A Trust Region and Affine Scaling Method for Nonlinearly Constrained Minimization

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Abstract. A nonlinearly constrained minimization problem can be solved by the exact penalty approach involving nondifferentiable functions $\sum_i |c_i(x)|$ and $\sum_i \max(0, c_i(x))$. In this paper, a trust region approach based on a 2-norm subproblem is proposed for solving a nonlinear l_1 problem. The (quadratic) approximation and the trust region subproblem are defined using affine scaling techniques. Explicit sufficient decrease conditions based on the approximations are suggested for obtaining a limit point satisfying complementarity, Kuhn-Tucker conditions, and second order necessary conditions. The global convergence analysis of the method is presented in [14].

Key Words. nonlinearly constrained minimization, trust region, sufficient decrease conditions, affine scaling, exact penalty, nonlinear l_1 problem, Newton step

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1. Introduction: Trust Region Methods for Nonlinearly Constrained Problems. The simple and intuitive trust region idea has been successful for unconstrained minimization: $\min f(x)$. In order to compute an improved step, the original nonlinear objective function $f(x)$ is approximated by a simple function $\psi_k(s)$ (usually a quadratic which is a second-order approximation to $f(x)$) and a subproblem is proposed based on this approximation and a bound Δ_k on the step:

$$\min_{s\in\Re^n} \{\psi_k(s), \|s\| \le \Delta_k\}. \tag{1.1}$$

The accuracy of the approximation $\psi_k(s)$ to $f(x_k+s) - f(x_k)$ is controlled by a simple adjustment of the upper bound Δ_k (trust region size) on the step-length. A global solution of (1.1) yields a sufficient reduction of the approximation $\psi_k(s)$. Based on the reduction of this approximation, explicit sufficient decrease conditions on a step s_k have been established. For unconstrained minimization, the trust region method offers a clean theoretical analysis with strong convergence results (if the length of a step is measured with a 2-norm) and simple implementation.

Research on trust region methods for constrained minimization, particularly nonlinearly constrained minimization, is much more limited and less satisfactory [16]. Trust region methods for nonlinear *equality* constrained minimization have been considered, e.g., [23], [5], [21], [4]. However, these methods require some relaxation of the equality and 2-norm bound constraints so that the the trust region subproblem can yield a useful step. For general minimization problems with *inequality constraints*, the research results are even sparser. We are unaware of any trust region method with acceptance conditions on the steps, which yields convergence to a limit point satisfying the first and second-order necessary conditions.

In this paper, we propose a trust region method for solving a general nonlinearly constrained problem, possibly with both equality and inequality nonlinear constraints. The trust region method involves approximately solving a trust region subproblem with a 2-norm bound constraint and some consistent linear equality constraints. Hence it is possible to directly employ the trust region techniques used in unconstrained minimization. Explicit sufficient decreased conditions for optimality are proposed.

1.1. A Nonlinearly Constrained Problem and the Nondifferentiable Exact Penalty Function. Consider a general nonlinearly constrained minimization problem

$$\begin{aligned} & \min_{x\in\Re^n} f(x) \\ \text{subject to}\quad & c_i(x) = 0, \quad i = 1:m, \\ & c_i(x) \le 0, \quad i = m+1 : m+t. \end{aligned} \tag{1.2}$$

The exact penalty approach using nondifferentiable functions $\sum_{i=1}^{m}|c_i(x)|$ and $\sum_{i=m+1}^{m+t}\max(0, c_i(x))$ is an attractive technique for unifying minimization and achievement of feasibility [11]. Moreover, the constraint functions, appear unaltered in the penalty function and hence no additional nonlinearity is introduced. A general nonlinear programming problem involving equality and inequality constraints can be solved by a minimization of the following form:

$$\min_{x\in\Re^n} \nu f(x) + \sum_{i=1}^{m}|c_i(x)| + \sum_{i=m+1}^{m+t}\max(0, c_i(x)), \tag{1.3}$$

where ν is a penalty parameter. The apparent disadvantage of the use of (1.3) is the introduction of nondifferentiability which may cause slow global convergence and exhibit a local Maratos effect (e.g., [11]). However, we believe that the nondifferentiability in the functions $\sum_{i=1}^{m} |c_i(x)|$ and $\sum_{i=m+1}^{m+t} \max(0, c_i(x))$ is not any more difficult to handle than feasibility constraints. Indeed, as we intend to demonstrate in this paper, this nondifferentiability can be efficiently managed and the Maratos effect need not occur if the Hessians of $f(x)$ and $c(x)$ are used.

1.2. A Nonlinear l_1 Problem. In order to emphasize the main ideas of our approach, we ignore the penalty parameter adjustment issue. Since nondifferentiability occurs, for both $\sum_{i=1}^{m} |c_i(x)|$ and $\sum_{i=m+1}^{m+t} \max(0, c_i(x))$, when a function $c_i(x)$ is zero, it can essentially be dealt with in the same way. Therefore in our presentation we concentrate on a general nonlinear l_1 problem of the following formulation:

$$\min_{x \in \Re^n} \Upsilon(x) \stackrel{\text{def}}{=} \|c(x)\|_1 + f(x). \tag{1.4}$$

In (1.4), $\|c(x)\|_1 \stackrel{\text{def}}{=} \sum_{i=1}^{m} |c_i(x)|$ denotes the 1-norm of the vector $c(x)$. We assume the column vector function $c(x) \stackrel{\text{def}}{=} [c_1(x); \cdots; c_m(x)]$, where $c_i(x), 1 \le i \le m$, and $f(x)$ are twice continuously differentiable. (Our presentation will use many Matlab [15] notations and the semicolon here is used to create a column vector c from scalars c_i). We assume that, function values, gradients, and possibly Hessians can be computed. The objective is to compute a local minimizer of (1.4).

In addition to solving a general nonlinear programming problem, a nonlinear l_1 problem (1.4) is itself an important class of optimization problems. Many application problems arise directly as minimizing a function of this form, e.g., nonlinear data fitting [3, 2, 22, 12, 18, 19].

Techniques for handling the nondifferentiability in $\|c(x)\|_1$ and nonlinear constraints are closely related. This is reflected by the fact that existing methods for solving an l_1 problem (1.4) are closedly related to the sequential quadratic programming methods for general nonlinearly constrained problems (e.g., [20, 12, 18, 22, 2, 3]).

The nondifferentiability of $\|c(x)\|_1$ makes it difficult to extend the trust region idea to (1.4): an approximation becomes invalid as soon as a function $c_i(x)$ changes its sign. This may hinder a sufficient reduction of an approximation to $\Upsilon(x)$. Attempts have been made to generalize a trust region idea to the problem (1.4) using either a linear programming (LP) or quadratic programming (QP) problem as a trust region subproblem [12, 11]. Unfortunately, these trust region methods are not satisfactory: e.g., the method in [12] involves switching between two phases of the algorithm. Furthermore, as mentioned in [16], sufficient decrease conditions have not been formulated for methods using a LP or QP as a subproblem. These existing trust region methods do not guarantee convergence to a local minimizer for a nonlinear l_1 problem (1.4). The unsatisfactory state of trust region methods for the l_1 problem is not surprising: it is consistent with that of trust region methods for nonlinearly constrained optimization problems.

1.3. Some Features of the New Trust Region and Affine Scaling Method. In this paper, we propose a trust region method using an affine scaling technique for a nonlinear l_1 minimization (1.4). This new method is a further development of some of our previous research on solving simpler problems including linear l_1 [8], l_∞ [7], p-th norm minimization [13], and minimizing a nonlinear function with bound constraints [9].

One of the main ideas behind our new approach is to combine an affine scaling technique with the trust region idea to overcome nondifferentiability. Affine scaling methods have been developed for linear programming problems to approach solutions by going through the strictly feasible region (e.g.,[10, 1]). An implicit assumption for an interior point method for a linear programming problem is an initial point which is approximately centered in the strictly feasible region. Unlike the traditional simplex methods which follow the boundaries of the feasible regions, affine scaling methods generate approximate solutions in the strict interiors of the feasible regions and have proven to be computationally effective for large linear programming problems. Moreover, variations of affine scaling methods can have polynomial complexity for linear programming problems.

Two important properties of affine scaling methods for linear programs are of special importance. Firstly, *all* the constraints are considered when determining a step s_k. When computing a descent direction, an active set method ignores the constraints which are not active at x_k. Secondly, a better step can be computed if x_k is relatively centered in the feasible region.

For a nonlinearly constrained problem or a nonlinear l_1 problem (1.4), computing a relatively centered initially strictly feasible point is a very difficult task. Moreover, it is unclear whether it pays to do so, particularly when an initial point with much smaller objective function value than that of a "center" is available. In addition, it is more costly to maintain strict feasibility with an affine scaling technique when a problem is nonlinear (e.g., a line search is performed on linear approximations). This suggests that we need to be flexible when applying an affine scaling technique to a nonlinearly constrained problem. Specifically, in our strategy for solving a nonlinear l_1 problem (1.4), we allow an iterate x_k to be on a nondifferentiable curve, although attempts can be made to stay away from it using a backtracking technique. Moreover, we incorporate distance information related to the nondifferentiable curves when determining a new step, using affine scaling technique.

The proposed trust region and affine scaling method works in a surprisingly similar fashion to a trust region method for unconstrained minimization. At each iteration, an affine scaling (diagonal) matrix $\mathcal{M}_k$ is chosen to ensure that a step can be generated for both complementarity and dual feasibility. Based on second-order information, a trust region subproblem is approximately solved to determine a step. A second-order approximation to the change of the objective function is sufficiently decreased and the agreement between the approximation and the original nonlinear l_1 function is measured. Finally, according to the agreement measurement, the trust region size Δ_k is adjusted in a simple fashion to ensure sufficient reduction of the nonlinear l_1 function at each iteration.

The main cost of each iteration is an evaluation of the functions/gradients/Hessians of $f(x)$ and $c(x)$ and computing an approximate solution to a 2-norm trust region subproblem with consistent linear equality constraints. Hence the techniques for solving a 2-norm trust region subproblem, developed for unconstrained minimization, can be readily applied.

The emphasis of this paper is on the motivation and derivation of the new trust region and affine scaling method (subsequently referred to as TRASM). This method has comparable properties to a trust region method for unconstrained minimization. Explicit sufficient decrease conditions for complementarity, dual feasibility and second order optimality are proposed based on function reduction. Some preliminary computational results will be reported; more detailed computational investigation is delayed to a subsequent

σ	10	100	1000
SL$_1$QP	12	193	2000
TRASM	12	14	24

TABLE 1
Rosenbrock Problem: Number of Iterations Using SL$_1$QP and TRASM

paper. Although the contribution of this paper is mostly theoretical, the real objective of our research is to develop a method which can compute a solution to a nonlinear l_1 problem (1.4) efficiently and reliably, particularly in the large-scale setting. Based on the success of the trust region method for unconstrained minimization and the affine scaling method for linear programming problems, we believe that our new affine scaling trust region method has a great potential for achieving computational efficiency.

To illustrate the performance of algorithm TRASM, we consider the following Rosenbrock equations problem:

$$
\begin{aligned}
c_1(x) &= \sigma(x_2 - x_1^2) = 0, \\
c_2(x) &= 1 - x_1 = 0,
\end{aligned}
$$

with the starting point $x = (-1.2, 1)$. Using a successive quadratic programming approach with the l_1 exact penalty function to measure the progress, the performance is increasingly poor as the positive parameter σ becomes large as reported in [11]. In Table 1, we list the number of iterations taken by SL$_1$QP and a preliminary implementation of algorithm TRASM.

The trajectory of the iterates generated by TRASM is graphed in FIG. 1.

