ESSAYS ON DIGITAL MEDIA PLATFORMS AND MARKET MANIPULATION

A Dissertation

Presented to the Faculty of the Graduate School

of Cornell University

in Partial Fulfillment of the Requirements for the Degree of

Doctor of Philosophy

by

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May 2022

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Cornell University 2022

This dissertation presents three essays on digital platforms and market manipulation, and investigates the economic impact of social media platforms on manipulation and coordination in the financial markets, as well as their influence on marketing and business models.

This first essay, "Media Trading Groups and Short Selling Manipulation", models how chatroom traders, forming a coalition via social media platforms, influence the stock price in the presence of large and strategic short sellers. The economic consequences of this dynamic game are studied in a micro-founded quasi-competitive equilibrium framework, which is new to the literature. Various equilibrium phenomena arise, including price bubbles, short squeezes, forced liquidations, and precautionary savings by the large trader. Media groups discipline the large trader's incentive to short sell, but it can either increase or decrease market efficiency. Additionally, it uniformly improves social welfare under the belief-neutral welfare criterion.

The second essay, "Index Design: Hedging and Manipulation", studies optimal index design to both facilitate hedging and alleviate illegal manipulation in a competitive equilibrium paradigm, modified to deal with manipulation. Specifically, a large trader is trading both derivatives and assets, and effectively hides her trades behind the competitive market clearing mechanism. Unlike

the strategic game paradigm, a volume-weighted average pricing (VWAP) index both introduces basis risk and encourages manipulation because of the additional randomness in volume weight and the greater price impact enjoyed by the large trader. In contrast, an equal-weighted average pricing (EWAP) index both preserves market completeness and discourages manipulation.

The third essay, "A Model of Influencer Economy", studies an influencer economy in which brand owners and sellers depend on influencers to attract consumers (who cares about influencer affinity as well as product quality), while competing in both influencer and product markets. As technologies governing marketing outreach improve, the equilibrium features non-monotonicities in influencer market concentration, payoffs, and distributional inequality. Influencer heterogeneity and horizontal product differentiation are substitutes; small style differences complement vertical product differentiation while large differences substitute. Moreover, assortative matching between sellers and influencers occurs under endogenous influence, with the maximum horizontal differentiation principle recovered in the limit of costless style selection. Meanwhile, the sellers' bargaining power counteracts the influencers' tendency to over-invest in influence power and they jointly determine the direction and magnitude of the sub-optimal acquisition. Finally, regulations for balanced seller-influencer matching can encourage seller competition under single dimensional seller-influencer heterogeneity. But uni-directional exclusivity contracts are welfare-improving for sufficiently differentiated products and uncongested influencers' markets.

BIOGRAPHICAL SKETCH

Siguang Li received a B.S. in Lanscaping and Architecture from Southwest Jiaotong University in 2007, and a Ph.D. in Economics from Peking University in 2013, and a M.A. in Economics in 2020. Before studying at Cornell University, he was a PER fellowship visiting scholar at Columbia University in 2016. He is expected to receive his Ph.D. in Economics from Cornell University in May 2022. In July 2022, he will join Hong Kong University of Science and Technology as a tenure-track Assistant Professor in Finance. His current research focuses on market manipulation, digital economy, financial technology and innovation, and mechanism & information design such as a mechanism design approach to blockchain design.

This document is dedicated to my wife, Xin, and our two boys, Chengzhi and Chengrui.

ACKNOWLEDGEMENTS

Life is a journey, and the people along this journey makes it meaningful and exciting. A self-evident fact is that I would not be writing these words today without their support.

First, I am deeply indebted to my advisors Robert Jarrow, Lin William Cong, and Yongmiao Hong. Bob, your encouragement of my initial interest in finance, and the continuous guidance through research, best exemplified by giving me the opportunity to explore new knowledge, was formative. Your devotion, keenness, and humbleness toward research and life influences me imperceptibly. I hope that I have made you proud. Will, you have always pushed me to the best economist and scholar possible. I am always impressed by your sharp and invaluable feedback, and thanks for exposing me to the front-edge research in Fintech. Meanwhile, you are very considerate and supportive of me. Given the tough job market this year, I would not get the offer from Hong Kong University of Science and Technology (Guangzhou Campus) without your strong support. Yongmiao, thank you for guiding me through the transition from coursework to research. I have been given the opportunity to TA for your graduate econometric course, and even more importantly, the opportunity to talk with you on research in Uris Hall. These communications were insightful and expanded my research horizon greatly.

Second, my strong gratitude belongs to the Department of Economics at Cornell University, and the excellent faculty here. I would like to thank Tommaso Denti, Panle J. Barwick, David Ng, Yi Chen, Levon Barseghyan, Francesca Molinari, Kristoffer Nimark, Gideon Saar, Murillo Campello, Warren Bailey and Jennifer Wissink for their feedback and suggestions over the years. I would also like to thank the administrative staff for their patience and their assistance.

Third, Qingmin Liu and Xi Weng were particularly instrumental in leading me to pursue my graduate studies at Cornell University. Qingmin, you strong support took me to a new level of understanding about both research and my personal life. Xi, thank you for being my best friend and teacher, almost since we met. As I prepare to pursue my academic career and step into that role myself, you are my best role models.

Fourth, I thank Jaden Chen and Haokun Sun for their availability whenever I want to talk. I also benefited greatly from their insightful discussions on research projects. I thank Qi Wu and Yue Wang for their invaluable help and guidance through the job market process. Futhermore, it would also be remiss of me to not thank Muye Chen, Zihan Hu, Judy Zhang, Yimeng Tang, Bowen Tan, Qinshu Xue, Bin Zhao, Zhou Fan, Chenyang Li, Wentong Chen, Yu Wang, Christa Deneault, Molly Ingram and Hyuk-Soo Kwon for their friendship, conversations, and commiserations. I also thank my friends and teachers for their information and help during my job market year, including but not limited to, Tong Yu, Xunan Feng, Hongyan Fang, Shaorong Li, Xinqiao Ping, Jian Ni, Shijun He, Qing Liu, Tianyu Wang and Bin Chen.

Last but not least, I want to thank my wife for taking care of me and our boys; I thank my boys for the joy and their companion; I thank my parents for their love and support; I thank my three sisters for taking care of my parents. You are my source of strength and reminds me who I am and where my roots are.

TABLE OF CONTENTS

CHAPTER 1

MEDIA TRADING GROUPS AND SHORT SELLING MANIPULATION

1.1 Introduction

Although the American securities' industry folklore includes many stories of price bubbles, market manipulations, corners, and short squeezes, the recent high profile GameStop trading craze and other "meme" episodes have again raised issues regarding the need for new security market regulations to deal with market manipulation, corners, and short squeezes because of the newly emerged media trading groups.[1] This paper constructs a model of such market episodes to provide some insights into the relevant issues of market efficiency and the need for regulations.

We construct a competitive equilibrium model with strategic trading to address these concerns. In our model, chatroom traders can form a coalition via social media platforms to influence the stock price by their collective trades.[2]

[1] See, *Wall Street Journal*, February 4, 2021, "Regulators Say Market Infrastructure Was Resilient in GameStop Frenzy"; *Wall Street Journal*, February 27, 2021, "GameStop Resurgence Reinforces New Reality for Hedge Funds"; and *Wall Street Journal*, June 3, 2021, "Meme Rally Lifts Unlikely Winners".

[2] What differentiates a social media group from the classical literature on coordination in financial markets, such as Morris and Shin (1998) with currency attacks? For example, when Soros "broke the Bank of England", he effectively coordinated hedge funds and other financial actors to challenge the UK's currency peg. More generally, large investors make public announcements of their short positions, explaining why they shorted a particular asset, to attract more investors to move prices to profit quicker from their trades. There are two key differences. One, the general-purpose technology like digital media platforms lowers the physical communication costs and makes coordination easier. Two, there exist less incentive conflicts in trading among chatroom day traders. In particular, large investors have competing interests when they trade in multiple markets, and thus they engage in strategic information transmission and they can only trust other investors' actions, rather than their words, as in the observational learning literature (see, e.g., Bikhchandani et al. (1992)). In contrast, chatroom day traders have less competing interests, and thus it is more likely for them to share their private information truthfully

Large short sellers exist who strategically short the stock. These large short sellers' trades have an impact on the price which enables them to manipulate the market. The economic consequences of this interaction are analyzed. As such, our model serves as a first approximation to the recent GameStop media frenzy episode.

Various equilibrium phenomena arise in our model, including price bubbles, short squeezes, forced liquidations, and precautionary savings by the large trader. The market dynamics in our incomplete information game proceed as follows. First, the large trader chooses an optimal short position, given her beliefs about media group formation, and the stock market clears in a competitive equilibrium. Second, if a media group forms, a coordinator is selected to increase the media group's price impact, and the market clears again. This price increase results in the shorts receiving a margin call, which may require them to partially liquidate their positions. If so, the stock price increases again, and a price bubble and short squeeze results. In the final time period, all participants liquidate their positions. This uncertainty about media group formation generates a non-trivial trade-off for the large trader between the size of her manipulative short position and precautionary savings to guard against a potential short squeeze due to an unexpected price bubble.

Central to this equilibrium is the mechanism underlying the media group's coalition formation and the selection of a media delegate to choose a media impact parameter. A (hypothetical) media impact parameter proxies for the visibility of the media group's participation in buying the stock. The stock price in the next period increases if the media group's impact parameter is large enough.

and benefit from the "wisdom of the crowd".

The sequence underlying this coordination mechanism is as follows. First, nature determines whether a media group forms with a certain probability. Second, if formed, a trader is selected to post the media impact parameter's value, and observing this parameter, each small trader decides whether or not to promote the stock himself on social platforms. If enough small traders choose to participate, the stock price increases in the next period due to the price impact of the media group's trade. In conjunction, when coordination is successful, the media group coalition has a price impact similar to that of a large trader.

Given the probability of a media group forming, there are three possible cases. First, if the large trader's initial wealth is large enough and the cost of a forced liquidation is low, then the large trader's optimal short position is given by an unconstrained optimum. In this case, the media group has no direct deterrence effect on short selling and a price bubble does not occur. Second, if the large trader's initial wealth is too low and that the cost of a forced liquidation is high, the optimal position is given by the maximum affordable short position without triggering a forced liquidation. Here, the existence of a media group has the strongest deterrence effect on short selling. A small price bubble occurs. Third, if the large trader's initial wealth and the cost for a forced liquidation are both low enough, the larger trader tolerates a certain degree of short squeezes to reap the high profits from shorting. Here, the existence of a media group has only a weak deterrence effect on short selling, but when a forced liquidation occurs, a larger price bubble arises.

With respect to allocational efficiency, media groups reduce allocational efficiency when the average trader is over-optimistic, and the stock is overpriced before the short selling occurs. In contrast, media groups improve allocational efficiency when the average trader is over-pessimistic, and the stock is underpriced before the short selling manipulation. If media groups are not overly optimistic, our theorems show that media group coalitions provide an important deterrent to short seller bear raids and thus enhance market efficiency. Depending upon the circumstance regulations to prohibit media group coalition formation may not be needed. Using a belief-neutral welfare criterion, we show that media groups discipline the large trader's incentive to short sell, and it uniformly improves social welfare. Again, the implication is that increased regulations to prohibit media group coalition formation are probably unnecessary.

An outline for this paper is as follows. Section 1.2 reviews the literature. Section 1.3 presents the model. Section 1.4 provides a micro-foundation for media visibility, and explains how the media group presence generates short squeezes. Section 1.5 establishes the existence, uniqueness, and comparative statics of the quasi-competitive equilibrium, as well as its empirical implications. Section 1.6 extends the model to include generalizations of the model's assumptions, and the conclusion is given in section 2.8.

1.2 Literature Review

The existing literature dealing with speculation and manipulation can be broadly categorized as either trade-based or information-based, and our paper is more closely related to the first strand.[3] Trade-based manipulation functions through "runs", i.e., buying or selling alone (Hart, 1977; Fischel and Ross, 1991;

[3]See Putniņš (2020) for a complete survey, which further divide manipulation into information-based, trade-based, action-based, order-based, and benchmark manipulation.

Jarrow, 1992; Allen and Gale, 1992; Allen and Gorton, 1992; Chakraborty and Yılmaz, 2004a,b; Goldstein and Guembel, 2008; Williams and Skrzypacz, 2020),[4] whereas information-based manipulation involves spreading misleading information (Vila, 1989; Kumar and Seppi, 1992; Benabou and Laroque, 1992; Gerard and Nanda, 1993; Bagnoli and Lipman, 1996; Van Bommel, 2003; Chakraborty and Yılmaz, 2008). Our paper adds to this vast literature by modeling media trading groups.

Second, our paper is also related to the literature on bubble creation via short sale constraints under heterogenous beliefs, and it has been shown that short sale constraints with heterogeneous beliefs creates bubbles (Miller, 1977; Bhattacharya and Spiegel, 1991; Santos and Woodford, 1997; Scheinkman and Xiong, 2003; Hong et al., 2006; Scheinkman, 2013).[5] Van Wesep and Waters (2021) investigate bubbles and crashes under their behavioral assumptions. Our paper contributes to the literature by providing a new equilibrium foundation and studying implications of market corners and short squeezes (Kyle, 1984; Villa, 1987; Pirrong, 1993; Shleifer and Vishny, 1997; Cooper and Donaldson, 1998; Allen et al., 2006).[6] An exception is Brunnermeier and Pedersen (2005) who study how predatory trading can generate forced liquidations and short squeezes for financially distressed traders. Our model features a dynamic and strategic interaction between the various market participants including a quantity impact on the price where price bubbles are created by media groups' coordinated trading.

Last, the idea that small traders can jointly change their market power via

[4] Other relevant papers include Cherian and Jarrow (1995); Cherian and Kuriyan (1995); Fishman and Hagerty (1995); Khanna and Sonti (2004); Aggarwal and Wu (2006).

[5] See Jarrow (2015) for a complete survey.

[6] Case studies involving corners and squeezes include Jegadeesh (1993); Jordan and Jordan (1996); Merrick Jr et al. (2005).

media trading groups is similar to that underlying the labor union, and a useful reference is Taschereau-Dumouchel (2020). Another paper is Kovbasyuk and Pagano (2015), in which some investors have incentives to communicate their positions to others. Our contributions to the literature are twofold. First, to our knowledge, we are the first paper to model how media groups work, and we provide a micro-foundation for a media group's price impact. Second, our model is dynamic with strategic trading and a quasi-competitive equilibrium, which is new to the literature.

1.3 The Model

This section presents the basic model. Extensions are explored in subsequent sections.

1.3.1 The Set-up

We use a multiple-stage model with the following four stages, including *shorting*, *coalition formation*, *margin adjustment*, and *payoff realization*. Trading occurs in the first three stages, and all payoffs occur in the final stage. There are two types of traders: a continuum of small traders (chatroom day traders) of measure $I > 0$, and a large trader (hedge fund). The collection of small traders may or may not form a media group coalition.

The randomness in the economy is given by a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ (details specified shortly). There are two assets traded in the economy: a money market account (mma) paying a riskless return and a risky stock. Without loss of generality, we set the mma return to be zero, which implies that the stock is denominated in units of the mma. The stock shares are infinitely divisible with $N > 0$ shares outstanding. Markets are assumed frictionless except for various short sale restrictions to be introduced below.

The stock's re-trade value in stage 4 (i.e., the *payoff realization* stage) is affected by price impacts of both a media group coalition of small traders (acting in concert like a large trader) and a large short trader. Formally, the stage 4 stock price is given by

$$S := h_2(z)\xi_\kappa > 0. \tag{1.1}$$

Here, $z \in \mathcal{Z} = (-\infty, 0]$ represents the shares shorted by the large trader and the function $h_2 : \mathcal{Z} \to \mathbb{R}_+$ measures the negative effect (i.e., price pressure) of the large trader's short selling,[7] which is assumed to be both increasing and differentiable in z (i.e., decreasing in $|z|$) and $h_2(0) = 1$.[8] Moreover, ξ_κ is the re-trade value of the stock considering a media group, where $\kappa \in \mathcal{K} = [0, \bar{\kappa}]$ measures (proxies for) the price impact of the media group coalition's trades (i.e., the magnitude of a media frenzy) on the fundamental value of the stock.[9]

The idea behind the media group's price impact parameter κ is as follows. The stronger the media group participation, the more optimistic the small traders' beliefs are about the future stock price, and the higher the re-trade value

[7]In addition to supply/demand effects, a price impact $h_2(z)$ could be due to changes in the firm's economic decisions, including investment, financing decisions, cash holdings, and other corporate policies (Jiang et al., 2020). Indeed, an increase in short selling leads to lower share prices, a reduction in investment and equity offerings (Grullon et al., 2015), an increase in the cost of external financing (Meng et al., 2020), excessive corporate cash holdings (Wang, 2018), and more costly cash dividends (Chen et al., 2019).

[8]Note that the monotonicity is the key assumed property because a monotonic function is almost everywhere differentiable (Royden and Fitzpatrick, 2010, Sect. 6.1).

[9]A micro-foundation justification for κ is in section 1.4.1.

ξ_κ in the payoff realization stage (ignoring the price suppressing effect $h_2(z)$). For analytic simplicity, we assume that

$$\xi_\kappa = h_1(\kappa)\xi$$

where $\xi : \Omega \to \mathbb{R}_+$is a non-negative random variable and $h_1(\kappa)$ measures the media group's price impact, which is increasing, continuous, and differentiable in κ with $h_1(0) = 1$. Henceforth,

$$S = h_1(\kappa)h_2(z)\xi. \tag{1.2}$$

A competitive equilibrium in the stock market corresponds to a time 0 price, denoted $s(\kappa, z) : \mathcal{K} \times \mathcal{Z} \to \mathbb{R}_+$, such that the aggregate demand for the stock equals the outstanding supply.

For clarity of the presentation, we first consider a complete market.[10] Since there is only one stock trading, the stock's evolution must be a binomial process. Given this observation, we let $\Omega = \{H, L\}$, $\mathcal{F} = 2^\Omega$, $\mathbb{P}(H) = 1 - \mathbb{P}(L) = p$, and

$$\xi(\omega) = \begin{cases} \bar{\xi}, & if\ \omega = H \\ \underline{\xi}, & if\ \omega = L \end{cases}$$

and $\bar{\xi} > \underline{\xi}$.

It is well known that $\bar{\xi} > \underline{\xi}$ is both a necessary and sufficient condition for a binomial model to be complete. To avoid arbitrage, we require that $s(\kappa, z) \in [h_1(\kappa)h_2(z)\underline{\xi}, h_1(\kappa)h_2(z)\bar{\xi}]$. This will be shown to always hold in our model's equilibrium. To simplify the subsequent notation, we normalize the stock's payoffs such that $\bar{\xi} = 1$ and $\underline{\xi} = 0$ below.

[10] An incomplete market is studied in section 1.6.3.

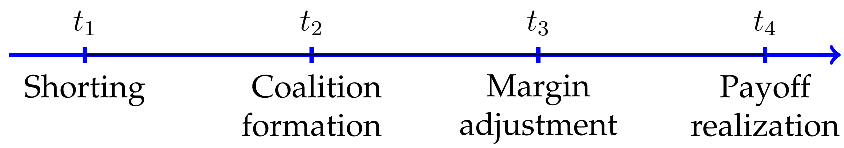

Figure 1.1: Timeline of the media trading group game

To include a time dynamic into this static model, we introduce four possible stages at time 0 as illustrated in Figure 1.1. These stages correspond to shocks to the market's equilibrium resulting from actions taken by the various traders. Note that, in each stage, the market price needs to adjust to clear the market. The timeline of the incomplete information game is given as follows. When necessary, we use the subscript s_j and z_j to denote the stock price and the large trader's shorts in stage $j \in \{1, 2, 3, 4\}$.

(i) In stage 1, the large trader shorts $z_1 < 0$ shares of the stock with a prior belief that a media group occurs with probability $\lambda \in [0, 1]$ in stage 2. The stock market clears with a price $s(0, z_1)$. See section 1.6.1 for a discussion on how the media group formation can depend on the large trader's short position $z_1 \in \mathcal{Z}$. Note that in stage 1, the continuum of small traders act as price takers since the media group has not yet been formed. Consequently, their beliefs regarding the media group's price impact parameter κ does not affect the competitive equilibrium in stage 1.[11]

(ii) In stage 2, a media group forms with probability $\lambda \in [0, 1]$. When a media group is formed, a coordinator proposes a target for the media group's

[11]This can be justified in the following sense. First, unlike the large trader, small traders do not have the necessary resources to evaluate the presence of media groups. Second, a single small investor, even with a correct estimate of λ, cannot impact the aggregate asset demand and the market clearing price.

price impact parameter $\kappa \in \mathcal{K}$. Then, the small traders decide to support or reject the coordinator's proposal, and the coalition's media price impact parameter κ aggregates those of the individual small traders. Note that the stage 1 short position z_1 chosen by the large trader is observable. A more detailed discussion of κ is in section 1.4.1.

(iii) In stage 3, an adjustment to the large trader's short position occurs if the large trader's margin falls below a certain threshold, and a "margin call" results. In this case, the large trader's broker buys stock to liquidate some of the large trader's short position. The modified short position is denoted by z_3 where $z_1 < z_3 \leq 0$ and the market price is $s(\kappa, z_3)$.[12] The liquidation is the result of a short squeeze and a price bubble occurs.

In stage 3, a forced purchase by the large trader entails both endogenous and exogenous costs. The endogenous costs are the direct wealth loss from buying back the stock to cover some shorts at an inflated price. The exogenous cost per share, $c > 0$, arises due to a transaction cost charged by the broker to execute the large trader's rebalancing. In the baseline model, we assume that the exogenous cost is zero (i.e., $c = 0$), although the more general case $c > 0$ is discussed in section 1.6.4.

[12] A margin call refers specifically to a broker's demand that an investor deposit additional money or securities into the account so that it is brought up to the minimum value. A detailed description can be found here. https://www.investopedia.com/terms/m/margincall.asp.

1.3.2 The Small Traders

Small traders are assumed to be risk neutral with heterogenous subjective beliefs. They are risk neutral because they are interpreted as day traders. Each trader $i \in I$ holds a different belief denoted by $\mathbb{P}_i(\bar{\xi}) = 1 - \mathbb{P}_i(\underline{\xi}) = \theta_i$ where the parameter $\theta_i \in [0, 1]$ has the cumulative distribution function $F(\theta_i)$ across the small traders' beliefs with the density $f(\theta_i) > 0$ for all $\theta_i \in [0, 1]$.[13] Obviously, $F(0) = 0$ and $F(1) = 1$. This distribution function measures the small traders' beliefs in the economy from pessimistic to optimistic. Here, the ith small trader's subjective probability measure $\mathbb{P}_i$ differs from the actual statistical probability measure $\mathbb{P}$.

To simplify the analysis, we impose the following assumption.

Assumption 1.3.1 (Small Traders' Demands and Wealths)**.**

For all $i \in I$,

i) trader i's trading size $d_i \in [0, 1]$, and

ii) trader i's initial wealth $W_i \geq \sup_{\kappa \in \mathcal{K}} h_1(\kappa)$.

Assumption 2.3.1 (i) implies that small traders cannot short the stock and can purchase a maximum of 1 share. The one share maximum is without loss of generality because the same analysis follows replacing the trading size limit with any $\bar{d} > 0$. Assumption 2.3.1 (ii) implies that the small trader's initial wealth W_i is large enough so that, under all possible equilibrium prices $s(\kappa, z)$ (to be shown later), he can purchase any shares desired within the trading limit

[13] Here, we use I to denote both the measure and the index set of small traders.

without debt financing.

The small traders are price takers, who solve the following problem:

$$d_i^*(s) = \arg \max_{d_i \in [0,1]} d_i \left(\mathbb{E}_i[S] - s\right) + W_i.$$

$$s.t.\ d_i s \leq W_i.$$

Fix $\kappa \in \mathcal{K}$ and $z \in \mathcal{Z}$, then

$$\begin{aligned}\mathbb{E}_i[S] = \mathbb{E}_i[h_1(\kappa)h_2(z)\xi] &= \theta_i[1 \cdot h_1(\kappa)h_2(z)] + (1-\theta_i)[0 \cdot h_1(\kappa)h_2(z)] \\ &= \theta_i h_1(\kappa)h_2(z)\end{aligned}$$

is the expected payoff from the risky assets, and $(W_i - d_i s)$ is the investment in the mma. Given s, the optimal stock position for the small trader i is given by[14]

$$d_i^*(s) = \mathbb{1}_{[s,\infty)}\big(\mathbb{E}_i[h_1(\kappa)h_2(z)\xi]\big) = \begin{cases} 1, & \text{if } \theta_i h_1(\kappa)h_2(z) \geq s \\ 0, & \text{otherwise.} \end{cases} \tag{1.3}$$

Alternatively stated, the ith small trader buys one share of the stock if and only if

$$\theta_i \geq \frac{s}{h_1(\kappa)h_2(z)}.$$

Fixing s, given the distribution of beliefs across the small traders, the aggregate market demand is

$$\int_{i \in I} d_i^*(s) = I \int_{\frac{s}{h_1(\kappa)h_2(z)}}^{1} dF(\theta_i) = I\left[1 - F\left(\frac{s}{h_1(\kappa)h_2(z)}\right)\right].$$

[14]The indicator function $\mathbb{1}_A(x)$ equals 1 when $x \in A$, and equals 0 when $x \notin A$.

1.3.3 The Large Trader

The large trader is also assumed to be risk neutral with a subjective belief $\mathbb{P}_L(\bar{\xi}) = \theta_L$, but unlike the small traders, the large trader is not a price taker. We impose the following assumption so that the large trader always desires to short the stock at time 0.

Assumption 1.3.2 (Large Trader Shorts the Stock)**.**

$$I\Big(1 - F\big(h_1(\bar{\kappa})\theta_L\big)\Big) > N.$$

From the large trader's perspective, this condition states that the initial price for the stock in the shorting stage (i.e., stage 1) is too high compared to its future payoff in the payoff realization stage (i.e., stage 4). We show later that Assumption 1.3.2 implies that $z^* < 0$.[15] Intuitively, Assumption 1.3.2 requires that the number of small traders whose subjective beliefs (θ_i) are more optimistic than those of the large trader (θ_L) outnumbers the total supply of the stock N, even when the large trader believes that the media group has the strongest price impact on the stock in stage 4 (i.e., $h_1(\kappa) = h_1(\bar{\kappa})$).

The large trader's problem is to solve

[15] See the proof of Proposition 1.5.1.

$$
\begin{aligned}
z_1^* = \arg\max_{z_1 \in \mathcal{Z}} \Big\{ & W_L - z_1 s(0, z_1) + \mathbb{E}_L^{\lambda}[z_3(z_1) S] \\
& - \mathbb{E}\big[|z_3(z_1) - z_1| s(\kappa_3, z_3(z_1)) \mathbb{1}\{z_3(z_1) \neq z_1\}\big] \Big\} \qquad (1.4) \\
s.t. \ & |z_1 s(0, z_1)|\, m \leq W_L
\end{aligned}
$$

where $\mathbb{E}_L^{\lambda}[\cdot]$ is the expectation under the large trader's subjective belief $\mathbb{P}_L$, given the probability of a media group forming is $\lambda \in [0, 1]$. Note that z_3 is the (revised) shorting position in stage 3 (see Eq. (1.9)). Here, in Eq. (1.4), W_L is the large trader's initial wealth, the second term is the initial revenue gain from selling the shorting position in stage 1, the third term is the cash outflow in the final period, and the fourth term is the cash outflow when a forced liquidation occurs.

The left side of the budget constraint represents the required margin held by the large trader for a short position of size $z_1 < 0$ with m representing the margin multiplier. In current markets, $m = 1.5$.[16] This constraint makes the large trader act as if she is risk averse, and causes her to reduce her shorts.

The large trader is not a price taker because the stock price in the shorting stage $s(0, z_1)$ depends on the short shares $z_1 < 0$. Knowing this, the large trader can exert a quantity impact on the market price by changing her trade size. Note that the revised short position z_3 enters the expected payoff in the payoff realization stage, because the margin constraint might be violated due to the

[16]The regulatory bodies in United States, including both the New York Stock Exchange (i.e., NYSE) and the Financial Industry Regulatory Authority (i.e., FINRA), require that investors have an initial short margin of 150 % and must maintain a margin of at least 125% of the total value of the short securities. More details can be found at `https://www.finra.org/rules-guidance/key-topics/margin-accounts`.

price impact of the media group in stage 2. The media group's impact on the price does not change the margin constraint in the shorting stage which occurs before the media group's existence.

1.3.4 Market Clearing Stock Price

This section determines the time 0 equilibrium stock price $s(\kappa, z)$. We call this a quasi-competitive equilibrium price because all traders except the large trader are price takers. Given κ and z, the equilibrium clearing condition is

$$N + |z| = I\left[1 - F\left(\frac{s(\kappa, z)}{h_1(\kappa)h_2(z)}\right)\right] \tag{1.5}$$

where $z < 0$. Note that the large trader's shorts increase the supply of shares in the stock market. Let $G(\cdot) = F^{-1}(\cdot)$, which is well defined because $F(\theta)$ is strictly increasing (i.e., $F'(\theta) = f(\theta) > 0$, $\forall\theta \in [0, 1]$). We can solve equation (1.5) for the equilibrium stock price $s(\kappa, z)$.

Lemma 1.3.1. *Given the large trader's short position $z < 0$ and the coalition's impact parameter κ, the equilibrium stock price is*

$$s(\kappa, z) = h_1(\kappa)h_2(z)\theta(z) \tag{1.6}$$

where

$$\theta(z) := G\left(\frac{I - N + z}{I}\right). \tag{1.7}$$

Proof. It follows directly from equation (1.5). □

In this lemma, the quantity $\theta(z)$ can be interpreted as the belief of the "marginal" small trader, whose trade determines the equilibrium price. To see this, note that the profit per share from holding the stock to the ith trader is

$$\mathbb{E}_i[S] - s(\kappa, z) = h_1(\kappa)h_2(z)\big(\theta_i - \theta(z)\big).$$

So, any trader whose belief θ_i is more optimistic than $\theta(z)$ holds the stock, while anyone more pessimistic does not. Hence, the trader with a belief equal to $\theta(z)$ is just indifferent to buying, i.e., the marginal trader. Therefore, the equilibrium price is the expected stage 4 price to the marginal small trader.

This lemma implies the following comparative statics:

(i) $\frac{d\theta(z)}{dz} = \frac{1}{If((I-N+z)/I)} > 0$. The larger the short position of the large trader ($|z| \uparrow$), the smaller the beliefs of the marginal trader ($\theta(z) \downarrow$). This follows directly from equation (1.7).

(ii) $\frac{\partial s(\kappa,z)}{\partial |z|} < 0$. The more the large trader shorts ($|z| \uparrow$), the lower is the equilibrium stock price ($s(\kappa, z) \downarrow$). Note that $z < 0$ and $\frac{\partial s(\kappa,z)}{\partial |z|} = -\frac{\partial s(\kappa,z)}{\partial z} < 0$ because both $h_2(z)$ and $\theta(z)$ are nonnegative and increasing in z.

(iii) $\frac{\partial s(\kappa,z)}{\partial \kappa} > 0$. The stronger the coalition strength ($\kappa \uparrow$), the larger the stock price ($s(\kappa, z) \uparrow$). This follows directly from $h_1(\kappa)$ increasing in κ.

The equilibrium price has two components. The first, $h_1(\kappa)h_2(z)$, measures the price impact due to the small traders' coalition as well as the large trader's short selling. The second, $\theta(z)$, measures the impact of the large trader's short position on the marginal trader. The fact that $\frac{d\theta(z)}{dz} > 0$ implies that increased short selling generates more liquidity because a lower price attracts more small traders (i.e., $\theta(z) \downarrow$). Note that $s(\kappa, z)$ can be *forward-looking* when the pair (κ, z) is *forward-looking* and reflects the large trader's prior belief about future media group presence.

1.4 Media Trading Groups

Recently, a high-profile media group of day traders generated a frenzy with GameStop's stock. Day traders pushed "Buy GameStop" on Reddit and chat forums, causing the stock price to increase more than 300%. Trading in GameStop became a high-profile battle between online forum day traders and hedge fund short sellers. Other examples of similarly affected stocks include Nio, AMC, Plug Power, Blackberry, and Bed Bath & Beyond. According to the Wall Street Journal,[17]

> Behind the swings, many see ordinary investors, stuck at home in the pandemic, swapping tips and hatching trading strategies on online forums like Reddit's WallStreetBets—often buying things Wall Street has bet against. Many tout their long-shot wagers with the expression "YOLO," or, "You only live once".

As noted in the Wall Street Journal, chatroom traders gather in online communities such as TikTok, Twitter, Reddit, and the messaging platform Discord. They piggyback on each other's ideas and trades, and they coordinate via these popular social media, thereby affecting the stock price. The next section provides a model of this phenomena via the construction of the market impact parameter κ.

[17] See Wall Street Journal, January 26, 2021, "BlackBerry, AMC and Other Reddit YOLO Favorites That Aren't GameStop".

1.4.1 A Micro-foundation for Media Visibility

A media group is a coalition of small traders who implicitly choose a trader to follow in social media forums, who in turn determines the media group's price impact parameter $\kappa \in \mathcal{K}$. The selected trader, say the jth, must be sufficiently optimistic about the stock's final stage value such that regardless of the large trader's short position, he will buy the stock, i.e. $\theta_j > \underset{z<0}{sup}\, \theta(z)$.

The idea underlying this coordination mechanism is similar to the actions of a labor union that delegates their bargaining power to a union leader (Taschereau-Dumouchel, 2020). Suppose each small trader can exert some effort $e_i \in \{0, \bar{e}\}$ with a sufficiently small cost $\eta_i > 0$ to promote the stock in their personal network, i.e., recommend the stock to friends and share news on the stock's fundamental.

On the aggregate level, the coalition's impact parameter κ simply aggregates those of the small traders, that is,

$$\kappa = \int_{i \in I} e_i dF(\theta_i) \tag{1.8}$$

where $\sup \mathcal{K} := \bar{\kappa} = I\bar{e}$. This impact parameter quantifies the enthusiasm among the small traders for buying and promoting the stock.

When a media group forms, it selects a coordinator who posts a target impact parameter $\kappa > 0$, which is observed by the day traders. We assume that whenever the aggregate media impact is large enough and exceeds a certain threshold (i.e., $\kappa > \kappa_0 \geq 0$), all participating small traders get an additional expressive utility $u > \eta$ for promoting the stock. Here, κ_0 is normalized to zero. The idea of an expressive utility comes from retail investors expressing anger and discontent against institutional shorts.[18] In essence, small traders get an additional utility from a future price bubble (due to $h_1(\kappa)$) by trading against the shorts. When no media group is formed, each individual small investor can still promote the stock on his personal network (i.e., $e > 0$) and incur the cost η. Here, however, there is no expressive utility gain, since each single small trader has a size of measure zero. The coordination game is as follows.

(i) With probability $\lambda \in [0,1]$, nature determines whether a media group forms.[19]

(ii) Depending on whether a media group is formed or not,

- if a media group is formed,
  (a) a coordinator is selected who posts a target impact parameter $\kappa > 0$;
  (b) small traders observing $\kappa > 0$ select a promotion effort e_i;
- if a media group is not formed,
  (a) no target on κ is set (i.e., $\kappa = 0$);
  (b) small traders select a promotion effort e_i;

(iii) The coalition's level of κ is aggregated according to equation (1.8) and observed. If $\kappa > 0$, an expressive utility is generated for all participants.

[18]Expressive utility is not new to the literature. It is related to the literature on taste and preference heterogeneity, including those on "sin" stocks (see, for instance, Hong and Kacperczyk (2009)), loyalty and patriotism (Cohen, 2009; Morse and Shive, 2011), corporate social responsibility (CSR) (Fama and French, 2007; Friedman and Heinle, 2016), and environmental, social and governance (ESG) issues (Martin and Moser, 2016; Riedl and Smeets, 2017; Oehmke and Opp, 2020; Goldstein et al., 2021). From a modeling perspective, it is similar to that of anticipatory utility (Banerjee et al., 2019).

[19]Section 1.6.1 discusses how to endogenize λ as a function of the shorts z.

(iv) The decision for the small traders in this game is how much effort to exert and the decision for the coordinator is the choice of κ.

We have two comments on the coordination game. First, the expressive utility $u(\kappa)$ can depend on κ in a monontonic way. Second, the setup can be extended to include negative efforts, that is, an investor can make an effort to dissuade others from participating in the media group and share negative news and rumors on the stock's fundamental.

For the media group coalition, we now prove that there exists a Nash equilibrium such that if a media group is formed, all small traders exert the maximum effort. In other words, it negates the conjecture that, traders with optimistic beliefs desire a high κ, whereas traders with pessimistic beliefs would desire a low κ.

Lemma 1.4.1 (Unanimous Support)**.** *Assume that*

$$\frac{dz_3}{d\kappa} \geq \sup_{(\kappa,z)\in[0,\bar{\kappa}]\times[z^{\intercal}(0),z^{\intercal}(\bar{\kappa})]} \Big\{ -\frac{h_1'(\kappa)}{h_1(\kappa)}\frac{h_2(z)}{h_2'(z)} \Big\}$$

There exists a Nash Equilibrium $(\{e_i^*\}_{i\in I}, \kappa^*)$ *such that*

i) if there is no media group, $e_i^* = 0$ *for all* $i \in I$ *and* $\kappa^* = 0$*;*

ii) if there is a media group, $e_i^* = \bar{e}$ *for all* $i \in I$ *and* $\kappa^* = \bar{\kappa}(= I\bar{e})$*.*

Proof. See Appendix A.1.1. □

The assumed condition in Lemma 1.4.1 ensures that the coordinator has an incentive to choose $\kappa^* = \bar{\kappa}$. See Lemma 1.5.1 for a detailed discussion. To understand this result, note the following facts.

(i) The perfectly-aligned incentive comes from two facts. First, there exists no cost related to the aggregate level κ, which means that all small traders with positive stock holdings benefit from a large κ, regardless of their subjective beliefs θ_i, while small traders with zero stock holdings weakly benefit. Second, on the individual level, both the expressive utility and the effort cost also do not depend on the aggregate κ. Together, these two observations imply that all small traders can be well represented by a single trader.

(iii) Anger expressing. Critical to this construction is anger expressing, a form of expressive utility. We bundle the collective decisions to the target variable κ, to circumvent the problem that a single small trader has zero mass.

(iv) Equilibrium multiplicity and refinement. Multiple equilibria exist in the coordination game. For instance, an equilibrium exists in which the coordinator proposes $\kappa^* = 0$ and all small traders choose no effort $e_i = 0$. In this case, we can exclude this equilibrium by maximizing the small traders' aggregate welfare. In particular, from lemma 1.5.1, we have $\frac{\partial S}{\partial \kappa} \geq 0$, which implies that the proposal $\kappa^* = \bar{\kappa}$ is welfare maximizing for the coalition.

(v) The cost η and expressive utility u are introduced to obtain the previous equilibrium refinement. If we remove both η and u, there still exists a Nash equilibrium such that without media groups all small traders choose $e_i^* = 0$, and with media groups all small traders choose $e_i^* = \bar{e}$ (and the small traders who do not hold stocks weakly prefer $\bar{e}$ since the utility from anger expressing no longer exists).

(vi) The coalition's aggregate choice κ^* can be extended to depend on the short position $z \in \mathcal{Z}$, that is, a larger short position (i.e., $|z| \uparrow$) can lead to a stronger pushback from the coalition (i.e., $\kappa^* \uparrow$). See section 1.6.2.

(vii) Note that there exists no "tragedy of the commons" here.

1.4.2 Bubbles and Short Squeezes

To understand how media groups shape the incentive for bubbles and short squeezes, we present an intuitive discussion before proceeding with the formal equilibrium analysis. Consider the following example with a small probability of a media group forming, say, $\lambda = \frac{1}{100}$. Let the large trader have an initial wealth W_L, which is relatively small so that she wants to short to the maximal amount z_m (i.e., $W_L = m|z_m|s(0, z_m)$) when there is no media group (i.e., $\lambda = 0$).

Suppose now a media coalition forms. Consider the large trader's situation.

(i) In stage 1, the large trader moves knowing the market price $s(0, z)$ and the price impact of her trades. With probability $\frac{1}{100}$, the stage 3 stock price jumps to $s(\bar{\kappa}, \check{z})$, and with remaining probability $\frac{99}{100}$, it remains unchanged at $s(0, z)$. Since λ is small, she aggressively shorts a large position $z_1 < 0$ as if acting under the case with no media group (i.e., $z_1 \approx z_m$). The stock price clears at $s_1 = s(0, z_1)$.

(ii) In stage 2, when a media group forms, according to Lemma 1.4.1, the media group selects $\kappa = \bar{\kappa}$ and the stock price clears at $s_2 = s(\bar{\kappa}, z_1) > s(0, z_1) = s_1$. The media-driven price surge creates a price bubble and a "bear trap" because the liability per share in stage 4 increases to

$$S(\omega) = h_1(\bar{\kappa})h_2(z_1)\xi(\omega) > h_1(0)h_2(z_1)\xi(\omega) = h_2(z_1)\xi(\omega).$$

(iii) In stage 3, due to the price bubble in stage 2, the large trader's margin account falls below the broker's required margin. A margin call occurs and shorts are liquidated (i.e., $W_L < m|z_1|s(\bar{\kappa}, z_1)$, and the short position is revised to $z_3 = \check{z}(z_1)$ such that $z_3 \in (z_1, 0]$ and

$$W_L = \underbrace{m|z_3|s(\bar{\kappa}, z_3)|}_{\text{new margin}} + \underbrace{|z_3 - z_1|s(\bar{\kappa}, z_3)}_{\text{liquidation cost}}. \tag{1.9}$$

The large trader's scramble to reduce shorts increases the size of the bubble, and the liability in stage 4 further increases to

$$S(\omega) = h_1(\bar{\kappa})h_2(z_3)\xi(\omega) > h_1(\bar{\kappa})h_2(z_1)\xi(\omega).$$

The stock price is now $s_3 = s(\bar{\kappa}, z_3) > s(\bar{\kappa}, z_1) = s_2$.

With the aid of lemma 1.3.1, the welfare dynamics are as follows. Let "CS_j^L" stand for the large trader's profits and "CS_j^i" the surplus for the ith small trader in stage $j \in \{1, 2, 3, 4\}$. Recall that the marginal small trader's belief is $\theta(z)$.

- In stage 1, for those active small traders with $\theta_i \geq \theta(z_1)$ (ignoring W_i),

$$CS_1^i = \mathbb{E}_i[S] - s(0, z_1) = \theta_i h_1(0)h_2(z_1) - s(0, z_1) = h_2(z_1)\big(\theta_i - \theta(z_1)\big)$$

  because $h_1(0) = 1$. For the large trader,[20]

$$\begin{aligned} CS_1^L &= \mathbb{E}_L^\lambda[z_3 S] - z_1 s(0, z_1) + W_L \\ &\approx z_1\big(h_1(0)h_2(z_1)\theta_L - s(0, z_1)\big) + W_L = |z_1|h_2(z_1)\big(\theta(z_1) - \theta_L\big) + W_L. \end{aligned}$$

[20] The approximate equality comes from the fact that when $\lambda \to 0$, we have $z_3 \to z_1$ and $\mathbb{E}_L^\lambda[S] \to h_2(z_1)\theta_L$.

- In stage 2, the large trader does not change her position, and the media group chooses $\kappa^* = \bar{\kappa}$. Here, all small traders benefit weakly from this,

$$CS_2^i = h_1(\bar{\kappa})h_2(z_1)\big(\theta_i - \theta(z_1)\big)\mathbb{1}\big(\theta_i \geq \theta(z_1)\big) \geq CS_1^i.$$

In contrast, the large trader is hurt by the price bubble,

$$\begin{aligned} CS_2^L &= z_1\big(\mathbb{E}_L[S] - s(0,z_1)\big) + W_L = z_1\big(h_1(\bar{\kappa})h_2(z_1)\theta_L - s(0,z_1)\big) + W_L \\ &= |z_1|h_2(z_1)(\theta(z_1) - h_1(\bar{\kappa})\theta_L) + W_L < CS_1^L. \end{aligned}$$

This is because the media group's coordinated choice of κ creates a price bubble, which increases the large trader's future liability. However, the short profits remain unchanged because they only depend on the stock price $s_1 = s(0, z_1)$ whereas $s_2 = s(\bar{\kappa}, z_1) > s(0, z_1) = s_1$.

- In stage 3, the stock price only adjusts if a short squeeze happens. In this case the stock price increases to $s_3 := s(\bar{\kappa}, z_3) = h_1(\bar{\kappa})h_2(z_3)\theta(z_3)$. The new marginal small trader is $\theta(z_3)$ where $0 \geq \theta(z_3) > \theta(z_1)$. The small traders with beliefs $\theta_i \in [\theta(z_1), \theta(z_3)]$ are selling their stocks to meet the demand of the large trader to reduce shorts. In addition, the large trader is further hurt by the short squeeze because

$$CS_3^L = \underbrace{z_3\mathbb{E}_L[S]}_{\text{total Liability}} \underbrace{-z_1 s(0,z_1)}_{\text{initial Gains}} - \underbrace{s(\bar{\kappa},z_3)(z_3 - z_1)}_{\text{liquidation cost}} + W_L$$

Given that $(z_3 - z_1)\big(s(0,z_1) - s(\bar{\kappa},z_3)\big) < 0$ and thus

$$-z_1 s(0,z_1) - s(\bar{\kappa},z_3)(z_3 - z_1) < -z_3 s(0,z_1),$$

we further have

$$
\begin{aligned}
CS_3^L &< z_3 h_1(\bar{\kappa}) h_2(z_3)\theta_L - z_3 s(0, z_1) + W_L \\
&= |z_3| h_2(z_1) \left(\theta(z_1) - \theta_L h_1(\bar{\kappa}) \frac{h_2(z_3)}{h_2(z_1)} \right) + W_L \\
&< |z_3| h_2(z_1) \left(\theta(z_1) - h_1(\bar{\kappa})\theta_L \right) + W_L < CS_2^L < CS_1^L.
\end{aligned}
$$

Note that $h_2(z)$ is increasing in z and thus $\frac{h_2(z_3)}{h_2(z_1)} > 1$. Small traders benefit and increase their profits because

$$
CS_3^i = h_1(\bar{\kappa}) h_2(z_3) \big(\theta_i - \theta(z_1) \big) > CS_2^i > CS_1^i.
$$

1.5 Equilibrium Analysis

This section presents the formal equilibrium analysis and the equilibrium characterization. Two important cases are presented: the no media group case ($\lambda = 0$) and the sure media group case ($\lambda = 1$). Moreover, we discuss the empirical implications of our analysis, including market discipline, efficiency, and policy.

1.5.1 Equilibrium Concept

The equilibrium used below is a Subgame Perfect Nash Equilibrium (SPNE), which is characterized by the following requirements.

(i) In stage 1, the large trader's short position z_1^* solves problem (1.4), given the equilibrium stock price $s_j(\kappa, z)$, the short position function $z_3(z_1)$ in stage 3, and the small traders' equilibrium strategies d_i^* (in each stage j) and their supporting strategies e_i^* in stage 2.

(ii) In stage $j \in \{1, 2, 3\}$, given the tuple (z_j, κ_j, s_j), the i-th small trader chooses the optimal stock position $d_i^*(s)$ given by Eq. (2.2). Furthermore, the stock price s_j clears the market, that is, $\int_{i \in \mathcal{I}} d_i^*(s_j) = N + |z_j|$.

(iii) In stage 2, whenever a media group forms, the coordinator's choice of κ^* maximizes the expected value of his stock position. The i-th small trader chooses e_i^* to maximize his utility in the coordination game, and the equilibrium impact parameter satisfies $\kappa^* = \int_{i \in \mathcal{I}} e_i^* dF(\theta_i)$.

(iv) In stage 3, whenever a forced liquidation occurs, the short position z_3 is revised according to Eq. (1.9).

Some comments are in order. First, imposing market-clearing when the small traders are in the market leads to a forward-looking price. Indeed, this is the case after the realization of the media trading group in stage 2. Second, there exists a dynamic inconsistency in the stock price in the shorting stage (i.e., stage 1), because the price at which the large trader sells the asset does not reflect his forward-looking beliefs about the likelihood of liquidation, the likelihood of a trading group arising, or the heterogeneity in small traders' beliefs. However, this can still be a good approximation because small traders often lack the expertise to evaluate the existence of a media coalition and they might fail to anticipate it before it arises. In other words, small traders under-value the stock before the media coalition formation.[21]

[21]Note that the main result on equilibrium trading pattern (i.e., Proposition 1.5.1) still holds when the small traders can consistently predict the probability of media group presence. In particular, it pushes the stock price higher in the shorting stage, which makes shorting even more lucrative.

1.5.2 Existence and Characterization

Given that the short position z is observed by the small traders, backward induction can be employed to solve for the equilibrium. Denote $\widehat{W}_L(\kappa, z) := m|z|s(\kappa, z)$ to be the minimal wealth above which no margin calls occur.

(i) In stage 3, the large trader's short position $z_1 < 0$ (determined in stage 1) and the media group's choice of κ (determined in stage 2), are given. If the large trader's initial wealth is large enough ($W_L \geq \widehat{W}_L(\kappa, z_1)$), then she can post more margin after a media driven price bubble without liquidating shorts. If $W_L < \widehat{W}_L(\kappa, z_1)$, there will be liquidations by the broker to restore the maintenance margin. The revised short position after liquidation, $z_3(z_1)$, as a function of the stage 1 short position z_1, is the solution to equation (1.9), restated here for convenience:

$$\underbrace{m|z_3|s(\kappa, z_3)}_{\text{new margin}} + \underbrace{s(\kappa, z_3)(z_3 - z_1)}_{\text{liquidation cost}} = W_L.$$

By monotonicity of $h_2(z_1)$, the stock price increases to $s(\kappa, z_3)$.

