COMPLETE SETS OF ORTHOGONAL F-SQUARES OF PRIME POWER ORDER

by

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Abstract

This paper represents a contribution to the statistical and mathematical theory of orthogonality of latin squares and of F-squares. The results are useful in designing experiments wherein several sets of treatments are applied either simultaneously or sequentially on the same set of experimental units. The main theorem relates sets of single degree of freedom contrasts in an s^{2m} factorial (s a prime power) to the s^m treatments in a latin square of order s^m; necessary and sufficient conditions for sets of s^m-1 single degree of freedom contrasts, to form the treatment contrasts in sets of orthogonal latin squares, are given. The one-to-one correspondence between factorial effects and treatments in F-squares and latin squares is demonstrated. Theorems are presented on the decomposition of latin squares into F-squares. Under certain conditions, a complete set of orthogonal latin squares may be obtained from a given pair. An example is used to demonstrate the theoretical results.

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1. Introduction and Definitions. This paper represents a contribution to the statistical and mathematical theory of orthogonal latin and F-squares. The results will be useful in designing experiments either simultaneously or sequentially using the same set of experimental units. After some preliminary definitions below, we present our main theorem relating sets of single degrees of freedom in an s^{2m} factorial (s a prime power) to the s^m treatments in a latin square. We demonstrate the necessary and sufficient conditions for these s^m-1 single degrees of freedom to construct a latin square and for another set of s^m-1 single degrees of freedom to form a second latin square of order s^m which is orthogonal to the first. The complete set of orthogonal latin squares is constructed in this manner.

In the next section we demonstrate the one-to-one correspondence between factorial effects and treatments in F-squares and latin squares. Three theorems are presented showing how to decompose latin squares into F-squares with differing numbers of symbols. These three theorems are called decomposition theorems, whereas theorems relating to construction of F-squares with fewer symbols are denoted as composition theorems (see Mandeli [1975]).

In section 4 a theorem is given which shows, for s a prime power, and for a pair of orthogonal latin squares of order s, how to obtain the complete set of orthogonal latin squares of order s. In the last section of the paper, the complete set of orthogonal latin squares of order 8 is used to illustrate the results.

An F-square has been defined by Hedayat [1969] and Hedayat and Seiden [1970] as follows:

Definition 1.1. Let $A = [Q_{ij}]$ be an $n \times n$ matrix and let $\Sigma = (A_1, A_2, \cdots, A_m)$ be the ordered set of m distinct elements or symbols of A. In addition, suppose that for each $k = 1, 2, \cdots, m$, A_k appears exactly λ_k times ($\lambda_k \geq 1$) in each row and column of A. Then A will be called a frequency square or, more concisely, an F-square on Σ of order n and frequency vector $(\lambda_1, \lambda_2, \cdots, \lambda_m)$.

The notation we use to denote this F-square differs somewhat from that given by Hedayat and Seiden [1970]. We call such a square an $F\left(A_1^{\lambda_1}, A_2^{\lambda_2}, \cdots, A_m^{\lambda_m}\right)$. Note that $\lambda_1 + \lambda_2 + \cdots + \lambda_m = n$ and that when $\lambda_k = 1$ for all k and $m = n$, a latin square of order n results.

As with latin squares, one may consider orthogonality of a pair of F-squares of the same order. The above cited authors have given the following definition to cover this situation:

Definition 1.2. Given an F-square $F_1\left(A_1^{\lambda_1}, A_2^{\lambda_2}, \cdots, A_k^{\lambda_k}\right)$ and an F-square $F_2\left(B_1^{u_1}, B_2^{u_2}, \cdots, B_t^{u_t}\right)$, we say F_2 is an orthogonal mate for F_1 (and write $F_2 \perp F_1$), if upon superposition of F_2 on F_1, A_i appears $\lambda_i u_j$ times with B_j. Note that when $\lambda_i = 1 = u_j$ for all i and j and $\sum_{i=1}^{k} \lambda_i = n = \sum_{j=1}^{t} u_j$, we have the familiar definition of the orthogonality of two latin squares of order n. In passing we mention that we shall denote the complete set of n-1 orthogonal latin squares of order n, by the notation $OL(n,n-1)$.

