemma 2.2 in [LS2] relied on sharp estimates (see [JN1]) for the size of singular sets of Gromov-Hausdorff limits of manifolds satisfying (5.1.2) and a two-sided Ricci curvature bound. The analogous estimates for $\mathbb{F}$-limits of noncollapsed Ricci flows are so far unavailable.

The basic idea for proving Theorems 35 and 36 is similar to that in the case of Ricci-limit spaces [CCT]. We first consider the model case where (M^{2n}, g, J, f) is a smooth, complete gradient Kähler-Ricci soliton which isometrically splits a factor of $\mathbb{R}$: $(M, g, f) = (M' \times \mathbb{R}, g' + dy^2, f' + \frac{y^2}{4})$, where (M', g', f') is a gradient Ricci soliton of dimension $2n - 1$. Then $z := 2\langle \nabla f, J \nabla y \rangle$ satisfies

$$\nabla z = J \nabla y - 2Rc(J \nabla y) = J \nabla y,$$

since $Rc(J \nabla y) = JRc(\nabla y) = 0$. In particular, ∇z is parallel and pointwise orthogonal to ∇y. It follows that z is a coordinate for another factor of $\mathbb{R}$ split by (M, g).

We now outline the steps we will take to implement this idea in the singular setting:

- We show that any point $y \in X$ close in $\mathbb{F}$-distance to a metric soliton which either splits $\mathbb{R}^{2k+1}$ or is static and splits a factor of $\mathbb{R}^{2k-1}$ is well-approximated by sequences of points $(y_i, t_i) \in M_i \times (-T_i, 0]$ which are $(\epsilon'(\epsilon), r_i)$-selfsimilar for some $r_i \in [r_1, r_2]$, and either $(2k+1, \epsilon'(\epsilon), r_i)$-split, or else $(\epsilon'(\epsilon), r_i)$-static and $(2k-1, \epsilon'(\epsilon), r_i)$-split, where $\lim_{\epsilon \to 0} \epsilon'(\epsilon) = 0$. The converse was shown in [B5], so it suffices to consider only the weak quantitative strata.

- We construct a parabolic regularization q associated to the conjugate heat kernel based at any almost-selfsimilar point (x_0, t_0), which satisfies estimates similar to $4\tau(f - W)$, where f is the potential function for a shrinking GRS.

- Given a strong $(2k+1, \epsilon, r)$-splitting map $y = (y_1, ..., y_{2k+1})$ based at an almost-selfsimilar point (x_0, t_0), we show that the functions

$$z_i := \frac{1}{2}\langle \nabla q, J \nabla y_i \rangle$$

are almost-splitting functions whose gradients are almost-orthogonal to those of y_i, and use appropriate linear combinations of these functions and y_i to conclude that (x_0, t_0) is $(2k+2, \epsilon'(\epsilon), r)$-split.

- The previous step gives $\widehat{S}^{\epsilon'(\epsilon),2k+1}_{r_1,r_2} \subseteq \widehat{S}^{\epsilon,2k}_{r_1,r_2}$, hence

$$S^{2k+1} = \bigcup_{\epsilon \in (0,1)} \widehat{S}^{\epsilon'(\epsilon),2k+1}_{0,\epsilon} \subseteq \bigcup_{\epsilon \in (0,1)} \widehat{S}^{\epsilon,2k}_{0,\epsilon} = S^{2k}.$$

Remark 19. *Because tangent flows correspond to singular Ricci solitons which are smooth outside a subset of codimension four, it may be shown directly that the gradient flow of* $z := 2\langle \nabla f, J \nabla y_i \rangle$ *starting at almost every point in the regular set is well-defined for all time. However, it does not seem possible to show directly (without considering smooth, closed Ricci flow approximants) that the flow of* ∇z *on the regular set is complete, since the available* ϵ*-regularity theorem proved in* [B4, B5] *uses Nash entropy rather than volume ratio. While it is straightforward to show that volume and distance are preserved by the flow of* ∇z*, it is harder to show that the Nash entropy is preserved. For this reason, we employ Bamler's strategy of establishing an infinitesimal symmetry (see Theorem 15.50 in* [B5]*) of the heat kernel on the singular soliton using smooth Ricci flow approximants to prove the completeness of* ∇z*. Then Theorem 15.29 of* [B5] *gives the improved splitting of the singular soliton as a metric flow, which is the desired result.*

In Section 2, we review definitions from [B3, B5] relevant to our methods and results. In Section 3, we construct a parabolic regularization of approximate Ricci soliton potentials. In Section 4, we use these regularizations to construct almost-splitting maps on Kähler-Ricci flows, and finish the proof of Theorems 35 and 36. In Section 5, we construct isometric actions on tangent flows, and prove Theorem 37. In Section 6, we show that if a limiting metric soliton isometrically splits a factor of $\mathbb{R}^k$, this can be used to find almost-split points in the approximating Ricci flows, which completes the proof of Theorem 35.

5.2 Strong Almost-GRS Potentials

In this section, we construct parabolic regularizations of potential functions associated to conjugate heat kernels based at almost-selfsimilar points of a Ricci flow. These functions still satisfy the almost-soliton identities, but also satisfy additional estimates which will be useful in Section 5.

Definition 33. *A strong (ϵ, r)-soliton potential based at $(x_0, t_0) \in M \times I$ is a function $h \in C^\infty(M \times [t_0 - \epsilon^{-1}r^2, t_0 - \epsilon r^2])$ such that if $W := \mathcal{N}_{x_0,t_0}(r^2)$, then*

(i) $\square(4\tau(h - W)) = -2n$,

(ii) $r^{-2}\displaystyle\int_{t_0-\epsilon^{-1}r^2}^{t_0-\epsilon r^2}\int_M \left|\tau(R + |\nabla h|^2) - (h - W)\right| d\nu_t dt \leq \epsilon,$,

(iii) $\displaystyle\int_{t_0-\epsilon^{-1}r^2}^{t_0-\epsilon r^2}\int_M \tau\left|Rc + \nabla^2 h - \frac{1}{2\tau}g\right|^2 d\nu_t dt \leq \epsilon,$

(iv) $\displaystyle\sup_{t\in[t_0-\epsilon^{-1}r^2, t_0-\epsilon r^2]}\int_M \left|\tau(R + 2\Delta h - |\nabla h|^2) + h - n - W\right| d\nu_t \leq \epsilon,$

(v) $\displaystyle\int_M \left(h - \frac{n}{2}\right) d\nu_t = W$ *for all* $t \in [t_0 - \epsilon^{-1}r^2, t_0 - \epsilon r^2]$.

The following proposition is an analogue of Bamler's construction (Theorem 12.1 of [B5]) of strong almost-splitting maps which approximate weak almost-splitting maps.

Proposition 26. *For any $\epsilon > 0$, $Y < \infty$, the following holds whenever $\delta \leq \overline{\delta}(\epsilon, Y)$. Suppose $(M^n, (g_t)_{t\in[-T,0]})$ is a closed Ricci flow with $\mathcal{N}_{x_0,t_0}(r^2) \geq -Y$. Assume $(x_0, t_0) \in M \times I$ is (δ, r)-selfsimilar, and set $q := 4\tau(f - W)$, where $d\nu = d\nu_{x_0,t_0} = (4\pi\tau)^{-\frac{n}{2}}e^{-f}dg$ and $W := \mathcal{N}_{x_0,t_0}(r^2)$. Then there exists a function $q' \in C^\infty(M \times [t_0 - \epsilon^{-1}r^2, t_0 - \epsilon r^2])$ such that $f' := \frac{1}{4\tau}q' + W$ is a strong (ϵ, r)-soliton potential based at (x_0, t_0), and*

$$\int_{t_0-\epsilon^{-1}r^2}^{t_0-\epsilon r^2}\int_M |\nabla(f - f')|^2 d\nu_t dt + \sup_{t\in[t_0-\epsilon^{-1}r^2, t_0-\epsilon r^2]}\int_M (f - f')^2 d\nu_t \leq \epsilon. \tag{5.2.1}$$

Proof. Without loss of generality, we can assume $r = 1$ and $t_0 = 0$. Bamler's on-diagonal upper bounds for the heat kernel (Theorem 7.1 in [B4]) imply $f \le \Lambda(Y)$ on $M \times [-\epsilon^{-1}, -\epsilon]$ if $\delta \le \overline{\delta}$. Fix $Z \in (\Lambda, \infty)$ to be determined, and let $\chi \in C^\infty(\mathbb{R})$ be such that $\chi(s) = s$ for $s \in (-\infty, \frac{1}{2}Z]$, $|\chi'| \le 2$, $|\chi''| \le 10Z^{-1}$, and $\chi(s) = Z$ for all $s \in [Z, \infty)$. Set $\widetilde{q} := \chi \circ q$, which satisfies $\widetilde{q} \le 4\tau(\Lambda + Y)$.

Step 1: (Bound the truncation errors) We first apply Proposition 6.5 of [B5] to obtain

$$\nu_t(\{q \ge Z\}) \le e^{-\frac{Z}{8\tau}} \int_{\{q \ge Z\}} e^{\frac{q}{8\tau}} d\nu_t \le C(Y) e^{-\frac{Z}{8\tau}} \int_{\{q \ge Z\}} e^{\frac{1}{2}f} d\nu_t \le C(Y) e^{-\frac{Z}{8\tau}}$$

for all $t \in [-\frac{1}{2}\delta^{-1}, -\delta]$. Thus, for any $p \in [1, \infty)$,

$$\begin{aligned}
\int_{\{q \ge Z\}} q^p d\nu_t &= \int_{\{q \ge Z\}} \left(p \int_0^{q(x)} r^{p-1} dr \right) d\nu_t(x) = p \int_0^\infty \int_{\{q \ge Z\}} 1_{\{r \le q(x)\}} r^{p-1} d\nu_t(x) dr \\
&= p \int_Z^\infty r^{p-1} \nu_t(\{q \ge r\}) dr + p Z^p \nu_t(\{q \ge Z\}) \\
&\le C(Y,p) \int_Z^\infty r^{p-1} e^{-\frac{r}{8\tau}} dr + C(Y,p) Z^p e^{-\frac{Z}{8\tau}}
\end{aligned}$$

In particular, we have

$$\int_{\{q \ge Z\}} q^p d\nu_t \le C(Y,p,\epsilon) e^{-\frac{\epsilon Z}{10}}.$$

for all $t \in [-10\epsilon^{-1}, -\frac{1}{10}\epsilon]$. Using (2.7.1), we can estimate

$$\begin{aligned}
\int_{-10\epsilon^{-1}}^{-\frac{1}{10}\epsilon} \int_M |\nabla (q - \widetilde{q})|^2 d\nu_t dt &\le 2 \int_{-10\epsilon^{-1}}^{-\frac{1}{10}\epsilon} \int_{\{q \ge Z/2\}} |\nabla q|^2 d\nu_t dt \\
&\le \left(\int_{-10\epsilon^{-1}}^{-\frac{1}{10}\epsilon} \int_M |\nabla q|^4 d\nu_t dt \right)^{\frac{1}{2}} \left(\int_{-10\epsilon^{-1}}^{-\frac{1}{10}\epsilon} \nu_t(\{q \ge Z/2\}) dt \right)^{\frac{1}{2}} \\
&\le C(Y,\epsilon) e^{-\frac{\epsilon Z}{20}}.
\end{aligned}$$

Next, we estimate

$$\int_M (q - \widetilde{q})^2 d\nu_t \le \int_{\{q \ge Z/2\}} 2q^2 d\nu_t \le C(Y,\epsilon) e^{-\frac{\epsilon Z}{10}}$$

for any $t \in [-10\epsilon^{-1}, -\frac{1}{10}\epsilon]$, and apply (2.7.1) to obtain

$$\int_{-10\epsilon^{-1}}^{-\frac{1}{10}\epsilon}\int_M |\Delta(q-\widetilde{q})|d\nu_t dt \le \int_{-10\epsilon^{-1}}^{-\frac{1}{10}\epsilon}\int_M |1-(\chi' \circ q)|\,|\Delta q| d\nu_t dt + \int_{-10\epsilon^{-1}}^{-\frac{1}{10}\epsilon}\int_M |\chi'' \circ q|\,|\nabla q|^2 d\nu_t dt$$
$$\le \left(\int_{-10\epsilon^{-1}}^{-\frac{1}{10}\epsilon}\int_M \left(4|\Delta q|^2 + |\nabla q|^4\right) d\nu_t dt\right)^{\frac{1}{2}} \left(\int_{-10\epsilon^{-1}}^{-\frac{1}{10}\epsilon} \nu_t(\{q \ge Z/2\})dt\right)^{\frac{1}{2}}$$
$$\le C(Y,\epsilon)e^{-\frac{\epsilon Z}{20}}.$$

Step 2: (Estimate the error by parabolic regularization) By Step 1, we can choose $t^* \in [-2\epsilon^{-1}, -2\epsilon^{-1}+1]$ such that

$$\int_M |\nabla(q-\widetilde{q})|^2 d\nu_{t^*} + \int_M |q-\widetilde{q}|^2 d\nu_{t^*} + \int_M |\Delta(q-\widetilde{q})| d\nu_{t^*} \le C(Y,\epsilon)e^{-\frac{\epsilon Z}{20}}.$$

Let $q' \in C^\infty(M \times [t^*, 0])$ solve $\square q' = -2n$, with $q'(\cdot, t^*) := \widetilde{q}$. Because $-\Lambda \le \widetilde{q} \le Z$, the maximum principle gives $-8\epsilon^{-1}(\Lambda+Y) - 4n\epsilon^{-1} \le q' \le Z$ on $M \times [t^*, 0]$, so if $Z \ge 8\epsilon^{-1}(\Lambda+Y)$, then (2.7.2) and the almost-selfsimilar inequalities (Definition 17(i)) imply

$$\frac{d}{dt}\int_M (\widetilde{q}-q')^2 d\nu_t = 2\int_M (\widetilde{q}-q')(\square\widetilde{q}+2n)d\nu_t - 2\int_M |\nabla(\widetilde{q}-q')|^2 d\nu_t$$
$$\le C(\epsilon,Y)Z\int_M \left(|\square q + 2n| + |1-\chi' \circ q| + (\chi'' \circ q)|\nabla q|^2\right) d\nu_t$$
$$- 2\int_M |\nabla(\widetilde{q}-q')|^2 d\nu_t$$
$$\le C(\epsilon,Y)Z\nu_t(\{q \ge Z/2\}) + (\nu_t(\{q \ge Z/2\}))^{\frac{1}{2}}\left(\int_M |\nabla q|^4 d\nu_t\right)^{\frac{1}{2}}$$
$$+ \Psi(\delta|\epsilon, Y, Z) - 2\int_M |\nabla(\widetilde{q}-q')|^2 d\nu_t$$

so we may integrate in time, using Step 1, Hölder's inequality, and (2.7.1) to obtain

$$\sup_{t \in [t^*, -\frac{1}{10}\epsilon]}\int_M (\widetilde{q}-q')^2 d\nu_t + \int_{t^*}^{-\frac{1}{10}\epsilon}\int_M |\nabla(\widetilde{q}-q')|^2 d\nu_t dt \le \Psi(\delta|Y,\epsilon,Z) + C(Y,\epsilon)Ze^{-\frac{\epsilon Z}{20}}.$$

Combining this with the estimates of Step 1 gives

$$\sup_{t \in [t^*, -\frac{1}{10}\epsilon]}\int_M (q-q')^2 d\nu_t + \int_{t^*}^{-\frac{1}{10}\epsilon}\int_M |\nabla(q-q')|^2 d\nu_t dt \le \Psi(\delta|Y,\epsilon,Z) + C(Y,\epsilon)Ze^{-\frac{\epsilon Z}{20}},$$

hence (5.2.1) holds if we choose $Z \geq \underline{Z}(\epsilon, Y)$ and then $\delta \leq \overline{\delta}(\epsilon, Y, Z)$.

Step 3: (Show f' is an almost-soliton potential function) We now consider a quantity analogous to Perelman's differential Harnack quantity:

$$w' := \tau\left(R + \frac{1}{2\tau}\Delta q' - \frac{1}{16\tau^2}|\nabla q'|^2\right) + \frac{1}{4\tau}q' - n + W,$$

so that

$$\begin{aligned} \square(16\tau(w' - W)) &= -\square\left(16\tau^2 + 8\tau\Delta q' - |\nabla q'|^2 + 4q' - 16\tau n\right) \\ &= 32\tau^2\left|Rc + \frac{1}{4\tau}\nabla^2 q' - \frac{1}{2\tau}g\right|^2. \end{aligned} \tag{5.2.2}$$

We can write

$$\begin{aligned} w' - W &= (w' - w) + (w - W) \\ &= \frac{1}{2}\Delta(q' - q) - \frac{1}{16\tau}\left(|\nabla q'|^2 - |\nabla q|^2\right) + \frac{1}{4\tau}(q' - q) + (w - W). \end{aligned}$$

By Step 1, we can estimate

$$\int_M \left||\nabla q'|^2 - |\nabla q|^2\right| d\nu_t \leq C\int_{\{q \geq Z/2\}} |\nabla q|^2 d\nu_t \leq C(Y,\epsilon)e^{-\frac{\epsilon Z}{20}},$$

and so (also using the almost-selfsimilar identities)

$$\int_M 16\tau|w' - W|d\nu_t \leq C(Y,\epsilon)e^{-\frac{\epsilon Z}{20}} + C(\epsilon)\delta. \tag{5.2.3}$$

