THE LAPLACIAN ON HYPERBOLIC RIEMANN SURFACES AND MAASS FORMS

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This thesis concerns the spectral theory of the Laplacian on Riemann surfaces of finite type, with emphasis on the quotients of the upper half plane by congruence subgroups.

In a first part we show, following Otal, that on a Riemann surface M of genus g with n punctures there are at most $2g - 2 + n$ eigenvalues λ with $\lambda \leq 1/4$.

In a second part, we focus on arithmetic surfaces. This subject is treated by Maass in a paper that is difficult to read. We work out some examples of his construction of Maass forms.

BIOGRAPHICAL SKETCH

Yasemin Kara was born in Istanbul. After earning her B.S and M.S. degrees in mathematics from Bogazici University, she continued her graduate study at Cornell University.

To my parents

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TABLE OF CONTENTS

CHAPTER 1

THE HYPERBOLIC LAPLACIAN AND BESSEL FUNCTIONS

Let M be a finite area hyperbolic surface; that is, M is a 2 dimensional, oriented, complete Riemannian manifold with constant curvature -1. The hyperbolic surface M can be identified with the quotient surface $\mathbb{H}/\Gamma$ where $\mathbb{H}$ is the Poincaré upper half-plane and Γ is a subgroup of $PSL(2,\mathbb{R})$ acting freely and properly discontinously on $\mathbb{H}$. We consider $\mathbb{H}$ with the metric and measure defined as

$$ds^2 = \frac{dx^2 + dy^2}{y^2} \quad \text{and} \quad d\mu_{\mathbb{H}} = \frac{dxdy}{y^2}.$$

In the above, $PSL(2,\mathbb{R})$ is the group of orientation preserving isometries of $\mathbb{H}$, and Γ is the group of deck transformations of the universal cover $p : \mathbb{H} \longrightarrow M$.

A complex manifold X of complex dimension one is called a Riemann surface, or equivalently X is called a Riemann surface if it is a real two dimensional oriented manifold equipped with a conformal structure. All Riemann surfaces are orientable since all complex manifolds are orientable when considered as real manifolds. By the Uniformization theorem, any simply connected Riemann surface is conformally equivalent to either the Riemann sphere, or the complex plane $\mathbb{C}$ or the upper half plane $\mathbb{H}$. According to their universal covers, Riemann surfaces are called elliptic, parabolic and hyperbolic respectively. The only example of an elliptic Rimeann surface is the Riemann sphere itself. The parabolic Riemann surfaces are the complex plane $\mathbb{C}$, the cylinder $\mathbb{R}^2/\mathbb{Z}$, and the tori $\mathbb{R}^2/\Gamma$ where Γ is a lattice isomorhic to $\mathbb{Z}^2$. The remaining Riemann surfaces are hyperbolic. The hyperbolic Riemann surfaces have constant curvature -1. Hence, almost all Riemann surfaces are hyperbolic surfaces.

For any n-dimensional Riemannian manifold (M, g), the Laplace-Beltrami operator on $C^k(M)$, $k \geq 2$, is defined as

$$\Delta_M f = \mathrm{div}_M(\mathrm{grad}_M f)$$

for any $f \in C^k(M)$ where div_M, grad_M are with respect to the Riemannian metric g. Moreover, if the manifold (M, g) is oriented then

$$\Delta_M f = \mathrm{div}_M(\mathrm{grad}_M f) = *d * df$$

where $*$ is the Hodge star operator depending also on the metric g. A complex number λ is called an *eigenvalue* of the Laplace-Beltrami operator if

$$\Delta_M f = \lambda f$$

for some non-zero $f \in C^k(M)$ where $k \geq 2$; and such an f is called the *eigenfunction* of the Laplacian. By the regularity theorems for elliptic operators, it follows that $f \in C^\infty(M)$. Furthermore, since Δ_M is a symmetric and nonnegative operator, λ is real. We will give a more geometric and intuitive definition of the Laplace-Beltrami operator in the next section.

1.1 The hyperbolic Laplacian

There is a Laplacian Δ_M acting on the functions on any Riemannian manifold M. The Laplacian $\Delta_M f(x)$ associates to a function f the difference between $f(x)$ and the average of its values on a small ball around x:

$$\Delta_M f(x) = \lim_{r\to 0} \frac{C_n}{r^2} \left(\frac{1}{\mathrm{vol}\, B_r(x)} \int_{B_r(x)} f(y)\mu_M(dy) - f(x) \right),$$

where $n = \dim M$, C_n is a constant depending on n and μ_M is the Riemannian measure on M.

On all oriented Riemannian manifolds there is a Hodge $*$ operator, which takes k-forms on M to $n-k$ forms. In terms of the Hodge operator, we have $\Delta_M(f) = *d*d$.

This is especially easy to use for conformal metrics on Riemann surfaces, for then for 1-forms ϕ we have $(*\phi)(v) = \phi(-iv)$ so that on 1-forms the Hodge $*$ depends only on the conformal structure and not on the metric. For 2-forms, the star operator divides by the metric.

In particular, on $\mathbb{H}$ with the metric

$$\frac{dx^2+dy^2}{y^2} \quad \text{we have} \quad \Delta_{\mathbb{H}} = y^2\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right).$$

This differential is invariant under $\operatorname{Aut}\mathbb{H}$: if $\alpha \in \operatorname{Aut}\mathbb{H}$, i.e., if α is a Moebius transformation, then

$$\Delta_{\mathbb{H}}(\alpha^* f) = \alpha^*(\Delta_{\mathbb{H}} f).$$

So all Riemann surfaces uniformized by $\mathbb{H}$ carry a natural hyperbolic Laplacian.

We will be interested in the spectral theory of $\Delta_{\mathbb{H}}$ and of Δ_X for various hyperbolic Riemann surfaces X.

1.2 The hyperbolic Laplacian and Bessel functions

Let us apply separation of variables to the equation $\Delta_{\mathbb{H}} f = \lambda f$. Separation of variables is always a bit unmotivated: we will look for solutions of the form

$$f(x,y) = \sqrt{y}g(x)h(y).$$

If we substitute the expression for f in the equation for eigenfunctions of the Laplacian, after a bit of manipulation we find

$$\frac{g''(x)}{g(x)} + \left(-\frac{1}{4y^2} + \frac{h'(y)}{yh(y)} + \frac{h''(y)}{h(y)}\right) = \frac{\lambda}{y^2}.$$

In the standard way, we see that $x \mapsto g''(x)/g(x)$ is a function of y alone, hence a constant which we call $-l^2$. Multiplying through by y^2, we are led to the differential equation

$$y^2h''(y) + yh'(y) - h(y)\left(\frac{1}{4} + \lambda + l^2y^2\right) = 0.$$

This is a perfectly good differential equation, but if we want to turn it into a "standard" equation with tabulated solutions, set $\alpha u = y$, and $k(u) = h(\alpha u)$; we find

$$u^2k''(u) + uk'(u) - k(u)\left(\frac{1}{4} + \lambda + l^2\alpha^2u^2\right) = 0.$$

Thus if we choose $\alpha = 1/l$ and set $1/4 + \lambda = \nu^2$, we find the modified Bessel equation

$$u^2k''(u) + uk'(u) - (u^2 + \nu^2)k(u) = 0.$$

If $u \mapsto k(u)$ is a solution of this equation, then for any $l > 0$ we have

$$\Delta_{\mathbb{H}}\left(\sqrt{y}e^{ilx}k(ly)\right) = \lambda\sqrt{y}e^{ilx}k(ly) \quad \text{where } \lambda = \nu^2 - \frac{1}{4}. \tag{1.1}$$

1.3 Modified Bessel functions

We have seen that the *modified Bessel equation*

$$z^2w'' + zw' - (z^2 + \nu^2)w = 0 \tag{1.2}$$

appears when looking for solutions of $\Delta_{\mathbb{H}}f = \lambda f$ in the upper half-plane. Here we will find solutions to this equation that are especially interesting because for these solutions, $x \mapsto w(x + iy)$ tends to 0 as x tends to ∞ for fixed y.

1.4 The formal power series

In this section we will assume that ν is not a half-integer. As in every elementary differential equations class, write

$$w(z) = z^{\mu}(a_0 + a_1 z + a_2 z^2 + \dots), \quad a_0 \neq 0,$$

and substitute the series in the equation (1.2). Examining the lowest degree terms yields $\mu = \pm\nu$, and then to the recurrence relation

$$m(m+2\nu)a_m = a_{m-2} \text{ when } \mu = \nu, \quad m(m-2\nu)a_m = a_{m-2} \text{ when } \mu = -\nu$$

leading to $a_i = 0$ when i is odd in both cases, and

$$a_{2k} = \frac{\Gamma(1+\nu)}{4^k\, k!\, \Gamma(k+\nu+1)} a_0 \text{ when } \mu = \nu, \quad a_{2k} = \frac{\Gamma(1-\nu)}{4^k\, k!\, \Gamma(k-\nu+1)} a_0 \text{ when } \mu = -\nu$$

leading to the power series

$$w_1(z) = a_0 \Gamma(1+\nu) z^{\nu} \sum_{k=0}^{\infty} \frac{1}{k!\, \Gamma(k+\nu+1)} \left(\frac{z}{2}\right)^{2k}$$

and

$$w_2(z) = a_0 \Gamma(1-\nu) z^{-\nu} \sum_{k=0}^{\infty} \frac{1}{k!\, \Gamma(k-\nu+1)} \left(\frac{z}{2}\right)^{2k}.$$

1.5 A first integral representation

Define the function K_ν by the formula

$$K_\nu(z) = \frac{1}{2} \int_0^{\infty} e^{-\frac{z}{2}\left(t+\frac{1}{t}\right)} t^{\nu} \frac{dt}{t} = \frac{1}{2} \int_{-\infty}^{\infty} e^{-z \cosh s} e^{\nu s} ds. \tag{1.3}$$

The change of variables $t = e^s$ leads from the first integral to the second. The integral converges (extremely rapidly) for $\operatorname{Re} z > 0$, and for all complex values of ν.

The change of variables $u = -s$ shows that $K_\nu = K_{-\nu}$. We will be particularly interested in the case where ν is purely imaginary: $\nu = i\tau$. Then

$$K_{-i\tau} = K_{i\tau} = \overline{K_{-i\tau}},$$

so $K_{i\tau}$ is a real function.

Let us check that K_ν does satisfy equation (1.2); this is easier in the second form of the equation. We have

$$K'_\nu(z) = \frac{1}{2}\int_{-\infty}^{\infty} -\cosh s e^{\nu s} e^{-z\cosh s} ds$$
$$K''_\nu(z) = \frac{1}{2}\int_{-\infty}^{\infty} (\cosh s)^2 e^{\nu s} e^{-z\cosh s} ds$$

so, using $\cosh^2 s - \sinh^2 s = 1$, we get

$$z^2K''_\nu(z)+zK'_\nu(z)-(z^2+\nu^2)K_\nu(z) = \frac{1}{2}\int_{-\infty}^{\infty}\left(z^2(\sinh s)^2 - z\cosh s - \nu^2\right) e^{-z\cosh s} e^{\nu s} ds \tag{1.4}$$

Now a miracle happens: we have

$$\frac{\partial}{\partial s}\left(\left(\nu + z\sinh s\right) e^{\nu s} e^{-z\cosh s}\right) = \left(z\cosh s + \nu^2 - z^2(\sinh s)^2\right) e^{\nu s} e^{-z\cosh s}$$

and hence the integral on the right of equation (1.4) vanishes identically, since

$$\left(\nu + z\sinh s\right) e^{\nu s} e^{-z\cosh s}$$

vanishes at $\pm\infty$.

Thus K_ν is some superposition of w_1 and w_2; suppose that $\operatorname{Re}\nu > 0$ so that w_2 is dominant as $z \to 0$; let us compute the coefficient of w_2 by computing

$$\lim_{z\to 0} z^\nu K_\nu(z) = \lim_{z\to 0}\frac{z^\nu}{2}\int_0^\infty e^{-\frac{z}{2}\left(t+\frac{1}{t}\right)} t^\nu \frac{dt}{t}.$$

Note that we know that the limit exists, at least if $\nu \notin \frac{1}{2}\mathbb{Z}$. Make the change of variables $zt = 2u$ to find

$$K_\nu(z) = \frac{1}{2}\int_0^\infty e^{-\left(u+\frac{z^2}{4u}\right)} z^{-\nu}(2u)^\nu \frac{du}{u}$$

that can be simply evaluated at $z = 0$ to find

$$\lim_{z\to 0} z^\nu K_\nu(z) = \frac{2^\nu}{2}\int_0^\infty e^{-u}u^\nu \frac{du}{u} = 2^{\nu-1}\Gamma(\nu).$$

1.6 Some reminders about the Γ-function

In this section it will be helpful to remember that the Γ-function has poles at the negative integers $0, -1, -2, \dots$ with residue $\frac{(-1)^m}{m!}$ at $-m$, and that

$$\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin \pi z}. \tag{1.5}$$

One of the things one can derive from this is

$$\left|\Gamma\left(\frac{1}{2} + it\right)\right| = \sqrt{\frac{\pi}{\cosh \pi t}}$$

when $t \in \mathbb{R}$, so decreases very rapidly as $t \to \pm\infty$. Then the functional equation

$$\Gamma(z-1) = \frac{\Gamma(z)}{z-1}$$

easily shows that on lines $\operatorname{Re} z = -n + 1/2$ the function $|\Gamma(z)|$ decreases (exponentially fast) for each n and decreases with n (like $1/n!$).

1.7 Another integral representation

We will now find another integral representation of the Bessel K-function:

$$K_\nu(z) = \frac{1}{8\pi i}\int_{c-i\infty}^{c+i\infty} \Gamma\left(\frac{s+\nu}{2}\right)\Gamma\left(\frac{s-\nu}{2}\right)\left(\frac{z}{2}\right)^{-s} ds, \tag{1.6}$$

when $c > |\operatorname{Re}\nu|$. This integral is defined and converges when $\operatorname{Re} z > 0$, and the difficulty in extending it consists of defining $z^{-s} = e^{-s\log z}$. For instance, we can define $\log z$ in the complement of the negative real axis; the function is then defined there.

Suppose $\nu \notin \mathbb{Z}$. The 1-form

$$\omega := \Gamma\left(\frac{s+\nu}{2}\right)\Gamma\left(\frac{s-\nu}{2}\right)\left(\frac{z}{2}\right)^{-s} ds$$

has poles at the points $\nu, \nu-2, \nu-4, \ldots$ and $-\nu, -\nu-2, -\nu-4, \ldots$.

At the point $s = \nu - 2k$ the residue of ω is

$$(-1)^k\Gamma(\nu-k)\frac{2}{k!}\left(\frac{z}{2}\right)^{2k-\nu} = \frac{2\pi}{\sin\pi\nu}\,\frac{1}{k!\Gamma(1-\nu+k)}\left(\frac{z}{2}\right)^{2k-\nu}$$

by (1.5) and at the point $-\nu-2k$ the residue of ω is

$$(-1)^k\Gamma(-\nu-k)\frac{2}{k!}\left(\frac{z}{2}\right)^{2k+\nu} = -\frac{2\pi}{\sin\pi\nu}\,\frac{1}{k!\Gamma(1+\nu+k)}\left(\frac{z}{2}\right)^{2k+\nu}.$$

Thus the integral on a path γ going from $-\infty$ to itself, surrounding all the poles counterclockwise (The bound for $|\Gamma|$ on lines $\operatorname{Re} z = -n + 1/2$ justifies this) will give

$$\frac{1}{8\pi i}\int_\gamma \omega = \frac{\pi}{2\sin\pi\nu}\left(-\left(\frac{z}{2}\right)^{\nu}\sum_{k=0}^{\infty}\frac{1}{k!\Gamma(1+\nu+k)}\left(\frac{z}{2}\right)^{2k} + \left(\frac{z}{2}\right)^{-\nu}\sum_{k=0}^{\infty}\frac{1}{k!\Gamma(1-\nu+k)}\left(\frac{z}{2}\right)^{2k}\right).$$

Furthermore, it is not difficult to deform the contour to go from $c - i\infty$ to $c + i\infty$ so long as $c > |\operatorname{Re}\nu|$. This shows that the integral does satisfy the modified Bessel function for $\nu \notin \mathbb{Z}$.

To check that it coincides with the function defined by (1.3), it is enough to compute the coefficient of $z^{-\nu}$ since both functions are elements of the 1-dimensional vector space of solutions of (1.2) that decrease at infinity. We find

$$2^{\nu}\frac{\pi}{2\sin\pi\nu\Gamma(1-\nu)} \quad \text{here, and} \quad 2^{\nu-1}\Gamma(\nu) \quad \text{there.}$$

These are indeed equal by (1.5).

1.8 The asymptotic development

The function K_ν satisfies the asymptotic development as $x \to \infty$

$$K_\nu(x) = \sqrt{\frac{\pi}{2x}}\, e^{-x}(1 + o(1)). \tag{1.7}$$

This is a case of Laplace's method, applied to the formula

$$K_\nu(x) = \frac{1}{2}\int_0^\infty e^{-\frac{x}{2}\left(t+\frac{1}{t}\right)} t^\nu \frac{dt}{t}.$$

We will split the integral into two integrals, one is from 0 to 1, and the other is from 1 to ∞. Let $g(t) = t^{\nu-1}$, then at $t = 1$, $g(t) = (1+t-1)^{\nu-1} = 1+(\nu-1)(t-1)+\ldots$, giving $g(t) \sim 1$. Now, let $h(t) = -\frac{1}{2}\left(t + \frac{1}{t}\right)$ which has the following Taylor series at $t = 1$:

$$h(1+u) = -1 - \frac{1}{2}u^2 + \ldots$$

Then the integral

$$\frac{1}{2}\int_1^\infty e^{-\frac{x}{2}\left(t+\frac{1}{t}\right)} t^\nu \frac{dt}{t}$$

has the asymptotic development

$$\frac{1}{4}\Gamma\left(\frac{1}{2}\right) e^{-x}\left(\frac{x}{2}\right)^{-1/2} = \frac{1}{2}\sqrt{\frac{\pi}{2x}} e^{-x}.$$

If we make the change of variables $s = 1/t$, the integral from 0 to 1 becomes an integral from 1 to ∞,

$$\frac{1}{2}\int_0^1 e^{-\frac{x}{2}\left(t+\frac{1}{t}\right)} t^\nu \frac{dt}{t} = \frac{1}{2}\int_1^\infty e^{-\frac{x}{2}\left(s+\frac{1}{s}\right)} s^{-\nu} \frac{ds}{s}.$$

The Laplace method again also yields the same aymptotic development for this integral. Therefore, we obtain

$$K_\nu(x) = \sqrt{\frac{\pi}{2x}}\, e^{-x}(1 + o(1)), \quad x \to \infty.$$

1.9 Fourier series and Bessel functions

Let $f : \mathbb{H} \longrightarrow \mathbb{C}$ be a function growing at most polynomially and satisfying

$$f(z+T) = f(z) \quad \text{and} \quad \Delta_{\mathbb{H}} f = -\left(\frac{1}{4} + r^2\right) f$$

where $z = x + iy$ and $r > 0$. Then f can be written as a Fourier series of the form

$$f(x+iy) = a_0 y^{1/2+ir} + b_0 y^{1/2-ir} + \sum_{n \neq 0} a_n \sqrt{y} K_{ir}\left(\frac{2\pi y|n|}{T}\right) e^{2i\pi nx/T}.$$

In order to see this, first we note that both $y^{1/2+ir}$ and $y^{1/2-ir}$ are eigenfunctions of $\Delta_{\mathbb{H}}$. Moreover, we know that the solutions of $\Delta_{\mathbb{H}}$ are given by (1.1) as

$$\Delta_{\mathbb{H}}\left(\sqrt{y} e^{ilx} K_\nu(ly)\right) = \lambda \sqrt{y} e^{ilx} K_\nu(ly) \quad \text{where } \lambda = \nu^2 - \frac{1}{4}$$

and K_ν is the solution of the modified Bessel equation. Solving for ν in terms of $r > 0$ yields $\nu = ir$. Since f is periodic with period T, we get $l = 2\pi/T$. Taking all possible superpositions of these solutions leads to the Fourier expansion of f.

CHAPTER 2

THE SPECTRAL THEOREM

Let M be a Riemann surface with finite area and of type (g, n) i.e. M is obtained from a compact Riemann surface of genus g by removing n points. Starting from this chapter, we will consider $-\Delta_M$ so that the eigenvalues are contained in the interval $[0, \infty)$. That is, a real number $\lambda \geq 0$ is called an *eigenvalue* of the Laplace-Beltrami operator if

$$\Delta_M f + \lambda f = 0$$

for some non-zero $f \in C^\infty(M)$; and such an f is called the *eigenfunction* of the Laplacian.

If M is a compact, connected Riemann surface, then the spectrum of the Laplacian is a discrete set $\{0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_i \leq \dots\}$ with finite dimensional eigenspaces, and $L^2(M)$ is the direct sum of these eigenspaces. However, if M is noncompact i.e. $n > 0$, then the Laplacian also has continuous spectrum $[1/4, \infty)$. Moreover, this spectrum can be described completely in terms of Eisenstein series.

2.1 Continuous Spectrum and Eisenstein Series

The function y^s satisfies $\Delta_{\mathbb{H}} y^s + s(1-s)y^s = 0$. Let c be a cusp of M, and identify M with $\mathbb{H}/\Gamma$, where $z \longmapsto z+1$ belongs to Γ and represents a loop surrounding c. Let Γ_∞ be the subgroup generated by $z \longmapsto z+1$.

Definition 1. *The Eisentein series corresponding to c is defined as*

$$E_c(z, s) = \sum_{\gamma \in \Gamma / \Gamma_\infty} (\operatorname{Im} \gamma(z))^s.$$

Theorem 1. *(i) $E_c(z,s)$ converges locally uniformly for* $\operatorname{Re} s > 1$ *and is invariant under* Γ.

(ii) For each s, $E_c(z,s)$ satisfies $\Delta E_c(z,s) + s(1-s)E_c(z,s) = 0$.

(iii) $E_c(z,s)$ admits a meromorphic continuation to the whole complex plane $\mathbb{C}$. It has finitely many simple poles in the interval $(1/2, 1]$ and has no poles on the line $\operatorname{Re} s = \frac{1}{2}$.

(iv) For each s, $E_c(z, 1-s)$ can be written as a sum of Eisenstein series corresponding to other cusps with coefficients depending on s.

In particular, we can consider the function

$$g_{c,r}(z) = E_c\left(z, \frac{1}{2} + ir\right) \text{ so that}$$

$$-\Delta_M g_{c,r}(z) = \left(\frac{1}{4} + r^2\right) g_{c,r}(z).$$

where $r \in \mathbb{R}$. Even though the functions $g_{c,r}$ are eigenfunctions of the Laplacian, they are not in $L^2(M)$. For each cusp c define a subspace $\mathcal{E}_c(M)$ of $L^2(M)$ as follows: For each $\phi \in C_0^\infty(\mathbb{R})$, consider

$$\int_{-\infty}^{\infty} \phi(r) g_{c,r}(z) dr;$$

such a function is in $L^2(M)$. Let $\mathcal{E}_c(M)$ be the closure of the span of all such integrals. For distinct cusps, the spaces $\mathcal{E}_c(M)$ are orthogonal. Now consider the inverse of Laplacian restricted to the orthogonal complement of the direct sum of $\mathcal{E}_c(M)$ and $\mathbb{C}$, i.e.

$$\left(-\Delta_M|_{((\bigoplus_c \mathcal{E}_c(M)) \bigoplus \mathbb{C})^\perp}\right)^{-1}.$$

This operator is compact, so $\left(-\Delta_M|_{((\bigoplus_c \mathcal{E}_c(M)) \bigoplus \mathbb{C})^\perp}\right)^{-1}$ has a discrete spectrum in the interval $[0, \infty)$ accumulating at 0. Therefore, $-\Delta_M|_{(\bigoplus_c \mathcal{E}_c(M))}$ has a discrete spectrum in the interval $[0, \infty)$ accumulating at ∞. A very detailed discussion of the spectral theory of the Laplacian on M can be found in [7].

As a summary, for a Riemann surface with finite area and of type (g, n) the Laplacian has a discrete spectrum in the interval $[0, \infty)$ and has continous spectrum $[1/4, \infty)$ with spectral density equal to the number of cusps. An element of $L^2(M)$ such that $-\Delta_M f = \lambda f$ is called a cusp form if it vanishes at all cusps. Phillips and Sarnak conjecture that the space of cusp forms is trivial for generic groups i.e. of a countable union of real-analytic hypersurfaces in moduli space [11]. However, for arithmetic surfaces $\mathbb{H}/\Gamma(N)$ the space of cusp forms is not empty. We will show how they are constructed in the following chapters.

CHAPTER 3

SMALL EIGENVALUES

Let M be a hyperbolic Riemann surface with finite area and of type (g, n). By the Uniformization theorem $\mathbb{H}$ is the universal cover of M and let $\lambda_0(\mathbb{H})$ be the bottom of the spectrum of the Laplacian on $\mathbb{H}$. It is characterized as the infimum of the Rayleigh quotients of g

$$\inf \frac{\int_{\mathbb{H}} |\operatorname{grad}_{\mathbb{H}} g|^2 d\mu_{\mathbb{H}}}{\int_{\mathbb{H}} |g|^2 d\mu_{\mathbb{H}}},$$

where the infimum can be taken over all the functions $g : \mathbb{H} \longrightarrow \mathbb{R}$ that are compactly supported and C^∞. It is known that the bottom of the spectrum of the Laplacian for the hyperbolic plane and for the annulus is equal to $1/4$ [9].

In the spectrum of the Laplacian for M, the eigenvalues are called small if $0 < \lambda \leq 1/4$. Small eigenvalues are important since they have a significant role on how the closed geodesics behave asymptotically on a compact Riemann surface. Therefore, the study of the existence and the number of these eigenvalues have gained a considerable amount of attraction over years. A detailed history of the subject can be found in Buser [2] and Chavel [3]. Let us give a brief summary about small eigenvalues here. Randol [12] proves that any compact Riemann surface has a finite covering space possessing arbitrarily many small eigenvalues. This article uses the Selberg trace formula; and Randol shows the existence of the eigenfunctions corresponding to these small eigenvalues without constructing them. Buser [1] actually constructs compact Riemann surfaces with $g \geq 2$ for which the first $2g - 2$ eigenvalues are less than ϵ. In the same article, he also proves that $\lambda_{4g-2} > 1/4$ for any Riemann surface with $g \geq 2$. Buser proves the following theorems in [1].

Theorem 1. *For any $\delta > 0$, there exists a compact Riemann surface with $g \geq 2$ such that*

$$\lambda_{2g-3} < \delta.$$

Theorem 2. *For any $n \in \mathbb{N}$ and for any arbitrarily small $\epsilon > 0$, there exists a compact Riemann surface M with genus $g \geq 2$ such that*

$$\lambda_n \leq \frac{1}{4} + \epsilon.$$

Theorem 3. *For any compact Riemann surface with $g \geq 2$,*

$$\lambda_{4g-2} > \frac{1}{4}.$$

Randol also shows [13] that for a compact Riemann surface with genus $g \geq 2$ if $\lambda_1, \lambda_2, \ldots, \lambda_{2g-3}$ are sufficently small, then $\lambda_{2g-2} > 1/4$. This result was known to Buser though it was not published. Moreover, Schmutz [15] conjectured that a closed Riemann surface of genus g has at most $2g-2$ small eigenvalues and he proved the conjecture for $g = 2$. Finally, Otal and Rosas [10] proved this conjecture in 2009. They actually proved the following theorem:

Theorem 4. *Let M be a hyperbolic surface with finite area and of type (g, n), i.e. with genus g and n punctures. Then the $(2g-2+n)$-th eigenvalue λ_{2g-2+n} is greater than $\frac{1}{4}$.*

In the proof of the above theorem, Otal and Rosas used topological methods: a version of Borsuk-Ulam theorem whereas the previous theorems were proven by using the Minimax principles and Cheeger's inequality in general.

Now we will give a proof of a slightly different version of Theorem 1.

Proof. For any given $1 > \epsilon > 0$, we can choose a Riemann surface M of genus g with n punctures such that all the geodesics of the maximal multicurve Γ have length ϵ. Note that $|\Gamma| = 3g - 3 + n$ [6]; let $\mathcal{S}$ be the set of components of the complement of Γ, then note that $|\mathcal{S}| = 2g - 2 + n$.

Let $\mathbf{B}$ be the band model for the hyperbolic plane, i.e.

$$\mathbf{B} = \{z \in \mathbb{C} \text{ such that } |\operatorname{Im} z| < \pi/2\}$$

with the hyperbolic metric $(|dz|/\cos \operatorname{Im} z)$. Each curve $\gamma \in \Gamma$ has a neighbourhood A_γ isomorphic to the subset

$$\left\{ y \in \mathbf{B} \text{ such that } |y| < \frac{\pi}{2} - \frac{\epsilon}{2} \right\} / \epsilon\mathbb{Z}$$

of the cylinder $\mathbf{B}/\epsilon\mathbb{Z}$. Furthermore, denote by S_{th}(S thick) the component of $M - \cup_\gamma A_\gamma$ corresponding to $S \in \mathcal{S}$.

Now, consider the functions in $L^2(M)$ which are equal to some constants on the components $M - \cup_\gamma A_\gamma$ and on each A_γ they interpolate linearly between the values at the ends. In other words, for each $\mathbf{a} \in \mathbb{R}^{\mathcal{S}}$ associate an element of $f_{\mathbf{a}} \in L^2(M)$ constructed as follows where $\mathbf{a} : \mathcal{S} \longrightarrow \mathbb{R}$, $\mathbf{a}(S) = a_S$:

$$f_{\mathbf{a}}(p) = \begin{cases} a_S, & p \in S_{th} \\ \alpha_\gamma y + \beta_\gamma, & p \in A_\gamma \end{cases}$$

where the real constants α_γ and β_γ are chosen to make $f_{\mathbf{a}}$ continuous. Denote by E_ϵ the space of functions $f_{\mathbf{a}}$. The elements of E_ϵ have distributional derivatives in $L^2(M)$.

Claim 5. *We will prove that as* $\epsilon \longrightarrow 0$,

$$\sup_{f \in E_\epsilon - \{0\}} \frac{\int_M |\operatorname{grad} f|^2 d\mu_M}{\int_M |f|^2 d\mu_M} \longrightarrow 0.$$

Proof.

$$\begin{aligned}
\int_M |\operatorname{grad} f_{\mathbf{a}}|^2 d\mu_M &= \int_M df_{\mathbf{a}} \wedge *df_{\mathbf{a}} \\
&= \left| \sum_\gamma \int_0^\epsilon \int_{-\pi/2+\epsilon/2}^{\pi/2-\epsilon/2} \alpha_\gamma dy \wedge (-\alpha_\gamma dx) \right| \\
&\leq \sum_\gamma \epsilon |\alpha_\gamma|^2 (\pi - \epsilon) \longrightarrow 0 \text{ with } \epsilon.
\end{aligned}$$

Furthermore, we will show that $\int_M |f_{\mathbf{a}}|^2 d\mu_M$ is bounded below and does not depend on ϵ leading to the proof of the claim.

$$\begin{aligned}
\int_M |f_{\mathbf{a}}|^2 d\mu_M &\geq \sum_\gamma \int_0^\epsilon \int_{-\pi/2+\epsilon/2}^{\pi/2-\epsilon/2} \frac{|\alpha_\gamma y + \beta_\gamma|^2}{\cos^2 y} dx dy \\
&= \sum_\gamma \epsilon \int_{-\pi/2+\epsilon/2}^{\pi/2-\epsilon/2} \frac{|\alpha_\gamma y + \beta_\gamma|^2}{\cos^2 y} dy
\end{aligned}$$

Since the integrand on the interval $(-\pi/2 + \epsilon/2, \pi/2 - \epsilon/2)$ is positive, at least for one γ, then the value of the integral only depends on the behaviour of the integrand at the lower and the upper limits of the integral. By substituting $u = \pi/2 - y$ and splitting the integral into two integrals for $a \in (\epsilon/2, \pi - \epsilon/2)$, the integral becomes

$$\begin{aligned}
\int_M |f_{\mathbf{a}}|^2 d\mu_M &\geq \sum_\gamma \epsilon \int_{\epsilon/2}^{\pi-\epsilon/2} \frac{|\alpha_\gamma(\pi/2 - u) + \beta_\gamma|^2}{\cos^2(\pi/2 - u)} du \\
&= \sum_\gamma \epsilon \int_{\epsilon/2}^{a} \frac{|\alpha_\gamma(\pi/2 - u) + \beta_\gamma|^2}{\sin^2(u)} du \\
&+ \sum_\gamma \epsilon \int_{a}^{\pi-\epsilon/2} \frac{|\alpha_\gamma(\pi/2 - u) + \beta_\gamma|^2}{\sin^2(u)} du
\end{aligned}$$

Now let $u = \pi - v$ in the second integral, then we have

$$\begin{aligned}
\int_M |f_{\mathbf{a}}|^2 d\mu_M \geq & \sum_\gamma \epsilon \int_{\epsilon/2}^{a} \frac{|\alpha_\gamma(\pi/2 - u) + \beta_\gamma|^2}{\sin^2(u)} du \\
& + \sum_\gamma \epsilon \int_{\epsilon/2}^{\pi - a} \frac{|\alpha_\gamma(v - \pi/2) + \beta_\gamma|^2}{\sin^2(\pi - v)} dv \\
& = \sum_\gamma \epsilon \int_{\epsilon/2}^{a} \frac{|\alpha_\gamma(\pi/2 - u) + \beta_\gamma|^2}{\sin^2(u)} du \\
& + \sum_\gamma \epsilon \int_{\epsilon/2}^{\pi - a} \frac{|\alpha_\gamma(v - \pi/2) + \beta_\gamma|^2}{\sin^2(v)} dv
\end{aligned}$$

By using $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots$ and letting $a = \pi/2$, we have

$$\begin{aligned}
\int_M |f_{\mathbf{a}}|^2 d\mu_M \geq & \sum_\gamma \epsilon \int_{\epsilon/2}^{\pi/2} \frac{|\alpha_\gamma(\pi/2 - u) + \beta_\gamma|^2}{u^2} du \\
& + \sum_\gamma \epsilon \int_{\epsilon/2}^{\pi/2} \frac{|\alpha_\gamma(v - \pi/2) + \beta_\gamma|^2}{v^2} dv \\
& = \sum_\gamma \epsilon \left(\frac{|a_S|^2}{\epsilon/2} + \frac{|a_{S'}|^2}{\epsilon/2} \right) + O(1)
\end{aligned}$$

where a_S and $a_{S'}$ are the end values for each γ. Each a_S contributes three times in the above sum, so we have

$$\int_M |f_{\mathbf{a}}|^2 d\mu_M \geq \sum_{S \in \mathcal{S}} 6|a_S|^2 + O(1)$$

$\square$

Then the result follows from the following generality from functional analysis:

Min-max principle: If Q is a positive semi-definite quadratic form on a Hilbert space H, and $E \subset H$ is an m-dimensional subspace such that

$$\frac{|Q(v)|}{|v|^2} < \lambda$$

for all $v \neq 0$ in E, then Q has at least m eigenvalues $\leq \lambda$. $\square$

Now we will give the outline of the proof of Theorem 4 in the case $n = 0$.

Sketch of the proof of theorem 4:

Proof. Let M be a compact Riemann surface with $g \geq 2$. Denote by E the real vector space spanned by the eigenfunctions of the Laplacian with eigenvalue $\leq 1/4$. Let m be the dimension of E. For $f \in E \setminus \{0\}$, let

$$Z(f) = \{p \in M | f(p) = 0\}$$

be the nodal set of f. Otal and Rosas [10] describe the topology of $Z(f)$. The functions $f \in E \setminus \{0\}$ are real-analytic, so

Proposition 6. *The nodal set $Z(f)$ is the union of a locally finite graph with vertices of even multiplicity and of some isolated points.*

When M is compact, $Z(f)$ is the union of a compact graph and a finite set of isolated points. Then to each $f \in E \setminus \{0\}$, they associate a compact and incompressible subsurface of M which is called the characteristic surface of f described as follows.

The nodal graph of f, denoted by $N(f)$, is defined as the union of the connected components of its nodal set. If f changes sign, then $N(f) \neq \emptyset$. Moreover, the sign of f is well-defined on each connected component of $M \setminus N(f)$. We view the components of $N(f)$ that are contained in discs embedded in M as trivial (including isolated points). Let $N'(f)$ be the subset of $N(f)$ obtained by removing the trivial components of $N(f)$. We notice that each connected component of $M \setminus N'(f)$ is the union of an essential component of $M \setminus N(f)$ and a finite number of pairwise disjoint discs. Each component of $M \setminus N'(f)$ is given the sign of f on the essential components of $M \setminus N(f)$. The union of components of $M \setminus N'(f)$

on which f has a positive sign (respectively a negative sign) is denoted by $C^+(f)$ (respectively $C^-(f)$). If f does not change sign, then we define $C^+(f)$ or $C^-(f)$ to be M according to the sign of f. Notice that if each component of $N(f)$ is contained in a disc, then either $C^+(f)$ or $C^-(f)$ is empty. Furthermore, we see that the surfaces $C^+(f)$ and $C^-(f)$ are incompressible by construction. Now, let $S^+(f)$ (resp. $S^-(f)$) be the union of the components of $C^+(f)$ (resp. $C^-(f)$) which are not discs or annuli. The sets $S^+(f)$ and $S^-(f)$ are still incompressible.