2. <u>One-to-One Correspondence Between Factorial Effects and Orthogonal Latin Squares</u>. Federer <u>et al</u>. [1969] discuss a technique for constructing a set of mutually orthogonal latin squares and denote it as "factorial complete confounding construction of $OL(s^m, s^m-1)$ sets" for s a prime power. Let the s^{2m} factorial be of the form $s^m \times s^m = \prod_{i=1}^{m} X_i \times \prod_{i=1}^{m} Y_i$, when each X_i and each Y_i are at s levels, 0, 1, $\cdots$, s-1,

and where the X_i denote main effects of one set of m factors and the Y_i denote main effects of the second set of m factors. Let the row effects of an s^m-row by s^m-column square be associated (completely confounded) with main effects X_i and all interactions of the X_i; likewise, let the column effects of the square be completely confounded with the main effects Y_i and all interactions of the Y_i. Thus, $(s^m-1)/(s-1)$ effects involving the X_i will be confounded with rows and $(s^m-1)/(s-1)$ effects involving the Y_i will be confounded with the column effects.

Following Raktoe and Federer [1960] we utilize their method of decomposing the $(s^{2m}-1)/(s-1)$ effects each with $(s-1)$ degrees of freedom plus the mean with one degree of freedom into s^{2m} single degree of freedom contrasts. We then set up an $s^m \times s^m$ square for the s^{2m} combinations such that the s^m-1 single degree of freedom contrasts forming the row effects are closed under multiplication. (By being closed under multiplication we mean that the product of any two effects is a third effect contained within the set or is the mean (1) effect.) Likewise, the s^m-1 single degree of freedom contrasts completely confounded with column effects are also closed under multiplication. In an $OL(s^m, s^m-1)$ set there are s^m+1 entities, i.e. rows, columns, and the treatments from the s^m-1 mutually orthogonal latin squares. We assign the s^{2m} factorial effects to the rows, columns, and the s^m-1 sets of treatments as follows:

Source of variation	Degrees of freedom
Correction for the mean	1
Rows of the square: a set of s^m-1 single degree of freedom contrasts from an s^{2m} factorial, which is closed under multiplication	s^m-1
Columns of the square: a set of s^m-1 single degree of freedom contrasts from an s^{2m} factorial, which is closed under multiplication	s^m-1
Latin square number one treatments: a set of s^m-1 single degree of freedom contrasts from an s^{2m} factorial, which is closed under multiplication	s^m-1
⋮	
Latin square number s^m-1 treatments: a set of s^m-1 single degree of freedom contrasts from an s^{2m} factorial, which is closed under multiplication	s^m-1
Total	s^{2m}

To construct latin square number j, say, from the above analysis of variance table, we first construct an $s^m \times s^m$ square consisting of the s^{2m} treatment combination such that main effects $X_1, X_2, \cdots, X_m$ and their interactions are confounded with rows and main effects $Y_1, Y_2, \cdots, Y_m$ and their interactions are confounded with columns; then we call each of the s^m levels of the set of s^m-1 effects, under latin square number j's treatments in the above analysis of variance table, a "treatment" A_k where $k = 1, 2, \cdots, s^m$ and we put A_k next to all treatment combinations, in the $s^m \times s^m$ square of row-column intersection, which makes up the k^{th} level of the set of s^m-1 effects under latin square number j's treatments in the above analysis of variance. An example and details of this construction procedure is illustrated in section 5.

Theorem 2:1: If a set of $s^m-1 = n-1$ single degree of freedom effects from an $s^{2m} = n^2$ factorial treatment design are unconfounded with row and column effects and are closed under multiplication, they can be used to construct a latin square of order s^m. Likewise, if given another set of $s^m-1 = n-1$ single degree of freedom effects unconfounded with row and column effects and closed under multiplication, then this set can be used to construct a second latin square of order s^m orthogonal to the first. And the entire set of $(s^m-1)^2$ single degree of freedom effects unconfounded with row and column effects can be used to construct the complete set of orthogonal latin squares of order s^m, i.e. the $OL(s^m, s^m-1)$ set.

Conversely, given a latin square of order s^m, the $s^m-1 = n-1$ degrees of freedom associated with the treatments can be partitioned into $s^m-1 = n-1$ single degree of freedom effects which are unconfounded with rows and columns and which are closed under multiplication.