(ii) In stage 2, the media group forms with probability $\lambda \in [0, 1]$. Without a media group, $\kappa^* = 0$. With a media group, by lemma 1.5.1, the optimal media choice is $\kappa^* = \bar{\kappa}$, regardless of the margin adjustment in stage 3.

(iii) In stage 1, the large trader anticipates that a media group forms with probability λ and with a media impact parameter κ^*, as well as the subsequent actions by the broker, including a margin call and potential liquidations.

If she anticipates no liquidations in stage 3, her problem is

$$
\begin{aligned}
z_1^* &= \arg\max_{z_1 \in \mathcal{Z}} z_1 \Big(\mathbb{E}_L^{\lambda}[S] - s(0, z_1) \Big) + W_L \\
&= \arg\max_{z_1 \in \mathcal{Z}} z_1 \Big(\big[\lambda h_1(\bar{\kappa}) h_2(z_1) + (1-\lambda) h_1(0) h_2(z_1) \big] \theta_L - s(0, z_1) \Big) + W_L \\
&= \arg\max_{z_1 \in \mathcal{Z}} z_1 h_2(z_1) \Big(\big[\lambda h_1(\bar{\kappa}) + 1 - \lambda \big] \theta_L - \theta(z_1) \Big) + W_L \\
& s.t. \;\; m|z_1| s(\bar{\kappa}, z_1) \leq W_L.
\end{aligned}
$$

If she anticipates liquidations, there are two cases.

(a) If there is no media group (with probability $1-\lambda$), there will be no liquidations and the profit is

$$
z_1 \Big(\mathbb{E}_L \big[\xi h_2(z_1) \big] - s(0, z_1) \Big) + W_L.
$$

(b) If there is a media group (with probability λ), there will be liquidations and the profit is

$$
\underbrace{z_3 \mathbb{E}_L[\xi h_1(\bar{\kappa}) h_2(z_3)]}_{\text{Total Liability}} \underbrace{- z_1 s(0, z_1)}_{\text{Initial Gains}} - \underbrace{s(\bar{\kappa}, z_3)(z_3 - z_1)}_{\text{Liquidation Cost}} + W_L
$$

where z_3 is defined in equation (1.9). Note that we use the fact that in equilibrium $\kappa^* = \bar{\kappa}$ (see lemma 1.5.1).

To complete the argument,we need to introduce some notations. Define $V_1(z_1)$ to be the expected profit for the large trader when there are no forced liquidations, i.e.

$$
V_1(z_1) = z_1 h_2(z_1) \Big(\big[\lambda h_1(\bar{\kappa}) + 1 - \lambda \big] \theta_L - \theta(z_1) \Big) + W_L. \tag{1.10}
$$

Define $z^{\dagger}(\lambda)$ to be the optimal solution to V_1 whenever it admits a unique optimizer, i.e.,

$$\left.\frac{\partial V_1}{\partial z_1}\right|_{z_1=z^{\dagger}(\lambda)} = 0. \tag{1.11}$$

Denote by $z^{\intercal}(\kappa)$ the maximum short position without triggering a forced liquidation, given the media group's price impact parameter κ. This is the solution to the following expression:

$$mh_1(\kappa)|z^{\intercal}(\kappa)|s\big(\kappa, z^{\intercal}(\kappa)\big) = W_L. \tag{1.12}$$

Next, define

$$V_2(z_1) := \lambda\Big(h_1(\bar{\kappa})\big[z_3h_2(z_3) - z_1h_2(z_1)\big]\theta_L - \lambda s(\bar{\kappa}, z_3)(z_3 - z_1)\Big)\mathbb{1}(z_1 < z^{\intercal}\big(\bar{\kappa}\big)) \tag{1.13}$$

where $z_3(z_1)$ is given in equation (1.9). Note that, by lemma 1.5.2 below, $V_2(z_1)$ is continuous at $z_1 = z^{\intercal}(\bar{\kappa})$ and $V_2(z_1) < 0$ for $z_1 < z^{\intercal}(\bar{\kappa})$. Moreover, note that

$$\mathbb{E}[U_L^1] = V_1(z_1) + V_2(z_1).$$

When $z_1 \in [z^{\intercal}(\bar{\kappa}), 0]$,

$$\mathbb{E}[U_L^1] = V_1(z_1).$$

When $z_1 \in (-\infty, z^{\intercal}(\bar{\kappa}))$,

$$\mathbb{E}[U_L^1] = \underbrace{\lambda z_3h_1(\bar{\kappa})h_2(z_3)\theta_L + (1-\lambda)z_1h_2(z_1)\theta_L}_{\text{expected liability}}\underbrace{-\lambda s(\bar{\kappa}, z_3)(z_3 - z_1)}_{\text{liquidation cost}}\underbrace{-z_1s(0, z_1)}_{\text{initial gain}}+W_L. \tag{1.14}$$

Thus, $V_2(z)$ is the additional impact of liquidations on the large trader's expected profit.

The following lemmas are needed to characterize the equilibrium.

Lemma 1.5.1. *Assume that* $\frac{dz_3}{d\kappa} \geq \sup_{(\kappa,z)\in[0,\bar{\kappa}]\times[z^{\intercal}(0),z^{\intercal}(\bar{\kappa})]} \Big\{ -\frac{h_1'(\kappa)}{h_1(\kappa)}\frac{h_2(z)}{h_2'(z)} \Big\}$.

Then, the media group coordinator selects $\kappa^* = \bar{\kappa}$ *whenever a media coalition is formed.*

Proof. See Appendix A.1.2. □

The condition in lemma 1.5.1 that

$$\frac{dz_3}{d\kappa} \geq \sup_{(\kappa,z)\in[0,\bar{\kappa}]\times[z^{\intercal}(0),z^{\intercal}(\bar{\kappa})]} \Big\{ -\frac{h_1'(\kappa)}{h_1(\kappa)}\frac{h_2(z)}{h_2'(z)} \Big\} \tag{1.15}$$

requires some explanation. There are two cases which imply this condition.

(i) $h_2(z) = 1, \forall z \in \mathcal{Z}$. Here, the large trader's shorts have zero price impact. The right side of equation (1.15) approaches $-\infty$ and the condition is always satisfied.

(ii) $\frac{dz_3}{d\kappa} \geq 0$, i.e. the larger the media group's price impact parameter κ, the larger the number of shorts liquidated. The derivative $\frac{dz_3}{d\kappa}$ can be obtained by differentiating equation (1.9).

Lemma 1.5.2 (Properties of $V_2(z_1)$)**.**

i) $V_2(z_1)$ *is continuous; ii)* $V_2(z_1) < 0$ *for* $z_1 < z^{\intercal}(\bar{\kappa})$*; and iii)* $V_2'\big(z^{\intercal}(\bar{\kappa})_-\big) > 0$ *where* $V_2'(a_-) = \lim_{z\to a, z<a} \frac{dV_2(z)}{dz}$.

Proof. See Appendix A.1.3. □

The continuity of $V_2(z_1)$ plays a key role in ensuring the existence of equilibrium.

Proposition 1.5.1 (Equilibrium Price Bubbles and Short Squeezes)**.** *Suppose that* $h_2(z_1) = O\left(|z_1|^{-1+\delta}\right)$ *for some (sufficiently small)* $\delta > 0$ *as* $z_1 \to -\infty$.[22] *Assume condition (1.15) in lemma 1.5.1. Then, there exists a sub-game perfect Nash Equilibrium (SPNE) such that:*

i) the large trader shorts $z_1^* < 0$ *shares of the stock;*

ii) if a media group forms, the media coordinator selects $\kappa^* = \bar{\kappa}$ *and all small traders choose effort* $e_i^* = \bar{e}$*; if there is no media group,* $\kappa^* = 0$ *and all small traders choose effort* $e_i^* = 0$*;*

iii) when liquidations occur, the revised position is given by $z_3(z_1^*)$*;*

iv) the market clearing stock price in stage j *is given by* $s_j = s(\kappa_j, z_j)$.

Proof. See Appendix A.1.4. □

The condition $h_2(z_1) = O\left(|z_1|^{-1+\delta}\right)$ guarantees that the decay in the stock price in stage 4 due to the shorts is such that for large $|z_1|$ the margin constraint remains binding.

This proposition shows that an equilibrium exists and that the equilibrium is such that if no media group forms, then the small players do not promote buying the stock. But, when a media group forms, the small players exert maximum effort to promote buying the stock and to create a price bubble and short squeeze.

[22] $f(x) = O(g(x))$ for some real-valued functions f and g means that $\lim_{x\to\infty} \frac{f(x)}{g(x)} < \infty$.

1.5.3 Two Cases

This section presents two special cases. The first, no media groups ($\lambda = 0$), characterizes markets before the existence of social media and the possible formation of media groups. The second, sure media groups ($\lambda = 1$), characterizes a market where a media group exists and it explains how the large short trader behaves in such a circumstance. Here, the threat of margin calls and liquidations dominates and pre-empts aggressive shorting.

Case I ($\lambda = 0$)

This section discusses the case $\lambda = 0$, where there is no media coalition formation. Here, the incentive for the large trader to short is the strongest, and for brevity, this discussion is in section 1.5.3. Market efficiency is discussed in section 1.5.6. The welfare implications are discussed next. Traditional wisdom suggests that shorting hurts small traders. However, this is not necessarily true. It depends on their beliefs and the price impacts of both shorting and the media group.

- **Before short selling.** In stage 0, by equation (1.7), the stock market price $s(0,0)$ coincides with the marginal trader's belief $\theta(0)$, because $h(0,0) = h_1(0)h_2(0) = 1$, i.e.,

  $$s(0,0) = \theta(0) = G\big((I-N)/I\big).$$

  The total small trader surplus is

  $$CS_0 := \int_{\theta(0)}^{1} \Big(\mathbb{E}_\theta[S] - s(0,0)\Big) dF(\theta) = \int_{\theta(0)}^{1} \big(\theta - \theta(0)\big) dF(\theta).$$

- **After short selling.** In stage 1 and thereafter, the stock price $s(0, z_1^*)$ and the marginal trader's belief $\theta(z_1^*)$ satisfy

$$s(0, z_1^*) = h_2(z_1^*)\theta(z_1^*) \quad \text{with} \quad \theta(z_1^*) = G\big((I - N + z_1^*)/I\big).$$

The total small trader surplus is

$$CS_1 := h_2(z_1^*) \int_{\theta(z_1^*)}^{1} (\theta - \theta(z_1^*)) dF(\theta).$$

The change in this surplus is

$$\Delta CS = CS_1 - CS_0 = h_2(z_1^*) \int_{\theta(z_1^*)}^{1} \big(\theta - \theta(z_1^*)\big) dF(\theta) - \int_{\theta(0)}^{1} (\theta - \theta(0)) dF(\theta). \tag{1.16}$$

Define $c : \mathcal{Z} \to \mathbb{R}$ such that

$$c(z) = \frac{\int_{\theta(0)}^{1} \big(\theta - \theta(0)\big) dF(\theta)}{\int_{\theta(z)}^{1} \big(\theta - \theta(z)\big) dF(\theta)},$$

and let

$$\tilde{\theta}(z) := \theta(0) + \frac{h_2(z)(\theta(0) - \theta(z))}{1 - h_2(z)}.$$

Lemma 1.5.3 (Small Trader Surplus)**.**

i) If $h_2(z_1^*) \geq c(z_1^*)$, then $\Delta CS \geq 0$. The small traders are better off.

Otherwise, $\Delta CS < 0$. The small traders are worse off.

ii) Choose h_2 such that the corresponding z_1^* satisfies $h_2(z_1^*) = c(z_1^*)$,[23] *so that $\Delta CS = 0$.*

For each individual trader, the most optimistic $\theta_i \in \big(\tilde{\theta}(z_1^*), 1\big]$ are worse off, while those with beliefs $\theta_i \in \big[\theta(z_1^*), \tilde{\theta}(z_1^*)\big]$ are better off.

[23] The condition $h_2(z_1^*) = c(z_1^*)$ introduces a fixed point issue because z_1^* depends on h_2. The fixed-point issue further reduces to a certain continuity property of the optimizer z_1^* and that the set of functions h_2 is big enough. To focus on the economic intuition, we directly assume that such a fixed point of h_2 exists.

Proof. See Appendix A.1.5. □

Lemma 1.5.3 shows that whether short selling hurts small traders depends on the relative magnitude of the negative price impact from shorting, measured by $h_2(z_1^*) < h_2(0)$, versus the liquidity effect of attracting more small traders into the market, measured by $\theta(z_1^*) < \theta(0)$, When the price suppressing effect is weak, short selling provides more liquidity and the small traders benefit. In contrast when the price impact is strong, which is a "bear raid,"[24] small traders are hurt. Note that the most optimistic small traders are always hurt because of a deteriorating expected value, while the less optimistic small traders benefit from the extra trading opportunity provided by the shorts. Indeed, the small traders with $\theta_i \in (\theta(z_1^*), \theta(0))$ now buy stocks because the stock price is more affordable. Short selling leads to a strict positive surplus for this later group.

Case II ($\lambda = 1$**)**

When $\lambda = 1$, a media group forms, and by Lemma 1.5.1 and Lemma 1.4.1, we have the media impact parameter is at its largest value, $\kappa^* = \bar{\kappa}$. The large trader fully anticipates this media coalition, and consequently, a short squeeze never occurs. The large trader holds cash to avoid subsequent forced liquidations after a margin call. This is the precautionary motive.

Proposition 1.5.2 (Precautionary Saving Motive)**.** *If* $\lambda = 1$ *(a sure media coalition), then* $z_1^* \in [z^{\intercal}(\bar{\kappa}), 0]$.

[24]A bear raid is an unlawful practice used by short sellers to push a target company's stock price lower by either concerted short selling or spreading ill-disposed rumors.

Proof. See Appendix A.1.6. □

Recall that $z^{\intercal}(\bar{\kappa})$ is the maximal short position without triggering liquidations when the media coalition's choice is $\bar{\kappa}$. Proposition 1.5.2 thus states that the large trader anticipates and avoids the adverse liquidation costs after such a margin call.

1.5.4 Uniqueness

Uniqueness of the equilibrium requires additional structure. Instead of imposing this structure directly on $h_2(z)$ and $F(\theta)$, we assume the following sufficient conditions.

Assumption 1.5.1 (Concavity of the Large Trader's Objective Function)**.**

(i) $|z|h_2(z)\theta(z)$ *is weakly decreasing in* $z \in \mathcal{Z}$*;*

(ii) $zh_2(z)\big(h_1(\kappa)\theta_L - \theta(z)\big)$ *is concave in* $z \in \mathcal{Z}$*,* $\forall \kappa \in \mathcal{K}$*; and*

(iii) $V_2(z)$ *is weakly concave in* z *for all* $z \in \big(-\infty, z^{\intercal}(\bar{\kappa})\big)$*.*

Assumption 1.5.1 (i) says that the budget set $m|z_1|s(0, z_1) \leq W_L$ is more likely to bind when the large trader shorts more (i.e., z_1 decreases). Claim (ii) is equivalent to assuming that $V_1(z_1)$, the expected payoff without liquidations, is concave in the short position z_1. To see this define

$$\kappa(\lambda) := \inf\{\kappa : h_1(\kappa) = \lambda h_1(\bar{\kappa}) + (1-\lambda)h_1(0)\},$$

and note that $\{\kappa(\lambda) : \lambda \in [0,1]\} = [0,\bar{\kappa}]$. Last, claim (iii) implies that $V_2(z)$ is weakly concave for all $z \in \mathcal{Z}$ because by lemma 1.5.1 (iii) we have $V_2'(z^{\intercal}(\bar{\kappa})_-) > 0$, which, together with $V_2'(z) = 0$ for all $z \in (z^{\intercal}(\bar{\kappa}), 0)$, implies concavity on the domain $\mathcal{Z}$. In conjunction, these conditions imply that $V_1(z) + V_2(z)$ is concave, and $V_2'(z) > 0$ for $z < z^{\intercal}(\bar{\kappa})$.

Moreover, by lemma 1.5.1 (ii), $V_2(z) < 0$ holds, and thus the concavity of $V_2(z)$ is equivalent to assuming a convex liquidation cost $-V_2(z)$, that is, when the short position exceeds $z^{\intercal}(\bar{\kappa})$, the larger the initial short position, the higher the marginal loss induced by a forced liquidation.

Assumption 1.5.1 is non-empty, as illustrated by the following example.

Example 1.5.1 (Large Traders' Objective Function)**.**

This example shows that Assumption 1.5.1 is non-empty. Let $h_2(z) = 1, \forall z \in \mathcal{Z}$, i.e., there is no price impact from the large trader's short selling. To ensure property (ii), let $\theta_i \sim$ Uniform$([0,1])$, i.e. $F(\theta) = \theta, \forall \theta \in [0,1]$. This yields

$$zh_2(z)\big(h_1(\kappa)\theta_L - \theta(z)\big) = z\Big(h_1(\kappa)\theta_L - (I - N + z)/I\Big).$$

The quadratic function on the right side of this expression is concave.

To get (i), restrict $\mathcal{Z} = \big(-\frac{1}{2}(I-N), 0\big]$. Then,

$$\frac{d}{dz}\Big(-zh_2(z)\theta(z)\Big) = \frac{d}{dz}\Big(z(I - N + z)/I\Big) = \frac{-2z - (I-N)}{I} \leq 0$$

for $z \in \mathcal{Z}$. This concludes the example.

Lemma 1.5.4 proves uniqueness and characterizes the solution for $\lambda \in$

$(0, 1)$.[25]

Lemma 1.5.4 (Uniqueness and Forced Liquidations)**.**

Fix $\lambda \in (0, 1)$. Assume both conditions in Proposition 1.5.1 and Assumption 1.5.1 hold. Then, there exists a unique SPNE such that:

i) in stage 1, the large trader's optimal short position is

$$z_1^* = \begin{cases} z^\dagger(\lambda), & \text{if } W_L \geq \widehat{W}_L\big(\bar{\kappa}, z^\dagger(\lambda)\big) \\ \in \big(z^\dagger(\lambda), z^\intercal(\bar{\kappa})\big)\,, & \text{if } W_L < \widehat{W}_L\big(\bar{\kappa}, z^\dagger(\lambda)\big) \;\&\; V_1'(z^\intercal(\bar{\kappa})) + V_2'\big(z^\intercal(\bar{\kappa})_-\big) < 0 \\ z^\intercal(\bar{\kappa}), & \text{if } W_L < \widehat{W}_L\big(\bar{\kappa}, z^\dagger(\lambda)\big) \;\&\; V_1'(z^\intercal(\bar{\kappa})) + V_2'\big(z^\intercal(\bar{\kappa})_-\big) \geq 0, \end{cases}$$

ii) in stage 2, $\kappa^* = \bar{\kappa}$ whenever a media group arises; otherwise, $\kappa^* = 0$,

iii) in stage 3, when a margin call and a forced liquidation occur, the revised position $z_3(z_1^*)$ is given by equation (1.9). Otherwise, it remains unchanged (i.e., $z_3 = z_1^*$).

Proof. See Appendix A.1.7. □

Figure 1.2 illustrates the equilibrium in Lemma 1.5.4. The solid black line refers to the case where the initial wealth $W_L \geq \widehat{W}_L\big(\bar{\kappa}, z^\dagger(\lambda)\big)$. The dashed black line corresponds to the optimal short position $z^\dagger(\lambda)$ for this situation. This is the case where the initial wealth is sufficient to cover any future margin calls generated by the formation of a media group coalition.

The other two solid lines correspond to the cases where the cost of forced liquidations matters. The kink is generated by the additional liquidation cost

[25]Note that we still cannot eliminate the multiplicity of equilibrium for the coordination game in section 1.4.1, which is the typical case in the coordination game literature.

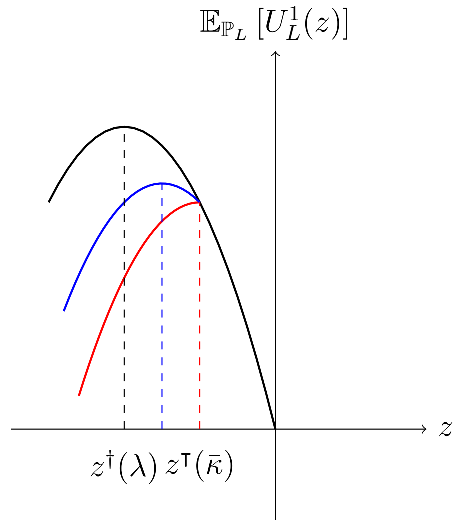

Figure 1.2: Equilibrium Illustration

$V_2(z)$. The blue line refers to the case where $V_1'\big(z^{\intercal}(\bar{\kappa})\big) + V_2'\big(z^{\intercal}(\bar{\kappa})_-\big) < 0$ and the red line refers to the case where $V_1'\big(z^{\intercal}(\bar{\kappa})\big) + V_2'\big(z^{\intercal}(\bar{\kappa})_-\big) \geq 0$. The optimal short positions are noted by the dashed lines. Such a liquidation is inevitable when a media driven price bubble occurs. In both cases the initial wealth is insufficient to cover the price increase (i.e., $W_L < \widehat{W}_L\big(\bar{\kappa}, z^{\dagger}(\lambda)\big)$). Here, if the liquidation cost is not too large (i.e., $V_1'\big(z^{\intercal}(\bar{\kappa})\big) + V_2'\big(z^{\intercal}(\bar{\kappa})_-\big) < 0$), the large trader tolerates a certain degree of forced liquidation and limits her short position slightly. This difference is the size of precautionary savings. If the liquidation cost is large enough (i.e., $V_1'\big(z^{\intercal}(\bar{\kappa})\big) + V_2'\big(z^{\intercal}(\bar{\kappa})_-\big) \geq 0$), then sufficient precautionary savings is taken such that a margin call induced liquidation never occurs. The short position is $z^{\intercal}(\bar{\kappa})$.

To complete the uniqueness result, we now characterize the optimal short position when $\lambda \in \{0, 1\}$.

Lemma 1.5.5 (Equilibrium characterization for $\lambda \in \{0, 1\}$)**.** *Assume both condi-*

tions in Proposition 1.5.1 and Assumption 1.5.1 hold. Then,

i) when $\lambda = 0$, $z_1^*(0) = max\left\{z^\dagger(0), z^\intercal(0)\right\}$;

ii) when $\lambda = 1$, $z_1^*(1) = max\left\{z^\dagger(1), z^\intercal(\bar{\kappa})\right\}$.

Proof. See Appendix A.1.8. □

This lemma shows that when a media group does not exist, the large trader's shorts only depend on their initial wealth, the price impact function, and expected profits. But, when the media group is sure to form, the large trader anticipates its impact and shorts to maximize her expected profits given the existing market conditions and the likelihood of a margin call and liquidation under the premise that a forced liquidation never occurs.

1.5.5 Comparative Statics

We are interested in understanding how the large trader's short position z_1^* depends on both λ and κ. To this end, we introduce Assumption 1.5.2.

Assumption 1.5.2 (Monotonicity)**.**

$V_0(z) := \frac{-zh_2(z)}{If((I-N+z)/I)[h_2(z)+zh_2'(z)]}$ *is decreasing in* z.

Assumption 1.5.2 imposes a joint condition on the small traders' density, $f(\cdot)$, and the media groups price impact $h_2(\cdot)$. Assumption 1.5.2 is trivially satisfied in Example 3.6.1. To see this note that $h_2(z) = 1$, $h_2'(z) = 0$, and $f(\cdot) = 1$. Thus, $V_0(z) = -\frac{z}{I}$, which is decreasing in z. We can now state the lemma.

Lemma 1.5.6 (Monotonicity)**.**

The unconstrained optimum, $z^{\dagger}(\lambda)$*, is increasing in* λ*, and the maximum affordable short position,* $z^{\intercal}(\kappa)$*, is increasing in* κ*.*

Proof. See Appendix A.1.9. □

Lemma 1.5.6, in conjunction with Lemma 1.5.4 and Lemma 1.5.5, generate the implication that media groups discourage shorting. Indeed, the large investor has a strong incentive to short the stock whenever: i) she believes that a media group is unlikely to form, or the coalition's choice of κ is small; and ii) the coalition's price impact is small (i.e. $h_1(\cdot) \downarrow$). We summarize these insights in the following corollary 1.5.1.

Corollary 1.5.1 (Short Selling Incentives)**.**

i) the incentive for short selling is strongest when there is no media coalition (i.e., $\lambda = 0$*); Similarly, the incentive for shorting is minimal under a sure media group (i.e.,* $\lambda = 1$*).*

ii) when the large trader's initial wealth W_L *is large, the short position* z_1^* *is increasing in the probability of the media group forming* λ*.*

Proof. See Appendix A.1.10. □

1.5.6 Empirical Implications

This section discusses empirical implications, including market discipline, efficiency and policy recommendations regarding media groups.

Market Discipline

This section discusses the discipline effect of media groups and the welfare implications of their existence. The discipline effect refers to the deterrence of the media group on a large trader's incentive to aggressively short a targeted stock. The equilibrium in Lemma 1.5.4 illustrates three possible cases, herein called strong deterrence, weak deterrence, and no direct deterrence, depending on the initial wealth of the large trader and the magnitude of the endogenous cost from a liquidation after a margin call.

If the large trader has sufficient initial wealth, then the presence of a media group never triggers a liquidation, which means that the optimal short position is given by an unconstrained optimizer. This is the situation of no direct deterrence.[26] If the large trader has insufficient wealth to cover margin calls, then media groups significantly modify the large trader's strategic behavior. Indeed, their potential existence lowers the expected profit per shorted stock because with probability $\lambda > 0$, the price bubble due to a media frenzy can generate a short squeeze and a bear trap.

We define three cases with respect to media groups:

[26] Note that the optimal short position still depends on the probability of media coalitions' presence, i.e., $z^{\dagger}(\lambda)$ is increasing in λ.

- strong deterrence. If $W_L < \widehat{W}_L\big(\bar{\kappa}, z^{\dagger}(\lambda)\big)$ & $V_1'(z^{\intercal}(\bar{\kappa})) + V_2'\big(z^{\intercal}(\bar{\kappa})_-\big) \geq 0$, the incentive for precautionary savings is so strong that margin calls and liquidation are preempted by a reduced short position, that is, $z_1^* = z^{\intercal}(\bar{\kappa})$.
- weak deterrence. If $W_L < \widehat{W}_L\big(\bar{\kappa}, z^{\dagger}(\lambda)\big)$ & $V_1'(z^{\intercal}(\bar{\kappa})) + V_2'\big(z^{\intercal}(\bar{\kappa})_-\big) < 0$, the large trader anticipates and willingly tolerates a certain amount of forced liquidations to gain large profits from shorting when no media group forms, that is $z_1^* \in \big(z^{\dagger}(\lambda), z^{\intercal}(\bar{\kappa})\big)$.
- no direct deterrence. If $W_L \geq \widehat{W}_L\big(\bar{\kappa}, z^{\dagger}(\lambda)\big)$, the large trader has sufficient wealth so that margin calls do not distort the optimal short position $z^{\dagger}(\lambda)$.

When is weak or strong deterrence likely to occur? Proposition 1.5.3 partially addresses this question.

Proposition 1.5.3 (Beliefs and Deterrence). *Fix $W_L < \widehat{W}_L(\bar{\kappa}, z^{\dagger}(1))$ so that $z^{\dagger}(1) < z^{\intercal}(\bar{\kappa})$.[27] Then, there exists a cutoff probability $\bar{\lambda} \in (0, 1]$ such that: i) weak deterrence results if $\lambda \in (0, \bar{\lambda})$, and ii) strong deterrence results if $\lambda \in [\bar{\lambda}, 1]$. Furthermore, forced liquidations only occur for $\lambda \in (0, \bar{\lambda})$.*

Proof. See Appendix A.1.11. □

The key economic insight from this proposition is straightforward. When the probability of the media group forming increases, strong deterrence is more likely to occur. This complements Lemma 1.5.6, which proves the monotonicity of $z^{\dagger}(\lambda)$. Second, as the probability of the media group forming λ increases,

[27] We impose the restriction that $W_L < \widehat{W}_L(\bar{\kappa}, z^{\dagger}(1))$, because when λ increases, $z^{\dagger}(\lambda)$ increases (i.e., $|z^{\dagger}(\lambda)| \downarrow$). Thus, there exists a cutoff $\tilde{\lambda}$ above which $z^{\dagger}(\lambda)$ is feasible again, and a more complicated pattern involving "no deterrence", "weak deterrence", "strong deterrence, "and "no deterrence" might occur.

the observed pattern of a “short squeeze” is non-monotonic. In particular, as λ increases from 0 to 1, we first see “no forced liquidations,” then “forced liquidations” (after a media frenzy), and lastly “no forced liquidation” again. We do not observe forced liquidations when $\lambda = 0$ and $\lambda \geq \bar{\lambda}$. In the former case, a short squeeze does not occur simply because there exists no media group. In contrast, in the latter case, the large trader is so concerned about a media driven price bubble and short squeezes that she chooses a very conservative short position to avoid the costs of a forced liquidation. In other words, short squeezes only occur when a media group forming is small but non-zero.

Allocational Efficiency

Given the discipline effects of media groups, the next question to address is whether media groups increase allocational market efficiency. The notion of allocational market efficiency employed measures the distance between the market equilibrium price implied by a martingale measure and the actual probability for the stock. The expected stock payoff under the actual probability generates the “fundamental value,” that is, $\mathbb{E}_{\mathbb{P}}[\xi] = p$, because the mma return is zero. The martingale probability corresponds to the marginal trader’s belief $\theta(z)$, i.e.

$$\mathbb{Q}(\omega = H) = \theta(z) := G\left(\frac{I - N + z}{I}\right)$$

where $z < 0$ is the short position. Hence, we define an efficiency loss function as

$$L(p, \theta(z)) := -\big|p - \theta(z)\big|. \tag{1.17}$$

The larger the deviation from the stock’s objective probability, the bigger the allocational efficiency loss. The idea is that when the market price deviates from its fundmental value, there is a welfare loss due to distorted investment decisions arising from incorrect cost of capital determinations. Recall that $\theta(z_1^*(0))$ is the martingale measure when there is shorting but no media groups, and $\theta(z_1^*(\lambda))$ is the martingale measure when there are both shorting and media groups.[28]

Hence, media groups exhibit increased allocational efficiency if and only if

$$L\big(p, \theta(z_1^*(0))\big) < L\Big(p, \theta\big(z_1^*(\lambda)\big)\Big).$$

We can now prove the following proposition.

Proposition 1.5.4 (Allocational Efficiency)**.** *Media groups decrease allocational efficiency when $p < \theta_L < \theta(0)$, and they increase allocational efficiency when $p \geq \theta(0) > \theta_L$.*[29]

Proof. See Appendix A.1.12. □

As stated in the proposition, media groups can either increase or decrease allocational efficiency. This is because the stock valuation, as measured by $\theta(0)$, where there are no media group or short selling, already deviates from the fundamental value $\mathbb{E}_{\mathbb{P}}[\xi] = p$. The marginal investor's belief can be either too optimistic (small traders are overly optimistic when $I\big[1 - F(p)\big] \gg N$), or too pessimistic when $I\big[1-F(p)\big] \ll N$). In the former case, shorting improves efficiency

[28]Note that, proposition 1.5.4 also holds when we consider the revised position z_3.

[29]When $\theta(0) > p > \theta_L$, media groups can either increase or decrease allocational efficiency, which further depends on model specifications, including price impact functions h_1, h_2 and small traders' belief distribution $f(\cdot)$. Specifically, fix the model structure, media groups can only increase allocational efficiency when $p \geq p^* := \frac{1}{2}(\theta(z_1^*(0)) + \theta(z_1^*(\lambda))$. Note that $p^* \in (\theta_L, \theta(0))$.

because it helps to correct the over-pricing. This implies that media groups' influence impedes the self-correction of the stock market via shorting and they decrease allocational efficiency. In the latter case, the stock is already undervalued and shorting further suppresses the stock price, decreasing efficiency. Here is where the media groups increase allocational efficiency. Whether or not media groups increase or decrease allocational efficiency depends on one's view of the marginal investor's belief. Only if the average small trader is overly optimistic are regulatory restrictions beneficial.

The Belief-neutral Efficiency

With heterogeneous and distorted beliefs, Brunnermeier et al. (2014) propose a belief-neutral welfare criterion, which asserts that an allocation is belief-neutral efficient if the allocation is obtainable as the optimum from a social planner's welfare function under a reasonable set of beliefs. Using this criterion, allocations can be categorized into one of three types: i) uniformly belief-neutral efficient allocations; ii) uniformly belief-neutral inefficient allocations; and iii) those that are neither uniformly efficient nor uniformly inefficient across all reasonable beliefs.

In this approach the social planner is agnostic about the actual probability measure, which corresponds to $\mathbb{P}(\cdot)$ here. To implement this approach, assume that $\mathbb{P}$ is unknown to the social planner. To simplify analysis, we use the utilitarian social welfare function,

$$W(E^h[u_L], \left\{E_0^h[U_S^i]\right\}_{i\in\mathcal{I}}) = E^h[u_L] + \int_{i\in\mathcal{I}} E^z[u_S^i]dF(\theta_i) \tag{1.18}$$

where risk-neutral preferences imply that $u_L(w) = w$ and $u_S^i(w) = w$. Note that the expectation is taken with respect to $\mathbb{P}^h$, a convex combination of the large trader's belief $\mathbb{P}_L$ and small traders' beliefs. We define the set of reasonable beliefs to be all $\mathbb{P}^h \in \Pi^h$ such that

$$\mathbb{P}^h = h^L \mathbb{P}_L + \sum_i h^i \mathbb{P}_S^i, \quad \text{where } h^i \geq 0, \quad h_L \geq 0 \text{ and } \sum_i h^i + h^L = 1.$$

We use this criterion to compare the ex-ante welfare between two regimes where shorting initially occurs (i.e., in stage 1 at time 0): (i) the one without media group presence, and (ii) the other with potential media threats. First, note that with/without media trading groups, regardless of the probability measure the social planner adopts, (internal) transfers among risk-neutral investors via re-trades from market clearing do not directly impact social welfare, which implies

$$\begin{aligned} W(E^h[u_L], \{E_0^h[U_S^i]\}_{i\in\mathcal{I}}) &= E^h[z_1^* S] + \int_{\theta_i \geq \theta(z_1^*)} E^h[S] dF(\theta_i) \\ &= N \cdot E^h[S] = N \cdot E^h[h_2(z_1^*)\xi] = N \cdot E^h[\xi] + N \cdot E^h\left[\left(h_2(z_1^*) - 1\right)\xi\right]. \end{aligned} \tag{1.19}$$

The first equality in Eq. (1.19) cancels all transfers among investors so that welfare only depends on the stock value under the social planner's belief. The last equality further decomposes it into an exogenous value term $N * E^h[\xi]$ independent of the presence of media trading groups, although it does depend on the social planner's belief $\mathbb{P}^h$, and a second term $N * E^h\left[\left(h_2(z_1^*) - 1\right)\xi\right]$, which depends on both the initial short position and the social planner's belief.

Now, fix an arbitrary belief $\mathbb{P}^h$. Note that because $z_1^*(0) < z_1^*(\lambda) < 0$ (i.e., there exists less shorting when media groups exist)[30] and $h_2(z)$ is strictly increasing,

[30] A formal proof of this can be found in the proof of Proposition 1.5.4.

social welfare is improved in the regime with potential media groups, regardless of the belief adopted by the social planner. In other words, if shorting has a price suppressing effect and the related real effects are negative as described in our framework, the discipline effect of media trading groups reduces the large trader's shorting position, leading to an increase in social welfare uniformly. Hence, we have shown that media groups improve social welfare under this welfare criterion by disciplining short selling, implying regulatory intervention of media groups is not warranted.

This result is valid regardless of whether the price impact of media groups is transitory or long-lasting as long as the discipline effect on the initial short position remains. This is because the price impact of media groups, if permanent, would only contribute to additional welfare gains. Moreover, the belief-neutral welfare comparison still holds if the real effects of shorting are not uniformly positive over all possible beliefs for the social planner.

1.6 Extensions

This section discusses endogenous media coalition presence, endogenous media impact, incomplete markets, liquidity costs, and costs to using social media.

1.6.1 Endogenous Media Coalition Presence

To endogenize the media coalition presence probability with respect to a particular large trader's short position, one can use rational inattention theory (Sims, 2003; Hirshleifer and Teoh, 2003; DellaVigna and Pollet, 2009), which analyzes how a decision-maker with limited attention allocates it across various sources of information (i.e., which large traders to watch). There is a growing body of evidence that attention is, in fact, limited (Chetty et al., 2009). This can be addressed in our model in a reduced form fashion by assuming that the larger the shorts by the large trader, the higher is the probability of a media group forming, i.e., $\lambda = \lambda(z)$ is strictly decreasing in z. Interested readers can refer to Bordalo et al. (2013) and Persson (2018). We do not pursue the modeling details here since it is equivalent to modeling endogenous media impact in section 1.6.2.

1.6.2 Endogenous Media Impact

This section presents a setting in which an equilibrium exists such that the media impact parameter depends on the shorting position $z < 0$. Consider the case in which the ith small trader gains an additional expressive utility only when the coalition's proposal gets sufficient support and he has invested in the stock.

Lemma 1.6.1 (Endogenous Media Impact)**.**

There exists a SPNE such that:

i) when there is no media group, $\kappa^* = 0$ and $e_i^* = 0$ for all $i \in I$;

ii) when a media group forms, the coordinator selects the target $\kappa^* = I(1 - F(\theta(z)))\bar{e}$; the ith small trader chooses $e_i^* = \bar{e}$ when his belief $\theta_i \geq \theta(z)$, and $e_i^* = 0$ when $\theta_i < \theta(z)$.

Consequently, the equilibrium media impact parameter $\kappa^*(z) = I(1 - F(\theta(z)))\bar{e}$ is *strictly decreasing in $z < 0$, i.e., $\frac{d\kappa^*(z)}{dz} < 0$.*

Proof. See Appendix A.1.13. □

1.6.3 Incomplete Markets

In the baseline model, the market is assumed to be complete. However, this assumption is unnecessary. In an incomplete market, we have traded a mma and a risky stock, whose exogenous initial re-trade value $\xi : \Omega \to \mathbb{R}_+$. Here, the random variable ξ can take on a continuum of possible values, making the market incomplete. As before, there is a continuum of small traders with total mass $I > 0$. Trader $i \in I$ is risk neutral and her belief, indexed as θ_i, is defined uniquely by $\theta_i := \mathbb{E}_{\mathbb{P}_i}[\xi]$ Note the change for θ_i in a complete market which represents a probability, whereas here it is an expectation.

Everything else remains unchanged. In particular, the belief type θ_i has the cumulative distribution function $F(\cdot)$ with a density $f(\cdot)$ everywhere positive, and we impose the exogenous trade size restriction, Assumption 2.3.1, for the small traders. Then, given the initial wealth W_i and the stock price s, the small trader i's optimization problem is[31]

$$d_i^*(s) = \arg\max_{d_i \in [0,1]} d_i \Big(\mathbb{E}_i[h_1(\kappa)h_2(z)\xi] - s \Big) + W_i,$$

which implies

$$d_i^*(s) = \mathbb{1}\big(\theta_i h_1(\kappa)h_2(z) \geq s\big) = \mathbb{1}\left(\theta_i \geq \frac{s}{h_1(\kappa)h_2(z))}\right)$$

[31]Here, $\mathbb{E}_i[\cdot] = \mathbb{E}_{\mathbb{P}_i}[\cdot]$.

and

$$\int_{i\in I} d_i^*(s) = I\left[1 - F\left(\frac{s}{h_1(\kappa)h_2(z)}\right)\right].$$

Given the large trader's short position $z < 0$, and the exogenous stock supply $N < I$, the stock market clearing condition is again

$$N - z = \int_{i\in I} d_i^*(s) = I\left[1 - F\left(\frac{s}{h_1(\kappa)h_2(z)}\right)\right],$$

which implies the quasi-competitive equilibrium price $s(\kappa, z)$ satisfies

$$s(\kappa, z) = h_1(\kappa)h_2(z)G\big((I - N + z)/I\big) =: h_1(\kappa)h_2(z)\theta(z).$$

Now, everything follows as in a complete market model.

1.6.4 Liquidity Costs

We now consider the extension where when the broker liquidates some of the large trader's shorts to meet the margin constraint, there is an exogenous cost per share equal to $c > 0$. Introducing this liquidity cost implies a greater wealth loss from a liquidation and thus more shorts are needed to be closed after a margin adjustment. Indeed, with the cost $c > 0$ for a margin adjustment, the revised short position after a liquidation satisfies

$$W_L = \underbrace{|z_3|s(\kappa, z_3)|m}_{\text{new margin}} + \underbrace{s(\kappa, z_3)|z_3 - z_1|}_{\text{endogenous cost}} + \underbrace{c|z_3 - z_1|}_{\text{exogenous cost}} \tag{1.20}$$

where $z_1 < 0$ is the initial short position. Compared to equation (1.9) in the baseline model, the liquidation costs consist of both an endogenous and an exogenous part. If we fix κ and the initial short position z_1, the quantity z_3 needs to be revised more (i.e., more shorts are closed) so that the maintenance margin can be restored, which further implies that the large trader's loss increases because the stock price bubble increases even more.

In turn, this implies a stronger incentive for precautionary savings and a strong deterrence is more likely to occur. To see this, note that

$$V_2(z_1) = \Big(\lambda[z_3 h_1(\bar{\kappa})h_2(z_3) - z_1 h_1(\bar{\kappa})h_2(z_1)]\theta_L - \lambda[s(\bar{\kappa}, z_3) + c](z_3 - z_1)\Big)\mathbb{1}\big(z_1 < z^{\intercal}(\bar{\kappa})\big).$$

Compared to equation (1.13), the large trader faces a larger liquidation cost whenever a margin adjustment occurs. Note that, with λ fixed, this does not affect the optimal short position $z^{\dagger}(\lambda)$ when the initial wealth is large enough (i.e., $W_L \geq \widehat{W}_L\big(\bar{\kappa}, z^{\dagger}(\lambda)\big)$) because the liquidation cost only matters when the budget constraint is binding. However, it does imply that strong deterrence is more likely to occur when $W_L < \widehat{W}_L\big(\bar{\kappa}, z^{\dagger}(\lambda)\big)$. Indeed,

$$\begin{aligned} V_2'\big(z^{\intercal}(\bar{\kappa})\big) &= \lambda h_1(\bar{\kappa}) \underbrace{\big[1 - z_3'\big(z^{\intercal}(\bar{\kappa})\big)\big]}_{K_1 > 0} \\ &\times \underbrace{\Big\{\big(\theta(z^{\intercal}(\bar{\kappa})) - \theta_L\big)h_2\big(z^{\intercal}(\bar{\kappa})\big) - z_1 h_2'\big(z^{\intercal}(\bar{\kappa})\big)\Big\}}_{K_2 > 0} + \lambda h_1(\bar{\kappa}) \underbrace{c\left(1 - z_3'\big(z^{\intercal}(\bar{\kappa})\big)\right)}_{K_3 > 0}. \end{aligned}$$

This further implies that[32]

$$\frac{d\left[V_1'\big(z^{\intercal}(\bar{\kappa})\big) + V_2'\big(z^{\intercal}(\bar{\kappa})_-\big)\right]}{dc} > 0.$$

Thus, compared to the initial model, $V_1'\big(z^{\intercal}(\bar{\kappa})\big) + V_2'\big(z^{\intercal}(\bar{\kappa})_-\big) \geq 0$ is more likely to hold, and thus strong deterrence is more likely to occur.

[32]Note that $z^{\intercal}(\bar{\kappa})$ and $V_1'\left(z^{\intercal}(\bar{\kappa})\right)$ are both independent of $c > 0$. By differentiating equation (1.20), we get $z_3'\big(z^{\intercal}(\bar{\kappa})\big) = \frac{c/h_1(\bar{\kappa}) + h_2\theta}{c/h_1(\bar{\kappa}) + h_2\theta + K_0}$ where $K_0 := -m\big(h_2 + h_2' z^{\intercal}(\bar{\kappa})\big)\theta - m h_2 z^{\intercal}(\bar{\kappa})\theta'(z^{\intercal}(\bar{\kappa}))$. Finally, if we modify lemma 1.5.2 to include $c > 0$, we have $V_2'(z^{\intercal}(\bar{\kappa})_-) > 0$. Note that $\theta(z^{\intercal}(\bar{\kappa})) > \theta_L$, otherwise it is suboptimal to short in stage 1. This, combined with the fact that $V_2'(z^{\intercal}(\bar{\kappa})_-) > 0$, implies that K_1, K_2 and K_3 are all positive.

1.6.5 Media Visibility Costs

In the baseline model, a media group can choose the level of media visibility without incurring any costs. We make this assumption to simplify the analysis and it is a reasonable first approximation for the direct communication costs when using online chatrooms. However, there may be other opportunity costs associated with increasing the media group's visibility with respect to a targeted stock, e.g., the time spent in chat rooms. Moreover, it can take the form of negative real effects independent of the randomness in the stock. This is because any excessive scrutiny received makes the managerial incentives unnecessarily conservative and decreases the underlying firm's value. It could also be the cost of achieving consensus when the media group is more visible.

We can easily incorporate these other costs into the previous model. Assume that the cost of media impact κ is given by $C_M(\kappa) = \frac{1}{2}c_M\kappa^2$ where $c_M \geq 0$. Given the large trader's short position $z < 0$, the marginal small trader's belief type is $\theta(z) = G\left(\frac{I-N+z}{I}\right)$. We can change the objective function of the media group to

$$
\begin{aligned}
U_C(\kappa|z) &= \int_{i\in I}\Big(d_i^*\mathbb{E}_{\mathbb{P}_i}[S] - d_i^* s(\kappa, z) + W_i\Big)di - C_M(\kappa) \\
&= h_1(\kappa)h_2(z)\int_{\theta(z)}^{1}(\theta_i - \theta(z))f(\theta_i)d\theta_i - C_M(\kappa) + \int_{i\in I} W_i di \\
&= h_1(\kappa)h_2(z)\Big[1 - F\big(\theta(z)\big)\Big]\Big(E[\theta|\theta \geq \theta(z)] - \theta(z)\Big) - C_M(\kappa) + \int_{i\in I} W_i di.
\end{aligned}
$$

Hence, the solution is

$$\kappa^*(z) = \arg\max_{\kappa\in\mathcal{K}} U_C(\kappa|z).$$

If we take $h_1(\kappa) = 1 + a\kappa$, $h_2(z) = 1$, $\mathcal{K} = \mathbb{R}_+$ and $\theta \sim$ Uniform$\big([0,1]\big)$, we can embed this mapping $\kappa^*(z)$ for stage 2 in the previous equilibrium analysis. The main difference is that the larger the media visibility cost (i.e., $c_M \uparrow$), the lower the coalition's media impact parameter (i.e., $\kappa^*(z) \downarrow$), which implies that a price bubble and short squeeze are less severe and thus the shorts are more aggressive.

1.7 Conclusion

This paper studies how chatroom traders unionized via social media platforms influence the stock price through impact trading. The economic consequences of this coalition formation is analyzed in a micro-founded quasi-competitive equilibrium framework with strategic trading. A large trader strategically shorts a targeted stock, fully anticipating the price impact of her trade. Meanwhile, small traders may form a media group to coordinate their purchases and exert efforts to promote the stock on social platforms and individual networks. Various equilibrium phenomena arise, including price bubbles, short selling, short squeezes, forced liquidations, and precautionary savings. We show that media groups can discipline the large trader's incentive to short sell, but it can be either allocational efficiency increasing or decreasing. Using a belief-neutral welfare criterion, we show that media groups discipline the large trader's incentive to short sell, and it uniformly improves social welfare.

CHAPTER 2

INDEX DESIGN: HEDGING AND MANIPULATION

2.1 Introduction

This paper investigates how to construct an index of an asset's price, for the payment of cash-settled derivatives, such that: (1) the derivative can be used to hedge the underlying asset price risk; (2) the derivative can be used for price discovery of the underlying asset; and (3) the index is not easily manipulated. All of these three concerns, including *hedging, price discovery, and manipulation*, defined herein, are important in the creation of an index. Among these, the first requirement, hedging, is often the primary reason for the creation of the index, and this problem is inherently dynamic. Moreover, the third requirement, manipulation-proof, is important because if an index is easily manipulated, then the price process is not based on fundamentals, and it cannot be used to hedge the underlying asset price risk. Last, the second one, price discovery, is also important (Duffie et al., 2017), but it is often not the primary purpose for the creation of the index. In this paper, we focus on the first and the third reasons, i.e., hedging and manipulation.

A canonical example of index construction is the transformation from the London Interbank Offered Rate (Libor) to the Secured Overnight Financing Rate (SOFR) for their use in interest rate derivatives. SOFR is seen preferable to Libor because it is transaction based, rather than using quoted borrowing rates, especially after the Libor scandal.[1] But, unlike the construction of the Libor index

[1]For instance, see, *Wall Street Journal*, February 3, 2017, "CFTC Fines Royal Bank of Scotland

which is essentially equally weighted, the SOFR index is volume weighted.