Using (2.7.1) and Step 2, we have

$$\begin{aligned} \int_{t}^{-\frac{\epsilon}{2}}\int_M \left||\nabla q'|^2 - |\nabla q|^2\right| d\nu_t dt &\leq \int_{t}^{-\frac{\epsilon}{2}}\int_M |\langle\nabla(q' - q), \nabla(q' - q) + 2\nabla q\rangle| d\nu_t dt \\ &\leq \int_{t}^{-\frac{\epsilon}{2}}\int_M |\nabla(q' - q)|^2 d\nu_t dt + C(Y,\epsilon)\left(\int_{t}^{-\frac{\epsilon}{2}}\int_M |\nabla(q' - q)|^2 d\nu_t dt\right)^{\frac{1}{2}} \\ &\leq \Psi(\delta|\epsilon, Y, Z) + C(Y,\epsilon)Ze^{-\frac{\epsilon Z}{40}}, \end{aligned} \tag{5.2.4}$$

$$\int_{t}^{\ \frac{\epsilon}{2}} \left| \int_M \Delta(q^0 \quad q) d\nu_t \right| dt = \int_{t}^{\ \frac{\epsilon}{2}} \left| \int_M \langle \nabla f, \nabla (q^0 \quad q) \rangle d\nu_t \right| dt \tag{5.2.5}$$

$$\left(\int_{t}^{\ \frac{\epsilon}{2}} \int_M |\nabla f|^2 d\nu_t dt \right)^{\frac{1}{2}} \left(\int_{t}^{\ \frac{\epsilon}{2}} \int_M |\nabla (q^0 \quad q)|^2 d\nu_t dt \right)^{\frac{1}{2}}$$

$$\Psi(\delta|\epsilon, Y, Z) + C(Y,\epsilon) Z e^{\ \frac{\epsilon Z}{40}}.$$

Choose a cutoff function $\zeta \in C^{\infty}([t\ ,\ \ \frac{1}{2}\epsilon])$ satisfying $\zeta(t)|[t\ ,\ \ \epsilon] \quad 1$, $|\zeta^0| \quad 4\epsilon^{\ 1}$, and $\zeta(\ \ \frac{\epsilon}{2}) = 0$. Then (5.2.2) implies

$$\int_{t}^{\ \frac{\epsilon}{2}} \zeta \frac{d}{dt} \int_M \tau(w^0 \quad W) d\nu_t dt = 2 \int_{t}^{\ \frac{\epsilon}{2}} \int_M \zeta \tau^2 \left| Rc + \frac{1}{4\tau} \nabla^2 q^0 \quad \frac{1}{2\tau} g \right|^2 d\nu_t dt. \tag{5.2.6}$$

On the other hand, integration by parts gives

$$\int_{t}^{\ \frac{\epsilon}{2}} \zeta \frac{d}{dt} \int_M \tau(w^0 \quad W) d\nu_t dt = \quad \int_{t}^{\ \frac{\epsilon}{2}} \zeta^0 \int_M \tau(w^0 \quad W) d\nu_t dt \quad \int_M \tau(w^0 \quad W) d\nu_t\ . \tag{5.2.7}$$

By combining (5.2.6),(5.2.7) with estimates (5.2.3),(5.2.4),(5.2.5), and Steps 1,2, we obtain

$$\int_{t}^{\ \epsilon} \int_M \tau^2 \left| Rc + \frac{1}{4\tau} \nabla^2 q^0 \quad \frac{1}{2\tau} g \right|^2 d\nu_t dt \quad \Psi(\delta|\epsilon, Y, Z) + C(Y,\epsilon) Z e^{\ \frac{\epsilon Z}{40}}. \tag{5.2.8}$$

Thus, property (iii) of strong almost-soliton potential functions holds if we choose $Z \quad \underline{Z}(\epsilon, Y)$ and $\delta \quad \overline{\delta}(\epsilon, Y, Z)$. In the sense of distributions,

$$|\tau(w^0 \quad W)| \quad 2\tau^2 \left| Rc + \frac{1}{4\tau} \nabla^2 q^0 \quad \frac{1}{2\tau} g \right|^2,$$

so we can use (5.2.8) to estimate

$$\int_M |\tau(w^0 \quad W)| d\nu_t \quad \int_M |\tau(w^0 \quad W)| d\nu_t \quad = \int_{t}^{t} \int_M \quad |\tau(w^0 \quad W)| d\nu_s ds$$

$$\Psi(\delta|Y, \epsilon, Z) + C(Y,\epsilon) Z e^{\ \frac{\epsilon Z}{40}}$$

for all $t \in [t\ ,\ \ \epsilon]$. Combining this with (5.2.3) gives property (iv) of strong almost-soliton functions if we choose $Z \quad \underline{Z}(\epsilon, Y)$ and $\delta \quad \overline{\delta}(\epsilon, Y, Z)$, while (ii) follows by combining $(iii), (iv)$. To verify (v), we note that

$$\frac{d}{dt} \left(\tau \int_M \left(f^0 \quad \frac{n}{2} \right) d\nu_t \quad \tau W \right) = \frac{1}{4} \int_M (\quad q + 2n) d\nu_t = 0,$$

and moreover

$$\left|\int_M \left(f' - \frac{n}{2}\right) d\nu_{-1} - W\right| \leq \int_M |q' - q| d\nu_{-1} \leq \Psi(\delta|Y, \epsilon) + C(Y, \epsilon) Z e^{-\frac{\epsilon Z}{40}},$$

so we can add a small constant to q to obtain f' satisfying (v), without affecting properties $(i) - (iv)$ or (5.2.1) □

In the case where (x_0, t_0) is also almost-static, the scalar and Ricci curvature terms are small, and $4\tau(h - W)$ is a regularization of Bamler's almost-radial function (see Proposition 13.1 of [B5] when $k = 0$).

Definition 34. *A strong (ϵ, r)-radial function based at $(x_0, t_0) \in M \times I$ is a function $h \in C^\infty(M \times [t_0 - \epsilon^{-1} r^2, t_0 - \epsilon r^2])$ such that if $W := \mathcal{N}_{x_0,t_0}(r^2)$, then*

(i) $\square q = -2n$,

(ii) $r^{-4} \int_{t_0 - \epsilon^{-1} r^2}^{t_0 - \epsilon r^2} \int_M \left| |\nabla q|^2 - 4q \right| d\nu_t dt \leq \epsilon$,

(iii) $r^{-2} \int_{t_0 - \epsilon^{-1} r^2}^{t_0 - \epsilon r^2} \int_M |\nabla^2 q - 2g|^2 d\nu_t dt \leq \epsilon$,

(iv) $\int_M q d\nu_t = 2n\tau$ *for all* $t \in [t_0 - \epsilon^{-1} r^2, t_0 - \epsilon r^2]$.

Given these definitions, we can rephrase Proposition 26, and give a criterion for the existence of strong (ϵ, r)-radial functions. Moreover, we will establish slightly improved estimates, which will be useful for the proof of Theorem 37.

Proposition 27. *For any $\epsilon > 0$, $Y < \infty$ and $p \in [1, \infty)$, the following holds if $\delta \leq \overline{\delta}(\epsilon, Y, p)$ and $\alpha \leq \overline{\alpha}(\epsilon, Y)$. Suppose $(M^n, (g_t)_{t \in I}, (x_0, t_0))$ is a closed, pointed Ricci flow satisfying $\mathcal{N}_{x_0,t_0}(r^2) \geq -Y$. Assume (x_0, t_0) is (δ, r)-selfsimilar and h is a strong (δ, r)-soliton potential. Then*

$$\sup_{t \in [t_0 - \epsilon^{-1} r^2, t_0 - \epsilon r^2]} \int_M |\nabla h|^p d\nu_t \leq C(Y, \epsilon, p), \tag{5.2.9}$$

$$\int_{t_0-\epsilon^{-1}r^2}^{t_0-\epsilon r^2}\int_M \left(\tau\left|Rc+\nabla^2 h-\frac{1}{2\tau}g\right|^2+r^{-2}\left|\tau(R+|\nabla h|^2)-(h-W)\right|\right)e^{\alpha f}d\nu_t dt\le\epsilon. \tag{5.2.10}$$

If in addition (x_0,t_0) *is* (δ,r)*-static, then* $q:=4\tau(h-W)$ *is a strong* (ϵ,r)*-radial function satisfying*

$$\int_{t_0-\epsilon^{-1}r^2}^{t_0-\epsilon r^2}\int_M \left(r^{-2}|\nabla^2 q-2g|^2+r^{-4}\left||\nabla q|^2-4q\right|\right)e^{\alpha f}d\nu_t dt\le\epsilon.$$

Proof. By time translation and parabolic rescaling, we can assume $r=1$ and $t_0=0$. Fixing $T\in[\epsilon^{-1},\delta^{-1}]$, we use properties $(ii),(v)$ of strong almost-soliton potentials to get

$$\int_{-T}^{-\epsilon}\int_M|\nabla q|^2 d\nu_t dt\le\int_{-T}^{-\epsilon}\int_M 16\tau\left(\tau(R+|\nabla h|^2)-(h-W)\right)d\nu_t dt+16T^3\delta^{-1}+\int_{-T}^{-\epsilon}\int_M 4q\,d\nu_t dt$$
$$\le 16T\delta+16T^3\delta+8nT^2\le 10nT^2$$

assuming $\epsilon\le\overline{\epsilon}$. If we choose $T=2p\epsilon^{-1}$, then we can therefore find $\widehat{t}\in[-2p\epsilon^{-1}-1,-2p\epsilon^{-1}]$ such that

$$\int_M|\nabla q|^2 d\nu_{\widehat{t}}\le 10nT^2,$$

so the hypercontractivity of the heat kernel (Theorem 12.1 in [B4]) gives

$$\sup_{t\in[-2\epsilon,-\frac{\epsilon}{2}]}\int_M|\nabla q|^p d\nu_t\le CT^2=C(p,\epsilon).$$

By Cauchy's inequality, (5.2.10) will follow from

$$\int_{-\epsilon^{-1}}^{-\epsilon}\int_M\left(\tau\left|Rc+\nabla^2 h-\frac{1}{2\tau}g\right|^2+|\tau(R+|\nabla h|^2)-(h-W)|\right)e^{\alpha f}d\nu_t dt\le C(Y,\epsilon)$$

if $\alpha\le\overline{\alpha}(\epsilon,Y)$. Fix a cutoff function $\zeta\in C^\infty([-2\epsilon^{-1},-\frac{1}{2}\epsilon])$ such that $\zeta(-2\epsilon^{-1})=\zeta(-\frac{1}{2}\epsilon)=0$, $\zeta|[-\epsilon^{-1},-\epsilon]\equiv 1$, and $|\zeta'|\le 4\epsilon^{-1}$. We compute (recalling the definition of w' from the proof of Proposition 26)

$$\frac{d}{dt}\int_M\tau(w'-W)e^{\alpha f}=\int_M\square\left(\tau(w'-W)\right)e^{\alpha f}d\nu_t-\tau\int_M(w'-W)\square^*\left((4\pi\tau)^{-\frac{n}{2}}e^{-(1-\alpha)f}\right)dg_t$$
$$=2\tau^2\int_M\left|Rc+\nabla^2 h-\frac{1}{2\tau}g\right|^2 e^{\alpha f}d\nu_t$$
$$+\alpha\tau^2\int_M(w'-W)\left(R+(1-\alpha)|\nabla f|^2-\frac{n}{2\tau}\right)e^{\alpha f}d\nu_t.$$

Multiplying both sides by ζ and integrating, then rearranging gives (assuming $\alpha \le \overline{\alpha}(Y,\epsilon)$)

$$\int_{-\epsilon^{-1}}^{-\epsilon}\int_M \tau^2 \left|Rc + \nabla^2 h - \frac{1}{2\tau}g\right|^2 e^{\alpha f} d\nu_t \le C(Y,\epsilon)\left(\int_{-2\epsilon^{-1}}^{-\frac{1}{2}\epsilon}\int_M |w' - W|^2 d\nu_t dt\right)^{\frac{1}{2}}$$
$$+ C(Y,\epsilon)\left(\int_{-2\epsilon^{-1}}^{-\frac{1}{2}\epsilon}\int_M |w' - W|^2 d\nu_t dt\right)^{\frac{1}{2}}\left(\int_{-2\epsilon^{-1}}^{-\frac{1}{2}\epsilon}\int_M (R^2 + |\nabla f|^4 + 1)e^{2\alpha f} d\nu_t dt\right)^{\frac{1}{2}}.$$

The L^2 Poincare inequality and property (iv) of Definition 4.3 give

$$\int_{-2\epsilon^{-1}}^{-\frac{1}{2}\epsilon}\int_M |w' - W|^2 d\nu_t dt \le C(Y,\epsilon)\int_{-2\epsilon^{-1}}^{-\frac{1}{2}\epsilon}\int_M \left(R^2 + |\nabla h|^4 + 2|\nabla^2 h|^2 + h^2 + 1\right) d\nu_t dt$$
$$\le C(Y,\epsilon)\int_{-2\epsilon^{-1}}^{-\frac{1}{2}\epsilon}\int_M \left(|Rc|^2 + |\nabla h|^4 + 1\right) d\nu_t dt.$$

The L^4 estimate for $|\nabla h|$ is a consequence of (5.2.9), while the Ricci curvature is bounded using (2.7.1).

Now suppose $(x_0, 0)$ is also $(\delta, 1)$-static. Then q clearly satisfies properties $(i),(iv)$, while $(iii),(iv)$ follow from combining properties $(iii),(ii)$, respectively, of strong almost-soliton potentials with the almost-static inequalities. The remaining inequality follows from the improved estimate for strong almost-soliton potentials and the estimate

$$\int_{-2\epsilon^{-1}}^{-\frac{1}{2}\epsilon}\int_M |Rc|^2 e^{\alpha f} d\nu_t + \sup_{t\in[-2\epsilon^{-1},-\frac{1}{2}\epsilon]}\int_M R e^{\alpha f} d\nu_t \le \Psi(\delta|Y,\epsilon),$$

which itself follows from Cauchy's inequality, the almost-static inequalities, and (2.7.1),(2.7.2). □

Remark 20. *It is also possible to construct regularized versions of Bamler's almost radial functions (c.f. Section 13 of [B5]) when $k > 0$ near points which are almost-selfsimilar, almost-static, and almost-split, but we will not need this.*

5.3 Improved Splitting for Noncollapsed Kähler-Ricci Flows

Near a point which is almost-selfsimilar and almost-split, we obtain estimates on the Ricci curvature in the direction of the almost-splitting.

Lemma 21. *Suppose $(M^n,(g_t)_{t\in I},(x,0))$ is a closed, pointed Ricci flow, $\mathcal{N}_{x,0}(1)\geq -Y$, $y\in C^\infty(M\times[-\delta^{-1},-\delta])$ is a strong $(1,\delta,1)$-splitting map, and $(x,0)$ is $(\delta,1)$-selfsimilar. Then*

$$\int_{-\epsilon^{-1}}^{-\epsilon}\int_M |Rc(\nabla y)|d\nu_t dt \leq \Psi(\delta|\epsilon,Y).$$

Proof. Suppose by way of contradiction there exist $\epsilon>0$, $Y<\infty$, a sequence $\delta_i\searrow 0$ and closed Ricci flows $(M_i,(g_{i,t})_{t\in(-\delta_i^{-1},0]})$ along with $(\delta_i,1)$-selfsimilar points $(x_i,0)$ and strong $(1,\delta_i,1)$-splitting maps $y_i\in C^\infty(M_i\times[-\delta_i^{-1},-\delta_i])$ based at $(x_i,0)$ such that

$$\liminf_{i\to\infty}\int_{-\epsilon^{-1}}^{-\epsilon}\int_M |Rc_{g_i}(\nabla y_i)|d\nu_t^i dt>0,$$

where ν^i is the conjugate heat kernel of $(M_i,(g_{i,t})_{t\in[-\delta_i^{-1},0]})$ based at $(x_i,0)$. By passing to a subsequence, we can assume $\mathbb{F}$-convergence

$$(M_i,(g_{i,t})_{t\in[-\delta_i^{-1},0]},(\nu_t^i)_{t\in[-\delta_i^{-1},0]}) \xrightarrow[i\to\infty]{\mathbb{F}} (\mathcal{X},(\mu_t)_{t\in(-\infty,0]})$$

on compact time intervals, where $\mathcal{X}$ is a future-continuous metric soliton; moreover, $d\mu_t=(4\pi\tau)^{-\frac{n}{2}}e^{-f}dg_t$, on $\mathcal{R}$, where $f\in C^\infty(\mathcal{R})$ satisfies $Rc+\nabla^2 f=\frac{1}{2\tau}g$ on $\mathcal{R}$. Let (U_i) be a precompact exhaustion of $\mathcal{R}$, and let $\psi_i:U_i\to M_i$ be time-preserving diffeomorphisms such that $\psi_i^* g_i\to g$ and $\psi_i^* K^i(x_i,0;\cdot,\cdot)\to(4\pi\tau)^{-\frac{n}{2}}e^{-f}$ in $C^\infty_{loc}(\mathcal{R})$, where K^i is the conjugate heat kernel of $(M_i,(g_{i,t})_{t\in(-\delta_i^{-1},0]})$. By Theorem 15.50 of [Bam3], we have a splitting of Ricci flow spacetimes $\mathcal{R}=\mathcal{R}'\times\mathbb{R}$ and of metric flows $\mathcal{X}=\mathcal{X}'\times\mathbb{R}$, and $\psi_i^* y_i\to y_\infty$ in $C^\infty_{loc}(\mathcal{R})$, where $y_\infty:\mathcal{X}\to\mathbb{R}$ is the projection onto the $\mathbb{R}$-factor. In particular, we have $Rc_{g_\infty}(\nabla y_\infty)=0$, hence $Rc_{\psi_i^* g_i}(\nabla(\psi_i^* y_i))\to 0$ in $C^\infty_{loc}(\mathcal{R})$ as $i\to\infty$. Let $K\subseteq\mathcal{R}_{[-\epsilon^{-1},-\epsilon]}$ be an arbitrary compact subset, and set $K_t := X_t \cap K$, so that

$$
\begin{aligned}
&\limsup_{i\to\infty}\int_{\epsilon}^{\epsilon^{-1}}\int_M |Rc_{g_i}(\nabla y_i)|d\nu_t^i dt \\
&\quad\leq \limsup_{i\to\infty}\left(\int_{\epsilon}^{\epsilon^{-1}}\int_{\psi_{i,t}(K_t)}|Rc_{g_i}(\nabla y_i)|d\nu_t^i dt+\int_{\epsilon}^{\epsilon^{-1}}\int_{M_i\setminus\psi_{i,t}(K_t)}|Rc_{g_i}(\nabla y_i)|d\nu_t^i dt\right)\\
&\quad\leq \limsup_{i\to\infty}\left(\int_{\epsilon}^{\epsilon^{-1}}\int_{K_t}|Rc_{\psi_i^* g_i}(\nabla(\psi_i^* y_i))|\psi_i^* K^i(x_i,0;\cdot,\cdot)d(\psi_i^* g_{i,t})dt\right)\\
&\quad+\limsup_{i\to\infty}\left(\int_{\epsilon}^{\epsilon^{-1}}\int_{M_i\setminus\psi_{i,t}(K_t)}|Rc_{g_i}(\nabla y_i)|^{\frac{3}{2}}d\nu_t^i dt\right)^{\frac{2}{3}}\left(\int_{\epsilon}^{\epsilon^{-1}}\nu_t^i\left(M_i\setminus\psi_{i,t}(K_t)\right)dt\right)^{\frac{1}{3}}\\
&\quad\leq \limsup_{i\to\infty}\left(\int_{\epsilon}^{\epsilon^{-1}}\int_{M_i}|Rc_{g_i}(\nabla y_i)|^{\frac{3}{2}}d\nu_t^i dt\right)^{\frac{2}{3}}\left(\int_{\epsilon}^{\epsilon^{-1}}\left(1-\nu_t^i(\psi_{i,t}(K_t))\right)dt\right)^{\frac{1}{3}}\\
&\quad\leq \left(\int_{\epsilon}^{\epsilon^{-1}}(1-\mu_t(K_t))\,dt\right)^{\frac{1}{2}}\limsup_{i\to\infty}\left(\int_{\epsilon}^{\epsilon^{-1}}\int_{M_i}|Rc_{g_i}(\nabla y_i)|^{\frac{3}{2}}d\nu_t^i dt\right)^{\frac{2}{3}}.
\end{aligned}
$$

We then estimate

$$
\int_{\epsilon}^{\epsilon^{-1}}\int_{M_i}|Rc_{g_i}(\nabla y_i)|^{\frac{3}{2}}d\nu_t^i dt \leq \left(\int_{\epsilon}^{\epsilon^{-1}}\int_M |Rc_{g_i}|^2 d\nu_t^i dt\right)^{\frac{3}{4}}\left(\int_{\epsilon}^{\epsilon^{-1}}\int_M |\nabla y_i|^4 d\nu_t^i dt\right)^{\frac{1}{4}} \leq C(Y,\epsilon)
$$

for sufficiently large $i \in \mathbb{N}$. Since $K \subseteq \mathcal{R}_{[-\epsilon^{-1},-\epsilon]}$ was arbitrary, we obtain

$$
\limsup_{i\to\infty}\int_{\epsilon}^{\epsilon^{-1}}\int_M |Rc_{g_i}(\nabla y_i)|d\nu_t^i dt = 0,
$$

a contradiction. □

Remark 21. *This proof is easily adapted to show L^p bounds on $Rc(\nabla y)$ for any $p \in [1,2)$, but fails for $p = 2$. This creates additional technical difficulties in the proof of Proposition 28.*

Next, we use the estimates for strong almost-soliton potential functions and strong almost-splitting maps to construct new almost splitting maps on a Kähler-Ricci flows. We observe that many of these estimates would fail without the use of the parabolic approximation h.