Proof: It can be shown that the set, say G, of s^{2m} single degree of freedom effects is an abelian group of s^{2m} elements under multiplication where the inverse of an effect $X^{X_1} Y^{X_2}$ would be $X^{s-X_1} Y^{s-X_2}$. Since we are in mod s, our group has the special property that $X^s = \prod_{i=1}^{s} X = 1$, where 1 is the mean effect and X is any effect in the set of s^{2m} effects. By relabelling the effects, we can say that we have an abelian group under addition (instead of multiplication) with the special property that $\sum_{i=1}^{s} X = 0$ for all $X \epsilon G$ $\left(\text{in place of } \prod_{i=1}^{s} X = 1 \text{ for all } X\epsilon G\right)$. This can be done by using the symbol 0 to denote the mean effect (instead of 1) and by using the symbol + (instead of juxtaposition) to denote the interaction of effects. Thus, we write $X_1 + Y_2 + Y_2 + X_3$, for example, in place of $X_1 Y_2^2 X_3$.

If H is an abelian group of s^{2m} elements under addition with the property that $\sum_{i=1}^{s} X = 0$ for all $X\epsilon H$ and if K is another abelian group of s^{2m} elements under addition with the property that $\sum_{i=1}^{s} X = 0$ for all $X\epsilon K$, then H is isomorphic to K.

Now if we can show that $\sum_{i=1}^{s} X = 0$ for every $X \epsilon GF(s^{2m})$, then $GF(s^{2m})$ will be isomorphic to our group G, and hence G would be a finite field of s^{2m} elements. Since $GF(s^{2m})$ has s^{2m} elements, $GF(s^{2m})$ has characteristic s and Z_s, the field of integers mod s, in a subfield of $GF(s^{2m})$. However in Z_s, $\sum_{i=1}^{s} 1 = 0$, and therefore $X\left(\sum_{i=1}^{s} 1\right) = X(0)$ for all $X \epsilon GF(s^{2m})$. Since $X\left(\sum_{i=1}^{s} 1\right) = \sum_{i=1}^{s} X = 0$ for all $X \epsilon GF(s^{2m})$, the group G is isomorphic to $GF(s^{2m})$. Yet H_1 be a set of s^m-1 single d.f. effects unconfounded with rows and columns, which is closed under addition together with the mean effect 0. In the same manner as above, it can be shown that H_1 is isomorphic to the finite field $GF(s^m)$ with s^m elements. Hence the set H_1 can be denoted by the ordered set $\left\{0, 1, x, x^2, \cdots, x^{s^a-2}\right\}$. A one-to-one correspondence exists between a latin square of order s^m, say L_1 and the addition table for H_1 (see, e.g., Mann [1949] and Raghavarao [1971]). Thus, a one-to-one correspondence exists between the "treatments" of the latin square L_1 and the set H_1. Now let H_2 be a second set of s^m-1 single degree of freedom effects unconfounded with rows and columns that is closed under addition with the condition that no element in this set is in H_1. Again, as for H_1, H_2 is isomorphic to the finite field with s^m elements $GF(s^m)$. Hence H_2 must be equal to $x^r H_1$ for some $1 \leq r \leq s^m-2$. That is, H_2 is just H_1 multiplied by some non-zero element of H_1, say x^r, and since a field is a group under addition $x^r H_1$, i.e. H_2, is simply a permutation of the ordered set H_1. A one-to-one correspondence exists between a latin square of order s^m, say L_2, orthogonal to L_1, and the addition table for H_2 (Mann [1949] and Raghavarao [1971]). Thus, a one-to-one correspondence exists between the "treatments" of the latin square L_2 and the set H_2. It can be shown (Mann [1949]) that the complete set of $(s^m-1)^2$ single degree of freedom effects unconfounded with rows and columns can be decomposed into (s^m-1) sets of (s^m-1) effects that are closed under addition. Hence we have s^m-1 sets $H_1, H_2, \cdots, H_{s^m-1}$, each respectively in one-to-one correspondence with the "treatments" of the latin square $L_1, L_2, \cdots, L_{s^m-1}$ of order s^m, where $L_1 \perp L_2 \perp \cdots \perp L_{s^m-1}$. So the complete set of $(s^m-1)^2$ single degree of freedom effects unconfounded with rows and columns is in one-to-one correspondence with the "treatments" in the $OL(s^m, s^m-1)$ set.