This paper is motivated by two crucial insights. First, when an index is created to avoid manipulation, its construction can inhibit the ability of the derivative to hedge price risks. Indeed, to see this, consider the following. Let the information set revealed (the randomness realized) by the underlying asset price be denoted $\mathcal{F}$. We want the index to reflect only the risk in $\mathcal{F}$. However, if the index reflects additional randomness, then there is basis risk. An example is the VWAP-constructed SOFR index. The SOFR index uses volume weights and because the volume for any given transaction is random and different from the randomness contained in interest rates, these weights introduce basis risk to the index.

Second, the market price determination mechanism is also critical to an analysis of these issues. Specifically, there are two different approaches that can be used to understand market clearing.

1. The competitive market paradigm, modified to deal with manipulation. Here, most traders are price takers, and an equilibrium is based on equating supply and demand, where the demand is optimally determined.
2. A strategic game paradigm. Traders act strategically, having information about other traders, and a Nash equilibrium is the basis for a market clearing mechanism.

Which of these two price determination mechanisms better approximates security markets is important for index design issues. In large markets, e.g., interest

$85 Million Over Attempted Manipulation of Interest-Rate Benchmark".

rate markets, one can argue that the competitive market paradigm is a better approximation because most traders probably act as price takers, except for a few large financial institutions. This paper takes this later perspective and uses the competitive market paradigm, modified for the existence of large traders, to understand index design.

In such a market, we show that an EWAP-type index based on transaction data can both effectively alleviate market manipulation and facilitate hedging. The logic is straightforward. In a competitive market paradigm, there exists a large number of price takers. When the index is EWAP constructed, the price impact on the index by a single large trader is limited and vanishes in the limit with a continuum of price takers. In contrast, a VWAP-type index gives the large trader a big price impact, and thus it begets a strong incentive for index manipulation to reap benefits from index related derivatives. This also destroys perfect hedging and market completeness. Our main result holds under various scenarios such that when the large trader's derivative position is exogenous, when the derivative position is endogenous, when the large trader has market power on both the asset and derivative markets, and also in a continuous time model.

An outline for this paper is as follows. Section 2.2 presents a literature review. The model's setup is contained in Section 2.3 for a multiple round model. Sections 2.4 and 2.5 present the model's implications, with Section 2.6 providing various extensions. Section 2.7 studies the continuous time model and Section 2.8 concludes.

2.2 Literature Review

The existing literature is static and uses the strategic game paradigm (Eisl et al., 2017; Baldauf et al., 2018; Coulter et al., 2018; Duffie and Dworczak, 2020). Hence, it ignores hedging and concentrates on manipulation. In doing so, it neglects the essential reason why the index is created. As a result, a basic result is such that an EWAP-type index, ceteris paribus, is more easily manipulated than a VWAP-type. This result depends crucially on the market clearing mechanism. As argued by Duffie and Dworczak (2020), small transactions should be discarded because they are cheap to manipulate. In contrast, under the competitive equilibrium paradigm, transactions by small traders jointly determine the martingale measure and adding them to the index greatly limits the large trader's price impact. Moreover, as argued above, the VWAP-type construction introduces basis risk due to the use of volume weights, which is ignored in the existing literature. An exception is Zhang (2020), who studies financial benchmark manipulation in a dynamic setting, but does not focus on index design. Instead, he proposes a measurement of manipulation incentives for regulation.

Our paper's contributions are twofold. First, it is dynamic, and thus suitable for analyzing hedging. Even our static model has a dynamic element to it, and it aggregates both the asset price before and after manipulation if it exists. Second, it is based on the competitive market paradigm, which can better approximate security markets than the alternative strategic game paradigm. Our basic result is in stark contrast with the existing literature, and we show VWAP is more easily manipulated than is EWAP. Moreover, the VWAP construction destroys complete markets and introduces basis risk (i.e., the intermediary's hedge fails), and the EWAP construction preserves complete markets and thus is consistent with perfect hedging.

Our paper is also remotely related to two more strands of literature. First, it is related to the empirical work investigating index manipulation, including Libor manipulation (Abrantes-Metz et al., 2012; Bonaldi, 2017; Gandhi et al., 2019), VIX manipulation (Griffin and Shams, 2018), and FX manipulation (Evans et al., 2018). Second, it is also related to papers on trade-based manipulation, which functions through buying and selling alone (Jarrow, 1992; Allen and Gale, 1992; Allen and Gorton, 1992; Chakraborty and Yılmaz, 2004b, 2008; Goldstein and Guembel, 2008; Williams and Skrzypacz, 2020).

2.3 The Setup

This section introduces the basic model structure to study index manipulation by a large trader in an otherwise competitive market.

2.3.1 The Basic Model

We construct a model where multiple rounds of trading occur in sequence between time 0 and time 1. There are three types of traders: a continuum of small traders of measure $\mathcal{I} > 0$, an intermediary (a broker), and a large trader. The randomness in the economy is given by the probability space $(\Omega, \mathcal{F}, \mathbb{P})$. There are three assets traded in the economy: a money market account (mma) paying a riskless return, a commodity (asset), and a derivative on the asset. Without loss of generality, we set the mma return to be zero, which implies that the asset and the derivative are both denominated in units of the mma. The time 0 asset price is denoted S_0. The asset's liquidation value at time 1 is denoted as $S_1 = \xi$ with $\xi : \Omega \to \mathbb{R}_+$, a non-negative random variable. The asset has an exogenous supply of $N > 0$ shares, which are infinitely divisible.

A derivative on the asset trades at time 0 with price C_0. The derivative is cash settled at time 1 with a payoff given by a function $C_1 : \mathbb{R}_+ \to \mathbb{R}_+$ of an asset market index I_t for $t \in \{0, 1\}$. The index's value at time 0 is fixed at unity. The index's value at time 1 is based on the average asset price in the trading rounds occuring between times 0 and 1. In this sense, one should think of the time period [0,1] as corresponding to a single trading interval, whose successive trades determine the index's value at time 1 for cash settlement of the derivative.[2] The asset market is assumed to be complete, so that the derivative is a redundant security and its time 0 price is uniquely determined by no-arbitrage considerations. The market is assumed frictionless except for various short sale restrictions on the small traders that are introduced shortly.

To purpose of the model is to understand how a large trader, who holds an initial position in the derivative, can manipulate the equilibrium asset price at time 0 to affect the market index that determines the derivative's time 1 payoff.[3] To capture this scenario, we assume that the large trader purchases $z > 0$ derivatives at time 0 from the intermediary. The intermediary is unconstrained in buying and selling shares of the asset and derivative, believes the market is complete, and enforces the derivative's arbitrage-free pricing. The intermediary,

[2] This interpretation is most easily seen in the continuous time model in Section 2.7 when considering trades over an instant in time.

[3] An example to keep in mind is an interest rate cap with Libor the underlying index.

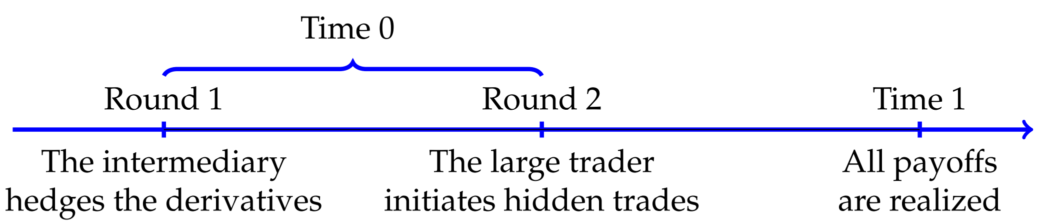

Figure 2.1: Timeline of the index manipulation game

being a broker, holds no inventories and hedges his short derivative position by buying $z\Delta$ shares of the asset at time 0 where Δ corresponds to the derivative's hedge ratio.

The smaller traders participate in the asset market, but they cannot sell short. The large trader can also purchase shares of the asset. An equilibrium in the asset market corresponds to a sequence of time 0 prices, denoted $S_0(z, x) : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ where z is the large trader's derivative holdings and x is the large trader's asset holdings such that the aggregate demand for the asset equals its outstanding supply, including the demands of the intermediary and small traders.

To capture the construction of the market index and potential market manipulation, we introduce two trading rounds at time 0 as illustrated in Figure 2.1. These trading rounds correspond to changes to the market's demand resulting from actions taken by the large trader. In particular, we have the following:

- Round 1 corresponds to the initial equilibrium in the asset market where the small traders and intermediary, acting as price takers, clear the market. The large trader purchases $z > 0$ derivatives from the intermediary at the arbitrage-free price C_0. The intermediary believes the market is complete and hedges her short derivative position by buying Δz shares of the asset where Δ is the hedge ratio. The initial equilibrium asset price is denoted $S_0(z, 0)$ where 0 corresponds to the asset holdings of the large trader in round 1. At time 0, the asset market index is determined from past trading intervals, and fixed at $I_0 = 1$.

- Round 2 has the large trader purchasing $x > 0$ shares of the asset, perturbing the equilibrium asset price to $S_0(z, x)$. The small traders necessarily retrade due to this perturbation. The time 1 index value is defined as an average asset price over the trading rounds at times 0 and 1, i.e.

$$\begin{aligned} I_1 &= \frac{1}{2}S_1 + \frac{1}{2}\big(wS_0(z,x) + (1-w)S_0(z,0)\big) \\ &= \frac{1}{2}\big(S_1 + S_0(z,0)\big) + \frac{w}{2}\big(S_0(z,x) - S_0(z,0)\big) \end{aligned}$$

  for some given weight $0 \leq w \leq 1$ on the second round of trading at time 0. The exact weighting of the time 0 and time 1 prices is not important in the subsequent analysis, as long as the weights (1/2) are fixed.

- Two different indexes will be analyzed below depending upon the choice of the weight w.

  – A volume-weighted average pricing (VWAP) index. Here, the weight on round 2, $w = x/N$, corresponds to the volume of the large trader's transaction relative to the total volume of all traders' trades.

  – An equal-weighted average pricing (EWAP) index. Here, $w = 1/\mathfrak{c} = 0$,[4] which corresponds to an equal weighting of the transactions where the number of the small traders' retrades in round 1 correspond to an uncountably infinity (one trade from each trader in $\mathcal{I}$)

[4]We denote the cardinality of the continuum by $\mathfrak{c}$.

relative to the large trader's single trade. Under EWAP, the large trader can only initiate a hidden trade just once and thus the weight assigned to her trading is zero, i.e., $w = 0$. As discussed in Duffie and Dworczak (2020), EWAP mechanisms are susceptible to gaming by the large trader via order splitting, that is, a trader can submit mutiple small orders several times, rather than one large transaction. Because the large trader can only trade once in our model, this type of strategic order splitting is excluded.[5]

- At time 1, the cash-settled derivatives payoff is (by construction)

$$C_1\big(2I_1 - S_0(z,0)\big) = C_1\Big(\xi + w\big(S_0(z,x) - S_0(z,0)\big)\Big). \tag{2.1}$$

  We note that the functional form of equation (2.1) is not crucial to the subsequent analysis as long as the payoff is increasing in $S_0(z,x)$. The affine transformation of I_1 is only included to make the payoff simpler in appearance.

To be consistent with a complete market, we assume that the asset's evolution is a binomial process. We need to introduce some additional notation. With respect to the probability space, let $\Omega = \{H, L\}$, $\mathcal{F} = 2^{\Omega}$, and $\mathbb{P}(H) = 1 - \mathbb{P}(L) = p$. The asset's time 1 liquidation value is given by $S_1(\omega) = \xi(\omega)$ where

$$\xi(\omega) = \begin{cases} \bar{\xi}, & if\ \omega = H \\ \underline{\xi}, & if\ \omega = L \end{cases}$$

[5]If we allow for mutiple trades (and thus order splitting) by the large trader between time 0 and 1, the basic results hold if either: (i) the large trader can only trade a finite number of times since $w = n/\mathfrak{c} = 0$ for all $n < \infty$; or, (ii) she can only trade a countably infinite times since a countably infinite set has a measure zero; or, (iii) she can submit an uncountably infinite number of transaction orders but there is a fixed cost of submitting an order; or, (iv) the transactions are submitted by well-identified traders, such as the case for Libor and the Platts' Market on Close process for energy price benchmarks.

and $\bar{\xi} > \underline{\xi}$.

It is well known that $\bar{\xi} > \underline{\xi}$ is a necessary and sufficient condition for the binomial model to be complete. And, to avoid arbitrage, we require that $S_0(z,x) \in (\underline{\xi}, \bar{\xi})$. This will be shown to always hold in our model's equilibrium.

2.3.2 The Small Traders

Small traders are assumed to be risk neutral with heterogenous subjective beliefs. The risk neutral assumption is relaxed in a subsequent section. Each trader $i \in [0, \mathcal{I}]$ holds a different belief denoted by $\mathbb{P}_i(\bar{\xi}) = 1 - \mathbb{P}_i(\underline{\xi}) = \theta_i$, where the parameter $\theta_i \in [0,1]$ has the cumulative distribution function $F(\theta_i)$ with the density $f(\theta_i) > 0$ for all $\theta_i \in [0,1]$. Obviously, $F(0) = 0$ and $F(1) = 1$. This cumulative distribution function measures the distribution of the small trader's beliefs in the economy from pessimistic to optimistic. Here, the ith small trader's subjective probability measure $\mathbb{P}_i$ can differ from the actual statistical probability measure $\mathbb{P}$.

We impose the following assumption.

Assumption 2.3.1 (Small Traders Trading Constraints and Wealths)**.**

For all small traders $i \in [0, \mathcal{I}]$,

i) the trading quantity $d_i \in [0,1]$, and
ii) the initial wealth $W_i \geq \bar{\xi}$.

Assumption 2.3.1 (i) states that the small traders cannot short the asset and can buy at most one share of the asset. The one share maximum is without loss of generality and can be replaced with any $\bar{d} > 0$. Assumption 2.3.1 (ii) states that the small trader has sufficient initial wealth to purchase any shares desired within the trading limit without shorting the mma.

Small traders are price takers who solve the following optimization problem

$$d_i^\star(S_0) = \underset{d_i \in [0,1]}{\arg\max} \left(d_i \mathbb{E}_{\mathbb{P}_i}[S_1] - d_i S_0 + W_i\right)$$
$$s.t.\ d_i S_0 \leq W_i$$

Here, $\mathbb{E}_{\mathbb{P}_i}[\cdot]$ denotes expectation under $\mathbb{P}_i$,

$$\mathbb{E}_{\mathbb{P}_i}[S_1] = \mathbb{E}_{\mathbb{P}_i}[\xi] = \theta_i \bar{\xi} + (1 - \theta_i)\underline{\xi},$$

and $W_i - d_i s_0$ corresponds to the investment in the mma.

Given the asset price S_0, the ith small trader's optimal asset position is given by[6]

$$d_i^\star\left(S_0\right) = \mathbb{1}_{[S_0, \bar{\xi}]}\left(\mathbb{E}_{\mathbb{P}_i}[S_1]\right) = \begin{cases} 1, & \text{if } \theta_i \bar{\xi} + (1-\theta_i)\underline{\xi} \geq S_0 \\ 0, & \text{otherwise.} \end{cases} \tag{2.2}$$

Alternatively, the small trader i buys one share of asset if and only if

$$\theta_i \geq \frac{S_0 - \underline{\xi}}{\bar{\xi} - \underline{\xi}}.$$

Fixing S_0, given the distribution of beliefs across the small traders, the aggregate market demand is

$$\int_{i \in [0, \mathcal{I}]} d_i^\star(S_0) = \mathcal{I}\left[1 - F\left(\frac{S_0 - \underline{\xi}}{\bar{\xi} - \underline{\xi}}\right)\right]. \tag{2.3}$$

[6]Note that, the indicator function $\mathbb{1}_A(x)$ equals 1 when $x \in A$, and equals 0 when $x \notin A$.

2.3.3 The Large Trader

The large trader does not trade in the asset market at time 0 in round 1. Instead, the large trader only purchases $z > 0$ shares of the derivative in round 1. In the baseline model, the large trader's derivative position is not strategic. This implies that the large trader is a price taker with respect to the derivative's time 0 price C_0. We will relax this restriction and endogenize the initial derivative position in section 2.6.2 below. In equilibrium, the price S_0 and the risk neutral probabilities $\mathbb{Q}$ depend on z, although the large trader is not optimizing over z.

However, the large trader purchases assets in round 2. Fully anticipating the impact of her trades on the asset price in round 2, she strategically manipulates both the asset price and index to benefit from her existing position in the derivative. As such, her trades are hidden and unknown to either the small traders or intermediary.

2.3.4 The Intermediary and the Derivative Market

The intermediary holds a belief given by $\mathbb{P}$. Given the binomial process for the asset price, the intermediary believes that the asset market is complete. The intermediary acts as a broker, buying and selling derivatives at time 0 in round 1, and holding no inventory. He determines the arbitrage-free price of the derivative. Because the intermediary holds no inventory, he hedges his derivative position in the asset market.

The intermediary believes that the derivative's time 1 payoff is determined by the index value $I_1 = \frac{1}{2}(S_1 + S_0(z,0))$, which fails to include the hidden trades of the large trader. The reason for this belief is that the large trader purposefully hides her illegal manipulative trades.[7] This model is created in order to design an index such that this sort of hidden illegal manipulation is precluded.

Given the intermediary is unconstrained in buying and selling the asset and derivative, the intermediary's actions determine the derivative's price. The intermediary sees the payoff to the derivative from expression (2.1) as

$$C_1\big(2I_1 - S_0(z,0)\big) = C_1(\xi).$$

The standard binomial derivative pricing model (Shreve, 2005, Chap. 1) yields

$$C_0 = \mathbb{E}_{\mathbb{Q}}\big[C_1(\xi)\big] = \frac{C_1(\bar{\xi})(S_0 - \underline{\xi}) + C_1(\underline{\xi})(\bar{\xi} - S_0)}{\bar{\xi} - \underline{\xi}} \tag{2.4}$$

where

$$S_0(z,0) = \mathbb{E}_{\mathbb{Q}}[\xi], \tag{2.5}$$

and

$$\mathbb{Q}(H) = 1 - \mathbb{Q}(L) = \frac{S_0 - \underline{\xi}}{\bar{\xi} - \underline{\xi}}.$$

Standard "delta-hedging" gives

$$C_0 = \Delta S_0(z,0) + m$$

where m is the position in the mma and

$$\Delta = \partial C_0 / \partial S_0(z,0) = \frac{C_1(\bar{\xi}) - C_1(\underline{\xi})}{\bar{\xi} - \underline{\xi}}. \tag{2.6}$$

[7]We impose a behavioral assumption that the intermediary acts naively as if there is no manipulation when doing the hedging. Ideally, this assumption can be removed by building a more complex model in which the intermediary anticipates the possibility of manipulation in a rational expectation equilibrium. See Section 2.6.1 for a detailed discussion.

2.4 The Round 1 Asset Market Equilibrium

This section determines the equilibrium in the asset market at time 0 in round 1. Recall that the aggregate supply of shares in the asset market is $N > 0$. Thus, the asset market clearing condition is

$$N = z\Delta + \mathcal{I}\left[1 - F\left(\frac{S_0 - \underline{\xi}}{\bar{\xi} - \underline{\xi}}\right)\right] \tag{2.7}$$

where $z\Delta$ corresponds to the intermediary's demands and $\mathcal{I}\left[1 - F\left(\frac{S_0 - \underline{\xi}}{\bar{\xi} - \underline{\xi}}\right)\right]$ represents the aggregate demand of the small traders. Note that the equilibrium price S_0 depends on the derivative position z of the large trader because of the hedging action of the intermediary.

Denote by $G(\cdot) = F^{-1}(\cdot)$ the inverse of the cumulative distribution function $F(\cdot)$. Note that $G(\cdot)$ is well defined because $F(\theta)$ is strictly increasing for all $\theta \in [0, 1]$.

Proposition 2.4.1 (Round 1 Equilibrium Asset Price)**.**

Given the large trader's derivative position $z > 0$, the equilibrium price is

$$S_0(z, 0) = \underline{\xi} + (\bar{\xi} - \underline{\xi})G\left(\frac{z\Delta + \mathcal{I} - N}{\mathcal{I}}\right) \tag{2.8}$$

where the martingale measure is

$$\mathbb{Q}(H|z) = 1 - \mathbb{Q}(L|z) = G\left(\frac{z\Delta + \mathcal{I} - N}{\mathcal{I}}\right). \tag{2.9}$$

The proof follows directly from expressions (2.7) and (2.5). The following comparative statics will prove useful below.

- The asset price is increasing in the large trader's derivative position, i.e.

$$\frac{\partial S_0(z,0)}{\partial z} = \frac{\Delta\left(\bar{\xi} - \underline{\xi}\right)}{\mathcal{I} f\left(\frac{z\Delta + \mathcal{I} - N}{\mathcal{I}}\right)} > 0.$$

- The equilibrium asset price $S_0(z,0)$ can be either concave, linear, or convex in z. The properties of $S_0(z,0)$ are implied by the function $G(\cdot)$, or equivalently $F(\theta)$. For instance, if $F(\theta) = \sqrt{\theta}$, which is concave, then $G(\theta) = \theta^2$ and the asset price $S_0(z,0)$ is convex in z.

- The curvature of $S_0(z,0)$ is related to the degree of pessimism/optimism across the small traders. For $F(\theta)$ concave, there are more small traders holding pessimistic beliefs. When the derivative position of the large trader z increases by one unit, the effective asset supply is reduced by Δ. Hence, when z is close to zero, if $F(\theta)$ is concave, a tiny price increase implies that more pessimistic small traders will not purchase the asset.

2.5 The Round 2 Asset Market Equilibrium

In round 2, the large trader strategically buys the asset to affect the asset index and the future value of her derivative position. This section determines the perturbed equilibrium asset price.

2.5.1 The Equilibrium

Let $x(S_0)$ be the optimal asset demand of the large trader given the price S_0. The large trader trades at the asset price $S_0(z,x)$, determined in the perturbed equilibrium, which depends on her quantity purchased $x \geq 0$. The equilibrium clearing condition is modified to

$$N = z\Delta + \int_{i\in[0,\mathcal{I}]} d_i^\star\big(S_0(z,x)\big) + x \tag{2.10}$$

where $z\Delta$ is fixed since the intermediary does not retrade.

Note that this is the same equilibrium computed in round 1 but with a modified supply of shares $(N - x)$ outstanding. In this perturbed equilibrium, the small traders retrade and reduce their holdings relative to the initial equilibrium since $S_0(z,x) > S_0(z,0)$. This implies that the aggregate asset demanded by the small traders is revised to

$$\int_{i\in[0,\mathcal{I}]} d_i^\star\big(S_0(z,x)\big) = \mathcal{I}\left[1 - F\left(\frac{S_0(z,x) - \underline{\xi}}{\bar{\xi} - \underline{\xi}}\right)\right].$$

Combining this with equation (2.10), the large trader successfully determines the market clearing price

$$S_0(z,x) = \underline{\xi} + (\bar{\xi} - \underline{\xi})G\left(\frac{z\Delta + x + \mathcal{I} - N}{\mathcal{I}}\right). \tag{2.11}$$

2.5.2 Index Design, Manipulation, and Market Completeness

Assuming the large trader doesn't optimize over z in round 1,[8] she acts as a price taker with respect to the impact of her derivative trades on the asset. Because the large trader acts as a price taker in the derivative market at round 1, she values the asset and derivative's payoffs using the martingale measure $\mathbb{Q}(\omega|z)$.

In round 2 she anticipates that her trade size x will affect the asset price, and strategically manipulates the asset index. Denote by W_L the large trader's

[8] The derivative position z is endogenized in the general equilibrium analysis in section 2.6.2.

initial wealth. She is risk neutral and to simplify the analysis, we assume that she cannot short sell the asset (i.e., $x \geq 0$). Hence, the large trader chooses

$$
\begin{aligned}
x^{\star}(z) = \arg\max_{x \geq 0} & \Big\{ x\mathbb{E}_{\mathbb{Q}}[\xi] + z\mathbb{E}_{\mathbb{Q}}\Big[C_1\Big(\xi + w\big(S_0(z,x) - S_0(z,0)\big)\Big)\Big] \\
& + W_L - xS_0(z,x) - zC_0\Big\} \\
s.t. \quad & xS_0(z,x) + zC_0 \leq W_L.
\end{aligned}
$$

In the objective function, the first term is the time 0 value of the investment in the asset, the second one is the value of the derivative position, and the last one is the value of the mma position. The following theorem can now be proven.

Theorem 2.5.1 (Index Design and Manipulation)**.**

(i) Under the EWAP index weighting, $x^{\star}_{EW}(z) = 0$ and there is no index manipulation.

(ii) If $\frac{z}{N}\mathbb{E}_{\mathbb{Q}}[C_1'(\xi)] > 1$, then under the VWAP index weighting, $x^{\star}_{VW}(z) > 0$ and the large trader manipulates the index (i.e., $I_1 > \frac{1}{2}\left(S_0(z,x) + S_1\right)$).

Proof. See Appendix B.1.1. □

To understand this theorem, note the following two points.

1. By assertion (i), under the EWAP index weighting, $w = 0$, and the linkage to the derivative market position's impact doesn't exist. It is not profitable for the large trader to manipulate the asset market.

2. By assertion (ii), under the VWAP index weighting, $w > 0$, and if the derivative's payoff is sufficiently sensitive to the index change (i.e., $\frac{z}{N}\mathbb{E}_{\mathbb{Q}}[C_1'(\xi)] > 1$), the gains from the derivative market offsets the losses due to buying the asset. And, the large trader manipulates the asset price to inflate the index's value to profit from her position in the derivative market.

Hence, the implication for index design is that the volume-weighted index is more likely to promote index manipulation. In contrast, the equal weighted index limits the influence of the large trader's asset purchases, making market manipulation unprofitable. The volume weighted index also destroys market completeness.

Theorem 2.5.2 (Index Design and Market Completeness)**.**

(i) Under the EWAP index weighting, the market is complete.

(ii) Since $x^\star_{VW} > 0$, under the VWAP index weighting, the market is incomplete.

This result follows due to the randomness introduced into the derivative's payoff by the large trader's round 2 transactions on the volume weighted index, which depends on $x > 0$. These transactions, unanticipated and unknown to the intermediary (the only agent hedging in our model), destroys his round 1 hedge. For instance, the intermediary was under-hedged if the payoff to the derivative is convex in the index at time 1,[9]

$$\Delta_x := \frac{C_1\big(\bar{\xi} + w\big(S_0(z,x) - S_0(z,0)\big)\big) - C_1\big(\underline{\xi} + w\big(S_0(z,x) - S_0(z,0)\big)\big)}{\bar{\xi} - \underline{\xi}}$$

[9]Here, we use the convexity of $C_1(\cdot)$ in the inequality.

$$> \frac{C_1(\bar{\xi}) - C_1(\underline{\xi})}{\bar{\xi} - \underline{\xi}} = \Delta$$

where the correct hedge ratio, given $x > 0$, is given on the left side of the preceding expression. In contrast, the equal weighted index does not depend on the larger trader's perturbation to the market's trading volume. Furthermore, the incomplete market result for the VWAP mechanism in theorem 2.5.2 follows from the behavioral assumption that the intermediary is naive. The large trader's transaction size being hidden is crucial to this conclusion and index manipulation alone (if anticipated) does not necessarily render the derivative market incomplete.[10]

These two insights into the EWAP and VWAP indexes stand in contrast to the implications in the existing literatures, see the discussion in section 2.2. The reason is twofold. First, the existing models analyzing manipulation are not based on the competitive market paradigm where traders are price takers and not aware of the actions of large traders. This non-strategic behavior is different from that evidenced in the Nash equilibrium based existing models.[11] Second, the existing models are not dynamic, and consequently, ignore market completeness considerations with respect to index construction. This is a serious omission because often indexes are constructed purposefully to facilitate hedging (market completeness). And, the introduction of another random variable - volume - into the index's realized randomness destroys the ability of the index to hedge the payoffs to the underlying asset due to its fundamental risk (ξ in the above model).

[10]See section 2.6.1 for more discussion.

[11]Here, Nash equilibrium should be interpreted more generally as including both Bayesian Nash equilibrium and Perfect Bayesian Nash equilibrium.

We conclude this section with the following remark.

Remark 2.5.1 (Large Trader's Knowledge)**.**

The above result generalizes to the large trader not knowing $S_0(z,x)$, but only knowing that $\frac{\partial S_0(z,x)}{\partial x} > 0$. This follows because the result only depends on the first derivative of the perturbed price function to prove that it is optimal to manipulate.

2.5.3 Welfare Issues

This section explores the impact of index manipulation, when it occurs, on the market participants.

- The small traders make profits because $S_0(z, x^\star_{VW}) > S_0(z,0)$ and they are long the asset. But, only those smaller traders who are in the market, i.e., those with subjective beliefs $\theta_i \in \left[\frac{S_0(z,0)-\underline{\xi}}{\bar{\xi}-\underline{\xi}}, \frac{S_0(z,x^\star_{VW})-\underline{\xi}}{\bar{\xi}-\underline{\xi}}\right)$.
- The large trader benefits. Although the large trader loses value on her asset position by the amount

$$x^\star_{VW}\big(\mathbb{E}_{\mathbb{Q}}[\xi] - S_0(z, x^\star_{VW})\big) = x^\star_{VW}\big(S_0(z,0) - S_0(z, x^\star_{VW})\big) < 0,$$

  the gains in her derivative position

$$z\left(\mathbb{E}_{\mathbb{Q}}\left[C_1\big(\xi + w\big(S_0(z,x) - S_0(z,0)\big)\big)\right] - C_0\right) > 0$$

  exceed these losses. She loses value on the asset trade because her transaction increases the purchase price above $\mathbb{E}_{\mathbb{Q}}[\xi] = S_0(z,0)$.
- The intermediary loses the amount

$$z\left(\mathbb{E}_{\mathbb{Q}}\left[C_1\big(\xi + w\big(S_0(z,x) - S_0(z,0)\big)\big)\right] - C_0\right) < 0$$

because her trade was not fully hedged.

2.6 Robustness and Discussions

The remainder of this paper shows that these two implications are robust to relaxations of this simple model's structure, including the extensions to continuous time and risk aversion.

2.6.1 A Sophisticated Intermediary

In the baseline model, the intermediary is not aware of index manipulation ex ante because the large trader purposefully hides illegal trades. In this subsection, we discuss how this assumption can be supported in a rational expectation equilibrium. To this end, we can introduce a prior belief for the intermediary over the large trader's trade size. The strategic action of the large trader further depends on some private information about being detected, which is not explicitly modeled. Consider the following scenario in which with a sufficiently small probability, say $\lambda = \frac{1}{100}$, the intermediary believes that the large trader can hide her trade and manipulate the index, and with complementary probability $1 - \lambda = \frac{99}{100}$, the large trader cannot.

For the EWAP mechanism, independent of λ and the large trader's private information, the large trader does not have an incentive to game the index (i.e., $x^{\star}_{EW} = 0$) since the effective index weight assigned to her trade is zero. Knowing this, the intermediary employs a hedging strategy which coincides with that in the baseline model and the derivative market is complete in the sense of perfect hedging.

For the VWAP mechanism, when the large trader's private information reveals that she will be detected in round 2, the large trader simply does not trade since it is not profitable. In contrast, when detection is unlikely, the large trader trades strategically and benefits from index manipulation. This is because, before the realization of manipulation detection, the intermediary acts based on his prior, and the intermediary's hedging strategy and derivative pricing are well approximated by the baseline model with λ sufficiently small. Thus, the derivative market is incomplete by the same reasoning as that in theorem 2.5.2. To summarize: (i) given that the intermediary is sophisticated, markets with manipulation possible are more likely to be incomplete when there exists uncertainty over the large trader's trade size; and, (ii) the naive intermediary assumption can be viewed as the limit case when the intermediary's belief over the probability of manipulation is small (i.e., $\lambda \to 0$).

2.6.2 Optimal Derivative Selection by the Large Trader

This section relaxes the assumption that the large trader's derivative position z is exogenous. The appropriate equilibrium concept is that of a rational expectation equilibrium, where the large trader's optimal derivative position is consistent with the equilibrium. Here, we restrict the choice set for the large trader's derivative quantity to be binary, i.e. $\mathcal{C} = \{0, \hat{z}\}$. The remaining setup is unchanged. Specifically, in round 1, the large trader chooses the optimal derivative position $z^{\star} \in \mathcal{C}$, still acting as a price taker in the derivative market. In round

2, the large trader manipulates the asset price and all traders take her optimal derivative position $z^\dagger$ as given. In equilibrium, the consistency requirement is that $z^\star = z^\dagger$.

As in section 2.4, the large trader's sole goal is to manipulate the asset to benefit from her derivative position. The large trader knows the impact of her trading on the index (and that on the payoff to the derivative). The large trader's wealth is constrained by W_L dollars and she can not short sell the asset (i.e., $x \geq 0$).

Let us explain the equilibrium construction. We use backward induction.

- In round 2, her objective is to choose

$$x^\star(z, z^\dagger) = \arg\max_{x \geq 0} U(x, z) \tag{2.12}$$
$$s.t.\ xS_0(z^\dagger, x) + zC_0(z^\dagger) \leq W_L$$

  where

$$U(x, z) := x\mathbb{E}_{\mathbb{Q}}[\xi] + z\mathbb{E}_{\mathbb{Q}}\left[C_1\Big(\xi + w\big[S_0(z^\dagger, x) - S_0(z^\dagger, 0)\big]\Big)\right]$$
$$+ W_L - xS_0(z^\dagger, x) - zC_0(z^\dagger)$$

  and the martingale measure is $\mathbb{Q}(\omega|z^\dagger)$.

  Note the logic behind the notation $x^\star(z, z^\dagger)$. The first argument z shows up because the large trader is forward looking, while the second argument $z^\dagger$ comes from the price taking behavior assumption.

- In round 1, the large trader predicts her optimal asset holdings $x^\star\left(z, z^\dagger\right)$

in round 2. Hence, the round 1 solution is

$$z^\star = \arg\max_{z\in\mathcal{C}} U(x^\star\big(z, z^\dagger\big), z) \quad (2.13)$$
$$s.t.\ x^\star\big(z, z^\dagger\big) S_0\Big(z^\dagger, x^\star\big(z, z^\dagger\big)\Big) + zC_0(z^\dagger) \le W_L.$$

Second, we give a definition modified from the standard rational expectation equilibrium.

Definition 2.6.1 (Rational Expectation Equilibrium)**.**

The tuple $\big(z^\dagger, \mathbb{Q}(\cdot|z^\dagger), S_0(z^\dagger, x), C_0(z^\dagger), \Delta, x^\star(z^\star, z^\dagger), z^\star\big)$ *is a rational expectation equilibrium if it satisfies:*
i) Individual utility maximization. All traders solve their optimization problems problems. Specifically,

1) the smaller traders' demands satisfy equation (2.2) with $S(z^\dagger, 0)$ *and* $S_0(z^\dagger, x^\dagger)$ *in rounds 1 and 2;*

2) the intermediary hedges the derivative with price $C_0(z^\dagger)$ *in round 1; and*

3) the large trader optimally chooses $z^\star$ *in round 1, and* $x^\star(z^\star, z^\dagger)$ *in round 2.*
ii) Market clearing.

1) $N = z^\dagger\Delta + \int_{i\in\mathcal{I}} d_i^\star\big(S_0(z^\dagger, 0)\big)$ *in round 1, and*

2) $N = z^\dagger\Delta + x^\dagger + \int_{i\in\mathcal{I}} d_i^\star\big(S_0(z^\dagger, x^\dagger)\big)$ *in round 2.*
iii) Consistency. Specifically, $z^\star = z^\dagger$ *and thus* $x^\star(z^\star, z^\dagger) =: x^\dagger$*.*

Remark 2.6.1 (Hidden Trades)**.**

This definition 2.6.1 modifies the traditional rational expectation equilibrium by allowing the existence of hidden trades by the large trader in round 2. The hidden trades *can be viewed as the large trader's private information.*

The following proposition now follows.

Proposition 2.6.1 (REE Equilibrium)**.**

Assume that $\hat{z} < \frac{W_L}{C_0(\hat{z})}$ *and* $\frac{\hat{z}}{N}\mathbb{E}_{\mathbb{Q}}\left[C_1'(\xi)\right] > 1$ *where the martingale measure is* $\mathbb{Q}(\cdot|z^{\dagger})$.

Then, $\hat{z}$ *is a rational expectation equilibrium (i.e.,* $z^{\dagger} = \hat{z}$*) such that:*

i) $x^{\star}_{EW}(\hat{z}) = 0$ *under EWAP; and*

ii) $x^{\star}_{VW}(\hat{z}) > 0$ *under VWAP (index manipulation occurs).*

Proof. See Appendix B.1.2. □

This proposition 2.6.1 verifies that the index design implications from the exogenous z case (Theorem 2.5.1) still apply when z is endogenized. The restriction of a binary choice set can be relaxed. If instead the choice set is $\mathcal{C} = [0, \bar{z}]$, for an arbitrary $z^{\dagger}$ we can still use the equilibrium construction (see, equation (2.12) and (2.13) to find $z^{\star} \in \mathcal{C}$). Now, the consistency condition that $z^{\star} = z^{\dagger}$ becomes a fixed point problem. By choosing a suitable functional form for $F(\theta)$, we can solve this fixed point problem analytically.

2.6.3 Market Power

This section relaxes the assumption that the large trader is a price taker in the derivative market. Knowing that her derivative holdings z affects the martingale measure in the asset market, the large trader strategically chooses the optimal level of derivatives, fully anticipating that he can also manipulate the asset price by hidden trades, x, in the round 2 equilibrium.

Here, we modify the large trader's objective function by using the actual probability measure $\mathbb{P}$ to determine the appropriate present values. Again, the problem is solved via backward induction.

- Step 1: In round 2, z is taken as given, and the large trader determines the optimal hidden trade

$$\begin{aligned} x^{\star}(z) = \arg\max_{x \geq 0} \Big\{ & x\mathbb{E}_{\mathbb{P}}[\xi] + z\mathbb{E}_{\mathbb{P}}\left[C_1\Big(\xi + w\big[S_0(z,x) - S_0(z,0)\big]\Big)\right] \\ & + W_L - xS_0(z,x) - zC_0(z)\Big\} \\ s.t.\ & xS_0(z,x) + zC_0(z) \leq W_L. \end{aligned}$$

- Step 2: In round 1, the large trader anticipates his best action $x^{\star}(z)$ in round 2, and solves

$$\begin{aligned} z^{\star} \in \arg\max_{z \geq 0} \Big\{ & x^{\star}(z)\mathbb{E}_{\mathbb{P}}[\xi] + z\mathbb{E}_{\mathbb{P}}\Big[C\big(\xi + w * [S_0(z, x^{\star}(z)) - S_0(z,0)]\big)\Big] \\ & + W_L - xS_0\big(z, x^{\star}(z)\big) - zC_0(z)\Big\} \\ s.t.\ & x^{\star}(z)S_0\big(z, x^{\star}(z)\big) + zC_0(z) \leq W_L. \end{aligned}$$

The optimal solution is characterized by the pair $\big(z^{\star}, x^{\star}(z^{\star})\big)$.

The notations $C_0(z), S_0(z,0)$ and $S_0(z,x)$ capture the effects of the large trader's derivative position z on the martingale measure $\mathbb{Q}(\cdot|z)$ and the market prices of the asset and the index.

The following proposition characterizes the equilibrium. To simplify the notation, we normalize $\bar{\xi} = 1$, $\underline{\xi} = 0$, and $\mathcal{I} = N$. To facilitate an analytic solution, we add the following assumption.

Assumption 2.6.1 (Parametric Structures)**.**

(i) the heterogeneous beliefs $\theta_i \sim$ *Uniform*$([0, 1])$*;*

(ii) $C_1(x) = ax$*, where* $a > 0$*; and*

(iii) $W_L \geq (1 + p^2)N$.

Given the assumption, we can now prove the following proposition.

Proposition 2.6.2 (Equilibrium with Market Power)**.**

Given Assumption 2.6.1,

(i) Under EWAP, $x^\star_{EW} = \frac{pN}{3}$ *and* $z^\star_{EW} = \frac{pN}{3a}$*. The intermediary is fully hedged.*

(ii) Under VWAP, $x^\star_{VW} > \frac{pN}{3}$ *and* $z^\star_{VW} \in \left(\frac{pN}{3a}, \frac{pN}{3a}\frac{1}{(1-2p/3)}\right)$*;*

In particular, $x^\star_{VW} > x^\star_{EW}$ *and* $z^\star_{VW} > z^\star_{EW}$*. The intermediary is not fully hedged.*

Proof. See Appendix B.1.3. □

First, we introduced Assumption 2.6.1 to make the problem tractable, hence, this is a strongly parametrized result. Under the EWAP type index, the large trader holds both assets and derivatives and the market is complete.

Second, compared to the baseline model where the large trader is a price taker in the asset market, proposition 2.6.1 illustrates that the large trader exploits both markets under VWAP type indices ($z^\star_{VW} > z^\star_{EW}$) and the intermediary's hedge fails.

2.6.4 Small Traders with CARA Utility

This section relaxes the assumption that the small traders are risk neutral. Instead, we assume that they have CARA utility functions, i.e., $U_i(w) = -e^{-\lambda w}$, $\forall i \in [0, \mathcal{I}]$, and that their subjective beliefs coincide with the actual probability measure, i.e., $\mathbb{P}_i(\cdot) = \mathbb{P}(\cdot)$. To simplify the notation, we let $\mathcal{I} = 1$. We also remove the position limits on the small trader's asset purchases.

The ith small trader's problem solves

$$d_i^\star(S_0) = \arg\max_{d_i \in \mathbb{R}} \mathbb{E}\left[U_i^1\right] = -pe^{-\lambda\left(W_i - d_i S_0 + d_i \bar{\xi}\right)} - (1-p)e^{-\lambda\left(W_i - d_i S_0 + d_i \underline{\xi}\right)}.$$

The optimal solution is characterized by the first order condition, i.e.,

$$\frac{\partial \mathbb{E}[U_i^1]}{\partial d_i} = p\lambda e^{-\lambda\left(W_i - d_i S_0 + d_i \bar{\xi}\right)}(\bar{\xi} - S_0) - (1-p)\lambda e^{-\lambda\left(W_i - d_i S_0 + d_i \underline{\xi}\right)}(S_0 - \underline{\xi}) = 0.$$

The solution is

$$d_i^\star(S_0) = \frac{1}{\lambda(\bar{\xi} - \underline{\xi})}\left[\log\left(\frac{p}{1-p}\right) + \log\left(\frac{\bar{\xi} - S_0}{S_0 - \underline{\xi}}\right)\right]$$

When the large trader is absent, the asset market clearing condition is

$$N = z\Delta + \int_{i \in [0,1]} d_i^\star(S_0) = z\Delta + \frac{1}{\lambda(\bar{\xi} - \underline{\xi})}\left[\log\left(\frac{p}{1-p}\right) + \log\left(\frac{\bar{\xi} - S_0}{S_0 - \underline{\xi}}\right)\right].$$

The equilibrium asset price is

$$S_0(z, 0) = \underline{\xi} + (\bar{\xi} - \underline{\xi})\left[1 + e^{\lambda(\bar{\xi} - \underline{\xi})(N - \Delta z)}(1-p)/p\right]^{-1}$$

With the aid of proposition 2.4.1, the martingale measure is:

$$\mathbb{Q}(H|z) = 1 - \mathbb{Q}(L|z) = \frac{S_0(z,0) - \underline{\xi}}{\bar{\xi} - \underline{\xi}} = \frac{1}{1 + e^{\lambda(\bar{\xi}-\underline{\xi})(N-\Delta z)}(1-p)/p}.$$

In round 2, when the large trader buys $x > 0$ shares, the outstanding supply as viewed by the small traders is reduced to $N - x$, and the perturbed asset price is

$$S_0(z,x) = \underline{\xi} + (\bar{\xi} - \underline{\xi}) \left[1 + e^{\lambda(\bar{\xi}-\underline{\xi})(N-x-\Delta z)}(1-p)/p\right]^{-1} \tag{2.14}$$

Some comparative statics are provided in the next lemma.

Lemma 2.6.1 (Equilibrium Price Properties)**.**

i) $\frac{\partial S_0(z,x)}{\partial x} > 0$*;*

ii) $\frac{\partial^2 S_0(z,x)}{\partial x^2} > 0$*; and*

iii) $\frac{\partial^2 S_0(z,x)}{\partial z \partial x} < 0$.

Proof. See Appendix B.1.4. □

First, property (i), the monotonicity of $S_0(z,x)$ over x, extends a key result in the baseline model (Theorem 2.5.1) to the exponential utility function case. A detailed explanation is given in Remark 2.5.1.

Second, property (ii) establishes the concavity of the perturbed price function. When the price approaches $\bar{\xi}$ ($\underline{\xi}$, respectively), then all small traders will become net suppliers (buyers) of the asset, which limits the price impact when the large trader initiates a big buy (sell) order.

Third, property (iii) says that the marginal price impact measured by $\frac{\partial S_0(z,x)}{\partial x}$ is decreasing in the large trader's derivative position z. The larger the derivative position z, the smaller the marginal price impact.

2.7 A Continuous Time Model

This section extends the static model to continuous time. Since many of the concepts are similar, the descriptions will be brief, unless otherwise necessary.

2.7.1 The Setup

For the continuous time model, we follow the notation in Jarrow (2017). Given is a continuous time economy with continuous trading over a finite time horizon $t \in [0, T]$ and a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in[0,T]}, \mathbb{P})$ where the filtration is generated by a Brownian motion B_t with $B_0 = 0$. Traded in this economy are three assets: a riskless mma, a asset S_t, and a derivative whose payoff depends on a asset index (to be specified shortly). The mma has unit value for all t. The asset pays a liquidating dividend ξ at time T without any intermediate cash flows over $[0, T)$. We let

$$S_T = \xi = e^{\mu T - \frac{1}{2}\sigma^2 T + \sigma B_T}$$

where μ and $\sigma > 0$ are constants. The derivative, with more structures imposed on the market, is redundant in this economy (to be shown below).

The Large Trader and the Intermediary

As before, the large trader enters the market at time 0 with a purchase of z shares of a derivative that matures at time T^*. Let x_t be the large trader's cumulative share holdings in the asset, with $x_0 = 0$. We assume that the large trader trades only a finite number of times within the trading interval. She does this to hide her trades, and she only buys the asset. Then, x_t is nondecreasing and of finite variation. We assume that x_t is right continuous. Denote the set of trading times by $0 < \tau_1 < \cdots < \tau_K < T$ where K is an integer and the large trader cannot trade at time T. Every time the large trader trades, the equilibrium is perturbed, because it effectively reduces the available supply of shares to $N - x_{\tau_j}$ at time τ_j. For simplicity, we assume that the large trader only trades once at time $T^\star$, just before the derivative matures. The purpose of this trade is to manipulate the asset index and profit from her position in the derivative. Since there is only one trade, $x_{T^\star -} = 0$ and $\Delta x_{T^\star} = x_{T^\star}$.

The intermediary has a belief $\mathbb{P}$. He sells z shares of the derivative to the large trader at time 0, and hedges these shares by buying $z\Delta_0$ shares of the asset, where Δ_0 is the appropriate hedge ratio. He is unaware of the large trader's asset purchases at time $T^\star$.

The Small Traders

There is a mass $i \in [0, \mathcal{I}]$ of identical small traders who are risk averse with homogeneous beliefs $\mathbb{P}$ and state dependent utility functions $U(\omega, y)$ in terminal wealth $y \in \mathbb{R}_+$. We assume that the utility function is appropriately jointly measurable, and for almost every $\omega \in \Omega$, it is: (i) continuous and differentiable in y, (ii) strictly increasing and strictly concave in y, (iii) satisfies the Inada condition, $\lim_{y \to 0} U'(\omega, y) = \infty$ and $\lim_{y \to \infty} U'(\omega, y) = 0$, and (iv) has an asymptotic elasticity, $AE(U, \omega) = \limsup_{y \to \infty} \frac{yU'(\omega,y)}{U(\omega,y)} < 1$.

The small traders only trade in the asset market and act as price takers. They have initial endowments e_M, e_S in the mma and asset, respectively. The standard structure is imposed on a small trader's trading strategies (see Jarrow (2017)). That is, given the evolution of the asset price S_t,[12] a trading strategy is a stochastic process (m_t, d_t) such that the asset position d_t is a predictable process integrable with respect to S and the mma holding m_t is an optional process. A trading strategy's wealth process, starting from an initial wealth

$$W_0 = e_M + e_S S_0$$

is

$$W_t = m_t + d_t S_t.$$

We assume all trading strategies are self-financing

$$W_t = W_0 + \int_0^t d_u dS_u$$

and such that $W_t \geq 0$ for all for all $t \in [0, T]$. The value function of the trader's optimization problem is

$$v(W_0) = \underset{(m,d) \in \mathcal{C}(W_0, S)}{sup} \mathbb{E}_{\mathbb{P}}[U(W_T)]$$

[12]Note that S_t reads as $S_t(z, x)$. However, in the continuous trading economy, z is exogenously given and thus we suppress it in the notation. Occasionaly, we also suppress x when it causes no confusion.

where

$$\mathcal{C}(W_0,S)=\{(m,d):W_0=e_M+e_SS_0,\ W_t=m_t+d_tS_t=W_0+\int_0^t d_u dS_u\geq 0,\ \forall\, t\in[0,T]\}.$$

Finally, to guarantee that a solution exists, we assume that there exists a $y>0$ such that $v(y)<\infty$.

The Asset Index

The asset index, I_t, is defined by

$$I_t := S_{t-}(x_{t-}) + w\big[S_t(x_t) - S_{t-}(x_{t-})\big] \tag{2.15}$$

for all $t\in[0,T]$ where $I_0=1$ and $w\in[0,1]$ is a weighting function. Without loss of generality, we drop the division by $\frac{1}{2}$ as employed in the static model. This index has the interpretation that is the average transaction price over any time interval $[t,t+dt]$. It is only when the large trader transacts that there is more than one transaction price over this instantaneous interval corresponding to the quick succession of trades generated by the large trader's asset purchases. Under the EWAP weighting, $w=0$; while for the VWAP weighting, $w=\frac{x_t-x_{t-}}{N_t}$.