Proposition 28. *Suppose $h \in C^\infty(M \times [-\delta^{-1}, -\delta])$ is a strong $(\delta,1)$-soliton potential function at $(x_0,0)$ satisfying*

$$\int_{-\delta^{-1}}^{-\delta}\int_M |\nabla(h-f)|^2 d\nu_t dt \le \delta,$$

and assume $y \in C^\infty(M \times [-\delta^{-1}, -\delta])$ is a strong $(1,\delta,1)$-splitting map, where $(x_0,0)$ is $(\delta,1)$-selfsimilar. If $\delta \le \overline{\delta}(Y,\epsilon,p)$, then the following hold, where $q := 4\tau(h-W)$ and $z := \frac{1}{2}\langle \nabla q, J\nabla y\rangle$:

(i) $\displaystyle\int_{-\epsilon^{-1}}^{-\epsilon}\int_M |\nabla^2 q|^2\left(|\nabla y|^{2p}+|\nabla q|^{2p}\right)d\nu_t dt \le C(Y,\epsilon,p)$ *for each $p \in \mathbb{N}$,*

(ii) $\displaystyle\int_{-\epsilon^{-1}}^{-\epsilon}\int_M \left||\nabla z|^2-1\right| d\nu_t dt \le \Psi(\delta|Y,\epsilon)$,

(iii) $\displaystyle\int_{-\epsilon^{-1}}^{-\epsilon}\int_M |\langle\nabla z, \nabla y\rangle|\, d\nu_t dt \le \Psi(\delta|Y,\epsilon)$,

(iv) $\displaystyle\int_{-\epsilon^{-1}}^{-\epsilon}\int_M |\square z| d\nu_t dt \le \Psi(\delta|Y,\epsilon)$,

(v) $\displaystyle\int_{-\epsilon^{-1}}^{-\epsilon}\int_M |\nabla z - J\nabla y|^2 d\nu_t dt \le \Psi(\delta|Y,\epsilon)$.

In particular, for $\delta \le \overline{\delta}(Y,\epsilon)$, (y,z) is a weak $(2,\epsilon,1)$-splitting map.

Proof. (i) Observe that

$$\square|\nabla q|^{2(p+1)} \le (p+1)|\nabla q|^{2p}\square|\nabla q|^2 = -2(p+1)|\nabla q|^{2p}|\nabla^2 q|^2.$$

Upon integration, (5.2.9) lets us estimate

$$2(p+1)\int_{-\epsilon^{-1}}^{-\epsilon}\int_M |\nabla^2 q|^2|\nabla q|^{2p} d\nu_t dt \le -\int_M |\nabla q|^{2(p+1)}d\nu_t\Big|_{t=-2\epsilon^{-1}}^{t=-\frac{1}{2}\epsilon} \le C(Y,\epsilon,p)$$

assuming $\delta \le \overline{\delta}(Y,\epsilon,p)$. Next, we estimate

$$\begin{aligned}\square\left(|\nabla y|^{2(p+1)}|\nabla q|^2\right) &\le 2(p+1)|\nabla^2 y|^2|\nabla y|^{2p}|\nabla q|^2 - 2|\nabla y|^{2(p+1)}|\nabla^2 q|^2 - 2\langle \nabla |\nabla y|^{2(p+1)}, \nabla |\nabla q|^2\rangle \\ &\le -2|\nabla y|^{2(p+1)}|\nabla^2 q|^2 + 8(p+1)|\nabla^2 y| \cdot |\nabla y|^{2p+1}|\nabla^2 q| \cdot |\nabla q|\end{aligned}$$

Integration on $M \times [-2\epsilon^{-1}, -\frac{1}{2}\epsilon]$ then gives

$$\begin{aligned}&\int_{-2\epsilon^{-1}}^{-\frac{1}{2}\epsilon}\int_M |\nabla^2 q|^2|\nabla y|^{2p} d\nu_t dt \\ &\qquad\le -\int_M |\nabla y|^{2(p+1)}|\nabla q|^2 d\nu_t \Big|_{t=-2\epsilon^{-1}}^{t=-\frac{1}{2}\epsilon} \\ &\qquad\quad + 8(p+1)\left(\int_{-2\epsilon^{-1}}^{-\frac{1}{2}\epsilon}\int_M |\nabla^2 y|^2|\nabla y|^{2(2p+1)} d\nu_t dt\right)^{\frac{1}{2}}\left(\int_{-2\epsilon^{-1}}^{-\frac{1}{2}\epsilon}\int_M |\nabla^2 q|^2|\nabla q|^2 d\nu_t dt\right)^{\frac{1}{2}} \\ &\qquad\le C(Y,\epsilon,p)\end{aligned}$$

assuming $\delta \le \overline{\delta}(Y,\epsilon,p)$, where we used Bamler's estimates for strong almost-splitting maps (Proposition 12.21 of [Bam3]).

(ii) For any $t \in [-2\epsilon^{-1}, -\frac{1}{2}\epsilon]$,

$$\begin{aligned}\int_M (\langle \nabla q, \nabla y\rangle - 2y)\, d\nu_t &= -(4\pi\tau)^{-\frac{n}{2}}4\tau\int_M \langle \nabla e^{-f}, \nabla y\rangle dg_t + 4\tau\int_M \langle \nabla(h-f), \nabla y\rangle d\nu_t \\ &= 4\tau\int_M (\Delta y) d\nu_t + 4\tau\int_M \langle \nabla(h-f), \nabla y\rangle d\nu_t,\end{aligned}$$

so we can estimate

$$\begin{aligned}\int_{-2\epsilon^{-1}}^{-\frac{1}{2}\epsilon}\left|\int_M (\langle \nabla q, \nabla y\rangle - 2y)\, d\nu_t\right| dt &\le 4\tau n\int_{-2\epsilon^{-1}}^{-\frac{1}{2}\epsilon}\int_M |\nabla^2 y|^2 d\nu_t dt \\ &\quad + 4\tau\left(\int_{-2\epsilon^{-1}}^{-\frac{1}{2}\epsilon}\int_M |\nabla(h-f)|^2 d\nu_t dt\right)^{\frac{1}{2}}\left(\int_{-2\epsilon^{-1}}^{-\frac{1}{2}\epsilon}\int_M |\nabla y|^2 d\nu_t dt\right)^{\frac{1}{2}} \\ &\le \Psi(\delta|Y,\epsilon).\end{aligned}$$

Using the L^1-Poincare inequality and Lemma 21,

$$
\begin{aligned}
&\int_{-2\epsilon^{-1}}^{-\frac{1}{2}\epsilon}\int_M |\langle \nabla q, \nabla y\rangle - 2y|\, d\nu_t dt \\
&\quad\le \int_{-2\epsilon^{-1}}^{-\frac{1}{2}\epsilon}\left|\int_M (\langle \nabla q, \nabla y\rangle - 2y)\, d\nu_t\right| dt + C\int_{-2\epsilon^{-1}}^{-\frac{1}{2}\epsilon} \tau \int_M |\nabla(\langle \nabla q, \nabla y\rangle - 2y)|\, d\nu_t dt \\
&\quad\le \Psi(\delta|Y,\epsilon) + C(\epsilon)\int_{-2\epsilon^{-1}}^{-\frac{1}{2}\epsilon}\int_M \left|(4\tau Rc + \nabla^2 q - 2g)(\nabla y)\right| d\nu_t dt + C(\epsilon)\int_{-2\epsilon^{-1}}^{-\frac{1}{2}\epsilon}\int_M |Rc(\nabla y)| d\nu_t dt \\
&\quad\quad + C(\epsilon)\left(\int_{-2\epsilon^{-1}}^{-\frac{1}{2}\epsilon}\int_M |\nabla^2 y|^2 d\nu_t dt\right)^{\frac{1}{2}}\left(\int_{-2\epsilon^{-1}}^{-\frac{1}{2}\epsilon}\int_M |\nabla q|^2 d\nu_t dt\right)^{\frac{1}{2}} \\
&\quad\le \Psi(\delta|Y,\epsilon) + C(\epsilon)\left(\int_{-2\epsilon^{-1}}^{-\frac{1}{2}\epsilon}\int_M |4\tau Rc + \nabla^2 q - 2g|^2 d\nu_t dt\right)^{\frac{1}{2}}\left(\int_{-2\epsilon^{-1}}^{-\frac{1}{2}\epsilon}\int_M |\nabla y|^2 d\nu_t dt\right)^{\frac{1}{2}} \\
&\quad\le \Psi(\delta|Y,\epsilon).
\end{aligned}
$$

Then Hölder's inequality and (5.2.9) give

$$
\begin{aligned}
&\int_{-2\epsilon^{-1}}^{-\frac{1}{2}\epsilon}\int_M (\langle \nabla q, \nabla y\rangle - 2y)^2\, d\nu_t dt \\
&\quad\le \left(\int_{-2\epsilon^{-1}}^{-\frac{1}{2}\epsilon}\int_M |\langle \nabla q, \nabla y\rangle - 2y|\, d\nu_t dt\right)^{\frac{1}{2}}\left(\int_{-2\epsilon^{-1}}^{-\frac{1}{2}\epsilon}\int_M |\langle \nabla q, \nabla y\rangle - 2y|^3\, d\nu_t dt\right)^{\frac{1}{2}} \\
&\quad\le \Psi(\delta|Y,\epsilon)\left(\int_{-2\epsilon^{-1}}^{-\frac{1}{2}\epsilon}\int_M (|\nabla q|^3|\nabla y|^3 + |y|^3) d\nu_t dt\right)^{\frac{1}{2}} \\
&\quad\le \Psi(\delta|Y,\epsilon).
\end{aligned}
$$

Next, we compute

$$\square(\langle \nabla q, \nabla y\rangle - 2y)^2 = -2|\nabla(\langle \nabla q, \nabla y\rangle - 2y)|^2 - 4(\langle \nabla q, \nabla y\rangle - 2y)\langle \nabla^2 q, \nabla^2 y\rangle.$$

Fix a cutoff function $\zeta \in C^\infty([-2\epsilon^{-1}, -\frac{1}{2}\epsilon])$ such that $\zeta(-2\epsilon^{-1}) = \zeta(-\frac{1}{2}\epsilon) = 0$, $\zeta|[-\epsilon^{-1}, -\epsilon] \equiv 1$, and $|\zeta'| \le 4\epsilon^{-1}$. Then

$$
\begin{aligned}
\int_{-2\epsilon^{-1}}^{-\frac{1}{2}\epsilon}\int_M \zeta'(\langle \nabla q, \nabla y\rangle - 2y)^2 d\nu_t dt = & -\int_{-2\epsilon^{-1}}^{-\frac{1}{2}\epsilon}\zeta\left(\frac{d}{dt}\int_M (\langle \nabla q, \nabla y\rangle - 2y)^2 d\nu_t\right) dt \\
=& 2\int_{-2\epsilon^{-1}}^{-\frac{1}{2}\epsilon}\int_M \zeta|\nabla(\langle \nabla q, \nabla y\rangle - 2y)|^2 d\nu_t dt \\
& + 4\int_{-2\epsilon^{-1}}^{-\frac{1}{2}\epsilon}\int_M \zeta(\langle \nabla q, \nabla y\rangle - 2y)\langle \nabla^2 q, \nabla^2 y\rangle d\nu_t dt,
\end{aligned}
$$

so that (using part (i) and Proposition 12.21 of [B5])

$$
\begin{aligned}
\int_{-\epsilon^{-1}}^{-\epsilon}\int_M \left|\nabla\left(\langle \nabla q, \nabla y\rangle - 2y\right)\right|^2 d\nu_t dt
&\le 4\epsilon^{-1}\int_{-2\epsilon^{-1}}^{-\frac{1}{2}\epsilon}\int_M \left(\langle \nabla q, \nabla y\rangle - 2y\right)^2 d\nu_t dt + 4\int_{-2\epsilon^{-1}}^{-\frac{1}{2}\epsilon}\int_M |\nabla q||\nabla^2 q||\nabla y||\nabla^2 y| d\nu_t dt \\
&\quad + 4\int_{-2\epsilon^{-1}}^{-\frac{1}{2}\epsilon}\int_M |\nabla^2 q||\nabla^2 y||y| d\nu_t dt \\
&\le \Psi(\delta|Y,\epsilon) + 4\left(\int_{-2\epsilon^{-1}}^{-\frac{1}{2}\epsilon}\int_M |\nabla^2 q|^2|\nabla q|^2 d\nu_t dt\right)^{\frac{1}{2}}\left(\int_{-2\epsilon^{-1}}^{-\frac{1}{2}\epsilon}\int_M |\nabla^2 y|^2|\nabla y|^2 d\nu_t dt\right)^{\frac{1}{2}} \\
&\quad + 4\left(\int_{-2\epsilon^{-1}}^{-\frac{1}{2}\epsilon}\int_M |\nabla^2 q|^2 d\nu_t dt\right)^{\frac{1}{2}}\left(\int_{-2\epsilon^{-1}}^{-\frac{1}{2}\epsilon}\int_M |\nabla^2 y|^2|y|^2 d\nu_t dt\right)^{\frac{1}{2}} \\
&\le \Psi(\delta|Y,\epsilon).
\end{aligned}
$$

Integrating

$$
\square\left(|\nabla y|^2|\nabla q|^4\right) \le -2|\nabla^2 y|^2|\nabla q|^4 - 16\langle \nabla^2 y(\nabla y), \nabla^2 q(\nabla q)\rangle|\nabla q|^2
$$

against the conjugate heat kernel, and applying part (i) and (5.2.9) gives

$$
\begin{aligned}
\int_{-\epsilon^{-1}}^{-\epsilon}\int_M |\nabla^2 y|^2|\nabla q|^4 d\nu_t dt
&\le \left|\int_M |\nabla y|^2|\nabla q|^2 d\nu_t\Big|_{t=-\epsilon^{-1}}^{t=-\epsilon}\right| \\
&\quad + 8\left(\int_{-\epsilon^{-1}}^{-\epsilon}\int_M |\nabla^2 y|^2|\nabla y|^2 d\nu_t dt\right)^{\frac{1}{2}}\left(\int_{-\epsilon^{-1}}^{-\epsilon}\int_M |\nabla^2 q|^2|\nabla q|^6 d\nu_t dt\right)^{\frac{1}{2}} \\
&\le C(Y,\epsilon)
\end{aligned}
$$

assuming $\delta \le \overline{\delta}(Y,\epsilon)$. We can use Hölder's inequality to estimate

$$
\begin{aligned}
\int_{-\epsilon^{-1}}^{-\epsilon}\int_M \left|(\nabla^2 q - 2g)(\nabla y)\right|^2 d\nu_t dt
&\le 2\int_{-\epsilon^{-1}}^{-\epsilon}\int_M \left|\nabla\left(\langle \nabla q, \nabla y\rangle - 2y\right)\right|^2 d\nu_t dt + 2\int_{-\epsilon^{-1}}^{-\epsilon}\int_M |\nabla^2 y|^2|\nabla q|^2 d\nu_t dt \\
&\le \Psi(\delta|Y,\epsilon) + \left(\int_{-\epsilon^{-1}}^{-\epsilon} |\nabla^2 y|^2|\nabla q|^4 d\nu_t dt\right)^{\frac{1}{2}}\left(\int_{-\epsilon^{-1}}^{-\epsilon}\int_M |\nabla^2 y|^2 d\nu_t dt\right)^{\frac{1}{2}} \\
&\le \Psi(\delta|Y,\epsilon)
\end{aligned}
$$

and we also obtain the rough estimate

$$\int_{\epsilon^{-1}}^{\epsilon}\int_M \left|(\nabla^2 q+2g)(\nabla y)\right|^2 d\nu_t dt \leq 2\int_{\epsilon^{-1}}^{\epsilon}\int_M \left(|\nabla^2 q|^2|\nabla y|^2+4|\nabla y|^2\right) d\nu_t dt \leq C(Y,\epsilon).$$

Now use Hölder's inequality, and combine estimates:

$$\begin{aligned}\int_{\epsilon^{-1}}^{\epsilon}\int_M \left||\nabla^2 q(\nabla y)|^2-4\right| d\nu_t dt &\leq \int_{\epsilon^{-1}}^{\epsilon}\int_M \left\langle(\nabla^2 q+2g)(\nabla y),(\nabla^2 q-2g)(\nabla y)\right\rangle d\nu_t dt+4\int_{\epsilon^{-1}}^{\epsilon}\int_M \left|1-|\nabla y|^2\right| d\nu_t dt\\ &\leq \Psi(\delta|Y,\epsilon).\end{aligned}$$