3. <u>One-to-One Correspondence Between Factorial Effects and Orthogonal F-squares and the Decomposition of Latin Squares into F-squares.</u>

By a generalization of the method of constructing orthogonal latin squares, if a set of $s^{m-r}-1$ single degree of freedom effects in the s^{2m} factorial treatment design is closed under multiplication then it can be used to construct an $F\left(A_1^{s^r}, A_2^{s^r}, \cdots, A_{s^{m-r}}^{s^r}\right)$-square, where $0 \leq r \leq m-1$. This F-square can be constructed by first constructing an $s^m \times s^m$ square consisting of the s^{2m} treatment combinations such that main effects $X_1, X_2, \cdots, X_m$ and their interactions are confounded with rows and main effects $Y_1, Y_2, \cdots, Y_m$ and their interactions are confounded with columns; then we call each of the s^{m-r} levels of a set of $s^{m-r}-1$ effects single degree of freedom effects, unconfounded with rows and columns that are closed under multiplication, a "treatment" A_k where $k = 1, 2, \cdots, s^{m-r}$ and we put A_k next to all treatment combinations, in the $s^m \times s^m$ square of row-column intersections, which make up the $k^{\underline{th}}$ level of the set of the $s^{m-r}-1$ effects mentioned above. An example and details of the construction procedure are given in section 5.

We now have the following theorem that relates factorial effects to F-squares.

<u>Theorem 3.1</u>: <u>There is a one-to-one correspondence between a set of $(s-1)$ single degree of freedom effects, unconfounded with rows and columns, that is closed under multiplication in the s^{2m} factorial treatment design and an $F\left(A_1^{s^{m-1}}, A_2^{s^{m-1}}, \cdots, A_s^{s^{m-1}}\right)$ square. Also, there is a one-to-one correspondence between the $\frac{(s^m-1)^2}{s-1}$ sets of this type and the complete set of $\frac{(s^m-1)^2}{s-1}$ orthogonal $F\left(A_1^{s^{m-1}}, A_2^{s^{m-1}}, \cdots, A_s^{s^{m-1}}\right)$ squares.</u>

Proof: Each set of s-1 single degree of freedom effects, unconfounded with rows and columns, that is closed under multiplication forms an effect with s-1 degrees of freedom and the $\frac{(s^m-1)^2}{s-1}$ sets of this type form $\frac{(s^m-1)^2}{s-1}$ effects, each with s-1 degrees of freedom in the s^{2m} factorial treatment design. By Hedayat, Raghavarao, and Seiden [1975] these $\frac{(s^m-1)^2}{s-1}$ effects can be used to construct the complete set of $\frac{(s^m-1)^2}{s-1}$ orthogonal $F\left(A_1^{s^{m-1}}, A_2^{s^{m-1}}, \cdots, A_s^{s^{m-1}}\right)$ squares.

Q.E.D.

In the following three theorems, which are called decomposition theorems, we show how to decompose an $OL(s^m, s^m-1)$ set into a set of orthogonal F-squares.

Theorem 3.2: Each latin square in the set of orthogonal latin squares $OL(s^m, s^m-1)$ can be decomposed into $\frac{s^m-1}{s-1}$ orthogonal $F\left(A_1^{s^{m-1}}, A_2^{s^{m-1}}, \cdots, A_s^{s^{m-1}}\right)$ squares; that is, the entire $OL(s^m, s^m-1)$ set can be decomposed into $\frac{(s^m-1)^2}{s-1}$ orthogonal $F\left(A_1^{s^{m-1}}, A_2^{s^{m-1}}, \cdots, A_s^{s^{m-1}}\right)$ squares.

The proof follows immediately from Theorems 2.1 and 3.1 .

The next theorem decomposes the $OL(s^m, s^m-1)$ set into F-squares with s^{m-1} symbols and s symbols.

Theorem 3.3: Each latin square $F(A_1, A_2, \cdots, A_{s^m})$ in the set $OL(s^m, s^m-1)$ can be decomposed into $1F\left(A_1^s, A_2^s, \cdots, A_{s^{m-1}}^s\right) + s^{m-1}F\left(A_1^{s^{m-1}}, A_2^{s^{m-1}}, \cdots, A_s^{s^{m-1}}\right)$ orthogonal F-squares for $m \geq 2$. Hence the entire $OL(s^m, s^m-1)$ set can be decomposed into $(s^m-1)F\left(A_1^s, A_2^s, \cdots, A_{s^{m-1}}^s\right) + s^{m-1}(s^m-1)F\left(A_1^{s^{m-1}}, A_2^{s^{m-1}}, \cdots, A_s^{s^{m-1}}\right)$ orthogonal F-squares for $m \geq 2$.