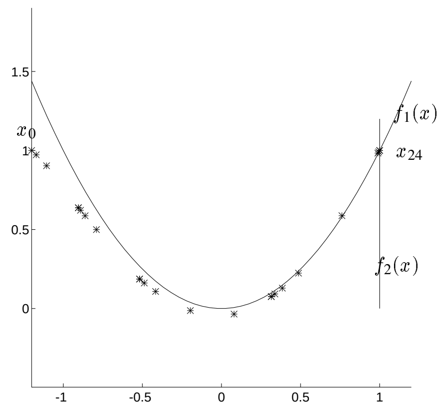

FIG. 1. *Trajectory of Algorithm TRASM for the Rosenbrock Problem with* $\sigma = 1000$

1.4. Organization, Notations and Assumptions. The presentation of this paper is organized as follows:

The global convergence analysis of TRASM is presented in [14]. A more succinct description of the algorithm is also given there.

Notationally, we generally use $(\cdot)$ to emphasize a dependence relation, e.g., $\lambda_k(D_k)$ depends on D_k. Our presentation follows many **Matlab** [15] notations. For example, a semicolon ; in $[c_1; c_2]$ is used to create a column vector $\begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$ while , in $[c_1, c_2]$ is used to create a row vector $[c_1 c_2]$. Matrices can be generated by submatrices in the same fashion. In addition, given any $d \in \Re^m$, $\text{diag}(d)$ denotes an m-by-m diagonal matrix with the vector s defining the diagonal entries in their natural order; $|d|$ denotes a vector of the same dimension with the ith component equal $|d_i|$. Moreover, for any nonsingular matrix $A \in \Re^{m\times m}$ and any $k > 0$, A^{-k} denotes the inverse of A^k, where A^k is the k-th power of A. The sign function is defined as below:

$$\text{sgn}(c_i) \stackrel{\text{def}}{=} \begin{cases} 1 & \text{if } c_i > 0, \\ 1 & \text{if } c_i = 0, \\ -1 & \text{if } c_i < 0. \end{cases} \tag{1.5}$$

We make the following smoothness and compactness assumptions throughout the presentation: Given an initial point $x_0 \in \Re^n$, we assume that the level set $\mathcal{L}$ of $\Upsilon(x)$ is compact, where $\mathcal{L} \stackrel{\text{def}}{=} \{x : x \in \Re^n \text{ and } \Upsilon(x) \leq \Upsilon(x_0)\}$. The functions $f(x)$ and $c(x)$ are assumed to be at least continuously differentiable in $\mathcal{L}$.

2. Mathematical Background on Nonlinear l_1 Problems. Characterizations in terms of necessary and sufficient conditions for a local minimizer of a nonlinear l_1 problem (1.4) have been established, e.g., [11]. These characterizations have been described in many different forms. The followings are the most often seen, e.g., Theorem 14.2.1, 14.2.2, 14.2.3 and Lemma 14.3.1 in [11].

Let $J^* \stackrel{\text{def}}{=} J(x^*)$ with $J(x) \stackrel{\text{def}}{=} [\nabla c_1(x), \cdots, \nabla c_m(x)] \in \Re^{n\times m}$.

Necessary Conditions: If x^* is a local minimizer of a nonlinear l_1 problem (1.4) then there exists $y^* \in \Re^m$ with $|y^*| \leq 1$ and $y_i^* = \text{sgn}(c_i^*)$ if $c_i^* \neq 0$, such that

$$\nabla f^* + J^* y^* = 0.$$

Furthermore, if $\{J_i^* : c_i^* = 0\}$ are linearly independent, then, for any s with $J_i^{*T} s = 0$, for all i with $c_i^* = 0$,

$$s^T \nabla^2 f^* s + \sum_{i=1}^{m} s^T y_i^* \nabla^2 c_i^* s \geq 0.$$

Sufficient Conditions: Assume that $[x^*; y^*]$ satisfies the first-order necessary conditions, i.e.,

$$\nabla f^* + J^* y^* = 0,$$

with $|y^*| \leq 1$ and $y_i^* = \mathrm{sgn}(c_i^*)$ if $c_i^* \neq 0$. Furthermore, $\{J_i^* : c_i^* = 0\}$ are linearly independent and for any s with $J_i^{*T} s = 0$, for all i with $c_i^* = 0$,

$$s^T \nabla^2 f^* s + \sum_{i=1}^{m} y_i^* s^T \nabla^2 c_i^* s > 0.$$

Then x^* is a strict isolated minimizer of (1.4).

Based on the definition (1.5) of sgn(c), it is straightforward to verify that the following is equivalent to the first-order necessary conditions

$$\begin{cases} J(x)\lambda = -(\nabla f(x) + J(x)\mathrm{sgn}(c)), \\ -\mathrm{diag}(|c|^{\frac{1}{2}})\lambda = 0, \\ -2e \leq \mathrm{diag}(\mathrm{sgn}(c))\lambda \leq 0, \end{cases} \tag{2.1}$$

where $e \stackrel{\text{def}}{=} [1; \cdots; 1] \in \Re^m$. (Recall that sgn$(0) = 1$). We express the first-order necessary conditions in this form to simplify our subsequent presentation. The first two equations in (2.1) are referred to as the *complementarity condition* and the inequality $-2e \leq \mathrm{diag}(\mathrm{sgn}(c))\lambda \leq 0$ is called *dual feasibility*. Furthermore, we say that the *strict complementarity condition* is satisfied at a point x if $|c| + \min\{|\lambda|, |\mathrm{diag}(\mathrm{sgn}(c))\lambda + 2e|\} > 0$.

3. Affine Scaling and Trust Regions. Compared to a line search based algorithm, a trust region method uses Hessian information in a natural way: the negative curvature information is exploited by solving a trust region subproblem with a 2-norm bound constraint. Hence, it is reasonable to assume that the Hessians of the involved functions ($f(x)$ and $c_i(x)$ for an l_1 problem (1.4)) are available. Two essential components of the trust region idea include a second-order approximation (e.g., a simple quadratic) to the change of the original objective function and defining a subproblem, which is computationally tractable and whose solution yields a sufficient reduction of the approximation. We seek an approximation for which it is possible to perform one-dimensional minimization and a subproblem which minimizes some quadratic function under a 2-norm bound on the step.

For unconstrained minimization, the nonlinear objective function can be approximated by a simple quadratic function based on Taylor's theorem. Unfortunately, the nonlinear l_1 function (1.4) is much more complicated; a simple quadratic is not readily available for the second-order approximation to the nonlinear l_1 function due to nondifferentiability. Therefore it is of both theoretical and computational interest to develop a second-order approximation. We explore some possibilities below.

Let $J_k \stackrel{\text{def}}{=} J(x_k)$. Define $\nabla^2\|c\|_1 \stackrel{\text{def}}{=} \sum_{i=1}^n \text{sgn}(c_i)\nabla^2 c_i$. If x_k is at a differentiable point, i.e., $|c(x_k)| > 0$, then $\nabla^2\|c_k\|_1$ is the Hessian of $\|c(x_k)\|_1$ at x_k. Based on Taylor's expansion up to the second-order, we can consider the quadratic function $\mathcal{Q}_k(s)$ below:

$$\mathcal{Q}_k(s) \stackrel{\text{def}}{=} \nabla f_k^T s + \text{sgn}(c_k)^T J_k^T s + \frac{1}{2} s^T \nabla^2 f_k s + \frac{1}{2} s^T \nabla^2\|c_k\|_1 s. \tag{3.1}$$

If there is some $c_i(x_k) = 0$, there is no neighborhood within which $\mathcal{Q}_k(s)$ is a second-order approximation (the objective function is not differentiable at this point). Assume $|c_k| > 0$ and that $f(x)$ and $c_i(x)$ are. twice continuously differentiable. Then, the objective function $\Upsilon(x)$ for (1.4) is tiwce continuously differentiable at x_k. Nonetheless, $\mathcal{Q}_k(s)$ is a valid second-order approximation only when $\text{sgn}(c_{ki} + \nabla c_{ki}^T s + \frac{1}{2} s^T \nabla^2 c_{ki} s)$ remains the same as $\text{sgn}(c_{ki})$ for all i. In order to maintain validity of approximation $\mathcal{Q}_k(s)$ to $\Upsilon(x_k + s) - \Upsilon(x_k)$, it is possible that only small steps can be taken and thus fail to yield a sufficient progress.

A nonlinear l_1 function can also be approximated by piecewise linear or quadratic functions. Consider the piecewise linear function $\phi_k(s)$ where

$$\phi_k(s) \stackrel{\text{def}}{=} \nabla f_k^T s + \|c_k + J_k^T s\|_1 - \|c_k\|_1. \tag{3.2}$$

It is clear that $\phi_k(s)$ is a linear approximation to $\Upsilon(x_k + s) - \Upsilon(x_k)$, i.e.,

$$\Upsilon(x_k + s) - \Upsilon(x_k) = \phi_k(s) + O(\|s\|_2^2).$$

The nonlinear function $\Upsilon(x_k + s) - \Upsilon(x_k)$ can also be approximated by the following piecewise quadratic:

$$\nabla f_k^T s + \frac{1}{2} s^T \nabla^2 f_k s + \|c_k + J_k^T s + [\frac{1}{2} s^T \nabla^2 c_{k1} s; \cdots; \frac{1}{2} s^T \nabla^2 c_{km} s]\|_1 - \|c_k\|_1.$$

However, these approximations are computationally complicated: A one-dimensional minimization of the above piecewise quadratic is not a straightforward task.

Due to complexity, these piecewise functions are unappealing as objective functions in a trust region subproblem. Subsequently, we develop a simple quadratic which is asymptotically a second-order approximation.

3.1. Affine Scaling and Second-Order Approximation. Next we investigate how affine scaling can help us maintain simple second-order approximations to the nonlinear l_1 function.

Let $D(x)$ represent the following distance measurement to nondifferentiable curves $c(x) = 0$:

$$D(x) \stackrel{\text{def}}{=} \text{diag}\{|c(x)|^{\frac{1}{2}}\}, \tag{3.3}$$

and $D_k \stackrel{\text{def}}{=} D(x_k)$.

The complementarity condition in (2.1) specifies that,

$$\begin{bmatrix} J(x) \\ -D \end{bmatrix} \lambda = -\begin{bmatrix} \nabla f(x) + J(x)\text{sgn}(c(x)) \\ 0 \end{bmatrix}.$$

Hence, at any local minimizer of (1.4), the orthogonal projection of $[\nabla f(x) + J(x)\text{sgn}(c(x)); 0]$ on to the null space of $[J(x)^T, -D(x)]$ is zero.

Let us regard the function $\Upsilon(x)$ as a function in the augmented space $[s;w] \in \Re^{m+n}$: $\Upsilon(x,w) \stackrel{\text{def}}{=} \Upsilon(x)$. (The variable w does not appear in the definition of the function $\Upsilon(x)$.) Then the gradient of $\Upsilon(x,w)$ equals $[\nabla f(x) + J(x)\text{sgn}(c(x)); 0]$ (we subsequently refer it as the the argumented gradient of $\Upsilon(x)$). Since minimizing function $\Upsilon(x,w)$ in the null space of $[J_k^T, -D_k]$ will result in the orthogonal projection of the argumented gradient $[\nabla f_k + J_k\text{sgn}(c_k); 0]$ approaching zero, it is reasonable to search for an improvement $[s;w]$ in $\text{null}([J_k^T, -D_k])$. We observe that, assuming $J_k^T s - D_k w = 0$, the simple linear function $(\nabla f_k + J_k\text{sgn}(c_k))^T s$ becomes a first-order approximation to $\Upsilon(x_k + s) - \Upsilon(x_k)$. This suggests that we consider solving:

$$
\begin{array}{lll}
 & & \displaystyle\min_{s,w} \; (\nabla f_k + J_k\text{sgn}(c_k))^T s + 0^T w \\
(3.4) & \text{subject to} & J_k^T s - D_k w = 0 \\
 & & \left\| \begin{bmatrix} s \\ w \end{bmatrix} \right\|_2 \le \Delta_k.
\end{array}
$$

The importance of the equality constraints in (3.4) goes beyond satisfaction of the complementarity condition. It imposes, up to the first-order, a control over the manner in which the function $c(x)$ changes: the changes are correlated to the quantity $|c_k|^{\frac{1}{2}}$ which is a measurement of the distance to nondifferentiable curves. In this regard, the role of the scaling matrix D_k is similar to the affine scaling transformation used in linear programming.