The Derivative Market

Recall that $x_0=0$ is the time 0 asset holding position by the large trader. Given $S_t(x_0)$ is the equilibrium price (to be defined and determined below), we know there exists a martingale measure $\mathbb{Q}_0$ such that

$$S_t(x_0)=\mathbb{E}_{\mathbb{Q}_0}\left[\xi\middle|\mathcal{F}_t\right].$$

Here, $\mathbb{Q}_0$ is the martingale measure for the time $t = 0$ equilibrium (see the next section).

Using the martingale representation theorem, under the probability measure $\mathbb{Q}_0$, we can write $S_t(x_0)$ as a stochastic integral with respect to a Brownian motion, called $\widetilde{B}_t$, i.e.,

$$dS_t(x_0) = \sigma S_t(x_0) d\widetilde{B}_t. \tag{2.16}$$

It is well known that under this evolution the market is complete because this is the Black-Scholes Merton model. Hence, by the second fundamental theorem of asset pricing, $\mathbb{Q}_0$ is unique.

Consider a derivative with maturity T^* whose payoff is $C_{T^\star}\big(I_{T^\star}\big)$.

Assuming the intermediary is unaware of the hidden trades of the large trader, she believes the payoff to the derivative is $C_{T^\star}\big(S_{T^\star}(x_0)\big)$ at time T^* where $I_{T^\star} = S_{T^\star}(x_0)$ and the evolution for $S_t(x_0)$ is given by expression (2.16). To hedge, the intermediary holds

$$\Delta_t = \frac{\partial C_t}{\partial S_t(x_0)}$$

shares of the asset, where $C_t = \mathbb{E}_{\mathbb{Q}_0}\Big[C_{T^\star}\big(S_{T^\star}(x_0)\big)\Big|\mathcal{F}_t\Big]$. Note that the hedge ratio Δ_t changes over time, and the intermediary needs to rebalance continuously. The time $t = 0$ price for the derivative is

$$C_0 = \mathbb{E}_{\mathbb{Q}_0}\left[C_{T^\star}\big(S_{T^\star}(x_0)\big)\right].$$

We emphasize that the intermediary, because his trades are unrestricted, determines the price of the derivative.

2.7.2 Equilibrium Price Process

To simplify the notation, we assume that the supply of shares is N at time $t = 0$ before the first trade by the intermediary and increases/decreases exactly by $z(\Delta_t - \Delta_0)$ units at time t (i.e., $N_t = N + z(\Delta_t - \Delta_0)$) when the intermediary rebalances their hedge so that the supply of shares is fixed across time at N.

Definition 2.7.1 (Radner Equilibrium)**.**

An equilibrium is a price process $S_t(x_0) = S_t(0)$ *and demands* $(m_t^i, d_t^i)_{i \in [0,\mathcal{I}]}$ *such that*

i) (Individual Optimization)

(m_t^i, d_t^i) *are optimal for all* $i \in [0, \mathcal{I}]$ *and*

ii) (Market clearing)

$\int_{i \in [0,\mathcal{I}]} d_t^i + z\Delta_t = N_t$ *for all* $t \in [0, T]$.

This is the standard Radner equilibrium used in continuous time and continuous trading models. The notation $x_0 = 0$ in the price process indicates that this is the equilibrium price process at time $t = 0$ with the large trader not in the asset market. Second, we omitted the market clearing condition for the derivatives, which trivially holds.

When the larger trader trades at time T^*, the equilibrium is perturbed, because it effectively reduces the available supply of shares to $N - x_{T^*}$ at time T^*. The trade is $x_{T^*} - x_{T^*-} = x_{T^*} \geq 0$. The equilibrium price process adjusts to this change and the price process is modified for the remaining time interval $[T^*, T]$. Note that we have two asset prices at time $T^\star$: $S_{T^\star-}(x_{T^\star-})$ and $S_T(x_T)$. The left limit $S_{T^\star-}(x_{T^\star-})$ is the asset price just an instant before the large trader's trade, and $S_{T^\star}(x_{T^\star})$ is the asset price after the large trader's trade.

Under the above structure, the following lemma follows (see Jarrow (2017), Corollary 10).

Lemma 2.7.1 (Initial Equilibrium)**.**

Let large trader's holdings in the derivative be $z > 0$ at time 0.

The equilibrium price process exists and is given by

$$S_t(x_0) = \mathbb{E}_{\mathbb{Q}_0}[\xi|\mathcal{F}_t]$$

where

$$\frac{d\mathbb{Q}_0}{d\mathbb{P}} = \frac{U'(\xi(N - z\Delta_0))}{\mathbb{E}_{\mathbb{P}}\left[U'\left(\xi(N - z\Delta_0)\right)\right]}$$

defines an equivalent martingale measure $\mathbb{Q}_0$.

Alternatively written,

$$S_t(x_0) = \mathbb{E}_{\mathbb{P}}\left[\frac{U'(\xi(N - z\Delta_0))\xi}{\mathbb{E}_{\mathbb{P}}[U'(\xi(N - z\Delta_0))|\mathcal{F}_t]}\middle|\mathcal{F}_t\right]. \tag{2.17}$$

When the large trader perturbs the equilibrium at time T^* by purchasing $x_{T^\star}$ shares, the equilibrium price process changes to

$$S_t(x_{T^\star}) = \mathbb{E}_{\mathbb{P}}\left[\frac{U'(\xi(N - z\Delta_0 - x_{T^\star})\xi}{\mathbb{E}_{\mathbb{P}}[U'(\xi(N - z\Delta_0 - x_{T^\star}))\,|\mathcal{F}_t]}\middle|\mathcal{F}_t\right] \tag{2.18}$$

for $t \geq T^*$.

2.7.3 Index Design and Manipulation

Given the equilibrium price process, we can now investigate index design and market manipulation.

The large trader's manipulation problem at time $T^{\star}-$, the instant before time $T^{\star}$, is to maximize

$$\begin{aligned} U_L(x_{T^*}) :=& x_{T^\star}\Big(\mathbb{E}_{\mathbb{Q}_0}\left[\xi - S_{T^\star}(x_{T^\star})\big|\mathcal{F}_{T^\star-}\right]\Big) + z\left(\mathbb{E}_{\mathbb{Q}_0}\left[C_{T^\star}(I_{T^\star})\big|\mathcal{F}_{T^\star-}\right] - C_0\right) \\ &= x_{T^\star}\big(S_{T^\star-}(x_{T^\star-}) - S_{T^\star}(x_{T^\star})\big) + z\big(C_{T^\star}(I_{T^\star}) - C_0\big) \\ &= x_{T^\star}\big(S_{T^\star-}(x_0) - S_{T^\star}(x_{T^\star})\big) + z\big(C_{T^\star}(I_{T^\star}) - C_0\big). \end{aligned} \tag{2.19}$$

Here, we use the fact that $x_{T^\star-} = x_0$. The intial and perturbed asset price processes are given by equations (2.17) and (2.18).

We next explore when the large trader can manipulate the asset price and index. First, we give an example where manipulation cannot occur.

Example 2.7.1 (No Manipulation)**.**

Consider an economy in which the small traders have a state independent logarithmic utility function, i.e., $U_i(y) = \ln y$ for all $i \in [0,\mathcal{I}]$. From lemma 2.7.1, given the large trader's trade size $x > 0$,

$$S_t(x) = \mathbb{E}_{\mathbb{P}}\left[\frac{U'(\xi(N - z\Delta_0 - x))\xi}{\mathbb{E}_{\mathbb{P}}[U'(\xi(N - z\Delta_0 - x))\,|\mathcal{F}_t]}\Big|\mathcal{F}_t\right] = \frac{1}{\mathbb{E}_{\mathbb{P}}[\xi^{-1}|\mathcal{F}_t]}.$$

Since the price process is independent of x, this economy is immune to price manipulation.

Lemma 2.7.2 illustrates gives a sufficient condition when the large trader can manipulate the asset price.

Lemma 2.7.2 (A Sufficient Condition for Manipulation)**.**

Assume that

$\mathbb{E}_{\mathbb{P}}\left[\left|U''(\xi(N-z\Delta_0-x))\xi^2\middle|\mathcal{F}_t\right|\right]<\infty$*, and*

$\mathbb{E}_{\mathbb{P}}\left[\left|U''(\xi(N-z\Delta_0-x))\xi\middle|\mathcal{F}_t\right|\right]<\infty, \forall x\in[0,N-z\Delta_0]$.

The large trader can initiate an upward price manipulation at time t if

$$\mathbb{E}_{\mathbb{P}}\left[U''(\xi(N-z\Delta_0-x))\xi^2\middle|\mathcal{F}_t\right]\mathbb{E}_{\mathbb{P}}\left[U'(\xi(N-z\Delta_0-x))\middle|\mathcal{F}_t\right] < \mathbb{E}_{\mathbb{P}}\left[U'(\xi(N-z\Delta_0-x))\xi\middle|\mathcal{F}_t\right]\mathbb{E}_{\mathbb{P}}\left[U''(\xi(N-z\Delta_0-x))\xi\middle|\mathcal{F}_t\right]. \quad (2.20)$$

Proof. See Appendix B.1.5. □

Now, we give an example in which condition (2.20) in lemma 2.7.2 is satisfied.

Example 2.7.2 (Preferences Satisfying the Sufficient Condition)**.**

Let the small traders have a state independent exponential utility function, i.e., $U(y)=-e^{-\lambda y}$.

Assume $\xi(\omega)>0,\ for\ all\ \omega\in\Omega$*, a non-constant random variable.*

Note that $|U''(\cdot)|\leq\lambda^2$ *and thus the regularity conditions reduce to* $\mathbb{E}_{\mathbb{P}}[\xi|\mathcal{F}_t]<\infty$ *and* $\mathbb{E}_{\mathbb{P}}[\xi^2|\mathcal{F}_t]<\infty$*. Moreover, equation (2.20) reduces to*

$$E\left[e^{-\lambda\xi(N-z\Delta_0-x)}\xi^2\middle|\mathcal{F}_t\right]E\left[e^{-\lambda\xi(N-z\Delta_0-x)}\middle|\mathcal{F}_t\right]>\left(E\left[e^{-\lambda\xi(N-z\Delta_0-x)}\xi\middle|\mathcal{F}_t\right]\right)^2$$

which follows from the Cauchy-Schwartz inequality and that ξ is non-constant.

Unfortunately, in light of lemma 2.7.1, exponential utility does not satisfy the Inada condition. However, we can circumvent this by constructing

$$\tilde{U}(y) = (1-\delta)(-e^{-\lambda y}) + \delta \ln y$$

for a sufficiently small $\delta > 0$*. Note that for any* $\delta > 0$*, the Inada condition is satisfied, since* $\lim_{y\to 0} \frac{1}{y} \to \infty$*.*

And, equation (2.20) reduces to

$$\begin{aligned}
&(1-\delta)^2\left\{E\left[e^{-\lambda\xi(N-z\Delta_0-x)}\xi^2\middle|\mathcal{F}_t\right]E\left[e^{-\lambda\xi(N-z\Delta_0-x)}\middle|\mathcal{F}_t\right]-\left(E\left[e^{-\lambda\xi(N-z\Delta_0-x)}\xi\middle|\mathcal{F}_t\right]\right)^2\right\}\\
&\quad+\frac{(1-\delta)\delta\lambda^2}{(N-z\Delta_0-x)}\left\{E\left[e^{-\lambda\xi(N-z\Delta_0-x)}\xi^2\middle|\mathcal{F}_t\right]E[\xi^{-1}|\mathcal{F}_t]-E\left[e^{-\lambda\xi(N-z\Delta_0-x)}\xi\middle|\mathcal{F}_t\right]\right\}\\
&\quad+\frac{(1-\delta)\delta\lambda}{(N-z\Delta_0-x)^2}\left\{E\left[e^{-\lambda\xi(N-z\Delta_0-x)}\middle|\mathcal{F}_t\right]-E[\xi^{-1}|\mathcal{F}_t]E\left[e^{-\lambda\xi(N-z\Delta_0-x)}\xi\middle|\mathcal{F}_t\right]\right\}>0
\end{aligned}$$

where the first term dominants when δ *is sufficiently small.*

Now, we can answer the question of index design and manipulation.

Proposition 2.7.1 (Index Design and Manipulation)**.**

Assume condition (2.20) holds at time $T^\star$*. Then,*

i) Under EWAP, $x^\star_{EW} = 0$*. There is no market manipulation;*

ii) Under VWAP, $x^\star_{VW} > 0$ *if* $\frac{z}{N}C'_{T^\star}\big(S_{T^\star}(x_0)\big) > 1$*. Market manipulation occurs.*

Proof. See Appendix B.1.6. □

This proposition confirms, in a continuous time model, that a volume weighted index admits manipulation, whereas an equal weighted index does not. We can also easily see that the volume weighted index destroys the complete market and the hedging effectiveness of the asset index.

Theorem 2.7.1 (Index Design and Market Completeness)**.**

(i) Under the EWAP index weighting, the market is complete.

(ii) Since $x^{\star}_{VW} > 0$, under the VWAP index weighting, the market is incomplete.

This follows because the payoff to the derivative at time T^* when $x^{\star}_{VW} > 0$ is $C_{T^\star}\Big(S_{T^\star-}(x_{T^\star-})+w\big(S_{T^\star}(x_{T^\star})-S_{T^\star-}(x_0)\big)\Big)$ and not $C_{T^\star}\big(S_{T^\star}(x_0)\big)$. The additional randomness introduced by the perturbed asset price and its effect on the index destroys the hedge.

2.8 Conclusion

This paper studies optimal index design to both facilitate hedging and alleviate illegal manipulation in a competitive equilibrium paradigm, modified to deal with manipulation. Specifically, a large trader is trading both derivatives and assets, and effectively hides his trades behind the competitive market clearing mechanism. Unlike the strategic game paradigm, a VWAP-type index both introduces basis risk and encourages market manipulation because of the additional randomness in volume weight and the greater price impact enjoyed by the large trader. In contrast, ceteris paribus, an EWAP-type index both preserves market completeness and discourages market manipulation.

Our analysis has important policy implications for index design. When the market price determination mechanism is best approximated by a competitive market paradigm, the VWAP fixings can be problematic for two reasons. On the one hand, the large traders are naturally endowed with a large price impact, which begets market manipulation. This is the economic content of theorem 2.5.1. On the other hand, the additional randomness from the volume weights can introduce basis risk and render exact hedging unachievable, because it will be impossible to synthetically construct a VWAP index using traded securities. For hedging, a transaction-based equal-weighted index is best.

CHAPTER 3

A MODEL OF INFLUENCER ECONOMY

3.1 Introduction

The past decade has witnessed the rise of the influencer economy (also known as "Wang Hong economy" and more recently dubbed by the media as the "creator economy") which prominently features social media marketing, testimonial endorsements, and product placements from people and organizations who have a purported expertise or social influence.[1] The phenomenal growth of the influencer economy is further accelerated by the recent COVID-19 pandemic (e.g., Sinha, 2021). In this new form of digital economy, influencers come in variety and from diverse background, and include content creators, celebrities and idols, and key opinion leaders (KOL) (Williams, 2016).[2] They enjoy their

[1]The influencer or creator economy generally refers to the independent businesses and side hustles launched by self-employed individuals who make money off of their knowledge, skills, or following. CBInsights (2021) provides an excellent introduction to the industry. According to a Benchmark survey (Geyser, 2021), from a mere $1.7 billion in 2016, influencer marketing has grown to a size of $9.7 billion in 2020, and is set to jump to approximately $13.8 Billion in 2021. There are approximately 50M creators today, according to SignalFire. In particular, in the Chinese market alone in 2019, there were already more than 6500 influencer-related companies and the total market value exceeded 10 billion CNY. Many influencers generate as much as seven-figure incomes. For example, Wei Ya, one famous influencer, made a fortune from her over 1 billion CNY in live streamed pre-sales on Single's day (Nov 11, the black Friday equivalent in China) alone in 2019, as reported at `https://wk.askci.com/details/a0f1a24536ab46ac9da04fd494686476/`. Famous Instagram influencers like Huda Kattan or Eleonora Pons net up to 6 figures per post. The top writers on Substack can rake in as much as $1M USD annually. Youtube paid out $30B to creators in the last 3 years (CBInsights, 2021).

[2]Content creators derive from "YouTube stars" marketed by YouTube in as early as 2011 (Lorenz, 2019). Now, it can be anyone who creates any form of content online, including TikTok videos and Clubhouse audios. For instance, on Twitch, daily users can watch live streams video games played by others via Streamlabs, and tips paid out on Twitch alone is estimated to be $141 million. Unlike live stream creators, Internet celebrities on Instagram and the like can post about or live stream special travel or dining experience, or simply routine daily lives. Many rely on physical attributes alone without actively creating content. For instance, Instagram enables

own fan base who are drawn to their talent, charisma, wisdom, appearance, etc., and profit by helping brand owners and service providers promote various products to the fans.[3] Influencers and (social) digital platforms coexist in a symbiotic relationship, with Multi-channel networks (MCNs) serving as bridge among these participants and helping linking the upstream content production and downstream e-commence. Understandably, existing studies have focused on the relationship between influencers and platforms or MCNs.

However, we have little theoretical understanding about the industrial organization of the influencer economy. How does technology affects the bargaining between sellers and influencers? How influencers shape product differentiation and pricing? How are influencers and brand owners matched and how to regulate the process? We answer these questions by developing a novel game-theoretic model in which sellers depend on influencers to acquire customers and compete in both the product market and influencers' labor market.

Specifically, we model three important groups of agents in the influencer economy, sellers (who are also producers), influencers, and consumers, allowing pair-wise group interactions through the product market, the influencers' labor market, and the social media platforms (for influencers to connect with consumers). Consumers are uniformly located on the unit circle in $\mathbb{R}^2$ with consumption utilities determined by both the true quality of the product and the

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brand owners to sell products through idols who attract consumers simply seeking to see them. Similarly, KOLs can target specific demographic in an interactive manner, making product sales more engaging by sharing their own thoughts and ideas.

[3]Influencers touch almost all aspects of life, including entertainment, fashion, food, movies, music, sports, etc., and increasingly utilize short videos (low cost and easy to spread). While there are many ways to monetize the influence, such as compensation for content creation or interaction with fans, influencers' largest income are still from commercials and e-commerce traffic direction.

style, status, identity, etc.—things that draw people towards influencers on social media like Instagram, or Da Ren on Alibaba. Sellers or brand owners (not on the unit circle) depend on influencers to sell products to consumers. All agents interact in four sequential stages. First, influencers' type and power are set. Second, sellers make production decisions. Third, sellers decide which influencer(s) to hire in the labor market. Finally, consumers choose which influencer to follow and consume the products the influencer promotes.

We begin with a monopolist seller and abstract from seller competition and seller-influencer matching to highlight the impact of general purpose technologies such as digital social platforms on influencers' labor market. Interestingly, as technologies governing marketing outreach improve, the equilibrium features non-monotonicities in influencer market concentration, payoffs, and distributional inequality. These results are driven by two factors. First, the optimal hiring for joint profit maximization is non-monotonic. When the background technology is very costly, only the best influencer can break even and get picked and a sub-population gets the product. As the cost continues to decrease, the best influencer does not dominate. More people have local influence and others get hired too. At some point, all these influencers are in competition and again only the best becomes dominant. Thus, non-monotonic concentration in influencers' market follows when the seller's incentive is sufficiently aligned with the seller-influencer(s) group's joint profit.

Second, the seller's incentive for endogenous bargaining power building distorts the hiring decision, causing the seller to hire the weak influencer even though it reduces their joint profit, whenever the seller's bargaining power is not very large. It naturally generates non-monotonicities in influencers' payoffs and distributional inequality. Initially, when the technology cost is very big, both influencers are paid very little because of the low joint profit. In contrast, when the technology cost is sufficiently small, both influencers are offered minimal wages because influencers are perfect substitutes, and thus the income gap is almost zero. Hence, we only see high payoffs for influencers and a big distributional inequality for intermediate technology cost range.

We then consider the setting in which two sellers compete in both labor market (two influencers) and the product market. Consistent with the current influencer market practice, we focus on "balanced matching" or mutual exclusivity contracts in the labor market, in which each seller can hire only one influencer. Under one-dimension heterogeneity, we fully characterize the price competition equilibrium with heterogeneity in either product quality, influencer power, or influencer style. Furthermore, to focus on pure strategy equilibrium that exist, we analyze two special cases of multiple-dimension heterogeneity. In the first case, sellers enjoy local monopoly power regardless of product quality when influencers' styles are sufficiently distinct. In the second case, it features market dominance, and to gain market dominance, the style difference between influencers needs to be suffciently small, and both the influencers' power gap and product quality gap must be sufficiently large simultaneously.

We move backward to endogenize sellers' production. Specifically, we investigate how influencer heterogeneity affects horizontal and vertical product differentiation, compared to traditional economies. We find that influencer heterogeneity and horizontal product differentiation are substitutes, mainly driven by the desire to avoid competition. When influencers' style difference is small, sellers differentiate products to reduce competition. When influencers' style difference is large, sellers hires influencers and have no incentive to differentiate products because more differentiated markets yield lower substitutability among products, therefore giving sellers less elastic demand for the products. This also implies that the well-established principle of maximum differentiation found in the literature (e.g., d'Aspremont et al., 1979; De Frutos et al., 1999) would no longer hold in an influencer economy.

Surprisingly, when it comes to vertical differentiation, small style differences complement while large differences substitute, mainly due to the incentive to grab the whole market and beat the competitor. To see it, note that when influencers' style difference increases, the return from investing in high quality also increases. When influencers' style difference is sufficiently large (small), both groups can (cannot) break even and choose high (low) quality and thus vertical differentiation is minimal. Only for intermediate style difference, can vertical differentiation be observed because the investment profit is only big enough to support one group investing to break even.

Next, we allow the influencers to endogenize their influence either in power or type. Under endogenous power acquisition, we show that socially inefficient under-investment and over-investment in influence can arise due to externality and endogenous bargaining power issue. In particular, influencers ignore the positive externality on consumer welfare in a uncongested influencer market, as well as the negative externality on other influencers in a congested influencer market. Similarly, a big/small bargaining power empowers/reduces the incentive for power acquisition. These two forces jointly determine the direction and magnitude of the sub-optimal acquisition. Furthermore, under endogenous style selection, assortative matching between sellers and influencers occur under endogenous influence, with the maximum horizontal differentiation principle recovered in the limit of costless style selection, mainly driven by the fact that the seller-influencer group's profit is supermodular in product quality and influencer power in a uncongested market.

Finally, to better understand the welfare implication of exclusivity contracts in this emerging industry and to guide regulatory policies geared towards balancing the power of influencers and sellers, we extend the analysis to "unbalanced matching". We find that regulations for balanced seller-influencer matching can encourage seller competition under single dimensional seller-influencer heterogeneity. But uni-directional exclusivity contracts are welfare-improving for sufficiently differentiated products and uncongested influencers' markets.

Literature— Previous research has focused on online platforms and has studied issues on revenue sharing rules (Bhargava, 2021; Jain and Qian, 2021), disclosure by internet influencers (Mitchell, 2021), influencer cartels (Hinnosaar and Hinnosaar, 2021), and firms' optimal affiliation with influencers (Pei and Mayzlin, 2019). Our study adds foremost to the emerging literature on digital platforms and the influencer or creator economy. We abstract away from the intermediation by platforms, but focus on seller competition and seller-influencer matching, which in turn affects product differentiation and endogenous influence acquisition.

Our study is also related to the broad literature on marketing and industrial organization, including classical articles such as Salop (1979). We add by analyzing the interaction of the two in a fast emerging influencer economy. Studies on advertising, have focused on the aggregate and cross-sectional levels of advertising and its welfare implications (Becker and Murphy, 1993; Spence and Owen, 1977; Butters, 1978; Dixit and Norman, 1978; Grossman and Shapiro, 1984; Nichols, 1985; Stegeman, 1991; Nelson, 1974; Johnson and Myatt, 2006). Most models assume no media or only focus on the informational effects or nuisance costs on viewers of advertisements (e.g., Johnson, 2013). Moreover, most studies do not endogenize locations of media stations, and the ones that do (e.g., Gal-Or and Dukes, 2003; Dukes, 2004) typically take sellers' product differentiation as exogenous. We study influencers whose matching with sellers are affected by the consumer base, and analyze endogenous product differentiation and influencers' style choices simultaneously. We do not focus on the level of advertising or its informational role, but on the complementarity between the multiple dimensions of consumer utility from following influencers and consuming products.

More recently, Amaldoss and He (2010) study how firms strategically target consumers to avoid intense price competition. Several marketing articles analyze how how firms compete in the effort of hiring influencers, including advertising intensity, competitive targeting of influencers in a network, and the network structure and its influence on prices, firm profits, and consumer surplus (Galeotti and Goyal, 2009; Katona, 2018). In particular, Fainmesser and Galeotti (2021) analyze search quality, advice transparency, and influencer strategy in the market for online influence. We differ in our focus on the interaction of seller-influencer matching and product market competition. In addition, we add to the discussion on exclusivity contracting in that while earlier studies have analyzed the link between uni-directional exclusivity contracts and bargaining (e.g., Gal-Or, 1997; Dukes and Gal-Or, 2003), we contribute by contrasting uni-directional with mutual exclusivity contracts in their impact on welfare in the new influencer economy.

The rest of the paper is organized as follows: Section 3.2 lays out the baseline model. Section 3.3 discusses influencer-induced consumption and technology. Section 3.4 solves price competition equilibrium with endogenous influencer hiring. Section 3.5 endogenizes seller's production stage. Section 3.6 investigates endogenous influencer influencer and its welfare implications. Section 3.7 extends the analysis to unbalanced matching.

3.2 Model Setup

There are $J \in \mathbb{Z}^+$ influencers, and the jth influencer's style type is denoted by $\theta_j \in \mathbf{S}^1$, where $\mathbf{S}^1 := \{s \in \mathbb{R}^2 : s_1^2 + s_2^2 = 1\}$ is the unit circle in $\mathbb{R}^2$. To capture distinctions among celebrities, macro-influencers, and micro-influencers, we allow them to differ in their own influence power $I_j \in \mathbb{R}_+$ which measures how easily consumers derive utility from affinity with them. An influencer's influence is also affected by a technology parameter, $c > 0$, that governs her outreach. This technology cost can be interpreted as how quickly the influence decreases on the circle away from the influencer. Traditional advertising channels through TVs, newspapers, etc., can be viewed as exerting the influnce with an extremely large c since the production of TV commercials, for example, would be tremendously costly due to limited airtime and labor- and capital-intensive outreach. Overall, the influencer-specific power and the common technology jointly determine the effective consumer base.

There are $K \in \mathbb{Z}^+$ risk-neutral sellers in the economy, and the kth seller sells a product of common consumption value y_k for consumers, with $k \in \{1, \cdots, K\}$. The sellers traditionally use advertisements to market their products but in an influencer economy, they work with influencers for marketing and outreach (and even direct sales). We denote the kth seller's utility or profit as U_k.

Consumer agents derive utilities from two sources. First, they enjoy having similar "style" as cerntain influencers. Style could refer to identity, fashion taste, and other things that draw people towards influencers on Instagram, Tiktok, Alibaba, etc., among the recent proliferation of social networks, digital platforms, and broadcasting channels. At the same time, they derive regular consumption value based on the quality of the goods.[4] Specifically, the ith consumer's style is given by $x_i \in \mathbb{R}^2$, which follows a uniform distribution on $\mathbf{S}^1$. Moreover, $\forall x_1, x_2 \in \mathbf{S}^1$, we define the norm $\|x_1 - x_2\|$ to be the distance along the short arc on the unit circle. The total mass of consumers is 2π.

Recall that x_i is the ith consumer's style type and y is the consumption value of the target product. Without consumption, the utility is normalized such that $u_i(x_i, y) = 0$. With a unit consumption (we assume that he faces a discrete choice regarding whether to buy a single unit of the product), consumer i gets:

$$u_i(x_i, y) = y * \left(1 - \frac{c}{I} * \|x_i - \theta\|\right) - p,$$

where p is the unit price charged for the consumption good and we assume $y \in \mathbb{R}^{++}$. Here, brand owners and product sellers depend on influencers to sell the goods, with only consumers having $\|x_i - \theta\| \leq I$ enter the demand function. In the baseline model, we assume a small fixed cost $\varepsilon > 0$ of hiring a new influencer

[4]For further institutional background, see, e.g., https://wearesocial.com/blog/2020/01/the-dawn-of-a-new-influencer-economy.

and turning on the influence power. ε helps break an indifference to avoid the seller's hiring multiple influencers yet keeping them idle, and is taken to be infinitesimal later.

Importantly, a consumer's utility depends on the proximity to the influencer they follow (i.e., the distance between x_i and θ). Using products advertised by influencers closer by yields a higher utility. Specifically, the ith consumer's utility function is given by:

$$u_i(x_i, y) = \begin{cases} y * \left(1 - \frac{c}{I}\|x_i - \theta\|\right) - p, & \text{if a unit good is consumed,} \\ 0, & \text{otherwise.} \end{cases} \tag{3.1}$$

Our setting adequately captures the many reasons typically given for engaging celebrities in advertising campaigns: grabbing attention, persuasion through expertise, and global outreach (Moeran, 2003).[5] I can represent attention grabbing, either through vacuous "human pseudo-events" in the words of American historian Daniel Boorstin or through skills or performance unrelated to the products; expertise and global, cross-cultural outreach can manifest through the combination of location θ and power I.

Timeline. In an influencer economy, influencers' type and power are first set. The sellers then decide on the products and subsequently hire influencers. Finally, the consumers choose which influencer to follow and consume the products offered. In Sections 3.3 and 3.4, we take the influencers' type and power,

[5] Advertising through conventional technology, e.g., through TV/newspaper, are often extremely costly (high c). We are cognizant that such a cost is used to indicate the existence of a price premium that can assure contractual performance in competitive equilibrium (Klein and Leffler, 1981). However, in an influencer economy with digital platforms and the proliferation of social-commercial network apps, c is relatively low and would not serve such a function for disclosing the presence of a large sunk "selling" costs.

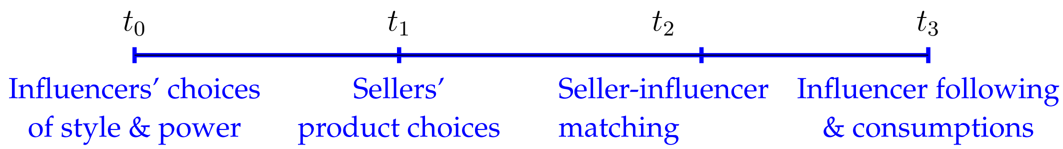

Figure 3.1: Timeline of the influencer economy

as well as the sellers' products as given, in order to focus on the sellers' hiring of influencers and influencers' impact on consumption. In Section 3.5, we allow the sellers to endogenize the products for sale. Our main findings are robust to having product decisions following seller-influencer matching. But in practice, firms decide on their business operations before exploring marketing channels, which is what our setup captures. Finally, in Section 3.6, we endogenize influencers' power (which can be interpreted as skill training over the intermediate term) and type (which can be interpreted as culture, talent, or interest cultivated over the long run, perhaps through childhood education).

Matching and bargaining protocols. We use a general (bilateral) Nash bargaining protocol for the negotiation once influencers are hired by sellers.[6] Specifically, denote by γ and $(1-\gamma)$ the bargaining power assigned to the seller and the influencer respectively. Once sellers and influencers are matched, they have exogenous options outside the match, e.g., from revisiting the influencer market, which we normalize to zero. Anticipating such bargaining processes, sellers and influencers endogenously match. Our baseline setup focuses on one-to-one match, which can be interpreted as that in practice, the seller-influencer contracts either feature mutual exclusivity clauses or they are all allowed to

[6]Gal-Or (1999) and Dukes and Gal-Or (2003) discuss the advantages of this modeling approach as illustrated in the commercial media and healthcare industries.

have multiple relationships so that the matching is balanced. This negotiation-based approach is realistic, and is also common occurrence in negotiations for advertising price in the media industry (Dukes and Gal-Or, 2003; Gal-Or, 1997).

The joint matching and bargaining problems are non-trivial. In specifying the protocols, we strive to balance tractability, transparency, convention in the literature, coherence with our non-repeated game set-up, and realism. In fact, many key results are independent on how the surplus is divided between matched sellers and influencers, as long as they care about group surplus. One can alternatively specify equilibria that gives all net surplus to the sellers or the influencers. We discuss unbalanced matching and the welfare implications of exclusivity in Section 3.7 where the sellers have exclusivity and non-compete clauses imposed on the influencers, which are also commonly observed in the influencer markets, especially in its nascent stage.

3.3 Influencer-Induced Consumption and Technology

We start with a monopolist seller offering a homogenous unit-consumption product, which is marketed by influencer(s) and sold to consumers. The abstraction from seller competition and seller-influencer matching allows us to highlight the impact of general purpose technologies on the influencer economy.

Fix an index set $\mathcal{J} \in \{1, \cdots, J\}$, we denote by $\Pi_{\mathcal{J}}$ the profit when all influencers in $\mathcal{J}$ are hired before advertising costs are deducted. For instance, $\Pi_{\{1\}}$ is the monopoly profit when only influencer $i \in \{1, 2\}$ is hired, and $\Pi_{\{1,2\}}$ the profit when both influencers 1 and 2 are hired. For ease of reference, denote by w_j the wage for influencer $j \in \mathcal{J}$ hired, and by U_k the kth seller's profit for all $k \in \{1, \cdots, K\}$.

3.3.1 Single-Influencer Benchmark and Bargaining Power

Before analyzing the general industrial organization of the influencer economy, let us consider the special case of one influencer to understand the formation of consumer base.

Lemma 3.3.1. *Given $J = K = 1$ and the values of (I, y, c), the potential consumer base is given by all $x \in \mathbf{S}^1$ such that $\|x - \theta\| \leq \min\left\{\frac{I}{c}, \pi\right\}$. There are two cases.*

(i) When $\frac{I}{c} \leq 2\pi$, the seller sets a monopoly price $p^* = \frac{y}{2}$, and the total revenue is $\Pi = \frac{yI}{2c}$. Only consumers with $\|x - \theta\| \leq \frac{I}{2c}$ are served.

(ii) When $\frac{I}{c} > 2\pi$, the seller set a price $p^* = y(1 - c\pi/I)$ and $\Pi = 2\pi p^* = 2\pi y(1 - c\pi/I)$.

Furthermore, the monopolist seller hires the influencer only when $\Pi \geq \varepsilon$. The seller's payoff and the influencer's wage are given by $U_1 = \gamma(\Pi - \varepsilon)$ and $w_1 = (1 - \gamma)(\Pi - \varepsilon)$.

Proof. See Appendix C.1.1. □

Lemma 3.3.1 illustrates how the seller targets a specific demographic by tapping into the influencer. Depending on the cost of background technology and influencers' ability, a monopolist seller may choose to target a subpopulation or

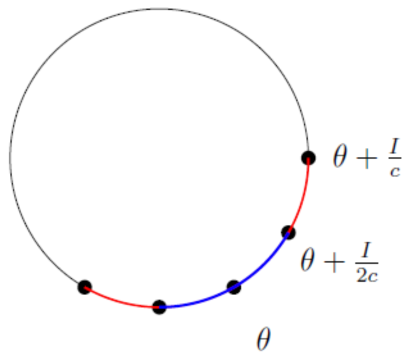

Figure 3.2: The circular market

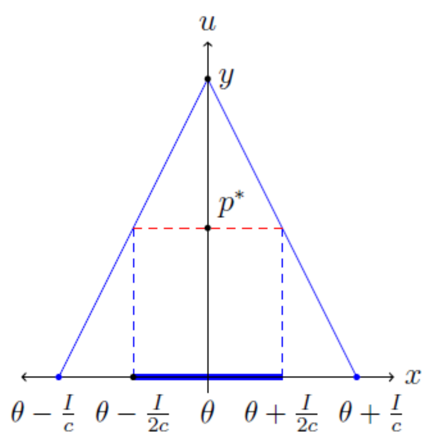

Figure 3.3: The monopolist pricing

the whole demographic of consumers. The seller only enters the market when the revenue is high enough. Figure 3.2 illustrates the consumer base in Lemma 3.3.1 when the background technology cost is big. Figure 3.3 illustrates the monopolist pricing in Lemma 3.3.1. Note that the vertical axis in Figure 3.3 corresponds to consumer utility. The monopolist price p^* is marked in red dashed line, and consumers served corresponds to the thick blue line.

3.3.2 Influencers's Competition and Technological Advances

Next, we examine the case with two representative influencers to understand how general purpose technologies such as digital social platforms and the Internet impact influencers' labor market. Without any loss of generality, we assume two heterogeneous influencers ($J = 2$) separated in type by $\|\theta_1 - \theta_2\| = \pi$, and that $I_1 \geq I_2$ (Influencer 1 is *strong* and Influencer 2 is *weak*). Furthermore, we impose Assumption 3.3.1 to ensure that at least one influencer is hired under equilibrium.

Assumption 3.3.1 (Sufficiently small technology cost)**.** $c < \frac{yI_1}{2\varepsilon}$.

Since $\Pi_{\{j\}}$ is fully characterized in Lemma 3.3.1, we present the joint profit $\Pi_{\{1,2\}}$ next.

Lemma 3.3.2 (Joint revenue function $\Pi_{\{1,2\}}$)**.** *For* $c \leq \frac{yI_1}{2\varepsilon}$*, the joint revenue function* $\Pi_{\{1,2\}}$ *is:*

$$\Pi_{\{1,2\}} = \begin{cases} \frac{y(I_1+I_2)}{2c}, & \text{if } \frac{I_1+I_2}{2\pi} < c \\ 2\pi y\left(1 - \frac{c\pi}{I_1+I_2}\right), & \text{if } c < \frac{I_1+I_2}{2\pi} \end{cases} \quad \text{whenever} \quad \frac{I_2}{I_1} \leq \frac{\pi y - \varepsilon}{\epsilon};$$

otherwise, $\Pi_{\{1,2\}} = 2\pi y\left(1 - \frac{c\pi}{I_1+I_2}\right)$.

Proof. See Appendix C.1.2. □

For ease of reference, define $\bar{c} = \frac{I_1}{I_2} * \frac{(I_1+I_2)\varepsilon}{2\pi^2 y}$.

Lemma 3.3.3 (Hiring for Joint Profit Maximization)**.** *Assume a sufficiently small fixed hiring cost such that* $\frac{\varepsilon}{y}(1+\frac{I_1}{I_2}) < \pi$*. To maximize the seller-influencers joint profit,* *(i) only the strong influencer is hired when* $c \in \left(\frac{yI_2}{2\varepsilon}, \frac{yI_1}{2\varepsilon}\right) \cup (0, \bar{c})$.

(ii) both influencers are hired when $c \in \left[\bar{c}, \frac{yI_2}{2\varepsilon}\right]$.

Proof. See Appendix C.1.3. □

Given the suboptimality of hiring the weak influencer alone, whether to hire both influencers boils down to whether $\Pi_{\{1,2\}} - 2\varepsilon \geq \Pi_{\{1\}} - \varepsilon$. Lemma 3.3.3 shows that the joint-profit-maximizing market structure is non-monotonic in the technology cost. When the cost is too high (traditional advertising is very costly), joint profit maximization requires the seller to choose a small number of high ability influencers. For example, a seller may end up picking one influencer because hiring a second-best influencer(s) cannot break even. In this case, only the best influencer gets picked and a sub-population gets the product. One caveat is that this region vanishes if $I_2 = I_1$. As the background technology cost goes down, more people have local influence, and a seller should pick more influencers. Thus, the best influencer does not dominate, and others get hired too. As the cost continues to decrease, at some point, all these influencers are in competition and again only the best becomes dominant.

Next, we investigate the seller's incentive to hire influencers, and show how the incentive misalignment ubiquitously drives over-employment, as illustrated by Proposition 3.3.4 below. Define $\underline{c} = \frac{I_2}{I_1} * \frac{(I_1+I_2)\varepsilon}{2\pi^2 y}$, and $\underline{\gamma} := \inf_{c<\underline{c}} \frac{\Pi_{\{1,2\}}(c)-2\varepsilon}{\Pi_{\{1\}}(c)-\varepsilon}$.[7]

Lemma 3.3.4 (Inefficient over-employment & Bargaining power building)**.** *Whenever* $\gamma < 1$, *there exists inefficient over-employment when* $c \in (\bar{c} - \delta, \bar{c})$ *for*

[7]Note that $\underline{\gamma} > \frac{\Pi_{\{1\}}(\underline{c})-2\varepsilon}{\Pi_{\{1\}}(\underline{c})-\varepsilon}$, and thus $\underline{\gamma} \to 1$ when $\varepsilon \to 0$.

$\delta > 0$ sufficiently small. Furthermore, whenever $\gamma \leq \underline{\gamma}$, there exists over-employment for all $c \leq \underline{c}$.

Proof. See Appendix C.1.4. □

The inefficient over-employment is best illustrated when the competition between influencers is intense such that the profit gap between hiring both influencers and only hiring a strong influencer is so small that hiring the weak influencer cannot offset the additional fixed cost. However, even if it generates a negative welfare, the monopolist seller does have an incentive to hire the weak influencer, that is, by hiring the weak influencer, it places the seller in an advantageous position when bargaining with the strong seller. The seller, after hiring the weak influencer, is willing to share the net profit increment between that generated by hiring both influencers and that by hiring the weak influencer (i.e., $(\Pi_{\{1,2\}} - 2\varepsilon) - (\Pi_{\{2\}} - \varepsilon)$). In contrast, if she hires only the strong influencer, the seller needs to give up a big portion of the profit (i.e., $(1-\gamma)(\Pi_{\{1\}} - \varepsilon)$). The smaller the seller's exogenous bargaining power, the stronger the incentive for over-employment. In particular, when the seller's bargaining power $\gamma \leq \underline{\gamma}$, both influencers are hired even when one influencer can serve the entire market.

In contrast, when the seller enjoys a sufficiently big bargaining power, the incentive for pursuing bargaining superiority through over-employment is alleviated and thus more aligned with the seller-influencers' joint profit. A direct ramification is that the observed concentration of influencers' market can be non-monotonic when γ gets close to 1.

Proposition 3.3.1 (Non-monotonicity in market concentration and influencers' payoffs). *Assume that the fixed hiring cost is sufficiently small (i.e.,* $\frac{I_1}{I_2} * \frac{\varepsilon}{\pi y}\left(1 + \frac{I_1}{I_2}\right) \leq 1$*). Then, as the technology cost* c *decreases, we have:*

(i) The concentration of influencers' market is non-monotonic when γ *is sufficiently large;*

(ii) The seller's payoff always increases;

(iii) Total payoffs for influencers first increase and then decrease for sufficiently small γ*;*

(iv) For sufficiently small γ*, the distributional inequality between influencers' payoffs is increasing in the technology cost* c *for small* c*, and decreasing for large* c*.*

Proof. See Appendix C.1.5. □

The proposition highlights that there is a strong interaction between bargaining power and the technology costs and the impact of technological advances is more nuanced. The interesting non-monotonicity results in Proposition 3.3.1 is driven by two forces. First, the optimal hiring decision to maximize the joint profit for the seller and influencer(s) is non-monotonic in the general technology parameter c, as illustrated by Lemma 3.3.3, mainly because influencers are perfect substitutes when the general technology is sufficiently cheap. Second, the seller's incentive for building bargaining power distorts the hiring decision, causing the seller to hire the weak influencer even though it reduces their joint profit.

Let us discuss the intuition behind the non-monotonicity result. First, claim

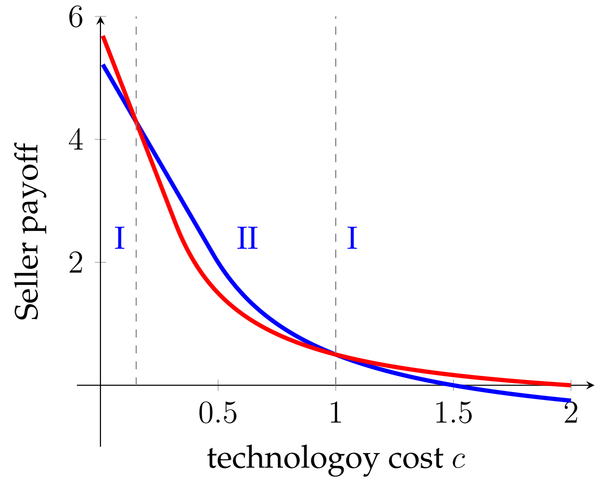

Figure 3.4: Non-monotonic market structure

(i) says that when the seller enjoys a big bargaining power, his incentive is almost perfectly aligned with the group's joint profit and thus a non-monotonic market structure is observed. Figure 3.4 illustrates this when $y = 1, \varepsilon = 0.5, I_1 = 2, I_2 = 1$, and $\gamma = 1$. The blue line corresponds to the seller's profit when both influencers are hired, while the red one is the seller's profit when only one influencer (i.e., the strong influencer) is hired. Hence, in Region I, when the general technology is very cheap or very expensive (i.e., $c > 1$ or $c < 0.15$), only one is hired; and in Region II (i.e., $c \in (0.15, 1)$, both influencers are hired.

Second, claim (ii) holds because the seller's payoff is just the upper envelope of two payoff functions U_1 and $\hat{U}_1$, which corresponds to the two cases when the seller hires both influencers and when she only hires the strong influencer, and note that they are both continuous and strictly decreasing in the general technology parameter c. In Figure 3.4, it corresponds to the upper envelope of the blue curve and the red curve, and the seller's payoff obviously increases when the general technology is becoming cheaper.

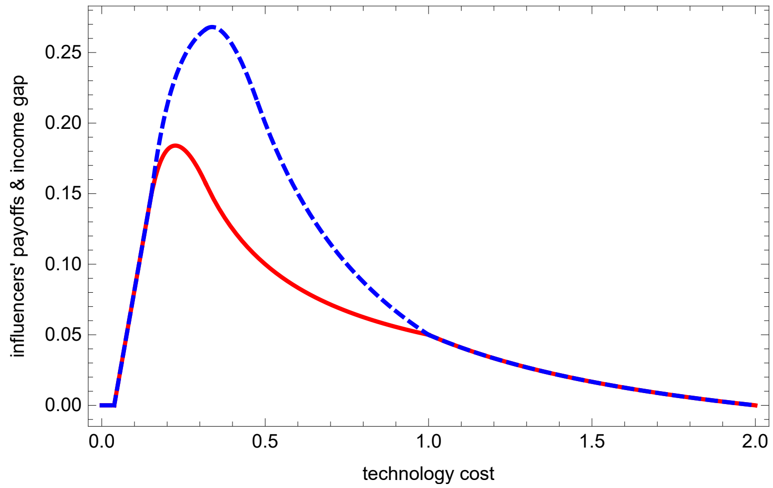

Figure 3.5: Non-monotonicity in influencers' payoffs and distributional inequality

Third, claim (iii) says that when the seller's bargaining power is not large enough, she has a strong incentive for bargaining power building by hiring both influencers. However, the over-employment can actually drive down influencers' wages, and in the extreme case when turning on influence is very cheap, both influencers are strong and are perfect substitute. Thus, influencers are paid very little even when the general technology is sufficiently cheap because influencers are then perfect substitutes for each other.

Figure 3.5 illustrates the dynamics of total payoffs for both influencers and distributional inequality when the background technology cost decreases. The parameters are specified as follows: $y = 1$, $I_1 = 2$, $I_2 = 1$, $\varepsilon = \frac{1}{2}$ and $\gamma = 0.8$. The blue line corresponds to the dynamics of total payoffs for both influencers. Initially, when the background technology is very costly, only influencer 1 is hired and the total revenue is very low, which bounds influencer 1's compensation. When the technology cost is reduced, more influencers are hired, and the total revenue increases. If the bargaining power is exogenously fixed, this transforms into a big increase in influencers' payoffs. Finally, when the background technology is very cheap, both the strong and the weak influencers can produce a big revenue if anyone is hired. This also implies that influencers are perfect substitutes, and by hiring both influencers, the seller can use fierce competition between influencers for bargaining power building and thus offer a minimal wage to influencers. Hence, total payoffs for both influencers first increase and then decrease when the general technology cost decreases.

The red line corresponds to the distributional inequality between influencers' payoffs, which is defined as the wage gap $w_1 - w_2$ between two influencers. Initially, when the technology cost is big, both influencers are paid very little because of the low joint profit. In contrast, when the technology cost is sufficiently small, both influencers are offered minimal wages, and thus the income gap is zero. Hence, we only see a big distributional inequality for intermediate technology cost range. Note that the non-monotonicity in concentration is related to the long-term sustainability and the unequal distribution emphasized in CBInsights (2021).

3.3.3 The Seller's Preferred Influencer Difference

This subsection establishes an interesting result on the seller's preference over the angle between the two influencers' style location as opposed to taking that exogenously given in section **??**. For example, the seller might prefer working with influencers who overlap somewhat even though this results in an overall less total market coverage. This would basically be in line with some of the intuitions from the paper such as Lemma 3.3.4.