Proof: This will be proved if we can prove that each of the (s^m-1) sets of (s^m-1) effects unconfounded with rows and columns that are closed under addition has a subset of $(s^{m-1}-1)$ effects that is also closed under addition. This in turn will be proved if we can prove that the group H_i $(i = 1, 2, \cdots, s^m-1)$ of s^m elements, where each H_i is one of the above (s^m-1) sets plus the mean (0) effect, has a subgroup of s^{m-1} elements. This is just a result of the First Sylow Theorem in modern algebra which tells us that H_i contains subgroups of s^k elements for $k = 1, 2, \cdots, m$.

A further decomposition of the $OL(s^m, s^m-1)$ set is given in Theorem 3.4.

Theorem 3.4. If p divides m, then each latin square $F(A_1, A_2, \cdots, A_{s^m})$ in the set of orthogonal latin squares $OL(s^m, s^m-1)$ can be decomposed into $\frac{s^m-1}{s^p-1}$ orthogonal $F\left(A_1^{s^{m-p}}, A_2^{s^{m-p}}, \cdots, A_{s^p}^{s^{m-p}}\right)$ squares, i.e., the entire $OL(s^m, s^m-1)$ set can be decomposed into $\frac{(s^m-1)^2}{s^p-1}$ orthogonal $F\left(A_1^{s^{m-p}}, A_2^{s^{m-p}}, \cdots, A_{s^p}^{s^{m-p}}\right)$ squares.

Proof: This will be proved if we can prove that if p divides m then each of the (s^m-1) sets of (s^m-1) effects unconfounded with rows and columns that are closed under addition can be decomposed into $\frac{s^m-1}{s^p-1}$ sets of (s^p-1) effects such that each set is closed under addition and the intersection of any two sets is empty. This in turn will be proved if we can prove that the group H_i $(i = 1, 2, \cdots, s^m-1)$ of s^m elements, where each H_i is one of the above (s^m-1) sets plus the mean (0) effect, has $\frac{s^m-1}{s^p-1}$ subgroups of s^p elements such that if K_j is one such subgroup and $K_{j'}$ is another such subgroup $(j \neq j')$, then $K_j \cap K_{j'} = \{0\}$ and every $X \epsilon H_i$ is in some K_j. The proof of this is very similar to the proof of Theorem 2.1. For further details the reader is referred to Mann [1949].

4. Constructing the Complete Set of Orthogonal Latin Squares From Two Orthogonal Latin Squares.

Theorem 2.1 dealt with first confounding s^m-1 single degree of freedom effects that are closed under addition with rows and confounding s^m-1 different single degree of freedom effects that are closed under addition with columns and then constructing the complete set of orthogonal latin squares $OL(s^m, s^m-1)$ from the row-column interactions. Theorem 4.1 presented below first starts with two orthogonal latin squares of order s^m and then from them finds the effects associated with rows, columns, and the remaining s^m-3 mutually orthogonal latin squares.

Theorem 4.1: *Given a pair of orthogonal latin squares of side* s^m *for* s *a prime and* m *a positive integer and given that the levels of an effect* X *correspond to the treatments in one latin square and that the levels of an effect* Y *correspond to the treatments in the second latin square, then the generalized interaction of* X *and* Y, *i.e.*

$$X \times Y = \sum_{u=1}^{s^m-1} XY^u \bmod(s^m) \quad ,$$

produces the effects associated with rows, columns, and the remaining s^m-3 *mutually orthogonal latin squares.*

For the proof the reader is referred to Mann [1949].

5. Example. As an example consider the OL(2^3, 2^3-1) = OL(8,7) set. We may relate the complete set of 7 orthogonal latin squares of order 8 to a $2^{2(3)} = 2^6$ factorial treatment design. Let the six main effects be A, B, C, D, E, and F each at two levels 0 and 1. We set up a 8 x 8 square consisting of the $2^6 = 64$ treatment combinations, confounding three main effects and their interactions with rows and three main effects and their interactions with columns. Let us confound main effects A, B, C and their interactions AB, AC, BC, ABC with rows and let us confound main effects D, E, F and their interactions DE, DF, EF, DEF with columns. Then we have the square in figure 1.

