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# ROUNDING OFF TO POWERS OF TWO IN THE ECONOMIC LOT SCHEDULING PROBLEM

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The following approach is often taken in solving the economic lot scheduling problem [2,3,4,5,6,7,8,9,10,11,12,13,14, 20]. One first finds approximate order intervals for the products by assuming that each product is produced in equal amounts and at equal intervals of time, and minimizing a cost function that has setup and holding costs. The second step in the procedure is to round off the order intervals $T^*_n$ thus computed to integer multiples of $\beta$ for some positive number $\beta$. The third step is to use these order intervals to find a feasible schedule.

In the first step, that of finding approximate order intervals for the products, most authors completely ignore the setup time. The choice of order intervals is guided solely by the setup and holding costs. Exceptions to this rule are Fujita and Dobson [5,10]. The more successful approach, that taken by Dobson, was also used in the context of production planning models by Jackson, Maxwell, and Muckstadt [16]. According to this approach, the setup and holding costs are minimized subject to a single constraint that states on the average, the total amount of time spent in setting up the machine does not exceeed the total amount of time available for setups. A precise formulation of this problem is given in the sequel.

With regard to the second step, a number of authors have recommended requiring that the order intervals be powers of two times $\beta$ rather than allowing them to be arbitrary integer multiples of $\beta$ [5,11,12,13]. Powers of two are chosen for two reasons. First, it has been observed empirically that power-of-two policies are almost always optimal within the class of policies in which all products are reordered at equal intervals of time, and that they are near-opitmal when they are ot optimal [7,8,13,19]. In addition, the special structure of power-of-two policies makes he third step easier and makes them easier to implement on the factory floor [18].

Many different types of methods have been used to perform the third step, that of finding a feasible schedule. We will not discuss these methods in detail. Methods that require that the order intervals be equal tend to be complex and often do not guarantee feasibility. This is largely due to the fact that the problem of detemining whether a feasible equal-order-interval schedule exists for a given set of order intervals is NP-complete [Hsu83]. The third step is much simpler when the order intervals are allowed to be unequal [3,5,17]

The purpose of this note is twofold. First, we propose a new way of performing the second step, that of rounding off the reorder intervals to powers of two. Second, we show that the cost of implementing the power-of-two order intervals is at most 6% higher than the cost of implementing the approximate order intervals computed in the first step. Of course, the average difference will be much smaller. Therefore the advantages of power-of-two order intervals can be had at what is usually a negligibly small cost.

## The First Step: Finding Approximate Order Intervals.

The mathematical formulation of the first step of the solution procedure, that of finding approximate order intervals for the products, takes on the following form:

$$\text{(P)} \qquad \min: \quad c(\mathcal{T}) \equiv \sum_n \left[ \frac{K_n}{T_n} + H_n T_n \right] \tag{1}$$

$$\text{such that:} \quad \sum_n \frac{\tau_n}{T_n} \le \rho \ , \tag{2}$$

where $\mathcal{T} \equiv (T_n, 1 \le n \le N)$ , $N$ is the number of products, $\rho$ is the fraction of the total operating time that is available for setups, and $K_n$ , $H_n$ , $\tau_n$ , and $T_n$ are respectively the setup cost, holding cost coefficient, setup time, and order interval for product $n$ . It is assumed that $H_n > 0$ and that $K_n + \tau_n > 0$ for all $n$ .

Let $\mathcal{T}^* \equiv (T_n^*, 1 \le n \le N)$ be the solution to (P). Using Lagrangean relaxation, it is easily verified that

$$T_n^{*2} = \frac{K_n + \lambda\tau_n}{H_n} \tag{3}$$

where $\lambda \equiv \nu^+$ is the positive part of $\nu$ and $\nu$ solves

$$\rho = \sum_n \frac{\tau_n \sqrt{H_n}}{\sqrt{K_n + \nu\tau_n}} . \tag{4}$$

The Second Step: Rounding Off the Order Intervals.