Consider a bounded sequence $\{d_k\} \in \Re^n$. Let $[s_k; w_k]$ denote the orthogonal projection of the augmented vector $[d_k; 0] \in \Re^{m+n}$ to the null space of $[J_k^T, -D_k]$. Then

$$
\begin{aligned}
\begin{bmatrix} s_k \\ w_k \end{bmatrix} &= \left(\begin{bmatrix} d_k \\ 0 \end{bmatrix} - \begin{bmatrix} J_k \\ -D_k \end{bmatrix} \lambda_k \right), \\
&= -\begin{bmatrix} d_k - J_k\lambda_k \\ D_k\lambda_k \end{bmatrix},
\end{aligned}
$$

where λ_k is the least squares solution to the following $(m+n)$-by-m linear system

$$
\begin{bmatrix} J_k \\ -D_k \end{bmatrix} \lambda_k \stackrel{\text{LS}}{=} \begin{bmatrix} d_k \\ 0 \end{bmatrix}.
$$

Therefore

$$
J_k^T s_k = D_k w_k = -D_k^2 \lambda_k.
$$

Geometrically, this means that, given any direction sequence $\{d_k\} \in \Re^n$, sufficiently large stepsize can be taken along the orthogonal projection of augmented direction $[d_k; 0]$ before the first-order approximation $(\nabla f_k + J_k\text{sgn}(c_k))^T s$ is invalidated due to a change of a sign. In particular, letting $d_k = -(\nabla f_k + J_k\text{sgn}(c_k))$, sufficiently large progress can be made along the projected gradient direction s_k to satisfy the complementarity condition.

If we consider minimization in the augmented space $[s;w] \in \Re^{m+n}$, the natural ball trust region,

$$
\left\| \begin{bmatrix} s \\ w \end{bmatrix} \right\|_2 \le \Delta_k,
$$

corresponds to an ellipsoid in the original space $x \in \Re^n$. The affine scaling affects the shape of the ellipsoid, e.g., see FIG.2.

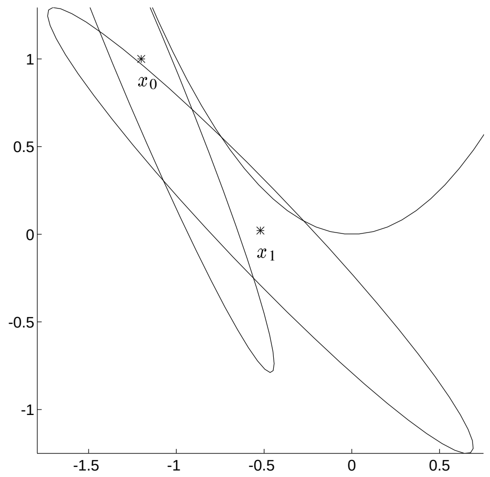

FIG. 2. *Trust Regions at x_0 and x_1 for the Rosenbrock Problem with $\sigma = 10$*

Notice that the scaling matrix $D_k = \text{diag}(|c_k|^{\frac{1}{2}})$ is different from an affine scaling matrix used in linear programming (e.g., [10]). There are three reasons for our choice. Firstly, the squareroot guarantees sufficient progress. Secondly, for an l_1 problem (1.4), there is a need to cross a nondifferentiable curve but the scaling diag$(|c_k|)$ prohibits it. Thirdly, the squareroot scaling is consistent with taking a Newton step locally (see §3.3).

We would like to point out that the constraints $J_k^T s - D_k w = 0$ are generally easy to satisfy: let $w = D_k^{-1} J_k^T s$ for any s assuming that $D_k > 0$. Moreover, the gradient components in the objective function of (3.4), corresponding to w variables, are zero. The structure of the problem can be computationally exploited depending on problem size and sparsity.

Next we derive a simple second-order approximation to the change $\Upsilon(x_k + s) - \Upsilon(x_k)$. If there exists $c_{ki} = 0$, the quadratic $\mathcal{Q}_k(s)$ in (3.1) is not a second-order approximation to this change. Even if $|c_k| > 0$, the quadratic $\mathcal{Q}_k(s)$ is a second-order approximation only when $\text{sgn}(c_{ki} + \nabla c_{ki}^T s + \frac{1}{2} s^T \nabla^2 c_{ki} s)$ equals $\text{sgn}(c_{ki})$. This can prohibit a large stepsize being taken and suggests a need to regulate the change of the function c_i up to second-order, e.g., let $d \in \Re^n,\ h \in \Re^m$,

$$\nabla c_{ki}^T d + \frac{1}{2} d^T \nabla^2 c_{ki} d - D_{kii} h_i = 0, \quad \text{for all } i = 1, \cdots, m. \tag{3.5}$$

Satisfying the quadratic constraints exactly (3.5) is a challenging task. However, as described below, the quadratic constraints can be satisfied, asymptotically, up to second-order by following an (explicit) parametric quadratic curve.

Let $[Z_k; \bar{Z}_k]$ be an orthonormal basis for the null space of $[J_k^T, -D_k]$, where $Z_k \in \Re^{n \times n}$ and $\bar{Z}_k \in \Re^{m \times n}$. Assume further that

$$\begin{bmatrix} J_k \\ -D_k \end{bmatrix} = \begin{bmatrix} Y_k \\ \bar{Y}_k \end{bmatrix} R_k, \quad \text{where } Y_k \in \Re^{n \times m}, \bar{Y}_k \in \Re^{m \times m} \text{ and } \begin{bmatrix} Y_k \\ \bar{Y}_k \end{bmatrix} \text{ is orthonormal}. \tag{3.6}$$

Hence $\begin{bmatrix} Y_k \\ \bar{Y}_k \end{bmatrix}$ is an orthonormal basis of $\begin{bmatrix} J_k \\ -D_k \end{bmatrix}$.

We consider the parametric quadratic curve $\mathcal{P}_k$ below:

$$\mathcal{P}_k \stackrel{\text{def}}{=} \{ \begin{bmatrix} Z_k \\ \bar{Z}_k \end{bmatrix} z + \begin{bmatrix} Y_k \\ \bar{Y}_k \end{bmatrix} R_k^{-T} \vartheta_k(Z_k z) : z \in \Re^n \}.$$

where $\vartheta_k(Z_k z)$ denotes the *correction* below:

$$\vartheta_k(s) \stackrel{\text{def}}{=} - \begin{bmatrix} \frac{1}{2} s^T \nabla^2 c_{k1} s \\ \vdots \\ \frac{1}{2} s^T \nabla^2 c_{km} s \end{bmatrix}, \quad \text{for any } s \in \Re^n. \tag{3.7}$$

In terms of the s variables, the quadratic curve $\mathcal{P}_k$ can be described by

$$\mathcal{P}_k = \{ u_k(s) : u_k(s) \stackrel{\text{def}}{=} s + Y_k R_k^{-T} \vartheta_k(s),\ J_k^T s - D_k w = 0 \}. \tag{3.8}$$

Computationally, the quadratic curve $\mathcal{P}_k$ can be followed by first computing s satisfying $J_k^T s - D_k w = 0$ and then correcting s with $Y_k R_k^{-T} \vartheta_k(s)$ (the orthonormal basis may not be explicitly required). Our correction technique is a slight modification of those used for nonlinear equality constrained minimization [6, 4]; we do not require evaluation of the functions $c_i(x)$ at $x_k + s_k$. Instead, the Hessians of $c(x)$ at x_k are used. This implies that we need to store the Hessians $\nabla^2 c_{ki}$, which can be done for many classes of large sparse problems. Our method can certainly be modified to evaluate the functions $c(x)$ at $x_k + s_k$ instead of storing $\nabla^2 c_{ki}$. We choose to store the Hessians in this presentation.

We now examine the quadratic $\mathcal{Q}_k(u)$ in (3.1) on the quadratic curve (3.8). Firstly, we observe that the quadratic $\mathcal{Q}_k(u)$ is a quartic of variable s where $J_k^T s - D_k w = 0$.

Let λ_k denote a least squares solution to

$$\begin{bmatrix} J_k \\ -D_k \end{bmatrix} \lambda \stackrel{\text{LS}}{=} - \begin{bmatrix} \nabla f_k + J_k \text{sgn}(c_k) \\ 0 \end{bmatrix}. \tag{3.9}$$

From the above definition of λ_k and (3.6), $\lambda_k = -R_k^{-1}[Y_k^T, \bar{Y}_k^T] \begin{bmatrix} \nabla f_k + J_k \text{sgn}(c_k) \\ 0 \end{bmatrix}$. Using $u - s = Y_k R_k^{-T} \vartheta_k(s)$, we have

$$(\nabla f_k + J_k \text{sgn}(c_k))^T (u - s) = \frac{1}{2} s^T \sum_{i=1}^m \lambda_{ki} \nabla^2 c_{ki} s. \tag{3.10}$$

A simple verification reveals that $\|u - s\|_2 = O(\|s\|_2^2)$ under some mild assumptions.

LEMMA 3.1. *Assume that the level set $\mathcal{L} = \{x : \Upsilon(x) \le \Upsilon(x_0)\}$ is compact. Let $\{x_k\}$ be any sequence in $\mathcal{L}$. Assume that* $\begin{bmatrix} J_k \\ -D_k \end{bmatrix}$ *has full rank. Then*

1. *If $f(x)$ and $c_i(x)$ are continuously differentiable, then the multiplier function λ_k is bounded, i.e., there exists χ_λ such that*

$$\|\lambda_k\|_2 \leq \chi_\lambda;$$

2. *If $f(x)$ and $c_i(x)$ are twice continuously differentiable and $J_k^T s - D_k w = 0$, then*

$$\|u - s\|_2 = \| \begin{bmatrix} Y_k \\ \bar{Y}_k \end{bmatrix} R_k^{-T} \vartheta_k(s) \|_2 = O(\|s\|_2^2).$$

For coherence of presentation, we delay the proofs of all the theorems in this section until §3.5.

When the Hessian $\nabla^2 c_{ki}$ of $c_i(x)$ are available, we denote $\psi_k(s)$ as the simple quadratic:

$$\psi_k(s) \stackrel{\text{def}}{=} \nabla f_k^T s + \text{sgn}(c_k)^T J_k^T s + \frac{1}{2} s^T \nabla^2 f_k s + \frac{1}{2} s^T \sum_{i=1}^{m} \lambda_{ki} \nabla^2 c_{ki} s. \tag{3.11}$$

It is then clear that

$$\begin{aligned} \mathcal{Q}_k(u) &= \mathcal{Q}_k(s) + \frac{1}{2} s^T \sum_{i=1}^{m} \lambda_{ki} \nabla^2 c_{ki} s + o(\|s\|_2^2) \\ &= \psi_k(s) + o(\|s\|_2^2), \end{aligned}$$

Unfortunately, the quadratic $\psi_k(s)$ is asymptotically a second-order approximation only when the sign of each quadratic $c_{ki} + \nabla c_{ki}^T u + \frac{1}{2} u^T \nabla^2 c_{ki} u$ is the same as that of c_{ki}. But controlling the signs of these quadratics along the parametric curve $\mathcal{P}_k$ is difficult because $\mathcal{Q}_k(s)$ is a quartic. Next we attempt to overcome this difficulty.