Rows	Columns 1	2	3	4	5	6	7	8
1	000000	000100	000010	000110	000001	000101	000011	000111
2	100000	100100	1000010	100110	100001	100101	100011	100111
3	010000	010100	010010	010110	010001	010101	010011	010111
4	110000	110100	110010	110110	110001	110101	110011	110111
5	001000	001100	001010	001110	001001	001101	001011	001111
6	101000	101100	101010	101110	101001	101101	101011	101111
7	011000	011100	011010	011110	011001	011101	011011	011111
8	111000	111100	111010	111110	111001	111101	111011	111111

Figure 1

We obtain the following analysis of variance table relating F-squares and latin squares to the effects in the 2^6 factorial treatment design:

	Source of variation	d.f.		
	CFM	1		
	ROWS	7		
	A		1	
	B		1	
	AB		1	
	C		1	
	AC		1	
	BC		1	
	ABC		1	
	COLUMNS	7		
	D		1	
	E		1	
	DE		1	
	F		1	
	DF		1	
	EF		1	
	DEF		1	
	LATIN SQUARE NUMBER ONE TREATMENTS	7		
$F_1(A_1^2, A_2^2, A_3^2, A_4^2)$ treatments	AD $= F_1(A_1^4, A_2^4)$ treatments		1	3
	BE $= F_2(A_1^4, A_2^4)$ treatments		1	
	ABDE $= F_3(A_1^4, A_2^4)$ treatments		1	
	CF $= F_4(A_1^4, A_2^4)$ treatments		1	
	ACDF $= F_5(A_1^4, A_2^4)$ treatments		1	
	BCEF $= F_6(A_1^4, A_2^4)$ treatments		1	
	ABCDEF $= F_7(A_1^4, A_2^4)$ treatments		1	

	LATIN SQUARE NUMBER TWO TREATMENTS	7		
	ADEF $= F_8(A_1^4, A_2^4)$ treatments		1	
$F_2(A_1^2, A_2^2, A_3^2, A_4^2)$ treatments	BD $= F_9(A_1^4, A_2^4)$ treatments		1	3
	ABEF $= F_{10}(A_1^4, A_2^4)$ treatments		1	
	CDE $= F_{11}(A_1^4, A_2^4)$ treatments		1	
	ACF $= F_{12}(A_1^4, A_2^4)$ treatments		1	
	BCE $= F_{13}(A_1^4, A_2^4)$ treatments		1	
	ABCDF $= F_{14}(A_1^4, A_2^4)$ treatments		1	
	LATIN SQUARE NUMBER THREE TREATMENTS	7		
	AEF $= F_{15}(A_1^4, A_2^4)$ treatments		1	
$F_3(A_1^2, A_2^2, A_3^2, A_4^2)$ treatments	BCF $= F_{16}(A_1^4, A_2^4)$ treatments		1	3
	ABCE $= F_{17}(A_1^4, A_2^4)$ treatments		1	
	ABDF $= F_{18}(A_1^4, A_2^4)$ treatments		1	
	BDE $= F_{19}(A_1^4, A_2^4)$ treatments		1	
	ACD $= F_{20}(A_1^4, A_2^4)$ treatments		1	
	CDEF $= F_{21}(A_1^4, A_2^4)$ treatments		1	
	LATIN SQUARE NUMBER FOUR TREATMENTS	7		
	ADF $= F_{22}(A_1^4, A_2^4)$ treatments		1	
$F_4(A_1^2, A_2^2, A_3^2, A_4^2)$ treatments	ABCF $= F_{23}(A_1^4, A_2^4)$ treatments		1	3
	BCD $= F_{24}(A_1^4, A_2^4)$ treatments		1	
	ABE $= F_{25}(A_1^4, A_2^4)$ treatments		1	
	BDEF $= F_{26}(A_1^4, A_2^4)$ treatments		1	
	CEF $= F_{27}(A_1^4, A_2^4)$ treatments		1	
	ACDE $= F_{28}(A_1^4, A_2^4)$ treatments		1	