For ease of reference, denote by $\beta := \|\theta_1 - \theta_2\|$ the style difference between the two influencers, where θ_j is the style location of influencer $j \in \{1, 2\}$. To simplify the reasoning, we remove the fixed hiring cost (i.e., $\varepsilon = 0$), which only plays a crucial role in the discussion of technological advances. This restriction implies that it is always optimal to hire both influencers. Given the bilateral Nash bargaining protocol, the payoffs for the seller and the two influencers are

$$w_1 = (1-\gamma)(\Pi_{\{1,2\}} - \Pi_{\{1\}}), \quad w_2 = (1-\gamma)(\Pi_{\{1,2\}} - \Pi_{\{2\}}),$$

and

$$U_1 = \Pi_{\{1,2\}} - w_1 - w_2 = (2\gamma - 1)\Pi_{\{1,2\}} - (1-\gamma)(\Pi_{\{1\}} + \Pi_{\{2\}}) \tag{3.2}$$

Note that only $\Pi_{\{1,2\}}$ depends on β, the style difference angle between the two influencers. Denote by β^S the style difference angle preferred by the seller.

Corollary 3.3.1 (Seller-preferred Influencers' Style Difference)**.** *Assume that* $\frac{I_1}{c} = \frac{I_2}{c} \in (0, \pi)$.[8] *Then,*

$$\beta^S = \begin{cases} \in [I, \pi], & \text{if} \quad \gamma > \frac{1}{2} \\ \in [0, \pi], & \text{if} \quad \gamma = \frac{1}{2} \\ 0, & \text{if} \quad \gamma < \frac{1}{2} \end{cases}$$

Proof. See Appendix C.1.6. □

Corollary 3.3.1 says that the seller prefers a zero style difference angle (i.e., $\theta_1 = \theta_2$) when her bargaining power parameter γ is small, and prefers a large

[8]This condition is imposed to recycle Lemma 3.7.1, and can be relaxed.

style difference angle when γ is big, and is indifferent for any style difference angle for a certain intermediate bargaining parameter.

The intuition behind Corollary 3.3.1 is as follows. Note that a larger style difference angle implies a bigger market coverage and a smaller over-lapping in consumer base for different influencers. When $\gamma > 0$ is small, a larger β can generate a bigger revenue but only a very limited fraction of total revenue, determined by the bargaining power, flows to the seller. Furthermore, given a small γ, a smaller β decreases the market coverage and generates less revenue, but it also implies intense competition and perfect substitution between the two influencers and leads to a better bargaining position for the seller under the bilateral Nash bargaining protocol. In the limit where $\beta = 0$, the seller gets the total revenue of working with a sole influencer in the product market. In other words, a small style difference angle betters the cash show to the seller although it decreases the market coverage. In contrast, for a big $\gamma > 0$, there is no tension between bargaining and revenue generating, because the effect on revenue from the market coverage change always dominates that from influencers' wage changes. Hence, the seller's payoff increases with the total revenue from working with both influencers, and thus a large style difference angle is more desirable.

3.3.4 General Characterizations with Multiple Influencers

Next, consider a setting with many influencers with the background technology fixed, which we normalize to one. Then, we study how influencer competition affects the monopolist seller's payoff, influencers' wages, and total welfare.

Residual Multilateral Bargaining Protocol. We propose a residual bargaining protocol to handle the multilateral bargaining problem. Specifically, given the bargaining parameter $\gamma \in [0,1]$, when a group of influencers are hired $\mathcal{J} \in \{1, \cdots, J\}$, the seller only bargains with influencer $j \in \mathcal{J}$ for the "residual profit gap" $\Pi_{\mathcal{J}} - \Pi_{\mathcal{J}/\{j\}}$, which leads to

$$w_j = (1-\gamma)(\Pi_{\mathcal{J}} - \Pi_{\mathcal{J}/\{j\}} - \varepsilon)_+, \quad \text{and} \quad U_1 = \Pi_{\mathcal{J}} - \sum_{j \in \mathcal{J}} w_j. \tag{3.3}$$

where $\Pi_{\mathcal{J}}$ is the monopolist profit when the seller hires the influencer group $\mathcal{J}$, excluding all fixed hiring cost.[9] Here, a_+ is defined as $a_+ := \max\{a, 0\}$. Note that, the term

$$\Pi_{\mathcal{J}} - \Pi_{\mathcal{J}/\{j\}} = \big(\Pi_{\mathcal{J}} - |\mathcal{J}|\varepsilon\big) - \big(\Pi_{\mathcal{J}/j} - |\mathcal{J}/j|\varepsilon\big)$$

measures the incremental change in residual profit gap from hiring influencer j.

The residual bargaining protocol has several desirable properties: (i) the allocation is unique and feasible because $\sum_{j\in\mathcal{J}} w_j + U_1 \leq \Pi_{\mathcal{J}}$; (ii) it is efficient because $\sum_{j\in\mathcal{J}} w_j + U_1 = \Pi_{\mathcal{J}}$; (iii) it cannot be blocked by a coalition of $S \subseteq \mathcal{J}$ for any $\gamma \in [0,1]$ because $\sum_{j\in S} w_j + U_1 \geq \Pi_S$ and $\sum_{j \in S} w_j \geq 0$; and (iv) it is consistent with the bilateral Nash bargaining in the baseline model.[10] In essence, residual bargaining protocol is an equilibrium refinement. Note that property (iii) is applicable to all reasonable refinements because otherwise the allocation is blocked by excluding influencer j from the group $\mathcal{J}$, which further implies that $w_j \leq \Pi_{\mathcal{J}} - \Pi_{\mathcal{J}/\{j\}}$. In this sense, the residual bargaining protocol is the natural candidate satisfying property (iii) and property (iv).

[9]The seller only shares the residual profit because of a credible threat that if influencer j rejects the offer, the seller switch to hiring the remaining group $\mathcal{J}/\{j\}$ and divide the profit $\Pi_{\mathcal{J}/\{j\}}$.

[10]It can also be generated from coalitional Nash bargaining (Compte and Jehiel, 2010)

To rule out the trivial case that no influencer is hired, we assume that $\frac{yI}{2} \geq \varepsilon$. For simplicity, we also assume that all influencers have the same power.

The limit case with $\varepsilon \to 0$. When the fixed hiring cost is infinitesimal, we show that all influencers are hired and each influencer receives a minimal wage when the joint profit function $\Pi_{\mathcal{J}}$ satisfies a specific property (S) as below.

Definition 3.3.1 (Property S)**.** *The monopolist profit $\Pi_{\mathcal{J}}$ is a submodular set function if for any $S \subseteq T \subset \{1, \cdots, J\}$ and for any $j \in S$, $\Pi_S - \Pi_{S/\{j\}} \geq \Pi_T - \Pi_{T/\{j\}}$.*

Property (S) requires that the profit loss of excluding influencer $j \in S$ is smaller when the group of influencers hired is larger.

Lemma 3.3.5 (Multiple Influencers)**.** *Under residual bargaining protocol, when property (S) holds, the seller's payoff is maximized when all influencers are hired, and the seller's payoff and influencers' wages are given by Eq. (3.3) with $\mathcal{J} = \{1, \cdots, J\}$. Additionally, each influencer's wage is minimized when all influencers are hired.*

Proof. See Appendix C.1.7. □

Furthermore, we can show that Lemma 3.3.5 always hold, regardless of Property (S), as long as there are at least two influencers for each influencer type.

Lemma 3.3.6 (Duplicate Influencers)**.** *Fix $\varepsilon = 0$.*

i) When there exists duplicate influencers for each influencer type, the seller gets the maximum revenue and all influencers, hired or not, get zero utility.

ii) In a large market with many influencers (i.e., $J \to \infty$), when influencers are randomly drawn, the seller gets all revenue and influencers get zero utility with a high probability.

Proof. See Appendix C.1.8. □

The intuition is simple. When there exists no hiring cost, the seller can afford hiring a large number of influencers so that each influencer is perfectly substitutable and thus she gets all the revenue. Now, we turn to the case with non-negligible hiring cost $\varepsilon > 0$.

The case with $\varepsilon > 0$. A non-negligible hiring cost implies that only a finite number of influencers, rather than all influencers, are hired. Note that the number of influencers hired is capped at $\left[\frac{2\pi y}{\varepsilon}\right]$.[11] Define $\bar{\Pi}_J = \max_{|\mathcal{J}|=J} \Pi_{\mathcal{J}}$. We can show that $\bar{\Pi}_J$ is maximized when all $J \geq 1$ influencers are equally distanced, that is, the distance between any two neighboring influencers equals $2\pi/J$.

Lemma 3.3.7 (Equally distanced Influencers)**.** *If the seller is restricted to hiring J influencers and can freely choose style locations for all influencers, then $\bar{\Pi}_J$ is achieved when all neighboring influencers hired are equally distanced (i.e., $\|\theta_j, \theta_{j+1}\| = \frac{2\pi}{J}$ for all $j \in \{1, \cdots, J\}$).*

Proof. See Appendix C.1.9. □

[11]The floor function $[x]$ return the greatest integer greater than or equal to $x \in \mathbb{R}$.

With the aid of Lemma 3.3.7,

$$\bar{\Pi}_J = \begin{cases} \frac{yIJ}{2}, & \text{if} \quad J \leq [\frac{2\pi}{I}] \\ 2\pi y \left(1 - \frac{\pi}{IJ}\right), & \text{if} \quad J > [\frac{2\pi}{I}] \end{cases} \tag{3.4}$$

Eq. (3.4) follows from the fact that all influencers are equally distanced and that the cutoff consumer indifferent between two neighboring influencers always receives a zero utility.[12] Denote $\bar{J} = \arg\max_{J \geq 0} \left(\bar{\Pi}_J - J\varepsilon\right)$. The interpretation of $\bar{n}$ is as follows. When the seller can freely select style locations for all influencers, $\bar{n}$ maximizes the joint profit net of the fixed hiring costs. By Eq. (3.4), we can explicitly solve the formula for $\bar{n}$ as below.

$$\bar{J} = \pi\sqrt{\frac{2y}{I\varepsilon}}$$

When $\bar{n}$ is not an integer, it should be understood such that

$$\bar{J} \in \left\{ \left[\pi\sqrt{2y/I\varepsilon}\right], \left[\pi\sqrt{2y/I\varepsilon}\right] + 1 \right\}.$$

Consider the asymptotics when all influencers are equally distanced and the number of influencer increases.

Lemma 3.3.8 (Asymptotics with many influencers). *Fix $\varepsilon > 0$. For a sufficiently large $\gamma > 0$, when the number of influencers $J = \bar{J} * m$ where $m \in \mathbb{N}$ or $J \to \infty$, the optimal number of influencers hired is given by $J^* = \bar{J}$. Given the residual bargaining protocol, the seller and influencers' payoffs are given by*

$$w_1^{\bar{J}} = (1-\gamma)(\bar{\Pi}_{\bar{J}} - \Pi_{\{1,\cdots,\bar{J}\}/\{1\}} - \varepsilon), \quad w_j^{\bar{J}} = w_1^{\bar{J}}, \quad \textit{and} \quad U_1 = \bar{\Pi}_{\bar{J}} - \bar{J}\varepsilon - \bar{J} * w_1^{\bar{J}}.$$

Proof. See Appendix C.1.10. □

[12]See Lemma 3.3.7 and its proof.

By comparing Lemma 3.3.6 and Lemma 3.3.8, we get several insights. First, when the hiring cost is non-negligible, the number of influencers hired is finite. Second, even with endogenous bargaining power building, the fixed cost guarantees all influencers can receive a positive wage. Third, when the seller enjoys a large bargaining power, the hiring plan always maximizes the joint profit for the seller and influencers in a large market.

3.4 Price Competition and Influencer Hiring

Now, we consider seller competition. In our influencer economy, sellers compete in both influencers' labor market and the product market. For tractability and transparency of the main mechanism, we focus on $J = K = 2$ for the remainder of the paper. We normalize the technology cost parameter $c = 1$ and take $\varepsilon = 0$ for simplicity.[13] We consider "balanced matching" in which the two sellers each can only hire one influencer. A seller-influencer group is then characterized by the 3-tuple (y_m, θ_m, I_m) for $m = 1, 2$. We discuss "unbalanced

[13]This assumption invites further explanation. The fixed hiring cost serves as an entry cost, and may lead to an endogenous entry problem. Recall that in a standard Bertrand price competition with endogenous entry, the seller offering a lower price always gets the whole market and the resulting payoff does not depend on the losing side seller's price. In contrast, unlike the previous Bertrand price competition, *sellers' payoffs depend on the proposed prices by all sellers* in our model setup. Depending on the equilibrium competition outcome, there are three cases: i) both sellers hire influencers with probability one, which occurs when equilibrium profits are sufficiently large; ii) one strong seller hires an influencer with probability one, the other seller hires an influencer with a probability less than one, which occurs when one seller-influencer group is much stronger; and iii) both sellers hire influencers with probability less than one, which occurs when competition between two groups are very intense. To circumvent the technical complexity, we either assume the equilibrium profits are sufficiently large, which reduces to heterogeneity among different seller-influencer groups, to cover the hiring cost, or simply discard the fixed hiring cost. We take the second approach because the fixed hiring cost is only crucial to the non-monotonicities result in Section **??** and do not generate extra insights in remaining sections.

matching" where one seller can hire multiple influencers in Section 3.7.

3.4.1 Seller-Influencer Group Heterogeneity

We examine how product quality, influencer style, and influence power each affect the product competition in the influencer economy. For simplicity, we first consider single-dimensional heterogeneity in endogenously formed seller-influencer groups. Note that when either the sellers or the influencers are homogeneous, the endogenous matching becomes trivial, allowing us to isolate the impact of the heterogeneity on influencer consumer-base acquisition and product market competition.

Heterogeneous product quality. To understand how product quality affects the competition, we set $I_1 = I_2 = I$ and $\theta_1 = \theta_2 = \theta$, but w.l.o.g., assume that $y_1 \geq y_2$. Proposition 3.4.1 fully characterizes the equilibrium. Obviously, when $y_1 = y_2$, the competition is most intense, and Bertrand competition leads to a zero profit equilibrium and both influencer-seller groups set their own price to zero. However, as long as $y_1 > y_2$, there exists an equilibrium in which both groups obtain a positive profit. The intuition is clear. When $y_1 > y_2$, even Group 2 sets $p_2^* = 0$, the product by Group 1 is still more attractive for consumers with his style type close to the influencer, i.e., $\|x - \theta\| \to 0$. Thus, Group 1 has an incentive to set a positive price. This, in turn, implies that Group 2 can get a positive profit if Group 1 decides not to set $p_1^* = 0$ by attracting consumers not targeted by Group 1.

Denote by $k(j)$ the matched seller's identity for influencer $j \in \{1, 2\}$ under equilibrium.

Proposition 3.4.1. *Assume that $I_1 = I_2 = I$, $\theta_1 = \theta_2 = \theta$ and $y_1 \geq y_2$. There exists an equilibrium in which $k(j) = j$ for $j = 1, 2$. After matching, the two seller-influencer groups choose their prices such that $(p_1^C, p_2^C) = \left(\frac{2y_1(y_1-y_2)}{4y_1-y_2}, \frac{y_2(y_1-y_2)}{4y_1-y_2}\right)$. Group 1 targets consumers with type $\|x - \theta\| \leq \frac{2Iy_1}{4y_1-y_2}$ and group 2 targets type $\frac{2Iy_1}{4y_1-y_2} < \|x - \theta\| \leq \frac{3Iy_1}{4y_1-y_2}$. Correspondingly, their profits are given by*

$$\Pi_1^C = \frac{8Iy_1^2(y_1 - y_2)}{(4y_1 - y_2)^2}, \quad \textit{and} \quad \Pi_2^C = \frac{2Iy_1y_2(y_1 - y_2)}{(4y_1 - y_2)^2}. \tag{3.5}$$

and payoffs for sellers and influencers are given by

$$U_1 = \gamma\Pi_1^C,\ U_2 = \gamma\Pi_2^C,\ w_1 = (1-\gamma)\Pi_1^C,\ \textit{and}\ w_2 = (1-\gamma)\Pi_2^c$$

Proof. See Appendix C.1.11. □

Figure 3.6 illustrates the equilibrium described in Proposition 3.4.1. Specifically, the seller with a high-quality product targets those high value consumer demographic sufficiently close to the influencer's style, as illustrated with blue lines. In contrast, the group with a low-quality product targets those consumers relatively far from the influencer's style, and we illustrate it with red lines.

Furthermore, from Proposition 3.4.1, when $\frac{y_1}{y_2} \to 1$, it converges to the Bertrand competition, which features $(p_1^C, p_2^C) = (0, 0)$. In contrast, when $y_1 \gg y_2$, it converges to an equilibrium with $(p_1^C, p_2^C) = (y_1/2, y_2/4)$, which means that the product with a high quality is priced at its monopoly price, while

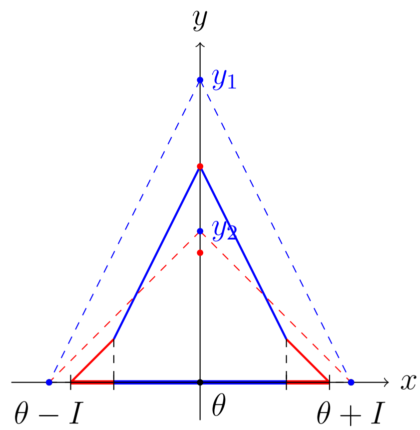

Figure 3.6: Heterogenous product quality

the product of a low quality is priced at a monopoly price in the residual market after removing the market share taken by the strong seller.

Heterogenous influencer power. We now move on to examine the impact of influencer power on product prices. We let $y_1 = y_2$ and $\theta_1 = \theta_2$. Obviously, when $I_1 = I_2$, the only equilibrium outcome sustained is the Bertrand price competition. However, when $I_1 > I_2$, there exists an equilibrium in which both groups obtain a positive profit. The reason is as follows. Note that $p_1 \geq p_2$, otherwise the second influencer-seller group will be priced out of the market. Moreover, the first group never wants to set price $p_1 = 0$ because consumers with type $x \in \mathbf{S}^1$ such that $I_2 < \|x - \theta\| \leq I_1$ always prefers Group 1 when $p_1 = 0$. Thus, group 1 can always ensure a positive profit by slightly increase the price. This in turn implies that the second group can get a positive profit by attracting consumers close to θ.

Proposition 3.4.2. *Assume that $y_1 = y_2 = y$, $\theta_1 = \theta_2 = \theta$ and that $I_1 \geq I_2$. There exists an equilibrium in which $k(j) = j$ for $j \in \{1, 2\}$. After matching, the two seller-*

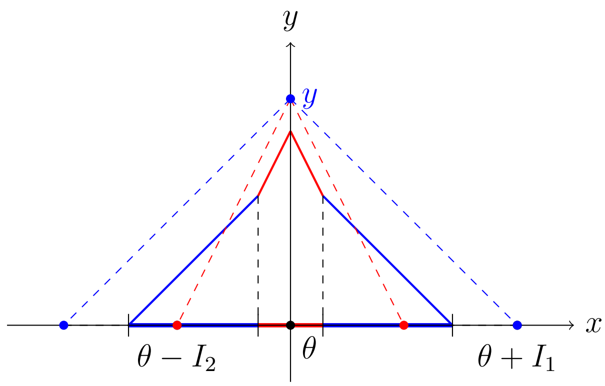

Figure 3.7: Heterogenous influence power

influencer groups set $(p_1^C, p_2^C) = \left(\frac{2y(I_1-I_2)}{4I_1-I_2}, \frac{y(I_1-I_2)}{4I_1-I2}\right)$. *Furthermore, group 1 targets consumers with types* $\frac{I_1 I_2}{4I_1-I_2} < \|x-\theta\| \leq \frac{I_1(2I_1+I_2)}{4I_1-I_2}$ *and group 2 targets consumers with types* $\|x-\theta\| \leq \frac{I_1 I_2}{4I_1-I_2}$. *Correspondingly, their profits are given by*

$$\Pi_1^C = \frac{4I_1^2(I_1-I_2)y}{(4I_1-I_2)^2}, \quad \textit{and} \quad \Pi_2^C = \frac{I_1 I_2(I_1-I_2)y}{(4I_1-I_2)^2}$$

and payoffs for sellers and influencers are given by

$$U_1 = \gamma\Pi_1^C,\ U_2 = \gamma\Pi_2^C,\ w_1 = (1-\gamma)\Pi_1^C,\ \textit{and}\ w_2 = (1-\gamma)\Pi_2^C.$$

Proof. See Appendix C.1.12. □

Figure 3.7 illustrates the equilibrium described in Proposition 3.4.2. Specifically, the group with a more powerful influencer targets the consumer demographic sufficiently far away from the influencer in style, as illustrated with blue lines. This is because they want to avoid tough price competition with the other group. Indeed, they can afford this because of the strong influencer power. In contrast, the group with a relatively weak influencer targets those consumers sufficiently close to the influencer's style, and we depict it with red lines.

The following corollary characterizes the two limit cases that $\frac{I_1}{I_2} \to \infty$ and that $\frac{I_1}{I_2} \to 1$.

Corollary 3.4.1. *When $\frac{I_1}{I_2} \to \infty$, the pricing strategies are given by $(p_1^C, p_2^C) = (\frac{y}{2}, \frac{y}{4})$, and when $\frac{I_1}{I_2} \to 1$, $(p_1^C, p_2^C) = (0, 0)$, the Bertrand price competition outcome.*

Note the difference in competition mode as it depends on how the power difference arises. When an influencer-seller group is more powerful because of the consumption value of the product, the stronger group focuses on attracting consumers with a taste similar to that of the influencer. In contrast, when the group's power comes from how easily the influencer attracts followers, the stronger group focuses on those consumers not reachable by the weaker group and sacrifice the loyal followers in the sense of taste proximity.

Heterogenous influencer style. Now, we consider how heterogeneity in influencers' style type affects market power and competition. Specifically, we study the equilibrium when there only exists heterogeneity in influencers' style, and we fix the other two dimensions (i.e., $y_1 = y_2$ and $I_1 = I_2$) to remove confounding effects.

For ease of reference, we define $\beta := \|\theta_1 - \theta_2\|$ and $\beta_0 := \frac{2}{67}(-7 + 5\sqrt{10})I \approx 0.263I$.

Proposition 3.4.3. *When $\beta \geq \beta_0$, there exists a pure strategy equilibrium such that $k(j) = j$. After matching, the two seller-influencer groups set prices such that*

$$p_1^C = p_2^C = \begin{cases} \frac{y}{5I}(2I + \beta), & \text{if } \beta_0 \leq \beta \leq \frac{6}{7}I \\ y * \left(1 - \frac{\beta}{2I}\right), & \text{if } \frac{6}{7}I < \beta \leq I \end{cases}$$

The two groups' profits are given by

$$\Pi_1^C = \Pi_2^C = \begin{cases} \frac{3y}{50I} * (2I+\beta)^2, & \textit{if } \beta_0 \leq \beta \leq \frac{6}{7}I \\ y * \beta\left(1 - \frac{\beta}{2I}\right), & \textit{if } \frac{6}{7}I < \beta \leq I \end{cases}$$

Moreover, for $\beta < \beta_0$, there exists no pure strategy equilibrium.

Whenever an equilibrium exists,

$$U_1 = \gamma\Pi_1^C,\ U_2 = \gamma\Pi_2\ w_1 = (1-\gamma)\Pi_1^c,\ \textit{and}\ w_2 = (1-\gamma)\Pi_2^c$$

Proof. See Appendix C.1.13. □

3.4.2 Multiple-dimension Heterogeneity

To focus on pure strategy equilibria that exists, we only extend the analysis to two cases of multi-dimensional heterogeneity. We investigate in the first case local monopoly when influencers' styles are sufficiently distinct. We then examine in a second case where one seller-influencer group dominates the entire market. Specifically, assume $y_1 \geq y_2$ and $I_1 \geq I_2$. We will later show that such a configuration can follow from the assortative matching and endogenous power acquisition between sellers and influencers.

Proposition 3.4.4 (Two Local Monopoly Sellers)**.** *If $\|\theta_1 - \theta_2\| \geq \frac{I_1+I_2}{2}$, the equilibrium matching is assortative, that is, $k(j) = j$ for $j \in \{1,2\}$. Furthermore, the two influencer-seller groups are serving distinct subsets of consumers and their products are priced at their own monopoly prices, i.e., $p_k^* = \frac{y_k}{2}$. Profits are given by $\Pi_1 = \frac{y_1 I_1}{2}$ and $\Pi_2 = \frac{y_2 I_2}{2}$. Finally, $U_k = \gamma\Pi_k$ and $w_k = (1-\gamma)\Pi_k$ for $k \in \{1,2\}$.*

Proof. See Appendix C.1.14. □

Proposition 3.4.4 reveals two key messages. First, when influencers' styles are sufficiently distinct, the competition is minimal and sellers enjoy local monopoly power, regardless of their product quality. Second, we establish assortative matching under regulated matching, which means that a seller with high quality products is matched with a more powerful seller. More discussions can be found in Section 3.6. Moreover, since influencers only help expose products and attract consumers and does not increase the common value of consumption y, the product with a better quality is always priced higher.

Next, we establish a market dominance result, i.e., a single seller-influencer group takes the entire product market. Note that when $\|\theta_1 - \theta_2\| < \frac{I_1+I_2}{2}$, none of the influencer-seller groups can get monopoly profits when the influence power and product quality do not differ between the two groups. However, when both the influence power and product quality of one group are sufficiently large relative to those of the other group, the market can be dominated by a monopolist. Assume that $y_1 > y_2$ and $I_1 > I_2$. Recall that $\beta = \|\theta_1 - \theta_2\|$.

Proposition 3.4.5 (Market Dominance). *If $\frac{\beta}{I_1} \leq \min\{\frac{1}{2} - \frac{y_2}{y_1}, \frac{1}{2} - \frac{I_2}{I_1}\}$, then the second influencer-seller group has no market share and quits. Specifically, Seller 1 hires Influencer 1 and set the price at $p_1^* = \frac{y_1}{2}$ and Group 1's profit is $\frac{y_1 I_1}{2}$. Under equilibrium, payoffs for Seller 1 and Influencer 1 are given by $U_1 = \frac{\gamma y_1 I_1}{2}$ and $w_1 = \frac{(1-\gamma) y_2 I_2}{2}$. Additionaly, $U_2 = w_2 = 0$.*

Proof. See Appendix C.1.15. □

To gain market dominance, the influencers' style difference needs to be sufficiently small, and their power gap and product quality gap must be sufficiently large. Furthermore, note that it is not sufficient to just have much greater influence power or much better product to force out the rival group in the price competition. Suppose group 1 has way superior product, group 2 can still compete to gain some market share because the influencer he works with creates what is similar to a sufficiently large product differentiation ($\theta_1 \neq \theta_2$). We elaborate on this substitution by style difference for product differentiation in Section 3.5.

3.5 Style Heterogeneity and Product Differentiations

Having understood how sellers hire influencers and price products, we now endogenize sellers' production stage. Specifically, we investigate how influencer heterogeneity affects horizontal and vertical product differentiations, compared to traditional economies.

3.5.1 Horizontal Product Differentiation

Our first observation is that influencers' style difference and product specialization are intuitively substitutes. To illustrate the insight, we compare two economies with and without influencers. The former is what the baseline model describes. We normalize the technology cost parameter $c = 1$ and take $\varepsilon = 0$ for simplicity. Both influencers have identical influence power $I_1 = I_2 = I$, but they are heterogeneous in their style, i.e., $\theta_1 \neq \theta_2$. In particular, θ_j are drawn from the unit circle $\mathbf{S}^1$ with a known density $f(\theta)$. Hence, both influencer's type and influence power are exogenously given, although there exists uncertainty.

The classical setup in Salop (1979) adequately describes an economy without influencers: Suppose that without influencers, the seller can enter the market selling their products by paying a fixed cost $F_H > 0$, and once they are in the market, they can costly change their style locations. In particular, consumers uniformly distributed on the unit circle $\mathbf{S}^1$ demands one unit and receives a utility of $u_i = y - p - t\|x_i - \alpha\|$, where $\|x - \theta\|$ is the distance from consumer i to the seller, who selects a location α. Here, denote by y and p the product quality and the price charged. To simplify the analysis, we consider the case with two identical sellers competing through product specialization. In light of Eq. (3.1), the transportation cost setting in Salop (1979) is equivalent to an influencer economy setting with the exogenous influence power given by $\frac{y}{t}$.

Now, we turn to show the equilibrium construction in the influencer economy. Specifically, sellers can either choose to hire an influencer and accept his style type, or they can pay a big fixed cost and chooses any arbitrary location on the unit circle $\mathbf{S}^1$. As in Salop (1979), each seller can only select one style for her product, which means that she can only hire one influencer, or pay the fixed cost and select a location only once. Furthermore, to mostly recycle existing results, we can focus on the knife-edge case in which $I = \frac{y}{t} \in (0, \pi)$, although main insights still go through and are best understood in a more general sense. After realizations of influencers' style types/locations, sellers face a trade-off between incurring the fixed cost and accepting the potentially undesirable style locations by influencers.

Given the realizations of θ_j, recall that $\beta := \|\theta_1 - \theta_2\|$ and $\beta_0 = \frac{2}{67}(-7 + 5\sqrt{10})I$. By Proposition 3.4.3, if both sellers accept influencers' locations, the price competition outcome yields profits such that $\Pi_1(\beta) = \Pi_2(\beta)$, that is,

$$\Pi_1(\beta) = \begin{cases} \frac{3y}{50I} * (2I+\beta)^2, & \text{if } \beta \in [\beta_0, \frac{6}{7}I] \\ y * \left(\beta - \frac{\beta^2}{2I}\right), & \text{if } \beta \in (\frac{6}{7}I, I) \\ \frac{1}{2}yI, & \text{if } \beta \geq I \end{cases} \tag{3.6}$$

Moreover, for $\beta < \beta_0$, there exists no pure strategy equilibrium. To avoid the non-existence issue of equilibrium, we assume that $\beta \geq \beta_0$ and that

$$0 < F_H < \Pi_1(I) - \Pi_1(\beta_0), \tag{3.7}$$

which implies that there exists a unique β^* such that $\Pi_1(\beta^*) = \Pi_1(I) - F_H$ holds.

Proposition 3.5.1 (Horizontal Product Differentiation)**.** *Assume that the fixed cost F_H is relatively small (i.e., Eq. (3.7) holds), then there exists an equilibrium such that:*
(i) When $\beta \geq \beta^*$, each seller hires one influencer and accepts his style location.
(ii) When $\beta < \beta^*$, one seller hires an influencer and accepts his style location, and the other seller pays the fixed cost F_H and select a location such that $\|\alpha_1 - \alpha_2\| \geq I$.

Proof. See Appendix C.1.16. □

Figure 3.8 illustrates the equilibrium in Proposition 3.5.1. The left subfigure entails a case with large style difference. The style locations for Influencers 1 and 2 are marked in blue nodes, while the two gray nodes illustrates the optimal style difference under the maximum differentiation principle (i.e., $\|\alpha_1 - \alpha_2\| = I \to \pi$). The blue arc corresponds to influencers style difference β. In this case, both sellers hire influencers to save the fixed cost in product

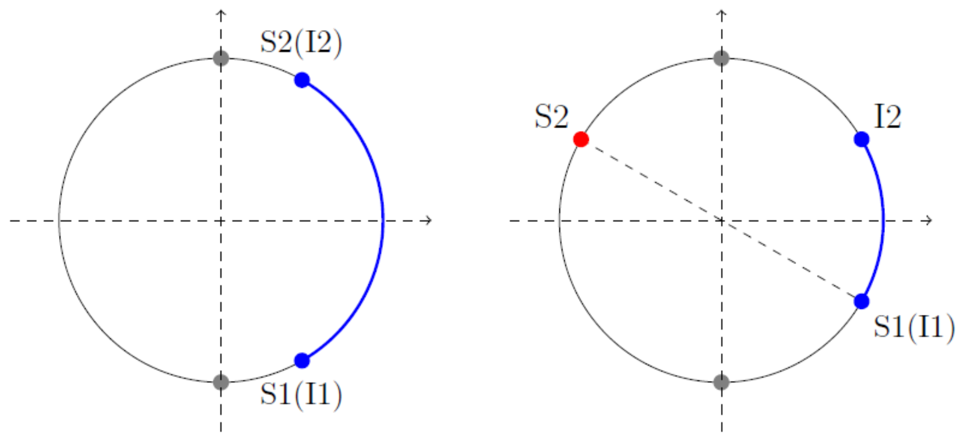

Figure 3.8: Influencer style difference and product differentiation

differentiation and adapt to their corresponding influencer's style, and maximum differentiation principle fails. The right figure illustrates the equilibrium when influencers' style difference is small. In this case, one seller, say Seller 1, hires an influencer, say influencer 1. Given this fact, Seller 2 chooses product differentiation and incur the fixed cost F_H to avoid the toughness of price competition with Seller 1. In the figure, Seller 2's location is marked in red. Hence, maximum differentiation principle is restored.

Corollary 3.5.1 (Failure of Maximum Differentiation Principle)**.** *Whenever $\beta \in (\beta^*, I)$, the maximum differentiation principle fails, and thus influencers' style difference partly substitutes product differentiation. Furthermore, when influencers' style differentiation is small (i.e., $\beta_0 \leq \beta < \beta^*$), sellers engage in product differentiation. On the opposite, there exists no product differentiation when influencers' style difference is large (i.e., $\beta > \beta^*$).*

The main message in Corollary 3.5.1 is that the well-established principle of maximum differentiation found in the literature (e.g., d'Aspremont et al., 1979;

De Frutos et al., 1999) would no longer hold in an influencer economy. More differentiated markets yield lower substitutability among products, therefore giving firms less elastic demand for the products. When influencers' style difference is small, sellers differentiate products to reduce competition. When influencers' style difference is large, sellers hires influencers and have no incentive to differentiate products.

3.5.2 Vertical Product Differentiation

The relationship between influencers' difference and vertical product differentiation (i.e., product quality differentiation) is more complex. When influencers are homogeneous and the influencer-seller matching is fixed, sellers have an incentive to differentiate product quality to avoid price competition (Shaked and Sutton, 1982). Consequently, we might observe heterogeneous product quality between sellers. Yet, when we allow for seller competition, heterogeneous product quality also arises, regardless of influencers' heterogeneity. This is mainly due to the incentive to grab the whole market and beat the competitor, rather than to avoid Bertrand style competition. Furthermore, vertical product differentiation is non-monotonic in the influencers' style difference.

Consider an endogenous balanced matching. Two influencers have identical influence powers and they might differ in their style locations (i.e., $I_1 = I_2$ and $\theta_1 \neq \theta_2$). Denote $\beta = \|\theta_1 - \theta_2\|$. The two sellers offer products with identical quality $y_1 = y_2 = \underline{y}$ ex ante. To reduce notation, we assume $I_1 = I_2 = 1$ and $\underline{y} = 1$. Influencers can pay a fixed cost F_V to invest in R&D to increase the product quality to $y > 1$. We use "NI" and "I" to denote "No Investing in high quality" and "Investing in high quality". Recall that $\beta_0 = \frac{2}{67}(-7+5\sqrt{10})I \approx 0.263$. From Proposition 3.4.3, for $\beta \in [\beta_0, \frac{5}{6}I]$,

$$\Pi_1^C = \frac{3y}{50I} * (2I+\beta)^2 = yA(\beta), \quad \text{where} \quad A(\beta) = \frac{3}{50}(2+\beta)^2.$$

For ease of reference, denote

$$V_1(\beta, y) = \Pi_{I,NI}^1 - \Pi_{NI,NI} = \Pi_{I,NI}^1 - A(\beta)$$
$$V_2(\beta, y) = \Pi_{I,I} - \Pi_{I,NI}^2 = yA(\beta) - \Pi_{I,NI}^2,$$

where $\Pi_{I,NI}^1 = \frac{y(1+2y)\left(2+8y+4y^2+\beta(4+3y)\right)^2}{(1+y)(8+19y+8y^2)^2}$, and $\Pi_{I,NI}^2 = \frac{(2+y)\left(4+8y+2y^2+\beta y(3+4y)\right)^2}{(1+y)(8+19y+8y^2)^2}$

Lemma 3.5.1 (Non-monotonic Vertical Differentiation)**.** *Assume that: i)* $\beta_0 \leq \beta \leq \frac{5}{6}$*; ii)* $\overline{y}(1-\beta_0) \leq \underline{y}$*; and iii)* $V_1(\beta_0, y) < F_V < V_2(\frac{5}{6}, y)$*. Then, there exists* $\overline{\beta}$ *and* $\underline{\beta}$ *such that:*

(i) $\beta \geq \overline{\beta}$*, there exists one Nash Equilibrium* (I, I)*;*

(ii) $\underline{\beta} \leq \beta < \overline{\beta}$*, there are two Nash Equilibrium:* (I, NI) *and* (NI, I)*;*

(iii) $\beta_0 \leq \beta < \underline{\beta}$*, there exists one Nash Equilibrium* (NI, NI)*.*

Proof. See Appendix C.1.17. □

The regularity conditions in Lemma 3.5.1 have three. One, condition i) ensures the existence of pure strategy equilibrium before investment and reduce unnecessary complications in profits calculation. Two, condition ii) ensures that under asymmetric investment (i.e., (I, NI)), the seller-influencer group with higher product quality does not dominate and force the other group quit the

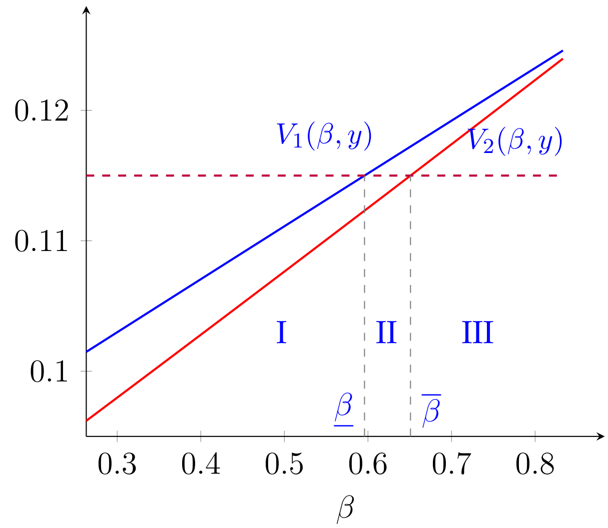

Figure 3.9: Vertical product differentiation

market. Three, condition iii) focuses on the most interesting cost range in which non-monotonicity arises.[14]

Lemma 3.5.1 characterizes a non-monotonic relationship between influencers' style difference and vertical product differentiation. Figure 3.9 illustrates the equilibrium configuration in Lemma 3.5.1 with $y_1 = 5/4$ and $y_2 = 1$. Specifically, the horizontal axis β corresponds to the difference in influencers' style locations. The two functions, $V_1(\beta, y)$ and $V_2(\beta, y)$, corresponding to the blue and red solid line in the figure, measures the profit gap between "Investing in high quality" and "No Investing", given the other group chooses "No Investing" and "Investing in high quality". Fix the cost of investment $F_V > 0$ (i.e., the purple dashed line). There are three regions, "I", "II" and "III", divided by two cutoffs $\overline{\beta}$ and $\underline{\beta}$. In region III, $V_2(\beta, y) > F_V$, and thus both groups choose to invest in high quality. In region II, $V_2(\beta, y) < F_V$ and $V_1(\beta, y) > F_V$, which means only

[14] For $F_V \leq V_1(\beta_0, y)$, (I, I) is the unique Nash equilibrium. Similarly, for $F_V > V_1(\frac{5}{6}, y)$, (NI, NI) is the unique Nash equilibrium.

one groups chooses to invest in high quality. In region I, both groups chooses not to invest in high quality.

Lemma 3.5.1 implies that, when influencers' difference is small, as the style difference β increases, we first see more vertical differentiation (thus influencers' difference and vertical product quality differentiation are complements). In contrast, when influencers' difference is large, as the style difference increases, we see less vertical differentiation (thus these two are substitutes). This result is intuitive: For small β (i.e, influencers have similar style locations), the competition is very intense, which greatly limits the return from investing in high quality. Thus, both groups choose low quality and the vertical differentiation is minimal. For intermediate β, the competition is less intense which improves the investment return. However, the investment profit is only big enough to support one group investing, and if both groups invest, then one group cannot break even, leading to the observed vertical differentiation. Last, for large β, the competition is very minimal and even when both groups invest, they can break even. We only need to note that the investment profit is strictly increasing in the underlying influencer difference. Hence, we observe no vertical product differentiation again.

The following result follows directly:

Proposition 3.5.2. *When influencers' style difference is relatively small, it is a complement with vertical product differentiation; When influencers' style difference is large, it becomes a substitute for vertical differentiation.*

3.6 Endogenous Influence and Welfare Implications

We now move to t_0 to allow the influences to endogenize their influence either in power or type. First, we allow endogenous power acquisition and show that socially inefficient under-investment and over-investment in influence can arise due to externality and endogenous bargaining power issue. Second, we show that maximum style differentiation and assortative seller-influencer matching hold in the long run.

3.6.1 Inefficient Power Acquisition

This section discusses influencers' incentives to endogeneize their power selection. Indeed, influencers have a strong incentive to compete through power/style differentiation to secure a more favorable outside option. The intuition is best illustrated by Example 3.6.1 below.

Example 3.6.1. *Assume $\gamma < 1$. Consider the perfectly symmetric case with endogenous hiring (i.e., $y_1 = y_2 =: y$, $\theta_1 = \theta_2 =: \theta$, and $I_1 = I_2 =: I$). Influencer 2 can get a wage of*

$$w_2 = \frac{\gamma I_1 I_2 (I_1 - I_2) y}{(4I_1 - I_2)^2} \tag{3.8}$$

Initially, $I_1 = I_2 = I$ and thus $w_2 = 0$. However, if influencer 2 can commit to influence power reduction by choosing a small I_2, we can optimize over Equation (3.8) to get

$$I_2^* = \frac{4}{7} I, \text{ and } w_2 = \frac{yI}{48}.$$

In other words, Influencer 2 reduces influence power to avoid price competition.[15]

A related question is, if we allow influencers' ability to be endogenized, how the cost affect the acquisition of ability/skill of the influencer? Maybe power would go up because influencers face more competition. How would welfare be affected? Would this create an arms race of potential influencers all wasting effort to acquire ability, but in the end the actual consumers they influence is limited (which is social sub-optimal)? Arms race in influence depends on how much additional utility is purely due to style preference, because the utility from the goods is bounded above. So the marketing arms race could severe. If this is played out in stages, this means many influencers spend effort to acquire power and too many endogenously become influencers.

We can show that there might exist socially sub-optimal investment in power acquisition. To simplify the analysis, we focus on a very specific example with a monopolist seller and 2 identical influencers. Initially, $I_1 = I_2 = \pi$, $y_1 = y_2 = y$. Influencers can pay a fixed cost $C_T > 0$ to increase influence power to 2π before they are hired by sellers. The question is when arms race is welfare optimal? We use "I" and "NI" to denote the influencer strategies entailing investing in power acquisition and not investing (keeping the power at π) respectively.

Proposition 3.6.1 (Power Acquisition & Inefficient Arms Race)**.** *Assume that $I_1 = I_2 = \pi$ and $\|\theta_1 - \theta_2\| = \pi$.*

The Nash Equilibrium for influencers to invest in power acquisition are given by:

(i) (NI, NI) is a Nash Equilibrium when $C_T > \frac{1}{3}(1-\gamma)y\pi$;

[15]Given that Influencer 2's voluntary power reduction, influencer 1 has no incentive to reduce power.

(ii) (I, NI) *and* (NI, I) *are Nash Equilibrium when* $C_T \in (\frac{1}{6}(1-\gamma)y\pi, \frac{1}{3}(1-\gamma)y\pi]$*;*

(iii) (I, I) *is a Nash Equilibrium when* $C_T \leq \frac{1}{6}(1-\gamma)y\pi$*.*

The optimal decision rule to maximize total welfare is given by

(i) when $C_T > \frac{1}{6}y\pi$*,* (NI, NI) *is optimal, i.e., no influencer should invest;*

(ii) when $\frac{1}{12}y\pi < C_T \leq \frac{1}{6}y\pi$*,* (I, NI) *(or* (NI, I)*) is optimal, i.e., only one influencer should invest;*

(iii) when $C_T \leq \frac{1}{12}y\pi$*,* (I, I) *is optimal, i.e., both influencers should invest.*

Proof. See Appendix C.1.18. □

Proposition 3.6.1 shows that power acquisition can exhibit over-investment, efficient investment, or under-investment relative to a socially efficient benchmark. To see it, we can check the incentive misalignment between power acquisition and total welfare. For instance, both influencers invest in power acquisition when $C_T \leq \frac{1}{6}(1-\gamma)y\pi$. In contrast, total welfare maximization requires $C_T \leq \frac{1}{12}y\pi$ for both influencers to invest. This implies that: (i) when $\gamma = \frac{1}{2}$, these two conditions coincide, which implies that power acquisition is welfare optimal; ii) when $\gamma > \frac{1}{2}$, there exists insufficient power acquisition for $C_T \in \left(\frac{1}{6}(1-\gamma)y\pi, \frac{1}{12}y\pi\right]$; and iii) when $\gamma < \frac{1}{2}$, there exists over-investment in power acquisition for $C_T \in \left(\frac{1}{12}y\pi, \frac{1}{6}(1-\gamma)y\pi\right]$.

There are two forces driving this result. One, the incentive misalignment between power acquisition and total welfare. Power acquisition does not consider externality on consumer welfare, as well as on other influencers. When the influencer market is not crowded, power acquisition can improve welfare by increasing consumer utility and it can exhibit under-investment when the positive externality on consumer welfare is not internalized. In contrast, when the influencer market is congested and all influencers compete for better wages, the arm race of potential influencers leads to wasting effort because the actual consumers they influence is very limited. Two, the bargaining power division among the seller and influencers. The bargaining power parameter assigned to influencers, $1-\gamma$, can also distort the incentive for power acquisition. A large γ reduces the incentive for power acquisition, while a small γ (i.e., $1-\gamma\uparrow$) empowers the incentive for power acquisition. Hence, depending on the congestion of influencer market and the bargaining power, we might observe over-investment, efficient investment and under-investment in power acquisition.

3.6.2 Style Selection and Seller-influencer Matching

In this section, we allow influencers to choose influence style and study seller-influencer matching with endogenous styles. To this end, we first consider an example with costly style differentiation as illustrated in Example 3.6.2. Then, we formally recover the maximum differentiation principle in the limit case with costless style selection.

Example 3.6.2. *Consider $y_1 = y_2 =: y$, $I_1 = I_2 =: I$, and $\theta_1 \neq \theta_2$. To ensure the existence of a pure strategy equilibrium, we also assume that $\beta := \|\theta_1-\theta_2\| \geq \frac{2}{67}(-7+5\sqrt{10})I$. Now, influencer 2 can pay a cost, $C(b)$, to select his own style location θ_2^*, where $b := \|\theta_1-\theta_2^*\|$, and we assume $C(\beta)=0$, $C'(\beta)=0$ and $C''(b)>0$ for $b\geq\beta$. Then, the optimal style type satisfies $d^*\in(\beta, I)$, that is, influencer 2 always invests in* *style differentiation as long as $\beta < I$.*

A detailed proof of the assertion in Example 3.6.2 can be found in Appendix C.1.19. Note that maximum style differentiation fails even when style selection is costly.

Now, we consider the other case when style selection is costless. First, we present an assortative matching result when the maximal style differentiation principle holds. Specifically, rank sellers and influencers by their product quality and influence power so that $y_1 \geq y_2$ and $I_1 \geq I_2$. Denote by $k(j)$ the matched seller identity $k(j)$ for influencer $j = 1, 2$.

Lemma 3.6.1. *Assume that $\|\theta_1 - \theta_2\| \geq \frac{I_1+I_2}{2}$. Then $k(j) = j$ for $j = 1, 2$, that is, the strong (weak) seller is matched with the strong (weak) influencer.*

Proof. See Appendix C.1.20. □

Lemma 3.6.1 shows the emergence of assortative matching when influencers' style locations are exogenous given such that the maximal style differentiation principle holds. The seller with a more valuable good can offer to hire a more powerful influencer by proposing a higher wage because the seller-influencer group's total profit is *supermodular* in influencer power and product quality parameter.[16]

Next, we turn to the problem of endogenous style location selection. We prove a special case in which $\frac{I_1+I_2}{2} \leq \pi$.

[16]A twice-differentiable function $f : X \times Y \to \mathbb{R}$ is supermodular iff $\frac{\partial^2 f}{\partial x \partial y} \geq 0$ for all $(x, y) \in X \times Y$.

Proposition 3.6.2 (Maximal style differentiation and assortative matching)**.** *Assume that* $\frac{I_1+I_2}{2} \leq \pi$*. When style location selected is costless, the maximum style differentiation holds, that is,* $\|\theta_1^* - \theta_2^*\| \geq \frac{I_1+I_2}{2}$*, and there exists no overlapping in consumers served. Furthermore, assortative matching applies under endogenous style location selection.*

Proof. See Appendix C.1.21. □

Proposition 3.6.2 states that influencers follow the maximal style differentiation principle whenever possible under both seller competition and regulated matching, because it minimizes the competition among influencer-seller groups. Furthermore, assortative matching ensues under endogenous style location selection.

The result is also in stark contrast with Gal-Or and Dukes (2003) that discovers a minimum differentiation in commercial media markets. Overthere, product differentiation is taken as exogenous and thus the substitutability between style differentiation and product differentiation is absent.

3.7 Unbalanced Matching, Exclusivity, and Regulation

In practice, a seller often hire multiple influencers and require the influencers not to advertise rival sellers' products (e.g., Zietek, 2016). For example, a large survey of influencers by Mavrck (Katz, 2019) shows that the majority of influencers (61%) are receiving exclusivity requests from brands. In fact, exclusivity contracts have been prevalent in industries such as healthcare and infurance and have led to many antitrust cases (Gal-Or, 1999). However, policies are being introduced to better protect the influencers and to reduce market concentration through encouraging competition. The industry has also grown in awareness that exclusivity should be mutual.[17] This means that either both sides can contract with multiple counterparties or both sides have to exclusively collaborate—exactly our setting of balanced matching aims to capture.