From $\begin{bmatrix} J_k \\ -D_k \end{bmatrix} = \begin{bmatrix} Y_k \\ \bar{Y}_k \end{bmatrix} R_k$, it follows that

$$[J_k^T, -D_k] \begin{bmatrix} Y_k \\ \bar{Y}_k \end{bmatrix} R_k^{-T} \vartheta_k(s) = \vartheta_k(s). \tag{3.12}$$

From $u - s = Y_k R_k^{-T} \vartheta_k(s)$ and (3.12), we have

$$J_k^T (u - s) - \vartheta_k(s) = D_k \bar{Y}_k R_k^{-T} \vartheta_k(s). \tag{3.13}$$

Using Lemma 3.1, we have: for $1 \leq i \leq m$,

$$c_{ki} + J_{ki}^T u + \frac{1}{2} u^T \nabla^2 c_{ki} u = c_{ki} + J_{ki}^T s + O(|c_{ki}|^{\frac{1}{2}} \|s\|_2^2) + o(\|s\|_2^2). \tag{3.14}$$

This relation reveals that the sign of the quadratic $c_{ki} + \nabla c_{ki}^T u + \frac{1}{2} u^T \nabla^2 c_{ki} u$ is *asymptotically* determined by the linear term $c_{ki} + \nabla c_{ki}^T s$ up to second-order.

Let $\mathcal{F}_k$ denote the following first-order sign restricted region:

$$\mathcal{F}_k \stackrel{\text{def}}{=} \{s : \text{diag}(c_k)(c_k + J_k^T s) \geq 0\}, \tag{3.15}$$

By maintaining $s \in \mathcal{F}_k$, $\psi_k(s)$ is a second-order approximation *asymptotically*. This is summarized in Theorem 3.2 below (A proof is given in §3.5).

THEOREM 3.2. *Assume that the functions $f(x)$ and $c(x)$ are twice continuously differentiable, the level set $\mathcal{L}$ is compact and $\begin{bmatrix} J_k \\ -D_k \end{bmatrix}$ has full rank for $x_k \in \mathcal{L}$. In addition, assume that $\{x_k\}$ converges to x^*, $x_{k+1} = x_k + u_k(s_k)$, where $u_k(s_k) = s_k + Y_k R_k^{-T} \vartheta_k(s_k)$ and $J_k^T s_k - D_k w_k = 0$. If $s_k \in \mathcal{F}_k$, then*

$$\Upsilon(x_k + u_k(s_k)) - \Upsilon(x_k) = \psi_k(s_k) + o(\|s_k\|_2^2).$$

Since the quadratic $\psi_k(s)$ does not depend on the range space component of $[J_k^T, -D_k]$, it may be reasonable to solve the following problem based on the approximation $\psi_k(s)$ to obtain a step s:

$$\begin{aligned} & \min_{s,w} \psi_k(s) \\ \text{subject to} \quad & J_k^T s - D_k w = 0 \\ & \left\| \begin{bmatrix} s \\ w \end{bmatrix} \right\|_2 \leq \Delta_k. \end{aligned} \tag{3.16}$$

The 2-norm bound is only imposed on the null space component (of $[J_k^T, D_k]$) because the range space component converges to zero quadratically as the orthogonal space component converges to zero.

3.2. Towards Dual Feasibility. Based on complementarity, we have motivated the inclusion of affine scaling constraints $J_k^T s - D_k w = 0$. These affine scaling constraints further enable us to discover that the quadratic $\psi_k(s_k)$, with a linear sign restriction $s_k \in \mathcal{F}_k$, is asymptotically a second-order approximation to changes of the nonlinear l_1 function. To achieve optimality, we have to address dual feasibility.

Subsequently, we first demonstrate by example that the problem (3.16) based on the affine scaling matrix D_k alone is not sufficient for dual feasibility. Then we introduce a second scaling matrix $\mathcal{D}_k$ in (3.18) which can lead to dual feasibility. Finally, to satisfy complementarity and dual feasibility, we motivate (3.21) with an affine scaling matrix $\mathcal{M}_k$ chosen to satisfy first-order optimality.

As discussed in §1, we allow a nondifferentiable initial point. For example, an initial point x_k with $k = 0$, satisfying complementarity but not dual feasibility, is acceptable. Assume further that the Hessian $\nabla^2 f_k + \sum_{i=1}^m \lambda_{ki} \nabla^2 c_{ki}$ of $\psi_k(s)$ is positive definite. Then it is necessary to leave one of the nondifferentiable curves to achieve further decrease. But (3.16) yields zero as its solution and thus unable to provide any progress.

To illustrate this difficulty computationally, let us consider the following modified Rosenbrock problem.

$$\begin{aligned} c_1(x) &= \sigma(x_2 - x_1^2), \\ c_2(x) &= 1 - x_1, \\ c_3(x) &= \frac{1}{11} x_1 - x_2 + 1.2 - \frac{1}{11}, \end{aligned} \tag{3.17}$$

with the starting point $x_0 = (1 + 100\mathbf{eps}, 1 + 100\mathbf{eps})$ where **eps** is the machine epsilon in Matlab. FIG.3 illustrates the trajectory of the problem (3.17) using a trust region subproblem (3.16) with scaling matrix D_k. We observe that it takes many iterations to move away from x_0 (compare to FIG. 4 below).

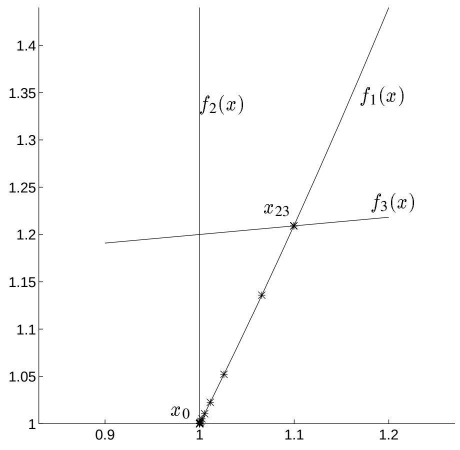

FIG. 3. *Trajectory for the Modified Rosenbrock Problem Using D_k Only*

When an affine scaling method is applied to a linear program, strict feasibility is usually assumed and maintained. Thus the situation described does not occur. Moreover, an affine scaling method for a linear program is very sensitive to a starting point and works well only when an initial point is relatively centered in the feasible region.

Next we investigate how to satisfy dual feasibility by considering another affine scaling matrix $\mathcal{D}_k$. Dual feasibility requires that $0 \le \text{diag}(c)\lambda(D) \le 2$ where $\lambda(D)$ solves

$$\begin{bmatrix} J(x) \\ -D(x) \end{bmatrix} \lambda = - \begin{bmatrix} \nabla f(x) + J(x)\text{sgn}(c(x)) \\ 0 \end{bmatrix}.$$

The notation $\lambda(D)$ is used to emphasize the dependence on the scaling matrix D.

The second affine scaling matrix $\mathcal{D}_k$ depends on dual multipliers $\lambda_k(D_k)$. Let ε be a small positive number, e.g., $\varepsilon = 10^{-3}$. Define the index set $\mathcal{V}$ as the indices of the functions c_i which are approaching zero, relative to the corresponding multipliers (with a tolerance ε), but the corresponding multipliers $\lambda_{ki}(D_k)$ predict that they should not:

$$\begin{aligned} \mathcal{V}_k = \{ j \ : \quad & \textbf{either} \quad (\lambda_{kj}(D_k)\text{sgn}(c_{kj}) > 0 \textbf{ and } |c_{kj}| < \varepsilon |\lambda_{kj}(D_k)|) \\ & \textbf{or} \quad (\lambda_{kj}(D_k)\text{sgn}(c_{kj}) < -2 \textbf{ and } |c_{kj}| < \varepsilon |2 + \lambda_{kj}(D_k)\text{sgn}(c_{kj})|) \}. \end{aligned}$$

Identify a function $c_{k\imath}$, which is the "closest" to a nondifferentiable curve among the functions in $\mathcal{V}_k$, i.e., $|c_{k\imath}| \stackrel{\text{def}}{=} \min(|c_{kj}| : j \in \mathcal{V}_k)$. Define the diagonal scaling matrix $\mathcal{D}_k$ as below:

$$\mathcal{D}_k \stackrel{\text{def}}{=} \begin{cases} D_k & \text{if } \mathcal{V}_k = \emptyset; \\ \mathcal{D}_{kii} = D_{kii}, \forall i \ne \imath \text{ and } \mathcal{D}_{k\imath\imath} = 1 & \text{otherwise.} \end{cases} \tag{3.18}$$

Lemma 3.3 below suggests that, to check whether the Kuhn-Tucker conditions are satisfied at a point x, we need only verify that the orthogonal projections of the argmented gradient, onto both the null spaces of $[J(x)^T, -D(x)]$ and $[J(x)^T, -\mathcal{D}(x)]$, are zero. (A proof can be found in §3.5).

LEMMA 3.3. *Assume that* $\begin{bmatrix} J(x) \\ -D(x) \end{bmatrix}$ *and* $\begin{bmatrix} J(x) \\ -\mathcal{D}(x) \end{bmatrix}$ *have full rank and and* $f(x)$ *and* $c(x)$ *are continuously differentiable at* x. *If the Kuhn-Tucker conditions (2.1) are satisfied at* x *then there exist* $\lambda(D)$ *and* $\lambda(\mathcal{D}) \in \Re^m$ *such that*

$$\begin{bmatrix} J(x) \\ -D(x) \end{bmatrix} \lambda(D) = \begin{bmatrix} \nabla f(x) + J(x) sgn(c(x)) \\ 0 \end{bmatrix} \quad and \quad \begin{bmatrix} J(x) \\ -\mathcal{D}(x) \end{bmatrix} \lambda(\mathcal{D}) = \begin{bmatrix} \nabla f(x) + J(x) sgn(c(x)) \\ 0 \end{bmatrix}.$$

On the other hand, assuming that the above conditions and the strict complementarity condition are satisfied at x, *then the Kuhn-Tucker conditions (2.1) are satisfied.*

Let $P_k(\mathcal{D}_k)$ denotes the orthogonal projector onto the null space of $[J_k^T, -\mathcal{D}_k]$. Assume that complementarity is satisfied at x_k but dual feasibility is violated. Then the projection $[g_k(\mathcal{D}_k); h_k(\mathcal{D}_k)]$ of the augmented gradient to the null space of $[J_k^T, -\mathcal{D}_k]$ is the same descent direction used in usual projected gradient type methods [11]. We emphasize, however, that our approach is distinctively different from a projected gradient approach since iterates are not forced to follow nondifferentiable hyperplanes and iterates can depart from nondifferentiable hyperplanes at any point, including the points satisfying complementarity.

To ensure that the projections are properly defined, we subsequently make the following assumption.

(AS.1) $\begin{bmatrix} J(x) \\ -D(x) \end{bmatrix}$ and $\begin{bmatrix} J(x) \\ -\mathcal{D}(x) \end{bmatrix}$ have full rank in $\mathcal{L}$.

It is easy to see that assumption (AS.1) holds if $\{\nabla c_i : c_i = 0\}$ are linearly independent in $\mathcal{L}$.