Claim 7. *For all $f \in E \setminus \{0\}$, $2 - 2g \leq \chi(S^+(f)) + \chi(S^-(f)) < 0$.*

Proof. We use the convention that $\chi(\emptyset) = 0$. First of all, $2 - 2g \leq \chi(S^+(f)) + \chi(S^-(f))$ is implied by the incompressibility of the surfaces $S^+(f)$ and $S^-(f)$. In order to prove $\chi(S^+(f)) + \chi(S^-(f)) < 0$, we notice that if $N(f) = \emptyset$, then either $S^+(f) = M$ or $S^-(f) = M$ and we are done. If $N(f) \neq \emptyset$, then we would like to show that $S^+(f) \cup S^-(f) \neq \emptyset$. Let $f \in E \setminus \{0\}$, then

$$\frac{\int_M |\operatorname{grad} f|^2 d\mu_M}{\int_M |f|^2 d\mu_M} \leq \frac{1}{4}.$$

The Rayleigh quotient is equal to $1/4$ if and only if f is an eigenfunction with eigenvalue $1/4$. If we denote the components of $M \setminus N'(f)$ by Y_i, $1 \leq i \leq k$, then $\exists j \in \{1, 2, \ldots, k\}$ such that the Rayleigh quotient of f on Y_j is

$$\frac{\int_{Y_j} |\operatorname{grad} f|^2 d\mu_M}{\int_{Y_j} |f|^2 d\mu_M} \leq \frac{1}{4}$$

since both quantities in the numerator and denominator of the Rayleigh quotient for the surface M equal the sum of the corresponding quantities over each Y_i by the disjointness of the components Y_i's. Assume Y_j is either a disc or an annulus and let $\pi_1(Y_j)$ be its fundamental group. Since the components of $M - N'(f)$ are incompressible, there is an injection from $\pi_1(Y_j)$ into $\pi_1(M)$. If we view $\pi_1(Y_j)$ as a subgroup of $\pi_1(M)$, then there exists a cover $\tilde{Y_j}$ of M corresponding to $\pi_1(Y_j)$.

This cover is either the universal cover $\tilde{M}$ of M or a cylinder. In both cases, it is well known that the bottom of the spectrum of Laplacian is $1/4$. Now, we embed Y_j into $\tilde{Y}_j$ and we extend $f|_{Y_j}$ on $\tilde{Y}_j$ by defining it to be 0 on the complement of the image of the embedding, call this function g. Since g has distributional derivatives in $L^2(Y_j)$, it is in the domain of the Laplacian for $\tilde{Y}_j$. On the other hand, the Rayleigh quotient of g is $\leq 1/4$ on $\tilde{Y}_j$. If its Rayleigh quotient is $< 1/4$, then we have a contradiction since the bottom of the spectrum of the Laplacian is $1/4$ for $\tilde{Y}_j$. If the Rayleigh quotient of g is equal to $1/4$, then it means an eigenfunction of the Laplacian on $\tilde{Y}_j$ with the eigenvalue $1/4$. This is a contradiction also since g is clearly not an eigenfunction. Therefore, Y_j cannot be a disc or an annulus. Hence, $S^+(f) \cup S^-(f) \neq \emptyset$. □

We previously defined E as the real vector space spanned by the eigenfunctions of the Laplacian with eigenvalue $\leq 1/4$. Let m be the dimension of E and our goal is to show that $m \leq 2g-2$. Let $\mathbb{S}(E)$ be the unit sphere of E for an arbitrary norm and $\mathbb{P}(E)$ be the projective space of E, i.e. $\mathbb{P}(E) = \mathbb{S}(E)/\sim_a$ where a is the antipodal map sending f to $-f$.

In the previous claim, we showed that $2-2g \leq \chi(S^+(f)) + \chi(S^-(f)) < 0$. Now we will partition $\mathbb{S}(E)$ according to $\chi(S^+(f)) + \chi(S^-(f))$. For each i, where $2-2g \leq i \leq -1$, define

$$\mathcal{S}_i = \{f \in \mathbb{S}(E) \mid \chi(S^+(f)) + \chi(S^-(f)) = i\}.$$

Then we have $\mathbb{S}(E) = \bigcup_{2-2g}^{-1} \mathcal{S}_i$ and $\mathbb{P}(E) = \bigcup_{2-2g}^{-1} \mathcal{P}_i$ where $\mathcal{P}_i = \mathcal{S}_i/\sim a$ since each $\mathcal{S}_i$ contains $-f$ whenever it contains f.

Claim 8. *For each i, $2-2g \leq i \leq -1$, the covering map $p_i : \mathcal{S}_i \mapsto \mathcal{P}_i$ is trivial.*

Proof. Since each component $S \in S^+(f) \cup S^-(f)$ has negative Euler characteristic, it contains a figure eight which cannot be isotoped to be disjoint from itself. Assume that $p_i : \mathcal{S}_i \longrightarrow \mathcal{P}_i$ is not trivial, then there is an isotopy f_t in $\mathcal{S}_i$ joining $f_0 = f$ to $f_1 = -f$. Choose a figure eight γ contained in one of the components of $S^+(f)$. Then this figure eight is moved by the isotopy to another figure eight contained in some component of $S^+(-f) = S^-(f)$. This gives a contradiction since $S^+(f)$ and $S^-(f)$ are disjoint. $\square$

We will complete the proof of the theorem by showing that $m \leq 2g-2$ where m is the dimension of the space E. Consider the double covering $p : \mathbb{S}(E) \longrightarrow \mathbb{P}(E)$. Denote its class in $H^1(\mathbb{P}(E), \mathbb{Z}_2)$ by α. The Čech cohomolgy class corresponding to $p_i : \mathcal{S}_i \mapsto \mathcal{P}_i$ is $\alpha|_{\mathcal{P}_i} = 0$ since these coverings are trivial. Since we have at most $2g-2$ $\mathcal{P}_i$, then $\alpha^{2g-2} = 0$ by Lemma 8 in [16]. On the other hand, the order of α in the $\mathbb{Z}_2$ cohomology ring of $\mathbb{P}(E)$ is m, $m \leq 2g-2$.

$\square$

CHAPTER 4

BACKGROUND ON ALGEBRAIC NUMBER THEORY

In this chapter, we will give some background information about quadratic fields and in general we will follow [14], [5].

4.1 Quadratic number fields

Definition 2. *A degree two extension K over the field $\mathbb{Q}$ is called a quadratic number field.*

Every quadratic field K can be written as

$$K = \mathbb{Q}[x]/(x^2 - d)$$

where d is a squarefree integer. We call K a real quadratic field if $d > 0$ and an imaginary quadratic field if $d < 0$. In this chapter, we will only work with real quadratic fields. Therefore, we assume $d > 0$ from now on. We can view K as a subfield of the complex numbers $\mathbb{C}$. Let $\sqrt{d}$ denote the positive square root of d. Then, there are two embeddings i.e. two injective homomorphisms from K into $\mathbb{C}$ defined as

$$\sigma_1(a + bx) = a + b\sqrt{d},$$
$$\sigma_2(a + bx) = a - b\sqrt{d}.$$

We note that since d is positive, the images of these embeddings lie in $\mathbb{R}$ and

$$K = \mathbb{Q}[x]/(x^2 - d) \cong \mathbb{Q}(\sqrt{d})$$

where $\mathbb{Q}(\sqrt{d}) = \{a + b\sqrt{d} \mid a, b \in \mathbb{Q}\}$.

Let $\alpha = a + bx \in K$, the norm and trace of α are defined as

$$\mathrm{N}(\alpha) = \sigma_1(\alpha)\sigma_2(\alpha) = a^2 - db^2 \text{ and } \mathrm{Tr}(\alpha) = \sigma_1(\alpha) + \sigma_2(\alpha) = 2a.$$

4.2 Ring of integers

Let $\mathcal{O}_K$ denote the subset of K consisting of elements of K which are integral over $\mathbb{Z}$. It is easy to show that $\mathcal{O}_K$ is a ring and it is called the ring of integers of K. Moreover, $\mathcal{O}_K$ is a free $\mathbb{Z}$-submodule of K with rank 2. We can precisely describe its integral basis which depends on d.

Theorem 9. *Let $K = \mathbb{Q}(\sqrt{d})$ be a quadratic field where d is a squarefree integer. Then*

(i) If $d \equiv 2$ or $d \equiv 3 \pmod 4$, then

$$\mathcal{O}_K = \mathbb{Z}[d] = \{a + b\sqrt{d} \mid a, b \in \mathbb{Z}\}$$

(ii) If $d \equiv 1 \pmod 4$,

$$\mathcal{O}_K = \mathbb{Z}[(1+\sqrt{d})/2] = \left\{a + b\left(\frac{1+\sqrt{d}}{2}\right) \mid a, b \in \mathbb{Z}\right\}$$

Note that if K is a real quadratic field, we can embed K into $\mathbb{R}^2$ by using the embeddings σ_1 and σ_2 as

$$\begin{aligned} \sigma : K &\longrightarrow \mathbb{R}^2 \\ \sigma(a + b\sqrt{x}) &= (\sigma_1(a + b\sqrt{x}), \sigma_2(a + b\sqrt{x})) \\ &= (a + b\sqrt{d}, a - b\sqrt{d}). \end{aligned}$$

Remark 10. *The image of $\mathcal{O}_K$ under σ is a lattice in $\mathbb{R}^2$. Moreover, if we consider an ideal $\mathfrak{a}$ of $\mathcal{O}_K$, the image of $\mathfrak{a}$ under σ becomes a sublattice of $\sigma(\mathcal{O}_k)$ in $\mathbb{R}^2$.*

4.3 The norm of an ideal

Let K be a field over $\mathbb{Q}$ with degree n and $\mathcal{O}_K$ be its ring of integers.

Definition 3. *Suppose $\mathfrak{a}$ is a nonzero integral ideal of $\mathcal{O}_K$. Then the norm of $\mathfrak{a}$ is defined to be the cardinality of $\mathcal{O}_K/\mathfrak{a}$ and it is denoted by $\mathrm{N}(\mathfrak{a})$.*

Proposition 11. *Let $\mathfrak{a}$ and $\mathfrak{b}$ be nonzero integral ideals of $\mathcal{O}_K$. Then*
(i) $\mathrm{N}(\mathfrak{a})$ is finite,
(ii)$\mathrm{N}(\mathfrak{a}\mathfrak{b}) = \mathrm{N}(\mathfrak{a})\,\mathrm{N}(\mathfrak{b})$,
(iii) Let a be a nonzero element of $\mathcal{O}_K$ and $\mathfrak{a} = (a)$ be the principal ideal generated by a. Then $|\,\mathrm{N}(a)| = \mathrm{N}(\mathfrak{a})$.

4.4 Units of a real quadratic field

In this section we will describe the group U_K of units of the ring of integers $\mathcal{O}_K$ of a real quadratic field K. We first note that an element $\alpha \in \mathcal{O}_K$ is a unit if and only if $\mathrm{N}(\alpha) = \mp 1$.

By Dirichlet's unit theorem, $U_K \cong \mathbb{Z} \times \{\mp 1\}$. An element $u \in U_K$ is called a fundamental unit if every element in U_K can be written of the form $\mp u^n$ for $n \in \mathbb{Z}$. Now, consider the embedding of K into $\mathbb{R}$ via σ_1, i.e. $\sigma_1(a + b\sqrt{x}) = a + b\sqrt{d}$ and we identify the elements of K with their images under σ_1. Then each unit except 1 and -1 lies in one of the intervals $(-\infty, -1)$, $(-1, 0)$, $(0, 1)$, and $(1, \infty)$. If u is a fundamental unit lying in one of these interval then u^{-1}, $-u$, and $-u^{-1}$ are also fundamental units. It is easy to see that all these fundamental units lie in only one of the above intervals. Therefore, there is one $u \in (1, \infty)$. It is called the fundamental unit and let us denote it by ε. Hence, every element of U_K except

1 and -1 can be written as $\mp\varepsilon^{\mp n}$ where n is a positive integer. Since $\varepsilon > 1$, then $\varepsilon^{n+1} > \varepsilon^n$ for any positive integer implying that ε is the smallest positive unit in $(1, \infty)$. Finding units is equivalent to solving

$$\mathrm{N}(a + b\sqrt{d}) = \mp 1,$$

where $a + b\sqrt{d} \in \mathcal{O}_K$. Given any quadratic number field we can always find the fundamental unit explicitly; for example by using the continued fraction expansion of $\sqrt{d}$.

Finally, let us give the definition of a totally positive unit. A unit $\beta \in U_K$ is called a totally positive unit if its image under the both embeddings σ_1 and σ_2 is positive. The totally positive elements of U_K form a subgroup.

4.5 The splitting of prime ideals in real quadratic fields

In an algebraic number field K i.e. the degree of K over $\mathbb{Q}$ is finite, every element of $\mathcal{O}_K$ can be factored into a finite number of irreducible elements; however, this factorization is not necessarily unique. It is unique if and only if every irreducible element is a prime in $\mathcal{O}_K$. On the other hand, unique factorization holds for the ideals of $\mathcal{O}_K$.

Theorem 2. *Every nonzero ideal of $\mathcal{O}_K$ can be written as a product of prime ideals uniquely up to the order of the factors.*

In this section, our goal is to give a complete description of the set of all prime ideals of $\mathcal{O}_K$ for a quadratic field $K = \mathbb{Q}(\sqrt{d})$. For this purpose, it is enough to show how each ideal generated by a rational prime can be decomposed into prime ideals since each prime ideal can only divide one rational prime number.

Let p be a prime number, consider the ideal $p\mathcal{O}_K$. This ideal is not necessarily a prime ideal in $\mathcal{O}_K$. However, it splits into prime ideals of $\mathcal{O}_K$ uniquely by the unique factorization theorem. From general theory, we know that only three cases can occur:

- $p\mathcal{O}_K = \mathfrak{p}\mathfrak{p}'$, $\mathrm{N}(\mathfrak{p}) = \mathrm{N}(\mathfrak{p}') = p$, $\mathfrak{p} \neq \mathfrak{p}'$ (in this case, we say p splits in $\mathcal{O}_K$)
- $p\mathcal{O}_K = \mathfrak{p}$, $\mathrm{N}(\mathfrak{p}) = p^2$ (in this case, we say p remains prime in $\mathcal{O}_K$)
- $p\mathcal{O}_K = \mathfrak{p}^2$, $\mathrm{N}(\mathfrak{p}) = p$ (in this case, we say p ramifies in $\mathcal{O}_K$)

where $\mathfrak{p}'$ is the conjugate of the ideal $\mathfrak{p}$. We can determine explicitly when each case occurs depending on p and d.

Theorem 3. *Let $K = \mathbb{Q}(\sqrt{d})$ be a quadratic field with squarefree positive integer d. Then,*

(i) The odd primes p for which d is a quadratic residue $\mod p$ split in K. So does 2, if $d \equiv 1 \mod 8$.

(ii) The odd primes p for which d is not a quadratic residue $\mod p$ remain prime in K. So does 2, if $d \equiv 5 \mod 8$

(iii) The odd prime divisors of d ramify in K. So does 2, if $d \equiv 2$ or $3 \mod 4$.

We will follow [14] for the proof of this theorem.

Proof. First we assume that p is odd. We know that $\mathcal{O}_K \cong \mathbb{Z}[\sqrt{d}]$ if $d \equiv 2$ or $3 \pmod 4$, and $\mathcal{O}_K = \mathbb{Z}[(1+\sqrt{d})/2]$ if $d \equiv 1 \pmod 4$. Consider the element $\alpha = a + b\left(\dfrac{1+\sqrt{d}}{2}\right)$ in $\mathbb{Z}[(1+\sqrt{d})/2]$. If b is even, then $\alpha \in \mathbb{Z}[\sqrt{d}]$. If b is odd, then $a + (b+p)\left(\dfrac{1+\sqrt{d}}{2}\right)$ in $\mathbb{Z}[\sqrt{d}]$. Hence, for any square free d,

$$\mathcal{O}_K/p\mathcal{O}_K \cong \mathbb{Z}[\sqrt{d}]/(p).$$

By sending x to $\sqrt{d}$, it is also clear that

$$\mathbb{Z}[\sqrt{d}] \cong \mathbb{Z}[x]/(x^2 - d).$$

Therefore, we have

$$\mathcal{O}_K/p\mathcal{O}_k \cong \mathbb{Z}[x]/(p, x^2 - d) \cong (\mathbb{Z}[x]/(p))/(x^2 - d) \cong \mathbb{F}_p[x]/(x^2 - \bar{d})$$

where $\bar{d} \equiv d \pmod p$ i.e. we have

$$\mathcal{O}_K/p\mathcal{O}_K \cong \mathbb{F}_p[x]/(x^2 - \bar{d}).$$

From the above isomorphism, we see that p splits in $\mathcal{O}_K$ means $\mathcal{O}_K/p\mathcal{O}_K$ is a product of two fields i.e. $x^2 - d$ has two distinct linear factors in $\mathbb{F}_p[x]/(x^2 - \bar{d})$. This is equivalent to saying d is a quadratic residue $\mod p$. Similarly, p remains prime in $\mathcal{O}_K$ means $\mathcal{O}_K/p\mathcal{O}_K$ is a field i.e. $x^2 - d$ is irreducible in $\mathbb{F}_p[x]/(x^2 - \bar{d})$ i.e. d is a quadratic non-residue in $\mod p$. Finally, p ramifies in $\mathcal{O}_K$ means $\mathcal{O}_K/p\mathcal{O}_K$ contains a nilpotent elements i.e. $x^2 - d$ is a square in $\mathbb{F}_p[x]/(x^2 - \bar{d})$ i.e. $d \equiv 0 \mod p$.

Now consider the case where $p = 2$. If $d \equiv 2$ or $3 \mod 4$, then we have either

$$\mathcal{O}_K/2\mathcal{O}_K \cong \mathbb{F}_2[x]/(x^2) \text{ or } \mathcal{O}_K/2\mathcal{O}_K \cong \mathbb{F}_2[x]/(x^2 + 1) \cong \mathbb{F}_2[x]/(x + 1)^2,$$

meaning that 2 ramifies in $\mathcal{O}_K$. If $d \equiv 1 \mod 4$, then

$$\mathcal{O}_K/2\mathcal{O}_K \cong \mathbb{F}_2[x]/(x^2 - x - (d - 1)/4).$$

If $d \equiv 1 \mod 8$, then $x^2 - x + (d - 1)/4 \equiv x^2 + x \equiv x(x + 1)$ in $\mathbb{F}_2[x]$. Hence, 2 splits if $d \equiv 1 \mod 8$. If $d \equiv 5 \mod 8$, then $x^2 - x + (d - 1)/4 \equiv x^2 + x + 1$ which is irreducible in $\mathbb{F}_2[x]$, therefore 2 remains prime. $\square$

Now we would like to determine which positive integers occur as norms in $K = \mathbb{Q}(\sqrt{d})$.

Theorem 12. *Let $K = \mathbb{Q}(\sqrt{d})$ and assume that the narrow class number h_K^+ of K is 1. Then*

(i) A prime number p occur as the norm of an element in $\mathcal{O}_K$ if and only if either p splits in K or p ramifies in K.

(ii) Let $n = \prod_i = p_i^{k_i}$ be a positive integer. Then n is the norm of an element in $\mathcal{O}_K$ if and only if p_i is the norm of an element in $\mathcal{O}_K$ when k_i is odd.

For the proof of this theorem, see [5].

Let us close this chapter with a few more definitions. The set of 2×2 matrices with integer entries and determinant 1 forms a group. It is called the *modular group* and denoted by $SL(2, \mathbb{Z})$ i.e.

$$SL(2,\mathbb{Z}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a, b, c, d \in \mathbb{Z}, ad - bc = 1 \right\}.$$

Let N be a positive integer. The *principal cogruence subgroup of level N* is defined as

$$\Gamma(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{Z}) : \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \pmod{N} \right\}.$$

A subgroup Γ of $SL(2, \mathbb{Z})$ is called the congruence subgroup of level N if $\Gamma \supset \Gamma(N)$ for some positive integer N.

The surface $\Gamma(N)/\mathbb{H}$ is a finite area Riemann surface with genus g and n cusps. The numbers g and n can be computed in terms of N. The number of cusps $c(N)$ of $\Gamma(N)$ is given by

$$C(N) = \begin{cases} (1/2)N^2 \prod_{p|N}(1 - 1/p^2) & \text{if } N > 2, \\ 3 & \text{if } N = 2. \end{cases}$$

The genus g is given by

$$g(N) = \begin{cases} 1 + \frac{d(N)(N-6)}{12N} & \text{if } N > 2, \\ 0 & \text{if } N = 2 \end{cases}$$

where

$$d(N) = (1/2)N^3 \prod_{p|N} (1 - 1/p^2) \quad \text{if } n > 2.$$

The proofs can be found in [4].

CHAPTER 5

CONSTRUCTION OF WAVE-FORMS AND EIGENFUNCTIONS IN A SPECIAL CASE

Hans Maass introduced Maass wave forms in his article "Über eine neue Art von nichtanalytischen automorphen Funktionen und die Bestimmung Dirichletscher Reihen durch Funktionalgleichungen" in 1949 [8]. Maass wave forms are complex valued nonanalytic automorphic functions which satisfy the Hyperbolic Laplacian. He gave examples of wave forms associated to the zeta functions of real quadratic fields. In this chapter, we will explain the details of the construction of such wave forms explicitly.

The paper of Maass is quite difficult to read, and it seems it has not been widely read. A translation appears in the appendix. In this chapter we will carry out the central construction of Maass's paper in a specific case: the quadratic number field $\mathbb{Q}(\sqrt{5})$, which leads to wave forms on the modular surface $\mathbb{H}/\Gamma(5)$, where $\Gamma(5)$ is the principal congrence subgroup of level 5 i.e. the following subgroup of $SL(2,\mathbb{Z})$:

$$\Gamma(5) = \{M \in SL(2,\mathbb{Z})\,|M \equiv I \mod 5\}.$$

Our main goal is to prove the following theorem.

Theorem 13. *There are at least three linearly independent functions $g_0(z)$, $g_1(z)$ and $g_2(z)$ on $X_5 = \mathbb{H}/\Gamma(5)$ satisfying $-\Delta_{\mathbb{H}} f = \frac{1}{4} f$ and having at most polynomial growth at the cusps. They satisfy the additional symmetries*

$$g_\rho\left(-\frac{1}{z}\right) = \sum_{\sigma=0}^{2} c_{\rho\sigma} g_\sigma(z)$$

with

$$(c_{\rho\sigma}) = \frac{1}{\sqrt{5}} \begin{pmatrix} 1 & 2 & 2 \\ 1 & w^2 + w^{-2} & w + w^{-1} \\ 1 & w + w^{-1} & w^2 + w^{-2} \end{pmatrix}$$

where $w = e^{\frac{2\pi i}{5}}$ *and*

$$g_\rho(z+1) = e^{\frac{2\pi i}{5}} g_\rho(z)$$

5.0.1 The field $\mathbb{Q}(\sqrt{5})$ and its ring of integers

Consider the real quadratic field $K = \mathbb{Q}(\sqrt{5}) = \{a + b\sqrt{5} \mid a, b \in \mathbb{Q}\}$ and let $\mathcal{O}_K$ denote the ring of integers of K. We know that $\mathcal{O}_K$ is a free $\mathbb{Z}$-submodule of K of rank 2 with an integral basis $\{1, (1+\sqrt{5})/2\}$, i.e.

$$\mathcal{O}_K = \left\{ a + b\left(\frac{1+\sqrt{5}}{2}\right) \mid a, b \in \mathbb{Z} \right\}.$$

The embeddings of K into $\mathbb{C}$ are

$$\sigma_1(a + b\sqrt{5}) = a + b\sqrt{5},$$
$$\sigma_2(a + b\sqrt{5}) = a - b\sqrt{5}.$$

Since K is a real quadratic field, we can embed K into $\mathbb{R}^2$ via the canonical embedding $\sigma : K \longrightarrow \mathbb{R}^2$ which is given by

$$\sigma(a + b\sqrt{5}) = (\sigma_1(a + b\sqrt{5}), \sigma_2(a + b\sqrt{5})) = (a + b\sqrt{5}, a - b\sqrt{5}).$$

Note that $\sigma(1) = (1, 1)$ and $\sigma((1+\sqrt{5})/2) = ((1+\sqrt{5})/2, (1-\sqrt{5})/2)$. By using the canonical embedding of K into $\mathbb{R}^2$, we get a lattice of $\mathcal{O}_K$ with rank 2 in $\mathbb{R}^2$.

5.0.2 The ideal $(\sqrt{5})$

Consider the principal ideal of $\mathcal{O}_K$ which is generated by $\sqrt{5}$,

$$(\sqrt{5}) = \{r\sqrt{5} \mid r \in \mathcal{O}_K\} = \left\{a\sqrt{5} + b\left(\frac{5+\sqrt{5}}{2}\right) \mid a, b \in \mathbb{Z}\right\}.$$

The set $\{\sqrt{5}, (5+\sqrt{5})/2\}$ forms an integral basis for $(\sqrt{5})$ and we see that $\sigma(\sqrt{5}) = (\sqrt{5}, -\sqrt{5})$ and $\sigma((5+\sqrt{5})/2) = ((5+\sqrt{5})/2, (5-\sqrt{5})/2)$.

We note that $\mathcal{O}_K/(\sqrt{5})$ is a field with five elements and we choose the set $\{-2, -1, 0, 1, 2\} \subset \mathcal{O}_K$ to represent each class in $\mathcal{O}_K/(\sqrt{5})$.

Let us denote the group of units of $\mathcal{O}_K$ by $U_K = \{\pm\varepsilon^n \mid n \in \mathbb{Z}\}$ where ε is the fundamental unit of $\mathcal{O}_K$. The fundamental unit ε is $(1+\sqrt{5})/2$, which has norm -1. The group of totally positive units of $\mathcal{O}_K$ is generated by $\varepsilon^2 = (3+\sqrt{5})/2$. Now let

$$u = \left(\frac{3+\sqrt{5}}{2}\right)^2 = \frac{7+3\sqrt{5}}{2} \equiv 1 \mod (\sqrt{5}).$$

It is the smallest totally positive unit bigger than one and congruent to 1 modulo the ideal $(\sqrt{5})$). Now let us consider the equivalence relation $\sim_u$ on $\mathcal{O}_K$ where

$$\mu_1 \sim_u \mu_2 \quad \text{if and only if} \quad \exists k \in \mathbb{Z} \quad \text{such that} \quad \mu_1 = u^k \mu_2.$$

Note that $\mathcal{O}_K/\sim_u \; = \coprod_{\rho \in \mathcal{O}_K/(\sqrt{5})} (\mathcal{O}_K^{\rho}/\sim_u)$ where $\mathcal{O}_K^{\rho}$ consists of elements of $\mathcal{O}_K$ which are equivalent to ρ modulo $(\sqrt{5})$. This is true because if $\mu_1 \sim_u \mu_2$ and $\mu_1 \equiv \rho \mod (\sqrt{5})$ then $\mu_2 \equiv \rho \mod (\sqrt{5})$ since $u \equiv 1 \mod (\sqrt{5})$. Moreover, the norm on $\mathcal{O}_K$ induces a map $\mathrm{N} : \mathcal{O}_K/\sim_u \longrightarrow \mathbb{Z}$.

We also notice that $U_K/\sim_u = \{\overline{-1}, \overline{1}, \overline{\varepsilon}, \overline{-\varepsilon}, \overline{\varepsilon^2}, \overline{-\varepsilon^2}, \overline{\varepsilon^3}, \overline{-\varepsilon^3}\}$ and the elements in the set $\{1, -1, \varepsilon, -\varepsilon, \varepsilon^2, -\varepsilon^2, \varepsilon^3, -\varepsilon^3\}$ are congruent to the elements $\{1, -1, -2, 2, -1, 1, 2, -2\}$ respectively modulo the ideal $(\sqrt{5})$.

5.0.3 The ζ-functions $\zeta(s, \rho)$

Now let us define the following zeta functions that are associated to each ρ where $\rho \in \mathcal{O}_K/(\sqrt{5}) = \{0, 1, 2, -1, -2\}$.

$$\zeta_0(s, \rho) \; = \sum_{a \in (\mathcal{O}_K^\rho - \{0\})/\sim_u} \frac{1}{|\operatorname{N}(a)|^s}, \quad \zeta_1(s, \rho) \; = \sum_{a \in (\mathcal{O}_K^\rho - \{0\})/\sim_u} \operatorname{sgn}(\operatorname{N}(a)) \frac{1}{|\operatorname{N}(a)|^s}$$

Remark 14. *Since $\mathcal{O}_K$ is a principal ideal domain, each ideal in $\mathcal{O}_K$ is generated by an element of $\mathcal{O}_K$. Note that the norm of a principal ideal (a) is $\operatorname{N}((a)) = |\operatorname{N}(a)|$. Hence, we can think of the sums in the definition of the ζ-functions as sums over the ideals of $\mathcal{O}_K$ with some additional properties.*

Theorem 15. *The ζ-functions defined above can be written as Dirichlet series which converge absolutely on some half-plane as*

$$\zeta_0(s, \rho) \; = \sum_{\substack{n \equiv b_\rho \ (5) \\ n \neq 0}} \frac{a_{\rho,n}}{|n|^s}, \quad \zeta_1(s, \rho) \; = \sum_{\substack{n \equiv b_\rho \ (5) \\ n \neq 0}} \frac{(\operatorname{sgn} n) a_{\rho,n}}{|n|^s},$$

where b_ρ is $0, 1$ and -1 respectively for $\rho = 0, 1, 2$ and we can determine $a_{\rho,n}$ explicitly.

In order to express the ζ-functions as in Theorem 15 (i.e. to find the coefficients in the Dirichlet series), first we need to determine which integers occur as norms of elements in $\mathcal{O}_K$; this is equivalent to finding the norms of principal ideals in $\mathcal{O}_K$ by the Remark 14. Since every ideal can be written as products of prime ideals, we first give a complete description of the set of all prime ideals $\mathcal{O}_K$. For this purpose, it is enough to show how each rational prime can be decomposed into prime ideals since each prime ideal can only divide one rational prime number.

Remark 16. *Consider $K = \mathbb{Q}(\sqrt{5})$. Assume p is an odd prime. By using Theorem 3, we see that if $p \equiv 1$ or $4 \mod 5$, then p splits in $\mathcal{O}_K$. If $p \equiv 2$ or $3 \mod 5$ then p remains prime in $\mathcal{O}_K$. The only prime which ramifies in $\mathcal{O}_K$ is $p = 5$ and the prime number 2 remains prime in $\mathcal{O}_K$.*

For $n > 0$, let $A(n) = |\{\mathfrak{a} : \mathfrak{a} \text{ is an ideal of } \mathcal{O}_K \text{ with } \mathrm{N}(\mathfrak{a}) = n\}|$

Lemma 17. *Let $n = p_1^{l_1} \dots p_k^{l_k}$ where each p_i is a distinct prime number. Then*

$$A(n) = \begin{cases} 0 & \textit{if any } p_i \textit{ is inert and } l_i \textit{ is odd, otherwise} \\ \prod_i (l_i + 1) & \textit{where } p_i \textit{ splits} \end{cases}$$

Claim 18. *We claim that*

$$\zeta_k(s, \bar{1}) = \zeta_k(s, \overline{-1}), \quad \zeta_k(s, \bar{2}) = \zeta_k(s, \overline{-2}) \quad \textit{for} \quad k = 0, 1$$

and

$$\zeta_1(s, \bar{1}) = -\zeta_1(s, \bar{2}).$$

5.0.4 The θ-functions

The Fourier Transform

Consider the real vector space $\mathbb{R}^n$ with the standard inner product $\mathbf{x} \cdot \mathbf{y} = x_1 y_1 + \cdots + x_n y_n$ for $\mathbf{x} = (x_1, \cdots, x_n)$, $\mathbf{y} = (y_1, \cdots, y_n) \in \mathbb{R}^n$. Suppose $f \in L^1(\mathbb{R}^n)$

for the measure $|\,\mathrm{d}^n\,\mathbf{x}|$ induced by the standard inner product. Then the Fourier transform $\hat{f} : \mathbb{R}^n \to \mathbb{C}$ of f is defined by

$$\hat{f}(\mathbf{y}) = \int_{\mathbb{R}^n} f(\mathbf{x}) e^{-2\pi i \mathbf{x}\cdot\mathbf{y}} |\,\mathrm{d}^n\,\mathbf{x}|$$

Now, let $\mathcal{S}(\mathbb{R}^n)$ denote the Schwartz space on $\mathbb{R}^n$.