Nevertheless, to better understand the welfare implication of exclusivity contracts in this emerging industry and to guide regulatory policies geared towards balancing the power of influencers and sellers, we extend the analysis to "unbalanced matching." In such a setting, a seller can hire multiple influencers, but not the other way round, which is consistent with the contracting landscape in the early stages of influencer industry (e.g., Zietek, 2016). We then compare settings with and without balanced matching to derive two key results: First, balanced matching (mutual exclusivity contracting) is optimal under one-dimensional heterogeneity even when we allow sellers to compete for multiple influencers. Second, unbalanced matching (uni-directional exclusivity) can be optimal when influencers' style locations are sufficiently unique in uncongested influencer markets.

We begin analysis with useful lemma about the joint profit when a seller

[17]Influencers increasingly value long-term partnerships With brands rather than one-off exclusivity requests; they also expect to be compensated more when exclusivity is required. As early as 2008, the entertainment industry began to see the value behind full-time creators building multi-platform brands, and influencers started getting Hollywood agents to help negotiate bilateral exclusive contracts (Collectively, 2020). New York State Regulators, China's State Administration for Market Regulation, and the British Federal Trade Commissions are all increasing regulations regarding influencer contracting. See, e.g., https://www.ftc.gov/news-events/press-releases/2019/11/ftc-releases-advertising-disclosures-guidance-online-influencers; http://www.zhonglun.com/Content/2020/11-11/1619074334.html.

hires both influencers. Note that the joint profit in Lemma 3.3.2 is established under $\beta = \pi$ and that influencer power can be large. Recall that $\beta = \|\theta_1 - \theta_2\|$.

Lemma 3.7.1 (Joint Profit Function $\Pi_{\{1,2\}}$)**.** *The joint profit function is given by*

$$\Pi_{\{1,2\}} = \begin{cases} yI, & \text{if } \beta \geq I \\ \frac{(2I+\beta)^2}{8I}y, & \text{if } \beta \leq \frac{2I}{3} \\ \frac{(2I\beta-\beta^2)}{I}y, & \text{if } \beta \in \left(\frac{2I}{3}, I\right) \end{cases} \tag{3.9}$$

and the pricing strategy is given by

$$p_1^* = p_2^* = \begin{cases} \frac{y}{2}, & \text{if } \beta \geq I \\ y * \left(\frac{\beta}{4I} + \frac{1}{2}\right), & \text{if } \beta \leq \frac{2I}{3} \\ y * \left(1 - \frac{\beta}{2I}\right), & \text{if } \beta \in \left(\frac{2I}{3}, I\right) \end{cases}$$

Proof. See Appendix C.1.22. □

Lemma 3.7.1 helps us establish that the equilibrium outcome coincides with that in regulated matching when heterogeneity is single-dimensional. Suppose that $y_1 \geq y_2$ and $I_1 \geq I_2$, we have:

Lemma 3.7.2 (Equilibrium under unbalanced matching)**.** *Allowing uni-directional exclusivity contracts:*

(i) When the influencers or the sellers differ in a single dimension, the equilibrium coincides with that in Proposition 3.4.1, 3.4.2 and 3.4.3.

(ii) When influencers are sufficiently unique (i.e., $\beta \geq \frac{I_1+I_2}{2}$), seller 1 hires both influencers and offers prices at $p_1^* = p_2^* = \frac{y_1}{2}$. Payoffs for sellers and influencers satisfy:

$$U_1 = \frac{\gamma y_1(I_1 + I_2)}{2}, U_2 = 0,\ w_1 = \frac{(1-\gamma)y_1 I_1}{2} \text{ and } w_2 = \frac{(1-\gamma)y_1 I_2}{2}.$$

Proof. See Appendix C.1.23. □

How do different forms of exclusivity affect welfare? Intuitively, compared to balanced matching, unbalanced matching features a monopolist seller with the relatively higher product quality. On the one hand, it increases welfare by replacing the seller with a low quality product, whose magnitude depends on the quality gap between sellers. On the other hand market concentration and monopolist pricing decreases surplus for consumers attracted, and prices out a large fraction of potential consumers. Note that when the quality gap between two products decreases, the former effect vanishes. Thus, unbalanced matching hurts consumers and social welfare when products have similar quality.

Proposition 3.7.1 compares the efficiency between balanced and unbalanced matchings.

Proposition 3.7.1 (Exclusivity Contracting and Welfare)**.**

(i) (Congested influencer market or homogeneous product market). Unbalanced matching lowers total welfare under one dimension heterogeneity, including heterogeneous product quality, heterogeneous influencer power and heterogeneous influencers' style locations.

(ii) (Uncongested influencer market). When $\beta > \frac{I_1+I_2}{2}$ and $y_1 > y_2$, unbalanced matching dominates regulated matching in total welfare.

Proof. See Appendix C.1.24. □

The key messages in Proposition 3.7.1 is intuitive. First, both product quality gap and influencer style difference affect intensity of seller competition. When the influencer market is not crowded and influencers' styles are distinct, regulation on mutual exclusivity contracting does not help encourage competition because of the inevitable local market power derived from influencer heterogeneity; given the economy features monopoly pricing anyway, uni-directional exclusivity is welfare-improving because it allows the better product to dominate.

In contrast, when products are close to being homogeneous or influencers are too similar in style, requiring mutual exclusivity and balanced matching can improve consumer welfare. Note that unbalanced matching always features joint profit maximization and a high quality product domination, while balanced matching features greater price competition. Simply put, the quality improvement channel is shut down in a homogeneous product market, while regulation can improve competition when the influencer market is crowded.

3.8 Conclusion

We build a model of the influencer economy in which (i) sellers produce goods and compete for consumers through influencers, (ii) sellers and influencers are matched in influencers' labor market and engage in Nash bargaining, and (iii) influencers acquire influence to attract consumers who identify with their style in addition to value the products they promote. We derive five key insights:

First, as technologies governing marketing outreach improve, the equilibrium features non-monotonicities in influencer market concentration, payoffs, and distributional inequality. Second, influencer heterogeneity and horizontal product differentiation are substitutes. At the same time, small style differences complement vertical product differentiation while large differences substitute. Third, assortative matching between sellers and influencers occurs under endogenous influence, with the maximum horizontal differentiation principle recovered in the limit of costless style selection. Fourth, the sellers' bargaining power counteracts the influencers' tendency to over-invest in influence power and they jointly determine the direction and magnitude of the sub-optimal acquisition. Fifth, regulations for balanced seller-influencer matching can encourage seller competition under single dimensional seller-influencer heterogeneity. But uni-directional exclusivity contracts are welfare-improving for sufficiently differentiated products and uncongested influencers' markets.

For tractability and to focus on the industrial organization of the influencer economy, we have largely abstracted away from the inner working of platforms and MCNs. In this regard, our findings constitute initial benchmark results rather than foregone conclusions. In our setting, platforms can be viewed as powerful influencing segments with large bargaining power, which leaves much to be desired. Profit sharing and contracting between influencers and platforms remain a crucial topic in understanding the digital economy. The organization of MCNs such as Douyin and Weibo, and their heterogeneity also constitute interesting future research.

APPENDIX A

APPENDIX OF CHAPTER 1

A.1 Relevant Proofs and Calculations

A.1.1 Proof of Lemma 1.4.1

Proof. The assumed condition guarantees that the coordinator selected is willing to set $\kappa^* = \bar{\kappa}$. See the proof of Lemma 1.5.1. Then, the remaining proof consists of two parts.

(i) Consider the case without media groups. For the ith small trader, his action does not impact the aggregate level of κ because of the form (1.8). Thus, if he chooses to promote the stock in his personal network (i.e., $e_i = \bar{e}$), then the return from the stock does not change, regardless of whether he holds stocks. This implies that he gets a negative utility $-\eta < 0$, which is dominated by the action $e_i = 0$ whose payoff is zero.

(ii) Consider the case with media groups. First, note that by Lemma 1.5.1, the coordinator has an incentive to choose $\kappa^* = \bar{\kappa}$, given all the small traders follow the proposed equilibrium strategy. Now, given the proposed target $\kappa_0 = \bar{\kappa} = I\bar{e}$, we further have two cases to check for the small traders' incentives.

   (a) The ith trader does not hold stocks. Given all other small traders chooses $e_j^* = \bar{e}$, $\forall j \neq i$, then

$$\kappa = \int_{i \in I} e_j^* dF(\theta_j) = I\bar{e} \geq \bar{\kappa}$$

Hence, choosing $e_i^* = \bar{e}$ leads to a payoff given by $(u - \eta) \geq 0$, strictly dominating $e_i = 0$. Note that, since the ith trader does not invest in the stock, he only cares about anger expressing utility and the effort cost.

(b) The ith trader hold a positive amount of stocks. His incentive from anger expressing coincides with that detailed above as the case without stock holding. On the other hand, a single small trader has measure zero and thus his individual effort does not change the aggregate κ. This implies that it is weakly dominant to support the coalition's proposal $\kappa^* = \bar{\kappa}$, given all other small traders choose to support. Hence, it is optimal for the ith trader to support the proposal.

The proof concludes. □

A.1.2 Proof of Lemma 1.5.1

Proof. Fix κ, there are two cases.

i. There is no forced liquidation in stage 3 and thus all small traders weakly benefit from a large κ because $S = \xi h_1(\kappa) h_2(z_1)$, which is strictly increasing in κ by the monotonicity of $h_1(\cdot)$.

ii. There is a forced liquidation in stage 3 and the marginal trader is given by $\theta_i = \theta(z_3)$, where z_3 is defined in Eq. (1.9).

Now, by the formula, $S = \xi h_1(\kappa) h_2(z_3)$, we have

$$\frac{dS}{d\kappa} = \xi h_1'(\kappa) h_2(z_3) + \xi h_1(\kappa) h_2'(z_3) \frac{dz_3}{d\kappa} \geq 0.$$

which reduces to

$$\frac{dz_3}{d\kappa} \geq -\frac{h_1'(\kappa)}{h_1(\kappa)} \frac{h_2(z_3)}{h_2'(z_3)}$$

which holds under the assumed condition.

Now, the claim that $\kappa^* = \bar{\kappa}$ follows. □

A.1.3 Proof of Lemma 1.5.2

Proof. We check the properties one by one.

1. $V_2(z_1)$ is continuous. We just need to check $V_2(z_1)$ is continuous at $z_1 = z^{\intercal}(\bar{\kappa})$. Define $z_3\left(z^{\intercal}(\bar{\kappa})_-\right) := \lim_{z \to z^{\intercal}(\bar{\kappa}), z < z^{\intercal}(\bar{\kappa})} z_3(z)$ and by definition of $z^{\intercal}(\bar{\kappa})$, $z_3\left(z^{\intercal}(\bar{\kappa})_-\right) = z^{\intercal}(\bar{\kappa})$. Thus, by Eq. (1.13),

$$V_2\big(z^{\intercal}(\bar{\kappa})_-\big) = \lim_{z \to z^{\intercal}(\bar{\kappa}), z < z^{\intercal}(\bar{\kappa})} V_2(z) = 0 = V_2(z^{\intercal}(\bar{\kappa})).$$

2. $V_2(z_1) < 0$ for $z_1 < z^{\intercal}(\bar{\kappa})$. To see it, note that $V_2(z_1) = 0$ for $z_1 \geq z^{\intercal}(\bar{\kappa})$.

Moreover, for $z_1 < z^{\intercal}(\bar{\kappa})$, rearrange $V_2(z_1)$ from Eq. (1.13),

$$
\begin{aligned}
V_2(z_1) &= \lambda h_1(\bar{\kappa})[z_3 h_2(z_3) - z_1 h_2(z_1)]\theta_L - \lambda s(\bar{\kappa}, z_3)(z_3 - z_1) \\
&= \lambda\theta_L(z_3 - z_1)h_1(\bar{\kappa})h_2(z_3) + \lambda z_1\big(h_1(\bar{\kappa})h_2(z_3) - h_1(\bar{\kappa})h_2(z_1)\big)\theta_L \\
&- \lambda\theta(z_3)(z_3 - z_1)h_1(\bar{\kappa})h_2(z_3) \\
&= \lambda\big(\theta_L - \theta(z_3)\big)(z_3 - z_1)h_1(\bar{\kappa})h_2(z_3) + \lambda z_1 h_1(\bar{\kappa})\big(h_2(z_3) - h_2(z_1)\big)\theta_L
\end{aligned}
$$

Note that $z_1 < 0$, $z_1 < z_3$, $\theta_L \leq \theta(z_1) < \theta(z_3)$ by monotonicity of $\theta(z) := G\big((I - N + z)/I\big)$ from Eq. (1.7), and $h_1(\bar{\kappa})[h_2(z_3) - h_2(z_1)] > 0$ by the monotonicity of $h_2(z)$. Now, the claim follows.

3. $V_2'\left(z^{\intercal}(\bar{\kappa})_-\right) := \lim_{h\to 0, h>0} V_2'\big(z^{\intercal}(\bar{\kappa}) - h\big) > 0$. Note that $V_2(z_1) < 0$ for $z_1 < z^{\intercal}(\bar{\kappa})$ and that $V_2\big(z^{\intercal}(\bar{\kappa})\big) = 0$. Now, by the Mean Value Theorem, for any $h > 0$, there exists a $t_h \in (0, 1)$ such that

$$V_2'(z^{\intercal}(\bar{\kappa}) - t_h h)(-h) = V_2\big(z^{\intercal}(\bar{\kappa}) - h\big) - V_2\big(z^{\intercal}(\bar{\kappa})\big) < 0.$$

Now, taking the limit $h \to 0$ on both sides yields the result.

The proof concludes. □

A.1.4 Proof of Proposition 1.5.1

Proof. First, we show that $z_1^* < 0$. It consists of two parts.

(i) we show that $z_1^* \in (-\infty, 0]$. By the condition that $h_2(z_1) = O\left(|z_1|^{-1+\delta}\right)$ for some $\delta > 0$ as $z_1 \to -\infty$,

$$m|z_1|s(0, z_1) = m|z_1|h_2(z_1)\theta(z_1) \geq m\theta_L O\left(|z_1|^{\delta}\right)$$

which approaches ∞ as $z_1 \to -\infty$. Here, we use lemma 1.3.1 and the fact that $\theta(z_1) \geq \theta_L$, which follows because otherwise the large trader losses money for each share of stock shorted. To see it, note that $V_2(z_1) \leq 0$ by Lemma 1.5.2 and $V_1(z_1) < W_L$ because $\big[\lambda h_1(\bar{\kappa}) + 1 - \lambda\big]\theta_L - \theta(z_1) > 0$ if $\theta(z_1) < \theta_L$. This further implies that $\mathbb{E}[U_L^1] < W_L$ and thus shorting is suboptimal.

Since the right side is of $O\left(|z_1|^\delta\right)$, the feasible budget set of z_1 must be finite and thus compact. Moreover, by the continuity of $V_2(z_1)$ and $V_1(z_1)$, $\mathbb{E}[U_L^1]$ is continuous in z_1. By Weierstrass theorem, there exists an optimal $z_1^* \in (-\infty, 0]$.

(ii) we show that $z_1^* \neq 0$. Note that, when $z_1 \geq z^\intercal(\bar{\kappa})$, $\mathbb{E}[U_L^1] = V_1(z_1)$. Thus,

$$\begin{aligned}\frac{dV_1(z_1)}{dz_1} &= \left(z_1 h_2'(z_1) + h_2(z_1)\right)\Big(\big[\lambda h_1(\bar{\kappa}) + 1 - \lambda\big]\theta_L - \theta(z_1)\Big) \\ &\quad + z_1 h_2(z_1)\Big(\big[\lambda h_1(\bar{\kappa}) + 1 - \lambda\big]\theta_L - \theta(z_1)\Big)\end{aligned}$$

This implies that

$$V_1'(0_-) = \Big(\big[\lambda h_1(\bar{\kappa}) + 1 - \lambda\big]\theta_L - \theta(0)\Big) \leq h_1(\bar{\kappa}) - \theta(0) < 0. \tag{A.1}$$

Here, we used that $\theta(z) = G\left(\frac{I-N+z}{I}\right)$, and by assumption 1.3.2, $\theta(0) = G\left(\frac{I-N}{I}\right) > h_1(\bar{\kappa})\theta_L$. Hence, $z_1^* = 0$ is suboptimal.

Second, claim (iii) follows from Lemma 1.5.1 and Lemma 1.4.1.

Third, claim (iv) is trivial. The proof concludes. □

A.1.5 Proof of Lemma 1.5.3

Proof. i) The proof for the first claim follows directly from the definition of $c(z_1^*)$.

ii) In stage 1, for all small traders holding the stock (i.e., $\theta_i \geq \theta(z_1^*)$), define

$$CS_1^i := h_2(z_1^*)(\theta_i - \theta(z_1^*))$$

There are two cases. One, for any small trader with $\theta_i \in (\theta(z_1^*), \theta(0)]$, $CS_0^i = 0$ because they are not holding the stock. Hence, $\Delta CS^i = CS_1^i > 0$. Two, for any small trader with $\theta_i > \theta(0)$, he is active in both before and after short selling and his surplus in stage 1 is given by:

$$CS_1^i := h_2(z_1^*)\big(\theta_i - \theta(z_1^*)\big) = h_2(z_1^*)(\theta_i - \theta(0)) + h_2(z_1^*)(\theta(0) - \theta(z_1^*))$$

Similarly, define $CS_0^i := \theta_i - \theta(0)$. Hence,

$$\Delta CS^i = CS_1^i - CS_0^i = \big(h_2(z_1^*) - 1\big)\big(\theta_i - \theta(0)\big) + h_2(z_1^*)\big(\theta(0) - \theta(z_1^*)\big) < 0$$

if and only if

$$\theta_i > \theta(0) + \frac{h_2(z_1^*)\big(\theta(0) - \theta(z_1^*)\big)}{1 - h_2(z_1^*)}.$$

This concludes the proof. □

A.1.6 Proof of Proposition 1.5.2

Proof. For any $z_1 \in (-\infty, z^{\intercal}(\bar{\kappa}))$, a forced liquidation is sure to occur and leads to a revised position $z_3(z_1)$, as defined in Eq. (1.9). We show that z_1 is dominated by $\tilde{z}_1 := z_3(z_1) \in [z^{\intercal}(\bar{\kappa}), 0]$. Note that a position $z_1 \in (-\infty, z^{\intercal}(\bar{\kappa}))$ surely triggers a forced liquidation and the expected payoff is given by Eq. (1.14) and thus

$$\begin{aligned}
\mathbb{E}[U_L^1] &= z_3\mathbb{E}_L[\xi h_1(\bar{\kappa})h_2(z_3)] - z_1 s(0, z_1) - s(\bar{\kappa}, z_3)(z_3 - z_1) + W_L \\
&= z_3\mathbb{E}_L[\xi h_1(\bar{\kappa})h_2(z_3)] - z_3 s(\bar{\kappa}, z_3) + z_1\big(s(\bar{\kappa}, z_3) - s(0, z_1)\big) + W_L \\
&= z_3\mathbb{E}_L[\xi h_1(\bar{\kappa})h_2(z_3)] - z_3 s(0, z_3) + \{z_3 s(0, z_3) - z_3 s(\bar{\kappa}, z_3)\} \\
&\qquad + z_1\big(s(\bar{\kappa}, z_3) - s(0, z_1)\big) + W_L \\
&\le z_3\mathbb{E}_L[\xi h_1(\bar{\kappa})h_2(z_3)] - z_3 s(0, z_3) + \{z_3 s(0, z_1) - z_3 s(\bar{\kappa}, z_3)\} \\
&\qquad + z_1\big(s(\bar{\kappa}, z_3) - s(0, z_1)\big) + W_L \\
&\le z_3\mathbb{E}_L[\xi h_1(\bar{\kappa})h_2(z_3)] - z_3 s(0, z_3) + (z_1 - z_3)\big(s(\bar{\kappa}, z_3) - s(0, z_1)\big) + W_L \\
&\le z_3\mathbb{E}_L[\xi h_1(\bar{\kappa})h_2(z_3)] - z_3 s(0, z_3) + W_L
\end{aligned}$$

All these algebras are self-evident once we note the monotonicity of $s(\kappa, z_1)$ and that $z_1 < z_3(z_1) \le 0$. The proof concludes. □

A.1.7 Proof of Lemma 1.5.4

Proof. We check the optimality of the decisions using backward deduction.

- In stage 3, first, whenver $z_1 \ge z^{\intercal}(\bar{\kappa})$, the large trader has enough savings to post more margin because

$$\begin{aligned}
W_L = m|z^{\intercal}(\bar{\kappa})| s(\bar{\kappa}, z^{\intercal}(\bar{\kappa})) &= m h_1(\bar{\kappa})\big|z^{\intercal}(\bar{\kappa})\big| h_2(z^{\intercal}(\bar{\kappa}))\theta\big(z^{\intercal}(\bar{\kappa})\big) \\
&\ge m h_1(\bar{\kappa})|z_1| h_2(z_1)\theta\big(z_1)\big) = m|z_1| s(\bar{\kappa}, z_1)
\end{aligned}$$

Here, in the inequality, we use assumption 1.5.1 (i), the monotonicity of $|z|h_2(z)\theta(z)$. Note that, when $W_L \geq W_L\big(\bar{\kappa}, z^\dagger(\lambda)\big)$, $z^\intercal(\bar{\kappa}) \leq z^\dagger(\lambda) < 0$.

Similarly, when $z_1 < z^\intercal(\bar{\kappa})$, a forced liquidation occur surely because $W_L < m|z_1|s(\bar{\kappa}, z_1)$ by the same reasoning as above. Consequently, the revised short position is given by $z_3(z_1)$ defined in Eq. (1.9).

- In stage 2, $\kappa^* = \bar{\kappa}$ by Lemma 1.5.1 and Lemma 1.4.1.
- In stage 1, there are two cases.

  (i) When $W_L \geq \widehat{W}_L\big(\bar{\kappa}, z^\dagger(\lambda)\big)$ holds, there will be no forced liquidation in stage 3 and thus $\mathbb{E}[U_L^1] = V_1(z)$, which is defined in Eq. (1.10). By Assumption 1.5.1 (ii), $z^\dagger(\lambda)$, the optimal solution to $V_1(z)$, is well-defined and unique. This also implies that $z_1^* = z^\dagger(\lambda)$.

  (ii) When $W_L < \widehat{W}_L\big(\bar{\kappa}, z^\dagger(\lambda)\big)$ holds, a forced liquidation arises surely, and thus $\mathbb{E}[U_L^1] = V_1(z_1) + V_2(z_1)$. Note that $E[U_L^1]$ is continuous but non-differentiable at $z_1 = z^\intercal(\bar{\kappa})$.

  (a) The case that $V_1\big(z^\intercal(\bar{\kappa})\big) + V_2\big(z^\intercal(\bar{\kappa})_-\big) \geq 0$.

  When $z_1 \geq z^\intercal(\bar{\kappa})$, $V_2(z_1) = 0$ and thus $\mathbb{E}[U_L^1] = V_1(z_1)$. Note that $V_2(z_1)$ is not differentiable at $z_1 = z^\intercal(\bar{\kappa})$. Hence, by equation (A.1),

$$\frac{d\mathbb{E}[U_L^1]}{dz_1}\bigg|_{z_1=0_-} = V_1'(0_-) < 0$$

  Meanwhile, by the concavity of $V_1(z_1)$, Assumption 1.5.1 (ii), and by the definition of $z^\dagger(\lambda)$,

$$V_1'(z_1) < 0, \quad \forall z_1 \in (z^\dagger(\lambda), 0]$$

  Hence, since $z^\dagger(\lambda) < z^\intercal(\bar{\kappa}) < 0$, $V_1'(z) < 0$ for all $z \geq z^\intercal(\bar{\kappa})$.

This implies that $z_1^* \leq z^{\intercal}(\bar{\kappa})$ because

$$\frac{d\mathbb{E}[U_L^1]}{dz_1} = V_1'(z_1) < 0, \quad \forall z_1 > z^{\intercal}(\bar{\kappa})$$

Moreover, by the concavity of $V_1(z)+V_2(z)$ (see Assumption 1.5.1 (ii) and (iii)), $\forall z_1 < z^{\intercal}(\bar{\kappa})$,

$$\frac{d\mathbb{E}[U_L^1]}{dz_1} = V_1'(z_1) + V_2'(z_1) \geq V_1'\big(z^{\intercal}(\bar{\kappa})\big) + V_2'\big(z^{\intercal}(\bar{\kappa})_-\big) \geq 0$$

which implies that $z_1^* \geq z^{\intercal}(\bar{\kappa})$. Thus, $z_1^* = z^{\intercal}(\bar{\kappa})$.

(b) The case that $V_1\big(z^{\intercal}(\bar{\kappa})\big) + V_2\big(z^{\intercal}(\bar{\kappa})_-\big) < 0$.

First, note that $V_2(z) = 0$ for $z \geq z^{\intercal}(\bar{\kappa})$, and thus

$$\lim_{z_1 \to z^{\intercal}(\bar{\kappa})_+} \frac{d\mathbb{E}[U_L^1]}{dz_1} = V_1'\big(z^{\intercal}(\bar{\kappa})\big) < 0$$

because $V_1'(z_1) < 0$, $\forall z_1 \in (z^{\dagger}(\lambda), 0)$ and note that $z^{\intercal}(\bar{\kappa}) \in \big(z^{\dagger}(\lambda), 0\big)$.

Second, note that $V_2(z) < 0$ for $z < z^{\intercal}(\bar{\kappa})$, and thus

$$\lim_{z_1 \to z^{\intercal}(\bar{\kappa})_-} \frac{d\mathbb{E}[U_L^1]}{dz_1} = V_1'\big(z^{\intercal}(\bar{\kappa})\big) + V_2'\big(z^{\intercal}(\bar{\kappa})_-\big) < 0$$

Hence, both $\lim_{z_1 \to z^{\intercal}(\bar{\kappa})_+} \frac{d\mathbb{E}[U_L^1]}{dz} < 0$ and $\lim_{z_1 \to z^{\intercal}(\bar{\kappa})_-} \frac{d\mathbb{E}[U_L^1]}{dz} < 0$, and thus $z_1^* < z^{\intercal}(\bar{\kappa})$.

Moreover, by Assumption 1.5.1 (iii), $V_2'(z_1) > 0$ for all $z_1 < z^{\intercal}(\bar{\kappa})$, which implies that

$$V_1'(z^{\dagger}(\lambda)) + V_2'(z^{\dagger}(\lambda)) > V_1'(z^{\dagger}(\lambda)) = 0.$$

Thus, $z_1^* > z^{\dagger}(\lambda)$. To summarize, $z_1^* \in \big(z^{\dagger}(\lambda), z^{\intercal}(\bar{\kappa})\big)$.

The proof concludes. □

A.1.8 Proof of Lemma 1.5.5

Proof. First, by proposition 1.5.1, $z_1^*(\lambda) < 0$ exists for both $\lambda \in \{0, 1\}$. Second, note that $\mathbb{E}[U_L^1] = V_1(z_1)$, because for $\lambda = 0$, a media group never exists, whereas for $\lambda = 1$, by Proposition 1.5.2, $z_1^*(1) \in [z^{\intercal}(\bar{\kappa}), 0]$, and again a forced liquidation never occurs. This implies that, the optimal solution z_1^* is achieved either at the unconstrained optimizer $z^{\dagger}(\lambda)$, well defined by Eq. (1.11) under Assumption 1.5.1 (ii), or the maximum affordable short position $z^{\intercal}(\kappa)$, defined in Eq. (1.12). Thus, for $\lambda = 0$, $z_1^*(0) = \max\{z^{\intercal}(0), z^{\dagger}(0)\}$, and for $\lambda = 1$, $z_1^*(1) = \max\{z^{\intercal}(\bar{\kappa}), z^{\dagger}(1)\}$. The proof concludes. □

A.1.9 Proof of Lemma 1.5.6

Proof. First, we verify the monotonicity of $z^{\intercal}(\kappa)$. Recall that $z^{\intercal}(\kappa)$ is defined as

$$|z^{\intercal}| h_2(z^{\intercal}) \theta(z^{\intercal}) = \frac{W_L}{m h_1(\kappa)}$$

Hence, whenever $\kappa_1 > \kappa_2$, $h_1(\kappa_1) > h_1(\kappa_2)$ by monotonicity of $h_1(\cdot)$. Thus, the right side decreases. As a result, the left side needs to decrease. By Assumption 1.5.1 (i), $z^{\intercal}$ needs to increases (and $|\tilde{z}|$ decreases), which implies that $z^{\intercal}(\kappa_1) > z^{\intercal}(\kappa_2)$.

Second, we verify the monotonicity of $z^{\dagger}(\lambda)$. Recall that, $z^{\dagger}$ is the solution to Eq. (1.11), and we restate it here.

$$\left[h_2(z^{\dagger}) + z^{\dagger} h_2'(z^{\dagger})\right]\left(\left[\lambda h_1(\bar{\kappa}) + 1 - \lambda\right]\theta_L - \theta(z^{\dagger})\right) - \frac{z^{\dagger} h_2(z^{\dagger})}{I f\left(\frac{I - N + z^{\dagger}}{I}\right)} = 0$$

Rearranging it yields,

$$\theta(z^\dagger) - \big(\lambda h_1(\bar{\kappa}) + 1 - \lambda\big)\theta_L = \frac{-z^\dagger h_2(z^\dagger)}{If\left(\frac{I-N+z^\dagger}{I}\right)[h_2(z^\dagger) + z^\dagger h_2'(z^\dagger)]} =: V_0(z^\dagger).$$

or equivalently,

$$\theta(z^\dagger) - V_0(z^\dagger) = \big(\lambda h_1(\bar{\kappa}) + 1 - \lambda\big)\theta_L.$$

Note that $\theta(z)$ is strictly increasing in z and $V_0(z)$ is decreasing in z by Assumption 1.5.2, and thus the left side is increasing in $z^\dagger$. Thus, if λ increases (note that $h_1(\bar{\kappa}) > h_1(0) = 1$), the left side needs to increase, which leads to an increase in $z^\dagger$. The proof concludes. □

A.1.10 Proof of Corollary 1.5.1

Proof. i) By lemma 1.5.5, $z_1^*(0) = z^\dagger(0) \vee z^\intercal(0)$. Thus, by Lemma 1.5.6, both $z^\dagger(\lambda)$ and $z^\intercal(\kappa)$ achieve the minimum at $\lambda = 0$ and $\kappa = 0$. The proof for the case of $\lambda = 1$ is identical and thus omitted.

ii) Note that, if $W_L \geq m|z^\dagger(0)|s\big(0, z^\dagger(0)\big)|$, the budget set is never binding. Hence, $z_1^* = z^\dagger(\lambda)$. Now, it follows from the monotonicity in Lemma 1.5.6. □

A.1.11 Proof of Proposition 1.5.3

Proof. First, we calculate $V_1'\big(z^\intercal(\bar{\kappa})\big) + V_2'\big(z^\intercal(\bar{\kappa})_-\big)$.

- The derivative $V_1'\big(z^{\intercal}(\bar{\kappa})\big)$ can be obtained by plugging $z_1 = z^{\intercal}(\bar{\kappa})$ into

$$V_1'(z_1) = \underbrace{\big[z_1 h_2'(z_1) + h_2(z_1)\big]}_{M_1>0} \times \underbrace{\Big(\big[\lambda h_1(\bar{\kappa}) + 1 - \lambda\big]\theta_L - \theta(z_1)\Big)}_{M_2<0} \underbrace{- z_1 h_2(z_1)\frac{d\theta(z_1)}{dz_1}}_{M_3>0}$$

Note that, the concavity of $V_1(z_1)$, Assumption 1.5.1(ii), implies that $V_1'(z_1)$ is strictly decreasing. Also, by the definition of $z^{\dagger}(\lambda)$, we have $V_1'(z^{\dagger}) = 0$ and thus $V_1'\big(z^{\intercal}(\bar{\kappa})\big) < 0$ because $z^{\dagger}(\lambda) < z^{\intercal}(\bar{\kappa}) < 0$. Now, we evaluate these terms at $z_1 = z^{\intercal}(\bar{\kappa})$. i) The facts that $z_1 < 0$ and that $\frac{d\theta(z_1)}{dz_1} > 0$ imply that $M_3 > 0$. ii) The fact that the large trader is short selling implies that $M_2 < 0$, otherwise $z_1 = 0$ is preferable for the large trader. iii) Now, the aforementioned fact that $V_1'\big(z^{\intercal}(\bar{\kappa})\big) < 0$ implies that $M_1 > 0$.

- The derivative for $V_2'\big(z^{\intercal}(\bar{\kappa})_-\big)$ is a little bit involved. By Eq. (1.13),

$$\begin{aligned}
V_2'(z_1) &= \frac{d}{dz_1}\Big\{\lambda\big(\theta_L - \theta(z_3)\big)(z_3 - z_1)h_1(\bar{\kappa})h_2(z_3) + \lambda\theta_L z_1 h_1(\bar{\kappa})\Big(h_2(z_3) - h_2(z_1)\Big)\Big\} \\
&= \lambda h_1(\bar{\kappa})\Bigg\{(z_3 - z_1) \times \frac{d}{dz_1}\Big[\big(\theta_L - \theta(z_3)\big)h_2(z_3)\Big] + \big(\theta_L - \theta(z_3)\big)h_2(z_3) \\
&\quad \times \left(\frac{dz_3}{dz_1} - 1\right) + \theta_L\Big(h_2(z_3) - h_2(z_1)\Big) + z_1\theta_L\Big[h_2'(z_3)\frac{dz_3}{dz_1} - h_2'(z_1)\Big]\Bigg\}
\end{aligned}$$

By the fact that $z_3(z^{\intercal}(\bar{\kappa})) = z^{\intercal}(\bar{\kappa})$,

$$\begin{aligned}
V_2'\big(z^{\intercal}(\bar{\kappa})_-\big) = \lambda h_1(\bar{\kappa}) &\underbrace{\Big(1 - z_3'\big(z^{\intercal}(\bar{\kappa})\big)\Big)}_{M_4>0} \\
&\times \underbrace{\Big\{\Big(\theta\big(z^{\intercal}(\bar{\kappa})\big) - \theta_L\Big)h_2\big(z^{\intercal}(\bar{\kappa})\big) - z_1\theta_L h_2'\big(z^{\intercal}(\bar{\kappa})\big)\Big\}}_{M_5>0}
\end{aligned}$$

By lemma 1.5.2, $V_2'\big(z^{\intercal}(\bar{\kappa})_-\big) > 0$, and thus $M_4 > 0$ and $z_3'\big(z^{\intercal}(\bar{\kappa})\big) < 1$.

To summarize, M_1, M_3, M_4 and M_5 are all positive. Note that $h_1(\bar{\kappa}) \geq 1$. Hence, $V_1'\big(z^{\intercal}(\bar{\kappa})\big) + V_2'\big(z^{\intercal}(\bar{\kappa})_-\big)$ is both continuous and strictly increasing in λ.

Second, we come to check the claim in Proposition 1.5.3. There are three cases to consider.

i) When $\lambda = 0$, this is the case with the no media market structure, and thus $z^*(0) = z_1^* = z^\dagger(0) \vee z^\intercal(0)$. Note that this case differs from the other cases $\lambda \in (0,1]$ because media frenzy never occurs. Note that $z^\intercal(\bar{\kappa})$ is irrelevant when $\lambda = 0$.

ii) When $\lambda \in (0,1)$, note that $z^\dagger(\lambda) < z^\dagger(1) < z^\intercal(\bar{\kappa})$ by the monotonicity of $z^\dagger(\lambda)$ (see Lemma 1.5.6). By the fact that $V_1'(z_1)$ is decreasing (due to the concavity of $V_1(z)$), $V_1'\big(z^\intercal(\bar{\kappa})\big) < 0$. Thus,

$$\lim_{\lambda \to 0} V_1'\big(z^\intercal(\bar{\kappa})\big) + V_2'\big(z^\intercal(\bar{\kappa})_-\big) = \lim_{\lambda \to 0} V_1'\big(z^\intercal(\bar{\kappa})\big) < 0.$$

iii) When $\lambda = 1$ (i.e., the sure media group case), $z_1^*(1) = z^\dagger(1) \vee z^\intercal(\bar{\kappa}) = z^\intercal(\bar{\kappa})$ because $z^\dagger(1) < z^\intercal(\bar{\kappa})$. This implies

$$\lim_{\lambda \to 1} V_1'\big(z^\intercal(\bar{\kappa})\big) + V_2'\big(z^\intercal(\bar{\kappa})_-\big) \geq 0.$$

Hence, by the fact that $V_1'\big(z^\intercal(\bar{\kappa})\big) + V_2'\big(z^\intercal(\bar{\kappa})_-\big)$ is continuous and strictly increasing in λ, there exists $\bar{\lambda} \in (0,1]$ such that when $\lambda \geq \bar{\lambda}$, it always features strong deterrence (i.e., $V_1'(z^\intercal(\bar{\kappa})) + V_2'\big(z^\intercal(\bar{\kappa})_-\big) \geq 0$), and when $0 < \lambda < \bar{\lambda}$, it always features weak deterrence (i.e., $V_1'(z^\intercal(\bar{\kappa})) + V_2'\big(z^\intercal(\bar{\kappa})_-\big) < 0$). The proof concludes. □

A.1.12 Proof of Proposition 1.5.4

Proof. First, note that $\theta(z)$ is strictly increasing in z, because

$$\frac{d\theta(z)}{dz} = \frac{1}{If\left(\frac{I-N+z}{I}\right)} > 0.$$

Second, note that $z_1^*(0) = z^\dagger(0) \vee z^\intercal(0)$ (see Lemma 1.5.5) and that $z^\dagger(\lambda) \vee z^\intercal(0) < z_1^*(\lambda) \le 0$ (see Lemma 1.5.4). By Lemma 1.5.6, the monotonicity of $z^\dagger(\lambda)$, we have

$$z_1^*(0) < z_1^*(\lambda) < 0$$

which, together with the monotonicity of $\theta(z)$, implies

$$\theta(z_1^*(0)) < \theta(z_1^*(\lambda)) \le \theta(0).$$

Third, by the fact that the large trader is shorting the stock, we have $\theta(z_1^*(0)) \ge \theta_L$. Thus, if $p < \theta_L < \underline{\theta}(0)$, then we have

$$p < \theta_L < \theta(z_1^*(0)) < \theta(z_1^*(\lambda)) \le \theta(0).$$

which implies

$$L(p, \theta(z_1^*(0))) \ge L\Big(p, \theta\big(z_1^*(\lambda)\big)\Big).$$

This proves the case for bad discipline (i.e., efficiency reducing) and the proof for the other case is identical and thus omitted. The proof concludes. □

A.1.13 Proof of Lemma 1.6.1

Proof. The proof consists of three parts.

i. When there is no media coalition, there exists no anger expressing utility and thus it is suboptimal to exert effort (i.e., $e_i^* = 0$).

ii. When there is media coalition, there are two cases.

(a) The ith trader does not hold stocks. Since there is no expressive utility, it is suboptimal to exert effort, which leads to a negative utility $-\eta < 0$.

(b) The ith trader hold a positive amount of stocks. Given all other traders with stocks supporting the proposal (i.e., $\theta \geq \theta(z)$, the small investors with beliefs more optimistic than the marginal trader before the voting stage),[1] it is optimal to support since $(u - \eta) \geq 0$ and

$$\kappa = \int_{i \in I} e_j^* dF(\theta_j) = I\Big(1 - F\big(\theta(z)\big)\Big)\bar{e} \geq \kappa^*(z)$$

Note that if we allow for trading the instant before voting, the stock price needs to increase by the amount $(u - \eta)$ at that instant to prevent retrading, and then falls back by the same amount after voting since the expressive utility is non-transferable.

iii. The derivative can be computed as

$$\frac{d\kappa^*(z)}{dz} = -If(\theta(z))\theta'(z) = -\frac{f(\theta(z))}{If\big((I - N + z)/I\big)} < 0.$$

The proof concludes. □

[1]To eliminate the incentive of speculation, we assume that the small traders do not anticipate the coalition unless it is formed. This removes the incentive for speculation-based trading.

APPENDIX B

APPENDIX OF CHAPTER 2

B.1 Relevant Proofs and Calculations

B.1.1 Proof of Theorem 2.5.1

Proof. The objective function is

$$U_L(x) := x\mathbb{E}_{\mathbb{Q}}[\xi] + z\mathbb{E}_{\mathbb{Q}}\left[C_1\Big(\xi + w\big(S_0(z,x) - S_0(z,0)\big)\Big)\right] + W_L - xS_0(z,x) - zC_0.$$

(i) Under EWAP, $w = 0$. Taking the first derivative of the objective function with respect to x gives

$$\frac{dU_L(x)}{dx} = \mathbb{E}_{\mathbb{Q}}[\xi] - S_0(z,x) - x\frac{\partial S_0(z,x)}{\partial x}.$$

By equation (2.11), $S_0(z,x) = \underline{\xi} + (\bar{\xi} - \underline{\xi})G\left(\frac{z\Delta + x + \mathcal{I} - N}{\mathcal{I}}\right)$. This implies

$$\frac{\partial S_0(z,x)}{\partial x} = \frac{\bar{\xi} - \underline{\xi}}{\mathcal{I}f\left(\frac{z\Delta + x + \mathcal{I} - N}{\mathcal{I}}\right)} > 0$$

and thus

$$S_0(z,x) > S_0(z,0) = \mathbb{E}_{\mathbb{Q}}[\xi].$$

We used equation (2.5) in the last equality. Hence, $\frac{dU_L(x)}{dx} < 0$ and thus $x^{\star}_{EW}(z) = 0$.

(ii) Under VWAP, $w = \frac{x}{N}$.

$$\begin{aligned}
\frac{dU_L(x)}{dx} &= \mathbb{E}_{\mathbb{Q}}[\xi] - S_0(z,x) - x\frac{\partial S_0(z,x)}{\partial x} \\
&+ z\mathbb{E}_{\mathbb{Q}}\Big[C_1'\Big(\xi + w\big[S_0(z,x) - S_0(z,0)\big]\Big)\Big] \times D_1 \\
\frac{d^2U_L(x)}{dx^2} &= -2\frac{\partial S_0(z,x)}{\partial x} - x\frac{\partial^2 S_0(z,x)}{\partial x^2} \\
&+ z\mathbb{E}_{\mathbb{Q}}\Big[C_1''\Big(\xi + w\big[S_0(z,x) - S_0(z,0)\big]\Big)\Big] \times D_1^2 \\
&+ z\mathbb{E}_{\mathbb{Q}}\Big[C_1'\Big(\xi + w\big[S_0(z,x) - S_0(z,0)\big]\Big)\Big] \times D_2
\end{aligned}$$

where

$$\begin{aligned}
D_1 &= \left(\frac{1}{N}\big[S_0(z,x) - S_0(z,0)\big] + \frac{x}{N}\frac{\partial S_0(z,x)}{\partial x}\right), \\
D_2 &= \frac{2}{N}\frac{\partial S_0(z,x)}{\partial x} + \frac{x}{N}\frac{\partial^2 S_0(z,x)}{\partial x^2}.
\end{aligned}$$

Hence,

$$\begin{aligned}
\left.\frac{dU_L(x)}{dx}\right|_{x=0} &= 0, \\
\left.\frac{d^2U_L(x)}{dx^2}\right|_{x=0} &= 2\left.\frac{\partial S_0(z,x)}{\partial x}\right|_{x=0}\left(\frac{z}{N}\mathbb{E}_{\mathbb{Q}}[C_1'(\xi)] - 1\right).
\end{aligned}$$

If $\frac{z}{N}\mathbb{E}_{\mathbb{Q}}[C'(\xi)] > 1$, then when $x \to 0$ we have both $\left.\frac{dU_L(x)}{dx}\right|_{x=0} = 0$ and $\left.\frac{d^2U_L(x)}{dx^2}\right|_{x=0} > 0$. Thus, $x^{\star}_{VW} > 0$. □

B.1.2 Proof of Proposition 2.6.1

Proof. We verify that $z^{\dagger} = \hat{z}$ constitutes an equilibrium according to definition 2.6.1.

i) Under EWAP, $w = 0$. Given $z^{\dagger} = \hat{z}$, by the first part of Theorem 2.5.1, $x^{\star}_{EW} = 0$. Moreover, the derivative is always fully hedged and correctly priced (i.e., $\mathbb{E}_{\mathbb{Q}}[C_1(\xi)] = C_0$), and thus the large trader is indifferent between choosing $z = 0$ and $z = \hat{z}$. Last, by assumption, $\hat{z}C_0(\hat{z}) < W_L$ ensures that $\hat{z}$ is affordable. Hence, $z^{\star}(\{0, \hat{z}\}) = \hat{z}$ is an equilibrium.

ii) Under VWAP, $w = \frac{x}{N}$. Given $z^{\dagger} = \hat{z}$, by the second part of Theorem 2.5.1, there exists a profitable deviation such that $x > 0$. To see it, note that the assumption $\hat{z}C_0(\hat{z}) < W_L$ implies that $\hat{z}$ is affordable and the large trader's net wealth after derivative trading, $W_L - \hat{z}C_0(\hat{z}) > 0$, is positive. Moreover, the condition that $\frac{\hat{z}}{N}\mathbb{E}_{\mathbb{Q}}\left[C_1'(\xi)\right] > 1$ guarantees a profitable deviation.

Now, consider the problem in round 1. If the large trader chooses $z = 0$, he has no incentive to manipulate the asset price and the payoff is 0. If he chooses $z = \hat{z}$, he knows that he has a profitable index manipulation in round 2. Thus, $z = \hat{z}$ constitutes an equilibrium. The proof concludes. □

B.1.3 Proof of Proposition 2.6.2

Proof. For ease of reference, recall that

$$S_0(z,0) = G\left(z\Delta/N\right) = z\Delta/N, \quad S_0(z,x) = G\big((z\Delta + x)/N\big) = (z\Delta + x)/N, \tag{B.1}$$

$$\begin{aligned}
\mathbb{Q}(1|z) &= 1 - \mathbb{Q}(0|z) = S_0(z,0), \quad \Delta = C_1(1) - C_1(0) = a, \\
C_0(z) &= \mathbb{E}_{\mathbb{Q}}[C_1(\xi)] = (z\Delta/N)C_1(1) + (1 - z\Delta/N)C_1(0) = z\Delta^2/N \\
\mathbb{E}_{\mathbb{P}}[\xi] &= p, \quad \mathbb{E}_{\mathbb{P}}[C_1(\xi)] = pC_1(1) + (1-p)C_1(0) = pa = p\Delta.
\end{aligned}$$

(i) Under EWAP, $w = 0$.

In round 2, the large trader's objective can be simplified to

$$\begin{aligned} x^\star(z) &= \arg\max_{x\geq 0} x\big(\mathbb{E}_{\mathbb{P}}[\xi] - S_0(z,x)\big) + z\big(\mathbb{E}_{\mathbb{P}}[C_1(\xi)] - C_0(z)\big) + W_L \\ &= \arg\max_{x\geq 0} x \underbrace{\left[p - \frac{z\Delta + x}{N}\right]}_{M_1 \geq 0} + z(p - z\Delta/N)\Delta + W_L. \end{aligned}$$

Note that $p - z\Delta/N \geq 0$ (i.e., $z \leq pN/\Delta$) because otherwise the objective function in round 2 is smaller than W_L and investing only in the mma is optimal. Moreover, $M_1 \geq 0$ implies $x^\star \leq pN - z\Delta \leq pN$ and $S_0(z,x) \leq p$. Since $p - z\Delta/N \geq 0$,

$$C_0(z) \leq \mathbb{E}_{\mathbb{P}}[C_1(\xi)] = pC_1(1) + (1-p)C_1(0) = p\Delta.$$

This implies

$$xS_0(z,x) + zC_0(z) \leq p^2N + \frac{pN}{\Delta}p\Delta = 2p^2N \leq W_L.$$

Hence, under Assumption 2.6.1, the budget constraint is never binding and thus[1]

$$x^\star(z) = \begin{cases} \frac{p - z\Delta/N}{2/N}, & \text{if} \quad p - z\Delta/N > 0; \\ 0, & \text{otherwise}. \end{cases}$$

In round 1, by plugging in $x^\star(z)$, we get

$$\begin{aligned} z^\star &= \arg\max_{z\geq 0} x^\star(z)\left[p - \frac{z\Delta + x^\star(z)}{N}\right] + z(p - z\Delta/N)\Delta + W_L \\ &= \arg\max_{z\geq 0} \frac{N}{4}\left(p - \frac{z\Delta}{N}\right)^2 + z\Delta\left(p - \frac{z\Delta}{N}\right) + W_L. \end{aligned}$$

The optimal solution is given by $z^\star_{EW} = \frac{pN}{3\Delta}$ and thus $x^\star_{EW}(z^\star) = \frac{pN}{3}$.

(ii) Under VWAP, $w = \frac{x}{N}$.

[1] Note that $\frac{p - z\Delta/N}{2/N} \leq pN/2 < pN$.