Based on Lemma 3.3, first-order optimality can be achieved if a sufficient reduction of an approximation to $\Upsilon(x_k + s) - \Upsilon(x_k)$ is obtained in both the null space of $[J_k^T, -D_k]$ and that of $[J_k^T, -\mathcal{D}_k]$. If a quadratic, e.g., $\psi_k(s)$, is used as an objective function in a trust region subproblem, it is too costly to solve two trust region subproblems at each iteration. Instead, we ensure sufficient reduction by solving one subproblem involving either $[J_k^T, -D_k]$ or $[J_k^T, -\mathcal{D}_k]$. Selection of this subproblem can be done based on minimizing a first-order approximation, e.g., $\phi_k(s)$, along the two projected gradients directions $g_k(D_k)$ and $g_k(\mathcal{D}_k)$; we can choose the one which yields larger decrease of $\phi_k(s)$. (An example is given in §4.) This may still seem costly due to computation of $g_k(D_k)$ and $g_k(\mathcal{D}_k)$. A closer examination of the diagonal matrix $\mathcal{D}_k$ reveals that D_k and $\mathcal{D}_k$ differ, if at all, by at most one component. Depending on methods for computing $g_k(D_k)$, this structure (a rank-one update) can be utilized and makes the additional computation of $g_k(\mathcal{D}_k)$ negligible.

Subsequently, we denote $\mathcal{M}_k$, which can be either D_k or $\mathcal{D}_k$, as the scaling matrix under consideration. For notation simplicity, if a quantity depends on $\mathcal{M}_k$, this dependence is implicitly assumed. For example, we use P_k to denote the orthogonal projector onto the null space of $[J_k^T, -\mathcal{M}_k]$, while $P_k(\mathcal{D}_k)$ denotes the orthogonal projector onto the null space of $[J_k^T, -\mathcal{D}_k]$. Similarly, λ_k denotes a least squares solution to the following $(m+n)$-by-m linear system

$$\begin{bmatrix} J_k \\ -\mathcal{M}_k \end{bmatrix} \lambda \overset{\text{LS}}{=} - \begin{bmatrix} \nabla f_k + J_k \text{sgn}(c_k) \\ 0 \end{bmatrix}; \tag{3.19}$$

$[g_k; h_k]$ denotes the projection of the augmented gradient to the null space of $[J_k^T, -\mathcal{M}_k]$, i.e.,

$$\begin{bmatrix} g_k \\ h_k \end{bmatrix} \stackrel{\text{def}}{=} -P_k \begin{bmatrix} \nabla f_k + J_k \text{sgn}(c_k) \\ 0 \end{bmatrix}. \tag{3.20}$$

Likewise, we assume that the quadratic function $\psi_k(s)$ in (3.11) and the quadratic curve $\mathcal{P}_k$ in (3.8) are defined relative to the affine scaling $\mathcal{M}_k$: the multipliers λ_{ki} used in $\sum_{i=1}^m \lambda_{ki} \nabla^2 c_{ki}$ and Z_k in (3.8) depend on $\mathcal{M}_k$. It easy to verify that, if the first-order sign restriction is maintained, $\psi_k(s)$ is always at least a first-order approximation of $\Upsilon(x_k+u) - \Upsilon(x_k)$

$$\Upsilon(x_k+u) - \Upsilon(x_k) = \psi_k(s) + o(\|s\|).$$

Summarizing the discussion above, we can determine a step sufficient for complementarity and dual feasibility by minimizing a second-order approximation $\psi_k(s)$. In other words, we determine $\mathcal{M}_k$ and then solve

$$\begin{array}{rl} & \min\limits_{s,w} \ \psi_k(s) \\ \text{subject to} & J_k^T s - \mathcal{M}_k w = 0 \\ & \left\| \begin{bmatrix} s \\ w \end{bmatrix} \right\| \le \Delta_k, \end{array} \tag{3.21}$$

where $\mathcal{M}_k$ is either D_k or $\mathcal{D}_k$. It is straightforward to verify that $\mathcal{D}_k = D_k$ in a neighborhood of a Kuhn-Tucker point with strict complementarity. Hence, if first-order optimality is achieved asymptotically, second-order optimality can be obtained.

The effect of the scaling matrix $\mathcal{D}_k$ on the modified example is illustrate in FIG.4. The scaling matrix $\mathcal{D}_k$ is used only once in this example. It takes 7 iterations to reach the minimizer, compared to 25 iterations when using D_k only.

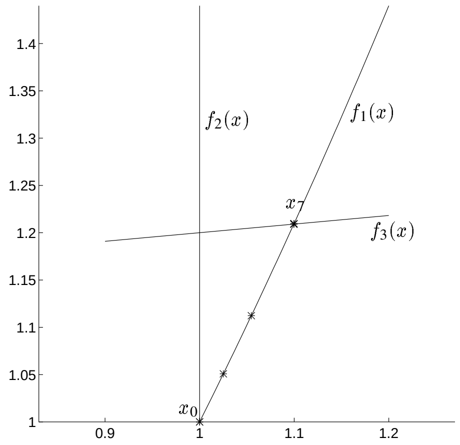

FIG. 4. *Trajectory for the Modified Rosenbrock Problem Using Scaling Matrices D_k and $\mathcal{D}_k$ (7 iterations, with $\sigma = 10$)*

In order for an iterative method to be truly efficient, it is desirable to achieve fast local convergence. In particular, we wish to define a subproblem to yield an appropriate Newton step asymptotically.

3.3. Trust Region Subproblem. In order to see whether the problem (3.16) is a good choice for a trust region subproblem, we need to further consider the local behavior of a solution of (3.16). In particular, we would like a solution of a trust region subproblem to be closely related to a Newton step (asymptotically).

Consider the following $(2m+n)$-by-$(2m+n)$ nonlinear systems of equations obtained from the complementarity condition:

$$\begin{aligned} &\mathrm{diag}(|c(x)|)\lambda = 0,\\ &J(x)\lambda + \nabla f(x) + J(x)\mathrm{sgn}(c) = 0. \end{aligned} \tag{3.22}$$

Assume that the Jacobian of (3.22) exists at $[x_k; \lambda_k]$. Assume that this Jacobian is nonsingular. Then a Newton step $[s; \delta]$ for (3.22) is defined by

$$\begin{bmatrix} \mathrm{diag}(\lambda_k \mathrm{sgn}(c_k))J_k^T & \mathrm{diag}(|c_k|) \\ \nabla^2 f_k + \sum_{i=1}^m \lambda_{ki}\nabla^2 c_{ki} + \nabla^2 \|c_k\|_1 & J_k \end{bmatrix} \begin{bmatrix} s \\ \delta \end{bmatrix} = - \begin{bmatrix} \mathrm{diag}(|c_k|)\lambda_k \\ J_k\lambda_k + \nabla f_k + J_k \mathrm{sgn}(c_k) \end{bmatrix}. \tag{3.23}$$

Assume that $\mathrm{diag}(|c_k|)$ is nonsingular. Then, we can

solve: $(J_k \mathrm{diag}(c_k)^{-1}\mathrm{diag}(\lambda_k)J_k^T + \nabla^2 f_k + \nabla^2\|c_k\|_1 + \sum_{i=1}^m \lambda_{ki}\nabla^2 c_{ki})s = -J_k\mathrm{sgn}(c_k) - \nabla f_k,$
update: $\lambda_{k+1} = -\mathrm{diag}(\lambda_k)\mathrm{diag}(c_k)^{-1}J_k^T s.$

Let s_k^N be this Newton step corresponding to x variables and $w_k^N = D_k^{-1}J_k^T s_k^N$. Then $[s_k^N; w_k^N]$ satisfies the equations $J_k^T s - D_k w = 0$.

Let $y_k = -\mathrm{diag}(c_k)^{-1}\mathrm{diag}(\lambda_k)J_k^T s_k^N$. From $\mathrm{diag}(c_k) = D_k^2 \mathrm{diag}(\mathrm{sgn}(c_k))$, we have

$$\mathrm{diag}(\mathrm{sgn}(c_k))\mathrm{diag}(\lambda_k)w_k^N = -D_k y_k.$$

Hence, assuming that Δ_k is sufficiently large so that $\| \begin{bmatrix} s_k^N \\ w_k^N \end{bmatrix} \| \le \Delta_k$, it follows that $[s_k^N; w_k^N]$ is a global solution to

$$\begin{aligned} &\min_{s,w} \psi_k(s) + \frac{1}{2}w^T \mathrm{diag}(\mathrm{sgn}(c_k))\mathrm{diag}(\lambda_k)w \\ \text{subject to}\quad & J_k^T s - D_k w = 0, \\ & \| \begin{bmatrix} s \\ w \end{bmatrix} \|_2 \le \Delta_k. \end{aligned}$$

Let

$$C_k \stackrel{\mathrm{def}}{=} \mathrm{diag}(|\lambda_k|). \tag{3.24}$$

Globally, since the term $w^T C_k w$ is convex, it is reasonable to solve the following trust region subproblem

$$\begin{aligned} &\min_{s,w} \psi_k(s) + \frac{1}{2}w^T C_k w, \\ \text{subject to}\quad & J_k^T s - \mathcal{M}_k w = 0, \\ & \| \begin{bmatrix} s \\ w \end{bmatrix} \|_2 \le \Delta_k. \end{aligned} \tag{3.25}$$

We subsequently choose (3.25) as *the trust region subproblem*. Notice that the sign restriction $s_k \in \mathcal{F}_k$ is not part of the trust region subproblem (3.16) since the additional linear inequality constraints will make it difficult to obtain a global solution of (3.16). Instead, we impose the sign restriction on the computed global solution of (3.16). This sign restriction will not prevent Newton steps s_k^N being used asymptotically (see [14]).

We further justify the trust region subproblem (3.25) by proving the equivalence of the second-order necessary conditions for a local minimizer of (3.25) and (1.4) (A proof of this lemma can be found in §3.5).

LEMMA 3.4. *Assume that x^* is a Kuhn-Tucker point of (1.4) satisfying strict complementarity condition.*

1. *If the second-order necessary conditions are satisfied at x^*, then for any $[s;w]$ satisfying $J^{*T}s - \mathcal{M}^* w = 0$,*

$$s^T(\nabla^2 f^* + \nabla^2 \|c^*\|_1 + \sum_{i=1}^{m} \lambda_i^* \nabla^2 c_i^*)s + w^T C^* w \geq 0;$$

   If the second-order sufficiency conditions are satisfied, the strict inequality holds.

2. *If, for any $[s;w]$ satisfying $J^{*T}s - \mathcal{M}^* w = 0$ and $\|s\| + \|w\| > 0$, the following holds:*

$$s^T(\nabla^2 f^* + \nabla^2 \|c^*\|_1 + \sum_{i=1}^{m} \lambda_i^* \nabla^2 c_i^*)s + w^T C^* w \geq 0,$$

   then the second-order necessary conditions are satisfied; If the strict inequality holds, then the second-order sufficient conditions are satisfied.

Next we assemble our algorithm.

3.4. A Trust Region and Affine Scaling Algorithm. From the discussion in §3.1, on page 9, if the affine scaling D_k is used in (3.25), the sign restriction $s_k \in \mathcal{F}_k$ will allow a large stepsize to be taken and hence achieve complementarity. Moreover, $\{\psi_k(s_k)\}$ is ultimately a second-order approximation to $\{\Upsilon(x_k + u_k) - \Upsilon(x_k)\}$ if, asymptotically, $s_k \in \mathcal{F}$ and $\mathcal{M}_k = D_k$ are maintained.