Poisson Summation Formula

Let Γ be a full rank lattice in $\mathbb{R}^n$ and $\Gamma^* \subseteq \mathbb{R}^n$ be its dual lattice i.e.

$$\Gamma^* = \{\boldsymbol{\mu} \in \mathbb{R}^n \text{ s.t. } \boldsymbol{\lambda} \cdot \boldsymbol{\mu} \in \mathbb{Z} \text{ for every } \boldsymbol{\lambda} \in \Gamma\}.$$

Let $g : \mathbb{R}^n \to \mathbb{C}$ be a Γ-periodic function and of class C^1 i.e. $g(\mathbf{x} + \boldsymbol{\lambda}) = g(\mathbf{x})$ for every $\mathbf{x} \in \mathbf{R}^n$ and $\boldsymbol{\lambda} \in \Gamma$. Then the Fourier coefficient of g for $\boldsymbol{\mu} \in \Gamma^*$ is defined by

$$c_{\boldsymbol{\mu}}(g) = \frac{1}{|\Gamma|} \int_{\mathcal{F}} g(\mathbf{x}) e^{-2i\pi \mathbf{x}\cdot\boldsymbol{\mu}} |\,\mathrm{d}^n\,\mathbf{x}|$$

where $|\Gamma|$ is the volume of the lattice and $\mathcal{F}$ is a fundamental domain of $\mathbb{R}^n/\Gamma$. Then

$$g(x) = \sum_{\boldsymbol{\mu} \in \Gamma^*} c_{\boldsymbol{\mu}}(g) e^{2\pi i \mathbf{x}\cdot\boldsymbol{\mu}}$$

Theorem 19. *(Poisson Summation Formula) Let $f \in \mathcal{S}(\mathbb{R}^n)$ and Γ be a full rank lattice in $\mathbb{R}^n$. Then for any $\mathbf{x} \in \mathbb{R}^n$*

$$\sum_{\boldsymbol{\lambda} \in \Gamma} f(\mathbf{x} + \boldsymbol{\lambda}) = \frac{1}{|\Gamma|} \sum_{\boldsymbol{\mu} \in \Gamma^*} \hat{f}(\boldsymbol{\mu}) e^{2\pi i \mathbf{x}\cdot\boldsymbol{\mu}}$$

Proof. Define a function $g : \mathbb{R}^n \to \mathbb{C}$ as

$$g(\mathbf{x}) = \sum_{\boldsymbol{\lambda} \in \Gamma} f(\mathbf{x} + \boldsymbol{\lambda}).$$

Since f is Schwartz class, g is C^∞ and a Γ-periodic function. Thus by using the Fourier inversion formula we get

$$\begin{aligned}
g(\mathbf{x}) &= \sum_{\boldsymbol{\mu}\in\Gamma^*} c_{\boldsymbol{\mu}}(g) e^{2\pi i \mathbf{x}\cdot\boldsymbol{\mu}} \qquad (5.1)\\
&= \sum_{\boldsymbol{\mu}\in\Gamma^*} \frac{1}{|\Gamma|}\left\{\int_{\mathcal{F}} g(\mathbf{y}) e^{-2\pi i \mathbf{y}\cdot\boldsymbol{\mu}} |\,\mathrm{d}^n\,\mathbf{y}|\right\} e^{2\pi i \mathbf{x}\cdot\boldsymbol{\mu}}\\
&= \frac{1}{|\Gamma|}\sum_{\boldsymbol{\mu}\in\Gamma^*} \left\{\int_{\mathcal{F}} \left(\sum_{\boldsymbol{\lambda}\in\Gamma} f(\mathbf{y}+\boldsymbol{\lambda})\right) e^{-2\pi i \mathbf{y}\cdot\boldsymbol{\mu}} |\,\mathrm{d}^n\,\mathbf{y}|\right\} e^{2\pi i \mathbf{x}\cdot\boldsymbol{\mu}}\\
&= \frac{1}{|\Gamma|}\sum_{\boldsymbol{\mu}\in\Gamma^*} \left\{\sum_{\boldsymbol{\lambda}\in\Gamma}\int_{\mathcal{F}} f(\mathbf{y}+\boldsymbol{\lambda}) e^{-2\pi i \mathbf{y}\cdot\boldsymbol{\mu}} |\,\mathrm{d}^n\,\mathbf{y}|\right\} e^{2\pi i \mathbf{x}\cdot\boldsymbol{\mu}}\\
&= \frac{1}{|\Gamma|}\sum_{\boldsymbol{\mu}\in\Gamma^*} \left\{\sum_{\boldsymbol{\lambda}\in\Gamma}\int_{\mathcal{F}} f(\mathbf{y}+\boldsymbol{\lambda}) e^{-2\pi i (\mathbf{y}+\boldsymbol{\lambda})\cdot\boldsymbol{\mu}} |\,\mathrm{d}^n\,\mathbf{y}|\right\} e^{2\pi i \mathbf{x}\cdot\boldsymbol{\mu}}\\
&= \frac{1}{|\Gamma|}\sum_{\boldsymbol{\mu}\in\Gamma^*} \left\{\int_{\mathbb{R}^n} f(\mathbf{y}) e^{-2\pi i \mathbf{y}\cdot\boldsymbol{\mu}} |\,\mathrm{d}^n\,\mathbf{y}|\right\} e^{2\pi i \mathbf{x}\cdot\boldsymbol{\mu}}\\
&= \frac{1}{|\Gamma|}\sum_{\boldsymbol{\mu}\in\Gamma^*} \hat{f}(\boldsymbol{\mu}) e^{2\pi i \mathbf{x}\cdot\boldsymbol{\mu}}
\end{aligned}$$

□

The θ-series

For each $\rho \in \mathcal{O}_K/(\sqrt{5})$, we define two θ- series as

$$\theta_0(t,t',\rho) = \sum_{\mu\equiv\rho\ (\sqrt{5})} e^{-\frac{\pi}{5}(\mu^2 t+\mu'^2 t')},$$

$$\theta_1(t,t',\rho) = \sum_{\mu\equiv\rho\ (\sqrt{5})} \mu\mu' e^{-\frac{\pi}{5}(\mu^2 t+\mu'^2 t')}$$

where μ' is the conjugate of μ. We would like to prove the following two relations:

$$\theta_0(t,t',\rho) = \frac{1}{\sqrt{5}}\frac{1}{\sqrt{tt'}} \sum_{\alpha\in\mathcal{O}_K/(\sqrt{5})} e^{2i\pi\,\mathrm{Tr}\left(\frac{\alpha'\rho}{5}\right)} \theta_0\left(\frac{1}{t'},\frac{1}{t},\alpha\right) \qquad (5.2)$$

and

$$\theta_1(t,t',\rho) = \frac{-1}{\sqrt{5}} \frac{1}{(\sqrt{tt'})^3} \sum_{\alpha \in \mathcal{O}_K/(\sqrt{5})} e^{2i\pi \operatorname{Tr}\left(\frac{\alpha'\rho}{5}\right)} \theta_1\left(\frac{1}{t'}, \frac{1}{t}, \alpha\right). \tag{5.3}$$

Now, consider $f : \mathbb{R}^2 \to \mathbb{R}$,

$$f(x_1, x_2) = e^{-\frac{\pi}{5}[t(x_1+a)^2 + t'(x_2+b)^2]} \tag{5.4}$$

for $t, t' > 0$. Let us compute the Fourier transform of f.

$$\hat{f}(y_1, y_2) = \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} e^{-\pi(t/5)(x_1+a)^2} e^{-2i\pi x_1 y_1} dx_1 \right) e^{-\pi(t'/5)(x_2+b)^2} e^{-2i\pi x_2 y_2} dx_2$$

Let $u = \sqrt{t/5}(x_1 + a)$, then $du = \sqrt{t/5}dx_1$ and $x_1 = u\sqrt{5/t} - a$. Similarly, let $v = \sqrt{t/5}(x_2 + a)$, then $dv = \sqrt{t/5}dx_2$ and $x_2 = v\sqrt{5/t} - b$. We see that

$$\begin{aligned}
\int_{-\infty}^{\infty} e^{-\pi(t/5)(x_1+a)^2} e^{-2i\pi x_1 y_1} dx &= \sqrt{5/t} \int_{-\infty}^{\infty} e^{-\pi u^2} e^{-2i\pi y_1 (u\sqrt{5/t} - a)} du \\
&= \sqrt{5/t} e^{2i\pi y_1 a} \int_{-\infty}^{\infty} e^{-\pi(u + i y_1 \sqrt{5/t})^2 - (5\pi y_1^2)/t} du \\
&= \sqrt{5/t} e^{2i\pi y_1 a} e^{-(5\pi y_1^2)/t} \int_{-\infty}^{\infty} e^{-\pi(u + i y_1 \sqrt{5/t})^2} du \\
&= \sqrt{5/t} e^{2i\pi y_1 a} e^{-(5\pi y_1^2)/t}
\end{aligned}$$

Therefore,

$$\hat{f}(y_1, y_2) = \frac{5}{\sqrt{tt'}} e^{2i\pi(y_1 a + y_2 b)} e^{-5\pi\left(\frac{y_1^2}{t} + \frac{y_2^2}{t'}\right)}$$

Now, we will apply the Poisson summation formula to the pair f and $\hat{f}$. We know that the images of $\mathcal{O}_K$ and the ideal $(\sqrt{5})$ under the canonical embedding σ are full rank lattices in $\mathbb{R}^2$. Let $\Gamma = \sigma((\sqrt{5}))$. We know that as lattices in $\mathbb{R}^2$,

$$\mathcal{O}_K = \mathbb{Z}\begin{pmatrix} 1 \\ 1 \end{pmatrix} + \mathbb{Z}\begin{pmatrix} \frac{1+\sqrt{5}}{2} \\ \frac{1-\sqrt{5}}{2} \end{pmatrix}$$

and

$$\Gamma = \mathbb{Z}\begin{pmatrix} \sqrt{5} \\ -\sqrt{5} \end{pmatrix} + \mathbb{Z}\begin{pmatrix} \frac{5+\sqrt{5}}{2} \\ \frac{5-\sqrt{5}}{2} \end{pmatrix}.$$

We also need to know the dual lattice of Γ to compute the right hand sight of the Poisson summation formula. Let B be the matrix whose columns are the basis vectors of Γ. Then we know that the columns of $(B^{-1})^T$ will span Γ^*. Let

$$B = \begin{pmatrix} \sqrt{5} & \frac{5+\sqrt{5}}{2} \\ -\sqrt{5} & \frac{5-\sqrt{5}}{2} \end{pmatrix}$$

Then, we see that

$$(B^{-1})^T = \begin{pmatrix} \frac{\sqrt{5}-1}{10} & 1/5 \\ \frac{-\sqrt{5}-1}{10} & 1/5 \end{pmatrix}$$

which means

$$\Gamma^* = \mathbb{Z}\begin{pmatrix} 1/5 \\ 1/5 \end{pmatrix} + \mathbb{Z}\begin{pmatrix} \frac{\sqrt{5}-1}{10} \\ \frac{-\sqrt{5}-1}{10} \end{pmatrix}$$

Also, note that $|\Gamma| = \det(B) = 5\sqrt{5}$. Consider the map $\phi : \frac{1}{5}\mathcal{O}_{\mathcal{K}} \to \Gamma^*$, given by

$$\phi((1/5)\boldsymbol{\lambda}) = \phi((1/5)\lambda, (1/5)\lambda') = ((1/5)\lambda', (1/5)\lambda)$$

where λ' is the conjugate of λ. Clearly, this map is an isomorphism.

Now, by Poisson summation formula

$$\begin{aligned}
\sum_{\boldsymbol{\mu}\equiv\boldsymbol{\rho}\ (\sqrt{5})} e^{-\pi\left(\frac{t\mu^2+t'(\mu')^2}{5}\right)} &= \sum_{\boldsymbol{\beta}\in\Gamma} e^{-\pi\left(\frac{t(\beta+\rho)^2+t'(\beta'+\rho')^2}{5}\right)} \\
&= \frac{1}{5\sqrt{5}} \sum_{\boldsymbol{\nu}\in\Gamma^*} \frac{5}{\sqrt{tt'}} e^{-5\pi\left(\frac{(\nu)^2}{t}+\frac{(\nu')^2}{t'}\right)+2i\pi(\nu\rho+(\nu')\rho')} \\
&= \frac{1}{\sqrt{5}} \frac{1}{\sqrt{tt'}} \sum_{\boldsymbol{\lambda}\in\mathcal{O}_K} e^{-\frac{\pi}{5}\left(\frac{\lambda^2}{t'}+\frac{(\lambda')^2}{t}\right)+2i\pi\,\mathrm{Tr}\left(\frac{\lambda'\rho}{5}\right)} \\
&= \frac{1}{\sqrt{5}} \frac{1}{\sqrt{tt'}} \sum_{\boldsymbol{\alpha}\in\mathcal{O}_K/(\sqrt{5})} e^{2i\pi\,\mathrm{Tr}\left(\frac{\alpha'\rho}{5}\right)} \sum_{\boldsymbol{\lambda}\equiv\boldsymbol{\alpha}\ (\sqrt{5})} e^{-\frac{\pi}{5}\left(\frac{\lambda^2}{t'}+\frac{(\lambda')^2}{t}\right)}
\end{aligned}$$

This completes the proof of (5.2).

By one of the properties of Fourier transform, we know that

$$\mathcal{F}\left(\frac{\partial^2 f}{\partial x_2 \partial x_1}\right)(y_1, y_2) = (2i\pi y_1)(2i\pi y_2)\hat{f}(y_1, y_2)$$

We apply this property to the function given in (5.4), and we get

$$\mathcal{F}\left(\frac{4\pi^2}{25} tt'(x_2+b)(x_1+a)f\right)(y_1, y_2) = -4\pi^2 y_1 y_2 \hat{f}(y_1, y_2) \tag{5.5}$$

Now, we apply the Poisson Summation Formula and the relation in (5.5):

$$
\begin{aligned}
&\frac{tt'}{25}\sum_{\boldsymbol{\beta}\in\Gamma}(\beta+\rho)(\beta'+\rho')e^{-\pi\left(\frac{t(\beta+\rho)^2+t'(\beta'+\rho')^2}{5}\right)}\\
&=\frac{tt'}{25}\sum_{\boldsymbol{\mu}\equiv\boldsymbol{\rho}\ (\sqrt{5})}\mu\mu' e^{-\pi\left(\frac{t\mu^2+t'(\mu')^2}{5}\right)}\\
&=-\left(\frac{\lambda'}{5}\right)\left(\frac{\lambda}{5}\right)\frac{1}{\sqrt{5}}\frac{1}{\sqrt{tt'}}\sum_{\boldsymbol{\lambda}\in\mathcal{O}_K} e^{-5\pi\left(\frac{(\lambda'/5)^2}{t}+\frac{(\lambda/5)^2}{t'}\right)+2i\pi((\lambda'/5)\rho+(\lambda/5)\rho')}\\
&=-\frac{1}{\sqrt{5}}\frac{1}{(\sqrt{tt'})^3}\sum_{\boldsymbol{\lambda}\in\mathcal{O}_K}\lambda\lambda' e^{-\frac{\pi}{5}\left(\frac{\lambda^2}{t'}+\frac{(\lambda')^2}{t}\right)+2i\pi\,\mathrm{Tr}\left(\frac{\lambda'\rho}{5}\right)}\\
&=-\frac{1}{\sqrt{5}}\frac{1}{(\sqrt{tt'})^3}\sum_{\boldsymbol{\alpha}\in\mathcal{O}_K/(\sqrt{5})} e^{2i\pi\,\mathrm{Tr}\left(\frac{\alpha'\rho}{5}\right)}\sum_{\boldsymbol{\lambda}\equiv\boldsymbol{\alpha}\ (\sqrt{5})}\lambda\lambda' e^{-\frac{\pi}{5}\left(\frac{\lambda^2}{t'}+\frac{(\lambda')^2}{t}\right)}
\end{aligned}
$$

This completes the proof of (5.3).

Corollary 20. *Notice that for fixed v,*

$$\theta_0(ue^{2v},ue^{-2v},\rho)-\delta(\rho)\ \textit{decays exponentially as}\ u\to\infty$$

from the definition and by the relation (5.2)

$$\theta_0(ue^{2v},ue^{-2v},\rho)\sim\frac{1}{u}\ \textit{as}\ u\to 0$$

where

$$\delta(\rho)=\begin{cases}1 & \rho=0\\ 0 & \rho\neq 0\end{cases}$$

5.0.5 The ξ-functions

Let us define ξ_0 and ξ_1 by

$$
\begin{aligned}
\xi_0(s,\rho)&=\left(\Gamma\left(\frac{s}{2}\right)\right)^2\left(\frac{5}{\pi}\right)^s\zeta_0(s,\rho),\\
\xi_1(s,\rho)&=\left(\Gamma\left(\frac{s+1}{2}\right)\right)^2\left(\frac{5}{\pi}\right)^s\zeta_1(s,\rho).
\end{aligned}
$$

Note that ζ-functions are absolutely convergent in a half plane.

Proposition 21. *The ξ-functions have the following integral representations:*

$$\xi_0(s,\rho) = 2\int_{v=-l}^{l}\int_{u=0}^{\infty} u^s \sum_{\substack{\mu \equiv \rho\ (\sqrt{5}) \\ \mu \neq 0}} e^{-\frac{\pi}{5}u(\mu^2 e^v + (\mu')^2 e^{-v})}\frac{dudv}{u},$$

$$\xi_1(s,\rho) = \frac{2\pi}{5}\int_{v=-l}^{l}\int_{u=0}^{\infty} u^{s+1} \sum_{\substack{\mu \equiv \rho\ (\sqrt{5}) \\ \mu \neq 0}} \mu\mu' e^{-\frac{\pi}{5}u(\mu^2 e^v + (\mu')^2 e^{-v})}\frac{dudv}{u}.$$

Proof. We know that

$$\Gamma(s) = \int_0^\infty e^{-t}t^{s-1}dt,$$

hence we get

$$\Gamma\left(\frac{s}{2}\right) = \int_0^\infty e^{-t}t^{s/2-1}dt.$$

Let $t = \frac{\pi}{5}\mu^2 x$, then $dt = \frac{\pi}{5}\mu^2 dx$. Therefore,

$$\begin{aligned}\Gamma\left(\frac{s}{2}\right) &= \int_0^\infty e^{-\frac{\pi}{5}\mu^2 x}\left(\frac{\pi}{5}\mu^2 x\right)^{s/2-1}\frac{\pi}{5}\mu^2 dx \\ &= \left(\frac{\pi}{5}\right)^{s/2}|\mu|^s\int_0^\infty e^{-(\pi/5)\mu^2 x}x^{s/2}\frac{dx}{x}\end{aligned}$$

By a similar computation above, we have

$$\left(\frac{5}{\pi}\right)^s\frac{1}{|\mu|^s}\frac{1}{|\mu'|^s}(\Gamma(s/2))^2 = \int_0^\infty\int_0^\infty e^{-\frac{\pi}{5}(\mu^2 x + (\mu')^2 y)}x^{s/2}y^{s/2}\frac{dxdy}{xy}$$

Let $x = ue^{2v}$, $y = ue^{-2v}$ which give us $xy = u^2$ and $x/y = e^{4v}$. Also, $2udu = xdy + ydx$, and $4e^{4v}dv = \frac{ydx - xdy}{y^2}$ from which we get

$$\frac{dxdy}{xy} = 4\frac{dudv}{u}.$$

Hence,

$$\left(\frac{5}{\pi}\right)^s \frac{1}{|\mu|^s}\frac{1}{|\mu'|^s}(\Gamma(s/2))^2 = 4\int_{-\infty}^{\infty}\left(\int_0^{\infty} u^s e^{-\frac{\pi}{5}u(\mu^2e^{2v}+(\mu')^2e^{-2v})}\frac{du}{u}\right)dv, \text{ i.e.}$$

$$\left(\frac{5}{\pi}\right)^s \frac{1}{|\operatorname{N}(\mu)|^s}(\Gamma(s/2))^2 = 4\int_{-\infty}^{\infty}\left(\int_0^{\infty} u^s e^{-\frac{\pi}{5}u(\mu^2e^{2v}+(\mu')^2e^{-2v})}\frac{du}{u}\right)dv$$

In order to get the ξ-function above we need to add $1/|\operatorname{N}(\mu)|^s$ for $\mu \in (\mathcal{O}_K^\rho - \{0\})/\sim_u$ on the left side of the integral. However, this is a sum over a pretty complicated quotient of a sublattice of $\mathcal{O}_K$. We want to write the ξ-function as an integral involving a θ-function and this requires us to have a sum over all elements $\mu \in \mathcal{O}_K^\rho - \{0\}$. For this purpose, we first change the order of the integration in the above integral and then we write the integral with respect to v that is from $-\infty$ to ∞ as a sum of integrals from $-l$ to l where $l = \frac{1}{2}\log u$ where $u = \left(\frac{1+\sqrt{5}}{2}\right)^4$ is the first totally positive unit which is congruent to 1 modulo the ideal $(\sqrt{5})$.

$$\begin{aligned}
&= 4\int_0^{\infty} u^s\left(\sum_{n=-\infty}^{n=\infty}\int_{-l}^{l} e^{-\frac{\pi}{5}u(\mu^2e^{2(v+2nl)}+(\mu')^2e^{-2(v+2nl)})}dv\right)\frac{du}{u}\\
&= 4\int_0^{\infty} u^s\left(\sum_{n=-\infty}^{n=\infty}\int_{-l}^{l} e^{-\frac{\pi}{5}u((\mu e^{2nl})^2e^{2v}+(\mu' e^{2nl})^2e^{-2v})}dv\right)\frac{du}{u}\\
&= 4\int_0^{\infty} u^s\left(\sum_{n=-\infty}^{n=\infty}\int_{-l}^{l} e^{-\frac{\pi}{5}u((\mu u^n)^2e^{2v}+(\mu' u^n)^2e^{-2v})}dv\right)\frac{du}{u}
\end{aligned}$$

In the last integral, we replaced l by $\frac{1}{2}\log u$ so that e^{2l} becomes u. Now we add $1/|\operatorname{N}(\mu)|^s$ for $\mu \in (\mathcal{O}_K^\rho - \{0\}) \sim_u$ on the left side of the integral.

$$\begin{aligned}
&\left(\frac{5}{\pi}\right)^s(\Gamma(s/2))^2\zeta_0(\rho, s) = \xi(s,\rho)\\
&= 4\int_0^{\infty} u^s\left(\int_{-l}^{l}\sum_{n=-\infty}^{n=\infty}\sum_{\mu\in(\mathcal{O}_K^\rho-\{0\})/\sim_u} e^{-\frac{\pi}{5}u((\mu u^n)^2e^{2v}+(\mu' u^n)e^{-2v})}dv\right)\frac{du}{u}
\end{aligned}$$

We see that if $\mu \in (\mathcal{O}_K^\rho - \{0\})/\sim_u$, then μu^n runs through all the elements η of the set $\mathcal{O}_K^\rho - \{0\}$ when n is an integer. Therefore, we have

$$\xi_0(s,\rho) = 4\int_{-l}^{l}\int_0^\infty u^s \sum_{\eta\equiv\rho\ (\sqrt{5}),\ \eta\neq 0} e^{-\frac{\pi}{5}u(\eta^2 e^{2v}+(\eta')^2 e^{-2v})}\frac{dudv}{u}$$

Similarly, we can prove the integral representation for $\xi_1(s,\rho)$.

$\square$

In the above sum, ρ can take 5 values. If $\rho \neq 0$, then the sum is over a translate of a sublattice of $\mathcal{O}_K$ and is equal to the $\theta_0(t,t',\rho)$ by definition. However, when $\rho = 0$ the above sum does not include a term for $\eta = 0$. Hence, it is equal to $\theta_0(t,t',0) - 1$. As a result,

$$\xi_0(s,\rho) = 4\int_0^\infty u^s \int_{-l}^{l} \left(\theta_0(ue^{2v}, ue^{-2v},\rho) - \delta(\rho)\right)\frac{dvdu}{u}$$

where

$$\delta(\rho) = \begin{cases} 1 & \rho = 0 \\ 0 & \rho \neq 0 \end{cases}$$

Now, we would like to prove ξ-functions are meromorphic and they satisfy a functional equation.

Proposition 22. *The ξ-functions defined above are meromorphic functions on $\mathbb{C}$ with simple poles at $s = 0$ and $s = 1$. They also satisfy*

$$\xi_0(1-s,\rho) = \frac{1}{\sqrt{5}}\sum_{\alpha \mod \sqrt{5}} e^{2i\pi \operatorname{Tr}\left(\frac{\alpha'\rho}{5}\right)}\xi_0(s,\alpha)$$

Proof. We will prove this by considering two cases:

Case 1: In this case, $\rho \neq 0$. First, we break the integral representation of ξ_0 into two integrals from 0 to 1 and from 1 to ∞. In the first integral, we make the change of variable $u \leftrightarrow 1/u$. Then we get

$$\begin{aligned}
\xi_0(s,\rho) &= 4\int_0^\infty u^s \left(\int_{-l}^{l} \theta_0(ue^{2v}, ue^{-2v}, \rho)dv\right)\frac{du}{u} \\
&= 4\int_1^\infty u^{-s}\left(\int_{-l}^{l} \theta_0(e^{2v}/u, e^{-2v}/u, \rho)dv\right)\frac{du}{u} + 4\int_1^\infty u^s\left(\int_{-l}^{l} \theta_0(ue^{2v}, ue^{-2v}, \rho)dv\right)\frac{du}{u}
\end{aligned}$$

Now we will apply the following relation for the θ_0-function

$$\theta_0(t,t',\rho) = \frac{1}{\sqrt{5}}\frac{1}{\sqrt{tt'}}\sum_{\alpha\in\mathcal{O}_K/(\sqrt{5})} e^{2i\pi\,\mathrm{Tr}\left(\frac{\lambda'\rho}{5}\right)}\theta_0\left(\frac{1}{t'}, \frac{1}{t}, \alpha\right)$$

in the first integral.

$$\begin{aligned}
&= \frac{4}{\sqrt{5}}\int_1^\infty u^{1-s}\int_{-l}^{l}\left(\sum_{\alpha\in\mathcal{O}_K/(\sqrt{5})} e^{2i\pi\,\mathrm{Tr}\left(\frac{\alpha'\rho}{5}\right)}\theta_0(ue^{2v}, ue^{-2v}, \alpha)\right)\frac{dvdu}{u} \\
&+ 4\int_1^\infty u^s\left(\int_{-l}^{l}\theta_0(ue^{2v}, ue^{-2v}, \rho)dv\right)\frac{du}{u}
\end{aligned}$$

In the above sum, the cases $\alpha = 0$ and $\alpha \neq 0$ are quite different. Since $\theta_0(ue^{2v}, ue^{-2v}\alpha) - \delta\left(\frac{\alpha}{\sqrt{5}}\right)$ is exponentially decreasing at infinity, the integrals above involving $\theta_0(ue^{2v}, ue^{-2v}, \alpha)$ define entire functions except when $\alpha = 0$. The function $\theta_0(ue^{2v}, ue^{-2v}, 0)$ leads to a pole at $s = 1$. In order to get this pole, we will add and subtract 1 to the function $\theta_0(ue^{2v}, ue^{-2v}, 0)$.

$$\begin{aligned}
&= \frac{4}{\sqrt{5}}\int_1^\infty u^{1-s}\int_{-l}^{l}\sum_{\alpha\in\mathcal{O}_K/(\sqrt{5}),\,\alpha\neq 0} e^{2i\pi\,\mathrm{Tr}\left(\frac{\alpha'\rho}{5}\right)}\theta_0(ue^{2v}, ue^{-2v}, \alpha)\frac{dvdu}{u} \\
&+ \frac{4}{\sqrt{5}}\int_1^\infty u^{1-s}\left(\int_{-l}^{l}[(\theta_0(ue^{2v}, ue^{-2v}, 0) - 1 + 1)]dv\right)\frac{du}{u} \\
&+ 4\int_1^\infty u^s\left(\int_{-l}^{l}\theta_0(ue^{2v}, ue^{-2v}, \rho)dv\right)\frac{du}{u}
\end{aligned}$$

After integrating the second integral above, we have

$$
\begin{aligned}
&= \frac{4}{\sqrt{5}} \int_1^\infty u^{1-s} \int_{-l}^{l} \sum_{\alpha \in \mathcal{O}_K/(\sqrt{5}),\ \alpha \neq 0} e^{2i\pi \operatorname{Tr}\left(\frac{\alpha' \rho}{5}\right)} \theta_0(ue^{2v}, ue^{-2v}, \alpha) \frac{dv du}{u} \\
&- \frac{8l}{\sqrt{5}(1-s)} + \frac{4}{\sqrt{5}} \int_1^\infty u^{1-s} \left(\int_{-l}^{l} (\theta_0(ue^{2v}, ue^{-2v}, 0) - 1) dv \right) \frac{du}{u} \\
&+ 4 \int_1^\infty u^s \left(\int_{-l}^{l} \theta_0(ue^{2v}, ue^{-2v}, \rho) dv \right) \frac{du}{u}
\end{aligned}
$$

Therefore, we have just shown that $\xi_0(s, \rho)$ where $\rho \neq 0$ is a meromorphic function whose only pole is at $s = 1$ with residue

$$
\frac{8l}{\sqrt{5}} = \frac{8 \log((3 + \sqrt{5})/2)}{\sqrt{5}}.
$$

Case 2: In this case, $\rho = 0$. First, we write the integral representation of $\xi_0(s, 0)$ and then split the integral into two integrals from 0 to 1 and from 1 to ∞. In the first integral, we make the substitution $u \leftrightarrow 1/u$. Then we get

$$
\begin{aligned}
\xi_0(s, 0) &= 4 \int_0^\infty u^s \left(\int_{-l}^{l} (\theta_0(ue^{2v}, ue^{-2v}, 0) - 1) dv \right) \frac{du}{u} \\
&= 4 \int_1^\infty u^{-s} \left(\int_{-l}^{l} (\theta_0(e^{2v}/u, e^{-2v}/u, 0) - 1) dv \right) \frac{du}{u} \\
&\quad + 4 \int_1^\infty u^s \left(\int_{-l}^{l} (\theta_0(ue^{2v}, ue^{-2v}, 0) - 1) dv \right) \frac{du}{u}
\end{aligned}
$$

The second integral above defines an entire function since $\theta_0(ue^{2v}, ue^{-2v}, 0) - 1$ decays exponentially at infinity. However, the first integral converges only for $\operatorname{Re} s > 1$ since $\theta_0(e^{2v}/u, e^{-2v}/u, 0)$ behaves like u at infinity. First, we will integrate -1 in the first integral and then we will apply the relation

$$
\theta_0(t, t', \rho) = \frac{1}{\sqrt{5}} \frac{1}{\sqrt{tt'}} \sum_{\alpha \in \mathcal{O}_K/(\sqrt{5})} e^{2i\pi \operatorname{Tr}\left(\frac{\lambda' \rho}{5}\right)} \theta_0 \left(\frac{1}{t'}, \frac{1}{t}, \alpha \right)
$$

to $\theta_0(e^{2v}/u, e^{-2v}/u, 0)$.

$$= -\frac{8l}{s} + 4\int_1^\infty u^{-s}\int_{-l}^{l}\theta_0(e^{2v}/u, e^{-2v}/u, 0)\frac{dvdu}{u} + 4\int_1^\infty u^s\left(\int_{-l}^{l}(\theta_0(ue^{2v}, ue^{-2v}, 0) - 1)dv\right)\frac{du}{u}$$

$$= -\frac{8l}{s} + \frac{4}{\sqrt{5}}\int_1^\infty u^{1-s}\int_{-l}^{l}\left(\sum_{\alpha\in\mathcal{O}_K/(\sqrt{5}),\ \alpha\neq 0} e^{2i\pi\operatorname{Tr}\left(\frac{\alpha'\rho}{5}\right)}\theta_0(ue^{2v}, ue^{-2v}, \alpha)\right)\frac{dvdu}{u}$$

$$+ 4\int_1^\infty u^s\int_{-l}^{l}(\theta_0(ue^{2v}, ue^{-2v}, 0) - 1)\frac{dvdu}{u}$$

As in the previous case, since $\theta_0(ue^{2v}, ue^{-2v}, 0) - 1$ is exponentially decreasing at infinity, the integrals above involving $\theta_0(ue^{2v}, ue^{-2v}, \alpha)$ define entire functions except when $\alpha = 0$ whereas the function $\theta_0(ue^{2v}, ue^{-2v}, 0)$ leads to a pole at $s = 1$. In order to get this pole, we will add and subtract 1 to the function $\theta_0(ue^{2v}, ue^{-2v}, 0)$.

$$= -\frac{8l}{s} + \frac{4}{\sqrt{5}}\int_1^\infty u^{1-s}\int_{-l}^{l}\sum_{\alpha\in\mathcal{O}_K/(\sqrt{5}),\ \alpha\neq 0} e^{2i\pi\operatorname{Tr}\left(\frac{\alpha'\rho}{5}\right)}\theta_0(ue^{2v}, ue^{-2v}, \alpha)\frac{dvdu}{u}$$

$$+ \frac{4}{\sqrt{5}}\int_1^\infty u^{1-s}\int_{-l}^{l}(\theta_0(ue^{2v}, ue^{-2v}, 0) - 1 + 1)\frac{dvdu}{u} + 4\int_1^\infty u^s\int_{-l}^{l}(\theta_0(ue^{2v}, ue^{-2v}, 0) - 1)\frac{dvdu}{u}$$

By integrating 1 in the second integral, we have

$$= -\frac{8l}{s} - \frac{8l}{\sqrt{5}(1-s)} + \frac{4}{\sqrt{5}}\int_1^\infty u^{1-s}\int_{-l}^{l}\sum_{\alpha\in\mathcal{O}_K/(\sqrt{5}),\ \alpha\neq 0} e^{2i\pi\operatorname{Tr}\left(\frac{\alpha'\rho}{5}\right)}\theta_0(ue^{2v}, ue^{-2v}, \alpha)\frac{dvdu}{u}$$

$$+ \frac{4}{\sqrt{5}}\int_1^\infty u^{1-s}\int_{-l}^{l}(\theta_0(ue^{2v}, ue^{-2v}, 0) - 1)\frac{dvdu}{u} + 4\int_1^\infty u^s\int_{-l}^{l}(\theta_0(ue^{2v}, ue^{-2v}, 0) - 1)\frac{dvdu}{u}$$

Remark 23. *Since $\theta_0(ue^{2v}, ue^{-2v}\rho) - \delta(\rho)$ is exponentially decreasing at infinity, the integrals above on the right hand side of the ξ-functions define entire functions. As a result, we have just proven that ξ-functions are meromorphic functions with poles at most at $s = 0$ and $s = 1$.*

Now, we would like to prove

$$\xi_0(1-s, \rho) = \frac{1}{\sqrt{5}}\sum_{\alpha\in\mathcal{O}_K/(\sqrt{5})} e^{2i\pi\operatorname{Tr}\left(\frac{\alpha'\rho}{5}\right)}\xi_0(s, \alpha)$$

Case1: $\rho \neq 0$ We already have an integral representation for $\xi_0(s,\rho)$. We substitute $s \leftrightarrow 1-s$ in this representation and get

$$\sqrt{5}\xi_0(1-s,\rho) = -\frac{8l}{s} + 4\int_1^\infty u^s \int_{-l}^{l} \sum_{\alpha\in\mathcal{O}_K/(\sqrt{5}),\, \alpha\neq 0} e^{2i\pi\,\mathrm{Tr}\left(\frac{\alpha'\rho}{5}\right)} \theta_0(ue^{2v}, ue^{-2v}, \alpha)\frac{dvdu}{u}$$
$$+ 4\int_1^\infty u^s \int_{-l}^{l} (\theta_0(ue^{2v}, ue^{-2v}, 0) - 1)\frac{dvdu}{u}$$
$$+ 4\sqrt{5}\int_1^\infty u^{1-s} \int_{-l}^{l} \theta_0(ue^{2v}, ue^{-2v}, \rho)\frac{dvdu}{u}$$

Now substitute u with $1/u$ in the last integral above resulting in

$$4\sqrt{5}\int_1^\infty u^{1-s} \int_{-l}^{l} \theta_0(ue^{2v}, ue^{-2v}, \rho)\frac{dvdu}{u} = 4\sqrt{5}\int_0^1 u^{s-1} \int_{-l}^{l} \theta_0(e^{2v}/u, ue^{-2v}/u, \rho)\frac{dvdu}{u}$$

In the relation

$$\theta_0(t,t',\rho) = \frac{1}{\sqrt{5}\sqrt{tt'}} \sum_{\alpha\in\mathcal{O}_K/(\sqrt{5})} e^{2i\pi\,\mathrm{Tr}\left(\frac{\alpha'\rho}{5}\right)} \theta_0\left(\frac{1}{t'}, \frac{1}{t}, \alpha\right)$$

let $t = e^{2v}/u$ and $t' = e^{-2v}/u$. Then we have

$$\theta_0(e^{2v}/u, e^{-2v}/u, \rho) = \frac{u}{\sqrt{5}} \sum_{\alpha\in\mathcal{O}_K/(\sqrt{5})} e^{2i\pi\,\mathrm{Tr}\left(\frac{\alpha'\rho}{5}\right)} \theta_0\left(ue^{2v}, ue^{-2v}, \alpha\right)$$

By applying the relation to $\theta_0(e^{2v}/u, ue^{-2v}/u, \rho)$, the third integral in $\sqrt{5}\xi_0(1-s,\rho)$ becomes

$$4\int_0^1 u^s \int_{-l}^{l} \sum_{\alpha\in\mathcal{O}_K/(\sqrt{5})} e^{2i\pi\,\mathrm{Tr}\left(\frac{\alpha'\rho}{5}\right)} \theta_0(ue^{2v}, ue^{-2v}, \alpha)\frac{dvdu}{u}$$

Now we can combine the first integral in $\sqrt{5}\xi_0(1-s,\rho)$ which is from 1 to ∞ with the last integral above to get an integral from 0 to ∞ for $\alpha \neq 0$. Therefore,

$$\sqrt{5}\xi_0(1-s,\rho) = \sum_{\alpha\in\mathcal{O}_K/(\sqrt{5}),\alpha\neq 0} e^{2i\pi\,\mathrm{Tr}\left(\frac{\alpha'\rho}{5}\right)} 4\int_0^\infty u^s \int_{-l}^{l} \theta_0(ue^{2v}, ue^{-2v}, \alpha)\frac{dvdu}{u}$$
$$-\frac{8l}{s} + 4\int_0^1 u^s \int_{-l}^{l} \theta_0(ue^{2v}, ue^{-2v}, 0)\frac{dvdu}{u}$$
$$+ 4\int_1^\infty u^s \int_{-l}^{l} (\theta_0(ue^{2v}, ue^{-2v}, 0) - 1)\frac{dvdu}{u}$$

Note that $\xi_0(s,0) = 4\int_0^\infty u^s \int_{-l}^{l} (\theta_0(ue^{2v}, ue^{-2v}, 0) - 1)\frac{dvdu}{u}$. In order to get $\xi_0(s,0)$, we will first add and subtract 1 in the second integral above and then integrate 1. The integral of 1 over the given region is just $8l/s$, and then we will combine the remaining integrals to get

$$\sqrt{5}\xi_0(1-s,\rho) = \sum_{\alpha\in\mathcal{O}_K/(\sqrt{5}),\alpha\neq 0} e^{2i\pi \operatorname{Tr}\left(\frac{\alpha'\rho}{5}\right)} 4\int_0^\infty u^s \int_{-l}^{l} \theta_0(ue^{2v}, ue^{-2v}, \alpha)\frac{dvdu}{u}$$
$$+ 4\int_0^\infty u^s \int_{-l}^{l} (\theta_0(ue^{2v}, ue^{-2v}, 0) - 1)\frac{dvdu}{u}$$

Note that when $\alpha = 0$, $e^{2i\pi \operatorname{Tr}\left(\frac{\alpha'\rho}{5}\right)} = 1$ leading to

$$\xi_0(1-s,\rho) = \frac{1}{\sqrt{5}} \sum_{\alpha\in\mathcal{O}_K/(\sqrt{5})} e^{2i\pi \operatorname{Tr}\left(\frac{\alpha'\rho}{5}\right)} \xi_0(s,\alpha)$$

Similar computations give the same relation for $\rho = 0$.