In round 2, note that $S_0(z,x) - S_0(z,0) = \frac{x}{N}$ and the large trader's objective is

$$\begin{aligned}
x^\star_{VW}(z) &= \arg\max_{x\geq 0} x * \Big(\mathbb{E}_{\mathbb{P}}[\xi] - S_0(z,x)\Big) \\
&\quad + z * \Big\{\mathbb{E}_{\mathbb{P}}\left[C_1\big(\xi + w[S_0(z,x) - S_0(z,0)]\big)\right] - C_0(z)\Big\} + W_L \\
&= \arg\max_{x\geq 0} x\Big[p - \frac{z\Delta + x}{N}\Big] + z\left\{pa\left(1 + \frac{x^2}{N^2}\right) + (1-p)a\frac{x^2}{N^2} - \frac{z\Delta^2}{N}\right\} + W_L \\
&= \arg\max_{x\geq 0} x\Big[p - \frac{z\Delta + x}{N}\Big] + \frac{x^2 z\Delta}{N^2} + z(p - z\Delta/N)\Delta + W_L.
\end{aligned}$$

Here, we use $a = \Delta$.

First, note that $z^\star \leq \frac{pN}{\Delta}$. Otherwise, suppose that $p - z\Delta/N < 0$. Then, all the terms that have x are all negative and thus $x^\star = 0$,[2] which further implies that $z^\star = \frac{pN}{2\Delta} < \frac{pN}{\Delta}$ by investigating the term $z(p - z\Delta/N)\Delta$, which is a contradiction.

Second, we show that the budget constraint is never binding by construction. Note that $x \leq N$, $S_0(z,x) \leq 1$ and $C_0(z) = \mathbb{E}_{\mathbb{Q}}[C_1(\xi)] = (z\Delta/N)C_1(1) \leq pa = p\Delta$ by the fact that $z^\star \leq \frac{pN}{\Delta}$ shown above.

$$x * S_0(z,x) + z * C_0(z) \leq N + \frac{pN}{\Delta}p\Delta = (1+p^2)N \leq W_L.$$

Hence, under Assumption 2.6.1, the constraint is not binding and we can just optimize over $x \geq 0$ to get

$$x^\star_{VW}(z) = \begin{cases} \frac{p - z\Delta/N}{2/N - 2z\Delta/N^2}, & \text{if} \quad p - z\Delta/N > 0; \\ 0, & \text{else.} \end{cases} \tag{B.2}$$

[2] Collecting all terms containing x, we get

$$x(p - z\Delta/N) - x^2/N + (x^2/N) * (z\Delta/N)$$

which is always negative when $p < z\Delta/N$ because $z\Delta/N < 1$.

In round 1, by plugging $x^\star_{VW}$ and ignoring W_L, we get

$$\begin{aligned} z^\star_{VW} &= \arg\max_{z\geq 0}\left\{\frac{1}{4(\frac{1}{N}-\frac{z\Delta}{N^2})}\left(p-\frac{z\Delta}{N}\right)^2 + z\Delta(p-z\Delta/N)\right\} \\ &= \arg\max_{z\geq 0}\underbrace{\left\{\frac{N}{4}\left(p-\frac{z\Delta}{N}\right)^2 + z\Delta(p-z\Delta/N)\right\}}_{H(z)} + \underbrace{\frac{N}{4}\left(p-\frac{z\Delta}{N}\right)^2\left[\frac{1}{1-z\Delta/N}-1\right]}_{K(z)\geq 0}. \end{aligned}$$

Here, $H(z)$ is the same as the objective under EWAP, which is a well-defined quadratic function and the maximizer is $z^\star = \frac{pN}{3\Delta}$.

Now, consider the non-negative term $K(z)$. Note that

$$\begin{aligned} K'(z) &= \frac{N}{4}\left(p-\frac{z\Delta}{N}\right)\frac{\Delta}{N}\frac{1}{(1-z\Delta/N)^2}\left(p-\frac{3z\Delta}{N}+\frac{2z^2\Delta^2}{N^2}\right) \\ &\propto \left(p-\frac{3z\Delta}{N}+\frac{2z^2\Delta^2}{N^2}\right) =: L(z) \end{aligned}$$

because all terms in the first row except the last one are positive. Note that $L(z)$ is strictly decreasing over $[0, \frac{3N}{4\Delta}]$ and strictly increasing over $(\frac{3N}{4\Delta}, \infty)$.

$$L(z^\star_{EW}) > 0 \quad \text{and} \quad L\left(\frac{pN}{3\Delta}\frac{1}{(1-2p/3)}\right) = -3p^2(1-p)/(3-2p)^2 < 0.$$

Hence, $H'(z^\star_{EW}) + K'(z^\star_{EW}) > 0$ and $H'\left(\frac{pN}{3\Delta}\frac{1}{(1-2p/3)}\right) + K'\left(\frac{pN}{3\Delta}\frac{1}{(1-2p/3)}\right) < 0$, since $H'(z^\star_{EW}) = 0$ and $H'(z)$ is strictly decreasing for $z > z^\star_{EW}$. This implies that the optimal derivative holding

$$z^\star_{VW} \in \left(\frac{pN}{3\Delta}, \frac{pN}{3\Delta}\frac{1}{(1-2p/3)}\right).$$

Finally, note that $\frac{dx^\star_{VW}(z)}{dz} = -\frac{N^2\Delta(1-p)}{2(N-z\Delta)^2} < 0$. Thus

$$x^\star_{VW}(z^\star_{VW}) > x^\star_{VW}\left(\frac{pN}{3\Delta}\frac{1}{(1-2p/3)}\right) = \frac{pN}{3} = x^\star_{EW}.$$

Here, we use the formula for $x^\star_{VW}$ in equation (B.2). □

B.1.4 Proof of Lemma 2.6.1

Proof. By taking derivative over equation (2.14) w.r.t. z, we get

$$\begin{aligned}
\frac{\partial S_0(z,x)}{\partial x} &= \left[1+e^{\lambda(\bar{s}-\underline{s})(N-\Delta z)}(1-p)/p\right]^{-2}\lambda(\bar{s}-\underline{s})^2(1-p)/p>0;\\
\frac{\partial^2 S_0(z,x)}{\partial x^2} &= -2\left[1+e^{\lambda(\bar{s}-\underline{s})(N-\Delta z)}(1-p)/p\right]^{-3}\lambda^2(\bar{s}-\underline{s})^3(1-p)^2/p^2<0;\\
\frac{\partial^2 S_0(z,x)}{\partial z\partial x} &= -2\left[1+e^{\lambda(\bar{s}-\underline{s})(N-\Delta z)}(1-p)/p\right]^{-3}\lambda^2\Delta(\bar{s}-\underline{s})^3(1-p)^2/p^2<0.
\end{aligned}$$

□

B.1.5 Proof of Lemma 2.7.2

Proof. It suffices to verify that $\frac{dS_t(x)}{dx}>0$. Now, recall that

$$S_t(x)=\mathbb{E}_{\mathbb{P}}\left[\frac{U'(\xi(N-z\Delta_0-x))\xi}{\mathbb{E}_{\mathbb{P}}[U'(\xi(N-z\Delta_0-x))|\mathcal{F}_t]}\Big|\mathcal{F}_t\right].$$

Calculating the derivative of $S_t(x)$ with respect to x yields the result. □

B.1.6 Proof of Proposition 2.7.1

Proof. Note that, when equation (2.20) holds, we have $\frac{dS_{T^\star}(x)}{dx}>0$.

Denote $x=x_{T^\star}$. Recall that $x_{T^\star-}=0$.

(i) Under EWAP, $w=0$ and thus $I_{T^\star}=S_{T^\star-}(0)$. The large trader's problem, equation (2.19), can be simplified to

$$\begin{aligned} x^{\star}_{EW} &= \arg\max_{x \geq 0} \mathbb{E}_{\mathbb{Q}_0}[U_L \,|\mathcal{F}_{T^\star -}] \\ &= \arg\max_{x \geq 0} \Big\{ x \big[S_{T^\star -}(0) - S_{T^\star}(x) \big] + z \Big[C_{T^\star}\big(S_{T^\star -}(0) \big) - C_0 \Big] \Big\} \end{aligned}$$

which implies

$$\frac{d\mathbb{E}_{\mathbb{Q}_0}[U_L \,|\mathcal{F}_{T^\star}]}{dx} = S_{T^\star -}(0) - S_{T^\star}(x) - x \frac{dS_{T^\star}(x)}{dx} < 0.$$

Here, we use the fact that

$$S_{T^\star}(0_+) = \lim_{x \to 0, x > 0} S_{T^\star}(x) = S_{T^\star}(0) = S_{T^\star -}(0).$$

This implies that $S_{T^\star -}(0) < S_{T^\star}(x)$, $\forall x > 0$. Thus, we have $x^{\star}_{EW} = 0$.

(ii) Under VWAP, $w = \frac{x}{N}$ and

$$I_{T^\star} = S_{T^\star}(0) + \frac{x}{N} \big(S_{T^\star}(x) - S_{T^\star -}(0) \big).$$

Now, we can reformulate the large trader's problem as (suppressing the argument x)

$$\begin{aligned} x^{\star}_{VW} &= \arg\max_{x \geq 0} \mathbb{E}_{\mathbb{Q}_0}[U_L \,|\mathcal{F}_{T^\star}] \\ &= \arg\max_{x \geq 0} \ \Big\{ x \big(S_{T^\star -} - S_{T^\star} \big) + z \left(C_{T^\star} \left[S_{T^\star -} + \frac{x}{N} (S_{T^\star} - S_{T^\star -}) \right] \right) - C_0 \Big\} \end{aligned}$$

Thus,

$$\begin{aligned} \frac{d\mathbb{E}_{\mathbb{Q}_0}[U_L \,|\mathcal{F}_{T^\star}]}{dx} &= S_{T^\star -} - S_{T^\star} - x \frac{dS_{T^\star}}{dx} \\ &\quad + z C'_{T^\star}(I_{T^\star}) \left[\frac{x}{N} \frac{dS_{T^\star}}{dx} + \frac{1}{N} (S_{T^\star} - S_{T^\star -}) \right] \end{aligned}$$

and

$$\frac{d^2\mathbb{E}_{\mathbb{Q}_0}[U_L\,|\mathcal{F}_{T^\star}]}{dx^2} = -x\frac{d^2S_{T^\star}}{dx^2} - 2\frac{dS_{T^\star}}{dx} + zC''_{T^\star}(I_{T^\star})\left[\frac{x}{N}\frac{dS_{T^\star}}{dx} + \frac{1}{N}(S_{T^\star} - S_{T^\star-})\right]^2 + zC'_{T^\star}(I_{T^\star})\left[\frac{2}{N}\frac{dS_{T^\star}}{dx} + \frac{x}{N}\frac{d^2S_{T^\star}}{dx^2}\right]$$

Note that when $x \to 0$, $\left.\frac{d\mathbb{E}^{\mathbb{Q}_0}[U_L|\mathcal{F}_{T^\star}]}{dx}\right|_{x=0} = 0$, and when $\frac{zc'_{T^\star}(S_{T^\star-})}{N} > 1$,

$$\left.\frac{d^2\mathbb{E}_{\mathbb{Q}_0}[U_L\,|\mathcal{F}_{T^\star}]}{dx^2}\right|_{x=0} = 2\frac{dS_{T^\star}}{dx}\left(\frac{zC'_{T^\star}(S_{T^\star-})}{N} - 1\right) > 0.$$

Thus, $x^*_{VW} > 0$. □

APPENDIX C

APPENDIX OF CHAPTER 3

C.1 Relevant Proofs and Calculations

C.1.1 Proof of Lemma 3.3.1

Proof. Given the price $p \geq 0$, the demand function is the whole unit circle, or determined by the marginal consumer such that $y * \left(1 - \frac{c}{I}\|x_i - \theta\|\right) - p = 0$. In the latter case,

$$D(p) = 2(1 - p/y) * I/c$$

which implies that $\Pi = D(p) * p$ and thus simple optimization yields $p^* = \frac{y}{2}$.

Note that, in the former case, $\Pi = p * 2\pi$ and thus we can increase the price p until the most distant consumer (i.e., $\|x - \theta\| = \pi$) becomes marginal, that is, $y * (1 - \pi c/I) - p^* = 0$, or $p^* = y * (1 - \pi c/I)$.

Finally, note that the whole unit circle is served only when the effective influence power is strong enough (or the background technology is cheap enough). This implies that it is optimal to serve the whole unit when

$$y * (1 - \pi c/I) \geq \frac{y}{2}$$

Rearranging it yields, $\frac{I}{c} \geq 2\pi$. Now, we can get the revenue expression of Π by simple algebra. The proof concludes. □

C.1.2 Proof of Lemma 3.3.2

Proof. Recall that $\Pi_{\{1,2\}}$ is the revenue when both influencers are hired by the monopolist seller, ignoring the fixed searching cost. When $c > \frac{I_1+I_2}{2\pi}$, the seller sets prices $p_1^* = p_2^* = \frac{y}{2}$ and influencer j attracts consumers with $x \in \mathbf{S}^1$ such that $\|x - \theta_j\| \leq \frac{I_j}{2c}$ for $j = 1, 2$. Note that there exists no overlapping in consumers served since $\frac{I_1}{2c} + \frac{I_2}{2c} \leq \pi$. Thus, the monopoly pricing is feasible and thus $\Pi_{\{1,2\}} = \Pi_{\{1\}} + \Pi_{\{2\}} = \frac{y(I_1+I_2)}{2c}$.

Now, consider the case that $c < \frac{I_1+I_2}{2\pi}$. First, note that both influencers are active and serves a non-zero size of consumers. Otherwise suppose w.l.o.g. that influencer 2 is serving no consumer. Then, for any positive price p_1, we can always charge a slightly higher price for all consumers sufficiently close to θ_2. Now, suppose influencer 1 serves consumers $x \in \mathbf{S}^1$ such that $\|x - \theta_1\| \leq s_1$ with $s_1 \in [0, \pi)$. Prices p_1 and p_2 are set such that the marginal consumer (i.e., $x \in \mathbf{S}^1$ & $\|x - \theta_1\| = s_1$) gets zero surplus, otherwise it is profitable to increase the product price at least for one influencer. Hence,

$$p_1 = y\left(1 - \frac{cs_1}{I_1}\right), \text{ and } p_2 = y\left(1 - \frac{c*(\pi - s_1)}{I_2}\right)$$

Then, we can write down the total revenue as a function of s_1, that is,

$$\Pi_{\{1,2\}} = 2p_1 * s_1 + 2p_2 * (\pi - s_1).$$

Maximizing the joint revenue function, it yields

$$p_1^* = p_2^* = y * \left(1 - \frac{c\pi}{I_1 + I_2}\right), \tag{C.1}$$

and

$$s_1 = \frac{I_1 \pi}{I_1 + I_2}, \text{ and } s_2 = \frac{I_2 \pi}{I_1 + I_2}. \tag{C.2}$$

Simple algebra yields $\Pi_{\{1,2\}} = 2\pi y \left(1 - \frac{c\pi}{I_1+I_2}\right)$ when $c < \frac{I_1+I_2}{2\pi}$.

Finally, note that the condition $\frac{I_2}{I_1} \leq \frac{\pi y - \varepsilon}{\epsilon}$ holds if and only if $\frac{I_1+I_2}{2\pi} < c \leq \frac{yI_1}{2\varepsilon}$. The proof concludes. □

C.1.3 Proof of Lemma 3.3.3

Proof. First, note that it is always sub-optimal to hire influencer 2 alone, which is dominated by hiring influencer 1 alone. Furthermore, when both influencers are hired, the total profit is given by $\Pi_{\{1,2\}} - 2\varepsilon$. When only the strong influencer 1 is hired, the total profit is $\Pi_{\{1\}} - \varepsilon$. Hence, the joint maximal profit for the seller and two influencers is given by

$$W = \max\{\Pi_{\{1,2\}} - 2\varepsilon, \Pi_{\{1\}} - \varepsilon\}$$

Second, note that it is optimal to hire influencer 1 alone when $c \in \left(\frac{yI_2}{2\varepsilon}, \frac{yI_1}{2\varepsilon}\right)$. To see it, given that $\frac{\varepsilon}{y}(1 + \frac{I_1}{I_2}) < \pi$, the condition that $c > \frac{yI_2}{2\varepsilon}$ implies that $c > \frac{I_1+I_2}{2\pi}$, and thus there exists no overlapping between consumers served if both influencers are served, that is, $\Pi_{\{1,2\}} = \Pi_{\{1\}} + \Pi_{\{2\}}$. This further implies that

$$\Pi_{\{1,2\}} - 2\varepsilon = \Pi_{\{1\}} - \varepsilon + \Pi_{\{2\}} - \varepsilon < \Pi_{\{1\}} - \varepsilon$$

and thus only the strong influencer 1 is hired.

Third, for $c \in [\bar{c}, \frac{yI_2}{2\varepsilon}]$, there are three cases.

- $c \in \left(\frac{I_1+I_2}{2\pi}, \frac{yI_2}{2\varepsilon}\right]$.

$$\Pi_{\{1,2\}} = \frac{y(I_1+I_2)}{2c}, \quad \Pi_{\{1\}} = \frac{yI_1}{2c}, \quad \text{and} \quad \Pi_{\{2\}} = \frac{yI_2}{2c}$$

- $c \in \left(\frac{I_1}{2\pi}, \frac{I_1+I_2}{2\pi}\right]$.

$$\Pi_{\{1,2\}} = 2\pi y\left(1 - \frac{c\pi}{I_1+I_2}\right), \quad \Pi_{\{1\}} = \frac{yI_1}{2c}, \quad \text{and} \quad \Pi_{\{2\}} = \frac{yI_2}{2c}$$

- $c \in \left(\frac{I_2}{2\pi}, \frac{I_1}{2\pi}\right]$.

$$\Pi_{\{1,2\}} = 2\pi y\left(1 - \frac{c\pi}{I_1+I_2}\right), \quad \Pi_{\{1\}} = 2\pi y\left(1 - \frac{c\pi}{I_1}\right), \quad \text{and} \quad \Pi_{\{2\}} = \frac{yI_2}{2c}$$

- $c \in \left(\overline{c}, \frac{I_2}{2\pi}\right]$.

$$\Pi_{\{1,2\}} = 2\pi y\left(1 - \frac{c\pi}{I_1+I_2}\right), \; \Pi_{\{1\}} = 2\pi y\left(1 - \frac{c\pi}{I_1}\right), \; \text{and } \Pi_{\{2\}} = 2\pi y\left(1 - \frac{c\pi}{I_2}\right)$$

  We can directly verify that

$$\Pi_{\{1,2\}} - 2\varepsilon \geq \Pi_{\{1\}} - \varepsilon$$

Fourth, for $c \in (0, \underline{c})$,

$$\Pi_{\{1,2\}} = 2\pi y\left(1 - \frac{c\pi}{I_1+I_2}\right), \quad \text{and} \quad \Pi_{\{1\}} = 2\pi y\left(1 - \frac{c\pi}{I_1}\right)$$

When $c = \overline{c}$, we have $\Pi_{\{1,2\}} - 2\varepsilon = \Pi_{\{1\}} - \varepsilon$ and it is easy to check that $\Pi_{\{1,2\}} - 2\varepsilon < \Pi_{\{1\}} - \varepsilon$ for all $c < \overline{c}$. The proof concludes. □

C.1.4 Proof of Lemma 3.3.4

Proof. The proof consists of two parts. First, we show that hiring both influencers are sub-optimal for all $c \leq \overline{c}$. Second, we prove there exists over-hiring for the two specific cases.

Optimal hiring for $c \leq \bar{c}$. Note that for $c \leq \bar{c}$, $\frac{I_1}{c} \geq 2\pi$ and the total welfare W_1 is given by

$$W_1 = 2\pi p_1 + 2 * \frac{\pi}{2}(y - p_1) - \varepsilon$$

Here, $p_1 = y\left(1 - \frac{c}{I_1}\pi\right)$. The first term $2\pi p_1$ is the seller-influencer group's joint profit, the second term $2 * \frac{\pi}{2}(y - p_1)$ is the consumer surplus, and the third term is the fixed hiring cost. This can be further simplified as

$$W_1 = \pi y + \pi y \left(1 - \frac{c}{I_1}\pi\right) - \varepsilon$$

Similarly, when both influencers are hired, the monopolist price and consumers served satisfy Eq (C.1) and (C.2). Actually, the monopolist pricing strategy coincides with that maximizing total welfare. The total welfare $\hat{W}_{1,2}$ is given by

$$\hat{W}_{1,2} = \big(2s_1p_1 + s_1(y - p_1)\big) + \big(2s_2p_2 + s_2(y - p_2)\big) - 2\varepsilon$$

Here, the first big term is the total welfare for consumers served by influencer 1, the second one by influencer 2, and the last term is the fixed hiring cost. By plugging the expressions from Eq (C.1) and (C.2), we can simplify it as

$$\hat{W}_{1,2} = \pi y + \pi y \left(1 - \frac{c\pi}{I_1 + I_2}\right) - 2\varepsilon$$

Note that when $c \leq 2\bar{c}$, $W_1 \geq \hat{W}_{1,2}$, and thus it is sub-optimal to hire both influencers when $c \leq \bar{c}$.

Case i). Note that $\Pi_{\{1,2\}} - 2\varepsilon = \Pi_{\{1\}} - \varepsilon$ and that social optimal hiring decision requires only influencer 1 is hired for all $c < \bar{c}$. First, if the monopolist seller only hire influencer 1, the Nash bargaining implies

$$U_1 = \gamma(\Pi_{\{1\}} - \varepsilon), \quad w_1 = (1 - \gamma)(\Pi_{\{1\}} - \varepsilon), \quad \text{and} \quad w_2 = 0$$

In contrast, if the seller hires both influencer 1 and 2, the payoffs for the seller and influencers are given by

$$\hat{U}_1 = \gamma(\Pi_{\{1,2\}} - 2\varepsilon) + (1-\gamma)(\Pi_{\{2\}} - \varepsilon), \quad w_2 = 0,$$
$$\text{and} \quad w_1 = (1-\gamma)\big((\Pi_{\{1,2\}} - 2\varepsilon) - (\Pi_{\{2\}} - \varepsilon)\big) \tag{C.3}$$

By definition of $\overline{c}$, $\gamma(\Pi_{\{1\}} - \varepsilon) = \gamma(\Pi_{\{1,2\}} - 2\varepsilon)$, which implies that $U_1(\overline{c}) < \hat{U}_1(\overline{c})$. Furthermore, $U_1(c) < \hat{U}_1(c)$ for all $c \in (\overline{c} - \delta, c)$ for $\delta > 0$ sufficiently small. Thus, it sufficies to show the payoffs in Eq. (C.3).

Given two influencers are hired, the seller bargains with influencer 1 over the surplus $(\Pi_{\{1,2\}} - 2\varepsilon) - (\Pi_{\{2\}} - \varepsilon)$, the marginal profit increment between hiring both influencers and only hiring influencer 2. If influencer 1 accepts this, Nash bargaining implies a payoff w_1 defined in Eq. (C.3). Indeed, on the off-equilibrium path (i.e, influencer 1 rejects the offer), the seller and influencer 2 shares the profit $\Pi_{\{2\}} - \varepsilon$. Similarly, the seller bargains with influencer 2 over the (negative) surplus $(\Pi_{\{1,2\}} - 2\varepsilon) - (\Pi_{\{1\}} - \varepsilon) < 0$, the marginal profit gap between hiring both influencers and only hiring influencer 1. Due to individual rationality, influencer 2 only accepts $w_2 \geq 0$, and thus $w_2 = 0$. Indeed, on the off-equilibrium path, the seller and influencer 1 shares $\Pi_{\{1\}} - \varepsilon$.

Case ii). Define $\underline{c} = \frac{I_2}{I_1} * \frac{(I_1+I_2)\varepsilon}{2\pi^2 y}$, which satisfies that $\Pi_{\{1,2\}} - 2\varepsilon < \Pi_{\{1\}} - \varepsilon$ and that $\Pi_{\{1,2\}} - 2\varepsilon < \Pi_{\{2\}} - \varepsilon$. When the seller hires both influencer 1 and 2, the bargaining payoffs are given by:

$$U_1 = \Pi_{\{1,2\}} - 2\varepsilon, \quad w_1 = 0, \quad \text{and} \quad w_2 = 0$$

When the seller only hires influencer 1, the Nash bargaining outcome is

$$\hat{U}_1 = \gamma(\Pi_{\{1\}} - \varepsilon), \quad \text{and} \quad w_1 = (1-\gamma)(\Pi_{\{1\}} - \varepsilon)$$

Hence, it is optimal to hire both influencers when $U_1 \geq \hat{U}_1$, which implies that

$$\gamma \leq \frac{\Pi_{\{1,2\}} - 2\varepsilon}{\Pi_{\{1\}} - \varepsilon}$$

Thus, by definition of $\underline{\gamma}$, $U_1(c) \geq \hat{U}_1(c)$ for all $c < \underline{c}$ when $\gamma \geq \underline{\gamma}$.

In addition, note that

$$\frac{\Pi_{\{1,2\}} - 2\varepsilon}{\Pi_{\{1\}} - \varepsilon} > \frac{\Pi_{\{1\}} - 2\varepsilon}{\Pi_{\{1\}} - \varepsilon} \geq \frac{\Pi_{\{1\}}(\underline{c}) - 2\varepsilon}{\Pi_{\{1\}}(\underline{c}) - \varepsilon}$$

Here, we use the fact that $\Pi_{\{1,2\}} > \Pi_{\{1\}}$, and that the term $\frac{\Pi_{\{1\}}-2\varepsilon}{\Pi_{\{1\}}-\varepsilon}$ is strictly decreasing in c. Hence, we have $\underline{\gamma} > \frac{\Pi_{\{1\}}(\underline{c})-2\varepsilon}{\Pi_{\{1\}}(\underline{c})-\varepsilon}$. The proof concludes. □

C.1.5 Proof of Proposition 3.3.1

Proof. i) It follows directly from the observation that when $\gamma = 1$, the optimal hiring decision coincides with the social optimal market structure, which is monotone. Hence, by continuity, the argument goes through for sufficiently large $\gamma \to 1$.

- First, for $c > \frac{yI_2}{2\varepsilon}$, it is optimal to only hire influencer 1. This is because hiring influencer 2 generates a negative profit, which cannot help the seller bargain with influencer 1. Thus,

$$\hat{U}_1 = \gamma(\Pi_{\{1\}} - \varepsilon), \quad w_1 = (1-\gamma)(\Pi_{\{1\}} - \varepsilon), \quad \text{and} \quad w_2 = 0.$$

- Second, for $\bar{c} \leq c \leq \frac{yI_2}{2\varepsilon}$, it is optimal to hire both influencers. To see it, consider the case that the seller hires both influencers. In this case, the seller only shares with influencer 1 the surplus $(\Pi_{\{1,2\}} - 2\varepsilon) - (\Pi_{\{2\}} - \varepsilon)$ by the same argument in the proof of Lemma 3.3.4 because influencer 2 is also hired. Similarly, the seller only shares with influencer 2 the surplus $(\Pi_{\{1,2\}} - 2\varepsilon) - (\Pi_{\{1\}} - \varepsilon)$. Thus,

  $w_1 = (1-\gamma)\big((\Pi_{\{1,2\}}-2\varepsilon)-(\Pi_{\{2\}}-\varepsilon)\big)$, and $w_2 = (1-\gamma)\big((\Pi_{\{1,2\}}-2\varepsilon)-(\Pi_{\{1\}}-\varepsilon)\big)$

  and

  $$\begin{aligned} U_1 &= \Pi_{\{1,2\}} - 2\varepsilon - w_1 - w_2 \\ &= \gamma(\Pi_{\{1,2\}} - 2\varepsilon) + (1-\gamma)(\Pi_{\{1\}} + \Pi_{\{2\}} - \Pi_{\{1,2\}}) \end{aligned}$$

  In contrast, if the seller only hires influencer 1, she gets

  $$\hat{U}_1 = \gamma(\Pi_{\{1\}} - \varepsilon)$$

  By definition of $\bar{c}$, $\Pi_{\{1,2\}} - 2\varepsilon = \Pi_{\{1\}} - \varepsilon$ for all $\bar{c} \leq c \leq \frac{yI_2}{2\varepsilon}$. This further implies that $U_1 \geq \hat{U}_1$, and thus it is optimal for the seller to hire both influencers.

- Third, for $\underline{c} < c < \bar{c}$, we have both

  $$\Pi_{\{1,2\}} - 2\varepsilon < \Pi_{\{1\}} - \varepsilon \quad \text{and} \quad \Pi_{\{1,2\}} - 2\varepsilon > \Pi_{\{2\}} - \varepsilon.$$

  In this case, when the seller hires both influencers, the payoffs are given by

  $$w_1 = (1-\gamma)\big((\Pi_{\{1,2\}} - 2\varepsilon) - (\Pi_{\{2\}} - \varepsilon)\big), \text{ and } w_2 = 0$$

and thus

$$U_1 = \gamma(\Pi_{\{1,2\}} - 2\varepsilon) + (1-\gamma)(\Pi_{\{2\}} - \varepsilon)$$

The payoff from hiring only influencer 1 is unchanged and given by $\hat{U}_1 = \gamma(\Pi_{\{1\}} - \varepsilon)$.

- Fourth, for $c \leq \underline{c}$, we have both

$$\Pi_{\{1,2\}} - 2\varepsilon < \Pi_{\{1\}} - \varepsilon \quad \text{and} \quad \Pi_{\{1,2\}} - 2\varepsilon \leq \Pi_{\{2\}} - \varepsilon.$$

By the residual surplus argument, it implies

$$U_1 = \Pi_{\{1,2\}} - 2\varepsilon, \quad w_1 = 0, \quad \text{and} \quad w_2 = 0.$$

To establish non-monotonicity, we just need to ensure the seller only hires influencer 1 for $\underline{c} < c < \overline{c}$, which requires that $U_1 < \hat{U}_1$, or equivalently

$$\Pi_{\{1,2\}} - 2\varepsilon < \gamma(\Pi_{\{1\}} - \varepsilon)$$

which reduces to

$$\gamma > \frac{\Pi_{\{1,2\}} - 2\varepsilon}{\Pi_{\{1\}} - \varepsilon}$$

Hence, if we take $\gamma \geq \sup_{c<\overline{c}} \left\{ \frac{\Pi_{\{1,2\}} - 2\varepsilon}{\Pi_{\{1\}} - \varepsilon} \right\} = \frac{\Pi_{\{1,2\}}(\overline{c}) - 2\varepsilon}{\Pi_{\{1\}}(\overline{c}) - \varepsilon} \in (0,1)$, then it is optimal to hire only influencer 1. The proof for part i) concludes. ■

ii) First, note that both U_1 and $\hat{U}_1$, the seller's payoff when both influencers are hired and only the strong influencer 1 is hired respectively, are strictly decreasing and continuous in c. Thus, the seller's payoff, $\max\{U_1, \hat{U}_1\}$, corresponds to the upper envelope of U_1 and $\hat{U}_1$, is also strictly decreasing in c and continuous. The proof for part ii) concludes. ■

iii) This directly follows from Lemma 3.3.4. Specifically, for $\gamma \leq \underline{\gamma}$, both influencers are hired for all $c \leq \underline{c}$. In this case, we have

$$w_1 = w_2 = 0, \quad \text{and} \quad w_1 - w_2 = 0.$$

Furthermore, for $\overline{c} < c < \frac{yI_2}{2\varepsilon}$, only the strong influencer 1 is hired. Thus,

$$w_1 = (1-\gamma)(\Pi_{\{1\}} - \varepsilon), \quad w_2 = 0, \quad \text{and} \quad w_1 - w_2 = (1-\gamma)(\Pi_{\{1\}} - \varepsilon).$$

Since $\Pi_{\{1\}}$ is strictly decreasing in c, we have both total wages for all influencers and the income gap between influencers are non-monotonic in the background technology cost parameter. The proof concludes for part iii) and iv). □

C.1.6 Proof of Corollary 3.3.1

Proof. Note that from Lemma 3.7.1, $\Pi_{\{1,2\}}$ is monotone in β. Thus, by Eq. (3.7), when $\gamma > \frac{1}{2}$, U_1 is strictly increasing in $\Pi_{\{1,2\}}$, and thus it achieves maximum for any $\beta \geq I$. The other two cases can be shown similarly. □

C.1.7 Proof of Lemma 3.3.5

Proof. First, note that it is always sub-optimal not to hire any influencer. Now, suppose only a subset $S \subsetneq \{1, \cdots, J\}$ of influencers are hired by the seller.

Consider one more influencer $a \notin S$ is hired. Then, $w_a = (1-\gamma)\left(\Pi_{S\cup\{a\}} - \Pi_S\right)$ and thus

$$\sum_{j\in S\cup\{a\}} w_j + U_1 = \Pi_S + \gamma\left(\Pi_{S\cup\{a\}} - \Pi_{\{S\}}\right) \geq \Pi_S \tag{C.4}$$

Furthermore, for $j \in S$,

$$w_j = (1-\gamma)(\Pi_{S\cup\{a\}} - \Pi_{(S\cup\{a\})/\{j\}}) \leq (1-\gamma)(\Pi_S - \Pi_{S/\{j\}}) \tag{C.5}$$

where the inequality follows from Property (S).

Thus, Eq. (C.4) and Eq. (C.5) jointly implies that the seller's payoff increases when influencer $a \notin S$ is hired. Since S is arbitrary, the seller's payoff is maximized only when $S = \{1, \cdots, J\}$. Meanwhile, note that each influencer's payoff decreases when an additional influencer is hired by Eq. (C.5). The proof concludes. □

C.1.8 Proof of Lemma 3.3.6

Proof. First, note that claim i) is straightforward. When $\varepsilon = 0$, the joint revenue is weakly increasing in the number of influencers hired. Furthermore, if there are at least two influencers hired for an influencer type, then all influencers of this type receives a zero wage because the incremental change in revenue is zero. Hence, the seller gets the maximum revenue by hiring a duplicate influencer for each influencer type.

Second, we come to show claim ii). From claim i), when the style type space is discrete and finite (i.e., $|\Theta| = N < \infty$ and denote all type elements by θ_l with

$l = 1, \cdots, N$), the seller gets all revenue when there exists at least two influencers for each type. Now, we show that, when influencers' type $\theta_j \in \Theta$ are i.i.d. drawn, with probability approaching one, this is indeed the case. Specifically, consider $J = 2n * N$. For the first $2N$ influencers, the probability of each influencer type gets 2 draws is bounded below by

$$\delta := \left(\prod_{l=1}^{N} \mathsf{Prob}(\theta = \theta_l) \right)^2$$

and thus the probability of getting 2 draws for all θ_l with $l \in \{1, \cdots, N\}$ is no less than

$$1 - (1 - \delta)^n \to 1$$

as $n \to \infty$.

Then, we can extend the analysis to the case in which influencers types θ_j are independently drawn from a continuous space $\mathbf{S}^1$ with a density function $f(\cdot)$ almost everywhere positive. Now, we can discrete the continuous style type space by choosing N style types equally distributed on the circle $\mathbf{S}^1$ so that the distance between any two successive types is exactly $1/N$. When N is sufficiently large, all types $\|\theta - \theta_j\| \leq \frac{1}{2N}$ yields approximately the same outcome as that of θ_j and the seller prices the product very close to y. Thus,

$$\mathsf{Prob}(\theta \approx \theta_j) = \int_{u \in \{s \in \mathbf{S}^1 : \|s - \theta_j\| \leq \frac{1}{2N}\}} f(s) ds.$$

Now, by the reasoning in the discrete case, the seller gets approximately all the total profits when $J \geq 2n * N$ as n is sufficiently large. Note that, the total welfare is also maximized.

□

C.1.9 Proof of Lemma 3.3.7

Proof. Suppose not. Then, there exist three neighboring influencers hired such that $\|\theta_{j-1}-\theta_j\| \neq \|\theta_j-\theta_{j+1}\|$. For simplicity, assume that $\|\theta_{j-1}-\theta_j\| < \|\theta_j-\theta_{j+1}\|$. There are two cases to consider.

Case i). There exists some consumers not served along the arc between θ_{j-1} and θ_{j+1}. Without loss of generality, we assume there exists consumers not served between θ_j and θ_{j+1}. Then, we can keep (p_{j-1}, p_j, p_{j+1}) unchanged, and shift θ_j to $\theta_j+\delta$ for a sufficiently small $\delta > 0$. This weakly increases the total revenue Π_J, because it weakly increases the size of consumers served by influencer $j-1$.

Case ii). All consumers are served along the arc between θ_{j-1} and θ_{j+1}. For ease of notation, define $x_{i,i+1} \in \mathbf{S}^1$ the cutoff type indifferent between purchasing from influencer i and from influencer $(i+1)$.

Lemma C.1.1. *The cutoff consumer type indifferent between following two neighboring influencers receive a zero utility.*

Proof. If not, suppose the cutoff type consumer $x_{j-1,j}$ receives a positive utility. Note that it cannot be the case that the consumer $x_{j,j+1}$ also receives a positive utility. Otherwise, we can increase p_j by a small amount without losing any consumers, which leads to a large total revenue. This implies that consumer $x_{j,j+1}$ receives a zero utility.

Now, we construct a hiring plan and a price scheme which generates more revenue. Consider the case that $p_{j-1} \geq p_j$ and the other case $p_{j-1} < p_j$ can be proved similarly. We shift θ_j to $\hat{\theta}_j = \theta_j + \delta$ and increase p_j to $\hat{p}_j = p_j + \frac{y}{I}\delta$ where $\delta > 0$ is small. Under the new hiring and pricing scheme, the cutoff consumer type $x_{j,j+1}$ remains unchanged, and the cutoff type $x_{j-1,j}$ shifts to $\hat{x}_{j-1,j} = x_{j-1,j} + \delta$. We can choose a sufficiently small $\delta > 0$ to ensure the consumer $\hat{x}_{j-1,j}$ still receives a positive utility. Now, all consumers between $x_{j-1,j}$ and $x_{j,j+1}$ either pay $p_{j-1} \geq p_j$ or $\hat{p}_j > p_j$. □

Lemma C.1.1, combined with the condition that $\|\theta_{j-1} - \theta_j\| < \|\theta_j - \theta_{j+1}\|$, implies that $p_{j-1} > p_{j+1}$ because influencer $(j+1)$ needs to serve more consumers than influencer $(j-1)$ to ensure both consumers $x_{j-1,j}$ and $x_{j,j+1}$ receive a zero utility. Denote by a_{j-1} and a_{j+1} the size of consumers served by influencer $(j-1)$ and $(j+1)$, respectively. Since $p_{j-1} > p_{j+1}$, $a_{j-1} < a_{j+1}$ by Lemma C.1.1.

However, the fact that $p_{j-1} > p_{j+1}$ implies there exists a price scheme more profitable. To see it, consider the new price scheme

$$(\hat{p}_{j-1}, \hat{p}_j, \hat{p}_{j+1}) = \left(p_{j-1} - \frac{y\delta}{I}, p_j, p_{j+1} + \frac{y\delta}{I}\right)$$

and

$$(\hat{\theta}_{j-1}, \hat{\theta}_j, \hat{\theta}_{j+1}) = \left(\theta_{j-1} + \frac{\delta}{2}, \theta_j + \delta, \theta_{j+1} - \frac{\delta}{2}\right)$$

Note that $\hat{a}_{j-1} = a_{j-1} + \delta$ and $\hat{a}_{j+1} = a_{j+1} - \delta$, and the size of consumers served other influencers remains unchanged, including influencer j. The total revenue change is given by

$$\begin{aligned}\Delta &= \hat{p}_{j-1}\hat{a}_{j-1} + \hat{p}_{j+1}\hat{b}_{j+1} - p_{j-1}a_{j-1} - p_{j+1}b_{j+1} \\ &= \left(p_{j-1} - \frac{y\delta}{I}\right) * (a_{j-1} + \delta) + \left(p_{j+1} + \frac{y\delta}{I}\right) * (b_{j+1} - \delta) - p_{j-1}a_{j-1} - p_{j+1}b_{j+1} \\ &= \frac{y\delta}{I}(b_{j+1} - a_{j+1}) + (p_{j-1} - p_{j+1})\delta + O(\delta^2)\end{aligned}$$

By the fact that $p_{j-1} > p_{j+1}$ and $a_{j-1} < a_{j+1}$, this generates a higher total revenue for a sufficiently small $\delta > 0$. This is a contradiction! Hence, it cannot be the case that all consumers are served but influencers are not equally distanced. The proof concludes. □

C.1.10 Proof of Lemma 3.3.8

Proof. Consider the first case that $J = \bar{J} * m$. Obviously, the seller can $\bar{J}$ equally-distance influencers and get the desired payoff $U_1 = \bar{\Pi}_{\bar{J}} - \bar{J} * w_j$. Indeed, this is optimal for the seller, given the residual bargaining protocol. Consider any alternative plan of hiring $n \in \mathbb{N}$ influencers ,

$$\bar{\Pi}_{\bar{J}} - \bar{J} * w_1^{\bar{J}} > \bar{\Pi}_n - \bar{n} * w_1^{\bar{n}}.$$

Note that when $\gamma = 1$, the above inequality trivially holds, by the definition of $\bar{J}$. By continuity, it also holds for all $\gamma \geq \gamma_n$ where $\gamma_n < 1$. Define $\bar{\gamma}(\varepsilon) = \sup_{1 \leq n \leq [2\pi/\varepsilon]} \gamma_n$. Thus, for all $\gamma \in [\bar{\gamma}(\varepsilon), 1]$, $J^* = \bar{J}$.

Now, consider the second case that $J \to \infty$. Fix $\delta > 0$. We can find J sufficiently large such that there exists a group of influencers $\theta_j (j \in \{1, \cdots, \bar{J}\})$

such that $\left|\|\theta_j - \theta_{j+1}\| - \frac{2\pi}{\bar{J}}\right| < \delta$. Then, we can use the same pricing scheme as in the discrete case that $J = m\bar{J}$. By the condition that $\left|\|\theta_j - \theta_{j+1}\| - \frac{2\pi}{\bar{J}}\right| < \delta$, the size of consumers served by each influencer and thus the revenue generated only deviate from those in the first case by an amount proportional to δ. Since $\delta > 0$ is arbitrary, the proof concludes. □

C.1.11 Proof of Proposition 3.4.1

Proof. First, note that the two influencers are symmetric for sellers, which means that sellers' incentives are trivial under regulated matching. Given this and the matched groups' profits, the influencer from the seller-influencer group with a bigger profit has no incentive to deviate when the bargaining power $(\gamma, 1-\gamma)$ is fixed. Thus, the other influencer also has no incentive to deviate, and we get $k(j) = j$ for $j \in \{1, 2\}$. Furthermore, given the group profit, payoffs for sellers and influencers just trivially follows from Nash bargaining.

Second, we construct an equilibrium in which $p_1^C \geq p_2^C \geq 0$ because the first influencer-seller group is stronger in the sense that it offers a product with a higher quality. This strategy implies that the first group is targeting the most valuable consumers and the consumer type boundary is pinned down by

$$y_1(1 - \|x^* - \theta\|/I) - p_1 = y_2(1 - \|x^* - \theta\|/I) - p_2$$

Obviously, since $y_1 \geq y_2$, all consumers with $\|x - \theta\| < \|x^* - \theta\|$ would purchase from Group 1. Furthermore, given this, the second group attracts those remaining consumers with $\|x-\theta\| \geq \|x^*-\theta\|$ and $\|x-\theta\| \leq \|x^{**}-\theta\|$ where

$$y_2(1-\|x^{**}-\theta\|/I)-p_2=0$$

Thus, we can calculate the demand (i.e., $q_1 = 2\|x^*-\theta\|$ for Group 1 and $q_2 = 2(\|x^{**}-\theta\|-\|x^*-\theta\|)$ for Group 2)

$$q_1(p_1,p_2) = 2I\left(1-\frac{p_1-p_2}{y_1-y_2}\right), \text{ and } q_2(p_1,p_2) = 2I\left(\frac{p_1-p_2}{y_1-y_2}-\frac{p_2}{y_2}\right)$$

and the profits as below.

$$\Pi_1 = 2Ip_1\left(1-\frac{p_1-p_2}{y_1-y_2}\right), \quad \Pi_2 = 2Ip_2\left(\frac{p_1-p_2}{y_1-y_2}-\frac{p_2}{y_2}\right),$$

Taking derivatives over Π_m with respect to p_m for $m=1,2$,

$$\frac{2I(-2p_1^C+p_2^C+y_1-y_2)}{y_1-y_2}=0, \text{ and } \frac{I(-4p_2^C y_1+2p_1^C y_2)}{(y_1-y_2)y_2}=0$$

Solving these two equations yields the desired solution. Moreover, the second order conditions are trivially satisfied and thus solutions given by the FOCs are optimal. By submitting the prices $(p_1^C, p_2^C) = \left(\frac{2y_1(y_1-y_2)}{4y_1-y_2}, \frac{y_2(y_1-y_2)}{4y_1-y_2}\right)$ into the demand functions and the profits, we obtain all the desired results after simple algebra manipulation. The proof concludes. □

C.1.12 Proof of Proposition 3.4.2

Proof. First, we focus on both groups' profits and skip the discussion on profit divisions between sellers and influencers and their incentives for matching, because the proof is almost the same as that in Proposition 3.4.1.

We construct an equilibrium which features $p_1^C \geq p_2^C \geq 0$. This is because, if $p_1 < p_2$, then the second influencer-seller group is priced out of the market because $\theta_1 = \theta_2$. Given that $p_1 \geq p_2$, consumers whose type is close to θ are attracted by the second group because the term $y\|x-\theta\|\left(\frac{1}{I_2}-\frac{1}{I_1}\right)$ only plays a secondary role compared the price gap $p_1 - p_2$. This yields the cutoff style type x^* such that all consumers with type x satisfying $\|x-\theta\| < \|x^*-\theta\|$ are served by the second group, i.e.,

$$y * (1 - \|x^* - \theta\|/I_1) - p_1 = y * (1 - \|x^* - \theta\|/I_2) - p_2$$

This leads to $\|x^* - \theta\| = \frac{I_1 I_2 (p_1 - p_2)}{y(I_1 - I_2)}$.

However, consumers with $\|x-\theta\| > \|x^*-\theta\|$ are attracted by the first group whenever it generates a positive utility, which implies a second cutoff type $x^{**} \in \mathbf{S}^1$ such that

$$y\left(1 - \|x^{**} - \theta\|/I_1\right) - p_1 = 0$$

which leads to $\|x^{**} - \theta\| \leq I_1\left(1 - p_1/y\right)$. By construction, consumers with a type satisfying $\|x^* - \theta\| < \|x - \theta\| \leq \|x^{**} - \theta\|$ are served by the first group.

Now, we can compute the profits for two influencer-seller groups, i.e.,

$$\Pi_1 = p_1 * \left((1 - p_1/y)I_1 - \frac{I_1 I_2 (p_1 - p_2)}{y(I_1 - I_2)}\right)$$
$$\Pi_2 = p_2 * \frac{I_1 I_2 (p_1 - p_2)}{y(I_1 - I_2)}$$

Obviously, the profit function Π_m is concave in p_m for $m = 1, 2$ and thus the optimal solution is fully characterized by the first-order conditions as below.

$$\frac{I_1}{y(I_1 - I_2)}\big(I_2(p_2 - y) + I_1(y - 2p_1)\big) = 0$$
$$\frac{I_1 I_2}{y(I_1 - I_2)}(p_1 - 2p_2) = 0$$

Solving these two equations, we get the desired solution (p_1^C, p_2^C), and by simple algebra, we can obtain the profits for both groups. The proof concludes. □

C.1.13 Proof of Proposition 3.4.3

Proof. First, we focus on both groups' profits and skip the discussion on profit divisions between sellers and influencers and their incentives for matching, because the proof is almost the same as that in Proposition 3.4.1.

Second, let us fix p_2 and consider the profit function of the first group. If these two prices are sufficiently close, then we would expect the following condition

$$y(1 - \|x^* - \theta_1\|/I) - p_1 = y(1 - \|x^* - \theta_2\|/I) - p_2$$

or equivalently,

$$p_2 - p_1 = (\|x^* - \theta_1\| - \|x^* - \theta_2\|) * y/I$$

For ease of reference, denote $s_m = \|x - \theta_m\|$ for $m = 1, 2$. When p_1 and p_2 are sufficiently close and not too small (to be discussed shortly), x^* lies in between θ_1 and θ_2 in the short arc, then we can solve these by utilitizing the fact that $s_1 + s_2 = \|\theta_1 - \theta_2\|$ and get

$$s_1 = \frac{1}{2}\big(\|\theta_1 - \theta_2\| + (p_2 - p_1) * I/y\big),$$
$$s_2 = \frac{1}{2}\big(\|\theta_1 - \theta_2\| - (p_2 - p_1) * I/y\big)$$

Note that this holds only when there exists no consumer unserved in the short arc between θ_1 and θ_2, that is,

$$I * (1 - p_1/y) + I * (1 - p_2/y) \geq \|\theta_1 - \theta_2\|,$$

which further reduces to $p_1 + p_2 \leq (2 - \frac{\|\theta_1 - \theta_2\|}{I}) * y$.