We note that, in (3.25), the exact Hessian requirement can be relaxed: let B_k approximate the Hessian of $\psi_k(s)$ at $s = 0$:

$$B_k \approx \nabla^2 f_k + \nabla^2 \|c_k\|_1 + \sum_{i=1}^{m} \lambda_{ki} \nabla^2 c_{ki},$$

and consider

$$\begin{array}{lll} & & \min_{s,w} (\nabla f_k + J_k \mathrm{sgn}(c_k))^T s_k + \frac{1}{2} s^T B_k s + \frac{1}{2} w^T C_k w \\ (3.26) & \text{subject to} & J_k^T s - \mathcal{M}_k w = 0 \\ & & \left\| \begin{bmatrix} s \\ w \end{bmatrix} \right\|_2 \leq \Delta_k, \end{array}$$

In order to achieve dual feasibility, the affine scaling $\mathcal{D}_k$ may need to be used while far away from a solution. Since $\mathcal{D}_{k_n} = 1$, the first-order sign restriction $s_k \in \mathcal{F}_k$ can prohibit a sufficient large stepsize being taken along a solution of the trust region subproblem (3.25). In order to overcome this difficulty and achieve optimality, we relax the restriction $s_k \in \mathcal{F}_k$ while far away from a solution and use piecewise approximations to measure progress if necessary. In other words, it is possible that a linear approximation to the nonlinear l_1 function is maintained far away from a Kuhn-Tucker point. Asymptotically, the second-order approximation $\psi_k(s)$ will be used.

In particular, we let the approximation $\Gamma_k(s)$ be either the piecewise linear $\phi_k(s)$ or its quadratic extension:

$$\Gamma_k(s) \stackrel{\text{def}}{=} \begin{cases} \text{either} & \phi_k(s), \\ \text{or} & \phi_k(s) + \frac{1}{2}s^T B_k s. \end{cases} \tag{3.27}$$

If $\Gamma_k(s) = \psi_k(s) + \frac{1}{2}s^T B_k s$, $B_k = \nabla^2 f_k + \nabla^2 \|c_k\|_1 + \sum_{i=1}^m \lambda_{ki} \nabla^2 c_{ki}$ and $s \in \mathcal{F}_k$, then $\Gamma_k(s) = \psi_k(s)$. It is clear that $\Gamma_k(s)$ is at least a first-order approximation to $\Upsilon(x_k + s) - \Upsilon(x_k)$ and, as a direct consequence of Theorem 3.2, we have Theorem 3.5 below (A proof can be found in §3.5).

THEOREM 3.5. *Assume that the functions $f(x)$ and $c(x)$ are twice continuously differentiable on the compact level set $\mathcal{L}$ and the full rank assumption (AS.1) holds. Assume that $\{x_k\}$ converges to x^*, where $x_{k+1} = x_k + u_k(s_k)$ with $u_k(s_k) = s_k + Y_k R_k^{-T} \vartheta_k(s_k)$ and $J_k^T s_k - \mathcal{M}_k w_k = 0$. Then*

$$\Upsilon(x_k + u_k) - \Upsilon(x_k) = \Gamma_k(s_k) + o(\|s_k\|_2).$$

Moreover, if, for sufficiently large k, $\mathcal{M}_k = D_k$, $B_k = \nabla^2 f_k + \nabla^2 \|c_k\|_1 + \sum_{i=1}^m \lambda_{ki} \nabla^2 c_{ki}$, $\Gamma_k(s) = \phi_k(s) + \frac{1}{2}s^T B_k s$ and $s_k \in \mathcal{F}_k$, then

$$\Upsilon(x_k + u_k) - \Upsilon(x_k) = \Gamma_k(s_k) + o(\|s_k\|_2^2).$$

We now propose a model trust region and affine scaling algorithm. This algorithm maintains an approximation $\Gamma_k(s)$ to the change of the original nonlinear l_1 function. This approximation is globally at least first-order and asymptotically second-order. The algorithm works in the usual fashion: compute a step, for example, based on the trust region subproblem (3.26), which yields a sufficient reduction of the approximation $\Gamma_k(s)$. At each iteration, an affine scaling matrix $\mathcal{M}_k$ can be selected based on reduction of a first-order approximation, e.g., $\phi_k(s)$, incurred by the projected gradients $g_k(D_k)$ and $g_k(\mathcal{D}_k)$. Then a solution $[p_k; q_k]$ to a trust region subproblem (3.26) is approximately computed and a step s_k is determined (from g_k and p_k) based on the reduction of the approximation $\Gamma_k(s)$. If s_k satisfies sign restriction $s_k \in \mathcal{F}_k$ and $\Gamma_k(s_k) = \psi_k(s_k)$, a corresponding point on the quadratic curve $\mathcal{P}_k$ (3.8) is computed in an attempt to achieve second-order approximation. If $\Gamma_k(s_k)$ approximates the change of the original objective function well, the step is taken and computation proceeds to the next iteration. Otherwise, the trust region size is reduced and the computation proceeds. This model algorithm is summarized in FIG. 5.

Comment: The model algorithm looks fairly simple and resembles a trust region algorithm for unconstrained minimization. The main cost of each iteration is computing, approximately, a solution of a trust region subproblem (3.25) for which computational techniques for unconstrained trust region methods can be analogously used. Note that the correction Step 3 is only performed when $B_k = \nabla^2 f_k + \nabla^2 \|c_k\|_1 +$

TRASM (Trust Region and Affine Scaling Method):
Let $0 < \zeta < \eta < 1$ and $0 < \gamma_1 < 1 < \gamma_2$.
For $k = 0, 1, \cdots$

Step 1 Choose $\mathcal{M}_k$ between D_k and $\mathcal{D}_k$;
Step 2 Compute s_k and $\Gamma_k(s_k)$;
Step 3 If $s_k \in \mathcal{F}_k$ and $\Gamma_k(s_k) = \psi_k(s_k)$, compute u_k on the curve $\mathcal{P}_k$; Otherwise, $u_k = s_k$;
Step 4 Compute

$$\rho_k = \frac{\Upsilon(x_k + u_k) - \Upsilon(x_k)}{\Gamma_k(s_k)};$$

If $\rho_k > \zeta$ then set $x_{k+1} = x_k + u_k$. Otherwise set $x_{k+1} = x_k$;
Step 5 Update Δ_k as below:
1. If $\rho_k \leq \zeta$ then set $\Delta_{k+1} \in (0, \gamma_1 \Delta_k]$.
2. If $\rho_k \in (\zeta, \eta)$ then set $\Delta_{k+1} \in [\gamma_1 \Delta_k, \Delta_k]$.
3. If $\rho_k \geq \eta$ then
   set $\Delta_{k+1} \in [\Delta_k, \gamma_2 \Delta_k]$.

FIG. 5. *A Trust Region and Afine Scaling Method for a Nonlinear l_1 Problem*

$\sum_{i=1}^m \lambda_{ki} \nabla^2 c_{ki}$ and the second-order approximation $\psi_k(s)$ is used. The correction step is ignored, i.e., $u_k = s_k$ if the Hessian of $c(x)$ is not available.

Our discussion also suggests that the sufficient progress for optimality can be measure by the reduction of approximation from the projected gradient $g_k(D_k)$, $g_k(\mathcal{D}_k)$ and a solution $[p_k; q_k]$ to the trust region subproblem (3.25). To connect to the trust region subproblem (3.26), we consider the augmented piecewise quadratic $\Psi_k(s, w)$ to the approximation function $\Gamma_k(s)$:

$$\Psi_k(s, w) \stackrel{\text{def}}{=} \phi_k(s) + \frac{1}{2} s^T B_k s + \frac{1}{2} w^T C_k w. \tag{3.28}$$

Next we propose to characterize the sufficient decrease conditions in terms of $\Gamma_k(s)$ and $\Psi_k(s, w)$. We first define a few important quantities based on the subproblem (3.26).

Given $s_k \in \Re^n$ and $w_k \in \Re^m$ such that $\mathcal{M}_k w_k - J_k^T s_k = 0$, we can compute the stepsizes α_k and $\breve{\alpha}_k$ which are the smallest and second smallest stepsizes along s_k to the hyperplane $c_k + J_k^T s = 0$:

$$\begin{aligned} \alpha_k &\stackrel{\text{def}}{=} \min\{-\tfrac{c_{k_i}}{\nabla c_{k_i}^T s_k} : -\tfrac{c_{k_i}}{\nabla c_{k_i}^T s_k} > 0,\ 1 \leq i \leq m\}, \\ \breve{\alpha}_k &\stackrel{\text{def}}{=} \min\{-\tfrac{c_{k_i}}{\nabla c_{k_i}^T s_k} : -\tfrac{c_{k_i}}{\nabla c_{k_i}^T s_k} > \alpha_k,\ 1 \leq i \leq m\}, \\ \alpha_k^\star &\stackrel{\text{def}}{=} \operatorname{argmin}_{\alpha \geq 0} \phi_k(\alpha s_k). \end{aligned} \tag{3.29}$$

Here $\alpha_k^\star$ denotes the optimal stepsize of $\phi_k(s)$ along s_k.

Let the superscript $\star$ denote the minimum value within the trust region. The superscript $*$ denotes the minimum value within the trust region along with the sign restriction $s_k \in \mathcal{F}_k$ defined by (3.15).

Specifically, denote these minimum values of $\Psi_k(s,w)$ as indicated below:

$$
\begin{aligned}
\Psi^\star_k[s_k,w_k] &\stackrel{\text{def}}{=} \Psi_k(\tau^\star_k s_k, \tau^\star_k w_k)\\
&\stackrel{\text{def}}{=} \min\nolimits_{\tau\geq 0}\{\Psi_k(\tau s_k,\tau w_k) : \tau \leq \check{\alpha}_k,\ \|\tau \begin{bmatrix} s_k \\ w_k \end{bmatrix}\|_2 \leq \Delta_k\},\\
\Psi^*_k[s_k,w_k] &\stackrel{\text{def}}{=} \Psi_k(\tau^*_k s_k, \tau^*_k w_k)\\
&\stackrel{\text{def}}{=} \min\nolimits_{\tau\geq 0}\{\Psi_k(\tau s_k,\tau w_k) : x_k+\tau s_k \in \mathcal{F}_k, \|\tau \begin{bmatrix} s_k \\ w_k \end{bmatrix}\|_2 \leq \Delta_k\}.
\end{aligned}
\tag{3.30}
$$

Based on previous discussions, we propose the sufficient decrease conditions below. The approximation Γ_k and s_k are chosen to satisfy:

> Let $0 < \beta_{cs}, \beta_{df}, \beta_q, \beta_{2nd} < 1, \beta_s > 0$. Let $[g_k; h_k]$ be the projected gradient defined by (3.20) and $[p_k; q_k]$ be a global solution to (3.26). Assume that there exists $w_k \in \Re^m$ such that $J_k^T s_k - \mathcal{M}_k w_k = 0$ and $\|s_k\|_2 \leq \beta_s \Delta_k$. (Recall that $p_k \stackrel{\text{def}}{=} p_k(\mathcal{M}_k)$ etc.)
>
> **(AC.1)** $\Gamma_k(s_k) < \beta_{cs}\Psi^*_k[g_k(D_k), h_k(D_k)]$;
>
> **(AC.2)** $\Gamma_k(s_k) < \beta_{df}\Psi^\star_k[g_k(\mathcal{D}_k), h_k(\mathcal{D}_k)]$;
>
> **(AC.3)** $\Gamma_k(s_k) < \beta_{2nd}\Psi^*_k[p_k, q_k]$. In addition, if $\Psi^*_k[p_k,q_k] \leq \beta_q \Psi^\star_k[g_k,h_k]$, then $B_k = \nabla^2 f_k + \nabla^2\|c_k\|_1 + \sum_{i=1}^m \lambda_{ki}\nabla^2 c_{ki}$, $\Gamma_k(s_k) = \psi_k(s_k)$ and $s_k \in \mathcal{F}_k$.