Having solved (P), the order intervals $T_n^*$ are rounded off to powers of two as follows. We define $z_n$ and the integers $p_n$ by

$$T_n^* \equiv z_n 2^{p_n} , \quad \sqrt{.5} \leq z_n < \sqrt{2} . \tag{5}$$

We assume that the products are indexed so that $z_n \leq z_{n+1}$ for all $n$ . For each $i$ , $1 \leq i \leq N$ , we consider a solution of the form $\mathcal{T}^i \equiv (T_n^i, 1 \leq n \leq N)$ where

$$T_n^i \equiv \alpha^i 2^{q_n^i} , \tag{6}$$

$$q_n^i \equiv \left\{ \begin{array}{ll} p_n - 1 , & n < i \\ p_n , & n \geq i \end{array} \right\} , \tag{7}$$

and $\alpha^i$ is a positive scalar. Note that the values of $q_n^i$ are chosen to make

$$\frac{\alpha^i}{2z^i} \le \frac{T_n^i}{T_n^*} \le \frac{\alpha^i}{z^i} \tag{8}$$

for all $n$ .

We wish to choose $\alpha^i$ so as to minimize $c(\mathcal{T}^i)$ subject to (2). Let

$$K^i \equiv \sum_n K_n 2^{-q_n^i} \quad \text{and} \quad H^i \equiv \sum_n H_n 2^{q_n^i} . \tag{9}$$

Then the cost of $\mathcal{T}^i$ can be written as $c(\mathcal{T}^i) = K^i/\alpha^i + H^i\alpha^i$ . The value of $\alpha^i$ that minimizes $c(\mathcal{T}^i)$ is clearly

$$\beta^i \equiv \sqrt{\frac{K^i}{H^i}} . \tag{10}$$

However (2) implies that $\alpha^i$ is greater than or equal to

$$\gamma^i \equiv \frac{1}{\rho} \sum_n \tau_n 2^{-q_n^i} . \tag{11}$$

Since $c(\mathcal{T}^i)$ is a convex function of $\alpha^i$ , the optimal value of $\alpha^i$ is given by

$$\alpha^i \equiv \max(\beta^i, \gamma^i) . \tag{12}$$

This identifies $N$ policies, one for each value of $i$. We propose that among these $N$ policies, the policy $\mathcal{T}^{i*}$ that minimizes $c(\mathcal{T}^i)$ over $i$ be chosen.

We now show that $\mathcal{T}^{i*}$ can be computed in $O(N)$ time. Clearly $q_n^1$, $K^1$, $H^1$, $\alpha^1$, and $c(\mathcal{T}^1)$ can be computed in $O(N)$ time. Since $q_n^i = q_n^{i+1}$ for all $n \neq i$, it is easy to verify that given $q_n^i$, $K^i$, $H^i$, and $\alpha^i$ we can compute $q_n^{i+1}$, $K^{i+1}$, $H^{i+1}$, $\alpha^{i+1}$ and $c(\mathcal{T}^{i+1})$ in constant time. Therefore $\mathcal{T}^{i*}$ can be computed in $O(N)$ time.

## The Cost Penalty

We now show that the increase in cost that results from rounding off the order intervals to powers of two is at most 6%, i.e., that $c(\mathcal{T}^{i*}) \leq \frac{1}{2}(\sqrt{2} + \sqrt{.5})\, c(\mathcal{T}*)$. There are two cases to consider, $\Sigma_n \tau_n/T_n < \rho$ and $\Sigma_n \tau_n/T_n = \rho$.

### Case 1: $\sum_n \frac{\tau_n}{T_n} < \rho$.

Suppose we were to increase the value of $\rho$ from its current value to $\sum_n \frac{\tau_n}{T_n^*}$. The cost $c(\mathcal{T}*)$ of $\mathcal{T}*$ is unaffected because $\mathcal{T}*$ is still feasible for (P). However the cost $c(\mathcal{T}^i)$ of $\mathcal{T}^i$ might increase. Therefore the ratio $c(\mathcal{T}^{i*})/c(\mathcal{T}*)$ will be higher for the new value of $\rho$ than it was for the original. Thus the cost penalty for Case 1 is bounded by the cost penalty for Case 2.