On the other hand, we can estimate

$$\begin{aligned}\int_{\epsilon^{-1}}^{\epsilon}\int_M \left||\nabla^2 q(\nabla y)|^2-|\nabla^2 q(J\nabla y)|^2\right| d\nu_t dt &\leq \left(\int_{\epsilon^{-1}}^{\epsilon}\int_M \left|\left|\nabla^2 q\left(\frac{\nabla y}{\sqrt{1+|\nabla y|^2}}\right)\right|^2-\left|\nabla^2 q\left(J\frac{\nabla y}{\sqrt{1+|\nabla y|^2}}\right)\right|^2\right| d\nu_t dt\right)^{\frac{1}{2}}\\ &\quad\cdot\left(\int_{\epsilon^{-1}}^{\epsilon}\int_M \left||\nabla^2 q(\nabla y)|^2-|\nabla^2 q(J\nabla y)|^2\right|\left(1+|\nabla y|^2\right) d\nu_t dt\right)^{\frac{1}{2}}.\end{aligned}$$

To estimate the first integral, we observe that

$$\begin{aligned}&\left|\left|\nabla^2 q\,(\nabla y)\right|^2-\left|\nabla^2 q\,(J\nabla y)\right|^2\right|\\ &=\left|\left(\left|\nabla^2 q\,(\nabla y)\right|^2-|(4\tau Rc-2g)(\nabla y)|^2\right)-\left(\left|\nabla^2 q\,(J\nabla y)\right|^2-|(4\tau Rc-2g)(J\nabla y)|^2\right)\right|\\ &\leq\left|\left\langle\left(-4\tau Rc+\nabla^2 q+2g\right)(\nabla y),\left(4\tau Rc+\nabla^2 q-2g\right)(\nabla y)\right\rangle\right|\\ &\quad+\left|\left\langle\left(-4\tau Rc+\nabla^2 q+2g\right)(J\nabla y),\left(4\tau Rc+\nabla^2 q-2g\right)(J\nabla y)\right\rangle\right|,\end{aligned}$$

so that

$$\begin{aligned}&\int_{\epsilon^{-1}}^{\epsilon}\int_M \left|\left|\nabla^2 q\left(\frac{\nabla y}{\sqrt{1+|\nabla y|^2}}\right)\right|^2-\left|\nabla^2 q\left(J\frac{\nabla y}{\sqrt{1+|\nabla y|^2}}\right)\right|^2\right| d\nu_t dt\\ &\leq 2\left(\int_{\epsilon^{-1}}^{\epsilon}\int_M |4\tau Rc+\nabla^2 q-2g|^2 d\nu_t dt\right)^{\frac{1}{2}}\left(\int_{\epsilon^{-1}}^{\epsilon}\int_M |4\tau Rc-\nabla^2 q-2g|^2 d\nu_t dt\right)^{\frac{1}{2}}\\ &\leq \Psi(\delta|Y,\epsilon).\end{aligned}$$

For the second integral, we only need the coarse upper bound

$$\int_{-\epsilon^{-1}}^{-\epsilon}\int_M \left||\nabla^2 q(\nabla y)|^2 - |\nabla^2 q(J\nabla y)|^2\right|\left(1+|\nabla y|^2\right)d\nu_t dt \leq \int_{-\epsilon^{-1}}^{-\epsilon}\int_M 4|\nabla^2 q|^2\left(|\nabla y|^4+|\nabla y|^2\right)d\nu_t dt \leq C(Y,\epsilon)$$

by part (i). Combining expressions, we finally obtain

$$\begin{aligned}\int_{-\epsilon^{-1}}^{-\epsilon}\int_M \left||\nabla z|^2 - 1\right| d\nu_t dt \leq{}& \frac{1}{4}\int_{-\epsilon^{-1}}^{-\epsilon}\int_M \left||\nabla^2 q(J\nabla y)|^2 - 4\right| d\nu_t dt + \frac{1}{4}\int_{-\epsilon^{-1}}^{-\epsilon}\int_M |\nabla^2 y|^2|\nabla q|^2 d\nu_t dt \\ &+\frac{1}{2}\left(\int_{-\epsilon^{-1}}^{-\epsilon}\int_M |\nabla^2 q|^2|\nabla q|^2 d\nu_t dt\right)^{\frac{1}{2}}\left(\int_{-\epsilon^{-1}}^{-\epsilon}\int_M |\nabla^2 y|^2|\nabla q|^2 d\nu_t dt\right)^{\frac{1}{2}} \\ \leq{}& \Psi(\delta|Y,\epsilon).\end{aligned}$$

(iii) Because $X \mapsto Rc(JX,X)$ and $X \mapsto g(JX,X)$ are skew-symmetric, we have

$$\begin{aligned}\langle\nabla z,\nabla y\rangle ={}& \frac{1}{2}\nabla^2 q(J\nabla y,\nabla y) + \frac{1}{2}\langle\nabla_{\nabla y}J\nabla y,\nabla q\rangle \\ ={}& \frac{1}{2}\left(4\tau Rc + \nabla^2 q - 2g\right)(J\nabla y,\nabla y) - \frac{1}{2}\nabla^2 y(\nabla y, J\nabla q),\end{aligned}$$

which allows us to estimate (again using Proposition 12.21 of [B5])

$$\begin{aligned}\int_{-\epsilon^{-1}}^{-\epsilon}\int_M |\langle\nabla z,\nabla y\rangle| d\nu_t dt \leq{}& \int_{-\epsilon^{-1}}^{-\epsilon}\int_M \left|\langle 4\tau Rc + \nabla^2 q - 2g, \nabla y \otimes J\nabla y\rangle\right| d\nu_t dt \\ &+\int_{-\epsilon^{-1}}^{-\epsilon}\int_M |\nabla^2 y|\cdot|\nabla y|\cdot|\nabla q| d\nu_t dt \\ \leq{}& \left(\int_{-\epsilon^{-1}}^{-\epsilon}\int_M \left|4\tau Rc + \nabla^2 q - 2g\right|^2 d\nu_t dt\right)^{\frac{1}{2}}\left(\int_{-\epsilon^{-1}}^{-\epsilon}\int_M |\nabla y|^4 d\nu_t dt\right)^{\frac{1}{2}} \\ &+\left(\int_{-\epsilon^{-1}}^{-\epsilon}\int_M |\nabla^2 y|^2|\nabla y|^2 d\nu_t dt\right)^{\frac{1}{2}}\left(\int_{-\epsilon^{-1}}^{-\epsilon}\int_M |\nabla q|^2 d\nu_t dt\right)^{\frac{1}{2}} \\ \leq{}& \Psi(\delta|Y,\epsilon).\end{aligned}$$

(iv) We compute

$$\begin{aligned}\square\langle\nabla q, J\nabla y\rangle ={}& 2Rc(\nabla q, J\nabla y) + \langle\square\nabla q, J\nabla y\rangle + \langle\nabla q, J\square\nabla y\rangle - 2\langle\nabla^2 q,\nabla(J\nabla y)\rangle \\ ={}& 2Rc(\nabla q, J\nabla y) - \langle Rc(\nabla q), J\nabla y\rangle - \langle\nabla q, JRc(\nabla y)\rangle - 2\langle\nabla^2 q,\nabla(J\nabla y)\rangle \\ ={}& -2\langle\nabla^2 q,\nabla(J\nabla y)\rangle,\end{aligned}$$

so that

$$\int_{-\epsilon^{-1}}^{-\epsilon}\int_M |\ z| d\nu_t dt \leq \left(\int_{-\epsilon^{-1}}^{-\epsilon}\int_M |\nabla^2 q|^2 d\nu_t dt\right)^{\frac{1}{2}} \left(\int_{-\epsilon^{-1}}^{-\epsilon}\int_M |\nabla^2 y|^2 d\nu_t dt\right)^{\frac{1}{2}} \leq \Psi(\delta|Y,\epsilon).$$

(v) Using part (ii) and Lemma 21, we estimate

$$\begin{aligned}
&\int_{-\epsilon^{-1}}^{-\epsilon}\int_M |\nabla z - J\nabla y|^2\, d\nu_t dt \\
&\qquad = \int_{-\epsilon^{-1}}^{-\epsilon}\int_M \left(|\nabla z|^2 + |\nabla y|^2 - 2\langle \nabla z, J\nabla y\rangle\right) d\nu_t dt \\
&\qquad \leq \int_{-\epsilon^{-1}}^{-\epsilon}\int_M \left(2 - \nabla^2 q(J\nabla y, J\nabla y)\right) d\nu_t dt + \int_{-\epsilon^{-1}}^{-\epsilon}\int_M |\nabla^2 y| |\nabla y| |\nabla q| d\nu_t dt + \Psi(\delta|Y,\epsilon) \\
&\qquad \leq \int_{-\epsilon^{-1}}^{-\epsilon}\int_M \left|4\tau Rc + \nabla^2 q - 2g\right| |\nabla y|^2 d\nu_t dt \\
&\qquad + \int_{-\epsilon^{-1}}^{-\epsilon}\int_M |Rc(\nabla y)| |\nabla y| d\nu_t dt + \Psi(\delta|Y,\epsilon) \\
&\qquad \leq \int_{-\epsilon^{-1}}^{-\epsilon}\int_M |Rc(\nabla y)| \left||\nabla y| - 1\right| d\nu_t dt + \int_{-\epsilon^{-1}}^{-\epsilon}\int_M |Rc(\nabla y)| d\nu_t dt + \Psi(\delta|Y,\epsilon) \\
&\qquad \leq \Psi(\delta|Y,\epsilon).
\end{aligned}$$

□

Next, we prove an elementary lemma which will allow us to form almost splitting maps using a linear combination of almost splitting maps along with the new almost-splitting maps constructed in Lemma 28.

Lemma 22. *Given $N, k \in \mathbb{N}$, there exists $C = C(N) < \infty$ such that the following holds. Suppose $(V, \langle\cdot,\cdot\rangle)$ is a real inner product space of dimension at most N, and let J be a complex structure on V compatible with the inner product: $\langle Jv, Jw\rangle = \langle v, w\rangle$ for all $v, w \in V$. If $v_1, ..., v_{2k+1} \in V$ are orthonormal, then there are $(a_{ij})_{1\leq j\leq 2k+1,\ 1\leq i\leq 2k+2}$ and $(b_{ij})_{1\leq j\leq 2k+1,\ 1\leq i\leq 2k+2}$ with $|a_{ij}| + |b_{ij}| \leq C$ such that*

$$\widetilde{v}_i := \sum_{j=1}^{2k+1} \left(a_{ij} v_j + b_{ij} J v_j\right), \qquad 1 \leq i \leq 2k+2,$$

are orthonormal.

Proof. It suffices to show the existence of $c(n) > 0$ such that for any orthonormal tuple $(v_1, ..., v_{2k+1})$ in V, there exists $i \in \{1, ..., 2k+1\}$ such that

$$w_i := Jv_i - \sum_{j=1}^{2k+1} \langle Jv_i, v_j \rangle v_j$$

satisfies $|w_i| \geq c(n)$. Suppose by way of contradiction there exist orthonormal tuples $(v_1^\alpha, ..., v_{2k+1}^\alpha)_{\alpha \in \mathbb{N}}$ such that $\max_{1 \leq i \leq 2k+1} |w_i^\alpha| \to 0$ as $\alpha \to \infty$. We can pass to subsequences so that $\lim_{\alpha \to \infty} v_i^\alpha = v_i^\infty$, where $(v_1^\infty, ..., v_{2k+1}^\infty)$ is an orthonormal tuple. Then, for each $i \in \{1, ..., 2k+1\}$, we have

$$0 = \lim_{\alpha \to \infty} w_i^\alpha = Jv_i^\infty - \sum_{j=1}^{2k+1} \langle Jv_i^\infty, v_j^\infty \rangle v_j^\infty .$$

That is, $Jv_1, ..., Jv_{2k+1}$ are in the $\mathbb{R}$-linear span of $v_1, ..., v_{2k+1}$. This means that the $\mathbb{R}$-linear span of $v_1, ..., v_{2k+1}$, equipped with the restriction of the complex structure, is a complex vector space of real dimension $2k+1$, a contradiction. □

We finally establish the improved splitting for Kähler-Ricci flows.

Proposition 29. *For any $\epsilon > 0$, $k \in \{0, ..., n\}$, $Y < \infty$, the following holds whenever $\delta \leq \overline{\delta}(\epsilon, Y)$. Suppose $(M^{2n}, (g_t)_{t \in I})$ is a Kähler-Ricci flow with $\mathcal{N}_{x_0,t_0}(r^2) \geq -Y$ for some $r > 0$. If $(x_0, t_0) \in M \times I$ is (δ, r)-selfsimilar and strongly $(2k+1, \delta, r)$-split, then (x_0, t_0) is weakly $(2k+2, \epsilon, r)$-split.*

Proof. By parabolic rescaling and time translation, we can assume $t_0 = 0$ and $r = 1$. For ease of notation, write $\nu_t := \nu_{x_0,0;t}$. Let $y : M \times [-\delta^{-1}, -\delta] \to \mathbb{R}^{2k+1}$ be a strong $(2k+1, \delta, 1)$-splitting map, and let $q \in C^\infty(M \times [-\delta^{-1}, -\delta])$ be a strong $(\delta, 1)$-soliton potential, both based at $(x_0, 0)$. For each $i \in \{1, ..., 2k+1\}$, Proposition 28 states that the functions $z_i := \frac{1}{2}\langle \nabla q, J \nabla y_i \rangle$ are weak $(1, \Psi(\delta|Y), 1)$-splitting maps based at $(x_0, 0)$ which satisfy

$$\int_{-\epsilon^{-1}}^{-\epsilon} \int_M |\nabla z_i - J \nabla y_i|^2 d\nu_t dt \leq \Psi(\delta|Y, \epsilon).$$

By replacing y with $A \circ y + b$ for some $A \in \mathbb{R}^{(2k+1)\times(2k+1)}$ with $|A - I_{2k+1}| \leq \Psi(\delta|Y,\epsilon)$ and $|b| \leq \Psi(\delta|Y,\epsilon)$, we can assume that

$$\int_{-\epsilon^{-1}}^{-\epsilon}\int_M \langle \nabla y_i, \nabla y_j \rangle d\nu_t dt = \delta_{ij}.$$

Similar to the proof of Proposition 10.8 in [B5], we consider the finite-dimensional real vector space V spanned by $\nabla y_i, J\nabla y_i \in \mathfrak{X}(M \times [-\epsilon^{-1}, -\epsilon])$, equipped with the restricted L^2 inner product

$$\langle X_1, X_2 \rangle_{L^2} = \int_{-\epsilon^{-1}}^{-\epsilon}\int_M \langle X_1, X_2 \rangle d\nu_t dt$$

and the obvious complex structure. Lemma 22 then provides a_{ij}, b_{ij}, where $1 \leq i \leq 2k+2$, $1 \leq j \leq 2k+1$, such that

$$\widetilde{V}_i := \sum_{j=1}^{2k+1} a_{ij}\nabla y_j + b_{ij} J \nabla y_j$$

are orthonormal in V, and $|a_{ij}|, |b_{ij}| \leq C(k)$. It follows that

$$\widetilde{y}_i := \sum_{j=1}^{2k+1} a_{ij} y_j + b_{ij} z_j$$

define a weak $(2k+2, \Psi(\delta|Y,\epsilon), 1)$-splitting map $\widetilde{y} = (\widetilde{y}_1, ..., \widetilde{y}_{2k+2})$ based at $(x_0, 0)$. □

Proof of Theorem 36. We first verify the existence of a lower bound of the Nash entropy on all of $P^*(x_\infty, A, -A^2)$. Given $y \in P^*(x_\infty, A, -A^2)$, we can find a sequence $(y_i, t_i) \in M_i \times (-T_i, 0]$ such that $(y_i, t_i) \xrightarrow[i\to\infty]{\mathfrak{C}} y_\infty$. Lemma 15.8 of [B5] implies that $(y_i, t_i) \in P^*(x_i, 2A, -(2A)^2)$, and Bamler's Nash entropy oscillation estimate (Corollary 5.11 in [B4]) then gives $\mathcal{N}_{y_i,t_i}(1) \geq -Y'(Y, A)$. Taking the limit as $i \to \infty$, we obtain (via Theorem 15.45 in [B5]) $\mathcal{N}_y(1) \geq -Y'(Y, A)$. The inclusion

$$\widehat{\mathcal{S}}^{\epsilon,2j+1}_{r_1,r_2} \cap P^*(x_\infty; A, -A^2) \subseteq \widehat{\mathcal{S}}^{\delta(\epsilon,Y,A),2j}_{r_1,r_2} \cap P^*(x_\infty; A, -A^2)$$

thus follows from Proposition 29. The remaining claim is then a consequence of the inclusions between quantitative strata and weak quantitative strata (Lemma 20.3 of [B5] and Proposition 33). □

Proof of Theorem 35. Taking $r_1 = 0$ and $r_2 = \epsilon$ in Theorem 36 gives

$$\widehat{\mathcal{S}}^{\epsilon,2j+1}_{0,\epsilon} \cap P^*(x_\infty, A, -A^2) \subseteq \widehat{\mathcal{S}}^{\delta(\epsilon,Y,A),2j}_{0,\epsilon} \cap P^*(x_\infty; A, -A^2).$$

Then taking the union over $\epsilon > 0$ gives

$$\mathcal{S}^{2j+1} \cap P^*(x_\infty, A, -A^2) \subseteq \mathcal{S}^{2j+2} \cap P^*(x_\infty, A, -A^2).$$

Finally, taking $A \nearrow \infty$ gives the claim. □

5.4 An Isometric Action on Tangent Flows

Suppose (M, g, J, f) is a (not necessarily complete) shrinking gradient Kähler-Ricci soliton, and let $\omega := g(J\cdot, \cdot)$ be the corresponding Kähler form. Then the Ricci soliton equation gives

$$\begin{aligned}(\mathcal{L}_{J\nabla f}J)(W) &= \mathcal{L}_{J\nabla f}(JW) - J([J\nabla f, W]) = \nabla_{J\nabla f}(JW) - \nabla_{JW}(J\nabla f) - J(\nabla_{J\nabla f}W - \nabla_W(J\nabla f)) \\ &= -J\nabla_{JW}\nabla f - \nabla_W\nabla f = JRc(JW) + Rc(W) - \frac{1}{2}(J^2W + W) = 0\end{aligned}$$

for any vector field $W \in \mathfrak{X}(\mathcal{R})$, and

$$\mathcal{L}_{J\nabla f}\omega = di_{J\nabla f}\omega = d\left(\omega(J\nabla f, \cdot)\right) = -d\left(g(\nabla f, \cdot)\right) = -d(df) = 0,$$

so that $J\nabla f$ is a real holomorphic Killing vector field on M. We now prove the completeness of the flow of this vector field for tangent flows.