Assume that the level set $\mathcal{L}$ is compact, $c(x)$ and $f(x)$ are twice continuously differentiable and the full rank assumption (AS.1) holds. We prove in [14] that, if the reduction of the approximation $\Gamma_k(s_k)$ satisfies (AC.1), every limit point satisfies the complementarity condition. If the reduction of the approximation $\Gamma_k(s_k)$ satisfies both (AC.1) and (AC.2) and the strict complementarity holds, every limit point satisfies the first-order necessary conditions. With the strict complementarity condition, if the reduction of the approximation $\Gamma_k(s_k)$ satisfies both (AC.1), (AC.2) and (AC.3), there exists at least one limit point at which the first and second-order necessary conditions are all satisfied.

There are different ways to construct algorithms in the model given. An example is described in §4.

3.5. Proofs of Lemma 3.1, Theorem 3.2, Lemma 3.3 and 3.4. We include the proofs for the lemmas and theorems in §3. Note that Lemma 3.1 is now stated, more generally, in terms of $\mathcal{M}_k$ instead of D_k.

Lemma 3.1. *Assume that the level set $\mathcal{L} = \{x : \Upsilon(x) \leq \Upsilon(x_0)\}$ is compact. Let $\{x_k\}$ be any sequence in $\mathcal{L}$. Assume that the full rank assumption (AS.1) holds. Then*

1. *If $f(x)$ and $c_i(x)$ are continuously differentiable, then the multiplier function λ_k is bounded, i.e., there exists χ_λ such that*

$$\|\lambda_k\|_2 \leq \chi_\lambda;$$

2. *If $f(x)$ and $c_i(x)$ are twice continuously differentiable and $J_k^T s - D_k w = 0$, then*

$$\|u - s\|_2 = \| \begin{bmatrix} Y_k \\ \bar{Y}_k \end{bmatrix} R_k^{-T} \vartheta_k(s) \|_2 = O(\|s\|_2^2).$$

Proof. We consider each result in turn.

1. Using the normal equations for (3.19),

$$\lambda_k = -(\begin{bmatrix} J_k \\ -\mathcal{M}_k \end{bmatrix}^T \begin{bmatrix} J_k \\ -\mathcal{M}_k \end{bmatrix})^{-1} \begin{bmatrix} J_k \\ -\mathcal{M}_k \end{bmatrix}^T \begin{bmatrix} \nabla f_k + J_k \mathrm{sgn}(c_k) \\ 0 \end{bmatrix}.$$

From the compactness of $\mathcal{L}$ and the fact that $\begin{bmatrix} J_k \\ -\mathcal{M}_k \end{bmatrix}$ has full rank, we immediately conclude that there exists $\chi_\lambda > 0$ such that

$$\|\lambda_k\|_2 \le \chi_\lambda;$$

2. By definition that $\begin{bmatrix} J_k \\ -\mathcal{M}_k \end{bmatrix} = \begin{bmatrix} Y_k \\ \bar{Y}_k \end{bmatrix} R_k$,

$$\begin{bmatrix} Y_k \\ \bar{Y}_k \end{bmatrix} R_k^{-T} \vartheta_k(s) = \begin{bmatrix} J_k \\ -\mathcal{M}_k \end{bmatrix} (\begin{bmatrix} J_k \\ -\mathcal{M}_k \end{bmatrix}^T \begin{bmatrix} J_k \\ -\mathcal{M}_k \end{bmatrix})^{-1} \vartheta_k(s).$$

Using the compactness assumption of the level set, the fact that $\begin{bmatrix} J_k \\ -\mathcal{M}_k \end{bmatrix}$ has full rank and $c(x)$ and $f(x)$ are twice continuously differentiable, it is easy to see that

$$\| \begin{bmatrix} Y_k \\ \bar{Y}_k \end{bmatrix} R_k^{-T} \vartheta_k(s) \|_2 = O(\|\vartheta_k(s)\|_2) = O(\|s\|_2^2).$$

The proof is completed. □

Theorem 3.2. *Assume that the functions $f(x)$ and $c(x)$ are twice continuously differentiable, the level set $\mathcal{L}$ is compact and $\begin{bmatrix} J_k \\ -D_k \end{bmatrix}$ has full rank for $x_k \in \mathcal{L}$. In addition, assume that $\{x_k\}$ converges to x^*, $x_{k+1} = x_k + u_k(s_k)$, where $u_k(s_k) = s_k + Y_k R_k^{-T} \vartheta_k(s_k)$ and $J_k^T s_k - D_k w_k = 0$. If $s_k \in \mathcal{F}_k$, then*

$$\Upsilon(x_k + u_k(s_k)) - \Upsilon(x_k) = \psi_k(s_k) + o(\|s_k\|_2^2).$$

Proof. From (3.14), for $c_i^* = 0$, we have

$$J_{ki}^T(u_k - s_k) + \frac{1}{2} s_k^T \nabla^2 c_{ki} s_k = o(\|\bar{u}_k - \bar{s}_k\|_2) = o(\|s_k\|_2^2). \tag{3.31}$$

Since $c_i^* = 0$ and $\|u_k - s_k\| = O(\|s_k\|_2^2)$, we have

$$\begin{aligned} & |c_{ki} + J_{ki}^T u_k + \frac{1}{2} u_k^T \nabla^2 c_{ki} u_k| \\ = \; & |c_{ki} + J_{ki}^T s_k| + o(\|s_k\|^2) \qquad \text{(using (3.31) and } \|u_k - s_k\|_2 = O(\|s_k\|_2^2)) \\ = \; & |c_{ki}| + \mathrm{sgn}(c_{ki}) J_{ki}^T s_k + o(\|s_k\|_2^2) \qquad \text{(since } s_k \in \mathcal{F}_k) \\ = \; & |c_{ki}| + \mathrm{sgn}(c_{ki}) J_{ki}^T s_k + \mathrm{sgn}(c_{ki}) J_{ki}^T (u_k - s_k) + \frac{1}{2} \mathrm{sgn}(c_{ki}) s_k^T \nabla^2 c_{ki} s_k + o(\|s_k\|_2^2) \qquad \text{(from (3.31))}. \end{aligned}$$

Using $\|u_k - s_k\| = O(\|s_k\|_2^2)$ again, we have

$$(3.32)\ |c_{ki} + J_{ki}^T u_k + \frac{1}{2}u_k^T \nabla^2 c_{ki} u_k| = |c_{ki}| + \text{sgn}(c_{ki}) J_{ki}^T u_k + \frac{1}{2}\text{sgn}(c_{ki}) u_k^T \nabla^2 c_{ki} u_k + o(\|s_k\|_2^2).$$

For components with $c_j^* \neq 0$ and k sufficiently large,

$$\text{sgn}(c_{kj})(c_{kj} + J_{kj}^T u_k + \frac{1}{2}u_k^T \nabla^2 c_{kj} u_k) > 0.$$

Since $c_i(x)$ are twice continuously differentiable, for any $1 \leq i \leq m$,

$$c_i(x_k + u_k) = c_{ki} + J_{ki}^T u_k + \frac{1}{2}u_k^T \nabla^2 c_{ki}(x_k + \zeta_k u_k) u_k$$

where $0 \leq \zeta_k \leq 1$. Since $\{x_k\}$ converges to x^* and $\nabla^2 c_i(x)$, $1 \leq i \leq m$, are continuous, we have

$$(3.33) \qquad c_i(x_k + u_k) = c_{ki} + J_{ki}^T u_k + \frac{1}{2}u_k^T \nabla^2 c_{ki} u_k + o(\|u_k\|^2), \quad 1 \leq i \leq m.$$

Moreover, using $\|u_k\|_2 = O(\|s_k\|_2)$, (3.32) and (3.33), we have

$$\begin{aligned}
\Upsilon(x_k + u_k) - \Upsilon(x_k) &= \nabla f_k^T u_k + \frac{1}{2}u_k^T \nabla^2 f_k u_k + \\
&\quad \|c_k + J_k^T u_k + \frac{1}{2}[u_k^T \nabla^2 c_{k1} u_k; \cdots; u_k^T \nabla^2 c_{km} u_k]\|_1 - \|c_k\|_1 + o(\|u_k\|_2^2) \\
&= \nabla f_k^T u_k + \frac{1}{2}u_k^T \nabla^2 f_k u_k + \text{sgn}(c_k)^T J_k^T u_k + \frac{1}{2}u_k^T \nabla^2 \|c_k\|_1 u_k + o(\|s_k\|_2^2).
\end{aligned}$$

From (3.10), $(\nabla f_k + J_k \text{sgn}(c_k))^T(u - s) = \frac{1}{2}s^T(\sum_{i=1}^m \lambda_{ki} \nabla^2 c_{ki})s$. Using defition (3.11) of $\psi_k(s)$, we have

$$\Upsilon(x_k + u_k) - \Upsilon(x_k) = \psi_k(s_k) + o(\|u_k\|_2^2).$$

The proof is completed. $\square$

Lemma 3.3. *Assume that* $\begin{bmatrix} J(x) \\ -D(x) \end{bmatrix}$ *and* $\begin{bmatrix} J(x) \\ -\mathcal{D}(x) \end{bmatrix}$ *have full rank and and $f(x)$ and $c(x)$ are continuously differentiable at x. If the Kuhn-Tucker conditions (2.1) are satisfied at x then there exist $\lambda(D)$ and $\lambda(\mathcal{D}) \in \Re^m$ such that*

$$\begin{bmatrix} J(x) \\ -D(x) \end{bmatrix} \lambda(D) = \begin{bmatrix} \nabla f(x) + J(x) sgn(c(x)) \\ 0 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} J(x) \\ -\mathcal{D}(x) \end{bmatrix} \lambda(\mathcal{D}) = \begin{bmatrix} \nabla f(x) + J(x) sgn(c(x)) \\ 0 \end{bmatrix}.$$

On the other hand, assuming that the above conditions and the strict complementarity condition are satisfied at x, then the Kuhn-Tucker conditions (2.1) are satisfied.

Proof. Assume that the Kuhn-Tucker conditions are satisfied at x. Then, using (2.1), there exists $\lambda(D)$ such that

$$\begin{cases} J(x)\lambda(D) + (\nabla f(x) + J(x)\text{sgn}(c)) = 0 \\ -D\lambda(D) = 0, \\ -2e \leq \text{diag}(\text{sgn}(c))\lambda(D) \leq 0, \end{cases}$$

Since $-2e \leq \text{diag}(\text{sgn}(c))\lambda(D) \leq 0$, $D = \mathcal{D}$ by definition (3.18). Thus, letting $\lambda(\mathcal{D}) = \lambda(D)$, we have

$$\begin{bmatrix} J(x) \\ -\mathcal{D} \end{bmatrix} \lambda(\mathcal{D}) = -\begin{bmatrix} \nabla f(x) + J(x)\text{sgn}(c) \\ 0 \end{bmatrix}.$$

On the other hand, assume now that the strict complementarity condition is satisfied, i.e., there exists $\lambda(D)$ such that

$$\begin{bmatrix} J(x) \\ -D \end{bmatrix} \lambda(D) = -\begin{bmatrix} \nabla f(x) + J(x)\text{sgn}(c) \\ 0 \end{bmatrix} \quad \text{and} \quad |c| + \min\{|\lambda|, |\text{diag}(\text{sgn}(c))\lambda + 2e|\} > 0.$$

In addition, there exists $\lambda(\mathcal{D})$ such that

$$\begin{bmatrix} J(x) \\ -\mathcal{D} \end{bmatrix} \lambda(\mathcal{D}) = -\begin{bmatrix} \nabla f(x) + J(x)\text{sgn}(c) \\ 0 \end{bmatrix}.$$