Case 2: $\sum_n \frac{\tau_n}{T_n} = \rho$ .

Suppose we were to use $\bar{\mathcal{T}}^i \equiv (\bar{T}^i_n \equiv \tau^i 2^{q^i_n},\ 1 \le i \le N)$ instead of using $\mathcal{T}^i$ . This would be tantamount to choosing $\alpha^i$ so as to make (2) tight for all $i$ , even though this may be a suboptimal choice of $\alpha^i$ . Therefore $c(\bar{\mathcal{T}}^i) \ge c(\mathcal{T}^i)$ for all $i$ .

An upper bound on $c(\mathcal{T}^{i*})/c(\mathcal{T}*)$ can be found as follows. The minimum of $c(\bar{\mathcal{T}}^i)$ over $i$ is an upper bound on the cost of $\mathcal{T}^{i*}$ . We use a weighted average of $c(\bar{\mathcal{T}}^i)$ over $i$ as an upper bound on the minimum over $i$ of $c(\bar{\mathcal{T}}^i)$ . The weight assigned to $i$ is $w_i$ where

$$
\begin{aligned}
w_i &\equiv \log_2(z_i/z_{i-1}) , \quad i \ne 1 ; \\
w_1 &\equiv \log_2(2z_1/z_N) .
\end{aligned}
\tag{15}
$$

Let

$$
z^i_n \equiv T^*_n\, 2^{-q^i_n} = \sqrt{\frac{\mu\tau_n}{H_n}}\, 2^{-q^i_n} . \tag{16}
$$

By (5) and (7),

$$
z^i_n = z_n \quad \text{if} \quad n \ge i \quad \text{and} \quad z^i_n = 2z_n \quad \text{if} \quad n < i . \tag{17}
$$

The following lemma is used in the proof.

Lemma 1. If $m < n$ then $\sum_i w_i \left[ \frac{z_m^i}{z_n^i} + \frac{z_n^i}{z_m^i} \right] \le (\sqrt{2} + \sqrt{.5})$ .

Proof. By (15) and (17)

$$\sum_i w_i \left[ \frac{z_m^i}{z_n^i} + \frac{z_n^i}{z_m^i} \right] = f\left[ \frac{z_m}{z_n} \right]$$

where

$$f(x) \equiv \left[x + \frac{1}{x}\right] \log_2 2x = \left[2x + \frac{1}{2x}\right] \log_2 \frac{1}{x} .$$

Since $m < n$ , (5) implies that $\frac{1}{2} \le \frac{z_m}{z_n} \le 1$ . On the interval $\left[\frac{1}{2} , 1\right]$ , the function $f$ is concave and attains its maximum value of $\sqrt{2} + \sqrt{.5}$ at $x = \sqrt{.5}$ . Therefore

$$\sum_i w_i \left[ \frac{z_m^i}{z_n^i} + \frac{z_n^i}{z_m^i} \right] \le (\sqrt{2} + \sqrt{.5}\ ) . \quad \Box$$

Theorem 1. $c(\mathcal{T}^{i*}) \le \frac{1}{2}(\sqrt{2} + \sqrt{.5})\ c(\mathcal{T}*) \cong 1.06\ c(\mathcal{T}*)$ .

Proof. Let $A_n \equiv \tau_n/T_n^*$ , $B_n \equiv H_n T_n^*$ , and $D_n \equiv K_n/T_n^*$ . Then by (9), (16), and (11) we have

$$c(\mathcal{T}^*) = \sum_n (B_n + D_n) ,$$

$$K^i = \sum_n D_n z_n^i , \quad H^i = \sum_n \frac{B_n}{z_n^i} ,$$

$$\rho = \sum_n A_n , \text{ and}$$

$$\gamma^i = \frac{\Sigma_n A_n z_n^i}{\Sigma_n A_n} = \sum_n a_n z_n^i$$

where $a_n \equiv \dfrac{A_n}{\Sigma_m A_m}$ . Note that $a_n \geq 0$ and $\Sigma_n a_n = 1$ . We can write $c(\bar{\mathcal{T}}^i)$ as

$$c(\bar{\mathcal{T}}^i) = \frac{K_i}{\gamma^i} + H^i \gamma^i$$

$$= \frac{\Sigma_n D_n z_n^i}{\Sigma_n a_n z_n^i} + \left[ \sum_n a_n z_n^i \right] \left[ \sum_n \frac{B_n}{z_n^i} \right] .$$