Proposition 30. *Suppose $(M_i^{2n}, (g_{i,t})_{t\in[-\epsilon_i^{-1},0]})$ are closed Kähler-Ricci flows, and that $(x_i, 0)$ are $(\epsilon_i, 1)$-selfsimilar, where $\epsilon_i \searrow 0$. Assume*

$$(M, (g_{i,t})_{t\in(-\epsilon_i^{-1},0)}, (\nu_{x_i,0;t})_{t\in[-\epsilon_i^{-1},0)}) \xrightarrow[i\to\infty]{\mathfrak{F}} (\mathcal{X}, (\nu_{x_\infty;t})_{t\in(-\infty,0)})$$

on compact time intervals where $\mathcal{X}$ is a metric soliton modeled on a singular shrinking Kähler-Ricci soliton $(X, d, \mathcal{R}_X, g_X, f_X)$ as in Theorem 17. Set $q := 4\tau(f_X - W) \in C^\infty(\mathcal{R})$, where $\mathcal{R} \subseteq \mathcal{X}$ is the regular part of the metric flow. Then $J\nabla q$ is complete, and the heat *kernel satisfies the following infinitesimal symmetry for all* $(x_1, x_0) \in \mathcal{R} \times \mathcal{R}$ *with* $\mathfrak{t}(x_0) < \mathfrak{t}(x_1)$*:*

$$\langle \nabla_{x_1} K(x_1; x_0), J\nabla q(x_1)\rangle + \langle \nabla_{x_0} K(x_1; x_0), J\nabla q(x_0)\rangle = 0.$$

Moreover, the flow of $J\nabla q$ *extends to a 1-parameter action by isometries on all of* X*.*

Remark 22. *This proof is modeled on Theorem 15.50 in* [B5].

Proof. Let (U_i) be a precompact exhaustion of $\mathcal{R}$,with open embeddings $\psi_i : U_i \to M_i \times (-\epsilon_i^{-1}, 0)$ realizing the $\mathbb{F}$-convergence on the regular part as in Theorem 16. By Proposition 26 and the proof of Theorem 15.69 in [B5], we can find almost-GRS potential functions $h_i \in C^\infty(M_i \times [-\epsilon_i^{-1}, -\epsilon_i])$ such that, $h_i \circ \psi_i \to f_X$ in $C^\infty_{loc}(\mathcal{R})$, where we identify $\mathcal{R} = \mathcal{R}_X \times (-\infty, 0)$ as in Theorem 17. Now fix $t_0 \in (-\infty, 0)$ and $u^\infty \in C^\infty_c(\mathcal{R}_{t_0})$, and let $u^i \in C^\infty(M_i \times \{t_0\})$ be an approximating sequence: $u^i \circ \psi_i \to u^\infty$ in $C^\infty_{loc}(\mathcal{R})$. Let $(\zeta^i_h)_{h\in(-\alpha,\alpha)}$ be the flow of $J_i\nabla q_i$, and define $u^{\prime,i} \in C^\infty(M_i \times \{t_0\} \times (-\alpha,\alpha))$ by $u^{\prime,i}(x,t_0,h) := u^i(\zeta^i_h(x), t_0)$, so that $u^{\prime,i}(\cdot, t_0, 0) = u^i(\cdot, t_0)$ and $\partial_h u^{\prime,i}(x,t_0,h) = \langle \nabla u^i, J_i\nabla q_i\rangle(\zeta^i_h(x), t_0)$. Next, let $u^{\prime\prime,i} \in C^\infty(M_i \times [t_0, 0] \times (-\alpha,\alpha))$ be given by $u^{\prime\prime,i}(\cdot, t_0, \cdot) = u^{\prime,i}$ and $\square u^{\prime\prime,i}(\cdot,\cdot,h) = 0$ for all $h \in (-\alpha,\alpha)$. Letting $\omega_i, \rho_i \in \Omega^2(M_i)$ denote the Kähler and Ricci 2-forms, respectively, we have

$$\langle \nabla^2 u^{\prime\prime,i}, g_i(J_i\cdot,\cdot)\rangle = \langle \nabla^2 u^{\prime\prime,i}, \omega_i\rangle = 0,$$

$$\langle \nabla^2 u^{\prime\prime,i}, Rc_{g_i}(J_i\cdot,\cdot)\rangle = \langle \nabla^2 u^{\prime\prime,i}, \rho_i\rangle = 0,$$

so we can estimate

$$\left|\square\left(\partial_h u^{\prime\prime,i} - \langle \nabla u^{\prime\prime,i}, J_i\nabla q_i\rangle\right)\right| \le 2\left|\langle \nabla^2 u^{\prime\prime,i}, \nabla(J_i\nabla q_i)\rangle\right| = 2\left|\left\langle \nabla^2 u^{\prime\prime,i}, \left(4\tau Rc(g_i) + \nabla^2 q_i - 2g_i\right)(J_i\cdot,\cdot)\right\rangle\right|$$
$$\le 2|\nabla^2 u^{\prime\prime,i}|\,|4\tau Rc(g_i) + \nabla^2 q_i - 2g_i|.$$

Let $\nu^i := \nu_{x_i,0}$, and integrate $\square|\nabla u^{\prime\prime,i}|^2 = -2|\nabla^2 u^{\prime\prime,i}|^2$ against ν^i to obtain

$$2\int_{t_0}^{t_1}\int_{M_i} |\nabla^2 u^{\prime\prime,i}|^2 d\nu^i_t dt \le \int_{M_i} |\nabla u^{\prime,i}|^2 d\nu^i_{t_0}.$$

However, we know that $u^{\prime,i}\circ\psi_i \to u^{\prime,\infty}$ in $C^\infty_{loc}(\mathcal{R}_{t_0})$, where $u^{\prime,\infty}(x) = u^\infty(\zeta_h^\infty(x))$, and (ζ_h^∞) is the (partially defined) flow of $J\nabla q$. In particular, we can estimate, for any $t_1 \in (t_0, 0)$,

$$\sup_{t\in[t_0,t_1]}\int_{M_i}\left|\partial_h u^{\prime\prime,i} - \langle\nabla u^{\prime\prime,i}, J_i\nabla q_i\rangle\right| d\nu_t^i \le \left(\int_{t_0}^{t_1}\int_{M_i}|4\tau Rc(g_i)+\nabla^2 q_i - 2g_i|^2 d\nu_t^i dt\right)^{\frac{1}{2}}\left(\int_{M_i}|\nabla u^{\prime,i}|^2 d\nu_{t_0}^i\right)^{\frac{1}{2}}$$
$$\le \Psi(\epsilon_i|t_0,t_1).$$

Because $u^{\prime\prime,i}\circ\psi_i \to u^{\prime\prime,\infty}$ in $C^\infty_{loc}(\mathcal{R}_{[t_0,0)})$, where $u^{\prime\prime,\infty}(x,h) = \int_{\mathcal{R}_{t_0}} K(x;y)u'(y,h)dg_{t_0}(y)$, we obtain $\partial_h u^{\prime\prime,\infty} = \langle\nabla u^{\prime\prime,\infty}, J\nabla q\rangle$. We can therefore compute, for all $x \in \mathcal{R}_{t_1}$,

$$\begin{aligned}
\partial_h|_{h=0}u^{\prime\prime,\infty}(x,0) = \frac{\partial}{\partial h}\Big|_{h=0}\int_{\mathcal{R}_{t_0}} K(x;y)u'(y,h)dg_{t_0}(y) &= \int_{\mathcal{R}_{t_0}} K(x;y)\langle\nabla u, J\nabla q\rangle(y)dg_{t_0}(y) \\
&= -\int_{\mathcal{R}_{t_0}} u(y)\left(\langle\nabla_y K(x;y), J\nabla q(y)\rangle + K(x;y)\mathrm{div}(J\nabla q)(y)\right)dg_{t_0}(y) \\
&= -\int_{\mathcal{R}_{t_0}} u(y)\langle\nabla_y K(x;y), J\nabla q(y)\rangle dg_{t_0}(y)
\end{aligned}$$

since

$$\mathrm{div}(J\nabla q) = \langle\nabla(J\nabla q), g\rangle = -\langle 2\omega - 4\tau\rho, g\rangle = 0.$$

On the other hand, we have

$$\partial_h|_{h=0}u^{\prime\prime,\infty}(x,0) = \langle\nabla u^{\prime\prime,\infty}(x,0), J\nabla q(x)\rangle = \int_{\mathcal{R}_{t_0}}\langle\nabla_x K(x;y), J\nabla q(x)\rangle u(y)dg_{t_0}(y),$$

and the infinitesimal symmetry follows.

By Theorem 15.45(c) in [B5], any $x \in \mathcal{R}_{t_1}$ satisfies $\lim_{\tau'\searrow 0}\frac{2}{\tau'}\int_{\frac{\tau'}{2}}^{\tau'}\mathcal{N}_x(\tau'')d\tau'' = 0$. Suppose $\gamma : I \to \mathcal{R}_{t_1}$ is an integral curve of $J\nabla q$, and fix $\tau' > 0$ sufficiently small so that if $t_0 := t_1 - \tau'$ and $t_0' := t_1 - \frac{1}{2}\tau'$, then there exists $x_0 \in \mathcal{R}_{t_0}$ which exists until time t_0'. Write $K(\gamma(s);\cdot) = (4\pi\tau)^{-\frac{n}{2}}e^{-f_s}$, where $f_s \in C^\infty(\mathcal{R}_{[t_0,t_0']})$, $s \in I$. By Theorem 14.54(b) of [B5], the completeness of $J\nabla q$ will follow from showing the following identity:

$$\frac{d}{ds}\int_{t_0}^{t_0'}\int_{\mathcal{R}_t} f_s e^{-f_s}dg_t dt = 0.$$

For $r > 0$, let $\eta_r \in C^\infty(\mathcal{R})$ be the cutoff functions from Lemma 3. Fix $\delta > 0$ and a cutoff $\overline{\eta}_\delta \in C^\infty([0,\infty))$ with $\overline{\eta}_\delta|[0,\delta] \equiv 0$ and $\overline{\eta}_\delta(a) = a$ for all $a \in [2\delta, \infty)$.

Claim: There exists $A = A(\delta) < \infty$ such that $\mathrm{supp}(\overline{\eta}_\delta \circ e^{-f_s}) \subset P^-(x_\infty, A, -A^2)$ for all $s \in I_\infty \cap [-\sigma^{-1}, \sigma^{-1}]$.

We recall the following Gaussian estimate for the conjugate heat kernel on $\mathcal{X}$ (Lemma 15.9 of [B5]):

$$e^{-f(y)} \leq C(T)\exp\left(-\frac{\left(d_{W_1}^{\mathcal{X}_{\mathfrak{t}(y)}}(\nu_{x_\infty;t_0}, \delta_y)\right)^2}{10\mathfrak{t}(y)}\right)$$

for all $y \in \mathcal{R}_{[-T,0)}$. We let $y = \gamma(s)$, and observe that $s \mapsto f(\gamma(s))$ is constant, so there exists $\Lambda \in (|t_0|^{\frac{1}{2}}, \infty)$ such that

$$d_{W_1}^{\mathcal{X}_{t_1}}(\nu_{x_\infty;t_0}, \delta_{\gamma(s)}) \leq \Lambda$$

for all $s \in I_\infty$. This implies $\gamma(I_\infty) \subset P^-(x_\infty, \Lambda, -\Lambda^2)$. We may therefore apply Lemma 15.9 of [B5] to conclude

$$f_s(y) \geq -C + \frac{1}{10\tau'}\left(d_{W_1}^{\mathcal{X}_t}(\nu_{\gamma(s);t}, \delta_y)\right)^2$$

for any $t \in [t_0, t_0']$ and $y \in \mathcal{R}_t$. Thus, there exists $A' = A'(\delta) < \infty$ such that

$$\mathrm{supp}(\overline{\eta}_\delta \circ e^{-f_s}) \subset P^-(\gamma(s), A', -(A')^2)$$

for all $s \in I_\infty \cap [-\sigma^{-1}, \sigma^{-1}]$. The Claim then follows from Proposition 3.40 of [B3], which describes inclusion properties of P^--parabolic neighborhoods.

By the Claim and Lemma 2.11(iv), we see that

$$\bigcup_{s \in [-\sigma^{-1}, \sigma^{-1}]} \mathrm{supp}((\overline{\eta}_\delta \circ e^{-f_s})\eta_r) \cap \mathcal{R}_{[t_0, t_0']}$$

is relatively compact in $\mathcal{R}_{[t_0,t_0']}$ for any fixed $\delta, r > 0$. Thus, for any $s_1, s_2 \in I_\infty$, the infinitesimal symmetry of K gives

$$\begin{aligned}
\int_{t_0}^{t_0'} \int_{\mathcal{R}_t} f_s(\overline{\eta}_\delta \circ e^{-f_s})\eta_r dg_t dt \Big|_{s=s_1}^{s=s_2} &= -\int_{s_1}^{s_2} \int_{t_0}^{t_0'} \int_{\mathcal{R}_t} \left\langle J\nabla q, \nabla\left(f_s(\overline{\eta}_\delta \circ e^{-f_s})\right)\right\rangle \eta_r dg_t dt ds \\
&= \int_{s_1}^{s_2} \int_{t_0}^{t_0'} \int_{\mathcal{R}_t} \left(\langle J\nabla q, \nabla \eta_r\rangle + \mathrm{div}(J\nabla q)\eta_r\right) f_s(\overline{\eta}_\delta \circ e^{-f_s}) dg_t dt ds \\
&= \int_{s_1}^{s_2} \int_{t_0}^{t_0'} \int_{\mathcal{R}_t} \langle J\nabla q, \nabla \eta_r\rangle f_s(\overline{\eta}_\delta \circ e^{-f_s}) dg_t dt ds. \qquad (5.4.1)
\end{aligned}$$

We recall that f_s is bounded uniformly (in s) from below on $\mathcal{R}_{[t_0,t_0']}$, so $f_s(\overline{\eta}_\delta \circ e^{-f_s})$ is uniformly bounded on $\mathcal{R}_{[t_0,t_0']}$. We note that

$$\bigcup_{s\in[-\sigma^{-1},\sigma^{-1}]} \mathrm{supp}\left(\overline{\eta}_\delta \circ e^{-f_s}\right) \cap \mathcal{R}_{[t_0,t_0']} \subseteq L$$

for any fixed $\delta > 0$, where $L \subseteq \mathcal{X}_{[t_0,t_0']}$ is compact. Integrating the estimate $|\nabla \sqrt{f - W}| \leq \frac{1}{2\sqrt{\tau}}$ along almost-minimizing curves in $\mathcal{R}_t$ we obtain $\sup_L |f| < \infty$, and so $\sup_L |\nabla f| < \infty$. Thus, we can bound the right hand side of (5.4.1) by

$$C\int_{t_0}^{t_0'} \int_{L\cap\mathcal{R}_t} |\nabla f| \cdot |\nabla \eta_r| dg_t dt \leq C\int_{t_0}^{t_0'} \int_{L\cap\mathcal{R}_t} |\nabla \eta_r| dg_t dt \leq Cr^2,$$

where $C < \infty$ is independent of $r > 0$, and the last inequality follows from the estimate for $\{r_{Rm} < r\}$ in Lemma 15.27 of [B5]. We can therefore take $r \to 0$ to obtain

$$\int_{t_0}^{t_0'} \int_{\mathcal{R}_t} f_{s_1}(\overline{\eta}_\delta \circ e^{-f_{s_1}}) dg_t dt = \int_{t_0}^{t_0'} \int_{\mathcal{R}_t} f_{s_2}(\overline{\eta}_\delta \circ e^{-f_{s_2}}) dg_t dt.$$

Finally, we take $\delta \searrow 0$ and appeal to the dominated convergence theorem to get the desired identity.

Now let $(\phi_s)_{s\in\mathbb{R}}$ be the flow on $\mathcal{R}_X$ generated by $J\nabla q$ (restricted to a time slice of $\mathcal{R}$). For any $x_1, x_2 \in X$, if $\epsilon > 0$ and $\gamma : [0,1] \to X$ is a curve with image in the regular set $\mathcal{R}_X$ of X and $\mathrm{length}(\gamma) < d(x_1,x_2) + \epsilon$, then $\phi_s \circ \gamma$ is a curve in $\mathcal{R}_X$ from $\phi_s(x_1)$ to $\phi_s(x_2)$ with $\mathrm{length}(\gamma)$, so $d(\phi_s(x_1), \phi_s(x_2)) < d(x_1,x_2) + \epsilon$; taking $\epsilon \searrow 0$, and replacing x_1, x_2 with $\phi_{-s}(x_1), \phi_{-s}(x_2)$ implies that $\phi_s : (\mathcal{R}_X, d) \to (\mathcal{R}_X, d)$ is an isometry for all $s \in \mathbb{R}$. We can therefore extend to a unique isometry $\phi_s : (X,d) \to (X,d)$, whose image is closed and contains $\mathcal{R}_X$, hence is bijective.

□

The proof strategy for the following proposition is roughly similar to that of Theorem 2 in [L1].