Remark 24. *Notice that we had defined $\xi_0(s,\rho)$ as*

$$\xi_0(s,\rho) = \left(\Gamma\left(\frac{s}{2}\right)\right)^2 \left(\frac{5}{\pi}\right)^s \zeta_0(s,\rho),$$

then

$$\zeta_0(s,\rho) = \xi_0(s,\rho)\left(\frac{\pi}{5}\right)^s \frac{1}{\Gamma(s/2)^2}.$$

We have just proven that $\xi_0(s,\rho)$ is a meromorphic function on the entire complex plane except simple poles at $s = 1$ and $s = 0$. Since $\frac{1}{\Gamma(s/2)}$ is an entire function of s with simple zeroes at $s = 0, -1/2, -1, \cdots$ and it vanishes nowhere else, then $\zeta_0(s,\rho)$ is a meromorphic function with a simple pole at $s = 1$.

□

5.1 Construction of wave forms

In this section, we will construct the wave forms corresponding the ζ-functions which we have defined before in (15). For each ρ, $0 \le \rho \le 2$, define a function $g_\rho(z)$ where $z = x + iy \in \mathbb{H}$ as follows:

$$g_\rho(z) = \log\left(\frac{1+\sqrt{5}}{2}\right)^4 \delta(\rho) y^{1/2} + \sum_{n\neq 0,\, n\equiv b_\rho\ (5)} a_n^\rho y^{1/2} K_0\left(\frac{2\pi|n|}{5}y\right) e^{\frac{2i\pi n}{5}x}$$

By definition, we know that $\delta(\rho) = 0$ only when $\rho = 0$, i.e. we have a constant term only if $\rho = 0$.

Claim 25. *We will show that*

$$g_\rho\left(-\frac{1}{z}\right) = \sum_{\rho=0}^{2} c_{\rho\sigma} g_\sigma(z)$$

Proof. Consider

$$\begin{aligned} g_\rho(y) &= \log\left(\frac{1+\sqrt{5}}{2}\right)^4 \delta(\rho) y^{1/2} + \sum_{n\neq 0,\, n\equiv b_\rho\ (5)} a_n^\rho y^{1/2} K_0\left(\frac{2\pi|n|}{5}y\right) \\ &= u_\rho(y) + F_\rho(y) \end{aligned}$$

First we want to prove

$$g_\rho\left(\frac{1}{y}\right) = \sum_{\rho=0}^{2} c_{\rho\sigma} g_\sigma(y)$$

for $y > 0$. By using the integral representation of K-Bessel function, we have

$$\begin{aligned} K_0\left(\frac{2\pi|n|}{5}y\right) &= \frac{1}{8\pi i}\int_{c-i\infty}^{c+i\infty} \Gamma\left(\frac{s}{2}\right)\Gamma\left(\frac{s}{2}\right)\left(\frac{2\pi|n|y}{(2)(5)}\right)^{-s} ds \\ &= \frac{1}{8\pi i}\int_{c-i\infty}^{c+i\infty} \Gamma^2\left(\frac{s}{2}\right)\left(\frac{5}{\pi}\right)^s \frac{1}{|n|^s y^s} ds \end{aligned}$$

where $c > 0$. Then we see that

$$F_\rho(y) = \frac{1}{8\pi i}\int_{c-i\infty}^{c+i\infty} \frac{\xi_\rho(s)}{y^{s-1/2}} ds$$

leading to

$$g_\rho(y) = u_\rho(y) + \frac{1}{8\pi i}\int_{c-i\infty}^{c+i\infty} \frac{\xi_\rho(s)}{y^{s-1/2}} ds.$$

From the above representation,

$$g_\rho\left(\frac{1}{y}\right) = u_\rho\left(\frac{1}{y}\right) + \frac{1}{8\pi i}\int_{c-i\infty}^{c+i\infty} \frac{\xi_\rho(s)}{y^{1/2-s}} ds$$

Now we use the identity

$$\xi_\rho(s) = \sum_{\sigma=0}^{2} c_{\rho\sigma}\xi_\sigma(1-s)$$

to get

$$g_\rho\left(\frac{1}{y}\right) = u_\rho\left(\frac{1}{y}\right) + \sum_{\sigma=0}^{2} c_{\rho\sigma}\frac{1}{8\pi i}\int_{c-i\infty}^{c+i\infty} \frac{\xi_\sigma(1-s)}{y^{1/2-s}} ds.$$

Now we make the substitution $s \leftrightarrow 1-s$ in the integral and obtain

$$\begin{aligned} g_\rho\left(\frac{1}{y}\right) &= u_\rho\left(\frac{1}{y}\right) + \sum_{\sigma=0}^{2} c_{\rho\sigma}\frac{1}{8\pi i}\int_{1-c-i\infty}^{1-c+i\infty} \frac{\xi_\sigma(s)}{y^{s-1/2}} ds \\ &= u_\rho\left(\frac{1}{y}\right) + \sum_{\sigma=0}^{2} c_{\rho l}\frac{1}{8\pi i}\int_{c-i\infty}^{c+i\infty} \frac{\xi_\sigma(s)}{y^{s-1/2}} ds \\ &\quad - \frac{1}{4}\sum_{\sigma=0}^{2} c_{\rho\sigma}\sum_{res}\frac{\xi_\sigma(s)}{y^{s-1/2}} \end{aligned}$$

In the above, we can replace the integral from $1-c-i\infty$ to $1-c+i\infty$ by the integral from $c-i\infty$ to $c+i\infty$ and by taking the residues of the integrand into account via the Phragmén-Lindelöf theorem.

We know that $\frac{\xi_\sigma(s)}{y^{s-1/2}}$ has simple poles at $s=0$ when $\sigma=0$ and at $s=1$ for all σ with residues $-8l = -8\log\left(\frac{3+\sqrt{5}}{2}\right)y^{1/2}$ and $\frac{8l}{\sqrt{5}} = \frac{8\log\left(\frac{3+\sqrt{5}}{2}\right)}{\sqrt{5}}y^{-1/2}$

respectively. Also, first row of the matrix $(c_{\rho\sigma})$ is $[1/\sqrt{5}, 2/\sqrt{5}, 2/\sqrt{5}]$. Then

$$\begin{aligned}
g_0\left(\frac{1}{y}\right) &= \log\left(\frac{1+\sqrt{5}}{2}\right)^4 y^{-1/2} + \sum_{l\sigma=0}^{2} c_{0\sigma}\frac{1}{8\pi i}\int_{c-i\infty}^{c+i\infty}\frac{\xi_\sigma(s)}{y^{s-1/2}}ds \\
&\quad - \frac{1}{4}\left(\frac{1}{\sqrt{5}}\left(-8\log\left(\frac{3+\sqrt{5}}{2}\right)y^{1/2} + \frac{8}{\sqrt{5}}\log\left(\frac{3+\sqrt{5}}{2}\right)y^{-1/2}\right)\right) \\
&\quad - \frac{1}{4}\left(\frac{2}{\sqrt{5}}\frac{8}{\sqrt{5}}\log\left(\frac{3+\sqrt{5}}{2}\right)y^{-1/2} + \frac{2}{\sqrt{5}}\frac{8}{\sqrt{5}}\log\left(\frac{3+\sqrt{5}}{2}\right)y^{-1/2}\right) \\
&= \log\left(\frac{1+\sqrt{5}}{2}\right)^4 y^{-1/2} + \sum_{\sigma=0}^{2} c_{0\sigma}\frac{1}{8\pi i}\int_{c-i\infty}^{c+i\infty}\frac{\xi_l(s)}{y^{s-1/2}}ds \\
&\quad + y^{1/2}\frac{1}{\sqrt{5}}\log\left(\frac{1+\sqrt{5}}{2}\right)^4 \\
&\quad + y^{-1/2}\left(-\frac{4}{5}\log\left(\frac{1+\sqrt{5}}{2}\right)^4 - \frac{1}{5}\log\left(\frac{1+\sqrt{5}}{2}\right)^4\right) \\
&= y^{1/2}\frac{1}{\sqrt{5}}\log\left(\frac{1+\sqrt{5}}{2}\right)^4 + \sum_{\sigma=0}^{2} c_{0\sigma}\frac{1}{8\pi i}\int_{c-i\infty}^{c+i\infty}\frac{\xi_\sigma(s)}{y^{s-1/2}}ds \\
&= \sum_{\sigma=0}^{2} c_{0\sigma}u_0(y) + \sum_{\sigma=0}^{2} c_{0\sigma}\frac{1}{8\pi i}\int_{c-i\infty}^{c+i\infty}\frac{\xi_\sigma(s)}{y^{s-1/2}}ds \\
&= \sum_{\sigma=0}^{2} c_{0\sigma}g_0(y)
\end{aligned}$$

We know that for $\rho = 1, 2$, $u_\rho(1/y) = u_\rho(y) = 0$. Hence, we need to show that

$$-\frac{1}{4}\sum_{\sigma=0}^{2} c_{\rho\sigma}\sum_{res}\frac{\xi_\sigma(s)}{y^{s-1/2}} = \frac{1}{\sqrt{5}}u_0(y) = \sum_{\sigma=0}^{2} c_{\rho\sigma}u_\rho(y)$$

for $\rho = 1, 2$ where

$$(c_{\rho\sigma}) = \frac{1}{\sqrt{5}}\begin{pmatrix} 1 & 2 & 2 \\ 1 & w^2+w^{-2} & w+w^{-1} \\ 1 & w+w^{-1} & w^2+w^{-2} \end{pmatrix}$$

and $w = e^{\frac{2i\pi}{5}}$. We will show the computation for $\rho = 1$ since the computation for

$\rho = 2$ is similar.

$$-\frac{1}{4}\sum_{l=0}^{2} c_{\rho\sigma} \sum_{res} \frac{\xi_\sigma(s)}{y^{s-1/2}} = -\frac{1}{4}\frac{1}{\sqrt{5}}\left(-8\log\left(\frac{3+\sqrt{5}}{2}\right)y^{1/2} + \frac{8}{\sqrt{5}}\log\left(\frac{3+\sqrt{5}}{2}\right)y^{-1/2}\right)$$
$$-\frac{1}{4}\frac{1}{\sqrt{5}}(w^2+w^{-2}+w+w^{-1})\frac{8}{\sqrt{5}}\log\left(\frac{3+\sqrt{5}}{2}\right)y^{-1/2}$$
$$= \frac{1}{\sqrt{5}}\log\left(\frac{1+\sqrt{5}}{2}\right)^4 y^{1/2} = \sum_{\sigma=0}^{2} c_{1\sigma}u_\sigma(y).$$

□

5.1.1 The ξ-functions

Let us define ξ_0 and ξ_1 by

$$\xi_0(s,\rho) = \Gamma\left(\frac{s-\frac{\pi n i}{\log\epsilon}}{2}\right)\Gamma\left(\frac{s+\frac{\pi n i}{\log\epsilon}}{2}\right)\left(\frac{5}{\pi}\right)^s \zeta_0(s,\rho),$$
$$\xi_1(s,\rho) = \Gamma\left(\frac{s+1-\frac{\pi n i}{\log\epsilon}}{2}\right)\Gamma\left(\frac{s+1+\frac{\pi n i}{\log\epsilon}}{2}\right)\left(\frac{5}{\pi}\right)^s \zeta_1(s,\rho).$$

Note that ζ-functions are absolutely convergent in a half plane.

Proposition 26. *The ξ-functions have the following integral representations:*

$$\xi_0(s,\rho) = 2\int_{v=-l}^{l}\int_{u=0}^{\infty} u^s \sum_{\substack{\mu\equiv\rho\ (\sqrt{5})\\ \mu\neq 0}} e^{-\frac{\pi}{5}u(\mu^2e^v+(\mu')^2e^{-v})}\frac{dudv}{u},$$
$$\xi_1(s,\rho) = \frac{2\pi}{5}\int_{v=-l}^{l}\int_{u=0}^{\infty} u^{s+1} \sum_{\substack{\mu\equiv\rho\ (\sqrt{5})\\ \mu\neq 0}} \mu\mu' e^{-\frac{\pi}{5}u(\mu^2e^v+(\mu')^2e^{-v})}\frac{dudv}{u}.$$

Proof. We know that

$$\Gamma(s) = \int_0^\infty e^{-t} t^{s-1} dt,$$

Let $c = \pi m i / \log \epsilon$. Then we get

$$\Gamma\left(\frac{s-c}{2}\right) = \int_0^\infty e^{-t} t^{\frac{s-c}{2}} \frac{dt}{t}.$$

Let $t = \frac{\pi}{5}\mu^2 x$, then $\frac{dt}{t} = \frac{dx}{x}$. Therefore,

$$\begin{aligned}\Gamma\left(\frac{s-c}{2}\right) &= \int_0^\infty e^{-\frac{\pi}{5}\mu^2 x} \left(\frac{\pi}{5}\mu^2 x\right)^{\frac{s-c}{2}} \frac{dx}{x} \\ &= \left(\frac{\pi}{5}\right)^{\frac{s-c}{2}} |\mu|^{s-c} \int_0^\infty e^{-(\pi/5)\mu^2 x} x^{\frac{s-c}{2}} \frac{dx}{x}\end{aligned}$$

By a similar computation above, we have

$$\left(\frac{5}{\pi}\right)^s \frac{1}{|\mu|^s} \frac{1}{|\mu'|^s} \left(\frac{|\mu|}{|\mu'|}\right)^c \Gamma\left(\frac{s-c}{2}\right) \Gamma\left(\frac{s+c}{2}\right) = \int_0^\infty \int_0^\infty e^{-\frac{\pi}{5}(\mu^2 x + (\mu')^2 y)} x^{\frac{s-c}{2}} y^{\frac{s+c}{2}} \frac{dxdy}{xy}$$

Let $x = ue^{2v}$, $y = ue^{-2v}$ which give us $xy = u^2$ and $x/y = e^{4v}$. Also, $2udu = xdy + ydx$, and $4e^{4v}dv = \frac{ydx - xdy}{y^2}$ from which we get

$$\frac{dxdy}{xy} = 4\frac{dudv}{u}.$$

Hence,

$$\left(\frac{5}{\pi}\right)^s \frac{1}{|\mu|^s} \frac{1}{|\mu'|^s} \left(\frac{|\mu|}{|\mu'|}\right)^c \Gamma\left(\frac{s-c}{2}\right) \Gamma\left(\frac{s+c}{2}\right) = 4\int_{-\infty}^\infty \left(\int_0^\infty u^s e^{-\frac{\pi}{5}u(\mu^2 e^{2v} + (\mu')^2 e^{-2v})} \frac{du}{u}\right) e^{-2vc} dv, \text{ i.}$$

$$\left(\frac{5}{\pi}\right)^s \frac{1}{|\operatorname{N}(\mu)|^s} \lambda_1(\mu)^m \Gamma\left(\frac{s-c}{2}\right) \Gamma\left(\frac{s+c}{2}\right) = 4\int_{-\infty}^\infty \left(\int_0^\infty u^s e^{-\frac{\pi}{5}u(\mu^2 e^{2v} + (\mu')^2 e^{-2v})} \frac{du}{u}\right) e^{-2vc} dv$$

In order to get the ξ-function above we need to add $\lambda_1(\mu)^m / |\operatorname{N}(\mu)|^s$ for $\mu \in$

$(\mathcal{O}_K^\rho - \{0\})/\sim_u$ on the left side of the integral. However, this is a sum over a pretty complicated quotient of a sublattice of $\mathcal{O}_K$. We want to write the ξ-function as an integral involving a θ-function and this requires to have a sum over all elements $\mu \in \mathcal{O}_K^\rho - \{0\}$. For this purpose, we first change the order of the integration in the above integral and then we write the integral with respect to v that is from $-\infty$ to ∞ as a sum of integrals from $-l$ to l where $l = \frac{1}{2}\log u$ where $u = \left(\frac{1+\sqrt{5}}{2}\right)^4$ is the first totally positive unit which is congruent to 1 modulo the ideal $(\sqrt{5})$.

$$= 4\int_0^\infty u^s \left(\sum_{n=-\infty}^{n=\infty} \int_{-l}^{l} e^{-\frac{\pi}{5}u(\mu^2 e^{2(v+2nl)} + (\mu')^2 e^{-2(v+2nl)})} e^{-2c(v+2nl)} dv\right) \frac{du}{u}$$

$$= 4\int_0^\infty u^s \left(\sum_{n=-\infty}^{n=\infty} \int_{-l}^{l} e^{-\frac{\pi}{5}u((\mu e^{2nl})^2 e^{2v} + (\mu' e^{2nl})^2 e^{-2v})} e^{-2cv} dv\right) \frac{du}{u}$$

$$= 4\int_0^\infty u^s \left(\sum_{n=-\infty}^{n=\infty} \int_{-l}^{l} e^{-\frac{\pi}{5}u((\mu u^n)^2 e^{2v} + (\mu' u^n)^2 e^{-2v})} e^{-2cv} dv\right) \frac{du}{u}$$

In the last integral, we replaced l by $\frac{1}{2}\log u$ so that e^{2l} becomes u. Now we add $1/|\operatorname{N}(\mu)|^s$ for $\mu \in (\mathcal{O}_K^\rho - \{0\}) \sim_u$ on the left side of the integral.

$$\left(\frac{5}{\pi}\right)^s (\Gamma(s/2))^2 \zeta_0(\rho, s) = \xi(s,\rho)$$

$$= 4\int_0^\infty u^s \left(\int_{-l}^{l} \sum_{n=-\infty}^{n=\infty} \sum_{\mu \in (\mathcal{O}_K^\rho - \{0\})/\sim_u} e^{-\frac{\pi}{5}u((\mu u^n)^2 e^{2v} + (\mu' u^n) e^{-2v})} e^{-2cv} dv\right) \frac{du}{u}$$

We see that if $\mu \in (\mathcal{O}_K^\rho - \{0\})/\sim_u$, then μu^n runs through all the elements η of the set $\mathcal{O}_K^\rho - \{0\}$ when n is an integer. Therefore, we have

$$\xi(s,\rho) = 4\int_{-l}^{l}\int_0^\infty u^s e^{-2cv} \sum_{\eta \equiv \rho\ (\sqrt{5}),\ \eta \neq 0} e^{-\frac{\pi}{5}u(\eta^2 e^{2v} + (\eta')^2 e^{-2v})} \frac{du dv}{u}$$

In the above sum, ρ can take 5 values. If $\rho \neq 0$, then the sum is over a translate of a sublattice of $\mathcal{O}_K$ and is equal to the $\theta_0(t, t', \rho)$ by definition. However, when

$\rho = 0$ the above sum does not include a term for $\eta = 0$. Hence, it is equal to $\theta_0(t, t', 0) - 1$. As a result,

$$\xi(s, \rho) = 4 \int_0^\infty u^s \int_{-l}^{l} \big(\theta_0(ue^{2v}, ue^{-2v}, \rho) - \delta(\rho)\big)\, e^{-2cv} \frac{dvdu}{u}$$

where

$$\delta(\rho) = \begin{cases} 1 & \rho = 0 \\ 0 & \rho \neq 0 \end{cases}$$

$\square$

APPENDIX A

This appendix consists of the translation of Hans Maass's article titled as "About a New Class of Nonanalytic Automorphic Functions and Determination of Dirichlet Series by Functional Equations" which was published in Math. Ann .121, 1949. The translation of the article was not available before.

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Zeta functions of rational and quadratic number fields have two important properties. On the one hand, they satisfy certain functional equations; on the other hand, they are linear combinations of special Dirichlet series with an Euler product development or they admit such a development themselves. The extend to which these zeta functions are defined by their functional equations is the starting point of a more general theory which is developed by E.Hecke[1]) using the Mellin transform

$$\Psi(s) = \int_0^\infty y^{s-1}\Phi(y)dy \tag{A.1}$$

and its inverse

$$\Phi(y) = \frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty} y^{-s}\Psi(s)ds. \tag{A.2}$$

This reversible integral transform establishes a remarkable relationship between solutions of Riemann functional equations, that can be developed into Dirichlet series, and automorphic functions. To apply the Hecke theory, the Γ-factors appearing in the functional equation must be of the form

$$\Gamma(s) \text{ or } \Gamma\left(\frac{s}{2}\right)\Gamma\left(\frac{s+1}{2}\right);$$

thus the zeta functions of real quadratic fields are not covered by the theory. This limitation is offset by the extraordinary importance of what the Hecke theory has achieved for the theory of functional equations of zeta functions for rational and imaginary quadratic number fields and culminates in an algebraic formulation of the problem of Euler product developments. So the question which arises is whether the functional equations of zeta functions of real quadratic fields can be treated in a similar way i.e. whether real quadratic zeta functions admit analogues of the modular functions associated to the zeta functions of real and imaginary quadratic fields. This is indeed the case and it is the class of functions g, satisfying the wave equation

$$\frac{\partial^2 g}{\partial x^2} + \frac{\partial^2 g}{\partial y^2} + \frac{r^2 + \frac{1}{4}}{y^2} g = 0 \quad \text{(r=Parameter)} \tag{A.3}$$

and that are invariant under certain noneuclidean transformations on the hyperbolic plane $y > 0$ with respect to the metric $y^{-2}(dx^2 + dy^2)$. The Mellin tranform and its inverse are the connection between the Dirichlet series and the automorphic wave functions so that we have a far-reaching analogue of the Hecke theory. This connection becomes apparent when we notice that the wave function g turns into a potential function when we formally replace the parameter r with $\frac{i}{2}$ in the series for g. The Dirichlet series do not transform directly into the wave functions, instead they are known by the equation (2) only on the line $x = 0$. We get the full knowledge of the wave functions only after a process similar to analytic continuation; for this we first construct a wave function g which coincides with the given values on the line $x = 0$. However, g is uniquely determined only when both g and $\dfrac{\partial g}{\partial x}$ are known on the line $x = 0$. This means every wave function g is associated with a pair of Dirichlet series. The functional equations corresponding to these series, which show an invariance under the substitution $s \mapsto 1 - s$, differ in a crucial way by the Γ-factors, which are given by

$$\Gamma\left(\frac{s+ir}{2}\right)\Gamma\left(\frac{s-ir}{2}\right) \text{ or } \Gamma\left(\frac{s+1+ir}{2}\right)\Gamma\left(\frac{s+1-ir}{2}\right). \tag{A.4}$$

This pairing of functional equations can indeed be observed in known examples[2]) and is explained by its relationship to the wave functions. As a variable of g we choose the complex number

$$\tau = x + iy$$

because the setting in which it is easiest to write noneuclidean motions is the upper half plane $y > 0$ where such motions are fractional linear transformations with real coefficients. So in the following we will consider complex valued nonanalytic automorphic functions $g(\tau)$ which satisfy the wave equation (3). We sometimes call the Hecke theory[2]) the "analytic case".

After this general introduction we give an overview of the main results of this article.

First we prove a general theorem about systems of functional equations which we will write down in full detail because it brings out the key features that underlie the entire theory.

Theorem 27. *Fix real numbers $\lambda > 0$ and $r \geqq 0$, a positive integer q, an $N \times N$ matrix $C = (c_{kl})$ with $C^2 = Id$, and integers $b_1, b_2, ..., b_N$.*

I. We want to know all systems of $2N$ functions

$$\varphi_1(s), \varphi_2(s), ..., \varphi_N(s); \quad \psi_1(s), \psi_2(s), ..., \psi_N(s)$$

with the following properties:

1. *The functions* $(s-1-ir)(s-1+ir)\varphi_k(s)$ *and* $\psi_k(s)$ $(k=1,2,...,N)$ *are entire functions of* s *of finite genus.*

2. *The following functional equations hold*

$$\xi_k(1-s) = \sum_{l=1}^{N} c_{kl}\xi_l(s),$$
$$\eta_k(1-s) = -\sum_{l=1}^{N} c_{kl}\eta_l(s), \quad (k=1,2,...,N) \tag{A.5}$$

if we set

$$\xi_k(s) = \left(\frac{\lambda}{\pi}\right)^{s} \Gamma\left(\frac{s-ir}{2}\right)\Gamma\left(\frac{s+ir}{2}\right)\varphi_k(s),$$
$$\eta_k(s) = \left(\frac{\lambda}{\pi}\right)^{s+1} \Gamma\left(\frac{s+1-ir}{2}\right)\Gamma\left(\frac{s+1+ir}{2}\right)\psi_k(s). \tag{A.6}$$

3. *In an appropriate half-plane the functions* $\varphi_k(s)$ *and* $\psi_k(s)$ *can be written as Dirichlet series*

$$\varphi_k(s) = \sum_{\substack{n \equiv b_k\ (q) \\ n \neq 0}} \frac{a_n^{(k)}}{|\, n\,|^s},$$
$$\psi_k(s) = \sum_{\substack{n \equiv b_k\ (q) \\ n \neq 0}} \frac{(\operatorname{sgn} n)a_n^{(k)}}{|\, n\,|^s}, \quad (k=1,2,...,N). \tag{A.7}$$

.

4. *The following equation holds*

$$\varrho_k\left(1-e^{\frac{2\pi i b_k}{q}}\right) = \sigma_k\left(1-e^{\frac{2\pi i b_k}{q}}\right) = 0, \quad (k=1,2,...,N) \tag{A.8}$$

if we put

$$\varrho_k = \sum_{l=1}^{N} c_{kl}\alpha_l, \quad \sigma_k = \sum_{l=1}^{N} c_{kl}\beta_l \tag{A.9}$$

and if we determine $\alpha_k,\ \beta_k$ *so that*

$$\varphi_k - \frac{\alpha_k}{s-1-ir} - \frac{\beta_k}{s-1+ir} \quad \textit{for } r>0 \textit{ or}$$
$$\varphi_k - \frac{\alpha_k}{s-1} - \frac{\beta_k}{(s-1)^2} \quad \textit{for } r=0, \tag{A.10}$$

are entire functions of s.

II. *Every system of functions with these properties corresponds bijectively to a system of* N *functions*

$$g_1(\tau), g_2(\tau), ..., g_N(\tau),$$

with the following properties via the integral transforms (1) *and* (2)*:*

1. *They satisfy the wave equation*

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{r^2+\frac{1}{4}}{y^2}\right) g_k(\tau) = 0 \ \ (k = 1, 2, ..., N) \tag{A.11}$$

and they are regular at every point in the upper half plane as functions of real variables x *and* y.

2. *There exist* κ_1, κ_2, A_1 *and* A_2 *such that*

$$\mid g_k(\tau) \mid \leq A_1 y^{\kappa_1} \ \textit{as}\ y \to \infty, \ \ \mid g_k(\tau) \mid \leq A_2 y^{-\kappa_2} \ \textit{as}\ y \to 0 \tag{A.12}$$

with positive constants $\varkappa_1$ *and* $\varkappa_2$ $(k = 1, 2, ..., N)$.

3. *The functions* g_k *satisfy*

$$g_k\left(\tau + \frac{\lambda}{q}\right) = e^{\frac{2\pi i b_k}{q}} g_k(\tau), \ \ (k = 1, 2, ..., N) \tag{A.13}$$

4. *The transformation formula*

$$g_k\left(-\frac{1}{\tau}\right) = \sum_{l=1}^{N} c_{kl} g_l(\tau), \ \ (k = 1, 2, ..., N) \tag{A.14}$$

holds.

A.0.2 Construction of the g_k

Starting from the Dirichlet series (7) we find the following representation for the system of functions II.

$$g_k(\tau) = u_k(y) + \sum_{\substack{n \equiv b_k\ (q) \\ n \neq 0}} a_n^{(k)} y^{\frac{1}{2}} K_{ir}\left(\frac{2\pi|n|}{\lambda} y\right) e^{\frac{2\pi i n}{\lambda} x} \tag{A.15}$$

with

$$u_k(y) = \begin{cases} M \varrho_k y^{\frac{1}{2}+ir} + \overline{M} \sigma_k y^{\frac{1}{2}-ir} & \text{for } r > 0, \\ M \left\{\varrho_k + \sigma_k \left(\log \frac{\lambda}{4\pi} - C\right)\right\} y^{\frac{1}{2}} + M \sigma_k y^{\frac{1}{2}} \log y & \text{for } r = 0. \end{cases} . \tag{A.16}$$

Here C is the Euler constant and

$$M = \frac{\sqrt{\lambda}}{4} \left(\frac{\lambda}{\pi}\right)^{\frac{1}{2}+ir} \Gamma\left(\frac{1}{2} + ir\right). \tag{A.17}$$

The Bessel function $K_\nu(z)$ which comes up in (15) satisfies the following differential equation if z is purely imaginary

$$z^2 \frac{d^2 w}{dz^2} + z \frac{dw}{dz} - (z^2 + \nu^2) w = 0 \tag{A.18}$$

and as $z \to \infty$ it has the following asymptotic behaviour[3])

$$K_\nu(z) \sim \sqrt{\frac{\pi}{2z}} e^{-z}. \tag{A.19}$$

A.0.3 Group invariance

Among all discontinous groups $\mathbf{G}\left(\frac{\lambda}{q}\right)$, which are generated by the two substitutions

$$\tau \mapsto \tau + \frac{\lambda}{q} \text{ and } \tau \mapsto -\frac{1}{\tau},$$

$\mathbf{G}(1)$ and $\mathbf{G}(2)$ are special because they are subgroups of the modular group $\mathbf{M}$. They are either $\mathbf{M}$ or so called the Theta group $\mathbf{T}$. In general, the transformation formulas 3. and 4. of the system II. define a homomorphism from the group $\mathbf{G}\left(\frac{\lambda}{q}\right)$ to the group generated by the matrices

$$(c_{kl}) \text{ and } \left(\delta_{kl}e^{\frac{2\pi i b_k}{q}}\right) \quad (\delta_{kl} = \text{Kronecker symbol}) \tag{A.20}$$

if the system of functions II. is linearly independent. The substitutions of $\mathbf{G}\left(\frac{\lambda}{q}\right)$ that get mapped to the identity matrix build a normal subgroup $\mathbf{N}$ of $\mathbf{G}\left(\frac{\lambda}{q}\right)$. Apparently, for all wave functions of the system II. we have

$$g_k(S\tau) = g_k(\tau) \text{ for } S \in \mathbf{N}. \tag{A.21}$$

In particular, this invariance is important if $\mathbf{N}$ has finite index in $\mathbf{G}\left(\frac{\lambda}{q}\right)$ i.e. if the group generated by the substitutions (20) is finite and if $\lambda = q$ or $\lambda = 2q$. In this case, by using the Siegel method we prove that there are only finitely many linearly independent automorphic wave functions for the group $\mathbf{N}$, which behave like $g_k(\tau)$ as $y \to \infty$ in all parabolic cusps of the fundamental domain of $\mathbf{N}$. Therefore, we get an important theorem that says the dimension of the linearly equivalent families of systems of functions I and II is finite. In particular, for the modular group $\mathbf{M}$ and the Thetagroup $\mathbf{T}$ we can find explicitly all wave functions for $r = 0$ that have the same behaviour in the parabolic cusps described above. This result corresponds to Theorem 2.

Theorem 28. *All solutions $\varphi(s)$ of the functional equation*

$$\xi(s) = \left(\frac{\lambda}{\pi}\right)^s \left(\Gamma\left(\frac{s}{2}\right)\right)^2 \varphi(s) = \xi(1-s), \tag{A.22}$$

which can be written as a Dirichlet series and for which $(s-1)^2\varphi(s)$ is an entire function of finite genus, form a linear family which is generated by $\zeta^2(s)$ in the case *of* $\lambda = 1$ *and by* $2^{-s}\zeta^2(s)$ *and* $(1+2^{1-2s})\zeta^2(s)$ *in the case of* $\lambda = 2$ *(*$\zeta(s)$ *=Riemann zeta function). By contrast, for neither* $\lambda = 1$ *nor for* $\lambda = 2$ *is there a non trivial entire function* $\varphi(s)$ *of finite genus that can be developed in a Dirichlet series and which satisfies*

$$\eta(s) = \left(\frac{\lambda}{\pi}\right)^{s+1}\left(\Gamma\left(\frac{s+1}{2}\right)\right)^2 \varphi(s) = -\eta(1-s)$$

A.0.4 Real quadratic fields

An important example of Theorem 1 is the class of zeta functions of the real quadratic field $R(\sqrt{D})$ with discriminat D, formed with any Größen character λ_1^n:

$$\zeta_0(s,\varrho,\mathfrak{a},\lambda_1^n,Q\sqrt{D}) = \frac{A^s}{\lambda_1^n(\mathfrak{a})}\sum_{\substack{\mu\,\equiv\,\varrho\ (\mathfrak{a}Q\sqrt{D}) \\ \mu\neq 0,\,(\mu)_{Q\sqrt{D}p_\infty}}} \frac{\lambda_1^n(\mu)}{|\mathtt{N}\mu|^s}$$

$$\zeta_1(s,\varrho,\mathfrak{a},\lambda_1^n,Q\sqrt{D}) = \frac{A^s}{\lambda_1^n(\mathfrak{a})}\sum_{\substack{\mu\,\equiv\,\varrho\ (\mathfrak{a}Q\sqrt{D}) \\ \mu\neq 0,\,(\mu)_{Q\sqrt{D}p_\infty}}} \operatorname{sgn}(\mathtt{N}\mu)\frac{\lambda_1^n(\mu)}{|\mathtt{N}\mu|^s}. \tag{A.23}$$

Here we sum over a complete system of non-vanishing, non-mod $Q\sqrt{D}p_\infty$ associated residue classes $\varrho \mod \mathfrak{a}Q\sqrt{D}$ where Q is an arbitrary non-negative integer, $\mathfrak{a}$ is an arbitrary ideal, ϱ is an arbitrary element of $\mathfrak{a}$ and we set $\mathtt{N}\mathfrak{a} = A$. The Größen character λ_1 is defined by

$$\lambda_1 = \left|\frac{\mu}{\mu'}\right|^{\frac{\pi i}{\log\varepsilon}}, \quad \epsilon = \text{ fundamental unit of } R(\sqrt{D}),\ \varepsilon > 1 \tag{A.24}$$

The functional equations

$$\xi_\nu(1-s,\varrho,\mathfrak{a},\lambda_1^n,Q\sqrt{D}) = \frac{(-1)^\nu}{Q\sqrt{D}} \sum_{\substack{\sigma \bmod \mathfrak{a}Q\sqrt{D} \\ \sigma\equiv 0\ (\mathfrak{a})}} e^{2\pi i \mathtt{S}\left(\frac{\varrho\sigma'}{AQD}\right)} \xi_\nu(s,\sigma,\mathfrak{a},\lambda_1^n,Q\sqrt{D}) \quad \text{(A.25)}$$

for $\nu = 0, 1$ with

$$\xi_0(s,\varrho,\mathfrak{a},\lambda_1^n,Q\sqrt{D}) = \left(\frac{QD}{\pi}\right)^s \Gamma\left(\frac{s-\frac{\pi n i}{\log\varepsilon}}{2}\right)\Gamma\left(\frac{s+\frac{\pi n i}{\log\varepsilon}}{2}\right)\zeta_0(s,\varrho,\mathfrak{a},\lambda_1^n,Q\sqrt{D})$$

$$\xi_1(s,\varrho,\mathfrak{a},\lambda_1^n,Q\sqrt{D}) = \left(\frac{QD}{\pi}\right)^{s+1} \Gamma\left(\frac{s+1-\frac{\pi n i}{\log\varepsilon}}{2}\right)\Gamma\left(\frac{s+1+\frac{\pi n i}{\log\varepsilon}}{2}\right)\zeta_1(s,\varrho,\mathfrak{a},\lambda_1^n,Q\sqrt{D}), \quad \text{(A.26)}$$

which are proved for $n = 0$ in [2]), allow us to apply Theorem 1 to the system of zeta function in (23) with

$$N = Q^2 D, \quad \lambda = q = QD, \quad r = \frac{\pi n}{\log\varepsilon} = cn. \quad \text{(A.27)}$$

One chooses the appropriate residue class $\varrho \bmod \mathfrak{a}Q\sqrt{D}$, $\varrho \equiv 0\ (\mathfrak{a})$ as an index instead of k so that the matrices in (20) coincide in our case with

$$\left(\frac{1}{Q\sqrt{D}}\right) e^{2\pi i \operatorname{Tr}\left(\frac{\varrho\sigma'}{AQD}\right)} \text{ and } \left(\delta_{\varrho\sigma} e^{\frac{2\pi i\, \mathrm{N}\,\varrho}{AQD}}\right). \quad \text{(A.28)}$$

Hecke[4]) showed that the principal congruence subgroup $\mathbf{M}(QD)$ of level QD gets mapped to the identity by the map from $\mathbf{M}$ to the group that is generated by the matrices (28), so that the wave functions corresponding to (23)

$$g(\tau,\varrho,\mathfrak{a},\lambda_1^n,Q\sqrt{D}) = 2l_Q\delta_n\left(\frac{\varrho}{\mathfrak{a}Q\sqrt{D}}\right)y^{\frac{1}{2}} + \frac{1}{\lambda_1^n(\mathfrak{a})}\sum_{\substack{\mu\equiv\varrho\ (\mathfrak{a}Q\sqrt{D}) \\ \mu\neq 0,\ (\mu)_{Q\sqrt{D}p_\infty}}} \lambda_1^n(\mu)y^{\frac{1}{2}}K_{icn}\left(\frac{2\pi|\mathtt{N}\mu|}{AQD}y\right)e^{\frac{2\pi i \mathtt{N}\mu}{AQD}x} \quad \text{(A.29)}$$

are invariant with respect to substitutions of $\mathbf{M}(QD)$. Here $l_Q = \frac{1}{2}\log\varepsilon_Q$ if $\varepsilon_Q(>1)$ generates the group of units in $R(\sqrt{D})$, which are congruent to 1 mod $Q\sqrt{D}p_\infty$

and

$$\delta_n(\mathfrak{b}) = \begin{cases} 1 & \text{for} \quad n = 0 \quad \text{an integral ideal } \mathfrak{b} \\ 0 & \text{otherwise.} \end{cases}$$

If we take into account the linear relations between the series (29), then in the case

$$D = 5, \quad Q = 1, \quad \mathfrak{a} = (1)$$

we are left with three linearly independent functions corresponding to the values $\varrho = 0, 1, 2$. Moreover, if $n = 0$ then the series are uniquely determined by part II, (1 to 4) of Theorem 27 up to a common constant factor.