	LATIN SQUARE NUMBER FIVE TREATMENTS	7		
$F_5(A_1^2,A_2^2,A_3^2,A_4^2)$ treatments	AF $= F_{29}(A_1^4,A_2^4)$ treatments		1	3
	BDF $= F_{30}(A_1^4,A_2^4)$ treatments		1	
	ABD $= F_{31}(A_1^4,A_2^4)$ treatments		1	
	CE $= F_{32}(A_1^4,A_2^4)$ treatments		1	
	ACEF $= F_{33}(A_1^4,A_2^4)$ treatments		1	
	BCDEF $= F_{34}(A_1^4,A_2^4)$ treatments		1	
	ABCDE $= F_{35}(A_1^4,A_2^4)$ treatments		1	
	LATIN SQUARE NUMBER SIX TREATMENTS	7		
$F_6(A_1^2,A_2^2,A_3^2,A_4^2)$ treatments	ADE $= F_{36}(A_1^4,A_2^4)$ treatments		1	3
	BF $= F_{37}(A_1^4,A_2^4)$ treatments		1	
	ABDEF $= F_{38}(A_1^4,A_2^4)$ treatments		1	
	BCDF $= F_{39}(A_1^4,A_2^4)$ treatments		1	
	ABCEF $= F_{40}(A_1^4,A_2^4)$ treatments		1	
	CD $= F_{41}(A_1^4,A_2^4)$ treatments		1	
	ACE $= F_{42}(A_1^4,A_2^4)$ treatments		1	
	LATIN SQUARE NUMBER SEVEN TREATMENTS	7		
$F_7(A_1^2,A_2^2,A_3^2,A_4^2)$ treatments	BEF $= F_{43}(A_1^4,A_2^4)$ treatments		1	3
	BCDE $= F_{44}(A_1^4,A_2^4)$ treatments		1	
	CDF $= F_{45}(A_1^4,A_2^4)$ treatments		1	
	AE $= F_{46}(A_1^4,A_2^4)$ treatments		1	
	ABF $= F_{47}(A_1^4,A_2^4)$ treatments		1	
	ABCD $= F_{48}(A_1^4,A_2^4)$ treatments		1	
	ACDEF $= F_{49}(A_1^4,A_2^4)$ treatments		1	
	TOTAL	64		

To construct latin square number one from effects AD, BE, ABDE, CF, ACDF, BCEF, and ABCDEF we let the symbols I, II, ···, VIII in the latin square be represented as shown on the following page. We now take the 8 x 8 square of the 2^6 treatment combinations (Figure 1) and put our "treatments" I, II, ···, VIII in the appropriate cells. We then get the following 8 x 8 latin square:

I	V	III	VII	II	VI	IV	VIII
V	I	VII	III	VI	II	VIII	IV
III	VII	I	V	IV	VIII	II	VI
VII	III	V	I	VIII	IV	VI	II
II	VI	IV	VIII	I	V	III	VII
VI	II	VIII	IV	V	I	VII	III
IV	VIII	II	VI	III	VII	I	V
VIII	IV	VI	II	VII	III	V	I

The remaining six latin squares are constructed in the same manner from their corresponding set of seven single degree of freedom effects in the analysis of variance table. The seven latin squares of order 8 constructed in this manner are pairwise orthogonal. Hence we have constructed the OL(8,7) set from the analysis of variance of the 2^6 factorial treatment design.

Each single degree of freedom effect can in turn be used to construct an $F(A_1^4, A_2^4)$ square by Theorem 3.1. To construct the F-square $F_1(A_1^4, A_2^4)$ from the AD effect in the analysis of variance table we let the symbols α and β in the $F_1(A_1^4, A_2^4)$ square be represented as follows:

Level of Effect	Combinations (see Illustration 1)
$(AD)_0$	I + II + III + IV $= \alpha$
$(AD)_1$	V + VI + VII + VIII $= \beta$

We now take the 8 x 8 square of the 2^6 treatment combinations (Figure 1) and put

Level of Effect	Combinations	
$(AD)_0, (BE)_0, (CF)_0, (ABDE)_0, (ACDF)_0, (BCEF)_0, (ABCDEF)_0$	000000+100100+010010+110110+001001+101101+011011+111111 =	I
$(AD)_0, (BE)_0, (CF)_1, (ABDE)_0, (ACDF)_1, (BCEF)_1, (ABCDEF)_1$	001000+101100+011010+111110+000001+100101+010011+110111 =	II
$(AD)_0, (BE)_1, (CF)_0, (ABDE)_1, (ACDF)_0, (BCEF)_1, (ABCDEF)_1$	010000+110100+000010+100110+011001+111101+001011+101111 =	III
$(AD)_0, (BE)_1, (CF)_1, (ABDE)_1, (ACDF)_1, (BCEF)_0, (ABCDEF)_0$	011000+111100+001010+101110+010001+110101+000011+100111 =	IV
$(AD)_1, (BE)_0, (CF)_0, (ABDE)_1, (ACDF)_1, (BCEF)_0, (ABCDEF)_1$	100000+000100+110010+010110+101001+001101+111011+011111 =	V
$(AD)_1, (BE)_0, (CF)_1, (ABDE)_1, (ACDF)_0, (BCEF)_1, (ABCDEF)_0$	101000+001100+111010+011110+100001+000101+110011+010111 =	VI
$(AD)_1, (BE)_1, (CF)_0, (ABDE)_0, (ACDF)_1, (BCEF)_1, (ABCDEF)_0$	110000+010100+100010+000110+111001+011101+101011+001111 =	VII
$(AD)_1, (BE)_1, (CF)_1, (ABDE)_0, (ACDF)_0, (BCEF)_0, (ABCDEF)_1$	111000+011100+101010+001110+110001+010101+100011+000111 =	VIII