By (3) we have $B_n = \lambda A_n + D_n$ , so

$$c(\mathcal{T}^*) = \sum_n 2D_n + \sum_n \lambda A_n \tag{18}$$

and

$$\sum_i w_i \, c(\bar{\mathcal{T}}^i) = \sum_{i,n} w_i \left[ \frac{\Sigma_n D_n z_n^i}{\Sigma_n a_n z_n^i} + \left[ \sum_n a_n z_n^i \right] \left[ \sum_n \frac{D_n}{z_n^i} \right] \right]$$

$$+ \sum_{i,n} w_i \lambda \left[ \sum_n a_n z_n^i \right] \left[ \sum_n \frac{A_n}{z_n^i} \right] . \qquad (19)$$

The remainder of the proof is divided into two parts. In Part 1 we show that the first term of (18) is within 6% of the first term of (19), and in Part 2 we show that the second term of (18) is within 6% of the second term of (19). Since $c(\mathcal{T}^{i*}) = \min_i c(\mathcal{T}^i) \le \min_i c(\bar{\mathcal{T}}^i) \le \Sigma_i \, w_i c(\bar{\mathcal{T}}^i)$, this will complete the proof.

Part 1.

Let $d_n \equiv \dfrac{D_n}{\Sigma_m D_m}$. Then $d_n \ge 0$ and $\Sigma_n d_n = 1$. We need to show that $D \le \sqrt{2} + \sqrt{.5}$ where

$$D \equiv \sum_i w_i \left[ \frac{\Sigma_n d_n z_n^i}{\Sigma_n a_n z_n^i} + \left[ \sum_n a_n z_n^i \right] \left[ \sum_n \frac{d_n}{z_n^i} \right] \right] .$$

By Jensen's inequality and by Lemma 1,

$$D \le \sum_i w_i \left[ \left[ \sum_n d_n z_n^i \right] \left[ \sum_n \frac{a_n}{z_n^i} \right] + \left[ \sum_n a_n z_n^i \right] \left[ \sum_n \frac{d_n}{z_n^i} \right] \right]$$

$$= \sum_{m<n} \left[ a_m d_n + d_m a_n \right] \sum_i w_i \left[ \frac{z_m^i}{z_n^i} + \frac{z_n^i}{z_m^i} \right] + \sum_n a_n d_n$$

$$\le \left[ \sqrt{2} + \sqrt{.5} \right] \left[ \sum_{m<n} \left[ a_m d_n + d_m a_n \right] + \sum_n a_n d_n \right]$$

$$\le \left[ \sqrt{2} + \sqrt{.5} \right] \left[ \sum_n a_n \right] \left[ \sum_n d_n \right] = \sqrt{2} + \sqrt{.5} \ . \qquad \square$$

Part 2.

We need to show that $A \le \frac{1}{2} (\sqrt{2} + \sqrt{.5})$ where

$$A \equiv \sum_i w_i \left[ \sum_n a_n z_n^i \right] \left[ \sum_n \frac{a_n}{z_n^i} \right] .$$

By Lemma 1,

$$A = \sum_{m<n} a_m a_n \sum_i w_i \left[ \frac{z_m^i}{z_n^i} + \frac{z_n^i}{z_m^i} \right] + \sum_n a_n^2$$

$$\le \frac{1}{2} \left[ \sqrt{2} + \sqrt{.5} \right] \left[ \sum_{m<n} 2 a_m a_n + \sum_n a_n^2 \right]$$

$$= \frac{1}{2} \left[ \sqrt{2} + \sqrt{5} \right]$$

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