Theorem 29. *The system of zeta functions*

$$\begin{aligned} \varphi_1(s) &= \zeta_0(s, 0, (1), 1, \sqrt{5}), \quad \psi_1(s) = \zeta_1(s, 0, (1), 1, \sqrt{5}), \\ \varphi_2(s) &= \zeta_0(s, 1, (1), 1, \sqrt{5}), \quad \psi_2(s) = \zeta_1(s, 1, (1), 1, \sqrt{5}), \\ \varphi_3(s) &= \zeta_0(s, 2, (1), 1, \sqrt{5}), \quad \psi_3(s) = \zeta_1(s, 2, (1), 1, \sqrt{5}) \end{aligned} \tag{A.30}$$

is uniquely determined by the conditions in part I. (1 to 4) of Theorem 27 with $N = 3, \ \ \lambda = q = 5, \ r = 0, \ b_1 = 0, \ b_2 = 1, \ b_3 = -1$ *and*

$$(c_{kl}) = \frac{1}{\sqrt{5}} \begin{pmatrix} 1 & 2 & 2 \\ 1 & \zeta^2 + \zeta^{-2} & \zeta^1 + \zeta^{-1} \\ 1 & \zeta^1 + \zeta^{-1} & \zeta^2 + \zeta^{-2} \end{pmatrix} \tag{A.31}$$

where $\zeta = e^{\frac{2\pi i}{5}}$.

Theorem 27 can be applied to the linear collection of wave functions of level Q since using the normal subgroup property of $\mathbf{M}(Q)$, one can determine a basis for the collection of the wave functions as in the analytic case. The basis consists of eigenfunctions of the substitutions $\tau \mapsto \tau + 1$. Therefore, it is reasonable to associate a pair of Dirichlet series

$$\varphi(s) = \sum_{n \neq 0} \frac{a_n}{|n|^s}, \quad \psi(s) = \sum_{n \neq 0} \frac{(\operatorname{sgn} n) a_n}{|n|^s} \tag{A.32}$$

to each wavefunction of level Q

$$g(\tau) = u(y) + \sum_{n \neq 0} a_n y^{\frac{1}{2}} K_{ir}\left(\frac{2\pi|n|}{Q} y\right) e^{\frac{2\pi i n}{Q} x} \tag{A.33}$$

by the method mentioned above because it is a linear process. The determination of $u(y)$ in terms of φ and ψ is done by using the residues of these functions.

Of particular interest is the collection of series that is analogous to the Eisentein series[5])

$$E(\tau, s; (a_1, a_2), Q) = \sum_{\substack{m_i \equiv a_i \ (Q) \\ (m_1, m_2) \neq (0,0)}} \frac{y^{\frac{s}{2}}}{|\ m_1\tau + m_2\ |^s} \tag{A.34}$$

which are at first only defined $\operatorname{Re} s > 2$ but they have analytic continuation as functions of s. The function values $E(\tau, 1+2ir; (a_1, a_2), Q)$, which we also want to call Eisenstein series, exist for all real r and represent the solutions of the wave equation which are invariant under the substitutions of $\mathbf{M}(Q)$. This can be easily seen by using the translation formulas

$$E(S\tau, 1+2ir; (a_1, a_2), Q) = E(\tau, 1+2ir; (a_1, a_2)S, Q) \text{ for } S \in \mathbf{M}. \tag{A.35}$$

Using the series $E(\tau, 1+2ir; (a_1, a_2), Q)$ we can, if $r > 0$, reduce an arbitrary wave function of level Q, which has a Fourier series similar to (33) in the parabolic cusps of a fundamendal domain of $\mathbf{M}(Q)$, to a cusp function i.e. to such a function which vanishes in all parabolic cusps. This is proved using the Hecke method, by passing to the series $E^*(\tau, 1+2ir; (a_1, a_2), Q)$ that comes from $E(\tau, 1+2ir; (a_1, a_2), Q)$, if we introduce the additional summation condition $(m_1, m_2) = 1$. This case is more complicated than the analytic case because in the reduction we have to notice that 2σ constants must be set to equal to 0, where σ denotes the number of the parabolic cusps of the fundamental domain of $\mathbf{M}(Q)$; because the terms

$u(y)$ in the σ Fourier series(of the kind (33)) of the given wave function on the parabolic points depend on two parameters. On the other hand, there are only σ linearly independent Eisenstein series $E(\tau, 1+2ir; (a_1, a_2), Q)$. These are sufficient because according to the certain bilinear relations we can only choose σ out of 2σ constants. In the case $r = 0$ the number of linearly independent Eisenstein series is in general smaller than σ and takes this value only for $Q = 1, 2, 3, 4, 6$. For other values of Q the applied methods above to solve the reduction problem are not sufficient. A more detailed investigation shows that among the Eisenstein series $E^*(\tau, 1; (a_1, a_2), 5)$ of level 5 there are exactly three that are linearly independent. Certain linear combinations of the series are identical with the wave functions of the system (30). Their behaviour with respect to the substitution $\tau \mapsto -\dfrac{1}{\tau}$ can be determined because of the known relations of the Eisenstein series at $Q = 5$. So for the functional eqautions of the zeta functions (30) there is a new proof which does not use the theta series of the two variables.

The Eisenstein series of level QD and the wave functions $g(\tau, \varrho, \mathfrak{a}, \lambda_1^n, Q\sqrt{D})$ are not linearly independent of each other. For example

$$\sum_{\{\mathfrak{a}\}} g(\tau, 0, \mathfrak{a}, 1, \sqrt{D})$$

can be represented by the Eisenstein series of level D where we sum over a complete system of representatives $\mathfrak{a}$ of the narrow ideal class of $R(\sqrt{D})$. The proof of the identity provides the Dirichlet class number formula for the real quadratic fields as a side result, as it comes out in the analytic case for the quadratic fields[4]).

The Hecke theory of T_n-operators, which is closely related to the problem of writing Dirichlet series as Euler products, can be translated into the wave functions of level Q without any significant modifications. In order to understand T_m^t-operators(s. T_n II) also for m that are not relatively prime to Q it is necessary to split the collection of all wave functions into subspaces which are characterized by their behaviour with respect to the operators $U, R_a \in \mathbf{M}$ and K defined by

$$U \equiv \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \; R_a \equiv \begin{pmatrix} \bar{a} & 0 \\ 0 & a \end{pmatrix} (Q) \text{ for } a\bar{a} \equiv 1 \; (Q), \;\; g(\tau) \mid K = g(-\bar{\tau}). \tag{A.36}$$

Similar to the analytic case we first build the subspaces $\mathfrak{F}_r(t, \chi, Q)$ of the wave functions of the character χ of the divisor t. The subspaces are the eigenfunctions $g(\tau)$ corresponding to the group of operators R_a with the eigenvalues $\chi(a)$:

$$g(\tau) \mid R_a = \chi(a) g(\tau) \tag{A.37}$$

Moreover these have the property that in their Fourier series there are only exponents whose greatest common divisor with Q is t. The fact that K commutes with the operators R_n and T_m^t allows us to split $\mathfrak{F}_r$ into two subspaces that consist of eigenfunctions of K corresponding to the eigenvalues 1 and-1:

$$\mathfrak{F}_r(t, \chi, Q) = \mathfrak{F}_r^{+1}(t, \chi, Q) + \mathfrak{F}_r^{-1}(t, \chi, Q). \tag{A.38}$$

The wave function $g(\tau)$ of level Q is an eigenfunction of the operator K corresponding to the eigenvalues 1 or -1 if and only if either the Dirichlet series $\psi(s)$ or $\varphi(s)$, corresponding to the function $g(\tau)$ in (32), vanishes identically. Every function in one of the subfamilies $\mathfrak{F}_r^{+1}(t, \chi, Q)$ and $\mathfrak{F}_r^{-1}(t, \chi, Q)$ corresponds to only one Dirichlet series. The linear families of the Dirichlet series that correspond to the families $\mathfrak{F}_r^{+1}(t, \chi, Q)$ and $\mathfrak{F}_r^{-1}(t, \chi, Q)$ are characterized by the fact that they belong to the system of functional equations with the Γ-factors

$$\Gamma\left(\frac{s+ir}{2}\right)\Gamma\left(\frac{s-ir}{2}\right) \;\text{ or }\; \Gamma\left(\frac{s+1+ir}{2}\right)\Gamma\left(\frac{s+1-ir}{2}\right).$$

The purpose of the decomposition (38) is to separate these two types. We can do the same thing for the linear family $\mathfrak{E}(Q)$ of the Eisenstein series of level Q:

$$\mathfrak{E}(Q) = \mathfrak{E}^{+1}(Q) + \mathfrak{E}^{-1}(Q). \tag{A.39}$$

The family of Dirichlet series corresponding to $\mathfrak{E}^{+1}(Q)$ or $\mathfrak{E}^{-1}(Q)$ is linearly equaivalent to the set of all L-series products

$$(t_1t_2)^{-s}L(s+ir,\chi_1)L(s-ir,\chi_2), \tag{A.40}$$

where t_i is an arbitrary divisor of Q and χ_i is an arbitrary character mod $\dfrac{Q}{t_i}$ with the restriction that χ_1 and χ_2 are both even or odd.

The operator theory applied to $\mathfrak{F}_r^{+1}(t,\chi,Q)$, $\mathfrak{F}_r^{-1}(t,\chi,Q)$ or any subspace which is invariant under operators T_m^t leads to the same result as in the analytic case. Basically it is the explanation of the following fact. Let $F^1(\tau), F^2(\tau), ..., F^{\varkappa}(\tau)$ be a basis of the invariant family $\mathfrak{E}_r$, which coincides with $\mathfrak{F}_r^{+1}(t,\chi,Q)$ or $\mathfrak{F}_r^{-1}(t,\chi,Q)$ or is contained in $\mathfrak{F}_r^{+1}(t,\chi,Q)$ or $\mathfrak{F}_r^{-1}(t,\chi,Q)$. The matrices $\lambda(m)$ that are built from the coefficients of the linear forms

$$F^{\varrho}(\tau) \mid T_m^t = \sum_{\sigma=1}^{\varkappa} \lambda_{\varrho\sigma}(m)F^{\sigma}(\tau) \tag{A.41}$$

satisfy the rule

$$\lambda(m_1)\lambda(m_2) = \sum_{\substack{d \mid m_1, m_2 \\ d>0}} \lambda\left(\frac{m_1m_2}{d^2}\right)\chi(d), \tag{A.42}$$

the matrix function

$$\Phi(s) = (\varphi_{\varrho\sigma(s)}) = \sum_{m=1}^{\infty} \lambda(m)(tm)^{-s} \tag{A.43}$$

has the Euler product

$$\Phi(s) = t^{-s}\prod_p (\lambda(1) - \lambda(p)p^{-s} + \lambda(1)\chi(p)p^{-2s})^{-1}. \tag{A.44}$$

From several relations between coefficients we obtain that the linear space generated by $\varkappa^2$ Dirichlet series φ has rank $\varkappa$ and is identical to a space $\mathfrak{D}$ which is associated to the space $\mathfrak{E}_r$. The "principal axes theorem" says that matrices $\lambda(p)$ are diagonalizable matrices if we make a clever choice of basis $F^{\varrho}(\tau),\ (\varrho = 1, 2, ..., \kappa)$ and if the prime number p is relatively prime to Q. The so called "principal axes theorem" can be proved in our case by applying the Petersson Principle of Metrization[6])to the wave functions. A certain normalization in the Euler products appearing on the prime factors that divide Q works by an elementary method that is also developed by Petersson(s.$K\ III$). New aspects in this study do not appear any longer so that the proofs can be described succinctly with regard to the detailed representations $T_n\ I.II$ and $K\ I, II, III$.

A.1 Systems of Functional Equations

For the proof of Theorem 1 and for later considerations we need estimates of the Bessel function $K_{ir}(z)$ for real $r > 0$ and positive z, which we want to derive first. We start with the integral representation W,6.15(4):

$$K_{ir}(z) = \frac{\Gamma\left(\frac{1}{2}\right)\left(\frac{1}{2}z\right)^{ir}}{\Gamma\left(\frac{1}{2}+ir\right)} \int_1^\infty e^{-zt}(t^2-1)^{ir-\frac{1}{2}} dt. \tag{A.45}$$

Here we substitute $z(t-1) = s$ and obtain

$$K_{ir}(z) = \sqrt{\frac{\pi}{2z}} e^{-z} \frac{1}{\Gamma\left(\frac{1}{2}+ir\right)} \int_0^\infty e^{-s} s^{ir-\frac{1}{2}} \left(1+\frac{s}{2z}\right)^{ir-\frac{1}{2}} ds. \tag{A.46}$$

With the help of the integral represenation of the Γ-function we get the following limit from (46)

$$\lim_{z\to\infty} K_{ir}(z)\sqrt{\frac{2z}{\pi}} e^z = 1 \tag{A.47}$$

and for arbitrary positive z the estimate is

$$\mid K_{ir}(z) \mid \leq C_1 z^{-\frac{1}{2}} e^{-z} \tag{A.48}$$

with the constant C_1 which only depends on r, where

$$C_1 = \frac{\pi}{\sqrt{2} \mid \Gamma\left(\frac{1}{2}+ir\right) \mid}. \tag{A.49}$$

For $r = 0$, the expression for $K_{ir}(z)\sqrt{\frac{2z}{\pi}}e^z$ is by (46) clearly a monotone increasing function of z, which explains

$$K_0(a)e^a\sqrt{\frac{a}{z}}e^{-z} \leq K_0(z) < \sqrt{\frac{\pi}{2z}}e^{-z} \text{ for } 0 < a \leq z. \tag{A.50}$$

Now let $\varphi_k(s), \psi_k(s)$ $(k = 1, 2, ...N)$ be a system of functions which satisfies the conditions $I, (1$ to $4)$ of Theorem 1. We prove that the system of functions defined by (15) has the properties $II, (1$ to $4)$. The convergences of the Dirichlet series (7) for at least one value of s is equivalent to the statement that for proper choice of constants C_2 and $\varkappa$ we get

$$\mid a_n^{(k)} \mid \leq C_2 \mid n \mid^{\varkappa} \quad (k = 1, 2, ...N). \tag{A.51}$$

Using (48) we conclude that

$$\mid g_k(\tau) - u_k(y) \mid \leq \sqrt{\frac{2\lambda}{\pi}} C_1 C_2 \sum_{n=1}^{\infty} n^{\varkappa - \frac{1}{2}} e^{-\frac{2\pi n}{\lambda} y}$$

holds. Hence for $y \to 0$ we have:

$$g_k(\tau) - u_k(y) = O\left(\int_0^{\infty} t^{\varkappa - \frac{1}{2}} e^{-\frac{2\pi y}{\lambda} t} dt\right) = O\left(y^{-\left(\varkappa + \frac{1}{2}\right)}\right)$$

if $\varkappa + \dfrac{1}{2} > 0$ which we may assume. The conditions (12) are satisfied with $\varkappa_1 > \dfrac{1}{2}$ and $\varkappa_2 = \varkappa + \dfrac{1}{2}$. The partial derivatives of the Fourier series of $g_k(\tau)$ all exist and can be computed by term by term differentiation because the formal derivatives of any order converge uniformly for $0 < \delta \leq y$ which can be seen easily. Hence we find that $g_k(\tau)$ satisfies the wave equation (11) and (13) is obvious due to the Fourier series of $g_k(\tau)$ and the condition (8). It only remains to prove (14). For this we need the integral representation

$$K_{\frac{\nu_1 - \nu_2}{2}}(y) = \frac{1}{8\pi i} \int_{\sigma - i\infty}^{\sigma + i\infty} \Gamma\left(\frac{t + \nu_1}{2}\right) \Gamma\left(\frac{t + \nu_2}{2}\right) \left(\frac{y}{2}\right)^{-t - \frac{\nu_1 + \nu_2}{2}} dt \tag{A.52}$$

where $y > 0$, $\sigma > -\operatorname{Re}\nu_1$, $\sigma > -\operatorname{Re}\nu_2$, which easily follows from the formula W,6.5(6) and their relation W,3.7(8) from which we conclude

$$y^{\frac{1+\nu_1+\nu_2}{2}} K_{\frac{\nu_1-\nu_2}{2}}\left(\frac{2\pi \mid n \mid}{\lambda} y\right) = \frac{1}{8\pi i} \int_{\sigma-i\infty}^{\sigma+i\infty} \left(\frac{\lambda}{\pi}\right)^{t+\frac{\nu_1+\nu_2}{2}} \frac{\Gamma\left(\frac{t+\nu_1}{2}\right)\Gamma\left(\frac{t+\nu_2}{2}\right)}{\mid n \mid^{t+\frac{\nu_1+\nu_2}{2}} y^{t-\frac{1}{2}}} dt \quad \text{(A.53)}$$

If we choose $\sigma > 1$ and big enough such that the line of integration lies inside the half plane of the absolute convergence of the Dirichlet series (7), then we have

$$F_k(y) = \sum_{\substack{n \equiv b_k\ (q) \\ n \neq 0}} a_n^{(k)} y^{\frac{1}{2}} K_{ir}\left(\frac{2\pi \mid n \mid}{\lambda} y\right) = \frac{1}{8\pi i} \int_{\sigma-i\infty}^{\sigma+i\infty} \frac{\xi_k(s)}{y^{s-\frac{1}{2}}} ds \qquad \text{(A.54)}$$

$$G_k(y) = \sum_{\substack{n \equiv b_k\ (q) \\ n \neq 0}} n\, a_n^{(k)} y^{\frac{3}{2}} K_{ir}\left(\frac{2\pi \mid n \mid}{\lambda} y\right) = \frac{1}{8\pi i} \int_{\sigma-i\infty}^{\sigma+i\infty} \frac{\eta_k(s)}{y^{s-\frac{1}{2}}} ds \qquad \text{(A.55)}$$

In both identities we replace the integral variable s by $1-s$ and apply the functional equation (5), then we obtain

$$F_k(y) = \sum_{l=1}^{N} c_{kl} \frac{1}{8\pi i} \int_{1-\sigma-i\infty}^{1-\sigma+i\infty} \frac{\xi_l(s)}{y^{\frac{1}{2}-s}} ds,$$

$$G_k(y) = -\sum_{l=1}^{N} c_{kl} \frac{1}{8\pi i} \int_{1-\sigma-i\infty}^{1-\sigma+i\infty} \frac{\eta_l(s)}{y^{\frac{1}{2}-s}} ds$$

After translating the line of integration by $1-\sigma$ to all positions and after taking into account the residues in the right way, similar to the analytic case we finally obtain

$$F_k^*\left(\frac{1}{y}\right) = \sum_{l=1}^{N} c_{kl} F_l^*(y), \quad G_k\left(\frac{1}{y}\right) = -\sum_{l=1}^{N} c_{kl} G_l(y) \quad (k = 1, 2, ..., N) \quad \text{(A.56)}$$

with

$$F_k^*(y) = u_k(y) + F_k(y), \qquad \text{(A.57)}$$

where $u_k(y)$ is defined by (16). To justify this method consider the condition $I, 1$ of Theorem 1. The functions

$$g_k\left(-\frac{1}{\tau}\right) = -\sum_{l=1}^{N} c_{kl} g_l(\tau) \quad (k = 1, 2, ..., N) \tag{A.58}$$

have now the following properties. They are solutions of the wave equation because in general if $g(\tau)$ is a solution then $g(S\tau)$ is also a solution if S is any unimodular substitution, and they and their first partial derivatives in x vanish at $x = 0$. This follows immediately from

$$g_k(\tau)_{x=0} = F_k^*(y), \quad g_k\left(-\frac{1}{\tau}\right)_{x=0} = F_k^*\left(\frac{1}{y}\right),$$

$$\frac{\partial}{\partial x} g_k(\tau)_{x=0} = \frac{2\pi i}{\lambda}\frac{1}{y} G_k(y), \quad \frac{\partial}{\partial x} g_k\left(-\frac{1}{\tau}\right)_{x=0} = -\frac{2\pi i}{\lambda}\frac{1}{y} G_k\left(\frac{1}{y}\right) \tag{A.59}$$

and the functional equations (56). Every solution $g(\tau)$ of the wave equation

$$\left(\frac{\partial^2 g}{\partial x^2} + \frac{\partial^2 g}{\partial y^2} + \frac{r^2 + \frac{1}{4}}{y^2}\right) g(\tau) = 0$$

with the initial values

$$g(\tau)_{x=0} = \frac{\partial}{\partial x} g(\tau)_{x=0} = 0$$

vanishes identically because $g(\tau)$ can be written as a power seires in x

$$g(\tau) = \sum_{n=0}^{\infty} c_n(y) x^n,$$

as a solution of elliptic differential equations and for the coefficients that depend on y we have the recursion formula

$$(n+2)(n+1)c_{n+2}(y) + c_n''(y) + \frac{r^2 + \frac{1}{4}}{y^2} c_n(y) = 0 \text{ for } n = 0, 1, 2, ...,$$

which in general implies $c_n(y) = 0$ because $c_0(y)$ and $c_1(y)$ vanish by assumption. With the functional equations (14) we proved one direction of the equaivalence in Theorem 1.

Vice versa let now $g_k(\tau)$ be a system of functions with the properties II, (1 to 4) of Theorem 1. By $II, 3$ the functions $g_k(\tau)$ are periodic in x with the period λ and therefore can be written as a Fourier series which necessarily takes the form

$$g_k(\tau) = \begin{Bmatrix} a_0^{(k)} y^{\frac{1}{2}+ir} + b_0^{(k)} y^{\frac{1}{2}-ir} \\ a_0^{(k)} y^{\frac{1}{2}} \log y + b_0^{(k)} y^{\frac{1}{2}} \end{Bmatrix}$$

$$\sum_{n\neq 0} \left\{ a_n^{(k)} y^{\frac{1}{2}} K_{ir}\left(\frac{2\pi \mid n \mid}{\lambda} y\right) + b_n^{(k)} y^{\frac{1}{2}} I_{ir}\left(\frac{2\pi \mid n \mid}{\lambda} y\right) \right\} e^{\frac{2\pi i n}{\lambda} x} \tag{A.60}$$

for $r > 0$ and $r = 0$ respectively. Here $K_\nu(z)$ and $I_\nu(z)$ are independent solutions of the differential equation (18). From the coefficient formula

$$a_n^{(k)} y^{\frac{1}{2}} K_{ir}\left(\frac{2\pi \mid n \mid}{\lambda} y\right) + b_n^{(k)} y^{\frac{1}{2}} I_{ir}\left(\frac{2\pi \mid n \mid}{\lambda} y\right) = \frac{1}{\lambda} \int_0^\lambda g_k(\tau) e^{-\frac{2\pi i n}{\lambda} x} dx \ (n \neq 0)$$

it can be seen $b_n^{(k)} = 0$ for $n \neq 0$ because $g_k(\tau)$ increases as $y \to \infty$ at most like a power of y while $I_{ir}\left(\frac{2\pi|n|}{\lambda} y\right)$ increase exponentially. The coefficients $a_n^{(k)}, b_n^{(k)}$ ($n \geq 0$) can be different than 0 by (13) only if $n \equiv b_k$ (q). The Fourier series of $g_k(\tau)$ is also of the form (15) and the numbers ϱ_k and σ_k which are defined by (16) satisfy the conditions (8). In the formula

$$a_n^{(k)} y^{\frac{1}{2}} K_{ir}\left(\frac{2\pi \mid n \mid}{\lambda} y\right) = \frac{1}{\lambda} \int_0^\lambda g_k(\tau) e^{-\frac{2\pi i n}{\lambda} x} dx \tag{A.61}$$

we plug in $y = \dfrac{c}{\mid n \mid}$ and conclude by (12) that as $\mid n \mid \to \infty$

$$a_n^{(k)} = O\left(\mid n \mid^{\varkappa_2 + \frac{1}{2}}\right), \tag{A.62}$$

holds by choosing a positive constant c so that $K_{ir}\left(\dfrac{2\pi}{\lambda} c\right) \neq 0$. The Dirichlet series (7) therefore have a half plane convergence and can be used to define the functions $\varphi_k(s)$ and $\psi_k(s)$. In order to verify the properties $I, 1$ and 2 we use the following integral formula W,13.21(8) which is inverse to (52)

$$\int_0^\infty K_\nu(t) t^{s-1} dt = 2^{s-2} \Gamma\left(\frac{s-\nu}{2}\right) \Gamma\left(\frac{s+\nu}{2}\right) \text{ for } \operatorname{Re} s > \mid \operatorname{Re} \nu \mid, \tag{A.63}$$

and after doing an easy substitution we obtain

$$4\int_0^\infty t^{\frac{1+\nu_1+\nu_2}{2}} K_{\frac{\nu_1-\nu_2}{2}}\left(\frac{2\pi \mid n \mid}{\lambda} t\right) t^{s-\frac{3}{2}} dt = \left(\frac{\lambda}{\pi}\right)^{s+\frac{\nu_1+\nu_2}{2}} \frac{\Gamma\left(\frac{s+\nu_1}{2}\right)\Gamma\left(\frac{s+\nu_2}{2}\right)}{\mid n \mid^{s+\frac{\nu_1+\nu_2}{2}}} \quad \text{(A.64)}$$

The representation

$$4\int_0^\infty F_k(t) t^{s-\frac{3}{2}} dt = \xi_k(s), \quad 4\int_0^\infty G_k(t) t^{s-\frac{3}{2}} dt = \eta_k(s) \quad \text{(A.65)}$$

follows immediately by term by term integration of the series in the integrand. We may do this because for $\operatorname{Re} s > \varkappa_2 + \frac{5}{2}$ since (62) implies

$$F_k(y) = O(y^{-\varkappa_2 - 1}), \quad G_k(y) = O(y^{-\varkappa_2 - 1}) \text{ for } y \to 0. \quad \text{(A.66)}$$

To get the representations for $\xi_k(s)$ and $\eta_k(k)$ which are true in the whole s-plane, we split the integral (65) into subintegrals over the intervals from $(0,1)$ to $(1,\infty)$ as in the analytical case. In the finite integrals we substitute $t \mapsto \frac{1}{t}$ and take into account the functional equations (56), which are equivalent to (14) after replacing $F_k\left(\frac{1}{y}\right)$ by $F_k^*\left(\frac{1}{y}\right) - u_k\left(\frac{1}{y}\right)$. These substitutions lead to the results

$$\frac{1}{4}\xi_k(s) = \int_1^\infty F_k(y) y^{s-\frac{3}{2}} dy + \sum_{l=1}^{N} \int_1^\infty F_l(y) y^{-\frac{1}{2}-s} dy \quad \text{(A.67)}$$

$$+\begin{cases} \dfrac{M\alpha_k}{s-1-ir} + \dfrac{\overline{M}\beta_k}{s-1+ir} - \dfrac{M\varrho_k}{s+ir} - \dfrac{\overline{M}\sigma_k}{s-ir} \\ \\ \dfrac{M\left(\alpha_k + \beta_k\left(\log\frac{\lambda}{4\pi} - C\right)\right)}{s-1} + \dfrac{M\beta_k}{(s-1)^2} - \dfrac{M\left(\varrho_k + \sigma_k\left(\log\frac{\lambda}{4\pi} - C\right)\right)}{s} + \dfrac{M\sigma_k}{s^2} \end{cases}$$

for $r > 0$ or $r = 0$ respectively, and

$$\frac{1}{4}\eta_k(s) = \int_1^\infty G_k(y) y^{s-\frac{3}{2}} dy - \sum_{l=1}^{N} \int_1^\infty G_l(y) y^{-\frac{1}{2}-s} dy. \quad \text{(A.68)}$$

The analytic properties of the functions $\varphi_k(s)$ and $\psi_k(s)$ that we assume in $I, 1$ in Theorem 1 can be seen immediately from (67) and (68). Also we can easily verify the functional equations (5). This proves Theorem 1. For the linear family of systems of functions with the properties of Theorem 1, one can prove under certain conditions a theorem about finiteness which says the following:

Theorem 30. *The maximal number of linearly independent systems of functions* φ_k, ψ_k $(k, 1, 2, .., N)$ *or* $g_k(\tau)$ $(k = 1, 2, ..., N)$ *which satisfy the conditions of Theorem* 1 *is finite if we assume* $\lambda = q$ *or* $\lambda = 2q$ *and if the group generated by the matrices*

$$\left(\delta_{kl} e^{\frac{2\pi i b_k}{q}}\right) \quad \textit{and} \quad C = (c_{kl})$$

is finite.

Proof. In the case of $\lambda = q$ or $\lambda = 2q$ the translation formulas (13) and (14) define a representation of order N of the modular group $\mathbf{M}$ and the theta group $\mathbf{T}$ respectively. The wave functions $g_k(\tau)$ are clearly invariant under the substitutions that get mapped to the identity by this representation and that build a normal sungroup $\mathbf{N}$ of finite index in $\mathbf{M}$ or $\mathbf{T}$ respectively if the group generated by matrices (20) is finite. Therefore, Theorem 4 follows from Theorem 5. □

Theorem 31. *Let* $\mathbf{G}$ *be a subgroup of the modular group of finite index. Let* $s_1, s_2, .., s_\sigma$ *be a complete system of nonequivalent parabolic cusps of a fundamental domain of* $\mathbf{G}$ *and let* A_ϱ *be an appropriate real unimodular substitution which sends* s_ϱ *to* ∞. *Then there are only finitely many linearly independent wave functions* $g(\tau)$ *which are invariant under the substitutions of* $\mathbf{G}$ *and which can be written as a series of the form*

$$g(A\varrho^{-1}\tau) = u_\varrho(y) + \sum_{n\neq 0} a_\varrho(n) y^{\frac{1}{2}} K_{ir}\left(\frac{2\pi \mid n \mid}{Q_\varrho} y\right) e^{\frac{2\pi i n}{Q_\varrho} x} \qquad \text{(A.69)}$$

in the parabolic cusps $(\varrho = 1, 2, ..., \sigma)$.

We did a proof using a method developed by C.L.Siegel[7]). First of all we may assume that $u_\varrho(y) = 0$ ($\varrho = 1, 2, ..., \sigma$) because these terms are of the form (16) hence they only depend linearly on 2σ parameters. In all parabolic cusps $g(\tau)$ vanishes, hence it is so-called a cusp function. Let $\tau \mapsto \tau + Q_\varrho$ ($Q_\varrho > 0$) be the generating group of the translations contained in $A_\varrho G A_\varrho^{-1}$. The Fourier series (69) shows the invariance of $g(\tau)$ with respect to substitutions

$$A_\varrho^{-1} \begin{pmatrix} 1 & Q_\varrho \\ 0 & 1 \end{pmatrix} A_\varrho$$

We denote by $\mathfrak{P}_\varrho$ the set of points which get sent by A_ϱ to the domain

$$-\frac{1}{2}Q_\varrho \leq x < \frac{1}{2}Q_\varrho,\ y \geq \varkappa.$$

For sufficiently large $\varkappa$ we can turn the set of points

$$\mathfrak{P} = \sum_{\varrho=1}^{\sigma} \mathfrak{P}_\varrho \tag{A.70}$$

into the fundamental domain

$$\mathfrak{F} = \mathfrak{P} + \mathfrak{B} \tag{A.71}$$

by adding a set of points $\mathfrak{B}$ such that the closure of $\mathfrak{B}$ lies in the upper half-plane. After we decide to pick a fixed value of $\varkappa$, we split $\mathfrak{B}$ into subsets of $\mathfrak{B}_\varrho$ in the following way:

$$\mathfrak{B} = \sum_{\varrho=1}^{\sigma} \mathfrak{B}_\varrho, \tag{A.72}$$

such that for any point $\tau = x + iy \in A_\varrho(\mathfrak{P}_\varrho + \mathfrak{B}_\varrho)$ the inequality

$$y \geq \varkappa_0 > 0 \tag{A.73}$$

is satisfied with $\varkappa_0$ being as large as possible for $\varrho = 1, 2, ..., \sigma$. Let M be the maximum of the absolute value of $g(\tau)$ in $\mathfrak{F}$. Because of the invariance of $g(\tau)$ in **G**

$$\mid g(\tau) \mid \leq M \tag{A.74}$$

holds in general. The equality is achieved at a point $\tau^* \in \mathfrak{F}$ which does not lie on the boundary circle $y = 0$ since $g(\tau)$ should vanish in the parabolic cusps. And τ^* may lie in $\mathfrak{P}_\iota + \mathfrak{B}_\iota$ so that for $\tau_0 = x_0 + iy_0 = A_\iota \tau^*$

$$y_0 \geq \varkappa_0, \mid g(A_\iota^{-1}\tau_0) \mid = M. \tag{A.75}$$

We now prove that $M = 0$ under the assumption

$$a_\varrho(n) = 0 \text{ for } \mid n \mid \leq m, \quad \varrho = 1, 2, ..., \sigma \tag{A.76}$$

for a sufficiently large m. With $\tau = x + \frac{i}{2} y_0$ we obviously have

$$a_\iota y_0^{\frac{1}{2}} K_{ir}\left(\frac{2\pi \mid n \mid}{Q_\iota} y_0\right) = \frac{1}{Q_\iota} \int_0^{Q_\iota} g(A_\iota^{-1}\tau) e^{-\frac{2\pi i n}{Q_\iota} x} dx \sqrt{2} \frac{K_{ir}\left(\frac{2\pi|n|}{Q_\iota} y_0\right)}{K_{ir}\left(\frac{\pi|n|}{Q_\iota} y_0\right)},$$

which implies the estimate

$$\begin{aligned} M = \mid g(A_\iota^{-1}\tau_0) \mid &\leq \sum_{|n|>m} \left| a_\iota(n) y_0^{\frac{1}{2}} K_{ir}\left(\frac{\pi \mid n \mid}{Q_\iota} y_0\right) \right| \\ &\leq 2\sqrt{2} M \sum_{n>m} \left| \frac{K_{ir}\left(\frac{2\pi n}{Q_\iota} y_0\right)}{K_{ir}\left(\frac{\pi n}{Q_\iota} y_0\right)} \right|. \end{aligned} \tag{A.77}$$

Due to asymptotic behaviour (19) of Bessel function $K_\nu(z)$ for large z we can conclude that

$$M \leq 4M \sum_{n>m} e^{-\frac{\pi n}{Q_\iota} y_0} \leq 4M \sum_{n>m} e^{-\frac{\pi n}{Q_\iota} \varkappa_0} \leq cMe^{-\frac{\pi n}{Q_\iota} \varkappa_0}$$

holds for sufficiently large m where c is a positive constant depending only on the decomposition $\mathfrak{F}$. If m also satisfies

$$m > \frac{Q_\varrho \log c}{\pi \varkappa_0} \quad (\varrho = 1, 2, ..., \sigma)$$

then we have $M = 0$, q.e.d.

In the case $r = 0$, using the estimate (50) we can give explicitly a sufficient condition for the identical vanishing of a cusp function for $\mathbf{G}$. From (77), it follows that for $m = 0$ we have

$$M \leq 2\sqrt{2}M \sum_{n>0} \frac{K_0\left(\frac{2\pi n}{Q_\iota} y_0\right)}{K_0\left(\frac{\pi n}{Q_\iota} y_0\right)} \leq \frac{\sqrt{2\pi}M}{\sqrt{a_\iota}K_0(a_\iota)e^{a_\iota}} \sum_{n>0} e^{-\frac{\pi n}{Q_\iota} y_0}$$

$$\leq \frac{\sqrt{2\pi}M}{\sqrt{a_\iota}K_0(a_\iota)e^{a_\iota}(e^{a_\iota}-1)}. \tag{A.78}$$

if we generally set $a_\varrho = \dfrac{\pi \varkappa_0}{Q_\varrho}$. In order to conclude that $M = 0$ we must assume

$$\sqrt{2\pi} < \sqrt{a_\varrho} K_0(a_\varrho) e^{a_\varrho}(e^{a_\varrho} - 1) \quad (\varrho = 1, 2, ..., \sigma) \tag{A.79}$$

On the right hand side of (79) there is a monotone increasing function of a_ϱ. The inequality is satisfied for $a_\varrho = \dfrac{3}{2}$ because by $W, S.699$ we have

$$K_0(\frac{3}{2})e^{\frac{3}{2}} = 0,9582101,,,, \; e^{\frac{3}{2}} - 1 = 3,4816891...$$

The identical vanishing of a cusp function for $\mathbf{G}$ hence follows from

$$\frac{\pi \varkappa_0}{Q_\varrho} > \frac{3}{2} \quad (\varrho = 1, 2, ..., \sigma). \tag{A.80}$$

We apply this result to the modular group and the theta group.