Illustration 1

our "treatments" α and β in the appropriate cells. Or alternatively, we could take the previously constructed latin square number one and replace "treatments" I, II, III, and IV by "treatment" α and "treatments" V, VI, VII, and VIII by "treatment" β. In either case we get the following $F_1(A_1^4, A_2^4)$ square:

$$\begin{array}{cccccccc}
\alpha & \beta & \alpha & \beta & \alpha & \beta & \alpha & \beta \\
\beta & \alpha & \beta & \alpha & \beta & \alpha & \beta & \alpha \\
\alpha & \beta & \alpha & \beta & \alpha & \beta & \alpha & \beta \\
\beta & \alpha & \beta & \alpha & \beta & \alpha & \beta & \alpha \\
\alpha & \beta & \alpha & \beta & \alpha & \beta & \alpha & \beta \\
\beta & \alpha & \beta & \alpha & \beta & \alpha & \beta & \alpha \\
\alpha & \beta & \alpha & \beta & \alpha & \beta & \alpha & \beta \\
\beta & \alpha & \beta & \alpha & \beta & \alpha & \beta & \alpha
\end{array}$$

The remaining forty eight $F(A_1^4, A_2^4)$ squares are constructed in the same manner from this corresponding single degree of freedom effect in the analysis of variance table. The forty nine $F(A_1^4, A_2^4)$ squares constructed in this manner are pairwise orthogonal. Hence each latin square in the OL(8,7) set decomposes into 7 orthogonal $F(A_1^4, A_2^4)$ squares and the entire OL(8,7) set decomposes into 49 orthogonal $F(A_1^4, A_2^4)$ squares.

In the preceding analysis of variance table, under Latin Square Number One Treatments, we see that the set of three effects, AD, BE, and ABDE is closed under multiplication and hence can be used to construct an $F_1(A_1^2, A_2^2, A_3^2, A_4^2)$ square. To construct this $F_1(A_1^2, A_2^2, A_3^2, A_4^2)$ square we let the symbols W, X, Y, and Z in the $F_1(A_1^2, A_2^2, A_3^2, A_4^2)$ be represented as follows:

Level of Effect	Combinations (see Illustration 1)
$(AD)_0, (BE)_0, (ABDE)_0$	I + II = W
$(AD)_0, (BE)_1, (ABDE)_1$	III + IV = X
$(AD)_1, (BE)_0, (ABDE)_1$	V + VI = Y
$(AD)_1, (BE)_1, (ABDE)_0$	VII + VIII = Z

We now take the 8 x 8 square of the 2^6 treatment combinations (Figure 1) and put our "treatments" W, X, Y, and Z in the appropriate cells. Or alternatively, we could take the previously constructed latin square number one and replace "treatments" I and II by "treatment" W, "treatments" III and IV by "treatment" X, "treatments" V and VI by Y, and "treatments" VII and VIII by Z. In either case we get the following $F_1(A_1^2, A_2^2, A_3^2, A_4^2)$ square:

W	Y	X	Z	W	Y	X	Z
Y	W	Z	X	Y	W	Z	X
X	Z	W	Y	X	Z	W	Y
Z	X	Y	W	Z	X	Y	W
W	Y	X	Z	W	Y	X	Z
Y	W	Z	X	Y	W	Z	X
X	Z	W	Y	X	Z	W	Y
Z	X	Y	W	Z	X	Y	W

Note that the set of seven effects corresponding to each latin square has such a subset of three effects that is closed under multiplication. Hence we see that each latin square in the OL(8,7) set decomposes into one $F(A_1^2, A_2^2, A_3^2, A_4^2)$ and four $F(A_1^4, A_2^4)$ squares. And so we can say that the entire OL(8,7) set decomposes into seven $F(A_1^2, A_2^2, A_3^2, A_4^2)$ squares and twenty eight $F(A_1^4, A_2^4)$ squares. This is a direct application of Theorem 3.3.

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