Proposition 31. *Let X be as in Proposition 6.1, and assume $Rc(g_X)=0$, so that $X=C(Y)$ is a metric cone with vertex $\{o\}$. Then the 1-parameter group of isometries $(\phi_s)_{s\in\mathbb{R}}$ of $C(Y)$ acts locally freely on the link Y.*

Remark 23. *The rough idea to assume by way of contradiction that a point $z\in C(Y)\setminus\{o\}$ is fixed by the action (ϕ_s), so that ϕ_s preserves the distance to z. Let q_i be a sequence of almost-radial functions based at $(x_i,0)$, and let $(z_i,-1)\in M_i\times[-\epsilon_i^{-1},0]$ converge to $(z,-1)$. At sufficiently small scales near $(z_i,-1)$, appropriate rescalings of q_i look like almost-splitting functions, so Proposition 28 gives almost-splitting functions y_i with $\nabla y_i\approx J\nabla q_i$. By a diagonal argument, after parabolic rescaling of flows, we get convergence of y_i to a function y_∞ on the tangent cone $C(Z)$ at z which induces a metric splitting. On the other hand, $\nabla y_i\approx J\nabla q_i$ implies that the flow of ∇y_∞ preserves the distance to the vertex of $C(Z)$, a contradiction.*

Proof. Fix a correspondence $\mathfrak{C}$ realized the $\mathbb{F}$-convergence to $\mathcal{X}$. It suffices to show that there is no point $z\in\partial B(o,1)$ satisfying $\phi_s(z)=z$ for all $s\in\mathbb{R}$. Suppose by way of contradiction such a point exists. For any $x\in\mathcal{R}_{C(Y)}$, we then have $d(\phi_s(x),z)=d(x,z)$ for all $s\in\mathbb{R}$. Choose a sequence $z_i\in M_i$ such that

$$(z_i,-1)\xrightarrow[i\to\infty]{\mathfrak{C}}(z,-1)\in C(Y)\times(-\infty,0)=\mathcal{X}_{<0}.$$

By Proposition 27, there is a sequence $\delta_i\searrow 0$ such that if $W_i:=\mathcal{N}_{x_i,0}(1)$, then $q_i:=4\tau(h_i-W_i)$ are strong $(\delta_i,1)$-conical functions based at $(x_i,0)$ which satisfy

$$\int_{-\delta_i^{-1}}^{-\delta_i}\int_{M_i}\left(\left|\nabla^2 q_i-2g_i\right|^2+\left||\nabla q_i|^2-4q_i\right|\right)e^{\alpha f_i}\,d\nu^i_{x_i,0;t}\,dt\le\delta_i \tag{5.4.2}$$

for some $\alpha>0$, where we have written $\nu^i_{x_i,0;t}=(4\pi\tau)^{-\frac{n}{2}}e^{-f_i}dg_{i,t}$. By the proof of Theorem 15.80 in [B5], we can therefore pass to a subsequence so that $q_i\circ\psi_i\to q_\infty:=d^2(\cdot,o)$ in $C^\infty_{loc}(\mathcal{R}_{<0})$ as $i\to\infty$, where ψ_i are as in Theorem 16. This implies

$$\liminf_{i\to\infty}\frac{1}{\tau}\int_{-1-2\tau}^{-1-\tau}\int_{M_i}(q_i)_+\,d\nu^i_{z_i,-1;t}\,dt\le\frac{1}{\tau}\int_{-1-2\tau}^{-1-\tau}\int_{\mathcal{R}_{C(Y)}}q_\infty\,d\nu_{z,-1;t}\,dt.$$

Claim: $\lim_{t \nearrow 1} \int_{R_t} q_\infty \, d\nu_{z, \ 1;t} = q_\infty(z) = 1.$

Now choose sequences $t_j \nearrow \ 1$, $y_j \in X$ such that (y_j, t_j) are H_n-centers of $(z, \ 1)$. Because $\min(q_\infty, 4)$ is 2-Lipschitz, then have

$$\left| \int_{R_t} \min(q_\infty, 4) d\nu_{z, \ 1;t_j} \quad \min(q_\infty(y_j), 4) \right| \quad 2\sqrt{H_n(1 + t_j)} \to 0$$

as $j \to \infty$. However, Claim 22.9(d) of [B5] implies that the natural topology agrees on X agrees with the product topology on $C(Y) \quad (\ \infty, 0)$; because $(y_j, t_j) \to (z, \ 1)$ in the natural topology, we have $y_j \to z$ in $C(Y)$, hence

$$\lim_{j \to \infty} q_\infty(y_j) = q_\infty(z) = 1.$$

We can therefore find $\gamma > 0$ such that, for any $\tau \in (0, 1)$, we have

$$\frac{1}{\tau} \int_{1 \ 2\tau}^{1 \ \tau} \int_{M_i} (q_i)_+ d\nu^i_{z_i, \ 1;t} dt \quad \gamma^2$$

for $i = i(\tau, \gamma) \in \mathbb{N}$ sufficiently large. Because $(z_i, \ 1) \xrightarrow[i \to \infty]{C} (z, \ 1)$, there exists $A < \infty$ such that $(z_i, \ 1) \in P(x_\infty; A, \ A^2)$ for all $i \in \mathbb{N}$. We can therefore use Bamler's conjugate heat kernel comparison theorem (Proposition 8.1 in [B5]) and (5.4.2) to obtain

$$\int_{1 \ (\delta_i^0) \ 1}^{1 \ \delta_i^0} \int_{M_i} \left(|\nabla^2 q_i \quad 2g_i|^2 + \left| |\nabla q_i|^2 \quad 4q_i \right| \right) d\nu^i_{z_i, \ 1;t} dt \quad \delta_i^0$$

for some sequence $\delta_i^0 \searrow 0$. We may then proceed as in the proof of Proposition 13.19 of [B5] to conclude that, for any $\epsilon > 0$,

$$\frac{1}{2\sqrt{a_i}} q_i : M_i \quad [\ 1 \quad (\gamma\beta)^2 \epsilon^{\ 1}, \ 1 \quad (\gamma\beta)^2 \epsilon] \to \mathbb{R}$$

are $(1, \epsilon, \gamma\beta)$-splitting maps for $\beta \quad \overline{\beta}(\epsilon)$, where

$$a_i := \int_{M_i} q_i d\nu^i_{z_i, \ 1;t_1 \ (\beta\gamma)^2}$$

satisfies

$$\frac{1}{2}\gamma^2 \le a_i \le C\int_{M_i} q_i e^{\alpha f_i} d\nu_t^i \le C\left(\int_{M_i} q_i^2 d\nu_t^i\right)^{\frac{1}{2}}\left(\int_{M_i} e^{2\alpha f_i} d\nu_t^i\right)^{\frac{1}{2}} \le C(Y,\epsilon,\beta,\gamma).$$

In fact, the lower bound follows from the estimate for $|\nabla q_i|$, (c.f. the proof of Proposition 12.1 of [B5]), while the upper bound follows from the L^2-Poincare inequality and and property (iv) of strong almost-radial functions.

We now apply Proposition 12.1 of [B5] to obtain strong $(1,\epsilon'',\gamma\beta)$-splitting maps y_i'' with

$$\beta^{-2}\int_{-1-(\beta\gamma)^2\epsilon''^{-1}}^{-1-(\beta\gamma)^2\epsilon''}\int_{M_i}\left|\nabla\left(\frac{q_i}{2\sqrt{a_i}}-y_i''\right)\right|^2 d\nu^i_{z_i,-1;t}dt \le \Psi(\beta|Y,\epsilon'')$$

for sufficiently large $i\in\mathbb{N}$. Next, apply Proposition 28 to obtain a weak $(1,\epsilon',\gamma\beta)$-splitting map y_i' satisfying

$$\beta^{-2}\int_{-1-(\beta\gamma)^2\epsilon'^{-1}}^{-1-(\beta\gamma)^2\epsilon'}\int_{M_i}\left|\frac{J\nabla q_i}{2\sqrt{a_i}}-\nabla y_i'\right|^2 d\nu^i_{z_i,-1;t}dt \le \Psi(\beta|Y,\epsilon')$$

for sufficiently large $i\in\mathbb{N}$, assuming $\epsilon''\le\overline{\epsilon}'(\epsilon'')$. Another application of Proposition 12.1 of [B5] yields strong $(1,\epsilon,\gamma\beta)$-splitting maps $y_i^\beta : M_i\times[-1-\beta^2\epsilon^{-1},-1-\beta^2\epsilon]\to\mathbb{R}$ satisfying

$$\beta^{-2}\int_{-1-(\beta\gamma)^2\epsilon^{-1}}^{-1-(\beta\gamma)^2\epsilon}\int_{M_i}\left|\frac{J\nabla q_i}{2\sqrt{a_i}}-\nabla y_i^\beta\right|^2 d\nu^i_{z_i,-1;t}dt \le \Psi(\beta|Y,\epsilon)$$

for large $i=i(\beta)\in\mathbb{N}$, assuming $\epsilon'\le\overline{\epsilon}'(\epsilon)$. We also pass to a subsequence so that $a_i\to a\in(0,\infty)$. Then

$$(\psi_i^{-1})_*\left(J_i\frac{\nabla q_i}{2\sqrt{a_i}}\right)\to V$$

in $C^\infty_{loc}(\mathcal{R}_{C(Y)})$, where $V:=\frac{1}{2\sqrt{a}}J\nabla q_\infty$.

Using Theorem 17, choose a sequence $\beta_j\searrow 0$ such that we have $\mathbb{F}$-convergence of the corresponding parabolic rescalings to a tangent flow of $\mathcal{X}$ based at $(z,-1)$:

$$\left(\mathcal{X}^{-1,\beta_j^{-1}},(\nu^{-1,\beta_j^{-1}}_{(z,0);t})_{t\in[-2,0]}\right)\xrightarrow[i\to\infty]{\mathbb{F}}\left(\mathcal{Y},(\nu_{y_\infty;t})_{t\in[-2,0]}\right),$$

where Y is a static metric flow modeled on a a Ricci flat cone $C(Z)$. By Theorem 2.16 of [B5], there is a precompact exhaustion (W_j) of $\mathcal{R}_{C(Z)}$ along with diffeomorphisms $\eta_j : W_j \to \mathcal{R}_{C(Y)}$ such that $\eta_j^*(\beta_j^{-2} g_{C(Y)}) \to g_{C(Z)}$ in $C^\infty_{loc}(\mathcal{R}_{C(Z)})$, and so that for any $\epsilon > 0$ and $D < \infty$,

$$\eta_j : \big(W_j \cap B(o_Z, D), d_{C(Z)}\big) \to \big(C(Y), \beta_j^{-1} d_{C(Y)}\big)$$

are ϵ-Gromov-Hausdorff maps to $B(z, \beta_j D)$ for sufficiently large $j = j(\epsilon, D) \in \mathbb{N}$. Define $\widetilde{g}_j := \beta_j^{-2} g_{C(Y)}$ and $V_j := \beta_j V$, so that $|V_j|_{\widetilde{g}_j} \le 10$ on $\mathcal{R}_{C(Y)} \cap B(o, 10)$; by $\Delta V_j = 0$ and elliptic regularity, we can pass to a subsequence so that $(\eta_j^{-1})_* V_j \to V_\infty$ in $C^\infty_{loc}(\mathcal{R}_{C(Z)})$. Moreover, if we define $r_j := \beta_j^{-1} d(z, \cdot)$, then $r_j \circ \eta_j \to r_\infty := d(o_Z, \cdot)$ locally uniformly on $\mathcal{R}_{C(Z)}$; this follows from the Gromov-Hausdorff convergence $(X, \beta_j^{-1} d, z) \to (C(Z), d_{C(Z)}, o_Z)$ (Theorem 2.16 of [B5]). Because r_j are Lipschitz, we know $V_j r_j$ is well-defined almost everywhere on $\mathcal{R}_{C(Y)}$. Because $r_j \circ \phi_s = r_j$, we may then conclude $V_j r_j = 0$ almost everywhere. Given $\chi \in C_c^\infty(\mathcal{R}_{C(Z)})$, we have

$$\begin{aligned}\int_{\mathcal{R}_{C(Z)}} \phi V_\infty r_\infty \, dg_{C(Z)} &= -\int_{\mathcal{R}_{C(Z)}} r_\infty \operatorname{div}(\phi V_\infty) dg_{C(Z)} = -\lim_{j \to \infty} \int_{\mathcal{R}_{C(Y)}} r_j \operatorname{div}\big((\phi \circ \eta_j^{-1}) V_j\big) \, dg_{C(Y)} \\ &= \lim_{j \to \infty} \int_{\mathcal{R}_{C(Y)}} (\phi \circ \eta_j^{-1}) V_j r_j dg_{C(Y)} = 0\end{aligned}$$

since $r_j \circ \eta_j \to r_\infty$ uniformly and $(\eta_j^{-1})_* V_j \to V_\infty$ in C^∞_{loc}. Thus $V_\infty r_\infty = 0$ almost everywhere in $\mathcal{R}_{C(Z)}$, so the flow of V_∞ preserves r_∞.

By Bamler's change of basepoint theorem (Theorem 6.40 in [B3]), we have

$$(M_i, (g_{i,t})_{t \in [-\delta_i^{-1}, -1]}, (\nu_{z_i, -1;t})_{t \in [-\delta_i^{-1}, -1]}) \xrightarrow[i \to \infty]{\mathbb{F}, \mathfrak{C}} (\mathcal{X}, (\nu_{z,-1;t})_{t \in (-\infty, -1]})$$

on compact time intervals. For each $j \in \mathbb{N}$, we can therefore choose $i(j) \in \mathbb{N}$ such that $\eta_j(W_j) \subseteq U_{i(j)}$,

$$\|\psi_{i(j)}^* g_{i(j)} - g_{C(Y)}\|_{C^j(U_{i(j)} \times [-2, -1-\beta_j^4], g_{C(Y)})} + \|\psi_{i(j)}^* q_{i(j)} - q_\infty\|_{C^j(U_{i(j)} \times [-2, -1-\beta_j^4], g_{C(Y)})} \le \beta_j^4,$$

$$d_{\mathbb{F}}\Big((M_{i(j)}, (g_{i(j),t})_{t \in [-\beta_j^{-4}, 0]}, (\nu_{z_{i(j)}, -1;t})_{t \in [-\beta_j^{-4}, -1]}), (\mathcal{X}_{[-\beta_j^{-4}, -1]}, (\nu_{z,-1;t})_{t \in [-\beta_j^{-4}, -1]})\Big) < \beta_j^4,$$

$$\int_{-1-\epsilon_j^{-1}\beta_j^2}^{-1-\epsilon_j\beta_j^2}\int_{M_{i(j)}}\left|\frac{J\nabla q_{i(j)}}{2\sqrt{a_{i(j)}}}-\nabla y_{i(j)}^{\beta_j}\right|^2 d\nu_{z_{i(j)},-1;t}dt \leq \epsilon_j\beta_j^2$$

for some sequence $\epsilon_j \searrow 0$, where we now view η_j as maps $W_j \times [-2,-1] \to \mathbb{R}_{C(Y)} \times [-2,-1]$ which are constant in time. . Define parabolic rescalings $\widehat{g}_{j,t} := \beta_j^{-2} g_{i(j),-1+\beta_j^2 t}$, $\widehat{\nu}_t^j := \nu_{z_{i(j)},-1;-1+\beta_j^2 t}$, $(1,\epsilon_j,1)$-splitting maps

$$\widehat{y}_j(\cdot,t) := \beta_j^{-1} y_{i(j)}^{\beta_j}(\cdot,-1+\beta_j^2 t)$$

based at $(z_{i(j)},0)$ in the rescaled flow, $\widehat{a}_j := \beta_j^{-2} a_{i(j)}$, and

$$\widehat{q}_j(\cdot,t) := \beta_j^{-2} q_{i(j)}(\cdot,-1+\beta_j^2 t),$$

Then

$$(M_{i(j)},(\widehat{g}_{j,t})_{t\in[-2,0]},(\widehat{\nu}_t^j)_{t\in[-2,0]}) \xrightarrow[i\to\infty]{\mathbb{F}} (\mathcal{Y},(\nu_{o_Z,0;t})_{t\in[-2,0]}),$$

and $\psi_{i(j)} \circ \eta_j$ realizes smooth convergence on $\mathcal{R}_{C(Z)}$; for example, $(\psi_{i(j)} \circ \eta_j)^* \widehat{g}_j \to g_{C(Z)}$ in $C^\infty_{loc}(\mathcal{R}_{C(Z)})$. so the proof of Theorem 15.50 in [B5] shows that $\mathcal{Y} = \mathcal{Y}' \times \mathbb{R}$ splits as a metric flow, and $(\psi_{i(j)} \circ \eta_j)^* \widehat{y}_j \to y_\infty$, where $y_\infty : \mathcal{Y} \to \mathbb{R}$ denotes the projection onto the $\mathbb{R}$-factor. On the other hand, our assumptions guarantee that

$$\left\|\frac{1}{2\sqrt{\widehat{a}_j}}(\psi_{i(j)}^{-1})^* \nabla^{\widehat{g}_j}\widehat{q}_j(\cdot,t) - V_j\right\|_{C^{j-1}(U_{i(j)},\widetilde{g}_j)} \leq C\beta_j^2,$$

which implies

$$\frac{1}{2\sqrt{\widehat{a}_j}}((\psi_{i(j)} \circ \eta_j)^{-1})^* \nabla^{\widehat{g}_j}\widehat{q}_j(\cdot,t) \to V_\infty$$

in $C^\infty_{loc}(\mathcal{R}_{C(Z)})$. Here, we again view η_j as a map $W_j \to U_{i(j)}$ which is constant in time. Let $\widehat{K}_j$ denote the heat kernel of the rescaled flows. For any compact subset $K \subseteq \mathcal{R}_{C(Z)}$, we then have

$$\int_{-2}^{-1}\int_K |V_\infty - \nabla y_\infty|^2 d\nu_{o_Z,-1;t}dt$$

$$= \lim_{j\to\infty}\int_{-2}^{-1}\int_K \left|\left((\psi_{i(j)}\circ\eta_j)^{-1}\right)_*\left(\frac{\nabla^{\widehat{g}_j}\widehat{q}_j}{2\sqrt{\widehat{a}_j}} - \nabla^{\widehat{g}_j}\widehat{y}_j\right)\right|^2 (\psi_{i(j)}\circ\eta_j)^*\widehat{K}_j(z_{i(j)},0;\cdot,t)d\left((\psi_{i(j)}\circ\eta_j)^*\widehat{g}_j\right)dt$$

$$= \lim_{j\to\infty}\int_{-2}^{-1}\int_{(\psi_{i(j)}\circ\eta_j)(K)}\left|\frac{\nabla^{\widehat{g}_j}\widehat{q}_j}{2\sqrt{\widehat{a}_j}} - \nabla^{\widehat{g}_j}\widehat{y}_j\right|^2 d\widehat{\nu}^j_t dt$$

$$\le \liminf_{j\to\infty}\beta_j^{-2}\int_{-1-2\beta_j^2}^{-1-\beta_j^2}\int_{M_{i(j)}}\left|\frac{\nabla^{g_{i(j)}}q_j}{2\sqrt{a_{i(j)}}} - \nabla^{g_{i(j)}}y_{i(j)}\right|^2 d\nu_{z_{i(j)},-1;t}dt = 0,$$

where we used that

$$\left((\psi_{i(j)}\circ\eta_j)^{-1}\right)_*\nabla^{\widehat{g}_j}\widehat{y}_j = \nabla^{(\psi_{i(j)}\circ\eta_j)^*\widehat{g}_j}(\psi_{i(j)}\circ\eta_j)^*\widehat{y}_j \to \nabla^{g_{C(Z)}}y_\infty$$

in $C^\infty_{loc}(\mathcal{R}_{C(Z)})$. Thus $V_\infty = \nabla y_\infty$. However, ∇y_∞ is a complete vector field on $\mathcal{R}_{C(Z)}$ which leaves any compact set in finite time, whereas the flow of V_∞ preserves any geodesic ball centered at o_Z, a contradiction. □

Proof of Theorem 37. This is a consequence of Propositions 30 and 31. □

5.5 Sequences of Noncollapsed Ricci Flows whose $\mathbb{F}$-Limits are Split Solitons

In this section, we show the qualitative equivalence of Bamler's notions of quantitative strata and weak quantitative strata, with one direction already established in [B5]. We begin with an elementary lemma concerning the 1-Wasserstein distance between product measures.