1. $\mathbf{G} = \mathbf{M}$. We have $\sigma = 1$. As a fundamental domain $\mathfrak{F}$ we choose $\mid \tau + \bar{\tau} \mid \leq 1$, $\mid \tau \mid \geq 1$ and $A_1 = Id$(identity matrix). Then $Q_1 = 1$, $\varkappa_0 = \dfrac{1}{2}\sqrt{3}$ and (80) is satisfied.

2. $\mathbf{G} = \mathbf{T}$. Here $\sigma = 2$. A fundamental domain $\mathfrak{F}$ is given by $\mid \tau + \bar{\tau} - 2 \mid \leq 2$, $\mid \tau \mid \geq 1$, $\mid \tau - 2 \mid \geq 1$. Let $\mathfrak{P}_1 + \mathfrak{B}_1$ be the intersection of $\mathfrak{F}$ with $\mid \tau - 1 \mid \geq \sqrt{2}$. Then this also determines $\mathfrak{P}_2 + \mathfrak{B}_2$. Let $A_1 = Id$ and A_2 be the noneuclidean rotation of order 2 with the fixed point $1 + i\sqrt{2}$. Obviously we have $A_2(\mathfrak{P}_2 + \mathfrak{B}_2) = \mathfrak{P}_1 + \mathfrak{B}_1$ so that $Q_1 = Q_2 = 2$ and $\varkappa_0 = 1$. Again condition (80) is satisfied. We obtain the following result:

Theorem 32. *There is no cusp function for the Theta group* $\mathbf{T}$ *and* $r = 0$ *which does not vanish identically.*

For $\mathbf{M}$ the theorem is obviously trivial if it holds for $\mathbf{T}$ because $\mathbf{T}$ is a subgroup of $\mathbf{M}$. In order to explain Theorem 3 we need a similar statement for special systems of cusp functions of level 5.

Theorem 33. *A system of cusp functions* $g_1(\tau), g_2(\tau), g_3(\tau)$ *for the congruence group* $\mathbf{M}(5)$ *and* $r = 0$ *vanish identically if for* $S \in \mathbf{M}$ *the transformation formulas of the form*

$$g_i(S\tau) = \sum_{k=1}^{3} a_{ik}(S) g_k(\tau) \quad (i = 1, 2, 3) \tag{A.81}$$

hold and the representation $(a_{ik}(S))$ *of the modular group* $\mathbf{M}/\mathbf{M}(5)$ *is unitary.*

For the proof we may assume that the matrix

$$\left(a_{ik} \left(\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \right) \right)$$

is a diagonal matrix. Otherwise, apply a unitary transformation to the system $g_k(\tau)$, $(k = 1, 2, 3)$. Hence,

$$g_k(\tau + 1) = \zeta_k g_k(\tau) \quad (k = 1, 2, 3) \tag{A.82}$$

with certain 5th roots of unity

$$\zeta_k = e^{\frac{2\pi i b_k}{5}}, \ 0 \leq b_k < 5.$$

The Fourier series of the functions are then of the form

$$g_k(\tau) = \sum_{n + \frac{b_k}{5} \neq 0} a_k(n) y^{\frac{1}{2}} K_0 \left(2\pi \left| n + \frac{b_k}{5} \right| y \right) e^{2\pi i \left(n + \frac{b_k}{5} \right) x}, \quad (k = 1, 2, 3). \tag{A.83}$$

Because we assume that the representation $(a_{ik}(S))$ is unitary, the function

$$\chi(\tau) = \sqrt{\sum_{k=1}^{3} g_k(\tau)\overline{g_k(\tau)}} \tag{A.84}$$

is invariant under the substitutions of $\mathbf{M}$. Furthermore, it vanishes in the parabolic cusp of the fundamental domain $\mid \tau + \bar{\tau} \mid \leq 1, \ \mid \tau \mid \geq 1$ of $\mathbf{M}$. Let M be the maximum of $\chi(\tau)$ in the fundamental domain. It is achieved at a finite point τ_0 in the fundamental domain. Obviously $\chi(\tau) \leq M$ holds for all τ. If we let

$$\mid g_k(\tau_0) \mid \leq \mid g_1(\tau_0) \mid, \quad (k = 1, 2, 3) \tag{A.85}$$

then

$$M \leq \sqrt{3} \mid g_1(\tau_0) \mid \text{ and } y \geq \frac{1}{2}\sqrt{3}, \tag{A.86}$$

if we set $\tau_0 = x_0 + iy_0$. Using the coefficient formula for the Fourier series we derive an estimate for $g_1(\tau_0)$ which implies $M = 0$ i.e. $\chi(\tau) = 0$. Let $\tau = x + i\vartheta y_0$ where ϑ is determined later and it is in the interval $0 < \vartheta < 1$. From the formula

$$\begin{aligned} & a_1(n) y_0^{\frac{1}{2}} K_0\left(2\pi \left|n + \frac{b_1}{5}\right| y_0\right) \\ & = \frac{1}{5}\int_1^5 g_1(\tau) e^{-2\pi i\left(n+\frac{b_1}{5}\right)x} dx \vartheta^{-\frac{1}{2}} \frac{K_0\left(2\pi \left|n + \frac{b_1}{5}\right| y_0\right)}{K_0\left(2\pi \left|n + \frac{b_1}{5}\right| \vartheta y_0\right)} \end{aligned} \tag{A.87}$$

we get the estimate

$$\mid a_1(n) \mid y_0^{\frac{1}{2}} K_0\left(2\pi \left|n + \frac{b_1}{5}\right| y_0\right) \leq \frac{M}{\sqrt{\vartheta}} \frac{K_0\left(2\pi \left|n + \frac{b_1}{5}\right| y_0\right)}{K_0\left(2\pi \left|n + \frac{b_1}{5}\right| \vartheta y_0\right)}$$

thus we obtain

$$\frac{M}{\sqrt{3}} \leq \mid g_1(\tau_0) \mid \leq \frac{M}{\sqrt{\vartheta}} \sum_{n+\frac{b_1}{5}\neq 0} a_k(n) \frac{K_0\left(2\pi \left|n + \frac{b_1}{5}\right| y_0\right)}{K_0\left(2\pi \left|n + \frac{b_1}{5}\right| \vartheta y_0\right)}$$

since by

$$2\pi \left|n + \frac{b_1}{5}\right| \vartheta y_0 \geq \frac{\pi\sqrt{3}}{5}\vartheta$$

and (50) we can conclude that

$$\frac{M}{\sqrt{3}} \leq \frac{M}{\sqrt{a}K_0(a)e^a}\sqrt{\frac{\pi}{2}}\sum_{n+\frac{b_1}{5}\neq 0} e^{-\pi\sqrt{3}\left|n+\frac{b_1}{5}\right|(1-\vartheta)} \tag{A.88}$$

holds if we set $a = \dfrac{\pi\sqrt{3}}{5}\vartheta$. The infinite series in (88) has the value

$$\frac{2}{\xi^5-1} \text{ for } b_1 = 0 \quad \text{and} \quad \frac{\xi^{5-b_1}+\xi^{b_1}}{\xi^5-1} \text{ for } b_1 > 0$$

with

$$\xi = e^{\frac{\pi\sqrt{3}}{5}(1-\vartheta)} = e^{\frac{\pi\sqrt{3}}{5}-a}.$$

In any case we get

$$\frac{M}{\sqrt{3}} \leq \frac{M}{\sqrt{a}K_0(a)e^a}\sqrt{\frac{\pi}{2}}\frac{\xi^4+\xi}{\xi^5-1} \tag{A.89}$$

because we have $2 < \xi^3+\xi^2 < \xi^4+\xi$. If we can find a in the interval $0 < a < \dfrac{\pi\sqrt{3}}{5}$ so that

$$\sqrt{a}K_0(a)e^a > \sqrt{\frac{3\pi}{2}\frac{\xi^4+\xi}{\xi^5-1}} \tag{A.90}$$

holds then $M = 0$. For $a = 0.16$ it follows from W. page 698 that $K_0(a)e^a = 2,3087874...$ and we conclude that

$$\sqrt{a}K_0(a)e^a > 0,9235 > 0,9199 > \sqrt{\frac{3\pi}{2}\frac{\xi^4+\xi}{\xi^5-1}}$$

indeed holds. This proves Theorem 7.

A.2 The Zeta Functions of Real Quadratic Fields

We apply the general result of the first paragraph to the zeta functions $\zeta_\nu(s, \varrho, \mathfrak{a}, \lambda_1^n, Q(D))$ ($\nu = 0, 1$), defined by (23), of the real quadratic field $R(\sqrt{D})$. We prove the functional equations (25) using the classical method that Hecke[2]) applied to the special case $n = 0$. Using the Γ-integrals, we can write the functions (26) in terms of the theta series

$$\vartheta_0(t, t', \varrho, \mathfrak{a}, Q\sqrt{D}) = \sum_{\mu \equiv \varrho \ (\mathfrak{a}Q\sqrt{D})} e^{-\frac{\pi}{AQD}(\mu^2 t + \mu'^2 t')},$$

$$\vartheta_1(t, t', \varrho, \mathfrak{a}, Q\sqrt{D}) = \sum_{\mu \equiv \varrho \ (\mathfrak{a}Q\sqrt{D})} \mu\mu' e^{-\frac{\pi}{AQD}(\mu^2 t + \mu'^2 t')} \tag{A.91}$$

We use the same notation in (91) as before. We set

$$\delta(\mathfrak{b}) = \begin{cases} 1 & \text{for an integral ideal } \mathfrak{b}, \\ 0 & \text{otherwise} \end{cases}$$

so

$$\xi_0(s, \varrho, \mathfrak{a}, \lambda_1^n, Q\sqrt{D})$$
$$= \frac{4}{\lambda_1^n(\mathfrak{a})} \int_0^\infty \left[\int_{-l_Q}^{l_Q} \left\{ \vartheta_0(ue^{2v}, ue^{-2v}\varrho, \mathfrak{a}, Q\sqrt{D}) - \delta\left(\frac{\varrho}{\mathfrak{a}Q\sqrt{D}}\right) \right\} e^{-2incv} dv \right] u^{s-1} du \tag{A.92}$$

$$\xi_1(s, \varrho, \mathfrak{a}, \lambda_1^n, Q\sqrt{D}) = \frac{4}{\lambda_1^n(\mathfrak{a})} \int_0^\infty \left[\int_{-l_Q}^{l_Q} \vartheta_1(ue^{2v}, ue^{-2v}\varrho, \mathfrak{a}, Q\sqrt{D}) e^{-2incv} dv \right] u^s du$$

for $\mathrm{Re}(s) > 1$. We get representations of ξ_ν, that hold for all s, by decomposing the integrals over u from 0 to 1 and 1 to ∞ and by applying the transformation formulas

$$\vartheta_0(t, t', \varrho, \mathfrak{a}, Q\sqrt{D}) = \frac{1}{Q\sqrt{D}\sqrt{tt'}} \sum_{\substack{\alpha \bmod \mathfrak{a}Q\sqrt{D} \\ \alpha \equiv 0 \ (\mathfrak{a})}} e^{2\pi i \operatorname{Tr}\left(\frac{\alpha'\varrho}{AQD}\right)} \vartheta_0\left(\frac{1}{t'}, \frac{1}{t}, \alpha, \mathfrak{a}, Q\sqrt{D}\right)$$

$$\vartheta_1(t, t', \varrho, \mathfrak{a}, Q\sqrt{D}) = \frac{-1}{Q\sqrt{D}(\sqrt{tt'})^?} \sum_{\substack{\alpha \bmod \mathfrak{a}Q\sqrt{D} \\ \alpha \equiv 0 \ (\mathfrak{a})}} e^{2\pi i \operatorname{Tr}\left(\frac{\alpha'\varrho}{AQD}\right)} \vartheta_1\left(\frac{1}{t'}, \frac{1}{t}, \alpha, \mathfrak{a}, Q\sqrt{D}\right) \tag{A.93}$$

to the integrands of the finite subintegrals and making the substitution $u \mapsto 1/u$. So we get

$$\xi_0(s,\varrho,\mathfrak{a},\lambda_1^n,Q\sqrt{D}) = -\frac{8l_Q\delta_n\left(\frac{\varrho}{\mathfrak{a}Q\sqrt{D}}\right)}{\lambda_1^n(\mathfrak{a})}\frac{1}{s} - \frac{8l_Q\delta_n(0)}{Q\sqrt{D}\lambda_1^n(\mathfrak{a})}\frac{1}{1-s}$$
$$+\frac{4}{\lambda_1^n(\mathfrak{a})}\int_1^\infty\left[\int_{-l_Q}^{l_Q}\left\{\vartheta_0(ue^{2v},ue^{-2v}\varrho,\mathfrak{a},Q\sqrt{D}) - \delta\left(\frac{\varrho}{\mathfrak{a}Q\sqrt{D}}\right)\right\}e^{-2incv}dv\right]u^{s-1}du$$
$$+\frac{4}{Q\sqrt{D}\lambda_1^n(\mathfrak{a})}\sum_{\substack{\alpha \bmod \mathfrak{a}Q\sqrt{D}\\ \alpha\equiv 0\ (\mathfrak{a})}} e^{2\pi i S\left(\frac{\alpha'\varrho}{AQD}\right)}\int_1^\infty\left[\int_{-l_Q}^{l_Q}\left\{\vartheta_0(ue^{2v},ue^{-2v},\alpha,\mathfrak{a},Q\sqrt{D})\right.\right.$$
$$\left.\left.-\delta\left(\frac{\alpha}{\mathfrak{a}Q\sqrt{D}}\right)\right\}e^{-2incv}dv\right]u^{-s}du \tag{A.94}$$

and

$$\xi_1(s,\varrho,\mathfrak{a},\lambda_1^n,Q\sqrt{D}) = \frac{4}{\lambda_1^n(\mathfrak{a})}\int_1^\infty\left[\int_{-l_Q}^{l_Q}\vartheta_1(ue^{2v},ue^{-2v}\varrho,\mathfrak{a},Q\sqrt{D}e^{-2incv}dv\right]u^s du$$
$$-\frac{4}{Q\sqrt{D}\lambda_1^n(\mathfrak{a})}\sum_{\substack{\alpha \bmod \mathfrak{a}Q\sqrt{D}\\ \alpha\equiv 0\ (\mathfrak{a})}} e^{2\pi i S\left(\frac{\alpha'\varrho}{AQD}\right)}\int_1^\infty\left[\int_{-l_Q}^{l_Q}\vartheta_1(ue^{2v},ue^{-2v},\alpha,\mathfrak{a},Q\sqrt{D})e^{-2incv}dv\right]u^{1-s}du. \tag{A.95}$$

Properties I, (1 to 4) of Theorem 1 with special data (27) can be now easily verified for the zeta function $\zeta_\nu(s,\varrho,\mathfrak{a},\lambda_1^n,Q\sqrt{D})$. For the associated wave function $g(\tau,\varrho,\mathfrak{a},\lambda_1^n,Q\sqrt{D})$ we have the relations

$$g(\tau+a,\varrho,\mathfrak{a},\lambda_1^n,Q\sqrt{D}) = e^{2\pi i\frac{a\mathrm{N}\varrho}{AQD}}g(\tau,\varrho,\mathfrak{a},\lambda_1^n,Q\sqrt{D})\ (a\text{ is whole rational}), \tag{A.96}$$

$$g(-\frac{1}{\tau},\varrho,\mathfrak{a},\lambda_1^n,Q\sqrt{D}) = \frac{1}{Q\sqrt{D}}\sum_{\substack{\sigma \bmod \mathfrak{a}Q\sqrt{D}\\ \sigma\equiv 0\ (\mathfrak{a})}} e^{2\pi i\,\mathrm{Tr}\left(\frac{\varrho\sigma'}{AQD}\right)}g(\tau,\sigma,\mathfrak{a},\lambda_1^n,Q\sqrt{D}). \tag{A.97}$$

Moreover, for an arbitrary positive integer m

$$g(\tau,\varrho,\mathfrak{a},\lambda_1^n,Q\sqrt{D}) = \frac{l_Q}{\sqrt{m}l_{mQ}}\sum_{\substack{\sigma \bmod \mathfrak{a}mQ\sqrt{D}\\ \sigma\equiv\varrho\ (\mathfrak{a}Q\sqrt{D})}} g(m\tau,\sigma,\mathfrak{a},\lambda_1^n,mQ\sqrt{D}) \tag{A.98}$$

which we can see from the Fourier series (29). With these three translation formulas we can determine, by method of Hecke[4]), the behaviour of $g(\tau, \varrho, \mathfrak{a}, \lambda_1^n, Q\sqrt{D})$ under arbitrary modular substitution $S \in \mathbf{M}$. Here we do not use any linear relations between $g(\tau, \varrho, \mathfrak{a}, \lambda_1^n, Q\sqrt{D})$. For

$$S = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathbf{M}$$

we get the translation formula

$$g(S\tau, \varrho, \mathfrak{a}, \lambda_1^n, Q\sqrt{D}) = \sum_{\substack{\sigma \bmod \mathfrak{a}Q\sqrt{D} \\ \sigma \equiv 0\ (\mathfrak{a})}} c_{\varrho\sigma}(S) g(\tau, \sigma, \mathfrak{a}, \lambda_1^n, Q\sqrt{D}) \tag{A.99}$$

with coefficients

$$c_{\varrho\sigma}(S) = \begin{cases} \dfrac{1}{cQ\sqrt{D}} \displaystyle\sum_{\substack{\alpha \bmod \mathfrak{a}cQ\sqrt{D} \\ \alpha \equiv \varrho\ (\mathfrak{a}Q\sqrt{D})}} e^{\frac{2\pi i}{cAQD}(a\mathrm{N}\alpha + \mathtt{S}\alpha\sigma' + d\mathrm{N}\sigma)} & \text{for } c > 0, \\ \delta_{\varrho\sigma} e^{\frac{b\mathrm{N}\varrho}{AQD}} & \text{for } c = 0,\ d = 1 \end{cases} \tag{A.100}$$

The unit root sums above were discussed by Hecke. For $S \in \mathbf{M}(QD)$ we get in particular $c_{\varrho\sigma}(S) = \delta_{\varrho\sigma}$, that is

$$g(S\tau, \varrho, \mathfrak{a}, \lambda_1^n, Q\sqrt{D}) = g(\tau, \varrho, \mathfrak{a}, \lambda_1^n, Q\sqrt{D}) \text{ for } S \in \mathbf{M}(QD). \tag{A.101}$$

The full family of wave functions $g(\tau, \varrho, \mathfrak{a}, \lambda_1^n, Q\sqrt{D})$ with fixed n, Q, D is obtained if $\mathfrak{a}$ runs through a full system of representatives of narrow ideal classes in $R\sqrt{D}$ and ϱ runs through the cosets mod $\mathfrak{a}Q\sqrt{D}$, which are contained in $\mathfrak{a}$; because

$$g(\tau, \varrho\beta, \mathfrak{a}\beta, \lambda_1^n, Q\sqrt{D}) = g(\tau, \varrho, \mathfrak{a}, \lambda_1^n, Q\sqrt{D}) \text{ for } \mathtt{N}\beta > 0. \tag{A.102}$$

The question about linear relations between the considered wave functions, besides the trivial ones

$$g(\tau, \varrho, \mathfrak{a}, \lambda_1^n, Q\sqrt{D}) = g(\tau, -\varrho, \mathfrak{a}, \lambda_1^n, Q\sqrt{D}), \tag{A.103}$$

is answered fully at this point only in cases that are given numerically. Since the maximum number of linearly independent series with fixed arguments $\mathfrak{a}, n, Q, D$ is less than Q^2D, we can apply Theorem 1 to this system of functions with $N < Q^2D$. Theorem 4 still holds. In particular we consider $D = 5,\ Q = 1$. Because the number of narrow ideal classes in $R(\sqrt{5})$ is 1, it is sufficient to assume that $\mathfrak{a} = 1$ and in view of (103) then $\varrho = 0, 1, 2$. The three series

$$g_n(\tau, \varrho) = g(\tau, \varrho, (1), \lambda_1^n, \sqrt{5}) \quad (\varrho = 0, 1, 2)$$

are linearly independent. Denoting the Fourier coefficients of these functions with exponent m by $a_\varrho(m)$ then we have by (26) for $m \neq 0$

$$a_\varrho = \sum_{\substack{\mu \equiv \varrho\ (\sqrt{5}) \\ (\mu)_{\sqrt{5}p_\infty}, \mathtt{N}\mu = m}} \lambda_1^n(\mu),$$

in particular

ϱ	0	1	2
$a_\varrho(1)$	0	2	0
$a_\varrho(-1)$	0	0	2
$a_\varrho(5)$	4	0	0

and the determinant of this coefficient scheme is different than 0. Let $\zeta = e^{\frac{2\pi i}{5}}$, then (96) and (97) become the translation formulas

$$g_n(\tau + 1, 0) = g_n(\tau, 0),\ g_n(\tau + 1, 1) = \zeta g_n(\tau, 1),\ g_n(\tau + 1, 2) = \zeta^{-1} g_n(\tau, 2),$$

$$\begin{pmatrix} g_n\left(-\frac{1}{\tau}, 0\right) \\ g_n\left(-\frac{1}{\tau}, 1\right) \\ g_n\left(-\frac{1}{\tau}, 2\right) \end{pmatrix} = \frac{1}{\sqrt{5}} \begin{pmatrix} 1 & 2 & 2 \\ 1 & \zeta^2 + \zeta^{-2} & \zeta + \zeta^{-1} \\ 1 & \zeta + \zeta^{-1} & \zeta^2 + \zeta^{-2} \end{pmatrix} \begin{pmatrix} g_n(\tau, 0) \\ g_n(\tau, 1) \\ g_n(\tau, 2) \end{pmatrix} \tag{A.104}$$

The corresponding translation formulas for the functions

$$g_n^*(\tau, 0) = \frac{1}{\sqrt{2}} g_n(\tau, 0),\ g_n^*(\tau, 1) = g_n(\tau, 1),\ g_n^*(\tau, 2) = g_n(\tau, 2) \tag{A.105}$$

define a unitary representation of the modular group $\mathbf{M}/\mathbf{M}(5)$ as we can easily check. This is important for the application of Theorem 7 to the system of functional equations (104).

A.3 The Eisenstein Series of Level Q

The question of all automorphic wave functions of level Q is significant for the theory of Dirichlet series, but it is also interesting itself. The easiest way to construct such functions is to build the Eisenstein series

$$E(\tau, s; (a_1, a_2), Q) = \sum_{\substack{m_i \equiv a_i \ (Q) \\ (m_1, m_2) \neq (0,0)}} \frac{y^{\frac{s}{2}}}{\mid m_1\tau + m_2 \mid^s} \tag{A.106}$$

where we sum over all nonvanishing pairs of rational integers (m_1, m_2) of the residue classes (a_1, a_2) mod Q. The series are absolutely convergent for $\mathrm{Re}\, s > 2$. The general term and hence E are the solutions of the wave equation

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{s(2-s)}{4y^2}\right) E = 0, \quad (\tau = x + iy) \tag{A.107}$$

and the translation formula

$$E(S\tau, s; (a_1, a_2), Q) = E(\tau, s; (a_1, a_2)S, Q) \text{ for } S \in \mathbf{M} \tag{A.108}$$

holds which implies in particular the invariance under the substitutions of $\mathbf{M}(Q)$. First we have to prove that the functions E are regular for $\mathrm{Re}\, s > 2$ can be extended analytically to the half plane for $\mathrm{Re}\, s < 2$ and on the vertical line $\mathrm{Re}\, s = 1$ they are regular. For this purpose we write the Eisenstein series as a Fourier series.

After a simple modification we get

$$E(\tau,s;(a_1,a_2),Q) = \delta\left(\frac{a_1}{Q}\right) y^{\frac{s}{2}} \zeta(s,a_2,Q) + \frac{y^{\frac{s}{2}}}{Q^s} \sum_{\substack{m_1 \equiv a_1\ (Q) \\ m_1 \neq 0}} \frac{1}{\mid m_1 \mid^s} \sum_{\substack{j \equiv a_2\ (Q) \\ j \bmod Qm_1}} f\left(\frac{\tau}{Q} + \frac{j}{Qm_1}, s\right) \tag{A.109}$$

with

$$\zeta(s,a,Q) = \sum_{\substack{n \equiv a\ (Q) \\ n \neq 0}} \frac{1}{\mid n \mid^s} \quad \text{and} \quad f(\tau,s) = \sum_{n+-\infty}^{n=\infty} \frac{1}{\mid \tau + n \mid^s}. \tag{A.110}$$

The Fourier series of a function $f(\tau, s)$ that is periodic in τ is

$$f(\tau,s) = \sum_{n=-\infty}^{n=\infty} \left\{ \int_{-\infty}^{\infty} \mid u + iy \mid^{-s} e^{-2\pi i n u} du \right\} e^{2\pi i n x}.$$

Here for $n \neq 0$, by W,6.16(1) we have

$$\int_{-\infty}^{\infty} \mid u{+}iy \mid^{-s} e^{-2\pi i n u} du = 2\int_0^{\infty} \frac{\cos 2\pi n u du}{(u^2+y^2)^{\frac{s}{2}}} = \frac{2\pi^{\frac{s}{2}}}{\Gamma\left(\frac{s}{2}\right)} \left(\frac{\mid n \mid}{y}\right)^{\frac{s-1}{2}} K_{\frac{s-1}{2}}(2\pi \mid n \mid y)$$

and for $n = 0$

$$\int_{-\infty}^{\infty} \mid u + iy \mid^{-s} du = 2\int_0^{\infty} \frac{du}{(u^2+y^2)^{\frac{s}{2}}} = \frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{s-1}{2}\right)}{\Gamma\left(\frac{s}{2}\right)} y^{1-s}$$

so that

$$f(\tau,s) = \sqrt{\pi}\frac{\Gamma\left(\frac{s-1}{2}\right)}{\Gamma\left(\frac{s}{2}\right)} y^{1-s} + \frac{2\pi^{\frac{s}{2}}}{\Gamma\left(\frac{s}{2}\right)} \sum_{n\neq 0} \left(\frac{\mid n \mid}{y}\right)^{\frac{s-1}{2}} K_{\frac{s-1}{2}}(2\pi \mid n \mid y) e^{2\pi i n x} \tag{A.111}$$

holds. We plug the above series into (109) and get

$$E(\tau,s;(a_1,a_2),Q)=\delta\left(\frac{a_1}{Q}\right)y^{\frac{s}{2}}\zeta(s,a_2,Q)+\frac{\sqrt{\pi}}{Q}\frac{\Gamma\left(\frac{s-1}{2}\right)}{\Gamma\left(\frac{s}{2}\right)}\zeta(s-1,a_1,Q)y^{1-\frac{s}{2}}$$

$$+\frac{2\pi^{\frac{s}{2}}}{Q^{\frac{s+1}{2}}\Gamma\left(\frac{s}{2}\right)}\sum_{n\neq 0}\left\{\sum_{\substack{m\equiv a_1\ (Q)\\ m\mid n}} e^{\frac{2\pi i n a_2}{Qm}}\mid m\mid^{1-s}\right\}\mid n\mid^{\frac{s-1}{2}} y^{\frac{1}{2}}K_{\frac{s-1}{2}}\left(\frac{2\pi\mid n\mid}{Q}y\right)e^{\frac{2\pi i n}{Q}x}. \tag{A.112}$$

Now the analytic continuation is done. In order to prove that the poles of the zeta and Gamma functions, which appear formally in (112), do not occur in $\operatorname{Re} s=1$, we must use the translation formulas of the $\zeta(s,a,Q)$ function. It is known that

$$\xi(s,a,Q)=\left(\frac{Q}{\pi}\right)^{\frac{s}{2}}\Gamma\left(\frac{s}{2}\right)\zeta(s,a,Q)=-\delta\left(\frac{a}{Q}\right)\frac{2}{s}-\frac{2}{\sqrt{Q}(1-s)}$$

$$+\int_1^\infty t^{\frac{s}{2}-1}\left\{\sum_{\substack{n\equiv a\ (Q)\\ n\neq 0}} e^{-\frac{\pi i n^2}{Q}}\right\}dt+\frac{1}{\sqrt{Q}}\sum_{b \bmod Q}\zeta^{ab}\int_1^\infty t^{\frac{1-s}{2}-1}\left\{\sum_{\substack{n\equiv b\ (Q)\\ n\neq 0}} e^{-\frac{\pi i n^2}{Q}}\right\}dt \tag{A.113}$$

if we set $\zeta=e^{\frac{2\pi i}{Q}}$ and therefore

$$\xi(1-s,a,Q)=\frac{1}{Q}\sum_{b \bmod Q}\zeta^{ab}\xi(s,b,Q). \tag{A.114}$$

Using the functional equations we can put the sum of the terms in the Fourier series (112) that are independent of x

$$u(y,s;(a_1,a_2),Q)=\delta\left(\frac{a_1}{Q}\right)y^{\frac{s}{2}}\zeta(s,a_2,Q)+\frac{\sqrt{\pi}}{Q}\frac{\Gamma\left(\frac{s-1}{2}\right)}{\Gamma\left(\frac{s}{2}\right)}\zeta(s-1,a_1,Q)y^{1-\frac{s}{2}} \tag{A.115}$$

in a form which the regularity of this expression at $s = 1$ can be seen immediately.

Namely,

$$u(y,s;(a_1,a_2),Q) = \frac{2y^{\frac{1}{2}}\pi^{\frac{s}{2}}}{Q^{\frac{s+1}{2}}\Gamma\left(\frac{s}{2}\right)}\left\{(\pi Q y)^{\frac{s-1}{2}}\Gamma\left(\frac{1-s}{2}\right)\zeta(1-s) + (\pi Q y)^{\frac{1-s}{2}}\Gamma\left(\frac{s-1}{2}\right)\zeta(s-1)\right\}$$

$$+ \frac{y^{\frac{1}{2}}}{Q}\left(\frac{\pi y}{Q^2}\right)^{\frac{s-1}{2}}\frac{\pi^{\frac{s}{2}}}{\Gamma\left(\frac{s}{2}\right)}\sum_{\substack{b \bmod Q \\ b \not\equiv 0\ (Q)}}\zeta^{a_2 b}\Gamma\left(\frac{1-s}{2}\right)\zeta(1-s,b,Q) \quad \text{for} \quad a_1 \equiv 0\ (Q)$$

(A.116)

and

$$u(y,s;(a_1,a_2),Q) = \frac{\sqrt{\pi}}{Q}\frac{\Gamma\left(\frac{s-1}{2}\right)}{\Gamma\left(\frac{s}{2}\right)}\zeta(s-1,a_1,Q)y^{1-\frac{s}{2}} \quad \text{for} \quad a_1 \not\equiv 0\ (Q) \qquad \text{(A.117)}$$

The following power seires at $s-1$

$$\zeta(s,b,Q) = \frac{2}{Q(s-1)} + \frac{2}{Q}\left(C - \sum_{a=1}^{Q-1}\zeta^{-ab}\log\left(2\sin\frac{a\pi}{Q}\right)\right) + ...$$

with the relation (114) allow us to compute the following limit:

$$\lim_{s\to 1}\Gamma\left(\frac{1-s}{2}\right)\zeta(1-s,b,Q) = -2\log\left(2\left|\sin\frac{b\pi}{Q}\right|\right) \quad \text{for} \quad b \not\equiv 0\ (Q). \qquad \text{(A.118)}$$

A simple computation gives

$$u(y,1;(a_1,a_2),Q) \qquad \text{(A.119)}$$

$$= \begin{cases} \frac{2}{Q}y^{\frac{1}{2}}\log y + \frac{2}{Q}\left(C + \log\frac{Q}{4\pi} - \sum_{b=1}^{Q-1}\zeta^{-a_2 b}\log\left(2\sin\frac{a\pi}{Q}\right)\right)y^{\frac{1}{2}} & \text{for } a_1 \equiv 0\ (Q) \\ -\frac{2}{Q}\log\left(2\left|\sin\frac{a_1\pi}{Q}\right|\right)y^{\frac{1}{2}} & \text{for } a_1 \not\equiv 0\ (Q) \end{cases}$$

and for $r > 0$ we get the following from (115):

$$\begin{aligned} u(y,1+2ir;(a_1,a_2),Q) =& \delta\left(\frac{a_1}{Q}\right)y^{\frac{1}{2}+ir}\zeta(1+2ir,a_2,Q) \\ &+ \frac{\sqrt{\pi}}{Q}\frac{\Gamma(ir)}{\Gamma\left(\frac{1}{2}+ir\right)}\zeta(2ir,a_1,Q)y^{\frac{1}{2}-ir}. \end{aligned} \qquad \text{(A.120)}$$

In order to determine the maximum number of linearly independent Eisenstein series, we consider the series

$$E^*(\tau, s; (a_1, a_2), Q) = \sum_{\substack{m_i \equiv a_i \ (Q) \\ (m_1, m_2) = 1}} \frac{y^{\frac{s}{2}}}{| \ m_1\tau + m_2 \ |^s} \tag{A.121}$$

which vanishes identically for $(a_1, a_2, Q) > 1$ if we set as usual that an empty sum has the value zero. Between the primitive series, which are charactarized by $(a_1, a_2, Q) = 1$, we have the linear relations

$$E^*(\tau, s; (a_1, a_2), Q) = \sum_{t \bmod Q} E(\tau, s; (ta_1, ta_2), Q)c(s, t, Q) \tag{A.122}$$

and

$$E(\tau, s; (a_1, a_2), Q) = \sum_{t \bmod Q} E^*(\tau, s; (ta_1, ta_2), Q)d(s, t, Q) \tag{A.123}$$

with

$$c(s, t, Q) = \sum_{\substack{tn \equiv 1 \ (Q) \\ n > 0}} \frac{\mu(n)}{n^s}, \quad d(s, t, Q) = \sum_{\substack{tn \equiv 1 \ (Q) \\ n > 0}} \frac{1}{n^s} \tag{A.124}$$

where $\mu(n)$ is the Möbius function. The proof can be done as in the analytic case(see [5])). The non-primitive series $E(\tau, s; (a_1, a_2), Q)$ with $(a_1, a_2, Q) = d > 1$

$$E(\tau, s; (a_1, a_2), Q) = d^{-s}E\left(\tau, s; \left(\frac{a_1}{d}, \frac{a_2}{d}\right), \frac{Q}{d}\right) \tag{A.125}$$

can be written in terms of E^* of level Q/d and these can be written again in terms of E^* of level Q because

$$E^*(\tau, s; (a_1, a_2), Q) = \sum_{\substack{b_i \bmod QQ' \\ b_i \equiv a_i \ (Q)}} E^*(\tau, s; (b_1, b_2), QQ') \tag{A.126}$$

in general. The linear equivalence of the series E and E^*, which can be seen from the identities (122) and (123) which are analytic in s, can be lost for special values of s if the coefficients $d(s,t,Q)$ become singular. On the line $\operatorname{Re} s = 1$ this is the case for $s = 1$. Indeed, the maximum number of linearly independent series E^* is less than the maximum number of linearly independent series E as we can see from the following considerations. The regularity of the function $c(s,t,Q)$ on the line $\operatorname{Re} s = 1$ including $s = 1$ follows from

$$c(s,t,Q) = \frac{1}{\varphi(Q)} \sum_{\chi} \frac{\chi(t)}{L(s,\chi)} \tag{A.127}$$

where we sum over all characters χ mod Q and φ is the Euler function and from the fact that the L-series

$$L(s,\chi) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s}$$

does not vanish for $\operatorname{Re} s = 1$. On the line $\operatorname{Re} s = 1$ the functions

$$d(s,t,Q) = \frac{1}{\varphi(Q)} \sum_{\chi} L(s,\chi)\chi(t) \tag{A.128}$$

are singular only for $(t,Q) = 1$ at the point s. Now we compute the term $u^*(y,s;(a_1,a_2),Q)$, which is independent of x, in the Fouruer series of $E^*(\tau,s;(a_1,a_2),Q)$. By (115), we have

$$u^*(y,s;(a_1,a_2),Q) = \sum_{t \bmod Q} u(y,s;(ta_1,ta_2),Q)c(s,t,Q)$$
$$= \delta\left(\frac{a_1}{Q}\right) y^{\frac{s}{2}} \sum_{t \bmod Q} \zeta(s,ta_2,Q)c(s,t,Q) + \frac{\sqrt{\pi}}{Q}\frac{\Gamma\left(\frac{s-1}{2}\right)}{\Gamma\left(\frac{s}{2}\right)} y^{1-\frac{s}{2}} \sum_{t \bmod Q} \zeta(s-1,ta_1,Q)c(s,t,Q). \tag{A.129}$$

If we assume $(a_1,a_2,Q) = 1$ then we may assume that $(a_2,Q) = 1$ when we compute the coefficients of $y^{\frac{s}{2}}$ in (129) because this term appears only for

$a_1 \equiv 0\ (Q)$. However, for $(a,Q)=1$ we have

$$\sum_{t \bmod Q} \zeta(s,ta,Q)c(s,t,Q) = \sum_{t \bmod Q} \sum_{\substack{tn \equiv 1\ (Q) \\ n>0}} \sum_{m \equiv ta\ (Q)} \frac{\mu(n)}{\mid mn \mid^s}$$

$$= \sum_{\substack{n>0,\ m \\ n \mid m,\, m \equiv a\ (Q)}} \frac{\mu(n)}{\mid m \mid^s} = \delta\left(\frac{a-1}{Q}\right) + \delta\left(\frac{a+1}{Q}\right)$$

so that

$$u^*(y,s;(a_1,a_2),Q) = \delta\left(\frac{a_1}{Q}\right)\left(\delta\left(\frac{a_2-1}{Q}\right)+\delta\left(\frac{a_2+1}{Q}\right)\right)y^{\frac{s}{2}} + \eta(s,a_1,Q)y^{1-\frac{s}{2}} \tag{A.130}$$

with some function $n(s,a_1,Q)$ which is regualar on the line $\mathrm{Re}\, s = 1$. We notice that the function $y^{\frac{1}{2}}\log y$ does not appear in $u^*(y,1;(a_1,a_2),Q$. The maximum number of linearly independent series $E^*(\tau,1;(a_1,a_2),Q)$ is therefore less than or equal to for the series $E(\tau,1;(a_1,a_2),Q)$. Moreover, for $r>0$, it follows from (130) that $y^{\frac{1}{2}+ir}$ appears in $u^*(y,1+2ir;(a_1,a_2),Q)$ if and only if $a_1 \equiv 0$ and $a_2 \equiv 1$ or $-1 \bmod Q$.