We show next, by contradiction, that for any i with $c_i(x) = 0$

$$-2 \leq \text{diag}(c_i)\lambda_i(D) \leq 0.$$

Assume otherwise. Then there exists $\imath$ with $\mathcal{D}_{\imath\imath} = 1$ and $\lambda_\imath(\mathcal{D}) = 0$. But

$$\begin{bmatrix} J(x) \\ -D \end{bmatrix} \lambda(\mathcal{D}) = -\begin{bmatrix} \nabla f(x) + J(x)\text{sgn}(c) \\ 0 \end{bmatrix}$$

and $D_{\imath\imath} = 0$ and $\lambda_\imath(\mathcal{D}) = 0$. This violates the strict complementarity assumption. Therefore $0 \leq \text{diag}(c_i)\lambda_i(D) \leq 2$. $\square$

Lemma 3.4. *Assume that x^* is a Kuhn-Tucker point of (1.4) satisfying strict complementarity condition.*

1. *If the second order necessary conditions are satisfied at x^*, then for any $[s; w]$ satisfying $J^{*T}s - \mathcal{M}^*w = 0$,*

$$s^T(\nabla^2 f^* + \nabla^2\|c^*\|_1 + \sum_{i=1}^m \lambda_i^* \nabla^2 c_i^*)s + w^T C^* w \geq 0.$$

   If the second order sufficiency conditions are satisfied, then the strict inequality holds;

2. *If, for any $[s; w]$ satisfying $J^{*T}s - \mathcal{M}^*w = 0$ and $\|s\| + \|w\| > 0$,*

$$s^T(\nabla^2 f^* + \nabla^2\|c^*\|_1 + \sum_{i=1}^m \lambda_i^* \nabla^2 c_i^*)s + w^T C^* w \geq 0,$$

   then the second order necessary conditions are satisfied; If the strict inequality holds, then the second order sufficient conditions are satisfied.

Proof. Since x^* is a Kuhn-Tucker point satisfying the strict complementarity, $D^* = \mathcal{D}^*$. We consider each case in turn.

1. First we assume that the second order necessary conditions are satisfied, i.e.,

$$s^T(\nabla^2 f^* + \nabla^2\|c^*\|_1 + \sum_{i=1}^m \lambda_i^* \nabla^2 c_i^*)s \geq 0, \quad \text{for any } s : \nabla c_i^{*T} s = 0, \forall c_i^* = 0.$$

For any $s \in \Re^n, \; w \in \Re^m$ with

$$J^{*T} s - D^* w = 0,$$

since $D^*_{ii} = |c^*_i|^{\frac{1}{2}}$, we have

$$\nabla c_i^{*T} s = 0, \text{ for any } c_i^* = 0.$$

Thus

$$s^T (\nabla^2 f^* + \nabla^2 \|c^*\|_1 + \sum_{i=1}^m \lambda_i^* \nabla^2 c_i^*) s \geq 0.$$

But C^* is positive semi-definite, hence

$$s^T (\nabla^2 f^* + \nabla^2 \|c^*\|_1 + \sum_{i=1}^m \lambda_i^* \nabla^2 c_i^*) s + w^T C^* w \geq 0.$$

Now assume that the second order sufficient conditions are satisfied. Assume that $D^* w - J^{*T} s = 0$ and $\|s\|_2 + \|w\|_2 > 0$. If $\|s\|_2 \neq 0$, using similar arguments as above, we immediately have $s^T (\nabla^2 f^* + \nabla^2 \|c^*\|_1 + \sum_{i=1}^m \lambda_i^* \nabla^2 c_i^*) s > 0$. If $\|s\|_2 = 0$, we have $\|w\|_2 > 0$. Partition w into components with $c_i^* = 0$, denoted by $w_{\mathcal{A}}$, and $c_i^* \neq 0$, denoted by $w_{\bar{\mathcal{A}}}$. Since $w_{\bar{\mathcal{A}}} = D^{*\,-1}_{\bar{\mathcal{A}}} J^{*T}_{\bar{\mathcal{A}}} s = 0$, we have $w_{\mathcal{A}} \neq 0$. Since strict complementarity holds, $w_{\mathcal{A}}^T C^*_{\mathcal{A}} w_{\mathcal{A}} > 0$. Thus we have

$$s^T (\nabla^2 f^* + \nabla^2 \|c^*\|_1 + \sum_{i=1}^m \lambda_i^* \nabla^2 c_i^*) s + w^T C^* w > 0.$$

2. Assume that s satisfies $J_i^{*T} s = 0$, for all $c_i^* = 0$. Let $w_i = 0$ if $c_i^* = 0$ and $w_i = D^{*\,-1}_{ii} J_i^{*T} s$ otherwise. Then $D^* w - J^{*T} s = 0$. Since $\lambda_i = 0$ if $c_i^* \neq 0$ and $w_i = 0$ otherwise, we have

$$s^T (\nabla^2 f^* + \nabla^2 \|c^*\|_1 + \sum_{i=1}^m \lambda_i^* \nabla^2 c_i^*) s = s^T (\nabla^2 f^* + \nabla^2 \|c^*\|_1 + \sum_{i=1}^m \lambda_i^* \nabla^2 c_i^*) s + w^T \mathrm{diag}(|\lambda^*|) w \geq 0.$$

If $s \neq 0$ and the strict inequality holds, we immediately see that the second order sufficient conditions hold.

The proof is completed. □

4. An Example of TRASM. In this section, we describe an example of the model algorithm to illustrate that it is possible to select $\mathcal{M}_k$ and s_k which satisfy the conditions (AC.1)-(AC.3).

As mentioned in §3.2, the scaling matrix $\mathcal{M}_k$ which defines the trust region subproblem (3.25) can be determined based on how well $g_k(D_k)$ and $g_k(\mathcal{D}_k)$ decrease the piecewise linear approximation $\phi_k(s)$. Hence, for example, we can select $\mathcal{M}_k$ as in Step 1 of EXAMPLE-TRASM.

In addition, we also observe that if, $\Psi_k^*[p_k, q_k] < \beta_1 \Psi_k^*[g_k, h_k]$ and

$$\Psi_k(s_k, w_k) < \beta_2 \Psi_k^*[p_k, q_k],$$

then the conditions (AC.1) and (AC.2) are satisfied. Hence we choose s_k and $\Gamma_k(s_k)$ as in Step 2.

An Example of TRASM:

Let $0 < \zeta < 1, 0 < \beta_1 < \beta_2 < 1$.

For $k = 0, 1, \cdots$

Step 1 % Determine $\mathcal{M}_k$.

For D_k and $\mathcal{D}_k$, compute $[g_k, h_k]$; Compute $\Gamma_k(g_k)$ as below:

if $\phi_k(\alpha_k^\star g_k) \le \Psi_k^\star[g_k, h_k]$

$\Gamma_k(\alpha_k^\star g_k) = \phi_k(\alpha_k^\star g_k)$

else

$\Gamma_k(\tau_k^\star g_k) = \Psi_k^\star(g_k, h_k) - \frac{1}{2} h_k^T C_k h_k$

$\alpha_k^\star = \tau_k^\star$

end

if $\Gamma_k(g_k(D_k)) \le \beta_1 \Gamma_k(g_k(\mathcal{D}_k))$

$\mathcal{M}_k = D_k$

else

$\mathcal{M}_k = \mathcal{D}_k$

end

Step 2 % Compute s_k and $\Gamma_k(s_k)$.

Compute a solution $[p_k; q_k]$ to the trust region subproblem (3.25); Compute $\Psi_k^*[p_k, q_k]$;

if $\Psi_k^*[p_k, q_k] \le \beta_2 \Psi_k^*[g_k, h_k]$

$s_k = \tau_k^* p_k$

else

$s_k = \alpha_k^\star g_k$

end

Step 3 If $s_k \in \mathcal{F}_k$ and $\Gamma_k(s_k) = \Psi_k(s_k, w_k) - \frac{1}{2} w_k^T C_k w_k$, compute u_k on the curve $\mathcal{P}_k$; Otherwise, $u_k = s_k$.

Step 4 Compute

$$\rho_k = \frac{\Upsilon(x_k + u_k) - \Upsilon(x_k)}{\Gamma_k(s_k)};$$

If $\rho_k > \mu$ then set $x_{k+1} = x_k + u_k$. Otherwise set $x_{k+1} = x_k$;

Step 5 Update Δ_k as in FIG.5.

FIG. 6. *A Sample Algorithm Construction*

Problem	m	n	[12](5 digits)	[3] (3 digits)	New (8 digits)
Bard	15	3	5	29	20
Elattar5.1	3	2		21	7
Elattar5.2	6	3	11	10	15
Elattar	51	6	10	88	19
Freudroth	2	2		12	6
Hettich	5	4	19		32
Jennrich	10	2		33	34
Kawalik	11	4	8	67	19 (*)
Madsen	3	2	19		35
Osborne1	33	5		241	26 (*)
Osborne2	65	11		257(F)	44
Powell	2	2		9	32
ExPowell	4	4			15
ExPowell	40	40			16
Rosenbrock($\sigma = 10$)	2	2		103	10
Rosenbrock($\sigma = 100$)	2	2			15
Rosenbrock($\sigma = 1000$)	2	2			19
Watson	31	4		62	20
Watson	31	6			16
Watson	33	9			25
Wood	6	4		45	8

FIG. 7. *Number of Function Evaluations*

A example of the model algorithm is given in FIG.6.

It is straightforward to verify that, at the end of each iteration,

$$\Gamma_k(s_k) \le \beta_2 \Psi_k^*[g_k(D_k), h_k(D_k)], \text{ and } \Gamma_k(s_k) \le \beta_2\beta_1 \Psi_k^*[g_k(\mathcal{D}_k), h_k(\mathcal{D}_k)].$$

In addition, condition (AC.3) is always satisfied.

We observe superlinear convergence in our preliminary experiment. Table 7 displays some computational results using TRASM for a selection of test problems. Description of these test problems can be found in either [12] or [17].

Our preliminary experimentation is done in Matlab, A global solution for the trust region subproblem (3.25) is approximated at each step. An accuracy of at least 10^{-8} is achieved for the problems tested. The results indicate that the method works well and holds great promise.

5. Conclusion. In this paper we propose a trust region and affine scaling algorithm for a nonlinear l_1 problem. Our discussion focuses on how to deal with the nondifferentiability arising from nondifferentiable functions $\|c(x)\|_1$. The method is applicable to a general problem of the form:

$$\min_{x\in\Re^n} \nu f(x) + \sum_{i=1}^{m} |c_i(x)| + \sum_{i=m+1}^{m+t} \max(0, c_i(x)).$$

Therefore, a nonlinearly constrained minimization problem (1.2) can be solved using the above exact penalty function.

In the presentation we emphasize the motivation of our new method. This new method utilizes a combination of the first-order approximation $\phi_k(s)$ and quadratic approximation $\psi_k(s)$, depending on an affine scaling diagonal matrix $\mathcal{M}_k$ (either D_k or $\mathcal{D}_k$). The subproblem (3.25) is defined using the affine scaling transformation $\mathcal{M}_k$. The subproblem consists of a quadratic as an objective function, consistent equality constraints and a 2-norm bound on the step. Moreover, sufficient decrease conditions are proposed to achieve complementarity, first-order and second-order optimality.

An example of the affine scaling and trust region algorithm is given with preliminary computational results indicating its feasibility. Although computational discussion of the method is limited in this paper, the real objective of our research is to develop a method which is capable of computing a solution for a nonlinearly constrained problem efficiently and reliably. We intend to carry out further computational investigation, particularly for large scale problems.

The global convergence analysis of the proposed method is presented in [14] with a more succinct description of the algorithm. The local analysis of the method will be reported subsequently.

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