Lemma 23. *Suppose $(X,d^X),(Y,d^Y)$ are metric spaces and $\mu_1,\mu_2\in\mathcal{P}(X)$, $\nu_1,\nu_2\in\mathcal{P}(Y)$. Then*

$$d_{W_1}^{X\times Y}(\mu_1\otimes\nu_1,\mu_2\otimes\nu_2)\le d^X_{W_1}(\mu_1,\mu_2)+d^Y_{W_1}(\nu_1,\nu_2).$$

Proof. Suppose q_X is a coupling of (μ_1, μ_2) and q_Y is a coupling of (ν_1, ν_2). Define $\sigma : X \times X \times Y \times Y \to X \times Y \times X \times Y$, $(x_1, x_2, y_1, y_2) \mapsto (x_1, y_1, x_2, y_2)$, and $q := \sigma_*(q_X \otimes q_Y)$. Then q is a coupling of $(\mu_1 \otimes \nu_1, \mu_2 \otimes \nu_2)$, so we can estimate

$$\begin{aligned} d_{W_1}^{X\times Y}(\mu_1 \otimes \nu_1, \mu_2 \otimes \nu_2) &\leq \int_{X\times Y\times X\times Y} d^{X\times Y}((x_1, y_1), (x_2, y_2))\, dq(x_1, y_1, x_2, y_2) \\ &\leq \int_{X\times X\times Y\times Y} \left(d^X(x_1, x_2) + d^Y(y_1, y_2)\right) dq_X(x_1, x_2) dq_Y(y_1, y_2) \\ &= \int_{X\times X} d^X(x_1, x_2) dq_X(x_1, x_2) + \int_{Y\times Y} d^Y(y_1, y_2) dq_Y(y_1, y_2). \end{aligned}$$

Taking the infimum over all such couplings q_X, q_Y gives the remaining claim. □

We now show that if a metric soliton splits a factor of $\mathbb{R}^k$, this can be used to extract a sequence of approximating points in smooth Ricci flows which are almost-selfsimilar and almost-split. This is analogous to the existence of almost-splitting maps in section 2.6 of [CC1] given Gromov-Hausdorff closeness to a metric product.

Proposition 32. *Suppose $(\mathcal{X}, (\mu_t)_{t\in(-\infty,0]})$ is a future continuous metric soliton satisfying the following:*

(a) *$(\mathcal{X}, (\mu_t)_{t\in(-\infty,0)})$ is an $\mathbb{F}$-limit of n-dimensional closed Ricci flows $(M_i, (g_{i,t})_{t\in(-\delta_i^{-1},0]}, (\nu_{x_i,0;t})_{t\in(-\delta^{-1},0]})$ as in Theorem 16, with $\mathcal{N}_{x_i,0}(1) \geq -Y$,*

(b) *$(\mathcal{X}, (\mu_t)_{t\in(-\infty,0)}) = (\mathcal{X}' \times \mathbb{R}^k, (\mu_t' \otimes \mu_{\mathbb{R}^k})_{t\in(-\infty,0)})$ as metric flow pairs for some metric soliton $(\mathcal{X}', (\mu_t')_{t\in(-\infty,0)})$, and this identification restricts to an isometry of Ricci flow spacetimes $\mathcal{R} = \mathcal{R}' \times \mathbb{R}^k$,*

(c) *Writing $d\mu_t = (4\pi\tau)^{-\frac{n}{2}} e^{-f} dg$ on $\mathcal{R}$ and $d\mu_t' = (4\pi\tau)^{-\frac{n}{2}} e^{-f'} dg'$ on $\mathcal{R}'$, we have $Rc(g) + \nabla^2 f = \frac{1}{2\tau} g$ on $\mathcal{R}$, $Rc(g') + \nabla^2 f' = \frac{1}{2\tau} g'$ on $\mathcal{R}'$, and $f = f' + \frac{1}{4\tau}|x|^2$ on $\mathcal{R}$.*

(d) *$\mathcal{N}_{(\mu_t)}(\tau) = W$ for all $\tau > 0$, where $W \in [-Y, 0]$.*

Then for any $\epsilon > 0$, $(x_i, 0)$ is $(\epsilon, 1)$-selfsimilar and $(k, \epsilon, 1)$-split for sufficiently large $i =$

$i(\epsilon) \in \mathbb{N}$. *If in addition,* $Rc(g) = 0$*, then* $(x_i, 0)$ *is also* $(\epsilon, 1)$*-static for sufficiently large* $i \in \mathbb{N}$.

Proof. By Nash entropy convergence (Theorem 15.45 of [B5]), assumption (d) and Proposition 7.1 of [B5] imply that $(x_i, 0)$ is $(\epsilon, 1)$-selfsimilar for sufficiently large $i \in \mathbb{N}$. If $Rc(g) = 0$, then $(x_i, 0)$ is $(\epsilon, 1)$-static for large $i \in \mathbb{N}$ by Claim 22.7 of [B5]. Suppose the remaining claim fails, so that there exists $\epsilon > 0$ such that, after passing to a subsequence, $(x_i, 0)$ is not $(k, \epsilon, 1)$-split. Fix a correspondence $\mathfrak{C}$ such that

$$(M_i^n, (g_{i,t})_{t\in(-\delta_i^{-1},0]}, (\nu_{x_i,0;t})_{t\in(-\delta_i^{-1},0]}) \xrightarrow[i\to\infty]{\mathbb{F},\mathfrak{C}} (\mathcal{X}, (\nu_{x_\infty;t})_{t\in(-\infty,0]}) \tag{5.5.1}$$

on compact time intervals. By passing to a subsequence, we can assume the convergence is time-wise for all times in some subset $I' \subset (-\infty, 0)$, where $|\mathbb{R} \setminus I'| = 0$.

Choose a sequence $t_j \nearrow 0$, and recall that

$$H_n|t| \geq \mathrm{Var}(\mu_t) = \mathrm{Var}(\mu'_t) + \mathrm{Var}(\mu_t^{\mathbb{R}^k}) \geq \mathrm{Var}(\mu'_t),$$

so we can find $z_j \in \mathcal{X}'_{t_j}$ such that $\mathrm{Var}(\delta_{z_j}, \mu'_{t_j}) < H_n|t_j|$. Then

$$d_{W_1}^{\mathcal{X}_{t_j}}(\delta_{(z_j,0^k)}, \mu_{t_j}) = d_{W_1}^{\mathcal{X}'_{t_j}\times\mathbb{R}^k}(\delta_{z_j}\otimes\delta_{0^k}, \mu'_{t_j}\otimes\mu_{t_j}^{\mathbb{R}^k}) \leq d_{W_1}^{\mathcal{X}'_{t_j}}(\delta_{z_j}, \mu'_{t_j}) + d_{W_1}^{\mathbb{R}^k}(\delta_{0^k}, \mu_{t_j}^{\mathbb{R}^k}) \leq 2H_n|t_j|.$$

Fix $\tau > 0$. Letting $e^\alpha \in \mathbb{R}^k$ be the standard basis vectors, we can find $z_{j,i}^\alpha, z_{j,i} \in M_i$ such that $(z_{j,i}^\alpha, t_0) \xrightarrow[i\to\infty]{\mathfrak{C}} (z_j, e^\alpha)$ and $(z_{j,i}, t_j) \xrightarrow[i\to\infty]{\mathfrak{C}} (z_j, 0^k)$, so that there are subsets $E_{j,i} \subset [-\tau^{-1}, 0]$ such that $\lim_{i\to\infty}|E_{j,i}| = 0$, $[-\tau^{-1}, t_j] \setminus E_{j,i} \subset I''^{,j}$ and

$$\lim_{i\to\infty} \sup_{t\in[-\tau^{-1},t_j]\setminus E_{j,i}} d_{W_1}^{Z_t}\Big((\varphi_t^i)_*\nu^i_{(z_{j,i}^\alpha,t_j);t}, (\varphi_t^\infty)_*\nu^i_{(z_j,e^\alpha);t}\Big) = 0$$

$$\lim_{i\to\infty} \sup_{t\in[-\tau^{-1},t_j]\setminus E_{j,i}} d_{W_1}^{Z_t}\Big((\varphi_t^i)_*\nu^i_{(z_{j,i},t_j);t}, (\varphi_t^\infty)_*\nu^i_{(z_j,0^k);t}\Big) = 0.$$

We can pass to further subsequences and use a diagonal argument to obtain subsets $E_j \subset [-j, 0]$ with $[t_j, 0] \subset E_j$ such that $|E_j| \leq 2^{-j}$, $[-\tau^{-1}, 0] \setminus E_j \subset I''^{,j}$, and $z'_j, z_j'^{,\alpha} \in M_j$ such that

$$\sup_{t\in[-j,0]\setminus E_j} d_{W_1}^{Z_t}\Big((\varphi_t^j)_*\nu^j_{(z'_j,t_j);t}, (\varphi_t^\infty)_*\nu_{(z_j,0^k);t}\Big) \leq 2^{-j},$$

$$\sup_{t\in[-j,0]\setminus E_j} d_{W_1}^{Z_t}\Big((\varphi_t^j)_*\nu^j_{(z_j^{\prime,\alpha},t_j);t},(\varphi_t^\infty)_*\nu_{(z_j,e^\alpha);t}\Big)\le 2^{-j}$$

We now show $(\nu^j_{(z_j',t_j);t})_{t\in[-j,t_j]}\xrightarrow[i\to\infty]{\mathfrak{C}}(\mu_t)_{t\in(-\infty,0)}$ uniformly on compact time intervals. In fact

$$d_{W_1}^{Z_t}\Big((\varphi_t^j)_*\nu^j_{(z_j',t_j);t},(\varphi_t^\infty)_*\mu_t\Big)\le d_{W_1}^{Z_t}\Big((\varphi_t^j)_*\nu^j_{(z_j',t_j);t},(\varphi_t^\infty)_*\nu_{(z_j,0^k);t}\Big)+d_{W_1}^{\mathcal{X}_t}(\nu_{(z_j,0^k);t},\mu_t)$$
$$\le 2^{-j}+2H_n|t_j|$$

for all $t\in[-j,0]\setminus E_j$. Similarly, for any $\tau>0$, for sufficiently large $j\in\mathbb{N}$, we have

$$d_{W_1}^{\mathcal{X}_{t_j}}(\delta_{(z_j,e^\alpha)},\mu'_{t_j}\otimes\nu^{\mathbb{R}^k}_{e^\alpha;t_j})\le d_{W_1}^{\mathcal{X}_{t_j}}(\delta_{z_j},\mu'_{t_j})+d_{W_1}^{\mathcal{X}_{t_j}}(\delta_{e^\alpha},\nu^{\mathbb{R}^k}_{e^\alpha;t_j})\le 2H_n|t_j|,$$

so by repeating the above reasoning with (z_j',t_j) replaced by $(z_j^{\prime,\alpha},t_j)$, we can also assume that

$$(\nu_{(z_j^{\prime,\alpha},t_j);t})_{t\in[-T_j,0]}\xrightarrow[j\to\infty]{\mathfrak{C}}(\mu'_t\otimes\nu^{\mathbb{R}^k}_{e^\alpha;t})_{t\in(-\infty,0)}$$

on compact time intervals. Because $(\nu_{x_j,0;t})_{t\in[-T_j,0]}\xrightarrow[j\to\infty]{\mathfrak{C}}(\mu_t)_{t\in(-\infty,0)}$, we can pass to a further subsequence to find subsets E_j' with $|E_j'|\le 2^{-j}$ such that

$$d_{W_1}^{Z_t}\Big((\varphi_t^\infty)_*\mu_t,(\varphi_t^j)_*\nu^j_{(x_j,0);t}\Big)\le 2^{-j}$$

for all $t\in[-j,0]\setminus E_j'$. It follows that

$$d_{W_1}^{g_{j,t}}(\nu^j_{(z_j',t_j);t},\nu^j_{(x_j,0);t})\le d_{W_1}^{Z_t}\Big((\varphi_t^j)_*\nu^j_{(z_j',t_j);t},(\varphi_t^\infty)_*\mu_t\Big)+d_{W_1}^{Z_t}\Big((\varphi_t^\infty)_*\mu_t,(\varphi_t^j)_*\nu^j_{(x_j,0);t}\Big)\le 2^{-j+1} \tag{5.5.2}$$

for all $t\in[-j,0]\setminus(E_j\cup E_j')$, where $|E_j\cup E_j'|\le 2^{-j+1}$.

Let $\psi_j:U_j\to V_j\subseteq M_j\times[-j,0]$ be time-preserving diffeomorphisms from a precompact exhaustion (U_j) of $\mathcal{R}$ realizing smooth convergence as in Theorem 16. Then $\psi_j^*K(z_j',t_j;\cdot,\cdot)\to(4\pi\tau)^{-\frac{n}{2}}e^{-f}$ in $C^\infty_{loc}(\mathcal{R})$ and $f=f'+\frac{1}{4\tau}|x|^2$, so we have the following convergence in $C^\infty_{loc}(\mathcal{R})$:

$$\psi_j^*K(z_j',t_j;\cdot,\cdot)\to(4\pi\tau)^{-\frac{n}{2}}e^{-f'-\frac{|x|^2}{4\tau}},$$

$$\psi_j^*K(z_j^{\prime,\alpha},t_j;\cdot,\cdot)\to(4\pi\tau)^{-\frac{n}{2}}e^{-f'-\frac{|x-e^\alpha|^2}{4\tau}}.$$

Writing $K(z_j', t_j; \cdot, \cdot) = (4\pi\tau_j)^{-\frac{n}{2}} e^{-f_j}$ and $K(z_j'^{,\alpha}, t_j; \cdot, \cdot) = (4\pi\tau_j)^{-\frac{n}{2}} e^{-f_j^\alpha}$, this means

$$\psi_j f_j \to f' + \frac{|x|^2}{4\tau}, \qquad \psi_j f_j^\alpha \to f' + \frac{|x - e^\alpha|^2}{4\tau}$$

in $C^\infty_{loc}(\mathcal{R})$.

Claim 1: For any $\delta' > 0$ when $j = j(\delta')$ is sufficiently large, (z_j', t_j) and $(z_j'^{,\alpha}, t_j)$ are $(\delta', 1)$-selfsimilar.

Because $\int_{\mathcal{R}'_{-\tau}} f' d\mu'_t - \frac{(n-k)}{2} = W$ for all $t \in (-\infty, 0)$, the proof of Corollary 15.47 in [B5] gives the following for any $\tau \in (0, \infty)$:

$$\lim_{j \to \infty} \mathcal{N}_{z_j';t_j}(\tau) = \int_{\mathcal{R}_{-\tau}} f d\mu_t - \frac{n}{2} = \int_{\mathcal{R}'_{-\tau} \times \mathbb{R}^k} \left(f' + \frac{|x|^2}{4\tau} \right) d(\mu'_t \otimes \mu_t^{\mathbb{R}^k}) - \frac{n}{2}$$
$$= \int_{\mathcal{R}'_{-\tau}} f' d\mu'_t - \frac{(n-k)}{2} =: W$$

and similarly,

$$\lim_{j \to \infty} \mathcal{N}_{z_j'^{,\alpha};t_j}(\tau) = \int_{\mathcal{R}'_{-\tau} \times \mathbb{R}^k} \left(f' + \frac{|x - e^\alpha|^2}{4\tau} \right) d(\mu'_t \otimes \mu_t^{\mathbb{R}^k}) - \frac{n}{2} = W.$$

The claim follows by choosing τ sufficiently large and appealing to Proposition 7.1 of [B5].

Choose $t \in [-1, -\frac{1}{2}] \cap I'$. Then

$$d_{W_1}^{g_{j,-1}} \left(\nu^j_{z_j'^{,\alpha}, t_j; -1}, \nu^j_{z_j', t_j; -1} \right) \leq d_{W_1}^{Z_t} \left((\varphi_t^j)_* \nu^j_{z_j'^{,\alpha}, t_j; t}, (\varphi_t^\infty)_* (\mu'_t \otimes \nu^{\mathbb{R}^k}_{(e^\alpha, 0^k); t}) \right)$$
$$+ d_{W_1}^{Z_t} \left((\varphi_t^j)_* \nu^j_{z_j', t_j; t}, (\varphi_t^\infty)_* (\mu'_t \otimes \mu_t^{\mathbb{R}^k}) \right) + d_{W_1}^{\mathbb{R}^k} \left(\nu^{\mathbb{R}^k}_{e^\alpha, 0; t}, \mu_t^{\mathbb{R}^k} \right).$$

Because $d_{W_1}^{\mathbb{R}^k} \left(\nu^{\mathbb{R}^k}_{e^\alpha, 0; t}, \mu_t^{\mathbb{R}^k} \right) \leq 1$ and the $\mathcal{F}$-convergence (5.5.1) is timewise at time t, we have

$$d_{W_1}^{g_{j,-1}} \left(\nu^j_{z_j'^{,\alpha}, t_j; -1}, \nu^j_{z_j', t_j; -1} \right) \leq 2$$

for sufficiently large $j \in \mathbb{N}$. Now fix $\delta > 0$ and $\beta > 0$ small. Taking $\delta' \leq \overline{\delta}'(\delta, n, Y)$ in Claim 1, we may therefore appeal to the proof of Proposition 10.8 in [B5] (up to Claim 10.31) to conclude that, setting $W_j' := \mathcal{N}_{z_j', t_j}(1)$, $W_j^\alpha := \mathcal{N}_{z_j'^{,\alpha}, t_j}(1)$, the functions

$$u_j^\alpha := 2\tau_j (f_j^\alpha - f_j') - 2\tau_j (W_j^\alpha - W_j')$$

satisfy the following properties for all $j = j(\delta) \in \mathbb{N}$ sufficiently large, assuming $\gamma \leq \overline{\gamma}$:

(i) $\int_{-\delta^{-1}}^{-\delta} \int_{M_j} \left(\tau_j^{\frac{1}{2}} |\partial_t u_j^\alpha| + |\nabla^2 u_j^\alpha|^2 \right) e^{\gamma f_j'} d\nu^j_{z_j', t_j; t} dt \leq \delta$,

(ii) $\int_{-\delta^{-1}}^{-\delta} \int_{M_j} \tau_j^{-1} |\nabla u_j^\alpha|^4 e^{2\gamma f_j'} d\nu^j_{z_j', t_j; t} dt \leq C(Y, \gamma)$,

(iii) $\int_{-\delta^{-1}}^{-\delta} \int_{M_j} |\langle \nabla u_j^\alpha, \nabla u_j^\beta \rangle - q^j_{\alpha\beta}| d\nu^j_{z_j', t_j; t} dt \leq \delta$ for some $q^j_{\alpha\beta} \in \mathbb{R}$.