Now let $r>0$, $(a_1,a_2)=1$ and $S = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ be a substitution of $\mathbf{M}$ so that $S^{-1}\infty$ is a given parabolic cusp of a fundamental domain of $\mathbf{M}(Q)$. Then $E^*(\tau,1+2ir;(a_1,a_2),Q)$ behaves at $\tau = S^{-1}\infty$ as

$$E^*(S^{-1}\tau,1+2ir;(a_1,a_2),Q) = E^*(\tau,1+2ir;(a_1,a_2)S^{-1},Q)$$

at $\tau=\infty$. In the Fourier series of $E^*(\tau,1+2ir;(a_1,a_2)S^{-1},Q)$, $y^{\frac{1}{2}+ir}$ appears if and only if

$$a_1 d - a_2 c \equiv 0, \quad -a_1 b + a_2 a \equiv \pm 1\ (Q),$$

i.e

$$a_1 \equiv \pm c, \quad a_2 \equiv \pm d \ (Q)$$

or equivalently, if the parabolic cusps $-\dfrac{a_2}{a_1}$ and $S^{-1}\infty = -\dfrac{d}{c}$ are equivalant in $\mathbf{M}(Q)$. Let $\sigma(Q)$ be the number of non-equivalent parabolic cusps of $\mathbf{M}(Q)$. In the case of $r > 0$ we prove that there are $\sigma(Q)$ linearly independent Eisenstein series E^*. It is known that the following formula holds:

$$\sigma(Q) = \begin{cases} 1 & \text{for} \quad Q = 1 \\ 3 & \text{for} \quad Q = 2 \\ \dfrac{Q^2}{2} \displaystyle\prod_{p|Q} \left(1 - \frac{1}{p^2}\right) & \text{for} \quad Q > 2. \end{cases} \tag{A.131}$$

Between the primitive series $E^*(\tau, 1 + 2ir; (a_1, a_2), Q)$,that are linearly equivalent to the collection of all Eisenstein series of level Q, there are only the following relations.

$$\begin{aligned} &E^*(\tau, 1 + 2ir; (a_1, a_2), Q) = E^*(\tau, 1 + 2ir; (b_1, b_2), Q) \\ &\text{for} \quad a_k \equiv b_k \ (Q) \quad \text{or} \quad a_k \equiv -b_k \ (Q) \qquad (k = 1, 2) \end{aligned} \tag{A.132}$$

Moreover, any linear combination of Eisenstein series vanishes identically if it is a cusp function. These facts follow immediately from the behaivour of the series E^* in the parabolic cusps if we notice in addition that two cusps $-\dfrac{a_2}{a_1}$ and $-\dfrac{b_2}{b_1}$ with relatively prime numerators and denominators are equivalent in $\mathbf{M}(Q)$ if and only if $a_k \equiv \pm b_k \ (Q) \ (k = 1, 2)$. Apparently, $\sigma(Q)$ is the maximal number of the linearly independent Eisenstein series .

The case $r = 0$ is much more complicated. If $\varphi(Q) = 1$ or 2, then there are $\sigma(Q)$ linearly independent series among the primitive Eisenstein series $E(\tau, 1; (a_1, a_2), Q)$ because by (119) $y^{\frac{1}{2}} \log y$ appears in the Fouruer series of $E(\tau, 1; (a_1, a_2), Q)$ if and only if $a_1 \equiv 0, \ a_2 \equiv \pm 1 \ (Q)$. Then the same considerations as in the case of $r > 0$

holds. Later we will see that for $r = 0$ and $\varphi(Q) > 2$ that is for $Q \neq 1, 2, 3, 4, 6$ the maximum number of linearly independent Eisenstein series is less than $\sigma(Q)$.

Now we formulate this subresult.

Theorem 34. *Let $r > 0$, Q arbitrary or $r = 0$, $\varphi(Q) \leq 2$(i.e. $Q = 1, 2, 3, 4, 6$). Then the maximal number of linearly independent Eisenstein series $E(\tau, 1 + 2ir; (a_1, a_2), Q)$ is equal to $\sigma(Q)$. Furthermore, in the linear collection of Eisenstein series, there are no cusp functions that do not vanish identically.*

To find the linear relations between the Eisenstein series in the case $r = 0$, it is useful to introduce the series

$$G(\tau, s; a_1, a_2, Q) = \frac{1}{Q} \sum_{b \bmod Q} \zeta^{-a_2 b} E(\tau, s; (a_1, b), Q) \tag{A.133}$$

which are linaerly equivalent to the Eisenstein series and have important symmetry properties for $s = 1$(see)[4]. By (112),

$$G(\tau, s; a_1, a_2, Q) = \delta\left(\frac{a_1}{Q}\right) \frac{1}{Q} y^{\frac{s}{2}} \sum_{n \neq 0} \frac{\zeta^{-a_2 n}}{\mid n \mid^s} + \delta\left(\frac{a_2}{Q}\right) \frac{\sqrt{\pi}}{Q} \frac{\Gamma\left(\frac{s-1}{2}\right)}{\Gamma\left(\frac{s}{2}\right)} \zeta(s-1, a_1, Q) y^{1-\frac{s}{2}}$$

$$+ \frac{2\pi^{\frac{s}{2}}}{Q^{\frac{s+1}{2}} \Gamma\left(\frac{s}{2}\right)} \sum_{n \neq 0} \left\{ \sum_{\substack{d_i \equiv a_i \ (Q) \\ d_1 d_2 = n}} \mid d_1 \mid^{\frac{1-s}{2}} \mid d_2 \mid^{\frac{s-1}{2}} \right\} y^{\frac{1}{2}} K_{\frac{s-1}{2}}\left(\frac{2\pi \mid n \mid}{Q} y\right) e^{\frac{2\pi i n}{Q} x}. \tag{A.134}$$

Using (119), we obtain

$$G(\tau, 1; a_1, a_2, Q) = \frac{2}{Q} \sum_{n \neq 0} \left\{ \sum_{\substack{d_i \equiv a_i \ (Q) \\ d_1 d_2 = n}} 1 \right\} y^{\frac{1}{2}} K_0\left(\frac{2\pi \mid n \mid}{Q} y\right) e^{\frac{2\pi i n}{Q} x} \tag{A.135}$$

$$+\begin{cases} \frac{2}{Q}y^{\frac{1}{2}}\log y + \frac{2}{Q}\left(C + \log\frac{Q}{4\pi}\right)y^{\frac{1}{2}} & \text{for } a_1 \equiv 0,\ a_2 \equiv 0\ (Q), \\ -\frac{2}{Q}\log\left(2\left|\sin\frac{a_2\pi}{Q}\right|\right)y^{\frac{1}{2}} & \text{for } a_1 \equiv 0,\ a_2 \not\equiv 0\ (Q), \\ -\frac{2}{Q}\log\left(2\left|\sin\frac{a_1\pi}{Q}\right|\right)y^{\frac{1}{2}} & \text{for } a_1 \not\equiv 0,\ a_2 \equiv 0\ (Q), \\ 0 & \text{for } a_1 \not\equiv 0,\ a_2 \not\equiv 0\ (Q) \end{cases}$$

which leads to the symmetry relations

$$\begin{aligned} G(\tau, 1; a_1, a_2, Q) &= G(\tau, 1; a_2, a_1, Q) \\ G(\tau, 1; a_1, a_2, Q) &= G(\tau, 1; -a_1, -a_2, Q). \end{aligned} \tag{A.136}$$

In order to capture all G-series that differ from each other, it is sufficient to restrict ourselves to

$$a_1 = 0,\ 0 \le a_2 \le \left[\frac{Q}{2}\right] \text{ and } a_1 = k,\ k \le a_2 \le Q - k\ \left(k = 1, 2, ..., \left[\frac{Q}{2}\right]\right). \tag{A.137}$$

A simple counting argument for the number $A(Q)$ of the specified pairs (a_1, a_2) gives the value

$$A(Q) = 1 + \left[\frac{Q}{2}\right]\left[\frac{Q+3}{3}\right]. \tag{A.138}$$

$A(Q)$ is an upper bound for the maximal number of linearly independent Eisenstein series. Due to

$$\begin{aligned} A(Q) &= \begin{cases} \sigma(Q) & \text{for } Q = 1, 2, 3 \\ \sigma(Q) + 1 & \text{for } Q = 4, 6 \end{cases} \\ A(Q) &< \sigma(Q) \quad \text{for } Q \ne 1, 2, 3, 4, 6 \end{aligned} \tag{A.139}$$

as one can easily see, by Theorem 8 there are no more relations besides (136) in the case $Q = 1, 2, 3$. For $Q = 4, 6$ there is exactly one relation (in each case $Q = 4$ and $Q = 6$) that does not appear in the symmetry relations. There are many such additional relations, in general, unless Q is a prime number. Because for any two positive integers t_1 and t_2 we have

$$\sum_{\nu \bmod t_1} G(\tau, 1; a_1 + t_2\nu, t_1 a_2, t_1 t_2) = \sum_{\nu \bmod t_1} G(\tau, 1; t_1 a_1, a_2 + t_2\nu, t_1 t_2). \quad \text{(A.140)}$$

The proof follows easily by comparing the coefficients in the Fourier series of each side of the equation (140). Let $A_n(a_1, a_2)$ be the Fourier coefficient of $\frac{Q}{2} G(\tau, 1; a_1, a_2, Q)$ of the exponent $n \neq 0$. By (135), $A_n(a_1, a_2)$ is the number that solves the system

$$d_1 \equiv a_1,\ d_2 \equiv a_2\ (Q),\ d_1 d_2 = n.$$

Therefore, the relation (140) is equivalent to

$$\sum_{\nu \bmod t_1} A_n(a_1 + t_2\nu, t_1 a_2) = \sum_{\nu \bmod t_1} A_n(t_1 a_1, a_2 + t_2\nu) \quad (Q = t_1 t_2)$$

The left hand side of the above equation is the number which solves the system

$$d_1 \equiv a_1\ (t_2),\ d_2 \equiv t_1 a_2\ (t_1 t_2),\ d_1 d_2 = n$$

and the right hand side is the solution of

$$d_1 \equiv t_1 a_1\ (t_1 t_2),\ d_2 \equiv a_2\ (t_2),\ d_1 d_2 = n.$$

These numbers are clearly the same which proves (140). Let $t_1 t_2 = Q$, then relation (140) is independent of the symmetry relations unless $t_1 = 1$ or $a_2 \equiv a_1\ (t_2)$. The proof of this is simple and omitted. Especially for $t_1 = t_2 = 2,\ a_1 = 0,\ a_2 = 1$ and $t_1 = 2,\ t_2 = 3,\ a_1 = 0, a_2 = 1$ we obtain

$$\begin{aligned} 2G(\tau, 1; 0, 1, 4) &= G(\tau, 1; 0, 2, 4) + G(\tau, 1; 2, 2, 4) \\ G(\tau, 1; 0, 1, 6) &= G(\tau, 1; 2, 3, 6). \end{aligned} \quad \text{(A.141)}$$

In general, for each composed level $Q = t_1 t_2$, there is at least one relation among the relations in (140) which is independent of the symmetry relations. On the other hand for any prime level $Q = q$, we have a complete system of relations in (136). At most one of the four series, that comes from $G(\tau, 1; a_1, a_2, Q)$ through replacing a_1, a_2 by $\pm a_1, \pm a_2$ or $\pm a_2, \pm a_1$, appears in the relation. To simplify notation we set $[a_1, a_2] = G(\tau, 1; a_1, a_2, Q)$. Let p be an arbitrary prime number. The Fourier coefficient of the exponent p is clearly different from zero only for the series $[1, p]$. So $[1, p]$ must not appear in the relation. By the Dirichlet Theorem about prime numbers in arithmetic progressions, p can represent an arbitrary coset that is relatively prime to q. Of course, $p = q$ is allowed. This explains that all series $[1, a]$(a arbitrary) do not appeaar in the relation. The Fourier coefficient of the exponent pp' is different than zero only for the series $[1, pp']$ if p and p' are arbitrary prime numbers. Because $[1, pp']$ does not appear in the relation, $[pp']$ cannot appear either. For a choice of p and p', $[p, p']$ is a given G-series. Our relation is empty, hence the completeness of the system of relations (136) is proved for $Q = q$. Therefore, Theorem 9 holds.

Theorem 35. *The maximal number of linearly independent Einsenstein series $E(\tau, 1; (a_1, a_2), Q)$ is equal to $\sigma(Q)$ for $Q = 1, 2, 3, 4, 6$ and less than $\sigma(Q)$ for all other Q, in particular equal to $\left(\frac{Q+1}{2}\right)^2$ for the prime level $Q \geq 3$ and less than or eqaul to $\left[\frac{Q}{2}\right]\left[\frac{Q+3}{2}\right]$ for all composed levels $Q \geq 4$.*

The question, whether a given function of level Q can be reduced to a cusp function using the Eisenstein series, has the answer "yes" under the assumptions of Theorem 8. The proof of this so called Reduction Theorem is based on Theorem 10.

Theorem 36. *Let $\mathbf{G}$ be a subgroup of the modular group $\mathbf{M}$ of finite index. Let the substitutions $A_\varrho \in \mathbf{M}$ ($\varrho = 1, 2, ...\sigma$) be such that $A_1^{-1}\infty, A_2^{-1}\infty, ..., A_\sigma^{-1}\infty$ is a complete system of nonequivalent parabolic cusps of a fundamental domain of $\mathbf{G}$.*

To each wave function $g(\tau)$, which is invariant under the substitutions of $\mathbf{G}$*, with* σ *power series*

$$g(A_{\varrho}^{-1}\tau) = u_{\varrho}(y) + \sum_{n\neq 0} a_{\varrho}(n) y^{\frac{1}{2}} K_{ir}\left(\frac{2\pi \mid n \mid}{Q_{\varrho}} y\right) e^{\frac{2\pi i n}{Q_{\varrho}} x} \tag{A.142}$$

we assign the vector $\mathfrak{u} = \{u_1(y), u_2(y), ..., u_{\varrho}(y)\}$*. In the linear collection of these vectors, there are at most* σ *vectors that are linearly independent.*

Proof. We assume a second automorphic function $h(\tau)$ with the power series

$$h(A_{\varrho}^{-1}\tau) = v_{\varrho}(y) + \sum_{n\neq 0} b_{\varrho}(n) y^{\frac{1}{2}} K_{ir}\left(\frac{2\pi \mid n \mid}{Q_{\varrho}} y\right) e^{\frac{2\pi i n}{Q_{\varrho}} x} \tag{A.143}$$

is given, and we apply the Green Theorem

$$\iint_{\mathfrak{B}} (V(\tau)\triangle U(\tau) - U(\tau)\triangle V(\tau)) dxdy = \int_{\mathfrak{R}} \left(V(\tau)\frac{\partial U(tau)}{\partial n} - U(\tau)\frac{\partial V(\tau)}{\partial n}\right) ds, \tag{A.144}$$

where $\mathfrak{R}$ is the boundary of a domain $\mathfrak{B}$ and n is the outside normal vector on the boundary of $\mathfrak{B}$. Especially, let $\mathfrak{B}$ be the domain that originates from the fundamental domain (71), if we cut the σ parabolic cusps along certain cross section. In $\mathfrak{P}_{\varrho}$, choose such a cross section that transforms by A_{ϱ} into the line $\mid x \mid \leq \frac{1}{2}Q_{\varrho},\ y = y_{\varrho}(\geq \varkappa)$. Since $g(\tau)$ and $h(\tau)$ are wave functions, the surface integral vanishes in (144). In the contour integral, only the contributions to the cross section is left since the boundary $\mathfrak{R}$ consists(if we neglect the cross sections) of pairwise equivalent pieces and the cross sections apparently cancel each other out. If we do a transformation of variables $\tau \rightarrow A_{\varrho}^{-1}\tau$ in the integral over the ϱth cross section, then we obtain

$$0 = \sum_{\varrho=1}^{\sigma} \int_{-\frac{1}{2}Q_{\varrho}}^{\frac{1}{2}Q_{\varrho}} \left(h(A_{\varrho}^{-1}\tau)\frac{\partial g(A_{\varrho}^{-1}\tau)}{\partial y} - g(A_{\varrho}^{-1}\tau)\frac{\partial h(A_{\varrho}^{-1}\tau)}{\partial y}\right)_{y=y_{\varrho}} dx$$

If we plug the Fourier series (142) and (143) into th the previous equation, then the integral reletion simplifies significantly. It becomes

$$0 = \sum_{\varrho=1}^{\sigma} \int_{-\frac{1}{2}Q_\varrho}^{\frac{1}{2}Q_\varrho} \left(v_\varrho(y_\varrho) \frac{du_\varrho(y_\varrho)}{dy_\varrho} - u_\varrho(y_\varrho) \frac{dv_\varrho(y_\varrho)}{dy_\varrho} \right) dx,$$

that is

$$0 = \sum_{\varrho=1}^{\sigma} Q_\varrho \left(v_\varrho(y_\varrho) \frac{du_\varrho(y_\varrho)}{dy_\varrho} - u_\varrho(y_\varrho) \frac{dv_\varrho(y_\varrho)}{dy_\varrho} \right).$$

Since the positions of the cross sections i.e. the coordinates $\frac{1}{\varrho}$, are independent of each other

$$c_\varrho = v_\varrho(y_\varrho) \frac{du_\varrho(y)}{dy} - u_\varrho(y_\varrho) \frac{dv_\varrho(y)}{dy} \tag{A.145}$$

must be constant. The c_ϱ satisfies the relation

$$\sum_{\varrho=1}^{\sigma} Q_\varrho c_\varrho = 0. \tag{A.146}$$

The fact that c_ϱ's are constant also follows immediately from the fact that $u_{\varrho(y)}$ and $v_\varrho(y)$ satisfy the differential equation

$$\frac{d^2 w}{dy^2} + \frac{r^2 + \frac{1}{4}}{y^2} w = 0.$$

From the representation

$$u_\varrho(y) = \begin{cases} a'_\varrho y^{\frac{1}{2}+ir} + a''_\varrho y^{\frac{1}{2}-ir} & \text{for } r > 0, \\ a'_\varrho y^{\frac{1}{2}} \log y + a''_\varrho y^{\frac{1}{2}} & \text{for } r = 0 \end{cases}$$

$$\tag{A.147}$$

$$v_\varrho(y) = \begin{cases} b'_\varrho y^{\frac{1}{2}+ir} + b''_\varrho y^{\frac{1}{2}-ir} & \text{for } r > 0, \\ b'_\varrho y^{\frac{1}{2}} \log y + b''_\varrho y^{\frac{1}{2}} & \text{for } r = 0 \end{cases}$$

it follows indeed that

$$c_\varrho = \begin{cases} 2ir(a'_\varrho b''_\varrho - a''_\varrho b'_\varrho) & \text{for } r > 0, \\ a'_\varrho b''_\varrho - a''_\varrho b'_\varrho & \text{for } r = 0. \end{cases} \tag{A.148}$$

So in general,

$$\sum_{\varrho=1}^{\sigma} Q_\varrho a'_\varrho b''_\varrho - a''_\varrho b'_\varrho) = 0 \tag{A.149}$$

holds. Denoting the column vectors with components

$$a'_1, a'_2, ..., a'_\varrho, a''_1, a''_2, ..., a''_\varrho$$

and

$$b'_1, b'_2, ..., b'_\varrho, b''_1, b''_2, ..., b''_\varrho$$

by $\mathfrak{a}$ and $\mathfrak{b}$, respectively, and introducing the matrix

$$X = \begin{pmatrix} & & & & Q_1 & & & \\ & & & & & Q_2 & & \\ & & & & & & \ddots & \\ & & & & & & & Q_\varrho \\ -Q_1 & & & & & & & \\ & -Q_2 & & & & & & \\ & & \ddots & & & & & \\ & & & -Q_\varrho & & & & \end{pmatrix} \tag{A.150}$$

where one has to put zeroes into the empty spaces, the bilinear relation (149) becomes

$$\mathfrak{a}' X \mathfrak{b} = 0 \tag{A.151}$$

Now let $g_1(\tau), g_2(\tau), ..., g_\mu(\tau)$ be a maximal system of automorphic wave functions for which the associated constant vectors $\mathfrak{a}_1, \mathfrak{a}_2, ..., \mathfrak{a}_\mu$ are linearly independent, then all $\mathfrak{a}_\varrho$'s solve the system of equations

$$\mathfrak{a}'_\iota X \xi = 0 \quad (\iota = 1, 2, ..., \sigma). \tag{A.152}$$

The rank of this system is μ, on the one hand side. On the other hand, the dimension of the space of solutions is at least μ, so that $\mu \leq 2\sigma - \mu$ or $\mu \leq \sigma$. This proves Theorem 10. □

Under the assumption of Theorem 8, there are $\sigma(Q)$ linearly independent Eisenstein series $E_1, E_2, ..., E_{\sigma(Q)}$. The vectors $\mathfrak{u}_1, \mathfrak{u}_2, ..., \mathfrak{u}_{\sigma(Q)}$ that are associated with the Eisenstein series in the sense of Theorem 10 are linearly independent, since the linear collection of the E_ϱ $(\varrho = 1, 2, ..., \sigma(Q))$ contains no cusp function that vanishes identically. By Theorem 10, the vectors u_ϱ $(\varrho = 1, 2, ..., \sigma(Q))$ form a basis of the linear collection of the possible vectors of this kind. This implies the following reduction theorem:

Theorem 37. *Let either $r > 0, Q$ arbitrary, or $r = 0, \varphi(Q) \leq 2$ (i.e. $Q = 1, 2, 3, 4$ or 6). Then for every wave function $g(\tau)$ that is invariant under $\mathbf{M}(Q)$ and has a Fourier series of type* (142) *in the parabolic cusps, there is a linear combination $\Lambda(\tau)$ of Eisenstein series of level Q, so that $g(\tau) - \Lambda(\tau)$ is a cusp function.*

For $r = 0$ we can determine the automorphic wave functions for the modular group and for the Theta group immediately, because by Theorem 6 there are no cusp functions for these groups and $r = 0$. Due to Theorem 11, every automorphic function of level 1 coincides with $E(\tau, 1; (0,0), 1)$ up to a constant factor. Nowe let $g(\tau)$ be a wave function for $r = 0$ that is invariant under $\mathbf{T}$. Because $\mathbf{M}(2) \subset \mathbf{T}$, there exists by Theorem 11, a linear combination $\Lambda(\tau)$ of Eisenstein series of level 2 , so that $f(\tau) = g(\tau) - \Lambda(\tau)$ is a cusp function of level 2. Since $\mathbf{T}$ admits the decomposition

$$\mathbf{T} = \mathbf{M}(2) + \mathbf{M}(2) \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

the cusp function $f(\tau) + f\left(-\frac{1}{\tau}\right)$ is invariant under $\mathbf{T}$, hence identically 0. It follows that $g(\tau) = \frac{1}{2}\left(\Lambda(\tau) + \Lambda\left(-\frac{1}{\tau}\right)\right)$. The automorphic function for $\mathbf{T}$ hence consists of linear combinations of the series sums

$$E(\tau,1;(a_1,a_2),2)+E(\tau,1;(a_2,a_1),2),$$

where $(a_1,a_2,2)=1$ may be assumed. The linearly independent functions

$$E_1(\tau)=E(\tau,1;(0,1),2)+E(\tau,1;(1,0),2),\quad E_2(\tau)=E(\tau,1;(1,1),2) \tag{A.153}$$

form therefore a basis of the linear collection of all $g(\tau)$. Let $d(n)$ be the number of positive divisors of n. Then by (112) and (119), one finds the series

$$\begin{aligned} E_1(\tau)+E_2(\tau) &= \frac{1}{2}E(\tau,1;(0,0),1)=y^{\frac{1}{2}}\log y+(C-\log 4\pi)y^{\frac{1}{2}} \\ &\quad +2\sum_{n\neq 0} d(n)y^{\frac{1}{2}}K_0(\pi\mid n\mid y)e^{2\pi inx} \end{aligned} \tag{A.154}$$

$$\begin{aligned} 3E_1(\tau)+2E_2(\tau) &= 3y^{\frac{1}{2}}\log y+(3(C-\log 4\pi)+\log 2)y^{\frac{1}{2}} \\ &\quad +2\sum_{n\neq 0} d(n)y^{\frac{1}{2}}K_0(\pi\mid n\mid y)e^{\pi inx}+4\sum_{n\neq 0} d(n)y^{\frac{1}{2}}K_0(\pi\mid n\mid y)e^{4\pi inx}. \end{aligned}$$

The φ-series that are assigned to the functions $E(\tau,1;(0,0),1)$, $E_1(\tau)+E_2(\tau)$, $3E_1(\tau)+2E_2(\tau)$ by the rules (32) and (33) are

$$8\zeta^2(s),\quad 4(2^{-s})\zeta^2(s),\quad 4(1+2^{1-2s})\zeta^2(s), \tag{A.155}$$

while all ψ-series vanish. This proves Theorem 2. Before we go into the special conditions of the case $Q=5,\ r=0$, we make some general considerations. Between the primitive Eisenstein series E and E^* of level Q there are by (122) and (127), the identities

$$\sum_{\substack{t \bmod Q \\ (t,Q)=1}} E^*(\tau,s;(ta_1,ta_2),Q)=\frac{1}{L(s,\chi_1)}\sum_{\substack{t \bmod Q \\ (t,Q)=1}} E^*(\tau,s;(ta_1,ta_2),Q) \tag{A.156}$$

where χ_1 denotes the unit character mod Q. For $s = 1$ we get

$$\sum_{\substack{t \bmod Q \\ (t,Q)=1}} E^*(\tau, 1; (ta_1, ta_2), Q) = 0 \tag{A.157}$$

We say that two parabolic cusps $-\dfrac{a_2}{a_1}$ and $-\dfrac{b_2}{b_1}$, with relatively prime numerators and denominators, are mod Q associated, if there is a number k that is relatively prime to Q so that

$$a_i \equiv kb_i \ (Q) \text{ for } i = 1, 2 \tag{A.158}$$

holds. The number of parabolic cusps, which are not equivalent with respect to $\mathbf{M}(Q)$, in a class of mod Q associated cusps is apparently equal to 1 for $Q = 1, 2$ and $\dfrac{1}{2}\varphi(Q)$ for $Q > 2$, so that for the number of classes of mod Q non-associated parabolic cusps, we get the expression

$$\sigma_0(Q) = Q \prod_{p|Q} \left(1 + \frac{1}{p}\right) \tag{A.159}$$

by (131). Let A and B be two substitutions in $\mathbf{M}$ with second rows (a_1, a_2) and (b_1, b_2). A_∞^{-1} and B_∞^{-1} are mod Q associated if and only if (158) is satisfied with $(k, Q) = 1$. In the Fourier series of the primitive series $E(\tau, 1; (a_1, a_2), Q)$ with $(a_1, a_2) = 1$ for the cusp B_∞^{-1}, $B \in \mathbf{M}$ the function $y^{\frac{1}{2}} \log y$ appears if and only if B_∞^{-1} and $-\dfrac{a_2}{a_1}$ are mod Q are assoiciated, because $E(\tau, 1; (a_1, a_2), Q)$ behaves, as $\tau \to B_\infty^{-1}$, like $E(\tau, 1; (a_1, a_2)B^{-1}, Q)$ as $\tau \to \infty$ and in the Fourier series for the cusp ∞, $y^{\frac{1}{2}} \log y$ appears if and only if $(a_1, a_2B^{-1}) \equiv (0, k)$ (Q) or $(a_1, a_2) \equiv (0, k)B$ (Q) holds with $(k, Q) = 1$. Now we determine $\sigma_0(Q)$ substitutions $B_\varrho \in \mathbf{M}$ in such a way that $B_\varrho^{-1}\infty$ $(\varrho = 1, 2, ..., \sigma_0(Q))$ are not mod Q associated. If we denote the second row of B_ϱ by $\underline{B}_\varrho$, then the series

$$E(\tau, 1; \underline{B}_\varrho, Q) \quad (\varrho = 1, 2, ..., \sigma_0(Q)) \tag{A.160}$$

are linearly independent, as one can see from their behaviour in the parabolic cusps. Among the $\sigma(Q)$ series

$$E^*(\tau, 1; k\underline{B}_\varrho, Q) \quad \left(k = 1 \text{ for } Q = 1, 2 \text{ and } 1 \leq k < \frac{Q}{2},\ (k, Q) = 1 \text{ for } Q > 2\right)$$

there are at most $\sigma(Q) - \sigma_0(Q)$ which are linearly independent, by the relation (157). Since the function $y^{\frac{1}{2}} \log y$ does not appear in the Fourier series of E^*, the E^*'s are linearly independent of the series (160).

We now restrict our attention to level $Q = 5$. Then we have $\sigma = 12$ and $\sigma_0 = 6$. The substitutions

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ 1 & 3 \end{pmatrix}, \begin{pmatrix} 1 & 3 \\ 1 & 4 \end{pmatrix}, \tag{A.161}$$

are denoted in the same order by $B_1, ..., B_6$. Such a choice is allowed because indeed the parabolic cusps $B_\varrho^{-1}\infty$ are not associated in mod 5. The system of relations (157) takes the form

$$E^*(\tau, 1; \underline{B}_\varrho, 5) + E^*(\tau, 1; 2\underline{B}_\varrho, 5) \quad (\varrho = 1, 2, ..., 6). \tag{A.162}$$

Since there are exactly nine linearly independent Eisenstein series for level 5, and since the six series in (160) are linearly independent, among the functions

$$F_\varrho(\tau) = E^*(\tau, 1; \underline{B}_\varrho, 5) \quad (\varrho = 1, 2, ..., 6) \tag{A.163}$$

we can have at most three that are linearly independent. The determination of the linear relations between the $F_\varrho(\tau)$ can be done in the following way. We write these functions in terms of G-series and notice that the system of relations (136) for the G-series of the prime level is complete. To do this computation, we need the values

$$c(1, k, 5) + c(1, -k, 5) = \left(\frac{k}{5}\right) \frac{1}{2L\left(1, \left(\frac{x}{5}\right)\right)} = \left(\frac{k}{5}\right) \frac{\sqrt{5}}{4 \log \varepsilon} \quad \left(\varepsilon = \frac{1 + \sqrt{5}}{2}\right), \tag{A.164}$$

which we plug into the representation

$$E^*(\tau,1;\underline{B}_\varrho,5)=E(\tau,1;\underline{B}_\varrho,5)\left(c(1,1,5)+c(1,-1,5)\right)$$
$$+E(\tau,1;2\underline{B}_\varrho,5)\left(c(1,2,5)+c(1,-2,5)\right),$$

so that we get

$$F_\varrho(\tau)=\frac{\sqrt{5}}{4\log\varepsilon}\left(E(\tau,1;\underline{B}_\varrho,5)-E(\tau,1;2\underline{B}_\varrho,5)\right). \tag{A.165}$$

Finally we notice that the functions $F_1(\tau),F_2(\tau),F_3(\tau)$ are linearly independent and the relations

$$\begin{aligned}F_4(\tau)&=\frac{\sqrt{5}-1}{2}(F_1(\tau)+F_3(\tau))-F_2(\tau)\\F_5(\tau)&=\frac{\sqrt{5}-1}{2}(F_2(\tau)+F_3(\tau))+F_2(\tau)\\F_6(\tau)&=\frac{\sqrt{5}-1}{2}(F_1(\tau)+F_2(\tau))-F_3(\tau)\end{aligned} \tag{A.166}$$

hold. Due to the translation formulas (107), which also hold for the E^*-series, we get

$$F_1(\tau+1)=F_1(\tau),\ F_2(\tau+1)=F_3(\tau),\ F_3(\tau+1)=F_4(\tau),$$
$$F_1(-\frac{1}{\tau})=F_2(\tau),\ F_2(-\frac{1}{\tau})=F_1(\tau),\ F_3(-\frac{1}{\tau})=F_6(\tau)$$

Hence, by (166),

$$(F_\varrho(\tau+1))=\begin{pmatrix}1&0&0\\0&0&1\\-\varepsilon'&-1&-\varepsilon'\end{pmatrix}(F_\varrho(\tau)),\ \left(F_\varrho\left(-\frac{1}{\tau}\right)\right)=\begin{pmatrix}0&1&0\\1&0&0\\-\varepsilon'&-\varepsilon'&-1\end{pmatrix}(F_\varrho(\tau)), \tag{A.167}$$

with $\varepsilon'=\dfrac{1-\sqrt{5}}{2}$ and $\varrho=1,2,3$.The eigenfunctions of the linear collection of the $F_\varrho(\tau)$ for the substitution $\tau\to\tau+1$ coincide up to constant factors with the functions

$$(G_\varrho(\tau)) = \begin{pmatrix} 2(\zeta - \zeta^{-1}) & 0 & 0 \\ \zeta^{-1} - 1 & -\sqrt{5}\zeta^{-1} & \sqrt{5} \\ 1 - \zeta & \sqrt{5}\zeta & -\sqrt{5} \end{pmatrix} (F_\varrho(\tau)) \quad \left(\zeta = e^{\frac{2\pi i}{5}}\right) \tag{A.168}$$

and transform by (167) according to the formulas

$$(G_\varrho(\tau+1)) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \zeta & 0 \\ 0 & 0 & \zeta^{-1} \end{pmatrix} (G_\varrho(\tau)), \ \left(G_\varrho\left(-\frac{1}{\tau}\right)\right) = \frac{1}{\sqrt{5}} \begin{pmatrix} 1 & 2 & 2 \\ 1 & \zeta^2 + \zeta^{-2} & \zeta^1 + \zeta^{-1} \\ 1 & \zeta^1 + \zeta^{-1} & \zeta^2 + \zeta^{-2} \end{pmatrix} (G_\varrho(\tau)) \tag{A.169}$$