Otherwise, $s_1 = I * (1 - p_1/y)$. Hence, to summarize,

$$s_1 = \begin{cases} \frac{1}{2}\big(\|\theta_1 - \theta_2\| + (p_2 - p_1) * I/y\big), & \text{if } p_1 + p_2 \leq (2 - \frac{\|\theta_1 - \theta_2\|}{I}) * y \\ I * (1 - p_1/y), & \text{otherwise} \end{cases}$$

However, when p_1 is sufficiently low, then $s_1 > \|\theta_1 - \theta_2\|$ occurs, and in this case, $s_1 - s_2 = \|x - \theta_1\| - \|x - \theta_2\| = \|\theta_1 - \theta_2\|$. This implies that the first group grabs the whole market when

$$p_1 \leq p_2 - \|\theta_1 - \theta_2\| * y/I$$

and the first group loses all consumers when

$$p_1 \geq p_2 + \|\theta_1 - \theta_2\| * y/I$$

Henceforth, we can write down the profit function for the first group as below

$$\Pi_1^C = \begin{cases} 0, & \text{if } p_1 \geq p_2 + \|\theta_1 - \theta_2\| * y/I \\ 2p_1 * (1 - p_1/y)I, & \text{if } p_1 \leq p_2 - \|\theta_1 - \theta_2\| * y/I \\ p_1 * \left((1 - p_1/y) * I + s_1\right), & \text{if } p_1 + p_2 \leq (2 - \frac{\|\theta_1 - \theta_2\|}{I}) * y \text{ and} \\ & p_1 \in \left(p_2 - \frac{\|\theta_1 - \theta_2\| * y}{I}, p_2 + \frac{\|\theta_1 - \theta_2\| * y}{I}\right) \\ 2p_1 * (1 - p_1/y)I, & \text{otherwise} \end{cases} \tag{C.6}$$

Note that there are two discontinuity points for the profit function above. Also note that the profit function Π_1^C is continuous at $p_1 = \left(2 - \frac{\|\theta_1 - \theta_2\|}{I}\right) * y - p_2$. Similarly, we can write down the profit function for the second group by symmetry. Obviously, it is suboptimal to choose $p_1 \geq p_2 + \|\theta_1 - \theta_2\| * y/I$, which leads to a zero profit. Then, we start with the third case in Equation (C.6) and we can derive the first order conditions as follows.

$$\frac{y\|\theta_1 - \theta_2\| + I(p_2 - 6p_1 + 2y)}{2y} = 0,$$
$$\frac{y\|\theta_1 - \theta_2\| + I(p_1 - 6p_2 + 2y)}{2y} = 0$$

Solving these two equations, we can get $p_1^* = p_2^* = \frac{y(2I+\|\theta_1-\theta_2\|)}{5I}$, and we can further get the profit for the first group as below, i.e.,

$$\Pi_1 = \frac{3y(2I + \|\theta_1 - \theta_2\|)^2}{50I}$$

when $p_2 - \|\theta_1 - \theta_2\| * y/I < p_1 < p_2 + \|\theta_1 - \theta_2\| * y/I$.

However, we need to check that $p_1^* + p_2^* \leq \left(2 - \frac{\|\theta_1-\theta_2\|}{I}\right) * y$ *indeed* holds, which further reduces to

$$\|\theta_1 - \theta_2\| \leq \frac{6}{7}I.$$

For any $\|\theta_1 - \theta_2\| > \frac{6}{7}I$, define $\hat{p}_1^* = \hat{p}_2^* = \left(1 - \frac{\|\theta_1-\theta_2\|}{2A}\right) * y$. Fix $p_2 = \hat{p}_2^*$, the term $p_1 * ((1 - p_1/y) * I + s_1)$ is strictly increasing in p_1 for $p_1 \leq \frac{y(2I+\|\theta_1-\theta_2\|)}{5I}$. Thus, the first (influencer-seller) group has no incentive to deviate downward. Meanwhile, for $p_1 > \hat{p}_1^*$, the term $p_1 * (1 - p_1/y) * I$ is strictly decreasing for $p_1 > \hat{p}_1^*$ since $\hat{p}_1^* \geq \frac{y}{2}$. This implies that the first group has no incentive to deviate upward. Thus, $(\hat{p}_1^*, \hat{p}_2^*)$ constitutes an equilibrium when $\|\theta_1 - \theta_2\| \in (\frac{6}{7}I, I)$ whenever no group is priced out of the market.

Finally, to finish the equilibrium construction, we need to ensure that the construction in the third case is also globally optimal, which means that the first group has no incentive to deviate by a big price cut as illustrated in the second case in Equation (C.6).

Specifically, to undercut the second group, Group 1 only needs to set the price $p_1 = p_2^* - \|\theta_1 - \theta_2\| * y/I$, which leads to a profit as below

$$\hat{\Pi}_1 = 2I * (p_2^* - \|\theta_1 - \theta_2\| * y/I) * \left(1 - \frac{p_2^* - \|\theta_1 - \theta_2\| * y/I}{y}\right)$$

Note that this is the most profitable deviation since $\Pi_1 = 2I(1-p_1/y)p_1$ is strictly decreasing for all $p_1 \leq p_2^* - \|\theta_1 - \theta_2\| * y/I$. To support the equilibrium, it requires

$$\hat{\Pi}_1 \leq \Pi_1$$

which leads to the condition in the proposition. The proof concludes. □

C.1.14 Proof of Proposition 3.4.4

Proof. The assortative matching and profits derivation follows directly from Lemma 3.6.1 (see Appendix C.1.20). Payoffs for sellers and influencers follows from the exogenous Nash bargaining argument. The proof concludes. □

C.1.15 Proof of Proposition 3.4.5

Proof. Consider the following equilibrium conjecture.

- Assortative matching in the labor market. $k(j) = j$ for $j \in \{1, 2\}$.

- Product market. Group 1, the matching between seller 1 and influencer 1, prices the product at $p_1^* = \frac{y_1}{2}$, and earns a total profit of $\Pi_1 = \frac{y_1 I_1}{2}$. Group 2 prices the product at $p_2^* = 0$.

Now, we verify that this constitutes an equilibrium.

First, given $p_1^* = \frac{y_1}{2}$, a consumer with type $x \in \mathbf{S}^1$ such that $\|x - \theta_2\| \leq I_2$, always prefers the product from group 1 to that from group 2 even when group 2 sets the price at zero, as long as the following two conditions hold, that is, for type $x = \theta_2$,

$$y_1 * \big(1 - \beta/I_1\big) - p_1^* \geq y_2 \tag{C.7}$$

and for type $\|x - \theta_2\| = I_2$ & $\|x - \theta_1\| = \beta + I_2$,

$$y_1 * \big(1 - (\beta + I_2)/I_1\big) - p_1^* \geq 0 \tag{C.8}$$

Simplifying these two equations yields the condition that $\frac{\beta}{I_1} \leq \min\{\frac{1}{2} - \frac{y_2}{y_1}, \frac{1}{2} - \frac{I_2}{I_1}\}$.

Second, given consumers' equilibrium choice and group 2's pricing strategy, group 1 has no incentive to deviate from the monopolist profit. Meanwhile, given the assumed condition and other participants' equilibrium strategy, $p_2^* = 0$ because she cannot attract any consumer by setting $p_2^* > 0$.

Third, anticipating the equilibrium profits, influencer 1 chooses to match with seller 1 and gets a payoff of $w_1 = \frac{(1-\gamma)y_1 I_1}{2}$. Instead, if she deviates to seller 2, influencer 1 can at most get $\hat{w} - 1 = \frac{(1-\gamma)y_2 I_1}{2} < w_1$. Given influencer 1's equilibrium matching choice, influencer 2 can only match with seller 2. The proof concludes. □

C.1.16 Proof of Proposition 3.5.1

Proof. First, by the formula of $\Pi_1(\beta)$ (i.e., Eq. (3.6)), we can check that $\Pi_1(\beta)$ is strictly increasing for all $\beta \in \big(\beta_0, I\big]$, which implies that

$$G(\beta) := \Pi_1(\beta) - \big(\Pi_1(I) - F_H\big)$$

has at most one solution for all $\beta \in \big(\beta_0, I\big]$. Indeed, note that $G(I) = F_H$ and $G(\beta_0) < 0$ by the assumed condition Eq. (3.7). Hence, there exists a unique β^* well defined. Further, by monotonicity, we have $G(\beta) > 0$ for $\beta > \beta^*$ and $G(\beta) < 0$ for $\beta \in [\beta_0, \beta^*)$.

Second, consider $\beta > \beta^*$ under regulated matching (i.e., 1-1 matching). By symmetry, we consider the incentive for seller 1. Given that seller 2 hires an influencer, say influencer 2, seller 1 can choose to hire influencer 1 and get a payoff of $\gamma\Pi_1(\beta)$, or pay a fixed cost and select a location to achieve maximum differentiation (i.e., $\|\alpha_1 - \alpha_2\| \geq I$. Here, $\alpha_2 = \theta_2$ is influencer 2's style location. This yields a payoff of $\gamma(\Pi_1(I) - F_H)$. However, note that $G(\beta) > 0$ for $\beta > \beta^*$. Thus, it is optimal to hire influencer 1 because $\gamma\Pi_1(\beta) \geq \gamma(\Pi_1(I) - F_H)$.

Third, consider $\beta \leq \beta^*$ under regulated matching. We consider an asymmetric equilibrium in which seller 1 hires influencer 1, and seller 2 turns on the influence by paying the fixed cost F_H. First of all, for seller 2, given that seller 1 already hires influencer 1, she can hire influencer 2 and engage in price competition, which yields a payoff of $\gamma\Pi_1(\beta)$. On the other hand, she can pay the fixed cost F_H and select a location to avoid competition and thus gets $\gamma(\Pi_1(I) - F_H)$. By the fact $G(\beta) < 0$ for $\beta \leq \beta^*$, she has an incentive to pay the fixed cost F_H. Furthermore, given that seller 2 chooses maximum differentiation, seller

1 has the incentive to hire influencer 1 and gets a profit of $\gamma\Pi_1(I)$. The proof concludes. $\square$

C.1.17 Proof of Lemma 3.5.1

Proof. Before we get started, note that under regulated matching the Nash bargaining always gives the seller a fraction γ of the seller-influencer group's total profit. Hence, we can directly focus on total profits of groups in the proof.

i) Profits for (I, I) and (NI, NI). First, under the assumed condition i) $\beta_0 \leq \beta \leq \frac{5}{6}$, Proposition 3.4.3 holds. Thus, when both groups choose to invest, each group gets a profit of

$$\Pi_{I,I} = yA(\beta).$$

and if they both choose not to invest, each group gets

$$\Pi_{NI,NI} = A(\beta).$$

ii) Profits for $(I, NI$ (and (NI, I)). Second, we compute profits for both seller-influencer groups when only one group, say, group 1, chooses to invest (and the other group choose not to invest). By the assumed condition ii), even if group 1 (seller 1 and influencer 1) chooses high quality y, group 2 can still attract some consumers even group 1 set a price at $p_1 = 0$. Now, denote by (p_1, p_2) the prices set by both groups.

For group 1, the size of consumers served is just $y_1(1 - \|x - \theta_1\|) - p_1 \geq 0$ or equivalently $(1 - p_1/y)$. For consumers between groups (along the short arc), the cutoff type $x^* \in \mathbf{S}^1$ satisfies

$$y(1 - \|x^* - \theta_1\|) - p_1 = 1 - (\beta - \|x^* - \theta_2\|) - p_2$$

Solving it yields

$$\|x^* - \theta_1\| = \frac{p_2 - p_1 + (y-1) + \beta}{y+1} =: s_1, \quad \text{and} \quad s_2 := \|x^* - \theta_2\| = \beta - s_1$$

We can further express profits for both groups as follows

$$\Pi^1_{I,NI} = (1 - p_1/y_1) * p_1 + p_1 * s_1, \quad \text{and} \quad \Pi^2_{I,NI} = (1 - p_2/y_2) * p_2 + p_2 * s_2$$

Taking first order conditions and solving them, we get

$$p_1^* = \frac{y_1(4y_1^2 + 8y_1y_2 + 3\beta y_1 y_2 + 2y_2^2 + 4\beta y_2^2)}{8y_1^2 + 19y_1y_2 + 8y_2^2} = \frac{2(1+2\beta)y + (8+3\beta)y^2 + 4y^3}{8 + 19y + 8y^2}$$

and

$$p_2^* = \frac{y_2(4y_2^2 + 8y_1y_2 + 3\beta y_1 y_2 + 2y_1^2 + 4\beta y_1^2)}{8y_1^2 + 19y_1y_2 + 8y_2^2} = \frac{4 + (8+3\beta)y + 2(1+2\beta)y^2}{8 + 19y + 8y^2}$$

We need to make sure that the cutoff type x^* gets a non-negative utility, that is,

$$y(1 - \|x^* - \theta\|) - p_1^* \geq 0,$$

which reduces to

$$\frac{y\big(2(5-6\beta)(1+y^2) + (22-25\beta)y\big)}{(1+y)(8+19y+8y^2)}$$

which trivially holds under the assumed condition i). Furthermore, we can directly verify that the second order conditions are satisfied.

Thus, profits for both groups can be further computed as

$$\Pi^1_{I,NI} = \frac{y(1+2y)\big(2 + 8y + 4y^2 + \beta(4+3y)\big)^2}{(1+y)\left(8 + 19y + 8y^2\right)^2}$$

and

$$\Pi^2_{I,NI} = \frac{y(2+y)\big(4+8y+2y^2+\beta(3y+4y^2)\big)^2}{(1+y)\,(8+19y+8y^2)^2}$$

iii) Nash Equilibrium Construction. To verify equilibrium, we state some properties first.

1. Fix y. $V_1(\beta, y) > V_2(\beta, y)$. Recall that $V_1(\beta, y) = \Pi^1_{I,NI} - \Pi_{NI,NI}$ and $V_2(\beta, y) = \Pi_{I,I} - \Pi^2_{I,NI}$.
2. Fix y. Both $V_1(\beta, y)$ and $V_2(\beta, y)$ are strictly increasing in β.

Given these two properties, we can verify the equilibrium. First, by the assumed condition iii), $F_V \in (V_1(\beta_0, y), V_2(5/6, y))$ and property 1), we have

$$V_1(5/6, y) > V_2(5/6, y) > F_V > V_1(\beta_0, y) > V_2(\beta, y)$$

which, together with the strict monotonicity of $V_1(\beta, y)$ and $V_2(\beta, y)$, implies that there exist $\underline{\beta} \in (\beta_0, 5/6)$ and $\overline{\beta} \in (\beta_0, 5/6)$ such that

$$F_V = V_1(\underline{\beta}, y) = V_2(\overline{\beta}, y), \quad \text{and} \quad \overline{\beta} > \underline{\beta}.$$

To summarize,

a) For $\beta \geq \overline{\beta}$, $V_2(\beta, y) \geq F_V$, or equivalently,

$$\Pi_{I,I} - \Pi^2_{I,NI} \geq F_V$$

Hence, given that influencer 1 chooses to invest, it is optimal for influencer 2 to invest. By symmetry, influencer 1 also chooses to invest, and thus (I, I) is a Nash Equilibrium.

b) For $\underline{\beta} \leq \beta < \overline{\beta}$, we have both $V_1(\beta, y) \geq F_V$ and $V_2(\beta, y) < F_V$, that is,

$$\Pi^1_{I,NI} - \Pi_{\{NI,NI\}} \geq F_V$$

and

$$\Pi_{\{I,I\}} - \Pi^2_{\{I,NI\}} < F_V$$

These two conditions read as follows. One, given that influencer 2 chooses not to invest, it is optimal for influencer 1 to invest. Two, given that influencer 1 chooses to invest, it is optimal for influencer 2 not to invest. Thus, (I, NI) is a Nash Equilibrium, so is (NI, I).

c) For $\beta < \underline{\beta}$, $V_1(\beta, y) < F_V$, or equivalently,

$$\Pi^1_{I,NI} - \Pi_{\{NI,NI\}} < F_V$$

which reads as, if influencer 2 does not invest, then it is optimal for influencer 1 not to invest. By symmetry, (NI, NI) is a Nash Equilibrium.

Now, it suffices to verify property 1) and 2) on $V_1(\beta, y)$ and $V_2(\beta, y)$. To this end, we write down the formulas and check them one by one.

- First, with y fixed, $V_1(\beta, y)$ is strictly increasing in β.

$$V_1(\beta, y) = \frac{(y-1)}{50(y+1)(8+19y+8y^2)}$$
$$\times \left\{1600y^5 + 32M_1 * y^4 + 4M_2 * y^3 + 25M_3 * y^2 + 8M_4 * y + 192(2+\beta)^2\right\}$$

where

$$M_1 = 251 + 51\beta - 6\beta^2, \quad M_2 = 3704 + 1604\beta - 99\beta^2$$
$$M_3 = 500 + 340\beta + 3\beta^2, M_4 = 623 + 548\beta + 62\beta^2$$

For instance, for M_1, there are two solutions, $\beta_1 \approx -3.49$ and $\beta_2 \approx 11.99$ and M_1 is strictly increasing for all $\beta \leq \frac{51}{12} = 4.25$. Furthermore, $M_1(0) = 251$, and thus $M_1 > 0$ and is strictly increasing for all $\beta \in (\beta_0, 5/6)$. Similarly, we can check that M_2, M_3 and M_4 are also positive and strictly increasing for $\beta \in (\beta_0, 5/6)$.

- Second, with y fixed, $V_2(\beta, y)$ is strictly increasing in β.

$$V_2(\beta, y) = \frac{(y-1)}{50(y+1)(8+19y+8y^2)}$$
$$\times \left\{1600 + 32M_1 * y + 4M_2 * y^2 + 25M_3 * y^3 + 8M_4 * y^4 + 192(2+\beta)^2 y^5\right\}$$

Hence, fix y, then $V_2(\beta, y)$ is strictly increasing in β.

- Third, with y fixed, $V_1(\beta, y) > V_2(\beta, y)$ for all $\beta \in (\beta_0, 5/6)$.

$$V_1(\beta, y) - V_2(\beta, y) = \frac{(y-1)^2}{50(y+1)(8+19y+8y^2)}$$
$$\times \left\{64 * M_5 + 40 * M_6 y + M_7 * y^2 + 40 * M_6 y^3 + 64 * M_5 y^4\right\}$$

where

$$M_5 = 13 - 12\beta - 3\beta^2, \quad M_6 = 97 - 88\beta - 22\beta^2, \quad M_7 = 6196 - 5604\beta - 1351\beta^2$$

By the same token, we can prove it as that for M_1. For instance, to show that $M_5 > 0$, note that there are two solutions $\beta_1 \approx -4.89$ and $\beta_2 \approx 0.89$. Hence, $M_5 > 0$ for all $\beta \in (-4.89, 0.89)$, and thus $M_5 > 0$ for all $\beta \in (\beta_0, 5/6)$. We can prove $M_6 > 0$ and $M_7 > 0$ in similar spirits.

All the proofs conclude. □

C.1.18 Proof of Proposition 3.6.1

Proof. The proof consists of three parts: i) initial equilibrium analysis; ii) Nash equilibrium construction; and iii) welfare analysis.

i) Initial Equilibrium Analysis. Initially, $I_1 = I_2 = \pi$. Because $\frac{I_1+I_2}{2} \leq \pi$, the seller, when hiring both influencers, set the price at $p_1^* = p_2^* = \frac{y}{2}$, which implies that *before* power acquisition,

$$\Pi^b_{\{1\}} = \frac{1}{2}y\pi, \quad \Pi^b_{\{2\}} = \frac{1}{2}y\pi, \quad \text{and} \quad \Pi^b_{\{1,2\}} = y\pi.$$

The Nash bargaining implies that

$$w_1^b = (1-\gamma)(\Pi_{\{1,2\}} - \Pi_{\{1\}}) = \frac{1}{2}(1-\gamma)y\pi.$$

Similarly, $w_2^b = \frac{1}{2}(1-\gamma)y\pi$.

ii) Nash Equilibrium Construction. First, we construct an equilibrium in which influencer 2 does not invest in power acquisition. Consider influencer 1's incentive to invest. If influencer 1 invests in power acquisition, then $I_1 + I_2 \geq 2\pi$ and thus by the proof of Lemma 3.3.2, the joint revenue function *after* power acquisition is given by[1]

$$\Pi^a_{\{1,2\}} = 2\pi y\left(1 - \frac{\pi}{I_1 + I_2}\right) = 2\pi y(1 - 1/3) = \frac{4}{3}\pi y.$$

The optimal pricing strategy that maximizes the joint revenue function is given by

$$p_1^a = p_2^a = y\left(1 - \frac{\pi}{I_1 + I_2}\right) = \frac{2y}{3}.$$

[1]Note that we assume that $c = 1$.

and consumers served by influencers satisfy $\|x - \theta_j\| \le s_j$ where

$$s_1 = \frac{I_1 \pi}{I_1 + I_2} = \frac{2\pi}{3}, \text{ and } s_2 = \frac{I_2 \pi}{I_1 + I_2} = \frac{\pi}{3}.$$

Similarly, if the seller only hires influencer j, profits are given by

$$\Pi^a_{\{1\}} = \frac{1}{2} y I_1 = y\pi, \quad \text{and} \quad \Pi^a_{\{2\}} = \frac{1}{2} y I_2 = \frac{1}{2} y\pi.$$

Hence, wages offered to influencers are given by

$$w^a_1 = (1-\gamma)(\Pi^a_{\{1,2\}} - \Pi^a_{\{2\}}) = (1-\gamma)\left(\frac{4}{3}y\pi - \frac{1}{2}y\pi\right) = \frac{5}{6}(1-\gamma)y\pi$$

and

$$w^a_2 = (1-\gamma)(\Pi^a_{\{1,2\}} - \Pi^a_{\{1\}}) = (1-\gamma)\left(\frac{4}{3}y\pi - y\pi\right) = \frac{1}{3}(1-\gamma)y\pi$$

Hence, given that influencer 2 does not invest, influencer 1 will invest in power acquisition when

$$w^a_1 - C_T \ge w^b_1 \implies C_T \le \frac{1}{3}(1-\gamma)y\pi$$

Second, we come to check influencer 2's incentive.

(1) By symmetry, when $C_T > \frac{1}{3}(1-\gamma)y\pi$, influencer 2 will not invest in power acquisition, that is, (NI, NI) is a Nash Equilibrium.

Given that influencer 1 chooses to invest, if influencer 2 also invests,

$$\hat{\Pi}_{\{1,2\}} = 2\pi y\left(1 - \frac{\pi}{I_1 + I_2}\right) = 2\pi y(1 - 1/4) = \frac{3}{2}y\pi$$

and

$$\hat{\Pi}_{\{1\}} = \Pi^a_{\{1\}} = y\pi.$$

Nash bargaining implies that

$$\hat{w}_2 = (1-\gamma)(\hat{\Pi}_{\{1,2\}} - \hat{\Pi}_{\{1\}}) = \frac{1}{2}(1-\gamma)y\pi$$

Hence, influencer 2 will not invest if and only if

$$\hat{w}_2 - C_T < w_2^a \implies C_T > \frac{1}{6}(1-\gamma)y\pi.$$

To summarize,

(2) When $\frac{1}{6}(1-\gamma)y\pi < C_T \leq \frac{1}{3}(1-\gamma)y\pi$, (I, NI) is a Nash Equilibrium.

(3) When $C_T \leq \frac{1}{6}(1-\gamma)y\pi$, (I, I) is a Nash Equilibrium.

iii) Welfare Analysis. Here, we show the optimal decision rule to maximize total welfare.

- First, when (NI, NI) is the outcome, the total welfare is achieved by letting influencer j serve consumers with $\|x-\theta_j\| \leq \frac{\pi}{2}$ and $x \in \mathbf{S}^1$, that is,

$$\begin{aligned} SW &= \int_{y(1-\|x-\theta_1^*\|/I_1)-\frac{y}{2}\geq 0} y(1-\|x-\theta_1^*\|/I_1)dx \\ &+ \int_{y(1-\|x-\theta_2^*\|/I_2)-\frac{y}{2}\geq 0} y(1-\|x-\theta_2^*\|/I_2)dx \\ &= \frac{3}{4}(yI_1 + yI_2) = \frac{3}{2}y\pi. \end{aligned}$$

- Second, when (I, NI) (or (NI, I)) is the outcome, total welfare is achieved by letting influencer j serve consumers with $\|x-\theta_j\| \leq \frac{I_j}{I_1+I_2}\pi$ for $j=1,2$, because the cutoff type consumer $x \in \mathbf{S}^1$ satisfies

$$y\left(1-\frac{\|x-\theta_1\|}{2\pi}\right) = y\left(1-\frac{\|x-\theta_2\|}{\pi}\right)$$

Then, total welfare is given by[2]

$$SW = \frac{5}{3} y\pi - C_T$$

- Third, wehn (I, I) is the outcome, total welfare is achieved by letting influencer j serve consumers with $\|x - \theta_j\| \leq \frac{\pi}{2}$. Total welfare is given by

$$SW = \frac{7}{4} y\pi - 2C_T$$

Hence, we can calculate the optimal decision rule by comparing total welfare under different outcomes, that is,

1. when $C_T \leq \frac{1}{12} y\pi$, (I, I) is optimal, i.e., both influencers should invest.
2. when $\frac{1}{12} y\pi < C_T \leq \frac{1}{6} y\pi$, (I, NI) (or (NI, I)) is optimal, i.e., only one influencer should invest.
3. when $C_T > \frac{1}{6} y\pi$, (NI, NI) is optimal, i.e., no influencer should invest.

The proof concludes. □

C.1.19 Proof of Example 3.6.2

Proof. By Proposition 3.4.3, influencer 2 is offered a wage of

$$w_2(b) = \begin{cases} \frac{3(1-\gamma)y}{50I} * (2I + b)^2, & \text{if } b \in [\beta_0, \frac{6}{7}I] \\ (1-\gamma)y * \left(b - \frac{b^2}{2I}\right), & \text{if } b \in (\frac{6}{7}I, I) \\ \frac{(1-\gamma)yI}{2}, & \text{if } b \geq I \end{cases}$$

[2] A detailed calculation is available upon request.

Thus, if he decides to adjust his own style to $b > \beta$, the cost-benefit analysis reduces to

$$w_2^N(b) = \begin{cases} (1-\gamma)\left(\frac{3y(2I+b)^2}{50I} - C(b)\right), & \text{if } b \in [\frac{2}{67}(-7+5\sqrt{10})I, \frac{6}{7}I] \\ (1-\gamma)\left(y\left(b - \frac{b^2}{2I}\right) - C(b)\right), & \text{if } b \in (\frac{6}{7}I, I) \\ (1-\gamma)\left(\frac{yI}{2} - C(b)\right), & \text{if } b \in [I, \infty) \end{cases}$$

Note that,

$$\lim_{b \to \beta} \frac{dw_2^N}{db} = (1-\gamma)\left(\frac{3y(2I+b)}{25I} - C'(\beta)\right) > 0,$$
$$\lim_{b \to I-} \frac{dw_2^N}{db} = -(1-\gamma)C'(I-d_0) < 0$$

Moreover, note that $\frac{dw_2^N}{db}$ is strictly decreasing in b, and thus there exists a unique maximizer $b^* \in (\beta, I)$. The proof concludes. □

C.1.20 Proof of Lemma 3.6.1

Proof. First, since influencers weakly prefer being hired, there are two possible cases under regulated matching: $k(j) = j$ and $k(j) = 2 - j$ where $j = 1, 2$. Second, given that $\|\theta_1 - \theta_2\| \geq \frac{I_1+I_2}{2}$, independent of the matching outcome, both seller-influencer groups can charge a monopolist price $p_1* = \frac{y_1}{2}$ and $p_2^* = \frac{y_2}{2}$. Thus, we can calculate payoffs for all sellers and influencers respectively, that is,

- $k(j) = j$. In this case,

$$U_1 = \frac{\gamma y_1 I_1}{2},\ U_2 = \frac{\gamma y_2 I_2}{2},\ w_1 = \frac{(1-\gamma) y_1 I_1}{2},\ \text{and } w_2 = \frac{(1-\gamma) y_2 I_2}{2}$$

- $k(j) = 2 - j$. In this case,

$$\hat{U}_1 = \frac{\gamma y_1 I_2}{2},\ \hat{U}_2 = \frac{\gamma y_2 I_1}{2},\ \hat{w}_1 = \frac{(1-\gamma) y_2 I_1}{2},\ \text{and } \hat{w}_2 = \frac{(1-\gamma) y_1 I_2}{2}$$

Note that the matching $k(j) = 2 - j$ is not stable because both seller 1 and influencer 1 can fully anticipate the payoffs when they match, and thus both are willing to form a match. Indeed, $k(j) = j$ for $j = 1, 2$ is a stable matching. Given that seller 1 is matched with influencer 1, both of them have no incentive to deviate because $U_1 \geq \hat{U}_1$ and $w_1 \geq \hat{w}_1$. Given this, seller 2 and influencer 2 forms a match. The proof concludes. □

C.1.21 Proof of Proposition 3.6.2

Proof. Consider the following equilibrium strategies conjectured as follows.

- On-equilibrium path, that is, $\|\theta_1 - \theta_2\| \geq \frac{I_1 + I_2}{2}$.

  The equilibrium matching outcome and payoffs are specified as that in Lemma 3.6.1.

- Off-equilibrium path, that is, $\|\theta_1 - \theta_2\| < \frac{I_1 + I_2}{2}$.

  Seller 1 is matched with influencer 1, and seller 2 is matched with influencer 2. Then, given the matching outcome, the two seller-influencer groups play a price competition game and if an equilibrium exists, payoff are specified according to the Nash bargaining. Whenever a pure strategy equilibrium does not exist, we assume that the payoffs for all sellers and influencers are bounded above by the worst equilibrium payoff, that is, the equilibrium with minimum total profits.

First, consider influencer 2. Under equilibrium, he gets a wage of $w_2 = \frac{(1-\gamma)y_2 I_2}{2}$. Instead, if he chooses θ_2 such that $\|\theta_1^* - \theta_2\| < \frac{I_1+I_2}{2}$. Then, given that seller 1 is matched with influencer 1, his wage $\tilde{w}_2$ cannot exceed w_2 because when he is matched with seller 2, their total profit is bounded above by $\frac{y_2 I_2}{2}$. More formally,

$$\Pi_2 \leq p_2(1-p_2/y_2)I_2 \leq p_2^m(1-p_2^m/y)I_2 = \Pi_2^m = \frac{y_2 I_2}{2}.$$

The first inequality says that influencer 2 can at most attract all consumers within his influence reach, and the second one says that it is weakly dominated by the monopoly price. Hence, he has an incentive to choose $\|\theta_1^* - \theta_2\| \geq \frac{I_1+I_2}{2}$. The same argument applies for influencer 1.

Second, given that $\|\theta_1^* - \theta_2^*\| \geq \frac{I_1+I_2}{2}$, by Lemma 3.6.1, both sellers have an incentive to accept the assortative matching outcome and the proposed equilibrium payoffs. The proof concludes. □

C.1.22 Proof of Lemma 3.7.1

Proof. First, the case for $\|\theta_1 - \theta_2\| \geq I$ is easy, and the optimal pricing strategy is given by $p_1^* = p_2^* = \frac{y}{2}$ and each influencer serves consumers with types $\|x - \theta_j\| \leq \frac{I}{2}$ for $j = 1, 2$. Note that there is no overlapping in consumers served by the two influencers, and the monopoly profit is achieved, which yields

$$\Pi_{\{1,2\}} = 2\Pi_{\{1\}} = 2 * \frac{1}{2}yI = yI.$$

Second, we come to show that in the case that $\|\theta_1 - \theta_2\| < I$, we the following three properties hold: i) no gap for all consumers between influencer 1 and 2 on the short arc; ii) both influencers are actively serving consumers on the short arc; and iii) the pricing must be symmetric such that $p_1 = p_2$. We show these three properties one by one.

i) Suppose there is a gap a positive measure of consumers are not served between θ_1 and θ_2 on the short arc. Then, $\|x - \theta_j\| < \frac{I}{2}$ for at least some $j \in \{1, 2\}$ because $\|\theta_1 - \theta_2\| < I$. This implies $p_j > \frac{I}{2}$. Moreover, for such an influencer j, the profit is given by $\Pi_{\{j\}} = p_j(1 - p_j/y)I$, which is strictly decreasing for all $p_j \in (\frac{I}{2}, y]$. This means that it is profitable to choose a price $\hat{p}_j = p_j - \varepsilon$ with $\varepsilon > 0$ sufficiently small.

ii) Suppose influencer 2 is not active on the short arc and θ_2 is on the right of θ_1 on the short arc (i.e. θ_2 is ahead of θ_1 in the clockwise direction on the circle). Then, we have

$$y(1 - \|\theta_2 - \theta_1\|/I) - p_1 \geq y - p_2$$

This further implies that for all $x \in \mathbf{S}^1$ on the right of θ_2 (and thus $\|x - \theta_1\| = \|x - \theta_2\| + \|\theta_1 - \theta_2\|$)

$$y(1 - \|\theta_2 - \theta_1\|/I - \|x - \theta_2\|/I) - p_1 \geq y(1 - \|x - \theta_2\|/I) - p_2$$

Thus, influencer 2 is not actively serving any consumer $x \in \mathbf{S}^1$. However, this is suboptimal because if we only let influencer 1 be active on the market, the maximum profit cannot exceed $\Pi_{\{1\}}$ (i.e., the monopoly profit from hiring a single influencer) and it is dominated by the profit function proposed in the lemma.

iii) Otherwise suppose $p_1 \neq p_2$ and assume w.l.o.g. $p_1 < p_2$. Consider a new pair of prices $(\hat{p}_1, \hat{p}_2) = (p_1 + \varepsilon, p_2 - \varepsilon)$ with $\varepsilon > 0$ sufficiently small, and we show that $(\hat{p}_1, \hat{p}_2)$ dominates (p_1, p_2). By the "no gap" result in (i) and that both influencers are active, all consumers between θ_1 and θ_2 on the short arc are served and denote by x^* the cutoff consumer type, and define $s_j = \|x^* - \theta_j\|$ for $j \in \{1, 2\}$. By the indifference condition for consumer x^*, we have

$$y(1 - s_1/I) - p_1 = y(1 - s_2/I) - p_2$$

and

$$s_1 + s_2 = \|\theta_1 - \theta_2\|$$

These two conditions together yields

$$s_j = \frac{1}{2}\left(\|\theta_1 - \theta_2\| + \frac{I}{y}(p_{3-j} - p_j)\right)$$

Under (p_1, p_2), the profit is given by

$$\Pi = p_1 s_1 + p_2 s_2 + p_1(1 - p_1/y)I + p_2(1 - p_2/y)I$$

and under $(\hat{p}_1, \hat{p}_2)$, the profit is given by

$$\hat{\Pi} = \hat{p}_1 \hat{s}_1 + \hat{p}_2 \hat{s}_2 + \hat{p}_1(1 - \hat{p}_1/y)I + \hat{p}_2(1 - \hat{p}_2/y)I$$

Note that

$$\begin{aligned}
&\hat{p}_1(1 - \hat{p}_1/y)I + \hat{p}_2(1 - \hat{p}_2/y)I \\
&\quad = I * \left(p_1 + p_2 - (p_1 + \varepsilon)^2/y - (p_2 - \varepsilon)^2/y\right) \\
&\quad = p_1 + p_2 - p_1^2/y - p_2^2/y + 2(p_2 - p_1)\varepsilon I/y + O(\varepsilon^2) \\
&\quad > p_1(1 - p_1/y)I + p_2(1 - p_2/y)I
\end{aligned}$$

Similarly, we can show that

$$\hat{p}_1\hat{s}_1 - p_1 s_1 = (p_1 + \varepsilon)\left(s_1 - \frac{I\varepsilon}{y}\right) - p_1 s_1 = \varepsilon s_1 - \frac{I\varepsilon}{y}p_1$$
$$\hat{p}_2\hat{s}_2 - p_2 s_2 = (p_2 - \varepsilon)\left(s_2 + \frac{I\varepsilon}{y}\right) - p_2 s_2 = -\varepsilon s_2 + \frac{I\varepsilon}{y}p_2$$

which implies that

$$\hat{p}_1\hat{s}_1 + \hat{p}_2\hat{s}_2 - (p_1 s_1 + p_2 s_2) = (s_1 - s_2)\varepsilon + (p_2 - p_1)I\varepsilon/y > 0$$

Thus, (p_1, p_2) is strictly dominated, and the contradiction implies that $p_1 = p_2$.

Third, we find the optimal pricing strategy and the profit function $\Pi_{\{1,2\}}$. By property (iii), we denote $p_1 = p_2 = p$ and the profit function is

$$\Pi = p * \|\theta_1 - \theta_2\| + 2p(1 - p/y)I \tag{C.9}$$

as long as

$$y\left(1 - \left\|\frac{\theta_1 + \theta_2}{2} - \theta_1\right\|/I\right) - p \geq 0$$

The unconstrained optimizer to the profit function is given by

$$p^\star = \frac{y * \|\theta_1 - \theta_2\|}{4I} + \frac{y}{2}$$

By plugging this into the IR condition for the consumer with type $\frac{\theta_1+\theta_2}{2}$,

$$\|\theta_1 - \theta_2\| \leq \frac{2I}{3}.$$

and we get the profit

$$\Pi_{\{1,2\}} = \frac{(2I + \|\theta_1 - \theta_2\|)^2 y}{8I}.$$

Whenever $\|\theta_1 - \theta_2\| \in (\frac{2I}{3}, I)$, $p^\star$ violates the IR condition for the cutoff type $\frac{\theta_1+\theta_2}{2}$. However, note that Π in Equation (C.9) is strictly increasing for all $p \in (0, p^\star]$. Thus, the optimal price is given by the IR condition for the cutoff type, which yields

$$p^* = y * \left(1 - \frac{\|\theta_1 - \theta_2\|}{2I}\right)$$

and the profit is given by

$$\Pi_{\{1,2\}} = \frac{\|\theta_1 - \theta_2\| * (2I - \|\theta_1 - \theta_2\|) * y}{I}.$$

The proof concludes. □

C.1.23 Proof of Lemma 3.7.2

Proof. The proof has two parts.

Part i) Again, we check three cases one by one.

- First, we consider heterogeneous product quality (i.e., $I_1 = I_2 = I, \theta_1 = \theta_2$ and $y_1 \geq y_2$). If the ex post matching is one-to-one, then it is characterized by Eq (3.5). In contrast, if optimal matching is such that both influencers are hired by seller 1, then

$$\hat{\Pi}_{\{1,2\}} = \frac{y_1 I}{2} = \hat{\Pi}_{\{1\}} = \hat{\Pi}_{\{2\}}$$

which implies that both influencers receive zero wages, that is, $\hat{w}_1 = \hat{w}_2 = 0$. This implies that influencer 2 will reject being hired by seller 1 together with influencer 1 since $w_2 = \frac{(1-\gamma)\Pi_2^C}{2} > \hat{w}_2$. Similarly, we can show that influencer 2 also rejects being hired together with influencer 1 by seller 2.

- Second, we consider heterogeneous influence power (i.e., $\theta_1 = \theta_2$, $y_1 = y_2$ and $I_1 \geq I_2$). Suppose seller 1 hires both influencers, then the optimal pricing strategy is $p_1^* = p_2^* = \frac{y_1}{2}$. Note that

$$\hat{\Pi}_{\{1,2\}} = \hat{\Pi}_{\{1\}} = \frac{y_1 I_1}{2}$$

  which implies $\hat{w}_2 = 0$ by the Nash bargaining argument. Since $w_2 = \frac{(1-\gamma)\Pi_2^C}{2} > \hat{w}_2$ influencer 2 rejects being hired together with influencer 1 by seller 1. Similarly, influencer 2 rejects the offer that seller 2 hires both influencers.

- We consider heterogeneous style difference (i.e., $y_1 = y_2 = y$, $I_1 = I_2 = I$ and $\theta_1 \neq \theta_2$). Here, we assume $\beta \geq \beta_0$ so that an equilibrium exists under regulated matching in Proposition 3.4.3. Since both sellers are identical, we compare regulated matching in Proposition 3.4.3 and the case when seller 1 hires both influencers.

  If the ex post matching is regulated matching, then

$$w_1 = w_2 = (1-\gamma)\Pi_1^C$$

  where

$$\Pi_1^C = \begin{cases} \frac{yI}{2}, & \text{if } \beta \geq I \\ y * \beta\left(1 - \frac{\beta}{2I}\right), & \text{if } \frac{6}{7}I < \beta < I \\ \frac{3y}{50I} * (2I+\beta)^2, & \text{if } \beta_0 \leq \beta \leq \frac{6}{7}I \end{cases}$$

  If the ex post matching is unbalanced matching, then

$$\hat{w}_1 = \hat{w}_2 = (1-\gamma)(\Pi_{\{1,2\}} - \Pi_1) = (1-\gamma)\left(\Pi_{1,2} - \frac{yI}{2}\right)$$

where

$$\Pi_{\{1,2\}} = \begin{cases} yI, & \text{if } \beta \geq I \\ \frac{(2I+\beta)^2}{8I} y, & \text{if } \beta \leq \frac{2I}{3} \\ \frac{(2I\beta - \beta^2)}{I} y, & \text{if } \beta \in \left(\frac{2I}{3}, I\right) \end{cases}$$

Now, it reduces to check that

$$\Pi_1^C \geq \Pi_{\{1,2\}} - \frac{yI}{2}$$

Note that it is equivalent to check the inequality under the case $y = 1$ and $I = 1$. We can check it case by case. For instance, when $\beta \in [\beta_0, \frac{2I}{3}]$, it reduces to

$$\frac{3}{50}(2+\beta)^2 \geq \frac{1}{8}(2+\beta)^2 - \frac{1}{2}$$

Because $\frac{3}{50} < \frac{1}{8}$, the inequality is most restrictive when $\beta = \frac{2}{3}$. It is easy to check that it indeed holds for $\beta = 23$ for all $\beta \in [0, \frac{2}{3}]$, so it is true for all $\beta \in [\beta_0, \frac{2}{3}]$. Similarly, we can check it for when $\beta \in [2/3, 6/7]$, $\beta \in (6/7, 1]$ and $\beta > 1$.

To summarize, $\hat{w}_1 < w_1$, and thus influencer 1 rejects being hired together inlfuencer 2 by seller 1. By symmetry, seller 2 cannot hire both influencers under equilibrium.

Part ii) There are three possible matching outcomes: 1) regulated matching, which features assortative matching such that $k(j) = j$ (see Lemma 3.6.1); 2) unbalanced matching with seller 1 hiring both influencers; and 3) unbalanced matching with seller 2 hiring both influencers.

- First, we consider regulated matching. In this case, $k(j) = j$ for $j = 1, 2$, and payoffs are given by:

$$U_1 = \frac{\gamma y_1 I_1}{2}, \quad U_2 = \frac{\gamma y_2 I_2}{2}, \quad w_1 = \frac{(1-\gamma)y_1 I_1}{2}, \quad \text{and } w_2 = \frac{(1-\gamma)y_2 I_2}{2}$$

- Second, we consider the case that seller 1 hires both influencers. Note that

  $$\Pi_{1,2} = \Pi_1 + \Pi_2, \quad \Pi_1 = \frac{y_1 I_1}{2} \quad \text{and} \quad \Pi_2 = \frac{y_1 I_2}{2}$$

  and thus

  $$U_1 = \frac{\gamma y_1 (I_1 + I_2)}{2}, \quad U_2 = 0, \quad w_1 = \frac{(1-\gamma)y_1 I_1}{2}, \quad \text{and } w_2 = \frac{(1-\gamma)y_1 I_2}{2}$$

- Second, we consider the case that seller 2 hires both influencers. Note that

  $$\Pi_{1,2} = \Pi_1 + \Pi_2, \quad \Pi_1 = \frac{y_2 I_1}{2} \quad \text{and} \quad \Pi_2 = \frac{y_2 I_2}{2}$$

  and thus

  $$U_1 = 0, \quad U_2 = \frac{\gamma y_2 (I_1 + I_2)}{2}, \quad w_1 = \frac{(1-\gamma)y_2 I_1}{2}, \quad \text{and } w_2 = \frac{(1-\gamma)y_2 I_2}{2}$$

  Obviously, influencers would choose unbalanced matching such that seller 1 hires both influencers.

The proof concludes. □

C.1.24 Proof of Proposition 3.7.1

Proof. The proof consists of two parts. Note that under unbalanced matching, the strong seller hiring both influencer always dominates the weak seller hiring both influencers because the product quality y enters the consumer utility in a product form, which implies total welfare and profits are proportional to the product quality.

- **Case i). Congested influencer market or homogeneous product market**

Denote by W_U the total welfare under (the best) unbalanced matching, and by W_R the total welfare under regulated matching. Now, it suffices to show that $W_U \leq W_R$ and we have three cases.

- Case 1). Heterogeneous influencer power (i.e., $y_1 = y_2 = y$, $\theta_1 = \theta_2 = \theta$ and $I_1 \geq I_2$). Total welfare only depends on the marginal consumer who is indifferent between consuming the product and zero consumption, which further depends on the equilibrium price.

  Under unbalanced matching, $p_1^* = p_2^* = \frac{y}{2}$, and the total size of consumer served is just I_1 (i.e., $\{x \in \mathbf{S}^1 : y\left(1 - \frac{1}{I_1}\|x - \theta\|\right) - p_1^* \geq 0\}$).

  In contrast, under regulated matching, the marginal consumer faces an equilibrium price given by $p_1^C = \frac{2y(I_1 - I_2)}{4I_1 - I_2}$ (see Proposition 3.4.2), which implies that the total size of consumer base is bigger than I_1 because

$$\frac{2y(I_1 - I_2)}{4I_1 - I_2} \leq \frac{y}{2}$$

  Note that the total welfare only depends on the size of consumer base. Hence, unbalanced matching lowers total welfare in the heterogeneous influencer power case, that is, $W_U \leq W_R$.

- Case 2). Heterogeneous influencers' style type (i.e., $y_1 = y_2 = y$, $I_1 = I_2 = I$ and $\theta_1 \neq \theta_2$). Here, we only focus on the case in which a pure strategy price competition equilibrium exists (see Proposition 3.4.3), and thus the equilibrium price under regulated matching is given by

$$p_1^C = p_2^C = \begin{cases} \frac{y}{5I}(2I+\beta), & \text{if } \beta \in [\frac{2}{67}(-7+5\sqrt{10})I, \frac{6}{7}I] \\ y * \left(1 - \frac{\beta}{2I}\right), & \text{if } \beta \in (\frac{6}{7}I, I) \\ \frac{y}{2}, & \text{if } \beta \geq I \end{cases}$$

In contrast, under unbalanced matching (see Lemma 3.7.1),

$$p_1^* = p_2^* = \begin{cases} \frac{y}{2}, & \text{if } \beta \geq I \\ y * \left(\frac{\beta}{4I} + \frac{1}{2}\right), & \text{if } \beta \leq \frac{2I}{3} \\ y * \left(1 - \frac{\beta}{2I}\right), & \text{if } \beta \in \left(\frac{2I}{3}, I\right) \end{cases}$$

We can directly verify that $p_1^C = p_2^C \leq p_1^* = p_2^*$ for $\beta \geq \beta_0$. Thus, total welfare is higher under regulated matching, because a consumer who purchases the product under unmatched matching is also willing to buy it under regulated matching. Hence, $W_U \leq W_R$.

- Case 3). Heterogeneous product quality (i.e., $I_1 = I_2 = I$, $\theta_1 = \theta_2 = \theta$ and $y_1 \geq y_2$).

  Under unbalanced matching, seller 1 hires both influencers and set a price at $p_1^* = p_2^* = \frac{y}{2}$. Influencer 2 is not active in the market. Hence, the total welfare is

$$W_U = \int_{\left\{x \in \mathbf{S}^1 : \|x-\theta\| \leq \frac{1}{2}I\right\}} y(1 - \|x-\theta\|/I)dx = \frac{3}{4}y_1 I.$$

  In contrast, under regulated matching, the equilibrium outcome is given by Proposition 3.4.1, and thus the total welfare is given by

$$W_R = \int_{R_1} y(1 - \|x-\theta\|/I_1)dx + \int_{R_2} y(1 - \|x-\theta\|/I_2)dx$$

where R_1 are consumers served by the first influencer-seller group targets consumers, that is, $R_1 = \{x \in \mathbf{S}^1 : \|x - \theta\| \leq \frac{2Iy_1}{4y_1 - y_2}\}$. Similarly, R_2 are consumers served by the second influencer-seller targets, that is, $R_2 = \{x \in \mathbf{S}^1 : \frac{2Iy_1}{4y_1 - y_2} < \|x - \theta\| \leq \frac{3Iy_1}{4y_1 - y_2}\}$.

We can further simplify the expression of W_R as[3]

$$W_R = \frac{Iy_1(14y_1^2 - 4y_1y_2 - y_2^2)}{(4y_1 - y_2)^2} = Iy_1 * f(x)$$

where $f(x) = (14x^2 - x - 1)/(4x - 1)^2$ and $x := y_1/y_2$.

Note that $f'(x) = -\frac{12(x-1)}{(4x-1)^3} < 0$, and thus $f(x)$ is strictly decreasing for $x > 1$. Moreover, also note that

$$\lim_{x \to 1} f(x) = 1, \quad \text{and} \quad \lim_{x \to \infty} f(x) = \frac{7}{8}.$$

which further implies

$$W_R \geq \frac{7}{8} Iy_1 > \frac{3}{4} Iy_1 = W_U.$$

The proof concludes for the first part. ■

- **Case ii). Uncongested influencer market**

First, note that when $\beta \geq \frac{I_1 + I_2}{2}$, each influencer charge a monopolist price $p_j^* = \frac{y}{2}$ when hired by a seller with product quality y, and serves a sub-population such that $x \in R_j$ and $R_j := \left\{x \in \mathbf{S}^1 : \|x - \theta_j\| \leq \frac{I}{2}\right\}$. Hence, there is zero over-lapping among consumers served by the two influencers.

[3] The skipped algebra is available upon request.

Under unbalanced matching, both influencers are hired by the strong seller 1. Furthermore, total welfare can be calculated as

$$W_U = \frac{3}{4}y_1(I_1 + I_2).$$

Similarly, under regulated matching, the total welfare equals

$$W_R = \frac{3}{4}y_1 I_1 + \frac{3}{4}y_2 I_2$$

Obviously, $W_U > W_R$ as long as $y_1 > y_2$. ■

All the proofs conclude. □

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