Moreover, for any $b > 0$, we can combine $(ii), (iii)$ to estimate

$$\begin{aligned} \int_{-2\epsilon^{-1}}^{-\frac{1}{2}\epsilon} \int_{M_j} |\langle \nabla u_j^\alpha, \nabla u_j^\beta \rangle - q^j_{\alpha\beta}| e^{\gamma f_j'} d\nu_t dt &\leq \frac{1}{b} \int_{-2\epsilon^{-1}}^{-\frac{1}{2}\epsilon} \int_{M_j} |\langle \nabla u_j^\alpha, \nabla u_j^\beta \rangle - q^j_{\alpha\beta}| d\nu_t dt \\ &\quad + b \int_{-2\epsilon^{-1}}^{-\frac{1}{2}\epsilon} \int_{M_j} |\langle \nabla u_j^\alpha, \nabla u_j^\beta \rangle - q^j_{\alpha\beta}| e^{2\gamma f_j'} d\nu^j_{z_j', t_j; t} \\ &\leq b^{-1}\delta + C(\epsilon, Y, \gamma) b. \end{aligned}$$

For any $\delta' > 0$, we can therefore choose $b \leq \overline{b}(\delta', Y, \gamma)$ and then $\delta \leq \overline{\delta}(\delta', Y, \gamma)$ so that

(iii$'$) $\int_{-2\epsilon^{-1}}^{-\frac{1}{2}\epsilon} \int_{M_j} |\langle \nabla u_j^\alpha, \nabla u_j^\beta \rangle - q^j_{\alpha\beta}| e^{\gamma f_j'} d\nu_t dt \leq \delta'$ for some $q^j_{\alpha\beta} \in \mathbb{R}$

whenever $j = j(\epsilon, \delta') \in \mathbb{N}$ is sufficiently large.

Claim 2: For any $\delta'' > 0$, twe have $|q^j_{\alpha\beta} - \delta_{\alpha\beta}| \leq \delta''$ for sufficiently large $j = j(\delta'') \in \mathbb{N}$.

Fix any compact set $K \subset \mathcal{R}_{[-2,-1]}$, and set $K_t := K \cap \mathcal{R}_t$ or $t \in [-2, -1]$. For sufficiently large j, we can estimate

$$\begin{aligned} &\int_{-2}^{-1} \int_{M_j} |\langle \nabla u_j^\alpha, \nabla u_j^\beta \rangle - \delta_{\alpha\beta}| d\nu^j_{z_j', t_j; t} dt \\ &\qquad = \int_{-2}^{-1} \int_{\psi_{j,t}(K_t)} |\langle \nabla u_j^\alpha, \nabla u_j^\beta \rangle - \delta_{\alpha\beta}| d\nu^j_{z_j', t_j; t} dt + \int_{-2}^{-1} \int_{M_j \setminus \psi_{j,t}(K_t)} |\langle \nabla u_j^\alpha, \nabla u_j^\beta \rangle - \delta_{\alpha\beta}| d\nu^j_{z_j', t_j; t} dt \\ &\qquad \leq \int_{-2}^{-1} \int_{K_t} 2\tau_j \left| \langle \nabla (f_j^\alpha - f_j), \nabla (f_j^\beta - f_j) \rangle \circ \psi_j - \delta_{\alpha\beta} \right| \left(K(z_j', t_j; \cdot, \cdot) \circ \psi_j \right) d(\psi_{j,t}^* g_{j,t}) dt \\ &\qquad\quad + \left(\int_{-2}^{-1} \int_{M_j \setminus \psi_j(K)} |\langle \nabla u_j^\alpha, \nabla u_j^\beta \rangle - \delta_{\alpha\beta}|^2 d\nu^j_{z_j', t_j; t} dt \right)^{\frac{1}{2}} \left(\int_{-2}^{-1} \nu^j_{z_j', t_j; t}(M \setminus \psi_j(K)) dt \right)^{\frac{1}{2}} \\ &\qquad \leq \sup_K \left| 2\tau_j \langle \nabla (f_j^\alpha - f_j), \nabla (f_j^\beta - f_j) \rangle \circ \psi_j - \delta_{\alpha\beta} \right| \\ &\qquad\quad + C(Y, \gamma) \left(1 - \int_{-2}^{-1} \nu^j_{z_j', t_j; t}(\psi_{j,t}(K_t)) dt \right)^{\frac{1}{2}}, \end{aligned}$$

where we used estimate (ii) to obtain the last inequality. Next, we observe that as $j \to \infty$, $2\tau_j \langle \nabla(f_j^\alpha - f_j), \nabla(f_j^\beta - f_j)\rangle \circ \psi_j$ converges in $C^\infty_{loc}(\mathbb{R})$ to

$$2\tau \left\langle \nabla\left(f' + \frac{|x - e^\alpha|^2}{4\tau}\right) - \nabla\left(f' + \frac{|x|^2}{4\tau}\right), \nabla\left(f' + \frac{|x - e^\beta|^2}{4\tau}\right) - \nabla\left(f' + \frac{|x|^2}{4\tau}\right)\right\rangle$$
$$= \big\langle (x - e^\alpha) - x, (x - e^\beta) - x\big\rangle = \delta_{\alpha\beta}$$

Moreover, we have

$$\int_{-2}^{-1} \nu^j_{z_j',t_j;t}(\psi_{j,t}(K_t))dt = \int_{-2}^{-1}\int_{K_t} \big(K(z_j', t_j; \cdot, t) \circ \psi_{j,t}\big)\, d(\psi_{j,t} g_{j,t})dt \to \int_{-2}^{-1} \mu_t(K)dt.$$

Combining expressions, we get

$$\begin{aligned}\limsup_{j\to\infty} |\delta_{\alpha\beta} - q^j_{\alpha\beta}| &\le \limsup_{j\to\infty}\int_{-2}^{-1}\int_{M_j} |\langle \nabla u_j^\alpha, \nabla u_j^\beta\rangle - \delta_{\alpha\beta}| d\nu^j_{z_j',t_j;t}dt \\ &\quad + \limsup_{j\to\infty}\int_{-2}^{-1}\int_{M_j} |\langle \nabla u_j^\alpha, \nabla u_j^\beta\rangle - q^j_{\alpha\beta}| d\nu^j_{z_j',t_j;t}dt \\ &\le C(\gamma, n)\left(1 - \int_{-2}^{-1} \mu_t(K)dt\right)^{\frac{1}{2}} + \delta'\end{aligned}$$

for any compact set $K \subseteq \mathbb{R}_{[-2,-1]}$. Choosing a compact exhaustion of $\mathbb{R}_{[-2,-1]}$ and $\delta' \le \overline{\delta}'(\delta'')$ then gives the claim.

By the W_1-distance estimate (5.5.2), we can apply Proposition 8.1 of [B5] with the following choice of parameters: $\alpha_0 = \gamma$, $\alpha_1 = 0$, $t_0 = t_j$, $t_1 = 0$, $D = 1$, $s \in [-\epsilon^{-1}, -\epsilon]$, $t_* = \frac{1}{2}\theta(1)\gamma\epsilon$, where θ is defined in the aforementioned proposition; then for $j \in \mathbb{N}$ sufficiently large, we have

$$d\nu^j_{x_j,0;s} \le C(Y,\gamma)e^{\gamma f_j'} d\nu^j_{z_j',t_j;s}$$

for all $s \in [-\epsilon^{-1}, -\epsilon]$. Combining this with $(i), (iii')$, the $(\delta, 1)$-selfsimilarity of (z_i', t_i), and taking δ sufficiently small gives the desired contradiction. □

Next, we verify that an $\mathbb{F}$-limit of metric solitons splitting $\mathbb{R}^k$ is also a metric soliton splitting $\mathbb{R}^k$.

Lemma 24. *Suppose $(\mathcal{X}_i, (\mu^i_t)_{t\in(-\infty,0)})$ is a sequence of metric solitons satisfying the following:*

- $(a)_i$ *Each pair $(\mathcal{X}_i, (\mu^i_t)_{t\in(-\infty,0)})$ is an $\mathbb{F}$-limit of n-dimensional closed Ricci flows as in Theorem 16, with Nash entropy lower bound $-Y$,*
- $(b)_i$ $(\mathcal{X}_i, (\mu^i_t)_{t\in(-\infty,0)}) = (\mathcal{X}^0_i \times \mathbb{R}^k, (\mu^{0,i}_t \otimes \mu_{\mathbb{R}^k})_{t\in(-\infty,0)})$ *as metric flow pairs for some metric solitons $(\mathcal{X}^0_i, (\mu^{0,i}_t)_{t\in(-\infty,0)})$, and this identification restricts to an isometry of Ricci flow spacetimes $\mathcal{R}_i = \mathcal{R}^0_i \times \mathbb{R}^k$,*
- $(c)_i$ *Writing $d\mu^i_t = (4\pi\tau)^{-\frac{n}{2}} e^{-f_i} dg_i$ on $\mathcal{R}_i$ and $d\mu^{0,i}_t = (4\pi\tau)^{-\frac{n}{2}} e^{-f^0_i} dg^0_i$ on $\mathcal{R}^0_i$, we have $Rc(g_i) + \nabla^2 f_i = \frac{1}{2\tau} g_i$ on $\mathcal{R}_i$, $Rc(g^0_i) + \nabla^2 f^0_i = \frac{1}{2\tau} g^0_i$ on $\mathcal{R}^0_i$, and $f_i = f^0_i + \frac{1}{4\tau}|x|^2$ on $\mathcal{R}_i$.*
- $(d)_i$ $\mathcal{N}_{(\mu_{i,t})}(\tau) = W_i$ *for all $\tau > 0$, where $W_i \in [-Y, 0]$.*

Assume that $(\mathcal{X}_i, (\mu^i_t)_{t\in(-\infty,0)})$ $\mathbb{F}$-converge to another metric flow pair $(\mathcal{X}_\infty, (\mu^\infty_t)_{t\in(-\infty,0)})$. Then $(\mathcal{X}, (\mu^\infty_t)_{t\in(-\infty,0)})$ is a metric soliton satisfying $(a)-(d)$ of Proposition 32.

If in addition $Rc(g^0_i) = 0$, then we have $(\mathcal{X}_\infty, (\mu^\infty_t)) = (\mathcal{X}'' \times \mathbb{R}^k, (\nu_{x'';t})_{t\in(-\infty,0)})$ as metric flow pairs for some static metric cone $\mathcal{X}''$ with vertex x''. Moreover, this identification restricts to an isometry of Ricci flow spacetimes $\mathcal{R}^\infty = \mathcal{R}'' \times \mathbb{R}^k$, and $Rc(g^\infty) = 0$, $Rc(g'') = 0$.

Proof. By passing to subsequences and using a diagonal argument, we may assume that

$$(M^n_i, (g_{i,t})_{t\in(-\epsilon_i^{-1},0]}, (x_i, 0))$$

is a sequence of closed, pointed Ricci flows such that $\mathcal{N}_{x_i,0}(1) \geq -Y$,

$$d_{\mathbb{F}}\Big((M_i, (g_{i,t})_{t\in(-\epsilon_i^{-1},0]}, (\nu_{x_i,0;t})_{t\in(-\epsilon_i^{-1},0]}), (\mathcal{X}_i, (\mu^i_t)_{t\in(-\epsilon_i^{-1},0]})\Big) \leq \epsilon_i,$$

where $\lim_{i\to\infty} \epsilon_i = 0$. By Proposition 32, we may moreover assume that $(x_i, 0)$ are $(\epsilon_i, 1)$-selfsimilar and strongly $(k, \epsilon_i, 1)$-split. In particular,

$$(M_i, (g_{i,t})_{t\in(-T_i,0]}, (\nu_{x^0_i,0;t})_{t\in(-T_i,0]}) \xrightarrow[i\to\infty]{\mathbb{F}} (\mathcal{X}_\infty, (\mu^\infty_t)_{t\in(-\infty,0]}).$$

Then $\mathcal{X}$ satisfies (a) by construction, (b) by Theorem 15.50 in [B5], and (d) by Proposition 7.1 in [B5] and the Nash entropy convergence Theorem 15.45 of [B5]. Moreover, we have $Rc(g) + \nabla^2 f = \frac{1}{2\tau}g$ on $\mathcal{R}$ by Theorem 15.69 of [B5]. By (b), we have

$$(4\pi\tau)^{-\frac{n}{2}}e^{-f}dg = \left((4\pi\tau)^{-\frac{n-k}{2}}e^{-f'}dg'\right) \otimes \left((4\pi\tau)^{-\frac{k}{2}}e^{-\frac{|x|^2}{4\tau}}\right) = (4\pi\tau)^{-\frac{n}{2}}e^{-f'-\frac{|x|^2}{4\tau}}dg$$

on $\mathcal{R}$, so that $f = f' + \frac{|x|^2}{4\tau}$. Thus $Rc(g), g, \nabla^2 f$ all split with respect to the decomposition $T\mathcal{R} = T\mathcal{R}' \oplus T\mathbb{R}^k$, so restricting $Rc(g) + \nabla^2 f = \frac{1}{2\tau}g$ to $T\mathcal{R}'$ gives $Rc(g') + \nabla^2 f' = \frac{1}{2\tau}g'$ on $\mathcal{R}'$.

Finally, suppose that $Rc(g_i') = 0$ for all $i \in \mathbb{N}$. By the proof of Claim 22.7 in [B5], we can also assume $(x_i, 0)$ are $(\epsilon_i, 1)$-static. Then Theorem 15.80 of [B5] guarantees the remaining claims. □

We may now apply the previous results to prove the reverse qualitative inclusion of quantitative singular strata as that proved in [B5]

Proposition 33. *For any $\epsilon > 0$, there exists $\delta = \delta(\epsilon, Y, A) > 0$ such that the following holds. Suppose $(\mathcal{X}, (\nu_{x_\infty;t})_{t\in(-T,0]})$ is a metric flow pair obtained as an $\mathbb{F}$-limit of noncollapsed Ricci flows as in Theorem 16 (with $\mathcal{N}_{x_i,0}(1) \geq -Y$). Assume $y_\infty \in \mathcal{X}_{<0} \cap P^*(x_\infty, A, -A^2)$ is (k, δ, r)-symmetric and that $(M, (g_t)_{t\in[-\delta^{-1},0]}, (y_0, 0))$ is a closed pointed Ricci flow such that $|\mathcal{N}_{y_0,0}(1) - \mathcal{N}_{y_\infty}(1)| < \delta$ and*

$$d_{\mathbb{F}}\left(((M,(g_t)_{t\in[-\delta^{-1},0]}), (\nu_{y_0,0;t})_{t\in[-\delta^{-1},0]}), (\mathcal{X}^{-t_0,r^{-1}}, (\nu_{y_\infty;t}^{-t_0,r^{-1}})_{t\in[-\delta^{-1},0]})\right) < \delta,$$

where $t_0 := \mathfrak{t}(y_\infty)$. Then one of the following holds:

(i) (y_0, t_0) is $(k, \epsilon, 1)$-split and $(\epsilon, 1)$-selfsimilar,

(ii) (y_0, t_0) is $(k-2, \epsilon, 1)$-split, $(\epsilon, 1)$-static, and $(\epsilon, 1)$-selfsimilar.

In particular, y_∞ is weakly $(k, \epsilon, 1)$-symmetric, and $\widehat{\mathcal{S}}^{\epsilon,k}_{r_1,r_2} \subseteq \mathcal{S}^{\delta(\epsilon,Y,A),k}_{r_1,r_2}$. Moreover, we have

$$\mathcal{S} = \bigcup_{\epsilon\in(0,1)} \mathcal{S}^{\epsilon,k}_{0,\epsilon}.$$

Proof. The hypotheses ensure that $\mathcal{N}_{y_\infty}(1) \geq -Y'(Y, A)$. By time translation and parabolic rescaling, we can assume $r = 1$ and $\mathfrak{t}(y_\infty) = 0$. Suppose by way of contradiction there is a sequence $\delta_i \searrow 0$ along with metric flows $\mathcal{X}^i$ each obtained as $\mathbb{F}$-limits of closed noncollapsed n-dimensional Ricci flows, $(k, \delta_i, 1)$-symmetric points $y_{\infty,i} \in \mathcal{X}^i_0$ with $\mathcal{N}_{y_{\infty,i}}(1) \geq -Y'$, closed pointed Ricci flows $(M_i, (g_{i,t})_{t\in[-\delta_i^{-1},0]}, (y_i, 0))$ satisfying $|\mathcal{N}_{y_i,0}(1) - \mathcal{N}_{y_{\infty,i}}(1)| < \delta_i$ and

$$d_{\mathbb{F}}\Big((M_i, (g_{i,t})_{t\in[-\delta_i^{-1},0]}, (\nu_{y_i,0;t})_{t\in[-\delta_i^{-1},0]}), (\mathcal{X}_i, (\nu_{y_{\infty,i};t})_{t\in[-\delta_i^{-1},0]})\Big) < \delta_i,$$

such that $(i), (ii)$ both fail for $(y_i, 0)$. Because $y_{\infty,i}$ are $(k, \delta_i, 1)$-symmetric, we can find metric flow pairs $(\mathcal{X}'_i, (\mu_t^{\prime,i})_{t\in[-\delta_i^{-1},0]})$ satisfying properties $(b)-(d)$ in Definition 18, along with $|\mathcal{N}_{\mu_t^{\prime,i}}(1) - \mathcal{N}_{y_{\infty,i}}(1)| < \delta_i$ and

$$d_{\mathbb{F}}\Big((\mathcal{X}_i, (\nu_{y_{\infty,i};t})_{t\in[-\delta_i^{-1},0]}), (\mathcal{X}'_i, (\mu_t^{\prime,i})_{t\in[-\delta_i^{-1},0]})\Big) < \delta_i.$$

Because $\mathcal{N}_{y_i,0}(1) \geq -Y' - \delta_i \geq -2Y'$, Theorem 16 lets us pass to a subsequence to obtain a future-continuous metric flow pair $(\mathcal{X}', (\mu'_t)_{t\in(-\infty,0]})$ such that

$$(M_i, (g_{i,t})_{t\in[-\delta_i^{-1},0]}, (\nu_{y_i,0;t})_{t\in[-\delta_i^{-1},0]}) \xrightarrow[i\to\infty]{\mathbb{F}} (\mathcal{X}', (\mu'_t)_{t\in(-\infty,0]})$$

uniformly on compact time intervals. By construction we also have

$$(\mathcal{X}'_i, (\mu_t^{\prime,i})_{t\in[-\delta_i^{-1},0]}) \xrightarrow[i\to\infty]{\mathbb{F}} (\mathcal{X}', (\mu'_t)_{t\in(-\infty,0]})$$

on compact time intervals, so by Lemma 24, $\mathcal{X}'$ also satisfies properties $(b)-(d)$ of Definition 18. By Proposition 32, we conclude that $(y_i, 0)$ satisfies one of (i) or (ii) when $i \in \mathbb{N}$ is sufficiently large, a contradiction. □

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