The wave functions $g(\tau, \varrho, (1), 1, \sqrt{5})$ $(\varrho = 0, 1, 2)$ for the system of the zeta functions $\zeta_\nu(\tau, \varrho, (1), 1, \sqrt{5})$ $(\nu = 0, 1; \varrho = 0, 1, 2)$ also satisfy the functional equations and one conjectures that they coincide with $G_\varrho(\tau)$ $(\varrho = 1, 2, 3)$ up to a common constant factor. Indeed, this is true and we will prove it now. The translation formulas for the system of functions

$$G_1^*(\tau) = \frac{1}{\sqrt{2}} G_1(\tau), \ G_2^*(\tau) = G_2(\tau), \ G_3^*(\tau) = G_3(\tau) \tag{A.170}$$

read

$$\big(G_\varrho^*(\tau+1)\big) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \zeta & 0 \\ 0 & 0 & \zeta^{-1} \end{pmatrix} (G_\varrho(\tau)), \ \left(G_\varrho^*\left(-\frac{1}{\tau}\right)\right) = \frac{1}{\sqrt{5}} \begin{pmatrix} 1 & \sqrt{2} & \sqrt{2} \\ \sqrt{2} & \zeta^2 + \zeta^{-2} & \zeta^1 + \zeta^{-1} \\ \sqrt{2} & \zeta^1 + \zeta^{-1} & \zeta^2 + \zeta^{-2} \end{pmatrix} \big(G_\varrho^*(\tau)\big) \tag{A.171}$$

and define apparently a unitary representation

$$(G_\varrho^*(S\tau)) = A_S(G_\varrho^*(\tau), \ \overline{A_s} A_S' = E \text{ for } S \in \mathbf{M} \tag{A.172}$$

of the modular group $\mathbf{M}/\mathbf{M}(5)$. Now let $g_1(\tau), g_2(\tau), g_3(\tau)$ be any system of solutions of the functional equations (171). The column vector $\mathfrak{g}(\tau)$ with components $g_{\varrho(\tau)}$ satisfies the translation formulas

$$\mathfrak{g}(S\tau) = A_S \mathfrak{g}(\tau) \text{ for } S \in \mathbf{M} \tag{A.173}$$

and admits a sereis representation of the form

$$\mathfrak{g}(\tau) = \mathfrak{u}(y) + \sum_{n\neq 0} \mathfrak{a}_n y^{\frac{1}{2}} K_0\left(\frac{2\pi \mid n \mid}{5} y\right) e^{\frac{2\pi i n}{5}x}. \tag{A.174}$$

Let column vector $\mathfrak{h}(\tau)$ with series

$$\mathfrak{h}(\tau) = \mathfrak{v}(y) + \sum_{n\neq 0} \mathfrak{b}_n y^{\frac{1}{2}} K_0\left(\frac{2\pi \mid n \mid}{5} y\right) e^{\frac{2\pi i n}{5}x}. \tag{A.175}$$

be another solution of (173). Since the representation matrices A_S are unitary, $\overline{\mathfrak{g}}'(\tau)d\mathfrak{h}(\tau)$ is invariant under $S \in \mathbf{M}$. In order to verify that $\mathfrak{u}(y)$ and $\mathfrak{v}(y)$ differ by only a constant factor, we use the Green theorem (144) in the form

$$\iint_{\mathfrak{B}} (\overline{\mathfrak{g}}'\triangle\mathfrak{h} - \mathfrak{h}'\triangle\overline{\mathfrak{g}})dxdy = \int_{\mathfrak{R}} \left(\overline{\mathfrak{g}}'\frac{\partial\mathfrak{h}}{\partial n} - \mathfrak{h}'\frac{\partial\overline{\mathfrak{g}}}{\partial n}\right) ds. \tag{A.176}$$

We form $\mathfrak{B}$ out of the modul figure by cutting off the parabolic cusp so that $\mathfrak{B}$ is described by the inequalities $(x^2+y^2) \geq 1,\ \mid x \mid \leq \frac{1}{2},\ y \leq y_0$. The surface integral vanishes, since the components of $\mathfrak{g}$ and $\mathfrak{h}$ are wave functions with equal numbers of waves. In the countor integral over $\mathfrak{R}$ the subintegrals over equivalent pieces of the boundary single out, due to the invariance property of $\overline{\mathfrak{g}}'d\mathfrak{h}$ that we mentioned above. Thus, it remains

$$\int_0^1 \left(\overline{\mathfrak{g}}'\frac{\partial\mathfrak{h}}{\partial y} - \mathfrak{h}'\frac{\partial\overline{\mathfrak{g}}}{\partial y}\right) dx = 0.$$

Taking into account the Fourier series (174) and (175), we obtain

$$\overline{\mathfrak{u}}'\frac{d\mathfrak{v}}{dy} - \mathfrak{v}'\frac{d\overline{\mathfrak{u}}}{dy} = 0. \tag{A.177}$$

Because of the two diagonal elements in the substitution A_S for $S = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, which are different from 1, we may assume a priori that

$$\mathfrak{u}(y) = \begin{pmatrix} u_1(y) \\ 0 \\ 0 \end{pmatrix},\ \text{and}\ \mathfrak{v}(y) = \begin{pmatrix} v_1(y) \\ 0 \\ 0 \end{pmatrix},$$

so that (177) becomes the condition

$$\overline{u_1(y)}\frac{dv_1(y)}{dy} - v_1(y)\frac{d\overline{u_1(y)}}{dy} = 0. \tag{A.178}$$

Under the condition $u_1(y) \neq 0$, we have $\left(\frac{v_1(y)}{\overline{u_1(y)}}\right)$ is constant. In particular, for $\mathfrak{g}(\tau) = \mathfrak{h}(\tau)$, this means that $\left(\frac{u_1(y)}{\overline{u_1(y)}}\right)$ is constant, hence $\left(\frac{v_1(y)}{u_1(y)}\right)$ is also constant, .q.e.d. By Theorem 7, $\mathfrak{g}(\tau)$ is determined unambigously as a solution of the functional equation (173) by $\mathfrak{u}(y)$ and is determined therefore generally up to a constant factor. An analogous result holds for $G_\varrho(\tau)$ $(\varrho = 1, 2, 3)$. This proves Teorem 3. In particular, we obtain the representation

$$\begin{aligned} g(\tau, 0, (1), 1, \sqrt{5}) &= C_0 G_1(\tau), \\ g(\tau, 1, (1), 1, \sqrt{5}) &= C_0 G_2(\tau), \\ g(\tau, 2, (1), 1, \sqrt{5}) &= C_0 G_3(\tau), \end{aligned} \tag{A.179}$$

with some constant C_0. Comparing the Fourier series of the functions(to each other), we see that C_0 has the value

$$C_0 = \frac{4\log\varepsilon}{\zeta - \zeta^{-1}}. \tag{A.180}$$

If we check the identities (179) explicitly in this way, then we obtain a new proof of the functional equation of the system of functions $g(\tau, \varrho, (1), 1, \sqrt{5})$, $(\varrho = 0, 1, 2,)$. The new proof does not use the theta series in two variables, but the relation

$$\sum_{N\mathfrak{a}=n} 1 = \sum_{d|n,\, d>0} \left(\frac{D}{d}\right) \quad (n > 0) \tag{A.181}$$

which is equivalent to the decomposition rule for the prime ideals in $R(\sqrt{D})$ for $D = 5$. Here $\mathfrak{a}$ runs through all integral ideals in $R(\sqrt{D})$ with norm n. Linear relation of the kind (179) can be proved also for arbitrary discriminants $D > 0$, with the help of (181). In general,

$$\sum_{\{a\}} g(\tau, 0, \mathfrak{a}, 1, \sqrt{D}) = \frac{Dl_1}{2l_0} \sum_{a=1}^{D-1} \left(\frac{D}{a}\right) G(\tau, 1; 0, a, D), \tag{A.182}$$

holds, where $\mathfrak{a}$ runs through a full system of representatives of the narrow ideal classes in $R(\sqrt{D})$ and $\dfrac{l_1}{l_0}$ is the number of strictly(totally) positive, mod $\sqrt{D}$ non-congruent units in $R(\sqrt{D})$. The proof of this identity is of particular interest, since it also provides us with the Dirichlet class number formula

$$h = -\frac{1}{2\log\varepsilon} \sum_{a=1}^{D-1} \log\sin\frac{a\pi}{D}. \tag{A.183}$$

On the one hand side, we get

$$\begin{aligned} \sum_{a=1}^{D-1} \left(\frac{D}{a}\right) G(\tau, 1; 0, a, D) = &-\frac{2}{D} \sum_{a=1}^{D-1} \left(\frac{D}{a}\right) \left(\log\sin\frac{a\pi}{D}\right) y^{\frac{1}{2}} \\ &+ \frac{8}{D} \sum_{n=1}^{\infty} \left\{ \sum_{d|n,\, d>0} \left(\frac{D}{d}\right) \right\} y^{\frac{1}{2}} K_0(2\pi n y) \cos(2\pi n x), \end{aligned} \tag{A.184}$$

from (135). On the other hand,

$$g(\tau, 0, \mathfrak{a}, 1, \sqrt{D}) = 2l_1 y^{\frac{1}{2}} + \frac{l_1}{l_0} \sum_{\substack{\mu \equiv 0 \ (\mathfrak{a}) \\ (\mu)_{p_\infty},\ \mu \neq 0}} y^{\frac{1}{2}} K_0\left(\frac{2\pi \mid N\mu \mid}{A} y\right) e^{-\frac{2\pi i N\mu}{A} x}, \tag{A.185}$$

holds by (29). Here $l_0 = \frac{1}{2}\log\varepsilon_0$ and ε_0, (> 1) generate the group of totally positive units in $R(\sqrt{D})$. In general, let $\mathfrak{K}_{\mathfrak{a}}$ be the narrow ideal class in $R(\sqrt{D})$, which is represented by $\mathfrak{a}$. If the norm of the base unit is $N\varepsilon = -1$, then we can put the series (185) into the form

$$g(\tau, 0, \mathfrak{a}, 1, \sqrt{D}) = 2l_1 y^{\frac{1}{2}} + \frac{4l_1}{l_0} \sum_{\mathfrak{b} \in \mathfrak{K}_{\mathfrak{a}^{-1}}} y^{\frac{1}{2}} K_0(2\pi N\mathfrak{b} y) \cos(2\pi N\mathfrak{b} x). \tag{A.186}$$

In the case, $N\varepsilon = 1$, we get

$$g(\tau, 0, \mathfrak{a}, 1, \sqrt{D}) = 2l_1 y^{\frac{1}{2}} - \frac{2l_1}{l_0}\left\{ \sum_{\mathfrak{b}\in\mathfrak{K}_{\mathfrak{a}^{-1}}} y^{\frac{1}{2}} K_0(2\pi N\mathfrak{b}y)e^{2\pi iN\mathfrak{b}x} + \sum_{\mathfrak{b}\in\mathfrak{K}_{\mathfrak{a}^{-1}}\sqrt{D}} y^{\frac{1}{2}} K_0(2\pi N\mathfrak{b}y)e^{-2\pi iN\mathfrak{b}x} \right\}. \tag{A.187}$$

Thus, in both cases, we get

$$\sum_{\{\mathfrak{a}\}} g(\tau, 0, \mathfrak{a}, 1, \sqrt{D}) = \frac{2l_1}{l_0}\left\{ hy^{\frac{1}{2}} \log\varepsilon + 2\sum_{n=1}^{\infty}\left(\sum_{N\mathfrak{b}=n} 1\right) y^{\frac{1}{2}} K_0(2\pi ny)\cos(2\pi nx)\right\}, \tag{A.188}$$

if we denote the number of ordinary ideal classes by h, so that, by (184),

$$\sum_{\{\mathfrak{a}\}} g(\tau, 0, \mathfrak{a}, 1, \sqrt{D}) = \frac{2l_1}{l_0}\left\{\left(h\log\varepsilon + \frac{1}{2}\sum_{a=1}^{D-1}\left(\frac{D}{a}\right)\log\sin\frac{a\pi}{D}\right) y^{\frac{1}{2}} + \frac{D}{4}\sum_{a=1}^{D-1}\left(\frac{D}{a}\right) G(\tau, 1; 0, a, D)\right\} \tag{A.189}$$

holds. This relation decomposes into the equations (182) and (183), since $g(\tau, 0, \mathfrak{a}, 1, \sqrt{D})$ and $G(\tau, 1; 0, a, D)$ are automorphic functions of level D.

Finally, we determine the linear collection of the pairs of Dirichlet series that are associated with the Eisenstein series for $r \geq 0$. The series $G(\tau, 1+2ir; a_1, a_2, Q)$ corresponds by (134) to the pairs of functions

$$\sum_{n\neq 0}\left\{\sum_{\substack{d_k \equiv a_k\ (Q)\\ d_1 d_2 = n}} |\, d_1\,|^{-ir} |\, d_2\,|^{ir}\right\} |\, n\,|^{-s}, \quad \sum_{n\neq 0}\operatorname{sgn} n\left\{\sum_{\substack{d_k \equiv a_k\ (Q)\\ d_1 d_2 = n}} |\, d_1\,|^{-ir} |\, d_2\,|^{ir}\right\} |\, n\,|^{-s},$$

up to a constant factor. This can be written as

$$\sum_{\substack{n \equiv a_1\ (Q)\\ n\neq 0}} \frac{1}{|\, n\,|^{s+ir}} \sum_{\substack{n \equiv a_2\ (Q)\\ n\neq 0}} \frac{1}{|\, n\,|^{s-ir}}, \quad \sum_{\substack{n \equiv a_1\ (Q)\\ n\neq 0}} \frac{\operatorname{sgn} n}{|\, n\,|^{s+ir}} \sum_{\substack{n \equiv a_2\ (Q)\\ n\neq 0}} \frac{\operatorname{sgn} n}{|\, n\,|^{s-ir}}. \tag{A.190}$$

Now let $(a_i, Q) = t_i > 0,\ a_i = t_i b_i$ and χ_i be an arbitrary character mod $\dfrac{Q}{t_i}$, $(i = 1, 2)$. Multiplying the products (190) by $\chi_1(b_1)\chi_2(b_2)$ and summing over the b_i mod $\dfrac{Q}{t_i}$, $(i = 1, 2)$, we obtain

$$\frac{1}{(t_1 t_2)^s} \sum_{n \neq 0} \frac{\chi_1(n)}{\mid n \mid^{s+ir}} \sum_{n \neq 0} \frac{\chi_2(n)}{\mid n \mid^{s-ir}},\ \frac{1}{(t_1 t_2)^s} \sum_{n \neq 0} \frac{(\operatorname{sgn} n)\chi_1(n)}{\mid n \mid^{s+ir}} \sum_{n \neq 0} \frac{(\operatorname{sgn} n)\chi_2(n)}{\mid n \mid^{s-ir}}, \tag{A.191}$$

up to a constant factor. All feasible pairs of functions of this sort are apparently linearly equivalent to the pairs (190). The functions (191) are identical with the L-series products

$$\begin{gathered} \frac{(1 + \chi_1(-1))(1 + \chi_2(-1))}{(t_1 t_2)^s} L(s + ir, \chi_1) L(s - ir, \chi_2), \\ \\ \frac{(1 - \chi_1(-1))(1 - \chi_2(-1))}{(t_1 t_2)^s} L(s + ir, \chi_1) L(s - ir, \chi_2). \end{gathered} \tag{A.192}$$

A.4 The Theory of T_n Opereators

Let $\mathbf{O}_n$ be the set of all substitutions

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

with integer entries and $ad - bc = n > 0$. Under the assumption $(n, Q) = 1$, we can choose the representatives S of the decomposition

$$\mathbf{O}_n = \sum_S \mathbf{M} S$$

so that

$$S \equiv \begin{pmatrix} 1 & 0 \\ 0 & n \end{pmatrix} \ (Q)$$

holds. Such a system of coset representatives is denoted by $\mathbf{V}_n$ and define the operator T_n for automorphic wave functions $F(\tau)$ of level Q by setting

$$F(\tau)T_n = \frac{1}{\sqrt{n}} \sum_{S\in\mathbf{V}_n} F(\tau) \mid S = \frac{1}{\sqrt{n}} \sum_{S\in\mathbf{V}_n} F(S\tau). \tag{A.193}$$

This function is independent of the choice of the system $\mathbf{V}_n$ and belongs again to the level Q. Let R_a be the operator defined by (36). Then the transformations

$$R_a \begin{pmatrix} a & bQ \\ 0 & d \end{pmatrix}, \tag{A.194}$$

where a runs through all positive divisors of n, b mod d varies and $ad = n$,which is a special system of representatives for $\mathbf{V}_n$, become suitable for carrying out the computations. As in the analytic case, one can prove the commutativity of the operators R_a and $T_n = T(n)$ as well as the multiplication rule

$$T(n)T(m) = \sum_{\substack{d \mid n,m \\ d>0}} T\left(\frac{nm}{d^2}\right) R_d \text{ for } (n,Q)=1,\ (m,Q)=1. \tag{A.195}$$

To define the T_n-operators for those n which are not relatively prime to the level Q, we decompose the linear space of automorphic functions into $\mathfrak{F}_r^{+1}(t,\chi,Q)$ and $\mathfrak{F}_r^{-1}(t,\chi,Q)$ for divisor t and character χ. As mentioned above, these consist of functions $F(\tau)$, which are multiplied by $\chi(n)$ under the operator R_n:

$$F(\tau) \mid R_n = \chi(n)F(\tau) \text{ for } (n,Q)=1, \tag{A.196}$$

and allows a Fourier type development of the form

$$F(\tau) = \delta\left(\frac{t}{Q}\right)u(y) + \sum_{\left(n,\frac{Q}{t}\right)=1} a(n)y^{\frac{1}{2}}K_0\left(\frac{2\pi \mid nt \mid}{Q}y\right)e^{\frac{2\pi int}{Q}x}. \tag{A.197}$$

Moreover, for the operator K defined by 36

$$F(\tau) \mid K = \begin{cases} F(\tau) \text{ for } F(\tau) \in \mathfrak{F}_r^{+1}(t,\chi,Q), \\ -F(\tau) \text{ for } F(\tau) \in \mathfrak{F}_r^{-1}(t,\chi,Q). \end{cases} \tag{A.198}$$

holds. Now let $Q = tt_1$ and q be a power product of prime divisors of level Q. Then we define the operator T_q^t corresponding to the divisor t for the function $F(\tau)$ by

$$F(\tau) \mid T_q^t = \frac{1}{\sqrt{q}} \sum_{l \bmod q} F\left(\frac{\tau + lt_1}{q}\right); \tag{A.199}$$

which disappears for $(q, t_1) > 1$, commutes with R_n and T_n for $(n, Q) = 1$ and maps $\mathfrak{F}_r^{\pm 1}(t, \chi, Q)$ to itself like T_n with $(n, Q) = 1$,. For any natural number m

$$T_m^t = T_q^t T_n, \text{ if } m = qn, \ (n, Q) = 1 \tag{A.200}$$

and q contains only prime factors of Q. The operator $T_m^t = T^t(m)$ applied to an arbitrary function $F(\tau) \in \mathfrak{F}_r^{\pm 1}(t, \chi, Q)$ has the effect:

$$F(\tau) \mid T_m^t = \frac{1}{\sqrt{m}} \sum_{\substack{ad = m \\ b \bmod d, \ d > 0}} \chi(a) F\left(\frac{a\tau + bt_1}{d}\right) \tag{A.201}$$

and satisfies the multiplication rule again

$$T^t(m_1) T^t(m_2) = \sum_{\substack{d \mid m_1, m_2 \\ d > 0}} T^t\left(\frac{m_1 m_2}{d^2}\right) \chi(d). \tag{A.202}$$

We now consider an arbitrary subspace $\mathfrak{G}_r$ of $\mathfrak{F}_r^1(t, \chi, Q)$ or $\mathfrak{F}_r^{-1}(t, \chi, Q)$, which is invariant with respect to the operators T_m^t.The functions $F^\varrho(\tau), \ (\varrho = 1, 2, ..., \varkappa)$ with the Fourier expansion

$$F^\varrho(\tau) = u^\varrho(y) + \sum_{(n, \frac{Q}{t}) = 1} a^\varrho(n) y^{\frac{1}{2}} K_{ir}\left(\frac{2\pi \mid nt \mid}{Q} y\right) e^{\frac{2\pi i n t}{Q} x} \tag{A.203}$$

may form a basis of $\mathfrak{G}_r$. The term $u^\varrho(y)$, which is independent of x, occurs only for $t = Q$ really, and of the form

$$u^\varrho(y) = \begin{cases} a_1^\varrho(0) y^{\frac{1}{2} + ir} + a_2^\varrho(0) y^{\frac{1}{2} - ir} \text{ for } r > 0, \\ a_1^\varrho(0) y^{\frac{1}{2}} \log y + a_2^\varrho(0) y^{\frac{1}{2}} \text{ for } r = 0. \end{cases} \tag{A.204}$$

By (201), one finds immediately

$$F^{\varrho}(\tau) \mid T_m^t = v^{\varrho}(y) + \sum_{\left(N, \frac{Q}{t}\right)=1} b^{\varrho}(N) y^{\frac{1}{2}} K_{ir}\left(\frac{2\pi \mid Nt \mid}{Q} y\right) e^{\frac{2\pi i N t}{Q} x} \tag{A.205}$$

with

$$b^{\varrho}(N) = \sum_{\substack{d \mid N, m \\ d > 0}} a^{\varrho}\left(\frac{Nm}{d^2}\right) \chi(d) \quad \text{for } \left(N, \frac{Q}{t}\right) = \left(m, \frac{Q}{t}\right) = 1. \tag{A.206}$$

The limitations for N and m are dropped if it is assumed $a^{\varrho}(n) = b^{\varrho}(n) = 0$ for $\left(N, \frac{Q}{t}\right) > 1$ at the beginning. If we make this assumption, then we have

$$v^{\varrho}(y) = \begin{cases} b_1^{\varrho}(0) y^{\frac{1}{2}+ir} + b_2^{\varrho}(0) y^{\frac{1}{2}-ir} \text{ for } r > 0, \\ b_1^{\varrho}(0) y^{\frac{1}{2}} \log y + b_2^{\varrho}(0) y^{\frac{1}{2}} \text{ for } r = 0. \end{cases} \tag{A.207}$$

when we set

$$b_1^{\varrho}(0) = m^{-ir} \sigma_{2ir}(m, \chi) a_1^{\varrho}(0) \quad \text{for } r \geq 0,$$

$$b_2^{\varrho}(0) = \begin{cases} m^{ir} \sigma_{-2ir}(m, \chi) a_2^{\varrho}(0) \quad \text{for } r > 0, \\ \left(\sum_{\substack{d \mid m \\ d > 0}} \chi(d) \log \frac{d^2}{m} \right) a_1^{\varrho}(0) + \sigma_0(m, \chi) a_2^{\varrho}(0) \quad \text{for } r = 0 \end{cases} \tag{A.208}$$

and generally

$$\sigma_k(m, \chi) = \sum_{\substack{d \mid m \\ d > 0}} \chi(d) d^k \tag{A.209}$$

is set. The claim (198) is expressed in the coefficient relations as

$$a^{\varrho}(-n) = \pm a^{\varrho}(n), \tag{A.210}$$

where the upper or lower sign respectively holds when $\mathfrak{G}_r$ is in $\mathfrak{F}_r^{+1}(t,\chi,Q)$ or $\mathfrak{F}_r^{-1}(t,\chi,Q)$. Because of the assumed invariance of the subspace $\mathfrak{G}_r$ regarding the operators T_m^t, there is a representation

$$F^\varrho|(\tau)T_m^t = \sum_{\sigma=1}^{\varkappa} \lambda_{\varrho\sigma}(m)F^\sigma(\tau) \quad (\varrho = 1,2,...,\varkappa) \tag{A.211}$$

with some constant coefficients $\lambda_{\varrho\sigma}(m)$. One gets the important relationship

$$\sum_{\substack{d \mid N,m \\ d>0}} a^\varrho\left(\frac{Nm}{d^2}\right)\chi(d) = \sum_{\sigma=1}^{\varkappa} \lambda_{\varrho\sigma}(m)a^\sigma(N) \quad \begin{pmatrix} m>0,\ N\neq 0, \\ \varrho = 1,2,...,\varkappa \end{pmatrix} \tag{A.212}$$

by comparing the Fourier coefficients in the expansion of both sides of (211). Apparently, we have

$$\sum_{\sigma=1}^{\varkappa} \lambda_{\varrho\sigma}(m)a^\sigma(N) = \sum_{\sigma=1}^{\varkappa} \lambda_{\varrho\sigma}(N)a^\sigma(m) \quad (m>0,\ N>0)\,, \tag{A.213}$$

from which

$$a^\varrho(m) = \sum_{\sigma=1}^{\varkappa} \lambda_{\varrho\sigma}(m)a^\sigma(1) \quad (m>0) \tag{A.214}$$

follows for $N=1$. Because the functions $F^\sigma(\tau)$ are linearly independent,there are whole rational numbers $N_1, N_2, ..., N_\varkappa$ such that the determinant is

$$|a^\varrho(N_\sigma)| \neq 0.$$

All N_σ may be supposed to be positive according to (210), so that a solution of the system (213) is possible after that of $\lambda_{\varrho\sigma}(m)$, if one lets the N values pass through $N_1, N_2, ..., N_\varkappa$:

$$\lambda_{\varrho\sigma}(m) = \sum_{\nu=1}^{\varkappa} b_{\varrho\sigma}^\nu a^\nu(m) \quad \text{for } m>0. \tag{A.215}$$

We use these equations to define $\lambda_{\varrho\sigma}(m)$ for $m<0$. By suitable choices of $u_{\varrho\sigma}(y)$, the $\varkappa^2$ functions

$$f_{\varrho\sigma}(\tau) = u_{\varrho\sigma}(y) + \sum_{m\neq 0} \lambda_{\varrho\sigma}(m)y^{\frac{1}{2}}K_{ir}\left(\frac{2\pi \mid mt \mid}{Q}y\right)e^{\frac{2\pi imt}{Q}x} \tag{A.216}$$

then represent a linearly equivalent system with the basis $F^{\varrho}(\tau)$ $(\varrho = 1, 2, ..., \varkappa)$. The linear equivalence carries over the functions $F^{\varrho}(\tau)$ and $f_{\varrho\sigma}(\tau)$ associated with the Dirichlet series

$$\varphi^{\varrho}(s) = \sum_{m=1}^{\infty} \frac{a^{\varrho}(m)}{(mt)^s}, \quad \varphi_{\varrho\sigma}(s) = \sum_{m=1}^{\infty} \frac{\lambda_{\varrho\sigma}}{(mt)^s}. \tag{A.217}$$

For the matrix $\lambda(m)$ formed with the coefficients $\lambda_{\varrho\sigma}(m)$,

$$\lambda(m_1)\lambda(m_2) = \sum_{\substack{d \,|m_1, m_2 \\ d > 0}} \lambda\left(\frac{m_1 m_2}{d^2}\right)\chi(d). \tag{A.218}$$

As in the analytic case, for the function matrix

$$\Phi(s) = (\varphi_{\varrho\sigma}(s)) = \sum_{m=1}^{\infty} \lambda(m)(mt)^{-s} \tag{A.219}$$

we obtain the Euler product

$$\Phi(s) = t^{-s} \prod_p (\lambda(1) - \lambda(p)p^{-s} + \chi(p)\lambda(1)p^{-2s})^{-1}. \tag{A.220}$$

A number of important theorems can now literally be proved as in the theory of Hecke operators. In particular, we obtain: The characteristic roots of the function matrix

$$B(\tau) = (f_{\varrho\sigma}(\tau)) \tag{A.221}$$

belong even to the linear space of $F^{\varrho}(\tau)$ and correspond to Dirichlet series with a product development of the kind (220) if one replaces herein $\lambda(1)$ and $\lambda(p)$ by matrices of first degree. Conversely, every function in $\mathfrak{G}_r$ associated to a Dirichlet series with a product development is a characteristic root of $B(\tau)$. The question whether it gives a system of $\varkappa$ linearly independent functions in $\mathfrak{G}_r$, to which a Dirichlet series with an Euler product development corresponds, is equivalent to the fact that $B(\tau)$ can be transformed to a diagonal form by using a constant matrix.

Necessary and sufficient condition is that there are $\varkappa$ different eigenfunctions of the operator ring produced by T_m^t, which do not differ each other only by constant factors. These eigenfunctions agree up to constant factors with the characteristic roots of $B(\tau)$. A satisfactory solution to this problem is available currently for $t = 1$ and $r > 0$. The families $\mathfrak{F}^{\pm}(t, \chi, Q)$ namely in the case of $r > 0$ decompose into two invariant subspaces of Eisenstein series and cusp functions. The Eisenstein series in $\mathfrak{F}^{+1}(t, \chi, Q)$ or $\mathfrak{F}^{-1}(t, \chi, Q)$ of the divisor $t = 1$ corresponds to the linear space of $L-$ product series $L(s+ir, \chi_1) \times L(s-ir, \chi_2)$ by (192), where χ_1 and χ_2 are characters mod Q mean, $\chi = \chi_1\chi_2$ and $\chi_1(-1) = \chi_2(-1) = 1$ or -1 holds. Since the Dirichlet series for these functions have an Euler product development, the associated Eisenstein series occur among the characteristic roots of $B(\tau)$. We can therefore look at only the cusp functions. Now let $\mathfrak{G}_r$ be permenantly an invariant subspace of cusp functions. By means of Petersson Metrization Principle we prove that all matrices $\lambda(n)$ with $(n, Q) = 1$ can be simultaneously transformed into a diagonal shape. So then the problem for the subspaces $\mathfrak{G}_r$ of the divisor $t = 1$ is completely solved since $\lambda(n)$ vanishes for $(n, Q) > 1$, where $t = 1$ is assumed.

Let $F_1(\tau)$ and $F_2(\tau)$ be two cusp functions and let $\mathfrak{F}$ be a fundamental domain for the group $\mathbf{M}(Q)$. The integral

$$(F_1(\tau), F_2(\tau)) = (F_1(\tau), F_2(\tau))_{\mathbf{M}(Q)} = \iint_{\mathfrak{F}} F_1(\tau)\overline{F_2(\tau)}\frac{dxdy}{y^2} \tag{A.222}$$

does not depend on the choice of $\mathfrak{F}$ and is called the scalar product of $F_1(\tau)$ and $F_2(\tau)$. Furthermore $F_1(\tau)$ and $F_2(\tau)$ are functions of the same character χ, then

$$(F_1(\tau)|T_n, F_2(\tau)) = \chi(n)(F_1(\tau), F_2(\tau)|T_n) \quad \text{for} \quad (n, Q) = 1. \tag{A.223}$$

Petersson's proof (KII) of the corresponding formula for the modular functions can be transferred verbatim to the existing case. Somewhat more shortly, one can proceed in this case if one notes that the left and right classes from $\mathbf{O}_n$ to $\mathbf{M}$

possess a common system $\mathbf{V}_n$ of representatives S:

$$\mathbf{O}_n = \sum_{S \in \mathbf{V}_n} \mathbf{M}S = \sum_{S \in \mathbf{V}_n} S\mathbf{M} \quad \text{with} \quad S \equiv \begin{pmatrix} 1 & 0 \\ 0 & n \end{pmatrix} (Q).$$

The reduction of the formula (223) in the case $(n = p)$(prime number) becomes then non-essential. H.Petersson proves and uses the existence of such a representative system $\mathbf{V}_n$ only for $n = p$. For any n, one can construct such a system in the following manner. Let $\mathbf{O}_{n,g}$ be the set of all substitutions

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad \text{with} \quad (a, b, c, d) = g,\ ad - bc = n.$$

It is obvious that

$$\mathbf{O}_{n,g} = \begin{pmatrix} g & 0 \\ 0 & g \end{pmatrix} \mathbf{O}_{\frac{n}{g^2},1} \quad \text{for} \quad g^2 | n \quad \text{and} \quad \mathbf{O}_n = \sum_{g^2|n,\ g>0} \mathbf{O}_{n,g}. \tag{A.224}$$

The representation

$$\mathbf{O}_{n,1} = \mathbf{M}S_n\mathbf{M} \quad \text{with} \quad S_n = \begin{pmatrix} 1 & 0 \\ 0 & n \end{pmatrix} \tag{A.225}$$

gives rise to a decomposition

$$\mathbf{O}_{n,1} = \sum_{i=1}^{\varrho(n)} \mathbf{M}S_nL_i^* \quad \text{with} \quad L_i^* \in \mathbf{M}. \tag{A.226}$$

Because of $S_n\mathbf{M}(n) \subset \mathbf{M}S_n$ it applies to any $Q_i \in \mathbf{M}(n)$

$$\mathbf{O}_{n,1} = \sum_{i=1}^{\varrho(n)} \mathbf{M}S_nL_i, \quad L_i = Q_iL_i^*. \tag{A.227}$$

With suitable choice of substitutions Q_i, $L_i \in \mathbf{M}(Q)$ can be achieved because the congruences

$$Q_i \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} (n), \quad Q_i \equiv L_i^{*-1}(Q)$$

are compatible each other since $(n, Q) = 1$. Denote the transpose matrix of L_i by L_i' and set $A_i = L_i' S_n L_i$, then it follows obviously

$$\mathbf{O}_{n,1} = \sum_{i=1}^{\varrho(n)} \mathbf{M} A_i = \sum_{i=1}^{\varrho(n)} A_i \mathbf{M}, \quad A_i \equiv S_n \ (Q); \tag{A.228}$$

because $\mathbf{O}_{n,1}$ does not change when all the matrices are replaced by the tansposes, moreover $A_i' = A_i$. In general $(g^2|n)$

$$\mathbf{O}_{\frac{n}{g^2},1} = \sum_{i=1}^{\varrho(\frac{n}{g^2})} \mathbf{M} A_i^{(g)} = \sum_{i=1}^{\varrho(\frac{n}{g^2})} A_i^{(g)} \mathbf{M}, \quad A_i^{(g)} \equiv \begin{pmatrix} 1 & 0 \\ 0 & \frac{n}{g^2} \end{pmatrix} \ (Q), \tag{A.229}$$

so that with

$$B_i^{(g)} = \begin{pmatrix} g & 0 \\ 0 & g \end{pmatrix} R_g A_i^{(g)}, \quad R_g \equiv \begin{pmatrix} \overline{g} & 0 \\ 0 & g \end{pmatrix} \ (Q), \quad R_g \in \mathbf{M} \tag{A.230}$$

finally we get

$$\mathbf{O}_{n,g} = \sum_{i=1}^{\varrho(\frac{n}{g^2})} B_i^{(g)} \mathbf{M} = \sum_{i=1}^{\varrho(\frac{n}{g^2})} \mathbf{M} B_i^{(g)}, \quad B_i^{(g)} \equiv S_n \ (Q). \tag{A.231}$$

We now choose a normalized orthogonal basis $F^{\varrho}(\tau)$ $(\varrho = 1, 2, ..., \varkappa)$ in $\mathfrak{G}_r$:

$$(F^{\varrho}(\tau), F^{\sigma}(\tau)) = \delta_{\varrho\sigma} \quad (= \text{Kronecker symbol})$$

and by (223) for $(n, Q) = 1$ we get

$$\lambda_{\varrho\sigma}(n) = (F^{\varrho}(\tau)|T_n, F^{\sigma}(\tau)) = \chi(n)(F^{\varrho}(\tau), F^{\sigma}(\tau)|T_n) = \chi(n)\overline{\lambda_{\sigma\varrho}(n)},$$

so

$$\lambda(n) = \chi(n)\overline{\lambda(n)}'. \tag{A.232}$$

Thus, there are the commutation relations

$$\left.\begin{aligned} \lambda(n)\overline{\lambda(n)}' &= \overline{\lambda(n)}'\lambda(n) \\ \lambda(n)\lambda(m) &= \lambda(m)\lambda(n) \end{aligned}\right\} \text{ for } (n, Q) = (m, Q) = 1 \tag{A.233}$$

The necessary and sufficient condition is that $\lambda(n)$ can be transformed simultaneously to a diagonal form by using a single unitary matrix. We thus obtain the following result:

Theorem 38. *In the linear space $\mathfrak{F}_r^{\pm 1}(1, \chi, Q)$ associated to the divisor* 1 *and character χ, in the case of $r > 0$ there exists a basis $F^\varrho(\tau)$ ($\varrho = 1, 2, ..., \varkappa$) consisting of eigenfunctions of the ring generated by the operators T_n with $(n, Q) = 1$. The functions $F^\varrho(\tau)$ with the Fourier expansion*

$$F^\varrho(\tau) = \delta\left(\frac{1}{Q}\right) u^\varrho(y) + \sum_{\substack{(n,Q) = 1 \\ n \neq 0}} a^\varrho(n) y^{\frac{1}{2}} K_{ir}\left(\frac{2\pi \mid n \mid}{Q} y\right) e^{\frac{2\pi i n}{Q} x}. \tag{A.234}$$

associated to the Dirichlet series

$$\varphi^\varrho(s) = \sum_{\substack{n = 1 \\ (n,Q) = 1}} \frac{a^\varrho(n)}{n^s} \tag{A.235}$$

have the Euler product development

$$\varphi^\varrho(s) = \prod_{(p,Q)=1} (1 - a^\varrho(p)p^{-s} + \chi(p)p^{-2s})^{-1}, \tag{A.236}$$

if we assume $a^\varrho(1) = 1$ ($\varrho = 1, 2, ..., \varkappa$), which means no restriction.

The investigation of subspaces $\mathfrak{F}_r^{\pm 1}(t, \chi, Q)$ for any divisor t and $r \geq 0$ leads to similar results as in the analytic case with the method developed by H.Petersson. However, it must be noted that the theory of Eisenstein series for $r = 0$ yet in no satisfying condition.

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